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Existence of positive solutions of elliptic equations with Hardy term

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Abstract. This paper is devoted to studying the existence of positive solutions of the problem:

$$\begin{cases} -\Delta u = \frac{u^p}{|x|^a} + h(x,u,\nabla u), & \text{in}\Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is an open bounded smooth domain with boundary $\partial\Omega$, and $1 < p < \frac{N-a}{N-2}$, $0 < a < 2$. Under suitable conditions of $h(x,u,\nabla u)$, we get *a priori* estimates for the positive solutions of problem ($*$). By making use of these estimates and topological degree theory, we further obtain some existence results for the positive solutions of problem ($*$) when $1 < p < \frac{N-a}{N-2}$.

Keywords: *a priori* estimates, Hardy term, positive solutions.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be an open bounded smooth domain with boundary $\partial\Omega$. We consider the following elliptic problem with Hardy term:

$$\begin{cases} -\Delta u = \frac{u^p}{|x|^a} + h(x,u,\nabla u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $0 < a < 2, 1 < p < \frac{N-a}{N-2}$. We mainly focus on the existence of solutions for problem (1.1). It is worth pointing out that problem (1.1) occurs in various branches of mathematical physics and biological models. Theoretically, when $a = 0$, there is no Hardy term in problem (1.1). As is known to all, that the processing without a Hardy term is much simpler than the processing with a Hardy term. When $h(x,u,\nabla u) = h(x,u)$, which means, no gradient terms appear in problem (1.1), in this case, problem (1.1) is reduced to the following problem:

$$\begin{cases} -\Delta u = u^p + h(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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problem of this type was raised as an important issue in the survey paper [14]. For the existence of solutions to problem of this type, it was studied by many authors with different methods and techniques: upper and lower solution method, mountain pass theorem, *a priori* estimates, fixed points theorem and so on. We recall the papers [2,7,13,15,24] and the references therein. In [7], Figueiredo, Gossez and Ubilla concerned with the existence of solutions based on weak upper and lower solution method. Besides, Ambrosetti and Rabinowitz proposed the mountain pass theorem in [2], and proved the existence of nontrivial solutions. In [13], the author concerned the existence and regularity of solutions based on *a priori* estimates and blow up method by imposing suitable conditions on the coefficients and $h(x,u)$. It is worth mentioning that when $h(x,u)=0$, problem (1.2) becomes:

$$\begin{cases} -\Delta u = u^p, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

There are a lot of work related to this subject. For the existence of solutions, we refer to the pioneering work of [5,9,10,22]. It is well known that the Sobolev exponent $2^* = \frac{2N}{N-2}$ serves as the dividing number for existence and non-existence of solutions to (1.3); please see [5] and [9]. It is pointed out that the proof in [5] is based on Pohozaev identity and moving planes method. While in [10], the proof is based on a scaling argument reminiscent to that used in the theory of Minimal Surfaces to get *a priori* bounds.

As for the problem (1.1) containing gradients term, variational methods can not directly be applied for the problem generally. Thus, some other methods are proposed, we refer to [1,8,19,25] and the references therein. Specifically, for the following problem:

$$\begin{cases} -\Delta u = h(x,u,\nabla u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The authors in [8] obtained the existence of positive solution through an iterative method based on mountain-pass techniques. In [1], the existence of solutions are obtained for this problem with convection term by using the Galerkin methods. It should be noted in particular that, the method Gidas and Spruck proposed in [10] is also applicable to the case with gradients term.

Recently, great attention has been focused on the study of the existence and non-existence solutions of the Hardy–Hénon equation:

$$-\Delta u = |x|^a u^p, \quad \text{in } \Omega. \tag{1.5}$$

Traditionally, the equation (1.5) is called Hardy (Hénon, or Lane–Emden) equation for $a < 0 (a > 0,\ a = 0)$. It is shown in [11] that for $a < -2, 1 < p < \frac{N+a}{N-2}$, equation (1.5) has no positive solutions in $\mathbb{R}^N$. Besides, in [21], Reichel and Zou proved that equation (1.5) do not admit any classical solutions in $\mathbb{R}^N$ if $1 < p < \frac{N+2+2a}{N-2}$ and $a > -2$. The non-existence results of Reichel and Zou was revisited by Phan and Souplet in [18], and a new proof of non-existence of bounded solutions in the case $N = 3$ is provided by using the technique introduced in [23]. For the Dirichlet boundary value problem of (1.5), Ni obtained the existence of multiplicity bounded positive solutions by using the upper and lower solution and approximation methods in [16]. Particularly, in [27], Zhu studied the following Hénon equation with perturbation terms in the unit ball B of $\mathbb{R}^N (N > 4)$:

$$\begin{cases} -\Delta u = |x|^\alpha |u|^{p-2} u + h(x), & \text{in } B, \\ u = 0, & \text{on } \partial B. \end{cases} \tag{1.6}$$

By applying the perturbation method in the unit sphere, the author obtained an infinite number of mutually different solutions to problem (1.6). The difficulty with this problem lies in the existence of the Hardy term, so we need to overcome the difficulties brought about by the Hardy term. It is worth pointing out that, in [27], the technique for handling $|x|^a$ is to impose special symmetry restrictions on u; the authors in [18] deal with the Hardy term $|x|^a$ by a change of variables and a doubling-rescaling argument. These methods provide us with good ideas for dealing with Hardy term.

In this paper, we focus on the existence of positive solutions for the problem (1.1) with Hardy term. Through the well-known Liouville-type theorem (see [10, 18]), a change of variable and doubling-rescaling argument (see [20]). We firstly get the decay estimates of the solutions, then we derive *a priori* bounds for positive solutions of problem (1.1). Motivated by the works above, we can show the existence of solutions combined with the topological degree theory under some assumptions.

Firstly, we propose the definition of weak solution.

Definition 1.1. We say that $u \in H_0^1(\Omega)$ is a weak solution of problem (1.1) if

$$\int_\Omega \nabla u \cdot \nabla \varphi dx = \int_\Omega \frac{u^p}{|x|^a} \varphi dx + \int_\Omega h(x, u, \nabla u)\varphi dx, \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Throughout this paper, we always denote by $\|.\|_q$ the norm of $L^q(\Omega)$ for any $q \geq 1$, which means $\|u\|_q = \|u\|_{L^q(\Omega)} = (\int_\Omega |u|^q dx)^{\frac{1}{q}}$, $1 \leq q < \infty$, and $\|u\|_\infty = \|u\|_{L^\infty(\Omega)} = \sup_\Omega |u|$, $q = \infty$.

Next, we introduce the assumptions required for this paper, to this end, we first introduce the following eigenvalue problem:

$$\begin{cases} -\Delta \varphi = \lambda \varphi, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

we denote by $\lambda_1(\Omega)$ the first eigenvalue of problem (1.1). Then we give the following hypotheses on $h(x, u, \nabla u)$:

(H_1) For $m > 0$, $h(x, m, \zeta)$ is Hölder continuous and $h(x, m, \zeta) \geq 0$.

(H_2) If $1 < p < \frac{N-a}{N-2}$, we assume that there exists a positive constant λ_0 such that $\lim_{m\to\infty} \frac{h(x,m,\zeta)}{m^p} = 0$, $\lim_{m\to 0} \frac{h(x,m,\zeta)}{m} = \lambda_0$, and $|h(x, m, \zeta)| \leq C(1 + m^p + \zeta^b)$ for $m > 0$, $1 < b < \frac{2p}{p+1} < p < \frac{N-a}{N-2}$, appropriate constant $C > 0$.

Remark 1.2. If $1 < p < \frac{N-a}{N-2}$, then $h(x, u, \nabla u) = \lambda_0 u + \frac{|u|^{b-1}u|\nabla u|^2}{1+|\nabla u|^2}$ satisfies (H_1) and (H_2).

Now, we are turning to state the main results.

Theorem 1.3. *Let $N \geq 3, 0 < a < 2$ and $1 < p < \frac{N+2}{N-2}$. There exists a constant $\bar{C} = \bar{C}(N, p, a) > 0$ such that the following hold:*

(1) Any nonnegative solution of problem (1.1) in $\Omega = \{x \in \mathbb{R}^N; 0 < |x| < \rho\} (\rho > 0)$ satisfies that:

$$|u(x)| \leq \bar{C}|x|^{-\frac{2-a}{p-1}} \quad \textit{and} \quad |\nabla u(x)| \leq \bar{C}|x|^{-\frac{p+1-a}{p-1}}, \qquad 0 < |x| < \frac{\rho}{2}.$$

(2) Any nonnegative solution of problem (1.1) in $\Omega = \{x \in \mathbb{R}^N; |x| > \rho\}$ $(\rho > 0)$ satisfies that:

$$|u(x)| \leq \bar{C}|x|^{-\frac{2-a}{p-1}} \quad \textit{and} \quad |\nabla u(x)| \leq \bar{C}|x|^{-\frac{p+1-a}{p-1}}, \qquad |x| > 2\rho.$$

The proof of Theorem 1.3 is based on a change of variable and a generalization of a doubling-rescaling arguments; see [18].

Theorem 1.4. *Assume that* $1 < p < \frac{N-a}{N-2}$, $0 < a < 2$, *and that* (H_1) *and* (H_2) *hold with* $\lambda_0 < \lambda_1(\Omega)$. *Then there exist two universal positive constants* $\widetilde{C}$ *and* $\widehat{C}$ *such that for any positive solution* $u \in C^2(\Omega\backslash\{0\}) \cap C(\overline{\Omega})$ *of problem* (1.1), *there holds* $\widetilde{C} \le \|u\|_{C(\Omega)} \le \widehat{C}$.

The proof of Theorem 1.4 is based on the well-known blow up technique introduced by Gidas and Spruck (see [10]) and adopted by Phan (see [18]).

Therefore, according to Theorem 1.4 and the Leray–Schauder degree theory, we can get the existence of positive solutions to problem (1.1).

Theorem 1.5. *Assume that* $1 < p < \frac{N-a}{N-2}$, $0 < a < 2$, $\frac{u^p}{|x|^a} + h(x,u,\nabla u) \in L^k$, *where* $k < \min\{\frac{N}{a}, \frac{N(p-1)}{(p+1-a)b}\}$, (H_1) *and* (H_2) *hold, then problem* (1.1) *has at least one solution.*

The rest of paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we concern the decay estimates of solutions. Section 4 is devoted to the proof of Theorem 1.4, in which we establish *a priori* estimates to problem (1.1) by blow up technique. In Section 5, we prove the existence of solutions for problem (1.1) by topology degree theory and give the proof of Theorem 1.5.

2 Preliminaries

In this section, we will give some lemmas which will be used to prove the main results.

Lemma 2.1. *Let* $u(x)$ *be a nonnegative* C^2 *solution of the following equation:*

$$-\Delta u = u^p, \qquad x \in \mathbb{R}^N,$$

where $N > 2, 1 < p < \frac{N+2}{N-2}$. *Then* $u(x) \equiv 0$.

Lemma 2.2. *Let* $\mathbb{R}^N_+$ *be the half space* $\{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}$. *Suppose that* $u(x)$ *is a nonnegative* $C^2(\mathbb{R}^N_+) \bigcap C^0(\{x \in \mathbb{R}^N : x_N \ge 0\})$ *solution of the following problem:*

$$\begin{cases} -\Delta u = u^p, & x \in \mathbb{R}^N_+, \\ u = 0, & x_N = 0, \end{cases}$$

where $1 < p < \frac{N+2}{N-2}$. *Then* $u(x) \equiv 0$.

Remark 2.3. Lemma 2.1 and Lemma 2.2 follow directly from [10, Theorem 1.2, Theorem 1.3].

Lemma 2.4 ([18])**.** *Let* $N \ge 2$, $a > -2$, $p > 1$. *If* $p < min\{p_s, p_s(a)\}$ *or* $p \le \frac{N+a}{N-2}$, $p_s = \frac{N+2}{N-2}$, $p_s(a) = \frac{N+2+2a}{N-2}$. *Then the following equation:*

$$-\Delta u = |x|^a u^p$$

has no positive solution in $\mathbb{R}^N$.

Lemma 2.5 (Hardy's inequality [4])**.** *Assume* $N \ge 3$ *and* $r > 0$. *Suppose that* $u \in H^1(B(0,r))$. *Then* $\frac{u}{|x|} \in L^2(B(0,r))$, *with the estimate*

$$\int_{B(0,r)} \frac{u^2}{|x|^2} dx \le \int_{B(0,r)} \left(|Du|^2 + \frac{u^2}{r^2}\right) dx.$$

Lemma 2.6 ([20])**.** *Let (X,d) be a complete metric space, $\emptyset \neq D \subset \Sigma \subset X$ with Σ closed. Furthermore, assume that $M : D \to (0,\infty)$ is bounded on compact subsets of D, and fix a real $K > 0$. If $y \in D$ is such that*

$$M(y)\,\mathrm{dist}(y,\Gamma) > 2k, \qquad \Gamma = \Sigma \backslash D,$$

then there exists $x \in D$ such that

$$M(x)\,\mathrm{dist}(x,\Gamma) > 2k, \qquad M(x) \geq M(y),$$

and

$$M(z) \leq 2M(x) \quad \text{for all} \quad z \in D \cap \overline{B\big(x,kM^{-1}(x)\big)}.$$

Lemma 2.7 (Leray–Schauder [3])**.** *Assume that X is a real Banach space, Ω is a bounded, open subset of X and $\Phi : [a,b] \times \overline{\Omega} \to X$ is given by $\Phi(\lambda,u) = u - T(\lambda,u)$ with T a compact map. Define*

$$T_\lambda(u) = T(\lambda,u), \quad u \in X,$$

$$\Phi_\lambda = I - T_\lambda, \quad \lambda \in [a,b],$$

$$\Sigma = \{(\lambda,u) \in [a,b] \times \overline{\Omega} : \Phi(\lambda,u) = 0\},$$

and note $\Sigma_\lambda = \{u \in \overline{\Omega} : (\lambda,u) \in \Sigma\}$. We also suppose that,

$$\Phi(\lambda,u) = u - T(\lambda,u) \neq 0, \qquad \forall (\lambda,u) \in [a,b] \times \partial\Omega.$$

If $\deg(\Phi_a,\Omega,0) \neq 0$, then we have,

(1) $\Phi(\lambda,u) = u - T(\lambda,u) = 0$ has a solution $u \in X$ in Ω for every $a \leq \lambda \leq b$.

(2) Furthermore, there exists a compact connected set $\mathcal{C} \subset \Sigma$ such that

$$\mathcal{C} \cap (\{a\} \times \Sigma_a) \neq \emptyset, \mathcal{C} \cap (\{b\} \times \Sigma_b) \neq \emptyset.$$

3 Decay estimates

In this section, we concern the decay estimates of solutions to the problem (1.1). We need the following lemma, which is an extensive of [20, Theorem 6.1] and [18, Lemma 2.1].

Lemma 3.1. *Let $N \geq 3, 1 < p < \frac{N+2}{N-2}, \alpha \in (0,1]$. Assume in addition that $c(x) \in C^\alpha(\overline{B_1})$ satisfies that,*

$$\|c(x)\|_{C^\alpha(\overline{B_1})} \leq C_1 \quad \text{and} \quad c(x) \geq C_2, \quad x \in \overline{B_1}, \tag{3.1}$$

for some $C_1, C_2 > 0$, where $B_1 = \{x \in \mathbb{R}^N; |x| < 1\}$. Then there exists a constant C, depending only on α, C_1, C_2, p, N such that, for any nonnegative classical solution u of

$$-\Delta u = \frac{u^p}{c(x)} + h(x,u,\nabla u), \quad x \in \overline{B_1}, \tag{3.2}$$

u satisfies that,

$$|u(x)|^{\frac{p-1}{2}} + |\nabla u(x)|^{\frac{p-1}{p+1}} \leq C\big(1 + \mathrm{dist}^{-1}(x,\partial B_1)\big), \qquad x \in B_1.$$

Proof. Arguing by contradiction. Denote $N_k = |u_k|^{\frac{p-1}{2}} + |\nabla u_k|^{\frac{p-1}{p+1}}$. We suppose that there exist a sequence of c_k, u_k, y_k verifying that (3.1), (3.2), and $N_k(y_k) > 2k\big(1 + \operatorname{dist}^{-1}(y_k, \partial B_1)\big) > 2k \operatorname{dist}^{-1}(y_k, \partial B_1)$. By Lemma 2.6, there exists x_k such that

$$N_k(x_k) \geq N_k(y_k), \qquad N_k(x_k) > 2k \operatorname{dist}^{-1}(x_k, \partial B_1).$$

and

$$N_k(z) \leq 2N_k(x_k), \quad \text{for all } z \text{ satisfying } |z - x_k| \leq kN_k^{-1}(x_k).$$

Consequently, we have that,

$$\lambda_k = N_k^{-1}(x_k) \to 0, \qquad k \to +\infty, \tag{3.3}$$

due to $N_k(x_k) \geq N_k(y_k) > 2k$. Next we let

$$v_k(y) = \lambda_k^{\frac{2}{p-1}} u_k(x_k + \lambda_k y), \qquad \widetilde{c_k}(y) = c_k(x_k + \lambda_k y),$$

noting that $|v_k(0)|^{\frac{p-1}{2}} + |\nabla v_k(0)|^{\frac{p-1}{p+1}} = 1$,

$$|v_k(y)|^{\frac{p-1}{2}} + |\nabla v_k(y)|^{\frac{p-1}{p+1}} \leq 2, \qquad |y| \leq k, \tag{3.4}$$

and

$$\begin{aligned} -\Delta v_k &= -\lambda_k^{\frac{2p}{p-1}} \Delta u_k(x_k + \lambda_k y) \\ &= \lambda_k^{\frac{2p}{p-1}} \left(\frac{u_k^p(x_k + \lambda_k y)}{c_k(x_k + \lambda_k y)} + h\big(x_k + \lambda_k y, u_k(x_k + \lambda_k y), \nabla u_k(x_k + \lambda_k y)\big) \right) \\ &= \frac{v_k^p}{c_k(x_k + \lambda_k y)} + \lambda_k^{\frac{2p}{p-1}} h(x_k + \lambda_k y, \lambda_k^{-\frac{2}{p-1}} v_k, \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k) \\ &= \frac{v_k^p}{\widetilde{c_k}(y)} + \lambda_k^{\frac{2p}{p-1}} h(x_k + \lambda_k y, \lambda_k^{-\frac{2}{p-1}} v_k, \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k). \end{aligned} \tag{3.5}$$

So we see that v_k satisfies the following equation:

$$-\Delta v_k = \frac{v_k^p}{\widetilde{c_k}(y)} + \lambda_k^{\frac{2p}{p-1}} h\Big(x_k + \lambda_k y, \lambda_k^{-\frac{2}{p-1}} v_k, \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k\Big) \tag{3.6}$$

where $|y| \leq k$. According to the condition (H_2) on $h(x, u, \nabla u)$, it implies that,

$$\lambda_k^{\frac{2p}{p-1}} h(x_k + \lambda_k y, \lambda_k^{-\frac{2}{p-1}} v_k, \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k) \leq C, \quad |y| \leq k,$$

for k large enough, we deduce that there exist a subsequence of v_k converges in $C^1_{loc}(\mathbb{R}^N)$ to a function $v(y) > 0$. Fix $y \in \mathbb{R}^N$ and denote $\mu_k = \lambda_k^{-\frac{2}{p-1}} v_k(y)$, $\xi_k = v_k^{-\frac{p+1}{2}} \nabla v_k(y)$, we may write that,

$$\lambda_k^{\frac{2p}{p-1}} h(x_k + \lambda_k y, \lambda_k^{-\frac{2}{p-1}} v_k, \lambda_k^{-\frac{p+1}{p-1}} \nabla v_k) = v_k^p \mu_k^{-p} h(x_k + \lambda_k y, \mu_k, \mu_k^{\frac{p+1}{2}} \xi_k).$$

Obviously, $\mu_k \to \infty$ as $k \to +\infty$ and ξ_k is bounded. Besides, if $\{x_k\}$ is bounded, condition (H_2) implies that,

$$v_k^p(y) \mu_k^{-p} h(x_k + \lambda_k y, \mu_k, \mu_k^{\frac{p+1}{2}} \xi_k) \to 0, \qquad k \to +\infty. \tag{3.7}$$

On the other hand, due to (3.1), we have $C_2 \le \widetilde{c_k} \le C_1$, and for each $R > 0$ and $k > k_0(R)$ large enough, the following holds:

$$|\widetilde{c_k}(y) - \widetilde{c_k}(z)| \le C_1 |\lambda_k (y - z)|^\alpha \le C_1 |y - z|^\alpha, \qquad |y|, |z| \le R. \tag{3.8}$$

Therefore, by the Arzelà–Ascoli theorem; see [26], there exists $\widetilde{c}$ in $C(\mathbb{R}^N)$, with $\widetilde{c} \ge C_2$ such that, after extracting a subsequence, $\widetilde{c_k} \to \widetilde{c}$ in $C_{loc}(\mathbb{R}^N)$. Now for each $R > 0$ and $1 < q < \infty$, by (3.4), (3.6) and interior elliptic L^q estimates, the sequence v_k is uniformly bounded in $W^{2,q}(B_R)$. Using standard embeddings and interior elliptic Schauder estimates, after extracting a subsequence, we may assume that $v_k \to v$ in $C^2_{loc}(\mathbb{R}^N)$. Moreover, (3.3) and (3.8) imply that $|\widetilde{c_k}(y) - \widetilde{c_k}(z)| \to 0$ as $k \to +\infty$, so that the function $\widetilde{c}$ is actually a constant $C > 0$. Therefore, we have that,

$$\frac{v_k^p}{\widetilde{c_k}(y)} \to C v^p, \qquad k \to +\infty. \tag{3.9}$$

According to (3.7) and (3.9), it follows that $v > 0$ is a classical solution of

$$-\Delta v = C v^p, \qquad y \in \mathbb{R}^N$$

and satisfying $|v(0)|^{\frac{p-1}{2}} + |\nabla v(0)|^{\frac{p-1}{p+1}} = 1$, this contradicts the Liouville-type theorem. □

By Lemma 3.1, we are ready to prove the decay estimates of solutions to problem (1.1) as follows.

Proof of Theorem 1.3. Assume either $\Omega = \{x \in \mathbb{R}^N; 0 < |x| < \rho\}$ and $0 < |x_0| < \frac{\rho}{2}$, or $\Omega = \{x \in \mathbb{R}^N; |x| > \rho\}$ and $|x_0| > 2\rho$. We denote $R_0 = \frac{1}{2}|x_0|$, and observe that, for all $y \in B_1$, $\frac{|x_0|}{2} < |x_0 + R_0 y| < \frac{3|x_0|}{2}$, so that $x_0 + R_0 y \in \Omega$ in either case. Let us thus define that,

$$U(y) = R_0^{\frac{2-a}{p-1}} u(x_0 + R_0 y).$$

Therefore,

$$\begin{aligned} -\Delta U(y) &= -R_0^{\frac{2p-a}{p-1}} \Delta u(x_0 + R_0 y) \\ &= R_0^{\frac{2p-a}{p-1}} \left(\frac{u^p(x_0 + R_0 y)}{|x_0 + R_0 y|^a} + h\big(x_0 + R_0 y, u(x_0 + R_0 y), \nabla u(x_0 + R_0 y)\big) \right) \\ &= \frac{U^p(y)}{|\frac{x_0}{R_0} + y|^a} + R_0^{\frac{2p-a}{p-1}} h\Big(x_0 + R_0 y, R_0^{-\frac{2-a}{p-1}} U(y), R_0^{-\frac{p+1-a}{p-1}} \nabla U(y)\Big) \\ &= \frac{U^p(y)}{c(y)} + R_0^{\frac{2p-a}{p-1}} h\Big(x_0 + R_0 y, R_0^{-\frac{2-a}{p-1}} U(y), R_0^{-\frac{p+1-a}{p-1}} \nabla U(y)\Big), \end{aligned} \tag{3.10}$$

where $c(y) = |\frac{x_0}{R_0} + y|^a$. Then U is a solution of

$$-\Delta U(y) = \frac{U^p(y)}{c(y)} + R_0^{\frac{2p-a}{p-1}} h\Big(x_0 + R_0 y, R_0^{-\frac{2-a}{p-1}} U(y), R_0^{-\frac{p+1-a}{p-1}} \nabla U(y)\Big), \qquad y \in B_1.$$

Notice that $|y + \frac{x_0}{R_0}| \in [1, 3]$ for all $y \in \overline{B_1}$ and $\|c(y)\|_{C^1(\overline{B_1})} \le C(a)$ according to Lemma 3.1, where $C(a)$ is a constant depending on a. Applying Lemma 3.1 again, we have that $|U(0)| + |\nabla U(0)| \le C$. Hence,

$$|u(x_0)| \le \bar{C} R_0^{-\frac{2-a}{p-1}}, \qquad |\nabla u(x_0)| \le \bar{C} R_0^{-\frac{p+1-a}{p-1}},$$

which yields the desired conclusion. □

4 *A priori* estimates

We will show *a priori* bounds for the positive solutions to problem (1.1) in this section. Owing to the well-known Liouville-type results (Lemma 2.1, Lemma 2.2 and Lemma 2.4), we can get *a priori* estimates. Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. To get the lower bound, we argue by contradiction. Assume that $\|u\|_{C(\Omega)} < \widetilde{C}$ holds for any $\widetilde{C} > 0$. Therefore, there exists a sequence solution $\{u_k\}$ of problem (1.1) such that

$$M_k = \sup_{x\in\Omega} u_k(x) \to 0, \quad \text{as } k \to +\infty.$$

Multiplying the first equation of problem (1.1) by u_k, and integrating the result over Ω, by Hölder inequality, Young's inequality with ε and Hardy's inequality, then we have that

$$\begin{aligned}
\int_\Omega |\nabla u_k|^2 dx &= \int_\Omega \frac{u_k^{p+1}}{|x|^a} + u_k h(x,u_k,\nabla u_k) dx \\
&= \int_\Omega \frac{u_k^a}{|x|^a}\cdot u_k^{p+1-a} dx + \int_\Omega u_k h(x,u_k,\nabla u_k) dx \\
&\le \Big(\int_\Omega \frac{u_k^2}{|x|^2}dx\Big)^{\frac{a}{2}} \Big(\int_\Omega u_k^{\frac{2(p+1-a)}{2-a}} dx\Big)^{\frac{2-a}{2}} + C\int_\Omega u_k(1+u_k^p+|\nabla u_k|^b)dx \\
&\le \varepsilon \int_\Omega \frac{u_k^2}{|x|^2}dx + C\Big(\int_\Omega u_k^{\frac{2(p+1-a)}{2-a}} dx + \int_\Omega u_k + u_k^{p+1} + u_k|\nabla u_k|^b\Big)dx \\
&\le C\Bigg[\varepsilon\int_\Omega (|\nabla u_k|^2 + u_k^2)dx + \int_\Omega u_k^{\frac{2(p+1-a)}{2-a}} dx + \int_\Omega u_k dx \\
&\quad + \int_\Omega u_k^{p+1}dx + \Big(\int_\Omega |\nabla u_k|^2 dx\Big)^{\frac{b}{2}}\Big(\int_\Omega u_k^{\frac{2}{2-b}}dx\Big)^{\frac{2-b}{2}}\Bigg] \\
&\le C\Bigg[\varepsilon\int_\Omega |\nabla u_k|^2 dx + \varepsilon\int_\Omega u_k^2 dx + \int_\Omega u_k^{\frac{2(p+1-a)}{2-a}} dx + \int_\Omega u_k dx \\
&\quad + \int_\Omega u_k^{p+1}dx + \varepsilon\int_\Omega |\nabla u_k|^2 dx + \int_\Omega u_k^{\frac{2}{2-b}}dx\Bigg].
\end{aligned} \tag{4.1}$$

Hence,

$$\begin{aligned}
&(1-\varepsilon C)\int_\Omega |\nabla u_k|^2 dx \\
&\le C\Big(\varepsilon\int_\Omega |u_k|^2 dx + \int_\Omega u_k^{\frac{2(p+1-a)}{2-a}} dx + \int_\Omega u_k dx + \int_\Omega u_k^{p+1}dx + \int_\Omega u_k^{\frac{2}{2-b}}dx\Big).
\end{aligned} \tag{4.2}$$

Let $\varepsilon \to 0$, then we have that

$$\begin{aligned}
\|\nabla u_k\|_2^2 &\le C\Big(\|u_k\|_{\frac{2(p+1-a)}{2-a}}^{\frac{2(p+1-a)}{2-a}} + \|u_k\|_1 + \|u_k\|_{p+1}^{p+1} + \|u_k\|_{\frac{2}{2-b}}^{\frac{2}{2-b}}\Big) \\
&\le C\|u_k\|_{C(\Omega)} \\
&= CM_k \to 0, \quad \text{as } k \to +\infty.
\end{aligned} \tag{4.3}$$

Further, let $u_k(x) = M_k U_k(x)$, obviously, $U_k(x)$ satisfies that,

$$\begin{cases} -\Delta U_k = \frac{M_k^{p-1} U_k^p}{|x|^a} + M_k^{-1} h(x, M_k U_k, M_k \nabla U_k), & \text{in } \Omega, \\ U_k > 0, & \text{in } \Omega, \\ U_k = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Owing to

$$\begin{aligned} \int_\Omega |\nabla U_k|^2 dx &= \int_\Omega \left(M_k^{p-1} \frac{U_k^{p+1}}{|x|^a} + U_k \frac{h(x, M_k U_k, M_k \nabla U_k)}{M_k} \right) dx \\ &= \int_\Omega M_k^{p-1} \frac{U_k^a}{|x|^a} U_k^{p+1-a} dx + \int_\Omega U_k \frac{h(x, M_k U_k, M_k \nabla U_k)}{M_k} dx \\ &\leq M_k^{p-1} \Big(\int_\Omega \frac{U_k^2}{|x|^2} dx \Big)^{\frac{a}{2}} \Big(\int_\Omega U_k^{\frac{2(p+1-a)}{2-a}} dx \Big)^{\frac{2-a}{2}} + \int_\Omega U_k^2 \frac{h(x, M_k U_k, M_k \nabla U_k)}{M_k U_k} dx \\ &\leq \left(M_k^{\frac{p-1}{a}} \int_\Omega \frac{U_k^2}{|x|^2} dx \right)^{\frac{a}{2}} \left(M_k^{\frac{p-1}{2-a}} \int_\Omega U_k^{\frac{2(p+1-a)}{2-a}} dx \right)^{\frac{2-a}{2}} + \int_\Omega U_k^2 \frac{h(x, M_k U_k, M_k \nabla U_k)}{M_k U_k} dx \\ &\leq \frac{a}{2} C M_k^{\frac{p-1}{a}} \int_\Omega (U_k^2 + |\nabla U_k|^2) dx + \frac{2-a}{2} M_k^{\frac{p-1}{2-a}} \int_\Omega U_k^{\frac{2(p+1-a)}{2-a}} dx \\ &\quad + \int_\Omega U_k^2 \frac{h(x, M_k U_k, M_k \nabla U_k)}{M_k U_k} dx. \end{aligned} \tag{4.5}$$

By (H_2) and the standard elliptic estimates; see [12], we can easily see that, the subsequence in U_k converges to a positive function v in $C^2(\Omega)$. Moreover, v satisfies

$$\begin{cases} -\Delta v = \lambda_0 v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

On the other hand, problem (4.6) has no positive solution due to $\lambda_0 < \lambda_1(\Omega)$. This reaches a contradiction. Consequently, there exists a universal constant $\widetilde{C} > 0$ such that for any positive solution u of problem (1.1), we have that,

$$\|u\|_{C(\Omega)} \geq \widetilde{C}. \tag{4.7}$$

To get the upper bound, we also proceed by contradiction. Assume that $\|u\|_{C(\Omega)} > \widehat{C}$ holds. Therefore, there exists a sequence of solutions u_k and a sequence of points $P_k \in \Omega$ such that

$$M_k = \sup_{x \in \Omega} u_k(x) = u_k(P_k) \to \infty, \quad \text{as } k \to +\infty.$$

We may assume that $P_k \to P \in \overline{\Omega}$ as $k \to +\infty$, and we divide the proof into the following two cases:

Case 1. $P \in \Omega \backslash \{0\}$ or $P \in \partial\Omega$. In this case, we rescale the solution as the following:

$$U_k(y) = \lambda_k^{\frac{2}{p-1}} u_k(P_k + \lambda_k y), \qquad \lambda_k = M_k^{-\frac{p-1}{2}}.$$

Therefore, we deduce that,

$$\begin{aligned}
-\Delta U_k(y) &= -\lambda_k^{\frac{2p}{p-1}}\Delta u_k(P_k+\lambda_k y)\\
&= \lambda_k^{\frac{2p}{p-1}}\left(\frac{u_k^p(P_k+\lambda_k y)}{|P_k+\lambda_k y|^a}+h\big(P_k+\lambda_k y,u_k(P_k+\lambda_k y),\nabla u_k(P_k+\lambda_k y)\big)\right)\\
&= \frac{U_k^p(y)}{|P_k+\lambda_k y|^a}+\lambda_k^{\frac{2p}{p-1}}h\big(P_k+\lambda_k y,\lambda_k^{-\frac{2}{p-1}}U_k(y),\lambda_k^{-\frac{p+1}{p-1}}\nabla U_k(y)\big).
\end{aligned} \tag{4.8}$$

Then U_k satisfies that,

$$\begin{cases}
-\Delta U_k=\frac{U_k^p}{|P_k+\lambda_k y|^a}+\lambda_k^{\frac{2p}{p-1}}h\Big(P_k+\lambda_k y,\lambda_k^{-\frac{2}{p-1}}U_k,\lambda_k^{-\frac{p+1}{p-1}}\nabla U_k\Big), & \text{in } \Omega_k,\\
U_k>0, & \text{in } \Omega_k,\\
U_k=0, & \text{on } \partial\Omega_k,
\end{cases} \tag{4.9}$$

where $\Omega_k=\lambda_k^{-1}(\Omega-\{P_k\})$. Notice that $\lambda_k=M_k^{-\frac{p-1}{2}}$, we can deduce that,

$$\begin{aligned}
&\left|\frac{U_k^p}{|P_k+\lambda_k y|^a}+\lambda_k^{\frac{2p}{p-1}}h(P_k+\lambda_k y,\lambda_k^{-\frac{2}{p-1}}U_k,\lambda_k^{-\frac{p+1}{p-1}}\nabla U_k)\right|\\
&\quad=\left|\frac{U_k^p}{|P_k+\lambda_k y|^a}+M_k^{-p}h(P_k+\lambda_k y,M_kU_k,M_k^{\frac{p+1}{2}}\nabla U_k)\right|\\
&\quad\le\left|\frac{U_k^p}{|P_k+\lambda_k y|^a}+C(M_k^{-p}+U_k^p+M_k^{\frac{(p+1)b}{2}-p}|\nabla U_k|^b)\right|\\
&\quad\le C,
\end{aligned} \tag{4.10}$$

and so we find that U_k is a solution of the equation:

$$-\Delta U_k=\frac{U_k^p}{|P_k+\lambda_k y|^a}+\lambda_k^{\frac{2p}{p-1}}h\Big(P_k+\lambda_k y,\lambda_k^{-\frac{2}{p-1}}U_k,\lambda_k^{-\frac{p+1}{p-1}}\nabla U_k\Big)$$

in a rescaled domain Ω_k. Since $U_k(0)=1, 0<U_k\le 1$, by elliptic estimates and standard embedding similar as that in [10], up to a subsequence, without loss of generality, still denoted by U_k, we can deduce that $\{U_k\}$ is convergent in $C_{loc}(\mathbb{R}^N)$. Hence, by the Arzelà–Ascoli theorem and standard diagonal argument, up to a subsequence, there exists a subsequence of $\{U_k\}$ and function $v\in C(\Omega)$, such that $U_k\to v$ uniformly on compact sets of Ω. In addition, v satisfies the equation $-\Delta v=lv^p$, where $1<p<\frac{N-a}{N-2}$, for some $l>0$ either in the whole space $\mathbb{R}^N$, or in a half-space with 0 boundary conditions. Clearly, this contradicts with the Lemma 2.1 and Lemma 2.2.

Case 2. $P=0$. In this case, we rescale the solution according to $U_k(y)=\lambda_k^{\frac{2-a}{p-1}}u_k(P_k+\lambda_k y)$, $\lambda_k=M_k^{-\frac{p-1}{2-a}}$. By a simple calculation, we infer that,

$$\begin{aligned}
-\Delta U_k(y) &= -\lambda_k^{\frac{2p-a}{p-1}}\Delta u_k(P_k+\lambda_k y)\\
&= \lambda_k^{\frac{2p-a}{p-1}}\left(\frac{u_k^p(P_k+\lambda_k y)}{|P_k+\lambda_k y|^a}+h\big(P_k+\lambda_k y,u_k(P_k+\lambda_k y),\nabla u_k(P_k+\lambda_k y)\big)\right)\\
&= \frac{U_k^p(y)}{|\frac{P_k}{\lambda_k}+y|^a}+\lambda_k^{\frac{2p-a}{p-1}}h\big(P_k+\lambda_k y,\lambda_k^{-\frac{2-a}{p-1}}U_k(y),\lambda_k^{-\frac{p+1-a}{p-1}}\nabla U_k(y)\big),
\end{aligned} \tag{4.11}$$

Then U_k satisfies that

$$\begin{cases} -\Delta U_k = \frac{U_k^p}{|\frac{P_k}{\lambda_k}+y|^a} + \lambda_k^{\frac{2p-a}{p-1}} h\Big(P_k + \lambda_k y, \lambda_k^{-\frac{2-a}{p-1}} U_k, \lambda_k^{-\frac{p+1-a}{p-1}} \nabla U_k\Big), & \text{in } \Omega_k, \\ U_k > 0, & \text{in } \Omega_k, \\ U_k = 0, & \text{on } \partial\Omega_k, \end{cases} \tag{4.12}$$

where $\Omega_k = \lambda_k^{-1}(\Omega - \{P_k\})$. Due to $\lambda_k = M_k^{-\frac{p-1}{2-a}}$, we can deduce that

$$\begin{aligned} &\left| \frac{U_k^p}{|\frac{P_k}{\lambda_k} + y|^a} + \lambda_k^{\frac{2p-a}{p-1}} h(P_k + \lambda_k y, \lambda_k^{-\frac{2-a}{p-1}} U_k, \lambda_k^{-\frac{p+1-a}{p-1}} \nabla U_k) \right| \\ &\quad = \left| \frac{U_k^p}{|\frac{P_k}{\lambda_k} + y|^a} + M_k^{-\frac{2p-a}{2-a}} h(P_k + \lambda_k y, M_k U_k, M_k^{\frac{p+1-a}{2-a}} \nabla U_k) \right| \\ &\quad \le \left| \frac{U_k^p}{|\frac{P_k}{\lambda_k} + y|^a} + C(M_k^{-\frac{2p-a}{2-a}} + M_k^{p-\frac{2p-a}{2-a}} U_k^p + M_k^{\frac{(p+1-a)b-(2p-a)}{2-a}} |\nabla U_k|^b) \right| \\ &\quad \le C, \end{aligned} \tag{4.13}$$

and thus we find that U_k is a solution of the following equation:

$$-\Delta U_k = \frac{U_k^p}{|\frac{P_k}{\lambda_k} + y|^a} + \lambda_k^{\frac{2p-a}{p-1}} h\left(P_k + \lambda_k y, \lambda_k^{-\frac{2-a}{p-1}} U_k, \lambda_k^{-\frac{p+1-a}{p-1}} \nabla U_k\right)$$

in a rescaled domain Ω_k containing $B(0, \rho\lambda_k^{-1})$ for some $\rho > 0$. Moreover, it follows from the estimate in Theorem 1.3 that the sequence $\frac{|P_k|}{\lambda_k} = |P_k| u_k^{\frac{p-1}{2-a}}(P_k)$ is bounded. We may thus assume that $\frac{P_k}{\lambda_k} \to x_0 \in \mathbb{R}^N$ as $k \to +\infty$. A similar limiting procedure as in Case 1 then produces a positive solution v of

$$-\Delta v = \frac{v^p}{|y + x_0|^a}, \qquad y \in \mathbb{R}^N, \tag{4.14}$$

where $0 < a < 2$, then by elliptic regularity, we obtain that u_k satisfy a local $W^{2,\widehat{q}}$ bound for $\frac{N}{2} < \widehat{q} < \frac{N}{|a|}$, so a local Hölder bound holds, and this is sufficient to pass the limit to obtain a solution of problem (4.14). After a space shift, this gives a contradiction with Lemma 2.4. Therefore, there exists a positive constant $\widehat{C}$ such that

$$\|u\|_{C(\Omega)} \le \widehat{C}. \tag{4.15}$$

(4.7) and (4.15) yield the desired conclusion of Theorem 1.4 and this completes the proof. □

5 Existence results

This section devotes to proving some existence results to problem (1.1). For the convenience of proving existence results, we consider the following problem with a parameter $t \in [0, 1]$,

$$\begin{cases} -\Delta u = \frac{u^p}{|x|^a} + h(x, u, \nabla u) + t(|u| + \lambda), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Fortunately, we have proved the boundedness of solutions firstly in Section 4, therefore, in this section, we only need to use the Leray–Schauder topological degree theory (see [3,6,17]) to prove the existence of solutions.

Proof of Theorem 1.5. We always assume that $h(x,u,\nabla u)$ satisfies (H_1) and (H_2). Let $X = C(\Omega)$, we denote that,

$$f(x,u) = \frac{u^p}{|x|^a} + h(x,u,\nabla u).$$

Given $u \in X$ and $t > 0$, let $\phi_t(u) = u$ be the unique solution of the problem (5.1). Then the solution to problem (1.1) is equivalent to a fixed point of the operator $\phi_0(u)$. Since $f \in L^k(\Omega)$ for $k < \min\{\frac{N}{a}, \frac{N(p-1)}{(p+1-a)b}\}$, we have $\phi_t(u) \in W^{2,r}(\Omega)$ for $r \in (\frac{N}{2}, \min\{\frac{N}{a}, \frac{N(p-1)}{(p+1-a)b}\})$. Therefore, $\phi_t : X \to X$ is compact. Observe that the right-hand sides in (5.1) are nonnegative for every $u \in X$, hence, ϕ_t has no fixed point beyond the nonnegative cone $K = \{u' \in X : u' > 0\}$ for any $t \geq 0$.

Let $\|u\|_X = \varepsilon$ for $\varepsilon > 0$ small. Assume $u = \phi_0(u)$, using L^p estimates, we have that,

$$\|u\|_\infty \leq C\|u\|_{2,r} \leq C\|f\|_r \leq C\|u\|_\infty^p,$$

where $\|\cdot\|_{2,r}$ denotes the norm in $W^{2,r}(\Omega)$. Furthermore, We can deduce that,

$$\|u\|_\infty \leq C\|u\|_\infty^p \leq C\varepsilon^{p-1}\|u\|_\infty.$$

This is a contradiction for ε sufficiently small due to the assumption $p > 1$. Hence $u \neq \phi_0(u)$ and the homotopy invariance of the topological degree implies

$$\deg(I - \phi_0, 0, B_\varepsilon) = \deg(I, 0, B_\varepsilon) = 1,$$

where I denotes the identity and $B_\varepsilon = \{u \in X : \|u\|_X < \varepsilon\}$.

Theorem 1.4 immediately implies $\phi_T(u) \neq u$ for T large and $u \in \overline{B_R} \bigcap K$, $\phi_t(u) \neq u$ for $t \in [0,T]$ and $u \in (\overline{B_R} \backslash B_R) \bigcap K$ (where $R > 0$ is lage enough), hence we have that,

$$\deg(I - \phi_0, 0, B_R) = \deg(I - \phi_T, 0, B_R) = 0.$$

Then we can obtain $\deg(I - \phi_0, 0, B_R \backslash \overline{B_\varepsilon}) = -1$, hence, there exist $u \in (B_R \backslash \overline{B_\varepsilon}) \bigcap K$ such that $\phi_0(u) = u$. Finally, the maximum principle implies the positivity of u. □

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Exponential decay for a Klein–Gordon–Schrödinger system with locally distributed damping

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Abstract. A coupled damped Klein–Gordon–Schrödinger equations are considered where Ω is a bounded domain of $\mathbb{R}^2$, with smooth boundary Γ and ω is a neighbourhood of $\partial\Omega$ satisfying the geometric control condition. The aim of the paper is to prove the existence, uniqueness and uniform decay for the solutions.

Keywords: Klein–Gordon–Schrödinger, localized damping, exponential stability, asymptotic behavior, existence and uniqueness.

2020 Mathematics Subject Classification: 35B40, 35B45, 35Q55.

1 Introduction

The aim of this paper is to study the following KGS system defined in Ω which is a bounded domain in $\mathbb{R}^2$

$$\begin{aligned} i\psi_t + \kappa\Delta\psi + i\alpha b(x)\psi &= \phi\psi\chi(\omega) \in \Omega\times(0,+\infty) \\ \phi_{tt} - \Delta\phi + \phi + \lambda(x)\phi_t &= -\operatorname{Re}\nabla\psi\chi(\omega) \in \Omega\times(0,+\infty) \qquad (1.1)\\ \psi = \phi &= 0, \quad \text{on } \Gamma\times(0,+\infty) \end{aligned}$$

with locally distributed damping and where Γ is a smooth boundary and ω is an open subset of Ω such that $\operatorname{meas}(\omega) > 0$ and satisfying the geometric control condition. Let $\alpha > 0$ and $\chi(\omega)$ to represent the characteristic function, that is $\chi = 1$ in ω and $\chi = 0$ in $\Omega\setminus\omega$. We also consider $b, \lambda \in L^\infty(\Omega)$ to be nonnegative functions such that

$$b(x) \geq b_0 > 0 \quad \text{a.e. in } \omega \quad \text{and} \quad \lambda(x) \geq \lambda_0 > 0 \quad \text{a.e. in } \omega,$$

in order for the nonlinearity ψ to exist where the damping terms

$$i\alpha b(x)\psi, \qquad \lambda(x)\phi_t$$

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are effective and reciprocally. If the damping is effective in the whole domain, i.e. $b(x) \geq b_0 > 0$ a.e. in Ω and $\lambda(x) \geq \lambda_0 > 0$ a.e. in Ω we can consider $\chi_\omega \equiv 1$ a.e. in Ω. The variable (complex) ψ stands for the dimensionless low frequency electron field, whereas (real) ϕ denotes the dimensionless low frequency density. This system describes the nonlinear interaction between high frequency electron waves and low frequency ion plasma waves in a homogeneous magnetic field, adapted to model the UHH (Upper Hybrid Heating) plasma heating scheme.

UHH is the dominant branch of the general Electron Cyclotron Resonance Heating (ECRH) scheme, which, for tokamaks and stellarators, constitutes a basic method of plasma build-up and heating. Moreover, ECRH is an attractive method to study transport mechanisms, since it allows for a very localised power deposition, thus influencing temperature and current profiles. The UHH scheme consists in injecting electromagnetic waves in the range $100-200GHz$, from the high field side towards the core of the device. Within this frequency range, the incident wave takes on the character of a longitudinal oscillation for the resonant electrons, which become highly energetic. With respect to the physical mechanism involved in the energy damping of the waves, UHH comprises of two stages:

1. Collisionless damping. The energy of the waves is transferred to the resonant electrons, through collisionless mechanisms, e.g. Landau damping. Subsequently, the electrons gain excessive kinetic energy, thus heated.

2. Collisional damping. The excessive electron energy is distributed over electrons and non-resonant ions, through Coulomb collisions, producing bulk heating of the plasma (equipartition).

Collisional damping is very crucial for the success of UHH. If collisions are infrequent, non-thermal distributions will occur, which may result in a reduction in the power delivered to the plasma. Therefore, it is important to determine the operational conditions for the device, under which UHH becomes effective, namely the collisions manage to distribute the excessive electron energy over the species at an exponential rate. The term $Re\nabla\psi$ is a consequence of the different low frequency coupling that was considered, i.e. the polarization drift instead of the ponderomotive force. The system focuses on the vital role of collisions by considering the non-homogeneous polarization drift for the low frequency coupling (see [12]).

By setting $\theta = \phi_t + \epsilon\phi$ where ϵ is a real positive constant to be specified later, the system (1.1) becomes

$$i\psi_t + \kappa\Delta\psi + i\alpha b(x)\psi = \phi\psi\chi(\omega), \tag{1.2}$$

$$\phi_t + \epsilon\phi = \theta, \tag{1.3}$$

$$\theta_t + (\lambda(x) - \epsilon)\theta - \Delta\phi + (1 - \epsilon(\lambda(x) - \epsilon))\phi = -\operatorname{Re}\nabla\psi\chi(\omega) \tag{1.4}$$

satisfying the following initial conditions

$$\psi(x,0) = \psi_0(x), \quad \phi(x,0) = \phi_0(x), \quad \theta(x,0) = \theta_0(x). \tag{1.5}$$

Therefore, one may set the energy equation of the problem by

$$E(t) := \frac{1}{2}\left\{ \|\psi\|^2_{L^2(\Omega)} + \kappa\|\nabla\psi\|^2_{L^2(\Omega)} + \int_\omega \phi|\psi|^2 + \|\phi\|^2_{L^2(\Omega)} + \|\nabla\phi\|^2_{L^2(\Omega)} + \|\phi_t\|^2_{L^2(\Omega)} \right\}. \tag{1.6}$$

Assumption 1.1. *We denote by ω the intersection of Ω with a neighborhood of $\partial\Omega$ in $\mathbb{R}^2$ and we will call it a neighborhood of the boundary of Ω. We assume that $b, \lambda \in L^\infty(\Omega)$ are nonnegative functions such that*

$$b(x) \geq b_0 > 0, \quad \text{a.e. in } \omega, \qquad \lambda(x) \geq \lambda_0 > 0, \quad \text{a.e. in } \omega.$$

In addition, if $b(x) \geq b_0 > 0$ a.e. in Ω then we can consider $\chi_\omega \equiv 1$ in Ω, and if $\lambda(x) \geq \lambda_0 > 0$ a.e. in Ω, then we can consider $\chi_\omega \equiv 1$ in Ω.

Definition 1.2 (Geometric control condition)**.** Let ω geometrically control Ω, i.e there exists $T_0 > 0$, such that every geodesic of Ω travelling with speed 1 and issued at $t = 0$, which enters the set ω in a time $t < T_0$. So, the couple (ω, T_0) satisfies the geometric control condition (GCC, in short) if every geodesic of Ω, traveling with speed 1 and issued at $t = 0$ enters the open set ω before the time T_0.

Assumption 1.3. *We assume that ω satisfies the geometric control condition. The standard example is when ω is a neighbourhood of $\overline{\Gamma(x_0)}$ where*

$$\Gamma(x_0) := \{x \in \Gamma; (x - x_0) \cdot \nu(x) > 0\}$$

and $\nu(x)$ is the unit outward normal at $x \in \Gamma$.

As a consequence of the previous assumption it follows that there exists a couple (ω, T_0), with $T_0 > 0$, such that the following observability inequalities occur:

$$\|\psi_0\|^2_{L^2(\Omega)} \leq \int_0^T \int_\omega |\psi(x,t)|^2 dxdt \tag{1.7}$$

for the following problem

$$\begin{cases} i\psi_t + \Delta\psi = 0 \in \Omega \times (0,T), \\ \psi = 0 \quad \text{on } \Gamma \times (0,T), \\ \psi(0) = \psi_0 \in L^2(\Omega) \end{cases} \tag{1.8}$$

and

$$\|\phi_1\|^2_{L^2(\Omega)} + \|\nabla\phi_0\|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |\phi_t(x,t)|^2 dxdt \tag{1.9}$$

with regards to the following problem

$$\begin{cases} \phi_{tt} - \Delta\phi = 0 \in \Omega \times (0,T), \\ \phi = 0 \quad \text{on } \Gamma \times (0,T), \\ \phi(0) = \phi_0 \in H_0^1(\Omega), \\ \phi_t(0) = \phi_1 \in L^2(\Omega) \end{cases} \tag{1.10}$$

for some positive constant $C = C(\omega, T_0)$ and for all $T > T_0$. The proof of (1.8) can be found in [13] and [18] while the proof of (1.10) is established in [3] and [15].

The aim of this work is to generalize the previous results of [21] by considering the damped structure $i\alpha b(x)\psi$ instead of $i\alpha\psi$ for the Schrödinger equation following the ideas of [1, 2]. Due to the right-hand side of the wave equation, i.e. $-\operatorname{Re}\nabla\psi\chi(\omega)$ the energy functional of the system depends upon the integral $\int \phi|\psi|^2$ which introduces a time that is required by the damping to smooth out the differences between the kinetic energies of the resonant electrons and non resonant ions. The presence of the damping terms in both equations of the system does not necessary guarantee that the energy $E(t)$ associated to the system is a non increasing function of the parameter t. Indeed in [12] where $b(x), \lambda(x)$ are effective in the whole of Ω and in [21] where $\lambda(x)$ is effective in ω the energy exponential rate depends upon the parameters of the system and t^*.

Our main task is to investigate the parametric energy decay for the system. Specifically, we seek necessary conditions, dependent on the parameters of the system b_0, λ_0, so that energy decay occurs at an exponential rate and therefore improve previous results by focusing on the ω. This ensures that, under specific plasma conditions, the energy of the coupled ion-electron wave is effectively dissipated to the plasma. In fact in Section 3 we will prove that the energy ia a non increasing function. For this purpose, we make use of the observability inequality for both, the wave and the Schrödinger equations. It is important to mention that the use of the observability inequality instead of the multiplier technique allows us to consider sharp regions ω satisfying the geometric control condition. Indeed, the inequalities given in (1.7) and (1.9) are proved by means of microlocal analysis and produce sharp regions when compared with the multiplier method. The main results of this paper are the following:

Theorem 1.4. *Let* $(\psi_0, \phi_0, \theta_0) \in \{H_0^1(\Omega) \cap H^2(\Omega)\}^2 \times H_0^1(\Omega)$ *and assuming that* $(\lambda_0 - \epsilon) > \frac{2}{3\alpha\kappa b_0}$, $(\lambda_0 - \epsilon), (1 - \epsilon(\lambda_0 - \epsilon)) > 0$ *hold then there exists a unique regular solution of* (1.2)–(1.4) *such that*

$$\begin{aligned} &\psi \in L^\infty(0, \infty; H_0^1(\Omega) \cap H^2(\Omega)), && \psi_t \in L^\infty(0, \infty; L^2(\Omega)), \\ &\phi \in L^\infty(0, \infty; H_0^1(\Omega) \cap H^2(\Omega)), && \phi_t \in L^\infty(0, \infty; H_0^1(\Omega)), \\ &\phi_{tt} \in L^\infty(0, \infty; L^2(\Omega)). \end{aligned}$$

Theorem 1.5. *Let* $(\psi_0, \phi_0, \theta_0) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ *and the assumptions of Theorem 1.4 hold, then there exists positive constant* C, ν, μ *such that the following decay rate holds*

$$E_\mu(t) \le Ce^{-\nu t} E(0), \qquad \forall\, t \ge 0$$

for every regular solution of the problem (1.1).

Let us recall the following known results. From Poincaré's inequality we have

$$\|u\|_{L^2(\Omega)} \le c \|\nabla u\|_{L^2(\Omega)}, \quad \text{for all } u \in H_0^1(\Omega),$$

and the Gagliardo–Nirenberg inequality for dimension $n = 2$

$$\|u\|_{L^4(\Omega)} \le c \|u\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}} \quad \text{for all } u \in H_0^1(\Omega). \tag{1.11}$$

Notation: Denote by $H^s(\Omega)$ both the standard real and complex Sobolev spaces on Ω. For *simplicity reasons* sometimes we use H^s, L^s for $H^s(\Omega)$, $L^s(\Omega)$, and $\|\cdot\|$, $(\cdot, \cdot)$ for the norm and the inner product of $L^2(\Omega)$ and $\|\cdot\|_\omega$, $(\cdot, \cdot)_\omega$ for the norm and the inner product of $L^2(\omega)$ respectively as well as $\int dx$ denotes the integration over the domain Ω. Finally, C is a general symbol for any positive constant.

2 Existence and uniqueness

In this section we derive a priori estimates for the solutions of the system (1.1). Let $\{\omega_\nu\}$ be a basis of $H_0^1(\Omega) \cap H^2(\Omega)$ formed by the real eigenfunctions of Δ such that the sequence

$\{\omega_\nu\}$ gives a Hilbert basis of L^2 (i.e. an orthonormal basis of L^2) and let V_m be a subset of $H_0^1(\Omega)\cap H^2(\Omega)$ generated by the first m vectors. Then, let $g_{im}\in\mathbb{C}$ and $h_{im}, k_{im}\in\mathbb{R}$ with

$$\psi_m(t)=\sum_{i=1}^m g_{im}(t)\omega_i, \qquad \phi_m(t)=\sum_{i=1}^m h_{im}(t)\omega_i, \qquad \theta_m(t)=\sum_{i=1}^m k_{im}(t)\omega_i$$

such that $\{(\psi_m(t),\phi_m(t),\theta_m(t))\}$ is the solution to the following Cauchy problem:

$$\begin{cases} i(\psi_{t,m},u)+\kappa(\Delta\psi_m,u)+i\alpha(b(x)\psi_m,u)=(\phi_m\psi_m\chi(\omega),u), \ \forall\ u\in V_m, \\ (\phi_{t,m},z)=(\theta_m,z)-\epsilon(\phi_m,z), \ \forall\ z\in V_m, \\ (\theta_{t,m},v)+((\lambda(x)-\epsilon)\theta_m,v)-(\Delta\phi_m,v)+((1-\epsilon(\lambda(x)-\epsilon))\phi_m,v) \\ \qquad =-\operatorname{Re}(\nabla\psi_m\chi(\omega),v)\ \forall\ v\in V_m, \\ \psi_m(0)=\psi_{0m}\to\psi_0, \quad \phi_m(0)=\phi_{0m}\to\phi_0\in H_0^1(\Omega)\cap H^2(\Omega), \\ \theta(0)=\theta_{0m}\to\theta_0\in H_0^1(\Omega). \end{cases} \tag{2.1}$$

Let $Y=(\psi_m,\phi_m,\theta_m)$ then (2.1) also reads

$$\begin{cases} (\psi_{t,m},u)=i\kappa(\Delta\psi_m,u)+\alpha(b(x)\psi_m,u)-i(\phi_m\psi_m\chi(\omega),u), \ \forall\ u\in V_m, \\ (\phi_{t,m},z)=(\theta_m,z)-\epsilon(\phi_m,z), \forall\ z\in V_m, \\ (\theta_{t,m},v)=-((\lambda(x)-\epsilon)\theta_m,v)+(\Delta\phi_m,v)-((1-\epsilon(\lambda(x)-\epsilon))\phi_m,v) \\ \qquad -\operatorname{Re}(\nabla\psi_m\chi(\omega),v)\ \forall\ v\in V_m, \\ \psi_m(0)=\psi_{0m}\to\psi_0, \quad \phi_m(0)=\phi_{0m}\to\phi_0\in H_0^1(\Omega)\cap H^2(\Omega), \\ \theta(0)=\theta_{0m}\to\theta_0\in H_0^1(\Omega). \end{cases} \tag{2.2}$$

Then the considered matrix is the identity and therefore one may write $Y_t=F(Y)$ with smooth F. Hence the Cauchy–Lipschitz theorem applies straightforward. Since, the approximate system (2.1) is a finite system of ordinary differential equations which has a solution in $[0,t_m[$ the extension of the solution to the whole interval $[0,T]$, for all $T>0$, is a consequence of the first estimate we are going to obtain. Let us consider $u=\overline{\psi_m}$ in the first equation of (2.1). Then by taking the imaginary part we have

$$\frac{1}{2}\frac{d}{dt}\|\psi_m\|^2+\alpha\int b(x)|\psi_m|^2=0 \tag{2.3}$$

and because

$$\int b(x)|\psi_m|^2\ge\int_\omega b(x)|\psi_m|^2\ge b_0\int_\omega|\psi_m|^2 \tag{2.4}$$

almost everywhere in ω we have

$$\frac{1}{2}\frac{d}{dt}\|\psi_m\|^2+\alpha b_0\|\psi_m\|_\omega^2\le 0. \tag{2.5}$$

Finally, multiplying by 2 and integrating over $(0,t)$ for $t\in[0,t_m)$ concludes in

$$\|\psi_m\|^2+2\alpha b_0\int_0^t\|\psi_m(s)\|_\omega^2 ds\le\|\psi_{m0}\|^2. \tag{2.6}$$

Then, since $\psi_{m0}\to\psi_0$ in $H_0^1(\Omega)\cap H^2(\Omega)$ we have

$$(\psi_m)\text{ is bounded in }L^\infty(0,\infty;L^2(\Omega)) \tag{2.7}$$

and for $C_1 = C(\|\psi_0\|) > 0$ we also have

$$\int_0^\infty \|\psi_m(s)\|_\omega^2 ds \leq C_1 = C(\|\psi_0\|). \tag{2.8}$$

Next, taking $u = -\overline{\psi_{t,m}}$ in the first equation of (2.1) and considering the real part produces

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\psi_m\|^2 + \alpha Im \int b(x)\psi_m\overline{\psi_{t,m}} = -\operatorname{Re}\int_\omega \phi_m\psi_m\overline{\psi_{t,m}} \tag{2.9}$$

and similarly with (2.4) we have

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\psi_m\|^2 + \alpha b_0 Im \int_\omega \psi_m\overline{\psi_{t,m}} \leq -\operatorname{Re}\int_\omega \phi_m\psi_m\overline{\psi_{t,m}}. \tag{2.10}$$

Now, substituting $u = \alpha b_0\psi_m$ in the first equation of (2.1), integrating over ω and taking the real part we have

$$\alpha b_0 \operatorname{Im}\int_\omega \psi_m\overline{\psi_{t,m}} = \alpha\kappa b_0\|\nabla\psi_m\|_\omega^2 + \alpha b_0\int_\omega \phi_m|\psi_m|^2$$

and substituting the expression into (2.10) we obtain

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\psi_m\|^2 + \alpha\kappa b_0\|\nabla\psi_m 0_\omega^2 + \alpha b_0\int_\omega \phi_m|\psi_m|^2 \leq -\operatorname{Re}\int_\omega \phi_m\psi_m\overline{\psi_{t,m}}. \tag{2.11}$$

Therefore, by taking into consideration that

$$\frac{d}{dt}\int_\omega \phi_m|\psi_m|^2 = \int_\omega \phi_{t,m}|\psi_m|^2 + 2\int_\omega \phi_m\psi_m\overline{\psi}_{t,m}$$

equation (2.11) becomes

$$\frac{1}{2}\frac{d}{dt}\{\kappa\|\nabla\psi_m\|^2 + \int_\omega \phi_m|\psi_m|^2\} + \alpha\kappa b_0\|\nabla\psi_m\|_\omega^2 + \alpha b_0\int_\omega \phi_m|\psi_m|^2 \leq \frac{1}{2}\operatorname{Re}\int_\omega \phi_{t,m}|\psi_m|^2. \tag{2.12}$$

Continuing with the second equation of the system (2.1), let $v = \theta_m$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{\|\theta_m\|^2 + \|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2\} + (\lambda_0-\epsilon)\|\theta_m\|_\omega^2 + \epsilon\|\nabla\phi_m\|^2 \\ + \epsilon(1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2 \leq -\operatorname{Re}\int_\omega \nabla\psi_m\theta_m.\end{aligned} \tag{2.13}$$

Then, adding equations (2.12) and (2.13) produces

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\kappa\|\nabla\psi_m\|^2 + \int_\omega \phi_m|\psi_m|^2 + \|\theta_m\|^2 + \|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2\} \\ &\quad + \alpha\kappa b_0\|\nabla\psi_m\|_\omega^2 + \alpha b_0\int_\omega \phi_m|\psi_m|^2 + (\lambda_0-\epsilon)\|\theta_m\|_\omega^2 + \epsilon\|\nabla\phi_m\|^2 \\ &\quad + \epsilon(1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2 \leq -\operatorname{Re}\int_\omega \nabla\psi_m\theta_m + \frac{1}{2}\operatorname{Re}\int_\omega \phi_{t,m}|\psi_m|^2,\end{aligned} \tag{2.14}$$

where

$$\frac{1}{2}\operatorname{Re}\int_\omega \phi_{t,m}|\psi_m|^2 = \frac{1}{2}\operatorname{Re}\int_\omega \theta_m|\psi_m|^2 - \frac{\epsilon}{2}\operatorname{Re}\int_\omega \phi_m|\psi_m|^2$$

and with the use of $\|u\|_4 \le c\|u\|^{1/2}\ \|\nabla u\|^{1/2}$ we have

$$\left|\int_\omega \theta_m \nabla\psi_m\right| \le \frac{\alpha\kappa b_0}{2}\|\nabla\psi_m\|_\omega^2 + \frac{1}{2\alpha\kappa b_0}\|\theta_m\|_\omega^2$$

$$\left|\frac{1}{2}\int_\omega \theta_m|\psi_m|^2\right| \le \|\theta_m\|_\omega\ \|\psi_m\|_{4,\omega}^2 \le \frac{(\lambda_0-\epsilon)}{4}\|\theta_m\|_\omega^2 + \frac{\alpha\kappa b_0}{4}\|\nabla\psi_m\|_\omega^2 + C.$$

Therefore, equation (2.14) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\kappa\|\nabla\psi_m\|^2 + \int_\omega \phi_m|\psi_m|^2 + \|\theta_m\|^2 + \|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2\right\}\\ &\quad + \frac{3\alpha\kappa b_0}{4}\|\nabla\psi_m\|_\omega^2 + (\alpha b_0+\epsilon)\int_\omega \phi_m|\psi_m|^2 + \left(\frac{3(\lambda_0-\epsilon)}{4} - \frac{1}{2\alpha\kappa b_0}\right)\|\theta_m\|_\omega^2\\ &\quad + \epsilon\|\nabla\phi_m\|^2 + \epsilon(1+\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2 \le C\end{aligned} \tag{2.15}$$

for $\frac{3(\lambda_0-\epsilon)}{4} - \frac{1}{2\alpha\kappa b_0} > 0$. Set $\beta_0 = \min\{\frac{3\alpha\kappa b_0}{4}, (\alpha b_0+\epsilon), (\frac{3(\lambda_0-\epsilon)}{4} - \frac{1}{2\alpha\kappa b_0}), \epsilon, (1-\epsilon(\lambda_0-\epsilon))\}$, with $\beta_0 > 0$ and

$$H_0(t) = \kappa\|\nabla\psi_m\|^2 + \int_\omega \phi_m|\psi_m|^2 + \|\theta_m\|^2 + \|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2.$$

Hence we have

$$\frac{d}{dt}H_0(t) + \beta_0 H_0(t) \le C. \tag{2.16}$$

Using Gronwall's Lemma we obtain

$$H_0(t) \le \frac{C}{\beta_0}(1-e^{-\beta_0 t}) + H_0(t)e^{-\beta_0 t}$$

and

$$H_0(t) \le H_0(t)e^{-\beta_0 t} + \frac{C}{\beta_0}.$$

Finally, using (1.11) we estimate the following integral

$$\int_\omega \phi_m|\psi_m|^2 \le \frac{\kappa}{2}\|\nabla\psi_m\|^2 + \frac{1}{2}\|\nabla\phi_m\|^2 + C$$

then

$$H_0(t) \ge \frac{\kappa}{2}\|\nabla\psi_m\|^2 + \|\theta_m\|^2 + \frac{1}{2}\|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2 - C$$

and finally gives

$$\frac{\kappa}{2}\|\nabla\psi_m\|^2 + \|\theta_m\|^2 + \frac{1}{2}\|\nabla\phi_m\|^2 + (1-\epsilon(\lambda_0-\epsilon))\|\phi_m\|_\omega^2 \le C.$$

Therefore, we have

$$\begin{aligned}&(\psi_m) \text{ is bounded in } L^\infty(0,\infty;H_0^1(\Omega)),\\ &(\theta_m) \text{ is bounded in } L^\infty(0,\infty;L^2(\Omega)),\\ &(\phi_m) \text{ is bounded in } L^\infty(0,\infty;H_0^1(\Omega)).\end{aligned} \tag{2.17}$$

Moving to the next estimate we take the time derivative of the first equation of (2.1) and by choosing $u = \overline{\psi_{t,m}}$ we obtain

$$i(\psi_{tt,m}, \overline{\psi_{t,m}}) + \kappa(\Delta\psi_{t,m}, \overline{\psi_{t,m}}) + i\alpha(b(x)\psi_{t,m}, \overline{\psi_{t,m}}) = (\phi_{t,m}\psi_m\chi(\omega), \overline{\psi_{t,m}}) + (\phi_m\psi_{t,m}\chi(\omega), \overline{\psi_{t,m}}).$$

Taking into consideration the imaginary part we have

$$\frac{1}{2}\frac{d}{dt}\|\psi_{t,m}\|^2 + \alpha\int b(x)|\psi_{t,m}|^2 \leq \int \phi_{t,m}\psi_m\overline{\psi_{t,m}}$$

where since $\|\psi_m\|_\infty \leq c\|\Delta\psi_m\|^{1/2}\|\psi_m\|^{1/2}$ we obtain

$$\left|\int_\omega \phi_{t,m}\psi_m\overline{\psi_{t,m}}\right| \leq \|\phi_{t,m}\|\ \|\psi_m\|_\infty\|\psi_{t,m}\|_\omega \leq \frac{\alpha b_0}{2}\|\psi_{t,m}\|^2_\omega + \frac{\epsilon\kappa}{4}\|\Delta\psi_m\|^2 + C(\|\phi_{t,m}\|, \|\psi_m\|).$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\|\psi_{t,m}\|^2 + \frac{\alpha b_0}{2}\|\psi_{t,m}\|^2_\omega \leq \frac{\epsilon\kappa}{4}\|\Delta\psi_m\|^2 + C(\|\phi_{t,m}\|, \|\psi_m\|). \tag{2.18}$$

Moving to the next energy estimate by choosing $u = \overline{\Delta\psi_{t,m}} + \epsilon\overline{\Delta\psi_m}$ for the first equation of (2.1) and taking the real part we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\kappa\|\Delta\psi_m\|^2 + 2\alpha\,\mathrm{Im}\int b(x)\psi_m\overline{\Delta\psi_m} - 2\,\mathrm{Re}\int_\omega \phi_m\psi_m\overline{\Delta\psi_m}\right\} + \kappa\epsilon\|\Delta\psi_m\|^2 \\
&+ 2\alpha\epsilon\,\mathrm{Im}\int b(x)\psi_m\overline{\Delta\psi_m} - 2\epsilon\,\mathrm{Re}\int_\omega \phi_m\psi_m\overline{\Delta\psi_m} = \alpha\,\mathrm{Im}\int b(x)\psi_{t,m}\overline{\Delta\psi_m} \\
&- \mathrm{Re}\int_\omega \phi_{t,m}\psi_m\overline{\Delta\psi_m} - \mathrm{Re}\int_\omega \phi_m\psi_{t,m}\overline{\Delta\psi_m} + \alpha\epsilon\,\mathrm{Im}\int b(x)\psi_m\overline{\Delta\psi_m} - \epsilon\,\mathrm{Re}\int \phi_m\psi_m\overline{\Delta\psi_m}.
\end{aligned} \tag{2.19}$$

Next, choosing $v = -\Delta\theta$ in the second equation of (2.1) produces

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|\Delta\phi_m\|^2 + (1+\epsilon^2)\|\nabla\phi_m\|^2 + \|\nabla\theta_m\|^2 - 2\int \lambda(x)\phi_{t,m}\Delta\phi_m\right\} \\
&\quad + \epsilon\left\{\|\Delta\phi\|^2 + (1+\epsilon^2)\|\nabla\phi\|^2 + \|\nabla\theta_m\|^2 - 2\int \lambda(x)\phi_{t,m}\Delta\phi_m\right\} \\
&\leq -\mathrm{Re}\int_\omega \Delta\psi_m\nabla\theta_m - \epsilon\int \lambda(x)\phi_{t,m}\Delta\phi_m.
\end{aligned} \tag{2.20}$$

Adding (2.18) with (2.19) and (2.20) produces

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}H_1(t) + \epsilon H_1 = {}& \alpha\,\mathrm{Im}\int b(x)\psi_{t,m}\overline{\Delta\psi_m} \\
& - \mathrm{Re}\int_\omega \phi_{t,m}\psi_m\overline{\Delta\psi_m} - \mathrm{Re}\int_\omega \phi_m\psi_{t,m}\overline{\Delta\psi_m} + \alpha\epsilon\,\mathrm{Im}\int b(x)\psi_m\overline{\Delta\psi_m} \\
& - \epsilon\,\mathrm{Re}\int \phi_m\psi_m\overline{\Delta\psi_m} - \mathrm{Re}\int_\omega \Delta\psi_m\nabla\theta_m - \epsilon\int \lambda(x)\phi_{t,m}\Delta\phi_m \\
& + \left(\epsilon - \frac{\alpha b_0}{2}\right)\|\psi_{t,m}\|^2 + C\|\nabla\phi_m\|^2\|\nabla\psi_m\|^2
\end{aligned}$$

where

$$\begin{aligned}
H_1(t) = {}& \|\psi_{t,m}\|^2 + \kappa\|\Delta\psi_m\|^2 + 2\alpha\,\mathrm{Im}\int b(x)\psi_m\overline{\Delta\psi_m} - 2\,\mathrm{Re}\int_\omega \phi_m\psi_m\overline{\Delta\psi_m} + \|\Delta\phi_m\|^2 \\
& + (1+\epsilon^2)\|\nabla\phi_m\|^2 + \|\nabla\theta_m\|^2 - 2\int \lambda(x)\phi_{t,m}\Delta\phi_m.
\end{aligned}$$

Set

$$
\begin{aligned}
F_1(t) = {} & \alpha \operatorname{Im} \int b(x)\psi_{t,m}\overline{\Delta\psi_m} - \operatorname{Re}\int_\omega \phi_{t,m}\psi_m\overline{\Delta\psi_m} - \operatorname{Re}\int_\omega \phi_m\psi_{t,m}\overline{\Delta\psi_m} \\
& + \alpha\epsilon \operatorname{Im}\int b(x)\psi_m\overline{\Delta\psi_m} - \epsilon\operatorname{Re}\int \phi_m\psi_m\overline{\Delta\psi_m} - \operatorname{Re}\int_\omega \Delta\psi_m\nabla\theta_m \\
& - \epsilon\int \lambda(x)\phi_{t,m}\Delta\phi_m + \frac{\epsilon\kappa}{4}\|\Delta\psi_m\|^2 + \left(\epsilon - \frac{\alpha b_0}{2}\right)\|\psi_{t,m}\|^2 \\
& + C(c_0, R, \epsilon, \kappa, \alpha, b_0, \|\theta_m\|, |\nabla\phi\|).
\end{aligned} \tag{2.21}
$$

Evaluating the terms in H_1 and F_1 we have

$$
\begin{aligned}
\left|\int_\omega \phi_{t,m}\psi_m\overline{\Delta\psi_m}\right| &\le \|\psi_m\|_\infty\|\phi_{t,m}\|\ \|\Delta\psi_m\| \le \frac{\kappa\epsilon}{8}\|\Delta\psi_m\|^2 + C(\kappa,\epsilon,\|\psi_m\|,\|\phi_{t,m}\|), \\
\left|\int_\omega \Delta\psi_m\nabla\theta_m\right| &\le \frac{\kappa\epsilon}{8}\|\Delta\psi_m\|^2 + \frac{2}{\kappa\epsilon}\|\nabla\theta_m\|^2, \\
\left|\int b(x)\psi_m\overline{\Delta\psi_m}\right| &\le \|b(x)\|_\infty\|\psi_m\|\ \|\Delta\psi_m\| \le \epsilon_1\|\Delta\psi_m\|^2 + C(\epsilon_1)(\|\psi_m\|,\|b(x)\|_\infty), \\
\left|\int b(x)\psi_{t,m}\overline{\Delta\psi_m}\right| &\le \|b(x)\|_\infty\|\psi_{t,m}\|\ \|\Delta\psi_m\| \le \frac{\kappa\epsilon}{8}\|\Delta\psi_m\|^2 + C(\kappa,\epsilon,\|b(x)\|_\infty)\|\psi_{t,m}\|^2 \\
\left|\int \phi_m\psi_m\overline{\Delta\psi_m}\right| &\le \|\phi_m\|_4\|\psi_m\|_4\|\Delta\psi_m\| \le \epsilon_2\|\Delta\psi_m\|^2 + C(\epsilon_2)(\|\psi_m\|,\|\phi_m\|,\|\nabla\psi_m\|,\|\nabla\phi_m\|), \\
\left|\int \phi_m\psi_{t,m}\overline{\Delta\psi_m}\right| &\le \|\phi_m\|_\infty\|\psi_{t,m}\|\ \|\Delta\psi_m\| \\
&\le \frac{\kappa\epsilon}{8}\|\Delta\psi_m\|^2 + \frac{\epsilon}{2}\|\Delta\phi_m\|^2 + \frac{\alpha b_0}{4}\|\psi_{t,m}\|^2 + C(\kappa,\epsilon,\alpha,b_0,c,\|\phi_m\|), \\
\left|\int \lambda(x)\phi_t\Delta\phi_m\right| &\le \|\lambda(x)\|_\infty\|\phi_{t,m}\|\ \|\Delta\phi_m\| \le \epsilon_3\|\Delta\phi_m\|^2 + C(\epsilon_3)(\|\lambda(x)\|_\infty,\|\phi_{t,m}\|).
\end{aligned}
$$

Therefore there exists a constant $\beta_1 > 0$ such that

$$
\beta_1 H_1(t) \le F_1 + C(\kappa,\epsilon,\alpha,b_0,\epsilon_1,\epsilon_2,\epsilon_3,\|\lambda(x)\|_\infty,\|b(x)\|_\infty,\|\phi_m\|,\|\psi_m\|,\|\nabla\psi_m\|,\|\nabla\phi_m\|,\|\phi_{t,m}\|)
$$

and

$$
\frac{d}{dt}H_1(t) + \beta_1 H_1(t) \le C. \tag{2.22}
$$

Employing Gronwall's Lemma we finally obtain

$$
\|\psi_{t,m}\|^2 + \|\Delta\psi_m\|^2 + \|\Delta\phi_m\|^2 + \|\nabla\theta_m\|^2 \le C. \tag{2.23}
$$

Hence,

$$
\begin{aligned}
&(\psi_m) \text{ is bounded in } L^\infty(0,\infty; H_0^1(\Omega)\cap H_0^2(\Omega)), \\
&(\theta_m) \text{ is bounded in } L^\infty(0,\infty; H_0^1(\Omega)), \\
&(\phi_m) \text{ is bounded in } L^\infty(0,\infty; H_0^1(\Omega)\cap H_0^2(\Omega)) \\
&(\psi_{t,m}) \text{ is bounded in } L^\infty(0,\infty; L^2(\Omega)).
\end{aligned} \tag{2.24}
$$

Therefore we may extract subsequences $\{\psi_\nu\} \subset \{\psi_m\}$, $\{\phi_\nu\} \subset \{\phi_m\}$ and $\{\theta_\nu\} \subset \{\theta_m\}$ such that

$$\begin{aligned} \psi_\nu &\rightharpoonup \psi \quad \text{for the weak star topology of } L^\infty(0,\infty; H_0^1(\Omega)\cap H_0^2(\Omega)), \\ \theta_\nu &\rightharpoonup \theta \quad \text{for the weak star topology of } L^\infty(0,\infty; H_0^1(\Omega)), \\ \phi_\nu &\rightharpoonup \phi \quad \text{for the weak star topology of } L^\infty(0,\infty; H_0^1(\Omega)\cap H_0^2(\Omega)) \\ \psi_{t,\nu} &\rightharpoonup \psi_t \quad \text{for the weak star topology of } L^\infty(0,\infty; L^2(\Omega)). \end{aligned} \tag{2.25}$$

These convergencies are sufficient to pass to the limit (on a standard manner) in (2.1) which results in

$$\begin{aligned} i\psi_t + \kappa\Delta\psi + i\alpha b(x)\psi &= \phi\psi\chi(\omega) && \text{in } L^\infty(0,\infty;L^2(\Omega)), \\ \phi_{tt} - \Delta\phi + \phi + \lambda(x)\phi_t &= -\operatorname{Re}\nabla\psi\chi(\omega) && \text{in } L^\infty(0,\infty;L^2(\Omega)). \end{aligned} \tag{2.26}$$

From [22, Lemma 4.1, Chapter II] we may derive that

$$\phi \in C(0,\infty; H_0^1(\Omega)\cap H_0^2(\Omega)) \quad \text{and} \quad \phi_t \in C(0,\infty; L^2(\Omega))$$

and since $\psi_t = \frac{1}{i}(-\kappa\Delta\psi - i\alpha b(x)\psi + \phi\psi\chi(\omega)) \in L^\infty(0,\infty;L^2(\Omega)$ using results in [16] we then obtain that

$$\psi \in C(0,\infty; H_0^1(\Omega)\cap H_0^2(\Omega)).$$

Let (ψ_1,ϕ_1) and (ψ_2,ϕ_2) be two solutions of the problem. Then by setting $z = \psi_1 - \psi_2$ and $w = \phi_1 - \phi_2$ the uniqueness of the solutions follows using the same above analysis.

This concludes the proof of Theorem 1.4. □

3 Uniform decay rates

In order to prove the energy decay of the system we derive some useful estimates.

Theorem 3.1. *Assume that Theorem 1.4 holds and let $C^* > 0$ denote a constant such that $|E(0)| \le C^*$. Then there exists a $t^* > 0$ such that for every $t \ge t^*$, $E(0) > 0$.*

Proof. Taking into consideration the assumptions of Theorem 1.4 and the result $\|\psi\| \le \epsilon^*$ for all $t \ge t^* > 0$ we evaluate the integral of the energy functional, that is

$$\left|\int_\omega \phi|\psi|^2 dx\right| \le c\|\phi\|\|\psi\|_4^2 \le \frac{1}{2}\|\phi\|^2 + \frac{c^2(\epsilon^*)^2}{2}\|\nabla\psi\|^2.$$

Therefore we have

$$E(t) \ge \frac{1}{2}\left[\|\psi\|^2 + \left(\kappa - \frac{c^2(\epsilon^*)^2}{2}\right)\|\nabla\psi\|^2 + \|\nabla\phi\|^2 + \|\phi_t\|^2\right], \quad \text{for } t \ge t^* \tag{3.1}$$

which completes the proof by choosing $\kappa > \frac{c^2(\epsilon^*)^2}{2}$. □

Proceeding with the analysis we take the inner product of (1.2) with $\overline{\psi_t + \alpha\psi}$, adding equation (2.3) and following similar steps as the ones for the a priori estimates we have

$$\frac{1}{2}\frac{d}{dt}\left\{\|\psi\|^2 + \kappa\|\nabla\psi\|^2 + \int_\omega \phi|\psi|^2\right\} + \kappa\alpha b_0\|\nabla\psi\|_\omega^2 + \alpha\int b(x)|\psi|^2 + \alpha b_0\int_\omega \phi|\psi|^2 = \frac{1}{2}\int_\omega \phi_t|\psi|^2. \tag{3.2}$$

Next, taking the inner product of (1.1) with ϕ_t gives

$$\frac{1}{2}\frac{d}{dt}\{\|\phi_t\|^2+\|\nabla\phi\|^2+\|\phi\|^2\}+\int\lambda(x)|\phi_t|^2=\operatorname{Re}\int_\omega\nabla\psi\phi_t. \tag{3.3}$$

Adding equations (2.5), (3.2) and (3.3) results in

$$\begin{aligned}E_t(t)+ab_0\|\psi\|_\omega^2+\kappa\alpha b_0\|\nabla\psi\|^2+\alpha\int b(x)|\psi|^2+\int\lambda(x)|\phi_t|^2\\+\alpha b_0\int_\omega\phi|\psi|^2=\frac{1}{2}\int_\omega\phi_t|\psi|^2+\operatorname{Re}\int_\omega\nabla\psi\phi_t.\end{aligned} \tag{3.4}$$

From equation (2.3) we have

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_\omega^2+\alpha\|\psi\|_\omega^2\le 0$$

from which we get

$$\|\psi\|_\omega\le\|\psi(0)\|_\omega e^{-\alpha t}=\epsilon^* \tag{3.5}$$

and therefore since

$$\lim_{t\to\infty}\sup\|\psi\|_\omega=0.$$

Next, evaluating the integrals

$$\begin{aligned}\left|\alpha b_0\int_\omega\phi|\psi|^2\right|&\le\frac{\epsilon^* c\alpha b_0}{2\kappa}\|\phi\|_\omega^2+\frac{\kappa\alpha b_0}{2}\|\nabla\psi\|_\omega^2,\\ \left|\operatorname{Re}\int_\omega\nabla\psi\phi_t\right|&\le\frac{1}{2\epsilon}\int\lambda(x)|\phi_t|^2+\frac{\epsilon}{2\lambda_0}\|\nabla\psi\|_\omega^2,\\ \left|\frac{1}{2}\int_\omega\phi_t|\psi|^2\right|&\le\frac{\epsilon^*}{8\epsilon}\int\lambda(x)|\phi_t|^2+\frac{\epsilon}{2\lambda_0}\|\nabla\psi\|_\omega^2.\end{aligned}$$

Therefore

$$\begin{aligned}E_t(t)\le&-\left(\frac{\kappa\alpha b_0}{2}-\frac{\epsilon}{\lambda_0}\right)\|\nabla\psi\|_\omega^2-\alpha\int b(x)|\psi|^2-\left(1-\frac{1}{2\epsilon}-\frac{\epsilon^*}{8\epsilon}\right)\int\lambda(x)|\phi_t|^2\\&+\frac{\epsilon^* c\alpha b_0}{2\kappa}\|\phi\|_\omega^2-\alpha b_0\|\psi\|_\omega^2.\end{aligned} \tag{3.6}$$

For $\mu>0$ let us define the perturbed energy

$$E_\mu(t)=E(t)+\mu p(t) \tag{3.7}$$

where

$$p(t)=(\psi(t),\psi(t))+(\phi_t(t),\phi(t))_\omega. \tag{3.8}$$

Lemma 3.2. *For $\mu, C>0$ we have that*

$$|E_\mu(t)-E(t)|\le\mu CE(t),\quad \textit{for all } t\ge t^*.$$

Proof. We have

$$|p(t)|\le\|\psi\|^2+\frac{1}{2}\|\phi_t\|^2+\frac{c_1}{2}\|\nabla\phi\|^2\le C^*E(t)$$

which completes the proof. □

Next, by taking the time derivative of $p(t)$ we obtain

$$\begin{aligned} p_t(t) &= 2\,\mathrm{Re}(\psi_t,\psi) + (\phi_{tt},\phi)_\omega + (\phi_t,\phi_t)_\omega \\ &\leq 2\,\mathrm{Re}(\psi_t,\psi) + (\phi_{tt},\phi)_\omega + \frac{1}{\lambda_0}\int_\omega \lambda(x)|\phi_t|^2 \\ &\leq 2\,\mathrm{Re}(\psi_t,\psi) + (\phi_{tt},\phi) + \frac{1}{\lambda_0}\int \lambda(x)|\phi_t|^2 \end{aligned}$$

which with the help of (1.1) we can deduce that

$$p_t(t) \leq\; -2\alpha\int b(x)|\psi|^2 - \|\nabla\phi\|^2 - \|\phi\|^2 + \frac{1}{\lambda_0}\int \lambda(x)|\phi_t|^2 - \int \lambda(x)\phi_t\phi - \mathrm{Re}\int_\omega \nabla\psi\phi. \quad (3.9)$$

Adding equations (3.6)–(3.9) gives

$$\begin{aligned} E_{t,\mu} &= E_t(t) + \mu p_t(t) \\ &\leq -\left(\frac{\kappa\alpha b_0}{2} - \frac{\epsilon}{\lambda_0}\right)\|\nabla\psi\|^2_\omega - \alpha(2\mu+1)\int b(x)|\psi|^2 \\ &\quad - \left(1 - \frac{1}{2\epsilon} - \frac{\epsilon^*}{8\epsilon} - \frac{\mu}{\lambda_0}\right)\int \lambda(x)|\phi_t|^2 \\ &\quad + \frac{\epsilon^* c\alpha b_0}{2\kappa}\|\phi\|^2_\omega - \mu\|\nabla\phi\|^2 - \mu\|\phi\|^2 - \mu\int\lambda(x)\phi_t\phi - \mathrm{Re}\,\mu\int_\omega \nabla\psi\phi - \alpha b_0\|\psi\|^2_\omega, \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} \left|\mu\int_\omega \nabla\psi\phi\right| &\leq \frac{c\mu}{2}\|\nabla\psi\|^2_\omega + \frac{\mu}{2}\|\nabla\phi\|^2, \\ \left|\mu\int \lambda(x)\phi_t\phi\right| &\leq \frac{c\mu\|\lambda\|_\infty}{2}\int \lambda(x)|\phi_t|^2 + \frac{\mu}{2}\|\nabla\phi\|^2 \end{aligned}$$

which concludes in

$$\begin{aligned} E_{t,\mu} &= E_t(t) + \mu p_t(t) \\ &\leq -\left(\frac{\kappa\alpha b_0}{2} - \frac{\epsilon}{\lambda_0} - \frac{c\mu}{2}\right)\|\nabla\psi\|^2_\omega - \alpha(2\mu+1)\int b(x)|\psi|^2 \\ &\quad - \left(1 - \frac{1}{2\epsilon} - \frac{\epsilon^*}{8\epsilon} - \frac{\mu}{\lambda_0} - \frac{c\mu\|\lambda\|_\infty}{2}\right)\int \lambda(x)|\phi_t|^2 - \mu\left(1 - \frac{\epsilon^* c\alpha b_0}{2\kappa\mu}\right)\|\phi\|^2. \end{aligned}$$

Therefore we will require the following expressions to be nonnegative

$$\begin{cases} \frac{\kappa\alpha b_0}{2} - \frac{\epsilon}{\lambda_0} - \frac{c\mu}{2} > 0, \\ 1 - \frac{1}{2\epsilon} - \frac{\epsilon^*}{8\epsilon} - \frac{\mu}{\lambda_0} - \frac{c\mu\|\lambda\|_\infty}{2} > 0, \\ 1 - \frac{\epsilon^* c\alpha b_0}{2\kappa\mu} > 0. \end{cases}$$

Therefore, choosing κ sufficiently large enough such that the following inequality holds

$$2\kappa\lambda_0 > \epsilon^* acb_0(2 + \lambda_0\|\lambda\|_\infty)$$

we may deduce that there exists a k such that

$$E_{t,\mu}(t) \leq -k\left[\int b(x)|\psi|^2 + \int \lambda(x)|\phi_t|^2\right] \quad (3.11)$$

and hence $E_\mu(t)$ would be a non increasing function.

Remark 3.3. The time t^* introduced in the energy decay analysis which is present through the constant ϵ^* has a specific physical meaning. This is the time so that the non-collisional integral $\int \phi|\psi|^2$ is absorbed by the collisional terms (see (3.1)). Therefore, t^* roughly signifies the time required by the collisional damping to smooth out the excessive difference of the kinetic energies of the resonant electrons and the non-resonant ions (equipartition). It is important to note that equation (3.6) is a non increasing function due to the positive term $\|\phi\|_\omega$ which depends heavily on the t^*.

Lemma 3.4. *For all $T > T_0$ there exists a positive constant $C = C(t)$ such that if (ψ, ϕ) is the regular solution of the system (1.1) where $(\psi_0, \phi_0, \phi_1) \in \{H_0^1(\Omega) \cap H^2(\Omega)\}^2 \times H_0^1(\Omega)$ we have*

$$E_\mu(0) \leq C \int_0^T \left[\int b(x)|\psi|^2 + \int \lambda(x)|\phi_t|^2 \right] dt. \tag{3.12}$$

Proof. We will prove this lemma by contradiction. Assume (3.12) is not true and let $(\psi_k(0), \phi_k(0), \phi_{t,k}(0))$ be a sequence of initial data where the corresponding solutions $(\psi_k, \phi_k, \phi_{t,k})$ with $E_{\mu,k}(0)$, uniformly bounded in k satisfy

$$\lim_{k\to+\infty} \frac{E_{\mu,k}(0)}{\int_0^T \left[\int b(x)|\psi_k|^2 + \int \lambda(x)|\phi_{t,k}|^2 \right] dt} = +\infty. \tag{3.13}$$

Since $E_{\mu,k}(t)$ is non increasing and $E_{\mu,k}(0)$ remains bounded we may obtain a subsequence, denoted again as (ψ_k, ϕ_k) for which we have

$$\begin{cases} \psi_k \rightharpoonup \psi \text{ weak star in } L^\infty(0,T;H_0^1(\Omega)), \\ \phi_k \rightharpoonup \phi \text{ weak star in } L^\infty(0,T;H_0^1(\Omega)), \\ \phi_{t,k} \rightharpoonup \phi_t \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \\ \psi_{t,k} \rightharpoonup \psi_t \text{ weak star in } L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{3.14}$$

By compactness results, see [14] we get

$$\begin{aligned} \psi_k \to \psi_k \text{ strongly in } L^\infty(0,T;L^2(\Omega)), \\ \phi_k \to \phi_k \text{ strongly in } L^\infty(0,T;L^2(\Omega)). \end{aligned} \tag{3.15}$$

Now, taking into consideration (3.13) and (3.14) we obtain

$$\begin{aligned} \lim_{k\to+\infty} \int_0^T \int b(x)|\psi_k|^2 dxdt = 0, \\ \lim_{k\to+\infty} \int_0^T \int \lambda(x)|\phi_{t,k}|^2 dxdt = 0. \end{aligned} \tag{3.16}$$

Setting

$$c_k := [E_{\mu,k}(0)]^{1/2} \quad \text{and} \quad \hat{\phi}_k = \frac{1}{c_k}\phi_k, \quad \hat{\psi}_k = \frac{1}{c_k}\psi_k$$

we infer that

$$\hat{E}_{\mu,k}(t) := \frac{E_{\mu,k}(t)}{c_k^2}$$

for which we have

$$\hat{E}_k(0) = 1. \tag{3.17}$$

Taking into consideration the following system

$$\begin{cases} i\hat{\psi}_{t,k} + \kappa\Delta\hat{\psi}_k + i\alpha b(x)\hat{\psi}_k = \hat{\phi}_k\psi_k\chi(\omega), \\ \hat{\phi}_{tt,k} - \Delta\hat{\phi}_k + \hat{\phi}_k + \lambda(x)\hat{\phi}_{t,k} = -\operatorname{Re}\nabla\hat{\psi}_k\chi(\omega), \\ \hat{\psi}_k = \hat{\phi}_k = 0 \in \Gamma\times(0,T), \\ \hat{\psi}_k(0) = \hat{\psi}_{0k},\ \hat{\phi}_k(0) = \hat{\phi}_{0k},\ \hat{\phi}_{t,k}(0) = \hat{\phi}_{1k} \text{ in } \Omega, \\ \hat{\phi}_{t,k} \to 0 \in L^2(0,T;L^2(\omega)) \end{cases} \tag{3.18}$$

and since $E_{\mu,k}(0) = 1$ we may deduce that for a subsequence $(\hat{\psi}_k, \hat{\phi}_k)$ it is true that

$$\begin{cases} \hat{\psi}_k \rightharpoonup \hat{\psi} \text{ weak star in } L^\infty(0,T;H_0^1(\Omega)), \\ \hat{\psi}_k \to \hat{\psi} \text{ strongly in } L^\infty(0,T;L^2(\Omega)), \\ \hat{\psi}_{t,k} \to \hat{\psi}_t \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \\ \hat{\phi}_k \rightharpoonup \hat{\phi} \text{ weak star in } L^\infty(0,T;H_0^1(\Omega)), \\ \hat{\phi}_{t,k} \rightharpoonup \hat{\phi}_t \text{ weak star in } L^\infty(0,T;L^2(\Omega)), \\ \hat{\phi}_k \to \hat{\phi} \text{ strongly in } L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{3.19}$$

From the (3.19), we obtain

$$\begin{aligned} \lim_{k\to+\infty}\int_0^T\int b(x)|\hat{\psi}_k|^2dxdt = 0, \\ \lim_{k\to+\infty}\int_0^T\int \lambda(x)|\hat{\phi}_{t,k}|^2dxdt = 0, \end{aligned} \tag{3.20}$$

and therefore by (3.20) and by the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$

$$\begin{aligned} \lim_{k\to+\infty}\int_0^T\int_\omega |\nabla\hat{\psi}_k|^2dxdt = 0, \\ \lim_{k\to+\infty}\int_0^T\int_\omega |\hat{\phi}_k\psi_k|^2dxdt = 0. \end{aligned} \tag{3.21}$$

Taking into consideration (3.20) and letting the limit $k \to +\infty$ for the system (3.18) we get for the wave equation

$$\begin{cases} \hat{\phi}_{tt} - \Delta\hat{\phi} + \hat{\phi} = 0 \text{ in } \Omega\ \times(0,T), \\ \hat{\phi} = 0 \in \Gamma\times(0,T), \\ \phi_t = 0 \text{ a.e.} \in \omega\times(0,T) \end{cases} \tag{3.22}$$

and for the Schrödinger equation

$$\begin{cases} i\hat{\psi}_t + \kappa\Delta\hat{\psi} = 0, \text{in } \Omega\ \times(0,T), \\ \hat{\psi} = 0 \text{ on } \Gamma\times(0,T). \end{cases} \tag{3.23}$$

Setting $\hat{\phi}_t = v$ equation (3.22) in the distributional sense becomes

$$\begin{cases} v_{tt} - \Delta v = 0 \in D'(\Omega\ \times(0,T)), \\ v = 0 \in \Gamma\times(0,T), \\ v = 0 \text{ a.e.} \in \omega\times(0,T). \end{cases} \tag{3.24}$$

From standard uniqueness results from equation (3.24) we may conclude that $v \equiv 0$, that is $\hat{\phi}_t \equiv 0$. Therefore for a.e. $t \in (0, T)$

$$\begin{cases} -\Delta\hat{\phi} = 0 \in \Omega, \\ \hat{\phi} = 0 \in \Gamma \end{cases} \tag{3.25}$$

which multiplying by $\hat{\phi}$ implies that $\hat{\phi} \equiv 0$. Following a similar procedure for the Schrödinger equation the uniqueness theorem concludes that $\hat{\psi} = 0$ a.e. $\in \Omega$.

In order to achieve a contradiction it is enough to prove that $\hat{E}_{\mu,k}(0) \to 0$ as $k \to +\infty$.

$$\begin{aligned} \hat{E}_{\mu,k}(0) = & \frac{1}{2}\Big\{\int((2\mu+1)|\hat{\psi}(x,0)|^2 + \kappa|\nabla\hat{\psi}(x,0)|^2 + |\hat{\phi}(x,0)|^2 + |\nabla\hat{\phi}(x,0)|^2 + |\hat{\phi}_t(x,0)|^2) \\ & + \int_\omega \phi(x,0)|\hat{\psi}(x,0)|^2 + 2\mu\int_\omega \hat{\phi}_t(x,0)\hat{\phi}(x,0)\Big\} \\ \le & \frac{1}{2}\Big\{\int((2\mu+1)|\hat{\psi}(x,0)|^2 + \left(\kappa + \frac{c}{2}\right)|\nabla\hat{\psi}(x,0)|^2 + \frac{3}{2}|\hat{\phi}(x,0)|^2 \\ & + (\mu c+1)|\nabla\hat{\phi}(x,0)|^2 + (\mu+1)|\hat{\phi}_t(x,0)|^2)\Big\} = E_{\mu,\hat{\psi}_k}(0) + E_{\mu,\hat{\phi}_k}(0). \end{aligned} \tag{3.26}$$

Our aim is to prove that $E_{\mu,\hat{\psi}_k}(0) \to 0$ and $E_{\mu,\hat{\phi}_k}(0) \to 0$ with the help of (1.7) and (1.9). For this purpose let $\hat{\psi}_k = v_k + w_k$ where $\hat{\psi}_k$ is the solution of the system (3.18) and v_k, w_k are the solutions of the following systems respectively,

$$\begin{cases} iv_{t,k} + \kappa\Delta v_k = 0 \in \Omega \times (0,T), \\ v_k = 0 \in \Gamma \times (0,T), \\ v_k(0) = \hat{\psi}_{0,k} \in \Omega \end{cases} \tag{3.27}$$

and

$$\begin{cases} iw_{t,k} + \kappa\Delta w_k = -i\alpha b(x)\hat{\psi}_k + \hat{\phi}_k\psi_k\chi(\omega), \in \Omega \times (0,T), \\ w_k = 0 \in \Gamma \times (0,T), \\ w_k(0) = 0 \in \Omega. \end{cases} \tag{3.28}$$

Similarly for the wave equation we obtain $\hat{\phi}_k = z_k + u_k$ which produces the following problems

$$\begin{cases} z_{tt,k} + \Delta z_k = 0 \in \Omega \times (0,T), \\ z_k = \hat{\phi}_{0k} \in \Gamma \times (0,T), \\ z_k(0) = \hat{\phi}_{0,k} \in \Omega, \\ z_{t,k}(0) = \hat{\phi}_{1k} \in \Omega \end{cases} \tag{3.29}$$

and

$$\begin{cases} u_{tt,k} + \Delta u_k = -\lambda(x)\hat{\phi}_{t,k} - \operatorname{Re}\nabla\hat{\psi}_k\chi(\omega) \in \Omega \times (0,T), \\ u_k = 0 \in \Gamma \times (0,T), \\ u_k(0) = 0 \in \Omega, \\ u_{t,k}(0) = 0 \in \Omega. \end{cases} \tag{3.30}$$

Therefore, it follows that

$$\begin{aligned}\hat{E}_{\mu,k} &\leq E_{\mu,\hat{\psi}_k}(0) + E_{\mu,\hat{\phi}_k}(0) = E_{\mu,v_k}(0) + E_{\mu,z_k}(0) \leq c_1 \int_0^T \int_\omega |v_k|^2 + c_2 \int_0^T \int_\omega |z_{t,k}|^2 \\ &\leq c_1 \Big(\int_0^T \int b(x)|\hat{\psi}_k|^2 + \int_0^T \int_\omega |w_k|^2 \Big) + c_2 \Big(\int_0^T \int \lambda(x)|\hat{\phi}_{t,k}|^2 + \int_0^T \int_\omega |x_k|^2 \Big).\end{aligned} \tag{3.31}$$

From equation (3.28) we have the following integral form

$$\hat{w}_k(t) = S(t)\hat{w}_k(0) + \int_0^T S(t-\tau)F(\tau)d\tau, \tag{3.32}$$

where $S(t)$ is the semigroup generated by

$$\begin{aligned} A : D(A) = H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega) &\to L^2(\Omega), \\ y &\to Ay = -i\Delta y \end{aligned}$$

and $F(t) = \hat{\phi}_k(t)\psi_k(t)\chi(\omega) - i\alpha b(x)\hat{\psi}_k(t)$. Thus, taking into consideration that $\|S(t)\|_{\mathcal{L}(L^2(\Omega))} \leq C$ we have

$$\|w_k\|^2 \leq c_1\|w_{0,k}\|^2 + c_2\Big(\int_0^T \|F(\tau)\|d\tau\Big)^2 \leq C(\|w_{0k}\|^2 + \|F\|^2_{L^1(0,T;L^2(\Omega))})$$

which with the help of the embedding $L^\infty(0,T;L^2(\Omega)) \to L^1(0,T;L^2(\Omega))$ and $w_k(0) = 0$ produces

$$\begin{aligned}\int_0^T \int_\omega |w_k|^2 &\leq \|w_k\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\leq C\|F\|^2_{L^1(0,T;L^2(\Omega))} \leq C \int_0^T \int |\hat{\phi}_k\psi_k\chi(\omega) - i\alpha b(x)\hat{\psi}_k|^2.\end{aligned} \tag{3.33}$$

Moving onto the wave equation we have the following integral form expression for the system (3.29)

$$U_k(t) = S(t)U_{0k} + \int_0^T S(t-\tau)F(\tau)d\tau$$

where

$$U_k = \begin{pmatrix} u_k \\ u_{t,k} \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 \\ -\lambda(x)\hat{\phi}_k - \operatorname{Re}\nabla\hat{\psi}_k\chi(\omega) \end{pmatrix}.$$

Evaluating the following integral

$$\begin{aligned}\int_0^T \int_\omega |u_{t,k}|^2 &\leq \|u_{t,k}\|^2_{L^\infty(0,T;L^2(\Omega))} \\ &\leq C\|F\|^2_{L^1(0,T;L^2(\Omega))} \leq C \int_0^T \int |-\lambda(x)\hat{\phi}_k - \operatorname{Re}\nabla\hat{\psi}_k\chi(\omega)|^2.\end{aligned} \tag{3.34}$$

Therefore, from equation (3.31) we obtain

$$\begin{aligned}\hat{E}_{\mu,k}(t) \leq C\Big(&\int_0^T \int b(x)|\hat{\psi}_k|^2 + \int_0^T \int |\hat{\phi}_k\psi_k\chi(\omega) - i\alpha b(x)\hat{\psi}_k|^2 \\ &+ \int_0^T \int \lambda(x)|\hat{\phi}_{t,k}|^2 + \int_0^T \int |-\operatorname{Re}\nabla\hat{\psi}_k\chi(\omega) - \lambda(x)\hat{\phi}_{t,k}|^2\Big),\end{aligned} \tag{3.35}$$

which taking into consideration (3.20) and (3.21) produces $\hat{E}_{\mu,k}(0) \to 0$ as $k \to +\infty$ and therefore contradicts the expression (3.17). □

Proof of Theorem 1.5. Continuing with the proof of Theorem 1.5 and by taking $T_0 > 0$ large enough from (3.11) we may deduce that

$$E_\mu(T_0) - E_\mu(0) \le -k\left[\int_0^{T_0} b(x)|\psi|^2 + \int \lambda(x)|\phi_t|^2\right] \tag{3.36}$$

and from Lemma 3.4 we also have

$$E_\mu(0) \le C\int_0^{T_0} D(t)$$

where

$$D(t) := \int b(x)|\psi|^2 + \int \lambda(x)|\phi_t|^2.$$

Therefore, we get

$$E_\mu(T_0) \le E_\mu(0) \le C\int_0^{T_0} D(t) \le -\frac{C}{k}E_\mu(T_0) + \frac{C}{k}E_\mu(0),$$

so

$$\left(1+\frac{C}{k}\right)E_\mu(T_0) \le \frac{C}{k}E_\mu(0).$$

Hence,

$$E_\mu(T_0) \le \nu E_\mu(0), \qquad 0 < \nu < 1.$$

Proceeding in a similar way from T to $2T$ and eventually to nT we have

$$E_\mu(nT) \le \nu^n E_\mu(0), \qquad \forall\, T > T_0.$$

Finally, let $t > T_0$ then $t = nT_0 + r$ for $0 \le r \le T_0$ and

$$E_\mu(t) \le E_\mu(t-r) = E_\mu(nT_0) \le \nu^n E_\mu(0) = \nu^{\frac{t-r}{T_0}} E_\mu(0) = e^{\frac{t-r}{T_0}\ln\nu} E_\mu(0).$$

Moreover, by Lemma 3.2 we have

$$E_\mu(t) \le 2E(t) \quad \text{for } t \ge 0$$

therefore

$$E_\mu(t) \le 2E(0)e^{\frac{t-r}{T_0}\ln\nu} \quad \text{for } t \ge 0$$

which completes the proof of Theorem 1.5. □

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Bifurcation analysis of fractional Kirchhoff–Schrödinger–Poisson systems in $\mathbb{R}^3$

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Abstract. In this paper, we investigate the bifurcation results of the fractional Kirchhoff–Schrödinger–Poisson system

$$\begin{cases} M([u]_s^2)(-\Delta)^s u + V(x)u + \phi(x)u = \lambda g(x)|u|^{p-1}u + |u|^{2_s^*-2}u & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi(x) = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $s, t \in (0,1)$ with $2t + 4s > 3$ and the potential function V is a continuous function. We show that the existence of components of (weak) solutions of the above equation associated with the first eigenvalue λ_1 of the problem

$$(-\Delta)^s u + V(x)u = \lambda g(x)u \quad \text{in } \mathbb{R}^3.$$

The main feature of this paper is the inclusion of a potentially degenerate Kirchhoff model, combined with the critical nonlinearity.

Keywords: Kirchhoff–Schrödinger–Poisson system, global bifurcation, first eigenvalue, fractional Laplacian, fixed point, whole space.

2020 Mathematics Subject Classification: 35B32, 35P30, 47J15, 35R11, 35Q55.

1 Introduction and main results

In this paper, we investigate the bifurcation result of the fractional Kirchhoff–Schrödinger–Poisson system

$$\begin{cases} M([u]_s^2)(-\Delta)^s u + V(x)u + \phi u = \lambda g(x)|u|^{p-1}u + |u|^{2_s^*-2}u, & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$\mathcal{P}$}$$

where $s, t \in (0,1)$ with $2t + 4s > 3$, $\lambda \in \mathbb{R}$, $p \in (0,1)$, $g(x) \in L^{\frac{6}{5-p}}(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3)$ is a weight function, the non-local coefficient $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ defined by $M(t) = a + bt$, where $a, b \geq 0$, and the Gagliardo semi-norm

$$[u]_s = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{1/2}.$$

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Here, we assume that $(-\Delta)^s$ is the fractional Laplacian which, up to a normalization constant, is denoted as

$$(-\Delta)^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^3 \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{3+2s}} dy, \qquad x \in \mathbb{R}^3,$$

for every $u \in C_0^\infty(\mathbb{R}^3)$, where $B_\varepsilon(x)$ is the ball of $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$ with radius $\varepsilon > 0$.

The Kirchhoff–Schrödinger–Poisson (KSP) system, including ($\mathcal{P}$) as a special model, describes the interaction of a quantum particle with an electromagnetic field. The (KSP) system consisting of a Schrödinger equation coupled with a Poisson equation and a Kirchhoff function has been studied extensively in various settings, such as Euclidean spaces, fractional spaces, and Heisenberg groups, due to its strong applications in physics. Some of the main topics of interest are the existence, multiplicity, and asymptotic behavior as well as the qualitative properties of the (weak) solutions such as regularity, symmetry, and concentration. For more information and references, one can consult the following papers [2,6,8–11,23].

The fractional (KSP) system is a generalization of the (KSP) system that involves fractional derivatives of order s in $(0,1)$. The fractional part of the system introduces new challenges and difficulties involving fractional derivatives and nonlocal and nonlinear properties. Various methods and techniques have been developed to deal with these problems, such as variational methods, the Nehari manifold, Ekeland variational principle, the concentration-compactness principle, and the mountain pass theorem. We refer the readers to [12,16,22,23]. Benci and Fortunato in [3] first introduced the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

to describe solitary waves with an electronic field. More recently, the authors in [16] used variational methods to obtain nonnegative solutions for an Schrödinger–Choquard–Kirchhoff type fractional p-Laplacian

$$\left(a + b\|u\|_s^{p(\xi-1)}\right)[(-\Delta)_p^s u + V(x)|u|^{p-2}u] = \lambda f(x,u) + \left(\int_{\mathbb{R}^N} \frac{|u|^{p^*_{\nu,s}}}{|x-y|^\mu} dy\right) |u|^{p^*_{\nu,s}-2}u \quad \text{in } \mathbb{R}^N,$$

where the nonlinearity f satisfies super-linear or sub-linear growth conditions and the parameter λ is large or small enough. In particular, it can be seen as a special case of the fractional Kirchhoff–Schrödinger–Poisson system.

On the other hand, bifurcation analysis is an important method of mathematics that studies how the qualitative behavior of solutions changes as a parameter varies, and moreover, a bifurcation point may correspond to the appearance or disappearance of the solutions or a change in their stability or symmetry. For instance, He, in [7], studied the nonhomogeneous semi-linear fractional Schrödinger equation with critical growth

$$\begin{cases} (-\Delta)^s u + u = u^{2_s^*-1} + \lambda(f(x,u)+h(x)), & x \in \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), \quad u(x) > 0 & x \in \mathbb{R}^N, \end{cases}$$

where $s \in (0,1)$, $N > 4s$ and $\lambda > 0$ is a parameter. They showed that there exists a positive bifurcation value of the parameter such that the problem has exactly two positive solutions for smaller values, no positive solutions for larger values, and a unique solution at the bifurcation value. Furthermore, many recent works investigate the bifurcation results for the fractional Kirchhoff or Schrödinger or Poisson equation under different assumptions on the potential functions and the non-linearities. Very recently, for $p \in (1,2)$ and λ is small, Ruiz [19] demonstrated the existence of a branch of positive radial solutions to the problem

$$\begin{cases} -\Delta u + u + \lambda \phi u = u_+^p \\ -\Delta \phi = u^2, \quad \lim_{|x|\to+\infty} \phi(x) = 0. \end{cases}$$

After that, in [24], Xu, Qin, and Chen established bifurcation results for positive solutions by using the local and global bifurcation techniques, a priori bounds for elliptic equation, and the properties of the principal eigenvalues to the Kirchhoff-type problem involving sign-changing weight functions

$$\begin{cases} -(a(x) + b(x)\|u\|^2)\Delta u = \lambda m(x) u + h(x) u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In [14], the bifurcation results and the regularity for the (weak) solutions of the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + l(x)\phi u = \lambda a(x)|u|^{p-1}u + f(\lambda, x, u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x) u^2, & \text{in } \mathbb{R}^3 \end{cases}$$

are proved, where a, l are weight functions and f satisfies the subcritical and critical growth condition, respectively.

Motivated by the above works, especially by [19], this paper is dedicated to investigating bifurcation results to the (weak) solutions of the (KSP) system ($\mathcal{P}$), while overcoming the challenges due to the lack of compactness in critical case as well as the degenerate nature of the Kirchhoff coefficient. To our knowledge, no such general results are provided for ($\mathcal{P}$).

More precisely, we put the hypotheses in the following:

- (V_1) $V \in C(\mathbb{R}^3)$ satisfies $\inf_{x\in\mathbb{R}^3} V(x) \geq V_0 > 0$, where $V_0 > 0$ is a constant;
- (V_2) $\text{meas}\{x \in \mathbb{R}^3 : -\infty < V(x) \leq \xi\} < +\infty$ for any $\xi \in \mathbb{R}$;
- $(M_1)'$ $M \in C(\mathbb{R}_0^+)$ satisfies that for any $\tau > 0$, there exists $\kappa = \kappa(\tau) > 0$, such that $M(t) \geq \tau$ for all $t \geq \tau$;
- (g_1) $g \in L^{6/(5-p)}(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $g(x) \geq 0$ a.e. in $\mathbb{R}^3$.

It is worth stressing that the degenerate case of Kirchhoff nonlinearity is included in the assumption of $(M_1)'$.

Before stating our main results, let us introduce some notations. Firstly, thanks to the Fourier transform, the fractional Sobolev space $H^s(\mathbb{R}^3)$ is defined by

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\xi|^{2s} + 1)|\widehat{u}|^2 d\xi < \infty \right\},$$

which is equipped with the standard norm and inter product

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} (|\xi|^{2s} + 1)|\widehat{u}|^2 d\xi \right)^{1/2}, \qquad \langle u, v \rangle = \int_{\mathbb{R}^3} (|\xi|^{2s} + 1)\widehat{u}\widehat{v} d\xi.$$

In fact, Plancherel's theorem in [5] guarantees that $\|u\|_2 = \|\widehat{u}\|_2$ and $\||\xi|^s \widehat{u}\|_2 = \|(-\Delta)^{\frac{s}{2}} u\|_2$, and then

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2 \right) dx \right)^{1/2}, \qquad \langle u, v \rangle = \int_{\mathbb{R}^3} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv \right) dx.$$

Furthermore, Proposition 3.4 and Proposition 3.6 in [5] imply that

$$\|(-\Delta)^{\frac{s}{2}}u\|_2^2 = \int_{\mathbb{R}^3} |\xi|^{2s}|\widehat{u}(\xi)|^2 d\xi = \frac{1}{C(s)}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy.$$

By virtue of [5, Theorem 6.5], the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$, with $p \in [2, 2_s^*]$, is continuous, where 2_s^* is the fractional critical Sobolev exponent, defined as $2_s^* = 6/(3-2s)$. Moreover, let $D^s(\mathbb{R}^3) = \{u \in L^{2_s^*}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx < \infty\}$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $[u]_s$. The continuous fractional Sobolev embedding $D^s(\mathbb{R}^3) \hookrightarrow L^{2_s^*}(\mathbb{R}^3)$ yields that there exists a best Sobolev constant

$$S_* = \inf_{u\in D^s(\mathbb{R}(^3)\setminus\{0\}} \frac{\int_{\mathbb{R}(^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx}{(\int_{\mathbb{R}^3} |u|^{2_s^*} dx)^{2/2^*}},$$

so that

$$\|u\|_{2_s^*} \le c[u]_s, \tag{1.1}$$

where $c = S_*^{-1/2}$. In this paper, the main solutions spaces E is the subspace of $H^s(\mathbb{R}^3)$, considered as

$$E = \left\{ u \in H^s(\mathbb{R}^3) : \|u\| = \left(\int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)|u|^2\right)dx\right)^{1/2} < \infty \right\}.$$

Obviously, E is a uniformly convex Banach space, see for instance [16].

Now, we state the main results of this paper in the following theorems.

Theorem 1.1. *Suppose that $s, p \in (0,1)$ and the hypotheses (V_1)–(V_2), $(M_1)'$ and (g_1) hold, equation $(\mathcal{P})$ has the unique bifurcation point $(0,0)$, and there exists an unbounded component $\mathcal{C}$ of positive weak solutions emanating from $(0,0)$.*

Notation:

- $\to$ and $\rightharpoonup$ denote the strong convergence and the weak convergence, respectively.
- $L^p(\mathbb{R}^3)$, $1 \le p \le +\infty$, denotes a Lebesgue space, and the norm in $L^p(\mathbb{R}^3)$ is denoted by $\|\cdot\|_p$.
- C, C_i are various positive constants.

2 Preliminaries

In this section, as preparation for proving the main results, we intend to introduce some fundamental notations, definitions and properties which are essential to the fractional setting.

Let $s, t \in (0,1)$, with $2t + 4s > 3$. Then, it follows that the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$ is continuous, due to the fact that $\frac{12}{3+2t} \le \frac{6}{3-2s} = 2_s^*$. For any $u \in H^t(\mathbb{R}^3)$, we define the linear functional $I_u : D^t(\mathbb{R}^3) \to \mathbb{R}$ by

$$I_u(v) = \int_{\mathbb{R}^3} u^2 v dx, \quad \text{for any } v \in D^t(\mathbb{R}^3).$$

Obviously, from the continuous embedding $H^s(\mathbb{R}^3) \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$ in the above, there exists $C_1 > 0$, such that

$$|I_u(v)| \leq \left(\int_{\mathbb{R}^3} |u^2|^{\frac{6}{3+2t}} dx\right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |v|^{\frac{6}{3-2t}} dx\right)^{\frac{3-2t}{6}} \leq cS_*^{-1/2}\|u\|_{H^t}^2[v]_t = c_0\|u\|_{H^t}^2[v]_t, \tag{2.1}$$

by (1.1) and the Hölder inequality, where $c_0 = cS_*^{-1/2}$. Hence, according to the Lax–Milgram theorem, for any $u \in H^t(\mathbb{R}^3)$, there exists a unique $\phi_u^t \in D^t(\mathbb{R}^3)$ satisfying

$$\int_{\mathbb{R}^3} u^2 v dx = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} v dx, \quad \text{for any } v \in D^t(\mathbb{R}^3), \tag{2.2}$$

which concludes ϕ_u^t is the (weak) solution of $(-\Delta)^t \phi_u^t = u^2$ in $\mathbb{R}^3$. Consequently, ϕ_u^t can be represented as

$$\phi_u^t = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy = c_t \frac{1}{|x|^{3-2t}} * u^2, \qquad x \in \mathbb{R}^3,$$

where $c_t = \Gamma(3-2t)/(\pi^{3/2}2^{2t}\Gamma(t))$ is the t-Riesz potential. Together with (2.1), taking ϕ_u^t as a test function of (2.2), we deduce that

$$[\phi_u^t]_t^2 = \int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq c_0\|u\|_{H^t}^2[\phi_u^t]_t, \qquad \int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq c_0^2\|u\|_{H^t}^4. \tag{2.3}$$

Now, substituting ϕ_u^t in problem $(\mathcal{P})$, it follows that the fractional Kirchhoff–Schrödinger–Poisson equation

$$M([u]_s^2)(-\Delta)^s u + V(x)u + \phi_u^t u = \lambda g(x)|u|^{p-1}u + |u|^{2_s^*-2}u \quad \text{in } \mathbb{R}^3.$$

Definition 2.1. We call that $u \in H^s(\mathbb{R}^3)$ is a (weak) *solution* of problem $(\mathcal{P})$, if for any $v \in E$, there holds

$$\int_{\mathbb{R}^3} \left(M([u]_s^2)(-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} v + V(x)uv + \phi_u^t uv\right) dx = \lambda \int_{\mathbb{R}^3} g(x)|u|^{p-1}uvdx + \int_{\mathbb{R}^3} |u|^{2_s^*-2}uvdx.$$

Furthermore, if there exist sequences $(\lambda_n)_n \subset \mathbb{R}$ and nontrivial (weak) solutions $(u_n)_n \subset E$ of problem $(\mathcal{P})$, such that $(\lambda_n, u_n)_n \to (\lambda, 0)$ as $n \to \infty$, then $(\lambda, 0)$ is a bifurcation point of problem $(\mathcal{P})$.

For more information on bifurcation, see, for instance [18]. Along this paper, let $(D^s(\mathbb{R}^3))^*$ be the dual space of $D^s(\mathbb{R}^3)$ and for each $u \in D^s(\mathbb{R}^3)$, let a functional $L: D^s(\mathbb{R}^3) \to (D^s(\mathbb{R}^3))^*$ be the weak formulation, defined by

$$\langle L(u), v\rangle = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx, \quad \text{for any } v \in E.$$

Note that, by using the Hölder inequality,

$$|\langle L(u), v\rangle| \leq [u]_s[v]_s, \qquad \langle L(u), u\rangle = [u]_s^2. \tag{2.4}$$

A simple observation of (2.4) yields that L is a bounded linear operator in $D^s(\mathbb{R}^3)$. Moreover, write for brevity,

$$\langle u, v\rangle_V = \int_{\mathbb{R}^3} V(x)uvdx, \qquad \|u\|_V = \left(\int_{\mathbb{R}^3} V(x)|u|^2 dx\right)^{1/2}, \quad \text{for any } u, v \in E.$$

Of course, arguing as (2.4), it follows that

$$|\langle u, v\rangle_V| \leq \|u\|_V\|v\|_V, \qquad \langle u, u\rangle_V = \|u\|_V^2.$$

Now, we are in the position to state some useful lemmas.

Lemma 2.2 ([13, Proposition 1.3]). *If X is uniformly convex and* (2.4) *holds, then L is of type* (S), i.e. *every sequence $(u_j)_j \in X$ such that*

$$u_j \rightharpoonup u, \qquad \langle L(u_j), u_j - u\rangle \to 0$$

has a subsequence that converges strongly to u in X.

Lemma 2.3 ([21, Lemma 2.3]). *For any $u \in H^s(\mathbb{R}^3)$, the function ϕ_u^t defined in* (2.2) *satisfies the next properties.*

(i_1) *ϕ_u^t is continuous with respect to u.*

(i_2) *$\phi_u^t \geq 0$ in $\mathbb{R}^3$ and $\phi_{\xi u}^t = \xi^2 \phi_u$ for any $\xi > 0$.*

(i_3) *If $u_n \rightharpoonup u$ in E and $u_n \to u$ in $L^p(\mathbb{R}^3)$, with $p \in [2, 2_s^*)$, as $n \to \infty$, then for any $v \in E$*

$$\int_{\mathbb{R}^3} \phi_{u_n}^t(x) u_n(x) v(x) dx = \int_{\mathbb{R}^3} \phi_u^t(x) u(x) v(x) dx + o(1),$$

and

$$\int_{\mathbb{R}^3} \phi_{u_n}^t(x) u_n^2 dx \to \int_{\mathbb{R}^3} \phi_u^t(x) u(x)^2 dx, \quad \text{as } n \to \infty.$$

Lemma 2.4 ([15, Lemma 1.1]). *Assume that $s \in (0,1)$ and (V_1)–(V_2) hold. If $p \in [2, 2_s^*]$, then the embeddings*

$$E \hookrightarrow H^s(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$$

are continuous, with $\min\{1, V_0\}[u]_s \leq \|u\|$, for all $u \in E$. Particularly, there exists a positive constant C_q, such that

$$\|u\|_q \leq C_q \|u\| \quad \text{for all } u \in E.$$

If $q \in [2, 2_s^*)$, the embedding $E \hookrightarrow L^q(\mathbb{R}^3)$ is compact. Furthermore, if $q \in [1, 2_s^*)$, then the embedding $E \hookrightarrow L^q(B_R)$ is compact for any $R > 0$.

Furthermore, to prove the main results, we need the following embedding theorem due to Lemma 2.1 in [4].

Lemma 2.5. *Let $s \in (0,1)$ and $w \in L^{3/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Then the embedding*

$$D^s(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3, w dx)$$

is continuous and compact, and $\|u\|_{2,w} \leq C_w [u]_s$, for all $u \in D^s(\mathbb{R}^3)$, with $C_w = S_*^{-1/2} \|w\|_{3/2}^{1/2} > 0$.

3 The subcritical case

In this section, we shall demonstrate the bifurcation results of the fundamental problem

$$M([u]_s^2)(-\Delta)^s u + V(x) u + \phi_u^t u = \lambda g(x) |u|^{p-1} u \quad \text{in } \mathbb{R}^3, \tag{3.1}$$

which is of significance in substantiating the proof of the main result. To this aim, let us consider the property of the first eigenvalue $\lambda_1(h)$ of the problem

$$(-\Delta)^s u + V(x) u = \lambda h(x) u, \tag{3.2}$$

where $h \in L^{3/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ is a strictly positive function.

Lemma 3.1. *The eigenvalue problem* (3.2) has the first eigenpair $(\lambda_1(h), u_1)$, where

$$0 < \lambda_1(h) = \min_{v \in E \setminus \{0\}} \frac{\|v\|^2}{\|v\|_{2,h}^2} = \min_{v \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^3}(|(-\Delta)^{\frac{s}{2}} v|^2 + V(x)|v|^2)dx}{\int_{\mathbb{R}^3} h(x)|v|^2 dx},$$

and the first eigenfunction u_1 has one sign. Furthermore, λ_1 is decreasing map with respect to h, i.e. if $0 < h_1 \le h_2 \in L^{3/2}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, then $\lambda_1(h_1) \ge \lambda_1(h_2)$.

Proof. Let $(v_k)_k \subset E \setminus \{0\}$ be a minimizing sequence of $\lambda_1(h)$ in Calculus of Variations. It can be normalized so that $\int_{\mathbb{R}^3} h(x)|v_k|^2 dx = 1$, and

$$\lambda_1(h) = \lim_{k \to \infty} \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_k|^2 dx + \int_{\mathbb{R}^3} V(x)|v_k|^2 dx \right). \tag{3.3}$$

Moreover, the fact that $[|v|]_s \le [v]_s$ for any $v \in E$ guarantees that $(|v_k|)_k$ is also a minimizing sequence, then we can further assume that v_k is positive. Since $\|v_k\|^2$ is a real convergent sequence in (3.3), we have

$$0 \le \|v_k\|^2 \le \lambda_1 + 1.$$

Consequently, the sequence $(v_k)_k$ is bounded in E. The reflexivity of E yields the existence of $0 \le \widehat{v} \in E$ such that $v_k \rightharpoonup \widehat{v}$ in E and $v_k \to \widehat{v}$ a.e. in $\mathbb{R}^3$, up a subsequence if necessary. Thanks to Lemma 2.5, we obtain that

$$\int_{\mathbb{R}^3} h|v_k|^2 dx \to \int_{\mathbb{R}^3} h|\widehat{v}|^2 dx \quad \text{as } k \to \infty. \tag{3.4}$$

Moreover, by the weak lower semi-continuity of the norm $\|\cdot\|$ and by (3.4), it follows that

$$0 \le \|\widehat{v}\| \le \liminf_{k \to \infty} \|v_k\|.$$

Thus, $\lambda_1 = \|\widehat{v}\|^2$ and $\widehat{v}$ is a critical point of $\psi(v) = \|v\|^2 / \|v\|_{2,h}^2$, i.e. for any $v \in E$

$$\int_{\mathbb{R}^3} \left((-\Delta)^{\frac{s}{2}} \widehat{v}\, (-\Delta)^{\frac{s}{2}} v + V(x)\widehat{v}v\right)dx \int_{\mathbb{R}^3} h(x)|\widehat{v}|^2 dx - \int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}} \widehat{v}|^2 + V(x)|\widehat{v}|^2\right)dx \int_{\mathbb{R}^3} h(x)\widehat{v}v dx = 0.$$

In conclusion, $\widehat{v}$ is the first eigenfunction corresponding to λ_1, provided that $\widehat{u} \not\equiv 0$.
Clearly, the definition of λ_1 implies at once that $\lambda_1(h_1) \ge \lambda_1(h_2)$. □

Proposition 3.2. *Let* $P = \{v \in E^* : v \ge 0\}$ *and let* $f(x) \in P$. *If* $(M_1)'$ *and* (V_1)–(V_2) *holds, then equation*

$$M([u]_s^2)(-\Delta)^s u + V(x)u = f(x) \quad \text{in } \mathbb{R}^3 \tag{3.5}$$

has a unique weak solution u in E. Furthermore, the operator $K : E^* \to E$, *defined by* $K(f) = u$, *where u is the unique weak solution of* (3.5), *is continuous.*

Proof. Of course, if $f \equiv 0$, then $u = 0$ is a unique (weak) solution of equation (3.5). Next, put $f \not\equiv 0$ and set $R = [u]_s^2 \ge 0$. Then, the problem (3.5) becomes

$$M(R)(-\Delta)^s u + V(x)u = f(x) \quad \text{in } \mathbb{R}^3. \tag{3.6}$$

Problem (3.6) has a variational structure and $J : E \to \mathbb{R}$, denoted as

$$J(u) = \frac{1}{2} M(R)[u]_s^2 + \int_{\mathbb{R}^3} V(x)|u|^2 dx - \langle f, u \rangle, \quad \text{for all } u \in E,$$

where $\langle \cdot, \rangle$ is the duality of E, is well defined and of class $C^1(E)$. It is easily deduced that the critical point of $J(u)$, defined by u_R, is a (weak) solution of (3.1). We first claim that J is coercive, bounded below, and sequentially weakly lower semi-continuous in E. Indeed, by Lemma 2.4 and $(M_1)'$, the Hölder inequality implies that

$$\begin{aligned} J(u) &\geq \frac{1}{2}M(R)[u]_s^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2dx - \|f\|_{E^*}\|u\| \\ &\geq \frac{1}{2}\min\{\kappa, V_0\}\|u\|^2 - C_f\|u\|. \end{aligned}$$

Consequently, $J(u) \to \infty$ as $\|u\| \to \infty$ and so J is coercive in E. Now, for any minimizing sequence $(u_n)_n$ in E, with $J(u_n) \to \inf_{u\in E} J(u)$ as $n \to \infty$, the coerciveness of J guarantees that there exists $K > 0$, such that $\|u_n\| \leq K$. Thus, for all n, it follows from the Hölder inequality that

$$|J(u_n)| \leq \max\left\{1, \frac{1}{2}M(R)\right\} \|u_n\|^2 + C_f\|u_n\| \leq \max\left\{1, \frac{1}{2}M(R)\right\} K^2 + C_fK,$$

which infers that

$$\inf_{u\in E} J(u_n) \geq -\max\left\{1, \frac{1}{2}M(R)\right\} K^2 - C_fK.$$

Hence, J is bounded below. Moreover, if $v_n \rightharpoonup v$ in E, in view of the weakly lower semi-continuity of $\|\cdot\|$,

$$J(v) \leq \liminf_{n\to\infty} \left(\frac{1}{2}M(R)[v_n]_s^2 + \int_{\mathbb{R}^3} V(x)|v_n|^2dx - \langle f, v_n\rangle\right),$$

We thus deduce that J is weakly lower semi-continuous. Consequently, it guarantees the existence of the unique global minimum u_R for the functional J in E, and moreover, u_R is obviously a (weak) solution of equation (3.6).

Next, let us turn to imply that u_R is also a (weak) solution of problem (3.5). Let $R_j \to R$ in $\mathbb{R}_+$ and let $(u_{R_j})_j$ be (weak) solutions of (3.5) with R replaced by R_j. Once again, by $(M_1)'$, the Hölder inequality and Lemma 2.4, we have

$$\min\{\kappa, V_0\}\|u_{R_j}\|^2 \leq M(R_j)[u_{R_j}]_s^2 + \|u_{R_j}\|_V^2 = \langle f, u_{R_j}\rangle \leq C_f\|u_{R_j}\|. \tag{3.7}$$

Thus, $\{u_{R_j}\}$ is bounded in E. The reflexivity of E, Lemmas 2.4 and 2.5 yield that, there exists $u \in E$, such that up to sequences, as $j \to \infty$,

$$(a)\ u_{R_j} \rightharpoonup u \text{ in } E; \quad (b)\ u_{R_j} \to u \text{ in } L^2(\mathbb{R}^3, wdx); \quad (c)\ u_{R_j} \to u \text{ in } L^q(\mathbb{R}^3) \text{ with } q \in [2, 2_s^*). \tag{3.8}$$

Recalling that $R_j \to R$ and $M \in C(\mathbb{R}^3)$ in the hypothesis $(M_1)'$, one has

$$\begin{aligned} &M(R)\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}vdx + \int_{\mathbb{R}^3} V(x)uvdx \\ &= \lim_{j\to\infty}\int_{\mathbb{R}^3} \left(M(R_j)(-\Delta)^{\frac{s}{2}}u_{R_j}(-\Delta)^{\frac{s}{2}}v + V(x)u_{R_j}v\right)dx \\ &= \langle f, v\rangle \quad \text{for any } v \in E, \end{aligned}$$

and so u is also a weak solution of (3.6). Moreover, taking the test function $v = u_R - u$ in the weak form of (3.6) and applying the Hölder inequality, we deduce that

$$\begin{aligned} 0 &= M(R)\langle L(u) - L(u_R), u - u_R\rangle + \langle u - u_R, u - u_R\rangle_V \\ &= M(R)\big([u]_s^2 - \langle L(u), u_R\rangle - \langle L(u_R), u\rangle + [u_R]_s^2\big) + \|u\|_V^2 - \langle u, u_R\rangle - \langle u_R, u\rangle + \|u_R\|_V^2 \\ &\geq M(R)\big([u]_s^2 - 2[u]_s[u_R]_s + [u_R]_s^2\big) + \|u\|_V^2 - 2\|u\|_V\|u_R\|_V + \|u_R\|_V^2 \\ &= M(R)([u]_s - [u_R]_s)^2 + (\|u\|_V - \|u_R\|_V)^2 \geq 0. \end{aligned} \tag{3.9}$$

We thus have $[u]_s^2 = [u_R]_s^2$ and $\|u\|_V = \|u_R\|_V$. Consequently,

$$\langle f, u - u_R\rangle = M(R)\langle L(u) - L(u_R), u - u_R\rangle + \langle u - u_R, u - u_R\rangle_V = 0,$$

and so $u = u_R$ a.e. in $\mathbb{R}^3$ due to the assumption that $f \not\equiv 0$. Hence,

$$u = u_R \quad \text{in } E, \tag{3.10}$$

and $u_{R_j} \rightharpoonup u_R$ in E due to (3.8)-(a). Now, we claim that

$$u_{R_j} \to u_R \quad \text{in } E. \tag{3.11}$$

From (3.8),

$$M(R_j)\left\langle L(u_{R_j}), u_{R_j} - u_R\right\rangle = \langle f, u_{R_j} - u_R\rangle - \int_{\mathbb{R}^3} V(x)u_{R_j}(u_{R_j} - u_R)dx \to 0, \quad \text{as } j \to \infty.$$

Combining with (2.4) and the fact that $D^s(\mathbb{R}^3)$ is a uniformly space, $u_{R_j} \to u_R$ in $D^s(\mathbb{R}^3)$ by applying Lemma 2.2, and moreover $u_{R_j} \to u_R$ in E by using (3.8)–(b). Therefore, the claim holds and the (weak) solution u_R of (3.6) is continuous with respect to R.

Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(R) = \frac{1}{M(R)}\langle f, u_R\rangle - \|u_R\|_V^2.$$

Note that, according to the continuity of mappings $R \mapsto \frac{1}{M(R)}$ by (M_1) and $R \mapsto u_R$, $h(R)$ is also a continuous mapping. Observe that $h(0) > 0$. In fact, we first claim that u_0, with $R = 0$, is not a constant. Otherwise, $\|u_0\|_V \leq C_d[u_0]_s = 0$ for some $C_d > 0$, due to Lemma 2.4, which implies in particular that $u_0 = 0$ a.e. in $\mathbb{R}^3$. Moreover, since u_0 is the (weak) solution of the problem

$$M(0)(-\Delta)^s u_0 + V(x)u_0 = f$$

and $f \not\equiv 0$, there is a contradiction with $u_0 = 0$ a.e. in $\mathbb{R}^3$. For such u_0,

$$h(0) = \frac{1}{M(0)}\langle f, u_0\rangle - \|u_0\|_V^2 = [u_0]_s^2 > 0.$$

Similarly, by using the same argument of (3.7) that u_R is bounded in E, there exists a positive constant C, such that

$$|h(R)| = \left|\frac{1}{M(R)}\langle f, u_R\rangle - \|u_R\|_V^2\right| \leq \frac{1}{\kappa}C_f\|u_R\| + \|u_R\|_V^2 \leq C_{f,\kappa}\|u_R\| + \|u_R\|^2 \leq C.$$

Now, denote $h_1(R) : \mathbb{R} \to \mathbb{R}$ as $h_1 = h(R) - R$. Combining all facts in the above, there exists $R_1 > C_f$, such that

$$h_1(0) = h(0) > 0 \quad \text{and} \quad h_1(R_1) = h(R_1) - R_1 < 0.$$

The intermediate value theorem yields at once the existence of zero-point for h_1. In other words, there exists $R > 0$, such that

$$R = h(R) = \frac{1}{M(R)}\langle f, u_R\rangle - \|u_R\|_V^2 = [u_R]_s^2.$$

Consequently, u_R is a weak solution of (3.1).

Consider the uniqueness of the (weak) solution of (3.1). Assume at first that there are distinct (weak) solutions $u_1, u_2 \in E$ of (3.1). Let $v = u_1 - u_2$ be the test function for the weak form of (3.1), which follows that

$$(a + b[u_1]_s^2)\,\langle L(u_1), u_1 - u_2\rangle + \langle u_1, u_1 - u_2\rangle_V = \int_{\mathbb{R}^3} f(u_1 - u_2)dx$$

and

$$(a + b[u_2]_s^2)\,\langle L(u_2), u_1 - u_2\rangle + \langle u_2, u_1 - u_2\rangle_V = \int_{\mathbb{R}^3} f(u_1 - u_2)dx$$

being u_1 and u_2 are the (weak) solutions of (3.1), where a, b are the constant given in the definition of Kirchhoff function M. As a consequence,

$$a\,\langle L(u_1) - L(u_2), u_1 - u_2\rangle + bJ_1(u_1, u_2) + \langle u_1 - u_2, u_1 - u_2\rangle_V = 0, \tag{3.12}$$

where

$$J_1(u_1, u_2) = [u_1]_s^2([u_1]_s^2 - \langle L(u_1), u_2\rangle) + [u_2]_s^2([u_2]_s^2 - \langle L(u_2), u_1\rangle).$$

By virtue of the Hölder inequality,

$$\begin{aligned} J_1(u_1, u_2) &\ge [u_1]_s^2([u_1]_s^2 - [u_1]_s[u_2]_s) + [u_2]_s^2([u_2]_s^2 - [u_2]_s[u_1]_s) \\ &\ge ([u_1]_s - [u_2]_s)([u_1]_s^3 - [u_2]_s^3) \ge 0. \end{aligned}$$

Then, clearly, by using the same argument of (3.9), from (3.12), $[u_1]_s = [u_2]_s$ and $\|u_1\|_V = \|u_2\|_V$. Similar to (3.10), it can be concluded that $u_1 = u_2$ in E.

Finally, it remains to prove that the operator K is continuous. Let $(f_j)_j \subset E^*$, $f \in E^*$ satisfy $f_j \to f$ strongly in E^* and $u_j, u \in E$ be the (weak) solutions of (3.1) corresponding to f_j and f, respectively. We only need to prove that $u_j \to u$ in E. Arguing as in the proof of (3.7) and (3.11), we conclude that $u_j \rightharpoonup u$ in E and $u_j \to u$ a.e. in $L^q(\mathbb{R}^3)$, with $q \in [2, 2_s^*)$, up to a sequence if necessary. Consequently,

$$\begin{aligned} M(u_j)\,\langle L(u_j), u_j - u\rangle &= \langle f_j, u_j - u\rangle - \langle u_j, u_j - u\rangle_V \\ &= \langle f_j - f, u_j\rangle + \langle f, u_j - u\rangle - \langle u_j, u_j - u\rangle_V \\ &\to 0, \quad \text{as } j \to \infty, \end{aligned}$$

which yields that $u_j \to u$ in E by Lemma 2.2. This completes the proof. □

We next prove the bifurcation results of (3.1). For any fixed λ, first denote the operator $N_\lambda :\ E \to E^*$ pointwise for all $u, v \in E$ as

$$\langle N_\lambda(u), v\rangle = \int_{\mathbb{R}^3} [\lambda g(x)|u|^{p-1}u - \phi_u^t u]v dx,$$

where $\langle \cdot, \rangle$ is the duality of E. We assert that $N_\lambda(u)$ is a compact operator. Suppose that $(u_j)_j$ is a bounded sequence in E. Lemma 2.4 yields that there exist a subsequence of $(u_j)_j$ (still defined by $(u_j)_j$) and $u \in E$, such that for any $R > 0$, as $j \to \infty$,

$$(a_1)\ u_j \rightharpoonup u \quad \text{in } E \qquad (a_2)\ u_j \to u \quad \text{in } L^q(\mathbb{R}^3),\ \text{with } q \in [2, 2_s^*) \qquad (a_3)\ u_j \to u \quad \text{a.e. in } \mathbb{R}^3. \tag{3.13}$$

By virtue of Lemma 2.3–(i_3), obviously it follows that

$$\sup_{\|v\| \le 1} \int_{\mathbb{R}^3} (\phi_{u_j}^t u_j - \phi_u^t u) v dx \to 0, \quad \text{as } j \to \infty.$$

Further, for all $R > 0$,

$$\begin{aligned} &\sup_{\|v\| \le 1} \left| \int_{\mathbb{R}^3} g(x)(|u_j|^{p-1}u_j - |u|^{p-1}u) dx \right| \qquad (3.14) \\ &\quad \le \sup_{\|v\| \le 1} \left| \int_{B_R} g(x)(|u_j|^{p-1}u_j - |u|^{p-1}u) dx \right| + \sup_{\|v\| \le 1} \left| \int_{\mathbb{R}^3 \setminus B_R} g(x)(|u_j|^{p-1}u_j - |u|^{p-1}u) dx \right|. \end{aligned}$$

Since $g \in L^{6/(5-p)}(\mathbb{R}^3)$, for any $\varepsilon > 0$, there is a constant $R > 0$ so large that

$$\begin{aligned} &\sup_{\|v\| \le 1} \int_{\mathbb{R}^3 \setminus B_R} g(x)(|u_j|^{p-1}u_j - |u|^{p-1}u) v dx \\ &\quad \le \sup_{\|v\| \le 1} \left(\int_{\mathbb{R}^3 \setminus B_R} |g(x)|^{\frac{6}{5-p}} dx \right)^{\frac{5-p}{6}} \left(\int_{\mathbb{R}^3 \setminus B_R} (|u_j|^p + |u|^p)^{\frac{6}{p}} dx \right)^{\frac{p}{6}} \|v\|_6 \\ &\quad \le \sup_{\|v\| \le 1} \|g\|_{L^{\frac{6}{5-p}}(\mathbb{R}^3 \setminus B_R)} 2^{\frac{6}{p}-1} (\|u_j\|_6^p + \|u\|_6^p) \|v\|_6 \\ &\quad \le 2^{\frac{6}{p}-1} c^{p+1} \|g\|_{L^{\frac{6}{5-p}}(\mathbb{R}^3 \setminus B_R)} (\|u_j\|^p + \|u\|^p) \sup_{\|v\| \le 1} \|v\| \\ &\quad \le \varepsilon/2. \end{aligned}$$

On the other hand, note that for all $R > 0$, the embedding $E \hookrightarrow L^q(B_R)$, with $q \in [1, 2_s^*)$, is compact by using Lemma 2.4. Hence, take a subsequence $(u_{j_k})_k \subset (u_j)_j$, such that $u_{j_k} \to u$ in $L^q(B_{R_\varepsilon})$ for all $q \in [1, 2_s^*)$, then up to a further subsequence, still denoted by $(u_{j_k})_k$, we have that $u_{j_k} \to u$ a.e. in B_{R_ε}. Thus, $g(x)|u_{j_k}|^{p+1} \to g(x)|u|^{p+1}$ a.e. in B_{R_ε}. Furthermore, for each measurable subset $B_E \subset B_{R_\varepsilon}$, with the help of (1.1), Lemma 2.4 and the Hölder inequality, we have

$$\int_{B_E} g(x)|u_{j_k}|^{p+1} dx \le \|g\|_{L^{\frac{6}{5-p}}(B_E)} \|u_{j_k}\|_6^{p+1} \le (cC)^{p+1} \|g\|_{L^{\frac{6}{5-p}}(B_E)},$$

being $(u_j)_j$ is bounded in E. Therefore, $(g(x)|u_{j_k}|^{p+1})_k$ is integrable and uniformly bounded in $L^1(B_{R_\varepsilon})$, since $g \in L^{6/(5-p)}(\mathbb{R}^3)$ by the assumption. The Vitali convergence theorem shows that

$$\lim_{k \to \infty} \int_{B_{R_\varepsilon}} g(x)|u_{j_k}|^{p+1} dx = \int_{B_{R_\varepsilon}} g(x)|u|^{p+1} dx, \tag{3.15}$$

and so $g(x)|u_j|^{p+1} \to g(x)|u|^{p+1}$ in $L^1(B_{R_\varepsilon})$, since the sequence $(u_{j_k})_k$ is arbitrary. Therefore,

$$\sup_{\|v\| \le 1} \left| \int_{B_R} g(x)(|u_j|^{p-1}u_j - |u|^{p-1}u) v dx \right| \to 0, \quad \text{as } j \to \infty,$$

and further (3.14) hold. Together with Proposition 3.2, we deduce that the operator $K \circ N_\lambda : E \to E$ is compact. For the fixed λ, let $K_\lambda : E \to E$, defined by

$$K_\lambda = \mathbb{I} - K \circ N_\lambda,$$

where $\mathbb{I}$ is the identity operator. Note that the zeros of K_λ are exactly the (weak) solutions of the problem (3.1).

Having completed all necessary preparations, now, we are ready to show Theorem 3.3.

Theorem 3.3. *Let $s,\ p \in (0,1)$. If (V_1)–(V_2), $(M_1)'$ and (g_1) hold, equation (3.1) has the unique bifurcation point $(0,0)$, and there exists an unbounded component $\mathcal{C}_0$ of (weak) solutions emanating from $(0,0)$.*

Proof. We first let $\lambda < 0$. For a fixed λ, consider the operator $H_1(r,\cdot) : E \to E$ as follows

$$H_1(r,u) = N_\lambda(r(\lambda g(x)|u|^{p-1}u - \phi_u^t u)), \qquad r \in [0,1].$$

We claim that there exists $\delta_1 > 0$, such that

$$u = H_1(r,u), \quad \text{for any } u \in B_{\delta_1}, u \not\equiv 0 \text{ and } r \in [0,1]. \tag{3.16}$$

Conversely, if there exists sequences $(u_n)_n$ and $(r_n)_n$, with $\|u_n\| \to 0$, $u_n \not\equiv 0$ and $r_n \in [0,1]$, such that $u_n = H_1(r_n,u_n)$. In other words, it follows that

$$\int_{\mathbb{R}^3} (M([u_n]_s^2)|(-\Delta)^{\frac{s}{2}} u_n|^2 + V(x)u_n^2 + r_n \phi_{u_n}^t u_n^2)dx = r_n \int_{\mathbb{R}^3} \lambda g(x)|u_n|^{p+1}dx \le 0 \tag{3.17}$$

by the definition of λ. Thanks to (M_1) and (V_1), we get $M([u_n]_s^2)[u_n]_s^2 + \|u_n\|_V^2 \ge 0$, and so $\|u_n\| = ([u_n]_s^2 + \|u_n\|_V^2)^{1/2} = 0$ by Lemma 2.3–(i_2). Of course, this is a contradiction with the assumption that $u_n \not\equiv 0$ in E and the claim is achieved. Therefore, we can choose $\varepsilon \in (0,\delta_1)$, such that

$$\deg(K_\lambda, B_\varepsilon, 0) = \deg(I - H_1(1,\cdot), B_\varepsilon, 0) = \deg(I - H_1(0,\cdot), B_\varepsilon, 0) = \deg(I, B_\varepsilon, 0) = 1 \tag{3.18}$$

by applying the homotopy invariance of H_1.

On the other hand, let $\lambda > 0$ and let $\psi \in E$, with $\psi > 0$. For this fixed λ and for any $r \in [0,1]$, denote $H_2(r,\cdot) : E \to E$ as

$$H_2(r,u) = N_\lambda(\lambda g(x)|u|^{p-1}u - \phi_u^t u + r\psi).$$

We claim that there exists $\delta_2 > 0$, such that $u \neq H_2(r,u)$ for any $u \in B_{\delta_2} \setminus \{0\}$ and for any $r \in [0,1]$. Let us argue by contradiction that if there exists a sequence $(v_j)_j \subset E$, with $v_j > 0$ and $\|v_j\| \to 0$, as $j \to \infty$, such that for any $r_j \in [0,1]$,

$$v_j = H_2(r_j, v_j), \tag{3.19}$$

which yields at once that

$$M([v_j]_s^2)(-\Delta)^s v_j + V(x)v_j + \phi_{v_j}^t v_j = \lambda g(x)|v_j|^{p-1}v_j + r_j\psi(x). \tag{3.20}$$

Moreover, there exists a positive constant C_0, such that $\|v_j\| \le C_0$ and $[v_j]_s \le \|v_j\| \le C_0$, being $\|v_j\| \to 0$ as $j \to \infty$. Furthermore, up to sequence,

$$v_j \to 0 \quad \text{a.e. in } \mathbb{R}^3 \tag{3.21}$$

by Lemma 2.4. Consequently, $M([v_j]_s^2) \le \max\{1, a + bC_0\} := C_0'$, and then

$$\lambda_1(g(x)(\max\{1, M([v_j]_s^2)\})^{-1}) \le \lambda_1((C_0')^{-1}g(x)).$$

For any $\varepsilon > 0$, taking the test function as a first eigenfunction $w_1 > 0$, by virtue of (1.1) and the Hölder inequality, since $g \in L^{6/(5-p)}(\mathbb{R}^3)$ by the assumption (g_1) and v_j is bounded in E, there exists $R_\varepsilon > 0$ so large that for all j

$$\begin{aligned}\int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} g(x)|v_j|^p w_1 dx &\le \|g\|_{L^{\frac{6}{5-p}}(\mathbb{R}^3 \setminus B_{R_\varepsilon})} \|v_j\|_6^p \|w_1\|_6 \\ &\le c^{p+1}\|g\|_{L^{\frac{6}{5-p}}(\mathbb{R}^3 \setminus B_{R_\varepsilon})} \|v_j\|^p \|w_1\| \le \varepsilon/2.\end{aligned} \tag{3.22}$$

Thus, arguing as the proof of (3.15),

$$\int_{B_{R_\varepsilon}} g(x)|v_j|^p w_1 dx = o(1) \quad \text{as } j \to \infty.$$

Similarly, according to the assumption (g_2), it is easily to see that $g(x)|v_j|w_1 \to 0$ in $L^1(B_{R_\varepsilon})$ as $j \to \infty$ and

$$\int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} g(x)v_j w_1 dx \le \|g\|_{L^{\frac{3}{2}}(\mathbb{R}^3 \setminus B_{R_\varepsilon})} \|v_j\|_6 \|w_1\|_6 \le c^2 \|g\|_{L^{\frac{3}{2}}(\mathbb{R}^3 \setminus B_{R_\varepsilon})} \|v_j\| \|w_1\| \le \varepsilon, \tag{3.23}$$

being $g \in L^{3/2}(\mathbb{R}^3)$ by the assumption. In conclusion, from (3.21), (3.22) and (3.23), there is R_ε so large that as $j \to \infty$

$$\begin{aligned}&\lambda \int_{\mathbb{R}^3} g(x)|v_j|^p w_1 dx - \lambda_1((C_0')^{-1}g(x)) \int_{\mathbb{R}^3} g(x)v_j w_1 dx - \int_{\mathbb{R}^3} \phi_{v_j} v_j w_1 dx \\ &= \lambda \int_{B_{R_\varepsilon}} g(x)|v_j|^p w_1 dx + \lambda \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} g(x)|v_j|^p w_1 dx - \lambda_1((C_0')^{-1}g(x)) \int_{B_{R_\varepsilon}} g(x)v_j w_1 dx \\ &\quad - \lambda_1((C_0')^{-1}g(x)) \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} g(x)|v_j| w_1 dx - C\|v_j\|^3 \|w_1\| \\ &\ge \lambda \int_{B_{R_\varepsilon}} g(x)|v_j|^p w_1 dx - \lambda_1((C_0')^{-1}g(x)) \int_{B_{R_\varepsilon}} g(x)|v_j| w_1 dx - C\varepsilon > 0.\end{aligned} \tag{3.24}$$

Since $\psi > 0$, (3.20) and (3.24) yield that as $n \to \infty$, we estimate

$$\begin{aligned}&\lambda_1(g(x)(\max\{1, M([v_j]_s^2)\})^{-1}) \int_{\mathbb{R}^3} g(x)v_j w_1 dx \\ &= \max\{M([v_j]_s^2), 1\} \left(\int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v_j (-\Delta)^{\frac{s}{2}} w_1 dx + \int_{\mathbb{R}^3} V(x)v_j w_1 dx \right) \\ &\ge M([v_j]_s^2) \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v_j (-\Delta)^{\frac{s}{2}} w_1 dx + \int_{\mathbb{R}^3} V(x)v_j w_1 dx \\ &= \lambda \int_{\mathbb{R}^3} \left(g(x)|v_j|^{p-1} v_j w_1 + r_j \psi(x) w_1 - \phi_{v_j}^t v_j w_1 \right) dx \\ &> \lambda_1((C_0')^{-1}g(x)) \int_{\mathbb{R}^3} g(x)v_j w_1 dx,\end{aligned}$$

and so

$$\{\lambda_1(g(x)\max\{1, M([v_j]_s^2)\}^{-1}) - \lambda_1((C_0')^{-1}g(x))\} \int_{\mathbb{R}^3} g(x)v_j w_1 dx > 0.$$

Since $\int_{\mathbb{R}^3} g(x)v_j w_1 dx > 0$, we have $\lambda_1(g(x)\max\{1, M([v_j]_s^2)\}^{-1}) > \lambda_1((C_0')^{-1}g(x))$. This is an obvious absurdum, and we proved the claim.

Hence, choosing $\varepsilon \in (0, \delta_2)$, we can find the homotopy invariance of H_2, i.e.

$$\deg(K_\lambda, B_\varepsilon, 0) = \deg(I - H_2(0,\cdot), B_\varepsilon, 0) = \deg(I - H_2(1,\cdot), B_\varepsilon, 0) = 0. \tag{3.25}$$

It follows from (3.18) and (3.25) that $(0,0)$ is a bifurcation point of ($\mathcal{P}$).

Now, it is sufficient to prove the existence of the unbounded component of (weak) solutions of (3.1). It is important to note that while the classical global bifurcation theorem [17, Theorem 1.3] is relevant to our argument, we cannot apply it directly because the operator K_λ lacks the differentiability at $u = 0$ and of odd-multiplicity eigenvalue. However, by modifying the global bifurcation theorem in Proposition 3.5 of [1] and replacing these conditions with the topological degree proofs for (3.18) and (3.25), we can derive an efficient version of [17, Theorem 1.3] for the assertion below.

For $\lambda_0 \neq 0$, we claim that $(\lambda_0, 0)$ is an isolated (weak) solution of (3.1). Set $\lambda < 0$. Similar to the analysis of (3.17), there are no nontrivial (weak) solutions of equation (3.1). Let $\lambda > 0$. Assume that there exists a sequence of (weak) solutions $(\lambda_n, u_n)_n \subset \mathbb{R} \times E$ of (3.1), such that $\lambda_n \to \lambda_0$ and $\|u_n\| \to 0$, as $n \to \infty$. Hence, arguing as (3.24), for any $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$, such that for any $n \geq N(\varepsilon)$,

$$\begin{aligned}
&\lambda_1(g(x)(\max\{1, M([v_j]_s^2)\})^{-1}) \int_{\mathbb{R}^3} g(x) v_j w_1 dx \\
&\quad \geq M([v_j]_s^2) \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} v_j (-\Delta)^{\frac{s}{2}} w_1 dx + \int_{\mathbb{R}^3} V(x) v_j w_1 dx \\
&\quad \geq (\lambda_0 - \varepsilon) \int_{\mathbb{R}^3} g(x) |v_j|^{p-1} v_j w_1 dx - \int_{\mathbb{R}^3} \phi_{v_j}^t v_j w_1 dx \\
&\quad > \lambda_1((C_0')^{-1} g(x)) \int_{\mathbb{R}^3} g(x) v_j w_1 dx,
\end{aligned}$$

which yields an absurdum $\lambda_1(g(x)(\max\{1, M([v_j]_s^2)\})^{-1}) > \lambda_1((C_0')^{-1} g(x))$. Therefore, $(0,0)$ is a unique bifurcation point of equation (3.1).

Furthermore, if $\mathcal{C}_0$ is bounded in $\mathbb{R} \times E$, by [17, Lemma 1.2] there is a bounded open set $\mathcal{O} \subset \mathbb{R} \times E$ such that $(0,0) \in \mathcal{O}$ and $\mathcal{O}$ contains nontrivial solution other than those in $B_\varepsilon \subset E$, with $\varepsilon > 0$ sufficiently small.

Now, we can argue as (1.11) of [17] to conclude that the existence of $\varepsilon > 0$ and values $\underline{\lambda}$ and $\overline{\lambda}$, such that $-\varepsilon < \underline{\lambda} < 0 < \overline{\lambda} < \varepsilon$ and $i(K_{\underline{\lambda}}, 0) = i(K_{\overline{\lambda}}, 0)$. Therefore, owing to (3.18) and (3.25), we have

$$1 = i(K_{\underline{\lambda}}, 0) = i(K_{\overline{\lambda}}, 0) = 0,$$

which is an obvious contradiction. Then, $\mathcal{C}_0$ is an unbounded component. □

4 Main result

To determine the bifurcation results of problem ($\mathcal{P}$), for any fixed λ, we define pointwise for $u, v \in E$, $T_\lambda : E \to E^*$ by

$$\langle T_\lambda(u), v \rangle = \int_{\mathbb{R}^3} \left\{ \lambda g(x) |u|^{p-1} u + |u|^{2_s^* - 2} u - \phi_u^t u \right\} v dx.$$

Suppose that $(u_n)_n \subset E$ is a bounded sequence in E. Then up to a subsequence, (3.13) also holds for some $u \in E$ by the reflexivity of E. Recalling the compactness result for the operator

N_λ, as shown in section 3, it remains to prove that for any $v \in E$,

$$\int_{\mathbb{R}^3} (|u_n|^{2_s^*-2}u_n - |u|^{2_s^*-2}u)v dx \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

Since $|u_n|^{2_s^*-2}u_n \in L^{(2_s^*)'}(\mathbb{R}^3)$, $v \in E$ and $E \subset L^{2_s^*}(\mathbb{R}^3)$, the definition of weak convergence yields at once that (4.1) is achieved. In conclusion, the operator $K \circ T_\lambda$ is also compact using Proposition 3.2.

Proof of Theorem 1.1. Let $H_\lambda : E \to E$ be defined as $H_\lambda(u) = K \circ T_\lambda(u)$, where K is the operator introduced by Proposition 3.2. Clearly, Theorem 3.3 guarantees the existence of the positive constants ε and δ, such that

$$\deg(K_\lambda, B_\delta, 0) = \begin{cases} 1, & \lambda \in (-\varepsilon, 0), \\ 0, & \lambda \in (0, \varepsilon). \end{cases}$$

We claim that for any λ, with $0 < \lambda < \varepsilon$, there exist δ_1, such that for any $r \in [0,1]$ and for the operator, defined by

$$\langle T_\lambda^r(u), v\rangle = \int_{\mathbb{R}^3} \left\{\lambda g(x)|u|^{p-1}u - \phi_u^t u + r|u|^{2_s^*-2}u\right\} v dx,$$

the problem

$$u - K \circ T_\lambda^r(u) = 0 \tag{4.2}$$

has no (weak) solutions with $\|u\| = \delta_1$. Otherwise, if there exists a sequence of nontrivial (weak) solutions $(u_n)_n$ of (4.2), with $\|u_n\| \to 0$ and $u_n > 0$, then it yields that

$$M([u_n]_s^2)[u_n]_s^2 + \|u_n\|_V^2 + \int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 dx = \int_{\mathbb{R}^3} \left\{\lambda g(x)|u_n|^p + r|u_n|^{2_s^*}\right\} dx.$$

Thanks to (3.24), taking the test function as the first eigenvalue w_1, we have

$$\begin{aligned} &\lambda_1(g(x)(\max\{1, M([u_n]_s^2)\})^{-1}) \int_{\mathbb{R}^3} g(x)u_n w_1 dx \\ &\quad = M([u_n]_s^2) \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} w_1 dx + \int_{\mathbb{R}^3} V(x)u_n w_1 dx \\ &\quad = \int_{\mathbb{R}^3} (\lambda g(x)|u_n|^{p-1}u_n w_1 - \phi_{u_n}^t u_n w_1 + |u_n|^{2_s^*-1} w_1) dx \\ &\quad > \lambda_1((C_0')^{-1} g(x)) \int_{\mathbb{R}^3} g(x) u_n w_1 dx, \end{aligned}$$

which implies an absurdum that $\lambda_1(g(x)(\max\{1, M([u_n]_s^2)\})^{-1}) > \lambda_1((C_0')^{-1}g(x))$. The claim holds. Hence, the homotopy invariance of the topological degree shows that for any $\lambda \in (0,\varepsilon)$ and $R \in (0, \delta_1)$

$$\deg(I - H_\lambda, B_R, 0) = \deg(K_\lambda, B_R, 0) = 0. \tag{4.3}$$

Fix $\lambda < 0$. Applying the same argument of (3.24), it follows that

$$\int_{\mathbb{R}^3} \{\lambda g(x)|u|^{p+1} + |u|^{2_s^*}\} dx \le 0.$$

Now similar to the analysis of (3.17), there are no nontrivial (weak) solutions of (4.2). Consequently, there exist $\varepsilon > 0$ and $\delta > 0$, with $\varepsilon \le \varepsilon_1$ and $\delta \le \delta_1$, such that for any $\lambda \le \varepsilon$ and for any $R \le \delta$

$$\deg(I - H_\lambda, B_R, 0) = \deg(K_\lambda, B_R, 0) = 1. \tag{4.4}$$

By utilizing (4.3) and (4.4), we get $(0,0)$ is a bifurcation point of equation ($\mathcal{P}$). Moreover, similar to the argument in Theorem 3.3, we imply that the existence of an unbounded component $\mathcal{C}$ of weak solutions of ($\mathcal{P}$). □

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The blow-up method applied to monodromic singularities

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Abstract. The blow-up method proves its effectiveness to characterize the integrability of the resonant saddles giving the necessary conditions to have formal integrability and the sufficiency doing the resolution of the associated recurrence differential equation using induction. In this work we apply the blow-up method to monodromic singularities in order to solve the center-focus problem. The case of nondegenerate monodromic singularities is straightforward since any real nondegenerate monodromy singularity can be embedded into a complex system with a resonant saddle. Here we apply the method to nilpotent and degenerate monodromic singularities solving the center problem when the center conditions are algebraic.

Keywords: monodromic singularity, blow-up, center problem, formal first integral.

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1 Introduction

The center-focus problem for systems of differential equations is one of the main unsolved problems in the qualitative theory of differential systems in the plane [23, 26]. For the non-degenerate monodromic singularities the center-focus problem is closely connected with integrability problem, see for instance references [37, 39]. The center-focus problem consists of providing the necessary and sufficient conditions under which a monodromic singularity has a neighborhood filled with periodic orbits. If the monodromic singularity is a non-degenerate singular point, i.e., its linear part has two purely imaginary eigenvalues, then the real differential system can be embedded in the complex plane and the singular point it transformed to a $1 : -1$ resonant saddle singular point, see [15, 16, 29, 30].

Indeed, the $1 : -1$ resonance can be generalized into a $p : -q$ resonance known as a $p : -q$ resonant singular point of a polynomial vector field in $\mathbb{C}^2$, see [19, 44].

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The characterization of the analytic integrability of several families of differential systems with a resonant saddle is studied in several works, see for instance [17–19, 28–30, 44] and references therein.

In order to find the necessary conditions of analytic integrability of a $p : -q$ resonant singular point there exist different algorithms. One of them is based on the transformation of the original system to its normal form through a series of invertible changes of variables [2]. Another algorithm propose directly the formal first integral, see [41,44]. Recently the blow-up method has been introduced to compute the necessary conditions, see [16].

Once the necessary conditions are obtained the second step is to prove their sufficiency. There is no general algorithm that works for all differential systems in order to prove the sufficiency. The sufficiency is guaranteed if, for instance, the system is Hamiltonian or time-reversible. Recall that a time-reversible system is invariant by certain symmetry. The existence of an explicit first integral well-defined in a neighborhood of the singular point guarantees also the existence of a center in a monodromic singular point. This first integral can be found through the knowledge of an integrating factor. The connections between integrating factors an analytic first integrals have been studied by different authors, see [8, 13, 31, 41] and references therein. Finally ad hoc methods to prove the sufficiency are used for some particular families, see for instance [12, 13, 19, 32, 34–36, 40, 44]. All these different algorithms to prove the sufficiency have been useless for certain differential systems. However, in [15] the blow-up method is used to prove the sufficiency doing the resolution of the associated recurrence differential equation using induction and all the open problems of previous works have been solved.

We remark that for an isolated singularity the existence of a formal first integral implies the existence of an analytic first integral, see [10,41]. Consequently, to prove the sufficiency is sufficient to prove the existence of a formal first integral. In [3] the formal integrability was studied through the existence of invariant analytic (sometimes algebraic) curves.

In this paper we use the blow-up method to approach the center-focus problem for nilpotent and degenerate monodromic singularities, also when there exists no formal integral. This method that was successfully applied for resonant saddles and nondegenerate monodromic singularities, is used here to determine necessary conditions. Also, it is also possible to prove the sufficiency when the center is formally integrable. We solve open cases and cases previously studied with very difficult techniques.

2 Blow-up method for monodromic non-degenerate singular points

A monodromic non-degenerate singular point at the origin of a differential system on $\mathbb{R}^2$ takes the form

$$\dot{u} = v + P(u,v), \qquad \dot{v} = -u + Q(u,v), \tag{2.1}$$

where $P(u,v)$ and $Q(u,v)$ are real analytic functions without constant and linear terms. Such singular point is a center, if and only if, the system has a first integral of the form

$$\Phi(u,v) = u^2 + v^2 + \sum_{k+l\geq 3} \phi_{kl} u^k v^l. \tag{2.2}$$

analytically defined around it, see [37,39]. Therefore, the center-focus problem reduces to the case of proving the existence of such analytic first integral. From this result straightforward emerge a method to determine the first necessary conditions to have a center, which consists in proposing a power series of the form (2.2). However the unique general method that enables us to prove the sufficiency for this first necessary conditions is to use the method developed in [15] solving the recurrence differential equation associated to the problem using induction.

The first step to apply the method is to complexify system (2.1) defining the complex variable $x = u + iv$ and system (2.1) becomes the equation $\dot{x} = ix + R(x, \bar{x})$. Considering also its complex conjugate equation we have the system

$$\dot{x} = ix + R(x, \bar{x}), \qquad \dot{\bar{x}} = -i\bar{x} + \bar{R}(x, \bar{x}).$$

If we define $y := \bar{x}$ as a new variable and $\bar{R}$ as a new function we obtain a complex system which is after the change of time $idt = dT$ written as

$$\dot{x} = x + G(x, y), \qquad \dot{y} = -y + H(x, y). \tag{2.3}$$

The power series (2.2) is now transformed into

$$\Psi(x, y) = xy + \sum_{i+j>2} \psi_{ij} x^i y^j,$$

verifying that $\mathcal{X}\Psi = \sum_{i=1} v_{2i+1}(xy)^{2i+2}$, where $\mathcal{X}$ is the vector field associated to system (2.3) and v_{2i+1} are polynomials in the parameters of the system. We note that if all the polynomials v_{2i+1} vanish then the power series $\Psi(x, y)$ is first integral of system (2.3). The singular point at the origin of system (2.3) is $1 : -1$ resonant saddle singular point and the values v_{2i+1} are the so-called *saddle constants*, see [41, 44].

When the $1 : -1$ resonant saddle singular point at the origin is generalized into the $p : -q$ resonant saddle singular point at the origin then the differential system is of the form

$$\dot{x} = p\,x + F_1(x, y), \qquad \dot{y} = -q\,y + F_2(x, y), \tag{2.4}$$

where F_1 and F_2 are analytic functions without constant and linear terms with $p, q \in \mathbb{Z}$ and $p, q > 0$, see [14, 33, 44]. In this case a $p : -q$ resonant saddle singular point is called a resonant center, if an only if, there exists a meromorphic first integral $\Psi = x^q y^p + \sum_{i+j>p+q} \psi_{ij} x^i y^j$ around it. We recall here that if $\Psi(x, y) \in \mathbb{C}[[x, y]]$, i.e, is a formal first integral in a neighborhood of the singularity, then there also exists an analytic first integral.

The blow-up method to detect formal integrability for a resonant singular point works as follows. We perform the blow-up $(x, y) \to (x, z) = (x, y/x)$ to system (2.4) which has a resonant singular point at the origin. So that the origin is replaced by the line $x = 0$, which contains two singular points that correspond to the separatrices of the resonant point at the origin of system (2.4). These two singular points are a $(p + q) : -p$ resonant saddle and a $(p + q) : -q$ resonant saddle that we call p_1 and p_2, respectively. The method is based on the following result.

Theorem 2.1. *The $p : -q$ resonant singular point at the origin of system (2.4) is analytically integrable if, and only if, either p_1 or p_2 is orbitally analytically linearizable.*

The proof is based on the fact that if the $p : -q$ resonant singular has an analytic first integral $\Psi(x, y)$ then both points p_1 or p_2 have also a well-defined analytic first integral given by $\Psi(x, zx)$. The sufficiency follows from Lemma 1 of [19] using the normal orbital form of the $p : -q$ resonant system (2.4) and the first integral of such normal orbital form. From Theorem 2.1 we deduce that the necessary conditions of integrability for the $p : -q$ resonant singular point generate the same ideal that the necessary integrability conditions of the singular points p_1 or p_2.

Hence we apply the blow-up $z = y/x$ and system (2.4) is transformed into the system

$$\dot{z} = -(p+q)z + x\mathcal{F}(x,z), \qquad \dot{x} = p\,x + x^2\mathcal{G}(x,z), \tag{2.5}$$

where $\mathcal{F}(0,0) = 0$ and $x = 0$ is an invariant line of the new system. Next we propose the power series $\tilde{\mathcal{H}} = \sum_{i\geq 1}^{\infty} f_i(z)x^i$, where $f_i(z)$ are arbitrary functions of z (in the case of formal integrability these functions must be polynomials). Let $\bar{\mathcal{X}}$ be the vector field associated to system (2.5). The lower terms of equation $\bar{\mathcal{X}}\tilde{\mathcal{H}} = 0$ give the differential equation for $f_1(z)$ given by $pf_1(z) - (p+q)zf_1'(z) = 0$ whose solution is $f_1(z) = c_1 z^{p/(p+q)}$. Taking into account that $f_1(z)$ must be a polynomial we take $c_1 = 0$ and consequently $f_1(z) = 0$. The power two of terms give the differential equation $2pf_2(z) - (p+q)zf_2'(z) = 0$ and its solution is $f_2(z) = c_2 z^{(2p)/(p+q)}$. Consequently, either $(2p)/(p+q) \in \mathbb{N}$ or we take $c_2 = 0$. Taking into account that $p,q \in \mathbb{Z}$ with $p,q > 0$ it always exists f_{k_0} such that $(k_0 p)/(p+q) \in \mathbb{N}$ (or $(k_0 q)/(p+q) \in \mathbb{N}$ for saddle point p_2). Finally, for each power of x of the equation $\bar{\mathcal{X}}\tilde{H} = 0$ we get the differential equation

$$k\,p\,f_k(z) - (p+q)\,z\,f_k{}'(z) + g_k(z) = 0, \tag{2.6}$$

where $g_k(z)$ depends on some previous functions $f_i(z)$ for $i = k_0, \ldots, k-1$. The solution of differential equation (2.6) is given by

$$f_k(z) = c_k\,z^{\frac{kp}{p+q}} + z^{\frac{kp}{p+q}} \int \frac{z^{-1-\frac{kp}{p+q}}}{p+q}\,g_k(z)\,dz, \tag{2.7}$$

where c_k is an arbitrary constant. From (2.7) it is easy to see that functions f_k in (2.7) are always polynomials except when appear logarithmic terms. If the origin is not a resonant center, always exists a value k_r such that for $k \geq k_r$ the functions $f_i(z)$ for $i \geq k_r$ can have logarithmic terms. In fact, the logarithmic term appears when there is a term s^{-1} in the integral of (2.7). This is the case when

$$-1 - \frac{k_r p}{p+q} + m_k = -1,$$

where m_k is the degree of the polynomial $g_k(s)$. So, we have $k_r = m_k(p+q)/p$. The coefficients of these logarithmic terms are the saddle constants of the original system (2.4).

Vanishing a certain number of saddle constants and checking that some of the next ones are zero we can apply the following procedure. First we apply the induction method to prove that the solution f_k of recursive equation (2.6) is always a polynomial to assure that system (2.5) has a formal first integral. Second, to prove the sufficiency of the original system (2.4) we can apply the following result.

Theorem 2.2. *Assume that system (2.5) has a formal first integral $\tilde{\mathcal{H}}(x,z)$. If the function $\tilde{H} = \tilde{\mathcal{H}}(x,y/x)$ is well-defined at the origin of system (2.4) then this system is analytic integrable in a neighborhood of the origin.*

The idea of the method is to study the connected singular points at infinity and if they are formally integrable and the first integral can be extended up to the origin then the origin is also formally integrable. The reason of why the coordinates $(x, z = y/x)$ are better than the original coordinates (x, y) is double. First because doing the blow-up we introduce x as a invariant curve of the new differential system and then we can propose an expansion passing through the origin in powers of x with coefficients as functions of z. The second is because the in the new variables (x, z) the coefficients functions of z are polynomials with perhaps some logarithmic terms, see [3]. This does not happens in the original variables, where the system may not have any invariant curve and if it does then the coefficient of the expansion do not have to be polynomial.

In this work we apply the same method to nilpotent and degenerate monodromic singularities in order to solve the center-focus problem. For degenerate monodromic singularities there is no general method to approach the center-focus problem. The method shows that the formal integrability of the points at infinity is intimately linked with the center problem at the origin even though the center at the origin is not formally integrable. The method determine center conditions for monodromic singularities which are algebraically solvable. In the following sections we solve several non trivial examples. The method can also be applied to systems that are not formally integrable at the monodromic singular point giving information for studying the center-focus problem.

3 Nilpotent monodromic singularities

In this section we consider different systems with a nilpotent singularity, and we study, using the blow-up method, the center-focus problem of such systems.

Proposition 3.1. *The nilpotent real cubic differential system*

$$\dot{x} = y + Ax^2y + Bxy^2 + Cy^3, \qquad \dot{y} = -x^3 + Px^2y + Kxy^2 + Ly^3. \tag{3.1}$$

is a center if and only if $P = B + 3L = (A + K)L = 0$.

Proof. In [9] was solved the center-focus problem of the nilpotent cubic system (3.1) constructing a Liapunov function and using different methods to prove the sufficiency. Indeed it is well-known that all the centers are analytically (hence formally) integrable, see [7]. Later in [22,27] the center-focus problem of such system is also solved using the fact that all the nilpotent centers are limit of non-degenerate centers. Here, we apply the blow-up method to solve it. Hence, applying the blow-up transformation

$$(x, y) \to (z, y) = (x/y, y) \tag{3.2}$$

system (3.1) becomes

$$\begin{aligned} \dot{z} &= 1 + Cy^2 + By^2z - Ly^2z + Ay^2z^2 - Ky^2z^2 - Py^2z^3 + y^2z^4, \\ \dot{y} &= y^3(L + Kz + Pz^2 - z^3), \end{aligned} \tag{3.3}$$

which has a regular point at the origin. Therefore system (3.3) is analytic integrable at the origin and the recursive differential equation do not generate logarithmic terms. Next, we propose the power series

$$\mathcal{H}(z, y) = \sum_{k=2}^{\infty} f_k(z) y^k. \tag{3.4}$$

We impose that $\dot{\mathcal{H}} = \dot{z}\partial\mathcal{H}/\partial z + \dot{y}\partial\mathcal{H}/\partial y = 0$ and equating to zero each coefficient of power of y we obtain the following recursive differential equation for the functions f_k

$$(k-1)(L+Kz+Pz^2-z^3)f_{k-1} + (C+Bz-Lz+Az^2-Kz^2-Pz^3+z^4)f'_{k-1} + f'_{k+1} = 0. \quad (3.5)$$

Solving for the first values of k we can take $f_k = 0$ for all k odd and for k even we find $f_2 = c_2$, where c_2 is an arbitrary integration constant that we can take $c_2 = 1$, then we have

$$f_4 = \frac{1}{6}(-12Lz - 6Kz^2 - 4Pz^3 + 3z^4) + c_4,$$
$$f_6 = \frac{1}{630}(P_6(z) - 60Pz^7) + c_6.$$

In order to have a polynomial in the original variables (x, y) we must to take $P = 0$. Then f_8 is a polynomial of degree 9 of the form

$$f_8 = \frac{1}{83160}\big(P_8(z) - 3696(B+3L)z^9\big) + c_8.$$

In this case we have to take $B + 3L = 0$. Taking $B = -3L$ then f_{10} is a polynomial of degree 15 given by

$$f_{10} = \frac{1}{83160}\big(P_{10}(z) - 5896800(A+K)Lz^{11}\big) + c_{10}.$$

If $(A+K)L = 0$ we have checked that some of the next f_k for k even are all of degree at most k. Now, we assume that f_s have degree s for $s = 2, 4, \ldots, k-1$ and solving the recursive equation (3.5) we obtain

$$f_{k+1}(z) = -\int (k-1)(L+Kz-z^3)f_{k-1} + (C-4Lz+(A-K)z^2+z^4)f'_{k-1} \quad (3.6)$$

where it is easy to see that the higher terms cancel, that is, if we introduce $f_{k-1}(z) = C_0 + C_1 z + \cdots + C_{k-1}z^{k-1}$ in (3.6) we get a polynomial for f_{k+1} of degree at most $k+1$. Consequently, we have proven the sufficiency since we have a formal first integral at the origin that in the original variables (x, y) is also formal for all the center cases. Here the blow-up method gives straightforward the necessary conditions and the sufficiency for all cases and in a unified method for all the center cases. □

Proposition 3.2. *Consider the nilpotent differential system*

$$\dot{x} = Ax^3 + By, \qquad \dot{y} = Cx^5 + Dx^2y, \quad (3.7)$$

where the unique monodromic condition is $(D-3A)^2 + 12BC < 0$. *It has a center at the origin if and only if* $3A + D = 0$.

Proof. The monodromic and center-focus problem of system (3.7) has been solved in [1]. Indeed, system (3.7) is a $(1,3)$-quasihomogeneous system and consequently, $V(x,y) = Cx^6 - 3Ax^3y + Dx^3y - 3By^2$ is an inverse integrating factor of (3.7). In fact such (p,q)-quasihomogeneous systems of degree r has a unique center condition given by

$$\int \frac{F_r(\varphi)}{G_r(\varphi)} d\varphi = 0 \quad (3.8)$$

where

$$G_r(\varphi) = p\,Q_{p+r}(\cos\varphi, \sin\varphi)\cos\varphi - q\,P_{q+r}(\cos\varphi, \sin\varphi)\sin\varphi,$$
$$F_r(\varphi) = P_{p+r}(\cos\varphi, \sin\varphi)\cos\varphi + Q_{q+r}(\cos\varphi, \sin\varphi)\sin\varphi.$$

using the weighted polar blow-up $(x, y) \to (\rho, \varphi)$ given by $x = r^p \cos\varphi$, $y = r^q \sin\varphi$. Nevertheless, the computation of condition (3.8) is very demanding and sometimes impossible, see [23,24]. However applying the blow-up method we compute the center condition in a straightforward way. We proceed in a similar way as in the previous example. After transformation (3.2) system (3.9) becomes

$$\begin{aligned}\dot{z} &= B + Ay^2z^3 - Dy^2z^3 - Cy^4z^6,\\ \dot{y} &= y^3z^2(D + Cy^2z^3).\end{aligned} \tag{3.9}$$

Since $B \neq 0$ by the monodromic condition we have that the origin of system (3.9) is also a regular point. Therefore system (3.9) has an analytic first integral around its origin. Hence, we look for a power series of the form (3.4). We compute $\dot{\mathcal{H}} = \dot{z}\partial\mathcal{H}/\partial z + \dot{y}\partial\mathcal{H}/\partial y$ for system (3.9) and equating to zero the coefficients of the same power of y yields the following recurrence differential equation

$$(k-4)Cz^5 f_{k-4} + (k-2)Dz^2 f_{k-2} - Cz^6 f'_{k-4} + (Az^3 - Dz^3)f'_{k-2} + Bf'_k = 0.$$

We take $f_k = 0$ for k odd and for k even we can take $f_2(z) = 1$ and

$$\begin{aligned}f_4(z) &= \frac{1}{3B}(-2Dz^3) + c_4,\\ f_6(z) &= \frac{1}{9B^2}(-3BC + 3AD + D^2)z^6 + c_6,\\ f_8(z) &= \frac{1}{27B^3}(P_8(z) - 2(3A + D)(AD - BC)z^9) + c_8,\end{aligned}$$

where P_8 is a polynomial of at most degree 8. In order to have a polynomial in the original variables (x, y) we must to take $(3A + D)(AD - BC) = 0$. So we impose $3A + D = 0$ because the other one is not compatible with the monodromic condition. In this case f_{10} takes the form

$$f_{10}(z) = \frac{1}{3B^2}(24ABc_8z^3 + 36A^2c_6z^6 - 3BCc_6z^6 - 4ACc_4z^9 + 3B^2c_{10}).$$

We can take all $c_4 = c_6 = c_8 = c_{10} = 0$ and then $f_{10}(z) = 0$ and also take $f_k = 0$ for all $k \geq 10$. Next we define

$$\begin{aligned}H &= f_2\Big(\frac{x}{y}\Big)y^2 + f_4\Big(\frac{x}{y}\Big)y^4 + f_6\Big(\frac{x}{y}\Big)y^6 + f_8\Big(\frac{x}{y}\Big)y^8\\ &= y^2 + \frac{2A}{B}x^3y - \frac{C}{3B}x^6,\end{aligned}$$

which is a polynomial first integral of system (3.7) and therefore it has a center at the origin. Here the computation of the necessary condition is straightforward unlike other known methods and our method also gives directly the sufficiency. □

Proposition 3.3. *The nilpotent differential system*

$$\dot{x} = y + x^2, \qquad \dot{y} = -x^3 + cx^4, \tag{3.10}$$

has not any analytic first integral at the origin and it has a center at the origin if and only if $c = 0$.

Proof. First we apply the blow-up transformation (3.2) and system (3.10) becomes

$$\dot{z} = 1 + yz^2 + y^2z^4 - cy^3z^5, \qquad \dot{y} = y^3z^3(-1 + cyz), \tag{3.11}$$

We propose a power series of the form (3.4) and impose that $\dot{\mathcal{H}} = \dot{z}\partial\mathcal{H}/\partial z + \dot{y}\partial\mathcal{H}/\partial y = 0$ for system (3.11) and we get the following recurrence differential equation

$$(k-3)cz^4 f_{k-3} - (k-2)z^3 f_{k-2} - cz^5 f'_{k-3} + z^4 f'_{k-2} + z^2 f'_{k-1} + f'_k = 0$$

and we can, as in previous case, take $f_2(z) = 1$, and $f_3(z) = c_3$,

$$f_4(z) = \frac{1}{2}z^4 + c_4, \qquad f_5(z) = \frac{1}{60}(45c_3z^4 - 24cz^5 - 20z^6) + c_5.$$

However, it is not possible to get a polynomial from f_5 in the original variables (x, y). Therefore the analytic first integral at infinity cannot be extended to the origin of system (3.10). This also implies system (3.10) has not an analytic first integral at the origin. Next we propose a power series of the form

$$\mathcal{V}(z, y) = \sum_{k=1}^{\infty} v_k(z)y^k. \tag{3.12}$$

and we impose that this $\mathcal{V}$ satisfies the equation

$$\dot{z}\partial\mathcal{V}/\partial z + \dot{y}\partial\mathcal{V}/\partial y - (\partial\dot{z}/\partial z + \partial\dot{y}/\partial y)\mathcal{V} = 0, \tag{3.13}$$

which is the equation of the inverse integrating factor. As an inverse integrating factor is not coordinates free (as happens for a first integral) and it is affected by the Jacobian of the transformation when we come back to the original coordinates. In this case the recurrence differential equation is

$$6cz^4v_{k-3} - 7z^3v_{k-2} - 2zv_{k-1} - cz^5v'_{k-3} + z^4v'_{k-2} + z^2v'_{k-1} + v'_k = 0.$$

Without loss of generality we now take $v_1 = 1$. Then $v_2 = z^2 + c_2$, and

$$\begin{aligned} v_3(z) &= \frac{1}{2}(2c_2z^2 + z^4) + c_3, \\ v_4(z) &= \frac{1}{20}(20c_3z^2 + 15c_2z^4 - 8cz^5) + c_4, \\ v_5(z) &= 5\frac{1}{420}(420c_4z^2 + 420c_3z^4 - 252c\,c_2z^5 + 35c_2z^6 + 12cz^7) + c_5. \end{aligned}$$

Taking into account that the inverse integrating factor for system (3.10) is obtained multiplying the power series (3.12) by the Jacobian of the transformation, we have to take $c = 0$ in a polynomial v_5 to ensure that V is polynomial in the original variables (x, y). Then

$$v_6 = \frac{1}{10080}(10080c_5z^2 + 12600c_4z^4 + 1680c_3z^6 + 525c_2z^8) + c_6.$$

Choosing $c_2 = c_3 = c_4 = c_5 = c_6 = 0$ then $v_6 = 0$ and we can choose $v_k = 0$ for all $k \geq 5$. Consequently,

$$\mathcal{V} = v_1\Big(\frac{x}{y}\Big)y + v_2\Big(\frac{x}{y}\Big)y^2 + v_3\Big(\frac{x}{y}\Big)y^3 + v_4\Big(\frac{x}{y}\Big)y^4 = y + x^2 + \frac{x^4}{2y}.$$

The inverse integrating factor of system (3.10) is obtained multiplying $\mathcal{V}$ by the Jacobian of the transformation

$$V = y\mathcal{V} = y^2 + x^2y + \frac{x^4}{2}.$$

For a general monodromic nilpotent singularity the existence of an inverse integrating factor in a neighborhood of singularity does not guarantees the existence of a center at this singularity, but for the nilpotent monodromic singularities with leading term $(y, -x^3)$ this is true, see the result in [6, 20]. System (3.10) with $c = 0$ was studied in [11], where it was proved that there exists no analytic first integral. Consequently, here we have used the blow–up method to find an inverse integrating factor of system (3.10) which gives the condition $c = 0$ implying that system (3.10) has a center at the origin if and only if $c = 0$. □

Proposition 3.4. *Consider the nilpotent differential system*

$$\dot{x} = y + ax^2 + 5xy^2, \qquad \dot{y} = -2x^3 + 3xy^2 - 4y^3, \tag{3.14}$$

where $a \in \mathbb{R}$. The first necessary condition of system (3.14) to have a center is $-98 + 47a^2 + 20a^4 = 0$. Moreover system (3.14) always has a focus at the origin.

Proof. System (3.14) has a monodromic singular point at the origin if and only if $|a| < 2$, see [27]. Applying the blow-up transformation (3.2) system (3.14) takes the form

$$\dot{z} = \ 1 + 9y^2z + ayz^2 - 3y^2z^2 + 2y^2z^4, \qquad \dot{y} = \ -y^3(4 - 3z + 2z^3). \tag{3.15}$$

We propose directly a power series of the form (3.12) and we impose that this $\mathcal{V}$ satisfies the equation (3.13). Recall that the transformation to the original variables (x, y) will be affected by the Jacobian of the transformation. In this case the recurrence differential equation is

$$\big(3(k-3)z - (4k-11) - 2(k-1)z^3\big)v_{k-2} - 2azv_{k-1} + \big(9z - 3z^2 + 2z^4\big)v'_{k-2} + az^2v'_{k-1} + v'_k = 0$$

and we can, as above, take $v_1(z) = 1$, and $v_2(z) = az^2 + c_2$, $v_3(z) = z + ac_2z^2 + z^4 + c_3$

$$\begin{aligned}
v_4(z) &= 5c_2z - \frac{3}{2}c_2z^2 + ac_3z^2 - 4az^3 + \frac{3}{4}az^4 + \frac{3}{2}c_2z^4 + c_4,\\
v_5(z) &= \frac{1}{60}(P_5(z) + 60z^6 - 15a^2z^6 + 10ac_2z^6) + c_5,\\
v_6(z) &= \frac{1}{1680}(P_7(z) - 525az^8 + 210a^3z^8 + 630c_2z^8 - 140a^2c_2z^8) + c_6,
\end{aligned}$$

where $P_i(z)$ are determined polynomials of degree i. Taking into account that in the original variable the inverse integrating factor is $V = y\mathcal{V}$ the coefficient in the term with z^8 in v_6 must be zero. Then, we have $-525a + 210a^3 + 630c_2 - 140a^2c_2 = 0$ which yields

$$c_2 = \frac{3(2a^3 - 5a)}{2(2a^2 - 9)},$$

if $2a^2 - 9 \neq 0$. Recall that if $2a^2 - 9 = 0$ this is not a monodromic case. Next, v_7 has the form

$$v_7(z) = \frac{P_8(z) + (14112 - 9904a^2 - 1376a^4 + 640a^6)z^9}{1680(2a^2 - 9)} + c_7,$$

where $P_8(z)$ is a determined polynomial of degree 8. The coefficient of the term with monomial z^9 must vanish, so,

$$16(-9+2a^2)(-98+47a^2+20a^4)=0.$$

The unique real roots of this polynomial satisfying the monodromic condition $|a|<2$ are $a=\pm 1.153741$. This last numerical value was obtained by Varin in [42] using the Bautin method after doing a generalized polar blow-up. The method developed in [42, 43] is not useful to compute the algebraic condition $-98+47a^2+20a^4=0$. Moreover, with our method we can distinguish between a center and a focus. If we compute more terms of the power series $\mathcal{V}$ the powers in z that must be zero have not a common root. Therefore, the origin of system (3.14) is always a focus. The algebraic necessary center condition $-98+47a^2+20a^4=0$ was also obtained in [27] using a more involved method based in the result that all the nilpotent centers are limit of non-degenerate centers. The fact that, under monodromy the origin of (3.14) is always a focus was also derived in Proposition 26 of [21]. Here we also use that the existence of a formal inverse integrating factor defined around a nilpotent monodromic singularity with leading term $(-y,x^3)$ is a necessary and sufficient condition to have a center at the singularity, see [6, 20]. □

4 Degenerate monodromic singularities

In this section we consider different systems with a degenerate singularity, and using the blow–up method we study the center-focus problem. The examples proposed here show the narrow relation between the center problem and the existence of a first integral for the singular points at infinity. The necessary conditions founded by the method do not always correspond to trivial cases of centers.

Proposition 4.1. *Consider the differential system*

$$\dot{x}=x^2y+ax^5+y^5, \qquad \dot{y}=-xy^2-x^5+bx^4y, \tag{4.1}$$

where a, $b\in\mathbb{R}$. System (4.1) has a center at the origin if and only if $5a+b=0$.

Proof. In [5] it is proved that the origin of system (4.1) is always monodromic. Moreover, system (4.1) has characteristic directions because the homogeneous polynomial $xq_n(x,y)-yp_n(x,y)$, where p_n and q_n are the lower homogeneous terms of system (4.1), has real roots. When the singular point has characteristic directions it is not possible to apply the Bautin method in order to solve the center-focus problem, see [25].

After applying the blow-up transformation (3.2) system (4.1) takes the form

$$\begin{aligned}\dot{z}&=y^2+2z^2+ay^2z^5-by^2z^5+y^2z^6\\ \dot{y}&=-yz(1-by^2z^3+y^2z^4).\end{aligned}\tag{4.2}$$

Now, we look for a power series of the form (3.4) and we compute $\dot{\mathcal{H}}=(\partial\mathcal{H}/\partial z)\dot{z}+(\partial\mathcal{H}/\partial y)\dot{y}$ for system (4.2). We obtain the following recursive differential equation

$$(k-2)z^4(b-z)f_{k-2}-kzf_k+(1+az^5-bz^5+z^6)f'_{k-2}+2z^2f'_k=0$$

Solving for the first values of k we can take $f_k = 0$ for all k odd and for k even we find $f_2 = 0$, $f_4 = z^2 + c_4$ where c_4 is an arbitrary integration constant, and

$$\begin{aligned} f_6 &= \frac{1}{6}(2 - 3az^5 - 3bz^5 + 2z^6) + c_6, \\ f_8 &= -\frac{1}{80}z^3(100a + 20b - 240bc_6 - 25a^2z^5 - 30abz^5 \\ &\quad - 5b^2z^5 + 20az^6 + 4bz^6 - 80zc_8 - 240c_6 z \log z), \end{aligned}$$

where c_6 and c_8 are arbitrary constants. Since f_8 must be a polynomial we have to impose $c_6 = 0$ and since it must be a polynomial in the original variables (x, y) we have to impose that the terms in z^9 vanish, that is, $5a + b = 0$. Under this restrictions we have that

$$f_{10} = c_8 z^2 \left(\frac{2}{3} + 4az^5 + \frac{2}{3}z^6 \right) + c_{10}z^5.$$

Then taking $c_8 = c_{10} = 0$ we get $f_{10} = 0$ and we can choose $f_k = 0$ for all $k \geq 10$. Consequently

$$\mathcal{H} = f_4\left(\frac{x}{y}\right)y^4 + f_6\left(\frac{x}{y}\right)y^6 + f_8\left(\frac{x}{y}\right)y^8 = \frac{x^6}{3} + 2ax^5y + x^2y^2 + \frac{y^6}{3},$$

which is a polynomial first integral of system (4.1). Therefore, when $5a + b = 0$ system (4.1) has a center at the origin. It remains to see that if $5a + b \neq 0$ then system (4.1) has a focus at the origin. From [5, Theorem 2.3] is derived the geometric criteria for proving that if $5a + b \neq 0$ then system (4.1) has a focus at the origin, see Proposition 3.19 in [5]. Here our blow-up method gives straightforward the necessary condition while for applying the geometric criteria the necessary condition is needed. □

Proposition 4.2. *Consider the differential system*

$$\dot{x} = x^2y + ax^3 + y^5, \qquad \dot{y} = -xy^2 + bx^2y - x^3, \tag{4.3}$$

where $a, b \in \mathbb{R}$. System (4.3) has a center at the origin if and only if $3a + b = 0$.

Proof. The origin of system (4.3) is monodromic if and only if, $(a - b)2 - 8 < 0$, see [5]. System (4.3) has also characteristic directions. Applying the blow-up (3.2) system (4.3) takes the form

$$\begin{aligned} \dot{z} &= y^2 + 2z^2 + az^3 - bz^3 + z^4 \\ \dot{y} &= -yz(1 - bz + z^2), \end{aligned} \tag{4.4}$$

after a scaling of time. Now, we compute $\dot{\mathcal{H}} = (\partial\mathcal{H}/\partial z)\dot{z} + (\partial\mathcal{H}/\partial y)\dot{y}$ for system (4.4) where $\mathcal{H}$ is a power series of the form (3.4) and we obtain the following recursive differential equation

$$-5z(1 - bz + z^2)f_k + f'_{k-2} + (2z^2 + az^3 - bz^3 + z^4)f'_k = 0.$$

Doing the computations of the first f_k we must to take $f_2 = f_3 = 0$ in order to be polynomials and

$$f_4 = c_4 e^{-\frac{2(3a+b)\arctan\left(\frac{a-b+2z}{\sqrt{8-(a-b)^2}}\right)}{\sqrt{8-(a-b)^2}}} z^2(2 + z(a - b + z)).$$

where in order to have a polynomial we have to take $3a + b = 0$ and without loss of generality $c_4 = 1$.

Next we must to take $f_5 = 0$ and $f_6 = (2+3z^3(2+4az+z^2)^{3/2}c_6)/3$, and taking $c_6 = 0$ we have $f_6 = 2/3$. Next, $f_7 = 0$ and $f_8 = c_8 z^4(2+4az+z^2)^2$. Then taking $c_8 = 0$ we obtain $f_8 = 0$ and we can choose $f_k = 0$ for all $k \geq 8$. Consequently

$$\mathcal{H} = f_4\Big(\frac{x}{y}\Big)y^4 + f_6\Big(\frac{x}{y}\Big)y^6 + f_8\Big(\frac{x}{y}\Big)y^8 = x^4 + 4ax^3y + 2x^2y^2 + \frac{2y^6}{3},$$

which is a polynomial first integral of system (4.3). Finally to see that for $3a + b \neq 0$ we have a focus at the origin, we use also the geometric criteria developed from [5, Theorem 2.3]. In Proposition 3.16 [5] is that system (4.3) has a focus if $3a + b \neq 0$. As in the example before the blow-up method gives the necessary condition directly. □

Proposition 4.3. *Consider the degenerate differential system*

$$\dot{x} = cx^2y + fxy^2 + dy^3, \qquad \dot{y} = \tilde{c}xy^2 + fy^3 + ax^5. \tag{4.5}$$

If the origin of system (4.5) is monodromic then it is a center if, and only if, $f = 0$.

Proof. In [38] Medvedeva studied the stability problem of the origin of system(4.5). The first non zero focal value of system (4.5) was given in [38] through a complicate and involved method using several blow-up transformations. The monodromy problem for system (4.5) was solved in [5] where the following result was given.

Lemma 4.4. *The origin of system (4.5) is monodromic if and only if one of the following conditions holds:*

a) $d\,a < 0$, $(\tilde{c} - c)(\tilde{c} - 2c) > 0$ *and* $d(\tilde{c} - c) < 0$

b) $d\,a < 0$, $\tilde{c} - c = 0$ *and* $cd > 0$.

c) $d\,a < 0$, $\tilde{c} - 2c = 0$ *and* $ca > 0$.

Applying the blow-up transformation (3.2) to system (4.5), the new differential system takes the form

$$\begin{aligned} \dot{z} &= d + cz^2 - \tilde{c}z^2 - ay^2z^6, \\ \dot{y} &= y\,(\tilde{c}z + ay^2z^5 + f) \end{aligned} \tag{4.6}$$

with the change of time $d\tau = y^2 dt$. From the monodromic condition we know that $d \neq 0$. System (4.6) has a regular point at the origin and consequently, an analytic first integral around the origin and the recursive differential equation do not generate logarithmic terms. Then the question is if this analytic first integral at infinity can be extended to the origin of the original system (4.5). In this case the recursive differential equation is

$$(k-2)az^5 f_{k-2} + k(\tilde{c}z + f)f_k - az^6 f'_{k-2} + (d + cz^2 - \tilde{c}z^2)f'_k = 0.$$

Then if $f_i = 0$ for $i = 1, \ldots, k-2$ we have that the value of f_k is

$$f_k = c_k e^{-\frac{kf\arctan\left(\frac{\sqrt{c-\tilde{c}}\,z}{\sqrt{d}}\right)}{\sqrt{c-\tilde{c}}\sqrt{d}}} (d + (c - \tilde{c})z^2)^{-\frac{k\tilde{c}}{2(c-\tilde{c})}}.$$

In order to have a well defined function in the original variables (x, y) we have to impose $f = 0$. Moreover, under the monodromic condition system (4.5) has a center at the origin since it is invariant with respect to the symmetry $(x, y, t) \to (-x, y, -t)$.

To finish the proof we see that if $f \neq 0$ then system (4.5) has a focus at the origin. We apply the geometrical criteria developed in [5, Theorem 2.3]. Consider the vector field

$$\mathcal{X}_c = (cx^2y + dy^3)\frac{\partial}{\partial x} + (\tilde{c}xy^2 + ax^5)\frac{\partial}{\partial y},$$

which has a center at the origin. Let $\mathcal{X}$ the vector field associated to system (4.5). Then we compute that

$$\mathcal{X} \wedge \mathcal{X}_c = fy^2(ax^6(\tilde{c} - c)x^2y^2 - dy^4)$$

which is semi-definite under the monodromic conditions of Lemma 4.4 and by Theorem 2.3 of [5] if $f \neq 0$ system (4.5) has a focus at the origin. □

Finally we consider the differential system

$$\begin{aligned} \dot{x} &= y^3 + 2ax^3y + 2x(\alpha x^4 + \beta xy^2), \\ \dot{y} &= -x^5 - 3ax^2y^2 + 3y(\alpha x^4 + \beta xy^2), \end{aligned} \tag{4.7}$$

where α, β, $a \in \mathbb{R}$. In [4] it was proven that system (4.7) with $\alpha\beta \neq 0$ is not orbitally reversible nor formally integrable. Moreover there are values of (α, β, a) with $a \neq 0$ and with the monodromic condition $|a| < 1/\sqrt{6}$ such that the origin of system (4.7) is a center. In fact the center condition is not algebraic in the parameters. In [23] it was also identified the center condition using the existence of an inverse integrating factor. Therefore the center problem is not algebraically solvable. As we will see, if we apply the blow-up method proposing a power series verifying the first integral equation we only find the algebraically solvable centers. So we will propose a power series satisfying the inverse integrating equation. Applying the blow-up (3.2) to system (4.7), the new differential system takes the form

$$\begin{aligned} \dot{z} &= 1 + 5ayz^3 + y^2z^6 - y^2z^5\alpha - yz^2\beta \\ \dot{y} &= -y^2z(3az + yz^4 - 3yz^3\alpha - 3\beta), \end{aligned} \tag{4.8}$$

after the scaling of time $d\tau = y^2dt$. Looking for a power series of the form (3.4) and computing the equation that satisfies a first integral we get only the center condition $\alpha = \beta = 0$ (the reader can follow the steps seeing the previous examples). Therefore the analytic first integral at infinity cannot always be extended to the origin of system (4.7). Next, we propose a power series of the form

$$\mathcal{F}(z, y) = y^{k_2} \sum_{k=0}^{\infty} v_k(z)y^k, \tag{4.9}$$

where $k_2 \in \mathbb{Q}$ and we impose that it satisfies the equation of certain inverse integrating factor $\dot{z}\partial\mathcal{F}/\partial z + \dot{y}\partial\mathcal{F}/\partial y = k_1(\partial\dot{z}/\partial z + \partial\dot{y}/\partial y)\mathcal{F}$, where $k_1 \in \mathbb{R}$. The recurrence differential equation is

$$\begin{aligned} &(-(k-2)z^5 - 3k_1z^5 - k_2z^5 + 3(k-2)z^4\alpha - 4k_1z^4\alpha + 3k_2z^4\alpha)v_{k-2} \\ &\quad + (-3(k-1)az^2 - 9ak_1z^2 - 3ak_2z^2 + 3(k-1)z\beta - 4k_1z\beta + 3k_2z\beta)v_{k-1} \\ &\quad + (z^6 - z^5\alpha)v'_{k-2} + (5az^3 - z^2\beta)v'_{k-1} + v'_k = 0 \end{aligned}$$

and we can take $v_0(z) = 1$, and

$$\begin{aligned} v_1(z) &= 3ak_1z^3 + ak_2z^3 + 2k_1z^2\beta - \frac{3}{2}k_2z^2\beta + c_1, \\ v_2(z) &= \frac{1}{6}k_2z^6 + \frac{1}{2}a^2(9k_1^2 + 6k_1(k_2-2) + (k_2-4)k_2)z^6 - \frac{3}{5}k_2z^5\alpha \\ &\quad - \frac{3}{2}c_1z^2\beta - \frac{3}{2}c_1k_2z^2\beta + 2k_1^2z^4\beta^2 + \frac{3}{8}k_2z^4\beta^2 + \frac{9}{8}k_2^2z^4\beta^2 \\ &\quad + \frac{1}{10}az^3\Big(10\,c_1(1+3k_1+k_2) \\ &\quad + (60k_1^2 + 3(7-5k_2)k_2 - k_1(28+25k_2))z^2\beta\Big) \\ &\quad + \frac{1}{10}k1(5z^6 + 8z^5\alpha + 20c_1z^2\beta - 5(1+6k_2)z^4\beta^2) + c_2. \end{aligned}$$

We do not write here the value of $v_3(z)$ due to its length. Now, choosing the values of k_1, k_2, c_1, c_2 and c_3 we impose that $v_3(z) = 0$. One solution is $k_1 = 12/13$ and $k_2 = 16/13$ and $c_1 = c_2 = c_3 = 0$ which implies $v_k = 0$ for all $k \geq 3$. Consequently,

$$\mathcal{F} = y^{k_2}\Big(v_0\Big(\frac{x}{y}\Big) + v_1\Big(\frac{x}{y}\Big)y + v_2\Big(\frac{x}{y}\Big)y^2\Big) = \frac{2x^6 + 12ax^3y^2 + 3y^4}{3y^{36/13}}.$$

The inverse integrating factor for system (3.10) is obtained by multiplying $\mathcal{V} = \mathcal{F}^{\frac{13}{12}}$ by the Jacobian of the transformation and the change of time made, i.e.

$$V = y^3\mathcal{V} = y^3\mathcal{F}^{\frac{13}{12}} = \left(y^2 + x^2y + \frac{x^4}{2}\right)^{\frac{13}{12}}.$$

For a degenerate singular point the existence of an inverse integrating factor defined around the singular point does not guarantee the existence of a center at the singular point. In fact for system (4.7) an extra nonalgebraic condition in the parameters is needed, see [4, 23].

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Homoclinic solutions for subquadratic Hamiltonian systems with competition potentials

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Abstract. In this paper, we consider of the following second-order Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \qquad \forall t \in \mathbb{R},$$

where $W(t,x)$ is subquadratic at infinity. With a competition condition, we establish the existence of homoclinic solutions by using the variational methods. In our theorem, the smallest eigenvalue function $l(t)$ of $L(t)$ is not necessarily coercive or bounded from above and $W(t,x)$ is not necessarily integrable on $\mathbb{R}$ with respect to t. Our theorem generalizes many known results in the references.

Keywords: Hamiltonian systems, homoclinic solutions, subquadratic potentials, competition condition, variational methods.

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1 Introduction

In this paper, we consider the following Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \qquad \forall t \in \mathbb{R}, \tag{1.1}$$

where $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $L \in C\left(\mathbb{R}, \mathbb{R}^{N^2}\right)$ is a symmetric matrix valued function and $\nabla W(t,x)$ denotes the gradient with respect to the x variable. A nontrivial solution $u(t)$ of problem (1.1) is homoclinic if $u(t) \to 0$, $\dot{u}(t) \to 0$ as $t \to \pm\infty$ and $u(t) \not\equiv 0$.

The importance of homoclinic solutions for Hamiltonian systems in studying the dynamic behavior has been recognized. In recent years, many mathematicians used the variational methods to show the existence and multiplicity of homoclinic solutions for systems (1.1) with different growth conditions on $W(t,x)$. In this paper, we only consider the subquadratic cases. In [5], Ding assumed

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(L') letting $l(t) \equiv \inf_{|q|=1}(L(t)q,q)$, there exists $\xi > 1$ such that

$$|t|^{-\xi} l(t) \to +\infty, \quad \text{as } |t| \to +\infty.$$

By (L'), Ding showed a compact embedding theorem from $H^1(\mathbb{R})$ to $L^p(\mathbb{R})$ for $p \in (1,+\infty]$. Under some other subquadratic conditions on $W(t,x)$ with respect to x, Ding obtained the existence and multiplicity of homoclinic solutions for systems (1.1). This result has been generalized by many mathematicians. For example, in [19], Zhang introduced condition

(L'') There exists a constant $l_0 > 0$ such that $l(t) + l_0 \geq 1$ for all $t \in \mathbb{R}$ and

$$\int_{\mathbb{R}} (l(t) + l_0)^{-1} \, dt < \infty. \tag{1.2}$$

By (L''), the embedding $H^1(\mathbb{R}) \hookrightarrow L^1(\mathbb{R})$ is compact. Obviously, (L'') is weaker than (L') and both of these two conditions yield that $l^{-1}(t)$ decays fast at infinity. When $l^{-1}(t)$ has a slow decay at infinity, it is difficult for us to obtain such compact embeddings. In this case, we can consider the decaying rate of $W(t,x)$ at infinity with respect to t. Let us consider the pure power nonlinearities with weight functions, i.e. $W(t,x) = a(t)|x|^\nu (\nu \in (1,2))$. In [23], Zhang and Yuan assumed that $a(t)$ belongs to $L^2(\mathbb{R},\mathbb{R}^+) \cap L^{\frac{2}{2-\nu}}(\mathbb{R},\mathbb{R}^+)$ to make sure the corresponding functional is well defined and show the convergence of the (PS) sequence. This condition is weakened by Sun, Chen and J. Nieto [12] by just requiring $a \in L^{\frac{2}{2-\nu}}(\mathbb{R},\mathbb{R}^+)$. In 2014, Lv and Tang [11] obtained homoclinic solutions for systems (1.1) with more general weight functions where $a \in L^p(\mathbb{R},\mathbb{R})$ for some $p \in \left(1, \frac{2}{2-\nu}\right]$. The readers are referred to [1–3, 6–10, 13–18, 20–22] for more details.

From above papers, we know that, the decaying rates of $l^{-1}(t)$ and $a(t)$ at infinity are important for us in finding homoclinic solutions of (1.1). There is an interesting question that whether systems (1.1) possesses homoclinic solutions when $a(t)$ is unbounded or $l(t)$ is oscillating (which means $\liminf_{|t|\to\infty} l(t) < +\infty$ and $\limsup_{|t|\to\infty} l(t) = +\infty$)? Motivated by the above analysis, we are encouraged to find a twisted condition between $l(t)$ and $a(t)$ which can be stated as follows:

$(W0)$ For $b \in [1,2]$ and $\mu \in (1,2)$, there exist $\gamma \in \left(b, \frac{2b}{2+b-b\mu}\right]$ and $k \in \left[0, \frac{\gamma-b}{b\gamma}\right]$ such that $\frac{a(t)}{(l(t))^k} \in L^\gamma(\mathbb{R})$.

More precisely, we obtain the following theorem.

Theorem 1.1. *Suppose that $(W0)$ holds for $b = 2$ and*

$(L1)$ *one of the following statements holds:*

(i) $L \in C^2(\mathbb{R},\mathbb{R}^{N^2})$ *and* $((L''(t) - \kappa L(t))x, x) \leq 0$ *for all* $|t| \geq \bar{r}_1$ *and* $x \in \mathbb{R}^N$;

(ii) $L \in C^1(\mathbb{R},\mathbb{R}^{N^2})$ *and* $|L'(t)x| \leq \kappa |L(t)x|$ *for all* $|t| \geq \bar{r}_1$ *and* $x \in \mathbb{R}^N$

with some $\kappa > 0$ *and* $\bar{r}_1 > 0$, *where* $L'(t) = (d/dt)L(t)$ *and* $L''(t) = (d^2/dt^2)L(t)$;

$(L2)$ *there exists* $M_0 > 0$ *such that* $l(t) \geq M_0$ *for all* $t \in \mathbb{R}$, *where* $l(t) \equiv \inf_{|u|=1}(L(t)u,u)$;

$(W1)$ $W(t,0) \equiv 0$, *there exists* $a \in C(\mathbb{R},\mathbb{R}^+)$ *such that* $|\nabla W(t,x)| \leq a(t)|x|^{\mu-1}$;

$(W2)$ *there exist* $\lambda \in (1,2)$, $\eta > 0$, $\zeta > 0$ *and open set* $\Omega \subset \mathbb{R}$ *such that*

$$W(t,x) \geq \eta |x|^\lambda, \qquad \forall (t,x) \in \Omega \times \mathbb{R}^N, |x| \leq \zeta.$$

Then system (1.1) possesses at least one nontrivial homoclinic solution.

$(L1)$ is assumed to show all the critical points of corresponding functional for systems

(1.1) are classical homoclinic solutions, which is introduced in [5]. In [11, 13, 18], the authors only considered the homoclinic solutions in sense of $u(t) \to 0$ as $|t| \to \infty$ while we consider the classical ones. To obtain the asymptotic behavior of the solutions at infinity, we can also consider the following condition

$(L3)$ there exist $\delta > 0$, $D > 0$, $q \in [1,2]$ and $r_0 > 0$ such that

$$\int_t^{t+\delta} \hat{l}^q(s)ds \le D$$

for all $|t| \ge r_0$, where $\hat{l}(t) \equiv \sup_{|u|=1}(L(t)u,u)$.

It is easy to see that $(L3)$ holds if all the eigenvalues of $L(t)$ are bounded from above. Then $(L3)$ can be seen as a generalization of the following bounded condition

$(L4)$ there exists $R > 0$ such that

$$(L(t)u,u) \le R|u|^2, \qquad \forall (t,u) \in \mathbb{R} \times \mathbb{R}^N.$$

Then we obtain the following theorem.

Theorem 1.2. *Suppose $(L2)$, $(L3)$, $(W1)$, $(W2)$ and $(W0)$ hold with $b = q$, then system (1.1) possesses at least one nontrivial homoclinic solution.*

Remark 1.3. In our theorems, condition $(W0)$ is a class of competition conditions between a and l. When $0 < \inf_{t\in\mathbb{R}} l(t) \le \sup_{t\in\mathbb{R}} l(t) < \infty$, $(W0)$ reduces to $a(t) \in L^\gamma(\mathbb{R})$, which is required in [12, 13, 18, 22]. There are examples satisfying the conditions of Theorems 1.1 and 1.2 but not the results in [2, 5, 7–14, 16–23].

Example 1.4 (Oscillating example for Theorem 1.1). Let $L(t) = l(t)Id_N$ and $W(t,x) = a(t)|x|^{\frac{8}{5}}$, where

$$l(t) = \begin{cases} \sin(\ln 2) + 1 & \text{for } |t| < 1, \\ t^{\frac{6}{7}}\left(\sin\left(\ln(t^2+1)\right) + 1\right) + 1 & \text{for } |t| \ge 1, \end{cases}$$

$$a(t) = t^{\frac{1}{20}}\left(\sin\left(\ln(t^2+1)\right) + 1\right)^{\frac{3}{10}}$$

and Id_N is the identity matrix of order N. It is easy to see that

$$\liminf_{|t|\to\infty} l(t) = 1, \quad \limsup_{|t|\to\infty} l(t) = +\infty, \quad \liminf_{|t|\to\infty} a(t) = 0 \quad \text{and} \quad \limsup_{|t|\to\infty} a(t) = +\infty.$$

Hence $l(t)$, $a(t)$ are neither coercive nor bounded from above and $l^{-1}(t)$, $(a(t))^p \notin L(\mathbb{R})$ for any $p \in (1,5]$. However, this example satisfies the conditions of Theorem 1.1 with $\gamma = 5$ and $k = \frac{3}{10}$. Here, we only need to show condition $(L1)$ is fulfilled while the other conditions can be easily checked. To check $(L1)$, we show (ii) holds, which can be verified by the following inequality

$$\left(\frac{6}{7}t^{-\frac{1}{7}}\sin\left(\ln(t^2+1)\right) + \frac{2t^{\frac{13}{7}}}{t^2+1}\cos\left(\ln(t^2+1)\right)\right)|x| \le \left(t^{\frac{6}{7}}\left(\sin\left(\ln(t^2+1)\right)+1\right)+1\right)|x|$$

for all $x \in \mathbb{R}^N$ and $|t|$ large enough.

Example 1.5 (Coercive example for Theorem 1.1)**.** There are also examples in which $l(t)$ and $a(t)$ are both coercive. Let $L(t) = \left(t^6+1\right) Id_N$ and $W(t,x) = t^{\frac{2}{5}}|x|^{\frac{3}{2}}$. If we choose $\gamma = 4$ and $k = \frac{1}{4}$, $(W0)$ is fulfilled. Moreover, other conditions of Theorem 1.1 can be easily checked. However this example does not satisfy the results in [2,5,7–14,17–23].

Example 1.6 (Oscillating example for Theorem 1.2)**.** Let

$$g(t) = \begin{cases} 2n^{\frac{8}{9}}\left(n^{\frac{8}{9}}+1\right)|t| - 2n^{\frac{17}{9}}\left(n^{\frac{8}{9}}+1\right), & n \le |t| < n + \frac{1}{2\left(n^{\frac{8}{9}}+1\right)}, \\ -2n^{\frac{8}{9}}\left(n^{\frac{8}{9}}+1\right)|t| + 2n^{\frac{17}{9}}\left(n^{\frac{8}{9}}+1\right) + 2n^{\frac{8}{9}}, & n + \frac{1}{2\left(n^{\frac{8}{9}}+1\right)} \le |t| \le n + \frac{1}{n^{\frac{8}{9}}+1}, \\ 0, & \text{otherwise} \end{cases} \tag{1.3}$$

and

$$m(t) = \begin{cases} 2n^{\frac{1}{72}}\left(n^{\frac{8}{9}}+1\right)|t| - 2n^{\frac{73}{72}}\left(n^{\frac{8}{9}}+1\right), & n \le |t| < n + \frac{1}{2\left(n^{\frac{8}{9}}+1\right)}, \\ -2n^{\frac{1}{72}}\left(n^{\frac{8}{9}}+1\right)|t| + 2n^{\frac{73}{72}}\left(n^{\frac{8}{9}}+1\right) + 2n^{\frac{1}{72}}, & n + \frac{1}{2\left(n^{\frac{8}{9}}+1\right)} \le |t| \le n + \frac{1}{n^{\frac{8}{9}}+1}, \\ 0, & \text{otherwise} \end{cases} \tag{1.4}$$

for all $n \in \mathbb{N} \cup \{0\}$. We see that $g(t), m(t) \ge 0$ and $g \notin L(\mathbb{R})$, $m \notin L(\mathbb{R})$. Let $a(t) = m(t) + e^{-|t|}$ and $L(t) = l(t) Id_N$, where $l(t) = \sqrt{g(t)+1}$. Obviously,

$$\liminf_{|t|\to\infty} l(t) = 1, \quad \limsup_{|t|\to\infty} l(t) = +\infty, \quad \liminf_{|t|\to\infty} a(t) = 0, \quad \limsup_{|t|\to\infty} a(t) = +\infty.$$

Choosing $q = 2$ and $\delta = \frac{1}{4}$, we deduce from the definitions of $\hat{l}$ and g that

$$\begin{aligned} \int_t^{t+\frac{1}{4}} \hat{l}^2(s) ds = \int_t^{t+\frac{1}{4}} l^2(s) ds = \int_t^{t+\frac{1}{4}} (g(s)+1) ds \\ \le \frac{1}{2}\left[\sum_{i=[|t|]-1,[|t|],[|t|]+1} \frac{i^{\frac{8}{9}}}{i^{\frac{8}{9}}+1}\right] + \frac{1}{4} \\ \le \frac{7}{4} \end{aligned}$$

for $|t|$ is large enough. Then $(L3)$ is checked. Moreover, $l^{-1}(t)$, $(a(t))^p \notin L(\mathbb{R})$ for any $p > 1$. Here we only give the proof for $(a(t))^p \notin L(\mathbb{R})$. It follows from the definition of $a(t)$ that

$$\begin{aligned} \int_{\mathbb{R}} a^p(s) ds &\ge \sum_{n=0}^{\infty} \int_n^{n+\frac{1}{2\left(n^{\frac{8}{9}}+1\right)}} m^p(s) ds \\ &= \sum_{n=0}^{\infty} \int_n^{n+\frac{1}{2\left(n^{\frac{8}{9}}+1\right)}} \left(2n^{\frac{1}{72}}\left(n^{\frac{8}{9}}+1\right)s - 2n^{\frac{73}{72}}\left(n^{\frac{8}{9}}+1\right)\right)^p ds \\ &= \sum_{n=0}^{\infty} \frac{n^{\frac{p}{72}}}{2(p+1)\left(n^{\frac{8}{9}}+1\right)} \\ &= +\infty \end{aligned}$$

which implies $(a(t))^p \notin L(\mathbb{R})$ for all $p > 1$. Finally, we show $(W0)$ is fulfilled with $b = q = 2$. Set $W(t,x) = a(t)|x|^{\frac{3}{2}}$. Choosing $\gamma = 4$ and $k = \frac{1}{4}$, from (1.3) and (1.4), we infer that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\frac{a(s)}{l^{\frac{1}{4}}(s)} \right)^4 ds \\
&\quad = \int_{\mathbb{R}} \frac{a^4(s)}{\sqrt{g(s)+1}} ds \\
&\quad \le \int_{\mathbb{R}} \frac{8\left(m^4(s) + e^{-4|s|}\right)}{\sqrt{g(s)+1}} ds \\
&\quad \le 8\int_{\mathbb{R}} \frac{m^4(s)}{\sqrt{g(s)}} ds + 8\int_{\mathbb{R}} e^{-4|s|} ds \\
&\quad \le 16 \sum_{n=0}^{\infty} n^{-\frac{63}{144}} \int_n^{n+\frac{1}{2\left(n^{\frac{8}{9}}+1\right)}} \left(2n^{\frac{1}{72}} \left(n^{\frac{8}{9}}+1\right) s - 2n^{\frac{73}{72}} \left(n^{\frac{8}{9}}+1\right) \right)^{\frac{7}{2}} ds \\
&\qquad + 16 \sum_{n=0}^{\infty} n^{-\frac{63}{144}} \int_{n+\frac{1}{2\left(n^{\frac{8}{9}}+1\right)}}^{n+\frac{1}{n^{\frac{8}{9}}+1}} \left(-2n^{\frac{1}{72}} \left(n^{\frac{8}{9}}+1\right) s + 2n^{\frac{73}{72}} \left(n^{\frac{8}{9}}+1\right) + 2n^{\frac{1}{72}} \right)^{\frac{7}{2}} ds + 4 \\
&\quad = \frac{32}{9} \sum_{n=0}^{\infty} \frac{n^{-\frac{7}{18}}}{n^{\frac{8}{9}}+1} + 4 \\
&\quad < +\infty.
\end{aligned}$$

Then all the conditions of Theorem 1.2 are satisfied. However, since a is not integrable or bounded, $l(t)$ is not bounded or coercive, our example does not satisfy the theorems in [2, 5, 8, 9, 11–14, 17, 18, 20–23].

2 Proof of Theorem 1.1

Set

$$E := \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^N) : \int_{\mathbb{R}} (|\dot{u}(t)|^2 + (L(t)u(t), u(t)))dt < \infty \right\}$$

with

$$(u,v) = \int_{\mathbb{R}} ((\dot{u}(t), \dot{v}(t)) + (L(t)u(t), u(t)))dt.$$

By $(L2)$, the embedding $E \hookrightarrow L^p\left(\mathbb{R}, \mathbb{R}^N\right)$ is continuous for all $p \in [2, +\infty]$. Hence, for any $p \in [2, +\infty]$,

$$\|u\|_p \le C_p \|u\|, \qquad \forall u \in E \tag{2.1}$$

for some $C_p > 0$. Furthermore, let $I : E \to \mathbb{R}$ be the functional of (1.1) defined by

$$I(u) = \int_{\mathbb{R}} \left(\frac{1}{2}|\dot{u}(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t)) \right) dt. \tag{2.2}$$

First, we give the following useful estimate.

Lemma 2.1. *Let $u \in E$. For any $\theta > 0$ and $q \in [1,2]$, the following inequality holds*

$$|u(t)| \le \theta^{\frac{1}{q^*}-1} \left(\int_t^{t+\theta} |u(s)|^q ds \right)^{\frac{1}{q}} + \theta^{\frac{1}{q^*}} \left(\int_t^{t+\theta} |\dot{u}(s)|^q \, ds \right)^{\frac{1}{q}}, \qquad \forall t \in \mathbb{R}. \tag{2.3}$$

Furthermore, if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, there holds

$$|\dot{u}(t)| \le \theta^{\frac{1}{q^*}-1} \left(\int_t^{t+\theta} |\dot{u}(s)|^q ds \right)^{\frac{1}{q}} + \theta^{\frac{1}{q^*}} \left(\int_t^{t+\theta} |\ddot{u}(s)|^q \, ds \right)^{\frac{1}{q}}, \qquad \forall t \in \mathbb{R}, \tag{2.4}$$

where $\frac{1}{q} + \frac{1}{q^*} = 1$ ($q^* = +\infty$, if $q = 1$).

Proof. For any $t,\ \tau \in \mathbb{R}$,

$$|u(t)| \le |u(\tau)| + \left| \int_\tau^t \dot{u}(s) ds \right|.$$

Integrating over $[t, t+\theta]$, we get

$$\begin{aligned} \theta |u(t)| &\le \int_t^{t+\theta} |u(\tau)| d\tau + \int_t^{t+\theta} \left| \int_\tau^t \dot{u}(s) ds \right| d\tau \\ &\le \theta^{\frac{1}{q^*}} \left(\int_t^{t+\theta} |u(s)|^q ds \right)^{\frac{1}{q}} + \theta \int_t^{t+\theta} |\dot{u}(s)| \, ds \\ &\le \theta^{\frac{1}{q^*}} \left(\int_t^{t+\theta} |u(s)|^q ds \right)^{\frac{1}{q}} + \theta^{\frac{1}{q^*}+1} \left(\int_t^{t+\theta} |\dot{u}(s)|^q \, ds \right)^{\frac{1}{q}}, \end{aligned}$$

which implies

$$|u(t)| \le \theta^{\frac{1}{q^*}-1} \left(\int_t^{t+\theta} |u(s)|^q ds \right)^{\frac{1}{q}} + \theta^{\frac{1}{q^*}} \left(\int_t^{t+\theta} |\dot{u}(s)|^q \, ds \right)^{\frac{1}{q}}.$$

Then we obtain (2.3). Similarly, we can also obtain (2.4). □

Lemma 2.2. *Suppose $(L2)$, $(W0)$–$(W2)$ hold, then $I \in C^1(E, \mathbb{R})$ and*

$$\langle I'(u), v \rangle = \int_{\mathbb{R}} [(\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))] dt. \tag{2.5}$$

Moreover, all the critical points of I are homoclinic solutions of (1.1) if $(L1)$ holds with $b = 2$ or $(L3)$ holds with $b = q$ respectively.

Proof. First, we show that I is well defined. By $(W1)$, we infer that

$$|W(t, u(t))| = \left| \int_0^1 (\nabla W(t, su(t)), u(t)) ds \right| \le \frac{1}{\mu} a(t) |u(t)|^\mu, \qquad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.6}$$

First, we consider a general case, i.e. $\gamma \in \left(1, \frac{2}{2-\mu}\right]$ and $k \in \left[0, \frac{\gamma-1}{\gamma}\right)$. For any $\Lambda \subset \mathbb{R}$, it follows from $(W0)$ and (2.1) that

$$\begin{aligned}
&\int_\Lambda a(t)|u(t)|^\mu dt\\
&\quad= \int_\Lambda \frac{a(t)}{(l(t))^k}(l(t))^k|u(t)|^\mu dt\\
&\quad\le \left(\int_\Lambda\left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_\Lambda (l(t))^{\frac{k\gamma}{\gamma-1}}|u(t)|^{\frac{\mu\gamma}{\gamma-1}}dt\right)^{\frac{\gamma-1}{\gamma}}\\
&\quad= \left(\int_\Lambda\left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_\Lambda (l(t))^{\frac{k\gamma}{\gamma-1}}|u(t)|^{\frac{2k\gamma}{\gamma-1}}|u(t)|^{\frac{(\mu-2k)\gamma}{\gamma-1}}dt\right)^{\frac{\gamma-1}{\gamma}}\\
&\quad\le \left(\int_\Lambda\left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}}\left[\left(\int_\Lambda l(t)|u(t)|^2dt\right)^{\frac{k\gamma}{\gamma-1}}\left(\int_\Lambda |u(t)|^{\frac{(\mu-2k)\gamma}{\gamma-1-k\gamma}}dt\right)^{\frac{\gamma-1-k\gamma}{\gamma-1}}\right]^{\frac{\gamma-1}{\gamma}}\\
&\quad\le \left(\int_\Lambda\left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}} C^{\mu-2k}_{\frac{(\mu-2k)\gamma}{\gamma-1-k\gamma}}\|u\|^\mu. \qquad (2.7)
\end{aligned}$$

When $k=\frac{\gamma-1}{\gamma}$, we have

$$\begin{aligned}
\int_\Lambda a(t)|u(t)|^\mu dt &= \int_\Lambda \frac{a(t)}{(l(t))^{\frac{\gamma-1}{\gamma}}}(l(t))^{\frac{\gamma-1}{\gamma}}|u(t)|^\mu dt\\
&\le \left(\int_\Lambda\left(\frac{a(t)}{(l(t))^{\frac{\gamma-1}{\gamma}}}\right)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\int_\Lambda l(t)|u(t)|^{\frac{\mu\gamma}{\gamma-1}}dt\right)^{\frac{\gamma-1}{\gamma}}\\
&\le \left(\int_\Lambda\left(\frac{a(t)}{(l(t))^{\frac{\gamma-1}{\gamma}}}\right)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\|u\|_\infty^{\frac{\mu\gamma}{\gamma-1}-2}\int_\Lambda l(t)|u(t)|^2dt\right)^{\frac{\gamma-1}{\gamma}}\\
&\le \left(\int_\Lambda\left(\frac{a(t)}{(l(t))^{\frac{\gamma-1}{\gamma}}}\right)^\gamma dt\right)^{\frac{1}{\gamma}}\|u\|_\infty^{\mu-\frac{2(\gamma-1)}{\gamma}}\|u\|^{\frac{2(\gamma-1)}{\gamma}}\\
&\le \left(\int_\Lambda\left(\frac{a(t)}{(l(t))^{\frac{\gamma-1}{\gamma}}}\right)^\gamma dt\right)^{\frac{1}{\gamma}} C_\infty^{\mu-\frac{2(\gamma-1)}{\gamma}}\|u\|^\mu. \qquad (2.8)
\end{aligned}$$

Since $\left(b,\frac{2b}{2+b-b\mu}\right]\subset\left(1,\frac{2}{2-\mu}\right]$ and $\left[0,\frac{\gamma-b}{b\gamma}\right]\subset\left[0,\frac{\gamma-1}{\gamma}\right]$ for all $b\in[1,2]$, (2.7) and (2.8) also hold when $\gamma\in\left(b,\frac{2b}{2+b-b\mu}\right]$ and $k\in\left[0,\frac{\gamma-b}{b\gamma}\right]$.

Choosing $\Lambda=\mathbb{R}$, we see I is well defined. Similar to Lemma 3.1 in [22], one shows $I\in C^1(E,\mathbb{R})$ and (2.5) holds. Finally, we show all the critical points of I are homoclinic solutions for (1.1), i.e. we need to show $u(t)\to 0$ and $\dot{u}(t)\to 0$ as $t\to\pm\infty$ if $u(t)$ is a critical point of I. We can easily deduce from (2.5) that $L(t)u-\nabla W(t,u)$ is the weak derivative of $\dot{u}$. Since $E\subset C^0\left(\mathbb{R},\mathbb{R}^N\right)$(the space of continuous functions), $W\in C^1\left(\mathbb{R}\times\mathbb{R}^N,\mathbb{R}\right)$ and $L\in C\left(\mathbb{R},\mathbb{R}^{N^2}\right)$, we know u is indeed in $C^2(\mathbb{R},\mathbb{R}^N)$. Obviously,

$$\int_t^{t+\theta}|u(s)|^q ds\le\theta^{\frac{2-q}{2}}\left(\int_t^{t+\theta}|u(s)|^2ds\right)^{\frac{q}{2}}\to 0\quad\text{as } |t|\to+\infty \qquad (2.9)$$

and

$$\int_t^{t+\theta}|\dot{u}(s)|^q ds\le\theta^{\frac{2-q}{2}}\left(\int_t^{t+\theta}|\dot{u}(s)|^2ds\right)^{\frac{q}{2}}\to 0\quad\text{as } |t|\to+\infty \qquad (2.10)$$

for any $\theta \in \mathbb{R}$. It follows from (2.3) that $u(t) \to 0$ as $|t| \to +\infty$. In order to prove $\dot{u}(t) \to 0$ as $|t| \to +\infty$, we show a useful estimate as follow. For any $b \in [1,2]$, it follows from $(W0)$ and (2.1) that

$$\begin{aligned}
&\int_{\mathbb{R}} |\nabla W(t,u(t))|^b dt \\
&\quad \le \int_{\mathbb{R}} a^b(t)|u(t)|^{b(\mu-1)} dt \\
&\quad = \int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^k}\right)^b (l(t))^{bk} |u(t)|^{b(\mu-1)} dt \\
&\quad \le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{b}{\gamma}} \left(\int_{\mathbb{R}} (l(t))^{\frac{bk\gamma}{\gamma-b}} |u(t)|^{\frac{b\gamma(\mu-1)}{\gamma-b}} dt\right)^{\frac{\gamma-b}{\gamma}} \\
&\quad \le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{b}{\gamma}} \left[\left(\int_{\mathbb{R}} l(t)|u(t)|^2 dt\right)^{\frac{bk\gamma}{\gamma-b}} \left(\int_{\mathbb{R}} |u(t)|^{\frac{b\gamma(\mu-1-2k)}{\gamma-b-bk\gamma}} dt\right)^{\frac{\gamma-b-bk\gamma}{\gamma-b}}\right]^{\frac{\gamma-b}{\gamma}} \\
&\quad \le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{b}{\gamma}} C_{\frac{b\gamma(\mu-1-2k)}{\gamma-b-bk\gamma}}^{b(\mu-1-2k)} \|u\|^{b(\mu-1)}. \qquad (2.11)
\end{aligned}$$

Similarly, when $k = \frac{\gamma-b}{b\gamma}$,

$$\begin{aligned}
\int_{\mathbb{R}} |\nabla W(t,u(t))|^b dt &\le \int_{\mathbb{R}} a^b(t)|u(t)|^{b(\mu-1)} dt \\
&= \int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^{\frac{\gamma-b}{b\gamma}}}\right)^b (l(t))^{\frac{\gamma-b}{\gamma}} |u(t)|^{b(\mu-1)} dt \\
&\le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^{\frac{\gamma-b}{b\gamma}}}\right)^\gamma dt\right)^{\frac{b}{\gamma}} \left(\int_{\mathbb{R}} l(t)|u(t)|^{\frac{b\gamma(\mu-1)}{\gamma-b}} dt\right)^{\frac{\gamma-b}{\gamma}} \\
&\le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^{\frac{\gamma-b}{b\gamma}}}\right)^\gamma dt\right)^{\frac{b}{\gamma}} \left[\|u\|_\infty^{\frac{b\gamma(\mu-1)}{\gamma-b}-2} \int_{\mathbb{R}} l(t)|u(t)|^2 dt\right]^{\frac{\gamma-b}{\gamma}} \\
&\le \left(\int_{\mathbb{R}} \left(\frac{a(t)}{(l(t))^{\frac{\gamma-b}{b\gamma}}}\right)^\gamma dt\right)^{\frac{b}{\gamma}} C_\infty^{b(\mu-1)-\frac{2(\gamma-b)}{\gamma}} \|u\|^{b(\mu-1)}. \qquad (2.12)
\end{aligned}$$

The following proof is divided into two cases.

Case 1. $(L1)$ holds with $b = 2$. Let $\mathcal{A}$ be the self-adjoint extension of $-(d^2/dt^2) + L(t)$ with $\mathfrak{D}(\mathcal{A}) \subset L^2(\mathbb{R}, \mathbb{R}^N)$. Since we have $(L2)$ and (i)(or (ii)) of $(L1)$, similar to Lemma 2.3 in [5], $\mathfrak{D}(\mathcal{A})$ is continuously embedded in $W^{2,2}(\mathbb{R}, \mathbb{R}^N)$. Making estimates as (2.9) and (2.10), it follows from (2.4) that $\dot{u}(t) \to 0$ as $|t| \to +\infty$ if $u \in \mathfrak{D}(\mathcal{A})$. Subsequently, we show all the critical points of I belong to $\mathfrak{D}(\mathcal{A})$. By (2.11) and (2.12) with $b = 2$, we see $\|\mathcal{A}u\|_{L^2}^2 = \int_{\mathbb{R}} |\nabla W(t,u(t))|^2 dt < \infty$. Then $u \in \mathfrak{D}(\mathcal{A})$, which shows u is a homoclinic solution for (1.1).

Case 2. $(L3)$ holds with $b = q$. Since $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, we deduce from $(L3)$ and (2.4) that

$$|\dot{u}(t)| \le \delta^{\frac{1}{q^*}-1} \left(\int_t^{t+\delta} |\dot{u}(s)|^q ds\right)^{\frac{1}{q}} + \delta^{\frac{1}{q^*}} \left(\int_t^{t+\delta} |\ddot{u}(s)|^q ds\right)^{\frac{1}{q}}.$$

By (2.10), we only need to consider $\int_t^{t+\delta}|\ddot{u}(s)|^q ds$. Similar to Lemma 3.1 in [22], (2.11) and (2.12), for any $\gamma \in \left(q, \frac{2q}{2+q-q\mu}\right]$ and $k \in \left[0, \frac{\gamma-q}{q\gamma}\right]$

$$\begin{aligned}
&\int_t^{t+\delta}|\ddot{u}(s)|^q ds \\
&\le 2^{q-1}\int_t^{t+\delta}\left(|\nabla W(s,u(s))|^q + |L(s)u(s)|^q\right) ds \\
&\le 2^{q-1}M_1\left(\int_t^{t+\delta}\left(\frac{a(s)}{(l(s))^k}\right)^\gamma ds\right)^{\frac{q}{\gamma}}\|u\|^{q(\mu-1)} + 2^{q-1}\int_t^{t+\delta}\left|(L(s)u(s))^T L(s)u(s)\right|^{\frac{q}{2}} ds \\
&= 2^{q-1}M_1\left(\int_t^{t+\delta}\left(\frac{a(s)}{(l(s))^k}\right)^\gamma ds\right)^{\frac{q}{\gamma}}\|u\|^{q(\mu-1)} + 2^{q-1}\int_t^{t+\delta}\left|(u(s))^T L^2(s)u(s)\right|^{\frac{q}{2}} ds \\
&= 2^{q-1}M_1\left(\int_t^{t+\delta}\left(\frac{a(s)}{(l(s))^k}\right)^\gamma ds\right)^{\frac{q}{\gamma}}\|u\|^{q(\mu-1)} + 2^{q-1}\int_t^{t+\delta}\left|(L^2(s)u(s),u(s))\right|^{\frac{q}{2}} ds \\
&\le 2^{q-1}M_1\left(\int_t^{t+\delta}\left(\frac{a(s)}{(l(s))^k}\right)^\gamma ds\right)^{\frac{q}{\gamma}}\|u\|^{q(\mu-1)} + 2^{q-1}\left[\sup_{s\ge t}|u(s)|^q\right]\int_t^{t+\delta}\hat{l}^q(s) ds \\
&\to 0 \quad \text{as } |t|\to+\infty,
\end{aligned}$$

where

$$M_1 = \begin{cases} C^{q(\mu-1-2k)}_{\frac{q\gamma(\mu-1-2k)}{\gamma-q-qk\gamma}}, & k\in\left[0,\frac{\gamma-q}{q\gamma}\right), \\ C_\infty^{q(\mu-1)-\frac{2(\gamma-q)}{\gamma}}, & k=\frac{\gamma-q}{q\gamma}. \end{cases}$$

Thus u is a homoclinic solution for (1.1). □

In the next lemma, we show the functional I satisfies the classical Palais–Smale ((PS) for short) condition. We say that I satisfies the (PS) condition, if any sequence $(u_i)_i$ in E such that

$$(I(u_i))_i \text{ is bounded and } I'(u_i)\to 0,$$

admits a convergent subsequence.

Lemma 2.3. *Under $(L2)$, $(W0)$ and $(W1)$, I satisfies the (PS) condition.*

Proof. Let $\{u_i\}_{i\in\mathbb{N}} \subset E$ be a sequence such that $\{I(u_i)\}_{i\in\mathbb{N}}$ is bounded and $I'(u_i)\to 0$ as $i\to+\infty$. Then there exists $B>0$ such that $|I(u_i)|\le B$. By (2.2), (2.7) and (2.8) with $\Lambda=\mathbb{R}$, we have

$$\|u_i\|^2 = 2I(u_i) + 2\int_{\mathbb{R}} W(t,u_i(t))\,dt \le 2B + \frac{2M_2}{\mu}\left(\int_{\mathbb{R}}\left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}}\|u_i\|^\mu,$$

where

$$M_2 = \begin{cases} C^{\mu-2k}_{\frac{(\mu-2k)\gamma}{\gamma-1-k\gamma}}, & k\in\left[0,\frac{\gamma-1}{\gamma}\right), \\ C_\infty^{\mu-\frac{2(\gamma-1)}{\gamma}}, & k=\frac{\gamma-1}{\gamma}, \end{cases}$$

which implies $\{u_i\}_{i\in\mathbb{N}}$ is bounded in E. Hence, there exists $u_0\in E$(up to passing to a subsequence) such that $u_i \rightharpoonup u_0$ in E and

$$\begin{aligned}
&\langle I'(u_i)-I'(u_0), u_i-u_0\rangle \\
&\quad = \|u_i-u_0\|^2 - \int_{\mathbb{R}}\left(\nabla W(t,u_i(t)) - \nabla W(t,u_0(t)), u_i(t)-u_0(t)\right) dt \to 0 \qquad (2.13)
\end{aligned}$$

as $i \to \infty$. Moreover, there exists $M_3 > 0$ such that

$$\sup_{j\in\mathbb{N}} \|u_i\|_\infty \le M_3 \quad \text{and} \quad \|u_0\|_\infty \le M_3. \tag{2.14}$$

For any $\varepsilon > 0$ it follows from $(W0)$ that there exists $T > 0$ such that

$$\left(\int_{|t|>T} \left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}} < \varepsilon. \tag{2.15}$$

It follows from $(W0)$ and Sobolev's compact embedding theorem in bounded domain that

$$\begin{aligned}
&\int_{|t|\le T} (\nabla W(t,u_i(t)) - \nabla W(t,u_0(t)), u_i(t) - u_0(t))\, dt \\
&\quad \le \int_{|t|\le T} a(t)\left(|u_i(t)|^{\mu-1} + |u_0(t)|^{\mu-1}\right)|u_i(t) - u_0(t)|\, dt \\
&\quad \le a_0\left(\left(\int_{|t|\le T}|u_i(t)|^\mu\right)^{\frac{\mu-1}{\mu}} + \left(\int_{|t|\le T}|u_0(t)|^\mu\right)^{\frac{\mu-1}{\mu}}\right)\left(\int_{|t|\le T}|u_i(t)-u_0(t)|^\mu\right)^{\frac{1}{\mu}} \\
&\quad \le \varepsilon
\end{aligned} \tag{2.16}$$

for i large enough, where $a_0 = \max_{|t|\le T} a(t)$. By $(W0)$, (2.7) and (2.8) with $\Lambda = \mathbb{R} \setminus [-T, T]$, one has

$$\begin{aligned}
&\int_{|t|>T} (\nabla W(t,u_i(t)) - \nabla W(t,u_0(t)), u_i(t) - u_0(t))\, dt \\
&\quad \le \int_{|t|>T} |\nabla W(t,u_i(t)) - \nabla W(t,u_0(t)\|u_i(t) - u_0(t)| dt \\
&\quad \le \int_{|t|>T} a(t)\left(|u_i(t)|^{\mu-1} + |u_0(t)|^{\mu-1}\right)(|u_i(t)| + |u_0(t)|)\, dt \\
&\quad \le 3\int_{|t|>T} a(t)\left(|u_i(t)|^\mu + |u_0(t)|^\mu\right) dt \\
&\quad \le 3M_2\left(\int_{|t|>T}\left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}}\left(\|u_i\|^\mu + \|u_0\|^\mu\right).
\end{aligned} \tag{2.17}$$

By the arbitrariness of ε, (2.15) and (2.17), we obtain

$$\int_{|t|>T} (\nabla W(t,u_i(t)) - \nabla W(t,u(t)), u_i(t) - u_0(t))\, dt \to 0 \quad \text{as } i \to +\infty. \tag{2.18}$$

Together with (2.16), we obtain

$$\int_{\mathbb{R}} (\nabla W(t,u_i(t)) - \nabla W(t,u(t)), u_i(t) - u_0(t))\, dt \to 0 \quad \text{as } i \to +\infty.$$

Consequently, we infer from (2.13) and (2.18) that $\|u_i - u_0\| \to 0$ as $i \to +\infty$. □

Proof of Theorem 1.1. By (2.2), (2.6) and (2.7) with $\Lambda = \mathbb{R}$, for any $u \in E$, we get

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W(t,u(t))dt \\
&\ge \frac{1}{2}\|u\|^2 - \frac{1}{\mu}\int_{\mathbb{R}} a(t)|u(t)|^\mu dt \\
&\ge \frac{1}{2}\|u\|^2 - \frac{M_2}{\mu}\left(\int_{\mathbb{R}}\left(\frac{a(t)}{(l(t))^k}\right)^\gamma dt\right)^{\frac{1}{\gamma}}\|u\|^\mu,
\end{aligned}$$

which implies that $I(u) \to +\infty$ as $\|u\| \to +\infty$. Thus I is bounded from below and satisfies the (PS) condition. Then there exists $\bar{u}$ such that $I(\bar{u}) = c = \inf_E I(u)$. We also need to show that $\bar{u} \not\equiv 0$. Letting $\varphi \in C_0^\infty(\Omega, \mathbb{R}^N) \setminus \{0\}$ and $s > 0$, it follows from (2.2) and $(W2)$ that

$$\begin{aligned} I(s\varphi) &= \frac{s^2}{2}\|\varphi\|^2 - \int_{\mathbb{R}} W(t, s\varphi(t))\, dt \\ &= \frac{s^2}{2}\|\varphi\|^2 - \int_{\Omega} W(t, s\varphi(t))\, dt \\ &\leq \frac{s^2}{2}\|\varphi\|^2 - \eta s^\lambda \int_{\Omega} |\varphi(t)|^\lambda\, dt, \end{aligned}$$

which implies $I(s\varphi) < 0$ when $s > 0$ small enough. Then we can deduce that $\inf_E I(u) < 0$, which implies that $\bar{u} \not\equiv 0$. □

Proof of Theorem 1.2. The only difference between Theorems 1.1 and 1.2 is the way to obtain the asymptotic behavior of the solutions for (1.1) at infinity. This has been shown in the proof of Lemma 2.2. The remaining part is similar to Theorem 1.1, we omit it here. □

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A modified zero energy critical point theory with applications to several nonlocal problems

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Abstract. In this paper, we devote ourselves to considering a modified zero energy critical point theory for a specific set of functionals denoted as Φ_μ, defined within the confines of a uniformly convex Banach space. Integrating the nonlinear generalized Rayleigh quotient approach with Ljusternik–Schnirelman category, we establish the nonexistence and multiplicity of zero energy critical points of the involved functionals. In particular, the modified zero energy critical point theory can be applied to more nonlocal problems. Our main results improve and complement the existing results in the related literature.

Keywords: Ljusternik–Schnirelman category, nonlinear generalized Rayleigh quotient, zero energy critical points.

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1 Introduction

In the past decades, researchers have used classical variational methods to deal with various nonlocal problems and obtained various properties of their solutions, such as existence, multiplicity, asymptotic behavior and so on. However, although the classical variational methods have been properly modified, it seems still difficult to be directly effective for some complicated or special nonlocal problems. Based on this situation, the research on new variational methods has aroused increasing interest. It is worth mentioning that in this process, the existence of the number and index theory makes the Ljusternik–Schnirelman category theory more widely used. For more detailed applications of this theory, we refer to [18, 33] and references therein. Along this direction, in this paper we employ the Ljusternik–Schnirelman category theory and the nonlinear generalized Rayleigh (NG-Rayleigh) quotient method to forge a critical point theory at zero energy levels for the energy functional (1.1). By means of this theory, we deal with several kinds of nonlocal problems, and present the nonexistence and multiplicity of their solutions at zero energy levels. More precisely, we consider $\Phi_\mu : X \to \mathbb{R}$

$$\Phi_\mu(u) := \frac{1}{\eta}N(u) - \frac{\mu}{\eta}A(u) - \frac{1}{\beta}B(u) + \frac{1}{\gamma}R(u), \tag{1.1}$$

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the setting is within X, a uniformly convex Banach space endowed with the norm $|\cdot|_X$, where $1 < \eta < \gamma < \beta$. Moreover, the functionals N, A, B, and R are considered to be homogeneous, nonnegative, and even, belonging to the class $C^1(X)$.

Throughout this paper, the following assumptions are imposed on the above-mentioned nonnegative even functionals:

($\mathcal{M}_1$) For any $u \in X\backslash\{0\}$, there exists $C > 0$ such that the following inequalities hold:

$$C\|u\|_X^\eta \geq A(u) > 0, \qquad C\|u\|_X^\beta \geq B(u) > 0, \qquad R(u) > 0, \qquad N(u) \geq C^{-1}\|u\|_X^\eta;$$

($\mathcal{M}_2$) $N(tu) = t^\eta N(u), A(tu) = t^\eta A(u), B(tu) = t^\beta B(u), R(tu) = t^\gamma R(u)$, for any $t > 0$;

($\mathcal{M}_3$) If $u_n \rightharpoonup u$ in X, then $A'(u_n) \to A'(u)$ and $B'(u_n) \to B'(u)$ in X^*. Moreover, for any $u_n, u \in X$, it holds that $R'(u_n)(u_n - u) \geq 0$.

($\mathcal{M}_4$) Let N be weakly lower semicontinuous, and there exists $C > 0$ such that for every $u_n, u \in X$, the inequality

$$(N'(u_n) - N'(u))(u_n - u) \geq C(\|u_n\|^{\eta-1} - \|u\|^{\eta-1})(\|u_n\| - \|u\|)$$

holds true.

According to assumption ($\mathcal{M}_1$), we know that $A(u) > 0$ for all $u \in X \setminus \{0\}$. Therefore, $\Phi_\mu(u) = 0$ is equivalent to

$$\mu = \mu_0(u) := \frac{N(u) - \frac{\eta}{\beta}B(u) + \frac{\eta}{\gamma}R(u)}{A(u)}, \quad \text{for any } u \in X \setminus \{0\},$$

where $\mu_0(u)$ is called the Rayleigh quotient, the functional is derived using the NG-Rayleigh quotient approach. For any $u \in X \setminus \{0\}$,

$$\mu_0'(u) = \frac{\Phi'_{\mu_0(u)}(u)}{A(u)} = 0,$$

if and only if $\Phi'_\mu(u) = 0$.

We will search for the critical points of μ_0 by considering the fibering map $t \mapsto \mu_0(tu)$. Obviously, $\mu_0(tu) \in C^2(0,\infty)$ for every $u \in X \setminus \{0\}, u \mapsto \mu_0'(tu) \in C^1(X \setminus \{0\})$ for every $t > 0$. In order to get the critical point of $\mu_0(tu)$, from ($\mathcal{M}_2$) it follows that

$$\mu_0(tu) = \frac{N(u)}{A(u)} - \frac{\eta}{\beta}\frac{B(u)}{A(u)}t^{\beta-\eta} + \frac{\eta}{\gamma}\frac{R(u)}{A(u)}t^{\gamma-\eta}, \quad \text{for any } u \in X \setminus \{0\},\ t > 0.$$

Let

$$\mu_0'(tu) = -(\beta - \eta)\frac{\eta}{\beta}\frac{B(u)}{A(u)}t^{\beta-\eta-1} + (\gamma - \eta)\frac{\eta}{\gamma}\frac{R(u)}{A(u)}t^{\gamma-\eta-1} = 0.$$

Then

$$t_0(u) = \left(\frac{\gamma - \eta}{\beta - \eta}\frac{\beta}{\gamma}\frac{R(u)}{B(u)}\right)^{\frac{1}{\beta-\gamma}},$$

that is $\mu_0'(t_0(u)u) = 0$. Since $\mu_0''(t_0(u)u) < 0$, we obtain that $t_0(u)$ is a non-degenerate global maximum point of $\mu_0(tu)$.

Define

$$\Lambda(u) := \mu_0(t_0(u)u) = \left(\frac{N(u)}{A(u)} - \frac{\eta}{\beta}\frac{B(u)}{A(u)}t_0(u)^{\beta-\eta} + \frac{\eta}{\gamma}\frac{R(u)}{A(u)}t_0(u)^{\gamma-\eta}\right)$$
$$= \frac{N(u)}{A(u)} + C_0\frac{R(u)^{\frac{\beta-\eta}{\beta-\gamma}}}{A(u)B(u)^{\frac{\gamma-\eta}{\beta-\gamma}}},$$

as a NG-Rayleigh quotient. It is obvious that $t_0(u)u$ can be considered as the zero energy critical point of Φ_μ, where $\mu = \Lambda(u)$. One may easily check that $\Lambda \in C^1(X\backslash\{0\})$,

$$\Lambda'(u)v = \frac{A(u)N'(u)v - N(u)A'(u)v}{A(u)^2}$$
$$+ C_0Q(u)\left(\frac{\beta-\eta}{\beta-\gamma}B(u)A(u)R'(u)v - \frac{\gamma-\eta}{\beta-\gamma}R(u)A(u)B'(u)v - B(u)R(u)A'(u)v\right),$$

for every $u \in X\backslash\{0\}, v \in X$, where

$$C_0 = \frac{\eta}{\gamma}\frac{\beta-\gamma}{\beta-\eta}\left(\frac{\gamma-\eta}{\beta-\eta}\frac{\beta}{\gamma}\right)^{\frac{\gamma-\eta}{\beta-\gamma}} > 0, \qquad Q(u) = \frac{R(u)^{\frac{\gamma-\eta}{\beta-\gamma}}}{A(u)^2B(u)^{\frac{\beta-\eta}{\beta-\gamma}}}.$$

To better utilize the Ljusternik–Schnirelman category theory, we first denote $\widetilde{\Lambda}$ as Λ on $S = \{u \in X\backslash\{0\} : \|u\| = 1\}$, where S is considered as a unit sphere in X and is a symmetric C^1 manifold. According to [28, Proposition 2.3], the critical point of $\widetilde{\Lambda}$ is also the critical point of Λ. Since $N(u), A(u), B(u), R(u)$ are even functionals, $\widetilde{\Lambda}$ is also an even functional. Now, let us recall the concept of Krasnoselskii genus. Given a set $F \subset S$, it is closed, nonempty and symmetric. We define

$$\gamma(F) := \inf\{n \in \mathbb{N} : \exists h : F \to \mathbb{R}^n \setminus \{0\} \text{ odd and continuous}\}.$$

to represent the Krasnoselskii genus of F. Setting

$$\mathcal{F}_n = \{F \subset S : F \text{ is compact, symmetric, and } \gamma(F) \geq n\},$$

for every $n \in \mathbb{N}$. Define the critical value of $\widetilde{\Lambda}$:

$$\mu_n := \inf_{F\in\mathcal{F}_n}\sup_{u\in F}\widetilde{\Lambda}, \text{ if } \widetilde{\Lambda} \text{ is bounded from below on } S.$$

It is well-known that the Krasnoselskii genus of the unit sphere in an infinite dimensional Banach space is infinite (cf. [13, Corollary 2.3]), namely, $\gamma(S) = \infty$.

Next, let us sketch some recent advances concerning the zero energy critical point theory. Recently, Quoirin et al. studied qualitative properties of zero energy critical points in [28], which means that at this point, the energy function and its derivatives are both zero. Furthermore, the authors in [28] established a new zero energy critical point theory using the NG-Rayleigh quotient method and Ljusternik–Schnirelman critical theory [2], and effectively applied it to several types of elliptic partial differential equations, resulting in the existence, nonexistence, and multiplicity of zero energy critical points. For more details on the NG-Rayleigh method, we refer to [19, 20] and references therein. Undoubtedly, the zero energy critical point theory established in paper [28] provides us with a new idea and perspective for solving nonlinear partial differential equations. As one of the advantages of this theory, it has a wide range of theoretical applications, that is, it can directly handle many types of nonlinear non-local problems, such as concave-convex problem, Schrödinger–Poisson problem, Kirchhoff-type problem, (p,q)-Laplace problem, and other elliptic problems. As a pioneer paper on zero energy critical point theory, the authors in [28] applied this theory to solve several local and non-local problems. For example, the following p-Laplacian problem with concave and convex nonlinearity was investigated in a bounded domain $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} -\Delta_p u = \mu|u|^{q-2}u + f|u|^{r-2}u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < q < p < r < p^*$, $f, g \in L^\infty(\Omega)$ with $g > 0$ in Ω, Δ_p is p-Laplaian operator, and $f > 0$ in some subdomain $\Omega' \subset \Omega$. With the help of the NG-Rayleigh quotient method and Ljusternik–Schnirelman theory, the authors obtained the existence, non-existence and multiplicity of zero energy solutions for concave and convex problems in [28].

On the other hand, the authors in [28] also considered the properties of the zero energy solution for the Schrödinger-Poisson system, which is physically meaningful. To elaborate, the authors conducted a comprehensive investigation into the intricacies of the following system:

$$\begin{cases} -\Delta u + \omega u + \mu\phi u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta u + a^2\Delta^2 u = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $p \in (2,3), \omega > 0$, and $a \geq 0$. In particular, the authors established the existence, non-existence, multiplicity and sign-changing properties of the zero energy radial solution of system (1.3). Subsequently, Quoirin et al. in [29] established the existence, multiplicity and bifurcation results of the critical points for a class of functionals with prescribed energy along the same technical route as in [28]. The authors first applied the corresponding critical point theory of prescribed energy to eigenvalue problems involving nonhomogeneous perturbations in [29], and its energy functional can be given by:

$$\Phi_\mu(u) = \frac{1}{p}(|\nabla u|_p^p - \mu|u|_p^p) - \frac{1}{r}|u|_r^r,\ u \in W_0^{1,p}(\Omega), \quad \text{where } 1 < r < p^*.$$

The authors made a noteworthy discovery regarding the Schrödinger–Poisson system. Specifically, for $c > 0$ (respectively, $c < 0$) and by choosing $p < r$ (respectively, $p > r$), the study revealed the existence of an infinite number of pairs $(\mu_{n,c}, u_{n,c}) \in \mathbb{R} \times W_0^{1,p}(\Omega) \setminus \{0\}$ satisfying $\Phi_{\mu_{n,c}}(\pm u_{n,c}) = c$ and $\Phi'_{\mu_{n,c}}(\pm u_{n,c}) = 0$. In other words, $\pm u_n$ represent prescribed energy critical points.

The authors next investigated the prescribed energy critical point of the Schrödinger–Bopp–Podolsky problem in [29]. The following represents the energy functional for this particular problem:

$$\Phi_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\omega}{2}\int_{\mathbb{R}^3} |u|^2 dx + \frac{\mu}{4}\int_{\mathbb{R}^3} \phi u^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx, \text{ where } p \in (2,3), \omega > 0.$$

The authors obtained the existence and multiplicity of the critical point with prescribed energy by the critical point theory. Besides, the authors also conducted relevant research on the concave-convex problem in [29], and we will not elaborate on it here.

Motivated by [28, 29], we are interested in making appropriate modifications and extensions according to the existing zero energy critical point theory so that it can be applied to some specific situations. Along this direction, we propose a suitable class of energy functional (1.1). We point out that the purpose of this paper is to address the problem of energy functional (1.1) in the following situation:

$$\Phi'_\mu(u) = 0, \qquad \Phi_\mu(u) = 0.$$

Let $\delta_0 = \inf_{X\setminus\{0\}} \Lambda(u)$. Then the primary result of our article can be stated as:

Theorem 1.1. *Suppose that $(\mathcal{M}_1)$–$(\mathcal{M}_4)$ hold.*

(i) If $\mu < \delta_0$, then there is no critical point having zero energy for the energy functional Φ_μ.

(ii) If $\mu > \delta_0$, then the energy functional Φ_μ has infinitely many zero energy critical points which change sign.

Remark 1.2. At the beginning, we attempt to investigate the qualitative properties for some nonlocal problems in $\mathbb{R}^N$ by employing Theorem 1.1. But the embedding $H^1_r(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact only for $2 < p < 2^*$, we cannot verify assumption $(\mathcal{M}_3)$ if problems under consideration are involved the critical exponents. Inspired by [28], we give a modified version of Theorem 3.1 in [28], so that we are able to deal with Schrödinger–Poisson systems with critical nonlinearity.

Remark 1.3. In order to deal with Kirchhoff-type problems with critical growth in bounded domains, we follow the idea of proof in Lemma 3.3 of [21] to detour the compact embedding theorem, hence assumption $(\mathcal{M}_3)$ can be verified, which leads to the nonexistence and multiplicity of zero energy critical points for the following Kirchhoff problem with critical nonlinearity.

Notations. Throughout this paper, the following notions are employed:

- Denote $|\cdot|_q$ for the $L^q(\mathbb{R}^N)$-norm for $q \in [1, \infty]$;
- Denote various positive constants by $C, C_0, C_1, C_2, C_3, \ldots$;
- Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with the norm $\|u\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^{\frac{1}{2}}$.

2 Proof of Theorem 1.1

In this proof, we refer to the technical approach demonstrated by [28] to prove Theorem 1.1. In order to supplement and enrich the zero energy critical point theory, we give a modified result, which can be applied to a wider range of nonlocal Lapalcian equations. Based on the Ljusternik–Schnirelman category (see [32, Theorem 5.7]), we only need to prove that $\widetilde{\Lambda}$ satisfies the Palais–Smale condition and is bounded from below on the unit sphere S.

Lemma 2.1. *Assume $(\mathcal{M}_1)$ holds, then $\widetilde{\Lambda}$ is bounded from below on S.*

Proof. From $(\mathcal{M}_1)$, for all $u \in S$,

$$\frac{N(u)}{A(u)} \geq \frac{C^{-1}\|u\|^\eta}{C\|u\|^\eta} = \frac{1}{C^2}, \qquad \frac{R(u)^{\frac{\beta-\eta}{\beta-\gamma}}}{B(u)^{\frac{\gamma-\eta}{\beta-\gamma}} A(u)} > 0.$$

Then

$$\widetilde{\Lambda} = \frac{N(u)}{A(u)} + C_0 \frac{R(u)^{\frac{\beta-\eta}{\beta-\gamma}}}{A(u)B(u)^{\frac{\gamma-\eta}{\beta-\gamma}}} > \frac{1}{C^2}.$$

Therefore, $\widetilde{\Lambda}$ is bounded from below on S. □

A crucial proposition required to validate the Palais–Smale condition for $\widetilde{\Lambda}(u)$ is as follows:

Lemma 2.2. *Assume that* $(u_n) \subset S$, $\widetilde{\Lambda}'(u_n) \to 0$, *then* $\Lambda'(u_n)(u_n - u) \to 0$ *as* $n \to \infty$.

Proof. Since $(u_n) \subset S = \{u \in X \backslash \{0\} : \|u\| = 1\}$, we see that $\|u_n\| = 1$. According to S is weakly closed and (u_n) is bounded in S, we can attain $u_n \rightharpoonup u$ in S. Let $\mathcal{T}_S(u) = \{v \in X : i'(u)v = 0\}$, at the point u, $\mathcal{T}_S(u)$ represents the tangent space to the set S, where $i(u) = \frac{1}{2}\|u\|^2$. Note that, for any $w \in X$ and any $n \in \mathbb{N}$, the pair $(t_n, v_n) \in \mathbb{R} \times \mathcal{T}_S(u_n)$ is uniquely identified, ensuring $w = v_n + t_n u_n$ and subsequently, $i'(u_n)w = i'(u_n)v_n + i'(u_n)t_n u_n$. According to the definition of $\mathcal{T}_S(u)$, we can obtain that $i'(u_n)v_n = 0, i'(u_n)u_n = \|u_n\|^2 = 1$. Therefore, $i'(u_n)w = t_n i'(u_n)u_n = t_n$. Then, (t_n) is bounded, consequently, (v_n) is also bounded. Since $\widetilde{\Lambda}'(u_n) \to 0$, namely, $|\Lambda'(u_n)v_n| \leq \varepsilon_n \|v_n\|$ for any $v_n \in \mathcal{T}_S(u_n)$ with $\varepsilon_n \to 0$, we have $\Lambda'(u_n)v_n \to 0$. According to the Lemma 2.1 of [28] we obtain that $\Lambda'(u_n)u_n = 0$. We conclude that $\Lambda'(u_n)w \to 0$ for any $w \in X$. Taking $w = u_n - u$, we get that $\Lambda'(u_n)(u_n - u) \to 0$, as $n \to \infty$. □

Lemma 2.3. *Assume that* $(\mathcal{M}_1), (\mathcal{M}_3), (\mathcal{M}_4)$ *hold, then* $\widetilde{\Lambda}$ *satisfies the Palais–Smale condition.*

Proof. Choose $(u_n) \subset S$ such that $(\widetilde{\Lambda}(u_n))$ is bounded and $\widetilde{\Lambda}'(u_n) \to 0$, i.e. $|\Lambda'(u_n)v| \leq \varepsilon_n \|v\|$ for any $v \in \mathcal{T}_S(u_n)$, with $\varepsilon_n \to 0$. By Lemma 2.2, together with the fact that $(u_n) \subset S$, $\widetilde{\Lambda}'(u_n) \to 0$, we know that

$$u_n \rightharpoonup u \quad \text{in } S, \qquad \Lambda'(u_n)(u_n - u) \to 0.$$

Then, for any $u_n, u \in S$,

$$\begin{aligned} &\Lambda'(u_n)(u_n - u) \\ &\quad = \frac{A(u_n)N'(u_n)(u_n - u) - N(u_n)A'(u_n)(u_n - u)}{A(u_n)^2} \\ &\qquad + C_0 Q(u_n) \left(\frac{\beta - \eta}{\beta - \gamma} B(u_n)A(u_n)R'(u_n)(u_n - u) \right. \\ &\qquad\qquad - \frac{\gamma - \eta}{\beta - \gamma} R(u_n)A(u_n)B'(u_n)(u_n - u) \\ &\qquad\qquad \left. - B(u_n)R(u_n)A'(u_n)(u_n - u) \right) \to 0. \end{aligned} \tag{2.1}$$

According to $(\mathcal{M}_1)$, we can infer that $A(u_n), B(u_n)$ is bounded, $N(u_n), R(u_n)$ is bounded away from zero. Since $(\widetilde{\Lambda}(u_n))$ is bounded, we know that $Q(u_n)$ is bounded. In the light of (M_3), we can obtain

$$A'(u_n)(u_n - u) \to 0, \qquad B'(u_n)(u_n - u) \to 0.$$

The above analysis leads to the following conclusion:

$$\left(N'(u_n) + R'(u_n)\right)(u_n - u) \to 0.$$

According to $(\mathcal{M}_3)$, we obtain $R'(u_n)(u_n - u) \geq 0$. From $(\mathcal{M}_4)$, we have

$$(N'(u_n) - N'(u))(u_n - u) \geq C(\|u_n\|^{\eta-1} - \|u\|^{\eta-1})(\|u_n\| - \|u\|) \geq 0$$

for every $u_n, u \in S$. Moreover, $u_n \rightharpoonup u$ in S, we have $N'(u)(u_n - u) \to 0$, then $N'(u_n)(u_n - u) \geq 0$. Therefore, we can conclude that $N'(u_n)(u_n - u) \to 0$. Since $(N'(u_n) - N'(u))(u_n - u) \to 0$, we obtain $\|u_n\| \to \|u\|$. Note that X is a reflexive Banach space and $u_n \rightharpoonup u$ in S, which imply that $u_n \to u$ in S. This completes the proof. □

Proof of Theorem 1.1.

(1) We prove that there is no critical point having zero energy when $\mu < \delta_0$. Note that u is a critical point of Λ, if and only if, $t_0(u)u$ is a zero energy critical point of Φ_μ with $\mu = \Lambda(u)$. In other words, this means that

$$\delta_0 = \inf_{X\setminus\{0\}} \Lambda(u) \leq \mu \leq \sup_{X\setminus\{0\}} \Lambda(u),$$

which yields the desired conclusion.

(2) According to Lemma 2.1, we know that $\widetilde{\Lambda}$ is bounded from below on S. Moreover, Lemma 2.3 implied that $\widetilde{\Lambda}$ satisfies the Palais–Smale condition. Note that $\hat{\gamma}(S) = \infty$, we get from Ljusternik–Schnirelman category (see [32, Theorem 5.7]) that there exists a sequence $(u_n) \subset S$ such that $\widetilde{\Lambda}'(u_n) = 0, \widetilde{\Lambda}(u_n) = \mu_n$. Therefore, the energy functional Φ has infinitely many zero energy sign changing critical points $(u_n) \subset S$.

3 Applications of Theorem 1.1

In this section, we shall prove the nonexistence of solutions and the existence of infinitely many solutions for three non-local problems, we confirm that these are just a small part of applications of Theorem 1.1.

3.1 Critical Schrödinger–Poisson system in the whole space

In this subsection, let us consider a Schrödinger–Poisson system with p-Laplacian:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda\phi u = |u|^{p^*-2}u + \mu|u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{3.1}$$

where $\lambda > 0$ is a constant, $12/7 < p < 3$, $p^* := 3p/(3-p)$, and $\Delta_p = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian. The p-Laplacian operator appears in nonlinear fluid dynamics, and the range of p is related to the velocity of the fluid and material. For more information on the physical origin of p-Laplacian, we refer to [9]. For any given $u \in W^{1,p}(\mathbb{R}^3)$, there exists a unique

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|}\mathrm{d}y, \qquad \phi_u \in D^{1,2}\left(\mathbb{R}^3\right),$$

satisfying $-\Delta\phi_u = |u|^2$ (see [17]).

The system (3.1) can be viewed as a perturbation of the system

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda\phi u = |u|^{q-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{3.2}$$

Du, Su and Wang first considered this system in [16], they established the existence of nontrivial solutions through the mountain pass theorem. Systems like (3.2) originate from quantum mechanics models [10, 12, 23], semiconductor theory [24, 25]. They described the interaction between quantum particles and electromagnetic fields. After the seminal work of Benci and Fortunato in [7,8], many researches have been conducted on systems such as (3.2) in the past few decades, as shown in [3,4,6,14,15,22,26,27,31] and their references.

Inspired by the above work, we are committed to studying the existence of solutions for system (3.1) in $\mathbb{R}^3$. Specifically, by using the nonlinear generalized Rayleigh quotient method and Ljusternik–Schnirelman theory, we obtain that there exist infinitely many zero energy sign changing weak solutions.

First of all, we give the variational framework for system (3.1). Let $W^{1,p}(\mathbb{R}^3)$ denote the usual Sobolev space equipped with the norm

$$\|u\| := \left(\int_{\mathbb{R}^3} |\nabla u|^p + |u|^p dx\right)^{\frac{1}{p}}.$$

One may easily get that the corresponding functional of (3.1) is $\Phi_\mu(u) : W^{1,p}(\mathbb{R}^3) \to \mathbb{R}$

$$\Phi_\mu(u) = \frac{1}{p}\int_{\mathbb{R}^3}(|\nabla u|^p + |u|^p)dx - \frac{\mu}{p}\int_{\mathbb{R}^3}|u|^p dx - \frac{1}{p^*}\int_{\mathbb{R}^3}|u|^{p^*}dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi u^2 dx.$$

It is standard to verify that Φ_μ is C^1. Then, for every $u \in W^{1,p}(\mathbb{R}^3)\backslash\{0\}$, we have

$$\mu_0(u) = \frac{\|u\|^p}{|u|_p^p} - \frac{p}{p^*}\frac{|u|_{p^*}^{p^*}}{|u|_p^p} + \frac{\lambda p}{4}\frac{\int_{\mathbb{R}^3}\phi u^2 dx}{|u|_p^p},$$

then

$$\mu_0(tu) = \frac{\|u\|^p}{|u|_p^p} - t^{p^*-p}\frac{p}{p^*}\frac{|u|_{p^*}^{p^*}}{|u|_p^p} + t^{4-p}\frac{\lambda p}{4}\frac{\int_{\mathbb{R}^3}\phi u^2 dx}{|u|_p^p}.$$

Let

$$\mu_0'(tu) = -t^{p^*-p-1}(p^*-p)\frac{p^*}{q}\frac{|u|_{p^*}^{p^*}}{|u|_p^p} + t^{3-p}(4-p)\frac{\lambda p}{4}\frac{\int_{\mathbb{R}^3}\phi u^2 dx}{|u|_p^p} = 0.$$

It is easy to see that

$$t_0(u) = \left(\frac{4-p}{p^*-p}\frac{p^*}{4}\frac{\lambda\int_{\mathbb{R}^3}\phi u^2 dx}{|u|_{p^*}^{p^*}}\right)^{\frac{1}{p^*-4}} > 0,$$

that is $\mu_0'(t_0(u)u) = 0$. Since $\mu_0''(t_0(u)u) < 0$, $t_0(u)$ is a nondegenerate global maximum point of $\mu_0(tu)$. Therefore, we have

$$\Lambda_1(u) = \mu_0(t_0(u)u) = \frac{\|u\|^p}{|u|_p^p} + \frac{p}{4}\frac{p^*-4}{p^*-p}\left(\frac{4-p}{p^*-p}\frac{p^*}{4}\right)^{\frac{4-p}{p^*-4}}\frac{(\lambda\int_{\mathbb{R}^3}\phi u^2 dx)^{\frac{p^*-p}{p^*-4}}}{|u|_p^p|u|_{p^*}^{p^*\frac{4-p}{p^*-4}}}.$$

For simplicity's sake, we call $\widetilde{\Lambda_1}$ the restriction of Λ_1 to S_1, where

$$S_1 = \{u \in W^{1,p}(\mathbb{R}^3)\backslash\{0\} : \|u\| = 1\}.$$

Now it is in a position to state our main result in this section as follows.

Theorem 3.1. *Let* $\delta_1 = \inf_{W^{1,p}(\mathbb{R}^3)\setminus\{0\}} \Lambda_1(u)$.

(i) If $\mu < \delta_1$*, then there is no nontrivial weak solution having zero energy for system* (3.1)*;*

(ii) If $\mu > \delta_1$*, then system* (3.1) *has infinitely many zero energy sign changing weak solutions.*

To prove Theorem 3.1, according to Ljusternik–Schnirelman category, we only need to prove that $\widetilde{\Lambda_1}$ is bounded from below and $\widetilde{\Lambda_1}$ satisfies the Palais-Smale condition.

Lemma 3.2. $\widetilde{\Lambda_1}$ *is bounded from below on* S_1.

Proof. Since the embedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is continuous, there exists $C > 0$ such that $C\|u\|^p \geqslant |u|_p^p$. Because $\frac{12}{7} < p < 3, p^* > 4$, one may check that

$$\widetilde{\Lambda_1}(u) = \frac{\|u\|^p}{|u|_p^p} + \frac{p}{4}\frac{p^*-4}{p^*-p}\left(\frac{4-p}{p^*-p}\frac{p^*}{4}\right)^{\frac{4-p}{p^*-4}} \frac{\left(\lambda\int_{\mathbb{R}^3}\phi u^2 dx\right)^{\frac{p^*-p}{p^*-4}}}{|u|_p^p |u|_{p^*}^{p^*\frac{4-p}{p^*-4}}} > \frac{1}{C}.$$

Therefore, $\widetilde{\Lambda_1}$ is bounded from below on S_1. □

In order to prove that $\widetilde{\Lambda_1}$ satisfies the Palais–Smale condition, we require the following proposition.

Lemma 3.3. *If* $u_n \rightharpoonup u$ *in* S_1*, then for any* $v \in W^{1,p}(\mathbb{R}^3)$ *there holds*

$$\int_{\mathbb{R}^3} |u_n|^{p-2}u_n v dx \to \int_{\mathbb{R}^3} |u|^{p-2}uv dx, \qquad \int_{\mathbb{R}^3} |u_n|^{p^*-2}u_n v dx \to \int_{\mathbb{R}^3} |u|^{p^*-2}uv dx, \qquad n \to \infty.$$

Proof. Since $u_n \rightharpoonup u$ in S_1, we derive that $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$. Note that the embedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $s \in [p, p^*]$, there exists a positive constant C such that

$$\int_{\mathbb{R}^3} \left||u_n|^{p-2}u_n\right|^{\frac{p}{p-1}} dx \le \int_{\mathbb{R}^3} \left||u_n|^{p-1}\right|^{\frac{p}{p-1}} dx = \int_{\mathbb{R}^3} |u_n|^p dx \le C\|u_n\|^p,$$

$$\int_{\mathbb{R}^3} \left||u_n|^{p^*-2}u_n\right|^{\frac{p^*}{p^*-1}} dx \le \int_{\mathbb{R}^3} \left||u_n|^{p^*-1}\right|^{\frac{p^*}{p^*-1}} dx = \int_{\mathbb{R}^3} |u_n|^{p^*} dx \le C\|u_n\|^{p^*}.$$

Therefore, $\{|u_n|^{p-2}u_n\}$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^3)$ and $\{|u_n|^{p^*-2}u_n\}$ is bounded in $L^{\frac{p^*}{p^*-1}}(\mathbb{R}^3)$. It follows from [34, Proposition 5.4.7] that $|u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u$ in $L^{\frac{p}{p-1}}(\mathbb{R}^3)$ and $|u_n|^{p^*-2}u_n \rightharpoonup |u|^{p^*-2}u$ in $L^{\frac{p^*}{p^*-1}}(\mathbb{R}^3)$. Thus for any $v \in W^{1,p}(\mathbb{R}^3)$ we have

$$\int_{\mathbb{R}^3} |u_n|^{p-2}u_n v dx \to \int_{\mathbb{R}^3} |u|^{p-2}uv dx, \qquad \int_{\mathbb{R}^3} |u_n|^{p^*-2}u_n v dx \to \int_{\mathbb{R}^3} |u|^{p^*-2}uv dx, \qquad n \to \infty,$$

as required. □

Lemma 3.4 (See [17, Proposition 2.1])**.** *For any* $u \in W^{1,p}(\mathbb{R}^3)$*, the following properties are applicable:*

(1) $\phi_u \ge 0$ *and for any* $t \in \mathbb{R}^+, \phi_{tu} = t^p\phi_u$.

(2) *There exists a positive constant* C *such that*

$$|\nabla\phi_u|_2^2 = \int_{\mathbb{R}^3} \phi_u |u|^p dx \le C\|u\|^{2p}.$$

(3) *In the case of $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^3)$, it follows that $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^{p-2} u_n \varphi dx \to \int_{\mathbb{R}^3} \phi_u u^{p-2} u \varphi dx, \quad \textit{for any } \varphi \in W^{1,p}(\mathbb{R}^3).$$

Lemma 3.5. *$\widetilde{\Lambda_1}$ satisfies the Palais–Smale condition.*

Proof. Choose a sequence $(u_n) \subset S_1$ such that $(\widetilde{\Lambda_1}(u_n))$ is bounded and $\widetilde{\Lambda_1}'(u_n) \to 0$, that is $|\Lambda_1'(u_n)v| \le \varepsilon_n \|v\|$ for any $v \in \mathcal{T}_{S_1}(u_n)$, with $\varepsilon_n \to 0$. With the help of Lemma 2.2, if there is a sequence $(u_n) \subset S_1$ and $\widetilde{\Lambda_1}'(u_n) \to 0$, we can obtain that $u_n \rightharpoonup u$ in S_1 and $\Lambda_1'(u_n)(u_n - u) \to 0$ as $n \to \infty$. Then, for any $u_n, u \in S_1$,

$$\begin{aligned}
&\Lambda_1'(u_n)(u_n-u) \qquad (3.3)\\
&= p\frac{|u_n|_p^p \int_{\mathbb{R}^3} \left(|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n-u) + |u_n|^{p-2}u_n(u_n-u)\right)dx - \|u_n\|^p \int_{\mathbb{R}^3} |u_n|^{p-2}u_n(u_n-u)dx}{|u_n|_p^{2p}}\\
&\quad + \left(4\frac{p^*-p}{q-4}|u_n|_{p^*}^{p^*}|u_n|_p^p \int_{\mathbb{R}^3} \phi u_n(u_n-u)dx - p^*\frac{4-p}{p^*-4}|u_n|_p^p \int_{\mathbb{R}^3} \phi u_n^2 dx \int_{\mathbb{R}^3} |u_n|^{p^*-2}u_n(u_n-u)dx\right.\\
&\quad \left. - p|u_n|_{p^*}^{p^*} \int_{\mathbb{R}^3} \phi u_n^2 dx \int_{\mathbb{R}^3} |u_n|^{p-2}u_n(u_n-u)dx\right) C_1 Q(u_n) \to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

where

$$C_1 = \lambda \frac{p}{4}\frac{p^*-4}{p^*-p}\left(\frac{4-p}{p^*-p}\frac{p^*}{4}\right)^{\frac{4-p}{p^*-4}}, \qquad Q(u_n) = \frac{\left(\int_{\mathbb{R}^3} \phi u_n^2 dx\right)^{\frac{4-p}{p^*-4}}}{|u_n|_p^{2p} |u_n|_{p^*}^{p^*\frac{p^*-p}{p^*-4}}}.$$

Next, we claim that

$$\int_{\mathbb{R}^3} \left(|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n-u) + |u_n|^{p-2}u_n(u_n-u)\right)dx \to 0, \quad \text{as } n \to \infty.$$

Therefore, we need to show that $|u_n|_{p^*}^{p^*}$, $|u_n|_p^p$, $\int_{\mathbb{R}^3} \phi u_n^2 dx$ and $Q(u_n)$ are bounded and

$$\int_{\mathbb{R}^3} |u_n|^{p-2}u_n(u_n-u)dx \to 0, \quad \int_{\mathbb{R}^3} |u_n|^{p^*-2}u_n(u_n-u)dx \to 0, \quad \int_{\mathbb{R}^3} \phi_{u_n} u_n(u_n-u)dx \to 0$$

as $n \to \infty$. Indeed, since $(u_n) \subset S_1$ and the embedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $s \in [p, p^*]$, we can obtain

$$0 < |u_n|_{p^*}^{p^*} \le C\|u_n\|^{p^*} = C, \qquad 0 < |u_n|_p^p \le C_1\|u_n\|^p = C_1.$$

This implied that $|u_n|_{p^*}^{p^*}$ and $|u_n|_p^p$ are bounded. By means of Lemma 3.4, we have

$$0 < \int_{\mathbb{R}^3} \phi u_n^2 dx \le C_2 \|u_n\|^4 = C_2.$$

Then $\int_{\mathbb{R}^3} \phi u_n^2 dx$ is bounded. We can deduce the boundedness of $Q(u_n)$ from the fact that $(\widetilde{\Lambda_1}(u_n))$ is bounded. According to Lemma 3.3, given $v = (u_n - u) \in S_1$, we can attain that

$$\int_{\mathbb{R}^3} \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n-u)dx \to 0, \qquad \int_{\mathbb{R}^3} \left(|u_n|^{p^*-2}u_n - |u|^{p^*-2}u\right)(u_n-u)dx \to 0,$$

as $n \to \infty$. Through Lemma 3.4, given $\varphi = u_n - u$, we obtain

$$\int_{\mathbb{R}^3} \left(\phi_{u_n} u_n - \phi_u u\right)(u_n-u)dx \to 0, \qquad n \to \infty.$$

Inasmuch as $u_n \rightharpoonup u$ in S_1, one can conclude that

$$\int_{\mathbb{R}^3} |u_n|^{p-2}u_n(u_n-u)dx \to 0, \int_{\mathbb{R}^3} |u_n|^{p^*-2}u_n(u_n-u)dx \to 0, \int_{\mathbb{R}^3} \phi_{u_n}u_n(u_n-u)dx \to 0,$$

as $n \to \infty$. Therefore, we obtain the following conclusion:

$$\int_{\mathbb{R}^3} \left(|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n-u) + |u_n|^{p-2}u_n(u_n-u)\right) dx \to 0, \quad \text{as } n \to \infty.$$

Notice that $p(u) = \int_{\mathbb{R}^3} |\nabla u|^p + |u|^p dx$, according to the Hölder inequality, one has

$$\begin{aligned}(p'(u_n) - p'(u), u_n - u) &\geq \|u_n\|^p + \|u\|^p - \|u_n\|^{p-1}\|u\| - \|u\|^{p-1}\|u_n\| \\ &= (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|) \geq 0.\end{aligned}$$

Owing to $u_n \rightharpoonup u$ in S_1, it follows that $(p'(u_n) - p'(u), u_n - u) \to 0$, as $n \to \infty$. Therefore, $\|u_n\| \to \|u\|$ in S_1. According to the uniform convexity of $W^{1,p}(\mathbb{R}^3)$, we can obtain that $u_n \to u$ in S_1. Consequently, there exits a sequence (u_n) such that $u_n \to u$ in S_1 up to a subsequence. Therefore, $\widetilde{\Lambda_1}$ satisfied the Palais–Smale condition. □

Proof of Theorem 3.1.

(i) We prove that there is no critical point having zero energy when $\mu < \delta_1$. A crucial observation is that u being a critical point of Λ_1 is equivalent to $t_0(u)u$ being a zero energy critical point of Φ_μ, where $\mu = \Lambda_1(u)$. In other words, this means that

$$\delta_1 = \inf_{W^{1,p}(\mathbb{R}^3)\setminus\{0\}} \Lambda_1(u) \leq \mu \leq \sup_{W^{1,p}(\mathbb{R}^3)\setminus\{0\}} \Lambda_1(u),$$

which yields the desired conclusion.

(ii) According to Lemma 3.2, we know that $\widetilde{\Lambda_1}$ is bounded from below on S_1. In the meantime, we obtain that $\widetilde{\Lambda_1}$ satisfies the Palais–Smale condition from Lemma 3.5. Note that $\hat{\gamma}(S_1) = \infty$, Ljusternik–Schnirelman category (see[32, Theorem 5.7]) yields that there exists a sequence $(u_n) \subset S_1$ such that $\widetilde{\Lambda_1}'(u_n) = 0, \widetilde{\Lambda_1}(u_n) = \mu_n$. Therefore, the energy functional Φ has infinitely many zero energy critical points $(u_n) \subset S_1$. Since $\widetilde{\Lambda_1}(u)$ is an even functional, $\widetilde{\Lambda_1}'(\pm u_n) = 0, \widetilde{\Lambda_1}(\pm u_n) = \mu_n$. Hence, system (3.1) possesses infinitely many zero energy sign changing weak solutions. □

3.2 Critical Schrödinger–Poisson system in bounded domains

The purpose of this subsection is to study the existence and nonexistence of solutions for a Schrödinger–Poisson system with critical nonlinearity in bounded domains. Here is the system under consideration:

$$\begin{cases} -\Delta u + \lambda\phi|u|^{q-2}u = \mu u - |u|^{2^*-2}u & \text{in } \Omega, \\ -\Delta\phi = |u|^q & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $\lambda = -1$, $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, μ is a real parameter, $1 < N/(N-2) < q < 2N/(N-2) = 2^*$. It is well known that problem (3.4) is equivalent to a nonlocal nonlinear problem related with famous Choquard equations in bounded domains. For more related results, for instance we refer to [1,5,11,30].

Now we start the analysis of problem (3.4). One may easily get that the corresponding functional of (3.4) is as follows:

$$\Phi_\mu(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\mu}{2}\int_\Omega |u|^2 dx - \frac{1}{2q}\int_\Omega \phi |u|^q dx + \frac{1}{2^*}\int_\Omega |u|^{2^*} dx, \qquad u \in H_0^1(\Omega).$$

This Hilbert space $H_0^1(\Omega)$ provides a suitable framework for our analysis, capturing the essential properties of functions under consideration. Given the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

It is apparent that the functional Φ_μ is C^1. Then, for every $u \in H_0^1(\Omega)\backslash\{0\}$,

$$\mu_0(u) = \frac{\|u\|^2}{|u|_2^2} - \frac{1}{q}\frac{\int_\Omega \phi|u|^q dx}{|u|_2^2} + \frac{2}{2^*}\frac{|u|_{2^*}^{2^*}}{|u|_2^2},$$

$$\mu_0(tu) = \frac{\|u\|^2}{|u|_2^2} - t^{2q-2}\frac{1}{q}\frac{\int_\Omega \phi|u|^q dx}{|u|_2^2} + t^{2^*-2}\frac{2}{2^*}\frac{|u|_{2^*}^{2^*}}{|u|_2^2}.$$

Let $\mu_0'(tu) = 0$, we obtain

$$t_0(u) = \left(\frac{2^*-2}{2q-2}\frac{2q}{2^*}\frac{|u|_{2^*}^{2^*}}{\int_\Omega \phi|u|^q dx}\right)^{\frac{1}{2q-2^*}} > 0,$$

that is $\mu_0'(t_0(u)u) = 0$. In the meantime, on account of $\mu_0''(t_0(u)u) < 0$, we can deduce that $t_0(u)$ is a nondegenerate global maximum point of $\mu_0(tu)$. As a result, we can obtain that

$$\Lambda_2(u) = \mu_0(t_0(u)u) = \frac{\|u\|^2}{|u|_2^2} + \frac{2}{2^*}\frac{2q-2^*}{2q-2}\left(\frac{2^*-2}{2q-2}\frac{2q}{2^*}\right)^{\frac{2^*-2}{2q-2^*}} \frac{(|u|_{2^*}^{2^*})^{\frac{2q-2}{2q-2^*}}}{|u|_2^2(\int_\Omega \phi|u|^q dx)^{\frac{2^*-2}{2q-2^*}}}.$$

For clarity, we call $\widetilde{\Lambda_2}$ the restriction of Λ_2 to S_2, where

$$S_2 = \{u \in H_0^1(\Omega)\backslash\{0\} : \|u\| = 1\}.$$

As for our central discovery in this subsection, it can be phrased as:

Theorem 3.6. *Let* $\delta_2 = \inf_{H_0^1(\Omega)\backslash\{0\}} \Lambda_2(u)$.

(i) If $\mu < \delta_2$*, then there is no nontrivial weak solution having zero energy in system* (3.4);

(ii) If $\mu > \delta_2$*, then system* (3.4) *has infinitely many zero energy sign changing weak solutions.*

As mentioned earlier, to achieve our goal, we only need to prove the boundedness from below of $\widetilde{\Lambda_2}$ on S_2 and verify that $\widetilde{\Lambda_2}$ meets the Palais–Smale condition.

Lemma 3.7. $\widetilde{\Lambda_2}$ *is bounded from below on* S_2.

Proof. According to the embedding theorem, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, there is a constant $C > 0$ such that $|u|_2^2 \le C\|u\|^2$. Note that $1 < N/(N-2) < q < 2^*$, one may verify that

$$\widetilde{\Lambda_2} = \frac{\|u\|^2}{|u|_2^2} + \frac{2}{2^*}\frac{2q-2^*}{2q-2}\left(\frac{2^*-2}{2q-2}\frac{2q}{2^*}\right)^{\frac{2^*-2}{2q-2^*}} \frac{(|u|_{2^*}^{2^*})^{\frac{2q-2}{2q-2^*}}}{|u|_2^2(\int_\Omega \phi|u|^q dx)^{\frac{2^*-2}{2q-2^*}}} > \frac{1}{C}.$$

That is to say, $\widetilde{\Lambda_2}$ is bounded from below on S_2. □

Lemma 3.8. *$\widetilde{\Lambda_2}$ satisfies the Palais–Smale condition.*

Proof. Choose a sequence $(u_n) \subset S_2$ such that $\widetilde{\Lambda_2}(u_n)$ is bounded and $\widetilde{\Lambda_2}'(u_n) \to 0$. In other words, for any $v \in \mathcal{T}_{S_2}(u_n)$, we have $|\Lambda_2'(u_n)v| \le \varepsilon_n\|v\|$, where $\varepsilon_n \to 0$.

With the assistance of Lemma 2.2, assuming a sequence $(u_n) \subset S_2$ satisfies $\widetilde{\Lambda_2}'(u_n) \to 0$, we can conclude that $u_n \rightharpoonup u$ in S_2 and $\Lambda_2'(u_n)(u_n - u) \to 0$ as $n \to \infty$. Then, for any $u_n, u \in S_2$,

$$\begin{aligned}\Lambda_2'(u_n)(u_n-u) = {}& 2\frac{|u_n|_2^2\int_\Omega \nabla u_n \nabla(u_n-u)dx - \|u_n\|^2\int_\Omega u_n(u_n-u)dx}{|u_n|_2^4} \\ &+ C_2 Q(u_n)\Bigg(2^*\frac{2q-2}{2q-2^*}|u_n|_2^2\int_\Omega \phi|u_n|^q dx \int_\Omega |u_n|^{2^*-2}u_n(u_n-u)dx \\ &- 2q\frac{2^*-2}{2q-2^*}|u_n|_{2^*}^{2^*}|u_n|_2^2\int_\Omega \phi|u_n|^{q-2}u_n(u_n-u)dx \\ &- 2|u_n|_{2^*}^{2^*}\int_\Omega \phi|u_n|^q dx\int_\Omega u_n(u_n-u)dx\Bigg) \to 0,\end{aligned} \tag{3.5}$$

where

$$C_2 = \frac{2}{2^*}\frac{2q-2^*}{2q-2}\left(\frac{2^*-2}{2q-2}\frac{2q}{2^*}\right)^{\frac{2^*-2}{2q-2^*}}, \qquad Q(u_n) = \frac{(|u_n|_{2^*}^{2^*})^{\frac{2^*-2}{2q-2^*}}}{|u_n|_2^4(\int_\Omega \phi|u_n|^q dx)^{\frac{2q-2}{2q-2^*}}}.$$

In order to prove Lemma 3.7, we first verify that $\int_\Omega \nabla u_n \nabla(u_n-u)dx \to 0$, as $n \to \infty$. Therefore, on the one hand, we need to show that $|u_n|_2^2, |u_n|_{2^*}^{2^*}, Q(u_n)$ is bounded. On the other hand, we need to prove

$$\int_\Omega u_n(u_n-u)dx \to 0, \int_\Omega \phi_{u_n}|u_n|^{q-2}(u_n-u)dx \to 0, \quad \text{as } n \to \infty.$$

In fact, we know that $u_n \not\equiv 0$ from $(u_n) \subset S_2$. According to Lemma 3.4, we can obtain

$$0 < \int_\Omega \phi_{u_n}|u_n|^q dx \le C_2\|u_n\|^{2q} = C_2.$$

Therefore, $\int_\Omega \phi_{u_n}|u_n|^q dx$ is bounded. On the basis of the embedding theorem, $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for $2 \le s \le 2^*$, we have

$$0 < |u_n|_2^2 \le C\|u_n\|^2 = C, \qquad 0 < |u_n|_{2^*}^{2^*} \le C\|u_n\|^{2^*} = C,$$

which implied that $|u_n|_2^2$ and $|u_n|_{2^*}^{2^*}$ are bounded. Note that $\widetilde{\Lambda_2}(u_n)$ is bounded, one may check that $Q(u_n)$ is bounded. Thanks to the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, one has

$$\int_\Omega u_n(u_n-u)dx \to 0, \quad \text{as } n \to \infty.$$

With the aid of Lemma 3.4, it holds

$$\int_\Omega \phi_{u_n}|u_n|^{q-2}u_n\varphi dx \to \int_\Omega \phi_u|u|^{q-2}u\varphi dx, \quad \text{for any } \varphi \in H_0^1(\Omega).$$

Let $\varphi = u_n - u$. Then we obtain

$$\int_\Omega \phi_{u_n}|u_n|^{q-2}(u_n-u)dx - \int_\Omega \phi_u|u|^{q-2}(u_n-u)dx \to 0, \quad \text{as } n\to\infty.$$

Since $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, we have $\int_\Omega \phi_u|u|^{q-2}(u_n-u)dx \to 0$. Consequently,

$$\int_\Omega \phi_{u_n}|u_n|^{q-2}(u_n-u)dx \to 0, \quad \text{as } n\to\infty.$$

The analysis leads to the following conclusion:

$$\int_\Omega \nabla u_n\nabla(u_n-u)dx + \int_\Omega |u_n|^{2^*-2}u_n(u_n-u)dx \to 0.$$

In order to obtain $\int_\Omega \nabla u_n\nabla(u_n-u)dx \to 0$, we will use the Hölder inequality to derive

$$\int_\Omega \nabla u_n\nabla(u_n-u)dx \ge 0, \qquad \int_\Omega |u_n|^{2^*-2}u_n(u_n-u)dx \ge 0.$$

For convenience, put $p_1(u) = \int_\Omega |\nabla u|^2dx$, $p_2(u) = \int_\Omega |u|^{2^*}dx$. One may check that

$$(p_1'(u_n) - p_1'(u), u_n-u) = \|u_n\|^2 + \|u\|^2 - \int_\Omega \nabla u_n\nabla u dx - \int_\Omega \nabla u\nabla u_n dx,$$

$$(p_2'(u_n) - p_2'(u), u_n-u) = \|u_n\|_{2^*}^{2^*} + \|u\|_{2^*}^{2^*} - \int_\Omega |u_n|^{2^*-2}u_n u dx - \int_\Omega |u|^{2^*-2}u u_n dx.$$

By virtue of the Hölder inequality, we have

$$\int_\Omega \nabla u_n\nabla u dx \le \left(\int_\Omega |\nabla u|^2dx\right)^{\frac12}\left(\int_\Omega |\nabla u_n|^2dx\right)^{\frac12} = \|u_n\|\|u\|,$$

$$\int_\Omega |u_n|^{2^*-2}u_n u dx \le \left(\int_\Omega |u_n|^{2^*}dx\right)^{\frac{2^*-1}{2^*}}\left(\int_\Omega |u|^{2^*}dx\right)^{\frac{1}{2^*}} = |u_n|_{2^*}^{2^*-1}|u|_{2^*},$$

$$\int_\Omega |u|^{2^*-2}u u_n dx \le \left(\int_\Omega |u|^{2^*}dx\right)^{\frac{2^*-1}{2^*}}\left(\int_\Omega |u_n|^{2^*}dx\right)^{\frac{1}{2^*}} = |u|_{2^*}^{2^*-1}|u_n|_{2^*}.$$

Therefore,

$$(p_1'(u_n) - p_1'(u), u_n-u) \ge (\|u_n\| - \|u\|)(\|u_n\| - \|u\|) \ge 0,$$

$$(p_2'(u_n) - p_2'(u), u_n-u) \ge (\|u_n\|^{2^*-1} - \|u\|^{2^*-1})(\|u_n\| - \|u\|) \ge 0.$$

Since $u_n \rightharpoonup u$ in S_2, we have $(p_1'(u), u_n-u) \to 0, (p_1'(u), u_n-u) \to 0$, as $n\to\infty$. From which it follows that

$$\int_\Omega \nabla u_n\nabla(u_n-u)dx \ge 0, \qquad \int_\Omega |u_n|^{2^*-2}u_n(u_n-u)dx \ge 0.$$

As a result, we can draw the following conclusion:

$$\int_\Omega \nabla u_n\nabla(u_n-u)dx \to 0, \quad \text{as } n\to\infty.$$

Moreover, since $(p_1'(u_n) - p_1'(u), u_n-u) \to 0$, we have $\|u_n\| \to \|u\|$ in S_2. By the uniform convexity of $H_0^1(\Omega)$, we deduce that $u_n \to u$ in S_2. Therefore, $\widetilde{\Lambda_2}$ satisfied the Palais–Smale condition. □

Proof of Theorem 3.6.

(i) We show that there is no critical point having zero energy when $\mu < \delta_2$. Note that u is a critical point of Λ, if and only if, $t_0(u)u$ is a zero energy critical point of Φ_μ with $\mu = \Lambda(u)$, this means the desired conclusion.

(ii) According to Lemma 3.5, we know that $\widetilde{\Lambda_2}$ is bounded from below on S_2. Moreover, $\widetilde{\Lambda_2}$ satisfies the Palais–Smale condition due to Lemma 3.7. Note that $\hat{\gamma}(S_2) = \infty$, it follows from Ljusternik–Schnirelman category (see [32, Theorem 5.7]) that there exists a sequence $(u_n) \subset S_2$ such that $\widetilde{\Lambda_2}'(u_n) = 0, \widetilde{\Lambda_2}(u_n) = \mu_n$. The remainder is the same as the proof of Theorem 3.1, here we omit it. □

3.3 Kirchhoff-type problems with critical growth

Now, let us consider the following Kirchhoff-type problem:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2)\Delta u = \mu u + |u|^4 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $a, b > 0$, μ is a real parameter.

Inspired by the works described above, in this paper we study the existence of zero energy solutions for a class of Kirchhoff problem with critical growth in bounded domains. In recent years, Kirchhoff-type equation is an extension of the classical D'Alembert's wave equation. It was firstly proposed by Kirchhoff in 1883. Various problems of Kirchhoff-type are usually named nonlocal problems in virtue of the appearance of the nonlocal term $a + b\int_\Omega |\nabla u|^2$ and have been extensively investigated up to now. In [28], Quoirin et al. investigated qualitative properties of solutions for a Kirchhoff-type problem with subcritical growth as an application of their zero energy critical point theory. However, their theory seems difficulty to deal with the problem like (3.6) involving the critical exponent. For this purpose, we explore a new strategy (Theorem 1.1) to solve this problem.

As usual, one can get that the corresponding functional of (3.6) is $\Phi_\mu : H_0^1(\Omega) \to \mathbb{R}$:

$$\Phi_\mu(u) = \frac{a}{2}\int_\Omega |\nabla u|^2 dx - \frac{\mu}{2}\int_\Omega |u|^2 dx - \frac{1}{6}\int_\Omega |u|^6 dx + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2.$$

It is evident that Φ_μ is C^1. Then, according to the previous preliminaries, for every $u \in H_0^1(\Omega)\backslash\{0\}$:

$$\mu_0(u) = \frac{a\|u\|^2}{|u|_2^2} - \frac{1}{3}\frac{|u|_6^6}{|u|_2^2} + \frac{b}{2}\frac{\|u\|^4}{|u|_2^2},$$

$$\mu_0(tu) = \frac{a\|u\|^2}{|u|_2^2} - \frac{t^4}{3}\frac{|u|_6^6}{|u|_2^2} + t^2\frac{b}{2}\frac{\|u\|^4}{|u|_2^2}.$$

Let $\mu_0'(tu) = 0$, we obtain

$$t_0(u) = \left(\frac{3}{4}\frac{b\|u\|^4}{|u|_6^6}\right)^{\frac{1}{2}} > 0,$$

and $t_0(u)$ is a nondegenerate global maximum point of $\mu_0(tu)$ via $\mu_0''(t_0(u)u) < 0$. Therefore, we have

$$\Lambda_3(u) = \frac{a\|u\|^2}{|u|_2^2} + \frac{3}{16}\frac{(b\|u\|^4)^2}{|u|_2^2 |u|_6^6}.$$

For simplicity, we call $\widetilde{\Lambda_3}$ the restriction of Λ_3 to S_3, where

$$S_3 = \{u \in H_0^1(\Omega)\backslash\{0\} : \|u\| = 1\}.$$

The main result we have derived in this section is expressed as:

Theorem 3.9. *Let $\delta_3 = \inf_{H_0^1(\Omega)\backslash\{0\}} \Lambda_3(u)$.*

(i) If $\mu < \delta_3$, then there is no nontrivial weak solution having zero energy in problem (3.6);

(ii) If $\mu > \delta_3$, then problem (3.6) has infinitely many zero energy sign changing weak solutions.

To verify this result, according to Ljusternik–Schnirelman category (see [32, Theorem 5.7]), it is necessary to prove that $\widetilde{\Lambda_3}$ is bounded below and satisfies the Palais–Smale condition.

Lemma 3.10. *$\widetilde{\Lambda_3}$ is bounded from below on S_3.*

Proof. Since the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, there exists $C > 0$ such that $C\|u\|^2 \geqslant |u|_2^2$. Hence, one can obtain that

$$\widetilde{\Lambda_3}(u) = \frac{a\|u\|^2}{|u|_2^2} + \frac{3}{16}\frac{b\|u\|^4}{|u|_2^2|u|_6^3} > \frac{a}{C},$$

which yields that $\widetilde{\Lambda_3}$ is bounded from below on S_3. □

Lemma 3.11. *If $(u_n) \subset S_3$, then $u_n \rightharpoonup u$ in $H_0^1(\Omega)\backslash\{0\}$ up to a subsequence. Therefore,*

$$\int_{\mathbb{R}^3} |u_n|^4 u_n \varphi dx \to \int_{\mathbb{R}^3} |u|^4 u \varphi dx, \quad as\ n \to \infty,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Proof. We employ the strategy outlined in Lemma 3.3 from [21]. In fact, it is related to a result from the Lebesgue Dominated Convergence Theorem. First we notice that

$$|u_n|^4 u_n \varphi \to |u|^4 u \varphi \quad \text{as } n \to \infty,$$

almost everywhere in the compact support Ω of φ, and

$$||u_n|^4 u_n \varphi \chi_\Omega| \leq |u_n|^5 |\varphi| \chi_\Omega,$$

where χ_Ω represents the characteristic function of Ω. Given that $u_n \to u$ in $L^s_{loc}(\mathbb{R}^3)$ for all $5 < s < 6$, utilizing the Hölder inequality yields

$$\int_\Omega |u_n|^5 |\varphi| dx \leq \left(\int_\Omega |u_n|^s dx\right)^{\frac{5}{s}} \left(\int_\Omega |\varphi|^{\frac{s}{s-5}} dx\right)^{\frac{s-5}{s}},$$

which means that $|u_n|^5 |\varphi| \chi_\Omega \in L^1(\mathbb{R}^3)$. According to Lebesgue dominated convergence theorem, it follows that

$$\int_{\mathbb{R}^3} |u_n|^4 u_n \varphi dx \to \int_{\mathbb{R}^3} |u|^4 u \varphi dx, \quad \text{as } n \to \infty.$$

The proof is now complete. □

Lemma 3.12. *$\widetilde{\Lambda_3}$ satisfied the Palais–Smale condition.*

Proof. Choosing a sequence $(u_n) \subset S_3$ such that $(\widetilde{\Lambda_3}(u_n))$ is bounded and $\widetilde{\Lambda_3}'(u_n) \to 0$, that is $|\Lambda_3'(u_n)v| \leq \varepsilon_n \|v\|$ for any $v \in \mathcal{T}_{S_3}(u_n)$, with $\varepsilon_n \to 0$. In view of Lemma 2.2, if there is a sequence $(u_n) \subset S_3$ and $\widetilde{\Lambda_3}'(u_n) \to 0$, we can acquire that $u_n \rightharpoonup u$ in S_3 and $\Lambda'(u_n)(u_n - u) \to 0$ as $n \to \infty$. Then, for any $u_n, u \in S_3$,

$$\begin{aligned}\Lambda'(u_n)(u_n - u) = 2a &\frac{|u_n|_2^2 \int_\Omega \nabla u_n \nabla(u_n - u)dx - \|u_n\|^2 \int_\Omega u_n(u_n - u)dx}{|u_n|_2^4} \\ &+ \frac{3b}{16} Q(u_n) \left(4|u_n|_2^2 |u_n|_6^6 \int_\Omega |\nabla u_n|^2 dx \int_\Omega \nabla u_n \nabla(u_n - u)dx \right. \\ &\left. - 6\|u_n\|^4 |u|_2^2 \int_\Omega |u_n|^4 u_n(u_n - u)dx - 2\|u_n\|^4 |u_n|_6^6 \int_\Omega u_n(u_n - u)dx \right) \to 0,\end{aligned}$$

as $n \to \infty$, where

$$Q(u_n) = \frac{b\|u_n\|^4}{|u_n|_2^4 (|u_n|_6^6)^{\frac{1}{2}}}.$$

Since $(u_n) \subset S_3$ and the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for $1 \leq s \leq 6$, there exists a constant $C > 0$ such that

$$0 < |u_n|_2^2 \leq C\|u_n\|^2 = C, \qquad 0 < |u_n|_6^6 \leq C\|u_n\|^6 = C.$$

Therefore, $|u_n|_2^2, |u_n|_6^6$ is bounded. Note that $(\widetilde{\Lambda_3}(u_n))$ is bounded, then $Q(u_n)$ is bounded. Since $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, one can easily deduce from the Sobolev embedding theorem that

$$\int_\Omega u_n(u_n - u)dx \to 0, \quad \text{as } n \to \infty.$$

By means of Lemma 3.11, given $\varphi = u_n - u$, then as $n \to \infty$,

$$\int_\Omega |u_n|^4 u_n(u_n - u)dx - \int_\Omega |u|^4 u(u_n - u)dx \to 0.$$

This yields that

$$\int_\Omega |u_n|^4 u_n(u_n - u)dx \to 0, \quad \text{as } n \to \infty,$$

which leads to the following conclusion:

$$\int_\Omega \nabla u_n \nabla(u_n - u)dx \to 0, \quad \text{as } n \to \infty,$$

which means that $\|u_n\| \to \|u\|$ in S_3. By the uniform convexity of $H_0^1(\Omega)$, it follows that $u_n \to u$ in S_3. In conclusion, $\widetilde{\Lambda_3}$ satisfied the Palais–Smale condition. □

Proof of Theorem 3.9.

(i) We prove that there is no critical point having zero energy when $\mu < \delta_3$. Note that u is a critical point of Λ_3, if and only if, $t_0(u)u$ is a zero energy critical point of Φ_μ with $\mu = \Lambda_3(u)$, this yields the desired conclusion.

(ii) According to Lemma 3.10, it follows that $\widetilde{\Lambda_3}$ is bounded from below on S_3. In the meantime, $\widetilde{\Lambda_3}$ satisfies the Palais–Smale condition as per the Lemma 3.12. Note that $\hat{\gamma}(S_3) = \infty$, Ljusternik–Schnirelman category (see [32, Theorem 5.7]) implied that there exists a sequence $(u_n) \subset S_3$ such that $\widetilde{\Lambda_3}'(u_n) = 0, \widetilde{\Lambda_3}(u_n) = \mu_n$. Analogous to the proof of Theorem 3.1, problem (3.6) possesses infinitely many zero-energy, sign-changing weak solutions. □

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Weighted Lorentz estimates for subquadratic quasilinear elliptic equations with measure data

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Abstract. In this work we mainly prove the following interior gradient estimates in weighted Lorentz spaces

$$g^{-1}\left[\mathcal{M}_1(\mu)\right] \in L^{q,r}_{w,loc}(\Omega) \implies |Du| \in L^{q,r}_{w,loc}(\Omega),$$

where $g(t) = ta(t)$ for $t \geq 0$ and $\mathcal{M}_1(\mu)(x)$ is the first-order fractional maximal function

$$\mathcal{M}_1(\mu)(x) := \sup_{r>0} \frac{r|\mu|(B_r(x))}{|B_r(x)|},$$

for a class of non-homogeneous divergence quasilinear elliptic equations with measure data in the subquadratic case

$$-\operatorname{div}\left[a\left((ADu\cdot Du)^{\frac{1}{2}}\right)ADu\right] = \mu \quad \text{in } \Omega,$$

whose model cases are the classical elliptic p-Laplacian equation with measure data

$$-\operatorname{div}\left(|Du|^{p-2}Du\right) = \mu \quad \text{for } 1 < p < 2$$

and the elliptic p-Laplacian equation with the logarithmic term and measure data

$$-\operatorname{div}\left(|Du|^{p-2}\log\left(1+|Du|\right)Du\right) = \mu \quad \text{for } 1 < p < 2.$$

It deserves to be specially noted that the subquadratic case is a little different from the superquadratic case since as an example, the modulus of ellipticity in the above-mentioned special cases tends to infinity when $|Du| \to 0$ for $1 < p < 2$.

Keywords: weighted, Lorentz, gradient, subquadratic, quasilinear elliptic, measure, data.

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1 Introduction

In this paper we mainly study the local gradient estimates in weighted Lorentz spaces for the following non-homogeneous quasilinear elliptic equations with right-hand side measure in divergence form

$$-\operatorname{div}\left[a\left((ADu\cdot Du)^{\frac{1}{2}}\right)ADu\right]=\mu \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ for $n\geq 2$, μ is a Borel measure with finite mass and $a:[0,\infty)\to[0,\infty)\in C^1[0,\infty)$ satisfies

$$-1<i_a:=\inf_{t>0}\frac{ta'(t)}{a(t)}\leq\sup_{t>0}\frac{ta'(t)}{a(t)}=:s_a<0 \quad \text{for any } t>0. \tag{1.2}$$

Moreover, the symmetric matrix $A(x)=\{a_{ij}(x)\}$ satisfies the following uniformly elliptic condition

$$\Lambda^{-1}|\xi|^2\leq A(x)\xi\cdot\xi\leq\Lambda|\xi|^2 \tag{1.3}$$

for every $\xi\in\mathbb{R}^n$, *a.e.* $x\in\mathbb{R}^n$ and some constant $\Lambda>0$. We remark that if $a(t)=t^{p-2}$ and A is the identity matrix I, then $i_a=s_a=p-2$ for $1<p<2$ and (1.1) is reduced to the classical elliptic p-Laplacian equation with right-hand side measure in divergence form

$$-\operatorname{div}\left(|Du|^{p-2}Du\right)=\mu \quad \text{for } 1<p<2. \tag{1.4}$$

It may be worthwhile to remark that another two natural examples of the functions a are $a(t)=t^{p-2}\log(1+t)$ for $1<p<2$, which makes (1.1) for $A=I$ is equal to

$$-\operatorname{div}\left(|Du|^{p-2}\log\left(1+|Du|\right)Du\right)=\mu,$$

and a more general example (see page 600 in [9] and page 314 in [46]), which is related to (p,q)-growth condition given by appropriate gluing of the monomials.

Define

$$g(t):=ta(t) \tag{1.5}$$

and

$$G(t):=\int_0^t g(\tau)\,d\tau=\int_0^t \tau a(\tau)\,d\tau \quad \text{for } t\geq 0. \tag{1.6}$$

From (1.2) we know that

$$g(t) \text{ is strictly increasing and continuous over } [0,+\infty), \tag{1.7}$$

and then

$$G(t) \text{ is increasing over } [0,+\infty) \text{ and strictly convex with } G(0)=0. \tag{1.8}$$

The partial differential equations involving measure data allow to consider various mathematical models in many interesting phenomena such as the blood flow in the heart [58] and state-constrained optimal control problems [23, 24]. The pointwise estimates of solutions to elliptic PDEs via suitable linear and nonlinear potentials of the right-hand side measure μ

were first investigated by Kilpeläinen & Malý [39,40], in which they obtained the pointwise estimates for u in terms of nonlinear Wolff potentials $W^{\mu}_{\beta,p}$ defined by

$$W^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|\,(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \quad \text{for } \beta \in \left(0, \frac{n}{p}\right],$$

where

$$|\mu|(B(x,\varrho)) := \int_{B(x,\varrho)} |\mu(y)|dy.$$

Remarkably, such estimates played an essential role in the nonlinear potential theory (see [38,60]). In more specific terms, Kilpeläinen & Malý [39,40] proved the following estimate

$$|u(x_0)| \leq C(n,p)\left[W^{\mu}_{1,p}(x,R) + \left(\fint_{B(x,R)} |u|^{\gamma} dx\right)^{\frac{1}{\gamma}}\right], \quad \gamma > p-1 \tag{1.9}$$

with $B(x,R) \subseteq \Omega$ for solution to the p-Laplacian equation with right-hand side measure (1.4). Afterwards, Trudinger & Wang [64] used a different approach to prove the pointwise estimate via the nonlinear Wolff potential for the p-Laplacian operators. Later, Duzaar & Mingione [35,51] extended (1.9) to the pointwise estimate at the gradient level

$$|Du(x_0)| \leq C(n,p)\left[\fint_{B(x_0,2R)} |Du|dx + W^{\mu}_{1/p,p}(x,2R)\right]$$

for solutions to the elliptic p-Laplacian equation (1.4) and more general case. In the subsequent papers, for the case $p \geq 2$ Kuusi & Mingione [44,45] made a deep study of the pointwise estimates for gradient

$$|Du(x_0)| \leq C(n,p)\left[\fint_{B(x_0,2R)} |Du|dx + C\left(I_1^{|\mu|}(x_0,2R)\right)^{\frac{1}{p-1}}\right]$$

of solutions to (1.4) and more general case in terms of the linear Riesz potential of the right-hand side $I_1^{|\mu|}(x,R)$ which is defined by

$$I_1^{|\mu|}(x,R) := \int_0^R \frac{|\mu|(B(x,\varrho))}{\varrho^{n-1}}\frac{d\varrho}{\varrho}.$$

In particular, we mention here that Duzaar & Mingione [33] obtained gradient estimates via linear Riesz potentials

$$|Du(x_0)| \leq C\fint_{B(x_0,2R)} |Du|dx + C\left[I_1^{|\mu|}(x_0,2R)\right]^{\frac{1}{p-1}}$$

for solutions of the general case of the elliptic p-Laplacian equation for $2-1/n < p < 2$. We remark that the lower bound $2-1/n$ on the exponent p is to ensure $W^{1,1}$-solutions (see [33]). It deserves to be specially noted that Dong, Nguyen, Phuc & Zhu [32,55,57] also studied the local and global pointwise gradient estimates for solutions to the quasilinear elliptic equation with measure data $-\operatorname{div} A(x,Du) = \mu$ in the case $1 < p \leq 2-1/n$, whose prototype is given by the elliptic p-Laplace equation (1.4). Moreover, an extension of the previous results to a class of general elliptic equations

$$-\operatorname{div}\left[a\left(|Du|\right)Du\right] = \mu$$

including the p-Laplacian equation has been recently given by Baroni [7], in which the author proved the following pointwise gradient estimates via the linear Riesz potential

$$g(|Du(x_0)|) \le Cg\left(\fint_{B(x_0,2R)} |Du|dx\right) + CI_1^{|\mu|}(x_0,2R).$$

Actually, Cianchi & Maz'ya [26–28] have proved Lipschitz regularity and sharp estimates for weak solutions of

$$-\operatorname{div}\left(a\left(|Du|\right)Du\right) = f, \tag{1.10}$$

which is first introduced and studied by Lieberman [46] as the most natural and best generalization of the p-Laplacian equation. In the meanwhile, the authors [5,6,10,21,25,30,31,52,65] also studied regularity estimates of weak solutions for the quasilinear elliptic equations (1.10).

In a general way we call w belongs to the class of the Muckenhoupt weights A_p for some $p > 1$ if $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$ almost everywhere satisfies

$$\left(\fint_{B_r} w(x)dx\right)\left(\fint_{B_r} w(x)^{\frac{-1}{p-1}}dx\right)^{p-1} \le C$$

for any ball B_r in $\mathbb{R}^n$. Moreover, we denote

$$A_\infty := \bigcup_{1<p<\infty} A_p \quad \text{and} \quad w(B_r) := \int_{B_r} w(x)dx.$$

Furthermore, the corresponding weighted Lebesgue space $L^p_w(B_r)$ consists of all functions h which satisfy

$$\|h\|_{L^p_w(B_r)} := \left(\int_{B_r} |h|^p\, w(x)dx\right)^{1/p} < \infty.$$

Now we give the following definition of weighted Lorentz spaces.

Definition 1.1. The weighted Lorentz space $L^{q,r}_w(\Omega)$ for any $0 < q < \infty$ and $0 < r \le \infty$ is the set of all measurable functions h satisfying

$$\|h\|_{L^{q,r}_w(\Omega)} < \infty,$$

where

$$\|h\|_{L^{q,r}_w(\Omega)} := \begin{cases} \left[q\displaystyle\int_0^\infty \lambda^{r-1} w\left(\{x\in\Omega : |h(x)| > \lambda\}\right)^{\frac{r}{q}} d\lambda\right]^{\frac{1}{r}} & \text{for } r<\infty, \\ \displaystyle\sup_{\lambda>0} \lambda\, w\left(\{x\in\Omega : |h(x)| > \lambda\}\right)^{\frac{1}{q}} & \text{for } r=\infty. \end{cases}$$

Actually, the weighted Lebesgue space $L^q_w(\Omega) = L^{q,q}_w(\Omega)$ and Marcinkiewicz space $\mathcal{M}^q(\Omega) = L^{q,\infty}(\Omega)$.

Lemma 1.2 (see [16,19,47,62,63])**.** *Assume that $w \in A_p$ for some $p > 1$. Then there exists a small positive constant $\sigma > 0$ such that*

$$C_1\left(\frac{|B_r|}{|B_R|}\right)^p \le \frac{w(B_r)}{w(B_R)} \le C_2\left(\frac{|B_r|}{|B_R|}\right)^\sigma$$

for any balls $B_r \subset B_R \subset \mathbb{R}^n$, where $C_2 > 1$ and $C_1 > 0$.

There are various kinds of Calderón–Zygmund type estimates for the elliptic equations of p-Laplacican type (see, for example, [3, 8, 17, 29, 41, 47, 48] and the references therein). More to the point, Mingione [50] first proved the local sharp estimates in Lorentz spaces for the solutions to the following p-Laplacian type elliptic equation with measure data

$$-\operatorname{div}\mathbf{a}(x, Du) = \mu \quad \text{in } \Omega. \tag{1.11}$$

Furthermore, Phuc [59] obtained the following global weighted norm inequality in Lorentz spaces for gradients of solutions to (1.11)

$$(\mathcal{M}_1(\mu))^{\frac{1}{p-1}} \in L^{q,r}_w(\Omega) \Longrightarrow |Du| \in L^{q,r}_w(\Omega)$$

for $2 - 1/n < p \le n$, any $q \in (0, +\infty)$ and $r \in (0, +\infty]$, where $\mathcal{M}_1(\mu)(x)$ is the first-order fractional maximal function

$$\mathcal{M}_1(\mu)(x) := \sup_{r>0} \frac{r|\mu|(B_r(x))}{|B_r(x)|}, \quad x \in \mathbb{R}^n.$$

Subsequently, Nguyen & Phuc [54, 56] obtained existence and global regularity estimates for gradients of solutions to quasilinear elliptic equations with measure data, whose prototypes are of the form $-\operatorname{div}\left(|Du|^{p-2}Du\right) = \delta|Du|^q + \mu$ for $1 < p \le 2 - 1/n$. In the meanwhile, Byun, Ok & Park [18] established the corresponding Calderón–Zygmund type estimates for quasilinear elliptic equations (1.11) with variable $p(x)$-growth involving measure data. Moreover, Byun, Cho & Youn [14] studied the existence of distributional solutions and the global Calderón–Zygmund type estimates to nonlinear elliptic problems (1.1) and more general case with the right-hand side Radon measure. Moreover, Avelin, Kuusi & Mingione [4] have investigated a limiting case of Calderón–Zygmund theory for a class of nonlinear elliptic equations modeled on the elliptic p-Laplacian equation with right-hand side measure (1.4). Motivated by the works mentioned above, our purpose of this paper is to establish the local weighted Lorentz gradient estimates for weak solutions of the problem (1.1) with the condition (1.2) in the case $-1/n < i_a \le s_a < 0$. More precisely, we shall prove that

$$g^{-1}\left[\mathcal{M}_1(\mu)\right] \in L^{q,r}_{w,loc}(\Omega) \implies |Du| \in L^{q,r}_{w,loc}(\Omega).$$

We now state the definition of weak solutions.

Definition 1.3. A function $u \in W^{1,G}_{loc}(\Omega)$ (see Definition 2.4) is a local weak solution of (1.1) if for any $\varphi \in W^{1,G}_0(\Omega) \cap L^\infty(\Omega)$ we have

$$\int_\Omega a\left((ADu \cdot Du)^{\frac{1}{2}}\right) ADu \cdot D\varphi dx = \int_\Omega \varphi d\mu.$$

In this work we shall assume that the coefficients of $A = \{a_{ij}\}$ are in the BMO space and their semi-norms are small enough. Higher integrability of solutions to various kinds of elliptic/parabolic PDEs with discontinuous coefficients of VMO/BMO type has been extensively studied by many authors (see [2, 15, 20, 41, 43]). We would like to point out that a function satisfies the small BMO condition if it satisfies the VMO condition. More precisely, we use the following small BMO condition.

Definition 1.4. We say that the matrix A of coefficients is (δ, R)-vanishing if

$$\sup_{0<r\le R}\sup_{x\in\mathbb{R}^n} \fint_{B_r(x)} \left|A(y) - \overline{A}_{B_r(x)}\right| dy \le \delta,$$

where

$$\overline{A}_{B_r(x)} = \fint_{B_r(x)} A(y)\, dy.$$

The main result of this work is stated as follows. First of all, we remark that the following conclusion is stated as a priori estimate for weak solutions. Actually, solutions to measure data problems (very weak solutions) are usually found by approximation procedures. So, they are often called SOLA (Solutions Obtained by Limiting Approximation). We can refer to the relevant existence theory in the papers [11–13, 37, 40]. In the following we shall mention a space $W^{1,f}(\Omega)$, where

$$f(t) := \int_0^t \frac{g(s)}{s} ds,$$

whose definition is just like Section 3.2 in [7]. More precisely, the exact definition of SOLA is given as follows: a function $u \in W^{1,f}_{loc}(\Omega)$ is a local SOLA of (1.1) if

$$\int_\Omega a\left((ADu \cdot Du)^{\frac{1}{2}}\right) ADu \cdot D\varphi dx = \int_\Omega \varphi d\mu$$

holds for any $\varphi \in C_0^\infty(\Omega)$, and moreover there exists a sequence of weak solutions $\{u_k\} \in W^{1,G}_{loc}(\Omega)$ of

$$-\operatorname{div}\left(a\left((ADu_k \cdot Du_k)^{\frac{1}{2}}\right) ADu_k\right) = \mu_k \quad \text{in } \Omega, \tag{1.12}$$

such that $u_k \to u$ in $W^{1,f}_{loc}(\Omega)$, where $\{\mu_k\} \in L^\infty(\Omega)$ converges weakly to μ in the sense of measure. In particular, we shall assume that $-1/n < i_a \le s_a < 0$ in the theorem below just like in the paper of Duzaar & Mingione [33], in which they supposed that $p > 2 - 1/n$ for the elliptic p-Laplacian equations and general case.

Now we shall give a concrete conclusion of this paper.

Theorem 1.5. *Suppose that $\mu \in L^\infty(\Omega)$ and $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of (1.1) in $\Omega \supset B_2$ for $-1/n < i_a \le s_a < 0$. Then we have*

$$g^{-1}[\mathcal{M}_1(\mu)] \in L^{q,r}_{w,loc}(\Omega) \Longrightarrow |Du| \in L^{q,r}_{w,loc}(\Omega)$$

for any $q \in (1, \infty)$ and $r \in (0, \infty]$, with the estimate

$$\|Du\|_{L^{q,r}_{w,loc}(B_1)} \le C \int_{B_2} (|Du| + 1)\, dx + C\|g^{-1}[\mathcal{M}_1(\mu))\|_{L^{q,r}_{w,loc}(B_2)},$$

where C is independent of u and μ.

2 Proof of the main result

In this section we shall finish the proof of the main result in this work, Theorem 1.5. First of all, we shall give some definitions on the general Orlicz spaces, which have been extensively studied in the area of analysis (see [1, 53]) and play a crucial role in many fields of mathematics including geometric, probability theory, stochastic analysis, Fourier analysis, partial differential equations and so on (see [61]).

Definition 2.1. A function G belongs to Φ, which consists of all increasing and convex functions $G : [0, +\infty) \to [0, +\infty)$, is said to be a Young function if

$$\lim_{t \to 0+} \frac{G(t)}{t} = \lim_{t \to +\infty} \frac{t}{G(t)} = 0.$$

Additionally, a Young function G is said to $G \in \Delta_2$ if there exists $M > 0$ such that

$$G(2t) \le MG(t) \quad \text{for any } t > 0. \tag{2.1}$$

Moreover, we call a Young function $G \in \nabla_2$ if there exists a number $a > 1$ such that

$$G(t) \le \frac{G(at)}{2a} \quad \text{for any } t > 0. \tag{2.2}$$

Example 2.2.

(1) $G_1(t) = (1+t)\log(1+t) - t \in \Delta_2$, but $G_1(t) \notin \nabla_2$.

(2) $G_2(t) = e^t - t - 1 \in \nabla_2$, but $G_2(t) \notin \Delta_2$.

(3) $G_3(t) = t^p \log(1+t) \in \Delta_2 \cap \nabla_2$ for $p > 1$.

Remark 2.3. Actually, if $G \in \Delta_2 \cap \nabla_2$, then we have

$$G(\theta_1 t) \le K\theta_1^{\beta_1} G(t) \quad \text{and} \quad G(\theta_2 t) \le 2a\theta_2^{\beta_2} G(t) \tag{2.3}$$

for every $t > 0$ and $0 < \theta_2 \le 1 \le \theta_1 < \infty$, where $\beta_1 = \log_2 M \ge \beta_2 = \log_a 2 + 1 > 1$.

Definition 2.4. Assume that G is a Young function. Then the Orlicz class $K^G(\mathbb{R}^n)$ is the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\int_{\mathbb{R}^n} G(|f|)\, dx < \infty.$$

The Orlicz space $L^G(\mathbb{R}^n)$ is the linear hull of $K^G(\mathbb{R}^n)$ and $W^{1,G}(\mathbb{R}^n) := \{f \in L^G(\mathbb{R}^n) \mid Df \in L^G(\mathbb{R}^n)\}$.

Moreover, in this work we need the following crucial lemmas, which will be used in the subsequent proof.

Lemma 2.5 ([1])**.** *Let G be a Young function satisfying $G \in \Delta_2 \cap \nabla_2$. Then*

(1) $K^G(\Omega) = L^G(\Omega)$.

(2) $C_0^\infty(\Omega)$ *is dense in* $L^G(\Omega)$.

(3) $L^{\beta_1}(\Omega) \subset L^G(\Omega) \subset L^{\beta_2}(\Omega) \subset L^1(\Omega)$ *with* $\beta_1 \ge \beta_2 > 1$ *as in* (2.3).

(4) If $f \in L^G(\mathbb{R}^n)$, *then*

$$\int_{\mathbb{R}^n} G(|f|)\, dx = \int_0^\infty |\{x \in \mathbb{R}^n : |f| > \mu\}|\; d\left[G(\mu)\right].$$

(5)

$$st \le \epsilon\, \widetilde{G}(s) + C(\epsilon) G(t) \quad \textit{for any } s, t \ge 0 \textit{ and } \epsilon > 0,$$

where $\widetilde{G}$ *is the conjugate function of* G

$$\widetilde{G}(t) := \sup_{s \ge 0} \{st - G(s)\} \quad \textit{for any } t \ge 0.$$

Now we shall recall the following results, which can be derived from Proposition 2.9 of [26], Lemma 1.9 and Lemma 2.4 of [65] and the change of variable.

Lemma 2.6. *Assume that $a(t)$ satisfies (1.2) for $s_a < 0$ and $G(t) = \int_0^t \tau a(\tau)\, d\tau$ for $t \geq 0$ is defined in (1.6).*

1. *For any $t > 0$ we find that*

$$\theta^{i_a} \leq \frac{a(\theta t)}{a(t)} \leq \theta^{s_a} \quad \text{and} \quad \theta^{2+i_a} \leq \frac{G(\theta t)}{G(t)} \leq \theta^{2+s_a} \quad \text{for any } \theta \geq 1. \tag{2.4}$$

2. *$G(t) \in \nabla_2 \cap \Delta_2$ and $\widetilde{G}(g(t)) \leq CG(t)$ for $t \geq 0$.*
3. *There exist $C = C(n, i_a, s_a) > 0$ and $\epsilon_0 = \epsilon_0(n, i_a, s_a) > 0$ we have*

$$G(|\xi - \eta|) \leq C(\epsilon) \left[a\left((A\xi \cdot \xi)^{\frac{1}{2}} \right) A\xi - a\left((A\eta \cdot \eta)^{\frac{1}{2}} \right) A\eta \right] \cdot (\xi - \eta) + \epsilon G(|\eta|)$$

for any $\xi, \eta \in \mathbb{R}^n$ and small positive constant $\epsilon \in (0, \epsilon_0)$.

Next, we can obtain the following important results for $s_a < 0$.

Lemma 2.7. *Assume that $a(t)$ satisfies (1.2) and $s_a < 0$, $G(t)$ is defined in (1.6) and*

$$V(z) = \sqrt{a(|z|)}z. \tag{2.5}$$

Then for any $\xi, \eta \in \mathbb{R}^n$ there exists $C = C(n, i_a, s_a) > 0$ we have

$$Ca(|\xi| + |\eta|) |\xi - \eta|^2 \leq |V(\xi) - V(\eta)|^2 \leq Ca(|\xi| + |\eta|) |\xi - \eta|^2, \tag{2.6}$$

$$C\sqrt{a(|\xi| + |\eta|)} |\xi - \eta|^2 \leq [V(\xi) - V(\eta)] \cdot (\xi - \eta) \leq C\sqrt{a(|\xi| + |\eta|)} |\xi - \eta|^2 \tag{2.7}$$

and

$$\left[a\left((A\xi \cdot \xi)^{\frac{1}{2}} \right) A\xi - a\left((A\eta \cdot \eta)^{\frac{1}{2}} \right) A\eta \right] \cdot (\xi - \eta) \geq C |V(\xi) - V(\eta)|^2. \tag{2.8}$$

Proof. We first find that

$$\begin{aligned} &V(\xi) - V(\eta) \\ &= \sqrt{a(|\xi|)}\xi - \sqrt{a(|\eta|)}\eta \\ &= (\xi - \eta) \int_0^1 \sqrt{a(|s\xi + (1-s)\eta|)}\, ds \\ &\quad + \frac{1}{2} \int_0^1 \frac{a'(|s\xi + (1-s)\eta|)}{|s\xi + (1-s)\eta|} \frac{1}{\sqrt{a(|s\xi + (1-s)\eta|)}} (s\xi + (1-s)\eta)\, [s\xi + (1-s)\eta] \cdot (\xi - \eta)\, ds. \end{aligned}$$

Then from (1.2) we deduce that

$$\begin{aligned} |V(\xi) - V(\eta)| &\leq |\xi - \eta| \int_0^1 \sqrt{a(|s\xi + (1-s)\eta|)} ds - \frac{i_a}{2} |\xi - \eta| \int_0^1 \sqrt{a(|s\xi + (1-s)\eta|)} ds \\ &= \left(1 - \frac{i_a}{2}\right) |\xi - \eta| \int_0^1 \sqrt{a(|s\xi + (1-s)\eta|)}\, ds. \end{aligned} \tag{2.9}$$

Similarly, we have

$$|V(\xi) - V(\eta)| \geq \left(1 + \frac{i_a}{2}\right) |\xi - \eta| \int_0^1 \sqrt{a(|s\xi + (1-s)\eta|)}\, ds, \tag{2.10}$$

$$[V(\xi)-V(\eta)]\cdot(\xi-\eta)\geq\left(1+\frac{i_a}{2}\right)|\xi-\eta|^2\int_0^1\sqrt{a\left(|s\xi+(1-s)\eta|\right)}\,ds \tag{2.11}$$

and

$$[V(\xi)-V(\eta)]\cdot(\xi-\eta)\leq\left(1-\frac{i_a}{2}\right)|\xi-\eta|^2\int_0^1\sqrt{a\left(|s\xi+(1-s)\eta|\right)}\,ds. \tag{2.12}$$

In view of the facts that $a(t)$ is strictly decreasing and $|s\xi+(1-s)\eta|\leq|\xi|+|\eta|$ for any $0\leq s\leq 1$, we find that

$$\int_0^1\sqrt{a\left(|s\xi+(1-s)\eta|\right)}ds\geq\int_0^1\sqrt{a\left(|\xi|+|\eta|\right)}ds=\sqrt{a\left(|\xi|+|\eta|\right)}, \tag{2.13}$$

which implies that the left-hand inequalities of (2.6) and (2.7) hold true. On the other hand, we define

$$s_0:=\frac{|\xi-\eta_0|}{|\xi-\eta|},$$

where η_0 is the minimum norm point on the line through ξ and η. Without loss of generality we may as well assume that $|\xi|\geq|\eta|>0$. It is easy to check that $s_0\geq\frac{1}{2}$. The following two cases shall be considered separately.

Case 1: $s_0\geq 1$. Then $|s\eta+(1-s)\xi|\geq|s\eta_0+(1-s)\xi|\geq|s0+(1-s)\xi|=(1-s)\,|\xi|\geq\frac{(1-s)}{2}\left(|\xi|+|\eta|\right)$ for any $s\in[0,1]$ and $|\xi|\geq|\eta|>0$. Furthermore, from Lemma 2.6 (1) and the decreasing property of $a(t)$ we conclude that

$$\begin{aligned}\int_0^1\sqrt{a\left(|s\eta+(1-s)\xi|\right)}ds&\leq\int_0^1\sqrt{a\left(\frac{(1-s)}{2}\left(|\xi|+|\eta|\right)\right)}ds\\&\leq C\sqrt{a\left(|\xi|+|\eta|\right)}\int_0^1(1-s)^{\frac{i_a}{2}}ds\\&\leq C\sqrt{a\left(|\xi|+|\eta|\right)}.\end{aligned} \tag{2.14}$$

Case 2: $\frac{1}{2}\leq s_0<1$. Recalling the definition of η_0 and choosing $s=\theta s_0$, we have

$$\begin{aligned}\int_0^1\sqrt{a(|s\eta+(1-s)\xi|)}\mathrm{d}s&\leq 2\int_0^{s_0}\sqrt{a(|s\eta+(1-s)\xi|)}\mathrm{d}s\\&\leq C\int_0^1\sqrt{a(|\theta\eta_0+(1-\theta)\xi|)}\mathrm{d}\theta\\&\leq C\int_0^1\sqrt{a\left((1-\theta)\,|\xi|\right)}d\theta,\end{aligned}$$

in view of the facts that $|\theta\eta_0+(1-\theta)\xi|\geq|\theta 0+(1-\theta)\xi|=(1-\theta)\,|\xi|$ for any $\theta\in[0,1]$ and $a(t)$ is decreasing. Similarly to Case 1, we find that

$$\int_0^1\sqrt{a\left(|s\eta+(1-s)\xi|\right)}ds\leq C\sqrt{a\left(|\xi|+|\eta|\right)}. \tag{2.15}$$

Therefore, from (2.9)–(2.15) we can conclude that the right-hand inequalities of (2.6) and (2.7) are true.

For the sake of clarity and brevity, we may as well assume that $A = I$ in the following proof. First of all, we can compute as follows

$$\xi a(|\xi|) - \eta a(|\eta|) = (\xi - \eta) \int_0^1 a(|s\xi + (1-s)\eta|)\, ds + \int_0^1 \frac{a'(|s\xi + (1-s)\eta|)}{|s\xi + (1-s)\eta|} (s\xi + (1-s)\eta) [s\xi + (1-s)\eta] \cdot (\xi - \eta)\, ds,$$

which implies that

$$\begin{aligned} &[\xi a(|\xi|) - \eta a(|\eta|)] \cdot (\xi - \eta) \\ &\geq |\xi - \eta|^2 \int_0^1 a(|s\xi + (1-s)\eta|)\, ds + i_a \int_0^1 a(|s\xi + (1-s)\eta|) \left| \frac{[s\xi + (1-s)\eta] \cdot (\xi - \eta)}{|s\xi + (1-s)\eta|} \right|^2 ds \\ &\geq (1 + i_a) |\xi - \eta|^2 \int_0^1 a(|s\xi + (1-s)\eta|)\, ds \end{aligned}$$

in view of (1.2). Then similarly to (2.13), we find that

$$\int_0^1 a(|s\xi + (1-s)\eta|)\, ds \geq a(|\xi| + |\eta|),$$

which implies that

$$[\xi a(|\xi|) - \eta a(|\eta|)] \cdot (\xi - \eta) \geq C a(|\xi| + |\eta|) |\xi - \eta|^2. \tag{2.16}$$

Thus, from (2.6) and (2.16) we can obtain (2.8) and then finish the proof. □

For a locally integrable function f in $\mathbb{R}^n$, we define its Hardy–Littlewood maximal function $\mathcal{M}(f)(x)$ as

$$\mathcal{M}(f)(x) := \sup_{r>0} \fint_{B_r(x)} |f(y)|\, dy.$$

If f is not defined outside a bounded domain Ω, then we let f be zero in the above definition if x leaves Ω. Moreover, we can obtain the following basic properties for the Hardy–Littlewood maximal functions.

Lemma 2.8 (see [42]).

1. *If $f \in L^1(\Omega)$, then we have the weak 1-1 estimate*

$$|\{x \in \Omega : (\mathcal{M}f)(x) > \lambda\}| \leq \frac{C_3}{\lambda} \int_\Omega |f(x)|\, dx \quad \text{for some constant } C_3 > 0. \tag{2.17}$$

2. *If $f \in L^G(\Omega)$ for $G \in \Delta_2 \cap \nabla_2$, then we have $\mathcal{M}f \in L^G(\Omega)$ with the estimates*

$$\frac{1}{C} \int_\Omega G(|f|)\, dx \leq \int_\Omega G(\mathcal{M}f)\, dx \leq C \int_\Omega G(|f|)\, dx.$$

In this paper we shall use the following version of the weighted Vitali covering lemma, which will be a crucial ingredient in obtaining our main result.

Lemma 2.9 ([59, Lemma 3.4]). *Assume that E and F are measurable sets, $E \subset F \subset B_1$, and that there exists an $\epsilon > 0$ such that $w(E) < \epsilon w(B_1)$ and that for all $x \in B_1$ and for all $r \in (0,1]$ with $w(E \cap B_r(x)) \geq \epsilon w(B_r(x))$ we have $B_r(x) \cap B_1 \subset F$. Then, we have*

$$w(E) \leq C\epsilon w(F).$$

Moreover, we shall also use the following standard arguments of measure theory.

Lemma 2.10 (see [22,59]). *Assume that $r \in (0,+\infty)$ and f is a nonnegative and measurable function in Ω. Let $m > 1$ be a constant. Then for $0 < q < \infty$ we have*

$$f \in L^{q,r}_w(\Omega) \text{ iff } S := \sum_{i \geq 1} m^{ir} \left[w\left(\left\{ x \in \Omega : f(x) > m^i \right\} \right) \right]^{\frac{r}{q}} < \infty$$

and

$$\frac{1}{C} S \leq \|f\|^r_{L^{q,r}_w(\Omega)} \leq C \left[(w(\Omega))^{\frac{r}{q}} + S \right],$$

where $C > 0$ is a constant depending only on m and w.

Furthermore, we shall prove the following important result, which involves a delicate argument and a new scaling procedure in the subquadratic case $s_a < 0$.

Lemma 2.11. *Assume that $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of (1.1) with (1.2) and $B_{2R} \subset \Omega$. If $v \in W^{1,G}(B_{2R})$ is the weak solution of*

$$\begin{cases} \operatorname{div} \left[a\left((ADv \cdot Dv)^{\frac{1}{2}} \right) ADv \right] = 0 & \text{in } B_{2R}, \\ v = u & \text{on } \partial B_{2R}, \end{cases} \tag{2.18}$$

then for any $\epsilon_1 > 0$ there exists a constant $C = C(n, i_a, s_a, \epsilon_1) > 1$ such that

$$\fint_{B_{2R}} |Du - Dv| \, dx \leq C g^{-1} \left(\frac{1}{\epsilon_1} \frac{|\mu|(B_{2R})}{(2R)^{n-1}} \right) + \epsilon_1 \fint_{B_{2R}} |Du| \, dx.$$

Proof. Without loss of generality we may as well assume that $R = 1$ by defining

$$\tilde{u}(x) = R^{-1} u(Rx), \quad \tilde{v}(x) = R^{-1} v(Rx) \quad \text{and} \quad \tilde{\mu}(x) = R\mu(Rx).$$

For $k \geq 1$ we define the following truncation operators (see [33,34,44,49])

$$T_k(s) := \max\{-k, \min\{k, s\}\} \quad \text{and} \quad \Phi_k(s) := T_1(s - T_k(s)), \; s \in \mathbb{R}.$$

Since u and v are weak solutions of (1.1) and (2.18) respectively, then we have

$$\int_{B_2} \left[a\left((ADu \cdot Du)^{\frac{1}{2}} \right) ADu - a\left((ADv \cdot Dv)^{\frac{1}{2}} \right) ADv \right] \cdot D\varphi dx = \int_{B_2} \varphi d\mu \tag{2.19}$$

for any $\varphi \in L^\infty(B_2) \cap W^{1,G}_0(B_2)$. Without loss of generality we may as well assume that

$$|\mu|(B_2) \leq \epsilon_1 \quad \text{and} \quad \fint_{B_2} |Du| dx \leq \frac{1}{\epsilon_1} \tag{2.20}$$

for any small constant $\epsilon_1 \in (0,1)$. If not, we can define

$$\tilde{u}(x) = \frac{u(x)}{\lambda}, \qquad \tilde{v}(x) = \frac{v(x)}{\lambda}, \qquad \tilde{\mu}(x) = \frac{\mu(x)}{g(\lambda)},$$

$$\tilde{a}(t) = \frac{a(\lambda t)}{a(\lambda)} \quad \text{and} \quad \widetilde{G}(t) = \frac{G(\lambda t)}{G(\lambda)},$$

where

$$\lambda = g^{-1}\left(\frac{1}{\epsilon_1}|\mu|(B_2)\right) + \epsilon_1 \fint_{B_2} |Du|dx.$$

Then, $\tilde{a}\,(t)$ satisfies (1.2) and $\tilde{u}(x) \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of

$$-\text{div}\left[\tilde{a}\left((AD\tilde{u}\cdot D\tilde{u})^{\frac{1}{2}}\right)AD\tilde{u}\right] = \tilde{\mu}.$$

Therefore, it is sufficient to prove the following inequality

$$\fint_{B_2} |Du - Dv|\,dx \le C \tag{2.21}$$

under the condition (2.20), where C is independent of ϵ_1. By choosing $\varphi = \Phi_k(u-v) \in L^\infty(B_2)\cap W^{1,G}_0(B_2)$ in (2.19) and using (2.8), we find that

$$\int_{C_k} |V(Du) - V(Dv)|^2\,dx \le C\int_{B_2} |\mu| dx \le C, \tag{2.22}$$

where $C_k := \{x \in B_2 : k < |u(x) - v(x)| \le k+1\}$. In the meantime, from (2.4), (2.6) and Young's inequality we find that

$$\begin{aligned}
|Du - Dv| &\le Ca^{-\frac{1}{2}}\left(|Du| + |Dv|\right)|V(Du) - V(Dv)| \\
&\le Ca^{-\frac{1}{2}}\left(|Du| + |Dv| + 1\right)|V(Du) - V(Dv)| \\
&\le C\left(|Du| + |Dv| + 1\right)^{-\frac{i_a}{2}}|V(Du) - V(Dv)| \\
&\le C\left(|Du - Dv|^{-\frac{i_a}{2}} + |Du|^{-\frac{i_a}{2}} + 1\right)|V(Du) - V(Dv)| \\
&\le C\,|V(Du) - V(Dv)|^{\frac{2}{2+i_a}} + \frac{1}{2}|Du - Dv| \\
&\quad + C|Du|^{-\frac{i_a}{2}}\,|V(Du) - V(Dv)| + |V(Du) - V(Dv)| \\
&\le C\,|V(Du) - V(Dv)|^{\frac{2}{2+i_a}} + \frac{1}{2}|Du - Dv| + C|Du|^{-\frac{i_a}{2}}\,|V(Du) - V(Dv)| + 1
\end{aligned}$$

for $i_a \in (-1/n, 0)$, which implies that

$$|Du - Dv| \le C\,|V(Du) - V(Dv)|^{\frac{2}{2+i_a}} + C|Du|^{-\frac{i_a}{2}}\,|V(Du) - V(Dv)| + 1$$

and then

$$\begin{aligned}
\int_{B_2} |Du - Dv|\,dx &\le C\int_{B_2} |V(Du) - V(Dv)|^{\frac{2}{2+i_a}} + 1dx \\
&\quad + C\left(\int_{B_2} |V(Du) - V(Dv)|^{\frac{2}{2+i_a}}\,dx\right)^{\frac{2+i_a}{2}}\left(\int_{B_2} |Du|dx\right)^{-\frac{i_a}{2}}
\end{aligned} \tag{2.23}$$

by using Hölder's inequality. Moreover, from Hölder's inequality, (2.22) and the definition of C_k we find that

$$\begin{aligned}
\int_{C_k} |V(Du) - V(Dv)|^{\frac{2}{2+i_a}}\,dx &\le C\,|C_k|^{1-\frac{1}{2+i_a}}\left(\int_{C_k} |V(Du) - V(Dv)|^2\,dx\right)^{\frac{1}{2+i_a}} \\
&\le C\,|C_k|^{1-\frac{1}{2+i_a}}\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \\
&\le C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}}\,\frac{1}{k^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})}}\left(\int_{C_k} |u - v|^{\frac{n}{n-\sigma}}\,dx\right)^{1-\frac{1}{2+i_a}}
\end{aligned}$$

for some $\sigma \in (-ni_a, 1) \subset (0,1)$. Therefore, we conclude that

$$\begin{aligned}
&\int_{B_2} |V(Du)-V(Dv)|^{\frac{2}{2+i_a}}\,dx \\
&\le \int_{C_0} |V(Du)-V(Dv)|^{\frac{2}{2+i_a}}\,dx + \sum_{k=1}^{\infty} \int_{C_k} |V(Du)-V(Dv)|^{\frac{2}{2+i_a}}\,dx \\
&\le C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{1}{k^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})}} \left(\int_{C_k} |u-v|^{\frac{n}{n-\sigma}}\,dx \right)^{1-\frac{1}{2+i_a}} \right\} \\
&\le C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \left[\sum_{k=1}^{\infty} \frac{1}{k^{\frac{n(1+i_a)}{n-\sigma}}} \right]^{\frac{1}{2+i_a}} \left(\sum_{k=1}^{\infty} \int_{C_k} |u-v|^{\frac{n}{n-\sigma}}\,dx \right)^{1-\frac{1}{2+i_a}} \right\} \\
&\le C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \left(\int_{B_2} |u-v|^{\frac{n}{n-\sigma}}\,dx \right)^{(1-\frac{1}{2+i_a})} \right\} \\
&\le C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \left(\int_{B_2} |Du-Dv|\,dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \right\}
\end{aligned} \tag{2.24}$$

by Sobolev's inequality and the fact that

$$\frac{n(1+i_a)}{n-\sigma} > 1,$$

since $\sigma \in (-ni_a, 1)$. Furthermore, from (2.20), (2.23), (2.24) and Young's inequality we obtain

$$\begin{aligned}
\int_{B_2} |Du-Dv|\,dx &\le C + C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \left(\int_{B_2} |Du-Dv|\,dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \right\} \\
&\quad + C\,[|\mu|(B_2)]^{\frac{1}{2}} \left(\int_{B_2} |Du|dx \right)^{-\frac{i_a}{2}} \left[1 + \left(\int_{B_2} |Du-Dv|\,dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \right]^{\frac{2+i_a}{2}} \\
&\le C + C\,[|\mu|(B_2)]^{\frac{1}{2+i_a}} \left\{ 1 + \left(\int_{B_2} |Du-Dv|\,dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \right\} \\
&\quad + C \left[[|\mu|(B_2)]^{\frac{1}{2}} \left(\int_{B_2} |Du|dx \right)^{-\frac{i_a}{2}} \right]^{\frac{2}{-i_a}} \\
&\quad + C \left(\int_{B_2} |Du-Dv|\,dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \\
&\le C + C\epsilon_1^{\frac{1+i_a}{-i_a}} + C \left(\int_{B_2} |Du-Dv|\,dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})} \\
&\le C + C \left(\int_{B_2} |Du-Dv|\,dx \right)^{\frac{n}{n-\sigma}(1-\frac{1}{2+i_a})},
\end{aligned}$$

which implies that (2.21) is true since

$$\frac{n}{n-\sigma}\left(1-\frac{1}{2+i_a}\right) < 1 \quad \text{for } i_a \in \left(-\frac{1}{n}, 0\right).$$

Thus, we finish the proof. □

We now switch to another comparison estimate for solutions to (1.1) and the homogeneous constant coefficient problem.

Lemma 2.12. *Assume that $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of (1.1) with $B_R \subset \Omega$ and (1.2). If $w \in W^{1,G}(B_R)$ is the weak solution of*

$$\begin{cases} \operatorname{div}\left[a\left((\overline{A}_{B_R}Dw\cdot Dw)^{\frac{1}{2}}\right)\overline{A}_{B_R}Dw\right]=0 & \text{in } B_R, \\ w=v & \text{on } \partial B_R, \end{cases} \tag{2.25}$$

then for any $\epsilon_1>0$ there exists a constant $C=C(n,i_a,s_a,\epsilon_1)>1$ such that

$$⨍_{B_R}|Du-Dw|\,dx \le Cg^{-1}\left(\frac{1}{\epsilon_1}\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right)+\epsilon_1⨍_{B_{2R}}|Du|\,dx. \tag{2.26}$$

Proof. If we select the test function $\varphi=v-w$, then after a direct calculation we can show the resulting expression as

$$I_1:=\int_{B_R}a\left((\overline{A}_{B_R}Dw\cdot Dw)^{\frac{1}{2}}\right)\overline{A}_{B_R}Dw\cdot Dwdx=\int_{B_R}a\left((\overline{A}_{B_R}Dw\cdot Dw)^{\frac{1}{2}}\right)\overline{A}_{B_R}Dw\cdot Dvdx=:I_2.$$

Using (1.3) and Lemmas 2.5–2.6, we find that

$$C\int_{B_R}G(|Dw|)\,dx\le I_1=I_2\le\tau\int_{B_R}G(|Dw|)\,dx+C(\tau)\int_{B_R}G(|Dv|)\,dx,$$

which implies that

$$\int_{B_R}G(|Dw|)\,dx\le C\int_{B_R}G(|Dv|)\,dx \tag{2.27}$$

by choosing τ small enough. Moreover, we apply Gehring's lemma (see Theorem 6.7 in [36]) to obtain the reverse Hölder type inequality

$$\left[⨍_{B_R}[G(|Dw|)]^{1+\delta_0}\,dx\right]^{\frac{1}{1+\delta_0}}\le C⨍_{B_{2R}}G(|Dw|)\,dx \tag{2.28}$$

for some positive constant $\delta_0>0$. On the other hand, we can also calculate the result of the expression $I_3=I_4$, where

$$I_3:=\int_{B_R}\left[a\left((ADv\cdot Dv)^{\frac{1}{2}}\right)ADu-a\left((ADw\cdot Dw)^{\frac{1}{2}}\right)ADw\right]\cdot(Dv-Dw)dx,$$
$$I_4:=-\int_{B_R}\left[a\left((ADw\cdot Dw)^{\frac{1}{2}}\right)ADw-a\left((\overline{A}_{B_R}Dw\cdot Dw)^{\frac{1}{2}}\right)\overline{A}_{B_R}Dw\right]\cdot(Dv-Dw)dx.$$

From Lemma 2.6 we find that

$$\epsilon\int_{B_R}G(|Dv|)\,dx+I_3\ge C\int_{B_R}G(|Dv-Dw|)\,dx.$$

Moreover, we first discover

$$\begin{aligned}|I_4|&\le\int_{B_R}a\left((ADw\cdot Dw)^{\frac{1}{2}}\right)|A-\overline{A}_{B_R}|\,|Dw|\,|Dw-Dv|\,dx\\&\quad+\int_{B_R}\left|a\left((ADw\cdot Dw)^{\frac{1}{2}}\right)-a\left((\overline{A}_{B_R}Dw\cdot Dw)^{\frac{1}{2}}\right)\right||\overline{A}_{B_R}Dw|\,|Dw-Dv|\,dx\\&=:I_{41}+I_{42}.\end{aligned}$$

Estimate of I_{41}. From (1.3), Lemma 2.6, Young's inequality and Hölder's inequality we find that

$$\begin{aligned} I_{41} &\le C \int_{B_R} a(|Dw|)\,|Dw|\,\left|A-\overline{A}_{B_R}\right|\,|Dw-Dv|\,dx \\ &\le \frac{\epsilon}{2\Lambda}\int_{B_R} G(|Dw-Dv|)\left|A-\overline{A}_{B_R}\right|dx + C(\epsilon)\int_{B_R}\widetilde{G}\left(a(|Dw|)\,|Dw|\right)\left|A-\overline{A}_{B_R}\right|dx \\ &\le \epsilon\int_{B_R} G(|Dw-Dv|)\,dx + C(\epsilon)\int_{B_R} G(|Dw|)\left|A-\overline{A}_{B_R}\right|dx \\ &\le \epsilon\int_{B_R} G(|Dw-Dv|)\,dx + C(\epsilon)\left\{\int_{B_R}\left[G(|Dw|)\right]^{1+\delta_0}dx\right\}^{\frac{1}{1+\delta_0}}\left[\int_{B_R}\left|A-\overline{A}_{B_R}\right|^{\frac{1+\delta_0}{\delta_0}}dx\right]^{\frac{\delta_0}{1+\delta_0}} \end{aligned}$$

for any $\epsilon > 0$, which implies that

$$\begin{aligned} I_{41} &\le \epsilon\int_{B_R} G(|Dw-Dv|)\,dx + C(\epsilon)\int_{B_{2R}} G(|Dw|)\,dx\left[\int_{B_R}\left|A-\overline{A}_{B_R}\right|dx\right]^{\frac{\delta_0}{1+\delta_0}} \\ &\le \epsilon\int_{B_R} G(|Dw-Dv|)\,dx + C(\epsilon)\delta^{\frac{\delta_0}{1+\delta_0}}\int_{B_{2R}} G(|Dv|)\,dx, \end{aligned}$$

where we used Definition 1.4 and (2.27)–(2.28).

Estimate of I_{42}. (1.2), (1.3), Lemma 2.6 and Lagrange's mean value theorem yield the bound

$$\begin{aligned} I_{42} &\le C\int_{B_R}\left|a'(\zeta)\right|\left|(ADw\cdot Dw)^{\frac{1}{2}} - \left(\overline{A}_{B_R}Dw\cdot Dw\right)^{\frac{1}{2}}\right|\,|Dw|\,|Dw-Dv|\,dx \\ &\le C\int_{B_R}\frac{|a(\zeta)|}{\zeta}\,\frac{\left|A-\overline{A}_{B_R}\right|\,|Dw|^2}{(ADw\cdot Dw)^{\frac{1}{2}} + \left(\overline{A}_{B_R}Dw\cdot Dw\right)^{\frac{1}{2}}}\,|Dw|\,|Dw-Dv|\,dx \\ &\le C\int_{B_R} a(|Dw|)\,|Dw|\,\left|A-\overline{A}_{B_R}\right|\,|Dw-Dv|\,dx, \end{aligned}$$

where ζ is between $(ADw\cdot Dw)^{\frac{1}{2}}$ and $\left(\overline{A}_{B_R}Dw\cdot Dw\right)^{\frac{1}{2}}$ satisfying

$$\Lambda^{-\frac{1}{2}}|Dw| \le \zeta, (ADw\cdot Dw)^{\frac{1}{2}}, \left(\overline{A}_{B_R}Dw\cdot Dw\right)^{\frac{1}{2}} \le \Lambda^{\frac{1}{2}}|Dw|.$$

And then, we have

$$I_{42} \le \epsilon\int_{B_R} G(|Dw-Dv|)\,dx + C(\epsilon)\delta^{\frac{\delta_0}{1+\delta_0}}\int_{B_{2R}} G(|Dv|)\,dx$$

for any $\epsilon > 0$, whose proof is totally similar to that of I_{41}. Thus, we choose ϵ small enough and combine the estimates of I_3 and I_4 to conclude that

$$\begin{aligned} \int_{B_R} G(|Dv-Dw|)\,dx &\le \epsilon\int_{B_R} G(|Dv|)\,dx + C(\epsilon)\delta^{\frac{\delta_0}{1+\delta_0}}\int_{B_{2R}} G(|Dv|)\,dx \\ &\le \epsilon_1^{2+s_a}\int_{B_{2R}} G(|Dv|)\,dx \end{aligned}$$

by selecting ϵ, δ small enough satisfying the last inequality. Since $\theta^{2+i_a}G(t) \le G(\theta t) \le \theta^{2+s_a}G(t)$ for any $\theta \ge 1$ and $t \ge 0$ by (2.4), we find that

$$\theta^{2+i_a}t \le G\left(\theta G^{-1}(t)\right) \le \theta^{2+s_a}t \quad \text{for any } \theta \ge 1,$$

which implies that

$$G^{-1}\left(\theta^{2+i_a}t\right) \le \theta G^{-1}(t) \le G^{-1}\left(\theta^{2+s_a}t\right) \quad \text{for any } \theta \ge 1.$$

In other words, we conclude that

$$\theta^{\frac{1}{2+s_a}} \le \frac{G^{-1}(\theta t)}{G^{-1}(t)} \le \theta^{\frac{1}{2+i_a}} \quad \text{for any } \theta \ge 1. \tag{2.29}$$

From Jensen's inequality and the reverse Hölder's inequality (see Lemma 4.2 in [7]) we deduce that

$$\begin{aligned} G\left(\fint_{B_R} |Dv - Dw| dx\right) &\le C \fint_{B_R} G\left(|Dv - Dw|\right) dx \\ &\le C\epsilon_1^{2+s_a} \fint_{B_R} G\left(|Dv|\right) dx \\ &\le C\epsilon_1^{2+s_a} G\left(\fint_{B_{2R}} |Dv| dx\right), \end{aligned}$$

which implies that

$$\fint_{B_R} |Dv - Dw| dx \le C\epsilon_1 \fint_{B_{2R}} |Dv| dx$$

by using (2.29). Finally, by using Lemma 2.11 and the above inequality we obtain

$$\begin{aligned} \fint_{B_R} |Du - Dw| dx &\le \fint_{B_R} |Du - Dv| dx + \fint_{B_R} |Dv - Dw| dx \\ &\le C g^{-1}\left(\frac{1}{\epsilon_1}\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right) + C\epsilon_1 \int_{B_R} |Du| dx + C\epsilon_1 \int_{B_{2R}} |Dv| dx \\ &\le C g^{-1}\left(\frac{1}{\epsilon_1}\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right) + C\epsilon_1 \int_{B_{2R}} |Du| dx + C\epsilon_1 \int_{B_{2R}} |Dv - Du| dx \\ &\le C g^{-1}\left(\frac{1}{\epsilon_1}\frac{|\mu|(B_{2R})}{(2R)^{n-1}}\right) + C\epsilon_1 \int_{B_{2R}} |Du| dx \end{aligned}$$

and then finish the proof. □

Additionally, we can get the following local Lipschitz regularity for the homogeneous constant coefficient problem.

Lemma 2.13 (see [7, Lemma 4.1])**.** *Let $w \in W^{1,G}(\Omega)$ be a weak solution to*

$$\operatorname{div}\left[a\left(\left(\overline{A}_{B_R} Dw \cdot Dw\right)^{\frac{1}{2}}\right) \overline{A}_{B_R} Dw\right] = 0 \quad \text{in } B_R \subset \mathbb{R}^n.$$

Then we can obtain the following De Giorgi type estimate

$$\sup_{B_{R/2}} |Dw| \le C \fint_{B_R} |Dw|\, dx.$$

The following crucial lemma, which shows how the upper level sets of $|Du|$ decay, allows us to build the interior gradient estimates.

Lemma 2.14. *Assume that* $\lambda > 0$. *There is a constant* $N = N(n, i_a, s_a) > 0$ *so that for any* $\epsilon > 0$, *there exists a small* $\delta = \delta(\epsilon) > 0$ *such that if* $u \in W^{1,G}_{loc}(\Omega)$ *is a local weak solution of* (1.1) *in* $B_{6r} \subset \Omega$ *for* $r \in (0,1]$ *with*

$$B_r \cap \left\{x \in B_1 : \mathcal{M}(|Du|)(x) \le \lambda\right\} \cap \left\{x \in B_1 : g^{-1}\left[\mathcal{M}_1(\mu)\right](x) \le \delta\lambda\right\} \neq \emptyset, \tag{2.30}$$

then we have

$$w\left(\left\{x \in B_r : \mathcal{M}(|Du|)(x) > N\lambda\right\}\right) < \epsilon w\left(B_r\right). \tag{2.31}$$

Proof. From (2.30), there exists a point $x_0 \in B_r$ such that

$$\fint_{B_\rho(x_0)} |Du|\, dx \le \lambda \quad \text{and} \quad g^{-1}\left(\rho \fint_{B_\rho(x_0)} d|\mu|\right) \le \delta\lambda \tag{2.32}$$

for all $\rho > 0$. Since $B_{4r} \subset B_{5r}(x_0)$, it follows from (2.32) that

$$\fint_{B_{4r}} |Du|\, dx \le \frac{|B_{5r}(x_0)|}{|B_{4r}|} \cdot \frac{1}{|B_{5r}(x_0)|} \int_{B_{5r}(x_0)} |Du|\, dx \le 2^n\lambda. \tag{2.33}$$

Since $t^{i_a+1}g(1) \le g(t) \le t^{s_a+1}g(1)$ for any $t \ge 1$ by Lemma 2.6, we know that $t^{\frac{1}{s_a+1}} \lesssim g^{-1}(t) \lesssim t^{\frac{1}{i_a+1}}$ for any $t \ge 1$. Similarly, we also see that $t^{\frac{1}{i_a+1}} \lesssim g^{-1}(t) \lesssim t^{\frac{1}{s_a+1}}$ for any $0 < t < 1$. In the same way, we also have

$$g^{-1}\left(4r \fint_{B_{4r}} d|\mu|\right) \le g^{-1}\left(\frac{4r}{5r} \cdot \frac{|B_{5r}(x_0)|}{|B_{4r}|} \cdot 5r \fint_{B_{5r}(x_0)} d|\mu|\right) \le C\delta\lambda.$$

Then we apply Lemma 2.12 to deduce that

$$\begin{aligned}\fint_{B_{3r}} |Du - Dw|\, dx &\le Cg^{-1}\left(\frac{4r}{\epsilon_1} \fint_{B_{4r}} d|\mu|\right) + \epsilon_1 \fint_{B_{4r}} |Du|\, dx \\ &\le C\frac{\delta\lambda}{\epsilon_1^{\frac{1}{i_a+1}}} + C\epsilon_1\lambda\end{aligned}$$

by choosing δ, ϵ_1 small enough satisfying $C\frac{\delta\lambda}{\epsilon_1^{\frac{1}{i_a+1}}} + C\epsilon_1\lambda \le \lambda$ in advance and then

$$\begin{aligned}\|Dw\|_{L^\infty(B_{3r})} &\le C\fint_{B_{4r}} |Dw|\, dx \\ &\le C\fint_{B_{4r}} |Du|\, dx + C\fint_{B_{4r}} |Du - Dw|\, dx \\ &\le N_1\lambda\end{aligned}$$

by Lemma 2.13 and (2.33), for some positive constant $N_1 \ge 1$. Now we shall claim that

$$\left\{x \in B_r : \mathcal{M}(|Du|)(x) > N\lambda\right\} \subset \left\{x \in B_r : \mathcal{M}(|Du - Dw|)(x) > N_1\lambda\right\} \tag{2.34}$$

for $N := \max\{3^n, 2N_1\}$. Actually, we take $x_1 \in \{x \in B_r : \mathcal{M}(|Du - Dw|)(x) \le N_1\lambda\}$. If $0 < \rho < r$, then we find that $B_\rho(x_1) \subset B_{2r}$ and so

$$\begin{aligned}\fint_{B_\rho(x_1)} |Du|\, dx &\le \fint_{B_\rho(x_1)} (|Dw| + |Du - Dw|)\, dx \\ &\le \mathcal{M}(|Du - Dw|)(x_1) + N_1\lambda \\ &\le 2N_1\lambda.\end{aligned}$$

On the other hand, if $\rho \geq r$, then $B_\rho(x_1) \subset B_{3\rho}(x_0)$. From (2.32), we deduce that

$$\fint_{B_\rho(x_1)} |Du|\,dx \leq \frac{|B_{3\rho}(x_0)|}{|B_\rho(x_1)|} \fint_{B_{3\rho}(x_0)} |Du|\,dx \leq 3^n \lambda \leq N\lambda.$$

Thus, the claim (2.34) is true. Then from Lemma 2.8 we estimate

$$\begin{aligned}\frac{1}{|B_r|}\left|\{x \in B_r : \mathcal{M}(|Du|) > N\lambda\}\right| &\leq \frac{1}{|B_r|}\left|\{x \in B_r : \mathcal{M}(|Du - Dw|) > N_1\lambda\}\right| \\ &\leq \frac{C}{N_1\lambda} \fint_{B_{3r}} |Du - Dw|\,dx \\ &\leq C\frac{\delta}{\epsilon_1^{\frac{1}{i_a+1}}} + C\epsilon_1,\end{aligned}$$

which implies that

$$w\left(\{x \in B_r : \mathcal{M}(|Du|) > N\lambda\}\right) \leq C\left(\frac{\delta}{\epsilon_1^{\frac{1}{i_a+1}}} + \epsilon_1\right)^\sigma w(B_r) < \epsilon w(B_r)$$

by Lemma 1.2 and choosing δ, ϵ_1 small enough satisfying the last inequality. Therefore, we finish the final proof of this lemma. □

Now we are ready to finish the proof of the main result, Theorem 1.5.

Proof. Let $u \in W^{1,G}_{loc}(\Omega)$ be the local weak solution of (1.1),

$$E = \{x \in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}$$

and

$$F = \{x \in B_1 : \mathcal{M}(|Du|) > \lambda\lambda_0\} \cup \left\{x \in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > \delta\lambda\lambda_0\right\} \quad \text{for any } \lambda \geq 1,$$

where

$$\lambda_0 = \frac{C_3}{N|B_1|}\left(\frac{C_2}{\epsilon}\right)^{\frac{1}{\sigma}} \int_{B_2} |Du| + 1dx. \tag{2.35}$$

It follows from the weak 1-1 estimate that

$$|\{x \in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}| \leq \frac{C_3}{N\lambda\lambda_0}\int_{B_1} |Du|dx < \left(\frac{\epsilon}{C_2}\right)^{\frac{1}{\sigma}} |B_1|,$$

which implies that

$$w\left(\{x \in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}\right) < \epsilon w(B_1)$$

by Lemma 1.2. Therefore, we apply Lemma 2.9 and Lemma 2.14 to have

$$w\left(E\right) \leq C\epsilon w\left(F\right). \tag{2.36}$$

Next, we divide into two cases.

Case 1: $r = +\infty$. From (2.36) we conclude that

$$\begin{aligned}\left[w\left(\{x \in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}\right)\right]^{\frac{1}{q}} &\leq C\epsilon^{\frac{1}{q}}\left[w\left(\{x \in B_1 : \mathcal{M}(|Du|)(x) > \lambda\lambda_0\}\right)\right]^{\frac{1}{q}} \\ &\quad + C\epsilon^{\frac{1}{q}}\left[w\left(\left\{x \in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > \delta\lambda\lambda_0\right\}\right)\right]^{\frac{1}{q}}\end{aligned}$$

for any $\lambda \geq 1$, which implies that

$$\begin{aligned}\|\mathcal{M}(|Du|)\|_{L^{q,\infty}_w(B_1)} &:= \sup_{\lambda>0} N\lambda\lambda_0 \left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}\right)\right]^{\frac{1}{q}} \\ &\le CN\epsilon^{\frac{1}{q}} \sup_{\lambda\ge 1} \lambda\lambda_0 \left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > \lambda\lambda_0\}\right)\right]^{\frac{1}{q}} + C\lambda_0 \\ &\quad + \frac{CN\epsilon^{\frac{1}{q}}}{\delta} \sup_{\lambda\ge 1} \delta\lambda\lambda_0 \left[w\left(\{x\in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > \delta\lambda\lambda_0\}\right)\right]^{\frac{1}{q}} \\ &\le C_4\epsilon^{\frac{1}{q}} \|\mathcal{M}(|Du|)\|_{L^{q,\infty}_w(B_1)} + C(\delta,\epsilon)\|g^{-1}\left(\mathcal{M}_1(\mu)\right)\|_{L^{q,\infty}_w(B_1)} + C\lambda_0.\end{aligned}$$

Then, by selecting ϵ small enough such that $C_4\epsilon^{\frac{1}{q}} = 1/2$ and using an approximation argument by choosing $|\nabla u|_k := \min\{\nabla u, k\}$ like the one in [8], we deduce that

$$\begin{aligned}\|Du\|_{L^{q,\infty}_w(B_1)} &\le \|\mathcal{M}(|Du|)\|_{L^{q,\infty}_w(B_1)} \\ &\le C\|g^{-1}\left(\mathcal{M}_1(\mu)\right)\|_{L^{q,\infty}_w(B_1)} + C\lambda_0 \\ &\le C\|g^{-1}\left(\mathcal{M}_1(\mu)\right)\|_{L^{q,\infty}_w(B_1)} + C\int_{B_2} |Du| + 1dx.\end{aligned}$$

Case 2: $0 < r < +\infty$. From (2.36) we find that

$$\begin{aligned}&\left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > N\lambda\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\quad = \left[w(E)\right]^{\frac{r}{q}} \le C\epsilon^{\frac{r}{q}}\left[w(F)\right]^{\frac{r}{q}} \\ &\quad \le C\epsilon^{\frac{r}{q}}\left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > \lambda\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\quad\quad + C\epsilon^{\frac{r}{q}}\left[w\left(\{x\in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > \delta\lambda\lambda_0\}\right)\right]^{\frac{r}{q}}.\end{aligned}$$

Actually, by applying an iteration procedure we can also prove

$$\begin{aligned}&\left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > N^m\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\quad \le C\epsilon^{\frac{mr}{q}}\left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > \lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\quad\quad + C\sum_{i=1}^{m}\epsilon^{\frac{ir}{q}}\left[w\left(\{x\in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > N^{m-i}\delta\lambda_0\}\right)\right]^{\frac{r}{q}}.\end{aligned} \tag{2.37}$$

Now we select ϵ small enough satisfying $N^r\epsilon^{\frac{r}{q}} < 1$ and then apply Lemma 2.10 to observe that

$$\begin{aligned}&\sum_{m=1}^{\infty} N^{mr}\lambda_0^r\left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > N^m\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\le C\sum_{m=1}^{\infty} N^{mr}\lambda_0^r \sum_{i=1}^{m}\epsilon^{\frac{ir}{q}}\left[w\left(\{x\in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > N^{m-i}\delta\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\quad + C\sum_{m=1}^{\infty} N^{mr}\epsilon^{\frac{mr}{q}}\lambda_0^r\left[w\left(\{x\in B_1 : \mathcal{M}(|Du|)(x) > \lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\le \frac{C}{\delta^r}\sum_{i=1}^{\infty}\epsilon^{\frac{ir}{q}}N^{ir}\sum_{m=i}^{\infty} N^{(m-i)r}\delta^r\lambda_0^r\left[w\left(\{x\in B_1 : g^{-1}\left(\mathcal{M}_1(\mu)\right)(x) > N^{m-i}\delta\lambda_0\}\right)\right]^{\frac{r}{q}} \\ &\quad + C\lambda_0^r\sum_{m=1}^{\infty} N^{mr}\epsilon^{\frac{mr}{q}} \\ &\le C\|g^{-1}\left(\mathcal{M}_1(\mu)\right)\|^r_{L^{q,r}_w(B_1)} + C\lambda_0^r < +\infty,\end{aligned}$$

which implies that

$$\|Du\|_{L^{q,r}_w(B_1)} \le \|\mathcal{M}(|Du|)\|_{L^{q,r}_w(B_1)} \le C\|g^{-1}\left(\mathcal{M}_1(\mu)\right)\|_{L^{q,r}_w(B_1)} + C\int_{B_2}|Du|+1dx.$$

Thus, this finishes the proof of the main result in this work. □

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Exact solution of the Susceptible–Exposed–Infectious–Recovered–Deceased (SEIRD) epidemic model

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Abstract. An exact solution of an initial value problem for the Susceptible–Exposed–Infectious–Recovered–Deceased (SEIRD) epidemic model is derived, and various properties of the exact solution are obtained. It is shown that the parametric form of the exact solution satisfies some linear differential system including a positive solution of an Abel differential equation of the second kind. In this paper Abel differential equations play an important role in establishing the exact solution of the SEIRD differential system, in particular the number of infected individuals can be represented in a simple form by using a positive solution of an initial value problem for an Abel differential equation. Uniqueness of positive solutions of an initial value problem to SEIRD differential system is also investigated, and it is shown that the exact solution is a unique solution in the class of positive solutions.

Keywords: exact solution, SEIRD epidemic model, initial value problem, linear differential system, Abel differential equation, uniqueness of positive solutions.

2020 Mathematics Subject Classification: 34A34.

1 Introduction

Recently there is an increasing interest in mathematical approach to the epidemic models. Since the pioneering work of Bernoulli [2], a vast literature and research papers has been published so far (cf. [4,5,9]), and studies of epidemic models have become one of the important areas in mathematical biology. In particular we mention Kermack and McKendrick [11] in which the Susceptible-Infectious-Recovered (SIR) epidemic model was proposed. Exact solutions of epidemic models have been investigated in recent years. We refer to Bohner, Streipert and Torres [3], Harko, Lobo and Mak [10] and Shabbir, Khan and Sadiq [16] and Yoshida [19] for SIR epidemic models, to Yoshida [18] for Susceptible–Infectious–Recovered–Deceased (SIRD) epidemic models, and to Yoshida [20] for Susceptible-Exposed-Infectious-Recovered (SEIR) epidemic models.

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The Susceptible–Exposed–Infectious–Recovered–Deceased (SEIRD) epidemic models have been an important and interesting subject to study (cf. [6, 12–15, 17]). However, there appears to be no known results about exact solutions of SEIRD epidemic models. The objective of this paper is to establish an exact solution of an initial value problem for SEIRD epidemic model. Our method is an adaptation of that used in Yoshida [20], and is based on the existence of unique positive solution of an initial value problem for Abel differential equations of the second kind. We refer the reader to Abel [1] and Davis [8] for Abel differential equations. Uniqueness of positive solutions of an initial value problem to SEIRD differential system is also studied, and we find that the exact solution is a unique solution in the class of positive solutions.

We study the Susceptible–Expose–Infectious–Recovered–Deceased (SEIRD) epidemic model

$$\frac{dS(t)}{dt} = -\beta S(t)I(t), \tag{1.1}$$

$$\frac{dE(t)}{dt} = \beta S(t)I(t) - \delta E(t), \tag{1.2}$$

$$\frac{dI(t)}{dt} = \delta E(t) - \gamma I(t) - \mu I(t), \tag{1.3}$$

$$\frac{dR(t)}{dt} = \gamma I(t), \tag{1.4}$$

$$\frac{dD(t)}{dt} = \mu I(t) \tag{1.5}$$

for $t > 0$, where β, γ, δ and μ are positive constants. The initial condition to be considered is the following:

$$S(0) = \tilde{S}, \quad E(0) = \tilde{E}, \quad I(0) = \tilde{I}, \quad R(0) = \tilde{R}, \; D(0) = \tilde{D}. \tag{1.6}$$

It is assumed throughout this paper that:

(A_1) $\tilde{I} > 0$;

(A_2) $\tilde{S} > \frac{\delta \tilde{E}}{\beta \tilde{I}}$;

(A_3) $\tilde{E} > \frac{\gamma+\mu}{\delta}\tilde{I}$;

(A_4) $\tilde{R} \geq 0$;

(A_5) $\tilde{D} \geq 0$ and $\tilde{D}$ satisfies

$$N - \tilde{R} > \tilde{S}e^{(\beta/\mu)\tilde{D}} + \tilde{D};$$

(A_6) $\tilde{S} + \tilde{E} + \tilde{I} + \tilde{R} + \tilde{D} = N$ (positive constant).

In Section 2 we obtain a parametric solution of an initial value problem for SEIRD differential system, and in Section 3 we derive an exact solution of an initial value problem for SEIRD differential system. Section 4 is devoted to various properties of the exact solution of SEIRD differential system. In Section 5 we show that there exists one, and only one, solution of an initial value problem for SEIRD differential system in the class of positive solutions.

2 Parametric solution of an initial value problem for SEIRD differential system

In this section we show that a positive solution of the initial value problem (1.1)–(1.6) can be represented in a parametric form.

Since

$$\frac{d}{dt}\left(S(t)+E(t)+I(t)+R(t)+D(t)\right)=\frac{dS(t)}{dt}+\frac{dE(t)}{dt}+\frac{dI(t)}{dt}+\frac{dR(t)}{dt}+\frac{dD(t)}{dt}=0$$

by (1.1)–(1.5), we obtain

$$S(t)+E(t)+I(t)+R(t)+D(t)=k \qquad (t\geq 0)$$

for some constant k. The hypothesis (A_6) implies

$$k=S(0)+E(0)+I(0)+R(0)+D(0)=\tilde{S}+\tilde{E}+\tilde{I}+\tilde{R}+\tilde{D}=N,$$

and therefore

$$S(t)+E(t)+I(t)+R(t)+D(t)=N \qquad (t\geq 0).$$

We state the following important lemma.

Lemma 2.1. *If* $(S(t),E(t),I(t),R(t),D(t))$ *is a solution of the SEIRD differential system* (1.1)–(1.5) *such that* $S(t)>0$ *for* $t>0$*, then*

$$D''(t)+(\delta+\gamma+\mu)D'(t)=\delta\mu\left(N-\tilde{R}+\frac{\gamma}{\mu}\tilde{D}-\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(t)}-\left(1+\frac{\gamma}{\mu}\right)D(t)\right) \tag{2.1}$$

for $t>0$.

Proof. We see from (1.1) and (1.5) that

$$D'(t)=\mu I(t)=\mu\left(\frac{S'(t)}{-\beta S(t)}\right)=-\frac{\mu}{\beta}(\log S(t))',$$

and integrating the above on $[0,t]$ gives

$$D(t)-\tilde{D}=-\frac{\mu}{\beta}(\log S(t)-\log\tilde{S}).$$

Hence we obtain

$$\log S(t)=-\frac{\beta}{\mu}(D(t)-\tilde{D})+\log\tilde{S}$$

and therefore

$$S(t)=\exp\left(\log\tilde{S}-\frac{\beta}{\mu}D(t)+\frac{\beta}{\mu}\tilde{D}\right)=\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(t)}. \tag{2.2}$$

It follows from (1.5) that $I(t)=D'(t)/\mu$, and hence $I'(t)=D''(t)/\mu$. Therefore, (1.3) implies that

$$\begin{aligned}E(t)&=\frac{1}{\delta}(I'(t)+(\gamma+\mu)I(t))\\&=\frac{1}{\delta}\left(\frac{D''(t)}{\mu}+(\gamma+\mu)\frac{D'(t)}{\mu}\right)\\&=\frac{1}{\delta\mu}\left(D''(t)+(\gamma+\mu)D'(t)\right).\end{aligned} \tag{2.3}$$

It is obvious that

$$R'(t) = \gamma I(t) = \gamma \frac{D'(t)}{\mu} = \frac{\gamma}{\mu} D'(t),$$

and hence

$$R(t) = \frac{\gamma}{\mu} D(t) + k$$

for some constant k. Letting $t = 0$ yields

$$k = \tilde{R} - \frac{\gamma}{\mu} \tilde{D},$$

and therefore

$$R(t) = \frac{\gamma}{\mu} D(t) + \tilde{R} - \frac{\gamma}{\mu} \tilde{D}. \tag{2.4}$$

We observe, using (1.5), (2.2)–(2.4), that

$$\begin{aligned}
\frac{D'(t)}{\mu} &= I(t) \\
&= N - S(t) - E(t) - R(t) - D(t) \\
&= N - \tilde{S} e^{(\beta/\mu)\tilde{D}} e^{-(\beta/\mu)D(t)} - \frac{1}{\delta\mu}\left(D''(t) + (\gamma+\mu)D'(t)\right) - \frac{\gamma}{\mu} D(t) - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - D(t)
\end{aligned}$$

which implies

$$\frac{1}{\delta\mu} D''(t) + \left(\frac{1}{\mu} + \frac{\gamma+\mu}{\delta\mu}\right) D'(t) = N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S} e^{(\beta/\mu)\tilde{D}} e^{-(\beta/\mu)D(t)} - \left(1 + \frac{\gamma}{\mu}\right) D(t).$$

Multiplying the above by $\delta\mu$ yields the desired identity (2.1). □

By a *solution* of the SEIRD differential system (1.1)–(1.5) we mean a vector-valued function $(S(t), E(t), I(t), R(t), D(t))$ of class $C^1(0,\infty) \cap C[0,\infty)$ which satisfies (1.1)–(1.5). Associated with every continuous function $f(t)$ on $[0,\infty)$, we define

$$f(\infty) := \lim_{t\to\infty} f(t).$$

Lemma 2.2. *Let $(S(t), E(t), I(t), R(t), D(t))$ be a solution of the SEIRD differential system (1.1)–(1.5) such that $S(t) > 0$, $E(t) > 0$, $I(t) > 0$ and $R(t) > 0$ for $t > 0$. Then there exists the limit $D(\infty)$.*

Proof. Since $I(t) > 0$ for $t > 0$, it follows from (1.5) that $D'(t) = \mu I(t) > 0$ for $t > 0$, and therefore $D(t)$ is increasing on $[0,\infty)$. It is easy to see that $D(t)$ is bounded from above in light of

$$D(t) = N - S(t) - E(t) - I(t) - R(t) < N \ (t > 0).$$

Hence there exists the limit $D(\infty)$. □

Theorem 2.3. *Let $(S(t), E(t), I(t), R(t), D(t))$ be a solution of the initial value problem (1.1)–(1.6) such that $S(t) > 0$, $E(t) > 0$, $I(t) > 0$ and $R(t) > 0$ for $t > 0$. Then the solution*

$(S(t), E(t), I(t), R(t), D(t))$ *can be represented in the following parametric form:*

$$S(\varphi(u)) = \tilde{S}e^{(\beta/\mu)\tilde{D}}u, \tag{2.5}$$

$$E(\varphi(u)) = \tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)} \int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv, \tag{2.6}$$

$$\begin{aligned} I(\varphi(u)) = {} & N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + \frac{\gamma+\mu}{\beta}\log u - \tilde{E}e^{-\delta\varphi(u)} \\ & - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)} \int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv, \end{aligned} \tag{2.7}$$

$$R(\varphi(u)) = -\frac{\gamma}{\beta}\log u + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}, \tag{2.8}$$

$$D(\varphi(u)) = -\frac{\mu}{\beta}\log u \tag{2.9}$$

for $e^{-(\beta/\mu)D(\infty)} < u \le e^{-(\beta/\mu)\tilde{D}}$, *where*

$$t = \varphi(u) = \int_u^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)}, \tag{2.10}$$

with $\psi(u)$ *satisfying the Abel differential equation of the second kind*

$$\psi'\psi - \frac{\delta+\gamma+\mu}{u}\psi = -\delta\,\frac{\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u}{u} \tag{2.11}$$

for $e^{-(\beta/\mu)D(\infty)} < u < e^{-(\beta/\mu)\tilde{D}}$, *and the following conditions*

$$\begin{aligned} \psi\big(e^{-(\beta/\mu)\tilde{D}}\big) &= \beta\tilde{I}, \\ \lim_{u\to e^{-(\beta/\mu)D(\infty)}+0} \psi(u) &= 0, \\ \psi(u) > 0 \quad & \text{in } (e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}}]. \end{aligned}$$

Proof. Since $D'(t) = \mu I(t) > 0$ for $t > 0$ in view of (1.5), we find that $D(t)$ is increasing on $[0, \infty)$. Then there exists the limit $D(\infty)$ by Lemma 2.2. It is easy to check that $u = u(t) = e^{-(\beta/\mu)D(t)}$ is decreasing on $[0, \infty)$, $e^{-(\beta/\mu)D(\infty)} < u \le e^{-(\beta/\mu)\tilde{D}}$ and $\lim_{t\to\infty} u(t) = e^{-(\beta/\mu)D(\infty)}$. Hence there exists the inverse function $\varphi(u) \in C^1(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}})$ of $u = u(t)$ such that

$$t = \varphi(u) \qquad \left(e^{-(\beta/\mu)D(\infty)} < u \le e^{-(\beta/\mu)\tilde{D}}\right),$$

$\varphi(u)$ is decreasing in $(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}}]$, $\lim_{u\to e^{-(\beta/\mu)D(\infty)}+0} \varphi(u) = \infty$, and $\varphi\big(e^{-(\beta/\mu)\tilde{D}}\big) = 0$. Substituting $t = \varphi(u)$ into (2.1) in Lemma 2.1 yields

$$\begin{aligned} & D''(\varphi(u)) + (\delta+\gamma+\mu)D'(\varphi(u)) \\ & \quad = \delta\mu\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(\varphi(u))} - \left(1 + \frac{\gamma}{\mu}\right)D(\varphi(u))\right) \end{aligned} \tag{2.12}$$

for $e^{-(\beta/\mu)D(\infty)} < u < e^{-(\beta/\mu)\tilde{D}}$. Differentiating both sides of $u = e^{-(\beta/\mu)D(\varphi(u))}$ with respect to u yields

$$\begin{aligned} 1 &= -\frac{\beta}{\mu}D'(\varphi(u))\varphi'(u)e^{-(\beta/\mu)D(\varphi(u))} \\ &= -\frac{\beta}{\mu}D'(\varphi(u))\varphi'(u)u, \end{aligned}$$

and therefore

$$D'(\varphi(u)) = -\frac{\mu}{\beta}\frac{1}{\varphi'(u)u}. \tag{2.13}$$

Since $D'(t) \in C^1(0,\infty)$ by means of (1.5) and $\varphi(u) \in C^1(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}})$, we see that $D'(\varphi(u)) \in C^1(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}})$, and consequently $1/(\varphi'(u)u) \in C^1(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}})$. We differentiate (2.13) with respect to u to obtain

$$D''(\varphi(u))\varphi'(u) = -\frac{\mu}{\beta}\left(\frac{1}{\varphi'(u)u}\right)',$$

and hence

$$D''(\varphi(u)) = -\frac{\mu}{\beta}\left(\frac{1}{\varphi'(u)u}\right)'\frac{1}{\varphi'(u)}. \tag{2.14}$$

It is obvious that

$$D(\varphi(u)) = -\frac{\mu}{\beta}\log u \tag{2.15}$$

in light of $u = e^{-(\beta/\mu)D(\varphi(u))}$. Combining (2.12)–(2.15), we get

$$\begin{aligned} -\frac{\mu}{\beta}\left(\frac{1}{\varphi'(u)u}\right)'\frac{1}{\varphi'(u)} + (\delta+\gamma+\mu)\left(-\frac{\mu}{\beta}\frac{1}{\varphi'(u)u}\right) \\ = \delta\mu\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u - \left(1+\frac{\gamma}{\mu}\right)\left(-\frac{\mu}{\beta}\log u\right)\right) \end{aligned}$$

or

$$\begin{aligned} \frac{\mu}{\beta}\left(-\frac{1}{\varphi'(u)u}\right)'\left(-\frac{1}{\varphi'(u)u}\right) - \frac{\mu}{\beta}\frac{\delta+\gamma+\mu}{u}\left(-\frac{1}{\varphi'(u)u}\right) \\ = -\delta\mu\frac{1}{u}\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + \frac{\mu}{\beta}\left(1+\frac{\gamma}{\mu}\right)\log u\right). \end{aligned} \tag{2.16}$$

Letting

$$\psi(u) := -\frac{1}{\varphi'(u)u}, \tag{2.17}$$

we observe that $\psi(u)$ satisfies (2.11). Since $t = \varphi(u) > 0$ for $e^{-(\beta/\mu)D(\infty)} < u < e^{-(\beta/\mu)\tilde{D}}$, we see from (1.5), (2.13) and (2.17) that

$$\psi(u) = \frac{\beta}{\mu}D'(\varphi(u)) = \beta I(\varphi(u)) > 0$$

in $(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}})$. If we define

$$\begin{aligned} \psi\left(e^{-(\beta/\mu)\tilde{D}}\right) &:= \lim_{u\to e^{-(\beta/\mu)\tilde{D}}-0}\psi(u) = \frac{\beta}{\mu}\lim_{u\to e^{-(\beta/\mu)\tilde{D}}-0}D'(\varphi(u)) \\ &= \frac{\beta}{\mu}\lim_{t\to+0}D'(t) = \frac{\beta}{\mu}\mu I(0) = \beta\tilde{I} > 0, \end{aligned}$$

then $\psi(u)$ is a positive continuous function in $(e^{-(\beta/\mu)D(\infty)}, e^{-(\beta/\mu)\tilde{D}}]$. It follows from (2.17) that

$$t = \varphi(u) = \int_{e^{-(\beta/\mu)\tilde{D}}}^{u}\varphi'(\xi)d\xi = \int_{u}^{e^{-(\beta/\mu)\tilde{D}}}\frac{d\xi}{\xi\psi(\xi)},$$

and therefore (2.10) holds. Since $\lim_{u\to e^{-(\beta/\mu)D(\infty)}+0} \varphi(u) = \infty$, it is necessary that $\lim_{u\to e^{-(\beta/\mu)D(\infty)}+0} \psi(u) = 0$.

Now we establish the representation formulae (2.5)–(2.9). We see from (2.2) and (2.15) that

$$\begin{aligned} S(\varphi(u)) &= \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(\varphi(u))} = \tilde{S}e^{(\beta/\mu)\tilde{D}}u, \\ D(\varphi(u)) &= -\frac{\mu}{\beta}\log u, \end{aligned}$$

which are the desired representations (2.5) and (2.9). Combining (1.1) with (1.2) yields the first order linear differential equation

$$E'(t) + \delta E(t) = -S'(t)$$

which implies

$$E(t) = \tilde{E}e^{-\delta t} - e^{-\delta t}\int_0^t e^{\delta\xi}S'(\xi)d\xi. \tag{2.18}$$

Differentiating (2.2), we obtain

$$S'(t) = -\frac{\beta}{\mu}\tilde{S}e^{(\beta/\mu)\tilde{D}}D'(t)e^{-(\beta/\mu)D(t)}. \tag{2.19}$$

Substitution of (2.19) into (2.18) gives

$$E(t) = \tilde{E}e^{-\delta t} + \frac{\beta}{\mu}\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_0^t e^{\delta\xi}D'(\xi)e^{-(\beta/\mu)D(\xi)}d\xi. \tag{2.20}$$

By changing the variables $D(\xi) = s$, we obtain

$$\begin{aligned} J := \int_0^t e^{\delta\xi}D'(\xi)e^{-(\beta/\mu)D(\xi)}d\xi &= \int_{\tilde{D}}^{D(t)} e^{\delta D^{-1}(s)}e^{-(\beta/\mu)s}ds \\ &= \int_{\tilde{D}}^{D(t)} e^{\delta\varphi(e^{-(\beta/\mu)s})}e^{-(\beta/\mu)s}ds \\ &= \frac{\mu}{\beta}\int_{D(t)}^{\tilde{D}} e^{\delta\varphi(e^{-(\beta/\mu)s})}\left(e^{-(\beta/\mu)s}\right)' ds \end{aligned}$$

in view of $D^{-1}(s) = \varphi(e^{-(\beta/\mu)s})$. Letting $v = e^{-(\beta/\mu)s}$ yields

$$J = \frac{\mu}{\beta}\int_{e^{-(\beta/\mu)D(t)}}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv. \tag{2.21}$$

Combining (2.20) with (2.21), we are led to

$$E(t) = \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{e^{-(\beta/\mu)D(t)}}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv. \tag{2.22}$$

Substituting $t = \varphi(u)$ into (2.22), we arrive at (2.6). We observe, using (2.4), that

$$R(\varphi(u)) = \frac{\gamma}{\mu}D(\varphi(u)) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} = -\frac{\gamma}{\beta}\log u + \tilde{R} - \frac{\gamma}{\mu}\tilde{D},$$

which is equal to (2.8). Since $I(\varphi(u)) = N - S(\varphi(u)) - R(\varphi(u)) - D(\varphi(u)) - E(\varphi(u))$, (2.7) follows from (2.5), (2.6), (2.8) and (2.9). □

Corollary 2.4. *Let $(S(t), E(t), I(t), R(t), D(t))$ be a solution of the initial value problem (1.1)–(1.6) such that $S(t) > 0$, $E(t) > 0$, $I(t) > 0$ and $R(t) > 0$ for $t > 0$. Then we obtain the following relations:*

$$S(t) = \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(t)}, \tag{2.23}$$

$$E(t) = \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{e^{-(\beta/\mu)D(t)}}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta D^{-1}(-(\mu/\beta)\log v)}dv, \tag{2.24}$$

$$I(t) = N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(t)} - \frac{\gamma+\mu}{\mu}D(t) - \tilde{E}e^{-\delta t} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{e^{-(\beta/\mu)D(t)}}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta D^{-1}(-(\mu/\beta)\log v)}dv, \tag{2.25}$$

$$R(t) = D(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} \tag{2.26}$$

for $t \geq 0$.

Proof. It is easy to see that

$$u = \varphi^{-1}(t) = e^{-(\beta/\mu)D(t)}, \tag{2.27}$$

$$\varphi(v) = D^{-1}(-(\mu/\beta)\log v) \tag{2.28}$$

in the proof of Theorem 2.3. Combining (2.5)–(2.8), (2.27) and (2.28), we are led to (2.23)–(2.26). □

3 Exact solution of an initial value problem for SEIRD differential system

In this section we establish an exact solution of an initial value problem for SEIRD differential system (1.1)–(1.5) by utilizing Theorem 2.3 in Section 2.

The following lemma follows from a result of Yoshida [18, Lemma 3] by replacing $\tilde{R}, \tilde{D}, \gamma, \mu$ by $\tilde{D}, \tilde{R}, \mu, \gamma$, respectively.

Lemma 3.1. *Under the hypothesis* (A_5), *the transcendental equation*

$$x = \frac{\mu}{\gamma+\mu}N - \frac{\mu}{\gamma+\mu}\tilde{R} + \frac{\gamma}{\gamma+\mu}\tilde{D} - \frac{\mu}{\gamma+\mu}\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)x}$$

has a unique solution $x = \alpha$ such that

$$\tilde{D} < \alpha < N$$

(*cf.* Figure 3.1).

We assume that the following hypothesis

(A_7) $\tilde{S} < \frac{\gamma+\mu}{\beta}e^{(\beta/\mu)(\alpha-\tilde{D})}$

holds in the rest of this paper. We note that (A_7) is equivalent to the following

(A_7') $\frac{\gamma+\mu}{\beta} > N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \frac{\gamma+\mu}{\mu}\alpha$

in view of $\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\alpha} = N - \tilde{R} + (\gamma/\mu)\tilde{D} - ((\gamma+\mu)/\mu)\alpha$.

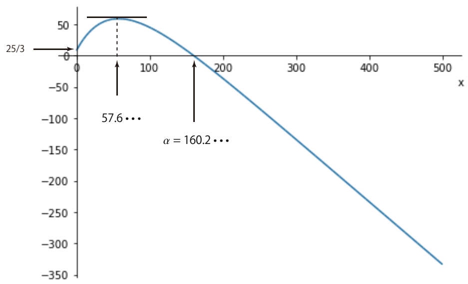

Figure 3.1: Variation of $(\mu/(\gamma+\mu))N - (\mu/(\gamma+\mu))\tilde{R} + (\gamma/(\gamma+\mu))\tilde{D} - (\mu/(\gamma+\mu))\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)x} - x$ for $N = 1000, \tilde{S} = 950, \tilde{R} = 0, \tilde{D} = 0, \beta = 0.2/1000$, $\gamma = 0.05$ and $\mu = 0.01$. In this case we find that $(\mu/(\gamma+\mu))(N-\tilde{S}) = 25/3$ and $0 < \alpha = 160.2\ldots < 1000$.

Remark 3.2. Combining (A_2) with (A_3), we have

$$\tilde{S} > \frac{\delta\tilde{E}}{\beta\tilde{I}} > \frac{\gamma+\mu}{\beta}.$$

Lemma 3.3. *There exists a unique positive solution $w(x)$ of the initial value problem for the Abel differential equation*

$$w'w + \frac{\beta(\delta+\gamma+\mu)}{\mu}w = \frac{\beta\delta}{\mu}\left(\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)x} - \frac{\beta(\gamma+\mu)}{\mu}x\right) \quad (\tilde{D} < x < \alpha), \tag{3.1}$$

subject to the initial condition

$$w(\tilde{D}) = \beta\tilde{I}. \tag{3.2}$$

Proof. Let

$$f(x) := N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)x} - \frac{\gamma+\mu}{\mu}x.$$

Since $f'(x) = 0$ for

$$x = \tilde{x} = \frac{\mu}{\beta}\log\left(\frac{\beta}{\gamma+\mu}\tilde{S}e^{(\beta/\mu)\tilde{D}}\right),$$

we see that $\tilde{D} < \tilde{x} < \alpha$ by means of (A_7) and Remark 3.2, and that $f'(x) > 0$ for $\tilde{D} < x < \tilde{x}$ and $f'(x) < 0$ for $\tilde{x} < x < \alpha$. Hence, $f(x)$ is increasing in $[\tilde{D}, \tilde{x})$ and decreasing in $(\tilde{x}, \alpha)$. Since $f(\tilde{D}) = N - \tilde{R} - \tilde{S} - \tilde{D} = \tilde{E} + \tilde{I} > 0$ and $\lim_{x\to\alpha-0} f(x) = 0$ by Lemma 3.1, it follows that $f(x) \in C[\tilde{D}, \alpha)$, $f(x) > 0$ in $[\tilde{D}, \alpha)$ and $\lim_{x\to\alpha-0} f(x) = 0$. Therefore there exists a unique positive solution $w(x)$ of the initial value problem (3.1), (3.2) by a result of Yoshida [20, Theorem 3] (*cf.* Figure 3.2). □

Lemma 3.4. *There exists a unique positive solution $\psi(u)$ of the initial value problem for the Abel differential equation*

$$\psi'\psi - \frac{\delta+\gamma+\mu}{u}\psi = -\delta\frac{\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u}{u} \quad (e^{-(\beta/\mu)\alpha} < u < e^{-(\beta/\mu)\tilde{D}}) \tag{3.3}$$

with the initial condition

$$\psi\left(e^{-(\beta/\mu)\tilde{D}}\right) = \beta\tilde{I} \tag{3.4}$$

(*cf*. Figure 3.3).

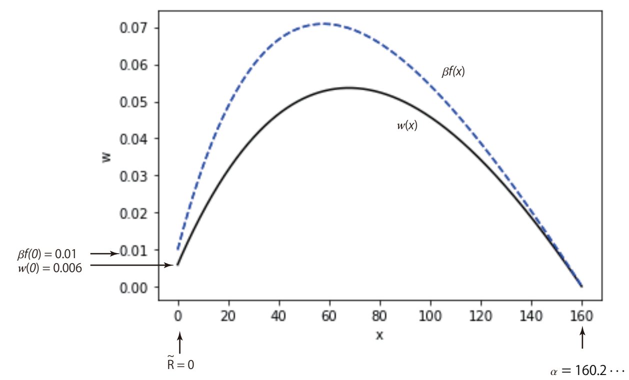

Figure 3.2: Variations of $\beta f(x)$ (dashed curve), and $w(x)$ (solid curve) obtained by the numerical integration of the initial value problem (3.1), (3.2), for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05, \delta = 0.2, \mu = 0.01$ and $\alpha = 160.2\ldots$ In this case we obtain $\beta f(0) = \beta N - \beta\tilde{S} = 0.01$ and $w(0) = \beta\tilde{I} = 0.006$.

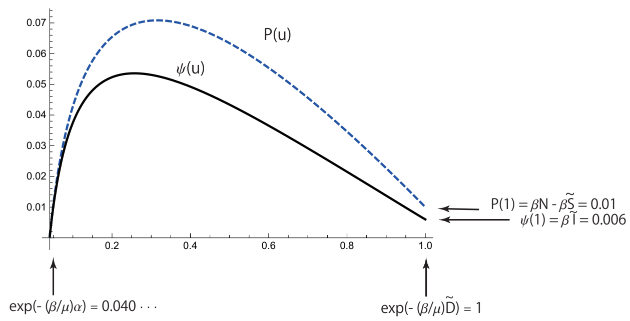

Figure 3.3: Variations of $P(u) := \beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma + \mu)\log u$ (dashed curve) and $\psi(u)$ (solid curve) obtained by the numerical integration of the initial value problem (3.3), (3.4) for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05, \delta = 0.2, \mu = 0.01$ and $\alpha = 160.2\ldots$ In this case we get $e^{-(\beta/\mu)\alpha} = 0.040\ldots$, $e^{-(\beta/\mu)\tilde{D}} = 1$, $P(1) = \beta N - \beta\tilde{S} = 0.01$ and $\psi(1) = \beta\tilde{I} = 0.006$.

Proof. Let $w(x)$ be a unique positive solution of the initial value problem (3.1), (3.2). We define $\psi(u)$ by

$$\psi(u) := w\left(-\frac{\mu}{\beta}\log u\right)$$

and find that

$$\psi'(u) = w'\left(-\frac{\mu}{\beta}\log u\right)\left(-\frac{\mu}{\beta}\frac{1}{u}\right),$$

and hence

$$
\begin{aligned}
\psi'(u)\psi(u) &= -\frac{\mu}{\beta}\frac{1}{u}w'\left(-\frac{\mu}{\beta}\log u\right)w\left(-\frac{\mu}{\beta}\log u\right) \\
&= -\frac{\mu}{\beta}\frac{1}{u}\Bigg[-\frac{\beta(\delta+\gamma+\mu)}{\mu}w\left(-\frac{\mu}{\beta}\log u\right) \\
&\quad +\frac{\beta\delta}{\mu}\left(\beta N-\beta\tilde{R}+((\beta\gamma)/\mu)\tilde{D}-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u+(\gamma+\mu)\log u\right)\Bigg] \\
&= \frac{\delta+\gamma+\mu}{u}\psi(u)-\delta\,\frac{\beta N-\beta\tilde{R}+((\beta\gamma)/\mu)\tilde{D}-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u+(\gamma+\mu)\log u}{u}
\end{aligned}
$$

for $e^{-(\beta/\mu)\alpha} < u < e^{-(\beta/\mu)\tilde{D}}$ by means of (3.1). Hence $\psi(u)$ satisfies (3.3). It is easily seen from (3.2) that

$$\psi\left(e^{-(\beta/\mu)\tilde{D}}\right) = w(\tilde{D}) = \beta\tilde{I}$$

and therefore (3.4) is satisfied. The uniqueness of $\psi(u)$ follows from that of $w(x)$. It can be shown that

$$\psi(u) > 0 \text{ in } (e^{-(\beta/\mu)\alpha}, e^{-(\beta/\mu)\tilde{D}}] \tag{3.5}$$

since $\psi(u) = w\big(-(\mu/\beta)\log u\big)$ and $w(x) > 0$ in $[\tilde{D}, \alpha)$. □

Lemma 3.5. *The unique positive solution $\psi(u)$ of the initial value problem* (3.3), (3.4) *satisfies the following relation*

$$
\begin{aligned}
\psi(u) &= \beta N-\beta\tilde{R}+\frac{\beta\gamma}{\mu}\tilde{D}-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u+(\gamma+\mu)\log u \\
&\quad -\beta\left(\tilde{E}e^{-\delta\varphi(u)}+\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\right)
\end{aligned} \tag{3.6}
$$

for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$, where

$$\varphi(u) := \int_u^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)}. \tag{3.7}$$

Conversely, the function $\psi(u)$ satisfying (3.5), (3.6) *is a solution of the initial value problem* (3.3), (3.4).

Proof. We note that (3.6) is some kind of integral equation of $\psi(u)$, in light of (3.7). Let $\psi(u)$ be the unique positive solution of the problem (3.3), (3.4), and define $z(u)$ by

$$z(u) := \psi(u) - P(u), \tag{3.8}$$

where

$$P(u) = \beta N-\beta\tilde{R}+\frac{\beta\gamma}{\mu}\tilde{D}-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u+(\gamma+\mu)\log u. \tag{3.9}$$

Dividing (3.3) by $\psi(u)$, we obtain

$$
\begin{aligned}
\psi'(u) &= \frac{\delta+\gamma+\mu}{u}-\delta\,\frac{P(u)}{u\psi(u)} = \frac{\gamma+\mu}{u}-\delta\,\frac{P(u)-\psi(u)}{u\psi(u)} \\
&= \frac{\gamma+\mu}{u}+\delta\,\frac{z(u)}{u\psi(u)}.
\end{aligned} \tag{3.10}
$$

On the other hand, differentiating (3.8) yields

$$\psi'(u) = -\beta\tilde{S}e^{(\beta/\mu)\tilde{D}} + \frac{\gamma+\mu}{u} + z'(u). \tag{3.11}$$

Combining (3.10) with (3.11), we get

$$z'(u) - \frac{\delta}{u\psi(u)}z(u) = \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}$$

or

$$z'(u) + \delta\varphi'(u)z(u) = \beta\tilde{S}e^{(\beta/\mu)\tilde{D}} \tag{3.12}$$

which is a linear differential equation of first order. It is clear that

$$\begin{aligned} z\big(e^{-(\beta/\mu)\tilde{D}}\big) &= \psi\big(e^{-(\beta/\mu)\tilde{D}}\big) - \left(\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S} - \frac{\beta(\gamma+\mu)}{\mu}\tilde{D}\right) \\ &= \beta\tilde{I} - \beta\big(N - \tilde{R} - \tilde{S} - \tilde{D}\big) \\ &= -\beta\tilde{E}. \end{aligned} \tag{3.13}$$

Now we solve the initial value problem (3.12), (3.13). Multiplying (3.12) by $e^{\delta\varphi(u)}$ yields

$$\big(e^{\delta\varphi(u)}z(u)\big)' = \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{\delta\varphi(u)}$$

and then integrating the above on $[u, e^{-(\beta/\mu)\tilde{D}}]$ gives

$$z\big(e^{-(\beta/\mu)\tilde{D}}\big) - e^{\delta\varphi(u)}z(u) = \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv.$$

Taking account of (3.13), we obtain

$$z(u) = -\beta\left(\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\right). \tag{3.14}$$

Combining (3.8) with (3.14), we observe that $\psi(u)$ satisfies (3.6) for $e^{-(\beta/\mu)\alpha} < u < e^{-(\beta/\mu)\tilde{D}}$. If $u = e^{-(\beta/\mu)\tilde{D}}$, then $\psi\big(e^{-(\beta/\mu)\tilde{D}}\big) = \beta\tilde{I}$ by (3.4) and the right hand side of (3.6) with $u = e^{-(\beta/\mu)\tilde{D}}$ is equal to $\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S} - (\beta(\gamma+\mu)/\mu)\tilde{D} - \beta\tilde{E} = \beta\tilde{I}$. Therefore (3.6) holds for $u = e^{-(\beta/\mu)\tilde{D}}$.

Conversely we suppose that the function $\psi(u)$ satisfies (3.5), (3.6), and let $e^{-(\beta/\mu)\alpha} < u < e^{-(\beta/\mu)\tilde{D}}$. Differentiating (3.6) with respect to u yields

$$\begin{aligned} \psi'(u) = &-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}} + \frac{\gamma+\mu}{u} - \beta\tilde{E}e^{-\delta\varphi(u)}\big(-\delta\varphi'(u)\big) \\ &-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}\left(e^{-\delta\varphi(u)}\big(-\delta\varphi'(u)\big)\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv - 1\right) \\ = &\frac{\gamma+\mu}{u} - \beta\delta\tilde{E}e^{-\delta\varphi(u)}\frac{1}{u\psi(u)} \\ &-\beta\delta\tilde{S}e^{(\beta/\mu)\tilde{D}}\frac{1}{u\psi(u)}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv. \end{aligned} \tag{3.15}$$

It follows from (3.6) that

$$-\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv = \psi(u) - P(u) + \beta\tilde{E}e^{-\delta\varphi(u)}. \tag{3.16}$$

We combine (3.15) with (3.16) to obtain

$$\begin{aligned}\psi'(u) &= \frac{\gamma+\mu}{u} - \beta\delta\tilde{E}e^{-\delta\varphi(u)}\frac{1}{u\psi(u)} + \delta\,\frac{\psi(u)-P(u)+\beta\tilde{E}e^{-\delta\varphi(u)}}{u\psi(u)}\\ &= \frac{\gamma+\mu}{u} - \delta\,\frac{P(u)-\psi(u)}{u\psi(u)}\\ &= \frac{\delta+\gamma+\mu}{u} - \delta\,\frac{P(u)}{u\psi(u)}\end{aligned}$$

and consequently, $\psi(u)$ satisfies (3.3) for $e^{-(\beta/\mu)\alpha} < u < e^{-(\beta/\mu)\tilde{D}}$. It is easy to see from (3.6) that

$$\begin{aligned}\psi\big(e^{-(\beta/\mu)\tilde{D}}\big) &= \beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S} - (\beta(\gamma+\mu)/\mu)\tilde{D} - \beta\tilde{E}\\ &= \beta N - \beta\tilde{R} - \beta\tilde{D} - \beta\tilde{S} - \beta\tilde{E}\\ &= \beta(N - \tilde{R} - \tilde{D} - \tilde{S} - \tilde{E}) = \beta\tilde{I}\end{aligned}$$

in view of $\varphi\big(e^{-(\beta/\mu)\tilde{D}}\big) = 0$, and therefore (3.4) is satisfied. □

Proposition 3.6. *Let $\psi(u)$ be the unique positive solution of the initial value problem* (3.3), (3.4)*, then we obtain the following inequalities*:

$$\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u > \psi(u) > 0, \tag{3.17}$$

$$\begin{aligned}&\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u\\ &\quad > \beta\left(\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\right) > 0\end{aligned} \tag{3.18}$$

for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$.

Proof. Since $\psi(u) > 0$ in $(e^{-(\beta/\mu)\alpha}, e^{-(\beta/\mu)\tilde{D}}]$, the relation (3.6) in Lemma 3.5 means

$$\begin{aligned}&\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u\\ &\quad > \beta\left(\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi(v)}dv\right)\end{aligned}$$

for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$. It is clear that

$$\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv > 0 \tag{3.19}$$

for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$, and therefore (3.18) follows. Since (3.19) holds, the relation (3.6) implies that

$$\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u > \psi(u) > 0$$

for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$, which is the desired inequality (3.17). □

Proposition 3.7. *Let $\psi(u)$ be the unique positive solution of the initial value problem* (3.3), (3.4)*, then we see that*

$$\lim_{u \to e^{-(\beta/\mu)\alpha}+0} \psi(u) = 0, \tag{3.20}$$

$$\lim_{u \to e^{-(\beta/\mu)\alpha}+0} \left(\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)} \int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv \right) = 0 \tag{3.21}$$

(*cf*. Figure 3.4).

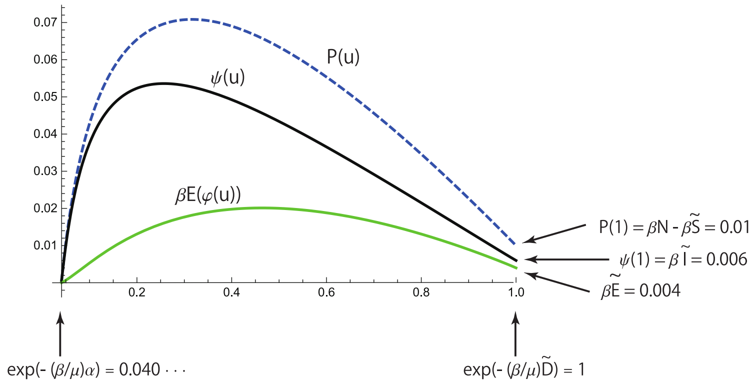

Figure 3.4: Variations of $P(u) = \beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma + \mu)\log u$ (dashed curve), $\beta E(\varphi(u))$ (green curve), and $\psi(u)$ (solid curve) obtained by the numerical integration of the initial value problem (3.3), (3.4) for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05, \delta = 0.2, \mu = 0.01$ and $\alpha = 160.2\ldots$ In this case we have $e^{-(\beta/\mu)\alpha} = 0.040\ldots$, $e^{-(\beta/\mu)\tilde{D}} = 1$, $P(1) = \beta N - \beta\tilde{S} = 0.01$, $\psi(1) = \beta\tilde{I} = 0.006$ and $\beta\tilde{E} = 0.004$. Moreover, $\lim_{u \to e^{-(\beta/\mu)\alpha}+0} P(u) = 0$, $\lim_{u \to e^{-(\beta/\mu)\alpha}+0} \psi(u) = 0$, and $\lim_{u \to e^{-(\beta/\mu)\alpha}+0} \beta E(\varphi(u)) = \lim_{u \to e^{-(\beta/\mu)\alpha}+0} (P(u) - \psi(u)) = 0$.

Proof. Since

$$\begin{aligned}
&\lim_{u \to e^{-(\beta/\mu)\alpha}+0} \left(\beta N - \beta\tilde{R} + \frac{\beta\gamma}{\mu}\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma + \mu)\log u \right) \\
&= \lim_{x \to \alpha - 0} \beta \left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)x} - \frac{\gamma + \mu}{\mu}x \right) = 0
\end{aligned}$$

by Lemma 3.1, Proposition 3.6 implies that (3.20) and (3.21) hold by taking the limit as $u \to e^{-(\beta/\mu)\alpha} + 0$ in (3.17) and (3.18). □

Lemma 3.8. *Let $\psi(u)$ be the unique positive solution of the initial value problem* (3.3), (3.4)*. Then there exists the inverse function $\varphi^{-1}(t) \in C^1(0, \infty)$ of the function*

$$t = \varphi(u) = \int_u^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)} \tag{3.22}$$

for $e^{-(\beta/\mu)\alpha} < u \leq e^{-(\beta/\mu)\tilde{D}}$, such that $\varphi^{-1}(t)$ is decreasing on $[0, \infty)$, $\varphi^{-1}(0) = e^{-(\beta/\mu)\tilde{D}}$ and $\lim_{t \to \infty} \varphi^{-1}(t) = e^{-(\beta/\mu)\alpha}$.

Proof. We easily see that $\varphi(u) \in C^1(e^{-(\beta/\mu)\alpha}, e^{-(\beta/\mu)\tilde{D}})$, $\varphi(u)$ is decreasing in $(e^{-(\beta/\mu)\alpha}, e^{-(\beta/\mu)\tilde{D}}]$ and $\varphi(e^{-(\beta/\mu)\tilde{D}}) = 0$. We divide (3.3) by $(\delta+\gamma+\mu)\psi(u)^2$ to obtain

$$\frac{1}{u\psi(u)} = \frac{\delta}{\delta+\gamma+\mu}\frac{P(u)}{u\psi(u)^2} + \frac{1}{\delta+\gamma+\mu}\frac{\psi'(u)}{\psi(u)}, \tag{3.23}$$

and therefore

$$\begin{aligned}\varphi(u) &= \int_u^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)} \\ &= \frac{\delta}{\delta+\gamma+\mu}\int_u^{e^{-(\beta/\mu)\tilde{D}}} \frac{P(\xi)}{\xi\psi(\xi)^2}d\xi + \frac{1}{\delta+\gamma+\mu}\int_u^{e^{-(\beta/\mu)\tilde{D}}} \frac{\psi'(\xi)}{\psi(\xi)}\,d\xi \\ &\geq \frac{1}{\delta+\gamma+\mu}\big(\log(\beta\tilde{I}) - \log\psi(u)\big),\end{aligned} \tag{3.24}$$

where $P(u)$ is defined by (3.8). We see that $\lim_{u\to e^{-(\beta/\mu)\alpha}+0}\log\psi(u) = -\infty$ in view of (3.20), and that $\lim_{u\to e^{-(\beta/\mu)\alpha}+0}\varphi(u) = \infty$ by taking the limit as $u \to e^{-(\beta/\mu)\alpha}+0$ in (3.24). Hence there exists the inverse function $\varphi^{-1}(t)$ which has the desired properties. □

The following is our main theorem.

Theorem 3.9. *The function* $(S(t), E(t), I(t), R(t), D(t))$ *defined by*

$$S(t) = \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t), \tag{3.25}$$

$$E(t) = \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv, \tag{3.26}$$

$$\begin{aligned}I(t) = {}& N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t) - \tilde{E}e^{-\delta t} \\ & - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv,\end{aligned} \tag{3.27}$$

$$R(t) = -\frac{\gamma}{\beta}\log\varphi^{-1}(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}, \tag{3.28}$$

$$D(t) = -\frac{\mu}{\beta}\log\varphi^{-1}(t) \tag{3.29}$$

is a solution of the initial value problem (1.1)–(1.6), *where* $\varphi(u)$ *and* $\varphi^{-1}(t)$ *are given in Lemma* 3.8.

Proof. First note that

$$\begin{aligned}(\varphi^{-1}(t))' &= \frac{1}{\varphi'(u)}\Big|_{u=\varphi^{-1}(t)} = -u\psi(u)\Big|_{u=\varphi^{-1}(t)} \\ &= -\varphi^{-1}(t)\psi(\varphi^{-1}(t)) = -\beta\varphi^{-1}(t)I(t)\end{aligned} \tag{3.30}$$

by taking account of (3.6) and (3.27). We see from (3.25) and (3.30) that

$$\begin{aligned}S'(t) &= \tilde{S}e^{(\beta/\mu)\tilde{D}}(\varphi^{-1}(t))' \\ &= -\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)I(t) \\ &= -\beta S(t)I(t)\end{aligned} \tag{3.31}$$

and therefore (1.1) follows. A direct calculation yields

$$\begin{aligned}
E'(t) = & -\delta\tilde{E}e^{-\delta t} \\
& + \tilde{S}e^{(\beta/\mu)\tilde{D}}\left(-\delta e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv + e^{-\delta t}\left(-e^{\delta t}\left(\varphi^{-1}(t)\right)'\right)\right) \\
= & -\delta\tilde{E}e^{-\delta t} - \delta\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv - \tilde{S}e^{(\beta/\mu)\tilde{D}}\left(\varphi^{-1}(t)\right)' \\
= & -\delta E(t) + \beta S(t)I(t)
\end{aligned} \tag{3.32}$$

in view of (3.26) and (3.31), and hence (1.2) is satisfied. An easy computation shows that

$$\begin{aligned}
I'(t) = & -\tilde{S}e^{(\beta/\mu)\tilde{D}}\left(\varphi^{-1}(t)\right)' + \frac{\gamma+\mu}{\beta}\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} \\
& - \left(\tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\right)' \\
= & \beta S(t)I(t) + \frac{\gamma+\mu}{\beta}(-\beta I(t)) - E'(t) \\
= & \beta S(t)I(t) - (\gamma+\mu)I(t) - \left(-\delta E(t) + \beta S(t)I(t)\right) \\
= & \delta E(t) - \gamma I(t) - \mu I(t)
\end{aligned} \tag{3.33}$$

in view of (3.30)–(3.32). Thus, (1.3) holds. It is easily seen from (3.30) that

$$R'(t) = -\frac{\gamma}{\beta}\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} = -\frac{\gamma}{\beta}(-\beta I(t)) = \gamma I(t)$$

which is the equation (1.4). Similarly we obtain

$$D'(t) = -\frac{\mu}{\beta}\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} = -\frac{\mu}{\beta}(-\beta I(t)) = \mu I(t)$$

which is the desired equation (1.5). It is easy to see that

$$\begin{aligned}
S(0) &= \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(0) = \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\tilde{D}} = \tilde{S}, \\
E(0) &= \tilde{E} + \tilde{S}e^{(\beta/\mu)\tilde{D}}\int_{\varphi^{-1}(0)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv = \tilde{E}, \\
I(0) &= N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S} + \frac{\gamma+\mu}{\beta}\left(-\frac{\beta}{\mu}\tilde{D}\right) - \tilde{E} \\
&= N - \tilde{R} - \tilde{S} - \tilde{D} - \tilde{E} = \tilde{I}, \\
R(0) &= -\frac{\gamma}{\beta}\left(-\frac{\beta}{\mu}\tilde{D}\right) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} = \tilde{R}, \\
D(0) &= -\frac{\mu}{\beta}\left(-\frac{\beta}{\mu}\tilde{D}\right) = \tilde{D}
\end{aligned}$$

in light of $\varphi^{-1}(0) = e^{-(\beta/\mu)\tilde{D}}$. Therefore, (1.6) is satisfied. □

Theorem 3.10. *Let* $(S(t), E(t), I(t), R(t), D(t))$ *be the exact solution* (3.25)–(3.29) *of the initial value problem* (1.1)–(1.6)*. Then,* $(\hat{S}(u), \hat{E}(u), \hat{I}(u), \hat{R}(u), \hat{D}(u))$ *defined by*

$$\left(\hat{S}(u), \hat{E}(u), \hat{I}(u), \hat{R}(u), \hat{D}(u)\right) := \left(S(\varphi(u)), E(\varphi(u)), I(\varphi(u)), R(\varphi(u)), D(\varphi(u))\right)$$

satisfies the linear differential system

$$\frac{d\hat{S}(u)}{du} = \frac{\hat{S}(u)}{u}, \tag{3.34}$$

$$\frac{d\hat{E}(u)}{du} - \frac{\delta}{u\psi(u)}\hat{E}(u) = -\frac{\hat{S}(u)}{u}, \tag{3.35}$$

$$\frac{d\hat{I}(u)}{du} - \frac{\gamma+\mu}{\beta}\frac{1}{u} = -\frac{\delta}{u\psi(u)}\hat{E}(u), \tag{3.36}$$

$$\frac{d\hat{R}(u)}{du} = -\frac{\gamma}{\beta}\frac{1}{u}, \tag{3.37}$$

$$\frac{d\hat{D}(u)}{du} = -\frac{\mu}{\beta}\frac{1}{u} \tag{3.38}$$

for $u \in (e^{-(\beta/\mu)\alpha}, e^{-(\beta/\mu)\tilde{D}})$*, and the initial condition*

$$\hat{S}\left(e^{-(\beta/\mu)\tilde{D}}\right) = \tilde{S}, \tag{3.39}$$

$$\hat{E}\left(e^{-(\beta/\mu)\tilde{D}}\right) = \tilde{E}, \tag{3.40}$$

$$\hat{I}\left(e^{-(\beta/\mu)\tilde{D}}\right) = \tilde{I}, \tag{3.41}$$

$$\hat{R}\left(e^{-(\beta/\mu)\tilde{D}}\right) = \tilde{R}. \tag{3.42}$$

$$\hat{D}\left(e^{-(\beta/\mu)\tilde{D}}\right) = \tilde{D}. \tag{3.43}$$

Proof. It follows from (3.30) that

$$\hat{I}(u) = I(\varphi(u)) = \frac{1}{\beta}\psi(u). \tag{3.44}$$

Since $S(t)$ satisfies (1.1), we obtain

$$S'(\varphi(u)) = -\beta S(\varphi(u))I(\varphi(u)) = -\beta\hat{S}(u)\hat{I}(u).$$

Therefore we arrive at

$$\begin{aligned}\frac{d\hat{S}(u)}{du} &= \frac{dS(t)}{dt}\Big|_{t=\varphi(u)}\varphi'(u) = S'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right)\\ &= \left(-\beta\hat{S}(u)\hat{I}(u)\right)\left(-\frac{1}{u\psi(u)}\right)\\ &= \frac{\hat{S}(u)}{u}\end{aligned}$$

in light of (3.44), and hence (3.34) holds. Using (1.2) and (3.44), we get

$$\begin{aligned}\frac{d\hat{E}(u)}{du} &= \frac{dE(t)}{dt}\Big|_{t=\varphi(u)}\varphi'(u) = E'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right)\\ &= \left(\beta\hat{S}(u)\hat{I}(u) - \delta\hat{E}(u)\right)\left(-\frac{1}{u\psi(u)}\right)\\ &= -\frac{\hat{S}(u)}{u} + \frac{\delta}{u\psi(u)}\hat{E}(u),\end{aligned}$$

which is equal to (3.35). We observe, using (1.3), that

$$
\begin{aligned}
\frac{d\hat{I}(u)}{du} &= \frac{dI(t)}{dt}\Big|_{t=\varphi(u)} \varphi'(u) \\
&= \left(\delta\hat{E}(u) - \gamma\hat{I}(u) - \mu\hat{I}(u)\right)\left(-\frac{1}{u\psi(u)}\right) \\
&= -\delta\frac{\hat{E}(u)}{u\psi(u)} + (\gamma+\mu)\frac{\hat{I}(u)}{u\psi(u)} \\
&= -\frac{\delta}{u\psi(u)}\hat{E}(u) + \frac{\gamma+\mu}{\beta}\frac{1}{u},
\end{aligned}
$$

and therefore (3.36) follows. We are led to

$$
\begin{aligned}
\frac{d\hat{R}(u)}{du} &= \frac{dR(t)}{dt}\Big|_{t=\varphi(u)} \varphi'(u) = R'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right) \\
&= \gamma\hat{I}(u)\left(-\frac{1}{u\psi(u)}\right) \\
&= -\frac{\gamma}{\beta}\frac{1}{u}
\end{aligned}
$$

by use of (1.4) and (3.44). Thus (3.37) is obtained. Similarly we have

$$
\begin{aligned}
\frac{d\hat{D}(u)}{du} &= \frac{dD(t)}{dt}\Big|_{t=\varphi(u)} \varphi'(u) = D'(\varphi(u))\left(-\frac{1}{u\psi(u)}\right) \\
&= \mu\hat{I}(u)\left(-\frac{1}{u\psi(u)}\right) \\
&= -\frac{\mu}{\beta}\frac{1}{u},
\end{aligned}
$$

which is the desired equation (3.38). It is easily seen that

$$
\begin{aligned}
\hat{S}\left(e^{-(\beta/\mu)\tilde{D}}\right) &= S\left(\varphi\left(e^{-(\beta/\mu)\tilde{D}}\right)\right) = S(0) = \tilde{S}, \\
\hat{E}\left(e^{-(\beta/\mu)\tilde{D}}\right) &= E\left(\varphi\left(e^{-(\beta/\mu)\tilde{D}}\right)\right) = E(0) = \tilde{E}, \\
\hat{I}\left(e^{-(\beta/\mu)\tilde{D}}\right) &= I\left(\varphi\left(e^{-(\beta/\mu)\tilde{D}}\right)\right) = I(0) = \tilde{I}, \\
\hat{R}\left(e^{-(\beta/\mu)\tilde{D}}\right) &= R\left(\varphi\left(e^{-(\beta/\mu)\tilde{D}}\right)\right) = R(0) = \tilde{R}, \\
\hat{D}\left(e^{-(\beta/\mu)\tilde{D}}\right) &= D\left(\varphi\left(e^{-(\beta/\mu)\tilde{D}}\right)\right) = D(0) = \tilde{D}.
\end{aligned}
$$

Hence, (3.39)–(3.43) are satisfied. □

Theorem 3.11. *Solving the initial value problem* (3.34)–(3.43)*, we obtain the parametric solution* (2.5)–(2.9) *for* $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$.

Proof. Since (3.34) is equivalent to

$$\frac{d}{du}\left(\frac{1}{u}\hat{S}(u)\right) = 0,$$

we have

$$\hat{S}(u) = ku$$

for some constant k. We see from (3.39) that

$$\hat{S}\left(e^{-(\beta/\mu)\tilde{D}}\right) = ke^{-(\beta/\mu)\tilde{D}} = \tilde{S}$$

which implies

$$k = \tilde{S}e^{(\beta/\mu)\tilde{D}}.$$

Therefore we obtain

$$\hat{S}(u) = \tilde{S}e^{(\beta/\mu)\tilde{D}}u. \tag{3.45}$$

It follows from (3.45) that

$$-\frac{\hat{S}(u)}{u} = -\tilde{S}e^{(\beta/\mu)\tilde{D}}$$

and hence (3.35) reduces to

$$\frac{d\hat{E}(u)}{du} - \frac{\delta}{u\psi(u)}\hat{E}(u) = -\tilde{S}e^{(\beta/\mu)\tilde{D}} \tag{3.46}$$

which can be rewritten as

$$\frac{d\hat{E}(u)}{du} + \delta\varphi'(u)\hat{E}(u) = -\tilde{S}e^{(\beta/\mu)\tilde{D}}. \tag{3.47}$$

Multiplying (3.47) by $e^{\delta\varphi(u)}$ gives

$$\frac{d}{du}\left(e^{\delta\varphi(u)}\hat{E}(u)\right) = -\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{\delta\varphi(u)},$$

and an integration of the above on $[u, e^{-(\beta/\gamma)\tilde{R}}]$ yields

$$\hat{E}(u) = e^{-\delta\varphi(u)}\left(\tilde{E} + \tilde{S}e^{(\beta/\mu)\tilde{D}}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\right),$$

which is equal to (2.6). Multiplying (3.36) by β, we have

$$\frac{d(\beta\hat{I}(u))}{du} - \frac{\gamma+\mu}{u} = -\frac{\beta\delta}{u\psi(u)}\hat{E}(u). \tag{3.48}$$

Define $z(u)$ by

$$z(u) := \beta\hat{I}(u) - \left(\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u\right),$$

then we obtain

$$\frac{dz(u)}{du} = \frac{d(\beta\hat{I}(u))}{du} + \beta\tilde{S}e^{(\beta/\mu)\tilde{D}} - \frac{\gamma+\mu}{u}. \tag{3.49}$$

Combining (3.48) with (3.49), we get

$$\begin{aligned}\frac{dz(u)}{du} &= \beta\tilde{S}e^{(\beta/\mu)\tilde{D}} - \frac{\beta\delta}{u\psi(u)}\hat{E}(u)\\ &= -\beta\left(-\tilde{S}e^{(\beta/\mu)\tilde{D}} + \frac{\delta}{u\psi(u)}\hat{E}(u)\right).\end{aligned} \tag{3.50}$$

It follows from (3.46) and (3.50) that

$$\frac{dz(u)}{du} = -\beta \frac{d\hat{E}(u)}{du},$$

and therefore

$$z(u) = -\beta\hat{E}(u) + k$$

for some constant k. Since

$$\begin{aligned} z\big(e^{-(\beta/\mu)\tilde{D}}\big) &= \beta\hat{I}\big(e^{-(\beta/\mu)\tilde{D}}\big) - \big(\beta N - \beta\tilde{R} - \beta\tilde{S} - \beta\tilde{D}\big) \\ &= \beta\tilde{I} - \big(\beta N - \beta\tilde{R} - \beta\tilde{S} - \beta\tilde{D}\big) = -\beta\tilde{E} \end{aligned}$$

and $-\beta\hat{E}\big(e^{-(\beta/\mu)\tilde{D}}\big) = -\beta\tilde{E}$, we see that $k = 0$, and therefore $z(u) = -\beta\hat{E}(u)$, i.e.,

$$\beta\hat{I}(u) = \big(\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u\big) - \beta\hat{E}(u),$$

which is equivalent to (2.7). Solving (3.37) yields

$$\hat{R}(u) = -\frac{\gamma}{\beta}\log u + k$$

for some constant k. The initial condition (3.42) implies

$$\hat{R}\left(e^{-(\beta/\mu)\tilde{D}}\right) = -\frac{\gamma}{\beta}\log e^{-(\beta/\mu)\tilde{D}} + k = \frac{\gamma}{\mu}\tilde{D} + k = \tilde{R}$$

and hence $k = \tilde{R} - (\gamma/\mu)\tilde{D}$. Hence we obtain

$$\hat{R}(u) = -\frac{\gamma}{\beta}\log u + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}.$$

Similarly we find that

$$\hat{D}(u) = -\frac{\mu}{\beta}\log u. \qquad \square$$

Remark 3.12. Let $I(t)$ be given by (3.27). Then $I(t)$ can be represented in the simple form

$$I(t) = \frac{1}{\beta}\psi\big(\varphi^{-1}(t)\big)$$

by taking account of (3.6) and (3.27).

4 Various properties of solution

This section is devoted to various properties of solution by investigating the exact solution of the initial value problem (1.1)–(1.6).

Theorem 4.1. *Let $D(t)$ be given by (3.29). Then we find that $D(\infty) = \alpha$,*

$$D(\infty) = N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(\infty)} - \frac{\gamma}{\mu}D(\infty), \tag{4.1}$$

and that $D(t)$ is an increasing function on $[0,\infty)$ such that

$$\tilde{D} \le D(t) < \alpha = D(\infty).$$

Proof. We easily see that

$$\begin{aligned} D(\infty) = \lim_{t\to\infty} D(t) &= \lim_{t\to\infty} -\frac{\mu}{\beta}\log \varphi^{-1}(t) \\ &= \lim_{u\to e^{-(\beta/\mu)\alpha}+0} -\frac{\mu}{\beta}\log u \\ &= \alpha. \end{aligned}$$

Since $\alpha = D(\infty)$, the identity (4.1) follows from the definition of α (see Lemma 3.1). In light of $e^{-(\beta/\mu)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\mu)\tilde{D}}$, we obtain

$$-\frac{\mu}{\beta}\log e^{-(\beta/\mu)\tilde{D}} \le D(t) < -\frac{\mu}{\beta}\log e^{-(\beta/\mu)\alpha}$$

or

$$\tilde{D} \le D(t) < \alpha = D(\infty).$$

It is easy to check that $D(t)$ is increasing on $[0,\infty)$ in view of the fact that $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$. □

Theorem 4.2. *Let $S(t)$ be given by (3.25). Then we deduce that*

$$S(\infty) = \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(\infty)}, \tag{4.2}$$

and that $S(t)$ is a decreasing function on $[0,\infty)$ such that

$$\tilde{S} \ge S(t) > \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\alpha} = S(\infty).$$

Proof. The identity (4.2) follows from

$$\begin{aligned} S(\infty) = \lim_{t\to\infty} S(t) &= \lim_{t\to\infty} \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) \\ &= \lim_{u\to e^{-(\beta/\mu)\alpha}+0} \tilde{S}e^{(\beta/\mu)\tilde{D}}u \\ &= \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\alpha} \\ &= \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(\infty)}. \end{aligned}$$

Since $e^{-(\beta/\mu)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\mu)\tilde{D}}$, we have

$$\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\alpha} < \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) \le \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\tilde{D}}.$$

Therefore we get

$$\tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)\alpha} < S(t) \le \tilde{S}.$$

Since $\varphi^{-1}(t)$ is decreasing on $[0,\infty)$, we observe that $S(t)$ is also decreasing on $[0,\infty)$. □

Theorem 4.3. *Let $R(t)$ be given by (3.28). Then we conclude that*

$$R(\infty) = \frac{\gamma}{\mu}D(\infty) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}, \tag{4.3}$$

and that $R(t)$ is an increasing function on $[0,\infty)$ such that

$$\tilde{R} \le R(t) < R(\infty).$$

Proof. We obtain

$$\begin{aligned} R(\infty) = \lim_{t\to\infty} R(t) &= \lim_{t\to\infty}\left(-\frac{\gamma}{\beta}\log\varphi^{-1}(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}\right) \\ &= \lim_{u\to e^{-(\beta/\mu)\alpha}+0}\left(-\frac{\gamma}{\beta}\log u + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}\right) \\ &= \frac{\gamma}{\mu}\alpha + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} \\ &= \frac{\gamma}{\mu}D(\infty) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}. \end{aligned}$$

Since $e^{-(\beta/\mu)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\mu)\tilde{D}}$, we get

$$\frac{\gamma}{\mu}\tilde{D} + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} \le R(t) < \frac{\gamma}{\mu}\alpha + \tilde{R} - \frac{\gamma}{\mu}\tilde{D},$$

or

$$\tilde{R} \le R(t) < \frac{\gamma}{\mu}D(\infty) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} = R(\infty).$$ □

Theorem 4.4. *Let $E(t)$ be given by (3.26). Then we find that*

$$E(\infty) = 0,$$
$$E(t) > 0 \quad \text{on } [0,\infty),$$

and $E(t)$ has the maximum $\max_{t\ge 0} E(t)$ at some $t = T_1 \in \{T;\, E'(T) = 0\}$, where

$$\begin{aligned} E'(T) = &\left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(T)\right)\psi(\varphi^{-1}(T)) \\ &- \delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(T) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(T)\right). \end{aligned}$$

Proof. We easily check that

$$\begin{aligned} E(\infty) &= \lim_{t\to\infty} E(t) \\ &= \lim_{u\to e^{-(\beta/\mu)\alpha}+0}\left(\tilde{E}e^{-\delta\varphi(u)} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta\varphi(u)}\int_u^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\right) \\ &= 0 \end{aligned}$$

in light of of (3.20) in Proposition 3.7. Since $e^{-(\beta/\mu)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\mu)\tilde{D}}$ $(t \ge 0)$ and $\hat{E}(u) > 0$ for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$ (cf. (3.19)), it is easily seen that $E(t) = \hat{E}(\varphi^{-1}(t)) > 0$ on $[0,\infty)$. The hypothesis (A_2) implies that the right differential derivative $E'_+(0)$ is positive because

$$E'_+(0) = \lim_{t\to+0} E'(t) = \lim_{t\to+0}\left(\beta S(t)I(t) - \delta E(t)\right) = \beta\tilde{S}\tilde{I} - \delta\tilde{E} > 0.$$

Since the definition of $E'_+(0)$ implies

$$0 < E'_+(0) = \lim_{t\to+0}\frac{E(t) - E(0)}{t} = \lim_{t\to+0}\frac{E(t) - \tilde{E}}{t},$$

we see that for $\varepsilon = (1/2)E'_+(0) > 0$ there exists a number $\delta_\varepsilon > 0$ such that

$$\left|\frac{E(t) - \tilde{E}}{t} - E'_+(0)\right| < \frac{1}{2}E'_+(0)$$

holds for $0 < t < \delta_\varepsilon$, and hence

$$\frac{1}{2}E'_+(0) < \frac{E(t) - \tilde{E}}{t}$$

or

$$E(t) > \tilde{E} + \frac{1}{2}E'_+(0)t > \tilde{E}$$

holds for $0 < t < \delta_\varepsilon$. Since $E(\infty) = 0$, there exists a number $\tilde{T}$ such that $E(\tilde{T}) = \tilde{E}$ and $E(t) \le \tilde{E}$ for $t \ge \tilde{T}$. Therefore there exists $\max_{0\le t\le \tilde{T}} E(t) = E(T_1)$ $(> \tilde{E})$ at some $t = T_1$ $(< \tilde{T})$. Since $E(t) \le \tilde{E}$ for $t \ge \tilde{T}$, we observe that $\max_{t\ge 0} E(t) = \max_{0\le t\le \tilde{T}} E(t) = E(T_1)$. It is obvious that $E'(T_1) = 0$. It can be shown from (3.25)–(3.27) and (3.44) that

$$\begin{aligned}
E'(t) &= -\delta E(t) + \beta S(t)I(t) \\
&= -\delta E(t) + \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi^{-1}(t)\psi(\varphi^{-1}(t)) \\
&= -\delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t) - I(t)\right) \\
&\quad + \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)\psi(\varphi^{-1}(t)) \\
&= \left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)\right)\psi(\varphi^{-1}(t)) \\
&\quad - \delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t)\right). \qquad (4.4)
\end{aligned}$$

□

Remark 4.5. If u_1 is a unique solution of the equation

$$\left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}u\right)\psi(u) = \delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + \frac{\gamma+\mu}{\beta}\log u\right),$$

then we get

$$T_1 = \varphi(u_1) = \int_{u_1}^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)}$$

in view of (3.22) (*cf.* Figure 4.1).

In case $E'(T_1) = 0$, we obtain $\beta S(T_1)I(T_1) = \delta E(T_1)$ by (1.2), and therefore $E(T_1) = (\beta/\delta)S(T_1)I(T_1)$. Hence, in Theorem 4.4 we see that

$$\max_{t\ge 0} E(t) = E(T_1) = \frac{\beta}{\delta}S(T_1)I(T_1).$$

Letting

$$\Psi(u) := \left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}u\right)\psi(u),$$

we observe that $\Psi(u)$ is a solution of the initial value problem for the Abel differential equation

$$\begin{aligned}
&\Psi'(u)\Psi(u) - \frac{\tilde{S}e^{(\beta/\mu)\tilde{D}}}{(\delta/\beta) + \tilde{S}e^{(\beta/\mu)\tilde{D}}u}\Psi(u)^2 - \frac{\delta+\gamma+\mu}{u}\left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}u\right)\Psi(u) \\
&= -\delta\left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}u\right)^2 \\
&\quad \times \frac{\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}u + (\gamma+\mu)\log u}{u} \qquad (4.5)
\end{aligned}$$

for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$, with the initial condition

$$\Psi\left(e^{-(\beta/\mu)\tilde{D}}\right) = \beta\left(\frac{\delta}{\beta} + \tilde{S}\right)\tilde{I}. \tag{4.6}$$

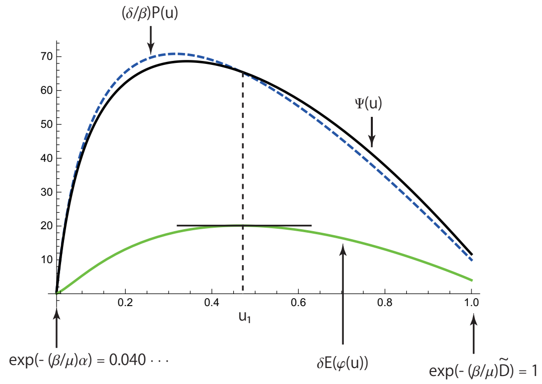

Figure 4.1: Variations of $(\delta/\beta)P(u) = \delta(N - \tilde{R} + (\gamma/\mu)\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + ((\gamma+\mu)/\beta)\log u)$ (dashed curve), $\delta E(\varphi(u))$ (green curve) and $\Psi(u)$ (solid curve) obtained by the numerical integration of the initial value problem (4.5), (4.6) for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05$, $\delta = 0.2$ and $\mu = 0.01$. In this case we see that there exists a unique u_1 such that $(\delta/\beta)P(u_1) = \Psi(u_1)$, and that T_1 is calculated by $T_1 = \varphi(u_1) = \int_{u_1}^{1} \frac{d\xi}{\xi\psi(\xi)}$, where $\psi(u)$ is a unique positive solution of the initial value problem $\psi'\psi - \frac{0.26}{u}\psi = -0.2\,\frac{0.2-0.19u+0.06\log u}{u}$ $(0.040\ldots < u < 1)$, $\psi(1) = 0.006$.

Theorem 4.6. *Let $I(t)$ be given by (3.27). Then we see that*

$$I(\infty) = 0,$$
$$I(t) > 0 \quad \text{on } [0,\infty),$$

and $I(t)$ has the maximum $\max_{t\ge 0} I(t)$ at some $t = T_2 \in \{T;\ I'(T) = 0\}$, where

$$I'(T) = -\frac{\delta+\gamma+\mu}{\beta}\psi(\varphi^{-1}(T)) + \delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(T) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(T)\right).$$

Proof. It follows from (3.20) and (3.30) that

$$\begin{aligned} I(\infty) &= \lim_{t\to\infty} I(t) = \lim_{t\to\infty}\frac{1}{\beta}\psi(\varphi^{-1}(t)) \\ &= \lim_{u\to e^{-(\beta/\mu)\alpha}+0}\frac{1}{\beta}\psi(u) \\ &= 0. \end{aligned}$$

Since $e^{-(\beta/\mu)\alpha} < \varphi^{-1}(t) \le e^{-(\beta/\mu)\tilde{D}}$ $(t \ge 0)$ and $\psi(u) > 0$ for $e^{-(\beta/\mu)\alpha} < u \le e^{-(\beta/\mu)\tilde{D}}$, we find that $I(t) = (1/\beta)\psi(\varphi^{-1}(t)) > 0$ on $[0,\infty)$. The hypothesis (A_3) implies that the right differential derivative $I'_+(0)$ is positive because

$$I'_+(0) = \lim_{t\to+0} I'(t) = \lim_{t\to+0}\left(\delta E(t) - \gamma I(t) - \mu I(t)\right) = \delta\tilde{E} - (\gamma+\mu)\tilde{I} > 0,$$

and therefore there exists a number $\delta_1 > 0$ such that $I(t) > \tilde{I}$ in $(0, \delta_1)$ as in the proof of Theorem 4.3. Since $I(\infty) = 0$, we can use the same arguments as in the proof of Theorem 4.3 to conclude that there exists the maximum $\max_{t\geq 0} I(t) = I(T_2)$ for some T_2. Then $I'(T_2) = 0$, and we obtain

$$\begin{aligned} I'(t) &= \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)\psi(\varphi^{-1}(t)) - \frac{\gamma+\mu}{\beta}\psi(\varphi^{-1}(t)) - E'(t) \\ &= \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)\psi(\varphi^{-1}(t)) - \frac{\gamma+\mu}{\beta}\psi(\varphi^{-1}(t)) \\ &\quad - \left[\left(\frac{\delta}{\beta} + \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)\right)\psi(\varphi^{-1}(t))\right. \\ &\quad \left. -\delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t)\right)\right] \\ &= -\frac{\delta+\gamma+\mu}{\beta}\psi(\varphi^{-1}(t)) \\ &\quad + \delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t)\right) \end{aligned}$$

in light of (3.30), (3.33), (3.44) and (4.4). □

Remark 4.7. In case u_2 is a unique solution of the equation

$$\frac{\delta+\gamma+\mu}{\beta}\psi(u) = \delta\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + \frac{\gamma+\mu}{\beta}\log u\right),$$

then we get

$$T_2 = \varphi(u_2) = \int_{u_2}^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)}$$

(*cf.* Figure 4.2). If $I'(T_2) = 0$, (1.3) implies that $\delta E(T_2) = (\gamma+\mu)I(T_2)$, and in Theorem 4.6 we see that

$$\max_{t\geq 0} I(t) = I(T_2) = \frac{\delta}{\gamma+\mu}E(T_2).$$

Theorem 4.8. *The function $E(t) + I(t)$ has the maximum*

$$\max_{t\geq 0}\,(E(t)+I(t)) = \tilde{S} + \tilde{E} + \tilde{I} - \frac{\gamma+\mu}{\beta}\left(1 + \log\tilde{S} - \log\frac{\gamma+\mu}{\beta}\right)$$

at

$$t = T_3 := \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}}\right) = \int_{(\gamma+\mu)/(\beta\tilde{S}e^{(\beta/\mu)\tilde{D}})}^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)} = S^{-1}\left(\frac{\gamma+\mu}{\beta}\right).$$

Moreover, $E(t) + I(t)$ is increasing in $[0, T_3)$ and is decreasing in (T_3, ∞).

Proof. We see from (3.26) and (3.27) that

$$E(t) + I(t) = N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t). \tag{4.7}$$

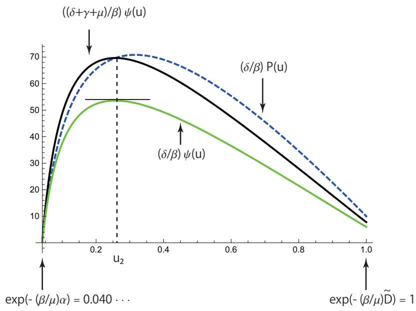

Figure 4.2: Variations of $(\delta/\beta)P(u) = \delta(N - \tilde{R} + (\gamma/\mu)\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + ((\gamma + \mu)/\beta)\log u)$ (dashed curve), $((\delta+\gamma+\mu)/\beta)\psi(u)$ (solid curve), and $(\delta/\beta)\psi(u)$ (green curve) obtained by the numerical integration of the initial value problem (4.5), (4.6) for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05$, $\delta = 0.2$ and $\mu = 0.01$. In this case we find that there exists a unique u_2 such that $(\delta/\beta)P(u_2) = ((\delta+\gamma+\mu)/\beta)\psi(u_2)$, and that T_2 is calculated by $T_2 = \varphi(u_2) = \int_{u_2}^{1} \frac{d\xi}{\xi\psi(\xi)}$, where $\psi(u)$ is the unique positive solution of the same initial value problem as in Figure 4.1.

Differentiating (4.7) with respect to t gives

$$\begin{aligned} E'(t) + I'(t) &= -\tilde{S}e^{(\beta/\mu)\tilde{D}}\left(\varphi^{-1}(t)\right)' + \frac{\gamma+\mu}{\beta}\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} \\ &= \left(-\tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\right)\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} \\ &= \left(-S(t) + \frac{\gamma+\mu}{\beta}\right)\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)}. \end{aligned}$$

Since

$$\frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} = -\psi\left(\varphi^{-1}(t)\right) < 0$$

by (3.30), we observe that $E'(t) + I'(t) = 0$ for

$$t = T_3 = \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}}\right) = S^{-1}\left(\frac{\gamma+\mu}{\beta}\right).$$

Note that

$$e^{-(\beta/\mu)\alpha} < \frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}} = \frac{\gamma+\mu}{\beta\tilde{S}}e^{-(\beta/\mu)\tilde{D}} < e^{-(\beta/\mu)\tilde{D}}$$

in view of (A_7) and Remark 3.2. In light of (3.22) we obtain

$$T_3 = \varphi\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}}\right) = \int_{(\gamma+\mu)/(\beta\tilde{S}e^{(\beta/\mu)\tilde{D}})}^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)} = S^{-1}\left(\frac{\gamma+\mu}{\beta}\right).$$

It is easy to check that $E'(t) + I'(t) > 0$ [resp. < 0] if and only if $t < T_3$ [resp. $> T_3$], because $\varphi^{-1}(t)$ is decreasing on $[0, \infty)$. Therefore we conclude that $E(t) + I(t)$ is increasing in $[0, T_3)$

and is decreasing in (T_3, ∞). It can be shown that

$$\begin{aligned}
\max_{t\geq 0} \big(E(t)+I(t)\big) &= N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(T_3) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(T_3) \\
&= N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}} + \frac{\gamma+\mu}{\beta}\log\left(\frac{\gamma+\mu}{\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}}\right) \\
&= N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \frac{\gamma+\mu}{\beta} + \frac{\gamma+\mu}{\beta}\left(\log\frac{\gamma+\mu}{\beta} - \log\tilde{S} - \frac{\beta}{\mu}\tilde{D}\right) \\
&= \tilde{S} + \tilde{E} + \tilde{I} - \frac{\gamma+\mu}{\beta}\left(1 + \log\tilde{S} - \log\frac{\gamma+\mu}{\beta}\right). \qquad \square
\end{aligned}$$

Remark 4.9. Since $u_3 = (\gamma+\mu)/\big(\beta\tilde{S}e^{(\beta/\mu)\tilde{D}}\big) = ((\gamma+\mu)/(\beta\tilde{S}))e^{-(\beta/\mu)\tilde{D}}$ is a unique solution of the equation

$$\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + \frac{\gamma+\mu}{\beta}\log u\right)' = 0,$$

we obtain

$$T_3 = \varphi(u_3) = \int_{((\gamma+\mu)/(\beta\tilde{S}))e^{-(\beta/\mu)\tilde{D}}}^{e^{-(\beta/\mu)\tilde{D}}} \frac{d\xi}{\xi\psi(\xi)}$$

(*cf.* Figure 4.3).

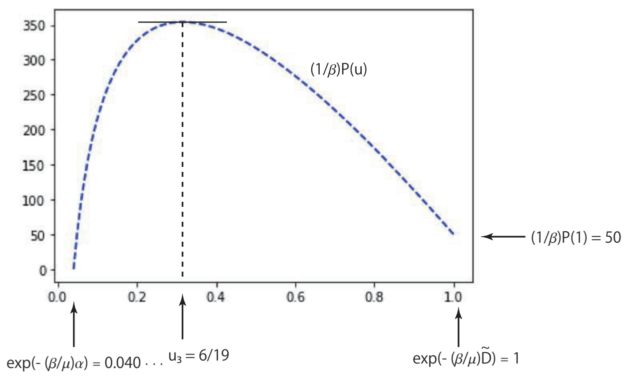

Figure 4.3: Variation of $(1/\beta)P(u) = N - \tilde{R} + (\gamma/\mu)\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}u + ((\gamma+\mu)/\beta)\log u$ (dashed curve) for $N = 1000, \tilde{S} = 950, \tilde{I} = 30, \tilde{E} = 20, \tilde{R} = 0, \beta = 0.3/1000, \gamma = 0.1$ and $\delta = 0.2$. In this case we observe that there exists a unique $u_3 = 6/19$ such that $(1/\beta)P'(u_3) = 0$, and that T_3 is calculated by $T_3 = \varphi(u_3) = \int_{6/19}^{1} \frac{d\xi}{\xi\psi(\xi)} = 41.9\ldots$, where $\psi(u)$ is the unique positive solution of the same initial value problem as in Figure 4.1.

Theorem 4.10. *The following relation holds:*

$$S(\infty) = \tilde{S} + \tilde{E} + \tilde{I} + \frac{\gamma+\mu}{\beta}\log\frac{S(\infty)}{\tilde{S}}.$$

Proof. Since $E(\infty) = I(\infty) = 0$, we obtain

$$\begin{aligned} S(\infty) &= N - R(\infty) - D(\infty) \\ &= N - \left(\frac{\gamma}{\mu}D(\infty) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D}\right) - D(\infty) \\ &= N - \tilde{R} - \tilde{D} + \frac{\gamma+\mu}{\beta}\left(\frac{\beta}{\mu}\tilde{D} - \frac{\beta}{\mu}D(\infty)\right) \\ &= \tilde{S} + \tilde{E} + \tilde{I} + \frac{\gamma+\mu}{\beta}\log\left(e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(\infty)}\right) \\ &= \tilde{S} + \tilde{E} + \tilde{I} + \frac{\gamma+\mu}{\beta}\log\frac{S(\infty)}{\tilde{S}} \end{aligned}$$

by use of (4.2) and (4.3). □

Theorem 4.11. *We find that*

$$S'(\infty) = E'(\infty) = I'(\infty) = R'(\infty) = D'(\infty) = 0.$$

Proof. Since $E(\infty) = I(\infty) = 0$, we see from (1.1)–(1.5) that

$$\begin{aligned} S'(\infty) &= -\beta S(\infty)I(\infty) = 0, \\ E'(\infty) &= \beta S(\infty)I(\infty) - \delta E(\infty) = 0, \\ I'(\infty) &= \delta E(\infty) - \gamma I(\infty) - \mu I(\infty) = 0, \\ R'(\infty) &= \gamma I(\infty) = 0, \\ D'(\infty) &= \mu I(\infty) = 0. \end{aligned}$$

□

Remark 4.12. The hypothesis (A_5) is satisfied if $\tilde{D} = 0$, since $N > \tilde{S} + \tilde{R}$.

Remark 4.13. It follows from Theorems 4.1–4.6 that $S(t) > 0, E(t) > 0, I(t) > 0$ for $t \geq 0$ and $R(t) > 0, D(t) > 0$ for $t > 0$ (*cf.* Figure 4.4).

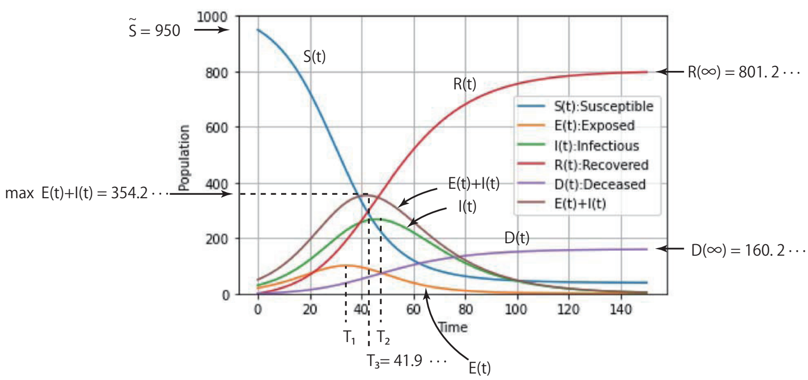

Figure 4.4: Variations of $S(t)$, $E(t)$, $I(t)$, $R(t)$, $D(t)$ and $E(t) + I(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.6) for $N = 1000, \tilde{S} = 950, \tilde{E} = 20, \tilde{I} = 30, \tilde{R} = 0, \tilde{D} = 0, \beta = 0.2/1000, \gamma = 0.05$, $\delta = 0.2$ and $\mu = 0.01$.

Remark 4.14. We note that

$$E'_+(0) + I'_+(0) = \beta \tilde{S} \tilde{I} - (\gamma + \mu) \tilde{I},$$

and that $E'_+(0) + I'_+(0) \leq 0$ is equivalent to $\tilde{S} \leq (\gamma + \mu)/\beta$. Let $E'_+(0) + I'_+(0) \leq 0$ be satisfied, and let $P(u)$ be given by (3.9). Then we see that

$$\frac{1}{\beta} P'(u) = -\tilde{S} e^{(\beta/\mu)\tilde{D}} + \frac{\gamma + \mu}{\beta} \frac{1}{u} = 0$$

at $u = ((\gamma + \mu)/(\beta \tilde{S})) e^{-(\beta/\mu)\tilde{D}}$ $(\geq e^{-(\beta/\mu)\tilde{D}})$ and that $(1/\beta)P(u)$ is increasing in $\left(e^{-(\beta/\mu)\alpha}, e^{-(\beta/\mu)\tilde{D}}\right]$, $\lim_{u \to e^{-(\beta/\mu)\alpha}+0} (1/\beta)P(u) = 0$ and $(1/\beta)P\left(e^{-(\beta/\mu)\tilde{D}}\right) = \tilde{E} + \tilde{I} > 0$. Since $\varphi^{-1}(t)$ is decreasing on $[0, \infty)$, $\varphi^{-1}(0) = e^{-(\beta/\mu)\tilde{D}}$ and $\lim_{t \to \infty} \varphi^{-1}(t) = e^{-(\beta/\mu)\alpha}$, we conclude that $E(t) + I(t) = (1/\beta)P\left(\varphi^{-1}(t)\right)$ is decreasing on $[0, \infty)$, $E(0) + I(0) = \tilde{E} + \tilde{I}$, and $E(\infty) + I(\infty) = 0$ (*cf.* Figure 4.5).

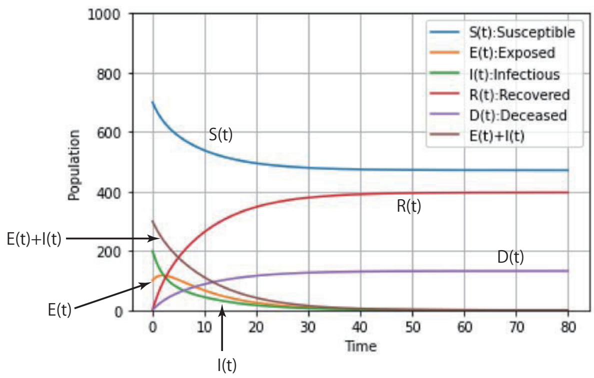

Figure 4.5: Variations of $S(t)$, $E(t)$, $I(t)$, $R(t)$, $D(t)$ and $E(t) + I(t)$ obtained by the numerical integration of the initial value problem (1.1)–(1.6) for $N = 1000, \tilde{S} = 700, \tilde{E} = 100, \tilde{I} = 200, \tilde{R} = 0, \tilde{D} = 0, \beta = 0.3/1000, \gamma = 0.3$, $\delta = 0.2$ and $\mu = 0.1$. In this case we find that $E'_+(0) = 22 > 0$, $I'_+(0) = -60 < 0$ and $E'_+(0) + I'_+(0) = -38 < 0$.

Remark 4.15. The function $D(t)$ given by (3.29) is a positive and increasing solution of the initial value problem for (2.1) with the initial conditions $D(0) = \tilde{D}$ and $D'_+(0) = \mu \tilde{I}$. In fact, it follows from Theorem 4.1 that $D(t)$ is an increasing function such that $D(t) > 0$ for $t > 0$. Since

$$\begin{aligned} D'(t) &= -\frac{\mu}{\beta} \frac{\left(\varphi^{-1}(t)\right)'}{\varphi^{-1}(t)} = -\frac{\mu}{\beta}\left(-\psi(\varphi^{-1}(t))\right) = \frac{\mu}{\beta}\psi(\varphi^{-1}(t)), \\ D''(t) &= -\frac{\mu}{\beta}\varphi^{-1}(t)\psi'(\varphi^{-1}(t))\psi(\varphi^{-1}(t)) \end{aligned}$$

in light of (3.30), we arrive at

$$\begin{aligned}
&D''(t) + (\delta+\gamma+\mu)D'(t) \\
&\quad = -\frac{\mu}{\beta}\varphi^{-1}(t)\left(\psi'(\varphi^{-1}(t))\psi(\varphi^{-1}(t)) - (\delta+\gamma+\mu)\frac{\psi(\varphi^{-1}(t))}{\varphi^{-1}(t)}\right) \\
&\quad = -\frac{\mu}{\beta}\varphi^{-1}(t)\left(-\delta\frac{\beta N - \beta\tilde{R} + ((\beta\gamma)/\mu)\tilde{D} - \beta\tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + (\gamma+\mu)\log\varphi^{-1}(t)}{\varphi^{-1}(t)}\right) \\
&\quad = \delta\mu\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t)\right) \\
&\quad = \delta\mu\left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D(t)} - \frac{\gamma+\mu}{\mu}D(t)\right)
\end{aligned}$$

in view of (3.3). It is easy to check that $D(0) = -(\mu/\beta)\log\varphi^{-1}(0) = -(\mu/\beta)\log e^{-(\beta/\mu)\tilde{D}} = \tilde{D}$ and

$$\begin{aligned}
D'_+(0) &= \lim_{\varepsilon\to+0} D'(\varepsilon) = \lim_{\varepsilon\to+0}\frac{\mu}{\beta}\psi(\varphi^{-1}(\varepsilon)) \\
&= \frac{\mu}{\beta}\psi(\varphi^{-1}(0)) = \frac{\mu}{\beta}\psi(e^{-(\beta/\mu)\tilde{D}}) = \frac{\mu}{\beta}\beta\tilde{I} = \mu\tilde{I}.
\end{aligned}$$

Remark 4.16. Let $D(t)$ be given by (3.29). Then the functions $S(t), E(t), I(t)$ and $R(t)$ given by (2.23)–(2.26) reduce to (3.25)–(3.28), respectively, since

$$e^{-(\beta/\mu)D(t)} = \varphi^{-1}(t), \qquad t = D^{-1}(-(\mu/\beta)\log\varphi^{-1}(t)) \quad \text{and} \quad \varphi(v) = D^{-1}(-(\mu/\beta)\log v).$$

Remark 4.17. If we suppose the hypothesis

(A'_4) $\tilde{R} \geq 0$ and $\tilde{R}$ satisfies

$$N - \tilde{D} > \tilde{S}e^{(\beta/\gamma)\tilde{R}} + \tilde{R},$$

then the transcendental equation

$$y = \frac{\gamma}{\gamma+\mu}N - \frac{\gamma}{\gamma+\mu}\tilde{D} + \frac{\mu}{\gamma+\mu}\tilde{R} - \frac{\gamma}{\gamma+\mu}\tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\gamma)y} \tag{4.8}$$

has a unique solution $y = \alpha_*$ such that

$$\tilde{R} < \alpha_* < N$$

(see Yoshida [18, Lemma 3]). Since the equation (4.8) reduces to the transcendental equation in Lemma 3.1 by the transformation $y = \tilde{R} - (\gamma/\mu)(\tilde{D} - x)$, we find that $\alpha_* = \tilde{R} - (\gamma/\mu)(\tilde{D} - \alpha)$. We define

$$\varphi_*(w) := \int_w^{e^{-(\beta/\gamma)\tilde{R}}} \frac{d\xi}{\xi\psi_*(\xi)}$$

for $e^{-(\beta/\gamma)\alpha_*} < w \leq e^{-(\beta/\gamma)\tilde{R}}$, where $\psi_*(\xi)$ is a unique positive solution of the initial value problem

$$\begin{aligned}
&\psi'_*(\xi)\psi_*(\xi) - \frac{\delta+\mu+\gamma}{\xi}\psi_*(\xi) \\
&\qquad = -\delta\,\frac{\beta N - \beta\tilde{D} + ((\beta\mu)/\gamma)\tilde{R} - \beta\tilde{S}e^{(\beta/\gamma)\tilde{R}}\xi + (\mu+\gamma)\log\xi}{\xi} \\
&\qquad\quad (e^{-(\beta/\gamma)\alpha_*} < \xi < e^{-(\beta/\gamma)\tilde{R}}), \\
&\psi_*(e^{-(\beta/\gamma)\tilde{R}}) = \beta\tilde{I}.
\end{aligned}$$

It follows from the transformation

$$\xi = e^{-(\beta/\gamma)\tilde{R}}e^{(\beta/\mu)\tilde{D}}u$$

that

$$\varphi_*(w) = \int_{e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\mu)\tilde{D}}w}^{e^{-(\beta/\mu)\tilde{D}}} \frac{du}{u\psi_*(e^{-(\beta/\gamma)\tilde{R}}e^{(\beta/\mu)\tilde{D}}u)},$$

where $e^{-(\beta/\mu)\alpha} < e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\mu)\tilde{D}}w \le e^{-(\beta/\mu)\tilde{D}}$. It is easy to check that $\psi_*(e^{-(\beta/\gamma)\tilde{R}}e^{(\beta/\mu)\tilde{D}}u)$ is a solution of the initial value problem (3.3), (3.4), and therefore

$$\psi_*\big(e^{-(\beta/\gamma)\tilde{R}}e^{(\beta/\mu)\tilde{D}}u\big) = \psi(u)$$

by the uniqueness of solutions of the initial value problem (3.3), (3.4). Hence we obtain

$$\varphi_*(w) = \varphi\big(e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\mu)\tilde{D}}w\big). \tag{4.9}$$

Let $\varphi_*^{-1}(t)$ and $\varphi^{-1}(t)$ be the inverse functions of

$$t = \varphi_*(w), \quad t = \varphi\big(e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\mu)\tilde{D}}w\big),$$

respectively, then we see that

$$\varphi_*^{-1}(t) = e^{-(\beta/\gamma)\tilde{R}}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) \quad (0 \le t < \infty). \tag{4.10}$$

It is easy to see that the hypothesis (A_7) is equivalent to

(A_8) $\tilde{S} < \dfrac{\mu+\gamma}{\beta}e^{(\beta/\gamma)(\alpha_*-\tilde{R})}$.

Let $\big(S_*(t), E_*(t), I_*(t), R_*(t), D_*(t)\big)$ be the exact solution of the initial value problem (1.1)–(1.6) by starting our arguments utilizing (1.4) instead of (1.5). Then we get

$$\begin{aligned}
S_*(t) &= \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi_*^{-1}(t),\\
E_*(t) &= \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-\delta t}\int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi_*(v)}dv,\\
I_*(t) &= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi_*^{-1}(t) + \frac{\mu+\gamma}{\beta}\log\varphi_*^{-1}(t)\\
&\qquad - \tilde{E}e^{-\delta t} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-\delta t}\int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi_*(v)}dv,\\
R_*(t) &= -\frac{\gamma}{\beta}\log\varphi_*^{-1}(t),\\
D_*(t) &= -\frac{\mu}{\beta}\log\varphi_*^{-1}(t) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}.
\end{aligned}$$

We observe, using (4.10), that

$$\begin{aligned}
S_*(t) &= \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi_*^{-1}(t) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}\big(e^{-(\beta/\gamma)\tilde{R}}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)\big)\\
&= \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) = S(t).
\end{aligned}$$

It follows from (4.9) and (4.10) that

$$\int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi_*(v)}dv = \int_{e^{-(\beta/\gamma)\tilde{R}}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} \exp\left(\delta\varphi\left(e^{(\beta/\gamma)\tilde{R}}e^{-(\beta/\mu)\tilde{D}}v\right)\right)dv$$
$$= e^{-(\beta/\gamma)\tilde{R}}e^{(\beta/\mu)\tilde{D}} \int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(w)}dw$$

and hence

$$E_*(t) = \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-\delta t}\int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi_*(v)}dv$$
$$= \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(w)}dw = E(t). \tag{4.11}$$

Since

$$N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi_*^{-1}(t) + \frac{\mu+\gamma}{\beta}\log\varphi_*^{-1}(t)$$
$$= N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t),$$

we deduce that $I_*(t) = I(t)$ in view of (4.11). It is easy to see that

$$R_*(t) = -\frac{\gamma}{\beta}\log\varphi_*^{-1}(t)$$
$$= -\frac{\gamma}{\beta}\left(-\frac{\beta}{\gamma}\tilde{R} + \frac{\beta}{\mu}\tilde{D} + \log\varphi^{-1}(t)\right)$$
$$= -\frac{\gamma}{\beta}\log\varphi^{-1}(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} = R(t),$$

and that

$$D_*(t) = -\frac{\mu}{\beta}\log\varphi_*^{-1}(t) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}$$
$$= -\frac{\mu}{\beta}\left(-\frac{\beta}{\gamma}\tilde{R} + \frac{\beta}{\mu}\tilde{D} + \log\varphi^{-1}(t)\right) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}$$
$$= -\frac{\mu}{\beta}\log\varphi^{-1}(t) = D(t).$$

Consequently we conclude that

$$\left(S_*(t), E_*(t), I_*(t), R_*(t), D_*(t)\right) \equiv \left(S(t), E(t), I(t), R(t), D(t)\right) \quad \text{on } [0,\infty).$$

Remark 4.18. The hypotheses (A_4') and (A_5) are equivalent to

(A_4'') $0 \le \tilde{R} < \dfrac{\gamma}{\beta}\log\left(1 + (\tilde{E}/\tilde{S}) + (\tilde{I}/\tilde{S})\right)$;

(A_5') $0 \le \tilde{D} < \dfrac{\mu}{\beta}\log\left(1 + (\tilde{E}/\tilde{S}) + (\tilde{I}/\tilde{S})\right)$,

respectively.

5 Uniqueness of positive solutions

This section is devoted to the uniqueness of positive solutions of the initial value problem (1.1)–(1.6). As a consequence we conclude that the exact solution (3.25)–(3.29) is the unique solution in the class of positive solutions.

A solution $(S(t), E(t), I(t), R(t), D(t))$ of the SEIRD differential system (1.1)–(1.5) is said to be *positive* if $S(t) > 0, E(t) > 0, I(t) > 0, R(t) > 0$ and $D(t) > 0$ for $t > 0$.

Theorem 5.1. *Let* $(S_i(t), E_i(t), I_i(t), R_i(t), D_i(t))$ $(i = 1, 2)$ *be solutions of the initial value problem* (1.1)–(1.6) *such that* $S_i(t) > 0, E_i(t) > 0, I_i(t) > 0, R_i(t) > 0$ *for* $t > 0$. *Then we find that*

$$(S_1(t), E_1(t), I_1(t), R_1(t), D_1(t)) \equiv (S_2(t), E_2(t), I_2(t), R_2(t), D_2(t)) \quad \text{on } [0, \infty). \tag{5.1}$$

Proof. First we note that $D_i(t) > 0$ for $t > 0$ $(i = 1, 2)$ since $D_i'(t) = \mu I_i(t) > 0$ for $t > 0$ and $D_i(0) = \tilde{D} \geq 0$. It follows from Lemma 2.1 that $D_i(t)$ $(i = 1, 2)$ satisfies (2.1) and the initial condition

$$D_i(0) = \tilde{D}, \qquad \lim_{\varepsilon \to +0} D_i'(\varepsilon) = \mu \tilde{I}$$

in view of (1.5) and (1.6). It is easy to see that

$$\boldsymbol{z}_i(t) := \big(D_i(t), D_i'(t)\big) \qquad (i = 1, 2)$$

are positive solutions of the initial value problem

$$\begin{aligned} \boldsymbol{y}'(t) &= \boldsymbol{f}(\boldsymbol{y}(t)), \qquad t > 0, \\ \boldsymbol{y}_+(0) &= \lim_{\varepsilon \to +0} \boldsymbol{y}(\varepsilon) = (\tilde{D}, \mu \tilde{I}), \end{aligned}$$

where $\boldsymbol{f}(\boldsymbol{y})$ is a function defined by

$$\boldsymbol{f}(\boldsymbol{y}) = \left(y_2, -(\delta + \gamma + \mu) y_2 + +\delta\mu \left(N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S} e^{(\beta/\mu)\tilde{D}} e^{-(\beta/\mu) y_1} - \left(1 + \frac{\gamma}{\mu}\right) y_1\right)\right)$$

for $\boldsymbol{y} = (y_1, y_2)$ such that $y_1 > 0$ and $y_2 > 0$. Since

$$\begin{aligned} \frac{\partial \boldsymbol{f}}{\partial y_1}(\boldsymbol{y}) &= \left(0, \beta\delta\tilde{S} e^{(\beta/\mu)\tilde{D}} e^{-(\beta/\mu) y_1} - \delta(\gamma + \mu)\right), \\ \frac{\partial \boldsymbol{f}}{\partial y_2}(\boldsymbol{y}) &= (1, -(\delta + \gamma + \mu)), \end{aligned}$$

we obtain

$$\left| \frac{\partial \boldsymbol{f}}{\partial y_k}(\boldsymbol{y}) \right| \leq \max \left\{ \beta\delta\tilde{S} e^{(\beta/\mu)\tilde{D}} + \delta(\gamma + \mu), 1 + (\delta + \gamma + \mu) \right\} \ (\equiv K) \ (k = 1, 2)$$

for $\boldsymbol{y} = (y_1, y_2)$ such that $y_1 > 0$ and $y_2 > 0$, where the magnitude of a vector $\boldsymbol{y}$, denoted by $|\boldsymbol{y}|$, is defined by

$$|\boldsymbol{y}| = |y_1| + |y_2| \quad \text{for } \boldsymbol{y} = (y_1, y_2) \in \mathbb{R}^2.$$

Therefore, $\boldsymbol{f}(\boldsymbol{y})$ satisfies a Lipschitz condition on $(0, \infty) \times (0, \infty)$ with Lipschitz constant K (see Coddington [7, p.248, Theorem 1]). Since

$$\boldsymbol{z}_i'(t) = \boldsymbol{f}(\boldsymbol{z}_i(t)), \qquad t > 0 \qquad (i = 1, 2),$$

integrating the above on $[\varepsilon, t]$ $(\varepsilon > 0)$ and then taking the limit as $\varepsilon \to +0$ yield

$$\boldsymbol{z}_i(t) - \lim_{\varepsilon \to +0} \boldsymbol{z}_i(\varepsilon) = \int_0^t \boldsymbol{f}(\boldsymbol{z}_i(s))ds, \qquad t > 0,$$

and we observe, using $\lim_{\varepsilon \to +0} \boldsymbol{z}_i(\varepsilon) = (\tilde{D}, \mu\tilde{I})$, that

$$\boldsymbol{z}_i(t) = (\tilde{D}, \mu\tilde{I}) + \int_0^t \boldsymbol{f}(\boldsymbol{z}_i(s))ds, \qquad t > 0.$$

Therefore we obtain

$$\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t) = \int_0^t (\boldsymbol{f}(\boldsymbol{z}_1(s)) - \boldsymbol{f}(\boldsymbol{z}_2(s)))\, ds, \qquad t > 0$$

and hence

$$|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)| \leq K \int_0^t |\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)|\, ds, \qquad t > 0$$

since $\boldsymbol{f}(\boldsymbol{y})$ satisfies a Lipschitz condition with Lipschitz constant K. Defining

$$W(t) := \int_0^t |\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)|\, ds,$$

we obtain

$$W'(t) - KW(t) \leq 0, \qquad t > 0,$$

or

$$\left(e^{-Kt}W(t)\right)' \leq 0, \qquad t > 0.$$

Since $e^{-Kt}W(t) \leq W(0) = 0$ $(t \geq 0)$, we see that $W(t) \leq 0$ $(t \geq 0)$. Hence

$$|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)| \leq KW(t) \leq 0, \qquad t > 0,$$

which yields

$$\boldsymbol{z}_1(t) = \boldsymbol{z}_2(t), \qquad t > 0.$$

Therefore we conclude that

$$D_1(t) \equiv D_2(t) \quad \text{on } (0, \infty).$$

Since $D_1(0) = D_2(0) = \tilde{D}$, we observe that

$$D_1(t) \equiv D_2(t) \quad \text{on } [0, \infty).$$

It follows from Corollary 2.4 that $S_i(t)$ $(i = 1, 2)$ can be represented by

$$S_i(t) = \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-(\beta/\mu)D_i(t)}$$

for $t \geq 0$. Since $D_1(t) = D_2(t)$ for $t \geq 0$, we deduce that $S_1(t) = S_2(t)$ for $t \geq 0$. Similarly we find that $E_1(t) = E_2(t)$ $(t \geq 0)$, $I_1(t) = I_2(t)$ $(t \geq 0)$ and $R_1(t) = R_2(t)$ $(t \geq 0)$. Consequently we conclude that (5.1) holds. □

Theorem 5.2. *Assume that the hypotheses* (A$_1$)–(A$_7$), (A$'_4$) *hold. The function* $(S(t), E(t), I(t), R(t), D(t))$ *given by*

$$\begin{aligned}
S(t) &= \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) = \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi_*^{-1}(t),\\
E(t) &= \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi(v)}dv\\
&= \tilde{E}e^{-\delta t} + \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-\delta t}\int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi_*(v)}dv,\\
I(t) &= N - \tilde{R} + \frac{\gamma}{\mu}\tilde{D} - \tilde{S}e^{(\beta/\mu)\tilde{D}}\varphi^{-1}(t) + \frac{\gamma+\mu}{\beta}\log\varphi^{-1}(t)\\
&\quad - \tilde{E}e^{-\delta t} - \tilde{S}e^{(\beta/\mu)\tilde{D}}e^{-\delta t}\int_{\varphi^{-1}(t)}^{e^{-(\beta/\mu)\tilde{D}}} e^{\delta\varphi_*(v)}dv\\
&= N - \tilde{D} + \frac{\mu}{\gamma}\tilde{R} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}\varphi_*^{-1}(t) + \frac{\mu+\gamma}{\beta}\log\varphi_*^{-1}(t)\\
&\quad - \tilde{E}e^{-\delta t} - \tilde{S}e^{(\beta/\gamma)\tilde{R}}e^{-\delta t}\int_{\varphi_*^{-1}(t)}^{e^{-(\beta/\gamma)\tilde{R}}} e^{\delta\varphi_*(v)}dv,\\
R(t) &= -\frac{\gamma}{\beta}\log\varphi^{-1}(t) + \tilde{R} - \frac{\gamma}{\mu}\tilde{D} = -\frac{\gamma}{\beta}\log\varphi_*^{-1}(t),\\
D(t) &= -\frac{\mu}{\beta}\log\varphi^{-1}(t) = -\frac{\mu}{\beta}\log\varphi_*^{-1}(t) + \tilde{D} - \frac{\mu}{\gamma}\tilde{R}
\end{aligned}$$

is a positive solution of the initial value problem (1.1)–(1.6)*, and is unique in the class of positive solutions.*

Proof. Combining Theorem 3.9, Remarks 4.13 and 4.17, we see that $(S(t), E(t), I(t), R(t), D(t))$ given above is a positive solution of the initial value problem (1.1)–(1.6). Uniqueness of positive solutions follows from Theorem 5.1. □

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Lagrange stability for a class of impulsive Duffing-type equations with low regularity

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Abstract. We discuss the Lagrange stability for a class of impulsive Duffing equation with time-dependent polynomial potentials. More precisely, we prove that under suitable impulses, all the solutions of the impulsive Duffing equation (with low regularity in time) are bounded for all time and that there are many (positive Lebesgue measure) quasi-periodic solutions clustering at infinity.

Keywords: boundedness, quasi-periodic solution, Moser's twist theorem, impulsive Duffing equation.

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1 Introduction

The stability theory plays a central role in differential equations for its practical value in real world applications. It is well known that the longtime behavior of a time-dependent nonlinear ordinary differential equation can be very intricate. For instance, the well-known Duffing equation,

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t),$$

is an example of dynamical system that exhibits chaotic behavior.

The generalized Duffing-type equation arises in a large class of practically important applied problems in mathematics, physical science and engineering such as the cubic–quintic Duffing oscillatory [9] and the Helmholtz–Duffing oscillator [8], which takes the form of

$$\ddot{x} = \sum_{j\in K} a_j\, x^j(t), \quad K \subset \mathbf{N} \text{ is finite.}$$

See [35] and the references therein for more details.

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1.1 Lagrange stability of Duffing-type equation

In contrast to "Lyapunov stability" that is related to the chaotic nature of the system, we pay special interest in this paper to the Lagrange stability of nonlinear systems, which means that all the solutions stay bounded for all time. The Lagrange stability refers roughly to the stability against the escape of a body from the system. We refer to the classical monograph [14] for more details about the Lagrange stability.

The study of Lagrange stability of Duffing-type equation dates back to Littlewood [16], Moser [20,21] and Morris [19]. In 1987, Dieckerhoff and Zehnder studied the Lagrange stability for the generalized Duffing-type equation with time-dependent polynomial potentials

$$\ddot{x} + x^{2n+1} + \sum_{i=0}^{2n} x^i p_i(t) = 0, \quad n \geq 1, \tag{1.1}$$

where $p_i \in C^\nu(\mathbf{T^1})$ $(0 \leq i \leq 2n)$ are 1-periodic with $\mathbf{T^1} = \mathbf{R}/\mathbf{Z}$, and proved that every solution $x(t)$ of (1.1) is bounded for all time, i.e., the solution $x(t)$ exists for all $t \in \mathbf{R}$ and $\sup_{t\in\mathbf{R}}(|x(t)| + |\dot{x}(t)|) < \infty$, if ν is the smallest integer satisfying

$$v > 1 + \frac{4}{n} + [\log_2 n] \to \infty \quad \text{as } n \to \infty.$$

There exist a lot of papers [12,15,17,18,32–34] devoting to the relaxation of the smoothness of p_i in (1.1) with respect to the t-variable when studying the Lagrange stability. However, there is an example in [31] showing that a continuous coefficient would result in an unbounded solution.

As we know, a abrupt change at certain instant during the evolution process falls into the scope of the impulsive dynamical system [1,13], which appear widely in applied mathematics. The appearance of impulse forces may cause complicated dynamic phenomenons and bring difficulties to study. There are many studies on the existence of periodic solutions of impulsive differential equations [2,7,10,22–24] via different approaches. See also [11,26] for the periodic solution of impulsive Duffing-type equation.

However, there are only few results on the Lagrange stability and the existence of quasi-periodic solutions for impulsive differential equations (see [3–5,25,30]). Coming back to the Duffing-type equation (1.1), the term $\sum_{i=0}^{2n} x^i p_i(t)$ can be regarded as the perturbation of $\ddot{x} + x^{2n+1} = 0$ (up to some transformations). Then the Lagrange stability of (1.1) show that all solutions of nonlinear equation $\ddot{x} + x^{2n+1} = 0$ is bounded under a periodic perturbation. It is very natural to ask the following question:

"what happens when the nonlinear equation $\ddot{x} + x^{2n+1} = 0$ is subject to both periodic perturbation and an impulse at the same time?"

Choosing different impulsive functions may have different effects on the solutions. It is also not surprising that an offhand choice of impulse force would destroy the Lagrange stability even though the coefficients p_i are sufficiently smooth. To establish the Lagrange stability of impulsive Duffing-type equation, one needs to be careful on the impulse such that the Poincaré map can be well organized in order to apply Moser's twist theorem after some symplectic transformations. We mention some progress in this respect. In 2019, [30] proved the boundedness of solutions and the existence of quasi-periodic solutions for the impulsive

Duffing equation

$$\begin{cases} \ddot{x} + x^{2n+1} + \sum_{i=0}^{2n} x^i p_i(t) = 0, \quad t \neq t_j, \quad n \geq 1, \\ \Delta x(t_j) := x(t_j^+) - x(t_j^-) = I_j(x(t_j^-), \dot{x}(t_j^-)), \\ \Delta \dot{x}(t_j) := \dot{x}(t_j^+) - \dot{x}(t_j^-) = J_j(x(t_j^-), \dot{x}(t_j^-)), \quad j = \pm 1, \pm 2, \dots \end{cases} \tag{1.2}$$

with the *low regularity in time*

$$p_i(t) \in C^\gamma(\mathbf{T^1}), \quad \gamma > 2 - \frac{1}{n},$$

and with the general sequences of impulsive functions $I_j, J_j : \mathbf{R^2} \to \mathbf{R}$, where $\mathbf{T^1} := \mathbf{R}/\mathbf{Z}$. Moreover, the following restricted conditions are needed: for $j = 1, \dots, k$,

(i) the jumps $I_j(x, y) = o(1)$ as $x^2 + y^2 \to +\infty$;

(ii) the jump map $\Phi_j : (x, y) \to (x, y) + (I_j(x, y), J_j(x, y))$ is an area-preserving homeomorphism,

which enables us to apply Moser's twist theorem. See [30, Remark 2.1] for the comparison of different types of impulse forces and their roles when studying the Lagrange stability.

For the particular case of cubic Duffing-type equation, [25] extended the Morris's boundedness result [19] to the impulsive Duffing equation

$$\begin{cases} \ddot{x} + x^3 + p(t) = 0, \quad t \neq t_j, \\ \Delta x(t_j) := x(t_j^+) - x(t_j^-) = I(x(t_j^-), \dot{x}(t_j^-)), \\ \Delta \dot{x}(t_j) := \dot{x}(t_j^+) - \dot{x}(t_j^-) = J(x(t_j^-), \dot{x}(t_j^-)), \quad j = \pm 1, \pm 2, \dots, \end{cases}$$

where $0 < t_1 < 1, t_{j+1} = t_j + 1$ for $j = \pm 1, \pm 2, \cdots$ and $p(t)$ is 1-periodic and integrable.

In 2020, [3] proposed some concrete and simple impulse forces, which do not satisfy the above conditions (i) and (ii), and proved the Lagrange stability and the existence of quasi-periodic solutions for the impulsive Duffing-type equation

$$\begin{cases} \ddot{x} + x^{2n+1} + \sum_{i=0}^{n} x^i p_i(t) = 0, \quad t \neq t_j, \quad n \geq 1, \\ \Delta x(t_j) = (\gamma_j - 1)\ x(t_j^-), \\ \Delta \dot{x}(t_j) = (\gamma_j^{n+1} - 1)\ \dot{x}(t_j^-)), \quad j = \pm 1, \pm 2, \dots, \end{cases} \tag{1.3}$$

where $\gamma_j > 0$ are some constants and the coefficients $p_i \in C^\infty(\mathbf{T^1})$ for technical simplicity.

In this paper, we pay special attention to the *sharp regularity* of the coefficients $p_i(t)$ in the Duffing-type equation, together with the impulse forces given by (1.3), to establish the Lagrange stability. More precisely, we will prove the Lagrange stability and the existence of quasi-periodic solutions for (1.3) with low regularity in time

$$p_i(t) \in C^\gamma(\mathbf{T^1}) \quad (i = 0, \dots, n), \quad \gamma > 1 - \frac{1}{n}.$$

1.2 Main result

To formulate our main result we have to introduce some notations and hypotheses. Let $\mathbf{R}, \mathbf{C}, \mathbf{N}$ and $\mathbf{Z}$ be the sets of all real numbers, complex numbers, natural numbers and integers, respectively. Denote by $\mathcal{T}$ the impulsive time sequence $\{t_j\}, j = \pm 1, \pm 2, \dots$, and denote by $\mathcal{A}$ the set of indexes j. We assume that the following condition **(H)** holds true.

(H) There exists a positive integer k such that $0 < t_1 < t_2 < \cdots < t_k < 1$, and that t_j's, γ_j's are 1-periodic in j in the sense that $t_{j+k} = t_j + 1, \gamma_{j+k} = \gamma_j$ for $j \le -(k+1)$ or $j \ge 1$; $t_{j+k+1} = t_j + 1, \gamma_{j+k+1} = \gamma_j$ for $-k \le j \le -1$.

The main result in this paper is the following theorem.

Theorem 1.1. *Suppose that condition* (**H**) *holds and that for each* $0 \le i \le n$*, there is* $p_i(t) \in C^\gamma(\mathbf{T^1})$ *with* $\gamma > 1 - \frac{1}{n}$*. In addition, assume that*

$$\prod_{j=1}^{k} \gamma_j = 1. \tag{1.4}$$

Then the time-1 map $\widetilde{P} : (x, \dot{x})_{t=0} \mapsto (x, \dot{x})_{t=1}$ *of* (1.3) *possesses many* (*positive Lebesgue measure*) *invariant closed curves whose radiuses tend to infinity, and thus every solution* $x(t)$ *of* (1.3) *is bounded for all time, i.e. it exists for all* $t \in \mathbf{R}$ *and*

$$\sup_{t\in\mathbf{R}} (|x(t)| + |\dot{x}(t)|) \le \widetilde{C} < +\infty,$$

where $\widetilde{C} = \widetilde{C}(x(0), \dot{x}(0))$ *depends on the initial data* $(x(0), \dot{x}(0))$.

Remark 1.2. In equation (1.3), the jump maps $\Phi_j : (x, y) \mapsto (x, y) + ((\gamma_j - 1)x, (\gamma_j^{n+1} - 1)y)$ are homeomorphisms which are not area-preserving (when $\gamma_j \ne 1$), and $|I_j(x, y)| = |(\gamma_j - 1)x| = O(|x|)$ (when $\gamma_j \ne 1$). Thus, the conditions (i) and (ii) mentioned above in [30] are not satisfied.

Remark 1.3. Equation (1.1) can be written as a Hamiltonian system with the Hamiltonian function $H = h_0(x, y) + R(x, y, t)$. It is essential to regard R as a relatively small perturbation with respect to h_0. See [15] for the detail. Otherwise, the stability might have been violated even without the impulse. Note also that the Duffing-type equation in (1.3) is simpler than (1.1) since the terms $p_i(t)x^i$ $(n + 1 \le i \le 2n)$ are absent. For the general case of equation (1.1) under the impulse given by (1.3), we refer to [3] for some discussions on the obstruction when establishing the Lagrange stability.

Remark 1.4. When using KAM theory to (1.2), one of the main difficulties is the estimation of "small property condition" of Moser's twist theorem. In [30], the difficult was overcome when the smoothness in time $p_i \in C^\gamma(\mathbf{T^1})$ with $\gamma > 2 - \frac{1}{n}$ is used. However, for equation (1.3), we observe that the *smoothness* can be relaxed to $C^\gamma(\mathbf{T^1})$ with $\gamma > 1 - \frac{1}{n}$, which is closely related to the almost sharp result in [34]. Our method is also based on the approximation techniques used in [34].

The paper is organized as follows. In Section 2, we establish the global existence of solutions for impulsive differential equations (1.3) and construct the associated time-one map. In Section 3, we introduce the action-angle variables and apply a preliminary symplectic transformation such that (1.3) becomes a nearly integrable Hamiltonian system. In Section 4, we introduce the approximate lemma to approximate the smooth periodic function by a real analytic function. In Section 5, we take further symplectic transformations such that Moser's twist theorem can be applied. In Section 6, we establish some estimates for the impulsive functions after the transformation. Finally, in Section 7, we prove Theorem 1.1 by employing Moser's twist theorem.

2 Global existence of solutions and time-one map

In this section, we establish the global existence of solutions for impulsive differential equations (1.3) and construct the associated time-one map. We begin with the general impulsive differential equation

$$\begin{cases} \dot{u} = F(t,u), \quad t \neq t_j, \\ \Delta u(t_j) := u(t_j^+) - u(t_j) = L_j(u(t_j)), \quad j \in \mathcal{A} \end{cases} \tag{2.1}$$

with the initial value condition

$$u(\tau^+) = u_0, \tag{2.2}$$

where $\tau \in \mathbf{R}, u_0 \in \mathbf{R^m}, m \in \mathbf{N}$, and where $u(\tau^+) = u(\tau)$ if $\tau \notin \mathcal{T}$. Suppose that

($\mathbf{H_1}$) The function $F : \mathbf{R} \times \mathbf{R^m} \mapsto \mathbf{R^m}$ is continuous, locally Lipschitz in the second variable.

($\mathbf{H_2}$) The function F is 1-periodic in the first variable. There exists a positive integer k such that $0 < t_1 < t_2 < \cdots < t_k < 1$, $t_{j+k} = t_j + 1$, $L_{j+k}(\cdot) = L_j(\cdot)$ for $j \leq -(k+1)$ or $j \geq 1$; $t_{j+k+1} = t_j + 1$, $L_{j+k+1}(\cdot) = L_j(\cdot)$ for $-k \leq j \leq -1$.

($\mathbf{H_3}$) The impulsive functions $L_j : \mathbf{R^m} \to \mathbf{R^m}$ are continuous for all $j \in \mathcal{A}$.

Lemma 2.1 ([30, Lemma 3.2]). *Assume that the conditions* ($\mathbf{H_1}$)–($\mathbf{H_3}$) *hold and that every jump equation*

$$u = v + L_j(v), \quad u \in \mathbf{R^m}, \qquad j = 1, \dots, k, \tag{2.3}$$

has a unique solution with respect to $v \in \mathbf{R^m}$. Assume in addition that all the solutions of the unforced equation $\dot{u} = F(t,u)$ exist for all $t \in \mathbf{R}$. Then the following conclusions hold true.

(a) *For any $\tau \in \mathbf{R}, u_0 \in \mathbf{R^m}$, there is a unique solution $u = u(t;\tau,u_0)$ of (2.1) satisfying the initial value condition (2.2), and it exists for all $t \in \mathbf{R}$.*

(b) *If the equation $\dot{u} = F(t,u)$ is conservative and the impulsive maps $\aleph_j : u \mapsto u + L_j(u) (j \in \mathcal{A})$ are homeomorphisms of $\mathbf{R^m}$, then for $t \in \mathbf{R} \setminus \mathcal{T}$, the map $Q_t : u_0 \mapsto u(t;\tau,u_0)$ is also a homeomorphism.*

(c) *The solution satisfies the elastic property. That is, for any $b_0 > 0$, there is $r_{b_0} > 0$ such that the inequality $|u_0| \geq r_{b_0}$ implies $|u(t;\tau,u_0)| \geq b_0$, for $t \in (\tau, \tau+1]$.*

In order to deduce a global existence result of the impulsive Duffing equation (1.3), by letting $y = \dot{x}$ and noting that $x(t_j^-) = x(t_j), y(t_j^-) = y(t_j)$, we can rewrite equation (1.3) as an equivalent system of the form

$$\begin{cases} \dot{x} = y, \quad t \neq t_j, \\ \dot{y} = -x^{2n+1} - \sum_{i=0}^{n} p_i(t) x^i, \quad t \neq t_j; \\ \Delta x(t_j) = I_j(x(t_j), y(t_j)) = (\gamma_j - 1) x(t_j), \\ \Delta y(t_j) = J_j(x(t_j), y(t_j)) = (\gamma_j^{n+1} - 1) y(t_j), \quad j = 1, 2, \dots, k. \end{cases} \tag{2.4}$$

For (2.4), each jump map

$$\widetilde{\aleph}_j : \begin{cases} u = x + I_j(x,y), \\ v = y + J_j(x,y), \quad j = 1, \dots, k \end{cases} \tag{2.5}$$

is a homeomorphism on $\mathbf{R}^2$. Note also that every solution $(x(t), y(t))$ of the unforced Duffing equation

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^{2n+1} - \sum_{i=0}^{n} p_i(t) x^i \end{cases}$$

satisfying the initial value condition $(x(t_0), y(t_0)) = (x_0, y_0)$ is unique and exists for all $t \in \mathbf{R}$. Then using the implicit function theorem and Lemma 2.1, we obtain the following corollary.

Corollary 2.2. *Suppose that condition* $(\mathbf{H})$ *holds and that for each* $0 \le i \le n, p_i(t) \in C^\gamma(\mathbf{T}^1)$ *with* $\gamma > 1 - \frac{1}{n}$. *Then the following statements hold.*

(a) *For any* $\tau \in \mathbf{R}$, $(x_0, y_0) \in \mathbf{R}^2$, *there is a unique solution*

$$(x(t), y(t)) = (x(t; \tau, x_0, y_0), y(t; \tau, x_0, y_0))$$

of (2.4) *satisfying the initial condition* $x(\tau^+) = x_0, y(\tau^+) = y_0$, *which exists for all* $t \in \mathbf{R}$.

(b) *The map* $Q_t : (x_0, y_0) \mapsto (x(t; \tau, x_0, y_0), y(t; \tau, x_0, y_0))$ *is continuous in* (x_0, y_0) *for* $t \in \mathbf{R} \setminus \mathcal{T}$.

(c) *The solution satisfies the elastic property. More precisely, for any* $b_0 > 0$, *there is* $r_{b_0} > 0$ *such that the inequalities* $|x_0| \ge r_{b_0}, |y_0| \ge r_{b_0}$ *implies that* $|x(t; \tau, x_0, y_0)| \ge b_0$ *and* $|y(t; \tau, x_0, y_0)| \ge b_0$ *for* $t \in (\tau, \tau + 1]$.

In order to deduce the time-one map of impulsive Duffing equation (2.4), we denote by $(x(t), y(t)) = (x(t; x_0, y_0), y(t; x_0, y_0))$ the solution of (2.4) satisfying the initial condition $(x(0), y(0)) = (x_0, y_0)$. Let

$$\begin{aligned}
\widetilde{P}_0 &: (x_0, y_0) \mapsto (x(t_1), y(t_1)) := (x_1, y_1), \\
\Phi_1 &: (x_1, y_1) \mapsto (x_1 + I_1(x_1, y_1), y_1 + J_1(x_1, y_1)) = (x(t_1^+), y(t_1^+)) := (x_1^+, y_1^+), \\
\widetilde{P}_1 &: (x_1^+, y_1^+) \mapsto (x(t_2), y(t_2)) := (x_2, y_2), \\
\Phi_2 &: (x_2, y_2) \mapsto (x_2 + I_2(x_2, y_2), y_2 + J_2(x_2, y_2)) = (x(t_2^+), y(t_2^+)) := (x_2^+, y_2^+), \\
&\vdots \\
\widetilde{P}_{k-1} &: (x_{k-1}^+, y_{k-1}^+) \mapsto (x(t_k), y(t_k)) := (x_k, y_k), \\
\Phi_k &: (x_k, y_k) \mapsto (x_k + I_k(x_k, y_k), y_k + J_k(x_k, y_k)) = (x(t_k^+), y(t_k^+) := (x_k^+, y_k^+), \\
\widetilde{P}_k &: (x_k^+, y_k^+) \mapsto (x(1), y(1)).
\end{aligned}$$

Then the time-one map $\widetilde{P} : (x_0, y_0) \mapsto (x(1), y(1))$ of (2.4) can be expressed by

$$\widetilde{P} = \widetilde{P}_k \circ \Phi_k \circ \cdots \circ \widetilde{P}_1 \circ \Phi_1 \circ \widetilde{P}_0.$$

Remark 2.3. Under the condition $(\mathbf{H})$, since the impulsive maps $\Phi_j : (x, y) \mapsto (x, y) + (I_j(x, y), J_j(x, y))$, $(j = 1, 2, \ldots, k)$ are homeomorphisms on $\mathbf{R}^2$, the time-one map $\widetilde{P}$ of (2.4) is also a homeomorphism on $\mathbf{R}^2$.

From Corollary 2.2 and Remark 2.3, we have the following result.

Corollary 2.4. *Suppose that the condition* $(\mathbf{H})$ *holds and that for each* $0 \le i \le n$ *there is* $p_i(t) \in C^\gamma(\mathbf{T}^1)$ *with* $\gamma > 1 - \frac{1}{n}$. *Then the time-one map* $\widetilde{P}$ *of* (2.4) *is a homeomorphism on* $\mathbf{R}^2$. *Moreover, for any* $b_0 > 0$, *there is* $r_{b_0} > 0$ *such that the inequalities* $|x_0| \ge r_{b_0}, |y_0| \ge r_{b_0}$ *implies that* $|x(1; x_0, y_0)| \ge b_0$ *and* $|y(1; x_0, y_0)| \ge b_0$.

3 Action-angle variables

In this section, we introduce the action-angle variables and apply a preliminary symplectic transformation such that (1.3) becomes a nearly integrable Hamiltonian system. The transformations are standard for the Duffing equation and can be found in [6,30,34] for instance. Let

$$x = AX, \tag{3.1}$$

$$Y = A^{-n}\dot{X} = A^{-n-1}\dot{x} = A^{-n-1}y, \tag{3.2}$$

$$\Delta X(t_j) = X(t_j^+) - X(t_j), \quad \Delta Y(t_j) = Y(t_j^+) - Y(t_j). \tag{3.3}$$

Then we see from equation (1.3) that

$$\begin{cases} \dot{X} = \frac{\partial H^*}{\partial Y}, \quad t \neq t_j, \\ \dot{Y} = -\frac{\partial H^*}{\partial X}, \quad t \neq t_j, \\ \Delta X(t_j) = (\gamma_j - 1)X(t_j) := \tilde{I}_j(X(t_j), Y(t_j)), \\ \Delta Y(t_j) = (\gamma_j^{n+1} - 1)Y(t_j) := \tilde{J}_j(X(t_j), Y(t_j)), \end{cases} \tag{3.4}$$

where $j = 1, 2, \ldots, k$ and

$$H^*(X, Y, t) = A^n \left(\frac{1}{2}Y^2 + \frac{1}{2(n+1)}X^{2(n+1)} \right) + \sum_{i=0}^{n} \frac{p_i(t)}{i+1} A^{i-n-1} X^{i+1}. \tag{3.5}$$

The similar formulation of (3.4) can be also found in Section 5 in [30].

Consider the auxiliary Hamiltonian system

$$\dot{X} = \frac{\partial H_0^*}{\partial Y}, \quad \dot{Y} = -\frac{\partial H_0^*}{\partial X}, \tag{3.6}$$

where

$$H_0^*(X, Y) = \frac{1}{2}Y^2 + \frac{1}{2(n+1)}X^{2(n+1)}.$$

Let $(X_0(t), Y_0(t))$ be the solution of (3.6) with initial $(X_0(0), Y_0(0)) = (1, 0)$. Then this solution is clearly periodic. Let T_0 be its minimal positive period. By the energy conservation, we have

(s_1) $(n+1)Y_0^2(t) + X_0^{2n+2}(t) \equiv 1$;

(s_2) $X_0(-t) = X_0(t)$, $Y_0(-t) = -Y_0(t)$;

(s_3) $\dot{X}_0(t) = Y_0(t)$, $\dot{Y}_0(t) = -X_0^{2n+1}(t)$;

(s_4) $X_0(t + T_0) = X_0(t)$, $Y_0(t + T_0) = Y_0(t)$.

We construct the following symplectic transformation

$$\Psi_0 : X = c^\alpha \lambda^\alpha X_0(\theta T_0), \quad Y = c^\beta \lambda^\beta Y_0(\theta T_0), \tag{3.7}$$

where $\alpha = \frac{1}{n+2}$, $\beta = 1 - \alpha = \frac{n+1}{n+2}$, $c = \frac{1}{\alpha T_0}$ and $(\lambda, \theta) \in \mathbf{R}^+ \times \mathbf{T}^1$ is the action-angle variables. By calculation, the Jacobian determinant det $\frac{\partial(X,Y)}{\partial(\theta,\lambda)} = 1$. Then the transformation Ψ_0 is indeed symplectic.

By (3.7), we have

$$\lambda = \frac{1}{c}[X^{2n+2} + (n+1)Y^2]^{\frac{n+2}{2n+2}}. \tag{3.8}$$

We claim that there exists the inverse function $\widetilde{X}_0^{-1}$ such that $\theta = \widetilde{X}_0^{-1}(c^{-\alpha}\lambda^{-\alpha}X)$. Indeed, from (3.7) we have $X_0(\theta T_0) = c^{-\alpha}\lambda^{-\alpha}X$. In the case of $\theta \in [0, \frac{1}{2}]$, by ($s_3$) we get $\frac{dX_0(\theta T_0)}{d\theta} = T_0 Y_0(\theta T_0) < 0$. Thus, we have

$$\theta = T_0^{-1} X_0^{-1}(c^{-\alpha}\lambda^{-\alpha}X).$$

In the case of $\theta \in (\frac{1}{2}, 1)$, by using (3.7), ($s_2$) and ($s_4$), we have

$$X = c^\alpha \lambda^\alpha X_0(\theta T_0) = c^\alpha \lambda^\alpha X_0(-\theta T_0) = c^\alpha \lambda^\alpha X_0((1-\theta)T_0).$$

Let $\xi = 1 - \theta$ and we have $\frac{dX_0(\xi T_0)}{d\xi} = T_0 Y_0(\xi T_0) < 0$ for $\xi \in (0, \frac{1}{2})$. Then we get $\xi = T_0^{-1} X_0^{-1}(c^{-\alpha}\lambda^{-\alpha}X)$ and thus

$$\theta = 1 - T_0^{-1} X_0^{-1}(c^{-\alpha}\lambda^{-\alpha}X).$$

From (3.4), we have that for $j = 1, 2, \ldots, k$

$$\begin{cases} X(t_j^+) = X(t_j) + \tilde{I}_j(X(t_j), Y(t_j)) = \gamma_j X(t_j), \\ Y(t_j^+) = Y(t_j) + \tilde{J}_j(X(t_j), Y(t_j)) = \gamma_j^{n+1} Y(t_j). \end{cases} \tag{3.9}$$

Then using (1.3), (3.7)–(3.9), we have that

$$\begin{aligned} \Delta\lambda(t_j) &= \lambda(t_j^+) - \lambda(t_j) \\ &= \frac{1}{c}[X^{2n+2}(t_j^+) + (n+1)Y^2(t_j^+)]^{\frac{n+2}{2n+2}} - \lambda(t_j) \\ &= \frac{1}{c}\{[X(t_j) + \widetilde{I}_j(X(t_j), Y(t_j))]^{2n+2} + (n+1)[Y(t_j) + \widetilde{J}_j(X(t_j), Y(t_j))]^2\}^{\frac{n+2}{2n+2}} - \lambda(t_j) \\ &= \frac{1}{c}\{[\gamma_j X(t_j)]^{2n+2} + (n+1)[\gamma_j^{n+1} Y(t_j)]^2\}^{\frac{n+2}{2n+2}} - \lambda(t_j) \\ &= \frac{1}{c}\{\gamma_j^{2n+2}[X^{2n+2}(t_j) + (n+1)Y^2(t_j)]\}^{\frac{n+2}{2n+2}} - \lambda(t_j) \\ &= \gamma_j^{n+2}\lambda(t_j) - \lambda(t_j) = (\gamma_j^{n+2} - 1)\lambda(t_j) \\ &=: J_j^*(\lambda(t_j), \theta(t_j)) \end{aligned} \tag{3.10}$$

for $j = 1, 2, \ldots, k$.

By using (3.7), we have that for $j = 1, 2, \ldots, k$ there is

$$X(t_j) = c^{\frac{1}{n+2}} \lambda^{\frac{1}{n+2}}(t_j) X_0(\theta(t_j)T_0), \qquad Y(t_j) = c^{\frac{n+1}{n+2}} \lambda^{\frac{n+1}{n+2}}(t_j) X_0(\theta(t_j)T_0), \tag{3.11}$$

and

$$X(t_j^+) = c^{\frac{1}{n+2}} \lambda^{\frac{1}{n+2}}(t_j^+) X_0(\theta(t_j^+)T_0), \qquad Y(t_j^+) = c^{\frac{n+1}{n+2}} \lambda^{\frac{n+1}{n+2}}(t_j^+) X_0(\theta(t_j^+)T_0). \tag{3.12}$$

Then using (3.10) and (3.12) we have that for $j = 1, 2, \ldots, k$,

$$\begin{aligned} X(t_j^+) &= c^{\frac{1}{n+2}}[\lambda(t_j) + J_j^*(\lambda(t_j), \theta(t_j))]^{\frac{1}{n+2}} X_0(\theta(t_j^+)T_0) \\ &= c^{\frac{1}{n+2}}[\lambda(t_j) + (\gamma_j^{n+2} - 1)(\lambda(t_j)]^{\frac{1}{n+2}} X_0(\theta(t_j^+)T_0) \\ &= \gamma_j c^{\frac{1}{n+2}} \lambda^{\frac{1}{n+2}}(t_j) X_0(\theta(t_j^+)T_0). \end{aligned} \tag{3.13}$$

Combining $\gamma_j > 0$, (3.9), (3.12) and (3.13), we have that for $j = 1, 2, \ldots, k$,

$$X(t_j) = c^{\frac{1}{n+2}} \lambda^{\frac{1}{n+2}}(t_j) X_0(\theta(t_j^+)T_0). \tag{3.14}$$

Similarly, by (1.3), (3.9), (3.10) and (3.12), we have that for $j = 1, 2, \dots, k$,

$$Y(t_j^+) = \gamma_j^{n+1} c^{\frac{n+1}{n+2}} \lambda^{\frac{n+1}{n+2}}(t_j) Y_0(\theta(t_j^+)T_0) = \gamma_j^{n+1} Y(t_j),$$

and thus

$$Y(t_j) = c^{\frac{n+1}{n+2}} \lambda^{\frac{n+1}{n+2}}(t_j) Y_0(\theta(t_j^+)T_0). \tag{3.15}$$

By (3.11), (3.14) and (3.15), we have $\theta(t_j^+) = \theta(t_j)$. Then, for $j = 1, 2, \dots, k$,

$$\Delta\theta(t_j) = \theta(t_j^+) - \theta(t_j) = 0 := I_j^*(\lambda(t_j), \theta(t_j)). \tag{3.16}$$

As a result, under the transformation Ψ_0, system (3.4) is changed into

$$\begin{cases} \dot{\theta} = \frac{\partial H}{\partial \lambda}, & t \neq t_j, \\ \dot{\lambda} = -\frac{\partial H}{\partial \theta}, & t \neq t_j, \\ \Delta\theta(t_j) = I_j^*(\lambda(t_j), \theta(t_j)), \\ \Delta\lambda(t_j) = J_j^*(\lambda(t_j), \theta(t_j)), & j = 1, 2, \dots, k, \end{cases} \tag{3.17}$$

where $I_j^*(\lambda(t_j), \theta(t_j)) = 0$, $J_j^*(\lambda(t_j), \theta(t_j)) = (\gamma_j^{n+2} - 1)\lambda(t_j)$ and $H(\lambda, \theta, t) = H_0(\lambda) + R(\lambda, \theta, t)$ with

$$H_0(\lambda) = d \cdot A^n \cdot \lambda^{\frac{2(n+1)}{n+2}}, \qquad d = \frac{1}{2(n+1)} c^{\frac{2(n+1)}{n+2}}$$

and

$$R(\lambda, \theta, t) = \sum_{i=0}^{n} \frac{p_i(t)}{i+1} A^{i-n-1} (c^\alpha X_0(\theta T_0))^{i+1} \lambda^{\alpha(i+1)}.$$

4 Approximation lemma

In this section, we make use of the Jackson–Moser–Zehnder approximate lemma (see [28,29,34] for the detail) to approximate the smooth periodic function R by a real analytic periodic function R_ε. Some estimates of R_ε and the remainder $R^\varepsilon = R - R_\varepsilon$ are also given for the later application.

Let $\mathbf{T}^1_\varepsilon = \{t \in \mathbf{C}/\mathbf{Z} : |\mathrm{Im}\, t| < \varepsilon\}$ for any $\varepsilon > 0$. By the Jackson–Moser–Zehnder lemma (see Lemma 6.1 in [30]), for each $p_i \in C^\gamma(\mathbf{T^1})$, $i = 0, 1, \dots, n$, and any $\varepsilon > 0$, there is a real analytic function (a complex value function $f(t)$ of complex variable t in some domain in $\mathbf{C}$ is called real analytic if it is analytic in the domain and is real for real argument t) $p_{i,\varepsilon}(t)$ from $\mathbf{T}^1_\varepsilon$ to $\mathbf{C}$ such that

$$\sup_{t \in \mathbf{T^1}} |p_{i,\varepsilon}(t) - p_i(t)| \leq C\varepsilon^\gamma \|p_i\|_{C^\gamma}$$

and

$$\sup_{t \in \mathbf{T}^1_\varepsilon} |p_{i,\varepsilon}(t)| \leq C\|p_i\|_{C^\gamma}.$$

Write

$$R(\lambda, \theta, t) = R_\varepsilon(\lambda, \theta, t) + R^\varepsilon(\lambda, \theta, t),$$

where

$$R_\varepsilon(\lambda, \theta, t) = \sum_{i=0}^{n} \frac{1}{i+1} A^{i-n-1} c^{\frac{i+1}{n+2}} X_0^{i+1}(\theta T_0) \lambda^{\frac{i+1}{n+2}} p_{i,\varepsilon}(t),$$

$$R^{\varepsilon}(\lambda,\theta,t)=\sum_{i=0}^{n}\frac{1}{i+1}A^{i-n-1}c^{\frac{i+1}{n+2}}X_0^{i+1}(\theta T_0)\lambda^{\frac{i+1}{n+2}}(p_i(t)-p_{i,\varepsilon}(t)).$$

Then, we have

$$H=H_0(\lambda)+R_{\varepsilon}(\lambda,\theta,t)+R^{\varepsilon}(\lambda,\theta,t), \tag{4.1}$$

where

$$H_0(\lambda)=d\cdot A^n\cdot\lambda^{\frac{2(n+1)}{n+2}},\qquad d=\frac{1}{2(n+1)}c^{\frac{2(n+1)}{n+2}}. \tag{4.2}$$

We introduce two definitions.

Definition 4.1. Given constants p and q, for a complex valued function $f=f(\lambda,\theta,t,A)$: $(\lambda,\theta,t)\in[1,+\infty)\times\mathbf{T^1}\times\mathbf{T}^1_{\varepsilon}\to\mathbf{C}$, where $A\gg1$ is a large constant, we say that

$$f=O_{\varepsilon}(A^p\lambda^q),$$

if f is C^{∞} in $(\lambda,\theta)\in[1,+\infty)\times\mathbf{T^1}$ and is analytic in $t\in\mathbf{T}^1_{\varepsilon}$ and for all nonnegative integers k and l, there is

$$\sup_{(\theta,t)\in\mathbf{T^1}\times\mathbf{T}^1_{\varepsilon}}|(D_{\lambda})^k(D_{\theta})^l f(\lambda,\theta,t,A)|<C_{k,l}A^p\lambda^{q-k},\quad\lambda\gg1,$$

where $C_{k,l}$ is a constant depending on k and l.

Definition 4.2. Given constants p and q, for a function $f=f(\lambda,\theta,t,A)$: $(\lambda,\theta,t)\in[1,+\infty)\times\mathbf{T^1}\times\mathbf{T^1}\to\mathbf{R}$, where $A\gg1$ is a large constant, we say that

$$f=O(A^p\lambda^q),$$

if f is C^{∞} in $(\lambda,\theta)\in[1,+\infty)\times\mathbf{T^1}$ and C^1 in $t\in\mathbf{T^1}$ and for all nonnegative integers k and l, there is

$$\sup_{(\theta,t)\in\mathbf{T^1}\times\mathbf{T^1}}|(D_{\lambda})^k(D_{\theta})^l f(\lambda,\theta,t,A)|<C_{k,l}A^p\lambda^{q-k},\quad\lambda\gg1,$$

where $C_{k,l}$ is a constant depending on k and l.

Lemma 4.3.

(i) *If* $f_1=O(A^{p_1}\lambda^{q_1}), f_2=O(A^{p_2}\lambda^{q_2})$, *then* $f_1\cdot f_2=O(A^{p_1+p_2}\lambda^{q_1+q_2})$;

(ii) *If* $f=O(A^p\lambda^{q_1}), g(\lambda)=O(\lambda^{q_2})$ *satisfy* $|g(\lambda)|\ge c\lambda^{q_2}$ *for* $\lambda\ge\lambda_0$, *and* $c>0, q_2>0$, *then* $f^*(\lambda,\theta,t):=f(g(\lambda),\theta,t)=O(A^p\lambda^{q_1q_2})$;

(iii) *If* $f=O(A^p\lambda^q), u=O(A^p\lambda^{q_1}), v=O(A^p\lambda^{q_2})$ *and* $q_1<1, q_2<0$, *then* $f^{**}(\lambda,\theta,t):=f(\lambda+u,\theta+v,t)=O(A^p\lambda^q)$.

Proof. (i). Since

$$(D^k_{\lambda}D^l_{\theta})(f_1\cdot f_2)=\sum_{i=0}^{k}\sum_{j=0}^{l}C^i_kC^j_l(D^{k-i}_{\lambda}D^{l-j}_{\theta}f_1)\cdot(D^i_{\lambda}D^j_{\theta}f_2),$$

by Definition 4.2, it follows that

$$f_1\cdot f_2=O(A^{p_1+p_2}\lambda^{q_1+q_2}).$$

(ii). Note that $(D_\lambda^k D_\theta^l) f(g(\lambda),\theta,t)$ is a sum of the terms

$$(D_\theta^l D_g^p f(g(\lambda),\theta,t)) \cdot (D_\lambda^{m_1} g) \cdot (D_\lambda^{m_2} g) \cdots (D_\lambda^{m_p} g)$$

with $\sum_{i=1}^p m_i = k$. Direct computation leads to the estimate

$$\sup_{(\theta,t)\in\mathbf{T}^1\times\mathbf{T}^1} |(D_\lambda^k)(D_\theta^l) f(g(\lambda),\theta,t))| \le C_{k,l} A^p \lambda^{q_1 q_2 - k},$$

and consequently

$$f^*(\lambda,\theta,t)) = f(g(\lambda),\theta,t)) = O(A^p \lambda^{q_1 q_2}).$$

(iii). We observe that $(D_\lambda^k)(D_\theta^l) f(\lambda+u,\theta+v,t)$ is a sum of the terms

$$D_\theta^l [(D_\phi^q D_\mu^p f(\mu,\phi))(D_\lambda^{m_1}\mu)(D_\lambda^{m_2}\mu)\cdots(D_\lambda^{m_p}\mu)(D_\lambda^{n_1}\phi)(D_\lambda^{n_2}\phi)\cdots(D_\lambda^{n_q}\phi)],$$

where $\mu = \lambda + u,\ \phi = \theta + v,\ 0 \le p+q \le k,\ \sum_{i=1}^p m_i + \sum_{i=1}^q n_i = k$, and

$$\begin{aligned} &D_\theta^l [(D_\phi^q D_\mu^p f(\mu,\phi))(D_\lambda^{m_1}\mu)(D_\lambda^{m_2}\mu)\cdots(D_\lambda^{m_p}\mu)(D_\lambda^{n_1}\phi)(D_\lambda^{n_2}\phi)\cdots(D_\lambda^{n_q}\phi)] \\ &\quad = \sum_{i=1}^{l} C_l^i (D_\theta^i D_\phi^q D_\mu^p f(\mu,\phi)) \cdot D_\theta^{l-i}[D_\lambda^{m_1}\mu)\cdots(D_\lambda^{m_p}\mu)(D_\lambda^{n_1}\phi)\cdots(D_\lambda^{n_q}\phi)], \end{aligned}$$

where $D_\theta^i D_\phi^q D_\mu^p f(\mu,\phi)$ is a sum of the terms

$$(D_\phi^{q+\tilde{q}} D_\mu^{p+\tilde{p}} f(\mu,\phi))(D_\theta^{\tilde{m}_1}\mu)(D_\theta^{\tilde{m}_2}\mu)\cdots(D_\theta^{\tilde{m}_p}\mu)(D_\theta^{\tilde{n}_1}\phi)(D_\theta^{\tilde{n}_2}\phi)\cdots(D_\theta^{\tilde{n}_q}\phi),$$

with $0 \le \tilde{p} + \tilde{q} \le i,\ \sum_{j=1}^{\tilde{p}} \tilde{m}_j + \sum_{j=1}^{\tilde{q}} \tilde{n}_j = i$. Noting that $u = O(A^p \lambda^{q_1})$, $v = O(A^p \lambda^{q_2})$, $q_1 < 1, q_2 < 0$, we have

$$\sup_{(\theta,t)\in\mathbf{T}^1\times\mathbf{T}^1} |(D_\lambda^k)(D_\theta^l) f(\lambda+u,\theta+v,t)| \le C_{k,l} A^p \lambda^{q-k}.$$

As a result, we obtain

$$f^{**}(\lambda,\theta,t) = f(\lambda+u,\theta+v,t) = O(A^p \lambda^q).$$ □

Let $\varepsilon = A^{-\nu}$, where $\nu > 0$ will be specified later. By Definition 4.1 and Definition 4.2, we have

$$R_\varepsilon(\lambda,\theta,t) = O_\varepsilon\big(A^{-1}\lambda^{\frac{n+1}{n+2}}\big), \tag{4.3}$$

$$R^\varepsilon(\lambda,\theta,t) = O\big(A^{-1-\nu\gamma}\lambda^{\frac{n+1}{n+2}}\big). \tag{4.4}$$

In the following, we will omit the constant d in $H_0(\lambda)$ (see (4.2)) without loss of generality.

5 Some transformations

Firstly, we look for a series of symplectic transformations $\Psi_1,\dots,\Psi_N$ such that $H^N := H \circ \Psi_1 \circ \cdots \circ \Psi_N = H_0^N + O(\varepsilon_0)$, $\varepsilon_0 = A^{-\delta}$, $\delta > 0$. The following lemma is similar to Lemma 7.1 in [30] and we refer to [30] for the proof.

Lemma 5.1. *Let $H(\lambda,\theta,t)$ be the same as (4.1). For $A \gg 1$, $\lambda \gg 1$, then there is a symplectic diffeomorphism Ψ_1 depending periodically on t of the form*

$$\Psi_1 : \begin{cases} \lambda = \tilde{\mu} + u_1(\tilde{\mu},\tilde{\phi},t), \\ \theta = \tilde{\phi} + v_1(\tilde{\mu},\tilde{\phi},t), \end{cases}$$

with $u_1 = O_\varepsilon\big(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}\big)$ and $v_1 = O_\varepsilon\big(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}}\big)$. Moreover the transformed Hamiltonian vector field $\Psi_1(X_H) = X_{H^1}$ is of the form

$$H^1(\tilde{\mu},\tilde{\phi},t) = H_0^1(\tilde{\mu},t) + \widetilde{R}_\varepsilon^1(\tilde{\mu},\tilde{\phi},t) + R^\varepsilon \circ \Psi_1(\tilde{\mu},\tilde{\phi},t),$$

where

$$H_0^1(\tilde{\mu},t) = H_0(\tilde{\mu}) + [R_\varepsilon](\tilde{\mu},t), \quad H_0 = dA^n \cdot \tilde{\mu}^{\frac{2(n+1)}{n+2}},$$

$$[R_\varepsilon] = O_\varepsilon\Big(A^{-1}\tilde{\mu}^{\frac{n+1}{n+2}}\Big), \quad \tilde{R}_\varepsilon^1(\tilde{\mu},\tilde{\phi},t) = O_{\frac{\varepsilon}{2}}\left(A^{-2-n}\right) + O_{\frac{\varepsilon}{2}}\Big(A^{u-1-n}\tilde{\mu}^{\frac{1}{n+2}}\Big),$$

$$R^\varepsilon \circ \Psi_1(\tilde{\mu},\tilde{\phi},t) = O\left(A^{-1-v\gamma}\tilde{\mu}^{\frac{n+1}{n+2}}\right).$$

Let

$$\tau > 0, \quad \nu < n(1+\tau), \tag{5.1}$$

and

$$\lambda \in \left[c_1 A^{(n+2)\tau}, c_2 A^{(n+2)\tau}\right], \quad c_2 > c_1 > 0.$$

Repeating the symplectic diffeomorphism in Lemma 5.1 for N times, we get N symplectic transformations $\Psi_1,\dots,\Psi_N$ such that

$$H^N(\mu,\phi,t) = H \circ \Psi_1 \circ \cdots \circ \Psi_N = H_0^N(\mu,t) + R_\varepsilon^N(\mu,\phi,t),$$

where

$$\mu \in \left[c_1 A^{(n+2)\tau}, c_2 A^{(n+2)\tau}\right], \quad c_2 > c_1 > 0,$$

$$H_0^N(\mu,t) = H_0(\mu) + H_1(\mu,t),$$

$$H_0 = d \cdot A^n \cdot \mu^{\frac{2(n+1)}{n+2}}, \quad H_1 = O_{\frac{\varepsilon}{2^N}}\left(A^{-1}\mu^{\frac{n+1}{n+2}}\right),$$

$$R_\varepsilon^N = O_{\frac{\varepsilon}{2^N}}\left(A^{-1-N(1+n)}\mu^{\frac{n+1-N(n+1)}{n+2}}\right) + O_{\frac{\varepsilon}{2^N}}\left(A^{-1+N(v-n)}\mu^{\frac{n+1-Nn}{n+2}}\right) + O\Big(A^{-1-\nu\gamma}\mu^{\frac{n+1}{n+2}}\Big).$$

Now the corresponding unforced equation in (3.17) can be changed into

$$\begin{cases} \dot{\phi} = \frac{\partial H^N}{\partial \mu} = \frac{\partial H_0(\mu)}{\partial \mu} + \frac{\partial H_1(\mu,t)}{\partial \mu} + \frac{\partial R_\varepsilon^N(\mu,\phi,t)}{\partial \mu}, \\ \dot{\mu} = -\frac{\partial H^N}{\partial \phi} = -\frac{\partial R_\varepsilon^N(\mu,\phi,t)}{\partial \phi}, \end{cases} \tag{5.2}$$

where

$$\frac{\partial H_0(\mu)}{\partial \mu} = d \cdot \frac{2n+2}{n+2} A^n \mu^{\frac{n}{n+2}}$$

and we omit the constant $d \cdot \frac{2n+2}{n+2}$ for simplicity in the following arguments. Define the diffeomorphism

$$\Psi: \quad \rho = \frac{\partial \mu^{\frac{2n+2}{n+2}}}{\partial \mu} = \frac{2n+2}{n+2}\mu^{\frac{n}{n+2}}, \qquad \phi = \phi, \tag{5.3}$$

and we get

$$\dot{\rho} = \frac{n(2n+2)}{(n+2)^2} \mu^{\frac{-2}{n+2}} \dot{\mu}.$$

Then we have

$$\dot{\rho} = O\left(A^{-1-N(n+1)} \mu^{\frac{n-1-N(n+1)}{n+2}}\right) + O\left(A^{-1+N(\nu-n)} \mu^{\frac{n-1-Nn}{n+2}}\right) + O\left(A^{-1-\nu\gamma} \mu^{\frac{n-1}{n+2}}\right),$$

$$\dot{\phi} = \rho + r(\rho,t) + O\left(A^{-1-N(n+1)} \mu^{\frac{-1-N(n+1)}{n+2}}\right) + O\left(A^{-1-N(\nu-n)} \mu^{\frac{-1-Nn}{n+2}}\right) + O\left(A^{-1-\nu\gamma} \mu^{\frac{-1}{n+2}}\right),$$

where $r(\rho,t) = \frac{\partial H_1(\mu,t)}{\partial \mu}$ with $\mu = \left(\frac{n+2}{2n+2}\rho\right)^{\frac{n+2}{n}}$. Thus

$$r(\rho,t) = O\left(A^{-1} \mu^{\frac{-1}{n+2}}\right) = O\left(A^{-1}\left(\frac{n+2}{2n+2}\rho\right)^{-\frac{1}{n}}\right) = O\left(A^{-1}\rho^{-\frac{1}{n}}\right).$$

Noting that $\mu \in [c_1 A^{(n+2)\tau}, c_2 A^{(n+2)\tau}]$, we have

$$\rho \in \left[c_1 \frac{2n+2}{n+2} A^{n\tau}, c_2 \frac{2n+2}{n+2} A^{n\tau}\right]. \tag{5.4}$$

It follows that

$$\dot{\rho} = O\left(A^{-1-N(n+1)+[n-1-N(n+1)]\tau}\right) + O\left(A^{-1+N(\nu-n)+(n-1-Nn)\tau}\right) + O\left(A^{-1-\nu\gamma+(n-1)\tau}\right),$$

$$\dot{\phi} = \rho + r(\rho,t) + O\left(A^{[-1-N(n+1)](1+\tau)}\right) + O\left(A^{-1+N(\nu-n)+(-1-Nn)\tau}\right) + O\left(A^{-1-\nu\gamma-\tau}\right).$$

When $N \gg 1$ and $\nu < n(1+\tau)$, we have

$$-1 + N(\nu-n) + (n-1-Nn)\tau = N[\nu - n(1+\tau)] + (n-1)\tau - 1 < 0,$$
$$-1 + N(\nu-n) + (-1-Nn)\tau = N[\nu - n((1+\tau)] - (1+\tau) < 0.$$

When $N \gg 1$ and $\tau > 0$, we have

$$-1 - N(n+1) + [n-1-N(n+1)]\tau < 0, \quad [-1-N(n+1)](1+\tau) < 0.$$

Note that $-1-\nu\gamma-\tau < -1-\nu\gamma+(n-1)\tau < n-1-\nu\gamma+(n-1)\tau$. Let

$$n - 1 - \nu\gamma + (n-1)\tau < 0, \tag{5.5}$$

Then, by (5.1) and (5.5), we have

$$\frac{(n-1)(1+\tau)}{\gamma} < \nu < n(1+\tau). \tag{5.6}$$

Since $\gamma > 1 - \frac{1}{n}$, we have $(n-1)/\gamma < n$. Then, when $\tau > 0$ and $\nu \in \left(\frac{(n-1)(1+\tau)}{\gamma}, n(1+\tau)\right)$, there is $\delta > 0$ and (5.2) can be changed into

$$\begin{cases} \dot{\phi} = \rho + r(\rho,t) + f(\rho,\phi,t) = \rho + r(\rho,t) + O(A^{-\delta}), \\ \dot{\rho} = g(\rho,\phi,t) = O(A^{-\delta}), \end{cases} \tag{5.7}$$

where $\phi \in \mathbf{T^1}, r(\rho,t) = O(A^{-1}\rho^{-\frac{1}{n}})$ and $\rho \in [c_3 A^{n\tau}, c_4 A^{n\tau}]$ for $c_4 > c_3 > 0$ given by (5.4).

Next we compute the transformed impulsive forces in (3.17). Based on the symplectic transformation Ψ_1 in Lemma 5.1, we see from the implicit function theorem that

$$\tilde{\mu} = \lambda + u(\lambda,\theta,t), \qquad \tilde{\phi} = \theta + v(\lambda,\theta,t).$$

Under the symplectic transformation Ψ_1, we see that the jumps $\Delta\theta(t_j)$ and $\Delta\lambda(t_j)$ in (3.17) can be changed into

$$\begin{cases} \Delta\tilde{\phi}(t_j) := \tilde{\phi}(t_j^+) - \tilde{\phi}(t_j) = \tilde{I}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j)), \\ \Delta\tilde{\mu}(t_j) := \tilde{\mu}(t_j^+) - \tilde{\mu}(t_j) = \tilde{J}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j)), \end{cases} \tag{5.8}$$

where $j = 1,2,\ldots,k$.

In the same way, under the symplectic transformation Ψ_2, the jumps $\Delta\tilde{\phi}(t_j)$ and $\Delta\tilde{\mu}(t_j)$ can be changed into new forms

$$\begin{cases} \Delta\bar{\phi}(t_j) := \bar{\phi}(t_j^+) - \bar{\phi}(t_j) = \bar{I}_j^*(\bar{\mu}(t_j),\bar{\phi}(t_j)), \\ \Delta\bar{\mu}(t_j) := \bar{\mu}(t_j^+) - \bar{\mu}(t_j) = \bar{J}_j^*(\bar{\mu}(t_j),\bar{\phi}(t_j)), \end{cases} \tag{5.9}$$

where $j = 1,2,\ldots,k$. Repeating this procedure by the symplectic transformations $\Psi_1,\ldots,\Psi_N$, the jumps in (3.17) are finally changed into

$$\begin{cases} \Delta\phi(t_j) := \phi(t_j^+) - \phi(t_j) = I_j^{**}(\mu(t_j),\phi(t_j)), \\ \Delta\mu(t_j) := \mu(t_j^+) - \mu(t_j) = J_j^{**}(\mu(t_j),\phi(t_j)), \end{cases} \tag{5.10}$$

where $j = 1,2,\ldots,k$. Combining (5.2) and (5.10), we see that (3.17) can be transformed into

$$\begin{cases} \dot{\phi} = \frac{\partial H_0(\mu)}{\partial \mu} + \frac{\partial H_1(\mu,t)}{\partial \mu} + \frac{\partial R_\varepsilon^N(\mu,\phi,t)}{\partial \mu}, \\ \dot{\mu} = -\frac{\partial R_\varepsilon^N(\mu,\phi,t)}{\partial \phi}, \quad t \neq t_j; \\ \Delta\phi(t_j) = I_j^{**}(\mu(t_j),\phi(t_j)), \\ \Delta\mu(t_j) = J_j^{**}(\mu(t_j),\phi(t_j)), \quad j = 1,2,\ldots,k. \end{cases} \tag{5.11}$$

Similarly, under Ψ defined by (5.3), system (5.11) can be transformed into

$$\begin{cases} \dot{\phi} = \rho + r(\rho,t) + f(\rho,\phi,t) = \rho + r(\rho,t) + O(A^{-\delta}), \\ \dot{\rho} = g(\rho,\phi,t) = O(A^{-\delta}), \quad t \neq t_j; \\ \Delta\phi(t_j) = I_j^{**1}(\rho(t_j),\phi(t_j)), \\ \Delta\rho(t_j) = J_j^{**1}(\rho(t_j),\phi(t_j)), \quad j = 1,2,\ldots,k, \end{cases} \tag{5.12}$$

where $\phi \in \mathbf{T^1}, r(\rho,t) = O(A^{-1}\rho^{-\frac{1}{n}})$ and $\rho \in [c_3 A^{n\tau}, c_4 A^{n\tau}]$.

It should be pointed out that, although we have not been able to formulate explicitly $I_j^{**1}(\rho(t_j),\phi(t_j))$ and $J_j^{**1}(\rho(t_j),\phi(t_j))$, we can implicitly express them. We will calculate the estimates of the impulsive functions $I_j^{**1}(\rho(t_j),\phi(t_j))$ and $J_j^{**1}(\rho(t_j),\phi(t_j))$ in next section.

6 Some estimates

In this section, we will establish some estimates for impulsive functions $I_j^{**1}(\rho,\phi)$ and $J_j^{**1}(\rho,\phi)$. To this end, we first give the estimates of $I_j^{**}(\mu,\phi)$ and $J_j^{**}(\mu,\phi)$. In this whole section and in the sequel, all the occurrences of j mean $j = 1,2,\ldots,k$.

Lemma 6.1. *Assume that the conditions in Theorem 1.1 are satisfied. Let $\mu(t_j)=\mu, \phi(t_j)=\phi$. We have the following estimates*

$$I_j^{**}(\mu,\phi)=O(A^{-1-n}\mu^{-\frac{n+1}{n+2}}),$$
$$J_j^{**}(\mu,\phi)=(\gamma_j^{n+2}-1)\mu+f_j(\mu,\phi)$$

with $f_j(\mu,\phi)=O(A^{-1-n}\mu^{\frac{1}{n+2}})$, where $I_j^{**}(\mu,\phi)$ and $J_j^{**}(\mu,\phi)$ are given by (5.10).

Proof. For $(\lambda,\theta)\in[c_1A^{(n+2)\tau},c_2A^{(n+2)\tau}]\times\mathbf{T^1}$, from Lemma 5.1, the symplectic diffeomorphism Ψ_1 is of the form

$$\Psi_1:\lambda=\tilde{\mu}+u_1(\tilde{\mu},\tilde{\phi},t),\qquad \theta=\tilde{\phi}+v_1(\tilde{\mu},\tilde{\phi},t),\tag{6.1}$$

where $(\tilde{\mu},\tilde{\phi})\in[c_1A^{(n+2)\tau},c_2A^{(n+2)\tau}]\times\mathbf{T^1}, u_1=O(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}), v_1=O(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}})$. By the implicit function theorem, we have

$$\tilde{\mu}=\lambda+u(\lambda,\theta,t),\qquad \tilde{\phi}=\theta+v(\lambda,\theta,t),\tag{6.2}$$

where $|u|<CA^{-1-n}\lambda^{\frac{1}{n+2}}$ and $|v|<CA^{-1-n}\lambda^{-\frac{n+1}{n+2}}$.

Next we show that

$$u=O\left(A^{-1-n}\lambda^{\frac{1}{n+2}}\right),\qquad v=O\left(A^{-1-n}\lambda^{-\frac{n+1}{n+2}}\right).\tag{6.3}$$

Indeed, we see from Lemma 5.1 that

$$\begin{cases}\lambda=\tilde{\mu}+\frac{\partial S_1}{\partial\theta}=\tilde{\mu}+\nu(\tilde{\mu},\theta,t),\\ \tilde{\phi}=\theta+\frac{\partial S_1}{\partial\tilde{\mu}}=\theta+g(\tilde{\mu},\theta,t),\end{cases}\tag{6.4}$$

where

$$\nu(\tilde{\mu},\theta,t)=O\left(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}\right),\qquad g(\tilde{\mu},\theta,t)=O\left(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}}\right).$$

From (6.2) and (6.4) we know

$$u=-\nu(\lambda+u,\theta,t).\tag{6.5}$$

If $\tilde{\mu}$ and λ are large, then $|D_{\tilde{\mu}}\nu|\le 1/2$, so that u is uniquely determined by the contraction principle. Moreover, the implicit function theorem implies that u is C^∞ with respect to $(\lambda,\theta)\in[c_1A^{(n+2)\tau},c_2A^{(n+2)\tau}]\times\mathbf{T^1}$. We claim that

$$u=O\left(A^{-1-n}\lambda^{\frac{1}{n+2}}\right).\tag{6.6}$$

Indeed, applying $(D_\lambda)^l$ to equation (6.5), the right hand side is a sum of the terms

$$(D_{\tilde{\mu}}^p)(D_\lambda^{j_1}(\lambda+u))(D_\lambda^{j_2}(\lambda+u))\cdots(D_\lambda^{j_p}(\lambda+u)),\tag{6.7}$$

with $1\le p\le l$ and $\sum_{i=1}^p j_i=l$. The highest order term is the one with $p=1$, namely $(D_{\tilde{\mu}}\nu)D_\lambda^n u$. Note that $|u|<CA^{-1-n}\lambda^{\frac{1}{n+2}}$. Assuming that for $j\le n-1$ the estimates $|D_\lambda^j u|<CA^{-1-n}\lambda^{\frac{1}{n+2}-j}$ hold true, then inductively, from (6.4) and (6.5) we can conclude that the same estimate holds true for $j=n$. In fact, from (6.4) we have

$$|D_{\tilde{\mu}}^p\nu|<CA^{-1-n}\lambda^{\frac{1}{n+2}-p},$$

which yields

$$|(1-D_{\tilde{\mu}}\nu)D_{\lambda}^{l}u| \leq CA^{-1-n}\lambda^{\frac{1}{n+2}-p}\lambda^{1-j_1}\cdots\lambda^{1-j_p} < CA^{-1-n}\lambda^{\frac{1}{n+2}-l}.$$

It follows that

$$|D_{\lambda}^{l}u| < CA^{-1-n}\lambda^{\frac{1}{n+2}-l}.$$

The estimates of $(D_\theta)^j(D_\lambda)^i u$ can be proved similarly. Thus, the claim (6.6) is valid. Similarly, one also has

$$v = O\left(A^{-1-n}\lambda^{-\frac{n+1}{n+2}}\right).$$

Under the symplectic transformation Ψ_1, the jumps $\Delta\theta(t_j)$ and $\Delta\lambda(t_j)$ in (3.17) can be changed into $\tilde{I}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j))$ and $\tilde{J}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j))$ (see (5.8)). Then using (3.17), (5.8), (6.1) and (6.2), we have

$$\begin{aligned}
\tilde{I}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j)) &= \tilde{\phi}(t_j^+) - \tilde{\phi}(t_j)\\
&= \theta(t_j^+) + v(\lambda(t_j^+),\theta(t_j^+),t_j) - \theta(t_j) - v(\lambda(t_j),\theta(t_j),t_j)\\
&= I_j^*(\lambda(t_j),\theta(t_j)) + v(\lambda(t_j) + J_j^*(\lambda(t_j),\theta(t_j)),\theta(t_j)\\
&\quad + I_j^*(\lambda(t_j),\theta(t_j)),t_j) - v(\lambda(t_j),\theta(t_j),t_j)\\
&= v[\gamma_j^{n+2}(\tilde{\mu}(t_j) + u_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j)),\tilde{\phi}(t_j) + v_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),t_j]\\
&\quad - v[\tilde{\mu}(t_j) + u_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),\tilde{\phi}(t_j) + v_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),t_j],\\
\tilde{J}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j)) &= \tilde{\mu}(t_j^+) - \tilde{\mu}(t_j)\\
&= \lambda(t_j^+) + u(\lambda(t_j^+),\theta(t_j^+),t_j) - \lambda(t_j) - u(\lambda(t_j),\theta(t_j),t_j)\\
&= J_j^*(\lambda(t_j),\theta(t_j)) + u(\lambda(t_j) + J_j^*(\lambda(t_j),\theta(t_j)),\theta(t_j)\\
&\quad + I_j^*(\lambda(t_j),\theta(t_j)),t_j) - u(\lambda(t_j),\theta(t_j),t_j)\\
&= (\gamma_j^{n+2}-1)\lambda(t_j) + u(\gamma_j^{n+2}\lambda(t_j),\theta(t_j),t_j) - u(\lambda(t_j),\theta(t_j),t_j)\\
&= (\gamma_j^{n+2}-1)\tilde{\mu}(t_j) + (\gamma_j^{n+2}-1)u_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j)\\
&\quad + u(\gamma_j^{n+2}(\tilde{\mu}(t_j) + u_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j)),\tilde{\phi}(t_j) + v_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),t_j)\\
&\quad - u(\tilde{\mu}(t_j) + u_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),\tilde{\phi}(t_j) + v_1(\tilde{\mu}(t_j),\tilde{\phi}(t_j),t_j),t_j)\\
&=: (\gamma_j^{n+2}-1)\tilde{\mu}(t_j) + \tilde{f}_j(\tilde{\mu}(t_j),\tilde{\phi}(t_j)).
\end{aligned}$$

It follows from

$$\begin{aligned}
u_1 &= O\left(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}\right), \qquad v_1 = O\left(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}}\right),\\
u &= O\left(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}\right), \qquad v = O\left(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}}\right)
\end{aligned}$$

and Lemma 4.3 that

$$\tilde{I}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j)) = O\left(A^{-1-n}\tilde{\mu}^{-\frac{n+1}{n+2}}\right), \qquad \tilde{f}_j(\tilde{\mu}(t_j),\tilde{\phi}(t_j)) = O\left(A^{-1-n}\tilde{\mu}^{\frac{1}{n+2}}\right). \tag{6.8}$$

Similarly, under the symplectic transformation Ψ_2, the jumps $\tilde{I}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j))$ and $\tilde{J}_j^*(\tilde{\mu}(t_j),\tilde{\phi}(t_j))$ can be changed into $\bar{I}_j^*(\bar{\mu}(t_j),\bar{\phi}(t_j))$ and $\bar{J}_j^*(\bar{\mu}(t_j),\bar{\phi}(t_j))$ (see (5.9)). Moreover, there are

$$\bar{I}_j^*(\bar{\mu}(t_j),\bar{\phi}(t_j)) = O\left(A^{-1-n}\bar{\mu}^{-\frac{n+1}{n+2}}\right)$$

and

$$\bar{J}_j^*(\bar{\mu}(t_j),\bar{\phi}(t_j)) = (\gamma_j^{n+2}-1)\bar{\mu}(t_j) + \bar{f}_j(\bar{\mu}(t_j),\bar{\phi}(t_j))$$

with

$$\bar{f}_j(\bar{\mu}(t_j),\bar{\phi}(t_j)) = O\Big(A^{-1-n}\bar{\mu}^{\frac{1}{n+2}}\Big).$$

Finally, by repeating this procedure and noting the fact that $\Psi = \Psi_m \circ \Psi_{m-1} \circ \cdots \circ \Psi_1$ transforms (3.17) into (5.11), we have

$$I_j^{**}(\mu(t_j),\phi(t_j)) = O\Big(A^{-1-n}\bar{\mu}^{-\frac{n+1}{n+2}}\Big)$$

and

$$J_j^{**}(\mu(t_j),\phi(t_j)) = (\gamma_j^{n+2}-1)\mu(t_j) + f_j(\mu(t_j),\phi(t_j))$$

with $f_j(\mu(t_j),\phi(t_j)) = O\big(A^{-1-n}\mu^{\frac{1}{n+2}}\big)$. This completes the proof of Lemma 6.1. □

Lemma 6.2. *Under the assumptions of Theorem 1.1, we have*

$$I_j^{**1}(\rho(t_j),\phi(t_j)) = O\Big(A^{-1-n}\rho^{-\frac{n+1}{n}}\Big),$$

and

$$J_j^{**1}(\rho(t_j),\phi(t_j)) = (\gamma_j^{n}-1)\rho(t_j) + \tilde{g}_j(\rho(t_j),\phi(t_j))$$

with $\tilde{g}_j(\rho(t_j),\phi(t_j)) = O\big(A^{-1-n}\rho^{-\frac{1}{n}}\big)$, *where* $I_j^{**1}(\rho(t_j),\phi(t_j))$ *and* $J_j^{**1}(\rho(t_j),\phi(t_j))$ *are given by (5.12).*

Proof. By (5.3), (5.12), Lemma 6.1 and Taylor's formula, we have

$$\begin{aligned} I_j^{**1}(\rho(t_j),\phi(t_j)) &= \phi(t_j^+) - \phi(t_j) = I_j^{**}(\mu(t_j),\phi(t_j)) \\ &= I_j^{**}\left(\left(\frac{n+2}{2n+2}\rho(t_j)\right)^{\frac{n+2}{n}},\phi(t_j)\right) \end{aligned}$$

and

$$\begin{aligned} J_j^{**1}(\rho(t_j),\phi(t_j)) &= \rho(t_j^+) - \rho(t_j) = \frac{2n+2}{n+2}\mu^{\frac{n}{n+2}}(t_j^+) - \rho(t_j) \\ &= \frac{2n+2}{n+2}[\mu(t_j) + J_j^{**}(\mu(t_j),\phi(t_j))]^{\frac{n}{n+2}} - \rho(t_j) \\ &= \frac{2n+2}{n+2}[\mu(t_j) + (\gamma_j^{n+2}-1)\mu(t_j) + f_j(\mu(t_j),\phi(t_j))]^{\frac{n}{n+2}} - \rho(t_j) \\ &= \frac{2n+2}{n+2}[\gamma_j^{n+2}\mu(t_j)]^{\frac{n}{n+2}}\left(1 + \frac{f_j(\mu(t_j),\phi(t_j))}{\gamma_j^{n+2}\mu(t_j)}\right)^{\frac{n}{n+2}} - \rho(t_j) \\ &= \gamma_j^n\rho(t_j)\left[1 + \frac{n}{n+2}\frac{f_j(\mu(t_j),\phi(t_j))}{\gamma_j^{n+2}\mu(t_j)}\left(1+\xi\frac{f_j(\mu(t_j),\phi(t_j))}{\gamma_j^{n+2}\mu(t_j)}\right)^{-\frac{2}{n+2}}\right] - \rho(t_j) \\ &= (\gamma_j^n-1)\rho(t_j) + \frac{\rho^{-\frac{2}{n}}(t_j) f_j\left(\left(\frac{n+2}{2n+2}\rho(t_j)\right)^{\frac{n+2}{n}},\phi(t_j)\right)}{\frac{n+2}{n}\left(\frac{n+2}{2n+2}\right)^{\frac{n+2}{n}}\gamma_j^2} \\ &\quad \times \left(1+\xi\frac{f_j\left(\left(\frac{n+2}{2n+2}\rho(t_j)\right)^{\frac{n+2}{n}},\phi(t_j)\right)}{\gamma_j^{n+2}\left(\frac{n+2}{2n+2}\rho(t_j)\right)^{\frac{n+2}{n}}}\right)^{-\frac{2}{n+2}} \end{aligned}$$

$$=: (\gamma_j^n - 1)\rho(t_j) + \tilde{g}_j(\rho(t_j), \phi(t_j))$$

by (6.8), where $\xi \in (0,1)$. Then by Lemma 4.3 and Lemma 6.1, we have

$$I_j^{**1}(\rho(t_j), \phi(t_j)) = O\Big(A^{-1-n}\rho^{\frac{n+2}{n}\cdot(-\frac{n+1}{n+2})}\Big) = O\Big(A^{-1-n}\rho^{-\frac{n+1}{n}}\Big),$$

$$\tilde{g}_j(\rho(t_j), \phi(t_j)) = O\Big(\rho^{-\frac{2}{n}}A^{-1-n}\rho^{\frac{n+2}{n}\cdot\frac{1}{n+2}}\Big) = O\Big(A^{-1-n}\rho^{-\frac{1}{n}}\Big).$$

This completes the proof of Lemma 6.2. □

7 Proof of Theorem 1.1

The following two lemmas are similar to Lemma 9.2 in [30] and Lemma 6.2 in [3], respectively. We refer to [30] and [3] for the proofs. Let $(\rho(t), \phi(t)) = (\rho(t,\rho,\phi), \phi(t,\rho,\phi))$ be the solution of (5.12) with the initial value $(\rho(0), \phi(0)) = (\rho, \phi)$. Let $\phi_1 = \phi(1), \rho_1 = \rho(1)$.

Lemma 7.1. *If all conditions of Theorem 1.1 hold, then the time one map Φ^1 of the flow Φ^t of (5.12) takes the form of*

$$\Phi^1 : \begin{cases} \phi_1 = \phi + \alpha(\rho) + F(\rho,\phi), \\ \rho_1 = \rho + G(\rho,\phi). \end{cases}$$

Moreover, $\dot{\alpha}(\rho) > 0$ and for any non-negative integers r, s with $r + s \le 5$,

$$\left|\frac{\partial^{r+s}F(\rho,\phi)}{\partial\rho^r\partial\phi^s}\right|, \quad \left|\frac{\partial^{r+s}G(\rho,\phi)}{\partial\rho^r\partial\phi^s}\right| < CA^{-\varepsilon_0},$$

where $\varepsilon_0 = \min(\tau,\delta) > 0$, $(\rho,\phi) \in [c_3A^{n\tau}, c_4A^{n\tau}] \times \mathbf{T^1}$, $c_4 > c_3 > 0$, $A \gg 1$, $\tau > 0$, $\delta > 0$.

Lemma 7.2. *Assume that the conditions of Theorem 1.1 are satisfied, then the time-1 map Φ^1 of (4.1) has the intersection property on $\Omega = \{(\rho,\phi) \mid \rho$ large enough, $\phi \in \mathbf{T^1}\}$, i.e. if Γ is an embedded circle in Ω homotopic to a circle $\rho =$ const. in Ω, then $\Phi^1(\Gamma)\bigcap\Gamma \neq \emptyset$. In particular, Φ^1 has the intersection property on $\Omega = \{(\rho,\phi) \mid c_3A^{n\tau} \le \rho \le c_4A^{n\tau}, \phi \in \mathbf{T^1}\}$, where $c_4 > c_3 > 0, \tau > 0$.*

Now let us state Moser's twist theorem. Let $\mathcal{D}$ be an annulus given by

$$\mathcal{D} \ : \ a \le r \le b, \quad 0 < a < b.$$

For convenience, we introduce for a function $h \in C^l(\mathcal{D})$ the norm

$$|h|_l = \sup_{\mathcal{D},\, m+n\le l}\left|\frac{\partial^{m+n}}{\partial r^m\partial\theta^n}\right|.$$

Theorem 7.3 (Moser's twist theorem). *Let $\alpha(r) \in C^l$ and $|\partial_r\,\alpha(r)| \ge \nu > 0$ on the annulus $\mathcal{D}$ for some l with $l \ge 5$, and ε be a positive number. Then there exists a $\delta > 0$ depending on $\varepsilon, l, \alpha(r)$, such that any area-preserving mapping*

$$M : \begin{cases} \theta_1 = \theta + 2\pi\alpha(r) + f(r,\theta), \\ r_1 = r + g(r,\theta) \end{cases}$$

of $\mathcal{D}$ into $\mathbf{R^2}$ with $f, g \in C^l$ and

$$|f|_l + |g|_l \le \nu\,\delta$$

possesses an invariant curve of the form

$$r = c + u(\xi), \qquad \theta = \xi + v(\xi)$$

in $\mathcal{D}$ where u, v are continuously differentiable, of period 2π and satisfy

$$|u|_1 + |v|_1 < \varepsilon,$$

and c is a constant in (a, b). Moreover, the induced mapping of this curve is given by

$$\xi \to \xi + \omega,$$

where ω is incommensurable with 2π, and satisfies infinitely many conditions

$$\left| \frac{\omega}{2\pi} - \frac{p}{q} \right| \geq \gamma q^{-\tau}$$

with some positive γ, τ, for all integers $q > 0$, p. In fact, each choice of ω in the range of $\alpha(r)$ and satisfying the above inequalities give rise to such an invariant curve.

Moser's twist theorem above can be found in [21, pp. 50–54] (see also [27]). It should be pointed out that the δ does not depend on ν. It should be also noted that the period 2π can be replaced by any period T. In addition, "any area-preserving mapping" can be relaxed to "any mapping which has intersection property".

We are now in a position to prove Theorem 1.1. From Lemma 7.1 and Lemma 7.2, by Moser's twist theorem, Φ^1 has an invariant curve $\widetilde{\Gamma}$ in the annulus $(\rho, \phi) \in [c_3 A^{n\tau}, c_4 A^{n\tau}] \times \mathbf{T^1}$, $c_4 > c_3 > 0$, $A \gg 1$, $\tau > 0$. It follows that the time-one map of the original system has an invariant curve $\widetilde{\Gamma}_{A_0}$. Choosing a sequence $A_0 = A_{m0} \to \infty$ as $m \to \infty$, we have that there are countable many invariant curves $\widetilde{\Gamma}_{A_{m0}}$, clustering at ∞. Therefore any solution of the original system is bounded. This completes the proof of Theorem 1.1.

Remark 7.4. Any solutions starting from the invariant curves $\widetilde{\Gamma}_{A_{m0}}$ $(m = 1, 2, \dots)$ are quasi-periodic with frequencies $(1, \omega_m)$ in time t, where $(1, \omega_m)$ satisfies Diophantine conditions and $\omega_m > C A_{m0}^{n\tau}$. Actually, the frequencies can form a positive Lebesgue set in $\mathbf{R}$.

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Multi-bump solutions of a Schrödinger–Bopp–Podolsky system with steep potential well

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Abstract. In this paper, we study the existence of multi-bump solutions for the following Schrödinger–Bopp–Podolsky system with steep potential well:

$$\begin{cases} -\Delta u + (\lambda V(x) + V_0(x))u + K(x)\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $p \in (4,6), a > 0$ and λ is a parameter. We require that $V(x) \geq 0$ and has a bounded potential well $\Omega = V^{-1}(0)$. Combining this with other suitable assumptions on Ω, V_0 and K, when λ is large enough, we obtain the existence of multi-bump-type solutions u_λ by using variational methods.

Keywords: Schrödinger–Bopp–Podolsky system, penalization method, variational methods.

2020 Mathematics Subject Classification: 35A15, 35B38, 35J60.

1 Introduction and main results

In this paper, we investigate the existence of multi-bump solutions for the following problem with steep potential well:

$$\begin{cases} -\Delta u + (\lambda V(x) + V_0(x))u + K(x)\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2\phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $p \in (4,6), a > 0$ and λ is a parameter.

To illustrate the significance of this article, we first introduce some background about Schrödinger–Bopp–Podolsky system. As mentioned in [10], problem (1.1) is a version of

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the Schrödinger–Bopp–Podolsky system, which is a Schrödinger equation coupled with a Bopp–Podolsky equation. It is worth mentioning that, Podolsky's theory is a second-order gauge theory for the electromagnetic field developed by Bopp [7], independently by Podolsky–Schwed [14]. For some more details about the Bopp–Podolsky equation, we refer to [5, 6, 15] and the references therein.

If $a = V_0(x) = 0, \lambda = K(x) = 1$, system (1.1) gives back the classical Schrödinger–Poisson system as follows:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

which has been first introduced by D'Aprile–Mugnai [9]. The authors studied the existence of radially symmetric solitary waves by using the variational approach method for the above question when $V(x)$ is a constant. In this system, the potential function V is regarded as an external potential, u and ϕ represent the wave functions associated with the particle and electric potential respectively. For more details on the physical aspects of this system, we refer the readers to [4, 8] and the references therein.

In the last decades, the classical Schrödinger–Poisson system has been widely studied under variant assumptions on V and f. By using variational methods, the existence, nonexistence, and multiplicity results are obtained in many papers. For example, when $f(u) = |u|^{p-1}u$ with $p \in (3, 5)$, Cerami and Vaira in [8] studied the following Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + u + K(x)\phi(x)u = a(x)f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3. \end{cases}$$

Without requiring any symmetry property on $K(x)$ and $a(x)$, they proved the existence of the positive ground state and bound state solutions by minimizing energy functional restricted to a Nehari manifold when $K(x)$ and $a(x)$ satisfy different assumptions. After that, Sun et al. in [18] extended the result to a general nonlinear term.

Note that, the steep potential well has been introduced by Bartsch and Wang [3] in the study of nonlinear Schrödinger equation. Our assumptions on V are similar to [11], in which Ding and Tanaka have proven the existence of multi-bump-type solutions for nonlinear Schrödinger equations. After that, more and more researchers have studied multi-bump-type solutions, we refer the readers to the papers [1, 12, 19]. In particular, Zhang and Ma in [21] considered the following system with steep potential well

$$\begin{cases} -\Delta u + (\lambda a(x) + a_0(x))u + K(x)\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

they obtained the existence of multi-bump solutions for (1.2) by using variational methods. Compared with [21], although our paper also studies the existence of multi-bump solutions, it studies a new system which has great significance.

If $a \neq 0$, system (1.1) is a Schrödinger–Bopp–Podolsky system. Based on variational methods, D'Avenia–Siciliano [10] first proved the existence and nonexistence results which depended on the parameters p and q to system

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi + \varepsilon^2\Delta^2\phi = 4\pi u^2, & x \in \mathbb{R}^3. \end{cases} \tag{1.3}$$

Later, for $p \in (2,3]$, Siciliano–Silva [17] obtained the existence and nonexistence of solutions to system (1.3) by means of the fiber map approach and the Implicit Function Theorem. Note that, the authors in [10] and [17] merely considered system (1.3) with subcritical growth, so Liu and Chen in [13] filled the gaps. More precisely, they studied the existence, nonexistence, and asymptotic behavior of ground state solutions to system (1.3) which involves a critical nonlinearity.

Recently, Wang et al. in [20] considered Schrödinger–Bopp–Podolsky system with general nonlinear term:

$$\begin{cases} -\Delta u + \omega u + q^2 \phi u = f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi + \varepsilon^2 \Delta^2 \phi = 4\pi u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.4}$$

where f is a continuous, superlinear, and subcritical nonlinearity. They proved the existence and multiplicity of sign-changing solutions of system (1.4) by using the method of invariant sets of descending flow incorporated with minimax arguments. In addition, the asymptotic behavior of sign-changing solutions was also established.

Motivated by all results mentioned above, it is quite natural to ask, does the system (1.1) have multi-bump solutions? In the present paper, we give an affirmative answer.

In this paper, we make the following assumptions:

- (V_1) $V(x) \in C\left(\mathbb{R}^3, \mathbb{R}^+\right)$ and $\Omega := \operatorname{int} V^{-1}(0)$ is a non-empty bounded set with smooth boundary. Moreover, there is a positive constant M_0 such that the measure of the set $A = \left\{x \in \mathbb{R}^3 : V(x) \le M_0\right\}$ is finite.
- (V_2) There is a $V_0(x) \in C\left(\mathbb{R}^3, \mathbb{R}\right)$ and a constant $M_1 > 1$ such that $|V_0(x)| \le M_1(V(x) + 1)$.
- (V_3) Ω possesses m connected components $\Omega_1, \ldots, \Omega_m$ such that $\overline{\Omega_j} \cap \overline{\Omega \backslash \Omega_j} = \emptyset$, and $\inf_{u \in H_0^1(\Omega_j), |u|_2 = 1} \int_\Omega \left[|\nabla u|^2 + V_0(x) u^2\right] \mathrm{d}x > 0$ for $j = 1, 2, \ldots, m$.

Now, we say something about (V_1): although A and M_0 in (V_1) are not explicitly mentioned in the article, they are used in the proof of Proposition 2.4. Note that the proof of Proposition 2.4 is very similar to Corollary 1.4 in [11], so it is omitted. In [11], Corollary 1.4 is proven by using Proposition 1.1, but the proof of Proposition 1.1 requires the use of A and M_0 to ensure the vanishing of the energy outside the sphere. Please see [11] for details. Therefore, the role of (V_1) is to ensure that Proposition 2.4 holds in our manuscript.

We also assume that

- (K) $K \in L^\infty\left(\mathbb{R}^3\right), K(x) \ge 0$ and $K \not\equiv 0$.

The main result of this paper reads as follows:

Theorem 1.1. *Assume that $(V_1), (V_2), (V_3)$ and (K) hold. Then, for any small $\nu > 0$ and any non-empty subset J of $\{1, 2, \ldots, m\}$, there exist $\Lambda = \Lambda(\nu)$ and $k(\nu) > 0$ such that, when $\lambda > \Lambda$ and $|K|_\infty \le k(\nu)$, (1.1) has a solution $u_\lambda \in H^1\left(\mathbb{R}^3\right)$ satisfying*

$$\left| \int_{\Omega_j} \left[|\nabla u_\lambda|^2 + (\lambda V(x) + V_0(x)) u_\lambda^2\right] \mathrm{d}x - \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c\left(\Omega_j\right) \right| \le \nu, \qquad j \in J$$

and

$$\int_{\mathbb{R}^3 \backslash \Omega_J} \left[|\nabla u_\lambda|^2 + (\lambda V(x) + V_0(x)) u_\lambda^2\right] \mathrm{d}x \le \nu,$$

where $\Omega_J = \bigcup_{j \in J} \Omega_j, c\left(\Omega_j\right)$ are some constants. Moreover, for any sequence of solutions $\{u_{\lambda_n}\}$ with $\lambda_n \to \infty$, going if necessary to a subsequence, u_{λ_n} converges strongly in $H^1\left(\mathbb{R}^3\right)$ to a function u satisfying $u(x) = 0$ for $x \in \mathbb{R}^3 \backslash \Omega_J$.

Remark 1.2. The constant $c\left(\Omega_j\right)$ in Theorem 1.1 is the least energy of all the nontrivial solutions for the following boundary value problem

$$-\Delta u + V_0(x)u = |u|^{p-2}u \quad \text{in } \Omega_j,\, u|_{\partial\Omega_j} = 0.$$

Hence under the assumption of (V_3), $c\left(\Omega_j\right) > 0$.

This paper is organized as follows. In Section 2, we give some variational frameworks. After that, we introduce a modified functional and verify the Palais–Smale condition. In Sections 4 and 5, we give some results on the Nehari manifold and the proof of Theorem 1.1 respectively.

2 Variational frameworks

We consider the following functional space

$$E := \left\{ u \in H^1\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x < \infty \right\}$$

with the inner product

$$(u,v)_E := \int_{\mathbb{R}^3} \left[\nabla u \nabla v + (V(x)+1)uv\right] \mathrm{d}x,$$

and the corresponding norm is $\|u\|_E = (u,u)_E^{1/2}$. It is easy to see that $(E, \|\cdot\|_E)$ is a Hilbert space and the embedding $E \hookrightarrow H^1\left(\mathbb{R}^3\right)$ is continuous. For any open set $D \subset \mathbb{R}^3$, we also define

$$E(D) = \left\{ u \in H^1(D) : \int_D V(x)u^2 \mathrm{d}x < \infty \right\},$$

$$\|u\|_{E(D)} = \int_D \left[|\nabla u|^2 + (V(x)+1)u^2\right] \mathrm{d}x.$$

Note that $\|\cdot\|_{E(D)}$ is equivalent to $\|\cdot\|_{H^1(D)}$ when D is bounded.

Now, we define $\mathcal{D}$ be the completion of $C_0^\infty\left(\mathbb{R}^3\right)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product

$$(u,v)_{\mathcal{D}} = \int_{\mathbb{R}^3} \left(\nabla u \nabla v + a^2 \Delta u \Delta v\right) \mathrm{d}x.$$

Then $\mathcal{D}$ is a Hilbert space, which is continuously embedded into $D^{1,2}\left(\mathbb{R}^3\right)$ and consequently into $L^6(\mathbb{R}^3)$. We denote that $L^q(\mathbb{R}^3)$ is the usual Lebesgue space with the standard norm $\|u\|_q := \left(\int_{\mathbb{R}^3} |u|^q \mathrm{d}x\right)^{\frac{1}{q}}$, $1 \le q < \infty$.

Proposition 2.1 (see [10]). *The space $\mathcal{D}$ is continuously embedded into $L^\infty(\mathbb{R}^3)$.*

By using the Lax–Milgram theorem, for every fixed $u \in E$, there exists a unique solution $\phi_u^a \in \mathcal{D}$ of the second equation in system (1.1). In order to explicitly write such solution (see [15]), we consider that

$$\mathcal{K}(x) = \frac{1 - e^{\frac{-|x|}{a}}}{|x|}.$$

As for $\mathcal{K}$, we have the following fundamental properties from [10].

Proposition 2.2 (see [10]). *For all $y \in \mathbb{R}^3, \mathcal{K}(\cdot - y)$ solves in the sense of distributions*

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi\delta_y.$$

Moreover,

(i) *if $f \in L^1_{loc}\left(\mathbb{R}^3\right)$ and for a.e. $x \in \mathbb{R}^3$, the map $y \in \mathbb{R}^3 \to \frac{f(y)}{|x-y|}$ is summated, then $\mathcal{K} * f \in L^1_{loc}\left(\mathbb{R}^3\right)$;*

(ii) *if $f \in L^p\left(\mathbb{R}^3\right)$ with $1 \leqslant p < \frac{3}{2}$, then $\mathcal{K} * f \in L^q\left(\mathbb{R}^3\right)$ for $q \in \left(\frac{3p}{3-2p}, +\infty\right]$.*

In both cases $\mathcal{K} * f$ solves

$$-\Delta\phi + a^2\Delta^2\phi = 4\pi f.$$

Then if we fix $u \in E$, the unique solution in $\mathcal{D}$ of the second equation in system (1.1) can be expressed by

$$\phi_u^a = \mathcal{K} * (Ku^2) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1 - e^{\frac{-|x-y|}{a}}}{|x-y|} K(y)u^2(y)\mathrm{d}y.$$

Now, let us summarize some properties of ϕ_u^a.

Proposition 2.3 (see [10]). *For every $u, v \in E$, the following statements are correct.*

(i) $\phi_u^a \geqslant 0$.

(ii) *For each $t > 0, \phi_{tu}^a = t^2\phi_u^a$.*

(iii) *If $u_n \rightharpoonup u$ in E, then $\phi_{u_n}^a \rightharpoonup \phi_u^a$ in $\mathcal{D}$.*

(iv) $\|\phi_u^a\|_{\mathcal{D}} \leqslant C\|u\|^2_{\frac{12}{5}} \leqslant C\|u\|_E^2$ *and* $\int_{\mathbb{R}^3} \phi_u^a|u|^2\,\mathrm{d}x \leqslant C\|u\|^4_{\frac{12}{5}} \leqslant C\|u\|_E^4$.

By using the classical reduction argument, system (1.1) can be reduced to a single equation:

$$-\Delta u + (\lambda V(x) + V_0(x))\,u + K(x)\phi_u^a u = |u|^{p-2}u, \qquad x \in \mathbb{R}^3. \tag{2.1}$$

From now on, the solutions of system (1.1) are equal to the solutions of equation (2.1). It is easy to see that the solutions of equation (2.1) can be regarded as critical points of the energy functional $I_\lambda : E \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (\lambda V(x) + V_0(x))\,u^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^a u^2\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\mathrm{d}x.$$

According to (V_1) and (V_3), it is easy to check that I_λ is a well defined C^1 functional in E. Moreover, $\forall\varphi \in E$, we have

$$\left\langle I'_\lambda(u), \varphi\right\rangle = \int_{\mathbb{R}^3}(\nabla u\nabla\varphi + (\lambda V(x) + V_0(x))\,u\varphi)\mathrm{d}x + \int_{\mathbb{R}^3} K(x)\phi_u^a u\varphi\mathrm{d}x - \int_{\mathbb{R}^3} |u|^{p-2}u\varphi\mathrm{d}x.$$

By assumption (V_3), there exist smoothly bounded open sets $\Omega'_1, \Omega'_2, \ldots, \Omega'_m \subset \mathbb{R}^3$ such that $\overline{\Omega_j} \subset \Omega'_j$ and $\overline{\Omega'_i} \cap \overline{\Omega'_j} = \emptyset$ for $i \neq j$. In the following proposition, which is one of the keys of our argument, we will give the positivity of the operator $-\Delta + (\lambda V(x) + V_0(x))$ acting on the space $E(D)$, where D is one of the following sets:

$$D = \mathbb{R}^3, \qquad \Omega'_j\ (j = 1, 2, \ldots, k), \quad \text{or} \quad \mathbb{R}^3 \setminus \bigcup_{j\in J} \Omega'_j \qquad (J \subset \{1, 2, \ldots, k\}).$$

Now, we define a norm $\|\cdot\|_{\lambda,D}$ on $E(D)$ for $\lambda \geq \Lambda_1$ by

$$\|u\|_{\lambda,D}^2 = \int_D \left[|\nabla u|^2 + (\lambda V(x) + V_0(x))\, u^2\right] \mathrm{d}x.$$

We write $\|\cdot\|_\lambda = \|\cdot\|_{\lambda,\mathbb{R}^3}$ for simplicity. From Corollary 1.3 in [11], we can get that there exist $C_{1,\lambda}, C'_{1,\lambda} > 0$ such that

$$C_{1,\lambda}\|u\|_{E(D)} \leq \|u\|_{\lambda,D} \leq C'_{1,\lambda}\|u\|_{E(D)} \quad \text{for } u \in E(D).$$

Proposition 2.4. (see [11]) *There exist $\delta_0, \nu_0 > 0$ such that for any set D and $u \in E(D)$*

$$\delta_0\|u\|_{\lambda,D}^2 \leq \|u\|_{\lambda,D}^2 - (p-1)\nu_0\|u\|_{L^2(D)}^2 \quad \text{for } \lambda \geq \Lambda_1.$$

3 Compactness condition

Since I_λ given in Section 2 does not satisfy the Palais–Smale condition easily, we modify it and establish the compactness conditions in this section. For $t \in \mathbb{R}$ and ν_0 given in Proposition 2.4, set

$$f(t) = \begin{cases} |t|^{p-2}t, & \text{if } |t| \leq \nu_0^{\frac{1}{p-2}}, \\ \nu_0 t, & \text{if } |t| \geq \nu_0^{\frac{1}{p-2}}, \end{cases}$$

and $F(t) = \int_0^t f(s)\mathrm{d}s$. Let $J \subset \{1, 2, \ldots, k\}$ and $\chi_J : \mathbb{R}^3 \to [0,1]$ be the characteristic function of $\Omega'_J := \bigcup_{j\in J}\Omega'_j$. We consider the penalized nonlinearity

$$g(x,t) = \chi_J(x)|t|^{p-2}t + (1-\chi_J(x))\, f(t).$$

Setting $G(t) = \int_0^t g(s)\mathrm{d}s$, we define $J_\lambda : E \to \mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \left(\lambda V(x) + V_0(x) u^2\right)\right) \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^a u^2 \mathrm{d}x - \int_{\mathbb{R}^3} G(x,u)\mathrm{d}x.$$

By using a standard method, one can see that J_λ is of class C^1 and its nontrivial critical points are nontrivial solutions of

$$-\Delta u + (\lambda V(x) + V_0(x))\, u + K(x)\phi_u^a(x) u = g(x,u) \quad \text{in } \mathbb{R}^3.$$

Since $f(t) = |t|^{p-2}t$ for $|t| \leq \nu_0^{\frac{1}{p-2}}$, a critical point u of J_λ solves the original problem (1.1) when it satisfies $|u(x)| \leq \nu_0^{\frac{1}{p-2}}$ for all $x \in \mathbb{R}^3 \backslash \Omega'_J$.

Next, we verify the Palais–Smale condition of J_λ. First of all, the following lemma can give the boundedness of the $(PS)_c$ sequence of J_λ.

Lemma 3.1. *For any $(PS)_c$ sequence $\{u_n\}_n \subset E$ of J_λ, there exists a positive constant $M(c)$ which is independent of $\lambda \geq \Lambda_1$ such that*

$$\limsup_{n\to\infty} \|u_n\|_\lambda^2 \leq M(c).$$

Proof. Due to $\{u_n\}_n$ is the $(PS)_c$ sequence of J_λ, we have

$$J_\lambda(u_n) - \frac{1}{p}\langle J'_\lambda(u_n), u_n\rangle = c + o(1) + \varepsilon_n \|u_n\|_\lambda,$$

where $\varepsilon_n \to \infty$ as $n \to \infty$. Then by using the fact $F(t) - \frac{1}{p}f(t)t \le (\frac{1}{2} - \frac{1}{p})\nu_0 t^2$ for $t \in \mathbb{R}$ and $\int_{\mathbb{R}^3} K(x)\phi_{u_n}^a u_n^2 \mathrm{d}x \ge 0$, we get

$$\begin{aligned}
c + o(1) + \varepsilon_n \|u_n\|_\lambda &= J_\lambda(u_n) - \frac{1}{p}\langle J'_\lambda(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\lambda^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\int_{\mathbb{R}^3} K(x)\phi_{u_n}^a u_n^2 \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3\setminus\Omega'_J}\left(F(u_n) - \frac{1}{p}f(u_n)u_n\right)\mathrm{d}x - \int_{\Omega'_J}\left(F(u_n) - \frac{1}{p}f(u_n)u_n\right)\mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\lambda^2 + \left(\frac{1}{4} - \frac{1}{p}\right)\int_{\mathbb{R}^3} K(x)\phi_{u_n}^a u_n^2 \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^3\setminus\Omega'_J}\left(F(u_n) - \frac{1}{p}f(u_n)u_n\right)\mathrm{d}x \\
&\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_\lambda^2 - \left(\frac{1}{2} - \frac{1}{p}\right)\nu_0\|u_n\|_{L^2}^2.
\end{aligned}$$

Using Proposition 2.4, we obtain

$$\left(\frac{1}{2} - \frac{1}{p}\right)\delta_0\|u_n\|_\lambda^2 \le c + o(1) + \varepsilon_n\|u_n\|_\lambda.$$

Hence, $\|u_n\|_\lambda$ is bounded as $n \to \infty$ and

$$\limsup_{n\to\infty}\|u_n\|_\lambda^2 \le M(c).$$ □

Now we have the following fact.

Lemma 3.2. *When $c > 0$, there exists $\Lambda_1 > 0$, such that J_λ satisfies the Palais–Smale condition at level c on E for $\lambda \ge \Lambda_1$ large enough.*

Proof. By using Lemma 3.1, we know that any $(PS)_c$-sequence $\{u_n\}_n$ is bounded in E. So, going if necessary to a subsequence, we may assume that

$$\begin{aligned}
u_n &\rightharpoonup u \quad \text{in } E \text{ and } H^1(\mathbb{R}^3), \\
u_n &\to u \quad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^3),\ 1 \le q < 6, \\
u_n &\to u \quad \text{a.e. in } \mathbb{R}^3.
\end{aligned}$$

Now we prove that $u_n \to u$ in E. Firstly, it is easy to check that $J'_\lambda(u) = 0$. In fact, by Proposition 2.3, we know that $\phi_{u_n}^a \rightharpoonup \phi_u^a$ in $\mathcal{D}$. For any $\varphi \in C_0^\infty(\mathbb{R}^3)$, since $K(x)u\varphi \in L^{\frac{6}{5}}(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} K(x)u\varphi\left(\phi_{u_n}^a - \phi_u^a\right)\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Similarly,

$$\begin{aligned}
\int_{\mathbb{R}^3} K(x)\varphi\phi_{u_n}^a(u_n - u)\mathrm{d}x &\le |K|_\infty\|\varphi\|_3\|\phi_{u_n}^a\|_6\|u_n - u\|_{L^2(\Omega_\varphi)} \\
&\le C\|u_n - u\|_{L^2(\Omega_\varphi)} \\
&\to 0
\end{aligned}$$

as $n \to \infty$, where Ω_φ is the support of φ. Consequently,

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(K(x)\phi_{u_n}^a u_n\varphi - K(x)\phi_u^a u\varphi\right)\mathrm{d}x\\&\quad=\int_{\mathbb{R}^3}K(x)u\varphi\left(\phi_{u_n}^a-\phi_u^a\right)\mathrm{d}x+\int_{\mathbb{R}^3}K(x)\varphi\phi_{u_n}^a\left(u_n-u\right)\mathrm{d}x\\&\quad\to 0\end{aligned}$$

as $n \to \infty$, thus we see that

$$\begin{aligned}\langle J_\lambda'(u_n)-J_\lambda'(u),\varphi\rangle &= \langle J_\lambda'(u_n),\varphi\rangle-\langle J_\lambda'(u),\varphi\rangle\\&=\int_{\mathbb{R}^3}(\nabla u_n\nabla\varphi+(\lambda V(x)+V_0(x))\,u_n\varphi)\mathrm{d}x+\int_{\mathbb{R}^3}K(x)\phi_{u_n}^a u_n\varphi\mathrm{d}x\\&\quad-\int_{\mathbb{R}^3}(\nabla u\nabla\varphi-(\lambda V(x)+V_0(x))\,u\varphi)\mathrm{d}x-\int_{\mathbb{R}^3}K(x)\phi_u^a u\varphi\mathrm{d}x\\&\quad-\int_{\mathbb{R}^3}g(x,u_n)\varphi\mathrm{d}x+\int_{\mathbb{R}^3}g(x,u)\varphi\mathrm{d}x\\&=o(1).\end{aligned}$$

So $J_\lambda'(u)=0$. Then we have

$$\begin{aligned}&\langle J_\lambda'\left(u_n\right)-J_\lambda'(u),u_n-u\rangle\\&\quad=\langle J_\lambda'\left(u_n\right),u_n-u\rangle-\langle J_\lambda'(u),u_n-u\rangle\\&\quad=\|u_n-u\|_\lambda^2+\int_{\mathbb{R}^3}\left(K(x)\phi_{u_n}^a u_n\left(u_n-u\right)-K(x)\phi_u^a u\left(u_n-u\right)\right)\mathrm{d}x\\&\qquad-\int_{\mathbb{R}^3\backslash\Omega_J'}\left(f\left(u_n\right)-f(u)\right)\left(u_n-u\right)\mathrm{d}x-\int_{\Omega_J'}\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)\left(u_n-u\right)\mathrm{d}x\\&\quad=\|u_n-u\|_\lambda^2+\int_{\mathbb{R}^3}K(x)\phi_{u_n}^a\left(u_n-u\right)^2\mathrm{d}x+\int_{\mathbb{R}^3}K(x)\left(\phi_{u_n}^a-\phi_u^a\right)u\left(u_n-u\right)\mathrm{d}x\\&\qquad-\int_{\mathbb{R}^3\backslash\Omega_J'}\left(f\left(u_n\right)-f(u)\right)\left(u_n-u\right)\mathrm{d}x-\int_{\Omega_J'}\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)\left(u_n-u\right)\mathrm{d}x\\&\quad=o(1)\end{aligned}$$

as $n\to\infty$. Because of $\max_{x\in\mathbb{R}}|f'(x)|\le(p-1)\nu_0$, by using the Mean Value Theorem, we get that

$$\int_{\mathbb{R}^3\backslash\Omega_J'}\left(f\left(u_n\right)-f(u)\right)\left(u_n-u\right)\mathrm{d}x\le(p-1)\nu_0\left\|u_n-u\right\|_2^2.$$

Noting that $u_n\to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^3)$, so we have

$$\int_{\Omega_J'}\left(|u_n|^{p-2}u_n-|u|^{p-2}u\right)\left(u_n-u\right)\mathrm{d}x=o(1)\quad\text{as } n\to\infty.$$

We also remark that $u_n\rightharpoonup u$ in $L^3\left(\mathbb{R}^3\right)$. Thus, by the uniqueness of limit, we have $|u_n-u|^{\frac{6}{5}}\rightharpoonup 0$ in $L^{\frac{5}{2}}\left(\mathbb{R}^3\right)$. Then according to $K\in L^\infty(\mathbb{R}^3)$ and $|u|^{\frac{6}{5}}\in L^{\frac{5}{3}}\left(\mathbb{R}^3\right)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3}K(x)\left(\phi_{u_n}^a-\phi_u^a\right)u\left(u_n-u\right)\mathrm{d}x&\le|K|_\infty\|\phi_{u_n}^a-\phi_u^a\|_6\left(\int_{\mathbb{R}^3}|u|^{\frac{6}{5}}|u_n-u|^{\frac{6}{5}}\mathrm{d}x\right)^{\frac{5}{6}}\\&=o(1)\quad\text{as } n\to\infty.\end{aligned}$$

Combining all these and the fact $\int_{\mathbb{R}^3} K(x)\phi_{u_n}^a (u_n - u)^2 \, dx \geq 0$, by using Proposition 2.4, we have

$$\delta_0 \|u_n - u\|_\lambda^2 \leq \|u_n - u\|_\lambda^2 - (p-1)\nu_0 \|u_n - u\|_2^2 + \int_{\mathbb{R}^3} K(x)\phi_{u_n}^a (u_n - u)^2 \, dx \leq o(1)$$

as $n \to \infty$, which completes the proof. □

Following the spirit of Lemma 3.2, we have

Lemma 3.3. *Suppose the sequences $\lambda_n \to \infty$ as $n \to \infty$ and $\{u_n\}_n$ in E satisfy*

$$J_{\lambda_n}(u_n) \leq c, \qquad \|\nabla J_{\lambda_n}(u_n)\|_{\lambda_n} \to 0.$$

Then, after passing to a subsequence, we have:

- (a) *$u_n \rightharpoonup u$ in E for some $u \in E$;*
- (b) *$u \equiv 0$ in $\mathbb{R}^3 \backslash \Omega_J$, and $u_j = u|_{\Omega_j} \in H_0^1(\Omega_j)$ solves $-\Delta v + V_0(x)v + K(x)\phi_u^a v = |v|^{p-2}v$ in Ω_j weakly for $j \in J$;*
- (c) *$\|u_n - u\|_{\lambda_n} \to 0$, consequently $u_n \to u$ in $H^1(\mathbb{R}^3)$;*
- (d) *For $n \to \infty$, u_n also satisfies:*
  - (1) $\int_{\mathbb{R}^3} \lambda_n V(x) u_n^2 dx \to 0$;
  - (2) $\int_{\mathbb{R}^3 \backslash \Omega_J'} \left(|\nabla u_n|^2 + (\lambda_n V(x) + V_0(x)) u_n^2\right) dx \to 0$;
  - (3) $\int_{\Omega_j'} \left(|\nabla u_n|^2 + (\lambda_n V(x) + V_0(x)) u_n^2\right) dx \to \int_{\Omega_j} \left(|\nabla u|^2 + V_0(x)u^2\right) dx, j = 1, \ldots, m.$

Proof. By a similar method of Lemma 3.1, we obtain that $\{u_n\}_n$ is bounded in E and $H^1(\mathbb{R}^3)$. So we could assume that for some $u \in E$,

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } E \text{ and } H^1(\mathbb{R}^3), \\ u_n &\to u \quad \text{in } L^q_{\text{loc}}(\mathbb{R}^3),\ 1 \leq q < 6, \\ u_n &\to u \quad \text{a.e in } \mathbb{R}^3. \end{aligned}$$

Let $C_m = \left\{x \in \mathbb{R}^3 : V(x) \geq \frac{1}{m}\right\}$. When n large enough such that $\lambda_n \leq 2(\lambda_n - \lambda_1)$, we have that

$$\begin{aligned} \int_{C_m} u_n^2 dx &\leq \frac{m}{\lambda_n} \int_{\mathbb{R}^3} \lambda_n V(x) u_n^2 dx \\ &\leq \frac{2m}{\lambda_n} \int_{\mathbb{R}^3} (\lambda_n - \lambda_1) V(x) u_n^2 dx \\ &\leq \frac{2m}{\lambda_n} \int_{\mathbb{R}^3} (\lambda_n - \lambda_1) V(x) u_n^2 dx + \frac{2m}{\lambda_n} \|u_n\|_{\lambda_1}^2 \\ &= \frac{2m}{\lambda_n} \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + (\lambda_n V(x) + V_0(x)) u_n^2\right) dx \\ &= \frac{2m}{\lambda_n} \|u_n\|_{\lambda_n}^2 \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, $u(x) = 0$ a.e. in $\bigcup_m C_m = \mathbb{R}^3 \backslash \overline{\Omega}$. For any $\varphi \in C_0^\infty(\Omega_j), j \in J$, we get

$$\langle J_{\lambda_n}'(u_n), \varphi \rangle = \int_{\Omega_j} \left(\nabla u_n \nabla \varphi + V_0(x) u_n \varphi + K(x)\phi_{u_n}^a u_n \varphi - |u_n|^{p-2} u_n \varphi\right) dx.$$

Due to $K(x)u\varphi \in L^{\frac{6}{5}}\left(\mathbb{R}^3\right), \Phi\left(u_n\right) \rightharpoonup \Phi(u)$ in $\mathcal{D}$ and $u_n \to u$ in $L^q_{\mathrm{loc}}\left(\mathbb{R}^3\right)$ for $1 \le q < 6$, for $n \to \infty$, we have

$$\begin{aligned}\int_{\Omega_j}\left(K(x)\phi^a_{u_n}u_n\varphi - K(x)\phi^a_u u\varphi\right)\mathrm{d}x &= \int_{\Omega_j} K(x)\phi^a_{u_n}\left(u_n-u\right)\varphi \mathrm{d}x + \int_{\Omega_j} K(x)\left(\phi^a_{u_n}-\phi^a_u\right)u\varphi \mathrm{d}x \\ &\to 0.\end{aligned}$$

Similar to the proof of Lemma 3.2, we have $\langle J'_{\lambda_n}(u_n) - J'_{\lambda_n}(u), \varphi\rangle \to 0$. Thus it follows from $\langle J'_{\lambda_n}\left(u_n\right), \varphi\rangle \to 0$ that

$$\int_{\Omega_j}\left(\nabla u \nabla \varphi + V_0(x)u\varphi + K(x)\phi^a_u u\varphi - |u|^{p-2}u\varphi\right)\mathrm{d}x = 0.$$

As a result, for $j \in J, u_j = u|_{\Omega_j} \in H^1_0\left(\Omega_j\right)$ solves $-\Delta v + V_0(x)v + K(x)\phi^a_u v = |v|^{p-2}v$ in Ω_j weakly. When $j \in \{1,2,\ldots,m\} \setminus J$, let $\varphi = u$, then we get

$$\int_{\Omega_j}\left(|\nabla u|^2 + V_0(x)u^2 + K(x)\phi^a_u u^2 - f(u)u\right)\mathrm{d}x = 0.$$

Because of $\varphi = u \in C^\infty_0\left(\Omega_j\right)$, we have

$$\int_{\Omega'_j}\left(|\nabla u|^2 + V_0(x)u^2 + K(x)\phi^a_u u^2 - f(u)u\right)\mathrm{d}x = 0.$$

From Proposition 2.4, $f(t)t \le \nu_0 t^2$ for $t \in \mathbb{R}$ and the fact that $K(x)\phi^a_u u^2 \ge 0$, we have

$$\begin{aligned}\delta_0\|u\|^2_{\Lambda_1,\Omega'_j} &\le \|u\|^2_{\Lambda_1,\Omega'_j} - (p-1)\nu_0\|u\|^2_{L^2\left(\Omega'_j\right)} \\ &\le \|u\|^2_{\Lambda_1,\Omega'_j} - \nu_0\|u\|^2_{L^2\left(\Omega'_j\right)} \\ &\le \int_{\Omega'_j}\left(|\nabla u|^2 + a_0(x)u^2 + K(x)\phi^a_u u^2 - f(u)u\right)\mathrm{d}x \\ &= 0.\end{aligned}$$

So that, $u = 0$ in Ω_j when $j \in \{1,2,\ldots,m\} \setminus J$ and we get (b).

In order to prove (c), we use the following fact:

$$\begin{aligned}o(1) &= \langle J'_{\lambda_n}\left(u_n\right) - J'_{\lambda_n}(u), u_n - u\rangle \\ &= \langle J'_{\lambda_n}\left(u_n\right), u_n - u\rangle - \langle J'_{\lambda_n}(u), u_n - u\rangle \\ &= \|u_n - u\|^2_{\lambda_n} + \int_{\mathbb{R}^3}\left(K(x)\phi^a_{u_n}u_n\left(u_n-u\right) - K(x)\phi^a_u u\left(u_n-u\right)\right)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3\setminus\Omega'_J}\left(f\left(u_n\right)-f(u)\right)\left(u_n-u\right)\mathrm{d}x - \int_{\Omega'_J}\left(\left|u_n\right|^{p-2}u_n - |u|^{p-2}u\right)\left(u_n-u\right)\mathrm{d}x \\ &= \|u_n - u\|^2_{\lambda_n} + \int_{\mathbb{R}^3}K(x)\phi^a_{u_n}\left(u_n-u\right)^2\mathrm{d}x + \int_{\mathbb{R}^3}K(x)\left(\phi^a_{u_n}-\phi^a_u\right)u\left(u_n-u\right)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3\setminus\Omega'_J}\left(f\left(u_n\right)-f(u)\right)\left(u_n-u\right)\mathrm{d}x - \int_{\Omega'_J}\left(\left|u_n\right|^{p-2}u_n - |u|^{p-2}u\right)\left(u_n-u\right)\mathrm{d}x.\end{aligned}$$

Similar to the proof of Lemma 3.2, we also have

$$\int_{\mathbb{R}^3\setminus\Omega'_J}\left(f\left(u_n\right)-f(u)\right)\left(u_n-u\right)\mathrm{d}x \le (p-1)\nu_0\left\|u_n-u\right\|^2_2,$$

$$\int_{\Omega'_J}\left(\left|u_n\right|^{p-2}u_n - |u|^{p-2}u\right)\left(u_n-u\right)\mathrm{d}x = o(1) \quad \text{as } n \to \infty$$

and

$$\int_{\mathbb{R}^3} K(x)\left(\phi_{u_n}^a-\phi_u^a\right) u\left(u_n-u\right) \mathrm{d}x=o(1) \quad \text{as } n \rightarrow \infty.$$

So we have

$$\delta_0\left\|u_n-u\right\|_{\lambda_n}^2 \leq\left\|u_n-u\right\|_{\lambda_n}^2-(p-1) \nu_0\left\|u_n-u\right\|_2^2+\int_{\mathbb{R}^3} K(x) \phi_{u_n}^a\left(u_n-u\right)^2 \mathrm{d}x \leq o(1).$$

This completes the proof of (c).

For (d), we use (c) and for sufficiently large $n, \lambda_n \leq 2\left(\lambda_n-\lambda_1\right)$. Then as $n \rightarrow \infty$, we have

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}^3} \lambda_n V(x) u_n^2 \mathrm{d}x \leq \int_{\mathbb{R}^3}\left(\lambda_n-\lambda_1\right) V(x) u_n^2 \mathrm{d}x & =\int_{\mathbb{R}^3}\left(\lambda_n-\lambda_1\right) V(x)\left(u_n-u\right)^2 \mathrm{d}x \\
& \leq \int_{\mathbb{R}^3}\left(\lambda_n-\lambda_1\right) V(x)\left(u_n-u\right)^2 \mathrm{d}x+\left\|u_n-u\right\|_{\lambda_1}^2=\left\|u_n-u\right\|_{\lambda_n}^2 \rightarrow 0.
\end{aligned}$$

Thus (1) in (d) is obtained. It is easy to show that (2), (3) in (d) also follows immediately from (1) in (d) and (c), and we obtain the conclusion. □

Lemma 3.4. *For any fixed $c>0$, there exists $\Lambda_c \geq \Lambda_1$ such that if u_λ is a critical point of J_λ satisfying $\left|J_\lambda\left(u_\lambda\right)\right| \leq c$ for $\lambda \geq \Lambda_c$, then $\left|u_\lambda\right| \leq \nu_0^{\frac{1}{p-2}}$ on $\mathbb{R}^3 \backslash \Omega_J^{\prime}$, ν_0 is defined in Proposition 2.4. In particular, u_λ solves the original problem (1.1).*

Proof. Since $u_\lambda \in E$ is a critical point of J_λ with $\left|J_\lambda\left(u_\lambda\right)\right| \leq c, u_\lambda$ is bounded in E uniformly for $\lambda \geq \Lambda_1$. And it satisfies the equation

$$-\Delta u_\lambda+\left(\lambda V(x)+V_0(x)\right) u_\lambda+K(x) \phi_{u_\lambda}^a u_\lambda=g\left(x, u_\lambda\right) \quad \text{in } \mathbb{R}^3.$$

Due to Lemma 5.1 in [2], $H_\lambda^{-1}:=\left(-\Delta+\left(\lambda V(x)+V_0(x)\right)\right)^{-1}$ is a well-defined bounded operator from $L^s\left(\mathbb{R}^3\right)$ to $L^r\left(\mathbb{R}^3\right)$ provided $1 \leq s \leq r \leq+\infty$ and $\frac{1}{s}-\frac{1}{r} \leq \frac{2}{3}$. And there exists a constant $C_{r, s}>0$ (independent of λ sufficiently large) such that

$$\left\|H_\lambda^{-1} g\right\|_r \leq C_{r, s}\|g\|_s, \qquad g \in L^s\left(\mathbb{R}^3\right).$$

Let $\chi_{\lambda, 0}$ be the characteristic function of the set $\left\{x \in \mathbb{R}^3:\left|u_\lambda(x)\right| \leq 1\right\}$ and define $v_{\lambda, 0}=\chi_{\lambda, 0} u_\lambda$, $w_{\lambda, 0}=u_\lambda-v_{\lambda, 0}=\left(1-\chi_{\lambda, 0}\right) u_\lambda$. Setting $l_{\lambda, 0}=g\left(\cdot, v_{\lambda, 0}\right)-K(\cdot) \phi_{u_\lambda}^a v_{\lambda, 0}$ and $h_{\lambda, 0}=g\left(\cdot, w_{\lambda, 0}\right)-K(\cdot) \phi_{u_\lambda}^a w_{\lambda, 0}$, we have $g\left(\cdot, u_\lambda\right)=l_{\lambda, 0}+h_{\lambda, 0}$. Since u_λ is bounded in $E, \phi_{u_\lambda}^a$ is bounded in L^{∞}. Thus, $l_{\lambda, 0}$ is bounded in $L^{\infty}\left(\mathbb{R}^3\right)$ uniformly in λ. Moreover, $h_{\lambda, 0}$ is bounded uniformly for λ in $L^{\frac{6}{p-1}}\left(\mathbb{R}^3\right)$. In fact,

$$\begin{aligned}
\left|\phi_{u_\lambda}^a(x)\right| & \leq \frac{1}{4 \pi}\left|\int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} u_\lambda^2(y) \mathrm{d}y\right| \\
& \leq c|K|_{\infty}\left(\int_{B_1(x)} \frac{u_\lambda^2(y)}{|x-y|} \mathrm{d}y+\int_{B_1^c(x)} \frac{u_\lambda^2(y)}{|x-y|} \mathrm{d}y\right) \\
& \leq c|K|_{\infty}\left(\left(\int_{B_1(x)} \frac{1}{|x-y|^2} \mathrm{~d}y\right)^{1/2}\left(\int_{B_1(x)} u_\lambda^4 \mathrm{~d}y\right)^{1/2}\right. \\
& \quad\left.+\left(\int_{B_1^c(x)} \frac{1}{|x-y|^4} \mathrm{~d}y\right)^{1/4}\left(\int_{B_1^c(x)}\left|u_\lambda\right|^{8/3} \mathrm{~d}y\right)^{4/3}\right) \\
& \leq c^{\prime}|K|_{\infty}.
\end{aligned}$$

In the set $|u_\lambda| \leq 1$, we have $|w_{\lambda,0}| = 0$; and in the set $|u_\lambda| > 1$, we have $|w_{\lambda,0}| = |u_\lambda - v_{\lambda,0}| = |(1-\chi_{\lambda,0})\, u_\lambda| = |u_\lambda| > 1$. So we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^3} |w_{\lambda,0}|^{\frac{6}{p-1}}\ \mathrm{d}x\right)^{\frac{p-1}{6}} &= \left(\int_{\{x:|u_\lambda|\leq 1\}} |w_{\lambda,0}|^{\frac{6}{p-1}}\ \mathrm{d}x + \int_{\{x:|u_\lambda|>1\}} |w_{\lambda,0}|^{\frac{6}{p-1}}\ \mathrm{d}x\right)^{\frac{p-1}{6}} \\
&\leq \left(0 + \int_{\{x:|u_\lambda|>1\}} |w_{\lambda,0}|^{6}\ \mathrm{d}x\right)^{\frac{p-1}{6}} \\
&= \left(\int_{\mathbb{R}^3} |w_{\lambda,0}|^{6}\ \mathrm{d}x\right)^{\frac{p-1}{6}}.
\end{aligned}$$

Therefore, combining this with Minkowski inequality, we have

$$\begin{aligned}
\|h_{\lambda,0}\|_{\frac{6}{p-1}} &\leq \|g(\cdot, w_{\lambda,0})\|_{\frac{6}{p-1}} + \left\|K(\cdot)\phi^a_{u_\lambda} w_{\lambda,0}\right\|_{\frac{6}{p-1}} \\
&\leq \left(\int_{\mathbb{R}^3} |u_\lambda|^6\ \mathrm{d}x\right)^{\frac{p-1}{6}} + |K|_\infty \left|\phi^a_{u_\lambda}\right|_\infty \left(\int_{\mathbb{R}^3} |w_{\lambda,0}|^{\frac{6}{p-1}}\ \mathrm{d}x\right)^{\frac{p-1}{6}} \\
&\leq \left(\int_{\mathbb{R}^3} |u_\lambda|^6\ \mathrm{d}x\right)^{\frac{p-1}{6}} + |K|_\infty \left|\phi^a_{u_\lambda}\right|_\infty \left(\int_{\mathbb{R}^3} |w_{\lambda,0}|^{6}\ \mathrm{d}x\right)^{\frac{p-1}{6}} \\
&\leq C\,\|u_\lambda\|_E^{p-1}.
\end{aligned}$$

Now we define $v_{\lambda,1} = H_\lambda^{-1} l_{\lambda,0}$ and $w_{\lambda,1} = H_\lambda^{-1} h_{\lambda,0}$ so that $u_\lambda = v_{\lambda,1} + w_{\lambda,1}$. Then, there exists $C_2 > 0$ such that

$$|v_{\lambda,1}|_\infty \leq C_2 \quad \text{and} \quad \|w_{\lambda,1}\|_{p_1} \leq C_2$$

uniformly in λ; here $p_1 = \infty$ if $p_0 = \frac{6}{p-1} > \frac{3}{2}$, and p_1 is arbitrarily close to and less than $\frac{3p_0}{3-2p_0}$ if $p_0 \leq \frac{3}{2}$. In the case $p_0 > \frac{3}{2}$ we are done. In the case $p_0 \leq \frac{3}{2}$, we have $5 \leq p < 6$. Thus, we can assume that there is a positive constant $\delta \leq 1$ such that $p = 6 - \delta$. Let $\chi_{\lambda,1}$ be the characteristic function of the set

$$A_\lambda = \left\{x \in \mathbb{R}^3 : |w_{\lambda,1}(x)| \leq C_2 + 1\right\}.$$

Setting

$$\begin{aligned}
\bar{v}_{\lambda,1} &= \chi_{\lambda,1} u_\lambda = \chi_{\lambda,1}\left(v_{\lambda,1} + w_{\lambda,1}\right), \\
\bar{w}_{\lambda,1} &= u_\lambda - \bar{v}_{\lambda,1} = \left(1 - \chi_{\lambda,1}\right)\left(v_{\lambda,1} + w_{\lambda,1}\right), \\
l_{\lambda,1} &= g\left(\cdot, \bar{v}_{\lambda,1}\right) - K(\cdot)\phi^a_{u_\lambda}\bar{v}_{\lambda,1}, \\
h_{\lambda,1} &= g\left(\cdot, \bar{w}_{\lambda,1}\right) - K(\cdot)\phi^a_{u_\lambda}\bar{w}_{\lambda,1}.
\end{aligned}$$

We see that $|l_{\lambda,1}|_\infty$ is bounded uniformly in λ. In addition, since the measure of the set A_λ^c is finite and $\|w_{\lambda,1}\|_{p_1} \leq C_2$, we have $h_{\lambda,1}$ is bounded in $L^{\frac{p_1}{p-1}}(\mathbb{R}^3)$. Now repeating the above argument with $v_{\lambda,2} = H_\lambda^{-1} l_{\lambda,1}$ and $w_{\lambda,2} = H_\lambda^{-1} h_{\lambda,1}$, we obtain a constant $C_3 > 0$ such that

$$|v_{\lambda,2}|_\infty \leq C_3 \quad \text{and} \quad \|w_{\lambda,1}\|_{p_2} \leq C_3,$$

where $p_2 = \infty$ if $\bar{p}_1 = \frac{p_1}{p-1} > \frac{3}{2}$, and p_2 is arbitrarily close to and less than $\frac{3\bar{p}_1}{3-2\bar{p}_1}$ if $\bar{p}_1 \leq \frac{3}{2}$. Using the assumption $p = 6 - \delta, 0 < \delta \leq 1$ and after a finite number of such steps we get a uniform bounded for $|u_\lambda|_\infty$.

According to the definition of g and uniform boundedness of $\left|\phi^a_{u_\lambda}\right|_\infty$, we obtain that $A(x) = \frac{g(x,u_\lambda(x))}{u_\lambda(x)} + K(x)\phi^a_{u_\lambda}$ is bounded in $L^\infty(\mathbb{R})$. Moreover, the negative part of $W_\lambda =$

$\lambda V + V_0 - A$ is bounded uniformly in λ. It follows from Theorem A.2.1 in [16] that the norm of W_λ^- in the Kato class K_3 is bounded uniformly in λ. Therefore, Theorem C.1.2 in [16] implies that there is a constant $C(r)$ which is independent of λ such that

$$|u_\lambda(x)| \leq C(r) \int_{B_r(x)} |u_\lambda| \, \mathrm{d}x,$$

where $B_r(x)$ is a ball in $\mathbb{R}^3$ centered at x with radius r. From Lemma 3.3(b), as $n \to \infty$

$$u_\lambda \to 0 \quad \text{in } L^2\left(\mathbb{R}^3 \backslash \overline{\Omega}\right).$$

Thus, choosing $r = \frac{1}{2}\operatorname{dist}\left(\Omega, \mathbb{R}^3 \backslash \Omega'\right)$, we have uniformly in $x \in \mathbb{R}^3 \backslash \Omega'$,

$$\begin{aligned} |u_\lambda(x)| &\leq C(r) \int_{B_r(x)} |u_\lambda| \, \mathrm{d}x \\ &\leq C(r) \left(\operatorname{meas} B_r(x)\right)^{\frac{1}{2}} |u_\lambda|_{2, B_r(x)} \\ &\leq C(r) \left(\operatorname{meas} B_r(x)\right)^{\frac{1}{2}} |u_\lambda|_{2, \mathbb{R}^3 \backslash \Omega} \\ &\to 0 \quad \text{as } \lambda \to \infty. \end{aligned}$$

□

4 Nehari manifold and minimax arguments

Consider the following nonlinear problems for $j \in J$,

$$\begin{cases} -\Delta u + V_0(x) u = |u|^{p-2} u, & \text{in } \Omega_j, \\ u = 0, & \text{on } \partial\Omega_j \end{cases}$$

and

$$\begin{cases} -\Delta u + \left(\lambda V(x) + V_0(x)\right) u = |u|^{p-2} u, & \text{in } \Omega_j', \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega_j' \end{cases}$$

with their corresponding functionals

$$\begin{aligned} I_j(u) &= \frac{1}{2} \int_{\Omega_j} \left(|\nabla u|^2 + V_0(x) u^2\right) \mathrm{d}x - \frac{1}{p} \int_{\Omega_j} |u|^p \mathrm{d}x; \qquad H_0^1\left(\Omega_j\right) \to \mathbb{R}, \\ I_{\lambda,j}(u) &= \frac{1}{2} \int_{\Omega_j'} \left(|\nabla u|^2 + \left(\lambda V(x) + V_0(x)\right) u^2\right) \mathrm{d}x - \frac{1}{p} \int_{\Omega_j'} |u|^p \mathrm{d}x; \qquad H^1\left(\Omega_j'\right) \to \mathbb{R}. \end{aligned}$$

It is easy to check that both I_j and $I_{\lambda,j}$ possess the mountain pass geometry and satisfy the (PS) condition. On the other hand, the infimum of I_j and $I_{\lambda,j}$ on their Nehari manifold

$$\begin{aligned} \mathcal{N}_j &= \left\{u \in H_0^1\left(\Omega_j\right) \backslash \{0\} : \left(\nabla I_j(u), u\right) = 0\right\}, \\ \mathcal{N}_{\lambda,j} &= \left\{u \in H^1\left(\Omega_j'\right) \backslash \{0\} : \left(\nabla I_{\lambda,j}(u), u\right) = 0\right\} \end{aligned}$$

are achieved by some $\omega_j \in \mathcal{N}$ and $\omega_{\lambda,j} \in \mathcal{N}_{\lambda,j}$ respectively. By a standard argument, we can see that $\omega_j, \omega_{\lambda,j}$ are critical points of I_j and $I_{\lambda,j}$ separately. The critical values $c_j = I_j\left(\omega_j\right)$ and $c_{\lambda,j} = I_{\lambda,j}\left(\omega_{\lambda,j}\right)$ are equal to the mountain pass value of their corresponding functional. Moreover, we also have the following lemma (see Lemma 3.1 in [11] and (3.8) for details).

Lemma 4.1. *The following statements hold:*

(a) *there is a* $\rho > 0$ *such* $0 < \rho \leq c_{\lambda,j} \leq c_j$ *for* $\lambda \geq \Lambda_1$ *sufficiently large;*

(b) $c_j = \max\limits_{r>0} I_j\left(rw_j\right), c_{\lambda,j} = \max\limits_{r>0} I_{\lambda,j}\left(rw_{\lambda,j}\right)$*;*

(c) $c_{\lambda,j} \to c_j$ *as* $\lambda \to \infty$*;*

(d)

$$c_j = \inf\left\{I_j(v) : v \in H_0^1\left(\Omega_j\right), \int_{\Omega_j} |v|^p \mathrm{d}x = \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_j\right\},$$
$$c_{\lambda,j} = \inf\left\{I_{\lambda,j}(v) : v \in H^1\left(\Omega_j'\right), \int_{\Omega_j'} |v|^p \mathrm{d}x = \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_{\lambda,j}\right\}.$$

In the following, we give a minimax argument for $J_\lambda(u)$. First of all, we fix $R \geq 2$ such that

$$\begin{aligned} &I_j\left(R\omega_j\right) < 0, \\ &R^2 \left\|w_j\right\|_{\lambda,\Omega_j'}^2 = R^p \left|w_j\right|_p^p \geq 2\left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_j \end{aligned} \tag{4.1}$$

for all $j \in J$. By relabeling the indices, we could assume $J = \{1, 2, \ldots, l\}$ $(l \leq m)$. We define $\gamma_0 : [0,1]^l \to E$,

$$\gamma_0\left(t_1, t_2, \ldots, t_l\right)(x) = \sum_{j=1}^{l} t_j R\omega_j(x), \tag{4.2}$$

$$\Gamma_J = \left\{\gamma \in C\left([0,1]^l, E\right); \gamma\left(t_1, t_2, \ldots, t_l\right) = \gamma_0\left(t_1, t_2, \ldots, t_l\right), \left(t_1, t_2, \ldots, t_l\right) \in \partial\left([0,1]^l\right)\right\}$$

and

$$b_{\lambda,J} = \inf_{\gamma \in \Gamma_J} \max_{t \in [0,1]^l} J_\lambda(\gamma(t)).$$

Obviously, $\Gamma_J \neq \emptyset$ since $\gamma_0 \in \Gamma_J$. Thus $b_{\lambda,j}$ is well defined.

According to Lemma 3.3 in [11], by using a topological degree argument we can get the following conclusion.

Lemma 4.2. *For any* $\gamma \in \Gamma_J$*, there is a* $t_\gamma \in [0,1]^l$ *such that for* $j \in J$

$$\int_{\Omega_j'} \left|\gamma\left(t_\gamma\right)(x)\right|^p \mathrm{d}x = \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_{\lambda,j}.$$

Lemma 4.3. $\sum_{j=1}^{l} c_{\lambda,j} \leq b_{\lambda,J} \leq \sum_{j=1}^{l} c_j + \mu$*, where*

$$\mu = \frac{R^4}{4} \sum_{j=1}^{l} \int_{\Omega_j} K(x) \phi^a_{\sum_{j=1}^{l} \omega_j}\left(\omega_j\right)^2 \mathrm{d}x. \tag{4.3}$$

Proof. According to Lemma 4.2, for any $\gamma \in \Gamma_J$, we have

$$\max_{t \in [0,1]^l} J_\lambda(\gamma(t)) \geq J_\lambda\left(\gamma\left(t_\gamma\right)\right) \geq J_{\lambda,\mathbb{R}^3 \backslash \Omega_J'}\left(\gamma\left(t_\gamma\right)\right) + \sum_{j=1}^{l} J_{\lambda,\Omega_j'}\left(\gamma\left(t_\gamma\right)\right),$$

where $J_{\lambda,\Omega_j'}(u)$ is defined by

$$J_{\lambda,\Omega_j'}(u)=\frac{1}{2}\int_{\Omega_j'}\left(|\nabla u|^2+\left(\lambda V(x)+V_0(x)u^2\right)\right)\mathrm{d}x+\frac{1}{4}\int_{\Omega_j'}K(x)\phi_u^a u^2\mathrm{d}x-\int_{\Omega_j'}G(x,u)\mathrm{d}x.$$

And the definition of $J_{\lambda,\mathbb{R}^3\backslash\Omega_j'}(u)$ is similar. According to Proposition 2.4 and the fact that $|G(x,t)|\le\frac{1}{2}\nu_0 t^2$ when $x\in\mathbb{R}^3\backslash\Omega_J'$, we get that

$$J_{\lambda,\mathbb{R}^3\backslash\Omega_j'}(u)\ge 0\quad\text{for } u\in E \text{ and } j\in J.$$

By using $\int_{\Omega_j'}K(x)\phi_u^a u^2\mathrm{d}x\ge 0$ and Lemma 4.1(d) we obtain

$$\begin{aligned}\max_{t\in[0,1]^l}J_\lambda(\gamma(t))&\ge\sum_{j=1}^{l}J_{\lambda,\Omega_j'}\left(\gamma\left(t_\gamma\right)\right)\ge\sum_{j=1}^{l}I_{\lambda,j}\left(\gamma\left(t_\gamma\right)\right)\\&\ge\sum_{j=1}^{l}\inf\left\{I_{\lambda,j}(v):v\in H^1\left(\Omega_j'\right),\int_{\Omega_j'}|v|^p\mathrm{d}x=\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_{\lambda,j}\right\}\\&=\sum_{j=1}^{l}c_{\lambda,j}.\end{aligned}$$

According to the arbitrary choice of γ, we have $\sum_{j=1}^{l}c_{\lambda,j}\le b_{\lambda,J}$. On the other hand,

$$\begin{aligned}b_{\lambda,J}&\le\max_{t\in[0,1]^l}J_\lambda\left(\gamma_0(t)\right)\\&=\max_{t\in[0,1]^l}\sum_{j=1}^{l}I_j\left(t_jR\omega_j\right)+\frac{1}{4}\sum_{j=1}^{l}\int_{\Omega_j}K(x)\phi^a_{\sum_{j=1}^{l}t_jR\omega_j}\left(t_jR\omega_j\right)^2\mathrm{d}x\\&\le\sum_{j=1}^{l}c_j+\frac{R^4}{4}\sum_{j=1}^{l}\int_{\Omega_j}K(x)\phi^a_{\sum_{j=1}^{l}\omega_j}\left(\omega_j\right)^2\mathrm{d}x.\end{aligned}$$

Thus, we get the conclusion. □

In the following, we denote $\sum_{j=1}^{l}c_j$ by c_J. It is easy to see that, for $\gamma\in\Gamma_J,\gamma(t)=\gamma_0(t)$ on $\partial[0,1]^l$. So, for $t=(t_1,t_2,\ldots,t_l)\in\partial[0,1]^l$, one has

$$J_\lambda(\gamma(t))=J_\lambda\left(\gamma_0(t)\right)\le\sum_{j=1}^{l}I_j\left(t_jR\omega_j\right)+\frac{R^4}{4}\sum_{j=1}^{l}\int_{\Omega_j}K(x)\phi^a_{\sum_{j=1}^{l}\omega_j}\left(\omega_j\right)^2\mathrm{d}x.$$

Choosing k small enough such that

$$\frac{R^4}{4}\sum_{j=1}^{l}\int_{\Omega_j}K(x)\phi^a_{\sum_{j=1}^{l}\omega_j}\left(\omega_j\right)^2\mathrm{d}x\le\frac{1}{2}\min_{j\in J}c_j$$

when $|K|_\infty\le k$. Due to Lemma 4.1(b), $I_j\left(t_jR\omega_j\right)\le c_j$ for $j\in J$. But on the other hand, because of $t=(t_1,t_2,\ldots,t_l)\in\partial[0,1]^l$, there must be some $j_0\in J,t_{j_0}\in\{0,1\}$. Thus $I_{j_0}\le 0$. Hence

$$J_\lambda(\gamma(t))\le\sum_{j=1}^{l}c_j-c_{j_0}+\frac{1}{2}\min_{j\in J}c_j\le\sum_{j=1}^{l}c_j-\frac{1}{2}\rho.$$

By Lemma 4.3 and $c_{\lambda,j}\to c_j$ for $j\in J$, we have $b_{\lambda,j}\ge\sum_{j=1}^{l}c_j-\frac{1}{4}\rho$ when λ is sufficiently large. Combining this and the Palais-Smale condition of J_λ, we conclude that $b_{\lambda,J}$ is a critical value of J_λ by using a standard deformation argument. Therefore, we have

Corollary 4.4. *There exists $k>0$ such that when $|K|_\infty\le k, b_{\lambda,J}$ is a critical value of J_λ for large λ.*

5 Proof of Theorem 1.1

In this section, we find the so-called multi-bump solution u_λ.

Firstly, we define

$$D_\lambda^\nu = \left\{ u \in E : \|u\|_{\lambda,\mathbb{R}^3\backslash\Omega_J'} \le \nu, \left| \|u\|_{\lambda,\Omega_j'} - \sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_j} \right| \le \nu, \quad j \in J \right\},$$

and

$$J_\lambda^c = \{u \in E : J_\lambda(u) \le c\}.$$

Then we have

Lemma 5.1. *For $0 < \nu < \frac{1}{3}\min_{j\in J}\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_j}$, there exist $k_1(\nu) > 0$ and $\sigma_0 > 0$, such that for $\lambda \ge \Lambda_1$ sufficiently large and $u \in \left(D_\lambda^{2\nu}\backslash D_\lambda^\nu\right) \cap J_\lambda^{c_J+\mu}$ we have*

$$\|\nabla J_\lambda(u)\|_\lambda \ge \sigma_0 \tag{5.1}$$

when $|K|_\infty < k(\nu)$. Here

$$\mu = \frac{R^4}{4}\sum_{j=1}^{l}\int_{\Omega_j} K(x)\phi^a_{\sum_{j=1}^l \omega_j}(\omega_j)^2\,\mathrm{d}x$$

is defined in (4.3).

Proof. If the conclusion is false, we can assume that there exists $u_n \in \left(D_{\lambda_n}^{2\nu}\backslash D_{\lambda_n}^\nu\right) \cap J_\lambda^{c_J+\mu}$ such that $\|\nabla J_{\lambda_n}(u_n)\|_{\lambda_n} \to 0, \lambda_n \to \infty$.

Since $(u_n) \subset J_{\lambda_n}^{c_J+\mu}$, according to Lemma 3.3, we have for some $u \in E$, $\|u_n - u\|_{\lambda_n} \to 0$ and

$$\begin{aligned} c_J + \mu &\ge \lim_{n\to\infty} J_{\lambda_n}(u_n) \\ &= \frac{1}{2}\int_{\Omega_J}\left(|\nabla u|^2 + V_0(x)u^2\right)\mathrm{d}x + \frac{1}{4}\int_{\Omega_J} K(x)\phi_u^a u^2 - \frac{1}{p}\int_{\Omega_J}|u|^p\mathrm{d}x \\ &= \sum_{j\in J}\frac{1}{2}\int_{\Omega_j}\left(|\nabla u|^2 + V_0(x)u^2\right)\mathrm{d}x + \frac{1}{4}\int_{\Omega_j} K(x)\phi_u^a u^2\mathrm{d}x - \frac{1}{p}\int_{\Omega_j}|u|^p\mathrm{d}x, \end{aligned}$$

where $u \equiv 0$ in $\mathbb{R}^3\backslash\Omega_J$, and $u_j = u|_{\Omega_j} \in H_0^1(\Omega_j)$ is the weak solutions of $-\Delta v + V_0(x)v + K(x)\phi_u^a v = |v|^{p-2}v$ in Ω_j for $j \in J$. Hence, if $u_j \ne 0, j \in J$ and $t_j u_j \in \mathcal{N}_j$, we have

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_j}\left(|\nabla u_j|^2 + V_0(x)u_j^2\right)\mathrm{d}x + \frac{1}{4}\int_{\Omega_j} K(x)\phi_u^a u_j^2\mathrm{d}x - \frac{1}{p}\int_{\Omega_j}|u_j|^p\,\mathrm{d}x \\ &\quad = \max_{t>0}\frac{t^2}{2}\int_{\Omega_j}\left(|\nabla u_j|^2 + V_0(x)u_j^2\right)\mathrm{d}x + \frac{t^4}{4}\int_{\Omega_j} K(x)\phi_u^a u_j^2\mathrm{d}x - \frac{t^p}{p}\int_{\Omega_j}|u_j|^p\,\mathrm{d}x \\ &\quad \ge \frac{t_j^2}{2}\int_{\Omega_j}\left(|\nabla u_j|^2 + V_0(x)u_j^2\right)\mathrm{d}x + \frac{t_j^4}{4}\int_{\Omega_j} K(x)\phi_u^a u_j^2\mathrm{d}x - \frac{t_j^p}{p}\int_{\Omega_j}|u_j|^p\,\mathrm{d}x \\ &\quad \ge I_j(t_j u_j) \ge c_j. \end{aligned}$$

Thus, we have two possibilities:

(1) there exist some $j_0 \in J$ such $u_{j_0} = u|_{\Omega_{j_0}} = 0$;

(2) $\frac{1}{2}\int_{\Omega_j}\left(|\nabla u_j|^2+V_0(x)u_j^2\right)\mathrm{d}x+\frac{1}{4}\int_{\Omega_j}K(x)\phi_u^a u_j^2\mathrm{d}x-\frac{1}{p}\int_{\Omega_j}|u_j|^p\,\mathrm{d}x\in[c_j,c_j+\mu]$.

When (1) occurs, by Lemma 3.3(d) we obtain

$$\left|\|u_n\|_{\lambda_n,\Omega'_{j_0}}-\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_{j_0}}\right|\to\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_{j_0}}\geq 3\nu,$$

which contradicts to the assumption $u_n\in D^{2\nu}_{\lambda_n}\backslash D^{\nu}_{\lambda_n}$.

If (2) occurs, by Lemma 3.3(b), it is easy to check

$$\begin{aligned}&\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega_j}\left(|\nabla u_j|^2+V_0(x)u_j^2\right)\mathrm{d}x+\left(\frac{1}{4}-\frac{1}{p}\right)\int_{\Omega_j}K(x)\phi_u^a u_j^2\mathrm{d}x\\&\quad=\frac{1}{2}\int_{\Omega_j}\left(|\nabla u_j|^2+V_0(x)u_j^2\right)\mathrm{d}x+\frac{1}{4}\int_{\Omega_j}K(x)\phi_u^a u_j^2\mathrm{d}x-\frac{1}{p}\int_{\Omega_j}|u_j|^p\,\mathrm{d}x.\end{aligned}$$

Thus,

$$\left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega_j}\left(|\nabla u|^2+V_0(x)u^2\right)\mathrm{d}x+\left(\frac{1}{4}-\frac{1}{p}\right)\int_{\Omega_j}K(x)\phi_u^a u^2\mathrm{d}x\in\left[c_j,c_j+\mu\right].$$

Since $\|u\|_E\leq M\,(c_J+\mu)$, we can choose $k_1(\nu)>0$ such that for $j\in J$,

(1) $\mu\leq 1$ and $\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}(c_j+\mu)}\leq\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_j}+\nu$;

(2) $\left[\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_j-\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}\left(\frac{1}{4}-\frac{1}{p}\right)\int_{\Omega_j}K(x)\phi_u^a u^2\mathrm{d}x\right]^{1/2}\geq\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_j}-\nu$, when $|K|_\infty<k_1(\nu)$.

Hence we have $\left|\left(\int_{\Omega_j}(|\nabla u|^2+V_0(x)u^2)\mathrm{d}x\right)^{1/2}-\sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1}c_j}\right|\leq\nu$. By Lemma 3.3 again we get that $u_n\in D^{2\nu}_{\lambda_n}$ as n is large, which is a contradiction. □

Lemma 5.2. *For $0<\nu<\frac{1}{3}\min_{j\in J}\left(\frac{1}{2}-\frac{1}{p}\right)c_j$, there exists $k(\nu)>0$, such that for $\lambda\geq\Lambda_1$ sufficiently large,* (1.1) *possesses a solution satisfying $u_\lambda\in D^\nu_\lambda$ when $|K|_\infty<k(\nu)$.*

Proof. If the conclusion is false, we assume that J_λ has no critical point in $D^\nu_\lambda\cap J_\lambda^{c_J+\mu}$, here μ is defined as that in Lemma 5.1. Since J_λ satisfies the Palais–Smale condition (see Lemma 3.2), there is a constant $\sigma_\lambda>0$ such that

$$\|\nabla J_\lambda(u)\|_\lambda\geq\sigma_\lambda,\qquad u\in D^\nu_\lambda\cap J_\lambda^{c_J+\mu}.$$

By (5.1) there holds, for $\lambda\geq\Lambda_1$ and $|K|_\infty\leq k_1(\nu)$,

$$\|\nabla J_\lambda(u)\|_\lambda\geq\sigma_0,\qquad u\in\left(D^{2\nu}_\lambda\backslash D^\nu_\lambda\right)\cap J_\lambda^{c_J+\mu}.$$

Combining these, we could define a Lipschitz continuous function $\theta:E\to[0,1]$ such that $\theta(u)=1$ for $u\in D^{3\nu/2}_\lambda;\theta(u)=0$ for $u\notin D^{2\nu}_\lambda$. Then, the vector field

$$V:J_\lambda^{c_J+\mu}\to E,V(u)=-\theta(u)\frac{\nabla J_\lambda(u)}{\|\nabla J_\lambda(u)\|_\lambda}$$

is well defined and Lipschitz continuous. And moreover

$$\|V(u)\|_\lambda\leq 1,\qquad u\in E. \tag{5.2}$$

Now we consider the associated gradient flow $\eta : [0,+\infty) \times J_\lambda^{c_J+\mu} \to J_\lambda^{c_J+\mu}$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}s}\eta = V(\eta), \qquad \eta(0,u) = u.$$

By a standard argument, one can show that

$$\frac{\mathrm{d}}{\mathrm{d}s} J_\lambda(\eta(s,u)) = -\theta(u) \left\| \nabla J_\lambda(u) \right\|_\lambda \leq 0 \tag{5.3}$$

and

$$\eta(s,u) = u, \qquad s \geq 0, \quad u \in J_\lambda^{c_J+\mu} \backslash D_\lambda^{2\nu}. \tag{5.4}$$

Recalling $\gamma_0 \in \Gamma_J$, a path which is defined by (4.2). Because of (4.1), we have that

$$\gamma_0(t) \notin D_\lambda^{2\nu}, \qquad t \in \partial[0,1]^l.$$

Therefore, by using (5.4), we have

$$\eta\left(s,\gamma_0(t)\right) = \gamma_0(t), \qquad t \in \partial[0,1]^l.$$

Thus, $\eta\left(s,\gamma_0(\cdot)\right) \in \Gamma_J$ for any $s \geq 0$.

Since $\operatorname{supp}\gamma_0 \subset \bigcup_{j\in J}\overline{\Omega_j}$ for $t \in [0,1]^l$, thus $J_\lambda\left(\gamma_0(t)\right)$ and $\|\gamma_0(t)\|^2_{\lambda,\Omega_j'}$ do not depend on $\lambda \geq 0$. Considering about

$$m_0 = \max\left\{ J_\lambda(u) : u \in \gamma_0\left([0,1]^l\right) \backslash D_\lambda^\nu \right\}, \tag{5.5}$$

we also have that m_0 does not depend on $\lambda \geq 0$. Furthermore, we claim that there exists $k(\nu) > 0$ such that

$$m_0 < c_J \tag{5.6}$$

when $|K|_\infty \leq k(\nu)$. In fact, for any $u = \sum_{j=1}^l t_j R\omega_j \in \gamma_0\left([0,1]^l\right) \backslash D_\lambda^\nu$, there must exists some $j_0 \in J$ such that

$$\left| t_{j_0} R \left\|\omega_{j_0}\right\|_{\lambda,\Omega_j'} - \sqrt{\left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_{j_0}} \right| > \nu.$$

According to the definition of ω_{j_0}, we know that $\|\omega_{j_0}\|^2_{\lambda,\Omega_j'} = \left(\frac{1}{2} - \frac{1}{p}\right)^{-1} c_{j_0}$. Thus, $\left|t_{j_0}R - 1\right| > \left(\frac{1}{2} - \frac{1}{p}\right)^{\frac{1}{2}} c_{j_0}^{-\frac{1}{2}}\nu$. So there exists $\delta(\nu) > 0$ such that

$$\frac{t_{j_0}^2 R^2}{2} \int_{\Omega_j} \left(\left|\nabla\omega_{j_0}\right|^2 + V_0(x)\omega_{j_0}^2\right) \mathrm{d}x - \frac{t_{j_0}^p R^p}{p} \int_{\Omega_j} \left|\omega_{j_0}\right|^p \mathrm{d}x < c_{j_0} - \delta(\nu).$$

And consequently,

$$\begin{aligned} J_\lambda(u) &= \sum_{j=1}^l I_j\left(t_j R\omega_j\right) + \frac{1}{4}\sum_{j=1}^l \int_{\Omega_j} K(x)\phi_u^a \left(t_j R\omega_j\right)^2 \mathrm{d}x \\ &< \sum_{j=1}^l c_j - \delta(\nu) + \frac{R^4}{4}\sum_{j=1}^l \int_{\Omega_j} K(x)\phi^a_{\sum_{j=1}^l \omega_j} \omega_j^2 \mathrm{d}x. \end{aligned}$$

Obviously, there is a $k(\nu) > 0$ such that

$$\frac{R^4}{4}\sum_{j=1}^{l}\int_{\Omega_j} K(x)\phi^a_{\sum_{j=1}^{l}\omega_j}\omega_j^2 dx < \frac{1}{2}\delta(\nu) \quad \text{for } |K|_\infty \le k(\nu).$$

Thus, $J_\lambda(u) < c_J - \frac{1}{2}\delta(\nu)$ and we show the claim.

Next, we prove that there is $k(\nu) > 0$ such that for some $S > 0$ and $|K|_\infty < k(\nu)$,

$$\max_{t\in[0,1]^l} J_\lambda\left(\eta\left(S,\gamma_0(t)\right)\right) \le \max\left\{m_0, c_J - \frac{1}{4}\sigma_0\nu\right\}. \tag{5.7}$$

If this is true, according to Lemma 4.3 and $\eta\left(S,\gamma_0(\cdot)\right) \in \Gamma_J$ we have

$$\sum_{j=1}^{l} c_{\lambda,j} \le b_{\lambda,J} \le \max_{t\in[0,1]^L} J_\lambda\left(\eta\left(S,\gamma_0(t)\right)\right) \le \max\left\{m_0, c_J - \frac{1}{4}\sigma_0\nu\right\} < c_J,$$

which contradicts to the fact $\sum_{j=1}^{l} c_{\lambda,j} \to c_J$. Thus, we obtain the lemma.

Next, we want to prove (5.7). Setting $u = \gamma_0(t) \in E$, if $u \notin D_\lambda^\nu$, because of (5.3) and (5.5), $J_\lambda(\eta(s,u)) \le J_\lambda(u) \le m_0$ for all $s \ge 0$. If $u \in D_\lambda^\nu$, we consider two possibilities:

(1) $\eta(s,u) \in D_\lambda^{3\nu/2}$ for all $s \in [0,S]$;

(2) $\eta(s,u) \in \partial D_\lambda^{3\nu/2}$ for some $s_0 \in [0,S]$.

When (1) occurs, we have $\theta(\eta(s,u)) = 1$ and $\|\nabla J_\lambda(\eta(s,u))\|_\lambda \ge \min\{\sigma_0,\sigma_\lambda\}$ when $|K|_\infty \le k_1(\nu)$ and $\lambda \ge \Lambda_1$ (see Lemma 5.1). Thus, setting $S = \frac{\sigma_0\nu}{2\min\{\sigma_0,\sigma_\lambda\}}$, by (5.3)

$$\begin{aligned} J_\lambda(\eta(S,u)) &= J_\lambda(u) + \int_0^S \frac{d}{ds} J_\lambda(\eta(s,u))ds \\ &= J_\lambda(u) - \int_0^S \theta(\eta(s,u))\left\|\nabla J_\lambda(\eta(s,u))\right\|_\lambda ds \\ &\le c_J + \mu - S\min\{\sigma_0,\sigma_\lambda\} \\ &= c_J + \mu - \frac{1}{2}\sigma_0\nu. \end{aligned} \tag{5.8}$$

When (2) occurs, there exist $0 < s_1 < s_2 \le S$ such that

$$\begin{aligned} &\eta\left(s_1,u\right) \in \partial D_\lambda^\nu, \\ &\eta\left(s_2,u\right) \in \partial D_\lambda^{3\nu/2}, \\ &\eta(s,u) \in D_\lambda^{3\nu/2}\backslash D_\lambda^\nu, s \in \left(s_1,s_2\right]. \end{aligned} \tag{5.9}$$

So we have, for some $j_0 \in J$,

$$\left\|\eta\left(s_2,u\right)\right\|_{\lambda,\mathbb{R}^3\backslash\Omega'_J} = \frac{3}{2}\nu \quad \text{or} \quad \left|\left\|\eta\left(s_2,u\right)\right\|_{\lambda,\Omega'_{j_0}} - \sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_{j_0}}\right| = \frac{3}{2}\nu.$$

We only see the latter case and the former one can be dealt with by a similar method. Following from (5.9), we have

$$\left|\left\|\eta\left(s_1,u\right)\right\|_{\lambda,\Omega'_{j_0}} - \sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_{j_0}}\right| \le \nu,$$

$$
\begin{aligned}
\|\eta(s_2,u)-\eta(s_1,u)\|_{\lambda,\Omega'_{j_0}} &\geq \left| \|\eta(s_2,u)\|_{\lambda,\Omega'_{j_0}} - \sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_{j_0}} \right| \\
&\quad - \left| \|\eta(s_1,u)\|_{\lambda,\Omega'_{j_0}} - \sqrt{\left(\frac{1}{2}-\frac{1}{p}\right)^{-1} c_{j_0}} \right| \\
&\geq \frac{1}{2}\nu.
\end{aligned}
$$

This implies $\|\eta(s_2,u)-\eta(s_1,u)\|_{\lambda} \geq \frac{1}{2}\nu$.

According to (5.2), $\left\|\frac{\mathrm{d}}{\mathrm{d}s}\eta\right\|_{\lambda} = \|V(\eta)\|_{\lambda} \leq 1$. Hence

$$
\frac{1}{2}\nu \leq \|\eta(s_2,u)-\eta(s_1,u)\|_{\lambda} \leq \left\| \int_{s_1}^{s_2} \frac{\mathrm{d}\eta}{\mathrm{d}s}\mathrm{d}s \right\|_{\lambda} \leq \int_{s_1}^{s_2} \left\| \frac{\mathrm{d}\eta}{\mathrm{d}s} \right\|_{\lambda} \mathrm{d}s \leq s_2 - s_1.
$$

According to (5.1), we have

$$
\begin{aligned}
J_\lambda(\eta(S,u)) &= J_\lambda(u) - \int_0^S \theta(\eta(s,u)) \,\|\nabla J_\lambda(\eta(s,u))\|_\lambda \,\mathrm{d}s \\
&\leq c_J + \mu - \int_{s_1}^{s_2} \sigma_0 \mathrm{d}s \\
&\leq c_J + \mu - \frac{1}{2}\sigma_0\nu.
\end{aligned} \tag{5.10}
$$

Then, we can choose $k(\nu) > 0$ such that $\mu \leq \frac{1}{4}\sigma_0\nu$ if $|K|_\infty \leq k(\nu)$. Combining with (5.8) and (5.10) we get (5.7). And hence J_λ possesses a critical point u_λ in D_λ^ν for $\lambda \geq \Lambda_1$ and $|K|_\infty \leq k(\nu)$. According to Lemma 3.4, we know that u_λ is a solution of (1.1). □

Proof of Theorem 1.1. Setting $u_{\lambda_n}(\lambda_n \to \infty)$ be a sequence of solutions of (1.1) obtained by the procedure above. Then, they are critical points of J_{λ_n} with critical value bounded by $c_J + \mu$. According to Lemma 3.3, we get the conclusion. □

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Necessary and sufficient conditions for one-dimensional variational problems with applications to elasticity

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Abstract. This paper deals with necessary and sufficient conditions for weak and strong minimizers of functionals $\Phi(u) = \int_a^b f(x, u(x), u'(x))\, dx$, where $u \in C^1([a, b], \mathbb{R}^N)$. We first derive conditions which are simpler than the known ones, and then apply them to several particular problems, including stability problems in the elasticity theory. In particular, we solve some open problems in [A. Majumdar, A. Raisch, Stability of twisted rods, helices and buckling solutions in three dimensions, *Nonlinearity* **27**(2014), 2841–2867] by finding optimal conditions for the stability of a naturally straight Kirchhoff rod under various types of endpoint constraints.

Keywords: minimizer, natural boundary conditions, conjugate points, field of extremals, elastica.

2020 Mathematics Subject Classification: 49K05, 74K10, 74G65, 34B15.

1 Introduction

This paper deals with necessary and sufficient conditions for local minimizers of one-dimensional variational problems for vector-valued functions. We consider the functional

$$\Phi : C^1([a, b], \mathbb{R}^N) \to \mathbb{R} : u \mapsto \int_a^b f(x, u(x), u'(x))\, dx, \tag{1.1}$$

where $-\infty < a < b < \infty$, $u = (u_1, u_2, \dots, u_N)$, and the Lagrangian[1]

$$f : [a, b] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} : (x, u, p) \mapsto f(x, u, p)$$

is sufficiently smooth ($f \in C^3$ or $f \in C^2$). We also fix a function $u^0 \in C^1([a, b], \mathbb{R}^N)$ and (possibly empty) subsets $I_a^{\mathcal{D}}, I_b^{\mathcal{D}}$ of the index set $I := \{1, 2, \dots, N\}$, and we look for conditions guaranteeing that u^0 is a local minimizer of Φ in the set

$$\mathcal{M} := \{u \in C^1([a, b], \mathbb{R}^N) : (u_i - u_i^0)(a) = 0 \text{ for } i \in I_a^{\mathcal{D}},\ (u_i - u_i^0)(b) = 0 \text{ for } i \in I_b^{\mathcal{D}}\}. \tag{1.2}$$

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[1] As in [8, pp. 11–12], by u we denote both the functions $[a, b] \to \mathbb{R}^N$ and the independent variable in $\mathbb{R}^N$, and by p we denote the last argument of f; see also similar notation $L(t, x(t), \dot{x}(t))$ vs. $L(t, x, v)$ in [15], for example.

This means that at $x = a$ we consider Dirichlet endpoint constraints for the components u_i with $i \in I_a^{\mathcal{D}}$, while the endpoints of the remaining components u_j with $j \in I \setminus I_a^{\mathcal{D}}$ are free; similarly for $x = b$. It is well known (see Proposition 2.1) that if u^0 is a local minimizer of this problem, then u^0 has to satisfy the natural boundary conditions

$$\frac{\partial f}{\partial p_j}(a, u^0(a), (u^0)'(a)) = 0 \quad \text{for } j \notin I_a^{\mathcal{D}} \quad \text{and} \quad \frac{\partial f}{\partial p_j}(b, u^0(b), (u^0)'(b)) = 0 \quad \text{for } j \notin I_b^{\mathcal{D}}.$$

We say that u^0 is a weak (or strong, resp.) local minimizer if there exists $\varepsilon > 0$ such that $\Phi(u^0) \leq \Phi(u)$ for any $u \in \mathcal{M}$ satisfying $\|u - u^0\|_{C^1} < \varepsilon$ (or $\|u - u^0\|_C < \varepsilon$, resp.), where $\|\cdot\|_{C^1}$ and $\|\cdot\|_C$ are the usual norms in C^1 and C, respectively (see Definition 2.2 and the subsequent comments for more details). If u^0 is a steady state of a mechanical system with potential energy Φ, and u^0 is a weak (or strong) local minimizer of Φ, then u^0 is stable with respect to perturbations which are small in C^1 (or C), respectively. On the other hand, if u^0 is not a minimizer, then u^0 is unstable.

If $I_a^{\mathcal{D}} = I_b^{\mathcal{D}} = I$, i.e. if one considers the Dirichlet endpoint constraints for all components and both ends, then necessary and sufficient conditions for u^0 to be a minimizer belong to the classical results in the calculus of variations, see [5,7,8], for example. They are based on the Jacobi theory (conjugate points) or the Weierstrass theory (field of extremals and excess function). In the general case such conditions are also known (see [15,16] and the references therein, and cf. also [17]); however, they use the notion of a *coupled point* which is more complicated than the classical notion of a *conjugate point*. This might be the reason why – as far as the author is aware – that general theory has not yet been applied in the elasticity theory, for example. In the scalar case, another approach to problems with variable endpoints (and a special class of Lagrangians) can be found in [12] but the conditions there are even more complicated than those in [15,16]. Reference [12] has been cited by several papers dealing with problems in the elasticity theory: Some of those papers use the complicated theory in [12] for scalar problems with special Lagrangians (see [10], for example), some use various ad-hoc estimates to obtain at least partial results in the vector-valued case (when the theory in [12] does not seem to apply, see [11], for example) and some refrain from considering variable endpoints because of the complexity of the theory in [12], see [3], for example, where the authors write: "…the application of the conjugate point test with nonclamped ends is a delicate issue …". Difficulties arising in a scalar problem with variable endpoints have also been analyzed in [14], for example.

The main purpose of this paper is to derive simple conditions for u^0 to be a minimizer, and to show how they can be applied to particular problems.

In Section 3 we derive necessary and sufficient conditions for weak minimizers by modifying the Jacobi theory (see Theorem 3.4 and also Remark 7.1 for the comparison of our conditions with those in [15,16]). In Section 4 we use the results from Section 3 to find optimal conditions for the stability of a naturally straight Kirchhoff rod under various types of endpoint constraints. The reasons for this particular application are the following:

- We show that our general results can easily be applied to vector-valued problems in the elasticity theory.
- We solve some open problems (and correct an erroneous result) in [11].
- We show how the choice of endpoint constraints influences the stability of the rod.

In Section 5 we use the Weierstrass theory to derive conditions for weak, strong and global minimizers, see Theorem 5.2. In this case we restrict our applications in Section 6 to the scalar case $N = 1$. The reason for this restriction is the following: If $N = 1$ and the Lagragian f is independent of its first variable x, then the phase plane analysis of the corresponding Du Bois-Reymond equation yields a very simple and efficient way to prove (or disprove) the existence of a suitable field of extremals; hence it is sufficient to verify the nonnegativity of the excess function in order to check our conditions. In particular, this approach does not require the verification of sufficient conditions based on the Jacobi theory and it can be used even if we do not know an explicit formula for u^0. In Section 6 we first determine the stability of a planar weightless inextensible and unshearable rod (see Example 6.3). This problem has already been analyzed in [1,10], for example, but our analysis is simpler than that in [10] and more complete than that in [1]. The notions of weak and strong minimizers are equivalent for functionals Φ in Section 4 and Example 6.3 (see Remark 4.2(vi) and Proposition 2.3, respectively). To illustrate various interesting features of minimizers in a more general case and demonstrate the applicability of our theory, in Example 6.5 we consider Lagrangians of the form $f(u, p) = u^2 + g(p)$, where g is a double-well function. In particular, the corresponding functional can possess both strong (even global) minimizers and minimizers which are weak but not strong.

Some of our results in the scalar case $N = 1$ have been obtained in the Master thesis [2].

2 Preliminaries

Throughout this paper we will use the symbols $\Phi, f, u^0, a, b, N, I, I_a^{\mathcal{D}}$ and $I_b^{\mathcal{D}}$ introduced in the Introduction. The partial derivatives of f will be denoted by $f_x, f_{u_i}, f_{p_i}, f_{p_i p_j}, \dots$

Given $\mathfrak{f} \in \{f, f_x, f_{u_i}, f_{p_i}, f_{p_i p_j}, \dots\}$, we will use the notation[2]

$$\mathfrak{f}^0(x) := \mathfrak{f}(x, u^0(x), (u^0)'(x)).$$

If $x \in \{a, b\}$ and W is a space of functions $[a, b] \to \mathbb{R}^N$, then we set

$$\begin{aligned} I_x^{\mathcal{N}} &:= I \setminus I_x^{\mathcal{D}}, \\ \mathbb{R}^N_{\mathcal{D},x} &:= \{\xi \in \mathbb{R}^N : \xi_i = 0 \text{ for } i \in I_x^{\mathcal{D}}\}, \\ \mathbb{R}^N_{\mathcal{N},x} &:= \{\xi \in \mathbb{R}^N : \xi_i = 0 \text{ for } i \in I_x^{\mathcal{N}}\}, \\ W_{\mathcal{D},x} &:= \{v \in W : v(x) \in \mathbb{R}^N_{\mathcal{D},x}\}, \\ W_{\mathcal{D}} &:= W_{\mathcal{D},a} \cap W_{\mathcal{D},b}. \end{aligned}$$

In particular, if $W = C^1 = C^1([a, b], \mathbb{R}^N)$, then

$$C^1_{\mathcal{D}} = \{v \in C^1([a, b], \mathbb{R}^N) : v_i(a) = 0 \text{ for } i \in I_a^{\mathcal{D}},\ v_i(b) = 0 \text{ for } i \in I_b^{\mathcal{D}}\} \tag{2.1}$$

is the space of C^1-test functions. (Notice that the set $\mathcal{M}$ in (1.2) satisfies $\mathcal{M} = u^0 + C^1_{\mathcal{D}}$.)

The norm in a general Banach space X will be denoted by $\|\cdot\|_X$; the norm in $W^{1,2}$ will also be denoted by $\|\cdot\|_{1,2}$. In particular, if $X = C^1 = C^1([a, b], \mathbb{R}^N)$ or $X = C = C([a, b], \mathbb{R}^N)$, then $\|u\|_{C^1} = \max_{x\in[a,b]} |u(x)| + \max_{x\in[a,b]} |u'(x)|$ or $\|u\|_C = \max_{x\in[a,b]} |u(x)|$, respectively, where $|u(x)|$ denotes the Euclidean norm of $u(x) \in \mathbb{R}^N$. We also set $B_\varepsilon := \{\xi \in \mathbb{R}^N : |\xi| < \varepsilon\}$.

[2]The superscript 0 in $\mathfrak{f}^0$ denotes evaluation of $\mathfrak{f}$ along the reference arc u^0; cf. similar notation $\hat{L}(t) = L(t, \hat{x}(t), \dot{\hat{x}}(t))$ in [15] or $\bar{\mathfrak{f}}(x) = \mathfrak{f}(x, u(x), u'(x))$ in [8, formulas (30), (39) in Section 2.3, pp. 114–116]. The advantages of our notation will become evident in Section 6: See the notation introduced in Theorem 6.1.

We will assume that u^0 is a critical point of Φ in the set $u^0 + C^1_{\mathcal{D}}$, i.e. $\Phi'(u^0)h = 0$ for any test function $h \in C^1_{\mathcal{D}}$, where Φ' denotes the Fréchet derivative of Φ. The following proposition is well known, but for the reader's convenience we explain the idea of its proof in the Appendix.

Proposition 2.1. *Let $f \in C^1$ and let u^0 be a critical point of Φ in $u^0 + C^1_{\mathcal{D}}$. Then u^0 is an extremal (i.e. it satisfies the Euler equations $\frac{d}{dx}(f^0_{p_i}) = f^0_{u_i}$, $i = 1, 2, \dots, N$), and u^0 also has to satisfy the natural boundary conditions*

$$f^0_{p_j}(a) = 0 \ \text{for} \ j \in I^{\mathcal{N}}_a \quad \text{and} \quad f^0_{p_j}(b) = 0 \ \text{for} \ j \in I^{\mathcal{N}}_b. \tag{2.2}$$

If $f_{p_i} \in C^1$ for $i = 1, 2, \dots, N$, and the strengthened Legendre condition

$$(\exists c^0 > 0) \qquad \sum_{i,j=1}^{N} f^0_{p_i p_j}(x)\xi_i\xi_j \geq c^0|\xi|^2, \qquad \xi \in \mathbb{R}^N, \ x \in [a,b], \tag{2.3}$$

is true, then $u^0 \in C^2$.

It is known that the Legendre condition (i.e. condition (2.3) with $c^0 = 0$) is necessary for u^0 to be a minimizer, but even the strengthened Legendre condition is not sufficient, in general. Assuming that

$$f \in C^3 \text{ satisfies (2.3), where } u^0 \in C^1([a,b], \mathbb{R}^N) \text{ is an extremal satisfying (2.2),} \tag{2.4}$$

and denoting $\sum_k = \sum_{k=1}^N$, we set

$$\Psi(h) := \int_a^b \mathfrak{F}(x, h(x), h'(x))\, dx, \quad h \in W^{1,2}([a,b], \mathbb{R}^N), \tag{2.5}$$

where

$$\mathfrak{F} = \mathfrak{F}(x,u,p) := \sum_{i,j} \Big(f^0_{p_i p_j}(x) p_i p_j + f^0_{p_i u_j}(x) p_i u_j + f^0_{u_i p_j}(x) u_i p_j + f^0_{u_i u_j}(x) u_i u_j \Big). \tag{2.6}$$

If $h \in C^1$, then $\Psi(h) = \Phi''(u^0)(h,h)$, i.e. Ψ is the second variation of Φ at u^0. In addition, if $h \in C^2$, then integration by parts yields

$$\Psi(h) = \int_a^b \sum_i (\mathcal{A}_i h) h_i \, dx + \sum_i (\mathcal{B}_i h) h_i \Big|_a^b, \tag{2.7}$$

where

$$\mathcal{A}_i h := -\frac{d}{dx}(\mathcal{B}_i h) + \mathcal{C}_i h, \quad \mathcal{B}_i h := \sum_j \Big(f^0_{p_i p_j} h'_j + f^0_{p_i u_j} h_j \Big), \quad \mathcal{C}_i h := \sum_j \Big(f^0_{u_i p_j} h'_j + f^0_{u_i u_j} h_j \Big). \tag{2.8}$$

Set also

$$\mathcal{A}h := (\mathcal{A}_1 h, \dots, \mathcal{A}_N h), \quad \mathcal{B}h := (\mathcal{B}_1 h, \dots \mathcal{B}_N h), \quad f_p := (f_{p_1}, \dots, f_{p_N}), \quad f_u = (f_{u_1}, \dots, f_{u_N}).$$

The (vector-valued) second-order linear differential equation $\mathcal{A}h = 0$ is called the *Jacobi* equation (for Φ and u^0): it will play a fundamental role in the study of positive definiteness of Ψ. Notice also that the Jacobi equation is the Euler equation for functional Ψ. More precisely, by using the symmetry relations $f_{p_i p_j} = f_{p_j p_i}$, $f_{p_i u_j} = f_{u_j p_i}$ and $f_{u_i u_j} = f_{u_j u_i}$ we obtain

$$\mathfrak{F}_{p_i}(x, h(x), h'(x)) = 2\mathcal{B}_i h(x), \quad \mathfrak{F}_{u_i}(x, h(x), h'(x)) = 2\mathcal{C}_i h(x), \tag{2.9}$$

hence

$$2\mathcal{A}_i h(x) = -\frac{d}{dx}\mathfrak{F}_{p_i}(x,h(x),h'(x)) + \mathfrak{F}_{u_i}(x,h(x),h'(x)). \tag{2.10}$$

Notice also that, given $h, w \in W^{1,2}$, (2.9) and the symmetry of the second-order derivatives of f mentioned above imply

$$\begin{aligned}\Psi'(h)w &= \int_a^b \sum_i \big(\mathfrak{F}_{p_i}(x,h(x),h'(x))w_i'(x) + \mathfrak{F}_{u_i}(x,h(x),h'(x))w_i(x)\big)\,dx \\ &= 2\int_a^b \sum_i \big(\mathcal{B}_i h\cdot w_i' + \mathcal{C}_i h\cdot w_i\big)\,dx = 2\int_a^b \sum_i \big(\mathcal{B}_i w\cdot h_i' + \mathcal{C}_i w\cdot h_i\big)\,dx = \Psi'(w)h.\end{aligned} \tag{2.11}$$

Definition 2.2. Let $w \in \mathcal{M}$, where $\mathcal{M}$ is a subset of $C^1([a,b],\mathbb{R}^N)$. The function w is called a *weak* or *strong local minimizer* in $\mathcal{M}$ if there exists $\varepsilon > 0$ such that $\Phi(v) \geq \Phi(w)$ for any $v \in \mathcal{M}$ satisfying $\|v-w\|_{C^1} < \varepsilon$ or $\|v-w\|_C < \varepsilon$, respectively.

Let $w \in \mathcal{N}$, where $\mathcal{N}$ is a subset of $W^{1,2}([a,b],\mathbb{R}^N)$. The function w is called a *local minimizer* in $\mathcal{N}$ if there exists $\varepsilon > 0$ such that $\Phi(v) \geq \Phi(w)$ for any $v \in \mathcal{N}$ satisfying $\|v\|_{1,2} < \varepsilon$.

If the inequalities $\Phi(v) \geq \Phi(w)$ in the definitions above are strict for $v \neq w$, then the minimizer w is called *strict*.

Since the adjectives *weak* and *strong* are not meaningful in the case of global minimizers, we often omit the word “local” in the notions of weak and strong local minimizers. Each strong minimizer is a weak minimizer but the opposite is not true, in general. For example, if $N = 1$ and $f(x,u,p) = p^2 + p^3$, then $u^0 \equiv 0$ is a weak but not strong minimizer of Φ in $u^0 + C^1_{\mathcal{D}}$ for any choice of $a, b, I_a^{\mathcal{D}}$ and $I_b^{\mathcal{D}}$ (see also Example 6.5 for a less trivial example). On the other hand, the following Proposition 2.3 and Remark 4.2(vi) show that in some cases the notions of weak and strong minimizers are equivalent. The choice of the class of Lagrangians in Proposition 2.3 is motivated by Example 6.3, where we consider the stability of a planar rod. Proposition 2.3 is true for any choice of $a, b, I_a^{\mathcal{D}}$ and $I_b^{\mathcal{D}}$; its proof is postponed to the Appendix.

Proposition 2.3. *Let $N = 1$ and $f(x,u,p) = (p-K)^2 + g(u)$, where $K \in \mathbb{R}$ and $g \in C^1(\mathbb{R})$. If $u^0 \in C^1$ is a weak minimizer, then it is a strong minimizer.*

The following proposition is a consequence of well known facts (see [5,8], for example). The assumptions in that proposition are much stronger than necessary, but the proposition will be sufficient for our purposes (see Remark 4.2(vi), Section 6 and the proof of Proposition 3.5).

Proposition 2.4.

(i) Let $f \in C^k$, $k \geq 2$.

If $u^0 \in C^1$ is a critical point of Φ in $u^0 + C^1_{\mathcal{D}}$ and (2.3) is true, then $u^0 \in C^k$ and u^0 satisfies the Du Bois-Reymond equation

$$\frac{d}{dx}\big(f^0 - (u^0)'\cdot f^0_p\big) = f^0_x \quad \text{in } [a,b]. \tag{2.12}$$

Conversely, if $u^0 \in C^2$ satisfies (2.12) and $(u^0)' \neq 0$ a.e., then u^0 is an extremal.

(ii) Let $f \in C^1$ satisfy the growth condition $(1+|p|)|f_p| + |f_u| \leq M(|u|)(1+|p|)^2$, where $M : [0,\infty) \to [0,\infty)$ is nondecreasing. Then $\Phi \in C^1(W^{1,2})$. In addition, if $u^0 \in W^{1,2}$ is a local minimizer of Φ in $u^0 + W^{1,2}_{\mathcal{D}}$, then there exists $C \in \mathbb{R}^N$ such that

$$f^0_p(x) = \int_a^x f^0_u(\xi)\,d\xi + C \quad \text{for a.e. } x \in [a,b].$$

3 Jacobi theory

In this section we will prove necessary and sufficient conditions for weak minimizers by modifying the classical Jacobi theory. Throughout this section we assume (2.4).

The following proposition is well known, but for the reader's convenience we provide its proof in the Appendix.

Proposition 3.1. *Assume* (2.4) *and let* Ψ *be defined by* (2.5).

(i) If Ψ *is positive definite in* $W^{1,2}_{\mathcal{D}}$*, then* u^0 *is a strict weak minimizer in* $u^0 + C^1_{\mathcal{D}}$.

(ii) If $\Psi(h) < 0$ *for some* $h \in W^{1,2}_{\mathcal{D}}$*, then* u^0 *is not a weak minimizer in* $u^0 + C^1_{\mathcal{D}}$.

We will consider the scalar case first. Assume that

$$h \text{ is a nontrivial solution of the Jacobi equation } \mathcal{A}h = 0. \tag{3.1}$$

Then the following classical result for problems with Dirichlet endpoint constraints is well known.

Theorem 3.2. *Assume* (2.4) *with* $N = 1$ *and* (3.1)*. Let* $I^{\mathcal{N}}_a = I^{\mathcal{N}}_b = \emptyset$ *and* $h(a) = 0$.

(i) If $h(y) = 0$ *for some* $y \in (a, b)$*, then* u^0 *is not a weak minimizer.*

(ii) If $h(y) \neq 0$ *for any* $y \in (a, b]$*, then* u^0 *is a strict weak minimizer.*

Our analogue in the case of variable endpoints is the following theorem.

Theorem 3.3. *Assume* (2.4) *with* $N = 1$ *and* (3.1)*. Let* $I^{\mathcal{N}}_a = I^{\mathcal{N}}_b = \{1\}$ *and* $\mathcal{B}h(a) = 0$.

(i) If $h(y) = 0$ *for some* $y \in (a, b]$ *or* $\mathcal{B}h(b)h(b) < 0$*, then* u^0 *is not a weak minimizer.*

(ii) If $h(y) \neq 0$ *for any* $y \in (a, b]$ *and* $\mathcal{B}h(b)h(b) > 0$*, then* u^0 *is a strict weak minimizer.*

In fact, a slight generalization of Theorem 3.3(ii) has been proved in [2]: The initial condition $\mathcal{B}h(a) = 0$ can be replaced with $\mathcal{B}h(a)h(a) \leq 0$. Unfortunately, the method of the proof in [2] does not seem to be easily extendable to the vector-valued case.

Theorems 3.2 and 3.3 are special cases of the following general theorem.

Theorem 3.4. *Assume* (2.4)*. Let* $h^{(1)}, \dots, h^{(N)}$ *be linearly independent solutions of the Jacobi equation* $\mathcal{A}h = 0$ *satisfying the initial conditions* $h(a) \in \mathbb{R}^N_{\mathcal{D},a}$*,* $\mathcal{B}h(a) \in \mathbb{R}^N_{\mathcal{N},a}$*. Set*

$$D(x) := \det(h^{(1)}(x), \dots, h^{(N)}(x)), \quad H := \operatorname{span}(h^{(1)}, \dots, h^{(N)}), \quad H_0 := \{h \in H : h(b) = 0\}.$$

(i) *If* $D(x) = 0$ *for some* $x \in (a, b)$ *or*

$$I^{\mathcal{N}}_b \neq \emptyset \;\; \text{and} \;\; \mathcal{B}h(b) \cdot h(b) < 0 \text{ for some } h \in H_{\mathcal{D},b},$$

then u^0 *is not a weak minimizer.*

(ii) *If* $D \neq 0$ *in* $(a, b]$ *and*

$$\text{either} \;\; I^{\mathcal{N}}_b = \emptyset \;\; \text{or} \;\; \mathcal{B}h(b) \cdot h(b) > 0 \text{ for any } h \in H_{\mathcal{D},b} \setminus \{0\},$$

then u^0 *is a strict weak minimizer.*

(iii) *Let* $D \neq 0$ *in* (a,b), $D(b) = 0$ *(hence* $H_0 \neq \{0\}$*), and* $I_b^{\mathcal{N}} \neq \emptyset$. *If*

$$\text{there exists } h \in H_0 \text{ such that } \mathcal{B}_i h(b) \neq 0 \text{ for some } i \in I_b^{\mathcal{N}}, \tag{3.2}$$

then u^0 *is not a weak minimizer. If* $I_b^{\mathcal{D}} = \emptyset$, *then* (3.2) *is always true.*

The proof of Theorem 3.4 is based on a modification of the classical Jacobi theory, and this is also true in the case of the corresponding proof in [16]. However, our conditions in Theorem 3.4 are simpler than those in [15,16], see Remark 7.1 in the Appendix.

In order to prove Theorem 3.4, we need some preparation. Given $y \in (a,b]$, let

$$X_y := \{h \in W^{1,2}([a,b],\mathbb{R}^N) : h(a) \in \mathbb{R}^N_{\mathcal{D},a},\ h(x) = 0 \text{ for } x \geq y\}$$

be endowed with the norm $\|h\|_{X_y} := (\int_a^b \sum_{i,j} f^0_{p_ip_j} h'_i h'_j \, dx)^{1/2}$ (which is equivalent to the standard norm in $W^{1,2}$ for $h \in X_y$ due to (2.3) and the boundary condition $h(b) = 0$), and let S_y denote the unit sphere in X_y. If $\tilde{y} \in (y,b]$, then $X_y \subset X_{\tilde{y}}$, hence $S_y \subset S_{\tilde{y}}$. Set also

$$\lambda_1 = \lambda_1(y) := \inf_{h \in S_y} \Psi(h) = 1 + \inf_{h \in S_y} \hat{\Psi}(h), \tag{3.3}$$

where

$$\hat{\Psi}(h) := \int_a^b \sum_{i,j} \Big(f^0_{p_iu_j} h'_i h_j + f^0_{u_ip_j} h_i h'_j + f^0_{u_iu_j} h_i h_j \Big)\, dx.$$

Since $S_y \subset S_{\tilde{y}}$ if $y < \tilde{y}$, the function λ_1 is nonincreasing. In addition, one can easily show that λ_1 is continuous, and the estimate

$$\begin{aligned} |h(x)| &= \Big| \int_x^y h'(\xi)\, d\xi \Big| \leq \Big(\int_x^y |h'(\xi)|^2 \, d\xi \Big)^{1/2} \sqrt{y-x} \\ &\leq \frac{1}{\sqrt{c^0}} \Big(\int_a^b \sum_{i,j} f^0_{p_ip_j} h'_i h'_j \, d\xi \Big)^{1/2} \sqrt{y-a} = \frac{1}{\sqrt{c^0}} \sqrt{y-a} \end{aligned}$$

for $h \in S_y$ and $x \in (a,y)$ implies $\lim_{y \to a+} \lambda_1(y) = 1$.

Proposition 3.5. *Let* D *be as in Theorem 3.4 and* $y \in (a,b]$.

(i) *If* $\lambda_1(y) = 0$, *then* $D(y) = 0$ *and* $\lambda_1(z) < 0$ *for* $z \in (y,b]$. *If* $D(y) = 0$, *then* $\lambda_1(y) \leq 0$.

(ii) *If* $h \in X_b$, *then* $\Psi(h) \geq \lambda_1(b)\|h\|^2_{X_b}$. *If* $\lambda_1(b) < 0$, *then there exists* $h \in X_b$ *such that* $\Psi(h) < 0$.

Proof. Let $\lambda_1(y) = 0$ and let B_y denote the closed unit ball in X_y. Since $\hat{\Psi}$ is weakly sequentially continuous, there exists $h_y \in B_y$ such that $\hat{\Psi}(h_y) = \inf_{B_y} \hat{\Psi} = -1$. We have $h_y \in S_y$ (otherwise $th_y \in B_y$ for some $t > 1$, and $\hat{\Psi}(th_y) = t^2 \hat{\Psi}(h_y) < \inf_{B_y} \hat{\Psi}$, which yields a contradiction). Since $\Psi(h_y) = \inf_{S_y} \Psi = 0$, h_y is a global minimizer of Ψ in X_y. Notice that $\mathfrak{F} \in C^1$ satisfies the growth condition

$$(1+|p|)|\mathfrak{F}_p(x,u,p)| + |\mathfrak{F}_u(x,u,p)| \leq C(1+|p|)(|u|+|p|) \leq 2C(1+|u|^2)(1+|p|^2),$$

where C depends only on the sup-norm of $f^0_{p_ip_j}, f^0_{p_iu_j}, f^0_{u_ip_j}, f^0_{u_iu_j}$, hence Proposition 2.4(ii) and (2.9) imply

$$2\mathcal{B}_i h_y(x) = \mathfrak{F}_{p_i}(x, h_y(x), h'_y(x)) = \int_a^x \mathfrak{F}_{u_i}(\xi, h_y(\xi), h'_y(\xi))\, d\xi + c_i = \int_a^x 2\mathcal{C}_i h_y \, d\xi + c_i \tag{3.4}$$

for a.e. $x \in [a,y]$. Since the right-hand side of (3.4) is a continuous function of x, $f \in C^3$ and (2.3) is true (hence the matrix $f^0_{p_ip_j}$ is invertible and the inverse matrix is a continuous function of x), we see that the restriction of h_y to $[a,y]$ is C^1. Denote this restriction by $\bar h_y$ and set $C^1_y := \{w \in C^1([a,y]) : w(a) \in \mathbb{R}^N_{\mathcal{D},a},\ w(y) = 0\}$, $\Psi_y(h) = \int_a^y \mathfrak{F}(x,h(x),h'(x))\,dx$. Then $\bar h_y$ is a critical point of Ψ_y in $\bar h_y + C^1_y = C^1_y$. Now Proposition 2.1, (2.10) and (2.9) imply that $\bar h_y$ is C^2, it satisfies the Jacobi equation $\mathcal{A}h = 0$ in $[a,y]$ and the natural boundary conditions $\mathcal{B}h(a) \in \mathbb{R}^N_{\mathcal{N},a}$. Since we also have $h_y(a) \in \mathbb{R}^N_{\mathcal{D},a}$, there exists $\alpha \in \mathbb{R}^N \setminus \{0\}$ such that $h_y = \sum_k \alpha_k h^{(k)}$ on $[a,y]$, where $h^{(k)}$ are as in Theorem 3.4. Since $h_y(y) = 0$, we have $D(y) = 0$.

Next assume on the contrary that $\lambda_1(y) = 0 = \lambda_1(z)$ for some $z \in (y,b]$. Then the minimizer h_y is a global minimizer of Ψ in X_z. Similarly as above we deduce that $h_y \in C^2([a,z])$ and h_y solves the Jacobi equation in $[a,z]$. Consequently, $h_y(y) = h'_y(y) = 0$, which yields a contradiction with the uniqueness of solutions of the initial value problem for the Jacobi equation.

Next assume that $D(y) = 0$. Then there exists $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{R}^N \setminus \{0\}$ such that $h := \sum_k \alpha_k h^{(k)}$ satisfies $h(y) = 0$, hence if we set $\tilde h(x) := h(x)$ for $x \le y$ and $\tilde h(x) := 0$ otherwise, then $\tilde h \in X_y$. In addition, using $\mathcal{A}_i h = 0$, $\mathcal{B}_i h(a) \in \mathbb{R}^N_{\mathcal{N},a}$, $h(a) \in \mathbb{R}^N_{\mathcal{D},a}$ and $h(y) = 0$ we obtain

$$\Psi(\tilde h) = \int_a^b \mathfrak{F}(x,\tilde h(x),\tilde h'(x))\,dx = \int_a^y \mathfrak{F}(x,h(x),h'(x))\,dx = \int_a^y \sum_i (\mathcal{A}_i h)h_i\,dx + \sum_i (\mathcal{B}_i h)h_i\Big|_a^y = 0,$$

hence $\lambda_1(y) \le 0$.

If $h \in X_b \setminus \{0\}$, then $\Psi(h) = \|h\|^2_{X_b}\Psi(h/\|h\|_{X_b}) \ge \lambda_1(b)\|h\|^2_{X_b}$ by the definition of λ_1. If $\lambda_1(b) < 0$, then the definition of λ_1 implies the existence of $h \in S_b$ such that $\Psi(h) < 0$. □

Proof of Theorem 3.4. We will show that

$$\text{the assumptions in (i) (or (iii)) imply } \Psi(h) < 0 \text{ for some } h \in W^{1,2}_{\mathcal{D}}, \tag{3.5}$$

while

$$\text{the assumptions in (ii) guarantee that } \Psi \text{ is positive definite in } W^{1,2}_{\mathcal{D}}, \tag{3.6}$$

hence the assertions in Theorem 3.4 will follow from Proposition 3.1.

(i) If $D(x) = 0$ for some $x \in (a,b)$, then Proposition 3.5(i) implies $\lambda_1(x) \le 0$ and $\lambda_1(b) < 0$, hence Proposition 3.5(ii) implies the existence of $h \in X_b \subset W^{1,2}_{\mathcal{D}}$ such that $\Psi(h) < 0$.

If $I^{\mathcal{N}}_b \ne \emptyset$ and $\mathcal{B}h(b) \cdot h(b) < 0$ for some $h \in H_{\mathcal{D},b} \subset W^{1,2}_{\mathcal{D}}$, then $\mathcal{A}h = 0$, $h_i(a) = 0$ for $i \in I^{\mathcal{D}}_a$ and $\mathcal{B}_i h(a) = 0$ for $i \in I^{\mathcal{N}}_a$, hence (2.7) implies

$$\Psi(h) = \mathcal{B}h \cdot h\Big|_a^b = \mathcal{B}h(b) \cdot h(b) < 0.$$

(ii) Assume that $D \ne 0$ in $(a,b]$. Then Proposition 3.5 implies $\lambda_1(b) > 0$ and $\Psi(h) \ge \lambda_1(b)\|h\|^2_{X_b}$ for $h \in X_b$. If $I^{\mathcal{N}}_b = \emptyset$, then $X_b = W^{1,2}_{\mathcal{D}}$, hence we are done.

Next assume that $I^{\mathcal{N}}_b \ne \emptyset$ and $\mathcal{B}\tilde h(b) \cdot \tilde h(b) > 0$ for any $\tilde h \in H_{\mathcal{D},b} \setminus \{0\}$ (hence $\mathcal{B}\tilde h(b) \cdot \tilde h(b) \ge c_1\|\tilde h\|^2_{1,2}$ for some $c_1 > 0$ due to $\dim H_{\mathcal{D},b} < \infty$), and let $h \in W^{1,2}_{\mathcal{D}}$ be fixed. Since $D(b) \ne 0$, there exists $\alpha \in \mathbb{R}^N$ such that $\tilde h := \sum_k \alpha_k h^{(k)}$ satisfies $\tilde h(b) = h(b)$. In particular, $\tilde h \in H_{\mathcal{D},b}$. Set $\hat h := h - \tilde h$. Then $\hat h \in X_b$, hence $\Psi(\hat h) \ge \lambda_1(b)\|\hat h\|^2_{X_b}$. In addition, $\Psi(\tilde h) = \mathcal{B}\tilde h(b) \cdot \tilde h(b) \ge c_1\|\tilde h\|^2_{1,2}$. Since Ψ is a quadratic functional, we have $\Psi''(\tilde h)(\hat h,\hat h) = 2\Psi(\hat h)$ and $\Psi''' = 0$. Using (2.11) and integration by parts we also obtain

$$\Psi'(\hat h)\tilde h = \Psi'(\tilde h)\hat h = 2\int_a^b \mathcal{A}\tilde h \cdot \hat h\,dx + 2\mathcal{B}\tilde h \cdot \hat h\Big|_a^b = 0,$$

hence there exists $c > 0$ such that

$$\Psi(h) = \Psi(\tilde{h} + \hat{h}) = \Psi(\tilde{h}) + \Psi'(\tilde{h})\hat{h} + \frac{1}{2}\Psi''(\tilde{h})(\hat{h}, \hat{h}) = \Psi(\tilde{h}) + \Psi(\hat{h}) \geq c\|h\|_{1,2}^2.$$

(iii) Let $h \in H_0$ and $\mathcal{B}_i h(b) \neq 0$ for some $i \in I_b^{\mathcal{N}}$. Then $\mathcal{A}h = 0$, $h(a) \in \mathbb{R}^N_{\mathcal{D},a}$, $\mathcal{B}h(a) \in \mathbb{R}^N_{\mathcal{N},a}$ and $h(b) = 0$, hence

$$\Psi(h) = \int_a^b \mathcal{A}h \cdot h\,dx + \mathcal{B}h \cdot h\Big|_a^b = 0.$$

Notice also that $h \neq 0$ due to $\mathcal{B}_i h(b) \neq 0$. Since $D \neq 0$ in (a, b), $D(b) = 0$, $\lim_{y \to a+} \lambda_1(y) = 1$ and λ_1 is continuous and nonincreasing, Proposition 3.5(i) implies $\lambda_1(b) = 0$, hence h is a global minimizer of Ψ in X_b. Choose $\tilde{h} \in C^1_{\mathcal{D}}$ with $\tilde{h}(a) = 0$, $\tilde{h}_j(b) = \delta_{ij}$ for $j = 1, 2, \ldots, N$. Then

$$\Psi'(h)\tilde{h} = 2\int_a^b \mathcal{A}h \cdot \tilde{h}\,dx + 2\mathcal{B}h \cdot \tilde{h}\Big|_a^b = 2\mathcal{B}_i h(b) \neq 0,$$

hence

$$\Psi(h + \varepsilon\tilde{h}) = \varepsilon\Psi'(h)\tilde{h} + o(\varepsilon) < 0$$

provided $|\varepsilon|$ is small enough and $\varepsilon\mathcal{B}_i h(b) < 0$.

If $I_b^{\mathcal{D}} = \emptyset$ and $h \in H_0 \setminus \{0\}$, then $\mathcal{A}h = 0$ and $h(b) = 0$, hence the uniqueness of the initial value problem for the Jacobi equation implies the existence of $i \in I_b^{\mathcal{N}} = I$ such that $\mathcal{B}_i h(b) \neq 0$. □

Remark 3.6.

(i) If Ψ is positive semidefinite but not positive definite, then there exists $h^* \in W^{1,2}_{\mathcal{D}} \setminus \{0\}$ such that $0 = \Psi(h^*) = \inf_{W^{1,2}_{\mathcal{D}}} \Psi$ and h^* can be determined from our analysis. For example, if $N = 1$ and $I_a^{\mathcal{D}} = I_b^{\mathcal{D}} = \emptyset$ (cf. Theorem 3.3), then h^* is a positive (or negative) solution of the Jacobi equation satisfying $\mathcal{B}h^*(a) = \mathcal{B}h^*(b) = 0$. If Φ depends smoothly on a parameter θ, u^0 is a critical point of Φ for any θ, and u^0 is (or is not, respectively) a weak minimizer for $\theta > \theta^*$ (or $\theta < \theta^*$, respectively), then the critical parameter θ^* corresponds to the case where h^* exists. (Such situation occurs, for example, in the study of stability of a twisted rod in Section 4.) In this case one can expect bifurcation for the problem $\Phi'(u) = 0$ at $\theta = \theta^*$ in the direction of h^*, cf. [6, Theorem 5.6].

(ii) Let $h^{(k)}$, $k = 1, 2, \ldots, N$, be as in Theorem 3.4, $\xi \in \mathbb{R}^N$ and $h^\xi := \sum_k \xi_k h^{(k)}$. Set $\mathfrak{A} := (a_{kl})_{k,l=1}^N$, where $a_{kl} = \mathcal{B}h^{(k)}(b) \cdot h^{(l)}(b)$, and

$$\Xi_{\mathcal{D}} := \{\xi \in \mathbb{R}^N : h^\xi(b) \in \mathbb{R}^N_{\mathcal{D},b}\}.$$

Then $\mathcal{B}h^\xi(b) \cdot h^\xi(b) = \mathfrak{A}\xi \cdot \xi$, i.e. the condition $\mathcal{B}h(b) \cdot h(b) > 0$ for any $h \in H_{\mathcal{D},b} \setminus \{0\}$ in Theorem 3.4(ii), for example, is equivalent to $\mathfrak{A}\xi \cdot \xi > 0$ for any $\xi \in \Xi_{\mathcal{D}} \setminus \{0\}$. In particular, if $I_b^{\mathcal{D}} = \emptyset$ (and $D(b) \neq 0$), then that condition is equivalent to the positive definiteness of the matrix $\mathfrak{A}$. Notice also that $a_{kl} = a_{lk}$ due to $2a_{kl} = \Psi'(h^{(k)})h^{(l)}$ and $\Psi'(h^{(k)})h^{(l)} = \Psi'(h^{(l)})h^{(k)}$.

(iii) Assertions (3.6) or (3.5) show that some of the assumptions in Theorem 3.4 are sufficient for the positivity or the negativity of Ψ, respectively. We will show that those assumptions are also necessary, at least in some cases.

Let Ψ be positive definite in $W^{1,2}_{\mathcal{D}}$. Since $X_b \subset W^{1,2}_{\mathcal{D}}$, Ψ is also positive definite in X_b and Proposition 3.5(i) implies $D \neq 0$ in $[a,b]$. If $I^{\mathcal{N}}_b \neq \emptyset$ and $h \in H_{\mathcal{D},b} \setminus \{0\}$, then $h \in W^{1,2}_{\mathcal{D}}$, $\mathcal{B}h(a) \in \mathbb{R}^N_{\mathcal{N},a}$, $\mathcal{A}h = 0$, hence

$$0 < \Psi(h) = \int_a^b \mathcal{A}h \cdot h\,dx + \mathcal{B}h \cdot h\Big|_a^b = \mathcal{B}h(b) \cdot h(b),$$

so that the assumptions in Theorem 3.4(ii) are satisfied. This fact and (3.6) show that the positive definiteness of Ψ in $W^{1,2}_{\mathcal{D}}$ and the assumptions of Theorem 3.4(ii) are equivalent.

Let $\Psi(\bar{h}) < 0$ for some $\bar{h} \in W^{1,2}_{\mathcal{D}}$ and

$$I^{\mathcal{D}}_b = \emptyset \quad \text{or} \quad I^{\mathcal{N}}_b = \emptyset. \tag{3.7}$$

Assume that the assumptions of Theorem 3.4(i) are not satisfied. Then $D \neq 0$ in (a,b) (hence $\lambda_1(b) \geq 0$ due to Proposition 3.5(i)) and either $I^{\mathcal{N}}_b = \emptyset$ or $\mathcal{B}h(b) \cdot h(b) \geq 0$ for any $h \in H_{\mathcal{D},b}$. If $I^{\mathcal{N}}_b = \emptyset$, then $W^{1,2}_{\mathcal{D}} = X_b$, hence $\Psi \geq 0$ in $W^{1,2}_{\mathcal{D}}$, which is a contradiction. Consequently, $I^{\mathcal{N}}_b \neq \emptyset$, $\mathcal{B}h(b) \cdot h(b) \geq 0$ for any $h \in H_{\mathcal{D},b}$ and $I^{\mathcal{D}}_b = \emptyset$ (due to (3.7)). If $D(b) \neq 0$, then there exists $\tilde{h} \in H_{\mathcal{D},b}$ such that $\tilde{h}(b) = \bar{h}(b)$. Set $\hat{h} := \bar{h} - \tilde{h} \in X_b$. Then similarly as in the proof of Theorem 3.4(ii) we obtain

$$0 > \Psi(\bar{h}) = \Psi(\tilde{h} + \hat{h}) = \Psi(\tilde{h}) + \Psi(\hat{h}) \geq \mathcal{B}\tilde{h}(b) \cdot \tilde{h}(b) + \lambda_1(b)\|\hat{h}\|^2_{X_b} \geq 0,$$

which is a contradiction. Consequently, $D(b) = 0$. Since $I^{\mathcal{D}}_b = \emptyset$ implies (3.2), all assumptions of Theorem 3.4(iii) are satisfied. These considerations and (3.5) show that if (3.7) is true, then the condition $\Psi(\bar{h}) < 0$ for some $\bar{h} \in W^{1,2}_{\mathcal{D}}$ is satisfied if and only if the assumptions of Theorem 3.4(i) or the assumptions of Theorem 3.4(iii) are satisfied. □

4 Stability of a twisted rod

In this section we use Theorem 3.4 in order to determine the stability of an unbuckled state of an inextensible, unshearable, isotropic Kirchhoff rod. Under suitable assumptions the strain energy of the rod is given by

$$\Phi(u) = \int_0^1 \left(\frac{A}{2}\big((u_1')^2 + (u_2')^2 \sin^2 u_1\big) + \frac{C}{2}(u_3' + u_2' \cos u_1)^2 + FL^2 \sin u_1 \cos u_2 \right) dx,$$

where u_1, u_2, u_3 are so called Euler angles describing the orientation of the director basis, $A, C > 0$ are constants, L is the rod-length and $F \in \mathbb{R}$ is an external terminal load; the rod is oriented horizontally (along the x axis), see [11, (9)]. The unbuckled state is given by $u^0(x) := (\frac{\pi}{2}, 0, 2\pi Mx)$ where M is a twist parameter. Notice that u^0 is an extremal satisfying the natural boundary conditions $f^0_{p_i}(x) = 0$ for $i = 1,2$ and $x = 0,1$. The stability of u^0 was studied in [11] under the Dirichlet boundary conditions $u_3(x) = u_3^0(x)$ for $x = 0,1$, and one of the following sets of boundary conditions for u_1, u_2:

$$u_1(0) = u_1(1) = \pi/2, \qquad u_2(0) = u_2(1) = 0, \tag{4.1}$$

$$u_1(0) = u_1(1) = \pi/2, \qquad u_2'(0) = u_2'(1) = 0, \tag{4.2}$$

$$u_1'(0) = u_1'(1) = 0, \qquad u_2'(0) = u_2'(1) = 0. \tag{4.3}$$

The results in [11] are essentially optimal in case (4.1), but the results in cases (4.2) and (4.3) are only partial, leaving several open problems. Notice that the Neumann boundary conditions are not the same as the natural boundary conditions in general (see [13] for related issues), but one can easily show (see Proposition 7.2 and Remark 7.3 in the Appendix) that the problem of stability of u^0 considered in [11] in cases (4.2) and (4.3) is equivalent to the question whether u^0 is a weak minimizer of Φ in $u^0 + C^1_{\mathcal{D}}$ with $I_0^{\mathcal{N}} = I_1^{\mathcal{N}} = \{2\}$ and $I_0^{\mathcal{N}} = I_1^{\mathcal{N}} = \{1,2\}$, respectively; hence we can use Theorem 3.4 in order to solve those problems. In fact, we will consider all possible subsets $I_0^{\mathcal{N}}, I_1^{\mathcal{N}}$ of $\{1,2\}$, and in each case we will find the borderline between the stability and instability (i.e. between the situations when u^0 is and is not a weak minimizer, respectively). On the other hand, we will always assume $3 \in I_0^{\mathcal{D}} \cap I_1^{\mathcal{D}}$, i.e. we will always consider the Dirichlet boundary conditions for the third component u_3.

In order to have a more graphic notation, given $I_0^{\mathcal{N}}, I_1^{\mathcal{N}} \subset \{1,2\}$, we denote the corresponding case by $\binom{c_0^1 c_1^1}{c_0^2 c_1^2}$, where $c_j^i = \mathcal{N}$ if $i \in I_j^{\mathcal{N}}$, $c_j^i = \mathcal{D}$ if $i \in I_j^{\mathcal{D}}$, $i = 1,2$, $j = 0,1$. For example, $\binom{\mathcal{D}\mathcal{D}}{\mathcal{N}\mathcal{N}}$ corresponds to the case $I_0^{\mathcal{N}} = I_1^{\mathcal{N}} = \{2\}$, i.e. (4.2), and $\binom{\mathcal{N}\mathcal{N}}{\mathcal{N}\mathcal{N}}$ corresponds to the case $I_0^{\mathcal{N}} = I_1^{\mathcal{N}} = \{1,2\}$, i.e. (4.3). Set also

$$\alpha := \frac{2\pi CM}{A}, \quad \beta := -\frac{FL^2}{A}, \quad \gamma := \sqrt{\left|\beta - \frac{1}{4}\alpha^2\right|}, \quad \delta := \frac{\alpha}{2}, \quad \theta := \frac{2\gamma\delta}{\gamma^2 + \delta^2}. \tag{4.4}$$

We will show that we may assume $\alpha > 0$, and for any $\binom{c_0^1 c_1^1}{c_0^2 c_1^2}$ with $c_j^i \in \{\mathcal{D}, \mathcal{N}\}$ we will find a function $g = g^{c_0^1 c_1^1}_{c_0^2 c_1^2} : (0,\infty) \to \mathbb{R} : \alpha \mapsto \beta$ which describes the borderline between stability and instability. In the particular cases (4.1), (4.2) and (4.3) we will also use the notation

$$g_D := g^{\mathcal{D}\mathcal{D}}_{\mathcal{D}\mathcal{D}}, \quad g_M := g^{\mathcal{D}\mathcal{D}}_{\mathcal{N}\mathcal{N}}, \quad \text{and} \quad g_N := g^{\mathcal{N}\mathcal{N}}_{\mathcal{N}\mathcal{N}},$$

respectively (the notation g_M reflects the fact that case (4.2) is called "Mixed" in [11, (13)]).

Proposition 4.1. *Let u^0 be as above, $\alpha > 0$, and let $I_0^{\mathcal{N}}, I_1^{\mathcal{N}} \subset \{1,2\}$ be fixed. Then there exists a continuous function $g : (0,\infty) \to \mathbb{R}$ having the properties mentioned above, i.e. if $\beta > g(\alpha)$ (or $\beta < g(\alpha)$, resp.), then u^0 is a strict weak minimizer (or is not a weak minimizer, resp.).*

(i) *Let $I_0^{\mathcal{D}} \cap \{1,2\} \neq \emptyset \neq I_1^{\mathcal{D}} \cap \{1,2\}$. Then*

$$g^{\mathcal{D}\mathcal{D}}_{\mathcal{N}\mathcal{D}} = g^{\mathcal{D}\mathcal{D}}_{\mathcal{D}\mathcal{N}} = g^{\mathcal{D}\mathcal{N}}_{\mathcal{D}\mathcal{D}} = g^{\mathcal{N}\mathcal{D}}_{\mathcal{D}\mathcal{D}}, \quad g^{\mathcal{D}\mathcal{N}}_{\mathcal{N}\mathcal{D}} = g^{\mathcal{N}\mathcal{D}}_{\mathcal{D}\mathcal{N}}, \quad g^{\mathcal{D}\mathcal{D}}_{\mathcal{N}\mathcal{N}} = g^{\mathcal{N}\mathcal{N}}_{\mathcal{D}\mathcal{D}} (= g_M), \tag{4.5}$$

$$\left.\begin{aligned} g_D(\alpha) &= \frac{\alpha^2}{4} - \pi^2, \quad g^{\mathcal{D}\mathcal{D}}_{\mathcal{N}\mathcal{D}}(\alpha) = \frac{\alpha^2}{4} - \frac{\pi^2}{4}, \\ g^{\mathcal{D}\mathcal{N}}_{\mathcal{N}\mathcal{D}}(\alpha) &= (k + \tfrac{1}{2})\pi(\alpha - (k + \tfrac{1}{2})\pi) \ \text{ if } \alpha \in [2k\pi, 2(k+1)\pi], \quad k = 0,1,2,\ldots, \\ g_M(\alpha) &= k\pi(\alpha - k\pi) \qquad\qquad \text{ if } \alpha \in [(2k-1)\pi, (2k+1)\pi],\ k = 0,1,2,\ldots. \end{aligned}\right\} \tag{4.6}$$

(ii) *Let either $I_0^{\mathcal{D}} \cap \{1,2\} = \emptyset$ or $I_1^{\mathcal{D}} \cap \{1,2\} = \emptyset$. Then*

$$g^{\mathcal{N}\mathcal{D}}_{\mathcal{N}\mathcal{D}} = g^{\mathcal{D}\mathcal{N}}_{\mathcal{D}\mathcal{N}}, \quad g^{\mathcal{N}\mathcal{N}}_{\mathcal{N}\mathcal{D}} = g^{\mathcal{N}\mathcal{N}}_{\mathcal{D}\mathcal{N}}, \quad g^{\mathcal{N}\mathcal{D}}_{\mathcal{N}\mathcal{N}} = g^{\mathcal{D}\mathcal{N}}_{\mathcal{N}\mathcal{N}}, \tag{4.7}$$

$$g_N(\alpha) = \inf\Big\{\beta \geq \frac{1}{2}\alpha^2 : (1-\theta^2)\cosh(2\gamma) + \theta^2\cos(2\delta) = 1\Big\} \in \Big[\frac{1}{2}\alpha^2, \alpha^2\Big],$$

$$g^{\mathcal{ND}}_{\mathcal{ND}}(\alpha) = \begin{cases} \sup\{\beta \in (\frac{1}{4}\alpha^2, \frac{1}{2}\alpha^2) : (\alpha^2 - 2\beta)\cosh(2\gamma) = 2\beta\} & \text{if } \alpha > 2, \\ \frac{1}{4}\alpha^2 & \text{if } \alpha = 2, \\ \sup\{\beta \in (\frac{1}{4}(\alpha^2-\pi^2), \frac{1}{4}\alpha^2) : (\alpha^2 - 2\beta)\cos(2\gamma) = 2\beta\} & \text{if } \alpha \in (0,2), \end{cases}$$

$$g^{\mathcal{NN}}_{\mathcal{ND}}(\alpha) = \inf\{\beta \geq \beta_\alpha : (\gamma^2 - \delta^2)\sinh(2\gamma) = 2\gamma\delta\sin(2\delta)\}, \quad \beta_\alpha := \begin{cases} \frac{1}{2}\alpha^2 & \text{if } \alpha \leq \pi, \\ g^{\mathcal{ND}}_{\mathcal{ND}}(\alpha) & \text{if } \alpha > \pi, \end{cases}$$

$$g^{\mathcal{ND}}_{\mathcal{NN}}(\alpha) = \begin{cases} \inf\{\beta \geq g^{\mathcal{ND}}_{\mathcal{ND}}(\alpha) : (\gamma^2 - \delta^2)\sinh(2\gamma) = -2\gamma\delta\sin(2\delta)\} & \text{if } \alpha \geq \alpha_0, \\ \inf\{\beta \geq g^{\mathcal{ND}}_{\mathcal{ND}}(\alpha) : \xi_1^2 \sin\xi_2\cos\xi_1 = \xi_2^2\sin\xi_1\cos\xi_2\} & \text{if } \alpha \in (\frac{1}{2}\pi, \alpha_0), \\ 0 & \text{if } \alpha \in (0, \frac{1}{2}\pi], \end{cases}$$

where $\xi_i := -\frac{1}{2}\alpha \pm \gamma$ *and* $\alpha_0 > 0$ *is defined by* $\alpha_0 = 2\sin\alpha_0$.

Remark 4.2. (i) If u^0 is a weak minimizer of Φ with given $I_0^{\mathcal{N}}, I_1^{\mathcal{N}}$ (and the borderline function g), then it remains a weak minimizer if we replace $I_x^{\mathcal{N}}$ with any subset of $I_x^{\mathcal{N}}$ for $x = 0, 1$, since the set $C^1_{\mathcal{D}}$ becomes smaller. Therefore the new borderline function $\tilde{g}$ has to satisfy $\tilde{g} \leq g$. In particular, $g_D \leq g \leq g_N$ for any borderline function g, $g^{\mathcal{ND}}_{\mathcal{ND}} \leq \min(g^{\mathcal{NN}}_{\mathcal{ND}}, g^{\mathcal{ND}}_{\mathcal{NN}})$, and $g^{\mathcal{ND}}_{\mathcal{ND}}(\alpha) \geq g^{\mathcal{DD}}_{\mathcal{ND}}(\alpha) = \frac{1}{4}(\alpha^2 - \pi^2)$. We also have $g_N(\alpha) \leq \alpha^2$ since the Cauchy inequality implies that the corresponding functional Ψ is positive definite for $\beta > \alpha^2$.

(ii) If $\alpha \in (0, \alpha_0)$ is fixed, then the function $\Xi(\beta) := \xi_1^2\sin\xi_2\cos\xi_1 - \xi_2^2\sin\xi_1\cos\xi_2$ appearing in the formula for $g^{\mathcal{ND}}_{\mathcal{NN}}$ in Proposition 4.1 has a unique root β^* in in the interval $[g^{\mathcal{ND}}_{\mathcal{ND}}(\alpha), \frac{1}{4}\alpha^2)$: This follows from our proof, since any root in that interval corresponds to the case when the corresponding functional Ψ is positive semidefinite but not positive definite, and the form of Ψ guarantees that, given α, this can happen only for one β. Consequently,

$$g^{\mathcal{ND}}_{\mathcal{NN}}(\alpha) = \sup\Big\{\beta < \frac{1}{4}\alpha^2 : \xi_1^2\sin\xi_2\cos\xi_1 = \xi_2^2\sin\xi_1\cos\xi_2\Big\} \quad \text{if } \alpha \in (0, \alpha_0).$$

In addition, our proof implies that if $\beta^* > g^{\mathcal{ND}}_{\mathcal{ND}}(\alpha)$, then Ξ changes sign at β^*. Similarly, if $\alpha > \alpha_0$ (or $\alpha > 0$, resp.), then the function $(\gamma^2 - \delta^2)\sinh(2\gamma) + 2\gamma\delta\sin(2\delta)$ (or $(\gamma^2 - \delta^2)\sinh(2\gamma) - 2\gamma\delta\sin(2\delta)$, resp.) has a unique root β^* in the interval $[g^{\mathcal{ND}}_{\mathcal{ND}}(\alpha), \infty)$ (or $[\beta_\alpha, \infty)$, resp.), and it changes sign at β^* if $\beta^* > g^{\mathcal{ND}}_{\mathcal{ND}}(\alpha)$ (or $\beta^* > \beta_\alpha$, resp.). In addition, the estimates in (i) guarantee that that root β^* satisfies $\beta^* \leq g_N(\alpha) \leq \alpha^2$. Analogous statements are true in the case of g_N.

(iii) Our definition of α and β in (4.4) implies that the borderline function g_M was estimated above and below in [11, Proposition 6] by functions

$$\overline{g_M}(\alpha) := \max(0, \alpha^2 - \pi^2) \quad \text{and} \quad \underline{g_M}(\alpha) := \pi^2(\alpha^2 - \pi^2)/(\alpha^2 + \pi^2),$$

respectively, see Figure 4.1. Let us also mention that the upper bound $\overline{g_N}(\alpha) := \frac{1}{4}\alpha^2$ for $g_N(\alpha)$ in [11, Proposition 5] is incorrect: The error is explained below.

(iv) The function $\hat{g}(\alpha) := \frac{1}{2}\alpha^2$ is a good approximation of functions g in Proposition 4.1(ii) for α large, see Table 4.1 and Figure 4.2. The functions $g^{\mathcal{ND}}_{\mathcal{NN}}, g^{\mathcal{NN}}_{\mathcal{ND}}$ oscillate between g_N and $g^{\mathcal{ND}}_{\mathcal{ND}}$, they intersect each other whenever $\alpha = k\pi$, $k = 1, 2, \ldots$, and then their common values equal $\hat{g}(\alpha)$ (and also $g_N(\alpha)$ if k is even). Similarly, $\min(g^{\mathcal{ND}}_{\mathcal{NN}}(\alpha), g^{\mathcal{NN}}_{\mathcal{ND}}(\alpha)) = g^{\mathcal{ND}}_{\mathcal{ND}}(\alpha)$ if $\alpha = (k + \frac{1}{2})\pi$, $k = 0, 1, 2, \ldots$. Similar behavior of functions $\tilde{g}(\alpha) = \frac{1}{4}\alpha^2$ and g_M, $g^{\mathcal{DN}}_{\mathcal{ND}}$, $g^{\mathcal{DD}}_{\mathcal{ND}}$ can be observed in Figure 4.1. The formulas for functions g in Proposition 4.1(ii) can

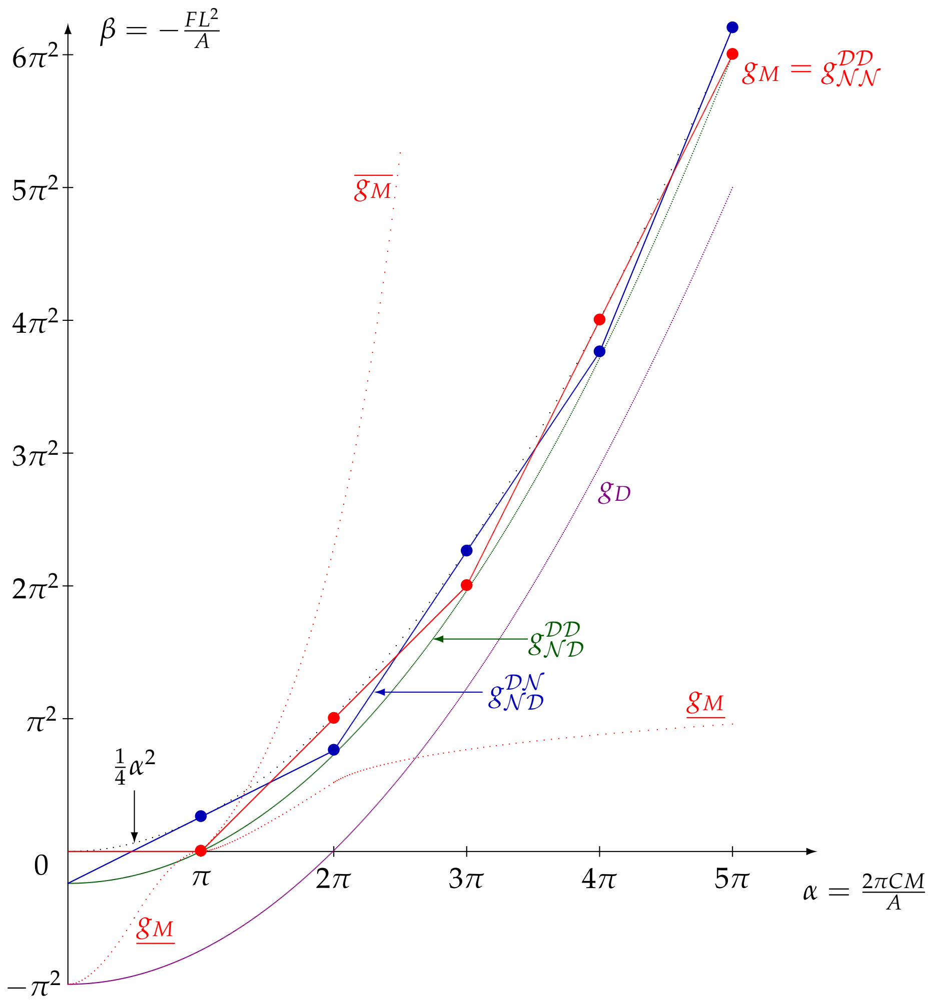

Figure 4.1: The case $I_0^{\mathcal{D}} \cap \{1,2\} \neq \emptyset \neq I_1^{\mathcal{D}} \cap \{1,2\}$.

be used in the numerical computations of g, but they also can be used in the study of the asymptotic or qualitative behavior of g. For example, they imply that $\lim_{\alpha \to 0+} \frac{g_N(\alpha)}{\alpha^2} = 1$, $\lim_{\alpha\to\infty}(\hat{g} - g^{\mathcal{ND}}_{\mathcal{ND}})(\alpha) = 0$, $g^{\mathcal{ND}}_{\mathcal{ND}}$ is $C^1 \setminus C^2$ at $\alpha = 2$, and g_N is $C \setminus C^1$ at $\alpha = 2k\pi$, $k = 1, 2 \ldots$.

(v) Numerical computations determining the borderlines for stability could be used also if we did not know the formulas for functions g in Proposition 4.1. If $\beta_0 < \beta_1$ and the problem with parameters (α_0, β_0) or (α_0, β_1) is unstable or stable, respectively, then one can set $\beta_2 := (\beta_0 + \beta_1)/2$ and numerically solve the Jacobi equations with suitable initial conditions and parameters (α_0, β_2) (by the Euler method, for example). If that problem is stable or unstable, then one can set $\beta_3 := (\beta_0 + \beta_2)/2$ or $\beta_3 := (\beta_2 + \beta_1)/2$, respectively, and solve the problem with parameters (α_0, β_3) etc. In fact, we used such general approach to compute the numerical values of functions g_N and $g^{\mathcal{ND}}_{\mathcal{ND}}$ first, and we verified a posteriori that the computed critical parameters correspond to the critical values determined by Proposition 4.1.

(vi) Let u^0 be a weak minimizer. Then a straightfoward modification of the proof of Proposition 2.3 shows that u^0 is also a strong minimizer. In fact, assume first that there exist $v^k \in W^{1,2}_{\mathcal{D}}$ such that $r_k := \|v^k\|_{1,2} \to 0$ and $\Phi(u^0 + v^k) < \Phi(u^0)$. Since $\Phi \in C^1(W^{1,2})$ is weakly sequentially lower semicontinuous, we can find a minimizer u^k of Φ in $\{u \in u^0 + W^{1,2}_{\mathcal{D}} : \|u - u^0\|_{1,2} \leq r_k\}$ and Lagrange multipliers $\lambda_k \leq 0$ such that $\Phi'(u^k)h = \lambda_k \Theta'(u^k)h$ for any

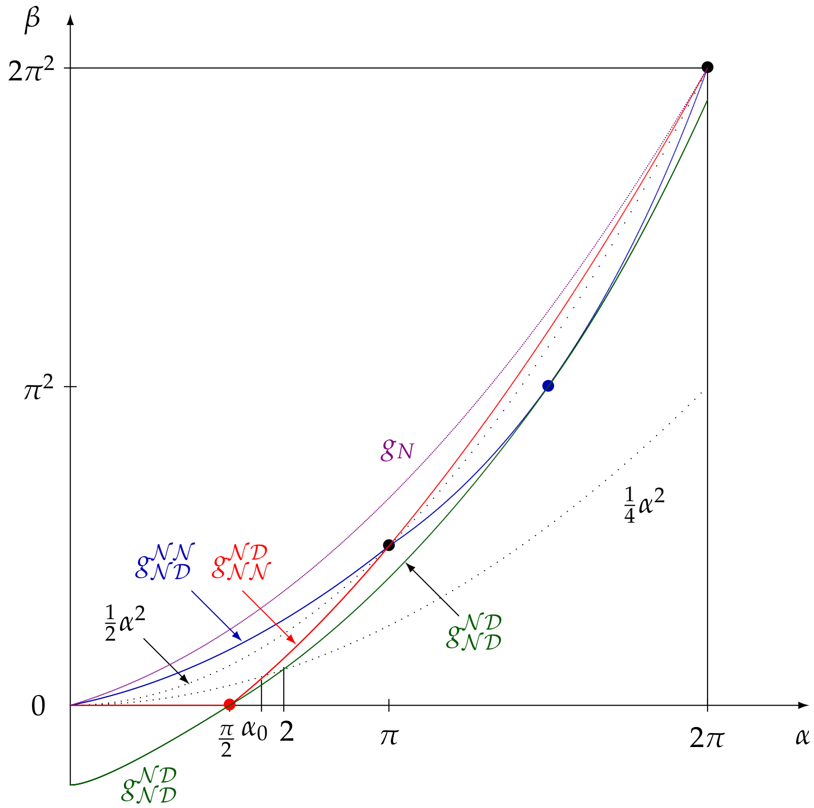

Figure 4.2: The case $I_0^{\mathcal{D}} \cap \{1,2\} = \emptyset$.

$h \in W^{1,2}_{\mathcal{D}}$, where $\Theta(u) = \|u - u^0\|^2_{1,2}$. The arguments in [5, Section 2.6] guarantee that $u^k \in C^2$ and u^k satisfy the Euler equations $(F_p^k(x))' = F_u^k(x)$, where $F_p^k(x) := F_p(\lambda_k, x, u^k(x), (u^k)'(x))$ (similarly F_u^k) and $F(\lambda, x, u, p) := f(x, u, p) - \lambda(|p - (u^0)'(x)|^2 + |u - u^0(x)|^2)$. These equations, the particular form of f, u^0, the positive definiteness of F^k_{pp} and the convergence $u^k \to u^0$ in $W^{1,2}$ guarantee that $\{u^k\}$ is a Cauchy sequence in $W^{2,1}$, hence in C^1, thus $u^k \to u^0$ in C^1. However, this contradicts our assumption that u^0 is a weak minimizer. Consequently, u^0 is a local minimizer in $u^0 + W^{1,2}_{\mathcal{D}}$. Next assume that there exist $v^k \in C^1_{\mathcal{D}}$ such that $\|v^k\|_C \to 0$ and $\Phi(u^0 + v^k) < \Phi(u^0)$. Then it is not difficult to show that there exists $c > 0$ such that $0 > \Phi(u^0 + v^k) - \Phi(u^0) \geq c\|v^k\|^2_{1,2} + o(1)$, hence $\|v^k\|_{1,2} \to 0$, which yields a contradiction and concludes the proof. □

Proof of Proposition 4.1. Notice that u^0 is a critical point of Φ for any choice of $I_0^{\mathcal{N}}, I_1^{\mathcal{N}} \subset \{1,2\}$. By Proposition 3.1, we have to determine the positivity of functional Ψ in $W^{1,2}_{\mathcal{D}}$. We have $\Psi(h) = \Psi_1(h_1, h_2) + \Psi_2(h_3)$, where

$$\Psi_1(h_1, h_2) = A \int_0^1 \big((h_1')^2 + (h_2')^2 - 2\alpha h_2' h_1 + \beta(h_1^2 + h_2^2)\big)\, dx, \quad \Psi_2(h_3) = C \int_0^1 (h_3')^2\, dx.$$

Since the positivity of Ψ does not change if we replace α by $-\alpha$ (consider $-h_1$ instead of h_1), we may assume $\alpha \geq 0$. Since the case $\alpha = 0$ is trivial, we assume $\alpha > 0$. Since Ψ_2 is positive definite in $W_0^{1,2}([0,1])$, it is sufficient to study the positivity of the functional

$$\tilde{\Psi}(h_1, h_2) := \frac{1}{2A}\Psi_1(h_1, h_2) = \frac{1}{2}\int_0^1 \big((h_1')^2 + (h_2')^2 - 2\alpha h_2' h_1 + \beta(h_1^2 + h_2^2)\big)\, dx \tag{4.8}$$

α/π	$g_N(\alpha)/\pi^2$	$g_{\mathcal{N}\mathcal{D}}^{\mathcal{N}\mathcal{N}}(\alpha)/\pi^2$	$\hat{g}(\alpha)/\pi^2$	$g_{\mathcal{N}\mathcal{N}}^{\mathcal{N}\mathcal{D}}(\alpha)/\pi^2$	$g_{\mathcal{N}\mathcal{D}}^{\mathcal{N}\mathcal{D}}(\alpha)/\pi^2$	$\Delta_{\max}(\alpha)/\pi^2$
0	0	0	0	0	-0.25	0.25
0.3	0.0842	0.0732	0.045	0.0000	-0.1222	0.2064
0.5	0.2137	0.1679	0.125	0.0000	0.0000	0.2137
0.7	0.3792	0.2820	0.245	0.1826	0.1533	0.2258
1.0	0.6717	0.5000	0.500	0.5000	0.4446	0.2271
1.3	1.0067	0.8197	0.845	0.8663	0.8129	0.1938
1.5	1.2549	1.1032	1.125	1.1440	1.1032	0.1516
1.7	1.5279	1.4334	1.445	1.4558	1.4305	0.0973
2.0	2.0000	2.0000	2.000	2.0000	1.9923	0.0076
2.5	3.2058	3.1274	3.125	3.1225	3.1225	0.0832
3.0	4.5759	4.5000	4.500	4.5000	4.4992	0.0767
3.5	6.1596	6.1248	6.125	6.1252	6.1248	0.0348
4.0	8.0000	8.0000	8.000	8.0000	7.9999	0.0001

Table 4.1: Numerical values of functions g and $\Delta_{\max} := g_N - g_{\mathcal{N}\mathcal{D}}^{\mathcal{N}\mathcal{D}}$ if $I_0^{\mathcal{D}} \cap \{1,2\} = \emptyset$.

in the space

$$\tilde{W}_{\mathcal{D}} := \{h \in W^{1,2}([0,1],\mathbb{R}^2) : h_i(j) = 0 \text{ for } i \in I_j^{\mathcal{D}},\ i = 1,2,\ j = 0,1\}. \tag{4.9}$$

In fact, Ψ is positive definite (or semidefinite, resp.) in $W_{\mathcal{D}}^{1,2}$ if and only if $\tilde{\Psi}$ is positive definite (or semidefinite, resp.) in $\tilde{W}_{\mathcal{D}}$. Therefore, in what follows, we will apply the Jacobi theory from Section 3 to the functional $\tilde{\Psi}$ with $\alpha > 0$. Notice that the assumptions in Theorem 3.4 depend only on the corresponding functional Ψ, and the conclusions can also be formulated in terms of Ψ, see (3.5), (3.6). We will use Theorem 3.4 in this way. More precisely, we will use assertions (3.5), (3.6) (with Ψ and $W_{\mathcal{D}}^{1,2}$ replaced by $\tilde{\Psi}$ and $\tilde{W}_{\mathcal{D}}$, respectively) to determine the positivity of $\tilde{\Psi}$ (hence the positivity of Ψ) and then we will use Proposition 3.1 (with $\Psi(h) = \Psi(h_1, h_2, h_3)$) to conclude that u^0 is (or is not) a minimizer of Φ.

Notice that the index sets for functional $\tilde{\Psi}$ satisfy $\tilde{I}_j^{\mathcal{D}} = I_j^{\mathcal{D}} \cap \{1,2\}$ and $\tilde{I}_j^{\mathcal{N}} = I_j^{\mathcal{N}} \cap \{1,2\} = I_j^{\mathcal{N}}$ for $j = 1,2$, hence we will use the notation $I_j^{\mathcal{N}}$ instead of $\tilde{I}_j^{\mathcal{N}}$. Similarly, the corresponding operators $\tilde{\mathcal{B}}_i$, $i = 1,2$ (cf. (2.8)), satisfy $\tilde{\mathcal{B}}_i(h_1,h_2) = \mathcal{B}_i(h_1,h_2,0)$ for $i = 1,2$, and – without fearing confusion – we will use the notation $\mathcal{B}_i h$ instead of $\tilde{\mathcal{B}}_i h$ and $\mathcal{B}h := (\mathcal{B}_1 h, \mathcal{B}_2 h)$ if $h = (h_1, h_2)$ and $i = 1,2$. The same applies to operators $\mathcal{C}_i$ and $\mathcal{A}_i$. Since

$$\mathcal{B}_1 h = h_1', \quad \mathcal{B}_2 h = -\alpha h_1 + h_2', \quad \mathcal{C}_1 h = \beta h_1 - \alpha h_2', \quad \mathcal{C}_2 h = \beta h_2, \tag{4.10}$$

the corresponding system of Jacobi equations is

$$\left.\begin{aligned} h_1'' + \alpha h_2' - \beta h_1 = 0, \\ h_2'' - \alpha h_1' - \beta h_2 = 0, \end{aligned}\right\} \quad \text{in } (0,1), \tag{4.11}$$

and the initial conditions for $h^{(1)}, h^{(2)}$ in Theorem 3.4 (with $N = 2$) are $h_i(0) = 0$ if $i \in \tilde{I}_0^{\mathcal{D}}$ and $i = 1,2$, $h_1'(0) = 0$ if $1 \in I_0^{\mathcal{N}}$, and $h_2'(0) = \alpha h_1(0)$ if $2 \in I_0^{\mathcal{N}}$.

The existence of continuous borderline functions g follows from the form of $\tilde\Psi$. Notice that if the index sets $\tilde I_0^{\mathcal D}$ and $\tilde I_1^{\mathcal D}$ are nonempty, then $h_1h_2(0) = h_1h_2(1) = 0$ for any $h \in \tilde W_{\mathcal D}$, hence

$$\int_0^1 h_2'h_1\,dx = -\int_0^1 h_1'h_2\,dx. \tag{4.12}$$

Identity (4.12) shows that the value of $\tilde\Psi$ does not change if we replace h_1 with h_2 and α with $-\alpha$. In general, the value of $\tilde\Psi$ does not change if we replace h_i with $\tilde h_i(x) = h_i(1-x)$ and α with $-\alpha$. These two observations guarantee (4.5) and (4.7).

Let us first consider the cases in Proposition 4.1(i), i.e. $\tilde I_0^{\mathcal D} \neq \emptyset \neq \tilde I_1^{\mathcal D}$. Then (4.12) guarantees $\int_0^1 2h_2'h_1\,dx = \int_0^1 (h_2'h_1 - h_1'h_2)\,dx$ and the Cauchy inequality implies that

$$\tilde\Psi \text{ is positive definite if } \alpha^2 < 4\beta. \tag{4.13}$$

Hence it is sufficient to study the case $\alpha^2 \geq 4\beta$.

Case $\binom{\mathcal{DD}}{\mathcal{DD}}$ has already been solved in [11, Proposition 3], but Theorem 3.4 enables us to to show $g_D(\alpha) = \frac{\alpha^2}{4} - \pi^2$ in a simpler way. Assume $\alpha^2 > 4\beta$. We can set $h^{(1)}(x) = (\sin\xi_1 x - \sin\xi_2 x, \cos\xi_1 x - \cos\xi_2 x)$ and $h^{(2)}(x) = (-\cos\xi_1 x + \cos\xi_2 x, \sin\xi_1 x - \sin\xi_2 x)$, where $\xi_{1,2} = -\frac12\alpha \pm \gamma$. The function D in Theorem 3.4 satisfies $D(x) = 2 - 2\cos(\xi_1 - \xi_2)x$, hence $D \neq 0$ in $(0,1]$ if and only if $|\xi_1 - \xi_2| < 2\pi$, i.e. if $\beta > g_D(\alpha)$. Consequently, if $\beta > g_D(\alpha)$, then u^0 is a strict weak minimizer (this remains true also if $4\beta = \alpha^2$ due to the monotonicity of $\tilde\Psi$ with respect to β), and if $\beta < g_D(\alpha)$, then u^0 is not a weak minimizer.

The remaining cases in Proposition 4.1(i) are $\binom{\mathcal{DD}}{\mathcal{ND}}$, $\binom{\mathcal{DN}}{\mathcal{ND}}$, and $\binom{\mathcal{DD}}{\mathcal{NN}}$. Assume $\alpha^2 > 4\beta$. Since $I_0^{\mathcal N} = \{2\}$, the initial conditions for $h^{(1)}, h^{(2)}$ in Theorem 3.4 are $h_1(0) = 0$ and $h_2'(0) = 0$. One can easily check that we can set $h^{(i)}(x) := (\sin\xi_i x, \cos\xi_i x)$, $i = 1,2$, where $\xi_{1,2} := -\frac12\alpha \pm \gamma$. The function D in Theorem 3.4 satisfies

$$D(x) = \sin(\xi_1 - \xi_2)x = \sin 2\gamma x = \sin\sqrt{\alpha^2 - 4\beta}\,x,$$

hence

$$\text{if } \alpha^2 - 4\beta > \pi^2, \text{ then } D(x) = 0 \text{ for some } x \in (0,1), \tag{4.14}$$

$$\text{if } 0 < \alpha^2 - 4\beta < \pi^2, \text{ then } D(x) \neq 0 \text{ in } (0,1]. \tag{4.15}$$

Theorem 3.4(i) (more precisely, assertion (3.5)) and (4.14) imply that

$$\tilde\Psi \text{ is not positive semidefinite if } \alpha^2 - 4\beta > \pi^2. \tag{4.16}$$

Let $I_1^{\mathcal N} = \emptyset$. If $0 < \alpha^2 - 4\beta < \pi^2$, then (4.15) and Theorem 3.4(ii) (more precisely, assertion (3.6)) guarantee that $\tilde\Psi$ is positive definite. If $0 = \alpha^2 - 4\beta < \pi^2$ and we replace β by $\tilde\beta := \beta - \varepsilon$ with $\varepsilon > 0$ small, then $0 < \alpha^2 - 4\tilde\beta < \pi^2$, hence the modified functional $\tilde\Psi^{\tilde\beta}$ (with β replaced by $\tilde\beta$) is positive definite, and the monotonicity of $\tilde\Psi$ with respect to β implies that $\tilde\Psi$ is positive definite as well. These facts together with (4.13) and (4.16) imply $g^{\mathcal{DD}}_{\mathcal{ND}}(\alpha) = \frac{\alpha^2}{4} - \frac{\pi^2}{4}$.

If $I_1^{\mathcal N} = \{2\}$ and $\alpha^2 > 4\beta$, then $H_{\mathcal D,b} = \{\tilde h \in \operatorname{span}(h^{(1)}, h^{(2)}) : \tilde h_1(1) = 0\}$ is spanned by $h := \sin\xi_2 h^{(1)} - \sin\xi_1 h^{(2)}$. We have

$$B := \mathcal{B}h(1)\cdot h(1) = h_2'(1)h_2(1) = (\xi_2 - \xi_1)\sin(\xi_2 - \xi_1)\sin\xi_1\sin\xi_2$$

and, assuming $\alpha \in [(2k-1)\pi, (2k+1)\pi]$, $k = 0,1,2,\dots$, $\alpha > 0$, we have $B > 0$ or $B < 0$ if and only if β is greater or less than $k\pi(\alpha - k\pi)$, respectively. Notice that

$$\alpha^2/4 \geq k\pi(\alpha - k\pi) \geq (\alpha^2 - \pi^2)/4. \tag{4.17}$$

These facts, Theorem 3.4(ii) and (4.13) imply that $\tilde{\Psi}$ is positive definite if $\beta > k\pi(\alpha - k\pi)$, $\beta \neq \alpha^2/4$. The assumption $\beta \neq \alpha^2/4$ can be removed by the same argument as above (by considering $\tilde{\beta} = \beta - \varepsilon$). If $\beta < k\pi(\alpha - k\pi)$, then $\alpha^2 > 4\beta$ due to (4.17), hence $B < 0$ and Theorem 3.4(i) imply that $\tilde{\Psi}$ is not positive semidefinite. Consequently, the formula for $g_M = g^{\mathcal{DD}}_{\mathcal{NN}}$ in (4.6) is true.

If $I_1^{\mathcal{N}} = \{1\}$, then we can use the same arguments as in the case $I_1^{\mathcal{N}} = \{2\}$ to show that the formula for $g^{\mathcal{DN}}_{\mathcal{ND}}$ in (4.6) is true. In particular, if $\alpha^2 > 4\beta$, then $H_{\mathcal{D},b} = \{\tilde{h} \in \text{span}(h^{(1)}, h^{(2)}) : \tilde{h}_2(1) = 0\}$ is spanned by $h := \cos\xi_2 h^{(1)} - \cos\xi_1 h^{(2)}$ and we have

$$B := \mathcal{B}h(1) \cdot h(1) = h_1'(1)h_1(1) = (\xi_1 - \xi_2)\sin(\xi_1 - \xi_2)\cos\xi_1\cos\xi_2,$$

hence assuming $\alpha \in [2k\pi, 2(k+1)\pi]$, $k = 0, 1, 2, \ldots$, we obtain $B > 0$ or $B < 0$ if and only if β is greater or less than $(k + \frac{1}{2})\pi(\alpha - (k + \frac{1}{2})\pi)$, respectively.

Next consider the cases in Proposition 4.1(ii), i.e. $\binom{\mathcal{ND}}{\mathcal{ND}}$, $\binom{\mathcal{NN}}{\mathcal{ND}}$, $\binom{\mathcal{ND}}{\mathcal{NN}}$ and $\binom{\mathcal{NN}}{\mathcal{NN}}$. Since $I_0^{\mathcal{N}} = \{1, 2\}$, the initial conditions for $h^{(1)}, h^{(2)}$ in Theorem 3.4 are $h_1'(0) = 0$ and $h_2'(0) = \alpha h_1(0)$. We will distinguish the following four subcases:

(ii-1) $\beta = \frac{1}{2}\alpha^2$,

(ii-2) $\beta = \frac{1}{4}\alpha^2$,

(ii-3) $\beta > \frac{1}{4}\alpha^2$ and $\beta \neq \frac{1}{2}\alpha^2$,

(ii-4) $\beta < \frac{1}{4}\alpha^2$.

(ii-1) Assume that $\beta = \frac{1}{2}\alpha^2$. We will show that $\tilde{\Psi}$ is positive definite (hence u^0 is a strict weak minimizer) in case $\binom{\mathcal{ND}}{\mathcal{ND}}$ and $\tilde{\Psi}$ is not positive semidefinite (hence u^0 is not a weak minimizer) in case $\binom{\mathcal{NN}}{\mathcal{NN}}$ if $\alpha \neq 2k\pi$. In addition, in case $\binom{\mathcal{NN}}{\mathcal{ND}}$, u^0 is or is not a weak minimizer if $\alpha \in ((2k-1)\pi, 2k\pi)$ or $\alpha \in (2k\pi, (2k+1)\pi)$, respectively, and the opposite is true in case $\binom{\mathcal{ND}}{\mathcal{NN}}$.

Recall that $\delta = \alpha/2$. If we set

$$\begin{aligned} h^{(1)}(x) &:= (e^{\delta x}(\cos(\delta x) - \sin(\delta x)), e^{\delta x}(\cos(\delta x) + \sin(\delta x))), \\ h^{(2)}(x) &:= (e^{-\delta x}(\cos(\delta x) + \sin(\delta x)), e^{-\delta x}(-\cos(\delta x) + \sin(\delta x))), \end{aligned}$$

then we obtain $D \equiv -2$, hence $\tilde{\Psi}$ is positive definite in case $\binom{\mathcal{ND}}{\mathcal{ND}}$ due to Theorem 3.4(ii).

Considering case $\binom{\mathcal{NN}}{\mathcal{NN}}$, one can check that the matrix $\mathfrak{A} = (a_{kl})$ in Remark 3.6(ii) satisfies

$$a_{11} = 4\delta e^{2\delta}\sin^2\delta, \quad a_{22} = -4\delta e^{-2\delta}\sin^2\delta, \quad a_{12} = a_{21} = -4\delta\sin\delta\cos\delta.$$

If $\delta \neq k\pi$, then choosing $\xi := (0, 1)$ and $h := \sum_{k=1}^2 \xi_k h^{(k)} = h^{(2)} \in H_{\mathcal{D},1} = H$ we obtain $\mathcal{B}h(1) \cdot h(1) = \mathfrak{A}\xi \cdot \xi = a_{22} < 0$, i.e. $\tilde{\Psi}$ is not positive semidefinite due to Theorem 3.4(i). Notice also that $\mathcal{B}h(0) = 0$, hence

$$\tilde{\Psi}(h) = \mathcal{B}h \cdot h\Big|_0^1 < 0. \tag{4.18}$$

If $\delta = k\pi$, then $\mathfrak{A} = 0$ (degenerate case). Already these facts contradict [11, Proposition 5] which claims the stability for $\beta > \frac{1}{4}\alpha^2$. In fact, the authors of [11] mention in their proof that

"We have not used any integration by parts . . . ", but they seem to use [11, (35)–(37)], and [11, (35)] does use an integration by parts requiring the boundary conditions $h_1h_2(0) = h_1h_2(1)$.

In case $\binom{\mathcal{ND}}{\mathcal{NN}}$ we set

$$h := e^{-\delta}(\cos\delta + \sin\delta)h^{(1)} - e^{\delta}(\cos\delta - \sin\delta)h^{(2)}.$$

Since at least one of the numbers $h_1^{(1)}(1)$ and $h_1^{(2)}(1)$ is non-zero, we have $\dim H_{\mathcal{D},1} \le 1$. Since $h_1(1) = 0$, we obtain $H_{\mathcal{D},1} = \mathrm{span}(h)$, and

$$\mathcal{B}h(1)\cdot h(1) = \mathcal{B}_2 h(1)\cdot h_2(1) = (-\alpha h_1 + h_2')(1)\cdot h_2(1) = 2\alpha\sin\alpha$$

due to $h_2(1) = 2$ and $h_2'(1) = \alpha\sin\alpha$. Consequently, $\mathcal{B}h(1)\cdot h(1) > 0$ if $\alpha \in (2k\pi, (2k+1)\pi)$ and $\mathcal{B}h(1)\cdot h(1) < 0$ if $\alpha \in ((2k-1)\pi, 2k\pi)$, so that our assertion follows from Theorem 3.4(ii) and Theorem 3.4(i), respectively.

Similarly, in case $\binom{\mathcal{NN}}{\mathcal{ND}}$ we set

$$h := e^{-\delta}(\cos\delta - \sin\delta)h^{(1)} + e^{\delta}(\cos\delta + \sin\delta)h^{(2)}.$$

Then $h_2(1) = 0$ and $H_{\mathcal{D},1} = \mathrm{span}(h)$;

$$\mathcal{B}h(1)\cdot h(1) = \mathcal{B}_1 h(1)\cdot h(1) = h_1'(1)h_1(1) = -2\alpha\sin\alpha \tag{4.19}$$

due to $h_1(1) = 2$ and $h_1'(1) = -\alpha\sin\alpha$. The rest of the proof is the same as in case $\binom{\mathcal{ND}}{\mathcal{NN}}$. Notice also that (similarly as in the case of (4.18)), (4.19) implies

$$\tilde{\Psi}(h) = \mathcal{B}h\cdot h\Big|_0^1 < 0 \tag{4.20}$$

provided $\alpha \in (2k\pi, (2k+1)\pi)$.

(ii-2) Assume that $\beta = \frac{1}{4}\alpha^2$. Set $\xi := -\frac{1}{2}\alpha$ and

$$\begin{aligned} h^{(1)}(x) &:= (\sin(\xi x) - \xi x\cos(\xi x), \cos(\xi x) + \xi x\sin(\xi x)),\\ h^{(2)}(x) &:= (\cos(\xi x) - \xi x\sin(\xi x), -\sin(\xi x) - \xi x\cos(\xi x)). \end{aligned}$$

Notice that the function D in Theorem 3.4 satisfies $D(x) = \xi^2x^2 - 1$, hence $D < 0$ in $[0,1]$ if $\alpha < 2$, and $D(x) = 0$ for some $x \in (0,1)$ if $\alpha > 2$. This shows that $\frac{1}{4}\alpha^2 < g_{\mathcal{ND}}^{\mathcal{ND}}(\alpha) \le \min\big(g_{\mathcal{ND}}^{\mathcal{NN}}(\alpha), g_{\mathcal{NN}}^{\mathcal{ND}}(\alpha), g_N(\alpha)\big)$ if $\alpha > 2$, i.e. u^0 cannot be a weak minimizer in any case.

Let $\alpha < 2$. Then u^0 is a strict weak minimizer in case $\binom{\mathcal{ND}}{\mathcal{ND}}$. Next consider case $\binom{\mathcal{NN}}{\mathcal{NN}}$. If $\beta = \alpha^2/2$, then (4.18) implies that $\tilde{\Psi}$ is not positive semidefinite. The monotonicity of $\tilde{\Psi}$ with respect to β shows that $\tilde{\Psi}$ cannot be positive semidefinite if $\beta = \alpha^2/4$ either, hence u_0 is not a weak minimizer. The same arguments show that u_0 is not a weak minimizer in case $\binom{\mathcal{NN}}{\mathcal{ND}}$, see (4.20). It remains to consider case $\binom{\mathcal{ND}}{\mathcal{NN}}$. Set

$$h := (\cos\xi - \xi\sin\xi)h^{(1)} - (\sin\xi - \xi\cos\xi)h^{(2)},$$

so that $h_1(1) = 0$. Then the restriction $\alpha < 2$ implies $h_2(1) = 1 - \xi^2 > 0$. Since $h_2'(1) = -\xi^2 + \xi\sin(2\xi)$, we see that $h_2'(1)h_2(1) > 0$ only if $\alpha < \alpha_0$, where α_0 is defined by $\alpha_0 = 2\sin\alpha_0$ ($\alpha_0 \approx 0.6\pi$).

(ii-3) Assume $\beta > \frac{1}{4}\alpha^2$, $\beta \neq \frac{1}{2}\alpha^2$, and set

$$\varphi(x) := e^{\gamma x}(\gamma^2 - \delta^2), \quad \psi_\pm(x) := e^{-\gamma x}(\gamma \pm \delta)^2.$$

Then we can take

$$\begin{aligned} h^{(1)}(x) &:= [(\varphi(x) + \psi_+(x))(\cos(\delta x) + \sin(\delta x)), (\varphi(x) + \psi_+(x))(-\cos(\delta x) + \sin(\delta x))], \\ h^{(2)}(x) &:= [(\varphi(x) + \psi_-(x))(\cos(\delta x) - \sin(\delta x)), (\varphi(x) + \psi_-(x))(\cos(\delta x) + \sin(\delta x))], \end{aligned}$$

and an easy computation yields

$$D(x) = 4(\gamma^2 - \delta^2)\Big((\gamma^2 - \delta^2)\cosh(2\gamma x) + \gamma^2 + \delta^2\Big). \tag{4.21}$$

The function D does not vanish in $(0, 1]$ if and only if $\gamma > \delta$ (i.e. $\beta > \frac{1}{2}\alpha^2$), or $\gamma < \delta$ and $\cosh(2\gamma) < \frac{\gamma^2+\delta^2}{\delta^2-\gamma^2}$. The last inequality can be written in the form

$$(\alpha^2 - 2\beta)\cosh(2\gamma) < 2\beta. \tag{4.22}$$

In case $\binom{\mathcal{NN}}{\mathcal{NN}}$, one has to consider the numbers a_{kl} in Remark 3.6(ii):

$$\begin{aligned} a_{11} &= 2\gamma(\varphi^2 - \psi_+^2)(1) + 2\delta(\varphi + \psi_+)^2(1)\cos(2\delta), \\ a_{22} &= 2\gamma(\varphi^2 - \psi_-^2)(1) - 2\delta(\varphi + \psi_-)^2(1)\cos(2\delta), \\ a_{12} = a_{21} &= -2\delta(\varphi + \psi_+)(\varphi + \psi_-)(1)\sin(2\delta). \end{aligned}$$

If $\gamma > \delta$ (i.e. $\beta > \frac{1}{2}\alpha^2$), then

$$a_{11}(\gamma + \delta)^{-2} + a_{22}(\gamma - \delta)^{-2} = 8(\gamma^2 + \delta^2)(\gamma - \theta\delta\cos(2\delta))\sinh(2\gamma) > 0,$$

hence the matrix $\mathfrak{A}$ is positive definite if and only if $a_{11}a_{22} > a_{12}^2$, which is equivalent to

$$(1 - \theta^2)\cosh(2\gamma) + \theta^2\cos(2\delta) > 1. \tag{4.23}$$

We used the assumption $\beta > \frac{1}{2}\alpha^2$ in order to derive (4.23), but this is not restrictive, since we know that u^0 can only be a weak minimizer of our problem in case $\binom{\mathcal{NN}}{\mathcal{NN}}$ when $\beta > \frac{1}{2}\alpha^2$. Hence in this case the condition (4.23) determines the domain of stability.

In cases $\binom{\mathcal{NN}}{\mathcal{ND}}$ and $\binom{\mathcal{ND}}{\mathcal{NN}}$, we set

$$h := (\varphi(1) + \psi_-(1))(\cos\delta + \sin\delta)h^{(1)} + (\varphi(1) + \psi_+(1))(\cos\delta - \sin\delta)h^{(2)}$$

and

$$h := (\varphi(1) + \psi_-(1))(\cos\delta - \sin\delta)h^{(1)} - (\varphi(1) + \psi_+(1))(\cos\delta + \sin\delta)h^{(2)},$$

respectively. Then $h_2(1) = 0$, $h_1(1) = D(1)$,

$$\mathcal{B}h(1) \cdot h(1) = h_1'h_1(1) = 4\gamma(\gamma^2 - \delta^2)D(1)\big((\gamma^2 - \delta^2)\sinh(2\gamma) - 2\gamma\delta\sin(2\delta)\big),$$

and $h_1(1) = 0$, $h_2(1) = -D(1)$,

$$\mathcal{B}h(1) \cdot h(1) = h_2'h_2(1) = 4\gamma(\gamma^2 - \delta^2)D(1)\big((\gamma^2 - \delta^2)\sinh(2\gamma) + 2\gamma\delta\sin(2\delta)\big),$$

respectively, where D is as in (4.21). Consequently, assuming that D does not vanish in $[0,1]$ (i.e. (4.22) is true), the stability conditions are

$$(\gamma^2-\delta^2)\sinh(2\gamma)-2\gamma\delta\sin(2\delta)>0 \tag{4.24}$$

and

$$(\gamma^2-\delta^2)\sinh(2\gamma)+2\gamma\delta\sin(2\delta)>0, \tag{4.25}$$

respectively. Notice that if $\beta=\frac{1}{2}\alpha^2$ (hence $\gamma=\delta$), then (4.24) and (4.25) are equivalent to the corresponding stability conditions in case (ii-1).

(ii-4) If $\beta<\frac{1}{4}\alpha^2$, then we can set

$$\begin{aligned} h^{(1)}(x) &:= (\xi_2\sin(\xi_1x)-\xi_1\sin(\xi_2x),\xi_2\cos(\xi_1x)-\xi_1\cos(\xi_2x)),\\ h^{(2)}(x) &:= (\xi_1\cos(\xi_1x)-\xi_2\cos(\xi_2x),-\xi_1\sin(\xi_1x)+\xi_2\sin(\xi_2x)), \end{aligned}$$

where $\xi_{1,2}=-\frac{1}{2}\alpha\pm\gamma$, and we obtain

$$D(x)=-2\beta+(\alpha^2-2\beta)\cos(2\gamma x). \tag{4.26}$$

If $\alpha^2-4\beta\geq\pi^2$, then D changes sign in $[0,1]$. Hence the condition $D>0$ in $[0,1]$ is equivalent to

$$\alpha^2-4\beta<\pi^2 \quad\text{and}\quad (\alpha^2-2\beta)\cos(2\gamma)>2\beta. \tag{4.27}$$

It is not difficult to check (cf. case (ii-2)) that if $\alpha<2$ or $\alpha>2$, then (4.27) or (4.22), respectively, is the (essentially optimal) sufficient condition for the stability in our problem in case $\binom{\mathcal{ND}}{\mathcal{ND}}$. If $\alpha=2$, then that sufficient condition is $\beta>1$.

Case (ii-2) shows that it remains to consider only case $\binom{\mathcal{ND}}{\mathcal{NN}}$ and $\alpha<\alpha_0$. Take

$$h:=(\xi_1\cos\xi_1-\xi_2\cos\xi_2)h^{(1)}-(\xi_2\sin\xi_1-\xi_1\sin\xi_2)h^{(2)}.$$

Then $h_1(1)=0$, $h_2(1)=-D(1)$ (where D is as in (4.26)), and

$$h_2'(1)=(\xi_1^2\sin\xi_2\cos\xi_1-\xi_2^2\sin\xi_1\cos\xi_2)(\xi_2-\xi_1).$$

Assuming $D>0$ in $[0,1]$ (i.e. (4.27)), the condition $h_2'h_2(1)>0$ is equivalent to

$$\xi_1^2\sin\xi_2\cos\xi_1>\xi_2^2\sin\xi_1\cos\xi_2. \tag{4.28}$$

Since $\xi_1=0$ if $\beta=0$, (4.28) can only be true if $\beta>0$. It is not difficult to see that $g^{\mathcal{ND}}_{\mathcal{NN}}(\alpha)=0$ for $\alpha\leq\frac{1}{2}\pi$ and $g^{\mathcal{ND}}_{\mathcal{NN}}(\alpha_0)=\frac{1}{4}\alpha_0^2$. If $\alpha>\alpha_0$, then (4.25) determines $g^{\mathcal{ND}}_{\mathcal{NN}}(\alpha)$.

The formulas for functions g in Proposition 4.1(ii) follow from the stability conditions (4.22), (4.23), (4.24), (4.25), (4.27), (4.28). □

Remark 4.3. Consider case $\binom{\mathcal{DD}}{\mathcal{NN}}$. We have $g^{\mathcal{DD}}_{\mathcal{NN}}(\alpha)=g_M(\alpha)>g^{\mathcal{DD}}_{\mathcal{ND}}(\alpha)$ except for $\alpha=\alpha_k:=(2k-1)\pi$, $k=1,2,\dots$. If $\alpha=\alpha_k$ and $\beta=g_M(\alpha)=g^{\mathcal{DD}}_{\mathcal{ND}}(\alpha)$, then the function D in Theorem 3.4 satisfies $D\neq0$ in $(0,1)$, $D(1)=0$, hence condition (3.2) cannot be satisfied (otherwise (3.5) would imply $\tilde\Psi(\bar h)<0$ for some $\bar h\in\tilde W_{\mathcal{D}}$, so that $\tilde\Psi(\bar h)<0$ also if β is slightly greater than $g_M(\alpha)$, which is a contradiction). For example, if $k=2$ (i.e. $\alpha=3\pi$, $\beta=2\pi^2$), then our proof shows that H_0 is spanned by $h(x):=(-\sin(\pi x)-\sin(2\pi x),\cos(\pi x)+\cos(2\pi x))$ and $\mathcal{B}_2h(1)=h_2(1)=h_1(1)=0$ which violates (3.2). This degeneracy seems to be also responsible for the non-smooth behavior of g_M at $\alpha=\alpha_k$. □

5 Field of extremals

In this section we modify the Weierstrass theory to provide necessary and sufficient conditions for weak, strong and global minimizers. Recall that $B_\varepsilon := \{\xi \in \mathbb{R}^N : |\xi| < \varepsilon\}$.

Definition 5.1. Let $f \in C^2$, $\tilde{\varepsilon} > 0$, and let $u^0 \in C^2$ be an extremal. The image $\mathcal{P}$ of a C^1-diffeomorphism $P : [a,b] \times B_{\tilde{\varepsilon}} \to [a,b] \times \mathbb{R}^N : (x,\alpha) \mapsto (x, \varphi(x,\alpha))$ is called a *field of extremals for* u^0 if $\varphi_x \in C^1$, $\varphi(\cdot,\alpha)$ is an extremal for each α, and $\varphi(\cdot,0) = u^0$. The *slope* of the field of extremals $\mathcal{P}$ is defined as $\psi : \mathcal{P} \to \mathbb{R}^N : (x,v) \mapsto \varphi_x(x,\alpha(x,v))$, where $\alpha(x,v)$ is defined by $\varphi(x,\alpha(x,v)) = v$.

It is known that in the case of the Dirichlet boundary conditions, the existence of a field of extremals $\varphi(x,\alpha)$ satisfying the self-adjointness condition (5.1), and the nonnegativity of the excess function

$$E(x,u,p,q) := f(x,u,q) - f(x,u,p) - (q-p) \cdot f_p(x,u,p)$$

for suitable (x,u,p,q) imply that u^0 is a strong minimizer. In addition, the existence of the field is guaranteed by the sufficient condition for the weak minimizer in Theorem 3.4(ii). In the general case we have the following analogue (see Theorem 6.1 for a simpler version in the scalar case $N = 1$):

Theorem 5.2. *Let $f \in C^2$, $\varepsilon > 0$, and let $u^0 \in C^2$ be an extremal satisfying (2.2).*

(i) *Let there exist a field of extremals $\mathcal{P}$ for u^0 satisfying the conditions*

$$\frac{\partial f_{p_i}(a,v,\psi(a,v))}{\partial v_j} = \frac{\partial f_{p_j}(a,v,\psi(a,v))}{\partial v_i} \quad \text{whenever} \;\; i,j \in I,\ v - u^0(a) \in B_\varepsilon, \tag{5.1}$$

$$f_p(a,v,\psi(a,v)) \cdot (v - u^0(a)) \le 0, \qquad \text{whenever} \;\; v - u^0(a) \in \mathbb{R}^N_{\mathcal{D},a} \cap B_\varepsilon, \tag{5.2}$$

$$f_p(b,v,\psi(b,v)) \cdot (v - u^0(b)) \ge 0, \qquad \text{whenever} \;\; v - u^0(b) \in \mathbb{R}^N_{\mathcal{D},b} \cap B_\varepsilon, \tag{5.3}$$

where ψ denotes the slope of the field. Assume also

$$E(x,v,\psi(x,v),q) \ge 0 \quad \text{for all} \;\; ((x,v),q) \in \mathcal{P} \times \mathbb{R}^N. \tag{5.4}$$

Then u^0 is a strong minimizer.

If (5.4) is only true for all $(x,v) \in \mathcal{P}$ and $q = q(x,v)$ satisfying $|q - \psi(x,v)| \le \eta$ for some $\eta > 0$, then u^0 is a weak minimizer.

If the field is global (i.e. $\mathcal{P} = [a,b] \times \mathbb{R}^N$) and (5.1), (5.2), (5.3) are true with B_ε replaced by $\mathbb{R}^N$, then u^0 is a global minimizer.

(ii) *Assume $I^{\mathcal{D}}_a = \emptyset$ and let there exist a field of extremals satisfying (5.1). If the reversed inequality "$\ge$" is true in (5.2), and the reversed strict inequality "$<$" is true in (5.3) for $v = u^0(b) + tw^0$, where $t \in (0,1)$ and $w^0 \in \mathbb{R}^N_{\mathcal{D},b}$ is fixed, then u_0 is not a weak minimizer.*

(iii) *Assume (2.4) and let the sufficient conditions for a weak minimizer in Theorem 3.4(ii) be satisfied. If $I^{\mathcal{D}}_a = \emptyset$ or $I^{\mathcal{N}}_a = \emptyset$ or*

$$\left.\begin{array}{l} f_{p_i}(a,u,p) \text{ for } i \in I^{\mathcal{D}}_a \text{ does not depend on } u_j, p_j \text{ with } j \notin I^{\mathcal{D}}_a, \\ f_{p_i u_j} = f_{p_j u_i} \text{ for } i,j \in I^{\mathcal{D}}_a, \end{array}\right\} \tag{5.5}$$

then a field of extremals satisfying (5.1), (5.2), (5.3) exists.

Remark 5.3. The well known Weierstrass necessary condition for minimizers asserts that the inequality $E(x,u^0(x),(u^0)'(x),q)\geq 0$ for all $q\in\mathbb{R}^N$ or $q=q(x)$ satisfying $|q-(u^0)'(x)|\leq\eta$ is necessary for u^0 to be a strong or weak minimizer, respectively, hence the nonnegativity conditions on E in Theorem 5.2 are not far from optimal. Similarly, Theorem 5.2(ii) shows that the sufficient conditions (5.2)–(5.3) in Theorem 5.2(i) are also necessary in some sense, at least if $I_a^{\mathcal{D}}=\emptyset$. □

The proof of part (iii) of Theorem 5.2 is quite technical and, in addition, we will not need that part in our examples (since we will prove the existence of the field required by Theorem 5.2(i)–(ii) by other arguments). Therefore the proof of part (iii) is postponed to the Appendix.

In what follows we assume that

$$\begin{aligned}&f\in C^2,\ u^0\in C^2 \text{ is an extremal,}\\&\mathcal{P}\text{ is a field of extremals for } u^0 \text{ with slope } \psi, \text{ and (5.1) is true.}\end{aligned}\tag{5.6}$$

Given $v\in C^1([a,b],\mathbb{R}^N)$ such that $\operatorname{graph}(v):=\{(x,v(x)):x\in[a,b]\}\subset\mathcal{P}$, we define the Hilbert invariant integral

$$I(v):=\int_a^b\big[f\big(x,v(x),\psi(x,v(x))\big)+\big(v'(x)-\psi(x,v(x))\big)\cdot f_p\big(x,v(x),\psi(x,v(x))\big)\big]\,dx.$$

The following proposition is well known, but for the reader's convenience we provide its proof in the Appendix.

Proposition 5.4. *Assume* (5.6). *Then there exists* $S\in C^2(\mathcal{P})$ *such that*

$$\begin{aligned}&I(v)=S(b,v(b))-S(a,v(a))\quad\textit{for any}\ \ v\in C^1([a,b],\mathbb{R}^N)\ \ \textit{with}\ \ \operatorname{graph}(v)\subset\mathcal{P},\\&S_v(x,v)=f_p(x,v,\psi(x,v))\quad\textit{for any}\ \ (x,v)\in\mathcal{P}.\end{aligned}\tag{5.7}$$

Proof of Theorem 5.2. (i) Let $u-u^0\in C^1_{\mathcal{D}}$, $\operatorname{graph}(u)\subset\mathcal{P}$, and let S be the function from Proposition 5.4. If u is close to u^0 in the sup-norm, then the assumptions (5.2)–(5.3) guarantee

$$S(a,u(a))-S(a,u^0(a))=\int_0^1 S_v(a,u^0(a)+t(u(a)-u^0(a)))\cdot(u(a)-u^0(a))\,dt\leq 0,$$

and similarly $S(b,u(b))-S(b,u^0(b))\geq 0$, hence $I(u^0)\leq I(u)$ due to Proposition 5.4. This fact and assumption (5.4) imply

$$\Phi(u)-\Phi(u^0)=\Phi(u)-I(u^0)\geq\Phi(u)-I(u)=\int_a^b E(x,u(x),\psi(x,u(x)),u'(x))\,dx\geq 0,$$

hence u^0 is a strong minimizer. The remaining assertions in (i) are obvious.

(ii) Choose $t_k\to 0+$ and let α_k be such that $\varphi(b,\alpha_k)=u^0(b)+t_kw^0$. Then $u^k:=\varphi(\cdot,\alpha_k)\to u^0$ in C^1, $u^k-u^0\in C^1_{\mathcal{D}}$ due to $I_a^{\mathcal{D}}=\emptyset$ and $w^0\in\mathbb{R}^N_{\mathcal{D},b}$, and, similarly as in (i), we obtain

$$\Phi(u^k)=I(u^k)=S(b,u^k(b))-S(a,u^k(a))<S(b,u^0(b))-S(a,u^0(a))=I(u^0)=\Phi(u^0),$$

hence u^0 is not a minimizer. □

6 Scalar examples with variable endpoints

Throughout this section (except for Remark 6.4) we assume $N = 1$ and $I_a^{\mathcal{D}} = I_b^{\mathcal{D}} = \emptyset$. Since we will often use Theorem 5.2, let us first reformulate it in this special case. Notice that the extremals in the field of extremals satisfy $\varphi_\alpha(x,\alpha) \neq 0$, hence we can assume $\varphi_\alpha > 0$ without loss of generality.

Theorem 6.1. *Let $N = 1$, $I_a^{\mathcal{D}} = I_b^{\mathcal{D}} = \emptyset$, $f \in C^2$ and let $u^0 \in C^2$ be an extremal satisfying* (2.2)*.*

(i) *Let there exist a field of extremals $\mathcal{P} = \{(x,\varphi(x,\alpha)) : x \in [a,b],\ \alpha \in (-\varepsilon,\varepsilon)\}$ for u^0 satisfying the conditions $\varphi_\alpha > 0$ and*

$$f_p^\alpha(a)\alpha \leq 0 \leq f_p^\alpha(b)\alpha, \qquad \alpha \in (-\varepsilon,\varepsilon), \tag{6.1}$$

where $f_p^\alpha(x) := f_p(x,\varphi(x,\alpha),\varphi_x(x,\alpha))$. Assume also

$$E(x,v,\psi(x,v),q) \geq 0 \quad \text{for all} \ \ ((x,v),q) \in \mathcal{P}\times\mathbb{R}. \tag{6.2}$$

Then u^0 is a strong minimizer.

If (6.2) *is only true for all $(x,v) \in \mathcal{P}$ and $q = q(x,v)$ satisfying $|q - \psi(x,v)| \leq \eta$ for some $\eta > 0$, then u^0 is a weak minimizer.*

If $\mathcal{P} = [a,b]\times\mathbb{R}$, then u^0 is a global minimizer.

(ii) *Let there exist a field of extremals satisfying $\varphi_\alpha > 0$. If, for $\alpha > 0$ or $\alpha < 0$, the reversed inequalities in* (6.1) *are true and one of them is strict (for example, if $f_p^\alpha(a) \geq 0 > f_p^\alpha(b)$ for $\alpha > 0$), then u_0 is not a weak minimizer.*

(iii) *Assume* (2.4) *and let the sufficient conditions for a weak minimizer in Theorem 3.3(ii) be satisfied. Then a field of extremals satisfying $\varphi_\alpha > 0$ and* (6.1) *exists.*

Remark 6.2. If $f_{up}^0 = 0$ and we set $P := f_{pp}^0$, $Q := f_{uu}^0$, then $\Psi(h) = \int_a^b (P(h')^2 + Qh^2)\,dx$ and the Jacobi equation has the form $-\frac{d}{dx}(Ph') + Qh = 0$. Notice also that if $P, Q > 0$, then Ψ is positive definite in $W^{1,2}$. Consequently, Remark 3.6(iii) implies that the sufficient conditions for a weak minimizer in Theorem 3.3(ii) are satisfied and Theorem 6.1(iii) implies the existence of a field of extremals satisfying $\varphi_\alpha > 0$ and (6.1). □

In the following examples we will consider Lagrangians $f = f(u,p)$ and we will use the phase plane analysis for the Du Bois-Reymond equation $f^0 - (u^0)'f_p^0 = C$.

Example 6.3. The study of the deformation of a planar weightless inextensible and unshearable rod (satisfying suitable boundary conditions) leads to the minimization of the functional

$$\Phi(u) = \int_0^1 \Big(\frac{1}{2}(u'-K)^2 + M\cos u\Big)\,dx, \qquad u \in C^1([0,1]), \tag{6.3}$$

where $K \in \mathbb{R}$, $M > 0$, and u denotes the angle between the tangent to the rod and a suitable vertical, see [10, (97)] and cf. also [1]. Functional Φ possesses multiple critical points, i.e. extremals satisfying the natural boundary conditions $u'(0) = u'(1) = K$; see [10] for their detailed analysis. Their stability was also analyzed in [10], but that analysis based on the approach from [12] is unnecessarily complicated. Somewhat simpler arguments were used in [1], but those arguments cannot be used for all critical points. We will show that Theorems 3.3 and 6.1 yield a very simple way to determine the stability of any critical point.

Proposition 2.3 implies that u^0 is a weak minimizer of Φ if and only if it is a strong minimizer. Therefore we will only speak about minimizers. Notice also that $f_{pp} = 1$ and the excess function satisfies $E(x,u,p,q) = \frac{1}{2}(q-p)^2 \geq 0$. Proposition 2.4 guarantees that any critical point of Φ is C^∞ and satisfies the Du Bois-Reymond equation $(u')^2 = 2M\cos u + C$, where C is a constant. Conversely, any non-constant solution of the Du Bois-Reymond equation is an extremal.

We consider the phase plane (u,v), where $v = u'$, and set

$$\phi_C := \{(u,v) : v^2 = 2M\cos u + C\}, \qquad C \in (-2M,\infty)$$

(see Figure 6.1). The considerations above show that given any non-constant critical point u^0, there exists $C^0 > -2M$ such that $(u^0(x),(u^0)'(x)) \in \phi_{C^0}$ for $x \in [0,1]$, $(u^0)'(0) = (u^0)'(1) = K$. On the other hand, if $(A_0,K),(A_1,K) \in \Phi_{C^0}$ for some $C^0 \in (2M,\infty)$, $A_0 \neq A_1$, and $u^0 \in C^1$ satisfies $(u^0(x),(u^0)'(x)) \in \phi_{C^0}$ for $x \in [0,1]$, $(u^0(0),(u^0)'(0)) = (A_0,K)$ and $(u^0(b),(u^0)'(b)) = (A_1,K)$ for some $b > 0$, then u^0 is a critical point if and only if $b = 1$ (the value of b is uniquely determined in this case since $(u^0)' \neq 0$). Similar assertion is true if $C^0 \in (-2M,2M]$ ($K \neq 0$ if $C^0 = 2M$), but this time one can have $(u^0(b),(u^0)'(b)) = (A_1,K)$ for multiple values of b (since u^0 need not be monotone), and one has to allow $A_1 = A_0$.

The phase plane analysis can be used to find critical points of Φ (see [2] for a particular case), but since those critical points are known (see [10], for example), we will restrict ourselves to the determination of their stability. More precisely, considering the case $K \geq 0$ (the case $K \leq 0$ being symmetric), we will show the following: A critical point of Φ is a minimizer if and only if either $u^0(x) \equiv (2k+1)\pi$ for some integer k or u^0 is a part of curve ϕ_{C^0} with $C^0 > 2M$ and $(u^0)''(0) < 0 < (u^0)''(1)$.

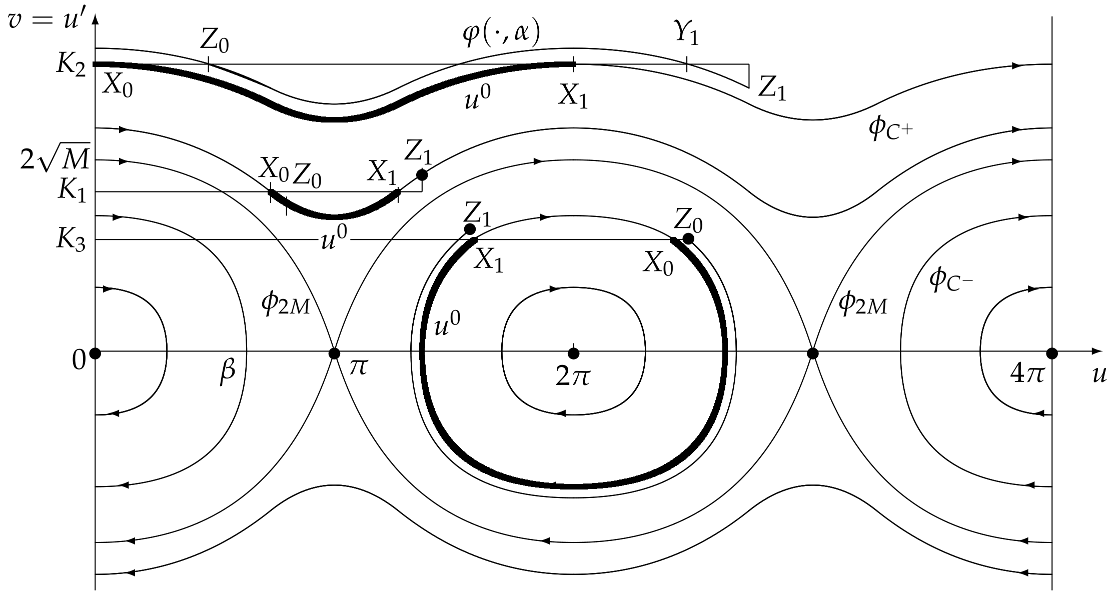

Figure 6.1: Phase plane and extremals for Example 6.3 and $0 \leq u \leq 4\pi$; $C^- < 2M < C^+$, $Z_i = (\varphi(i,\alpha),\varphi_x(i,\alpha))$, $i = 0,1$, $Y_1 = (A_1+\alpha,K)$, $X_i = (A_i,K) = (u^0(i),(u^0)'(i))$, $i = 0,1$.

Let us first consider a critical point u^0 being a part of curve ϕ_{C^0} with $C^0 > 2M$, and let (A_i,K) be as above. For symmetry reasons we may assume $K > 0$. Notice that $u'' = -M\sin u$,

$|(u^0)''(0)| = |(u^0)''(1)|$, and that $u^0(x)$ can also be defined (as an extremal, hence a part of ϕ_{C^0}) for $x \notin [0,1]$.

If $(u^0)''(0) < 0 < (u^0)''(1)$ (i.e. $u^0(0) \in (2k\pi, (2k+1)\pi)$ and $u^0(1) \in ((2m+1)\pi, (2m+2)\pi)$ for some $m \geq k$; see the extremal u^0 with $(u^0)'(0) = K_1$ in Figure 6.1), then $\varphi(x,\alpha) := u^0(x+\alpha)$, $x \in [0,1]$, $\alpha \in (-\varepsilon, \varepsilon)$, is a field of extremals for u^0 satisfying (6.1), hence Theorem 6.1(i) guarantees that u^0 is a minimizer. If $(u^0)''(0) > 0 > (u^0)''(1)$, then the same argument and Theorem 6.1(ii) show that u^0 is not a minimizer.

Next assume that $(u^0)''(0) \cdot (u^0)''(1) \geq 0$. We will show that u^0 is not a minimizer.

Assume $(u^0)''(0) < 0$, or $(u^0)''(0) = 0$ and $(u^0)'''(0) < 0$ (the cases $(u^0)''(0) > 0$, or $(u^0)''(0) = 0$ and $(u^0)'''(0) > 0$ are analogous). We necessarily have $A_1 = A_0 + 2k_0\pi$ for some $k_0 \in \{1, 2, \dots\}$. Let $\varphi(\cdot,\alpha)$ (with $|\alpha|$ being small) be the extremal with initial values $Z_0 := (\varphi(0,\alpha), \varphi_x(0,\alpha)) = (A_0 + \alpha, K)$ (see the extremal u^0 with $(u^0)'(0) = K_2$ in Figure 6.1). Then φ is a field of extremals for u^0, and $\varphi(\cdot,\alpha)$ is a part of the curve ϕ_{C^α}, where C^α is close to C^0, $C^\alpha > C^0$ if $\alpha > 0$.

Let $\alpha > 0$ be small. If u^1 and u^2 are extremals in ϕ_{C^0} and ϕ_{C^α}, respectively, and $u^1(0) = u^2(0) = 0$, then $u^1(b_1) = u^2(b_2) = 2\pi$ for some $0 < b_1 < b_2$ (due to $(u^2)' > (u^1)'$ whenever $u^2 = u^1$). This fact and the 2π-periodicity of the problem guarantee that $\varphi(b,\alpha) = A_1 + \alpha$ for some $b < 1$, hence $\varphi_x(1,\alpha) < (u^0)'(1)$, and Theorem 6.1(ii) implies that u^0 is not a minimizer.

Next consider the case $C^0 \in (-2M, 2M]$ and $K \geq 0$; $K \neq 0$ if $C^0 = 2M$. If $K > 0$ and $(u^0)''(0) > 0 > (u^0)''(1)$, then the same arguments as above guarantee that u^0 is not a minimizer. If $K = 0$ or $(u^0)''(0) < 0 < (u^0)''(1)$ (hence $A_1 < A_0$) or $(u^0)''(0) \cdot (u^0)''(1) \geq 0$ (hence $A_0 = A_1 = 2k\pi$), then choosing $\varphi(\cdot,\alpha)$ to be an extremal satisfying initial conditions $(\varphi(0,\alpha), \varphi_x(0,\alpha)) = (A_0 + \alpha, K)$ we see from the phase plane that $\varphi(\cdot,\alpha)$ and u^0 intersect in $(0,1)$ for any $\alpha \neq 0$ small (if, for example, $(u^0)''(0) < 0 < (u^0)''(1)$ and $\alpha > 0$ is small, then there exists $y \in (0,1)$ such that $\varphi(y,\alpha) = \min \varphi(\cdot,\alpha) < \min u^0$, and the inequalities $\varphi(0,\alpha) > u^0(0)$, $\varphi(y,\alpha) < u^0(y)$ imply that $\varphi(\cdot,\alpha)$ and u^0 intersect in $(0,y)$; see the extremal u^0 with $(u^0)'(0) = K_3$ in Figure 6.1). Consequently, $h := \varphi_x(\cdot,0)$ is a solution of the Jacobi equation satisfying $h(0) = 1$, $h'(0) = 0$, $h(y) = 0$ for some $y \in (0,1]$, and Theorem 3.3 guarantees that u^0 is not a minimizer.

Similar considerations as above can be used in the case of constant extremals $k\pi$, but we will use a different argument: If $u^0 \equiv (2k+1)\pi$, then $P = 1$, $Q = -M\cos u^0 = M$, and the solution $h(x) = e^{\sqrt{M}x} + e^{-\sqrt{M}x}$ of the Jacobi equation satisfies $h > 0$, $h'(0) = 0$, $h'(1) > 0$, hence u^0 is a minimizer. If $u^0 \equiv 2k\pi$, then $P = 1$, $Q = -M$ and the solution $h(x) = \cos(\sqrt{M}x)$ of the Jacobi equation satisfies $h(0) > 0$, $h'(0) = 0$ and either $h(x) = 0$ for some $x \in (0,1]$ or $h'(1) < 0$, hence u^0 is not a minimizer. □

Remark 6.4. The author of [9] considers the functional Φ in (6.3) with $K = 0$, $[a,b] = [-1/2, 1/2]$ (instead of $[a,b] = [0,1]$), and the Dirichlet boundary conditions $u(-1/2) = u(1/2) = 0$, see [9, (6)]. He considers the extremal u^0 satisfying $u^0(0) = \beta$ and $(u^0)'(0) = 0$, i.e. the extremal passing through the point $(\beta, 0)$ in Figure 6.1, and he provides explicit formulas for this extremal, its field of extremals φ and the derivative φ_α (see [9, (8),(9),(13),(14) and (16)]; functions u^0, φ and φ_α are denoted by θ, y and $\partial y/\partial\gamma$, respectively). The nonnegativity of the excess function then implies that u^0 is a strong minimizer. In [9, Introduction], the author claims that "Based on the Jacobian test, potential energy of Euler elasticas ... was proved to hold a weak minimum value...", but "...it is an open problem to find sufficient conditions for the potential energy for these Euler elasticas to hold a strong minimum." However, Proposition 2.3 shows that weak and strong minimizers of functional Φ in (6.3) are equivalent. In addition, Theorem 5.2(iii) implies that the positive definiteness of the second variation ψ in

$W_0^{1,2}(-1/2, 1/2)$ (i.e. the sufficient condition for a weak minimizer) guarantees the existence of the required field φ, hence the technical construction of the field in [9] is not necessary even if we do not consider Proposition 2.3. $\square$

Example 6.5. Consider the functional $\Phi(u) = \int_a^b f(u, u')\, dx$ in $C^1([a, b])$, where $f(u, p) = g(p) + u^2$ and g is a double-well function. More precisely, we will consider the following two cases (see Figure 6.2):

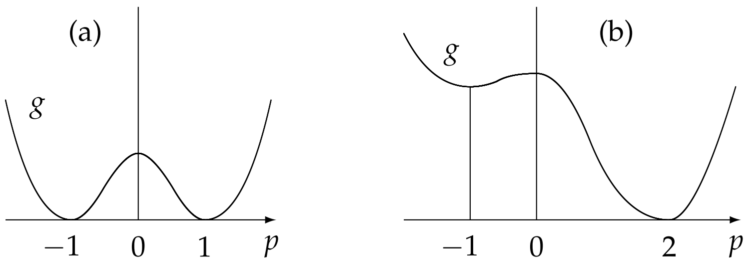

Figure 6.2: Graphs of g in the symmetric and non-symetric cases.

(a) $g(p) = (p^2 - 1)^2$ (hence $g'(p) = 4p(p^2 - 1)$, $g''(p) = 4(3p^2 - 1)$),
(b) $g(p) = \frac{1}{4}p^4 - \frac{1}{3}p^3 - p^2 + \frac{8}{3}$ (hence $g'(p) = (p + 1)p(p - 2)$, $g''(p) = 3p^2 - 2p - 2$).
Let us consider the symmetric case (a) first. The Du Bois-Reymond equation has the form

$$u^2 = C + h(u'), \qquad \text{where} \quad h(p) := 3p^4 - 2p^2,$$

see Figures 6.3 and 6.4 for the graph of h and the phase plane (u, u'), respectively. All minimizers have to satisfy $u'(a), u'(b) \in \{0, \pm 1\}$; the only constant extremal is $u \equiv 0$. Functional Φ does not possess local maximizers since $\Phi''(u^0)(1, 1) > 0$ for any u^0.

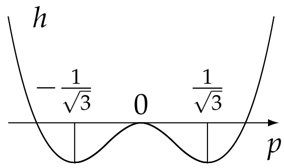

Figure 6.3: Graph of h in the symmetric case.

Since $f_{pp}(u, p) = 4(3p^2 - 1)$, the extremals in the region $|u'| \leq 1/\sqrt{3}$ (satisfying $(u^0)'(a) = (u^0)'(b) = 0$) cannot be local minimizers. The extremal u^* with $(u^*)'(a) = 1$ and $\min(u^*)' = 1/\sqrt{3}$ (see Figure 6.4) satifies $u^*(b^*) = 1$ for some $b^* > a$. If $b \in (a, b^*)$, then there exists a unique extremal u^0 satisfying $(u^0)'(a) = (u^0)'(b) = 1$ (and a unique extremal u^1 satisfying $(u^1)'(a) = (u^1)'(b) = -1$); in addition $(u^0)' > 1/\sqrt{3}$ (and $(u^1)' < -1/\sqrt{3}$). Since $P, Q > 0$ and the excess function $E = (q - p)^2((q + p)^2 + 2(p^2 - 1))$ considered as a function of q changes sign if $|p| < 1$, Remarks 6.2 and 5.3 show that the extremals u^0, u^1 are weak but not strong minimizers. (Remark 6.2 also guarantees the existence of a field of extremals, but this fact is not needed here: The Weierstrass necessary condition for strong minimizers in Remark 5.3 does not require the existence of a field of extremals.) Notice also that $\inf \Phi = 0$ is not attained (neither in C^1, nor in $W^{1,4}$): A minimizing sequence in C^1 can be obtained by suitable smooth approximation of piecewise C^1-functions u_ε satisfying $|u_\varepsilon'| = 1$ a.e. and $|u_\varepsilon| \leq \varepsilon$.

Next consider the nonsymmetric case (b). The Du Bois-Reymond equation has the form

$$u^2 = C + h(u'), \qquad \text{where} \quad h(p) := \frac{3}{4}p^4 - \frac{2}{3}p^3 - p^2,$$

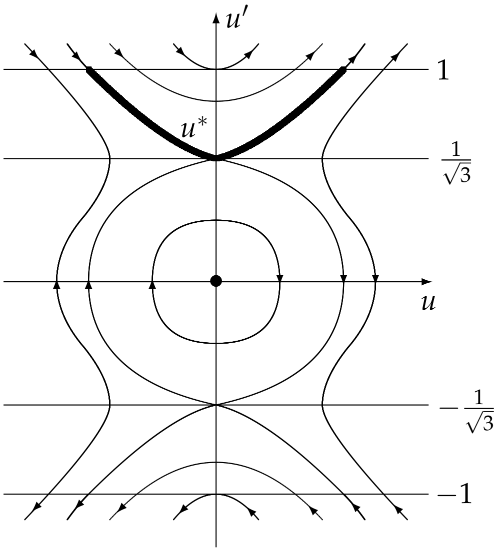

Figure 6.4: Phase plane in the symmetric case.

see Figures 6.5 and 6.6 for the graph of h and the phase plane (u, u'), respectively. All minimizers have to satisfy $u'(a), u'(b) \in \{0, -1, 2\}$; the only constant extremal is $u \equiv 0$.

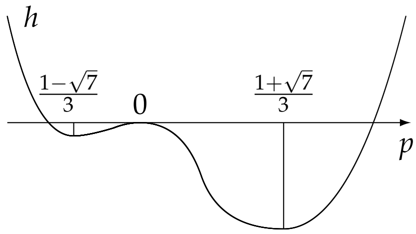

Figure 6.5: Graph of h in the non-symmetric case.

Since $f_{pp}(u, p) = 3p^2 - 2p - 2$, similarly as in case (a) we see that the extremals in the region $u' \in [\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}]$ are neither local minimizers nor local maximizers. The extremal u^* with $(u^*)'(a) = 2$ and $\min(u^*)' = \frac{1+\sqrt{7}}{3}$ (see Figure 6.6) satisfies $u^*(b^*) = 2$ for some $b^* > a$. If $b \in (a, b^*)$, then there exists a a unique extremal u^0 satisfying $(u^0)'(a) = (u^0)'(b) = 2$ and, as above, this extremal is a weak local minimizer. However, now $E = \frac{1}{12}(q-p)^2((\sqrt{3}(q+p) - \frac{2}{\sqrt{3}})^2 + 6p^2 - 4p - 13\frac{1}{3}) \geq 0$ for all q if $p \leq p_1$ or $p \geq p_2$, where $p_1 = \frac{1}{3}(1-\sqrt{21}) < -1$, $p_2 = \frac{1}{3}(1+\sqrt{21}) \in (\frac{1}{3}(1+\sqrt{7}), 2)$, and Remark 6.2 guarantees the existence of a field of extremals satisfying $\varphi_\alpha > 0$ and (6.1), hence u^0 is a strong local minimizer provided $\min(u^0)' > p_2$ (and it is not if $\min(u^0)' < p_2$). In fact, if $\min(u^0)' > p_2$, then Proposition 6.6 below shows the existence of a global field of extremals for u^0 satisfying the assumptions of Theorem 6.1(i), with slope $\psi > p_2$, hence u^0 is a global minimizer.

An analogous analysis as in the case $u' > \frac{1+\sqrt{7}}{3}$ shows that the extremals in the region $u' < \frac{1-\sqrt{7}}{3}$ are weak but not strong local minimizers. □

Proposition 6.6. *Let Φ and p_2 be as in Example 6.5(b), and let u^0 be a critical point of Φ satisfying $\min(u^0)' > p_2$. Then there exists a global field of extremals for u^0 satisfying the assumptions of Theorem 6.1(i), with slope $\psi > p_2$.*

Proof. Assume first $\alpha \geq 0$. Then we choose the extremals $u^\alpha := \varphi(\cdot, \alpha)$ in the global field such

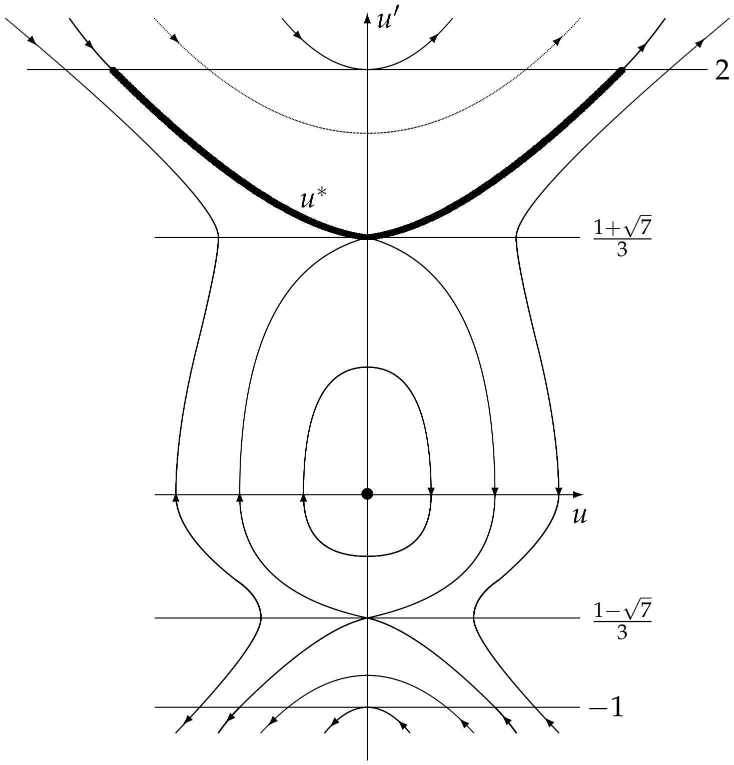

Figure 6.6: Phase plane in the non-symmetric case.

that $\varphi(\cdot,\alpha)$ is the solution of the Du Bois-Reymond equation with $(\varphi(a,\alpha),\varphi_x(a,\alpha)) = A(\alpha)$, where $A(\alpha) = (A_1(\alpha), A_2(\alpha)) : (0,\infty) \to \mathbb{R}^2$ is smooth,

$$A(\alpha) = \begin{cases} (u^0(a+\alpha),(u^0)'(a+\alpha)) & \text{if } \alpha \le b-a-\varepsilon, \\ (u^0(b)+\alpha-(b-a),2) & \text{if } \alpha \ge b-a+\varepsilon, \end{cases} \tag{6.4}$$

$$A_1'(\alpha) \ge 1, \quad A_2'(\alpha) > 0 \quad \text{for } \alpha \in (b-a-\varepsilon, b-a+\varepsilon), \text{ where } \varepsilon \in (0,(b-a)/2), \tag{6.5}$$

see Figure 6.7. Notice that $A_1'(b-a-\varepsilon) = (u^0)'(b-\varepsilon) > p_2 > 1$, $A_2'(b-a-\varepsilon) = (u^0)''(b-\varepsilon) > 0$, $A_1'(b-a+\varepsilon) = 1$, $A_2'(b-a+\varepsilon) = 0$, $A_1(b-a+\varepsilon) - A_1(b-a-\varepsilon) > 2\varepsilon$ (since $A_1(b-a+\varepsilon) = u^0(b)+\varepsilon$, $A_1(b-a-\varepsilon) = u^0(b-\varepsilon)$, $u^0(b) - u^0(b-\varepsilon) = (u^0)'(b-\theta\varepsilon)\varepsilon > p_2\varepsilon$), $A_2(b-a+\varepsilon) > A_2(b-a-\varepsilon)$, so that (6.5) can be satisfied.

Let us show that $\varphi_\alpha > 0$. Since $\varphi(x,\alpha) = u^0(x+\alpha)$ for $\alpha \le b-a-\varepsilon$ and $(u^0)' > 0$, we may assume $\alpha > b-a-\varepsilon$, hence $\varphi > 0$. Set $w(x,\alpha) = \varphi_\alpha(x,\alpha)$. Then (6.4)–(6.5) imply $w(a,\alpha) \ge 1$. Let h^{-1} denote the inverse of the increasing function $h|_{(p_2,\infty)}$. Since $\varphi(\cdot,\alpha)$ solves the Du Bois-Reymond equation, there exists $C(\alpha)$ such that $\varphi(x,\alpha)^2 = C(\alpha) + h(\varphi_x(x,\alpha))$. Consequently,

$$w_x = \frac{\partial}{\partial x}(\varphi_\alpha) = \frac{\partial}{\partial \alpha}(\varphi_x) = \frac{\partial}{\partial \alpha}(h^{-1}(\varphi^2 - C(\alpha)) = \underbrace{(h^{-1})'(\varphi^2 - C(\alpha))}_{>0}[2\varphi w - C'(\alpha)]. \tag{6.6}$$

If $w_x(a,\alpha) > 0$ (which is true for $\alpha < b-a+\varepsilon$ due to (6.4)–(6.5)), then $\varphi_x > 0$ and (6.6) guarantee $w_x(x,\alpha) > 0$ for $x > a$, hence $w(x,\alpha) \ge w(a,\alpha) \ge 1$. If $w_x(a,\alpha) = 0$ (which is true for $\alpha \ge b-a+\varepsilon$ due to (6.4)), then $(2\varphi w)(a,\alpha) = C'(\alpha)$ and

$$\frac{d}{dx}(2\varphi w - C'(\alpha))(a,\alpha) = 2\varphi_x w + 2\varphi w_x = 2\varphi_x w > 2p_2 > 0$$

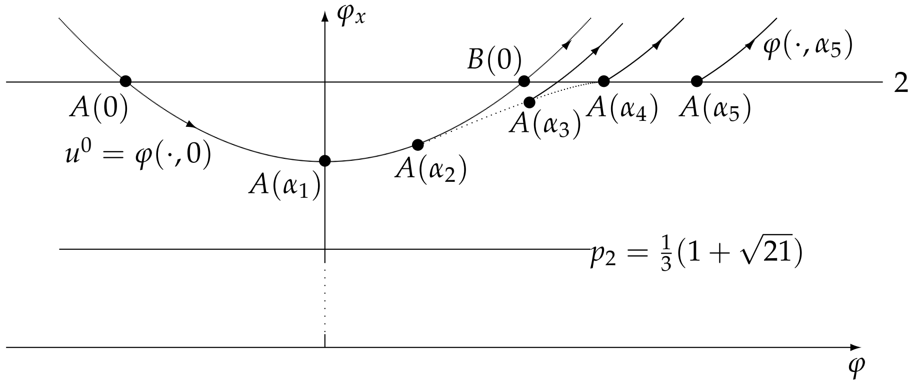

Figure 6.7: Global field of extremals: $A(\alpha) = (\varphi(a,\alpha),\varphi_x(a,\alpha))$, $B(\alpha) = (\varphi(b,\alpha),\varphi_x(b,\alpha))$, $(b-a)/2 = \alpha_1 < b-a-\varepsilon = \alpha_2 < \alpha_3 < \alpha_4 = b-a+\varepsilon < \alpha_5$.

hence $w_x(x,\alpha) > 0$ for $x > a$ close to a, and (6.6) implies $w_x(x,\alpha) > 0$ for all $x > a$. As before, this implies $w(x,\alpha) \geq 1$.

If $\alpha < 0$, then the choice of $\varphi(\cdot,\alpha)$ is symmetric: The extremal $\varphi(\cdot,\alpha)$ solves the Du Bois-Reymond equation in $[a,b]$ and $(\varphi(b,\alpha),\varphi_x(b,\alpha)) = B(\alpha) := (-A_1(-\alpha),A_2(-\alpha))$.

As an alternative to the technical construction of the global field above, we could also set $(\varphi(a,\alpha),\varphi_x(a,\alpha)) = A(\alpha)$, where

$$A(\alpha) = \begin{cases} (u^0(a+\alpha),(u^0)'(a+\alpha)) & \text{if } 0 \leq \alpha \leq b-a, \\ (u^0(b)+\alpha-(b-a),2) & \text{if } \alpha > b-a, \end{cases}$$

and analogously for $\alpha < 0$. Then the field $\varphi(\cdot,\alpha)$ is not sufficiently smooth if $|\alpha| = b-a$, but a simple generalization of Theorem 6.1 shows that this does not matter. In fact, denote $v^{\pm} := \varphi(\cdot,\pm(b-a))$. Let $u \in C^1([a,b])$; we want to show $\Phi(u) \geq \Phi(u^0)$. Approximating u suitably, we may assume that the set $\{x \in [a,b] : u(x) = v^+(x) \text{ or } u(x) = v^-(x)\}$ is finite. Set $\tilde{u} := \max(v^-,\min(v^+,u))$ and approximate $\tilde{u}$ by a sequence of C^1-functions u^k such that $\operatorname{graph}(u^k) \subset \mathcal{P}_1 := \{(x,\varphi(x,\alpha)) : x \in [a,b],\ |\alpha| \leq b-a\}$ and $u^k \to \tilde{u}$ in $W^{1,4}$. Then Theorem 6.1 shows that $\Phi(u^k) \geq \Phi(u^0)$, hence $\Phi(\tilde{u}) \geq \Phi(u^0)$ due to the continuity of Φ in $W^{1,4}$. Let $[x_1,x_2]$ be any maximal interval where $\tilde{u} = v^+$ (i.e. $u \geq v^+$) or $\tilde{u} = v^-$. Notice that either $x_1 = a$ or $u(x_1) = v^{\pm}(x_1)$, and either $x_2 = b$ or $u(x_2) = v^{\pm}(x_2)$. Denote $\Phi_{x_1}^{x_2}(u) = \int_{x_1}^{x_2} f(x,u(x),u'(x))\,dx$. Then the proof of Theorem 6.1 shows $\Phi_{x_1}^{x_2}(u) \geq \Phi_{x_1}^{x_2}(v^+)$ (if $u \geq v^+$ in $[x_1,x_2]$) or $\Phi_{x_1}^{x_2}(u) \geq \Phi_{x_1}^{x_2}(v^-)$, hence $\Phi(u) \geq \Phi(\tilde{u}) \geq \Phi(u^0)$. □

7 Appendix

Proof of Proposition 2.1. We will consider only the special case $N = 1$, $I_a^{\mathcal{D}} = \emptyset$, $I_b^{\mathcal{N}} = \emptyset$, but the arguments in our proof can also be used in the general case.

If $h \in C_{\mathcal{D}}^1 = \{\varphi \in C^1([a,b]) : \varphi(b) = 0\}$, then integration by parts yields

$$\begin{aligned} 0 = \Phi'(u^0)h &= \int_a^b (f_p^0(x)h'(x) + f_u^0(x)h(x))\,dx \\ &= gh\Big|_a^b + \int_a^b (f_p^0(x) - g(x))h'(x)\,dx, \end{aligned} \tag{7.1}$$

where $g(x) := \int_a^x f_u^0(\xi)\,d\xi$ is C^1. Considering test functions h with compact support in (a,b), the Du Bois-Reymond Lemma and (7.1) yield the existence of a constant C such that $f_p^0(x) = g(x) + C$, hence $f_p^0 \in C^1$ and the Euler equation $\frac{d}{dx}(f_p^0) = f_u^0$ is satisfied. This equation and the choice of h with $h(a) = 1$ in (7.1) imply

$$\begin{aligned} 0 = \Phi'(u^0)h &= \int_a^b (f_p^0(x)h'(x) + f_u^0(x)h(x))\,dx \\ &= f_p^0 h\Big|_a^b + \int_a^b \Big(-\frac{d}{dx}(f_p^0(x)) + f_u^0(x)\Big)h(x)\,dx = -f_p^0(a), \end{aligned}$$

which concludes the proof of the first part. If $f_p \in C^1$ and $f_{pp}^0 \geq c^0 > 0$, then the function $F(x,p) := f_p(x, u^0(x), p) - g(x) - C$ is C^1, $F(x, (u^0)'(x)) = 0$, $F_p(x, (u^0)'(x)) > 0$, hence the Implicit Function Theorem implies $u^0 \in C^2$. □

Proof of Proposition 2.3. The proof is based on an idea due to [4].

Let $u^0 \in C^1$ be a weak minimizer of Φ in $u^0 + C^1_{\mathcal{D}}$. Assume first that there exist $v^k \in W^{1,2}_{\mathcal{D}}$, $k = 1, 2, \dots$, such that $r_k := \|v^k\|_{1,2} \to 0$ and $\Phi(u^0 + v^k) < \Phi(u^0)$. Since Φ is weakly lower semicontinuous in $W^{1,2}$, there exists a minimizer u^k of Φ in the set $\{u \in u^0 + W^{1,2}_{\mathcal{D}} : \|u - u^0\|_{1,2} \leq r_k\}$, hence $\Phi(u^k) \leq \Phi(u^0 + v^k) < \Phi(u^0)$. Set $\Theta(u) := \|u - u^0\|^2_{1,2}$. Then there exists a Lagrange multiplier λ_k such that $\Phi'(u^k)h = \lambda_k \Theta'(u^k)h$ for any $h \in W^{1,2}_{\mathcal{D}}$ (where the derivatives are considered in $W^{1,2}$). Since $\Phi'(u^k)(u^k - u^0) \leq 0$, we have $\lambda_k \leq 0$. Standard theory implies that $u^0, u^k \in C^2$ solve the Euler equation

$$2(1 - \lambda_k)(u^k)'' = g'(u^k) - 2\lambda_k((u^0)'' + u^k - u^0),$$

which shows that the sequence u^k is bounded in C^2. Since $u^k \to u^0$ in $W^{1,2}$, the boundedness in C^2 implies $u^k \to u^0$ in C^1 which contradicts the fact, that u^0 is a weak minimizer. Consequently, u^0 is a local minimizer in $u^0 + W^{1,2}_{\mathcal{D}}$.

Next assume that there exist $v^k \in C^1_{\mathcal{D}}$ such that $\|v^k\|_C \to 0$ and $\Phi(u^0 + v_k) < \Phi(u^0)$. Since $\Phi'(u^0)h = \int_a^b (2((u^0)' - K)h' + g'(u^0)h)\,dx = 0$ for $h \in C^1_{\mathcal{D}}$, we have

$$\begin{aligned} 0 < \Phi(u^0) - \Phi(u^0 + v^k) &= \int_a^b [((u^0)' - K)^2 - ((u^0)' + (v^k)' - K)^2]\,dx + o(1) \\ &= -\int_a^b (v^k)'[(v^k)' + 2((u^0)' - K)]\,dx + o(1) \\ &= -\|v^k\|^2_{1,2} + \int_a^b g'(u^0)v^k\,dx + o(1) = -\|v^k\|^2_{1,2} + o(1), \end{aligned}$$

hence $v^k \to 0$ in $W^{1,2}$, which yields a contradiction. Consequently, u^0 is a strong minimizer. □

Proof of Proposition 3.1. Assume that $\Psi(h) \geq c\|h\|^2_{1,2}$ for some $c > 0$ and all $h \in W^{1,2}_{\mathcal{D}}$ and recall that $\Psi(h) = \Phi''(u^0)(h,h)$ if $h \in C^1$. If u^1 is close u^0 in C^1 and Ψ^1 denotes the functional Ψ with u^0 replaced by u^1, then one can easily check that $\Psi^1(h) = \Phi''(u^1)(h,h) \geq \frac{c}{2}\|h\|^2_{1,2}$ for $h \in C^1_{\mathcal{D}}$, and the Mean Value Theorem implies the existence of $\theta \in (0,1)$ such that

$$\Phi(u^0 + h) - \Phi(u^0) = \frac{1}{2}\Phi''(u^0 + \theta h)(h,h) \geq \frac{c}{4}\|h\|^2_{1,2}$$

whenever $h \in C^1_{\mathcal{D}}$ is small enough. Consequently, u^0 is a strict weak minimizer in $u^0 + C^1_{\mathcal{D}}$.

If $\Psi(h) < 0$ for some $h \in W^{1,2}_{\mathcal{D}}$, then the density of $C^1_{\mathcal{D}}$ in $W^{1,2}_{\mathcal{D}}$ and the continuity of Ψ in $W^{1,2}_{\mathcal{D}}$ guarantee the existence of $\tilde{h} \in C^1_{\mathcal{D}}$ such that $0 > \Psi(\tilde{h}) = \Phi''(u^0)(\tilde{h}, \tilde{h})$, which shows that u^0 is not a weak minimizer $u^0 + C^1_{\mathcal{D}}$. □

Proof of Theorem 5.2(iii). First assume that $I_a^{\mathcal{N}} = \emptyset$. If $I_b^{\mathcal{N}} = \emptyset$, then the assertion is well known (see [7] or [8], for example), hence we may assume $I_b^{\mathcal{N}} \neq \emptyset$. Our assumptions imply $D \neq 0$ in $(a,b]$ and $\mathcal{B}h(b) \cdot h(b) > 0$ for any $h \in H_{\mathcal{D},b} \setminus \{0\}$. We may also assume that f is defined and of class C^3 in an open neighbourhood of $\{(x, u^0(x), (u^0)'(x)) : x \in [a,b]\}$ in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ (see [2] for a detailed proof if $N = 1$). Consequently, there exists $\varepsilon > 0$ small such that u^0 can be extended (as an extremal) for $x \in [a-\varepsilon, a]$, f^0 satisfies (2.3) in $[a-\varepsilon, b]$, and the solutions $h^{(k)}$, $k = 1, 2, \dots, N$ of the Jacobi equation in $[a-\varepsilon, b]$ with initial conditions $h^{(k)}(a-\varepsilon) = 0$, $(h_i^{(k)})'(a-\varepsilon) = \delta_{ik}$, satisfy $D > 0$ in $(a-\varepsilon, b]$ and $\mathcal{B}h(b) \cdot h(b) > 0$ for any $h \in H_{\mathcal{D},b} \setminus \{0\}$ due to the continuous dependence of solutions of ODEs on initial values. Let $\varphi(\cdot, \alpha)$ be the extremal satisfying the initial conditions $\varphi(a-\varepsilon, \alpha) = u^0(a-\varepsilon)$, $\varphi_x(a, \alpha) = (u^0)'(a-\varepsilon) + \alpha$. The arguments in [7, 8] guarantee that such extremals define a field of extremals for u^0 (in $[a,b]$) satisfying (5.1). Condition (5.2) is empty, hence we only have to show that (5.3) is true. Thus assume that $v - u^0(b) \in \mathbb{R}^N_{\mathcal{D},b} \cap B_\varepsilon \setminus \{0\}$. We have $v = \varphi(b, \alpha)$ for some α small. Set $h^\alpha := \sum_k \alpha_k h^{(k)}$. If $i \in I_b^{\mathcal{D}}$, then $0 = \varphi_i(b, \alpha) - u_i^0(b) = h_i^\alpha(b) + o(\alpha)$, hence $h^\alpha = h^{\tilde\alpha} + o(\alpha)$ for some $h^{\tilde\alpha} \in H_{\mathcal{D},b} \setminus \{0\}$ and $\tilde\alpha = \alpha + o(\alpha)$. Since our assumptions imply $\mathcal{B}h^{\tilde\alpha}(b) \cdot h^{\tilde\alpha}(b) = \sum_{i \in I_b^{\mathcal{N}}} \mathcal{B}_i h^{\tilde\alpha}(b) h_i^{\tilde\alpha}(b) > 0$, we also have

$$
\begin{aligned}
f_p(b, v, \psi(b,v)) \cdot (v - u^0(b)) &= \sum_{i \in I_b^{\mathcal{N}}} f_{p_i}(b, \varphi(b,\alpha), \varphi_x(b,\alpha))(\varphi_i(b,\alpha) - u_i^0(b)) \\
&= \sum_{i \in I_b^{\mathcal{N}}} \big(\mathcal{B}_i h^\alpha(b) + o(\alpha)\big)\big(h_i^\alpha(b) + o(\alpha)\big) \\
&= \sum_{i \in I_b^{\mathcal{N}}} \big(\mathcal{B}_i h^{\tilde\alpha}(b) + o(\tilde\alpha)\big)\big(h_i^{\tilde\alpha}(b) + o(\tilde\alpha)\big) > 0.
\end{aligned}
$$

Next assume $I_a^{\mathcal{D}} = \emptyset$. Since our proof in this case uses similar arguments as in the case $I_a^{\mathcal{N}} = \emptyset$ (and a very detailed proof in the case $N = 1$ can be found in [2]), we will be brief. Given $\alpha \in \mathbb{R}^N$ small and $v = v(\alpha) := u^0(a) + \alpha$, the Implicit Function Theorem implies the existence of a unique $w = w(\alpha) \in \mathbb{R}^N$ close to $(u^0)'(a)$ such $f_p(a, v(\alpha), w(\alpha)) = 0$. Let $\varphi(\cdot, \alpha)$ be the extremal satisfying the initial conditions $\varphi(a, \alpha) = v(\alpha)$, $\varphi_x(a, \alpha) = w(\alpha)$. We claim that such extremals $\varphi(\cdot, \alpha)$ define the required field. In fact, the function P in Definition 5.1 is a C^1-diffeomorphism and $\varphi_x \in C^1$ due to the differentiability of solutions of ODEs on initial values and the fact that $h^{(k)} := \frac{\partial \varphi}{\partial \alpha_k}(\cdot, 0)$, $k = 1, \dots, N$, are linearly independent solutions of the Jacobi equation $\mathcal{A}h = 0$ satisfying the initial conditions $\mathcal{B}h(a) = 0$, hence $\det(h^{(1)}, \dots, h^{(N)}) \neq 0$ in $[a,b]$ due to our assumptions. Properties (5.1) and (5.2) follow from $f_p(a, v, \psi(a,v)) = 0$ and the proof of (5.3) is the same as in the case $I_a^{\mathcal{N}} = \emptyset$.

Finally assume (5.5). Let $h^{(1)}, \dots, h^{(N)}$ be solutions of the Jacobi equation $\mathcal{A}h = 0$ in $[a,b]$ satisfying the initial conditions

$$
\begin{aligned}
&h_i^{(k)}(a) = \eta\delta_{ik} \quad &&\text{for } k \in I_a^{\mathcal{D}}, i \in I, \quad &&(h_i^{(k)})'(a) = \delta_{ik} \quad &&\text{for } k \in I, i \in I_a^{\mathcal{D}}, \\
&h_i^{(k)}(a) = \delta_{ik} \quad &&\text{for } k \in I_a^{\mathcal{N}}, i \in I, \quad &&\mathcal{B}_i h^{(k)}(a) = 0 \quad &&\text{for } k \in I, i \in I_a^{\mathcal{N}},
\end{aligned}
$$

where $\eta \in [0,1]$. If $\zeta \geq 0$ is small, then

$$
\begin{aligned}
&h_i^{(k)}(a+\zeta) = (\eta+\zeta)\delta_{ik} + o(\zeta) \qquad &&\text{if } k, i \in I_a^{\mathcal{D}}, \\
&h_i^{(k)}(a+\zeta) = \delta_{ik} + O(\zeta) &&\text{otherwise,}
\end{aligned}
$$

hence $D(x) := \det(h^{(1)}(x), \dots, h^{(N)}(x)) > 0$ for $x \in [a, a+\zeta]$ and $\eta \in (0,1]$. If $\eta = 0$, then our assumptions imply $D(x) > 0$ for $x \in [a+\zeta, b]$ and $\mathcal{B}h(b) \cdot h(b) > 0$ for any $h := \sum_k \beta_k h^{(k)}$

satisfying $h_i(b) = 0$ for $i \in I_b^{\mathcal{D}}$ and $h \not\equiv 0$. Those properties remain true for $\eta > 0$ small and we fix such $\eta > 0$. Set $v_i(\alpha) = u_i^0(a) + \eta\alpha_i$ if $i \in I_a^{\mathcal{D}}$, $v_i(\alpha) = u_i^0(a) + \alpha_i$ if $i \in I_a^{\mathcal{N}}$ and $w_i(\alpha) = (u_i^0)'(a) + \alpha_i$ if $i \in I_a^{\mathcal{D}}$. The Implicit Function Theorem guarantees that there exist unique $w_i(\alpha)$ for $i \in I_a^{\mathcal{N}}$ (close to $(u_i^0)'(a)$) such that $f_{p_i}(a, v(\alpha), w(\alpha)) = 0$ for $i \in I_a^{\mathcal{N}}$ and α small. Let $\varphi(\cdot, \alpha)$ be extremals satisfying the initial conditions $\varphi(a, \alpha) = v(\alpha)$, $\varphi_x(a, \alpha) = w(\alpha)$. Then $\varphi_{\alpha_k}(a, 0) = h^{(k)}(a)$ and $\varphi_{x\alpha_k}(a, 0) = (h^{(k)})'(a)$, which shows that these extremals define a field of extremals for α small. The same arguments as above guarantee that properties (5.2),(5.3) are satisfied. Let us show that (5.1) is true. If $i, j \in I_a^{\mathcal{N}}$, then this follows from $f_{p_i}(a, v, \psi(a, v)) = f_{p_j}(a, v, \psi(a, v)) = 0$. Let $i \in I_a^{\mathcal{D}}$. If $j \in I_a^{\mathcal{N}}$, then the left-hand side in (5.1) is zero due to $f_{p_i u_j} = f_{p_i p_j} = 0$. If $j \in I_a^{\mathcal{D}}$, then that left-hand side equals $f_{p_i u_j}(a, v, \psi(a, v)) + \sum_{k \in I} f_{p_i p_k}(a, v, \psi(a, v))\psi_{k, v_j}(a, v)$. Since $f_{p_i u_j} = f_{p_j u_i}$, $f_{p_i p_k}(a, v, \psi(a, v)) = 0$ for $k \in I_a^{\mathcal{N}}$ and $\psi_{k, v_j}(a, v) = \frac{1}{\eta}\delta_{kj}$ if $k \in I_a^{\mathcal{D}}$, we see that that left-hand side equals to the right-hand side. □

Proof of Proposition 5.4. If $w = (w_1, \dots, w_N)$ depends on θ, then we denote $w_{i,\theta} := \frac{\partial w_i}{\partial \theta}$. By differentiating the identity $\varphi_x(x, \alpha) = \psi\big(x, \varphi(x, \alpha)\big)$ we obtain

$$\varphi_{j,xx} = \psi_{j,x} + \sum_k \psi_{j,v_k}\varphi_{k,x} = \psi_{j,x} + \sum_k \psi_{j,v_k}\psi_k.$$

If we substitute this relation into the Euler equations

$$\sum_j (f_{p_i p_j}\varphi_{j,xx} + f_{p_i u_j}\varphi_{j,x}) + f_{p_i x} - f_{u_i} = 0,$$

(where the arguments of the derivatives of f and φ are $\big(x, \varphi(x, \alpha), \varphi_x(x, \alpha)\big)$ and (x, α), respectively), then we obtain

$$\sum_j \Big(f_{p_i p_j}\big(\psi_{j,x} + \sum_k \psi_{j,v_k}\psi_k\big) + f_{p_i u_j}\psi_j\Big) + f_{p_i x} - f_{u_i} = 0, \tag{7.2}$$

where the arguments of the derivatives of f and ψ are $\big(x, v, \psi(x, v)\big)$ and (x, v), respectively. For $(x, v) \in \mathcal{P}$ we set

$$\begin{aligned} V(x, v) &:= f\big(x, v, \psi(x, v)\big) - f_p\big(x, v, \psi(x, v)\big) \cdot \psi(x, v), \\ W(x, v) &:= f_p\big(x, v, \psi(x, v)\big). \end{aligned} \tag{7.3}$$

We claim that

$$(W_{i,v_j} - W_{j,v_i})(x, v) = \frac{\partial f_{p_i}(x, v, \psi(x, v))}{\partial v_j} - \frac{\partial f_{p_j}(x, v, \psi(x, v))}{\partial v_i} = 0, \quad i, j \in I. \tag{7.4}$$

In fact, if f and φ are of class C^3, then setting $v = \varphi(x, \alpha)$ and $\psi(x, v) = \varphi_x(x, \alpha)$ in (7.4), the Euler equations imply that the d/dx-derivative of the resulting expression vanishes, hence the conclusion follows from (5.1). Such argument can also be used without the additional smoothness assumptions on f, φ, see the proof of [8, Proposition 6.1.1.4].

Now (7.4) and (7.2) imply $V_v = W_x$. This fact and (7.4) guarantee the existence of $S \in C^2(\mathcal{P})$ such that $S_x = V$ and $S_v = W$. Finally,

$$\begin{aligned} I(v) &= \int_a^b (V + W \cdot v')\,dx = \int_a^b (S_x + S_v \cdot v')\,dx = \int_a^b \frac{d}{dx} S\big(x, v(x)\big)\,dx \\ &= S\big(b, v(b)\big) - S\big(a, v(a)\big). \end{aligned}$$

□

Remark 7.1. Necessary and sufficient conditions for weak minimizers in [15, 16] are formulated in terms of (semi-)coupled points and seem to be more complicated than our conditions. In order to compare them, let us consider the scalar case with variable endpoints (i.e. $I_a^{\mathcal{D}} = I_b^{\mathcal{D}} = \emptyset$), and let h be the solution of the Jacobi equation satisfying the initial conditions $h(a) = 1$, $\mathcal{B}h(a) = 0$. Let us also denote $Q := f^0_{uu}$. Then our sufficient condition for a weak minimizer in Theorem 3.3 is equivalent to

$$h(y) \neq 0 \ \text{ for } \ y \in (a, b] \quad \text{and} \quad \mathcal{B}h(b) > 0, \tag{7.5}$$

while the sufficient condition for a weak minimizer in [15, 16] is equivalent to

$$-\mathcal{B}h(y) \neq \Big(\int_y^b Q\Big)h(y) \ \text{ for } \ y \in (a, b] \quad \text{and} \quad \int_a^b Q > 0. \tag{7.6}$$

The proofs of the sufficiency guarantee that (7.5) is equivalent to (7.6). Let us show this equivalence directly: For simplicity, consider just Lagrangians of the form $2f(x, u, p) = p^2 + Q(x)u^2$. Then $\mathcal{B}h = h'$ and the Jacobi equation has the form $h'' = Qh$. Let h be the solution of this equation with initial conditions $h(a) = 1$, $h'(a) = 0$.

First assume that (7.5) is true. Then integration by parts yields

$$\int_a^b Q = \int_a^b \frac{h''}{h} = \frac{h'}{h}\Big|_a^b + \int_a^b \frac{(h')^2}{h^2} > 0. \tag{7.7}$$

Assume to the contrary that $-h'(y) = (\int_y^b Q)h(y)$ for some $y \in (a, b]$. Then

$$-\int_y^b Q = \frac{h'(y)}{h(y)} = \frac{h'}{h}\Big|_a^y = \int_a^y \Big(\frac{h''}{h} - \frac{(h')^2}{h^2}\Big) = \int_a^y \Big(Q - \frac{(h')^2}{h^2}\Big). \tag{7.8}$$

Now (7.8) and (7.7) imply

$$\int_a^b Q = \int_a^y \frac{(h')^2}{h^2} < \frac{h'}{h}\Big|_a^b + \int_a^b \frac{(h')^2}{h^2} = \int_a^b Q,$$

which yields a contradiction.

Next assume that (7.5) fails, i.e. either $h(y) = 0$ for some $y \in (a, b]$ or $h'(b) \leq 0$, and assume also to the contrary (7.6) is true. If $h(y) = 0$ for some $y \in (a, b]$ and $h > 0$ on $[a, y]$, then $h'(y) < 0$, hence

$$-h'(a) = 0 < \Big(\int_a^b Q\Big)h(a),$$
$$-h'(y) > 0 = \Big(\int_y^b Q\Big)h(y),$$

so that there exists $z \in (a, y)$ such that $-h'(z) = (\int_z^b Q)h(z)$, which yields a contradiction. If $h > 0$ and $h'(b) \leq 0$, then

$$-h'(a) = 0 < \Big(\int_a^b Q\Big)h(a),$$
$$-h'(b) \geq 0 = \Big(\int_b^b Q\Big)h(b),$$

so that there exists $z \in (a, b]$ such that $-h'(z) = (\int_z^b Q)h(z)$ and we arrive at contradiction again.

The proof above shows that if y_1 is the first (= smallest) zero of h, then the smallest solution z_1 of the equation $-h'(z) = \big(\int_z^b Q\big)h(z)$ satisfies $z_1 < y_1$. The inequality $z_1 \leq y_1$ also follows from the proof of Theorem 3.4 and the corresponding proof in [16]. In fact, those proofs show that y_1 and z_1 correspond to the zeroes of the continuous nonincreasing functions $\lambda_1(y) = \inf_{S_y}\Psi$ and $\tilde\lambda_1(z) = \inf_{\tilde S_z}\Psi$, respectively, where S_y is the unit sphere in X_y (see (3.3)) and $\tilde S_z$ is the unit sphere in $\tilde X_z = \{h \in W^{1,2}([a,b]) : h(x) = h(z) \text{ for } x \geq z\}$. Since $X_y \subset \tilde X_y$ and the norm in X_y is equivalent to the norm in $W^{1,2}$, we have $\tilde\lambda_1 \leq \max\{C\lambda_1, 0\}$. □

The following proposition is motivated by [11] and Section 4. Given $u^0 \in C^1([a,b],\mathbb{R}^N)$, we will use the following notation (cf. (1.2)):

$$\begin{aligned}\mathcal{M} &:= u^0 + C^1_{\mathcal{D}} = \{u \in C^1([a,b]) : (u_i - u_i^0)(x) = 0 \text{ for } i \in I_x^{\mathcal{D}} \text{ and } x \in \{a,b\}\},\\ \mathcal{M}_{\mathcal{N}} &:= \{u \in \mathcal{M} : u_i'(x) = 0 \text{ for } i \in I_x^{\mathcal{N}} \text{ and } x \in \{a,b\}\}.\end{aligned}$$

Proposition 7.2. *Let $f \in C^1$ and let u^0 be a weak minimizer of Φ in $\mathcal{M}_{\mathcal{N}}$. Then u^0 is a weak minimizer in $\mathcal{M}$. Conversely, if u is a weak minimizer in $\mathcal{M}$ and $u^0 \in \mathcal{M}_{\mathcal{N}}$, then u^0 is a weak minimizer in $\mathcal{M}_{\mathcal{N}}$.*

Proof. For simplicity, we will prove the assertion only in the special case $N = 1$, $I_a^{\mathcal{D}} = \emptyset$, $I_b^{\mathcal{N}} = \emptyset$, but it will be clear from the proof that our arguments can also be used in the general case.

Hence assume first that u^0 is a weak minimizer of Φ in

$$\mathcal{M}_{\mathcal{N}} = \{u \in C^1([a,b]) : (u - u^0)(b) = 0,\ u'(a) = 0\}.$$

Then there exists $\varepsilon > 0$ such that u^0 is a (global) minimizer of Φ in the set

$$\mathcal{M}^\varepsilon_{\mathcal{N}} := \{u \in \mathcal{M}_{\mathcal{N}} : \|u - u^0\|_{C^1} < \varepsilon\}.$$

We will show that u^0 is a (global) minimizer in the set $\mathcal{M}^{\varepsilon/4}$, where

$$\mathcal{M}^\varepsilon := \{u \in \mathcal{M} : \|u - u^0\|_{C^1} < \varepsilon\},$$

hence u^0 is a weak minimizer of Φ in $\mathcal{M} = \{u \in C^1([a,b]) : (u - u^0)(b) = 0\}$.

Fix $u \in \mathcal{M}^{\varepsilon/4}$. Since $(u^0)'(a) = 0$, given $k \in \mathbb{N}$, there exists $\delta_k \in (0, 1/k)$ such that

$$|(u^0)'(x)| < 1/k \quad \text{for } x \in J_k := [a, a + \delta_k].$$

Since $\|u - u^0\|_{C^1} < \varepsilon/4$, we also have $|u'(x)| < \varepsilon/4 + 1/k$ for $x \in J_k$. Consequently, we can modify the function u in J_k such that the modified function $u^k \in C^1([a,b])$ satisfies $u^k = u$ on $[a + \delta_k, b]$, $(u^k)'(a) = 0$ and $|(u^k)'(x)| < \varepsilon/4 + 1/k$ for $x \in J_k$ (for example, we can choose $(u^k)'(x) = u'(\delta_k)(x - a)/(\delta_k - a)$ for $x \in J_k$). Then

$$|(u^k)' - (u^0)'| \leq |(u^k)'| + |(u^0)'| < \varepsilon/4 + 2/k \quad \text{on } J_k$$

and the Mean Value Theorem implies

$$|u^k - u^0| \leq |u^k - u| + |u - u^0| < \max_{J_k} |(u^k - u)'|\delta_k + \varepsilon/4 < (\varepsilon/2 + 2/k)/k + \varepsilon/4 \quad \text{on } J_k,$$

hence $u^k \in \mathcal{M}^\varepsilon_{\mathcal{N}}$ for k large, which implies $\Phi(u^k) \geq \Phi(u^0)$. Since $\Phi(u^k) \to \Phi(u)$, we have $\Phi(u) \geq \Phi(u^0)$.

The converse assertion is trivial. □

Remark 7.3. In [11, Propositions 5 and 6] the authors consider the function u^0 and the functional Φ from our Section 4, and they provide conditions guaranteeing that u^0 is a weak minimizer subject to the Neumann boundary conditions for some of its components (see (4.3) and (4.2) above). Proposition 7.2 shows that the Neumann boundary conditions do not play any role in such assertions, i.e. u^0 remains a weak minimizer if we replace "the Neumann boundary conditions" with "no boundary conditions". Consequently (see Proposition 2.1), u^0 then has to satisfy the corresponding natural boundary conditions (instead of the Neumann boundary conditions). The Neumann boundary conditions are different from the natural boundary conditions in general, but the first two components of the function u^0 in Section 4 satisfy both the Neumann and the natural boundary conditions. □

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Multiple solutions to a quasilinear periodic boundary value problem with impulsive effects

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Abstract. The authors investigate the multiplicity of solutions to a quasilinear periodic boundary value problem with impulsive effects. They use variational methods and some critical points theorems for smooth functionals, due to Ricceri, that are defined on reflexive Banach spaces. They obtain the existence of at least three solutions to the problem. The applicability of the results is illustrated with an example.

Keywords: three solutions, quasilinear periodic, boundary value problem, impulsive effects, variational methods.

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1 Introduction

We study the existence of at least three distinct classical solutions to the quasilinear periodic boundary value problem with impulsive effects

$$\begin{cases} -p(x')x'' + \alpha(t)x = \lambda f(t,x) + \mu g(t,x), & t \neq t_j, \quad \text{a.e. } t \in [0,1], \\ \Delta(h'(u'(t_j))) = I_j(u(t_j)), & j = 1,2,\dots,m, \\ x(1) - x(0) = x'(1) - x'(0) = 0, \end{cases} \tag{$P^{f,g}_{\lambda,\mu}$}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, $\lambda > 0$ and $\mu \geq 0$ are parameters, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1,\dots,m$, are continuous functions, $\Delta(h'(u'(t_j))) = h'(u'(t_j^+)) - h'(u'(t_j^-))$ with $h'(u'(t_j^\pm)) = \lim_{t\to t^\pm} h'(u'(t))$, and

$$h(y) = \int_0^y \left(\int_0^\tau p(\xi)d\xi \right) d\tau \quad \text{for every } y \in \mathbb{R}.$$

Recall that a function $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if it satisfies:

(a) $x \mapsto h(t,x)$ is measurable for every $x \in \mathbb{R}$;

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(b) $t \mapsto h(t,x)$ is continuous for a.e. $x \in [0,1]$;

(c) for every $\varepsilon > 0$ there exists a function $l_\varepsilon \in L^1([0,1])$ such that

$$\sup_{|x|\leq\varepsilon} |h(t,x)| \leq l_\varepsilon(t) \quad \text{for a.e. } t \in [0,1].$$

In this paper, and without further mention, we always assume that:

(Q_1) $p : \mathbb{R} \to (0,\infty)$ is continuous and nondecreasing on $[0,\infty)$, and there exist $M \geq m > 0$ such that

$$m \leq p(x) \leq M \quad \text{for all } x \in \mathbb{R}; \tag{1.1}$$

(Q_2) $\alpha \in C([0,1])$ and there exist $\alpha_1 \geq \alpha_0 > 0$ such that

$$\alpha_0 \leq \alpha(t) \leq \alpha_1 \quad \text{for all } t \in [0,1]; \tag{1.2}$$

(Q_3) $I_j : \mathbb{R} \to \mathbb{R}$ is nondecreasing and $I_j(0) = 0$ for $j = 1,\ldots,m$.

In recent years, impulsive differential equations have played an important role in modern applied mathematical models of real processes arising in phenomena studied in physics, ecology, biological systems, biotechnology, and industrial robotics. Many authors have applied variational methods to study the existence of multiple solutions of impulsive systems of the form (1.1) or its variations, and we refer the reader to [2–4, 6, 7, 12, 17, 20] and references cited therein for some recent results. For example, Bonanno and Livrea [3] studied the existence and multiplicity of solutions to the periodic boundary value problem

$$\begin{cases} -x'' + A(t)x = \lambda b(t)\nabla G(x), & t \in [0,T], \\ x(T) - x(0) = x'(T) - x'(0) = 0, \end{cases}$$

where $A(t) = (a_{i,j}(t))_{n\times n}$ is a positive definite matrix for all $t \in [0,T]$, $a_{i,j} \in C([0,T],\mathbb{R})$, $G \in C^1(\mathbb{R}^n,\mathbb{R})$, and $b \in L^1([0,T])\backslash\{0\}$ that is nonnegative a.e. In [6], by using a three critical points theorem due to Bonanno and Marano, the existence of at least three solutions to a quasilinear second order differential equation was discussed. Using the symmetric mountain pass theorem and genus properties of critical point theory, Shen and Liu [17] investigated the existence of infinitely many solutions to the second-order quasilinear periodic boundary value problem with impulsive effects

$$\begin{cases} -u(t)'' + b(t)u(t) - (|u(t)|^2)''u(t) = f(t,u), & t \in J, \\ \Delta(u'(t_j)) = I_j(u(t_j)), & j = 1,2,\ldots,m, \\ u(T) = u(0), \; u'(T) = u'(0), \end{cases}$$

where $b \in L^\infty(0,T;\mathbb{R})$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Using variational methods, Heidarkhani and Moradi [7] discussed the existence of at least one weak solution and infinitely many weak solutions to $(P^{f,g}_{\lambda,\mu})$ with $\mu = 0$ and $I_j \equiv 0$ for $j = 1,2,\ldots,m$.

Motivated by the above studies, in this paper, we establish new criteria to guarantee that the problem $(P^{f,g}_{\lambda,\mu})$ has at least three classical solutions for appropriate values of the parameters λ and μ. It is worth stressing that we only assume g to be a L^1-Carathéodory function which permits us to use variational methods. In addition, we obtain multiplicity results for two cases: (i) if the nonlinearity f is asymptotically quadratic, and (ii) if it is subquadratic as $|u| \to \infty$. Our approach is based on variational methods and a three critical points theorem due to Ricceri [14].

The remainder of this paper is organized as follows. Section 2 contains some preliminary lemmas, and Section 2.1 contains our main results and their proofs.

2 Preliminaries

Our main tool is a theorem of Ricceri [14, Theorem 2] which is recalled in Lemma 2.1 below. In what follows, we let X be a real Banach space, and as in [14], we let $\mathcal{W}_X$ denote the class of all functionals $\Phi : X \to \mathbb{R}$ having the property: If $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ with $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u. For example, if X is uniformly convex and $g : [0, \infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Lemma 2.1. *Let X be a separable and reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1-functional belonging to $\mathcal{W}_X$ that is bounded on bounded subsets of X and whose derivative admits a continuous inverse on X^*. Let $J : X \to \mathbb{R}$ be a C^1-functional with a compact derivative and assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, set*

$$\rho = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\}, \quad \sigma = \sup_{u\in\Phi^{-1}((0,\infty))} \frac{J(u)}{\Phi(u)},$$

and assume that $\rho < \sigma$. Then for each compact interval $[c, d] \subset (1/\sigma, 1/\rho)$ (with the conventions that $1/0 = \infty$ and $1/\infty = 0$), there exists $R > 0$ with the property: for every $\lambda \in [c, d]$ and every C^1-functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\gamma > 0$ such that for each $\mu \in [0, \gamma]$, the equation

$$\Phi'(u) = \lambda J'(u) + \mu\Psi'(u)$$

has at least three solutions in X whose norms are less than R.

We refer the reader to the papers [5, 8–10, 18, 19] in which Lemma 2.1 was successfully used to ensure the existence of at least three solutions to boundary value problems.

The following two results of Ricceri are taken from [15, Theorem 1] and [16, Proposition 3.1], respectively.

Lemma 2.2. *Let X be a reflexive real Banach space, $I \subseteq \mathbb{R}$ be an interval, and let $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous C^1 functional that is bounded on bounded subsets of X, and whose derivative admits a continuous inverse on X^*. Let $J : X \to \mathbb{R}$ be a functional with a compact derivative and assume that*

$$\lim_{\|x\|\to\infty} (\Phi(x) - \lambda J(x)) = \infty, \quad \textit{for all } \lambda \in I,$$

and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\in I} \inf_{x\in X} (\Phi(x) - \lambda(\rho - J(x))) < \inf_{x\in X} \sup_{\lambda\in I} (\Phi(x) - \lambda(\rho - J(x))).$$

Then there exist a nonempty open set $A \subseteq I$ and a positive number R with the property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) - \lambda J'(u) - \mu \Psi'(u) = 0$$

has at least three solutions in X whose norms are less than R.

Lemma 2.3. *Let X be a nonempty set and let Φ and Ψ be two real functions on X. Assume that there exist $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = J(x_0) = 0, \quad \Phi(x_1) > r, \quad \text{and} \quad \sup_{x \in \Phi^{-1}(-\infty, r]} J(x) < r\frac{J(x_1)}{\Phi(x_1)}.$$

Then for each ρ satisfying

$$\sup_{x \in \Phi^{-1}(-\infty, r]} J(x) < \rho < r\frac{J(x_1)}{\Phi(x_1)},$$

we have

$$\sup_{\lambda \geq 0} \inf_{x \in X} (\Phi(x) - \lambda(\rho - J(x))) < \inf_{x \in X} \sup_{\lambda \geq 0} (\Phi(x) - \lambda(\rho - J(x))).$$

We refer the reader to the paper of Sun *et al.* [18] in which Lemma 2.2 was successfully employed to ensure the existence of at least three solutions to boundary value problems.

To construct an appropriate function space and apply critical point theory, we introduce the following notations and results to be used in the proofs of our main results.

Let us define the Banach space E by

$$E = \left\{u : [0, 1] \to \mathbb{R} \,|\, u \text{ is absolutely continuous, } u(1) = u(0),\ u' \in L^2([0, 1]\right\},$$

equipped with the norm

$$\|u\| = \left(\int_0^1 \left(|u'|^2 + |u|^2\right) dt \right)^{\frac{1}{2}}.$$

Clearly, E is a Hilbert space with the dual space E^*.

For every $u \in E$, we define

$$\Phi(u) = \int_0^1 h(u'(t))dt + \frac{1}{2}\int_0^1 \alpha(t)|u(t)|^2 dt + \sum_{j=1}^{m} \int_0^{u(t_j)} I_j(\zeta) \mathrm{d}\zeta, \tag{2.1}$$

$$J(u) = \int_0^1 F(t, u(t))dt, \tag{2.2}$$

and

$$\Psi(u) = \int_0^1 G(t, u(t))dt, \tag{2.3}$$

where

$$F(t, x) = \int_0^x f(t, s)ds \quad \text{and} \quad G(t, x) = \int_0^x g(t, s)ds \quad \text{for all } x \in \mathbb{R}.$$

Standard arguments show that $I_\lambda := \Phi - \mu\Psi - \lambda J$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in E$ is given by

$$\begin{aligned}(\Phi' - \mu\Psi' - \lambda J')(u)(v) = &\int_0^1 h'(u'(t))v'(t)dt + \int_0^1 \alpha(t)u(t)v(t)dt \\ &+ \sum_{j=1}^{m} I_j(u(t_j))v(t_j) - \lambda \int_0^1 f(t, u(t))v(t)dt \\ &- \mu \int_0^1 g(t, u(t))v(t)dt, \quad \text{for all } v \in E.\end{aligned}$$

Furthermore, from the definition of Φ, we see that it is sequentially weakly lower semicontinuous.

Definition 2.4. By a weak solution of the problem ($P^{f,g}_{\lambda,\mu}$), we mean a function $u \in E$ such that

$$\int_0^1 h'(u'(t))v'(t)dt + \int_0^1 \alpha(t)u(t)v(t)dt + \sum_{j=1}^{m} I_j(u(t_j))v(t_j) - \lambda \int_0^1 f(t,u(t))v(t)dt - \mu \int_0^1 g(t,u(t))v(t)dt = 0,$$

for every $v \in E$.

By a classical solution of the problem ($P^{f,g}_{\lambda,\mu}$), we mean a function $u \in E$ such that $u(t)$ satisfies the equation in ($P^{f,g}_{\lambda,\mu}$) for a.e. $t \in [0,1] \setminus \{t_1, \dots, t_m\}$ and both the impulse condition and the boundary condition in ($P^{f,g}_{\lambda,\mu}$) hold.

Clearly, a critical point $u \in E$ of the functional I_λ is a weak solution of the problem ($P^{f,g}_{\lambda,\mu}$). Next , we show that u is indeed a classical solution.

Lemma 2.5. *If $u \in E$ is a critical point of I_λ, then u is a classical solution of* ($P^{f,g}_{\lambda,\mu}$).

Proof. Let $u \in E$ be a critical point for I_λ. Then, for any $v \in E$, it follows that

$$\begin{aligned} 0 &= \int_0^1 h'(u'(t))v'(t)dt + \int_0^1 \alpha(t)u(t)v(t)dt + \sum_{j=1}^{m} I_j(u(t_j))v(t_j) \\ &\quad - \lambda \int_0^1 f(t,u(t))v(t)dt - \mu \int_0^1 g(t,u(t))v(t)dt \\ &= \sum_{j=0}^{m+1} h'(u'(t))v(t)\Big|_{t=t_j^+}^{t_{j+1}^-} + \sum_{j=1}^{m} I_j(u(t_j))v(t_j) \\ &\quad - \int_0^1 [(h'(u'(t)))' - \alpha(t)u(t) + \lambda f(t,u(t)) + \mu g(t,u(t))]v(t)dt \\ &= \sum_{j=1}^{m} [-\Delta(h'(u'(t_j))) + I_j(u(t_j))]v(t_j) + h'(u'(1))v(1) - h'(u'(0))v(0) \\ &\quad - \int_0^1 [(h'(u'(t)))' - \alpha(t)u(t) + \lambda f(t,u(t)) + \mu g(t,u(t))]v(t)dt. \end{aligned}$$

That is, we have

$$\sum_{j=1}^{m} [-\Delta(h'(u'(t_j))) + I_j(u(t_j))]v(t_j) + h'(u'(1))v(1) - h'(u'(0))v(0) - \int_0^1 [(h'(u'(t)))' - \alpha(t)u(t) + \lambda f(t,u(t)) + \mu g(t,u(t))]v(t)dt = 0 \quad \text{for all } v \in E. \tag{2.4}$$

Without loss of generality, we assume that $v \in C_0^\infty(t_j, t_{j+1})$ and $v(t) = 0$ for $t \in [0, t_j] \cup [t_{j+1}, 1]$. Then, substituting into (2.4) gives

$$(h'(u'(t)))' - \alpha(t)u(t) + \lambda f(t,u(t)) + \mu g(t,u(t)) = 0 \quad \text{a.e. } t \in (t_j, t_{j+1}).$$

Thus, in view of the fact that $(h'(u'(t)))' = p(u'(t))u''(t)$, we see that u satisfies the equation in $(P^{f,g}_{\lambda,\mu})$. Now, by (2.4), we have

$$\sum_{j=1}^{m}[-\Delta(h'(u'(t_j))) + I_j(u(t_j))]v(t_j) + h'(u'(1))v(1) - h'(u'(0))v(0) = 0 \tag{2.5}$$

for all $v \in E$. Next we shall show that u satisfies the impulsive condition in $(P^{f,g}_{\lambda,\mu})$. If this is not the case, without loss of generality, we assume that there exists $j \in \{1, \dots, m\}$ such that

$$-\Delta(h'(u'(t_j))) + I_j(u(t_j)) \neq 0.$$

Let

$$v(t) = \prod_{i=0,\, i\neq j}^{m+1}(t - t_i).$$

Then,

$$\begin{aligned}
&\sum_{k=1}^{m}[-\Delta(h'(u'(t_k))) + I_k(u(t_k))]v(t_k) + h'(u'(1))v(1) - h'(u'(0))v(0)\\
&\quad= \sum_{k=1}^{m}[-\Delta(h'(u'(t_k))) + I_k(u(t_k))]\prod_{i=0,\, i\neq j}^{m+1}(t_k - t_i)\\
&\qquad + h'(u'(1))\prod_{i=0,\, i\neq j}^{m+1}(t_{m+1} - t_i) - h'(u'(0))\prod_{i=0,\, i\neq j}^{m+1}(t_0 - t_i)\\
&\quad= [-\Delta(h'(u'(t_j))) + I_j(u(t_j))]\prod_{i=0,\, i\neq j}^{m+1}(t_k - t_i) \neq 0,
\end{aligned}$$

which contradicts (2.5). Thus, u satisfies the impulse condition in $(P^{f,g}_{\lambda,\mu})$. Similarly, we can show that u satisfies the boundary condition in $(P^{f,g}_{\lambda,\mu})$. Therefore, u is a solution of $(P^{f,g}_{\lambda,\mu})$. □

We will also need the following lemma in the proof of our main result.

Lemma 2.6. *Let $S : E \to E^*$ be the operator defined by*

$$S(u)(v) = \int_0^1 h'(u'(t))v'(t)dt + \int_0^1 \alpha(t)u(t)v(t)dt + \sum_{j=1}^{m} I_j(u(t_j))v(t_j)$$

for every $u, v \in E$. Then S admits a continuous inverse on E^*.

Proof. For any $u \in E$, from conditions (Q_1)–(Q_3), it follows that

$$\begin{aligned}
S(u)(u) &= \int_0^1 h'(u'(t))u'(t)dt + \int_0^1 \alpha(t)|u(t)|^2dt + \sum_{j=1}^{m} I_j(u(t_j))u_i(t_j)\\
&\geq \min\{m, \alpha_0\}\|u\|^2,
\end{aligned}$$

which implies that S is coercive. Now, for any $u, v \in E$, we have

$$\begin{aligned}\langle S(u)-S(v), u-v\rangle &= \int_0^1 (h'(u'(t))-h'(v'(t)))(u'(t)-v'(t))dt \\ &+\int_0^1 \alpha(t)(u(t)-v(t))^2 dt \\ &+\sum_{j=1}^{m}\left(I_j(u(t))-I_j(v(t))\right)(u(t)-v(t)) \\ &\geq \min\{m,\alpha_0\}\|u-v\|^2.\end{aligned}$$

Thus, S is strongly monotone. Moreover, since E is reflexive, if $u_n \to u$ strongly in E as $n \to \infty$, it can be shown that $S(u_n) \to S(u)$ weakly in E^* as $n \to \infty$. Hence, S is demicontinuous. By [21, Theorem 26.A(d)], the inverse operator S^{-1} of S exists and is continuous. □

2.1 Main result

In this section, we state and prove our main results. Let

$$\lambda_1 = \inf_{u\in E\setminus\{0\}}\left\{\frac{\int_0^1 h(u'(t))dt+\frac{1}{2}\int_0^1 \alpha(t)|u(t)|^2 dt+\sum_{j=1}^{m}\int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta}{\int_0^1 F(t,u(t))\mathrm{d}t} : \int_0^1 F(t,u(t))\mathrm{d}t>0\right\}$$

and

$$\lambda_2 = \frac{1}{\max\{0,\ \lambda_0,\ \lambda_\infty\}},$$

where

$$\lambda_0 = \limsup_{u\to 0}\frac{\int_0^1 F(t,u(t))\mathrm{d}t}{\int_0^1 h(u'(t))dt+\frac{1}{2}\int_0^1 \alpha(t)|u(t)|^2 dt+\sum_{j=1}^{m}\int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta}$$

and

$$\lambda_\infty = \limsup_{\|u\|\to \infty}\frac{\int_0^1 F(t,u(t))\mathrm{d}t}{\int_0^1 h(u'(t))dt+\frac{1}{2}\int_0^1 \alpha(t)|u(t)|^2 dt+\sum_{j=1}^{m}\int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta}.$$

Theorem 2.7. *Assume that*

($\mathcal{A}_1$) *there exists a constant $\varepsilon > 0$ such that*

$$\max\left\{\limsup_{u\to 0}\frac{\max_{t\in[0,1]} F(t,u)}{|u|^2},\ \limsup_{|u|\to\infty}\frac{\max_{t\in[0,1]} F(t,u)}{|u|^2}\right\} < \varepsilon;$$

($\mathcal{A}_2$) *there exists a function $w \in E$ such that*

$$\int_0^1 h(w'(t))dt+\frac{1}{2}\int_0^1 \alpha(t)|w(t)|^2 dt+\sum_{j=1}^{m}\int_0^{w(t_j)} I_j(\zeta)\mathrm{d}\zeta \neq 0$$

and

$$\frac{8\varepsilon}{\min\{m,\alpha_0\}} < \frac{\int_0^1 F(t,w(t))\mathrm{d}t}{\int_0^1 h(w'(t))dt+\frac{1}{2}\int_0^1 \alpha(t)|w(t)|^2 dt+\sum_{j=1}^{m}\int_0^{w(t_j)} I_j(\zeta)\mathrm{d}\zeta}.$$

Then for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ such that for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for every $\mu \in [0,\gamma]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Remark 2.8. Under conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, it is true that $\lambda_1 < \lambda_2$ as is shown in the proof of Theorem 2.7 given below.

Proof of Theorem 2.7. Our aim is to apply Lemma 2.1 to the problem $(P^{f,g}_{\lambda,\mu})$. Take $X = E$; clearly, X is a separable and uniformly convex Banach space. From [13, Proposition 1.1] and its proof with $T = 1$ and $p = q = 2$, we have

$$\max_{t\in[0,1]} |u(t)| \le 2\|u\| \quad \text{for all } u \in X. \tag{2.6}$$

Let the functionals Φ, J, and Ψ be as given in (2.1)–(2.3). The functional Φ is C^1, and by Lemma 2.6, its derivative admits a continuous inverse on X^*. Moreover, Φ is sequentially weakly lower semicontinuous since Φ' is monotone (see the proof of Lemma 2.6). Since

$$\int_0^1 h(u'(t))dt = \int_0^1 \left(\int_0^{u'(t)} \left(\int_0^{\tau} p(\xi)d\xi \right) d\tau \right) dt,$$

from (1.1) and (1.2), it follows that

$$\begin{aligned}
\frac{1}{2}\min\{m,\alpha_0\}\|u\|^2 &\le \frac{m}{2}\int_0^1 |u'(t)|^2dt + \frac{\alpha_0}{2}\int_0^1 |u(t)|^2dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta \\
&\le \Phi(u) \\
&\le \frac{M}{2}\int_0^1 |u'(t)|^2dt + \frac{\alpha_1}{2}\int_0^1 |u(t)|^2dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta \\
&\le \frac{1}{2}\max\{M,\alpha_1\}\|u\|^2 + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta
\end{aligned} \tag{2.7}$$

for every $u \in X$. We then have

$$\lim_{\|u\|\to+\infty} \Phi(u) = \infty,$$

i.e., Φ is coercive. Now, let A be a bounded subset of X. Then there exist a constant $c > 0$ such that $\|u\| \le c$ for all $u \in A$. From (2.6), $\max_{t\in[0,1]} |u(t)| \le 2c$ for all $u \in A$. Thus, by the continuity of each I_j, we see that there exists $K > 0$ such that $\left|\sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta\right| < K$ for all $u \in A$. Then, by (2.7), we have

$$\Phi(u) \le \frac{1}{2}\max\{M,\alpha_1\}\|c\|^2 + K,$$

so Φ is bounded on each bounded subset of X.

To prove that $\Phi \in \mathcal{W}_X$, define

$$\Phi_1(u) = \int_0^1 h(u'(t))dt \quad \text{and} \quad \Phi_2(u) = \frac{1}{2}\int_0^1 \alpha(t)|u(t)|^2dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta$$

for all $u \in X$. Then,

$$\Phi(u) = \Phi_1(u) + \Phi_2(u) \quad \text{for all } u \in X.$$

As in (2.7), we have

$$\Phi_1(u) \geq d(u) := \frac{m}{2}\int_0^1 |u'(t)|^2 dt \quad \text{for all } u \in X. \tag{2.8}$$

Let $\{u_k\}$ be a sequence in X and let $u \in X$ be such that $u_k \rightharpoonup u$ and $\liminf_{k\to\infty} \Phi(u_k) \leq \Phi(u)$. To show that $\{u_k\}$ has a subsequence strongly converging to u, assume, to the contrary, that $\{u_k\}$ does not have such a subsequence. Then, there exist $\epsilon > 0$ and a subsequence $\{u_{k_n}\}$ of $\{u_k\}$ such that

$$\left\| \frac{u_{k_n} - u}{2} \right\| \geq \epsilon \quad \text{for all } n \in \mathbb{N}.$$

Note that $\{u_{k_n}\}$ converges uniformly to u by [13, Proposition 1.2]. Then, in view of the definition of $\|\cdot\|$, there exists $\epsilon_1 > 0$ such that

$$d\left(\frac{u_{k_n} - u}{2}\right) \geq \epsilon_1 \quad \text{for all } n \in \mathbb{N}.$$

Thus, from (2.8)

$$\Phi_1\left(\frac{u_{k_n} - u}{2}\right) \geq \epsilon_1 \quad \text{for all } n \in \mathbb{N}. \tag{2.9}$$

Now, the sequentially weakly lower semicontinuity of Φ implies that $\liminf_{n\to\infty} \Phi(u_{k_n}) = \Phi(u)$. Hence, there exists a subsequence $\{w_\ell\} = \{u_{k_{n_\ell}}\}$ of $\{u_{k_n}\}$ such that

$$\lim_{\ell\to\infty} \Phi(w_\ell) = \Phi(u).$$

Since $\{w_\ell\}$ converges uniformly to u and I_j, $j = 1\ldots, m$, are continuous, we see that

$$\lim_{\ell\to\infty} \Phi_2(w_\ell) = \Phi_2(u),$$

and so

$$\lim_{\ell\to\infty} \Phi_1(w_\ell) = \Phi_1(u). \tag{2.10}$$

It is clear that Φ_1 is sequentially weakly lower semicontinuous and that $(w_\ell + u)/2 \rightharpoonup u$ as $\ell \to \infty$. Then,

$$\Phi_1(u) \leq \liminf_{\ell\to\infty} \Phi_1\left(\frac{w_\ell + u}{2}\right). \tag{2.11}$$

By simple calculations and the nondecreasing nature of p, we have that for $y > 0$,

$$\begin{aligned} h''(\sqrt{y}) &= \frac{1}{4}y^{-1}p(\sqrt{y}) - \frac{1}{4}y^{-3/2}\int_0^{\sqrt{y}} p(\xi)d\xi \\ &\geq \frac{1}{4}y^{-1}p(\sqrt{y}) - \frac{1}{4}y^{-3/2}\sqrt{y}p(\sqrt{y}) = 0. \end{aligned}$$

Hence, $h(\sqrt{y})$ is convex. Moreover, $h(y)$ is continuous, strictly increasing for $y \geq 0$, and $h(0) = 0$. Thus, from [11, Theorem 2.1], we have

$$\frac{1}{2}\Phi_1(w_\ell) + \frac{1}{2}\Phi_1(u) \geq \Phi_1\left(\frac{w_\ell + u}{2}\right) + \Phi_1\left(\frac{w_\ell - u}{2}\right) \quad \text{for all } \ell \in \mathbb{N}.$$

Taking limit superior as $\ell \to \infty$ and using (2.8), (2.9), and (2.10) in the above inequality, we obtain

$$\Phi_1(u) - \epsilon_1 \geq \limsup_{\ell\to\infty} \Phi_1\left(\frac{w_\ell + u}{2}\right),$$

which contradicts (2.11). This shows that $\{u_k\}$ has a subsequence converging strongly to u. Therefore, $\Phi \in \mathcal{W}_X$.

The functionals J and Ψ are C^1-functionals with compact derivatives. Moreover, Φ has a strict local minimum 0 with $\Phi(0) = J(0) = 0$. Therefore, the regularity assumptions on Φ, J, and Ψ, as required in Lemma 2.1, are satisfied. In view of $(\mathcal{A}_1)$, there exist τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(t,u) \le \varepsilon |u|^2 \tag{2.12}$$

for every $t \in [0,1]$ and every u with $|u| \in [0,\tau_1) \cup (\tau_2, \infty)$. Since $F(t,u)$ is continuous on $[0,1] \times \mathbb{R}$, $F(t,u)$ is bounded on $[0,1] \times [\tau_1, \tau_2]$. Thus, we can choose $\eta > 0$ and $v > 2$ such that

$$F(t,u) \le \varepsilon |u|^2 + \eta |u|^v$$

for all $(t,u) \in [0,1] \times \mathbb{R}$. Then, from (2.6), we have

$$J(u) \le 4\varepsilon \|u\|^2 + \eta 2^v \|u\|^v \tag{2.13}$$

for all $u \in X$. Hence, from (2.7) and (2.13), we have

$$\limsup_{|u| \to 0} \frac{J(u)}{\Phi(u)} \le \frac{8\varepsilon}{\min\{m, \alpha_0\}}. \tag{2.14}$$

Moreover, by (2.12), for each $u \in X \setminus \{0\}$,

$$\begin{aligned} \frac{J(u)}{\Phi(u)} &= \frac{\int_{|u| \le \tau_2} F(t,u(t))\mathrm{d}t}{\Phi(u)} + \frac{\int_{|u| > \tau_2} F(t,u(t))dt}{\Phi(u)} \\ &\le \frac{\sup_{t \in [0,1], |u| \in [0,\tau_2]} F(t,u)}{\Phi(u)} + \frac{4\varepsilon \|u\|^2}{\Phi(u)} \\ &\le \frac{\sup_{t \in [0,1], |u| \in [0,\tau_2]} F(t,u)}{\frac{1}{2}\min\{m, \alpha_0\} \|u\|^2} + \frac{8\varepsilon}{\min\{m, \alpha_0\}}. \end{aligned}$$

Therefore,

$$\limsup_{\|u\| \to \infty} \frac{J(u)}{\Phi(u)} \le \frac{8\varepsilon}{\min\{m, \alpha_0\}}. \tag{2.15}$$

In view of (2.14) and (2.15), we have

$$\rho = \max\left\{0, \limsup_{\|u\| \to \infty} \frac{J(u)}{\Phi(u)}, \limsup_{u \to 0} \frac{J(u)}{\Phi(u)}\right\} \le \frac{8\varepsilon}{\min\{m, \alpha_0\}}. \tag{2.16}$$

Condition $(\mathcal{A}_2)$ together with (2.16) yield

$$\begin{aligned} \sigma &= \sup_{u \in \Phi^{-1}((0,\infty))} \frac{J(u)}{\Phi(u)} = \sup_{X \setminus \{0\}} \frac{J(u)}{\Phi(u)} \\ &\ge \frac{\int_0^1 F(t,w(t))dt}{\Phi(w(t))} = \frac{\int_0^1 F(t,w(t))dt}{\frac{1}{2}\max\{M, \alpha_1\}\|u\|^2 + \sum_{j=1}^m \int_0^{w(t_j)} I_j(\zeta)d\zeta} \\ &> \frac{8\varepsilon}{\min\{m, \alpha_0\}} \ge \rho. \end{aligned}$$

Thus, all the conditions of Lemma 2.1 are satisfied. With $\lambda_1 = 1/\sigma$ and $\lambda_2 = 1/\rho$, by Lemmas 2.1 and 2.5, for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ such that for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three classical solutions whose norms in X are less than R. □

The following result is another application of Lemma 2.1.

Theorem 2.9. *Assume that*

$$\max_{u\in E}\left\{\limsup_{u\to 0}\frac{\max_{t\in[0,1]}F(t,u)}{|u|^2},\ \limsup_{|u|\to\infty}\frac{\max_{t\in[0,1]}F(t,u)}{|u|^2}\right\}\le 0 \tag{2.17}$$

and

$$\sup_{u\in E}\frac{\int_0^1 F(t,u(t))\mathrm{d}t}{\int_0^1 h(u'(t))dt+\frac{1}{2}\int_0^1\alpha(t)|u(t)|^2dt+\sum_{j=1}^m\int_0^{u(t_j)}I_j(\zeta)d\zeta}>0. \tag{2.18}$$

Then for each compact interval $[c,d] \subset (\lambda_1, \infty)$, there exists $R > 0$ such that for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Proof. For any $\varepsilon > 0$, (2.17) implies that there exist τ_1 and τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(t,u) \le \varepsilon |u|^2$$

for every $t \in [0,1]$ and every u with $|u| \in [0,\tau_1) \cup (\tau_2, \infty)$. Since $F(t,u)$ is continuous on $[0,1] \times \mathbb{R}$, $F(t,u)$ is bounded on $[0,1] \times [\tau_1, \tau_2]$. Thus, as before, we can choose $\eta > 0$ and $v > 2$ so that

$$F(t,u) \le \varepsilon|u|^2 + \eta|u|^v$$

for all $(t,u) \in [0,1] \times \mathbb{R}$. Then, by the same process as in the proof of Theorem 2.7, we obtain (2.14) and (2.15). Since ϵ is arbitrary, (2.14) and (2.15) give

$$\max\left\{0,\ \limsup_{\|u\|\to+\infty}\frac{J(u)}{\Phi(u)},\ \limsup_{u\to 0}\frac{J(u)}{\Phi(u)}\right\}\le 0.$$

Then, with ρ and σ defined as in Lemma 2.1, we have $\rho = 0$, and by (2.18), we have $\sigma > 0$. In this case, $\lambda_1 = 1/\sigma$ and $\lambda_2 = \infty$. Thus, by Lemma 2.1 the theorem is proved. □

Remark 2.10. In condition $(\mathcal{A}_2)$ of Theorem 2.7, if we choose

$$w_0(t)=\begin{cases}\sigma, & t\in[0,1/4],\\ 2\sigma t+\sigma/2, & t\in[1/4,1/2],\\ -2\sigma t+5\sigma/2, & t\in[1/2,3/4],\\ \sigma, & t\in[3/4,1],\end{cases} \tag{2.19}$$

where $\sigma > 0$, then $w_0 \in E$, and $(\mathcal{A}_2)$ now takes the form

$(\widehat{\mathcal{A}_2})$ there exists a positive constant σ such that

$$\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta \neq 0$$

and

$$\frac{8\varepsilon}{\min\{m,\alpha_0\}}<\frac{\int_0^1 F(t,w_0(t))dt}{\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta}.$$

Next, we point out some results in which the function f is separable. To be precise, we consider the problem

$$\begin{cases} -p(x')x''+\alpha(t)x=\lambda\theta(t)f(x)+\mu g(t,x), & t\neq t_j, \quad \text{a.e. } t\in[0,1],\\ \Delta(h'(u'(t_j)))=I_j(u(t_j)), \quad j=1,2,\dots,m, \\ x(1)-x(0)=x'(1)-x'(0)=0, \end{cases} \qquad (\phi^{\theta}_{\lambda,\mu})$$

where $\theta:[0,1]\to\mathbb{R}$ is a nonzero function with $\theta\in L^1([0,1])$, $f:\mathbb{R}\to\mathbb{R}$ is a continuous function, and $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is an L^1-Carathéodory function. Let $F(t,x)=\theta(t)F(x)$ for every $(t,x)\in[0,1]\times\mathbb{R}$, where

$$F(x)=\int_0^x f(\xi)d\xi \quad \text{for all } x\in\mathbb{R}.$$

The following existence results are then consequences of Theorem 2.7.

Theorem 2.11. *Assume that*

$(\mathcal{A}_3)$ *there exists a constant $\varepsilon>0$ such that*

$$\sup_{t\in[0,1]}\theta(t)\cdot\max\left\{\limsup_{u\to 0}\frac{F(u)}{|u|^2},\limsup_{|u|\to\infty}\frac{F(u)}{|u|^2}\right\}<\varepsilon;$$

$(\mathcal{A}_4)$ *there exists a positive constant σ such that*

$$\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta \neq 0$$

and

$$\frac{8\varepsilon}{\min\{m,\alpha_0\}}<\frac{f(w_0(t))\int_0^1 \theta(t)dt}{\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta},$$

where w_0 is defined by (2.19).

Then for each compact interval $[c,d]\subset(\lambda_3,\lambda_4)$, where λ_3 and λ_4 are the same as λ_1 and λ_2, but with $\int_0^1 F(t,u(t))dt$ replaced by $\int_0^1 \theta(t)F(u(t))dt$, there exists $R>0$ such that for every $\lambda\in[c,d]$ and every L^1-Carathéodory function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$, there exists $\gamma>0$ such that for each $\mu\in[0,\gamma]$, the problem $(\phi^{\theta}_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Theorem 2.12. *Assume that there exists a positive constant σ such that*

$$\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta>0$$

and

$$\int_0^1 \theta(t)F(w_0(t))dt>0, \tag{2.20}$$

where w_0 is given by (2.19). In addition, assume that

$$\limsup_{u\to 0}\frac{F(u)}{|u|^2}=\limsup_{|u|\to\infty}\frac{F(u)}{|u|^2}=0. \tag{2.21}$$

Then for each compact interval $[c,d]\subset(\lambda_3,\infty)$, where λ_3 is the same as λ_1 but with $\int_0^1 F(t,u(t))\mathrm{d}t$ replaced by $\int_0^1 \theta(t)F(u(t))\mathrm{d}t$, there exists $R>0$ such that for every $\lambda\in[c,d]$ and every L^1-Carathéodory function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$, there exists $\gamma>0$ such that for each $\mu\in[0,\gamma]$, the problem $(\phi^{\theta}_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Proof. From (2.21), we easily see that $(\mathcal{A}_3)$ is satisfied for every $\varepsilon>0$. Moreover, using (2.20), by choosing $\varepsilon>0$ small enough, $(\mathcal{A}_4)$ will hold. Hence, the conclusion of this theorem follows from Theorem 2.11. □

As an example in which the hypotheses of Theorem 2.12 are satisfied, we have the following.

Example 2.13. Let $p(x)=4-\cot(x)$ for each $x\in\mathbb{R}$, $\alpha(t)=\theta(t)=1$ for every $t\in[0,1]$, $m=1$, $t_1=1/5$, $I_1(x)=x^3$ for each $x\in\mathbb{R}$, and

$$f(x)=\begin{cases}4x^3, & |x|\le 1,\\ 4x, & 1<|x|\le 2,\\ 8, & |x|\ge 2.\end{cases}$$

Then, it is easy to check that

$$F(x)=\begin{cases}x^4, & |x|\le 1,\\ 2x^2-1, & 1<|x|\le 2,\\ 8x-9, & x>2,\\ 8x+23, & x<-2.\end{cases}$$

By choosing $\sigma=1$, $w_0(t)$ becomes

$$w_0(t)=\begin{cases}1, & t\in[0,1/4],\\ 2t+1/2, & t\in[1/4,1/2],\\ -2t+5/2, & t\in[1/2,3/4],\\ 1, & t\in[3/4,1].\end{cases}$$

It is trivial to verify that

$$\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta>0,$$

$$\int_0^1 \theta(t)F(w_0(t))dt > 0,$$

and

$$\lim_{u\to 0}\frac{F(u)}{|u|^2} = \lim_{|u|\to\infty}\frac{F(u)}{|u|^2} = 0.$$

Hence, by Theorem 2.12, for each compact interval $[c,d] \subset (0,\infty)$, there exists $R > 0$ such that for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : [0,1]\times\mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem

$$\begin{cases} -p(x')x'' + x = \lambda f(x) + \mu g(t,x), & t \neq \frac{1}{5}, \quad \text{a.e. } t \in [0,1],\\ \Delta(h'(u'(\frac{1}{5}))) = I_1(u(\frac{1}{5})), \\ x(1) - x(0) = x'(1) - x'(0) = 0, \end{cases}$$

has at least three classical solutions whose norms in E are less than R.

The following theorem is a consequences of Lemma 2.3.

Theorem 2.14. *Assume that there exist three positive constants $1 \le \zeta < 2$, θ, and σ, with*

$$\theta < \sqrt{\frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{31\alpha_0\sigma^2}{48}}, \tag{2.22}$$

such that

($\mathcal{B}_1$) *$f(t,x) \ge 0$ for every $(t,x) \in \big([0,1/4]\times[0,\sigma]\big) \cup \big([3/4,1]\times[0,\sigma]\big) \cup \big([1/4,3/4]\times[\sigma,3\sigma/2]\big)$;*

($\mathcal{B}_2$) $$\frac{\int_0^1 \sup_{|u|\le\theta} F(t,u)dt}{\theta^2} < \frac{\min\{m,\alpha_0\}}{8}\,\frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\sigma)dt}{\frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{31\alpha_1\sigma^2}{48} + \sum_{j=1}^m \int_0^{w_0(t_j)} I_j(\zeta)\mathrm{d}\zeta};$$

($\mathcal{B}_3$) *there exists $p > 0$ and a positive constant q such that*

$$|F(t,u)| \le p|u|^{\zeta} + q \quad \textit{for all } (t,u) \in [0,1]\times\mathbb{R};$$

($\mathcal{B}_4$) *there exists $l > 0$ and a positive constant $\varrho \in \mathbb{R}$ such that*

$$G(t,u) \le lu^{\zeta} + \varrho \quad \textit{for all } (t,u) \in [0,1]\times\mathbb{R}.$$

Then there exist a nonempty open set $A \subset [0,\infty)$ and a positive number $R > 0$ such that for every $\lambda \in A$ and every L^1-Carathéodory function $g : [0,1]\times\mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that for each $\mu \in [0,\delta]$, the problem ($P^{f,g}_{\lambda,\mu}$) has at least three classical solutions whose norms in E are less than R.

Proof. Since the embeddings $E \hookrightarrow L^q$ $(q \ge 1)$ and $E \hookrightarrow L^\infty$ are compact (see Adams and Fournier [1]), there exists a positive constant C such that

$$|u|_{L^q([0,1])} \le C\|u\|.$$

For any $\lambda \ge 0$ and $u \in E$, from (Q_3), $(\mathcal{B}_3)$, and $(\mathcal{B}_4)$, we have

$$\begin{aligned} \Phi(u) - \lambda\Psi(u) &\ge \frac{1}{2}\min\{m,\alpha_0\}\|u\|^2 - \lambda\int_0^1 [F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))]dt \\ &\ge \frac{1}{2}\min\{m,\alpha_0\}\|u\|^2 - \lambda\left(\int_0^1 p|u|^{\zeta}dt + q\right) - \mu\left(l\int_0^1 |u(t)|^{\zeta}dt + \varrho\right) \\ &\ge \frac{1}{2}\min\{m,\alpha_0\}\|u\|^2 - \lambda pC_0\|u\|^{\zeta} - \mu lC_1\|u\|^{\zeta} - \lambda q - \mu\varrho. \end{aligned}$$

Since $\zeta < 2$,

$$\lim_{\|u\|\to+\infty} \Phi(u) - \lambda\Psi(u) = \infty \quad \text{for all } \lambda > 0.$$

Let w_0 be defined by (2.19) with σ given in the theorem. Now, $\min_{t\in[\frac{1}{4},\frac{3}{4}]}\{w_0(t)\} = \sigma$ and $\max_{t\in[\frac{1}{4},\frac{3}{4}]}\{w_0(t)\} = \frac{3\sigma}{2}$, so

$$\begin{aligned} J(w_0) &= \int_0^{\frac{1}{4}}\int_0^{\sigma} f(t,\xi)d\xi dt + \int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^{w_0(t)} f(t,\xi)d\xi dt + \int_{\frac{3}{4}}^{1}\int_0^{\sigma} f(t,\xi)d\xi dt \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^{\sigma} f(t,\xi)d\xi dt = \int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\sigma)dt. \end{aligned}$$

Moreover, simple calculations show that

$$\begin{aligned} \Phi(w_0) &= \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt + \sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta \\ &\leq \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{\alpha_1}{2}\int_0^1 |w_0(t)|^2dt + \sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)\mathrm{d}\zeta \\ &= \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{31\alpha_1\sigma^2}{48} + \sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)\mathrm{d}\zeta \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} \Phi(w_0) &= \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt + \sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta \\ &\geq \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{\alpha_0}{2}\int_0^1 |w_0(t)|^2dt \\ &= \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{31\alpha_0\sigma^2}{48}. \end{aligned} \tag{2.24}$$

Let $r = \frac{\min\{m,\alpha_0\}}{8}\theta^2$. Then, from (2.22) and (2.24), we have $\Phi(w_0) > r$. From the definition of Φ, (2.6), and (2.7), it follows that

$$\begin{aligned} \Phi^{-1}(-\infty, r] &= \{x \in E : \ \Phi(x) \leq r\} \\ &\subseteq \left\{x \in E : \ \max_{t\in[0,1]} |x(t)| \leq \sqrt{\frac{8r}{\min\{m,\alpha_0\}}}\right\} \\ &\subseteq \left\{x \in E : \ \max_{t\in[0,1]} |x(t)| \leq \theta\right\}. \end{aligned}$$

Therefore,

$$\sup_{u\in\Phi^{-1}((-\infty,r])} J(u) \leq \int_0^1 \sup_{|u|\leq\theta} F(t,u)dt.$$

Thus, from $(\mathcal{B}_2)$ and (2.23), we have

$$r\frac{J(w_0)}{\Phi(w_0)} = \frac{r}{\Phi(w_0)}\left(\int_0^1 F(t,w_0(t))dt\right)$$
$$\geq \frac{\frac{\min\{m,\alpha_0\}}{8}\theta^2\left(\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\sigma)dt\right)}{\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{31\alpha_1\sigma^2}{48}+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta}$$
$$> \int_0^1 \sup_{|u|\leq\theta} F(t,u)dt \geq \sup_{u\in\Phi^{-1}((-\infty,r])} J(u).$$

We can then fix ρ so that

$$\sup_{u\in\Phi^{-1}((-\infty,r])} J(u) < \rho < r\frac{J(w_0)}{\Phi(w_0)}.$$

From Lemma 2.3, we obtain

$$\sup_{\lambda\geq 0}\inf_{u\in E}(\Phi(u)-\lambda(\rho-J(u))) < \inf_{u\in E}\sup_{\lambda\geq 0}(\Phi(u)-\lambda(\rho-J(u))).$$

Hence, by Lemma 2.2, for each compact interval $[c,d]\subset(\lambda_1,\lambda_2)$, there exists $R>0$ such that for every $\lambda\in[c,d]$, and every L^1-Carathéodory function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, $\Phi'(u)-\lambda J'(u)-\mu\Psi'(u)=0$ has at least three solutions in E. Hence, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three classical solutions whose norms are less than R. □

2.2 Results and discussion

In this paper we investigate the existence of multiple solutions to a quasilinear periodic boundary value problem with impulsive effects. The main technique of proof involves variational methods and critical points theorems for smooth functionals. We obtain the existence of at least three solutions to the problem. The applicability of the results are illustrated by an example.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that there are no conflicts of interest in this paper.

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Radial solutions and a local bifurcation result for a singular elliptic problem with Neumann condition

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Abstract. We study the problem $-\Delta u = \lambda u - u^{-1}$ with a Neumann boundary condition; the peculiarity being the presence of the singular term $-u^{-1}$. We point out that the minus sign in front of the negative power of u is particularly challenging, since no convexity argument can be invoked. Using bifurcation techniques we are able to prove the existence of solution (u_λ, λ) with u_λ approaching the trivial constant solution $u = \lambda^{-1/2}$ and λ close to an eigenvalue of a suitable linearized problem. To achieve this we also need to prove a generalization of a classical two-branch bifurcation result for potential operators. Next we study the radial case and show that in this case one of the bifurcation branches is global and we find the asymptotical behavior of such a branch. This results allows to derive the existence of multiple solutions u with λ fixed.

Keywords: singular elliptic equation, positive solutions, radial solutions, variational bifurcation, two branches.

2020 Mathematics Subject Classification: 34B16, 35J20, 58E07.

1 Introduction

In the last decades several authors have studied semilinear elliptic problems with singular nonlinear term (with respect to the unknown function u). The model problem is the following:

$$\begin{cases} -\Delta u = \gamma u^{-q} + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $q > 0$ $\gamma \neq 0$ and f is a non linear term with standard growth conditions. Existence and multiplicity of solutions to problem (1.1) are usually investigated in terms of the behavior of f and the sign of γ.

A main aspect to be taken into account is the variational nature of (1.1): formally speaking solutions u of (1.1) are expected to be critical points of the functional

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\gamma}{1-q}\int_\Omega u^{1-q}\,dx - \int_\Omega F(x,u)\,dx,$$

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defined on $W_0^{1,2}(\Omega)$ and restricted to $\{u \geq 0\}$, where $F(x,s)$ is a primitive in s of $f(x,s)$ (if $q = 1$ a logarithm should be introduced). Unfortunately the presence of the singular term makes it problematic to give a rigorous formulation of the above ideas.

The majority of the known results concern the case $\gamma > 0$, where the term $u \mapsto -\frac{\gamma}{1-q}u^{1-q}$ is convex in the interval $]0, +\infty[$. This fact helps a lot, whether one tries to directly deal with I (by using some nonsmooth-critical-point theory) or to use an approximation scheme (by a sequence $I_n \to I$, I_n being $\mathcal{C}^1$ on $W_0^{1,2}(\Omega)$). For instance, if $f = 0$, the problem has a unique weak solution $\bar{u}$ in $W_0^{1,2}(\Omega)$ when $0 < q < 3$ and the solution is a minimizer for I. This result can be extended for all $q > 0$ dropping the request that $\bar{u} \in W_0^{1,2}(\Omega)$ (see [5, 9]). For a small non exhaustive list of multiplicity results for solutions of this kind of problems see [1, 3, 7, 11, 13–15, 17, 18, 26, 27] (and the references therein).

If we turn to $\gamma < 0$ the literature is scarcer: to the author's knowledge the main results are contained in [6, 12, 22, 27, 28]. In this situation solutions are "attracted" from the value zero and tend to develop "dead cores", so the formulation (1.1) needs to be modified in order to admit non strictly positive solutions. For instance the only solution for the case $f = 0$ is $u = 0$ (as one can easily see by multiplying the equation by u). Moreover a direct variational approach using the functional I seems difficult for the moment and the usual approach goes by perturbation methods.

We have found particularly interesting the paper [22] by Montenegro and Silva, where the authors use perturbation methods and show that there exist two nontrivial solutions when $\gamma = -1$, $0 < q < 1$, $f(x,u) = \mu u^p$, with $q < p <$ and $\mu > 0$ big enough. If we pass to $q = 1$, simple tests in the radial case suggest that the Dirichlet problem only has the trivial solution. As we said before solutions starting from zero are "forced to stick" at zero and not allowed to "emerge" (in contrast with the case of $q < 1$). On this respect see Remark 4.9.

For this reasons, in the case $q = 1$, we are lead to replace the Dirichlet condition with a Neumann one. In particular we have considered the problem

$$\begin{cases} -\Delta u = \lambda u - \dfrac{1}{u} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \nabla u \cdot \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where Ω is a bounded smooth open subset of $\mathbb{R}^N$ and ν denotes the unit normal defined on $\partial\Omega$. This corresponds to the problem of [22] with $q = p = 1$ (with Neumann condition).

In case $N = 1$ (1.2) is closely related to a problem studied by Del Pino, Manásevich, and Montero in 1992 (see [10]) who deal with an ODE, in the periodic case, with a more general, non autonomous, singular term $f(u,x)$ (singular in u and T-periodic in x). Using topological degree arguments they prove for instance that the equation:

$$-\ddot{u} = \lambda u - \frac{1}{u^\alpha}, \qquad u(x) > 0, \qquad u(x+T) = u(x),$$

where $\alpha \geq 1$, has a solution provided $\lambda \neq \frac{\mu_k}{4}$ for all k. Here μ_k denote the eigenvalues of a suitable linearized problem which arises in a natural way from the problem. In this case, which has a variational structure, the results of [10] can be derived from the existence of two global bifurcation branches which originate from trivial solutions of the linearized problem.

In this paper we present two types of results concerning problem (1.2). In Theorem (2.1) of Section 2 we prove the existence of two local bifurcation branches $(u_{1,\rho}, \lambda_{1,\rho})$ and $(u_{2,\rho}, \lambda_{2,\rho})$ of solutions of (1.2), such that $(u_{i,\rho}, \lambda_{1,\rho}) \to (\hat{u}, \hat{\lambda})$, as $\rho \to 0$, where $\hat{\lambda}/2$ is an eigenvalue of

$-\Delta$ with Neumann condition and $\hat{u}$ is the constant function: $\hat{u} \equiv \hat{\lambda}^{-1/2}$. The proof of (2.1) heavily relies on a variant of the well known abstract results on the existence of two bifurcation branches in the variational case (see [4, 19, 20, 25]). To the author surprise such a variant (see Theorem (3.1) seems not to be present in the literature so its proof is carried on in Section 3. It has to be said that proving (3.1) requires some additional technicalities compared to the standard version. Indeed in [4, 20] the proof goes by studying a suitable perturbed function f_ρ on the unit sphere S, while in our case S has to be replaced by a sphere-like set S_ρ also varying with ρ. This requires to construct suitable projections to show that all S_ρ's are homeomorphic to $S_0 = S$ (for ρ small). Apart from this the proof of (3.1) follows the ideas of [2].

In Section 4 we study the radial case in dimension $N = 2$ (the same could be probably done for $N \geq 3$) using ODE techniques and a continuation argument for the nodal regions of the solutions. In this way, following the ideas of [24], we are able to prove that one of the two branches (u_ρ, λ_ρ) is global and bounded in λ_ρ. This is done by proving that nodal regions of u_ρ cannot collapse along the branch and that $\lambda_\rho \to \bar{\lambda}$ as $\|u_\rho\| \to +\infty$, where $\bar{\rho}$ is an eigenvalue of another suitable linear problem. In this way – in the radial case – we can find a lower estimate in the number of solutions for a fixed λ, by counting the number of branches that cross λ.

2 A local bifurcation result for the singular problem

Let Ω be a bounded open subset of $\mathbb{R}^N$ with smooth boundary.

Theorem 2.1. *Let $\hat{\mu} > 0$ be an eigenvalue of the following Neumann problem:*

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \nabla u \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

(ν denotes the normal to $\partial\Omega$).

Then there exists $\rho_0 > 0$ such that for all $\rho \in \,]0, \rho_0[$ there exist two distinct pairs $(u_{1,\rho}, \lambda_{1,\rho})$ and $(u_{2,\rho}, \lambda_{2,\rho})$ such that, for $i = 1, 2$:

$$(u_{i,\rho}, \lambda_{i,\rho}) \text{ are solutions of (1.2)}, \; u_{i,\rho} \to \frac{1}{\sqrt{\mu/2}} \quad (\text{in } W^{1,2}(\Omega)\,)\,, \; \lambda_{i,\rho} \xrightarrow{\rho \to 0} \frac{\hat{\mu}}{2}.$$

Proof. We start by introducing some changes of variables. First of all notice that, for all $\lambda > 0$, Problem (1.2) has the constant solution $u(x) = \frac{1}{\sqrt{\lambda}}$. If we seek for solutions of the form $u = \frac{1}{\sqrt{\lambda}} + z$ we easily end up with the equivalent problem on z:

$$\begin{cases} -\Delta z = 2\lambda z - h_\lambda(z) & \text{in } \Omega, \\ \sqrt{\lambda} z > -1 & \text{in } \Omega, \\ \nabla z \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $h_\lambda : \,]-\frac{1}{\sqrt{\lambda}}, +\infty[\, \to \mathbb{R}$ is defined by

$$h_\lambda(s) = \frac{\lambda\sqrt{\lambda}s^2}{1 + \sqrt{\lambda}s}.$$

Now we consider another simple transformation: $v := \sqrt{\lambda}z$, so that (2.2) turns out to be equivalent to

$$\begin{cases} -\Delta v = 2\lambda\left(v - \frac{1}{2}h_1(v)\right) & \text{in } \Omega, \\ v > -1 & \text{in } \Omega, \\ \nabla v \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Now choose s_0 with $0 < s_0 < 1/2$ and a $\mathcal{C}^\infty$ cutoff function $\eta : \mathbb{R} \to [0,1]$ such that $\eta(s) = 1$ for $|s| \le s_0$, $\eta(s) = 0$, for $|s| \ge 2s_0$. Define $\tilde{h}_1 : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{h}_1(s) := \eta(s)\, h_1(s) \tag{2.4}$$

($\tilde{h}_1(s)$ is given the value zero for $s = -1$). Then $\tilde{h}_1 \in \mathcal{C}_0^\infty(\mathbb{R};\mathbb{R})$, $\tilde{h}_1'(0) = h_1''(0) = 0$, $\tilde{h}_1 = h_1$ on $[-s_0, s_0]$. Denote by $\tilde{H}_1 : \mathbb{R} \to \mathbb{R}$ the primitive function for $\tilde{h}_1$ (i.e. $\tilde{H}_1' = \tilde{h}_1$) such that $\tilde{H}_1(0) = 0$.

Now we apply the bifurcation theorem (3.1) with $\mathbb{H} := W^{1,2}(\Omega)$. $\mathbb{L} = L^2(\Omega)$, $\mathcal{H} = 0$, $\hat{\lambda} = \mu$, $\mathcal{H}_1(v) := \frac{1}{2}\int_\Omega \tilde{H}_1(v)\,dx$. In this way we get that there exists $\rho_0 > 0$ such that for all $\rho \in]0, \rho_0[$ there are two distinct pairs $(v_{1,\rho}, \mu_{1,\rho})$ and $(v_{2,\rho}, \mu_{2,\rho})$ which are *weak* solutions of

$$\begin{cases} -\Delta v = \mu\left(v - \frac{1}{2}\tilde{h}_1(v)\right) & \text{in } \Omega, \\ \nabla v \cdot \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

and such that

$$v_{i,\rho} \xrightarrow{\rho \to 0} 0 \quad (\text{in } W^{1,2}(\Omega)), \qquad \mu_{i,\rho} \xrightarrow{\rho \to 0} \mu_k, \qquad i = 1,2. \tag{2.6}$$

We claim the there exists a constant K such, for any $\mu \in [\hat{\mu}_k - 1, \hat{\mu}_k + 1]$ and any weak solution v of (2.5), v is bounded and:

$$\|v\|_\infty \le K\mu \left\| v - \frac{1}{2}\tilde{h}_1(v) \right\|_2 . \tag{2.7}$$

For this we use a standard bootstrap argument using the fact that the function $k(s) = (s - \frac{1}{2}\tilde{h}_1(s))$, appearing on the right hand side of (2.5), verifies

$$|k(s)| \le M|s| \qquad \forall s \in \mathbb{R} \tag{2.8}$$

for a suitable M (since $\tilde{h}_1'$ is bounded). Assume that v is a solution, i.e. $-\Delta v = \mu k(v)$, and $v \in L^q(\Omega)$ for some $q > 1$ (for sure this is true for $q = 2^*$). Then, by (2.8), $k(v) \in L^q(\Omega)$. From the standard Calderón–Zygmund theory (see e.g. Section 9.6 in [16]), we have $v \in W^{2,q}(\Omega)$. Then, using the Sobolev embedding Theorem, either $v \in L^{q_1}(\Omega)$ with $q_1 \le \frac{Nq}{N-2q}$ (if $2q \le N$) or $v \in \mathcal{C}^{0,\alpha}$ with $\alpha > 0$ (in the case $2q > N$). Iterating this argument a finite number of times we get the conclusion. Notice that we could go further and prove that v is $\mathcal{C}^\infty$ and is a classical solution.

Using (2.6) and (2.7) we get that $v_{i,\rho} \to 0$ in $L^\infty(\Omega)$ as $\rho \to 0$, so $|v_{i,\rho}| < s_0$, $i = 1,2$, for ρ_0 small. This implies that $\tilde{h}_1(v_{i,\rho}) = h_1(v_{i,\rho})$, and $v_{i,\rho}$ actually solve (2.3) with $\lambda_{i,\rho} := \frac{\mu_{i,\rho}}{2}$. Going backwards and setting $u_{i,\rho} := \frac{1}{\sqrt{\lambda_{i,\rho}}} + \sqrt{\lambda_{i,\rho}}v_{i,\rho}$, we find the desired solutions of (1.2).

□

3 A variant for the two bifurcation branches theorem for potential operators

Let $\mathbb{L}$ and $\mathbb{H}$ be two Hilbert spaces such that $\mathbb{H} \subset \mathbb{L}$ with a compact embedding $i : \mathbb{H} \to \mathbb{L}$. We use the notations $\|\cdot\|$, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_{\mathbb{L}}$, $\langle \cdot, \cdot \rangle_{\mathbb{L}}$ to indicate the norms and inner products in $\mathbb{H}$ and $\mathbb{L}$ respectively. Let $A : \mathbb{H} \to \mathbb{H}$ be a bounded linear symmetric operator such that

$$\langle Au, u \rangle \geq \nu \|u\|^2 - M \|u\|_{\mathbb{L}}^2 \qquad \forall u \in \mathbb{H} \tag{3.1}$$

where $\nu > 0$ and M are two constants. We say that $\lambda \in \mathbb{R}$ is an "eigenvalue for A" if there exists $e \in \mathbb{H} \setminus \{0\}$ with

$$\langle Ae, v \rangle = \lambda \langle e, v \rangle_{\mathbb{L}} \qquad \forall v \in \mathbb{H}$$

which corresponds to say that:

$$Ae = \lambda i^* e.$$

In this case we say that e is an "eigenvector" corresponding to λ.

It is well known that there exists a sequence (λ_n) of eigenvalues of A with $\lambda_n \leq \lambda_{n+1}$, $\lambda_n \to +\infty$, such that the corresponding eigenvectors generate $\mathbb{H}$. It is convenient to agreee that $\lambda_0 = -\infty$. We can suppose that for any $k \geq 1$ we are given an eigenvector e_k relative to λ_k with $\|e_n\|_{\mathbb{L}} = 1$, and

$$\langle e_n, e_m \rangle = \langle e_n, e_m \rangle_{\mathbb{L}} = 0 \quad \text{if } n \neq m.$$

If $\lambda \in \mathbb{R}$ we define

$$E_\lambda^- := \text{span}\{e_i : \lambda_i < \lambda\}, \quad E_\lambda^0 := \text{span}\{e_i : \lambda_i = \lambda\}, \quad E_\lambda^+ := \overline{\text{span}\{e_i : \lambda_i > \lambda\}}^{(\mathbb{H})}$$

($E_\lambda^0 = \{0\}$ if λ is not an eigenvalue). If $\lambda_n \leq \lambda \leq \lambda_{n+1}$ it is clear that

$$\sup_{\left\{u \in E_\lambda^- : \|u\|_{\mathbb{L}} = 1\right\}} \langle Au, u \rangle \leq \lambda_n, \qquad \inf_{\{u \in E^+\lambda : \|u\|_{\mathbb{L}} = 1\}} \langle Au, u \rangle \geq \lambda_{n+1},$$

while $\langle Au, u \rangle = \lambda$, if $u \in E_\lambda^0$.

Theorem 3.1 (Bifurcation). *Let $\mathcal{H} \in \mathcal{C}^1(\mathbb{H}; \mathbb{R})$, $\mathcal{H}_1 \in \mathcal{C}^1(\mathbb{L}; \mathbb{R})$ be such that*

$$\begin{aligned} &\mathcal{H}(0) = 0, \qquad \nabla \mathcal{H}(0) = 0, \qquad \lim_{u \to 0} \frac{\|\nabla \mathcal{H}(u)\|_{\mathbb{L}}}{\|u\|_{\mathbb{L}}} = 0 \\ &\mathcal{H}_1(0) = 0, \qquad \nabla_{\mathbb{L}} \mathcal{H}_1(0) = 0, \qquad \lim_{u \to 0} \frac{\|\nabla_{\mathbb{L}} \mathcal{H}_1(u)\|_{\mathbb{L}}}{\|u\|_{\mathbb{L}}} = 0. \end{aligned} \tag{3.2}$$

Notice that we are using the symbol ∇ to denote the gradient with respect to the inner product in $\mathbb{H}$ and $\nabla_{\mathbb{L}}$ for the corresponding gradient in $\mathbb{L}$.

Let $\hat{\lambda}$ be an eigenvalue for A. Then, for any $\rho > 0$ small, there exist $(u_{1,\rho}, \lambda_{1,\rho})$ and $(u_{2,\rho}, \lambda_{2,\rho})$ which solve the problem

$$Au + \nabla \mathcal{H}(u) = \lambda i^* \left(u + \nabla_{\mathbb{L}} \mathcal{H}_1(u)\right), \qquad u \neq 0, \tag{3.3}$$

such that $u_{1,\rho} \neq u_{2,\rho}$ and

$$u_{1,\rho} \xrightarrow{\mathbb{H}} 0, \qquad u_{2,\rho} \xrightarrow{\mathbb{H}} 0, \qquad \lambda_{1,\rho} \to \hat{\lambda}, \qquad \lambda_{2,\rho} \to \hat{\lambda} \quad \text{as } \rho \to 0. \tag{3.4}$$

Proof. We adapt the proof of Lemma 3.4 in [2]. Let $\hat{\lambda} = \lambda_i = \lambda_k$ with $\lambda_{i-1} < \lambda_i$ and $\lambda_k < \lambda_{k+1}$. We define $f : \mathbb{H} \to \mathbb{R}$ and $g : \mathbb{L} \to \mathbb{R}$ by

$$f(u) := \frac{1}{2} \langle Au, u \rangle + \mathcal{H}(u), \qquad g(u) := \frac{1}{2} \|u\|_{\mathbb{L}}^2 + \mathcal{H}_1(u).$$

Let $\mathcal{C} := \{u \in \mathbb{L} : 1 < \|u\|_{\mathbb{L}} < 2\}$. Moreover, if $0 < \rho < 1$ we define

$$f_\rho(u) := \frac{1}{\rho^2} f(\rho u), \qquad g_\rho(u) := \frac{1}{\rho^2} g(\rho u),$$
$$\mathcal{H}_\rho(u) := \frac{1}{\rho^2} \mathcal{H}(\rho u), \qquad \mathcal{H}_{1,\rho}(u) := \frac{1}{\rho^2} \mathcal{H}_1(\rho u),$$
$$\mathcal{S}_\rho := \{u \in \mathcal{C} : g_\rho(u) = 1\}.$$

In fact $f_\rho(u) = \frac{1}{2} \langle Au, u \rangle + \mathcal{H}_\rho(u)$ and $g_\rho(u) = \frac{1}{2} \|u\|_{\mathbb{L}}^2 + \mathcal{H}_{1,\rho}(u)$.

Since the result we are proving only involves the behaviour of $\mathcal{H}, \mathcal{H}_1$ near zero, we are allowed to modify $\mathcal{H}$ and $\mathcal{H}_1$ outside of a small ball around the origin. More precisely using (3.2) we can find R in $]0, 1/3[$ such that

$$\|\nabla \mathcal{H}(u)\| \leq \frac{\nu}{8} \|u\| \ \forall u \text{ with } \|u\| < 3R, \quad \|\nabla_{\mathbb{L}} \mathcal{H}_1(u)\| \leq \frac{1}{2} \|u\|_1 \ \forall u \text{ with } \|u\|_1 < 3R, \tag{3.5}$$

and define $\tilde{\mathcal{H}}(u) := \eta(\|u\|)\mathcal{H}(u)$, $\tilde{\mathcal{H}}_1(u) := \eta(\|u\|_1)\mathcal{H}_1(u)$, where $\eta : [0, +\infty[\to [0, 1]$ is a cutoff function with $\eta(s) = 1$ for $0 \leq s \leq R$, $\eta(s) = 0$ for $s \geq 3R$, and $\eta'(s) \leq 1$. Now since

$$\tilde{\mathcal{H}}(u) = \mathcal{H}(u) \quad \forall u \text{ with } \|u\| < R, \qquad \tilde{\mathcal{H}}_1(u) = \mathcal{H}_1(u) \quad \forall u \text{ with } \|u\|_{\mathbb{L}} < R,$$

then the conclusion of Theorem 3.1 holds for $\mathcal{H}, \mathcal{H}_1$ if and only if it holds for $\tilde{\mathcal{H}}, \tilde{\mathcal{H}}_1$. Indeed the first component u_ρ of a bifucation branch (for any of the two problems) eventually verifies $\|u_\rho\| < R$ and $\|u_\rho\|_1 < R$. So from now on we replace $\mathcal{H}$ with $\tilde{\mathcal{H}}$ and $\mathcal{H}_1$ with $\tilde{\mathcal{H}}_1$, maintaining the same notation. With simple computations we can deduce from (3.5) that the redefined functions verify:

$$\text{(a)} \quad |\nabla \mathcal{H}(u)| \leq \frac{\nu}{4} \|u\| \ \forall u \in \mathbb{H}, \qquad \text{(b)} \quad |\nabla_{\mathbb{L}} \mathcal{H}_1(u)| \leq \|u\|_{\mathbb{L}} \ \forall u \in \mathbb{L}. \tag{3.6}$$

From (a) in (3.6) we get

$$|\mathcal{H}(u)| \leq \frac{\nu}{4} \|u\|^2 \Rightarrow |\mathcal{H}_\rho(u)| \leq \frac{\nu}{4} \|u\|^2 \qquad \forall u \in \mathbb{H}, \forall \rho \in [0, 1]. \tag{3.7}$$

Using (3.1) and (3.7) we get that:

$$\|u\|^2 \leq \frac{4}{\nu} \left(f_\rho(u) + M \|u\|_{\mathbb{L}}^2 \right). \tag{3.8}$$

From (3.2) and (3.8) we easily get that, if $c \in \mathbb{R}$ and $\rho \to 0$:

$$\begin{aligned} &\sup_{u \in \mathcal{C}, f_\rho(u) \leq c} |\mathcal{H}_\rho(u)| \to 0, &&\sup_{u \in \mathcal{C}} |\mathcal{H}_{1,\rho}(u)| \to 0, \\ &\sup_{u \in \mathcal{C}, f_\rho(u) \leq c} \|\nabla \mathcal{H}_\rho(u)\| \to 0, &&\sup_{u \in \mathcal{C}} \|\nabla_{\mathbb{L}} \mathcal{H}_{1,\rho}(u)\|_{\mathbb{L}} \to 0. \end{aligned} \tag{3.9}$$

So if we extend the definition to $\rho = 0$ by letting $f_0(u) := \frac{1}{2}\langle Au, u\rangle$ and $g_0(u) := \frac{1}{2}\|u\|_{\mathbb{L}}^2$, then $(\rho, u) \mapsto f_\rho(u)$ is continuous on $[0, +\infty[\times\mathbb{H}$ and $(\rho, u) \mapsto g_\rho(u)$ is continuous on $[0, +\infty[\times\mathbb{L}$. We also define $\mathcal{S}_\rho$ for $\rho = 0$:

$$\mathcal{S}_0 := \{u \in \mathbb{L} : g_0(u) = 1\} = \left\{u \in \mathbb{L} : \|u\|_{\mathbb{L}}^2 = 2\right\}.$$

Notice that the critical values of f_0 on $\mathcal{S}_\rho$ are precisely the eigenvalues λ_n.

We claim that there exist $\bar\rho >$ such that the $\mathbb{L}$-closure of $\mathcal{S}_\rho$ is contained in $\mathcal{C}$ for all $\rho \in]0, \bar\rho]$ in other terms $\mathcal{S}_\rho$ is closed for $\rho > 0$ small. Indeed if the claim were false there would exist two sequences (ρ_n) and (u_n) such that $\rho_n \to 0$, $\rho_n \to 0$, $g_{\rho_n}(u_n) = \frac{\|u_n\|_{\mathbb{L}}^2}{2} + \frac{\mathcal{H}_1(\rho_n u_n)}{\rho^2} = 1$, and $\|u_n\|_{\mathbb{L}} \in \{1, 2\}$. From (3.2) we would have $\frac{\mathcal{H}_1(\rho_n u_n)}{\rho_n^2} = \frac{\mathcal{H}_1(\rho_n u_n)}{\|\rho_n u_n\|_{\mathbb{L}}^2}\|u_n\|_{\mathbb{L}}^2 \to 0$, so $\|u_n\|_{\mathbb{L}} \to \sqrt{2}$ which yields a contradiction for n large.

Let us split $\mathbb{H}$ as $\mathbb{H} = \mathbb{X}_1 \oplus \mathbb{X}_2 \oplus \mathbb{X}_3$, where

$$\mathbb{X}_1 := E_{\hat\lambda}^-, \qquad \mathbb{X}_2 := E_{\hat\lambda}^0, \qquad \mathbb{X}_3 := E_{\hat\lambda}^+$$

and consider the orthogonal projections $\Pi_i : \mathbb{H} \to \mathbb{X}_i$. $i = 1, 2, 3$. We also denote $\Pi_{13} := \Pi_1 + \Pi_3$. Given $\rho \in [0, \bar\rho]$ and $\delta \in]0, 1[$, we set

$$\mathcal{C}_\delta := \{u \in \mathcal{C} : \|\Pi_2(u)\|_{\mathbb{L}} \geq \delta\}, \qquad \mathcal{S}_{\rho,\delta} := \mathcal{S}_\rho \cap \mathcal{C}_\delta.$$

Since $\mathcal{S}_\rho$ is closed, then $S_{\rho,\delta}$ is a smooth manifold with boundary, the boundary being

$$\Sigma_{\rho,\delta} := \left\{u \in \mathcal{S}_\rho : \|\Pi_2(u)\|_{\mathbb{L}} = \delta\right\}.$$

Notice that $\mathcal{S}_{0,\delta} \neq \emptyset$ ($\delta < 1$). Let us indicate by $\bar f_\rho$ the restriction of f_ρ on $\mathcal{S}_{\rho,\delta}$.

We will use the notion of lower critical point for $\bar f_\rho$ (see [2,21] and the references therein): u is (lower) critical for $\bar f_\rho$ if and only there exist $\lambda, \mu \in \mathbb{R}$ such that $\mu \geq 0$, $\mu = 0$ if $\|\Pi_2(u)\|_{\mathbb{L}} > \delta$, and

$$\langle Au, v\rangle + \langle\nabla\mathcal{H}_\rho(u), v\rangle = \lambda\langle u + \nabla_{\mathbb{L}}\mathcal{H}_{1,\rho}(u), v\rangle_{\mathbb{L}} + \mu\langle\Pi_2(u), v\rangle_{\mathbb{L}} \quad \forall v \in \mathbb{H}. \tag{3.10}$$

Define $\Gamma : \mathcal{C}_\delta \times [1/2, 2] \to \mathcal{C}_\delta$ and $\varphi : [0, \bar\rho] \times \mathcal{C}_\delta \times [1/2, 2] \to \mathbb{R}$ by

$$\begin{aligned}\Gamma(u, t) &:= \frac{\delta\Pi_2(u)}{\|\Pi_2(u)\|_{\mathbb{L}}} + t\left(u - \frac{\delta\Pi_2(u)}{\|\Pi_2(u)\|_{\mathbb{L}}}\right)\\ &= t\Pi_{13}(u) + (\delta + t(\|\Pi_2(u)\|_{\mathbb{L}} - \delta))\frac{\Pi_2(u)}{\|\Pi_2(u)\|_{\mathbb{L}}},\\ \varphi(\rho, u, t) &:= g_\rho(\Gamma(u, t)).\end{aligned}$$

With easy computations:

$$\begin{aligned}\varphi(0, u, t) &= \frac{1}{2}\left(t^2\|\Pi_{13}(u)\|_{\mathbb{L}}^2 + (\delta + t(\|\Pi_2(u)\|_{\mathbb{L}} - \delta))^2\right)\\ &= \frac{1}{2}\left(t^2(\|u\|_{\mathbb{L}}^2 - 2\delta\|\Pi_2(u)\|_{\mathbb{L}} + \delta^2) + 2\delta t(\|\Pi_2(u)\|_{\mathbb{L}} - \delta) + \delta^2\right).\end{aligned}$$

Since $1 < \|u\|_{\mathbb{L}} < 2$ and $\|\Pi_2(u)\|_{\mathbb{L}} \geq \delta$, we have

$$\frac{t^2}{2} - 2\delta t^2 \leq \varphi(0, u, t) \leq \left(2 + \frac{\delta^2}{2}\right)t^2 + 2\delta t + \frac{\delta^2}{2}.$$

In particular:

$$2-2\delta \le \varphi(0,u,2), \qquad \varphi(0,u,1/2) \le \frac{1}{2}+\frac{\delta^2}{8}+\delta+\frac{\delta^2}{2} < \frac{1}{2}+2\delta.$$

We can choose $\delta_0 > 0$ so that $2-2\delta > 3/2$ and $\frac{1}{2}+2\delta < 3/4$ for all $\delta \in]0,\delta_0]$. From now on we consider $0<\delta\le\delta_0$. By (3.9), up to shrinking $\bar\rho$, we have

$$\sup_{u\in\mathcal{C}_\delta} \varphi(\rho,u,1/2) < 1, \qquad \inf_{u\in\mathcal{C}_\delta} \varphi(\rho,u,2) > 1 \qquad \forall \rho\in[0,\bar\rho].$$

Moreover,

$$\frac{\partial}{\partial t}\varphi(0,u,t) = t(\|u\|_{\mathbb{L}}^2 - 2\delta\,\|\Pi_2(u)\|_{\mathbb{L}} + \delta^2) + \delta(\|\Pi_2(u)\|_{\mathbb{L}} - \delta) \ge t(1-4\delta)$$

so, up to shrinking δ_0, we have $\frac{\partial}{\partial t}\varphi(0,u,t) \ge \frac{1}{4}$ for all $t\ge\frac{1}{2}$. Up to further shrinking $\bar\rho>0$ (again we use (3.9)), we have that $\rho\in[0,\bar\rho]$, $\delta\in]0,\delta_0]$, $u\in\mathcal{C}_\delta$ imply

$$\varphi(\rho,u,1/2)<1, \quad \varphi(\rho,u,2)>1, \quad \frac{\partial}{\partial t}\varphi(\rho,u,t)\ge\frac{1}{8} \quad \forall t\in[1/2,2].$$

We can therefore conclude that for all $u\in[0,\bar\rho]$ and $u\in\mathcal{C}_\delta$ there exists a unique $\bar t=\bar t(\rho,u)$ in $[1/2,2]$ such that $\varphi(\rho,u,\bar t(u,\rho))=1$, that is $\Gamma(u,\bar t(\rho,u))\in\mathcal{S}_{\rho,\delta}$. It is easy to check that $\bar t:[0,\bar\rho]\times\mathcal{C}_\delta\to[1/2,2]$ is continuous and so is $\Phi:[0,\bar\rho]\times\mathcal{C}_\delta\to\mathcal{S}_{\rho,\delta}$ defined by $\Phi(\rho,u):=\Gamma(u,\bar t(\rho,u))$. Notice that

$$t\in[1/2,2],\ u\in\mathcal{C}_\delta,\ \|\Pi_2(u)\|_{\mathbb{L}}=\delta \ \Rightarrow\ \|\Pi_2(\Gamma(u,t))\|_{\mathbb{L}}=\delta.$$

Therefore $\Phi(\rho,\cdot)$ maps $\{u\in\mathcal{C}_\delta, \|\Pi_2(u)\|_{\mathbb{L}}=\delta\}$ into $\Sigma_{\rho,\delta}$. Also notice that $\Phi(\rho,u)\circ\Phi(0,u)=u$ whenever $u\in\mathcal{S}_{\rho,\delta}$ and $\Phi(0,u)\circ\Phi(\rho,u)=u$ whenever $u\in\mathcal{S}_{0,\delta}$. We have thus proven that $\Phi(\rho,\cdot)|_{S_{0,\delta}}$ is a homeomorphism from $(S_{0,\delta},\Sigma_{0,\delta})$ to $(S_{\rho,\delta},\Sigma_{\rho,\delta})$ whose inverse is $\Phi(0,\cdot)|_{S_{\rho,\delta}}$.

Now let

$$a'_\rho := \sup_{(\mathbb{X}_1\oplus\mathbb{X}_2)\cap\Sigma_{\rho,\delta}} f_\rho \qquad a''_\rho := \inf_{(\mathbb{X}_2\oplus\mathbb{X}_3)\cap\mathcal{S}_{\rho,\delta}} f_\rho \tag{3.11}$$

$$b'_\rho := \sup_{(\mathbb{X}_1\oplus\mathbb{X}_2)\cap\mathcal{S}_{\rho,\delta}} f_\rho \qquad b''_\rho := \inf_{(\mathbb{X}_2\oplus\mathbb{X}_3)\cap\Sigma_{\rho,\delta}} f_\rho. \tag{3.12}$$

Notice that, by definition, $a''_\rho \le b'_\rho$. For $\rho=0$ it is easy to see that

$$a'_0 = \lambda_{i-1} + \frac{\delta^2}{2}(\hat\lambda-\lambda_{i-1}) < \hat\lambda = a''_0 = b'_0 = \hat\lambda < \lambda_{k+1} - \frac{\delta^2}{2}(\lambda_{k+1}-\hat\lambda) = b''_0$$

(recall that $0<\delta<1$). Let $\varepsilon_0>0$ with $\varepsilon_0<\hat\lambda-\lambda_{i-1}$. We claim that, if $\delta^2(\hat\lambda-\lambda_{i-1})<2\varepsilon_0$, then

$$\text{there exists no } u\in\Sigma_{0,\delta} \text{ with } u \text{ lower critical for } \bar f_0 \text{ and } \lambda_{i-1}+\varepsilon_0\le f_0(u). \tag{3.13}$$

By contradiction assume that such a u exists; then there exist $\lambda\in\mathbb{R}$ and $\mu\ge0$ such that (3.10) holds. Let $u_i=\Pi_i(u)$, $i=1,2,3$. Taking $v=u_2$ in (3.10) (with $\rho=0$) yields

$$\hat\lambda\,\|u_2\|_{\mathbb{L}}^2 = \langle Au_2,u_2\rangle = \langle Au,u_2\rangle = \lambda\,\langle u,u_2\rangle_{\mathbb{L}} + \mu\,\langle u_2,u_2\rangle_{\mathbb{L}} = (\lambda+\mu)\,\|u_2\|_{\mathbb{L}}^2.$$

Since $\|u_2\|_{\mathbb{L}} = \delta > 0$, we have $\lambda + \nu = \hat{\lambda}$, so $\lambda = \hat{\lambda} - \nu \le \hat{\lambda}$. Taking $v = u_3$:

$$\lambda_{k+1} \|u_3\|^2 \le \langle Au_3, u_3 \rangle = \langle Au, u_3 \rangle = \langle \lambda u + \mu u_2, u_3 \rangle = \lambda \|u_3\|_{\mathbb{L}}^2 \le \hat{\lambda} \|u_3\|_{\mathbb{L}}^2 .$$

Since $\lambda_{k+1} < \hat{\lambda}$, we have $u_3 = 0$. Then $u \in \mathbb{X}_1 \oplus \mathbb{X}_2 \cap \Sigma_{0,\delta}$, which implies

$$f_0(u) \le a_0' = \lambda_{i-1} + \frac{\delta^2}{2}(\hat{\lambda} - \lambda_{i-1}) < \lambda_{i-1} + \varepsilon_0$$

which gives a contradiction. Hence the claim is proven. Notice that (3.13) implies that the only critical value λ_0 of $\bar{f}_0$, with $\lambda_{i-1} + \varepsilon_0 \le \lambda_0 \le \lambda_{k+1} - \varepsilon_0$, is $\lambda_0 = \hat{\lambda}$. Indeed assume u_0 to be a critical point with $\bar{f}_0(u_0) = \lambda_0$: then, by (3.13), $u_0 \notin \Sigma_{0,\delta}$ so (3.10) holds with $\mu = 0$ which easily implies $\lambda_0 = \hat{\lambda}$.

From now on we fix $\varepsilon_0 > 0$ such that $5\varepsilon_0 < \min(\hat{\lambda} - \lambda_{i-1}, \lambda_{k+1} - \hat{\lambda})$ and $\delta > 0$ such that $\delta^2(\hat{\lambda} - \lambda_{i-1}) \le \varepsilon_0$ (so (3.13) holds with $\varepsilon_0/2$). Using (3.9) we can derive that, given $\varepsilon \in]0, \varepsilon_0]$ there exists $\rho(\varepsilon) \in]0, \bar{\rho}]$ such that, if $\rho \in]0, \rho(\varepsilon)]$:

$$\begin{aligned} a_\rho' &\le \lambda_{i-1} + \varepsilon_0 < \hat{\lambda} - 4\varepsilon < \hat{\lambda} - \varepsilon \le a_\rho'' \le \inf_{\mathbb{X}_2 \cap \mathcal{S}_{\rho,\delta}} f_\rho \\ &\le \sup_{\mathbb{X}_2 \cap \mathcal{S}_{\rho,\delta}} f_\rho \le b_\rho' \le \hat{\lambda} + \varepsilon < \hat{\lambda} + 4\varepsilon < \lambda_{k+1} - \varepsilon_0 \le b_\rho''; \end{aligned} \tag{3.14}$$

$$\begin{gathered} \text{there are no } u \in \Sigma_{\rho,\delta} \text{ with } u \text{ lower critical for } \bar{f}_\rho \text{ and} \\ f_\rho(u) \in [\lambda_{i-1} + \varepsilon_0, \lambda_{k+1} - \varepsilon_0]\,; \end{gathered} \tag{3.15}$$

$$\begin{gathered} \text{there are no } u \in \mathcal{S}_{\rho,\delta} \text{ with } u \text{ lower critical for } \bar{f}_\rho \text{ and} \\ f_\rho(u) \in \left[\lambda_{i-1} + \varepsilon_0, \hat{\lambda} - \varepsilon\right] \cup \left[\hat{\lambda} + \varepsilon, \lambda_{k+1} - \varepsilon_0\right]; \end{gathered} \tag{3.16}$$

$$\left|f_0(u) - f_\rho(\Phi(\rho, u))\right| < \varepsilon \qquad \forall u \in \mathcal{S}_{0,\delta} \text{ with } f_0(u) \le \lambda_{k+1} - \varepsilon_0; \tag{3.17}$$

$$\left|f_\rho(u) - f_0(\Phi(0, (u))\right| < \varepsilon \qquad \forall u \in \mathcal{S}_{\rho,\delta} \text{ with } f_\rho(u) \le \lambda_{k+1} - \varepsilon_0. \tag{3.18}$$

To prove (3.17) and (3.18) we use (3.8). If $\sigma \in [\varepsilon, 4\varepsilon]$, set $A_\rho^\sigma := \bar{f}_\rho^{\hat{\lambda}/2-\sigma}$, $B_\rho^\sigma := \bar{f}_\rho^{\hat{\lambda}/2+\sigma}$ i.e.:

$$A_\rho^\sigma = \left\{u \in S_{\rho,\delta} : f_\rho(u) \le \hat{\lambda}/2 - \sigma\right\}, \qquad B_\rho^\sigma = \left\{u \in S_{\rho,\delta} : f_\rho(u) \le \hat{\lambda}/2 + \sigma\right\}.$$

Moreover set $\tilde{A}_\rho^\sigma := \Phi(\rho, A_0^\sigma)$, $\tilde{B}_\rho^\sigma := \Phi(\rho, B_0^\sigma)$, $\hat{A}_\rho^\sigma := \Phi(0, A_\rho^\sigma)$, $\hat{B}_\rho^\sigma := \Phi(0, B_\rho^\sigma)$. From (3.17) and (3.18) (remind that $\Phi(\rho, \cdot)^{-1} = \Phi(0, \cdot)$) we get

$$\begin{aligned} &A_0^{4\varepsilon} \subset \Phi(0, A_\rho^{3\varepsilon}) \subset A_0^{2\varepsilon} \subset \Phi(0, A_\rho^{\varepsilon}), \qquad B_0^{\varepsilon} \subset \Phi(0, B_\rho^{2\varepsilon}) \subset B_0^{3\varepsilon}\Phi(0, B_\rho^{4\varepsilon}), \\ &A_\rho^{4\varepsilon} \subset \Phi(\rho, A_0^{3\varepsilon}) \subset A_\rho^{2\varepsilon} \subset \Phi(\rho, A_0^{\varepsilon}), \qquad B_\rho^{\varepsilon} \subset \Phi(\rho, B_0^{2\varepsilon}) \subset B_\rho^{3\varepsilon}\Phi(\rho, B_0^{4\varepsilon}). \end{aligned}$$

The above inclusions give rise to the following diagram in homology:

$$\begin{array}{ccccccc} H_q(B_\rho^\varepsilon, A_\rho^{4\varepsilon}) & \xrightarrow{i_1^*} & H_q(\tilde{B}_\rho^{2\varepsilon}, \tilde{A}_\rho^{3\varepsilon}) & \xrightarrow{i_2^*} & H_q(B_\rho^{3\varepsilon}, A_\rho^{2\varepsilon}) & \xrightarrow{i_3^*} & H_q(\tilde{B}_\rho^{4\varepsilon}, \tilde{A}_\rho^{\varepsilon}) \\ \downarrow \phi_1^* & & \phi_2^* \uparrow & & \downarrow \phi_3^* & & \phi_2^* \uparrow \\ H_q(\hat{B}_\rho^\varepsilon, \hat{A}_\rho^{4\varepsilon}) & \xrightarrow{j_1^*} & H_q(B_0^{2\varepsilon}, A_0^{3\varepsilon}) & \xrightarrow{j_2^*} & H_q(\hat{B}_\rho^{3\varepsilon}, \hat{A}_\rho^{2\varepsilon}) & \xrightarrow{j_3^*} & H_q(B_0^{4\varepsilon}, A_0^{\varepsilon}) \end{array}$$

where i_1, i_2, i_3, j_1, j_2, j_3 are embeddings and ϕ_1, ϕ_3 are restrictions of $\Phi(0, \cdot)$, while ϕ_2, ϕ_4 are restrictions of $\Phi(\rho, \cdot)$. It is clear that ϕ_i^* are isomorphisms. Notice that $i_2 \circ \phi_2 \circ j_1 \circ \phi_1$ is the embedding of $(B_\rho^\varepsilon, A_\rho^{4\varepsilon})$ in $(B_\rho^{3\varepsilon}, A_\rho^{2\varepsilon})$ and $j_3 \circ \phi_3 \circ i_2 \circ \phi_2$ is the embedding of $(B_0^{2\varepsilon}, A_0^{3\varepsilon})$ in $(B_0^{4\varepsilon}, A_0^{\varepsilon})$.

Since there are no critical values for $\bar{f}_0$ in $[\hat{\lambda}-4\varepsilon, \hat{\lambda}-\varepsilon] \cup [\hat{\lambda}+\varepsilon, \hat{\lambda}+4\varepsilon]$ (see (3.16)), then the pair $(B_\rho^\varepsilon, A_\rho^{4\varepsilon})$ is a deformation retract of the pair $(B_\rho^{3\varepsilon}, A_\rho^{2\varepsilon})$, so $i_2^* \circ \phi_2^* \circ j_1^* \circ \phi_1^*$ is an isomorphism. For analogous reasons $j_3^* \circ \phi_3^* \circ i_2^* \circ \phi_2^*$ is an isomorphism. It follows that $i_2^* \circ \phi_2^* : H_q(B_0^{2\varepsilon}, A_0^{3\varepsilon}) \to H_q(B_\rho^{3\varepsilon}, A_\rho^{2\varepsilon})$ is an isomorphism.

From the definitions (3.11) and (3.12) we have the inclusions:

$$(\mathcal{S}_{0,\delta} \cap (\mathbb{X}_1 \oplus \mathbb{X}_2), \Sigma_{0,\delta} \cap (\mathbb{X}_1 \oplus \mathbb{X}_2)) \subset (B_\rho^{3\varepsilon}, A_\rho^{2\varepsilon}) \subset (\mathcal{S}_{0,\delta} \setminus \mathbb{X}_3, \mathcal{S}_{\rho,\delta} \setminus (\mathbb{X}_2 \oplus \mathbb{X}_3))$$

which allow to repeat the arguments of [2] (see also the proof of Lemma 2.3 in [21]). To estimate the relative category:

$$\mathrm{cat}_{(B_\rho^{3\varepsilon}, A_\rho^{2\varepsilon})}(B_\rho^{3\varepsilon}) \geq 2 \qquad \forall \rho \in \,]0, \rho(\varepsilon)].$$

This implies that $\bar{f}_\rho$ hat at least two critical points $\bar{u}_{1,\rho}$, $\bar{u}_{2,\rho}$ with $\hat{\lambda}-3\varepsilon \leq f_\rho(\bar{u}_{i,\rho}) \leq \hat{\lambda}+2\varepsilon$. We have $\|\bar{u}_{i,\rho}\|_{\mathbb{L}}^2/2 + \mathcal{H}_{1,\rho}(\bar{u}_{i,\rho}) = 1$ and

$$\left\langle A\bar{u}_{i,\rho} + \nabla\mathcal{H}_\rho(\bar{u}_{i,\rho}), v\right\rangle = \lambda_{i,\rho}\left\langle \bar{u}_{i,\rho} + \nabla_{\mathbb{L}}\mathcal{H}_{1,\rho}(\bar{u}_{i,\rho}), v\right\rangle_{\mathbb{L}} \qquad \forall v \in \mathbb{H} \tag{3.19}$$

for a suitable Lagrange multiplier $\lambda_{i,\rho} \in \mathbb{R}$ (there is no μ, due to (3.15)). Taking $v = \bar{u}_{i,\rho}$ in (3.19):

$$\begin{aligned}
[\hat{\lambda}-2\varepsilon, \hat{\lambda}+3\varepsilon] \ni \bar{f}(\bar{u}_{i,\rho}) &= \frac{1}{2}\left\langle A\bar{u}_{i,\rho}, \bar{u}_{i,\rho}\right\rangle + \mathcal{H}_\rho(\bar{u}_{i,\rho}) \\
&= \mathcal{H}_\rho(\bar{u}_{i,\rho}) - \frac{1}{2}\left\langle \nabla\mathcal{H}_\rho(\bar{u}_{i,\rho}), \bar{u}_{i,\rho}\right\rangle + \frac{\lambda_{i,\rho}}{2}\left(\|\bar{u}_{i,\rho}\|_{\mathbb{L}}^2 + \left\langle \nabla_{\mathbb{L}}\mathcal{H}_{1,\rho}(\bar{u}_{i,\rho}), \bar{u}_{i,\rho}\right\rangle_{\mathbb{L}}\right) \\
&= \underbrace{\mathcal{H}_\rho(\bar{u}_{i,\rho}) - \frac{\left\langle \nabla\mathcal{H}_\rho(\bar{u}_{i,\rho}), \bar{u}_{i,\rho}\right\rangle}{2}}_{:=C_1(\rho)} + \lambda_{i,\rho}\left(1 + \underbrace{\left(\frac{\left\langle \nabla_{\mathbb{L}}\mathcal{H}_{1,\rho}(\bar{u}_{i,\rho}), \bar{u}_{i,\rho}\right\rangle_{\mathbb{L}}}{2} - \mathcal{H}_{1,\rho}(\bar{u}_{i,\rho})\right)}_{:=C_2(\rho)}\right)
\end{aligned}$$

By using (3.9) we obtain $C_1(\rho) \to 0$, $C_2(\rho) \to 0$, so for $\rho(\varepsilon)$ small enough we have $|\lambda_{i,\rho} - \hat{\lambda}| < 4\varepsilon$. We have thus proven that $\lambda_{1,\rho} \to \hat{\lambda}$ as $\rho \to 0$. Let $u_{i,\rho} := \rho\bar{u}_{1,\rho}$. Clearly $u_{i,\rho} \xrightarrow{\mathbb{L}} 0$ as $\rho \to 0$. By multiplying (3.19) by ρ and using the definitions of $\mathcal{H}_\rho$ and $\mathcal{H}_{1,\rho}$ we get that $(u, \lambda) = (u_{i,\rho}, \lambda_{i,\rho})$ verify (3.3). Taking the scalar product with $u_{i,\rho}$ in (3.3) gives $\left\langle Au_{i,\rho}, u_{i,\rho}\right\rangle \to 0$. Then, by (3.1), we have $u_{i,\rho} \xrightarrow{\mathbb{H}} 0$. □

4 A global bifurcation result for radial solutions

We consider the case $N = 2$ and $\Omega = B(0, R) = \left\{x \in \mathbb{R}^2 : \|X\| < R\right\}$. We look for radial solutions for Problem (2.2), i.e. $z(x, y) = w(\|(x, y)\|)$. Actually with similar arguments we could have considered the general case $N \geq 2$. Given $R > 0$, it is therefore convenient to introduce the Hilbert space

$$E := \left\{w : [0, R] \to \mathbb{R} : \int_0^R \rho\dot{w}^2\, d\rho < +\infty\right\}$$

endowed with $(v, w)_E := \int_0^R \rho\dot{v}\dot{w}\, d\rho + \int_0^R \rho v w\, d\rho$ and for $\lambda > 0$ the set

$$W_\lambda := \left\{w \in E : 1 + \sqrt{\lambda}w(\rho) > 0\right\}, \qquad \mathcal{W} := \left\{(w, \lambda) \in \mathbb{R} \times E : \lambda > 0, w \in W_\lambda\right\}.$$

It is clear that $\|w\|_\infty \le C\|w\|_E$, for a suitable constant C, so W is open in E and $\mathcal{W}$ is open in $\mathbb{R} \times E$. As well known the search for radial solutions leads to the equation

$$\begin{cases} \ddot{w} + \dfrac{\dot{w}}{\rho} = -\lambda w - \dfrac{\lambda w}{1+\sqrt{\lambda w}} =: f_\lambda(w), \\ \dot{w}(0) = \dot{w}(R) = 0. \end{cases} \tag{4.1}$$

By the above we mean that

$$(w,\lambda) \in \mathcal{W}, \qquad \int_0^R \rho \dot{w}\dot{\delta}\, d\rho = \int_0^R \rho f_\lambda(w)\delta\, d\rho \qquad \forall v \in E. \tag{4.2}$$

It is standard to check that "weak solutions", i.e. solutions to (4.2) actually solve (4.1) in a classical sense.

It is clear that $(0,\lambda)$ is a solution for (4.1) for any $\lambda \in \mathbb{R}$. We call "nontrivial " solution a pair (w,λ) with $w \neq 0$ such that (4.1) holds.

Remark 4.1. If (w,λ) is a nontrivial solution then $\lambda > 0$. To see this it suffices to multiply (4.1) by u and integrate over $[0,R]$. Actually this property is true in the general case (not just in the radial problem).

We shall use the following simple inequality.

Remark 4.2. Let $0 < a < b < +\infty$. We have

$$\frac{b-a}{b} \le \ln\left(\frac{b}{a}\right) \le \frac{b-a}{a}. \tag{4.3}$$

We have indeed

$$\ln\left(\frac{b}{a}\right) = \ln\left(1 + \frac{b-a}{a}\right) \le \frac{b-a}{a}$$

and

$$\ln\left(\frac{b}{a}\right) = -\ln\left(\frac{a}{b}\right) = -\ln\left(1+\frac{a-b}{b}\right) \ge -\frac{a-b}{b} = \frac{b-a}{b}.$$

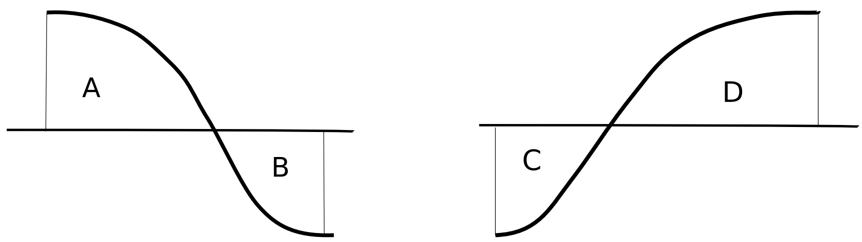

Figure 4.1: The different cases

Now let us suppose that a solution (w,λ) exists so we can find some properties and estimates on w. Arguing as in the proof of Lemma 2.2 in [8] we have that either $w = 0$ or $[0,R]$ can be split as the union of a finite number of subintervals $[r_{1,i}, r_{2,i}]$, $i = 1\ldots,k$, where w has one of the following behaviors (see Figure 4.1, we are skipping the index i):

(A) $w(r_1) > 0$, $\dot{w}(r_1) = 0$, $\dot{w} < 0$ in $]r_1, r_2]$, and $w(r_2) = 0$;

(B) $w(r_1) = 0$, $\dot{w} < 0$ in $[r_1, r_2[$, $\dot{w}(r_2) = 0$, and $w(r_2) < 0$;

(C) $w(r_1) < 0$, $\dot{w}(r_1) = 0$, $\dot{w} > 0$ in $]r_1, r_2]$, and $w(r_2) = 0$;

(D) $w(r_1) = 0$, $\dot{w} > 0$ in $[r_1, r_2[$, $\dot{w}(r_2) = 0$, and $w(r_2) > 0$.

So let $w : [r_1, r_2] \to \mathbb{R}$ be as in one of the above cases. Multiplying (4.1) by $\dot{w}$ gives

$$\frac{1}{2}\ddot{w}\dot{w}' + \frac{\dot{w}^2}{\rho} = \frac{d}{d\rho}F_\lambda(w)$$

where

$$F_\lambda(s) = \ln(1 + \sqrt{\lambda}s) - \sqrt{\lambda}s - \frac{\lambda}{2}s^2.$$

Let $p := \dot{w}^2$, the previous equation can be written as

$$\frac{1}{2}\dot{p} + \frac{p}{\rho} = \frac{d}{d\rho}F_\lambda(w)$$

which is equivalent to

$$\frac{d}{d\rho}(\rho^2 p) = 2\rho^2\frac{d}{d\rho}F_\lambda(w)\rho^2 = 2\rho^2\frac{d}{d\rho}F_1\left(\sqrt{\lambda}w\right).$$

We integrate between ρ_1 and ρ_2, where $r_1 \le \rho_1 \le \rho_2 \le r_2$:

$$\rho_2^2 p(\rho_2) - \rho_1^2 p(\rho_1) = 2\rho_2^2 F_\lambda(w(\rho_2)) - 2\rho_1^2 F_\lambda(w(\rho_1)) - \int_{\rho_1}^{\rho_2} 4\sigma F_\lambda(w(\sigma))\, d\sigma.$$

Notice that F_λ is increasing on $\left]-\frac{1}{\sqrt{\lambda}}, 0\right[$ and decreasing on $]0, +\infty[$, so

$\sigma \mapsto F_\lambda(w(\sigma))$ is increasing (decreasing) in cases (A) and (C) (in cases (B) and (D)).

We hence get, in cases (A) and (C):

$$-2(\rho_2^2 - \rho_1^2)F_\lambda(w(\rho_2)) \le -\int_{\rho_1}^{\rho_2} 4\sigma F_\lambda(w(\sigma))\, d\sigma \le -2(\rho_2^2 - \rho_1^2)F_\lambda(w(\rho_1))$$

while in cases (B) and (D):

$$-2(\rho_2^2 - \rho_1^2)F_\lambda(w(\rho_1)) \le -\int_{\rho_1}^{\rho_2} 4\sigma F_\lambda(w(\sigma))\, d\sigma \le -2(\rho_2^2 - \rho_1^2)F_\lambda(w(\rho_2)).$$

So in cases (A) and (C) we have

$$2\rho_1^2(F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1)) \le \rho_2^2 p(\rho_2) - \rho_1^2 p(\rho_1) \le 2\rho_2^2(F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1)) \tag{4.4}$$

and in cases (B) and (D):

$$2\rho_2^2(F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1)) \le \rho_2^2 p(\rho_2) - \rho_1^2 p(\rho_1) \le 2\rho_1^2(F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1)). \tag{4.5}$$

Now we estimate $w(\rho)$ – we need to take into account all the four cases (A), (B), (C), (D).

Case (A). We rename $\bar{\rho} := r_1$, $\rho_0 := r_2$ and let $h := w(\bar{\rho}) > 0$. We use (4.4) with $\rho_1 = \bar{\rho}$ and $\rho_2 = \sigma \in [\bar{\rho}, \rho_0]$:

$$2\bar{\rho}^2(F_\lambda(w(\sigma)) - F_\lambda(h)) \le \sigma^2\dot{w}(\sigma)^2 \le 2\sigma^2(F_\lambda(w(\sigma)) - F_\lambda(h)).$$

Then we take the square root and divide:

$$\sqrt{2}\frac{\bar{\rho}}{\sigma} \leq \frac{-\dot{w}(\sigma)}{\sqrt{F_\lambda(w(\sigma)) - F_\lambda(h)}} \leq \sqrt{2}$$

and now we integrate between $\bar{\rho}$ and $\rho \in [\bar{\rho}, \rho_0]$ getting

$$\sqrt{2}\bar{\rho}\ln\left(\frac{\rho}{\bar{\rho}}\right) \leq -\Phi_{\lambda,h}(w(\rho)) + \Phi_{\lambda,h}(h) \leq \sqrt{2}(\rho - \bar{\rho})$$

where $\Phi_{\lambda,h} : [0,h] \to \mathbb{R}$ is defined by

$$\Phi_{\lambda,h}(s) := \int_0^s \frac{d\xi}{\sqrt{F_\lambda(\xi) - F_\lambda(h)}}$$

(it is simple to check the the integral converges at $\xi = h$). So we deduce

$$\Phi^{-1}_{\lambda,h}\left(\Phi_{\lambda,h}(h) - \sqrt{2}\,(\rho - \bar{\rho})\right) \leq w(\rho) \leq \Phi^{-1}_{\lambda,h}\left(\Phi_{\lambda,h}(h) - \sqrt{2}\bar{\rho}\ln\left(\frac{\rho}{\bar{\rho}}\right)\right)$$

which we prefer to write as

$$\Phi^{-1}_{\lambda,h}\left(\Phi_{\lambda,h}(h) + \sqrt{2}\,(\bar{\rho} - \rho)\right) \leq w(\rho) \leq \Phi^{-1}_{\lambda,h}\left(\Phi_{\lambda,h}(h) + \sqrt{2}\bar{\rho}\ln\left(\frac{\bar{\rho}}{\rho}\right)\right). \tag{4.6}$$

In particular, taking $\rho = \rho_0$, which gives $w(\rho_0) = 0$, (and using (4.3)) we have

$$\sqrt{2}\frac{\bar{\rho}}{\rho_0}(\rho_0 - \bar{\rho}) \leq \sqrt{2}\bar{\rho}\ln\left(\frac{\rho_0}{\bar{\rho}}\right) \leq \Phi_{\lambda,h}(h) \leq \sqrt{2}\,(\rho_0 - \bar{\rho}). \tag{4.7}$$

Moreover taking $\rho_1 = \bar{\rho}$ and $\rho_2 = \rho_0$ in (4.4) we have:

$$\sqrt{2}\frac{\bar{\rho}}{\rho_0}\sqrt{-F_\lambda(h))} \leq -\dot{w}(\rho_0) \leq \sqrt{2}\sqrt{-F_\lambda(h))} \tag{4.8}$$

Case (B). We rename $\rho_0 := r_1$, $\bar{\rho} := r_2$ and let $h := w(\bar{\rho}) < 0$. We use (4.5) with $\rho_1 = \sigma \in [\rho_0, \bar{\rho}]$ and $\rho_2 = \bar{\rho}$:

$$2\bar{\rho}^2(F_\lambda(h) - F_\lambda(w(\sigma))) \leq -\sigma^2\dot{w}(\sigma)^2 \leq 2\sigma^2(F_\lambda(h) - F_\lambda(w(\sigma))).$$

We change sign and proceed as in case (A):

$$2\sigma^2(F_\lambda(w(\sigma)) - F_\lambda(h)) \leq \sigma^2\dot{w}(\sigma)^2 \leq 2\bar{\rho}^2(F_\lambda(w(\sigma)) - F_\lambda(h))).$$

Take the square root and divide:

$$\sqrt{2} \leq \frac{-\dot{w}(\sigma)}{\sqrt{F_\lambda(w(\sigma)) - F_\lambda(h)}} \leq \sqrt{2}\frac{\bar{\rho}}{\sigma}.$$

Integrate on $[\rho, \bar{\rho}_0]$:

$$\sqrt{2}(\bar{\rho} - \rho) \leq -\Phi_{\lambda,h}(h) + \Phi_{\lambda,h}(w(\rho)) \leq \sqrt{2}\bar{\rho}\ln\left(\frac{\bar{\rho}}{\rho}\right)$$

defining $\Phi_{\lambda,h} : [h, 0] \to \mathbb{R}$ as in case (A). Applying $\Phi^{-1}_{\lambda,h}$ we get that (4.6) holds in case (B) too. In particular, taking $\rho = \rho_0$ (and using (4.3)):

$$\sqrt{2}(\bar{\rho} - \rho_0) \le -\Phi_{\lambda,h}(h) \le \sqrt{2}\bar{\rho} \ln\left(\frac{\bar{\rho}}{\rho_0}\right) \le \sqrt{2}\frac{\bar{\rho}}{\rho_0}(\bar{\rho} - \rho_0) \tag{4.9}$$

and taking $\rho_1 = \rho_0$ and $\rho_2 = \bar{\rho}$ in (4.5) we have

$$\sqrt{2}\sqrt{-F_\lambda(h))} \le -\dot{w}(\rho_0) \le \sqrt{2}\frac{\bar{\rho}}{\rho_0}\sqrt{-F_\lambda(h))} \tag{4.10}$$

Case (C). We rename $\bar{\rho} := r_1$, $\rho_0 := r_2$ end let $h := w(\bar{\rho}) < 0$. Using (4.4) with $\rho_1 = \bar{\rho}$ and $\rho_2 = \sigma \in [\bar{\rho}, \rho_0]$ we obtain the same inequality of case (A). After taking the square root and dividing:

$$\sqrt{2}\frac{\bar{\rho}}{\sigma} \le \frac{\dot{w}(\sigma)}{\sqrt{F_\lambda(w(\sigma)) - F_\lambda(h)}} \le \sqrt{2}.$$

We integrate between $\bar{\rho}$ and $\rho \in [\bar{\rho}, \rho_0]$ getting

$$\sqrt{2}\bar{\rho} \ln\left(\frac{\rho}{\bar{\rho}}\right) \le \Phi_{\lambda,h}(w(\rho)) - \Phi_{\lambda,h}(h) \le \sqrt{2}(\rho - \bar{\rho})$$

with $\Phi_{\lambda,h} : [h, 0] \to \mathbb{R}$ defined as above. So we deduce

$$\Phi^{-1}_{\lambda,h}\left(\Phi_{\lambda,h}(h) + \sqrt{2}\bar{\rho} \ln\left(\frac{\rho}{\bar{\rho}}\right)\right) \le w(\rho) \le \Phi^{-1}_{\lambda,h}\left(\Phi_{\lambda,h}(h) + \sqrt{2}\,(\rho - \bar{\rho})\right). \tag{4.11}$$

In particular, taking $\rho = \rho_0$ (and using (4.3)):

$$\sqrt{2}\frac{\bar{\rho}}{\bar{\rho}_0}(\rho_0 - \bar{\rho}) \le \sqrt{2}\bar{\rho} \ln\left(\frac{\rho_0}{\bar{\rho}}\right) \le -\Phi_{\lambda,h}(h) \le \sqrt{2}\,(\rho_0 - \bar{\rho})\,. \tag{4.12}$$

Moreover taking $\rho_1 = \bar{\rho}$ and $\rho_2 = \rho_0$ in (4.4) we have

$$\sqrt{2}\frac{\bar{\rho}}{\rho_0}\sqrt{-F_\lambda(h))} \le \dot{w}(\rho_0) \le \sqrt{2}\sqrt{-F_\lambda(h))}. \tag{4.13}$$

Case (D). We rename $\rho_0 := r_1$, $\bar{\rho} := r_2$ and let $h := w(\bar{\rho}) > 0$. Using (4.5) with $\rho_1 = \sigma \in [\rho_0, \bar{\rho}]$ and $\rho_2 = \bar{\rho}$ we obtain the same inequalities of case (B). When we take the square root and divide:

$$\sqrt{2} \le \frac{\dot{w}(\sigma)}{\sqrt{F_\lambda(w(\sigma)) - F_\lambda(h)}} \le \sqrt{2}\frac{\bar{\rho}}{\sigma}.$$

Integrate on $[\rho, \bar{\rho}_0]$:

$$\sqrt{2}(\bar{\rho} - \rho) \le \Phi_{\lambda,h}(h) - \Phi_{\lambda,h}(w(\rho)) \le \sqrt{2}\bar{\rho} \ln\left(\frac{\bar{\rho}}{\rho}\right)$$

with the usual definition of $\Phi_{\lambda,h} : [h, 0] \to \mathbb{R}$. Applying $\Phi^{-1}_{\lambda,h}$ we obtain that (4.11) holds in case (D) too. In particular, taking $\rho = \rho_0$ (and using (4.3)):

$$\sqrt{2}\,(\bar{\rho} - \rho_0) \le \Phi_{\lambda,h}(h) \le \sqrt{2}\bar{\rho} \ln\left(\frac{\bar{\rho}}{\bar{\rho}_0}\right) \le \sqrt{2}\frac{\bar{\rho}}{\rho_0}(\bar{\rho} - \rho_0) \tag{4.14}$$

and taking $\rho_1 = \rho_0$ and $\rho_2 = \bar{\rho}_0$ in (4.5) we have

$$\sqrt{2}\sqrt{-F_\lambda(h))} \le \dot{w}(\rho_0) \le \sqrt{2}\frac{\bar{\rho}}{\rho_0}\sqrt{-F_\lambda(h))}. \tag{4.15}$$

Now we have

$$\sqrt{2}\Phi_{\lambda,h}(h) = \int_0^h \frac{d\xi}{\sqrt{F(\sqrt{\lambda}\xi) - F(\sqrt{\lambda}h)}} = \int_0^1 \frac{h\, d\sigma}{\sqrt{F(\sigma\sqrt{\lambda}h) - F(\sqrt{\lambda}h)}} = \frac{1}{\sqrt{\lambda}}\bar{\Phi}(\sqrt{\lambda}h)$$

where

$$\bar{\Phi}(s) := \int_0^1 \frac{s\, d\sigma}{\sqrt{F(\sigma s) - F(s)}} = \operatorname{sgn}(s)\int_0^1 \sqrt{\frac{s^2}{F(\sigma s) - F(s)}}\, d\sigma.$$

With simple computations:

$$\lim_{s\to 0} \frac{s^2}{F(\sigma s) - F(s)} = \frac{1}{1-\sigma^2}, \qquad \lim_{s\to +\infty} \frac{s^2}{F(\sigma s) - F(s)} = \frac{2}{1-\sigma^2},$$

and

$$\lim_{s\to -1^-} \frac{s^2}{F(\sigma s) - F(s)} = 0.$$

So we deduce that (see Figure 4.2)

$$\lim_{h\to 0^+} \Phi_{\lambda,h}(h) = \frac{\pi}{2\sqrt{2\lambda}}, \qquad \lim_{h\to +\infty} \Phi_{\lambda,h}(h) = \frac{\pi}{2\sqrt{\lambda}}, \tag{4.16}$$

$$\lim_{h\to 0^-} \Phi_{\lambda,h}(h) = -\frac{\pi}{2\sqrt{2\lambda}}, \qquad \lim_{h\to -1^+} \Phi_{\lambda,h}(h) = 0. \tag{4.17}$$

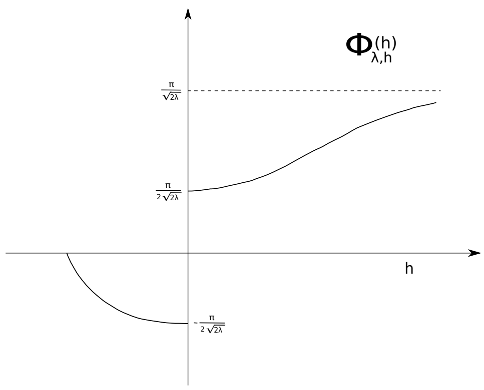

Figure 4.2: Graph of $\Phi_{\lambda,h}(h)$

To state the main result we need some notation, which we take from [8,23]. For $k \in \mathbb{N}$, $k \ge 1$, we consider

$$\begin{aligned}
\mathcal{S} &:= \{(w,\lambda) \in \mathcal{W} : (w,\lambda) \text{ is a solution to (4.1)}\} \\
\mathcal{S}_k^+ &:= \{(w,\lambda) \in \mathcal{S} :\ w \text{ has } k \text{ nodes in }]0,R[, w(0) > 0\}, \\
\mathcal{S}_k^- &:= \{(w,\lambda) \in \mathcal{S} :\ w \text{ has } k \text{ nodes in }]0,R[, w(0) < 0\}.
\end{aligned}$$

We also consider the two eigenvalue problems:

$$\ddot{w} + \frac{\dot{w}}{\rho} = -\mu w, \qquad \dot{w}(0) = \dot{w}(R) = 0. \tag{4.18}$$

$$\ddot{v} + \frac{\dot{v}}{\rho} = -\nu v, \qquad \dot{v}(0) = 0, v(R) = 0. \tag{4.19}$$

It is clear that $w \neq 0$ and $\mu \neq 0$ solve (4.18) if and only if, for some integer $k \geq 1$,

$$\mu = \mu_k := \left(\frac{y_k}{R}\right)^2 \tag{4.20}$$

where y_k denotes the k-th nontrivial zero of J_0' and J_0 is the first Bessel function, and

$$w = \alpha w_k, \quad \alpha \in \mathbb{R}, \qquad w_k(\rho) := J_0\left(\frac{y_k}{R}\rho\right). \tag{4.21}$$

For the sake of completeness we can agree that $\mu_0 = 0$ and $w_0(\rho) = J_0(0)$. In the same way $v \neq 0$ and ν solve (4.19) if and only if, for some integer $k \geq 1$:

$$\nu = \nu_k := \left(\frac{z_k}{R}\right)^2 \tag{4.22}$$

where z_k is the k-th zero of J_0 and

$$v = \alpha v_k, \quad \alpha \in \mathbb{R}, \qquad v_k(\rho) := J_0\left(\frac{z_k}{R}\rho\right). \tag{4.23}$$

Notice that $\nu_k < \mu_k < \nu_{k+1}$ for all k.

Theorem 4.3. *Let $\mu_k > 0$ be an eigenvalue for (4.18). Then $\mathcal{S}_k^+$ is a connected set and*

- $(0, \mu_k/2) \in \overline{\mathcal{S}_k^+}$;
- $0 < \inf\{\lambda \in \mathbb{R} : \exists w \in E \text{ with } (w,\lambda) \in \mathcal{S}_k^+\}$;
- $\sup\{\lambda \in \mathbb{R} : \exists w \in E \text{ with } (w,\lambda) \in \mathcal{S}_k^+\} < +\infty$;
- *$\mathcal{S}_k^+$ is unbounded and contains a sequence (w_n, λ_n) such that $\|w_n\|_E \to \infty$ and*

$$\lim_{n\to\infty} \lambda_n = \begin{cases} \mu_{k/2} & \text{if } k \text{ is even,} \\ \nu_{(k+1)/2} & \text{if } k \text{ is odd.} \end{cases} \tag{4.24}$$

Figure 4.3 somehow illustrates Theorem (4.3).

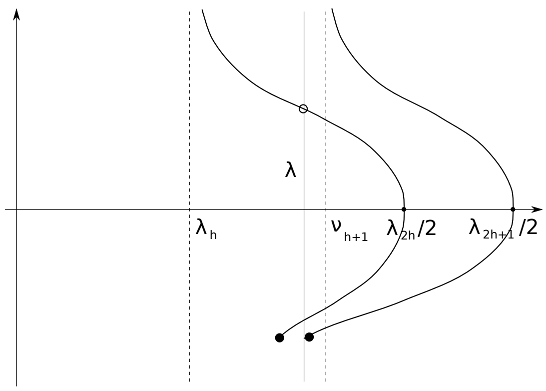

Figure 4.3: Bifurcation diagram

The proof of (4.3) will be obtained from some preliminary statements.

Remark 4.4. If $(w, \lambda) \in \mathcal{S}^+$ (resp. $(w, \lambda) \in \mathcal{S}^+$), and $0 = \rho_0 < \rho_1, \dots, \rho_k < \rho_{k+1=R}$, $\rho_1, \dots, \rho_k$ being the nodal points of w, then

$$\rho_{i+1} - \rho_i \geq \ (\leq) \ \frac{\pi}{4\sqrt{\lambda}} \quad \text{for } i \text{ even} \quad (\text{resp. for } i \text{ odd}). \tag{4.25}$$

This is easily seen using the right hand sides of the inequalities (4.7), (4.12), and (4.16).

Lemma 4.5. *For any integer k there esist two constants $\underline{\lambda}_k$ and $\overline{\lambda}_k$ such that*

$$(w, \lambda) \in \mathcal{S}_k^+ \cup \mathcal{S}_k^- \Rightarrow 0 < \underline{\lambda}_k \leq \lambda \leq \overline{\lambda}_k < +\infty. \tag{4.26}$$

Proof. Take any subinterval $[r_1, r_2]$ as in cases (A)–(D) and consider the first eigenvalue $\bar{\mu} = \bar{\mu}(r_1, r_2)$ for the mixed type boundary condition

$$\begin{cases} -(\rho \dot{w})' = \mu w \quad \text{on }]r_1, r_2[\\ \dot{w}(r_1) = 0, w(r_2) = 0 \qquad (\text{resp. } w(r_1) = 0, \dot{w}(r_2) = 0) \end{cases}$$

in cases (A), (C) (resp. cases (C), (D)). We can choose an eigenfunction $\bar{e}$ corresponding to $\bar{\mu}$ so that $z\bar{e} > 0$ in $]r_1, r_2[$. Multiplying (4.1) by $\bar{e}$ and integrating over $[r_1, r_2]$ yields

$$\bar{\mu} \int_{r_1}^{r_2} \rho z \bar{e} \, d\rho = \lambda \int_{r_1}^{r_2} \rho z \bar{e} \left(1 + \frac{1}{1 + \sqrt{\lambda z}} \right) d\rho.$$

This implies:

$$\lambda \int_{r_1}^{r_2} \rho z \bar{e} \, d\rho \leq \bar{\mu} \int_{r_1}^{r_2} \rho z \bar{e} \, d\rho \leq 2\lambda \int_{r_1}^{r_2} \rho z \bar{e} \, d\rho$$

which gives $\frac{\bar{\mu}}{2} \leq \lambda \leq \bar{\mu}$. Now since $]r_1, r_2[\subset]0, R[$ we have $\bar{\mu} \geq \bar{\mu}[0, R]$. On the other side since w has k nodal points we can choose r_1, r_2 such that $r_2 - r_1 \geq R/k$, which implies $\bar{\mu} \leq \sup_{b-a=R/k} \bar{\mu}(a, b) < +\infty$. This proves (4.26). □

Lemma 4.6. *Let (w_n, λ_n) be a sequence in $\mathcal{S}_k^+$. Then we can consider $0 < \rho_{1,n} < \dots < \rho_{k,n} < R$ to be the nodes of w_n and set $\rho_{0,n} := 0$, $\rho_{k+1,n} := R$; in ths way $wn(\rho) > 0$ on $]\rho_1, \rho_{i+1}[$ if i is even and $wn(\rho) < 0$ on $]\rho_1, \rho_{i+1}[$ if i is odd. The following facts are equivalent:*

(a)
$$\lim_{n \to \infty} \sup_{\rho \in [0,R]} w_n(\rho) = +\infty;$$

(b)
$$\lim_{n \to \infty} \inf_{\rho \in [0,R]} (1 + \lambda_n w_n(\rho)) = 0;$$

(c)
$$\lim_{n \to \infty} \sup_{\rho \in [\rho_{i,n}, \rho_{i+1,n}]} w_n(\rho) = +\infty \quad \textit{if i is even;}$$

(d)
$$\lim_{n \to \infty} \inf_{\rho \in [\rho_{i,n}, \rho_{i+1,n}]} (1 + \lambda_n w_n(\rho)) = 0 \quad \textit{if i is odd;}$$

(e)
$$\lim_{n \to \infty} \rho_{1+1,n} - \rho_{i,n} = 0 \quad \textit{if i is odd;}$$

Moreover, if any of the above holds, then (4.24) holds.

Proof. We can assume, passing to a subsequence that $\lambda_n \to \hat{\lambda} \in [\underline{\lambda}_k, \overline{\overline{\lambda}}_k]$. First notice that for all i even (corresponding to $w > 0$) we have

$$\rho_{i+1,n} - \rho_{i,n} \geq \frac{\pi}{4\sqrt{\underline{\lambda}_k}}$$

as we can infer from (4.7) or (4.14) and the behaviour of $\Phi_{\lambda,h}(h)$ in (4.16).

Let

$$h_{i,n} := \max_{\rho_{i,n} \le \rho_{i+1,n}} w(\rho) \text{ for } i \text{ even}, \qquad h_{i,n} := \min_{\rho_{i,n} \le \rho_{i+1,n}} w(\rho) \text{ for } i \text{ odd}.$$

Then for any i even:

$$h_{i,n} \to +\infty \Leftrightarrow \Phi_{\lambda_n, h_{i,n}}(h_{i,n}) \to \frac{\pi}{2\sqrt{\hat{\lambda}}} \Leftrightarrow \dot{w}(\rho_{i,n}) \to +\infty \Leftrightarrow \dot{w}(\rho_{i+1,n}) \to -\infty.$$

This can be deduced from (4.16), (4.8), and (4.15). In the same way, using (4.17), (4.10), and (4.13) we get that, for i odd:

$$1 + \sqrt{\lambda_n} h_{i,n} \to 0 \Leftrightarrow \Phi_{\lambda_n, h_{i,n}}(h_{i,n}) \to 0 \Leftrightarrow \dot{w}(\rho_{i,n}) \to -\infty \Leftrightarrow \dot{w}(\rho_{i+1,n}) \to +\infty.$$

Now we prove our claims. Let $\bar{i} \in \{0, \dots, k\}$ with $\bar{i}$ even (resp. odd) and suppose that $h_{\bar{i},n} \to +\infty$ (resp. $1 + \sqrt{\lambda_n} h_{\bar{i},n} \to 0$). Then $F_{\lambda_n}(h_{\bar{i},n}) \to +\infty$ (resp. $F_{\lambda_n}(h_{\bar{i},n}) \to -\infty$) and by (4.8), (4.15) ((4.10), (4.13)) we get that

$$\dot{w}_n(\rho_{\bar{i},n}) \to +\infty, \ \dot{w}_n(\rho_{\bar{i}+1,n}) \to -\infty \quad (\dot{w}_n(\rho_{\bar{i},n}) \to -\infty, \ \dot{w}_n(\rho_{\bar{i}+1,n}) \to +\infty)$$

which in turn implies

$$F_{\lambda_n}(h_{\bar{i}-1,n}) \to -\infty \text{ (resp. } +\infty), \ F_{\lambda_n}(h_{\bar{i}+1,n}) \to -\infty \text{ (resp. } +\infty)$$

(with the obvious exceptions when $\bar{i} - 1 < 0$ or $\bar{i} + 1 > k$). So we get

$$1 + \sqrt{\lambda_n} h_{\bar{i}-1,n} \to 0 \ (h_{\bar{i}-1,n} \to +\infty), \ 1 + \sqrt{\lambda_n} h_{\bar{i}+1,n} \to 0 \ (h_{\bar{i}+1,n} \to +\infty).$$

This shows that the property $|F_\lambda(h_{i,n})| \to +\infty$ "propagates" from the i-th interval to the previous and to the next one. From this it is easy to deduce that (a)–(d) are all equivalent. To prove that they are equivalent to (e) just use (4.7), (4.9), (4.12), (4.14), depending on the case, noticing that $\rho_{1,n} \ge \frac{\pi}{4\pi\underline{\lambda}_k}$, as from (4.25) (this would not be possible if we were considering $\mathcal{S}_k^-$).

Finally suppose that (w_n, λ_n) verifies any of (a)–(e). Then $\|w_n\|_\infty \to +\infty$. Let $\hat{w}_n := \frac{w_n}{\|w_n\|_\infty}$. We can suppose that $\hat{w}_n \rightharpoonup \hat{w}$ in E and that

$$\rho_{1,n} \to \rho_1, \ \rho_{2j-1,n} \to \rho_j, \ \rho_{2j,n} \to \rho_j \ 1 \le j \le k/2, \ \rho_{k,n} \to R \text{ if } k \text{ is odd},$$

where $0 = \rho_0 < \rho_1 < \cdots < \rho_h < \rho_h + 1 = R$ and $h = \lfloor k/2 \rfloor$ (so $\rho_1 = R$ when $k = 1$). It is not difficult to prove that $\hat{w}(\rho) > 0$ in $]\rho_i, \rho_{i+1}[$ if $i = 0, \dots, h$, $\hat{w}(\rho_1) = \cdots = \hat{w}(\rho_h) = 0$, $\hat{w}'(0) = 0$ and $\hat{w}'(R) = 0$ is k is even while $\hat{w}(R) = 0$ is k is odd. Moreover for any $i = 0, \dots, h$:

$$-(\rho \hat{w}')' = \hat{\lambda} \hat{w} \quad \text{on }]\rho_i, \rho_{i+1}[$$

Now we can rearrange $\hat{w}$ defining $\tilde{w} := \sum_{j=0}^h (-1)^j \alpha_j \hat{w} \mathbb{1}_{[\rho_j, \rho_{j+1}]}$, where $\alpha_1 = 1$ and $\alpha_j \hat{w}'_-(\rho_j) = \alpha_{j+1} \hat{w}'_+(\rho_j)$, $j = 1, \dots, h$. In this way $(\hat{\lambda}, \tilde{w})$ is an eigenvalue – eigenfunction pair relative for problem (4.21) if k is even and of (4.23) if k is odd. Since $\tilde{w}$ has $h = k/2$ nodal points for k even and $h + 1 = (k+1)/2$ if k is odd, then (4.24) holds. □

Proof of Theorem 4.3. If $\varepsilon \in]0, 1[$ we set

$$\mathcal{O}_\varepsilon := \left\{ (w, \lambda) \in E : \varepsilon < \lambda < \varepsilon^{-1}, 1 + \sqrt{\lambda} w(\rho) > \varepsilon, w(\rho) < \varepsilon^{-1} \ \forall \rho \in [0, R] \right\}.$$

Clearly $\mathcal{O}_\varepsilon$ is an open set with $\mathcal{O}_\varepsilon \subset \mathcal{W}$. Moreover, $(\mu_{k/2}, 0) \in \mathcal{O}_\varepsilon$ if ε is sufficiently small. Define $\tilde{h}_{\lambda,\varepsilon}$ as in (2.4) with $s_0 = \varepsilon$ and let $\tilde{h}_\lambda(s) := \tilde{h}_1(\sqrt{\lambda}s)$. Using [23] we get there that there exists a pair $(w_\varepsilon, \lambda_\varepsilon)$ in $\partial\mathcal{O}_\varepsilon$, with w_ε having k nodal points, which solves Problem (4.1) with $\tilde{h}_{\varepsilon,\lambda} := \tilde{h}_\varepsilon(\lambda, \cdot)$ instead of h_λ. Since $(w, \lambda) \in \partial\mathcal{O}_\varepsilon \Rightarrow \tilde{h}_\varepsilon(w, \lambda) = h_\lambda(w)$, we get that $(w_\varepsilon, \lambda_\varepsilon) \in \mathcal{S}_k^+$. For ε small we have $\varepsilon < \underline{\lambda}_k \leq \overline{\lambda}_k < \varepsilon^{-1}$ so we get $w_\varepsilon \in \partial\left\{1 + \sqrt{\lambda_\varepsilon}w > \varepsilon, w < \varepsilon^{-1}\right\}$ i.e. there exists a point $\rho_\varepsilon \in [0, R]$ such that

$$\text{either} \quad 1 + \sqrt{\lambda_\varepsilon}w_\varepsilon(\rho_\varepsilon) = \varepsilon \quad \text{or} \quad w_\varepsilon(\rho_\varepsilon) = \varepsilon^{-1}.$$

We can find a sequence $\varepsilon_n \to 0$ such that the corresponding $(w_n, \lambda_n) := (w_{\varepsilon_n}, \lambda_{\varepsilon_n})$ verify one of the above properties for all $n \in \mathbb{N}$. If the first one holds for all n, then (w_n, λ_n) verifies (b) of Lemma (4.6); in the second case (w_n, λ_n) verifies (a) of Lemma (4.6). Then by Lemma (4.6) $\|w_n\|_\infty \to \infty$ and (4.24) holds. This proves the theorem. □

Remark 4.7. As a consequence of Theorem (4.3) we get that for any $h \geq 1$ integer and any λ strictly between λ_h and $\lambda_{2h}/2$ there exists u such that (u, λ) solves Problem (1.2). The same is true for all λ strictly between ν_h and $\lambda_{2h-1}/2$.

Remark 4.8. The above proof fails if we follow the bifurcation branch (w_ρ, λ_ρ) with $w_\rho(0) < 0$. In this case it seems possible that the branch tends to a point $(\tilde{\lambda}, \tilde{w})$ where $\sqrt{\tilde{\lambda}}\tilde{w}(0) = -1$ (but $\sqrt{\tilde{\lambda}}\tilde{w}(0) > -1$ for $\rho > 0$). This phenomenon, if true, would be worth studying.

Remark 4.9. The computations of this section show that, if Ω is the ball, then there are no solutions for the Dirichlet problem. It is indeed impossible to construct a (nontrivial) solution (w, λ) for (4.1) with $w(R) = 0$.

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Existence and regularity of pullback attractors for a 3D non-autonomous Navier–Stokes–Voigt model with finite delay

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Dedicated to the memory of Isabel Morillo Montaño 'Beli', with love

Abstract. In this manuscript previous results [*Nonlinearity* **25**(2012), 905–930] are extended to a non-autonomous 3D Navier–Stokes–Voigt model in which a forcing term contains memory effects. Under suitable assumptions on the function driving the delay time, the existence and uniqueness of weak solution are proved. Existence and relationships among pullback attractors in several phase-spaces are analyzed for two possible choices of the attracted universes, namely, the standard one of fixed bounded sets, and another one given by a tempered condition. Some regularity results for these attractors are also established. Compactness and attraction norms are strengthened. Since the model does not have a regularizing effect, obtaining asymptotic compactness for the associated process is a more involved task. Our proofs rely on a sharp use of the energy equality, an energy method, bootstrapping arguments and by using bi-space attractors results.

Keywords: 3D Navier–Stokes–Voigt equations, delay terms, pullback attractors, bi-space attractors.

2020 Mathematics Subject Classification: 35Q30, 35Q35, 35B40, 35B41, 76F20.

1 Introduction and setting of the problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth enough (e.g. C^2) boundary $\partial\Omega$. We consider an arbitrary initial time $\tau \in \mathbb{R}$, and the following non-autonomous functional Navier–Stokes–

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Voigt problem:

$$\begin{cases} \dfrac{\partial}{\partial t}(u-\alpha^2\Delta u) - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(t) + g(t,u_t) \quad \text{in } \Omega\times(\tau,\infty), \\ \operatorname{div} u = 0 \quad \text{in } \Omega\times(\tau,\infty), \\ u = 0 \quad \text{on } \partial\Omega\times(\tau,\infty), \\ u(x,\tau) = u^\tau(x), \quad x\in\Omega, \\ u(x,\tau+s) = \phi(x,s), \quad x\in\Omega,\ s\in(-h,0), \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity, $\alpha > 0$ is a characterizing parameter of the elasticity of the fluid, $u = (u_1, u_2, u_3)$ is the velocity field of the fluid, p is the pressure, f is a non-delayed external force field, g is another external force containing some hereditary characteristics, and u^τ and $\phi(x, s-\tau)$ are the initial data in τ and $(\tau - h, \tau)$ respectively, where $h > 0$ is the time of memory effect. For each $t \geq \tau$, we denote by u_t the function defined a.e. on $(-h, 0)$ by the relation $u_t(s) = u(t+s)$, a.e. $s \in (-h, 0)$.

The Navier–Stokes–Voigt (NSV for short in the sequel) model of viscoelastic incompresible fluid, introduced by Oskolkov in [29], gives an approximate description of the Kelvin-Voigt fluid (see [22, 30]), and was proposed as a regularization of the 3D-Navier–Stokes equations for the purpose of direct numerical simulations in [2]. The extra regularizing term $-\alpha^2\Delta u_t$ changes the parabolic character of the equation, making it a well-posed (forward and backward) problem in 3D, but one does not observe any immediate smoothing of the solution, as expected in parabolic PDEs. Moreover, the generated semigroup is only asymptotically compact, similarly to damped hyperbolic systems. One of the studied topics about the problem is the inviscid question in some different senses. It is also worth observing that, when $\nu = 0$, the inviscid equation that one recovers is the simplified Bardina subgrid scale model of turbulence. The relationship between the original and inviscid models was also addressed in [2]. On other hand, some questions on the inviscid regularization were used for the study of a 2D surface quasi-geostrophic model in [21].

With respect to the non-delayed NSV model, the long-time behaviour of the solutions has been studied by different authors. Namely, in the autonomous case, the existence of a global compact attractor was proved by Kalantarov and Titi in [20]. Other related results have been also analyzed, as the Gévrey regularity of the global attractor (again for the autonomous model) when the force term is analytic of Gévrey type (see [19]), and the establishment of similar statistical properties (and invariant measures) as for the 3D-Navier–Stokes equations (cf. [23, 31]). Moreover, in the non-autonomous case, the existence of minimal pullback attractors in both V and $D(A)$ norms, and some regularity properties of these attractors, were obtained in [14]. We may also cite in this non-autonomous framework the paper [40], where the existence of uniform attractor for a NSV model is studied.

On the other hand, in many physical experiments, the inclusion of measurement devices to control properties of fluids (such as temperature, velocity, etc.) may incorporate additional external forces to the model including also delay effects (e.g. for a wind-tunnel model). In this sense, the study of 2D-Navier–Stokes models with delay terms – existence, uniqueness, stationary solutions, exponential decay, existence of attractors, et cetera – was initiated in the references [6–8] and, after that, many different questions, as dealing with unbounded domains, and models (for instance in three dimensions for modified terms) have been addressed (e.g., cf. [15, 17, 26, 28, 36] among others). In the past years, the asymptotic behaviour of the Navier–Stokes–Voigt equations with delays or with memory have been studied in [3, 12, 24, 34, 35, 38]. It is worth pointing out that, in [24], the authors establish the existence of pullback attractors in V norm for a three dimensional NSV model when the forcing term containing the delay is sublinear and only continuous. Since the uniqueness of solution is not guaranteed under these assumptions, they use the theory of multi-valued dynamical systems and similar arguments as in [28] for the proof of the asymptotic compactness of the process. In this work, we suppose more restrictive conditions on the delay operator that assure the uniqueness of solution, so we can apply the classical results of Dynamical Systems. However, in contrast with [24], we modify the phase-space enlarging the set of initial conditions. Moreover, for the associated single-valued process, we are able to obtain the existence of minimal pullback attractors, with richer compactness sections and not only in (roughly speaking) V norm, but also in $D(A)$ norm. Moreover, some regularity properties of these attractors are also successfully established. This analysis is carried out by applying similar techniques as in [14], but with the necessary modifications caused by the inclusion of a delay term.

As commented before, the difference between this model and the *standard* 2D-Navier–Stokes model is that there exists a regularizing effect in the 2D-Navier–Stokes model, while not here. For 2D-Navier–Stokes a continuous energy method can be applied thanks to the extra estimates that hold in higher norms (e.g., cf. [28]), which does not seem to hold for the NSV model. Some of the proofs in the previously cited references about NSV (e.g., cf. [20]) rely on splitting the problem in two, one with exponential decay, and the other with good asymptotic properties in the domain of a suitable fractional power of the Stokes operator. However, similarly as in [14], we will provide a simpler proof, which does not require the above mentioned technicalities, but a sharp use of the energy equality, and the energy method used by Rosa in [32]. Moreover, it is worth pointing out that our results in Section 3 do not use the regularity assumption on $\partial\Omega$ at all, and the force term may take values in V' instead of in L^2 as appears in [20].

The structure of the paper is the following. In Section 2 we recall some definitions of classical functional spaces to state our problem in an abstract form, basic properties and estimates of the involved operators. We also obtain a result on the existence, uniqueness and regularity of the weak solution for problem (1.1). We start Section 3 with a brief recall of the main definitions on the theory of minimal pullback attractors and bi-space attractors for non-autonomous dynamical systems within the framework of universes. Then, we prove the existence of pullback attractors in (roughly speaking) V norm and for two choices of the attracted universes, namely, the standard one of fixed bounded sets, and secondly, one given by a tempered growth condition. We also establish some relations among these families and improve compactness and attraction norm results. In Section 4, extra regularity for the obtained attractors will be deduced by using a bootstrapping argument that involves fractional powers of the Stokes operator. Finally, in Section 5, the problem of attraction in $D(A)$ norm is studied although it is more involved (namely it fits out from the standard theoretical results). Indeed under suitable assumptions, all attractors are proved to coincide.

2 Existence and uniqueness of solution

In this section we prove existence, uniqueness and regularity of the solutions to problem (1.1). These results will be obtained in a similar way as in [14], but with the necessary changes due to the inclusion of a delay term. We begin by stating the problem in an abstract setting, and to do so we recall several definitions of functional spaces, operators and some of their properties (for the details see [37]).

To start with, we consider the usual spaces in the variational theory of Navier–Stokes equations: H, the closure of $\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0\}$ in $(L^2(\Omega))^3$ with norm $|\cdot|$, and inner product $(\cdot,\cdot)$, and V, the closure of $\mathcal{V}$ in $(H_0^1(\Omega))^3$ with norm $\|\cdot\|$, and inner product $((\cdot,\cdot))$, that is, the L^2-product of gradients, thanks to the Poincaré inequality.

We will use $\|\cdot\|_*$ for the norm in V' and $\langle\cdot,\cdot\rangle$ for the duality $\langle V', V\rangle$. We consider every element $h \in H$ as an element of V', given by the equality $\langle h, v\rangle = (h, v)$ for all $v \in V$. It follows that $V \subset H \subset V'$, where the injections are dense and compact.

Let us define the linear continuous operator $A : V \to V'$ as $\langle Au, v\rangle = ((u, v))$ for all $u, v \in V$, and we denote $D(A) = \{u \in V : Au \in H\}$. By the regularity of $\partial\Omega$, one has that $D(A) = (H^2(\Omega))^3 \cap V$, and $Au = -P\Delta u$ for all $u \in D(A)$ is the Stokes operator (P is the ortho-projector from $(L^2(\Omega))^3$ onto H). On $D(A)$ we consider the norm $|\cdot|_{D(A)}$ defined by $|u|_{D(A)} = |Au|$. Observe that on $D(A)$ the norms $\|\cdot\|_{(H^2(\Omega))^3}$ and $|\cdot|_{D(A)}$ are equivalent, and $D(A)$ is compactly and densely injected in V. We will also denote by $\{w_j\}_{j\geq 1} \subset D(A)$ a Hilbert basis of H formed by normalized eigenfunctions of the Stokes operator A, with corresponding eigenvalues $\{\lambda_j\}_{j\geq 1}$ being $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and $\lim_{j\to\infty} \lambda_j = \infty$. Recall that the first eigenvalue of A satisfies

$$\lambda_1 = \inf_{v \in V\setminus\{0\}} \frac{\|v\|^2}{|v|^2}. \tag{2.1}$$

For the fractional powers of A, we have the following inclusions with continuous injection (cf. [33, Chapter III, Lemmas 2.4.2 and 2.4.3]):

$$D(A^\beta) \subset (L^{6/(3-4\beta)}(\Omega))^3, \quad \forall\, 0 \leq \beta < 3/4, \tag{2.2}$$

$$D(A^{3/4}) \subset (L^p(\Omega))^3, \qquad \forall\, 1 \leq p < \infty, \tag{2.3}$$

and

$$D(A^\beta) \subset (L^\infty(\Omega))^3, \quad \forall\, 3/4 < \beta \leq 1. \tag{2.4}$$

Now, we define

$$b(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

for every functions $u, v, w : \Omega \to \mathbb{R}^3$ for which the right-hand side is well defined. In particular, b has sense for all $u, v, w \in V$, and is a continuous trilinear form on $V \times V \times V$, i.e., there exists a constant $C_1 > 0$ such that

$$|b(u, v, w)| \leq C_1 \|u\| \|v\| \|w\|, \quad \forall u, v, w \in V. \tag{2.5}$$

Important properties concerning b are that

$$\begin{aligned} b(u, v, w) &= -b(u, w, v), \quad \forall u, v, w \in V, \\ b(u, v, v) &= 0, \quad \forall u, v \in V, \end{aligned} \tag{2.6}$$

and, using Agmon inequality (e.g. cf. [10]), we can assure that there exists a constant $C_2 > 0$ such that

$$|b(u, v, w)| \leq C_2 |Au|^{1/2} \|u\|^{1/2} \|v\| |w|, \quad \forall u \in D(A), v \in V, w \in H. \tag{2.7}$$

For any $u \in V$, we will use $B(u)$ to denote the element of V' given by $\langle B(u), w\rangle = b(u, u, w)$ for all $w \in V$. Thus, by (2.5),

$$\|B(u)\|_* \leq C_1 \|u\|^2, \quad \forall u \in V, \tag{2.8}$$

and in particular, by (2.7) and the identification of H' with H, if $u \in D(A)$, then $B(u) \in H$, with

$$|B(u)| \le C_2 |Au|^{1/2} \|u\|^{3/2}, \quad \forall u \in D(A). \tag{2.9}$$

In fact, from (2.4), one also deduces that if $u \in D(A^\beta)$ with $3/4 < \beta \le 1$, then $B(u) \in H$, and more exactly

$$|B(u)| \le C_{(\beta)} |A^\beta u| \|u\|, \quad \forall u \in D(A^\beta), \quad \forall\, 3/4 < \beta \le 1. \tag{2.10}$$

Analogously, if $0 \le \beta < 3/4$, from (2.2) one obtains that if $u \in D(A^\beta) \cap V$, $B(u) \in D(A^{\beta - 3/4})$, and more exactly

$$|A^{\beta-3/4} B(u)| \le C_{(\beta)} |A^\beta u| \|u\|, \quad \forall u \in D(A^\beta) \cap V, \quad \forall\, 0 \le \beta < 3/4. \tag{2.11}$$

Finally, in the case $\beta = 3/4$, from (2.3) one can see that if $u \in D(A^{3/4})$, then $B(u) \in D(A^{-\delta})$ for all $\delta > 0$, and more exactly

$$|A^{-\delta} B(u)| \le C_{(3/4,\delta)} |A^{3/4} u| \|u\|, \quad \forall u \in D(A^{3/4}), \quad \forall \delta > 0.$$

Now, we establish some appropriate assumptions on the term in (1.1) containing the delay.

Let $(X, \|\cdot\|_X)$ be a Banach space. We will denote $C_X = C([-h,0]; X)$, the space of continuous functions from $[-h, 0]$ into X, with the norm $\|\varphi\|_{C_X} = \max_{s \in [-h,0]} \|\varphi(s)\|_X$, and $L^2_X = L^2(-h, 0; X)$, where the norm will be denoted by $\|\cdot\|_{L^2_X}$. On the delay operator from (1.1), we consider that is well defined as $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$, and it satisfies the following assumptions:

(I) for all $\xi \in C_H$, the function $\mathbb{R} \ni t \mapsto g(t, \xi) \in (L^2(\Omega))^3$ is measurable,

(II) $g(t, 0) = 0$, for all $t \in \mathbb{R}$,

(III) there exists $L_g > 0$ such that for all $t \in \mathbb{R}$, and for all $\xi, \eta \in C_H$,

$$|g(t, \xi) - g(t, \eta)| \le L_g |\xi - \eta|_{C_H},$$

(IV) there exists $C_g > 0$ such that for all $\tau \le t$, and for all $u, v \in C([\tau - h, t]; H)$,

$$\int_\tau^t |g(s, u_s) - g(s, v_s)|^2 \, ds \le C_g^2 \int_{\tau - h}^t |u(s) - v(s)|^2 \, ds.$$

Examples of fixed, variable and distributed delay operators can be found, for instance, in [6, Section 3], [8, Sections 3.5 and 3.6], and [17, Section 3], and we omit them here just for the sake of brevity.

Observe that (I)–(III) imply that given $T > \tau$ and $u \in C([\tau - h, T]; H)$, the function $g_u : [\tau, T] \to (L^2(\Omega))^3$ defined by $g_u(t) = g(t, u_t)$ for all $t \in [\tau, T]$, is measurable and, in fact, belongs to $L^\infty(\tau, T; (L^2(\Omega))^3)$. Then, thanks to (IV), the mapping

$$\mathcal{G} : u \in C([\tau - h, T]; H) \to g_u \in L^2(\tau, T; (L^2(\Omega))^3)$$

has a unique extension to a mapping $\widetilde{\mathcal{G}}$ which is uniformly continuous from $L^2(\tau - h, T; H)$ into $L^2(\tau, T; (L^2(\Omega))^3)$. From now on, we will denote $g(t, u_t) = \widetilde{\mathcal{G}}(u)(t)$ for each $u \in L^2(\tau - h, T; H)$, and thus property (IV) will also hold for all $u, v \in L^2(\tau - h, T; H)$.

Since it will be used to deduce some estimates for the solutions of (1.1), we study the autonomous equation $u + \alpha^2 Au = \varphi$. From the Lax–Milgram lemma, we know that for each $\varphi \in V'$ there exists a unique $u_\varphi \in V$ such that $u_\varphi + \alpha^2 Au_\varphi = \varphi$. Therefore, the mapping

$$\mathcal{C} : u \in V \mapsto u + \alpha^2 Au \in V'$$

is linear and bijective, with $\mathcal{C}^{-1}\varphi = u_\varphi$. Moreover, by the definition of $D(A)$, we also have that $\mathcal{C}^{-1}(H) = D(A)$. Now, reasoning as in [14], we obtain that

$$\|u_\varphi\| \leq \alpha^{-2}\|\varphi\|_*, \quad \forall\varphi \in V', \tag{2.12}$$

and

$$|Au_\varphi| \leq 2\alpha^{-2}|\varphi|, \quad \forall\varphi \in H. \tag{2.13}$$

Let us consider that $u^\tau \in V$, $\phi \in L^2_H$, and $f \in L^2_{loc}(\mathbb{R}; V')$.

Definition 2.1. A weak solution to (1.1) is a function u that belongs to $L^2(\tau - h, T; H) \cap L^2(\tau, T; V)$ for all $T > \tau$, such that $u(\tau) = u^\tau$, $u(t) = \phi(t - \tau)$ a.e. $t \in (\tau - h, \tau)$, and satisfies

$$\frac{d}{dt}(u(t) + \alpha^2 Au(t)) + \nu Au(t) + B(u(t)) = f(t) + g(t, u_t), \quad \text{in } \mathcal{D}'(\tau, \infty; V'). \tag{2.14}$$

Observe that if u is a weak solution to (1.1), then $u(t) + \alpha^2 Au(t) \in L^2(\tau, T; V')$ for all $T > \tau$, and by (2.8), $\frac{d}{dt}(u(t) + \alpha^2 Au(t)) \in L^1(\tau, T; V')$ for all $T > \tau$. Therefore, by using (2.12) and reasoning as in [14], we can deduce that $u \in C([\tau, \infty); V)$, whence the initial datum $u(\tau) = u^\tau$ has full sense, and $u' \in L^2(\tau, T; V)$ for all $T > \tau$.

Furthermore, the following energy equality holds:

$$\frac{1}{2}\frac{d}{dt}(|u(t)|^2 + \alpha^2\|u(t)\|^2) + \nu\|u(t)\|^2 = \langle f(t), u(t)\rangle + (g(t, u_t), u(t)), \quad \text{a.e. } t > \tau. \tag{2.15}$$

Concerning the existence and uniqueness of weak solution to (1.1), we have the following result.

Theorem 2.2. *Let $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ satisfying (I)–(IV), be given. Then, for each $\tau \in \mathbb{R}$, $u^\tau \in V$ and $\phi \in L^2_H$, there exists a unique weak solution $u = u(\cdot; \tau, u^\tau, \phi)$ of (1.1).*

Moreover, if $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ and $u^\tau \in D(A)$, then u has the following regularity

$$u \in C([\tau, \infty); D(A)), \quad u' \in L^2(\tau, T; D(A)) \text{ for all } T > \tau, \tag{2.16}$$

and a.e. $t > \tau$ satisfies

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2 + \alpha^2|Au(t)|^2) + \nu|Au(t)|^2 + (B(u(t)), Au(t)) = (f(t) + g(t, u_t), Au(t)). \tag{2.17}$$

Proof. Uniqueness. Consider two weak solutions $u^{(1)}$ and $u^{(2)}$ to problem (1.1), corresponding to the same initial data, and denote $\hat{u} = u^{(1)} - u^{(2)}$. Observe that by (2.5) and (2.6),

$$\begin{aligned}|b(u^{(1)}(s), u^{(1)}(s), \hat{u}(s)) - b(u^{(2)}(s), u^{(2)}(s), \hat{u}(s))| &= |b(\hat{u}(s), u^{(1)}(s), \hat{u}(s))| \\ &\leq C_1\|u^{(1)}(s)\|\|\hat{u}(s)\|^2.\end{aligned}$$

Then, from the equation satisfied by $\hat{u}$ and the energy equality, it follows that

$$\begin{aligned}|\hat{u}(t)|^2 &+ \alpha^2\|\hat{u}(t)\|^2 + 2\nu\int_\tau^t \|\hat{u}(s)\|^2\,ds\\ &= -2\int_\tau^t b(\hat{u}(s),u^{(1)}(s),\hat{u}(s))\,ds + 2\int_\tau^t (g(s,u_s^{(1)}) - g(s,u_s^{(2)}),\hat{u}(s))\,ds\\ &\le 2C_1\int_\tau^t \|u^{(1)}(s)\|\|\hat{u}(s)\|^2\,ds + 2\int_\tau^t |g(s,u_s^{(1)}) - g(s,u_s^{(2)})||\hat{u}(s)|\,ds\end{aligned}$$

for all $t \ge \tau$. Now, by the Young inequality and the assumption (IV) on g, taking into account that $\hat{u}(s) = 0$ for $s \in (\tau - h, \tau)$, we obtain that

$$\begin{aligned}|\hat{u}(t)|^2 &+ \alpha^2\|\hat{u}(t)\|^2 + 2\nu\int_\tau^t \|\hat{u}(s)\|^2\,ds\\ &\le 2C_1\int_\tau^t \|u^{(1)}(s)\|\|\hat{u}(s)\|^2\,ds + \int_\tau^t |g(s,u_s^{(1)}) - g(s,u_s^{(2)})|^2\,ds + \int_\tau^t |\hat{u}(s)|^2\,ds\\ &\le 2C_1\int_\tau^t \|u^{(1)}(s)\|\|\hat{u}(s)\|^2\,ds + \lambda_1^{-1}(C_g^2+1)\int_\tau^t \|\hat{u}(s)\|^2\,ds\end{aligned}$$

for all $t \ge \tau$, and in particular

$$\|\hat{u}(t)\|^2 \le \alpha^{-2}\big(2C_1 + \lambda_1^{-1}(C_g^2+1)\big)\int_\tau^t \big(\|u^{(1)}(s)\| + 1\big)\|\hat{u}(s)\|^2\,ds$$

for all $t \ge \tau$. Thus, from the Gronwall lemma, we conclude uniqueness.

Existence. We will follow a Galerkin scheme similarly as in [14, Theorem 4]. Let $\{w_j\}_{j\ge1} \subset D(A)$ be the Hilbert basis of H formed by normalized eigenfunctions of the Stokes operator A introduced before.

For each integer $m \ge 1$, we pose the approximate problems of finding $u^m \in V_m := \mathrm{span}[w_1,\dots,w_m]$ with $u^m(t) = \sum_{j=1}^m \gamma_{m,j}(t)w_j$, where the coefficients $\gamma_{m,j}$ are required to satisfy the system

$$\begin{aligned}&\frac{d}{dt}(u^m(t) + \alpha^2 Au^m(t), w_j) + \nu((u^m(t),w_j)) + b(u^m(t),u^m(t),w_j)\\ &\quad = \langle f(t),w_j\rangle + (g(t,u_t^m),w_j), \quad \text{a.e. } t > \tau, \quad 1 \le j \le m,\end{aligned} \tag{2.18}$$

and the initial conditions

$$u^m(\tau) = P_m u^\tau \quad \text{and} \quad u^m(\tau+s) = P_m\phi(s) \quad \text{a.e. } s \in (-h,0),$$

where P_m is the orthogonal projector from H onto V_m. Observe that, by the choice of the basis $\{w_j\}_{j\ge1}$, the restriction $P_{m|_V}$ of P_m to V belongs to $\mathcal{L}(V)$, $\|P_{m|_V}\|_{\mathcal{L}(V)} \le 1$ for all $m \ge 1$, and $\lim_{m\to\infty}\|u^\tau - P_m u^\tau\| = 0$.

The above system of ordinary functional differential equations with finite delay fulfills the conditions for existence and uniqueness of local solution (see for example [18]).

Next, we will deduce a priori estimates that in particular assure that the solutions u^m do exist for all time $t \in [\tau - h, \infty)$.

Multiplying each equation in (2.18) by $\gamma_{m,j}(t)$ and summing from $j = 1$ to $j = m$, we obtain that a.e. $t > \tau$,

$$\begin{aligned}\frac{d}{dt}(|u^m(t)|^2 + \alpha^2\|u^m(t)\|^2) + 2\nu\|u^m(t)\|^2 &= 2\langle f(t),u^m(t)\rangle + 2(g(t,u_t^m),u^m(t))\\ &\le \nu\|u^m(t)\|^2 + \nu^{-1}\|f(t)\|_*^2 + |g(t,u_t^m)|^2 + |u^m(t)|^2,\end{aligned}$$

where we have used (2.6) to remove the nonlinear term b, and the Young inequality.

By integrating in time, from the assumptions on the delay operator g, in particular we deduce that

$$
\begin{aligned}
&|u^m(t)|^2 + \alpha^2 \|u^m(t)\|^2 \\
&\quad \leq |P_m u^\tau|^2 + \alpha^2 \|P_m u^\tau\|^2 + \nu^{-1} \int_\tau^t \|f(s)\|_*^2\, ds + C_g^2 \int_{\tau-h}^t |u^m(s)|^2\, ds + \int_\tau^t |u^m(s)|^2\, ds \\
&\quad \leq |u^\tau|^2 + \alpha^2 \|u^\tau\|^2 + C_g^2 \|\phi\|_{L^2_H}^2 + \nu^{-1} \int_\tau^t \|f(s)\|_*^2\, ds + \lambda_1^{-1}(C_g^2 + 1) \int_\tau^t \|u^m(s)\|^2\, ds
\end{aligned}
$$

for all $t \geq \tau$, and any $m \geq 1$. Now, by the Gronwall lemma we conclude that the sequence $\{u^m\}_{m\geq 1}$ is bounded in $C([\tau, T]; V)$ for all $T > \tau$. Moreover, since $u^m_\tau = P_m \phi$ converges to ϕ in $L^2(-h, 0; H)$, in particular, thanks to (IV), the sequence $\{g(\cdot, u^m_\cdot)\}_{m\geq 1}$ is bounded in $L^2(\tau, T; (L^2(\Omega))^3)$ for all $T > \tau$.

Now from (2.8), (2.18) and by the choice of the basis, we obtain that $v^m = \mathcal{C} u^m$ satisfies

$$
\|(v^m)'(t)\|_* \leq \nu \|u^m(t)\| + C_1 \|u^m(t)\|^2 + \|f(t)\|_* + \lambda_1^{-1/2} |g(t, u^m_t)|, \quad \text{a.e. } t > \tau,
$$

which implies that the sequence $\{dv^m/dt\}_{m\geq 1}$ is bounded in $L^2(\tau, T; V')$ for all $T > \tau$. Therefore, taking into account that $du^m/dt = \mathcal{C}^{-1}(dv^m/dt)$, we have that the sequence $\{du^m/dt\}_{m\geq 1}$ is bounded in $L^2(\tau, T; V)$ for all $T > \tau$.

Thus, by the compactness of the injection of V into H and the Ascoli–Arzelà theorem, we deduce that there exist a subsequence $\{u^{m'}\}_{m'\geq 1} \subset \{u^m\}_{m\geq 1}$ and a function $u \in W^{1,2}(\tau, T; V)$ for all $T > \tau$, with $u_\tau = \phi$, such that

$$
\begin{cases}
u^{m'} \overset{*}{\rightharpoonup} u \text{ weakly-star in } L^\infty(\tau, T; V), \\
u^{m'} \to u \text{ strongly in } C([\tau, T]; H), \\
u^{m'} \to u \text{ a.e. in } \Omega \times (\tau, T), \\
g(\cdot, u^{m'}_\cdot) \to g(\cdot, u_\cdot) \text{ strongly in } L^2(\tau, T; (L^2(\Omega))^3), \\
\dfrac{du^{m'}}{dt} \rightharpoonup \dfrac{du}{dt} \text{ weakly in } L^2(\tau, T; V), \\
\dfrac{dv^{m'}}{dt} = \mathcal{C}\left(\dfrac{du^{m'}}{dt}\right) \rightharpoonup \mathcal{C}\left(\dfrac{du}{dt}\right) \text{ weakly in } L^2(\tau, T; V'),
\end{cases}
\tag{2.19}
$$

for all $T > \tau$.

Now, using the same reasoning as in [14], we can obtain that $B(u^{m'}) \rightharpoonup B(u)$ weakly in $L^2(\tau, T; V')$, for all $T > \tau$. So, from all the convergences above, we can take limits in (2.18) and conclude that u satisfies (2.14).

Notice also that $u(\tau) = \lim_{m'\to\infty} u^{m'}(\tau) = \lim_{m'\to\infty} P_{m'} u^\tau = u^\tau$. Thus, u is the weak solution to (1.1).

Finally, the regularity property (2.16) and the identity (2.17) follow from the corresponding results proved in [14, Theorem 4] and the fact that, if $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$, then the function $f(\cdot) + g(\cdot, u_\cdot)$ belongs to $L^2_{loc}(\tau, \infty; (L^2(\Omega))^3)$. □

Remark 2.3. Observe that in the above proof, using the uniqueness of solution to the problem, for any $T > \tau$ we have that the whole sequence of the Galerkin approximations $\{u^m\}$ converges to u in $C([\tau, T]; H)$. Actually, all the convergences in (2.19), except the third one, hold for the whole sequence. Analogously, one also deduces that for any $t \in [\tau, T]$, $u^m(t) \rightharpoonup u(t)$ weakly in V.

In addition, if $u^\tau \in D(A)$ and $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$, then for any $T > \tau$ the sequence $\{u^m\}$ converges to u in $C([\tau, T]; V)$, and weakly-star in $L^\infty(\tau, T; D(A))$, for any $t \in [\tau, T]$, $u^m(t) \rightharpoonup u(t)$ in $D(A)$, and the sequence $\{du^m/dt\}$ converges to du/dt weakly in $L^2(\tau, T; D(A))$.

Remark 2.4. (i) The solution depends continuously on the initial data in the strong topology of $V \times L^2_H$. Moreover, when $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$, the solution depends continuously on the initial data in the strong topology of $D(A) \times L^2_V$. Indeed, this can be proved similarly to the proof of uniqueness of weak solution to (1.1), considering the difference of two solutions and using the Gronwall lemma.

(ii) The existence and uniqueness part of Theorem 2.2 do not need any regularity assumption on the boundary of the domain. In fact, this assumption is only required for the additional regularity results.

3 Existence of minimal pullback attractors in V norm

Before to start, let us recall some abstract definitions and results on pullback attractors and bi-space attractors theories. In fact, abstract existence results are omitted for the sake of brevity. For instance, they can be found in [4, 5, 13, 27] for pullback attractors (and references therein) and in [11] for bi-space pullback attractors (see also [1, 9, 39] for the autonomous bi-space attractors theory). They will be applied to a suitable dynamical system associated to (1.1), or to a restricted version involving more regularity or because of better properties.

Consider given a metric space (X, d_X), and let us denote $\mathbb{R}^2_d = \{(t, \tau) \in \mathbb{R}^2 : \tau \leq t\}$.

A process $\mathcal{U}$ on X is a mapping $\mathbb{R}^2_d \times X \ni (t, \tau, x) \mapsto \mathcal{U}(t, \tau)x \in X$ such that $\mathcal{U}(\tau, \tau)x = x$ for any $(\tau, x) \in \mathbb{R} \times X$, and $\mathcal{U}(t, r)(\mathcal{U}(r, \tau)x) = \mathcal{U}(t, \tau)x$ for any $\tau \leq r \leq t$ and all $x \in X$.

A process $\mathcal{U}$ is said to be continuous if for any pair $\tau \leq t$, the mapping $\mathcal{U}(t, \tau) : X \to X$ is continuous. It is said to be closed if for any $\tau \leq t$, and any sequence $\{x_n\} \subset X$, if $x_n \to x \in X$ and $\mathcal{U}(t, \tau)x_n \to y \in X$, then $\mathcal{U}(t, \tau)x = y$. It is clear that every continuous process is closed.

Let us denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$.

The process $\mathcal{U}$ is pullback $\widehat{D}_0$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{\mathcal{U}(t, \tau_n)x_n\}$ is relatively compact in X.

A process $\mathcal{U}$ on X being pullback $\widehat{D}_0$-asymptotically compact possesses a family of nonempty compact subsets of X, namely the atomized structure for the asymptotic behavior, the omega-limit family $\Lambda_X(\widehat{D}_0) = \{\Lambda_X(\widehat{D}_0, t) : t \in \mathbb{R}\}$ with

$$\Lambda_X(\widehat{D}_0, t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} \mathcal{U}(t, \tau) D_0(\tau)}^X.$$

It pullback attracts in X norm to $\widehat{D}_0$ (cf. [13, Proposition 3.4]), i.e.

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(\mathcal{U}(t, \tau)D_0(\tau), \Lambda_X(\widehat{D}_0, t)) = 0, \qquad \forall t \in \mathbb{R},$$

where $\mathrm{dist}_X(\cdot, \cdot)$ denotes the Hausdorff semi-distance in X. In fact, it is the minimal family of closed sections in X that attracts $\widehat{D}_0$. Moreover, if $\mathcal{U}$ is also a closed process on X, then (cf. [13, Proposition 3.5]) it is invariant, i.e. $\mathcal{U}(t, \tau)\Lambda_X(\widehat{D}_0, \tau) = \Lambda_X(\widehat{D}_0, t)$ for all $\tau \leq t$.

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 3.1. A process $\mathcal{U}$ on X is said to be pullback $\mathcal{D}$-asymptotically compact if it is $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

It is said that $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for $\mathcal{U}$ on X if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0(t, \widehat{D}) \leq t$ such that $\mathcal{U}(t,\tau)D(\tau) \subset D_0(t)$ for all $\tau \leq \tau_0(t, \widehat{D})$.

The suitable combination of the above two ingredients leads to

Definition 3.2. Given a metric space X, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a process $\mathcal{U}$ on X, a family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ is called a pullback $\mathcal{D}$-attractor for $\mathcal{U}$ if (i) $\mathcal{A}_{\mathcal{D}}(t)$ is compact in X for any $t \in \mathbb{R}$, (ii) $\mathcal{A}_{\mathcal{D}}$ pullback $\mathcal{D}$-attracts in X and (iii) it is invariant (i.e. $\mathcal{U}(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t)$ for any $\tau \leq t$).

Besides, it is said the minimal pullback $\mathcal{D}$-attractor for $\mathcal{U}$ on X if given any family $\widehat{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ of closed sets that pullback $\mathcal{D}$-attracts under $\mathcal{U}$, then $\mathcal{A}_{\mathcal{D}}(t) \subset C(t)$.

Without minimality, pullback attractors are not unique in general (cf. [27]). Minimality involves uniqueness and a clear candidate, after the definition of omega-limit families. Namely, the following result is well-known.

Theorem 3.3 (cf. [13, Theorem 3.11])**.** *Consider a closed process $\mathcal{U} : \mathbb{R}^2_d \times X \to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ which is pullback $\mathcal{D}$-absorbing for $\mathcal{U}$, and assume also that $\mathcal{U}$ is pullback $\widehat{D}_0$-asymptotically compact. Then, the family $\mathcal{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ defined by $\mathcal{A}_{\mathcal{D}}(t) = \overline{\bigcup_{\widehat{D} \in \mathcal{D}} \Lambda_X(\widehat{D}, t)}^X$ is the minimal pullback $\mathcal{D}$-attractor for $\mathcal{U}$ in X.*

Remark 3.4. Under the assumptions of Theorem 3.3, the family $\mathcal{A}_{\mathcal{D}}$ satisfies $\mathcal{A}_{\mathcal{D}}(t) \subset \Lambda_X(\widehat{D}_0, t)$ for any $t \in \mathbb{R}$. Actually, if $\widehat{D}_0 \in \mathcal{D}$, then $\mathcal{A}_{\mathcal{D}} = \Lambda_X(\widehat{D}_0)$. Moreover, if $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (ii)–(iii) in Definition 3.2. A sufficient condition for $\mathcal{A}_{\mathcal{D}} \in \mathcal{D}$ is to have that $\widehat{D}_0 \in \mathcal{D}$, the set $D_0(t)$ is closed for all $t \in \mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e., if $\widehat{D} \in \mathcal{D}$, and $\widehat{D}' = \{D'(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ with $D'(t) \subset D(t)$ for all t, then $\widehat{D}' \in \mathcal{D}$).

We will denote $\mathcal{D}_F(X)$ the universe of fixed nonempty bounded subsets of X, i.e., the class of all families $\widehat{D}$ of the form $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of X.

Now, it is easy to conclude the following result.

Corollary 3.5 (cf. [27, Corollaries 20 and 21])**.** *Under the assumptions of Theorem 3.3, if $\mathcal{D}$ contains $\mathcal{D}_F(X)$, then the minimal pullback attractor $\mathcal{A}_{\mathcal{D}_F(X)}$ also exists and $\mathcal{A}_{\mathcal{D}_F(X)}(t) \subset \mathcal{A}_{\mathcal{D}}(t)$ for all $t \in \mathbb{R}$. Moreover, if for some $T \in \mathbb{R}$, the set $\cup_{t \leq T} D_0(t)$ is bounded, then $\mathcal{A}_{\mathcal{D}_F(X)}(t) = \mathcal{A}_{\mathcal{D}}(t)$ for all $t \leq T$.*

Comparison results with different universes are also possible if the process $\mathcal{U}$ is well-posed in several metric spaces with a connection between them. Namely, Theorem 3.15 in [13] allows us to gain additional regularity about attractors. For the sake of brevity, we omit such statement. Nevertheless, we will recall another one with previous definitions, which will be analogously useful for our results (this is inspired from another study, cf. [25, Section 5]).

Theory of bi-space attractors (cf. [1,9,39] for autonomous setting and the references therein) is close to the previous results but joining extra regularity of the solution operator involving two spaces. Since our context is non-autonomous, we borrow some of these results from [11], settled in this framework also for closed processes. Consider given two metric spaces (X_i, d_{X_i}), $i = 1, 2$ (not necessarily related) and a process $\mathcal{U}$ on X_1. It is said (cf. [11, Definition 2.12]) that $\mathcal{U}$ is (X_1, X_2) closed if for any $\tau \leq t$ and $\{x_n\} \subset X_1 \cap X_2$ with $\mathcal{U}(t,\tau)x_n \in X_1 \cap X_2$, if $x_n \to x$ in X_2 and $\mathcal{U}(t,\tau)x_n \to y \in X_2$, then $x \in X_1$ and $\mathcal{U}(t,\tau)x = y$.

Given a parameterized-in-time family $\widehat{D}_0 \subset \mathcal{P}(X_1)$, a process $\mathcal{U}$ on X_1 is said (X_1, X_2) pullback $\widehat{D}_0$-asymptotically compact (cf. [11, Definition 2.4]) if for any $t \in \mathbb{R}$, sequence $\{\tau_n\} \subset (-\infty, t]$ and $\{x_n\} \subset X_1$ with $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$, the sequence $\{\mathcal{U}(t,\tau_n)x_n\}$ is relatively compact in X_2. Analogously to Definition 3.1, a process $\mathcal{U}$ on X_1 is said to be (X_1, X_2) pullback $\mathcal{D}$-asymptotically compact if it is (X_1, X_2) pullback $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.

We may run parallel the construction of a family with the desired properties of minimal pullback attractor for a universe $\mathcal{D}$ in $\mathcal{P}(X_1)$, provided that for any $\widehat{D} = \{D(s) : s \in \mathbb{R}\} \in \mathcal{D}$ and $t \in \mathbb{R}$ there exists $s_{\widehat{D},t} \leq t$ such that $\mathcal{U}(t,s)D(s) \subset X_2$ for all $s \leq s_{\widehat{D},t}$ (cf. [11, (2.2)]). In this case, data comes from X_1 and the arrival attracting space is X_2 with its corresponding metric [11, Definition 2.2].

Definition 3.6. Let be given a process $\mathcal{U}$ on X_1 and a universe $\mathcal{D}$ in $\mathcal{P}(X_1)$. The family $\widehat{\mathcal{A}}_{\mathcal{D}} = \{\widehat{\mathcal{A}}_{\mathcal{D}}(t) : t \in \mathbb{R}\}$ is called a (X_1, X_2) pullback $\mathcal{D}$-attractor if (i) $\widehat{\mathcal{A}}_{\mathcal{D}}(t) \subset X_1 \cap X_2$ is a nonempty compact set in X_2 for each $t \in \mathbb{R}$, (ii) it is pullback $\mathcal{D}$-attracting using the Hausdorff semidistance in X_2 and (iii) it is invariant. Besides, it is said minimal if for any other family $\widehat{C}$ of nonempty closed time-sections with values in X_2 and pullback $\mathcal{D}$-attracting in X_2, then $\widehat{\mathcal{A}}_{\mathcal{D}}(t) \subset \widehat{C}(t)$ for any $t \in \mathbb{R}$.

Similarly to Theorem 3.3, we may ensure the existence of the minimal (X_1, X_2) pullback $\mathcal{D}$-attractor under rather general conditions (cf. [11, Theorem 2.16]).

Theorem 3.7. *Let be given two metric spaces X_i, $i = 1, 2$, a process $\mathcal{U}$ on X_1, and a universe $\mathcal{D}$ in $\mathcal{P}(X_1)$. Suppose that there exists a family $\widehat{\mathcal{B}}_0$ in $\mathcal{P}(X_1)$ that is pullback $\mathcal{D}$-absorbing, such that for any $t \in \mathbb{R}$ there exists $s_{\widehat{\mathcal{B}}_0,t} \leq t$ such that $\mathcal{U}(t,s)\widehat{\mathcal{B}}_0(s) \subset X_2$ for any $s \leq s_{\widehat{\mathcal{B}}_0,t}$. If the process $\mathcal{U}$ is (X_1, X_2) closed and (X_1, X_2) pullback $\widehat{\mathcal{B}}_0$-asymptotically compact, then there exists $\widehat{\mathcal{A}}_{\mathcal{D}}$ the minimal (X_1, X_2) pullback $\mathcal{D}$-attractor for $\mathcal{U}$, and it is given by $\widehat{\mathcal{A}}_{\mathcal{D}}(t) = \overline{\cup_{\widehat{D}\in\mathcal{D}}\Lambda_{X_2}(\widehat{D}, t)}^{X_2} \subset \Lambda_{X_2}(\widehat{\mathcal{B}}_0, t)$.*

Remark 3.8. If $X_2 \subset X_1$ with continuous injection, the following consequences are immediate: (i) A process $\mathcal{U}$ on X_1 that is X_1 closed, it is also (X_1, X_2) closed. (ii) Given a universe $\mathcal{D}$ in $\mathcal{P}(X_1)$ and a process $\mathcal{U}$ (X_1, X_2) pullback $\mathcal{D}$-asymptotically compact, then $\Lambda_{X_1}(\widehat{D}) = \Lambda_{X_2}(\widehat{D})$ for any $\widehat{D} \in \mathcal{D}$ thanks to the minimality properties of omega-limit families and that a compact set in X_2 is compact in X_1. (iii) A process $\mathcal{U}$ that has a (X_1, X_2) pullback $\mathcal{D}$-attractor $\widehat{\mathcal{A}}_{\mathcal{D}}$, it also has a (X_1, X_1) pullback $\mathcal{D}$-attractor $\mathcal{A}_{\mathcal{D}}$ just using the embedding $X_2 \subset X_1$ (same arguments of minimality and compact sets than in (ii), even using different clousures). In this case we make an abuse of notation, identifying both families without any extra notation, gaining extra regularity in X_2 for the sections of the attractor.

In view of Theorem 2.2 and Remark 2.4 (i), we will apply the above abstract results in the phase-space $X = V \times L^2_H$, which is a Hilbert space with the norm $\|(u^\tau, \phi)\|^2_X = \|u^\tau\|^2 + \|\phi\|^2_{L^2_H}$ for a pair $(u^\tau, \phi) \in X$.

The first consequence after the Theorem 2.2 and Remark 2.4 (i) is the following

Corollary 3.9. *Let $f \in L^2_{loc}(\mathbb{R}; V')$, and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ satisfying (I)–(IV), be given. Then, the bi-parametric family of maps $S(t,\tau) : V \times L^2_H \to V \times L^2_H$, with $\tau \leq t$, given by*

$$S(t,\tau)(u^\tau, \phi) = (u(t; \tau, u^\tau, \phi), u_t(\cdot; \tau, u^\tau, \phi)), \tag{3.1}$$

where $u = u(\cdot;\tau,u^\tau,\phi)$ *is the unique weak solution to* (1.1)*, defines a continuous process on* $V \times L^2_H$.

We will need the following continuity result for the process S in a weak sense.

Proposition 3.10. *Let* $f \in L^2_{loc}(\mathbb{R};V')$, $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ *satisfying (I)–(IV), and* $\tau < t$ *be given. Then, for any sequence such that*

$$(u^{\tau,n},\phi^n) \rightharpoonup (u^\tau,\phi) \quad \text{weakly in } V \times L^2_V$$

and

$$\frac{d\phi^n}{ds} \rightharpoonup \frac{d\phi}{ds} \quad \text{weakly in } L^2_V,$$

the following convergences hold for the sequence of solutions $u(\cdot;\tau,u^{\tau,n},\phi^n)$ *towards the solution* $u(\cdot;\tau,u^\tau,\phi)$*:*

$$\begin{aligned}
&u(\cdot;\tau,u^{\tau,n},\phi^n) \overset{*}{\rightharpoonup} u(\cdot;\tau,u^\tau,\phi) && \text{weakly-star in } L^\infty(\tau,t;V),\\
&u(\cdot;\tau,u^{\tau,n},\phi^n) \to u(\cdot;\tau,u^\tau,\phi) && \text{strongly in } C([\tau-h,t];H),\\
&u(t;\tau,u^{\tau,n},\phi^n) \rightharpoonup u(t;\tau,u^\tau,\phi) && \text{weakly in } V,\\
&u(\cdot;\tau,u^{\tau,n},\phi^n) \rightharpoonup u(\cdot;\tau,u^\tau,\phi) && \text{weakly in } L^2(\tau-h,t;V).
\end{aligned} \tag{3.2}$$

Proof. Taking into account that $\{\phi^n\}$ is bounded in $W^{1,2}(-h,0;V) \subset C([-h,0];V)$, and the compactness of the injection of V into H, by the Ascoli–Arzelà theorem we deduce that $\phi^n \to \phi$ strongly in C_H. Therefore, the *a priori* estimates obtained for the Galerkin approximations in Theorem 2.2 also hold for the sequence of solutions $\{u(\cdot;\tau,u^{\tau,n},\phi^n)\}$, and then all the convergences in (3.2) hold. Finally, the fact that the whole sequence satisfies the above convergences is a consequence of the uniqueness of solution for the problem (cf. Remark 2.3). □

Now, we introduce an additional assumption on g in order to obtain some asymptotic estimates for the solutions to (1.1).

(V) Assume that $\nu\lambda_1 > C_g$, and that there exists a value $0 < \sigma < 2(\nu - \lambda_1^{-1}C_g)(\lambda_1^{-1} + \alpha^2)^{-1}$ such that for every $u \in L^2(\tau-h,t;H)$,

$$\int_\tau^t e^{\sigma s}|g(s,u_s)|^2\,ds \le C_g^2 \int_{\tau-h}^t e^{\sigma s}|u(s)|^2\,ds, \quad \forall t \ge \tau.$$

Lemma 3.11. *Consider given* $f \in L^2_{loc}(\mathbb{R};V')$ *and* $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ *satisfying conditions (I)–(V). Then, for any* $(u^\tau,\phi) \in V \times L^2_H$*, the following estimate holds for the solution* u *to* (1.1) *for all* $t \ge \tau$,

$$\|u(t)\|^2 \le \alpha^{-2}\max\{\lambda_1^{-1}+\alpha^2, C_g\}e^{\sigma(\tau-t)}\|(u^\tau,\phi)\|^2_{V\times L^2_H} + \alpha^{-2}\varepsilon^{-1}\int_\tau^t e^{\sigma(s-t)}\|f(s)\|_*^2\,ds, \tag{3.3}$$

where

$$\varepsilon = 2\nu - \sigma(\lambda_1^{-1}+\alpha^2) - 2\lambda_1^{-1}C_g > 0. \tag{3.4}$$

Proof. By the energy equality (2.15) and the Young inequality, we have

$$\begin{aligned}
&\frac{d}{dt}(|u(t)|^2 + \alpha^2\|u(t)\|^2) + 2\nu\|u(t)\|^2\\
&\quad \le \varepsilon\|u(t)\|^2 + \varepsilon^{-1}\|f(t)\|_*^2 + C_g|u(t)|^2 + C_g^{-1}|g(t,u_t)|^2, \quad \text{a.e. } t > \tau.
\end{aligned}$$

Thus,

$$\frac{d}{dt}(e^{\sigma t}|u(t)|^2 + \alpha^2 e^{\sigma t}\|u(t)\|^2) + e^{\sigma t}\big(2\nu - \varepsilon - \sigma(\lambda_1^{-1} + \alpha^2) - \lambda_1^{-1}C_g\big)\|u(t)\|^2 \\ \le e^{\sigma t}\varepsilon^{-1}\|f(t)\|_*^2 + e^{\sigma t}C_g^{-1}|g(t,u_t)|^2, \quad \text{a.e. } t > \tau,$$

and therefore, integrating in time above and using property (V), we obtain

$$e^{\sigma t}(|u(t)|^2 + \alpha^2\|u(t)\|^2) + \big(2\nu - \varepsilon - \sigma(\lambda_1^{-1} + \alpha^2) - \lambda_1^{-1}C_g\big)\int_\tau^t e^{\sigma s}\|u(s)\|^2\,ds \\ \le e^{\sigma\tau}(\lambda_1^{-1} + \alpha^2)\|u^\tau\|^2 + \varepsilon^{-1}\int_\tau^t e^{\sigma s}\|f(s)\|_*^2\,ds + C_g\int_{\tau-h}^t e^{\sigma s}|u(s)|^2\,ds \\ \le e^{\sigma\tau}\left((\lambda_1^{-1} + \alpha^2)\|u^\tau\|^2 + C_g\int_{-h}^0 |\phi(s)|^2\,ds\right) + \varepsilon^{-1}\int_\tau^t e^{\sigma s}\|f(s)\|_*^2\,ds + \lambda_1^{-1}C_g\int_\tau^t e^{\sigma s}\|u(s)\|^2\,ds$$

for all $t \ge \tau$, and from this last inequality and (3.4), in particular we deduce (3.3). □

From now on, being $\sigma > 0$ given in (V), we will assume that $f \in L^2_{loc}(\mathbb{R}; V')$ satisfies

$$\int_{-\infty}^0 e^{\sigma s}\|f(s)\|_*^2\,ds < \infty. \tag{3.5}$$

At the light of the previous result, we now define an appropriate concept of (tempered) universe for problem (1.1).

Definition 3.12. Denote by $\mathcal{D}_\sigma(V \times L^2_H)$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(V \times L^2_H)$ such that

$$\lim_{\tau\to-\infty}\left(e^{\sigma\tau}\sup_{(v,\varphi)\in D(\tau)}\|(v,\varphi)\|^2_{V\times L^2_H}\right) = 0.$$

According to the notation introduced in the previous section, we will denote by $\mathcal{D}_F(V \times L^2_H)$ the universe of fixed bounded sets in $V \times L^2_H$. Observe that trivially $\mathcal{D}_F(V \times L^2_H) \subset \mathcal{D}_\sigma(V \times L^2_H)$ and that $\mathcal{D}_\sigma(V \times L^2_H)$ is inclusion-closed.

Remark 3.13. Although from Lemma 3.11 it is easy to see that the family $\{\overline{B}_{V\times L^2_H}(0,\rho_\sigma(t)) : t \in \mathbb{R}\} \subset \mathcal{P}(V \times L^2_H)$ is pullback $\mathcal{D}_\sigma(V \times L^2_H)$-absorbing for the process S, where

$$\rho_\sigma^2(t) = 1 + \alpha^{-2}\varepsilon^{-1}(1 + \lambda_1^{-1}he^{\sigma h})e^{-\sigma t}\int_{-\infty}^t e^{\sigma s}\|f(s)\|_*^2\,ds,$$

we will need, in order to apply Proposition 3.10, to obtain a different pullback $\mathcal{D}_\sigma(V \times L^2_H)$-absorbing family.

Lemma 3.14. *Assume that $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V), and $f \in L^2_{loc}(\mathbb{R}; V')$ satisfies (3.5). Then, for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\sigma(V \times L^2_H)$, there exist $\tau_1(\widehat{D}, t, h) < t - 2h$ and functions $\{\rho_i\}_{i=1}^2$ such that for any $\tau \le \tau_1(\widehat{D}, t, h)$ and any $(u^\tau, \phi) \in D(\tau)$, it holds*

$$\|u(r;\tau,u^\tau,\phi)\|^2 \le \rho_1^2(t), \quad \forall\, r \in [t-2h, t], \tag{3.6}$$

$$\int_{t-h}^t \|u'(\theta;\tau,u^\tau,\phi)\|^2\,d\theta \le \rho_2^2(t), \tag{3.7}$$

where

$$\rho_1^2(t) = 1 + \alpha^{-2}\varepsilon^{-1}e^{-\sigma(t-2h)}\int_{-\infty}^{t} e^{\sigma s}\|f(s)\|_*^2\,ds, \tag{3.8}$$

$$\rho_2^2(t) = 4\alpha^{-4}h\rho_1^2(t)\left(\nu^2 + C_1^2\rho_1^2(t) + 2\lambda_1^{-2}C_g^2\right) + 4\alpha^{-4}\int_{t-h}^{t}\|f(s)\|_*^2\,ds, \tag{3.9}$$

and ε is given by (3.4).

Proof. Let $\tau_1(\widehat{D}, t, h) < t - 2h$ be such that

$$\alpha^{-2}\max\{\lambda_1^{-1}+\alpha^2, C_g\}e^{-\sigma(t-2h)}e^{\sigma\tau}\|(u^\tau,\phi)\|^2_{V\times L^2_H} \le 1 \quad \forall\, \tau \le \tau_1(\widehat{D},t,h),\ (u^\tau,\phi)\in D(\tau).$$

Consider fixed $\tau \le \tau_1(\widehat{D}, t, h)$ and $(u^\tau, \phi) \in D(\tau)$. The estimate (3.6) follows directly from (3.3), using the increasing character of the exponential.

Now, from (2.8), (2.14), (2.1) and the fact that A is an isometric isomorphism, we obtain that $v = Cu$ satisfies

$$\|v'(\theta)\|_* \le \nu\|u(\theta)\| + C_1\|u(\theta)\|^2 + \|f(\theta)\|_* + \lambda_1^{-1/2}|g(\theta,u_\theta)|, \quad \text{a.e. } \theta > \tau,$$

and therefore,

$$\|v'(\theta)\|_*^2 \le 4\nu^2\|u(\theta)\|^2 + 4C_1^2\|u(\theta)\|^4 + 4\|f(\theta)\|_*^2 + 4\lambda_1^{-1}|g(\theta,u_\theta)|^2, \quad \text{a.e. } \theta > \tau.$$

Integrating in time and using properties (II) and (IV), we deduce

$$\int_{t-h}^{t}\|v'(\theta)\|_*^2\,d\theta \le 4\nu^2\int_{t-h}^{t}\|u(\theta)\|^2\,d\theta + 4C_1^2\int_{t-h}^{t}\|u(\theta)\|^4\,d\theta$$
$$+ 4\int_{t-h}^{t}\|f(\theta)\|_*^2\,d\theta + 4\lambda_1^{-2}C_g^2\int_{t-2h}^{t}\|u(\theta)\|^2\,d\theta,$$

whence, by (2.12) and (3.6), the estimate (3.7) follows. □

Remark 3.15. Observe that $\lim_{t\to-\infty} e^{\sigma t}\rho_1(t) = 0$.

Corollary 3.16. *Under the assumptions of Lemma 3.14, the family $\widehat{D}_\sigma = \{D_\sigma(t) : t \in \mathbb{R}\} \subset \mathcal{P}(V \times L^2_H)$ defined by*

$$D_\sigma(t) = \left\{(w,\psi) \in V\times L^2_V : \exists \frac{d\psi}{ds} \in L^2_V,\ \|(w,\psi)\|_{V\times L^2_V} \le \widetilde{\rho}_\sigma(t),\ \left\|\frac{d\psi}{ds}\right\|_{L^2_V} \le \rho_2(t)\right\} \tag{3.10}$$

is pullback $\mathcal{D}_\sigma(V\times L^2_H)$-absorbing for the process S on $V\times L^2_H$ defined by (3.1), where $\widetilde{\rho}_\sigma(t)$ satisfies

$$\widetilde{\rho}_\sigma^2(t) = (1+h)\rho_1^2(t), \tag{3.11}$$

with $\rho_1(t)$ and $\rho_2(t)$ given by (3.8) and (3.9) respectively. Moreover, $\widehat{D}_\sigma \in \mathcal{D}_\sigma(V\times L^2_H)$.

Now, we prove that the process S is $(V\times L^2_H, V\times C_H)$ pullback $\widehat{D}_\sigma$-asymptotically compact. To this end, we will apply an energy method used by Rosa (cf. [32], see also [26] and [14]), which does not require any additional estimates on the solutions in higher norms in contrast with the *energy continuous method* (e.g., cf. [28]), or the method used in [20] with the fractional powers of the operator A. Our proof here relies on a sharp use of the differential equality that leads to the existence of an absorbing family, the use of weak limits in $V\times L^2_V$ in a diagonal argument, and the convergences established in Proposition 3.10.

Lemma 3.17. *Under the assumptions of Lemma 3.14, the process S defined by (3.1) is $(V \times L^2_H, V \times C_H)$ pullback $\widehat{D}_\sigma$-asymptotically compact, where $\widehat{D}_\sigma = \{D_\sigma(t) : t \in \mathbb{R}\}$ is defined in Corollary 3.16.*

Proof. Let us consider $t \in \mathbb{R}$, a sequence $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and a sequence $\{(u^{\tau_n}, \phi^n)\}$ with $(u^{\tau_n}, \phi^n) \in D_\sigma(\tau_n)$ for all n. We must prove that the sequence

$$\{S(t,\tau_n)(u^{\tau_n}, \phi^n)\} = \{(u(t;\tau_n,u^{\tau_n},\phi^n), u_t(\cdot;\tau_n,u^{\tau_n},\phi^n))\}$$

is relatively compact in $V \times C_H$.

First, we check the asymptotic compactness in the first component of S. By Corollary 3.16, for each integer $k \geq 0$, there exists $\tau_{\widehat{D}_\sigma}(k) \leq t-k$ such that $S(t-k,\tau)D_\sigma(\tau) \subset D_\sigma(t-k)$ for all $\tau \leq \tau_{\widehat{D}_\sigma}(k)$. From this and a diagonal argument, we can extract a subsequence $\{(u^{\tau_{n'}}, \phi^{n'})\} \subset \{(u^{\tau_n}, \phi^n)\}$ such that

$$S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}) \rightharpoonup (w^k,\psi^k) \quad \text{weakly in } V \times L^2_V, \tag{3.12}$$

$$\frac{d}{ds}u_{t-k}(\cdot;\tau_{n'},u^{\tau_{n'}},\phi^{n'}) \rightharpoonup \frac{d}{ds}\psi^k \quad \text{weakly in } L^2_V, \tag{3.13}$$

for all integer $k \geq 0$, where $(w^k,\psi^k) \in D_\sigma(t-k)$.

Now, applying Proposition 3.10 on each fixed interval $[t-k,t]$, we deduce that

$$\begin{aligned}
(w^0,\psi^0) &= (V \times L^2_V) - \text{weak} \lim_{n'\to\infty} S(t,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}) \\
&= (V \times L^2_V) - \text{weak} \lim_{n'\to\infty} S(t,t-k)S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}) \\
&= S(t,t-k)\left[(V \times L^2_V) - \text{weak} \lim_{n'\to\infty} S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})\right] \\
&= S(t,t-k)(w^k,\psi^k).
\end{aligned}$$

From (3.12) with $k=0$, we obtain in particular that $\|w^0\| \leq \liminf_{n'\to\infty}\|u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\|$. We will prove now that it also holds that

$$\limsup_{n'\to\infty}\|u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\| \leq \|w^0\|, \tag{3.14}$$

which combined with the weak converge of $u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})$ to w^0 in V, will imply the convergence in the strong topology of V.

Observe that, as we already used in Lemma 3.11, for any $\tau \in \mathbb{R}$ and $(u^\tau,\phi) \in V \times L^2_H$, the solution $u(\cdot;\tau,u^\tau,\phi)$, for short denoted $u(\cdot)$, satisfies the differential equality

$$\begin{aligned}
\frac{d}{dt}(e^{\sigma t}|u(t)|^2 + \alpha^2 e^{\sigma t}\|u(t)\|^2) &= \sigma e^{\sigma t}|u(t)|^2 + \alpha^2\sigma e^{\sigma t}\|u(t)\|^2 - 2\nu e^{\sigma t}\|u(t)\|^2 \\
&\quad + 2e^{\sigma t}\langle f(t),u(t)\rangle + 2e^{\sigma t}(g(t,u_t),u(t)), \quad \text{a.e. } t > \tau.
\end{aligned} \tag{3.15}$$

Since in particular $0 < \sigma < 2\nu(\lambda_1^{-1}+\alpha^2)^{-1}$, notice that $[\cdot]$, with $[v]^2 = (2\nu - \alpha^2\sigma)\|v\|^2 - \sigma|v|^2$, defines an equivalent norm to $\|\cdot\|$ in V.

We integrate the above expression in the interval $[t-k,t]$ for the solutions $u(\cdot;\tau_{n'},u^{\tau_{n'}},\phi^{n'})$

with $\tau_{n'} \leq t-k$, which yields

$$
\begin{aligned}
&|u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|^2 + \alpha^2\|u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\|^2 \\
&\quad = |u(t;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))|^2 + \alpha^2\|u(t;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\|^2 \\
&\quad = e^{-\sigma k}\left(|u(t-k;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|^2 + \alpha^2\|u(t-k;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\|^2\right) \\
&\qquad + 2\int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s),u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\rangle\,ds \\
&\qquad + 2\int_{t-k}^{t} e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds \\
&\qquad - \int_{t-k}^{t} e^{\sigma(s-t)}[u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))]^2\,ds. \qquad (3.16)
\end{aligned}
$$

On other hand, by (3.12), (3.13) and Proposition 3.10, we deduce that

$$u(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})) \rightharpoonup u(\cdot;t-k,w^k,\psi^k) \quad \text{weakly in } L^2(t-k,t;V).$$

From this, as $e^{\sigma(\cdot-t)}f(\cdot) \in L^2(t-k,t;V')$, it yields

$$
\begin{aligned}
\lim_{n'\to\infty}&\int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s),u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\rangle\,ds \\
&= \int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s),u(s;t-k,w^k,\psi^k)\rangle\,ds.
\end{aligned}
$$

Since $\int_{t-k}^{t} e^{\sigma(s-t)}[v(s)]^2\,ds$ defines an equivalent norm in $L^2(t-k,t;V)$, we also deduce from above that

$$\int_{t-k}^{t} e^{\sigma(s-t)}[u(s;t-k,w^k,\psi^k)]^2\,ds \leq \liminf_{n'\to\infty}\int_{t-k}^{t} e^{\sigma(s-t)}[u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))]^2\,ds.$$

Finally, again by (3.12), (3.13) and Proposition 3.10, it holds that

$$u(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})) \to u(\cdot;t-k,w^k,\psi^k) \quad \text{strongly in } L^2(t-k-h,t;H),$$

and therefore,

$$
\begin{aligned}
\lim_{n'\to\infty}&\int_{t-k}^{t} e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds \\
&= \int_{t-k}^{t} e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,w^k,\psi^k)),u(s;t-k,w^k,\psi^k))\,ds \qquad (3.17)
\end{aligned}
$$

From (3.16)–(3.17), taking into account (3.12) with $k=0$, the compactness of the injection of V into H, and (3.10), we conclude that

$$
\begin{aligned}
&|w^0|^2 + \alpha^2\limsup_{n'\to\infty}\|u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\|^2 \\
&\quad \leq e^{-\sigma k}(\lambda_1^{-1}+\alpha^2)\tilde{\rho}_\sigma^2(t-k) + 2\int_{t-k}^{t} e^{\sigma(s-t)}\langle f(s),u(s;t-k,w^k,\psi^k)\rangle\,ds \\
&\qquad + 2\int_{t-k}^{t} e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,w^k,\psi^k)),u(s;t-k,w^k,\psi^k))\,ds \\
&\qquad - \int_{t-k}^{t} e^{\sigma(s-t)}[u(s;t-k,w^k,\psi^k)]^2\,ds.
\end{aligned}
$$

Now, taking into account that $w^0 = u(t; t-k, w^k, \psi^k)$, integrating again in (3.15), we obtain

$$|w^0|^2 + \alpha^2\|w^0\|^2 = e^{-\sigma k}(|w^k|^2 + \alpha^2\|w^k\|^2) + 2\int_{t-k}^t e^{\sigma(s-t)}\langle f(s), u(s; t-k, w^k, \psi^k)\rangle\, ds$$
$$+ 2\int_{t-k}^t e^{\sigma(s-t)}(g(s, u_s(\cdot; t-k, w^k, \psi^k)), u(s; t-k, w^k, \psi^k))\, ds$$
$$- \int_{t-k}^t e^{\sigma(s-t)}[u(s; t-k, w^k, \psi^k)]^2\, ds.$$

Comparing the above two expressions, in particular we conclude that

$$|w^0|^2 + \alpha^2 \limsup_{n'\to\infty} \|u(t; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'})\|^2 \le e^{-\sigma k}(\lambda_1^{-1} + \alpha^2)\tilde{\rho}_\sigma^2(t-k) + |w^0|^2 + \alpha^2\|w^0\|^2.$$

But from Remark 3.15 and (3.11), we have that $\lim_{k\to\infty} e^{-\sigma k}\tilde{\rho}_\sigma^2(t-k) = 0$, so (3.14) holds, and we conclude that

$$u(t; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'}) \to w^0 \quad \text{strongly in } V.$$

Finally, we prove the asymptotic compactness in the second component of S. From (3.12) and (3.13) with $k = 0$, we have that

$$u_t(\cdot; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'}) \rightharpoonup \psi^0 \qquad \text{weakly in } L^2_V,$$
$$\frac{d}{ds}u_t(\cdot; \tau_{n'}, u^{\tau'_n}, \phi^{n'}) \rightharpoonup \frac{d}{ds}\psi^0 \quad \text{weakly in } L^2_V.$$

Thus, by applying the Ascoli–Arzelà theorem, we can deduce that there exists a subsequence (relabelled the same) such that $u_t(\cdot; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'})$ converges to ψ^0 in C_H. So, the proof is finished. □

As a consequence of the above results, we obtain the existence of minimal pullback attractors for the process S on $V \times L^2_H$ defined by (3.1).

Theorem 3.18. *Assume that $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V), and $f \in L^2_{loc}(\mathbb{R}; V')$ satisfies (3.5). Then, there exist the $(V \times L^2_H, V \times C_H)$ minimal pullback $\mathcal{D}_\sigma(V \times L^2_H)$ and $\mathcal{D}_F(V \times L^2_H)$-attractors $\{\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t) : t \in \mathbb{R}\}$ and $\{\mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}(t) : t \in \mathbb{R}\}$ respectively, both belonging to $\mathcal{D}_\sigma(V \times L^2_H)$, which means that they have compact sections in $V \times C_H$ and pullback attracts in this norm, and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}(t) \subset \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t) = \Lambda_{V\times C_H}(\widehat{D}_\sigma, t), \quad \forall\, t \in \mathbb{R}. \tag{3.18}$$

Moreover, if f satisfies the stronger requirement

$$\sup_{r\le 0}\left(e^{-\sigma r}\int_{-\infty}^r e^{\sigma s}\|f(s)\|_*^2\, ds\right) < \infty, \tag{3.19}$$

then both attractors coincide, i.e.,

$$\mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}(t) = \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t), \quad \forall\, t \in \mathbb{R}. \tag{3.20}$$

Proof. The process S is continuous on $V \times L^2_H$ by Corollary 3.9. By Remark 3.8, S is $(V \times L^2_H, V \times C_H)$ closed. There exists a pullback absorbing family $\widehat{D}_\sigma \in \mathcal{D}_\sigma(V \times L^2_H)$ by Corollary 3.16, and the process S is $(V \times L^2_H, V \times C_H)$ pullback $\widehat{D}_\sigma$-asymptotically compact by

Lemma 3.17. The existence of $\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}$ and $\mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}$ follows from Theorem 3.7 (actually Theorem 3.3 and Corollary 3.5 could also be applied, but using the bi-space attractors theory we strengthen compactness and attraction norm).

Moreover, the inclusion relation in (3.18) follows from Corollary 3.5.

The fact that $\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}$ belongs to $\mathcal{D}_\sigma(V\times L^2_H)$ is due to Remark 3.4, since the pullback absorbing family $\widehat{D}_\sigma \in \mathcal{D}_\sigma(V\times L^2_H)$ has closed sections and this universe is inclusion-closed.

Finally, the equality (3.20) is a consequence of Corollary 3.5, since $D_\sigma(t) \subset \overline{B}_{V\times L^2_V}(0,\widetilde{\rho}_\sigma(t))$ for all $t\in\mathbb{R}$, and the assumption (3.19) is equivalent to have that $\sup_{t\le T}\widetilde{\rho}_\sigma(t)$ is bounded for any $T\in\mathbb{R}$. □

Just splitted for the sake of clarity, with the same arguments as above, we obtain the following result, which relates the above attractors for the universes $\mathcal{D}_F(V\times L^2_H)\subset\mathcal{D}_\sigma(V\times L^2_H)$ with new ones for the universes $\mathcal{D}_F(V\times C_H)\subset\mathcal{D}_\sigma(V\times C_H)$.

Corollary 3.19. *Under the assumptions of Theorem 3.18 there exist the minimal pullback attractors $\mathcal{A}_{\mathcal{D}_F(V\times C_H)}$ and $\mathcal{A}_{\mathcal{D}_\sigma(V\times C_H)}$, both belonging to $\mathcal{D}_\sigma(V\times C_H)$, all time sections are compact subsets in $V\times C_H$, they attract in $V\times C_H$ norm, and the following relations hold:*

$$\mathcal{A}_{\mathcal{D}_F(V\times C_H)}(t)\subset\mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}(t)\subset\mathcal{A}_{\mathcal{D}_\sigma(V\times C_H)}(t)=\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t),\quad \forall\, t\in\mathbb{R}.$$

Proof. Observe that S is well-defined on $V\times C_H$ by Theorem 2.2 and closed by Remark 2.4 (i). Observe that $\widehat{D}_\sigma\subset\mathcal{P}(V\times C_H)$. Then the existence of attractors and its inclusion in $\mathcal{D}_\sigma(V\times C_H)$ follows from Theorem 3.3 and Remark 3.4.

The equality relation of pullback $\mathcal{D}_\sigma(V\times C_H)$ and $\mathcal{D}_\sigma(V\times L^2_H)$-attractors follows from [13, Theorem 3.15]. Indeed, observe that after an elapsed time h, by (3.3), $S(\cdot+h,\cdot)$ maps elements from $\mathcal{D}_\sigma(V\times L^2_H)$ into $\mathcal{D}_\sigma(V\times C_H)$.

The rest of inclusions follows from Corollary 3.5 or by minimality arguments. □

Remark 3.20. The stronger attraction and compactness properties of these results also apply to several previous ones concerning asymptotic behavior of PDE with delays (e.g., cf. [16]).

Remark 3.21. Observe that by the invariance of the minimal pullback attractors under the process S, and the regularity of the solutions, it is clear that the second component of any time section of $\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}$ and $\mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}$ lives in C_V. In fact, denoting $R^2_\sigma(t)=2\rho^2_1(t)$, from (3.6) it holds that

$$\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t)\subset\overline{B}_{V\times C_V}(0,R_\sigma(t)),\quad\forall\, t\in\mathbb{R}.$$

4 Regularity of the pullback attractors

The main goal of this paragraph is to provide some extra regularity for the attractors obtained in the previous section. This will be obtained by a bootstrapping argument, and making the most out of a representation of the solutions to the problem splitting it in two parts, the linear part with an exponential decay, and the nonlinear part with good enough estimates. In order to achieve these results, we will use the fractional powers of the Stokes operator, introduced in Section 2.

Observe that for every $\tau\in\mathbb{R}$, $(u^\tau,\phi)\in V\times L^2_H$, $f\in L^2_{loc}(\mathbb{R};V')$, and $g:\mathbb{R}\times C_H\to(L^2(\Omega))^3$ satisfying (I)–(IV), by Theorem 2.2, there exists a unique weak solution u to problem (1.1). Moreover, let us point out that the following representation of the solution holds:

$$u(t;\tau,u^\tau,\phi)=y(t;\tau,u^\tau,\phi)+z(t;\tau,0,0),\quad\forall\, t\ge\tau,$$

where $y = y(\cdot;\tau,u^\tau,\phi)$ and $z = z(\cdot;\tau,0,0)$ are solutions of

$$\begin{cases} y \in C([\tau,\infty);V) \cap L^2(\tau-h,T;H) \text{ for all } T > \tau, \\ \dfrac{d}{dt}(y(t)+\alpha^2 Ay(t)) + \nu Ay(t) = 0, \quad \text{in } \mathcal{D}'(\tau,\infty;V'), \\ y(\tau) = u^\tau, \\ y(t) = \phi(t-\tau) \quad \text{a.e. } t \in (\tau-h,\tau) \end{cases} \tag{4.1}$$

and

$$\begin{cases} z \in C([\tau,\infty);V) \cap L^2(\tau-h,T;H) \text{ for all } T > \tau, \\ \dfrac{d}{dt}(z(t)+\alpha^2 Az(t)) + \nu Az(t) = f(t) + g(t,u_t) - B(u(t)), \quad \text{in } \mathcal{D}'(\tau,\infty;V'), \\ z(\tau) = 0, \\ z(t) = 0 \quad \text{a.e.} t \in (\tau-h,\tau) \end{cases} \tag{4.2}$$

respectively.

The existence and uniqueness of weak solution to (4.1) and to (4.2) can be obtained reasoning as in the proof of Theorem 2.2.

For the problem (4.1) we have the following result.

Lemma 4.1. *For any $\tau \in \mathbb{R}$, $(u^\tau,\phi) \in V \times L^2_H$ and σ fulfilling that $0 < \sigma < 2(\nu - \lambda_1^{-1}C_g)(\lambda_1^{-1} + \alpha^2)^{-1}$, the solution $y = y(\cdot;\tau,u^\tau,\phi)$ of (4.1) satisfies*

$$\|y(t)\|^2 \le \alpha^{-2}(\lambda_1^{-1}+\alpha^2)e^{\sigma(\tau-t)}\|(u^\tau,\phi)\|^2_{V\times L^2_H} \quad \textit{for all } t \ge \tau. \tag{4.3}$$

Proof. It is analogous to the proof of (3.3), and we omit it. □

For the study of the problem (4.2), we will make use of the following lemma.

Lemma 4.2. *Let me given $F \in L^2_{loc}(\mathbb{R};D(A^{-\beta}))$ with $0 \le \beta \le 1/2$, $\tau \in \mathbb{R}$ and σ fulfilling that $0 < \sigma < 2(\nu - \lambda_1^{-1}C_g)(\lambda_1^{-1}+\alpha^2)^{-1}$. Then, the problem*

$$\begin{cases} z \in C([\tau,\infty);V) \cap L^2(\tau-h,T;H) \textit{ for all } T > \tau, \\ \dfrac{d}{dt}(z(t)+\alpha^2 Az(t)) + \nu Az(t) = F(t), \quad \textit{in } \mathcal{D}'(\tau,\infty;V'), \\ z(\tau) = 0, \\ z(t) = 0 \quad \text{a.e. } t \in (\tau-h,\tau) \end{cases}$$

has a unique solution z, which also satisfies $z \in C([\tau,\infty);D(A^{1-\beta}))$, and

$$|A^{1-\beta}z(t)|^2 \le \alpha^{-2}\varepsilon^{-1}\int_\tau^t e^{\sigma(s-t)}\left|A^{-\beta}F(s)\right|^2 ds \quad \textit{for all } t \ge \tau,$$

where ε is given by (3.4).

Proof. It can be done analogously as in [14, Lemma 26] with $z = 0$ in $(\tau-h,\tau)$. □

Now we can prove the following regularity result for the pullback attractors in V norm.

Theorem 4.3. *Consider given $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ satisfying conditions (I)–(V). Assume that $f \in L^2_{loc}(\mathbb{R}; D(A^{-\beta}))$ for some $0 \le \beta \le 1/2$, and that*

$$\sup_{r\le 0} \int_{r-1}^{r} \|f(s)\|_*^2 ds < \infty. \tag{4.4}$$

Then:

(1) If f also satisfies

$$\int_{-\infty}^{0} e^{\sigma s} |A^{-\beta} f(s)|^2 ds < \infty, \tag{4.5}$$

and

$$\begin{cases} \sup_{r\le 0} \int_{r-1}^{r} |A^{-1/4-\beta} f(s)|^2 ds < \infty, & \text{if } 0 < \beta < 1/4, \\ \sup_{r\le 0} \int_{r-1}^{r} |A^{-\delta} f(s)|^2 ds < \infty & \text{for some } 0 < \delta < 1/4, \text{ if } \beta = 0, \end{cases} \tag{4.6}$$

then, for any $t_1 < t_2$, the pullback attractor $\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)} = \mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}$ fulfills that

$$\bigcup_{t_1\le t\le t_2} \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t) = \bigcup_{t_1\le t\le t_2} \mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}(t) \text{ is a bounded subset of } D(A^{1-\beta}) \times C_{D(A^{1-\beta})}. \tag{4.7}$$

(2) If f also satisfies

$$\sup_{r\le 0} \int_{r-1}^{r} |A^{-\beta} f(s)|^2 ds < \infty, \tag{4.8}$$

then, for any $t_2 \in \mathbb{R}$, it holds that

$$\bigcup_{t\le t_2} \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t) = \bigcup_{t\le t_2} \mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}(t) \text{ is a bounded subset of } D(A^{1-\beta}) \times C_{D(A^{1-\beta})}. \tag{4.9}$$

Proof. Let us fix $t \in \mathbb{R}$ and $(v,\psi) \in \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t) = \mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}(t)$. By Remark 3.21 and (4.4), we see that

$$\bigcup_{r\le t} \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(r) \subset \overline{B}_{V\times C_V}(0, \widetilde{R}_\sigma(t)), \tag{4.10}$$

where $\widetilde{R}_\sigma^2(t) = 2 + 2\alpha^{-2}\varepsilon^{-1}e^{2\sigma h} \sup_{r\le t}(e^{-\sigma r}\int_{-\infty}^{r} e^{\sigma s}\|f(s)\|_*^2 ds)$, with ε given by (3.4).

Let $\{\tau_n\}_{n\ge 1} \subset (-\infty, t-h]$ be a sequence with $\tau_n \to -\infty$ as $n \to \infty$. By the invariance of $\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}$, for each $n \ge 1$ there exists $(u^{\tau_n}, \phi^n) \in \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(\tau_n)$ such that $(v,\psi) = S(t,\tau_n)(u^{\tau_n},\phi^n)$, and therefore,

$$(v,\psi) = Y(t,\tau_n)(u^{\tau_n},\phi^n) + Z(t,\tau_n)(0,0),$$

where

$$Y(t,\tau_n)(u^{\tau_n},\phi^n) = (y(t;\tau_n,u^{\tau_n},\phi^n), y_t(\cdot;\tau_n,u^{\tau_n},\phi^n))$$

and

$$Z(t,\tau_n)(0,0) = (z(t;\tau_n,0,0), z_t(\cdot;\tau_n,0,0))$$

are continuous processes on $V \times L^2_H$ associated to problems (4.1) and (4.2), respectively.

From (4.3) and (4.10) we deduce that $\|Y(t,\tau_n)(u^{\tau_n},\phi^n)\|_{V\times C_V} \to 0$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \|Z(t,\tau_n)(0,0) - (v,\psi)\|_{V\times C_V} = 0. \tag{4.11}$$

Let us denote $(u^n(r), u^n_r(\cdot)) = S(r, \tau_n)(u^{\tau_n}, \phi^n)$ for $r \geq \tau_n$ and $n \geq 1$. By (4.10) and the invariance of $\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}$,

$$(u^n(r), u^n_r(\cdot)) \in \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(r) \subset \overline{B}_{V\times C_V}(0, \widetilde{R}_\sigma(t)), \quad \forall \tau_n \leq r \leq t, \quad \forall n \geq 1. \tag{4.12}$$

Now we distinguish three cases.

Case 1. If $1/4 \leq \beta \leq 1/2$.

In this case, from (2.11), the continuous injection of V in $D(A^{3/4-\beta})$ and (4.12), we deduce that

$$\begin{aligned} |A^{-\beta}B(u^n(r))| &\leq C_{(3/4-\beta)}|A^{3/4-\beta}u^n(r)|\|u^n(r)\| \\ &\leq \widetilde{C}_{(3/4-\beta)}\|u^n(r)\|^2 \\ &\leq \widetilde{C}_{(3/4-\beta)}\widetilde{R}^2_\sigma(t), \quad \forall \tau_n \leq r \leq t, \quad \forall n \geq 1. \end{aligned}$$

Thus, if we assume (4.5), from Lemma 4.2, condition (V) on g, and the continuous injection of H in $D(A^{-\beta})$, we obtain that

$$\begin{aligned} |A^{1-\beta}z(\theta;\tau_n,0,0)|^2 &\leq 3\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\left(\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds + \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^4_\sigma(t)\right. \\ &\quad \left. + \int_{\tau_n}^t e^{\sigma(s-t)}|A^{-\beta}g(s,u^n_s)|^2\,ds\right) \\ &\leq 3\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\left(\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds + \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^4_\sigma(t)\right. \\ &\quad \left. + C_\beta C_g^2\lambda_1^{-1}\left(\int_{\tau_n-h}^{\tau_n} e^{\sigma(s-t)}\|\phi^n(s-\tau_n)\|^2\,ds + \int_{\tau_n}^t e^{\sigma(s-t)}\|u^n(s)\|^2\,ds\right)\right) \end{aligned}$$

for all $\theta \in [t-h, t]$, and then, from (4.12), we deduce that

$$\|Z(t,\tau_n)(0,0)\|^2_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}} \leq M^2_{\sigma,\beta}(t), \tag{4.13}$$

where

$$M^2_{\sigma,\beta}(t) = 6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\left(\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds + \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^4_\sigma(t) + 2C_\beta C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_\sigma(t)\right).$$

From (4.11), (4.13) and the weak lower semi-continuity of the norm, we deduce that (v,ψ) belongs to $\overline{B}_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}}(0, M_{\sigma,\beta}(t))$, and therefore (4.7) holds.

Moreover, if f satisfies (4.8), then (4.9) holds, and more exactly,

$$\bigcup_{t\leq t_2} \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t) \subset \overline{B}_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}}(0, \widetilde{M}_{\sigma,\beta}(t_2)), \quad \text{for all } t_2 \in \mathbb{R}, \tag{4.14}$$

where

$$\begin{aligned} \widetilde{M}^2_{\sigma,\beta}(t_2) = 6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\Bigg(&\sup_{t\leq t_2}\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2 ds \\ &+ \sigma^{-1}\widetilde{C}^2_{(3/4-\beta)}\widetilde{R}^4_\sigma(t_2) + 2C_\beta C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_\sigma(t_2)\Bigg). \end{aligned}$$

Case 2. If $0 < \beta < 1/4$.

In this case, if f satisfies (4.6), as $1/4 < 1/4+\beta < 1/2$, from (4.14) we have that

$$\bigcup_{r\le t}\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(r)\subset \overline{B}_{D(A^{3/4-\beta})\times C_{D(A^{3/4-\beta})}}(0,\widetilde{M}_{\sigma,1/4+\beta}(t)).$$

Thus, by (2.11) and (4.12), we obtain that

$$\begin{aligned}|A^{-\beta}B(u^n(r))| &\le C_{(3/4-\beta)}|A^{3/4-\beta}u^n(r)|\|u^n(r)\|\\ &\le C_{(3/4-\beta)}\widetilde{M}_{\sigma,1/4+\beta}(t)\widetilde{R}_\sigma(t),\quad \forall\tau_n\le r\le t,\quad \forall n\ge 1.\end{aligned}$$

Thus, if we assume (4.5), from Lemma 4.2 we deduce that

$$\|Z(t,\tau_n)(0,0)\|^2_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}}\le R^2_{\sigma,\beta}(t), \tag{4.15}$$

where

$$R^2_{\sigma,\beta}(t)=6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\bigg(\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds \\ +\sigma^{-1}C^2_{(3/4-\beta)}\widetilde{M}^2_{\sigma,1/4+\beta}(t)\widetilde{R}^2_\sigma(t)+2C_\beta C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_\sigma(t)\bigg).$$

Again, from (4.11), (4.15) and the weak lower semi-continuity of the norm, we deduce that (v,ψ) belongs to $\overline{B}_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}}(0,R_{\sigma,\beta}(t))$, and therefore (4.7) holds.

Moreover, if f satisfies (4.8), then (4.9) holds, and more exactly,

$$\bigcup_{t\le t_2}\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t)\subset \overline{B}_{D(A^{1-\beta})\times C_{D(A^{1-\beta})}}(0,\widetilde{R}_{\sigma,\beta}(t_2)),\quad \text{for all } t_2\in\mathbb{R}, \tag{4.16}$$

where

$$\widetilde{R}^2_{\sigma,\beta}(t_2)=6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\bigg(\sup_{t\le t_2}\int_{-\infty}^t e^{\sigma(s-t)}|A^{-\beta}f(s)|^2\,ds \\ +\sigma^{-1}C^2_{(3/4-\beta)}\widetilde{M}^2_{\sigma,1/4+\beta}(t_2)\widetilde{R}^2_\sigma(t_2)+2C_\beta C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_\sigma(t_2)\bigg).$$

Case 3. If $\beta=0$.

In this case, if f satisfies (4.6), as $0<\delta<1/4$, from (4.16) we see that

$$\bigcup_{r\le t}\mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(r)\subset \overline{B}_{D(A^{1-\delta})\times C_{D(A^{1-\delta})}}(0,\widetilde{R}_{\sigma,\delta}(t)).$$

So, by (2.10) and (4.12), we deduce that

$$\begin{aligned}|B(u^n(r))| &\le C_{(1-\delta)}|A^{1-\delta}u^n(r)|\|u^n(r)\|\\ &\le C_{(1-\delta)}\widetilde{R}_{\sigma,\delta}(t)\widetilde{R}_\sigma(t),\quad \forall\tau_n\le r\le t,\quad \forall n\ge 1.\end{aligned}$$

Thus, if we assume (4.5), from Lemma 4.2 we deduce that

$$\|Z(t,\tau_n)(0,0)\|^2_{D(A)\times C_{D(A)}}\le R^2_{\sigma,\delta,0}(t) \tag{4.17}$$

where

$$R^2_{\sigma,\delta,0}(t)=6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\bigg(\int_{-\infty}^t e^{\sigma(s-t)}|f(s)|^2\,ds+\sigma^{-1}C^2_{(1-\delta)}\widetilde{R}^2_{\sigma,\delta}(t)\widetilde{R}^2_\sigma(t)+2C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_\sigma(t)\bigg).$$

Again, from (4.11), (4.17) and the weak lower semi-continuity of the norm, we deduce that

$$(v, \psi) \in \overline{B}_{D(A)\times C_{D(A)}}(0, R_{\sigma,\delta,0}(t)),$$

and therefore (4.7) holds.

Moreover, if f satisfies (4.8), then (4.9) holds, and more exactly,

$$\bigcup_{t\le t_2} \mathcal{A}_{\mathcal{D}_\sigma(V\times L^2_H)}(t) \subset \overline{B}_{D(A)\times C_{D(A)}}(0, \widetilde{R}_{\sigma,\delta,0}(t_2)), \text{ for all } t_2 \in \mathbb{R},$$

where

$$\widetilde{R}^2_{\sigma,\delta,0}(t_2) = 6\alpha^{-2}\varepsilon^{-1}e^{\sigma h}\left(\sup_{t\le t_2}\int_{-\infty}^{t} e^{\sigma(s-t)}|f(s)|^2\,ds \right.$$
$$\left. + \sigma^{-1}C^2_{(1-\delta)}\widetilde{R}^2_{\sigma,\delta}(t_2)\widetilde{R}^2_{\sigma}(t_2) + 2C_g^2\lambda_1^{-1}\sigma^{-1}\widetilde{R}^2_\sigma(t_2)\right). \quad \square$$

5 Attraction in $D(A)$ norm

By the previous results, when $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^3)$, the restriction to $D(A)\times L^2_V$ of the process S defined by (3.1) is a process on $D(A)\times L^2_V$. Now, we will prove that under suitable assumptions on f and g, we can obtain the existence of minimal pullback attractors for S on $D(A)\times L^2_V$ and even more.

Proposition 5.1. *Assume that $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^3)$, and $g : \mathbb{R}\times C_H \to (L^2(\Omega))^3$ satisfying (I)–(IV), are given. Then, the restriction to $D(A)\times L^2_V$ of the bi-parametric family of maps $S(t,\tau)$, with $\tau \le t$, given by (3.1), is a continuous process on $D(A)\times L^2_V$.*

Proof. It is a consequence of Theorem 2.2 and Remark 2.4 (i). □

As in the previous section, we will need the following continuity result for the process S in a weak sense.

Proposition 5.2. *Let $f \in L^2_{loc}(\mathbb{R};(L^2(\Omega))^3)$, $g : \mathbb{R}\times C_H \to (L^2(\Omega))^3$ satisfying (I)–(IV), and $\tau < t$ be given. Then, for any sequence such that*

$$(u^{\tau,n}, \phi^n) \rightharpoonup (u^\tau, \phi) \text{ weakly in } D(A)\times L^2_{D(A)}$$

and

$$\frac{d\phi^n}{ds} \rightharpoonup \frac{d\phi}{ds} \text{ weakly in } L^2_{D(A)},$$

the following convergences hold for the sequence of solutions $u(\cdot;\tau,u^{\tau,n},\phi^n)$ towards the solution $u(\cdot;\tau,u^\tau,\phi)$:

$$u(\cdot;\tau,u^{\tau,n},\phi^n) \overset{*}{\rightharpoonup} u(\cdot;\tau,u^\tau,\phi) \text{ weakly-star in } L^\infty(\tau,t;D(A)),$$

$$u(\cdot;\tau,u^{\tau,n},\phi^n) \to u(\cdot;\tau,u^\tau,\phi) \text{ strongly in } C([\tau-h,t];V),$$

$$u(t;\tau,u^{\tau,n},\phi^n) \rightharpoonup u(t;\tau,u^\tau,\phi) \text{ weakly in } D(A),$$

$$u(\cdot;\tau,u^{\tau,n},\phi^n) \rightharpoonup u(\cdot;\tau,u^\tau,\phi) \text{ weakly in } L^2(\tau-h,t;D(A)).$$

Proof. It can be done analogously to that of Proposition 3.10. □

For the obtention of a pullback absorbing family for the process S restricted to $D(A) \times L^2_V$, we first have the following result.

Lemma 5.3. *Suppose that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ satisfies (4.4) and that $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V). Then, for any $\tau \in \mathbb{R}$, $(u^\tau, \phi) \in D(A) \times L^2_V$, and $0 < \underline{\sigma} < \sigma/3$, the solution $u = u(\cdot; \tau, u^\tau, \phi)$ of (1.1) satisfies*

$$|Au(t)|^2 \leq \alpha^{-2} \max\{\lambda_1^{-1} + \alpha^2, C_g\} e^{\sigma(\tau - t)} \|(u^\tau, \phi)\|^2_{D(A)\times L^2_V} + 2\alpha^{-2}\varepsilon^{-1} \int_\tau^t e^{\sigma(s-t)} |f(s)|^2\, ds$$
$$+ 4\alpha^{-2} C_\varepsilon C^3_{\underline{\sigma}} (\sigma - 3\underline{\sigma})^{-1} \left(e^{-3\underline{\sigma}(t-\tau)} \|(u^\tau, \phi)\|^6_{V \times L^2_H} + M^3_{t,\underline{\sigma}} \right) \tag{5.1}$$

for all $t \geq \tau$, where $\varepsilon > 0$ is given by (3.4),

$$C_\varepsilon = 27 C_2^4 (2\varepsilon^3)^{-1}, \tag{5.2}$$

$$C_{\underline{\sigma}} = \alpha^{-2} \max\left\{ \max\{\lambda_1^{-1} + \alpha^2, C_g\}, \left(2\nu - \underline{\sigma}(\lambda_1^{-1} + \alpha^2) - 2\lambda_1^{-1} C_g\right)^{-1} \right\}, \tag{5.3}$$

and

$$M_{t,\underline{\sigma}} = \sup_{r \leq t} \int_{-\infty}^r e^{\underline{\sigma}(s-r)} \|f(s)\|^2_* ds. \tag{5.4}$$

Proof. From Lemma 3.11, we have that

$$\|u(s)\|^2 \leq C_{\underline{\sigma}} \left(e^{\underline{\sigma}(\tau - s)} \|(u^\tau, \phi)\|^2_{V \times L^2_H} + M_{t,\underline{\sigma}} \right), \quad \forall \tau \leq s \leq t. \tag{5.5}$$

On the other hand, by (2.17),

$$\frac{d}{dt}(e^{\sigma t}\|u(t)\|^2 + \alpha^2 e^{\sigma t}|Au(t)|^2) + 2\nu e^{\sigma t}|Au(t)|^2 + 2e^{\sigma t}(B(u(t)), Au(t))$$
$$= \sigma e^{\sigma t}\|u(t)\|^2 + \alpha^2 \sigma e^{\sigma t}|Au(t)|^2 + 2e^{\sigma t}(f(t) + g(t, u_t), Au(t)), \quad \text{a.e. } t > \tau.$$

Thus, taking into account that $\|u(t)\|^2 \leq \lambda_1^{-1}|Au(t)|^2$,

$$2|\,(B(u(t)), Au(t))\,| \leq 2C_2 \|u(t)\|^{3/2} |Au(t)|^{3/2}$$
$$\leq C_\varepsilon \|u(t)\|^6 + \frac{\varepsilon}{2}|Au(t)|^2,$$
$$2|\,(f(t), Au(t))\,| \leq \frac{\varepsilon}{2}|Au(t)|^2 + \frac{2}{\varepsilon}|f(t)|^2,$$

and

$$2|(g(t, u_t), Au(t))| \leq \frac{C_g}{\lambda_1}|Au(t)|^2 + \frac{\lambda_1}{C_g}|g(t, u_t)|^2,$$

we deduce that

$$e^{\sigma t}(\|u(t)\|^2 + \alpha^2|Au(t)|^2) + (2\nu - \varepsilon - \sigma(\lambda_1^{-1} + \alpha^2) - \lambda_1^{-1}C_g) \int_\tau^t e^{\sigma s}|Au(s)|^2\, ds$$
$$\leq e^{\sigma\tau}(\lambda_1^{-1} + \alpha^2)|Au^\tau|^2 + 2\varepsilon^{-1} \int_\tau^t e^{\sigma s}|f(s)|^2\, ds + \lambda_1 C_g \int_{\tau - h}^t e^{\sigma s}|u(s)|^2\, ds$$
$$+ C_\varepsilon \int_\tau^t e^{\sigma s}\|u(s)\|^6\, ds$$
$$\leq e^{\sigma\tau}\left((\lambda_1^{-1} + \alpha^2)|Au^\tau|^2 + C_g\|\phi\|^2_{L^2_V}\right) + 2\varepsilon^{-1} \int_\tau^t e^{\sigma s}|f(s)|^2\, ds + \lambda_1^{-1} C_g \int_\tau^t e^{\sigma s}|Au(s)|^2\, ds$$
$$+ C_\varepsilon \int_\tau^t e^{\sigma s}\|u(s)\|^6\, ds$$

for all $t \geq \tau$.

From this inequality, since the choice of ε makes the term $\int_\tau^t e^{\sigma s}|Au(s)|^2\,ds$ disappear, using (5.5) we easily obtain (5.1). □

Definition 5.4. For any $\sigma, \underline{\sigma} > 0$, consider the universe $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$ formed by the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A) \times L^2_V)$ such that

$$\lim_{\tau\to-\infty}\left(e^{\sigma\tau}\sup_{(v,\varphi)\in D(\tau)}\|(v,\varphi)\|^2_{D(A)\times L^2_V}\right) = \lim_{\tau\to-\infty}\left(e^{\underline{\sigma}\tau}\sup_{(v,\varphi)\in D(\tau)}\|(v,\varphi)\|^2_{V\times L^2_H}\right) = 0.$$

Accordingly to the notation introduced in Section 3, $\mathcal{D}_F(D(A) \times L^2_V)$ will denote the class of families $\widehat{D} = \{D(t) = D : t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of $D(A) \times L^2_V$. Observe that the universe $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$, which is inclusion-closed, contains the universe $\mathcal{D}_F(D(A) \times L^2_V)$.

Remark 5.5. Under the additional assumption

$$\int_{-\infty}^0 e^{\sigma s}|f(s)|^2\,ds < \infty, \tag{5.6}$$

from Lemma 5.3 it is easy to see that, for $0 < \underline{\sigma} < \sigma/3$, the family $\{\overline{B}_{D(A)\times L^2_V}(0, \widetilde{R}_{\sigma,\underline{\sigma}}(t)) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A) \times L^2_V)$ is pullback $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$-absorbing for the process S on $D(A) \times L^2_V$, where

$$\widetilde{R}^2_{\sigma,\underline{\sigma}}(t) = 1 + 2\alpha^{-2}\varepsilon^{-1}(1+\lambda_1^{-1}he^{\sigma h})e^{-\sigma t}\int_{-\infty}^t e^{\sigma s}|f(s)|^2\,ds + (1+\lambda_1^{-1}h)4\alpha^{-2}C_\varepsilon C^3_{\underline{\sigma}}(\sigma-3\underline{\sigma})^{-1}M^3_{t,\underline{\sigma}}.$$

However, in order to apply Proposition 5.2, we need to obtain a different pullback $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$-absorbing family.

Lemma 5.6. *Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ satisfies (4.4) and (5.6), and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V). Then, for $0 < \underline{\sigma} < \sigma/3$ and for any $t \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$, there exist $\tau_2(\widehat{D}, t, h) < t - 2h$ and functions $\{\rho_i\}_{i=3}^4$ such that for any $\tau \leq \tau_2(\widehat{D}, t, h)$ and any $(u^\tau, \phi) \in D(\tau)$, it holds*

$$|Au(r;\tau,u^\tau,\phi)|^2 \leq \rho_3^2(t), \quad \forall r \in [t-2h, t], \tag{5.7}$$

$$\int_{t-h}^t |Au'(\theta;\tau,u^\tau,\phi)|^2\,d\theta \leq \rho_4^2(t), \tag{5.8}$$

where

$$\rho_3^2(t) = 1 + 2\alpha^{-2}\varepsilon^{-1}e^{-\sigma(t-2h)}\int_{-\infty}^t e^{\sigma s}|f(s)|^2\,ds + 4\alpha^{-2}C_\varepsilon C^3_{\underline{\sigma}}(\sigma-3\underline{\sigma})^{-1}M^3_{t,\underline{\sigma}}, \tag{5.9}$$

$$\rho_4^2(t) = 16\alpha^{-4}h\rho_3^2(t)\left(\nu^2 + C_2^2\lambda_1^{-3/2}\rho_3^2(t) + 2\lambda_1^{-2}C_g^2\right) + 16\alpha^{-4}\int_{t-h}^t |f(s)|^2\,ds, \tag{5.10}$$

where $\varepsilon, C_\varepsilon, C_{\underline{\sigma}}$ and $M_{t,\underline{\sigma}}$ are given by (3.4), (5.2), (5.3) and (5.4), respectively.

Proof. Let $\tau_2(\widehat{D}, t, h) < t - 2h$ be such that

$$\alpha^{-2}\max\{\lambda_1^{-1}+\alpha^2, C_g\}e^{-\sigma(t-2h)}e^{\sigma\tau}\|(u^\tau,\phi)\|^2_{D(A)\times L^2_V}$$
$$+ 4\alpha^{-2}C_\varepsilon C^3_{\underline{\sigma}}(\sigma-3\underline{\sigma})^{-1}e^{-3\underline{\sigma}(t-2h)}e^{3\underline{\sigma}\tau}\|(u^\tau,\phi)\|^6_{V\times L^2_H} \leq 1 \quad \forall\tau \leq \tau_2(\widehat{D},t,h),\ (u^\tau,\phi) \in D(\tau).$$

Consider fixed $\tau \leq \tau_2(\widehat{D}, t, h)$ and $(u^\tau, \phi) \in D(\tau)$. The estimate (5.7) follows directly from (5.1), using the increasing character of the exponential.

Now, from (2.9), (2.14) and (2.1), we obtain that $v = \mathcal{C}u$ satisfies

$$\begin{aligned}|v'(\theta)| &\leq \nu|Au(\theta)| + C_2|Au(\theta)|^{1/2}\|u(\theta)\|^{3/2} + |f(\theta)| + |g(\theta, u_\theta)| \\ &\leq \nu|Au(\theta)| + C_2\lambda_1^{-3/4}|Au(\theta)|^2 + |f(\theta)| + |g(\theta, u_\theta)|, \quad \text{a.e. } \theta > \tau,\end{aligned}$$

and therefore,

$$|v'(\theta)|^2 \leq 4\nu^2|Au(\theta)|^2 + 4C_2^2\lambda_1^{-3/2}|Au(\theta)|^4 + 4|f(\theta)|^2 + 4|g(\theta, u_\theta)|^2, \quad \text{a.e. } \theta > \tau.$$

Integrating in time above and using properties (II) and (IV) on g, we deduce

$$\begin{aligned}\int_{t-h}^t |v'(\theta)|^2\, d\theta \leq 4\nu^2 \int_{t-h}^t |Au(\theta)|^2\, d\theta + 4C_2^2\lambda_1^{-3/2} \int_{t-h}^t |Au(\theta)|^4\, d\theta \\ + 4\int_{t-h}^t |f(\theta)|^2\, d\theta + 4\lambda_1^{-2}C_g^2 \int_{t-2h}^t |Au(\theta)|^2\, d\theta,\end{aligned}$$

whence, by (2.13) and (5.7), the estimate (5.8) follows. □

Corollary 5.7. *Under the assumptions of Lemma 5.6, for $0 < \underline{\sigma} < \sigma/3$, the family $\widehat{D}_{\sigma,\underline{\sigma}} = \{D_{\sigma,\underline{\sigma}}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(D(A) \times L_V^2)$ defined by*

$$\begin{aligned}D_{\sigma,\underline{\sigma}}(t) = \Big\{(w, \psi) \in D(A) \times L^2_{D(A)} : \exists \frac{d\psi}{ds} \in L^2_{D(A)}, \\ \|(w, \psi)\|_{D(A)\times L^2_{D(A)}} \leq R_{\sigma,\underline{\sigma}}(t), \left\|\frac{d\psi}{ds}\right\|_{L^2_{D(A)}} \leq \rho_4(t)\Big\}\end{aligned} \tag{5.11}$$

is pullback $\mathcal{D}_\sigma(D(A) \times L_V^2) \cap \mathcal{D}_{\underline{\sigma}}(V \times L_H^2)$-absorbing for the process S on $D(A) \times L_V^2$ defined by (3.1), (and therefore $\mathcal{D}_F(D(A) \times L_V^2)$-absorbing too), where $R_{\sigma,\underline{\sigma}}(t)$ satisfies

$$R^2_{\sigma,\underline{\sigma}}(t) = (1 + h)\rho_3^2(t), \tag{5.12}$$

with $\rho_3(t)$ and $\rho_4(t)$ given by (5.9) and (5.10) respectively.

Now, we prove that the process S is $(D(A) \times L_V^2, D(A) \times C_V)$ pullback $\mathcal{D}_\sigma(D(A) \times L_V^2) \cap \mathcal{D}_{\underline{\sigma}}(V \times L_H^2)$-asymptotically compact. We will apply, under the natural necessary changes, the same energy method used in the proof of Lemma 3.17.

Lemma 5.8. *Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ satisfies (4.4) and (5.6), and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V). Then, for any $0 < \underline{\sigma} < \sigma/3$, the restriction to $D(A) \times L_V^2$ of the process S defined by (3.1) is $(D(A) \times L_V^2, D(A) \times C_V)$ pullback $\mathcal{D}_\sigma(D(A) \times L_V^2) \cap \mathcal{D}_{\underline{\sigma}}(V \times L_H^2)$-asymptotically compact.*

Proof. Let us fix $0 < \underline{\sigma} < \sigma/3$. Let be given $\widehat{D} \in \mathcal{D}_\sigma(D(A) \times L_V^2) \cap \mathcal{D}_{\underline{\sigma}}(V \times L_H^2)$, $t \in \mathbb{R}$, a sequence $\{\tau_n\} \subset (-\infty, t]$ with $\tau_n \to -\infty$, and a sequence $\{(u^{\tau_n}, \phi^n)\}$ with $(u^{\tau_n}, \phi^n) \in D(\tau_n)$ for all n. We must prove that the sequence

$$\{S(t, \tau_n)(u^{\tau_n}, \phi^n)\} = \{(u(t; \tau_n, u^{\tau_n}, \phi^n), u_t(\cdot; \tau_n, u^{\tau_n}, \phi^n))\}$$

is relatively compact in $D(A) \times C_V$.

First, we check the asymptotic compactness in the first component of S. By Corollary 5.7, for each integer $k \geq 0$, there exists $\tau_{\widehat{D}}(k) \leq t-k$ such that $S(t-k,\tau)D(\tau) \subset D_{\sigma,\underline{\sigma}}(t-k)$ for all $\tau \leq \tau_{\widehat{D}}(k)$. From this and a diagonal argument, we can extract a subsequence $\{(u^{\tau_{n'}},\phi^{n'})\} \subset \{(u^{\tau_n},\phi^n)\}$ such that

$$S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}) \rightharpoonup (w^k,\psi^k) \quad \text{weakly in } D(A)\times L^2_{D(A)}, \tag{5.13}$$

$$\frac{d}{ds}u_{t-k}(\cdot;\tau_{n'},u^{\tau_{n'}},\phi^{n'}) \rightharpoonup \frac{d}{ds}\psi^k \qquad \text{weakly in } L^2_{D(A)}, \tag{5.14}$$

for all integer $k \geq 0$, where $(w^k,\psi^k) \in D_{\sigma,\underline{\sigma}}(t-k)$.

Now, applying Proposition 5.2 on each fixed interval $[t-k,t]$, we deduce that

$$\begin{aligned}(w^0,\psi^0) &= (D(A)\times L^2_{D(A)}) - \text{weak} \lim_{n'\to\infty} S(t,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})\\ &= (D(A)\times L^2_{D(A)}) - \text{weak} \lim_{n'\to\infty} S(t,t-k)S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})\\ &= S(t,t-k)\left[(D(A)\times L^2_{D(A)}) - \text{weak} \lim_{n'\to\infty} S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})\right]\\ &= S(t,t-k)(w^k,\psi^k).\end{aligned}$$

From (5.13) with $k=0$, we obtain in particular that $|Aw^0| \leq \liminf_{n'\to\infty}|Au(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|$. We will prove now that it also holds that

$$\limsup_{n'\to\infty}|Au(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})| \leq |Aw^0|, \tag{5.15}$$

which combined with the weak converge of $u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})$ to w^0 in $D(A)$, will imply the convergence in the strong topology of $D(A)$.

Observe that, as we already used in Lemma 5.3, for any $\tau \in \mathbb{R}$ and $(u^\tau,\phi) \in D(A)\times L^2_V$, the solution $u(\cdot;\tau,u^\tau,\phi)$, for short denoted $u(\cdot)$, satisfies the differential equality

$$\begin{aligned}\frac{d}{dt}(e^{\sigma t}\|u(t)\|^2+\alpha^2 e^{\sigma t}|Au(t)|^2) &= \sigma e^{\sigma t}\|u(t)\|^2+\alpha^2\sigma e^{\sigma t}|Au(t)|^2-2\nu e^{\sigma t}|Au(t)|^2\\ &\quad -2e^{\sigma t}(B(u(t)),Au(t))+2e^{\sigma t}(f(t)+g(t,u_t),Au(t))\end{aligned} \tag{5.16}$$

a.e. $t > \tau$. Since in particular $0 < \sigma < 2\nu(\lambda_1^{-1}+\alpha^2)^{-1}$, notice that $[[\cdot]]$, with $[[v]]^2 = (2\nu-\alpha^2\sigma)|Av|^2-\sigma\|v\|^2$, defines an equivalent norm to $|\cdot|_{D(A)}$ in $D(A)$.

We integrate the above expression in the interval $[t-k,t]$ for the solutions $u(\cdot;\tau_{n'},u^{\tau_{n'}},\phi^{n'})$ with $\tau_{n'} \leq t-k$, which yields

$$\begin{aligned}&\|u(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\|^2+\alpha^2|Au(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|^2\\ &\quad = \|u(t;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\|^2+\alpha^2|Au(t;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))|^2\\ &\quad = e^{-\sigma k}\left(\|u(t-k;\tau_{n'},u^{\tau_{n'}},\phi^{n'})\|^2+\alpha^2|Au(t-k;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|^2\right)\\ &\qquad +2\int_{t-k}^t e^{\sigma(s-t)}(f(s),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds\\ &\qquad +2\int_{t-k}^t e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds\\ &\qquad -2\int_{t-k}^t e^{\sigma(s-t)}(B(u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds\\ &\qquad -\int_{t-k}^t e^{\sigma(s-t)}[[u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))]]^2\,ds.\end{aligned} \tag{5.17}$$

From (5.13), (5.14) and Proposition 5.2, in particular we have that

$$u(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\to u(\cdot;t-k,w^k,\psi^k)\quad \text{strongly in } C([t-k,t];V), \tag{5.18}$$

and also

$$u(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\rightharpoonup u(\cdot;t-k,w^k,\psi^k)\quad \text{weakly in } L^2(t-k,t;D(A)). \tag{5.19}$$

Then, it is not difficult to see that

$$\begin{aligned}\lim_{n'\to\infty}&\int_{t-k}^t e^{\sigma(s-t)}(B(u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds\\ &=\int_{t-k}^t e^{\sigma(s-t)}(B(u(s;t-k,w^k,\psi^k)),Au(s;t-k,w^k,\psi^k))\,ds.\end{aligned} \tag{5.20}$$

On other hand, as $e^{\sigma(\cdot-t)}f(\cdot)\in L^2(t-k,t;(L^2(\Omega))^3)$, it yields

$$\begin{aligned}\lim_{n'\to\infty}&\int_{t-k}^t e^{\sigma(s-t)}(f(s),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds\\ &=\int_{t-k}^t e^{\sigma(s-t)}(f(s),Au(s;t-k,w^k,\psi^k))\,ds.\end{aligned}$$

Moreover, from (5.18), in particular we also have that

$$u(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))\to u(\cdot;t-k,w^k,\psi^k)\quad \text{strongly in } L^2(t-k-h,t;H),$$

which jointly with (5.19), implies that

$$\begin{aligned}\lim_{n'\to\infty}&\int_{t-k}^t e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))),Au(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'})))\,ds\\ &=\int_{t-k}^t e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,w^k,\psi^k)),Au(s;t-k,w^k,\psi^k))\,ds\end{aligned}$$

Finally, as $\int_{t-k}^t e^{\sigma(s-t)}[[v(s)]]^2\,ds$ defines an equivalent norm in $L^2(t-k,t;D(A))$, we also deduce from above that

$$\begin{aligned}&\int_{t-k}^t e^{\sigma(s-t)}[[u(s;t-k,w^k,\psi^k)]]^2\,ds\\ &\qquad\le \liminf_{n'\to\infty}\int_{t-k}^t e^{\sigma(s-t)}[[u(s;t-k,S(t-k,\tau_{n'})(u^{\tau_{n'}},\phi^{n'}))]]^2\,ds.\end{aligned} \tag{5.21}$$

From (5.17), (5.20)–(5.21), taking into account (5.13) with $k=0$, the compactness of the injection of $D(A)$ into V, and (5.11), we conclude that

$$\begin{aligned}\|w^0\|^2&+\alpha^2\limsup_{n'\to\infty}|Au(t;\tau_{n'},u^{\tau_{n'}},\phi^{n'})|^2\\ &\le e^{-\sigma k}(\lambda_1^{-1}+\alpha^2)R^2_{\sigma,\underline{\sigma}}(t-k)+2\int_{t-k}^t e^{\sigma(s-t)}(f(s),Au(s;t-k,w^k,\psi^k))\,ds\\ &\quad+2\int_{t-k}^t e^{\sigma(s-t)}(g(s,u_s(\cdot;t-k,w^k,\psi^k)),Au(s;t-k,w^k,\psi^k))\,ds\\ &\quad-2\int_{t-k}^t e^{\sigma(s-t)}(B(u(s;t-k,w^k,\psi^k)),Au(s;t-k,w^k,\psi^k))\,ds\\ &\quad-\int_{t-k}^t e^{\sigma(s-t)}[[u(s;t-k,w^k,\psi^k)]]^2\,ds.\end{aligned}$$

Now, taking into account that $w^0 = u(t; t-k, w^k, \psi^k)$, integrating again in (5.16), we obtain

$$\begin{aligned}\|w^0\|^2 + \alpha^2|Aw^0|^2 = e^{-\sigma k}(\|w^k\|^2 + \alpha^2|Aw^k|^2) + 2\int_{t-k}^{t} e^{\sigma(s-t)}(f(s), Au(s; t-k, w^k, \psi^k))\, ds \\ + 2\int_{t-k}^{t} e^{\sigma(s-t)}(g(s, u_s(\cdot; t-k, w^k, \psi^k)), Au(s; t-k, w^k, \psi^k))\, ds \\ - 2\int_{t-k}^{t} e^{\sigma(s-t)}(B(u(s; t-k, w^k, \psi^k)), Au(s; t-k, w^k, \psi^k))\, ds \\ - \int_{t-k}^{t} e^{\sigma(s-t)}[[u(s; t-k, w^k, \psi^k)]]^2\, ds.\end{aligned}$$

Comparing the above two expressions, we conclude that

$$\|w^0\|^2 + \alpha^2 \limsup_{n'\to\infty} |Au(t; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'})|^2 \le e^{-\sigma k}(\lambda_1^{-1} + \alpha^2)R^2_{\sigma,\underline{\sigma}}(t-k) + \|w^0\|^2 + \alpha^2|Aw^0|^2.$$

But from (5.9) and (5.12), we have that $\lim_{k\to\infty} e^{-\sigma k} R^2_{\sigma,\underline{\sigma}}(t-k) = 0$, so (5.15) holds, and we conclude that

$$u(t; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'}) \to w^0 \quad \text{strongly in } D(A).$$

Finally, we prove the asymptotic compactness in the second component of S. From (5.13) and (5.14) with $k = 0$, we have that

$$\begin{aligned} u_t(\cdot; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'}) &\rightharpoonup \psi^0 \quad \text{weakly in } L^2_{D(A)}, \\ \frac{d}{ds}u_t(\cdot; \tau_{n'}, u^{\tau'_{n}}, \phi^{n'}) &\rightharpoonup \frac{d}{ds}\psi^0 \quad \text{weakly in } L^2_{D(A)}. \end{aligned}$$

Thus, by applying the Ascoli–Arzelà theorem, we can deduce that there exists a subsequence (relabeled the same) such that $u_t(\cdot; \tau_{n'}, u^{\tau_{n'}}, \phi^{n'})$ converges to ψ^0 in C_V. So, the proof is finished. □

Remark 5.9. Since $S : \mathbb{R}^2_d \times D(A) \times L^2_V \to D(A) \times L^2_V$ is a continuous process, by the regularity properties established in Theorem 2.2 and Remark 2.4 (i), $S : \mathbb{R}^2_d \times D(A) \times C_V \to D(A) \times C_V$ is a well-defined closed process. In particular, $\{\Lambda_{D(A)\times C_V}(\widehat{D}, t)\}_{t\in\mathbb{R}}$ is meaningful for any $\widehat{D} \in \mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$ by Lemma 5.8. Actually, by the embedding $C_V \subset L^2_V$, recalling Remark 3.8 (ii), it holds that $\Lambda_{D(A)\times C_V}(\widehat{D}, t) = \Lambda_{D(A)\times L^2_V}(\widehat{D}, t)$ for any $t \in \mathbb{R}$, which is therefore invariant for S.

In general, the pullback absorbing family $\widehat{D}_{\sigma,\underline{\sigma}}$ defined by (5.11) does not belong to the universe $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$, and we do not know whether or not S is pullback $\widehat{D}_{\sigma,\underline{\sigma}}$-asymptotically compact. Thus, we cannot apply Theorem 3.3 nor Theorem 3.7 to the family $\widehat{D}_{\sigma,\underline{\sigma}}$. Nevertheless, collecting the proved results, we may construct *by hand* a minimal pullback $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$-attractor in a better norm than the natural one for the phase-space $D(A) \times L^2_V$, namely in the $D(A) \times C_V$ norm.

Theorem 5.10. *Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ satisfies (4.4) and (5.6), and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V). Then, for any $0 < \underline{\sigma} < \sigma/3$, the family $\mathcal{A}_{\sigma,\underline{\sigma}} = \{\mathcal{A}_{\sigma,\underline{\sigma}}(t) : t \in \mathbb{R}\}$, given by*

$$\mathcal{A}_{\sigma,\underline{\sigma}}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}_\sigma(D(A)\times L^2_V)\cap\mathcal{D}_{\underline{\sigma}}(V\times L^2_H)} \Lambda_{D(A)\times C_V}(\widehat{D}, t)}^{D(A)\times C_V}, \quad \forall t \in \mathbb{R},$$

satisfies the following properties:

(a) $\lim_{\tau\to-\infty} dist_{D(A)\times C_V}(S(t,\tau)D(\tau),\mathcal{A}_{\sigma,\underline{\sigma}}(t)) = 0$ *for all* $t\in\mathbb{R}$ *and any* $\widehat{D}\in\mathcal{D}_\sigma(D(A)\times L^2_V)\cap\mathcal{D}_{\underline{\sigma}}(V\times L^2_H)$ *(pullback attraction).*

(b) $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ *is compact in* $D(A)\times C_V$ *for all* $t\in\mathbb{R}$.

(c) *It is minimal in the sense that if* $\widehat{C}=\{C(t):t\in\mathbb{R}\}\subset\mathcal{P}(D(A)\times C_V)$ *(resp.* $\widehat{C}=\{C(t):t\in\mathbb{R}\}\subset\mathcal{P}(D(A)\times L^2_V)$*) is a family of closed subsets of* $D(A)\times C_V$ *(resp.* $D(A)\times L^2_V$*) such that* $\lim_{\tau\to-\infty} dist_{D(A)\times C_V}(S(t,\tau)D(\tau),C(t))=0$ *(resp.* $\lim_{\tau\to-\infty} dist_{D(A)\times L^2_V}(S(t,\tau)D(\tau),C(t))=0$*) for all* $t\in\mathbb{R}$ *and any* $\widehat{D}\in\mathcal{D}_\sigma(D(A)\times L^2_V)\cap\mathcal{D}_{\underline{\sigma}}(V\times L^2_H)$*, then* $\mathcal{A}_{\sigma,\underline{\sigma}}(t)\subset C(t)$ *for all* $t\in\mathbb{R}$.

(d) $S(t,\tau)\mathcal{A}_{\sigma,\underline{\sigma}}(\tau)=\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ *for all* $\tau\le t$ *(invariance).*

In particular, $\mathcal{A}_{\sigma,\underline{\sigma}}$ *is the* $(D(A)\times L^2_V, D(A)\times C_V)$ *minimal pullback* $\mathcal{D}_\sigma(D(A)\times L^2_V)\cap\mathcal{D}_{\underline{\sigma}}(V\times L^2_H)$*-attractor for the process* $S:\mathbb{R}^2_d\times D(A)\times L^2_V\to D(A)\times L^2_V$.

Proof. It suffices to check (a)–(d).

Claim (a). The pullback $D(A)\times C_V$-attraction property is an easy consequence of Lemma 5.8 (see also Remark 5.9).

Claim (b). Consider a sequence $\{y^n\}\subset\mathcal{A}_{\sigma,\underline{\sigma}}(t)$. We will extract a converging subsequence $\{y^{n'}\}\subset\{y^n\}$ with $D(A)\times C_V-\lim_{n'} y^{n'}\in\mathcal{A}_{\sigma,\underline{\sigma}}(t)$.

By definition of $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ we may consider a sequence $\{x^n\}_n$ with $x^n\in\Lambda_{D(A)\times C_V}(\widehat{D}_n,t)$, where $\widehat{D}_n\in\mathcal{D}_\sigma(D(A)\times L^2_V)\cap\mathcal{D}_{\underline{\sigma}}(V\times L^2_H)$, with $\mathrm{dist}_{D(A)\times C_V}(x^n,y^n)<1/n$. For each $n\in\mathbb{N}$, this means that there exist sequences $\{z^{m,n}\}_m$ and $\{\tau^n_m\}_m$ with $\lim_m\tau^n_m=-\infty$, $z^{m,n}\in D_n(\tau^n_m)$ and $x^n=D(A)\times C_V-\lim_m S(t,\tau^n_m)z^{m,n}$. We may consider $m(n)$ such that

$$\mathrm{dist}_{D(A)\times C_V}(x^n,S(t,\tau^n_{m(n)})z^{m(n),n})<1/n,\quad\forall n\ge 1.$$

It is obvious that we are done if we obtain a subsequence $\{x^{n'}\}$ converging in $D(A)\times C_V$ since $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ is closed in $D(A)\times C_V$ and then $\lim_{n'}y^{n'}=\lim_{n'}x^{n'}\in\mathcal{A}_{\sigma,\underline{\sigma}}(t)$.

Now we rearrange the arguments of Lemma 5.8. For each integer $k\ge 0$, by the absorbing property established in Corollary 5.7, there exists $\tau_{\widehat{D}_n}(k)\le t-k$ such that

$$S(t-k,\tau)D_n(\tau)\subset D_{\sigma,\underline{\sigma}}(t-k),\quad\forall\tau\le\tau_{\widehat{D}_n}(k).$$

From this and a diagonal argument we can extract subsequences (the notation $\tau^{n'}_{m(n')}$ and $z^{m(n'),n'}$ is shorten for simplicity) $\{\tau_{n'}\}$ and $\{z^{n'}\}$ with $\tau_{n'}\to-\infty$ and $z^{n'}\in D_{n'}(\tau_{n'})$ such that

$$S(t-k,\tau_{n'})z^{n'}\rightharpoonup(w^k,\psi^k)\quad\text{weakly in } D(A)\times L^2_{D(A)},\text{ for all } k\ge 0,$$

where $(w^k,\psi^k)\in D_{\sigma,\underline{\sigma}}(t-k)$.

Now we can repeat verbatim the arguments from Lemma 5.8 to conclude that

$$D(A)\times C_V-\lim_{n'}S(t,\tau_{n'})z^{n'}=(w^0,\psi^0)$$

which is also the limit of $x^{n'}$ and $y^{n'}$, so $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ is relatively compact and closed, therefore compact in $D(A)\times C_V$.

Claim (c). The minimality among the families of closed subsets in $D(A)\times C_V$ is obvious, since $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ is the closure of omega-limit sets in this topology. For the case of $D(A)\times L^2_V$, observe that the omega-limit sets in this topology are those obtained in the $D(A) \times C_V$ topology (see Remark 5.9 (ii)). Besides, from (b), we have that $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ is compact in $D(A) \times C_V$, and therefore also compact (in particular closed) in $D(A) \times L^2_V$. So the minimality argument is analogous.

Claim (d). We prove it by double inclusion. Let us first check that $\mathcal{A}_{\sigma,\underline{\sigma}}$ is negatively invariant, that is,

$$\mathcal{A}_{\sigma,\underline{\sigma}}(t) \subset S(t,\tau)\mathcal{A}_{\sigma,\underline{\sigma}}(\tau), \quad \forall \tau \le t. \tag{5.22}$$

Consider $y \in \mathcal{A}_{\sigma,\underline{\sigma}}(t)$. Then $y = D(A) \times C_V - \lim_n y^n$ with $y^n \in \Lambda_{D(A)\times C_V}(\widehat{D}_n, t)$, where $\widehat{D}_n \in \mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$. As long as each $\Lambda_{D(A)\times C_V}(\widehat{D}_n, t)$ is invariant for the process S, there exists $x^n \in \Lambda_{D(A)\times C_V}(\widehat{D}_n, \tau)$ with $y^n = S(t,\tau)x^n$. Observe that $\mathcal{A}_{\sigma,\underline{\sigma}}(\tau)$ is compact (proved previously in (b)) in $D(A) \times C_V$. Therefore there exists a subsequence $\{x^{n'}\} \subset \{x^n\}$ with $D(A) \times C_V - \lim_{n'} x^{n'} = x \in \mathcal{A}_{\sigma,\underline{\sigma}}(\tau)$. In particular, by the $D(A) \times L^2_V$ continuity of $S(t,\tau)$ (in fact, it is also continuous in $D(A) \times C_V$) it holds that $y^{n'} = S(t,\tau)x^{n'}$ converges to $S(t,\tau)x$ in $D(A) \times L^2_V$. So, by the uniqueness of the limit, $y = S(t,\tau)x$ and (5.22) holds.

Let us check the converse inclusion

$$S(t,\tau)\mathcal{A}_{\sigma,\underline{\sigma}}(\tau) \subset \mathcal{A}_{\sigma,\underline{\sigma}}(t), \quad \forall \tau \le t.$$

Fix $\tau \le t$ and $x \in \mathcal{A}_{\sigma,\underline{\sigma}}(\tau)$. Then $x = D(A) \times C_V - \lim_n x^n$ with $x^n \in \Lambda_{D(A)\times C_V}(\widehat{D}_n, \tau)$, where $\widehat{D}_n \in \mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$. Using again the invariance property $\Lambda_{D(A)\times C_V}(\widehat{D}_n, t) = S(t,\tau)\Lambda_{D(A)\times C_V}(\widehat{D}_n, \tau)$, denote $y^n := S(t,\tau)x^n$. As long as $S(t,\tau)$ is continuous in $D(A) \times C_V$,

$$\mathcal{A}_{\sigma,\underline{\sigma}}(t) \supset \Lambda_{D(A)\times C_V}(\widehat{D}_n, t) \ni y^n = S(t,\tau)x^n \to S(t,\tau)x,$$

and since $\mathcal{A}_{\sigma,\underline{\sigma}}(t)$ is closed in $D(A) \times C_V$, we obtain that $S(t,\tau)x \in \mathcal{A}_{\sigma,\underline{\sigma}}(t)$, which concludes the positive invariance of $\mathcal{A}_{\sigma,\underline{\sigma}}$. □

Remark 5.11. Observe that [14, Theorem 35] can be improved analogously as we have proceeded here. The notation $X_{\sigma,\underline{\sigma}}$ coined in [14] -in a context without delay effects- for the analogous role of $\mathcal{A}_{\sigma,\underline{\sigma}}$ here, was used because at that moment we did not realize that this family already had compact sections and therefore it was the minimal pullback attractor (in several topologies).

Under the additional assumption

$$\sup_{r\le 0} \int_{r-1}^{r} |f(s)|^2 ds < \infty, \tag{5.23}$$

the pullback absorbing family $\widehat{D}_{\sigma,\underline{\sigma}}$ defined by (5.11) does belong to $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$, whence now we can apply Theorem 3.3, and actually we have the following result.

Theorem 5.12. *Assume that $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ satisfies (5.23), and $g : \mathbb{R} \times C_H \to (L^2(\Omega))^3$ fulfills conditions (I)–(V). Then, for any $0 < \underline{\sigma} < \sigma/3$,*

$$\mathcal{A}_{\sigma,\underline{\sigma}} = \mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}.$$

Actually, $\mathcal{A}_{\mathcal{D}_F(V\times L^2_H)}$ is the unique family of closed subsets in $D(A) \times L^2_V$ in the universe $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$ that is invariant for S and pullback $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$-attracting.

Proof. Consider a fixed value $\underline{\sigma} \in (0, \sigma/3)$.

Observe that under the above assumption on f, the family $\widehat{D}_{\sigma,\underline{\sigma}} = \{D_{\sigma,\underline{\sigma}}(t) : t \in \mathbb{R}\}$ defined by (5.11)–(5.12) belongs to $\mathcal{D}_\sigma(D(A) \times L^2_V) \cap \mathcal{D}_{\underline{\sigma}}(V \times L^2_H)$.

Let us prove the equality $\mathcal{A}_{\sigma,\underline{\sigma}} = \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}$ by double inclusion.

By Theorem 5.10, $\mathcal{A}_{\sigma,\underline{\sigma}}$ is well defined, and indeed, $\mathcal{A}_{\sigma,\underline{\sigma}}(t) \subset D_{\sigma,\underline{\sigma}}(t)$ for any $t \in \mathbb{R}$. By (5.23), for any fixed $t \in \mathbb{R}$, the set $\bigcup_{s \le t} D_{\sigma,\underline{\sigma}}(s)$ is bounded in $D(A) \times L^2_{D(A)}$ since $\sup_{s \le t} R^2_{\sigma,\underline{\sigma}}(s) < \infty$. In particular, from the invariance of $\mathcal{A}_{\sigma,\underline{\sigma}}$, we conclude that

$$\mathcal{A}_{\sigma,\underline{\sigma}}(t) \subset \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(t), \quad \forall t \in \mathbb{R}.$$

On the other hand, again by (5.23), from Theorem 4.3 we have that for any $\tau \in \mathbb{R}$, $\bigcup_{r \le \tau} \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(r)$ is a bounded subset of $D(A) \times L^2_V$, and therefore,

$$\operatorname{dist}_{D(A) \times L^2_V}(\mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(t), \mathcal{A}_{\sigma,\underline{\sigma}}(t)) \le \operatorname{dist}_{D(A) \times L^2_V}(S(t,\tau) \bigcup_{r \le \tau} \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(r), \mathcal{A}_{\sigma,\underline{\sigma}}(t)),$$

where the right-hand side goes to zero as $\tau \to -\infty$. So we conclude that

$$\mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(t) \subset \mathcal{A}_{\sigma,\underline{\sigma}}(t).$$

The final statement about uniqueness is a direct consequence of Remark 3.4. □

Remark 5.13. Observe that, in particular, if $f \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^3)$ satisfies (5.23), by Corollary 3.5 the minimal pullback attractor $\mathcal{A}_{\mathcal{D}_F(D(A) \times L^2_V)}$ does exist, and it also coincides with the family $\mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}$. They have compact sections in $D(A) \times C_V$ and pullback attract in this metric. Moreover, from Theorem 4.3 we have that

$$\bigcup_{t \le t_2} \mathcal{A}_{\sigma,\underline{\sigma}}(t) = \bigcup_{t \le t_2} \mathcal{A}_{\mathcal{D}_F(V \times L^2_H)}(t) \text{ is a bounded subset of } D(A) \times C_{D(A)} \text{ for any } t_2 \in \mathbb{R}.$$

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Dedicated to the memory of Pepe Real's widow Isabel Morillo Montaño 'Beli', with love, for so many years of friendship.

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Slow divergence integral in regularized piecewise smooth systems

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Abstract. In this paper we define the notion of slow divergence integral along sliding segments in regularized planar piecewise smooth systems. The boundary of such segments may contain diverse tangency points. We show that the slow divergence integral is invariant under smooth equivalences. This is a natural generalization of the notion of slow divergence integral along normally hyperbolic portions of curve of singularities in smooth planar slow–fast systems. We give an interesting application of the integral in a model with visible-invisible two-fold of type VI_3. It is related to a connection between so-called Minkowski dimension of bounded and monotone "entry-exit" sequences and the number of sliding limit cycles produced by so-called canard cycles.

Keywords: sliding limit cycles, piecewise smooth systems, regularization function, slow divergence integral, Minkowski dimension.

2020 Mathematics Subject Classification: 34E15, 34E17, 34C40.

1 Introduction

The notion of slow divergence integral [6, Chapter 5] has proved to be an important tool to study lower and upper bounds of limit cycles in smooth planar slow–fast systems (see e.g. [5–7, 10, 11] and references therein). In this paper by "smooth", we mean differentiable of class C^∞. One of the main goals of this paper is to define the slow divergence integral in regularized planar piecewise smooth (PWS) systems with sliding and to prove its invariance under smooth equivalences (by smooth equivalence we mean smooth coordinate change and a multiplication by a smooth positive function). This is a natural generalization of [25] where the slow divergence integral is defined only for a PWS two-fold bifurcation of type visible-invisible called VI_3 and the switching boundary is a straight line (for more details about two-fold singularity VI_3 see [28] and Sections 2 and 3). In this paper we define the slow

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divergence integral along sliding segments with a regular sliding vector field [16] (Section 2.1), and extend it to tangency points when only one vector field is tangent to switching boundary (Section 2.3), two-fold singularities of sliding type VV_1, II_1, VI_2 and VI_3 [28], and to a visible-invisible two-fold singularity when the sliding vector field vanishes at the two-fold point (Section 2.2).

Consider a smooth planar slow–fast system

$$X_{\epsilon,\lambda} = f_\lambda Y_\lambda + \epsilon Q_\lambda + O(\epsilon^2)$$

defined on open set $V \subset \mathbb{R}^2$, where $0 < \epsilon \ll 1$ is the singular perturbation parameter, $\lambda \sim \lambda_0 \in \mathbb{R}^l$, f_λ is a smooth family of functions and Y_λ and Q_λ are smooth families of vector fields. We suppose that $X_{0,\lambda}$ has a curve of singularities C_λ for all $\lambda \sim \lambda_0$ (Fig. 1.1). We further assume that $\nabla f_\lambda(p) \neq (0,0)$ for all $p \in \{(x,y) \in V \mid f_\lambda(x,y) = 0\}$ and that Y_λ has no singularities for each $\lambda \sim \lambda_0$. Then we have $C_\lambda = \{f_\lambda = 0\}$ and C_λ is a smooth family of 1-dimensional manifolds.

The orbits of the flow of $X_{0,\lambda}$ are located inside the leaves of a smooth 1-dimensional foliation $\mathcal{F}_\lambda$ on V tangent to Y_λ ($\mathcal{F}_\lambda$ is called the fast foliation of $X_{0,\lambda}$). If $p \in C_\lambda$, then $DX_{0,\lambda}(p)$ has one eigenvalue equal to zero, with eigenspace T_pC_λ, and the other one equal to $\operatorname{div} X_{0,\lambda}(p)$ (i.e., the trace of $DX_{0,\lambda}(p)$) with eigenspace $T_pl_{\lambda,p}$ ($l_{\lambda,p}$ is the leaf through p). We say that $p \in C_\lambda$ is normally hyperbolic if $\operatorname{div} X_{0,\lambda}(p) \neq 0$ (attracting when $\operatorname{div} X_{0,\lambda}(p) < 0$ and repelling when $\operatorname{div} X_{0,\lambda}(p) > 0$). We can define the notion of slow vector field on normally hyperbolic segments of C_λ. Let $p \in C_\lambda$ be a normally hyperbolic singularity and let $\bar{Q}_\lambda(p) \in T_pC_\lambda$ be the projection of $Q_\lambda(p)$ on T_pC_λ in the direction of $T_pl_{\lambda,p}$. $\bar{Q}_\lambda$ is called the slow vector field and its flow the slow dynamics. The time variable of the slow dynamics is the slow time $\bar{t} = \epsilon t$ where t is the time variable of the flow of $X_{\epsilon,\lambda}$. We point out that the classical definition of the slow vector field using center manifolds is equivalent to this definition. For more details see [6, Chapter 3].

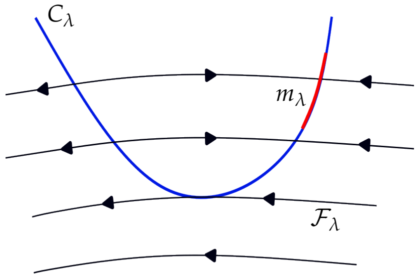

Figure 1.1: The dynamics of $X_{0,\lambda}$ with the curve of singularities C_λ (blue) and the fast foliation $\mathcal{F}_\lambda$. A normally hyperbolic segment $m_\lambda \subset C_\lambda$ (red) along which the slow divergence integral can be defined.

We define now the notion of slow divergence integral (see [6, Chapter 5]). If $m_\lambda \subset C_\lambda$ is a normally hyperbolic segment not containing singularities of $\bar{Q}_\lambda$, then the slow divergence integral along m_λ is defined by

$$I(m_\lambda) = \int_{\bar{t}_1}^{\bar{t}_2} \operatorname{div} X_{0,\lambda}(z_\lambda(\bar{t}))d\bar{t} \tag{1.1}$$

where $z_\lambda : [\bar{t}_1, \bar{t}_2] \to \mathbb{R}^2$, $z'_\lambda(\bar{t}) = \bar{Q}_\lambda(z_\lambda(\bar{t}))$ and $z_\lambda(\bar{t}_1)$ and $z_\lambda(\bar{t}_2)$ are the end points of m_λ (we parameterize m_λ by $\bar{t}$). This definition is independent of the choice of parameterization z_λ of

m_λ and the slow divergence integral is invariant under smooth equivalences (see [6, Section 5.3]).

If both eigenvalues of the linear part of $X_{0,\lambda}$ at $p \in C_\lambda$ are zero, then we say that p is a (nilpotent) contact point between C_λ and $\mathcal{F}_\lambda$. The slow divergence integral can also be defined along parts of C_λ that contain contact points, using its invariance under smooth equivalences and normal forms near contact points (see [6, Section 5.5]).

We come now to a natural question: can we define the notion of slow divergence integral if we replace the slow–fast system $X_{\epsilon,\lambda}$ with a regularized planar PWS system? In Section 2 we give a positive answer to the question. Instead of $X_{0,\lambda}$ we consider a λ-family of planar PWS systems (2.1) defined in Section 2. The switching boundary Σ_λ defined after (2.1) plays the role of the curve of singularities C_λ, and the Filippov sliding vector field Z_λ^{sl} on sliding subsets of Σ_λ (see (2.2)) plays the role of the slow vector field $\bar{Q}_\lambda$ on normally hyperbolic portions of C_λ (see Proposition 2.1). The function that will be integrated (Definition 2.2 in Section 2.1) is the divergence of a smooth slow–fast vector field visible in the scaling chart of a cylindrical blow-up applied to regularized PWS system (2.4) (for more details see [25] and the proof of Proposition 2.1). The notion of slow divergence integral in the PWS setting is well-defined when the sliding vector field Z_λ^{sl} has no singularities (see Definition 2.2).

We show that the slow divergence integral from Definition 2.2 is invariant under smooth equivalences (see Theorem 2.4 in Section 2.1).

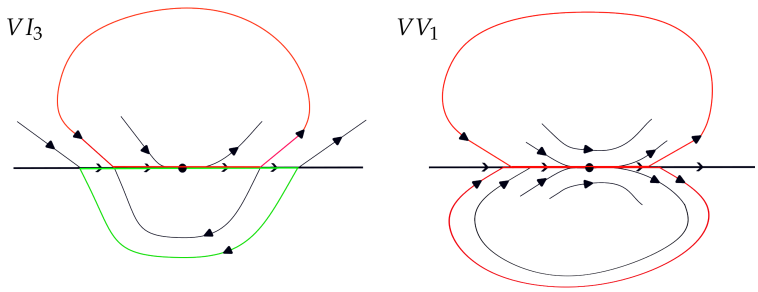

Figure 1.2: Limit periodic sets in planar PWS systems through two-fold points with sliding (the VI_3 case and the VV_1 case). They can be located in a region with invisible fold point (green) or in a region with visible fold point (red).

In Sections 2.2 and 2.3 we define the slow divergence integral near tangency points, as already mentioned above (tangency points in Σ_λ play the role of contact points between C_λ and $\mathcal{F}_\lambda$). We use the invariance of the slow divergence integral under smooth equivalence. The extension of the slow divergence integral to tangency points has proved to be crucial when we study the number of sliding limit cycles (i.e., isolated closed orbits with sliding segments) of a regularized planar PWS system produced by so-called canard limit periodic sets or canard cycles (for more details see [25]). In [25] only the VI_3 case has been studied, with canard cycles located in the region with invisible fold point (see the green closed curve in Fig. 1.2). Canard cycles contain both stable and unstable sliding portions of the discontinuity manifold (often called switching boundary). For example, it has been proved in [25] that the number of sliding limit cycles (produced by the canard cycles) of regularized quadratic PWS systems is unbounded.

A canard cycle can also be located in a region with visible fold point (for example, red closed curves in Fig. 1.2), and again the slow divergence integral associated to the segment of the switching boundary contained in the canard cycle plays an important role when studying sliding limit cycles (see [24]).

Besides sliding cycles, crossing limit cycles can exist for PWS systems and for example J. Llibre and co-workers have obtained upper bounds for a number of classes [13, 29, 32]. See also [2, 4, 17, 18, 20, 21, 30, 31] and references therein.

Section 3 is devoted to applications of the slow divergence integral from Section 2. In Section 3 we focus on the model used in [25] (visible-invisible two-fold VI_3) and read upper bounds of the number of sliding limit cycles and type of bifurcations near so-called generalized canard cycles (Fig. 3.1) from fractal properties of a bounded and monotone sequence in $\mathbb{R}$ defined using the slow divergence integral and the notion of slow relation function (see Section 3.1). The main advantage of this fractal approach is that, instead of computing the multiplicity of zeros of the slow divergence integral (like in [25]), it suffices to find the Minkowski dimension [14] of the sequence. There is a bijective correspondence between the multiplicity and the Minkowski dimension (see Section 4). A similar fractal approach has been used in [8, 22, 23, 26] where one deals with smooth slow–fast systems. See also [12, 35] and the references therein. We point out that there exist simple formulas for numerical computation of the Minkowski dimension of the sequence (see e.g. [23]). In Section 3.2 we state the main fractal results (Theorems 3.4–3.6), and in Section 4 we prove them.

For the sake of readability, in this paper we work in $\mathbb{R}^2$ using the Euclidean metric. We believe that the notion of slow divergence integral in regularized PWS systems on a smooth surface could also be defined. We point out that the slow divergence integral [6] is defined for slow–fast systems on a smooth surface.

2 The slow divergence integral in PWS systems with sliding

First we recall the basic definitions in PWS theory [9, 19] (switching boundary, sliding set, crossing set, sliding vector field, tangency point, two-fold singularity, etc.). Then we define the notion of slow divergence integral of a regularized PWS system along a sliding segment (not containing singularities of the sliding vector field) and prove that the integral is invariant under smooth equivalences (see Section 2.1). In Section 2.2 we extend the definition of the slow divergence integral to segments consisting of a stable sliding region, an unstable sliding region and a two-fold singularity between them. If the two-fold singularity is visible-invisible, then we assume that the sliding vector field is regular or has a hyperbolic singularity in the two-fold point. In Section 2.3 we define the slow divergence integral near a tangency point where the tangency (quadratic or more degenerate) appears only in one vector field.

Consider a λ-family of PWS systems in the plane

$$\dot{z} = \begin{cases} Z_\lambda^+(z) & \text{for } z \in \Sigma_\lambda^+, \\ Z_\lambda^-(z) & \text{for } z \in \Sigma_\lambda^-, \end{cases} \tag{2.1}$$

where $z = (x, y)$, $\lambda \sim \lambda_0 \in \mathbb{R}^l$ and the switching boundary is a smooth λ-family of 1-dimensional manifolds Σ_λ given by

$$\Sigma_\lambda = \{z \in \mathbb{R}^2 \mid h_\lambda(z) = 0\} \cap V,$$

with an open set V and a smooth family of functions h_λ such that $\nabla h_\lambda(z) \neq (0, 0)$, $\forall z \in \Sigma_\lambda$. The switching boundary Σ_λ separates the open set $\Sigma_\lambda^+ = \{z \in V \mid h_\lambda(z) > 0\}$ from the open set $\Sigma_\lambda^- = \{z \in V \mid h_\lambda(z) < 0\}$. We assume that the λ-family of vector fields $Z_\lambda^+ = (X_\lambda^+, Y_\lambda^+)$ (resp. $Z_\lambda^- = (X_\lambda^-, Y_\lambda^-)$) is smooth in the closure of the λ-family Σ_λ^+ (resp. Σ_λ^-). In this paper "smooth" means "C^∞-smooth".

The subset $\Sigma_\lambda^{sl} \subset \Sigma_\lambda$ consisting of all points $z \in \Sigma_\lambda$ such that

$$Z_\lambda^+(h_\lambda)(z) Z_\lambda^-(h_\lambda)(z) < 0$$

is called the sliding set, where $Z_\lambda^\pm(h_\lambda)(z) := \nabla h_\lambda(z) \cdot Z_\lambda^\pm(z)$ is the Lie-derivative of h_λ with respect to the vector field $Z_\lambda^\pm$ at z. A sliding point $z \in \Sigma_\lambda^{sl}$ is stable (resp. unstable) if $Z_\lambda^+(h_\lambda)(z) < 0$ and $Z_\lambda^-(h_\lambda)(z) > 0$ (resp. $Z_\lambda^+(h_\lambda)(z) > 0$ and $Z_\lambda^-(h_\lambda)(z) < 0$). We write $\Sigma_\lambda^{sl} = \Sigma_\lambda^s \cup \Sigma_\lambda^u$ where Σ_λ^s (resp. Σ_λ^u) is the set of all stable (resp. unstable) sliding points. In Σ_λ^s (resp. Σ_λ^u) the vector fields $Z_\lambda^\pm$ point toward (resp. away from) the switching boundary. We call the set $\Sigma_\lambda^{cr} \subset \Sigma_\lambda$ of all points $z \in \Sigma_\lambda$ such that

$$Z_\lambda^+(h_\lambda)(z) Z_\lambda^-(h_\lambda)(z) > 0$$

the crossing set.

At each point $z \in \Sigma_\lambda^{cr}$ the orbit of (2.1) crosses the switching boundary Σ_λ (it is the concatenation of the orbit of Z_λ^+ and the orbit of Z_λ^- through z). Along the sliding set Σ_λ^{sl}, the flow is given by the Filippov sliding vector field [16]

$$Z_\lambda^{sl}(z) = \frac{1}{(Z_\lambda^+ - Z_\lambda^-)(h_\lambda)} \left(Z_\lambda^+(h_\lambda) Z_\lambda^- - Z_\lambda^-(h_\lambda) Z_\lambda^+ \right)(z), \; z \in \Sigma_\lambda^{sl}. \tag{2.2}$$

The sliding vector field Z_λ^{sl} defined in (2.2) is tangent to Σ_λ^{sl}, i.e., $Z_\lambda^{sl}(z)$ is equal to the convex combination $\tau Z_\lambda^+(z) + (1-\tau) Z_\lambda^-(z)$ with

$$\tau = \tau_\lambda(z) = \frac{-Z_\lambda^-(h_\lambda)}{(Z_\lambda^+ - Z_\lambda^-)(h_\lambda)}(z) \in]0,1[. \tag{2.3}$$

We say that $z \in \Sigma_\lambda^{sl}$ is a pseudo-equilibrium of (2.1) if $Z_\lambda^{sl}(z) = 0$.

A point $z \in \Sigma_\lambda$ where $Z_\lambda^+(h_\lambda)(z) = 0$ or $Z_\lambda^-(h_\lambda)(z) = 0$ is a PWS singularity called tangency. We say that $z \in \Sigma_\lambda$ is a fold singularity (or a fold point) of Z_λ^+ (resp. Z_λ^-) if $Z_\lambda^+(h_\lambda)(z) = 0$ and $(Z_\lambda^+)^2(h_\lambda)(z) \neq 0$ (resp. $Z_\lambda^-(h_\lambda)(z) = 0$ and $(Z_\lambda^-)^2(h_\lambda)(z) \neq 0$). The fold point is visible if $(Z_\lambda^+)^2(h_\lambda)(z) > 0$ (resp. $(Z_\lambda^-)^2(h_\lambda)(z) < 0$) and invisible if $(Z_\lambda^+)^2(h_\lambda)(z) < 0$ (resp. $(Z_\lambda^-)^2(h_\lambda)(z) > 0$).

We say that $z \in \Sigma_\lambda$ is a two-fold singularity if z is a fold point of both $Z_\lambda^\pm$. A two-fold singularity $z \in \Sigma_\lambda$ is said to be visible-visible (VV) if z is visible in both $Z_\lambda^\pm$, invisible-invisible (II) if z is invisible in both $Z_\lambda^\pm$, and visible-invisible (VI) if z is visible in Z_λ^+ and invisible in Z_λ^- or visible in Z_λ^- and invisible in Z_λ^+. Following [28], there exist 7 (generic) cases for two-fold singularities taking into account the direction of the flow of $Z_\lambda^\pm$ and Z_λ^{sl}: 2 visible-visible cases (denoted by VV_1 and VV_2 in [28]), 2 invisible-invisible cases (II_1 and II_2) and 3 visible-invisible cases (VI_1, VI_2 and VI_3). For more details we refer to [1,19,27,28]. In Section 2.2 we define the notion of slow divergence integral near two-fold singularities of sliding type (VV_1, II_1, VI_2 and VI_3). The four sliding cases are illustrated in Fig. 2.2. We also treat a visible-invisible two-fold singularity where the sliding vector field points toward (or away from) the two-fold singularity on both sides (Fig. 2.3).

We consider a regularized PWS system [25]

$$\dot{z} = \phi(h_\lambda(z)\epsilon^{-1}) Z_\lambda^+(z) + (1 - \phi(h_\lambda(z)\epsilon^{-1})) Z_\lambda^-(z) \tag{2.4}$$

where $0 < \epsilon \ll 1$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth regularization function. We assume that ϕ is strictly monotone, i.e.,

$$\phi'(u) > 0 \quad \text{for all } u \in \mathbb{R}, \tag{2.5}$$

and ϕ has the following asymptotics for $u \to \pm\infty$:

$$\phi(u) \to \begin{cases} 1 & \text{for } u \to \infty, \\ 0 & \text{for } u \to -\infty. \end{cases} \tag{2.6}$$

Moreover, we assume that ϕ is smooth at $\pm\infty$ in the following sense: The functions

$$\phi_+(u) := \begin{cases} 1 & \text{for } u = 0, \\ \phi(u^{-1}) & \text{for } u > 0, \end{cases} \quad \phi_-(u) := \begin{cases} \phi(-u^{-1}) & \text{for } u > 0, \\ 0 & \text{for } u = 0, \end{cases}$$

are smooth at $u = 0$.

Using the asymptotics of ϕ given in (2.6), the system (2.4) becomes the PWS system (2.1) in the limit $\epsilon \to 0$. Combining this with the fact that ϕ is smooth at $\pm\infty$, we have that, for z kept in any fixed compact set in $V \setminus \Sigma_\lambda$, the right hand side of (2.4) is an $o(1)$-perturbation of the right hand side of (2.1) where $o(1)$ is a smooth function in (z, ϵ, λ) and tends to 0 as $\epsilon \to 0$, uniformly in (z, λ). Thus, the PWS system (2.1) describes the dynamics of (2.4), for $\epsilon > 0$ small, as long as z is kept uniformly away from Σ_λ.

Near Σ_λ^{sl}, the dynamics of (2.4), with $0 < \epsilon \ll 1$, is given by Proposition 2.1 (see also [33]).

2.1 Definition and invariance of the slow divergence integral

Proposition 2.1. *Suppose that the PWS system (2.1) has a stable (resp. unstable) sliding point $p \in \Sigma_{\lambda_0}^{sl}$. Then, for each $0 < \epsilon \ll 1$ and $\lambda \sim \lambda_0$, (2.4) has a locally invariant manifold near p with foliation by stable (resp. unstable) fibers, and the reduced dynamics on this manifold (when $\epsilon \to 0$) is given by sliding vector field Z_λ^{sl} defined in (2.2).*

Proof. Without loss of generality, we can assume that $\frac{\partial h_{\lambda_0}}{\partial y}(p) \neq 0$. Then the switching boundary Σ_λ (locally near p) is the graph of a smooth function $y = f_\lambda(x)$. Using $h_\lambda(x, y) = \epsilon \tilde{y}$, the system (2.4) multiplied by $\epsilon > 0$ becomes a slow–fast system

$$\begin{aligned} \dot{x} &= \epsilon \left(\phi(\tilde{y}) X_\lambda^+(x, f_\lambda(x)) + (1 - \phi(\tilde{y})) X_\lambda^-(x, f_\lambda(x)) + O(\epsilon \tilde{y}) \right), \\ \dot{\tilde{y}} &= \phi(\tilde{y}) Z_\lambda^+(h_\lambda)(x, f_\lambda(x)) + (1 - \phi(\tilde{y})) Z_\lambda^-(h_\lambda)(x, f_\lambda(x)) + O(\epsilon \tilde{y}). \end{aligned} \tag{2.7}$$

When $\epsilon = 0$, the curve of singularities of (2.7) is given by $\tilde{y} = \phi^{-1}(\tau_\lambda(x, f_\lambda(x)))$ where τ_λ is defined in (2.3). Each singularity $(x, \phi^{-1}(\tau_\lambda(x, f_\lambda(x))))$ is semi-hyperbolic with the nonzero eigenvalue equal to the divergence of the vector field (2.7), with $\epsilon = 0$, computed in that singularity:

$$(Z_\lambda^+ - Z_\lambda^-)(h_\lambda)(x, f_\lambda(x)) \phi' \left(\phi^{-1}(\tau_\lambda(x, f_\lambda(x))) \right). \tag{2.8}$$

The reason why the eigenvalue in (2.8) is nonzero is because $Z_\lambda^+(h_\lambda) Z_\lambda^-(h_\lambda) < 0$ and $\phi' > 0$ (see (2.5)). The curve of singularities is attracting (resp. repelling) if p is a stable (resp. unstable) sliding point. The result follows now from Fenichel's theory [15]. Notice that the reduced dynamics of (2.7) along the curve of singularities is given by the vector field

$$\left(\tau_\lambda X_\lambda^+ + (1 - \tau_\lambda) X_\lambda^- \right)(x, f_\lambda(x)). \tag{2.9}$$

We divided the x-component in (2.7) by ϵ and let $\epsilon \to 0$ with $\tilde{y} = \phi^{-1}(\tau_\lambda(x, f_\lambda(x)))$. We get the same expression (2.9) if we use the definition of the slow vector field introduced in Section 1. This completes the proof. □

Following [6, Chapter 5] or Section 1 in the smooth slow–fast system (2.7) one can define the notion of slow divergence integral along normally hyperbolic curve of singularities $\tilde{y} = \phi^{-1}(\tau_\lambda(x, f_\lambda(x)))$ when the sliding vector field in (2.9) has no singularities: it is the integral of the divergence in (2.8) where the variable of integration is the time variable of the flow of the sliding vector field. This is our motivation for the definition of the notion of slow divergence integral of regularized PWS system (2.4) (see also [25]).

Definition 2.2 (Slow divergence integral). Let $m_\lambda \subset \Sigma_\lambda^{sl}$ be a bounded segment (Fig. 2.1) not containing pseudo-equilibria of the PWS system (2.1). Let $z_\lambda : [t_1, t_2] \to \mathbb{R}^2$ be a solution of $z'(t) = Z_\lambda^{sl}(z(t))$ where $z_\lambda(t_1)$ and $z_\lambda(t_2)$ are the end points of m_λ (z_λ is a parameterization of m_λ). Then we define the slow divergence integral of regularized PWS system (2.4) associated to m_λ as

$$I(m_\lambda) = \int_{t_1}^{t_2} E_\lambda(z_\lambda(t))dt \tag{2.10}$$

where

$$E_\lambda(z) = (Z_\lambda^+ - Z_\lambda^-)(h_\lambda)(z)\phi'\left(\phi^{-1}(\tau_\lambda(z))\right), \qquad z \in \Sigma_\lambda^{sl}.$$

Remark 2.3. Note that the definition of the slow divergence integral given by (2.10) is independent of the choice of z_λ. Indeed, if $\hat{z}_\lambda$ is another solution to $z'(t) = Z_\lambda^{sl}(z(t))$ and $p \in m_\lambda$, then there exist $\tilde{t} \in [t_1, t_2]$ and $\bar{t}$ such that $z_\lambda(\tilde{t}) = \hat{z}_\lambda(\bar{t}) = p$. Then we have $z_\lambda(t) = \hat{z}_\lambda(t + \bar{t} - \tilde{t})$ due to uniqueness of solutions. Now, we get

$$\int_{t_1+\bar{t}-\tilde{t}}^{t_2+\bar{t}-\tilde{t}} E_\lambda(\hat{z}_\lambda(s))ds = \int_{t_1}^{t_2} E_\lambda(z_\lambda(t))dt,$$

where we use the change of variable $s = t + \bar{t} - \tilde{t}$.

If m_λ is stable (resp. unstable), then $I(m_\lambda)$ is negative (resp. positive).

The slow divergence integral from Definition 2.2 is invariant under smooth equivalences (Theorem 2.4.1 and Theorem 2.4.2). Theorem 2.4.3 tells us how to compute $I(m_\lambda)$ for an arbitrary parameterization of m_λ (see also [6, Proposition 5.3]).

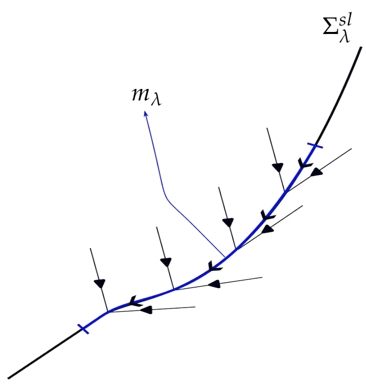

Figure 2.1: A segment $m_\lambda \subset \Sigma_\lambda^{sl}$ (blue).

Theorem 2.4 (Invariance of the slow divergence integral). *Let us denote the family of vector field in (2.4) by $Z_{\epsilon,\lambda}$ and let $m_\lambda \subset \Sigma_\lambda^{sl}$ be as in Definition 2.2. The following statements are true.*

1. *Let $T : V_w \to V_z \subset V$ ($w \mapsto z = T(w)$) be a smooth coordinate transformation, with open sets $V_w, V_z \subset \mathbb{R}^2$. Let $I(m_\lambda)$ be the slow divergence integral of $Z_{\epsilon,\lambda}$ along $m_\lambda \subset V_z$. Then the slow divergence integral $I(T^{-1}(m_\lambda))$ of the pullback of the vector field $Z_{\epsilon,\lambda}|_{V_z}$ along $T^{-1}(m_\lambda) \subset V_w$ is equal to $I(m_\lambda)$.*

2. *Let g be a smooth strictly positive function defined in a neighborhood of m_λ. Then the slow divergence integral of $Z_{\epsilon,\lambda}$ along m_λ is equal to the slow divergence integral of the equivalent vector field $g.Z_{\epsilon,\lambda}$ along m_λ.*

3. *Let $p_\lambda : [v_1, v_2] \to \mathbb{R}^2$ be a parameterization of m_λ. Then we have*

$$I(m_\lambda) = \int_{v_1}^{v_2} \frac{E_\lambda(p_\lambda(v))dv}{|\tilde{p}_\lambda(v)|},$$

where $\tilde{p}_\lambda$ is a smooth λ-family of nowhere zero functions satisfying

$$Z_\lambda^{sl}(p_\lambda(v)) = \tilde{p}_\lambda(v) p'_\lambda(v).$$

Proof. Statement 1. The pullback of the vector field $Z_{\epsilon,\lambda}|_{V_z}$ can be written as

$$T^*(Z_{\epsilon,\lambda}|_{V_z})(w) = \phi(h_\lambda \circ T(w)\epsilon^{-1})W_\lambda^+(w) + (1 - \phi(h_\lambda \circ T(w)\epsilon^{-1}))W_\lambda^-(w)$$

where $W_\lambda^\pm(w) = DT(w)^{-1}(Z_\lambda^\pm \circ T)(w)$. It is not difficult to see that the Lie-derivative of $h_\lambda \circ T$ with respect to the vector field $W_\lambda^\pm$ is given by

$$W_\lambda^\pm(h_\lambda \circ T)(w) = Z_\lambda^\pm(h_\lambda)(T(w)). \tag{2.11}$$

Using (2.11) and Definition 2.2 we find that the slow divergence integral of $T^*(Z_{\epsilon,\lambda}|_{V_z})$ along $T^{-1}(m_\lambda)$ is given by

$$I(T^{-1}(m_\lambda)) = \int_{t_1}^{t_2} E_\lambda(T(w_\lambda(t)))dt$$

where $w_\lambda : [t_1, t_2] \to T^{-1}(m_\lambda)$ is a solution of $w'(t) = W_\lambda^{sl}(w(t))$ (the Filippov sliding vector field along $T^{-1}(m_\lambda)$ is given by $W_\lambda^{sl}(w) = DT(w)^{-1}Z_\lambda^{sl}(T(w))$). Since $T \circ w_\lambda : [t_1, t_2] \to m_\lambda$ is a solution to $z'(t) = Z_\lambda^{sl}(z(t))$, the result follows.

Statement 2. From Definition 2.2 it follows that the slow divergence integral of $g.Z_{\epsilon,\lambda}$ along m_λ is equal to

$$I(m_\lambda) = \int_{\hat{t}_1}^{\hat{t}_2} E_\lambda(\hat{z}_\lambda(\hat{t}))g(\hat{z}_\lambda(\hat{t}))d\hat{t} \tag{2.12}$$

where $\hat{z}_\lambda : [\hat{t}_1, \hat{t}_2] \to m_\lambda$ and $\hat{z}'_\lambda(\hat{t}) = g(\hat{z}_\lambda(\hat{t}))Z_\lambda^{sl}(\hat{z}_\lambda(\hat{t}))$. We make in the integral in (2.12) the change of variable $t = \rho(\hat{t}) = \int_{\hat{t}_1}^{\hat{t}} g(\hat{z}_\lambda(v))dv$ with $\hat{t} \in [\hat{t}_1, \hat{t}_2]$. Then we have

$$\int_{\hat{t}_1}^{\hat{t}_2} E_\lambda(\hat{z}_\lambda(\hat{t}))g(\hat{z}_\lambda(\hat{t}))d\hat{t} = \int_0^{\rho(\hat{t}_2)} E_\lambda(\hat{z}_\lambda \circ \rho^{-1}(t))dt.$$

Since $(\hat{z}_\lambda \circ \rho^{-1})'(t) = Z_\lambda^{sl}((\hat{z}_\lambda \circ \rho^{-1})(t))$, $t \in [0, \rho(\hat{t}_2)]$, this integral is the slow divergence integral of $Z_{\epsilon,\lambda}$ associated to m_λ. This completes the proof of Statement 2.

Statement 3. The proof of Statement 3 is similar to the proof of Statement 2. □

Remark 2.5. It follows directly from Definition 2.2 that the slow divergence integral of $-Z_{\epsilon,\lambda}$ along m_λ is equal to the slow divergence integral of $Z_{\epsilon,\lambda}$ along m_λ multiplied by -1.

We will use the invariance of the slow divergence integral under smooth equivalences in Section 2.2 and Section 2.3.

If $m_\lambda \subset \Sigma_\lambda^{sl}$ contains pseudo-equilibria, then the slow divergence integral associated to m_λ is not well-defined.

2.2 The slow divergence integral near two-fold singularities

In this section we suppose that the sliding vector field Z_λ^{sl}, given by (2.2), is defined around a two-fold singularity. Our goal is to define the notion of slow divergence integral near such a two-fold singularity. Since the slow divergence integral (2.10) is invariant under smooth equivalences (Theorem 2.4), we use a normal form of (2.1), locally near the two-fold singularity, in which $h_\lambda(x,y) = y$ and the two-fold point corresponds to the origin $p = (0,0)$. Notice that such normal form coordinates exist because $\nabla h_\lambda(z) \neq (0,0)$, $\forall z \in \Sigma_\lambda$, in (2.1).

Using $h_\lambda(x,y) = y$ the two-fold p satisfies

$$Z_\lambda^\pm(h_\lambda)(0) = Y_\lambda^\pm(0) = 0, \quad (Z_\lambda^\pm)^2(h_\lambda)(0) = X_\lambda^\pm(0)\partial_x Y_\lambda^\pm(0) \neq 0, \tag{2.13}$$

and the sliding vector field Z_λ^{sl} near p can be written as

$$X_\lambda^{sl}(x) = \frac{\det Z_\lambda(x)}{(Y_\lambda^- - Y_\lambda^+)(x,0)} \tag{2.14}$$

where

$$\det Z_\lambda(x) := (X_\lambda^+ Y_\lambda^- - X_\lambda^- Y_\lambda^+)(x,0).$$

Remark 2.6. The notation $\det Z_\lambda$ comes from [1]. In [25] a similar notation has been used for $-(X_\lambda^+ Y_\lambda^- - X_\lambda^- Y_\lambda^+)$.

Since we assumed that the sliding vector field X_λ^{sl} is defined around the two-fold p, we find that $X_\lambda^+(0)X_\lambda^-(0) > 0$ if the folds have the same visibility (visible-visible or invisible-invisible) and $X_\lambda^+(0)X_\lambda^-(0) < 0$ if the folds have opposite visibility. We have $p \in \partial\Sigma_\lambda^s \cap \partial\Sigma_\lambda^u$. These properties follow directly from (2.13) and the definition of visible and invisible folds (see [1, Lemma 2.8]), and imply that $\partial_x(Y_\lambda^- - Y_\lambda^+)(0) \neq 0$ and $\partial_x Y_\lambda^+(0)\partial_x Y_\lambda^-(0) < 0$.

Using $\partial_x(Y_\lambda^- - Y_\lambda^+)(0) \neq 0$ it is clear that the sliding vector field in (2.14) has a removable singularity in $x = 0$ and

$$X_\lambda^{sl}(x) = \nu + O(x), \quad \nu = \frac{(\det Z_\lambda)'(0)}{\partial_x(Y_\lambda^- - Y_\lambda^+)(0)}. \tag{2.15}$$

From [1, Lemma 2.9] and [1, Corollary 2.10] it follows that $\nu \neq 0$ and $\operatorname{sgn}(\nu) = \operatorname{sgn}(X_\lambda^+(0))$ if the folds have the same visibility (VV_1 and II_1 in Fig. 2.2), and that $\nu \neq 0$ and $\operatorname{sgn}(\nu) = -\operatorname{sgn}\left(X_\lambda^+(0)(\det Z_\lambda)'(0)\right)$ if the folds have opposite visibility and $(\det Z_\lambda)'(0) \neq 0$ (VI_2 and VI_3 in Fig. 2.2). *If the folds have opposite visibility, we assume that $\nu \neq 0$ in (2.15) or $x = 0$ is a hyperbolic singularity of the sliding vector field X_λ^{sl} (or, equivalently, $x = 0$ is a zero of multiplicity 1 or 2 of the function $\det Z_\lambda$).* We refer to Fig. 2.3 (the multiplicity of the zero $x = 0$ of $\det Z_\lambda$ is 2).

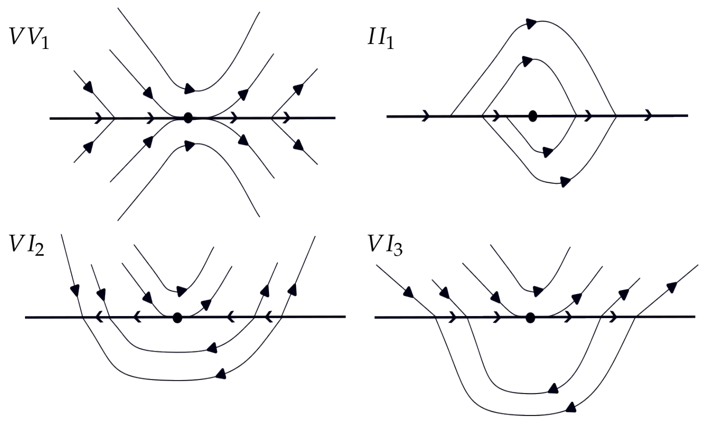

Figure 2.2: The different types of two-fold singularities with sliding: the folds in VV_1 and II_1 have the same visibility, while the folds in VI_2 and VI_3 have opposite visibility.

From $\partial_x(Y_\lambda^- - Y_\lambda^+)(0) \neq 0$ and $\partial_x Y_\lambda^+(0)\partial_x Y_\lambda^-(0) < 0$ it follows that the function τ_λ defined in (2.3) has the following property when $x \to 0$:

$$\lim_{x\to 0} \tau_\lambda(x,0) = \lim_{x\to 0} \frac{-Y_\lambda^-}{Y_\lambda^+ - Y_\lambda^-}(x,0) = \frac{\partial_x Y_\lambda^-(0)}{\partial_x(Y_\lambda^- - Y_\lambda^+)(0)} \in]0,1[. \tag{2.16}$$

Let us now compute the slow divergence integral along $[x_0, x_1]$, with $0 < x_0 < x_1$. We assume that x_1 is small enough such that $[x_0, x_1]$ does not contain any singularities of the sliding vector field X_λ^{sl}. We use Theorem 2.4.3. We take $p_\lambda(x) = (x, 0)$, $x \in [x_0, x_1]$, in Theorem 2.4.3. Then we have

$$\tilde{p}_\lambda(x) = \frac{\det Z_\lambda(x)}{(Y_\lambda^- - Y_\lambda^+)(x,0)},\ E_\lambda(p_\lambda(v)) = (Y_\lambda^+ - Y_\lambda^-)(x,0)\phi'\left(\phi^{-1}\left(\tau_\lambda(x,0)\right)\right).$$

This implies

$$I([x_0,x_1]) = \int_{x_0}^{x_1} \frac{|Y_\lambda^- - Y_\lambda^+|(Y_\lambda^+ - Y_\lambda^-)(x,0)}{|\det Z_\lambda|(x)} \phi'\left(\phi^{-1}\left(\frac{-Y_\lambda^-}{Y_\lambda^+ - Y_\lambda^-}(x,0)\right)\right) dx. \tag{2.17}$$

Finally, we define the slow divergence integral along $[0, x_1]$ (the left end point of $[0, x_1]$ is the two-fold point).

Definition 2.7. Let $m_\lambda = [0, x_1]$. Then the slow divergence integral along m_λ is defined as

$$I(m_\lambda) = \lim_{x_0 \to 0+} I([x_0, x_1])$$

where $I([x_0, x_1])$ is given in (2.17).

Remark 2.8. Notice that the function $x \mapsto \phi'\left(\phi^{-1}\left(\tau_\lambda(x,0)\right)\right)$ in (2.17) can be defined at $x = 0$ such that this function is smooth and positive on the segment m_λ (see (2.5), (2.6) and (2.16)). If the folds have the same visibility, then $I(m_\lambda)$ is well-defined (finite) because $\nu \neq 0$ in (2.15). Since we assume that the multiplicity of the zero $x = 0$ of $\det Z_\lambda$ does not exceed 2 when the folds have opposite visibility, $I(m_\lambda)$ is finite.

Remark 2.9. The slow divergence integral along $m_\lambda = [x_0, 0]$, with $x_0 < 0$, can be defined in a similar way: $I(m_\lambda) = \lim_{x_1 \to 0-} I([x_0, x_1])$ where $I([x_0, x_1])$ has the same form (2.17).

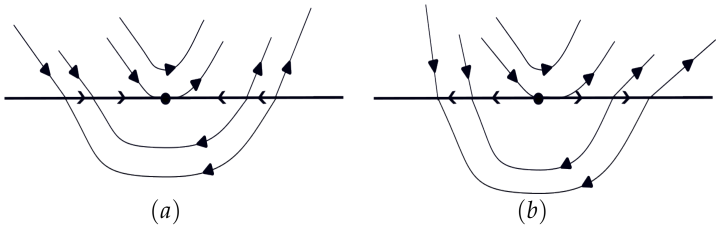

Figure 2.3: Non-generic visible-invisible two-fold singularities. The (extended) sliding vector field has a hyperbolic singularity at the two-fold points. (a) The sliding vector field points toward the two-fold singularity. (b) The sliding vector field points away from the two-fold singularity.

2.3 The slow divergence integral near one-sided tangency points

In this section we define the slow divergence integral near a tangency point $p \in \partial\Sigma_\lambda^s \cup \partial\Sigma_\lambda^u$ where both vectors $Z_\lambda^\pm(p)$ are nonzero and precisely one of them is tangent to Σ_λ at p (see e.g. Fig. 2.4). Like in Section 2.2, the switching boundary Σ_λ is locally given by $h_\lambda(x,y) = y$ and $p = (0,0)$. Since we suppose that $p \in \partial\Sigma_\lambda^s \cup \partial\Sigma_\lambda^u$, there is a side of p (without loss of generality we take $x > 0$) where the sliding vector field is defined, and given by (2.14). If the (nonzero) vector $Z_\lambda^+(0)$ (resp. $Z_\lambda^-(0)$) is tangent to Σ_λ, then $X_\lambda^+(0) \neq 0$, $Y_\lambda^+(0) = 0$ and $Y_\lambda^-(0) \neq 0$ (resp. $X_\lambda^-(0) \neq 0$, $Y_\lambda^-(0) = 0$ and $Y_\lambda^+(0) \neq 0$) and

$$X_\lambda^{sl}(x) = X_\lambda^+(0) + O(x) \quad \left(\text{resp. } X_\lambda^{sl}(x) = X_\lambda^-(0) + O(x)\right). \tag{2.18}$$

Since $X_\lambda^+(0) \neq 0$ (resp. $X_\lambda^-(0) \neq 0$), the sliding vector field X_λ^{sl} in (2.18) is regular near $x = 0$. Thus, the segment $[x_0, x_1]$, with $0 < x_0 < x_1$, does not contain any singularities of X_λ^{sl} if x_1 is small enough and we can define the slow divergence integral along $[x_0, x_1]$ exactly in the same way as in Section 2.2. The slow divergence integral is given by (2.17) and we use the same notation $I([x_0, x_1])$.

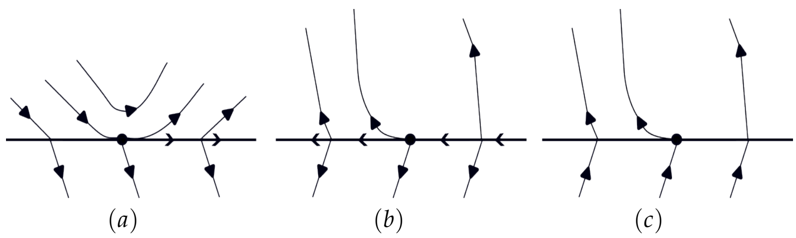

Figure 2.4: (a) The sliding vector field is defined on one side of the tangency point. (b) The sliding vector field is defined on both sides of the tangency point. (c) A crossing region around the tangency point (in this case the slow divergence integral near the tangency point is not defined).

We can now define the slow divergence integral near the tangency point p.

Definition 2.10. Let $m_\lambda = [0, x_1]$. Then the slow divergence integral along m_λ is defined as

$$I(m_\lambda) = \lim_{x_0 \to 0+} I([x_0, x_1]).$$

Remark 2.11. The slow divergence integral $I(m_\lambda)$ from Definition 2.10 is well-defined. Indeed, $\lim_{u\to\pm\infty}\phi'(u) = 0$ (due to the smoothness of ϕ at $\pm\infty$ given after (2.6)). Moreover, we have (a) $\lim_{u\to 1-}\phi^{-1}(u) = +\infty$, (b) $\lim_{u\to 0+}\phi^{-1}(u) = -\infty$ (see (2.6)) and finally (c) $\frac{-Y_\lambda^-}{Y_\lambda^+ - Y_\lambda^-}(x,0)$ tends to 1 (resp. 0) as $x\to 0+$ when $Z_\lambda^+(0)$ (resp. $Z_\lambda^-(0)$) is tangent to Σ_λ. It follows from (a), (b) and (c) that the integrand in (2.17) can be defined at $x=0$ (0 for $x=0$) and that the integrand is continuous on the segment m_λ. This implies that $I(m_\lambda)$ is well-defined.

3 Limit cycles and fractal analysis through visible-invisible two-fold VI_3

3.1 Model and assumptions

We consider a PWS system (2.1) where we assume that $\lambda\sim\lambda_0\in\mathbb{R}$, and $h_\lambda(x,y) = y$ (the switching boundary is the line $y=0$).

Assumption A. Suppose that there are $\eta_- < 0$ and $\eta_+ > 0$ such that the PWS system (2.1) for $\lambda=\lambda_0$ has stable sliding for all $x\in[\eta_-,0[$ (i.e., $Y_{\lambda_0}^+(x,0) < 0$ and $Y_{\lambda_0}^-(x,0) > 0$ for $x\in[\eta_-,0[$) and unstable sliding for all $x\in]0,\eta_+]$ (i.e., $Y_{\lambda_0}^+(x,0) > 0$ and $Y_{\lambda_0}^-(x,0) < 0$ for $x\in]0,\eta_+]$). Moreover, we assume that the Filippov sliding vector field X_λ^{sl} given by (2.14) is positive for $x\in[\eta_-,\eta_+]\setminus\{0\}$ and $\lambda=\lambda_0$.

Assumption A implies that $Y_{\lambda_0}^\pm(0) = 0$ and the origin $z=0$ is therefore a tangency point (see Section 2). We assume that $z=0$ for $\lambda=\lambda_0$ is a two-fold singularity. Moreover, we suppose that the two-fold singularity is visible from "above" and invisible from "below", i.e., the orbit of $Z_{\lambda_0}^+$ through $z=0$ is contained within $y>0$ near $z=0$, and the orbit of $Z_{\lambda_0}^-$ through $z=0$ is not contained within $y<0$ (Section 2).

Assumption B. We assume that the origin $z=0$ in the PWS system (2.1) is a visible-invisible two-fold for $\lambda=\lambda_0$: $Y_{\lambda_0}^\pm(0)=0$ and

$$\begin{cases} X_{\lambda_0}^+(0) & >0, \\ \partial_x Y_{\lambda_0}^+(0) & >0, \end{cases} \quad \begin{cases} X_{\lambda_0}^-(0) & <0, \\ \partial_x Y_{\lambda_0}^-(0) & <0. \end{cases} \tag{3.1}$$

Additionally, we assume that $(\det Z_{\lambda_0})'(0) < 0$ where $\det Z_\lambda$ is defined in (2.14).

Remark 3.1. From (3.1) it follows that $\partial_x(Y_{\lambda_0}^- - Y_{\lambda_0}^+)(0) < 0$. This, together with $(\det Z_{\lambda_0})'(0) < 0$ and (2.15), implies that $X_{\lambda_0}^{sl}(0) > 0$. Thus, $X_{\lambda_0}^{sl}(x) > 0$ for all $x\in[\eta_-,\eta_+]$ (see Assumption A).

Assumption B and the Implicit Function Theorem imply the existence of smooth λ-families of fold singularities $z_+ = (x_+(\lambda),0)$ from above and fold singularities $z_- = (x_-(\lambda),0)$ from below, for $\lambda\sim\lambda_0$, with $x_\pm(\lambda_0)=0$. The following assumption deals with non-zero velocity of the collision between z_+ and z_- for $\lambda=\lambda_0$ at the origin $z=0$:

$$x_+'(\lambda_0) - x_-'(\lambda_0) = \left(-\frac{\partial_\lambda Y_{\lambda_0}^+}{\partial_x Y_{\lambda_0}^+} + \frac{\partial_\lambda Y_{\lambda_0}^-}{\partial_x Y_{\lambda_0}^-}\right)(0) \neq 0$$

where $\partial_\lambda Y_{\lambda_0}^\pm$ means the partial derivative of $Y_\lambda^\pm$ w.r.t. λ, computed in $\lambda=\lambda_0$.

Assumption C. We assume that

$$\partial_\lambda Y_\lambda^- \partial_x Y_\lambda^+ \neq \partial_\lambda Y_\lambda^+ \partial_x Y_\lambda^- \tag{3.2}$$

at $(z, \lambda) = (0, \lambda_0)$.

We consider a regularized PWS system (2.4) with $h_\lambda(x, y) = y$:

$$\dot{z} = \phi(y\epsilon^{-2}) Z_\lambda^+(z) + (1 - \phi(y\epsilon^{-2})) Z_\lambda^-(z) \tag{3.3}$$

where $0 < \epsilon \ll 1$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth regularization function that satisfies the assumptions given after (2.4). More precisely, we have

Assumption D. We suppose that ϕ satisfies (2.5) and (2.6) and that ϕ is smooth at $\pm\infty$.

It is more convenient to write ϵ^{-2} in (3.3) instead of ϵ^{-1} so that we can directly use results from [25] (see Section 4).

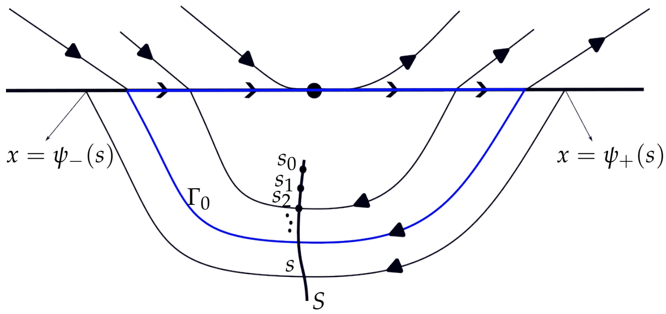

Figure 3.1: A fractal sequence $(s_n)_{n\in\mathbb{N}}$ near the canard cycle Γ_0.

Let S be an open section transversally cutting orbits of Z_λ^-, parametrized by a regular parameter $s \sim 0$ (Fig. 3.1). We assume that s increases as we approach the origin $z = 0$. For $\lambda = \lambda_0$, let Γ_s be the limit periodic set consisting of the orbit of $Z_{\lambda_0}^-$ connecting $(\psi_+(s), 0)$ and $(\psi_-(s), 0)$, and the segment $[\psi_-(s), \psi_+(s)] \subset \{y = 0\}$ (Fig. 3.1). We suppose that $[\psi_-(s), \psi_+(s)] \subset [\eta_-, \eta_+]$ for all $s \sim 0$. In [25] Γ_s is called a canard cycle. From the chosen parameterization of S it follows that $\psi_-'(s) > 0$ and $\psi_+'(s) < 0$. Following [25, Section 3], to study the number of limit cycles of (3.3) produced by Γ_s for $(\epsilon, \lambda) \sim (0, \lambda_0)$ one can use the slow divergence integral associated to the segment $[\psi_-(s), \psi_+(s)]$:

$$I(s) = \int_{\psi_-(s)}^{\psi_+(s)} \frac{(Y_{\lambda_0}^+ - Y_{\lambda_0}^-)^2(x,0)}{-\det Z_{\lambda_0}(x)} \phi' \left(\phi^{-1} \left(\frac{-Y_{\lambda_0}^-}{Y_{\lambda_0}^+ - Y_{\lambda_0}^-}(x,0) \right) \right) dx. \tag{3.4}$$

Remark 3.2. In (3.4) we use Definition 2.7 and Remark 2.9. Note that

$$I(s) = I(m_{\lambda_0}) + I(\tilde{m}_{\lambda_0})$$

where $m_{\lambda_0} = [0, \psi_+(s)]$ and $\tilde{m}_{\lambda_0} = [\psi_-(s), 0]$.

Remark 3.3. We suppose that Assumptions A through D are satisfied and write

$$\lambda = \lambda_0 + \epsilon\widetilde{\lambda}$$

with $\widetilde{\lambda} \sim 0$. We say that the cyclicity of the canard cycle Γ_0 inside (3.3) is bounded by $N \in \mathbb{N}$ if there exist $\epsilon_0 > 0$, $\delta_0 > 0$ and a neighborhood $\mathcal{U}$ of 0 in the $\widetilde{\lambda}$-space such that (3.3) has at most N limit cycles, each lying within Hausdorff distance δ_0 of Γ_0, for all $(\epsilon, \widetilde{\lambda}) \in]0, \epsilon_0] \times \mathcal{U}$. We call the smallest N with this property the cyclicity of Γ_0 and denote it by $\text{Cycl}(\Gamma_0)$.

If the slow divergence integral I in (3.4) has a simple zero at $s = 0$, then $\text{Cycl}(\Gamma_0) = 2$ and, for each small $\epsilon > 0$, the $\widetilde{\lambda}$-family in (3.3) undergoes a saddle-node bifurcation of limit cycles near Γ_0 when we vary $\widetilde{\lambda} \sim 0$. Under the same assumption on I, there is a smooth function $\widetilde{\lambda} = \widetilde{\lambda}(\epsilon)$, $\widetilde{\lambda}(0) = 0$, such that (3.3) with $\lambda = \lambda_0 + \epsilon\widetilde{\lambda}(\epsilon)$ has a unique (hyperbolic) limit cycle Hausdorff close to Γ_0 for each small $\epsilon > 0$.

If I has a zero of multiplicity $m \geq 1$ at $s = 0$, then $\text{Cycl}(\Gamma_0) \leq m + 1$. When $I(0) < 0$ (resp. $I(0) > 0$), then $\text{Cycl}(\Gamma_0) = 1$, and the limit cycle is hyperbolic and attracting (resp. repelling).

We refer the reader to [25, Theorem 3.1] and [25, Remark 3.4].

We say that the canard cycle Γ_0 is balanced if $s = 0$ is a zero of $I(s)$ defined in (3.4) ($I(0) = 0$). If Γ_0 is balanced, then there exists a unique function $G(s)$ satisfying $G(0) = 0$, $G'(0) > 0$ and

$$\int_{\psi_-(s)}^{\psi_+(G(s))} \frac{(Y^+_{\lambda_0} - Y^-_{\lambda_0})^2(x,0)}{-\det Z_{\lambda_0}(x)} \phi'\left(\phi^{-1}\left(\frac{-Y^-_{\lambda_0}}{Y^+_{\lambda_0} - Y^-_{\lambda_0}}(x,0)\right)\right) dx = 0 \tag{3.5}$$

for $s \sim 0$. This follows from the Implicit Function Theorem because $I(0) = 0$, $\psi'_-(s) > 0$, $\psi'_+(s) < 0$ and the integrand in (3.4) is negative for $x < 0$ and positive for $x > 0$ (see Assumptions A and D). We call G defined by (3.5) the slow relation function.

Assumption E. We suppose that Γ_0 is balanced and that $s = 0$ is an isolated zero of $I(s)$, meaning that $s = 0$ has a small neighborhood $]-\tilde{s}, \tilde{s}[$ ($\tilde{s} > 0$) that does not contain any other zero of $I(s)$.

Assumption E implies that I is either negative or positive for $s > 0$ (I is continuous). Using the above mentioned property of the integrand in (3.4) it can be easily seen that $0 < G(s) < s$ for $s > 0$ when I is negative and $G(s) > s$ for $s > 0$ when I is positive. Let $s_0 > 0$ be small and fixed. Thus, if I is negative (resp. positive), then the orbit of s_0

$$U_0 = \{s_0, s_1, s_2, \dots\} \tag{3.6}$$

defined by $s_{n+1} = G(s_n)$ (resp. $s_{n+1} = G^{-1}(s_n)$), $n \geq 0$, tends monotonically to the fixed point $s = 0$ of G. We want to study the Minkowski dimension of U_0.

Let us first define the notion of Minkowski (or box) dimension (see [14,34] and references therein). Let $U \subset \mathbb{R}^N$ be a bounded set. We define the δ-neighborhood of U:

$$U_\delta = \{x \in \mathbb{R}^N \mid d(x, U) \leq \delta\},$$

and denote by $|U_\delta|$ the Lebesgue measure of U_δ. The lower u-dimensional Minkowski content of U, for $u \geq 0$, is defined by

$$\mathcal{M}^u_*(U) = \liminf_{\delta \to 0} \frac{|U_\delta|}{\delta^{N-u}},$$

and analogously the upper u-dimensional Minkowski content $\mathcal{M}^{*u}(U)$ (we replace $\liminf_{\delta\to 0}$ with $\limsup_{\delta\to 0}$). We define lower and upper Minkowski dimensions of U:

$$\underline{\dim}_B U = \inf\{u \geq 0 \mid \mathcal{M}^u_*(U) = 0\}, \qquad \overline{\dim}_B U = \inf\{u \geq 0 \mid \mathcal{M}^{*u}(U) = 0\}.$$

We have $\underline{\dim}_B U \leq \overline{\dim}_B U$ and, if $\underline{\dim}_B U = \overline{\dim}_B U$, we call it the Minkowski dimension of U, and denote it by $\dim_B U$.

The upper Minkowski dimension is finitely stable. More precisely,

$$\overline{\dim}_B(U_1 \cup U_2) = \max\{\overline{\dim}_B U_1, \overline{\dim}_B U_2\},\ U_1, U_2 \subset \mathbb{R}^N.$$

If $U_1 \subset U_2$, then $\underline{\dim}_B U_1 \leq \underline{\dim}_B U_2$ and $\overline{\dim}_B U_1 \leq \overline{\dim}_B U_2$ ($\underline{\dim}_B$ and $\overline{\dim}_B$ are monotonic).

Furthermore, if $0 < \mathcal{M}_*^d(U) \leq \mathcal{M}^{*d}(U) < \infty$ for some d, then we say that U is Minkowski nondegenerate. In this case we have necessarily that $d = \dim_B U$. Recall also that the notion of being Minkowski nondegenerate is invariant under bi-Lipschitz maps. Namely, if Φ is a bi-Lipschitz map and U is Minkowski nondegenerate, then $\Phi(U)$ is also Minkowski nondegenerate (see [36, Theorem 4.1]).

We use these properties in Section 4.2.

Following [12], $\dim_B U_0$ exists, it is independent of the choice of $s_0 > 0$ and can take only the following discrete set of values: $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1$ (see also Theorem 4.1). The set U_0 is defined in (3.6).

3.2 Statement of results

In this section we consider the family (3.3) that satisfies Assumptions A through E and assume that $\lambda = \lambda_0 + \epsilon\widetilde{\lambda}$ with $\widetilde{\lambda} \sim 0$.

Theorem 3.4. *Let $s_0 > 0$ be small and fixed and let U_0 be the orbit of s_0 defined in (3.6). If $\dim_B U_0 = 0$, then the following statements hold.*

1. *(λ unbroken) There exists a smooth function $\widetilde{\lambda} = \widetilde{\lambda}(\epsilon)$, $\widetilde{\lambda}(0) = 0$, such that (3.3) with $\lambda = \lambda_0 + \epsilon\widetilde{\lambda}(\epsilon)$ has a unique (hyperbolic) limit cycle Hausdorff close to Γ_0 for each small $\epsilon > 0$.*

2. *(λ broken) We have that $\mathrm{Cycl}(\Gamma_0) = 2$ and, for every small $\epsilon > 0$, the $\widetilde{\lambda}$-family (3.3) undergoes a saddle-node bifurcation of limit cycles Hausdorff close to Γ_0.*

Theorem 3.4 will be proved in Section 4.1.

Theorem 3.5. *Let U_0 be the orbit of s_0 defined in (3.6), for a small $s_0 > 0$. If $\dim_B U_0 < 1$, then* $\mathrm{Cycl}(\Gamma_0) \leq \frac{2-\dim_B U_0}{1-\dim_B U_0}$.

Theorem 3.5 will be proved in Section 4.1.

Theorem 3.6. *Let U_0 be the orbit of s_0 defined in (3.6), for a small $s_0 > 0$, and $\dim_B U_0 = 0$. The following statements are true.*

1. *For $\widetilde{\lambda} = \widetilde{\lambda}(\epsilon)$ given in Theorem 3.4.1 and for each small $\epsilon > 0$, the Minkowski dimension of any spiral trajectory accumulating (in forward or backward time) on the unique limit cycle of (3.3) near Γ_0 is equal to 1.*

2. *For each small $\epsilon > 0$, the Minkowski dimension of any spiral trajectory accumulating (in forward or backward time) on the limit cycle of multiplicity 2 of (3.3), born in a saddle-node bifurcation of limit cycles Hausdorff close to Γ_0, is equal to $\frac{3}{2}$ and moreover, the spiral is Minkowski nondegenerate.*

Theorem 3.6 will be proved in Section 4.2. A small (Hausdorff) neighborhood of Γ_0 in which we consider spiral trajectories in Theorem 3.6.1 or Theorem 3.6.2 does not shrink to Γ_0 as $\epsilon \to 0$ (see Section 4.2).

4 Proof of Theorems 3.4–3.6

4.1 Proof of Theorems 3.4–3.5

Let $\tilde{s} > 0$ be small and fixed. Suppose that F is a smooth function on $[0, \tilde{s}[$, $F(0) = 0$ and $0 < F(s) < s$ for all $s \in]0, \tilde{s}[$. We define $H(s) := s - F(s)$ and the orbit of $s_0 \in]0, \tilde{s}[$ by H:

$$U := \{s_n = H^n(s_0) \mid n = 0, 1, \dots\}$$

where H^n denotes n-fold composition of H. It is clear that s_n tends monotonically to zero. We say that the multiplicity of the fixed point $s = 0$ of H is equal to m if $s = 0$ is a zero of multiplicity m of F ($F(0) = \dots = F^{(m-1)}(0) = 0$ and $F^{(m)}(0) \neq 0$). If $F^{(n)}(0) = 0$ for each $n = 0, 1, \dots$, then we say that the multiplicity of $s = 0$ of H is ∞.

Theorem 4.1 ([12])**.** *Let F be a smooth function on $[0, \tilde{s}[$, $F(0) = 0$ and $0 < F(s) < s$ for each $s \in]0, \tilde{s}[$. Let $H = \mathrm{id} - F$ and let U be the orbit of $s_0 \in]0, \tilde{s}[$ by H. Then $\dim_B U$ is independent of the initial point s_0 and, for $1 \leq m \leq \infty$, the following bijective correspondence holds:*

$$m = \frac{1}{1 - \dim_B U} \tag{4.1}$$

where m is the multiplicity of $s = 0$ of H (if $m = \infty$, then $\dim_B U = 1$).

If we denote by Φ the integrand in (3.4) and (3.5), then we have

$$I(s) = \int_{\psi_-(s)}^{\psi_+(s)} \Phi(x)dx = \int_{\psi_-(s)}^{\psi_+(G(s))} \Phi(x)dx + \int_{\psi_+(G(s))}^{\psi_+(s)} \Phi(x)dx = \int_{\psi_+(G(s))}^{\psi_+(s)} \Phi(x)dx$$

where in the last step we use (3.5). From $\psi_+'(s) < 0$, $\Phi(x) > 0$ for $x > 0$ and The Fundamental Theorem of Calculus it follows that there exists a negative smooth function $\Psi(s)$ such that

$$I(s) = \Psi(s)(s - G(s)).$$

This implies that $s = 0$ is a zero of multiplicity m of $I(s)$ if and only if $s = 0$ is a zero of multiplicity m of $s - G(s)$.

We will first suppose that the orbit U_0 in (3.6) is generated by the slow relation function G. If $\dim_B U_0 = 0$, then Theorem 4.1, with $H = G$, implies that the multiplicity of the fixed point $s = 0$ of G is 1. Thus, we have that $s = 0$ is a simple zero of I and Theorem 3.4.1 (resp. Theorem 3.4.2) follows directly from [25, Theorem 3.1] (resp. [25, Remark 3.4]). See also Remark 3.3. If $\dim_B U_0 < 1$, then the multiplicity of $s = 0$ of G is equal to $\frac{1}{1 - \dim_B U_0}$ (see (4.1)). Thus, $s = 0$ is a zero of multiplicity $\frac{1}{1 - \dim_B U_0}$ of I and [25, Remark 3.4]) implies that

$$\mathrm{Cycl}(\Gamma_0) \leq 1 + \frac{1}{1 - \dim_B U_0} = \frac{2 - \dim_B U_0}{1 - \dim_B U_0}.$$

This completes the proof of Theorem 3.5.

If U_0 is generated by G^{-1}, Theorem 3.4 and Theorem 3.5 can be proved in the same way as above (we use Theorem 4.1 with $H = G^{-1}$ and the fact that G and G^{-1} have the same multiplicity of the fixed point $s = 0$).

4.2 Proof of Theorem 3.6

The Minkowski dimension of spiral trajectories accumulating on a hyperbolic or non-hyperbolic limit cycle of planar vector fields (without parameters) has been studied in [35,37]. We prove Theorem 3.6 for spiral trajectories in a Hausdorff neighborhood of the canard cycle Γ_0 that does not shrink to Γ_0 as $\epsilon \to 0$.

We will first prove Theorem 3.6.1. We assume that $\dim_B U_0 = 0$ and $\widetilde{\lambda}(\epsilon)$ is given in Theorem 3.4.1. Let $\bar{V}$ be a fixed neighborhood of Γ_0. Then the unique limit cycle of (3.3) with $\widetilde{\lambda} = \widetilde{\lambda}(\epsilon)$ is located in $\bar{V}$ for each $\epsilon > 0$ small enough (see [25]). For such fixed $\epsilon > 0$, let Γ be any spiral trajectory in $\bar{V}$ accumulating on the limit cycle (in the forward time if the limit cycle is attracting or in the backward time if the limit cycle is repelling). We write $\Gamma = \tilde{\Gamma} \cup \bar{\Gamma}$ where $\bar{\Gamma}$ is the part of Γ sufficiently close to the limit cycle (we can apply the results of [35,37]) and $\tilde{\Gamma}$ is the rest of Γ (of finite length). It is clear that $\dim_B \tilde{\Gamma} = 1$ and $\overline{\dim}_B \bar{\Gamma} \geq 1$. Since $\underline{\dim}_B \leq \overline{\dim}_B$, $\underline{\dim}_B$ is monotonic and $\overline{\dim}_B$ is finitely stable (see Section 3.1), we have

$$\underline{\dim}_B \bar{\Gamma} \leq \underline{\dim}_B (\tilde{\Gamma} \cup \bar{\Gamma}) \leq \overline{\dim}_B (\tilde{\Gamma} \cup \bar{\Gamma}) = \max\{\overline{\dim}_B \tilde{\Gamma}, \overline{\dim}_B \bar{\Gamma}\} = \overline{\dim}_B \bar{\Gamma}. \tag{4.2}$$

Since the limit cycle is hyperbolic (see Theorem 3.4.1), [35, Theorem 10] implies that $\dim_B \bar{\Gamma} = \underline{\dim}_B \bar{\Gamma} = \overline{\dim}_B \bar{\Gamma} = 1$. Using (4.2) we obtain $\dim_B \Gamma = 1$. This completes the proof of Theorem 3.6.1.

The first part of Theorem 3.6.2 can be proved in the same way as Theorem 3.6.1. Since the non-hyperbolic limit cycle is generated by a saddle-node bifurcation of limit cycles we have $\dim_B \bar{\Gamma} = \underline{\dim}_B \bar{\Gamma} = \overline{\dim}_B \bar{\Gamma} = \frac{3}{2}$ (see [35, Theorem 10] and [37, Theorem 1]). To prove the claim about Minkowski nondegeneracy; first observe that $\mathcal{M}^{3/2}(\tilde{\Gamma}) = 0$ since $\dim_B(\tilde{\Gamma}) = 1 < 3/2$ so that this part does not affect the upper and lower Minkowski content of $\Gamma = \tilde{\Gamma} \cup \bar{\Gamma}$; hence, it is enough to show that $\bar{\Gamma}$ is Minkowski nondegenerate. To see this, we observe that $\bar{\Gamma}$ can be partitioned into finitely many pieces $\bar{\Gamma}_i$; $i = 1, \dots, k$ such that each $\bar{\Gamma}_i$ is bi-Lipschitz equivalent to $[0,1[\times U$ by the Lipschitz flow-box Theorem [3]. Note also that $\dim_B U = 1/2$ and it is Minkowski nondegenerate which implies that $[0,1[\times U$ is also Minkowski nondegenerate; see the proof of [37, Theorem 4(b)]. Finally, the finite stability of Minkowski dimension and of Minkowski nondegeneracy now complete the proof exactly as in the proof of [37, Theorem 4(b)].

Declarations

Ethical Approval Not applicable.

Competing interests The authors declare that they have no conflict of interest.

Authors' contributions All authors conceived of the presented idea, developed the theory, performed the computations and contributed to the final manuscript.

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Multiple positive radial solutions for Dirichlet problem of the prescribed mean curvature spacelike equation in a Friedmann–Lemaître–Robertson–Walker spacetime

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Abstract. In this paper, we consider the radially symmetric spacelike solutions of a nonlinear Dirichlet problem for the prescribed mean curvature spacelike equation in a Friedmann–Lemaître–Robertson–Walker spacetime. By using a conformal change of variable, this problem can be translated an equivalent problem in the Minkowski spacetime. By using the lower and upper solution method, fixed point, a priori bounds and topological degree method, we obtain the existence, nonexistence and multiplicity of radially symmetric spacelike solutions.

Keywords: topological degree, radially symmetric spacelike solutions, Dirichlet problem, prescribed mean curvature spacelike equation, Friedmann–Lemaître–Robertson–Walker spacetime.

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1 Introduction

Let $I \subseteq \mathbb{R}$ be an open interval in $\mathbb{R}$ with the metric $-dt^2$. Denote by $\mathcal{M}$ the $(N+1)$-dimensional product manifold $I \times \mathbb{R}^N$ with $N \geq 1$ endowed with the Lorentzian metric

$$g = -dt^2 + f^2(t)dx^2,$$

where $f \in C^\infty(I)$, $f > 0$, is called the *scale factor* or *warping function* in the related literature. Clearly, $\mathcal{M}$ is a Lorentzian warped product with base $(I, -dt^2)$, fiber $(\mathbb{R}^N, dx^2)$ and warping function f, we refer it as a Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime. In the fiber space $(\mathbb{R}^N, dx^2)$, the metric dx^2 is an arbitrary Riemannian metric in a Generalized FLRW spacetime. In cosmology, the FLRW spacetime is the accepted model for a spatially homogeneous and isotropic Universe. In this context, the warping function $f(t)$ is interpreted as the radius of the Universe at time t, and the sign of its derivative indicates if the Universe

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is expanding or contracting at given time, for more details of FLRW spacetime, we refer the reader to [11,21,22,27,34–37] and the references therein. Observe that for the particular case $f(t) \equiv 1$ we recover the Minkowski spacetime.

Given $f \in C^\infty(I)$, $f > 0$, for each $u \in C^\infty(\Omega)$, where Ω is a domain of $\mathbb{R}^N$, such that $u(\Omega) \subseteq I$, we can consider its graph $M = \{(x, u(x)) : x \in \Omega\}$ in the FLRW spacetime $\mathcal{M}$. The graph is spacelike whenever

$$|\operatorname{grad} u| < f(u) \quad \text{in } \Omega, \tag{1.1}$$

where $\operatorname{grad} u$ is the gradient of u in $\mathbb{R}^N$ and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$, in this case, the unit timelike normal vector field in the same time orientation of ∂_t is given by

$$A = \frac{f(u)}{\sqrt{f(u) - |\operatorname{grad} u|^2}} \left(\frac{1}{f^2(u)} \operatorname{grad} u + \partial_t \right),$$

and the corresponding mean curvature associated to A, is defined by

$$\frac{1}{N} \left\{ \operatorname{div} \left(\frac{\operatorname{grad} u}{f(u)\sqrt{f^2(u) - |\operatorname{grad} u|^2}} \right) + \frac{f'(u)}{\sqrt{f^2(u) - |\operatorname{grad} u|^2}} \left(N + \frac{|\operatorname{grad} u|^2}{f^2(u)} \right) \right\},$$

where div denotes the divergence operator of $\mathbb{R}^N$, $f'(u) := f' \circ u$, it can be seen as a quasilinear elliptic operator Q, because of (1.1). We are interested in the existence of spacelike graphs with a prescribed mean curvature function in the FLRW spacetime $\mathcal{M}$. The general problem of the curvature prescription is, given a function $H : I \times \mathbb{R}^N \to \mathbb{R}$, to obtain solutions of the quasilinear elliptic equation

$$Q(u) = H(u, x), \quad |\operatorname{grad} u| < f(u) \quad \text{in } \Omega, \tag{1.2}$$

and (1.2) is called the prescribed mean curvature spacelike equation in FLRW spacetime. Specially relevant is the case when H is constant, then it is called the prescribed constant mean curvature spacelike equation (if $H = 0$ it is also called the maximal spacelike graph equation).

In the recent years, most of the efforts have been directed to the prescribed mean curvature spacelike equation in Minkowski spacetime ($f(t) \equiv 1$), in this context, we mention the seminal work of R. Bartnik and L. Simon [1], E. Calabi [8], S.-Y. Cheng and S.-T. Yau [10] and A. E. Treibergs [39], in these papers, the spacelike graphs having the property that their mean curvature is zero or constant are considered. More recently, Dirichlet problems for prescribed mean curvature spacelike equation in Minkowski spacetime have been widely concerned by many scholars, and their attention is mainly focused on their positive solutions, we refer the reader to [3–6,12–16,23,24,28–32,41,42] and the references therein. In particular, based on the detailed analysis of time map, some exact multiplicity of positive solutions have been obtained in [24,42], for the radially symmetric solutions on a ball, some existence, nonexistence and multiplicity results have been established in [4,5], and some bifurcation results have been obtained in [14,28] via bifurcation technique, and when Ω is a general domain in $\mathbb{R}^N$, some existence and bifurcation results have been obtained in the papers [13,15,16,31]. In addition to, these concern discrete problems associated with the prescribed mean curvature spacelike equation in Minkowski spacetime, we refer the reader to [7,9,25,26] and the references therein.

In comparison with the study in Minkowski spacetime, the number of references devoted to the prescribed mean curvature spacelike equation in FLRW spacetime is appreciably lower. Only in the recent years, C. Bereanu, D. de la Fuente, A. Romero and P. J. Torres [2,20] have considered the existence and multiplicity of radially symmetric spacelike solutions of the Dirichlet problem by using the Schauder fixed point Theorem with approximation process, J. Mawhin and P. J. Torres [33, 38] have provided some sufficient conditions for the existence of radially symmetric spacelike solutions of the Neumann problem by the Leray–Schauder degree theory, G. Dai, A. Romero and P. J. Torres [17–19] have obtained the existence and multiplicity of radially symmetric spacelike positive solutions of the equation with 0-Dirichlet boundary condition on a ball and studied the global structure of the solution set via the Rabinowitz's global bifurcation method. Xu and Ma [40] have considered the differential and difference problems associated with the discrete approximation of radially symmetric spacelike solutions of the Dirichlet problem, by using lower and upper solutions, they proved the existence of solutions of the corresponding differential and difference problems, and based on the ideas of a prior bound showed the solutions of the discrete problem converge to the solutions of the continuous problem.

In this paper we are concerned with the mixed boundary value problem

$$\begin{cases} -(r^{N-1}\phi(v'))' = \lambda N r^{N-1}\left[\dfrac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - f(\varphi^{-1}(v))H(\varphi^{-1}(v),r)\right], & r\in(0,R),\\ |v'|<1, \quad r\in(0,R), \\ v'(0)=v(R)=0, \end{cases} \tag{1.3}$$

where $\phi(s) = \frac{s}{\sqrt{1-s^2}}$, and $\phi : (-1,1) \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, such an ϕ is called *singular*, λ is a positive parameter, R is a positive constant, $f \in C^\infty(I)$ and $f > 0$, I is an open interval in $\mathbb{R}$, $\varphi(s) = \int_0^s \frac{dt}{f(t)}$, φ^{-1} is the inverse function of φ, $H : I\times[0,R] \to \mathbb{R}$ is a continuous function. The aim of this paper is to investigate the intervals of the λ in which the (1.3) has zero, one or two positive radial solutions.

This study mainly motivated by the numerical approximation of radially symmetric spacelike solutions of the nonlinear Dirichlet problem for the prescribed mean curvature spacelike equation in FLRW spacetime:

$$\begin{cases} \operatorname{div}\left(\frac{\operatorname{grad} u}{f(u)\sqrt{f^2(u)-|\operatorname{grad} u|^2}}\right) + \frac{f'(u)}{\sqrt{f^2(u)-|\operatorname{grad} u|^2}}\left(N + \frac{|\operatorname{grad} u|^2}{f^2(u)}\right) = NH(u,|x|) & \text{in } \mathcal{B},\\ |\operatorname{grad} u| < f(u) \quad \text{in } \mathcal{B}, \\ u = 0 \quad \text{on } \partial\mathcal{B}, \end{cases} \tag{1.4}$$

where $\mathcal{B} = \{x\in\mathbb{R}^N : |x| < R\}$, $f\in C^\infty(I)$, $f>0$ and $H : I\times[0,+\infty)\to\mathbb{R}$ is the prescribed mean curvature function. We follow the method developed in [20], let us define the function $\varphi : I\to\mathbb{R}$ by $\varphi(s) = \int_0^s \frac{dt}{f(t)}$, and φ is an increasing diffeomorphism from I onto $J := \varphi(I)$ such that $\varphi(0) = 0$. Doing the change $v = \varphi(u)$ and taking radial coordinates, we can reduce the Dirichlet problem (1.4) to the mixed boundary value problem (1.3) with $\lambda = 1$, and the solutions of (1.3) with $\lambda = 1$ are just the radially symmetric spacelike solutions of (1.4).

We say that a function $v\in C^1[0,R]$ is a solution of (1.3) if $\|v'\|_\infty < 1$, $r^{N-1}\phi(v')\in C^1[0,R]$, and (1.3) is satisfied. For (1.3), since the graph associate to v is spacelike, i.e. $\|v'\|_\infty < 1$, we deduce that $\|v\|_\infty < R$, this implies the image of nonnegative v is in $[0,R]$, therefore, when discussing the nonnegative solutions of (1.3), we always assume $\varphi^{-1}([0,R]) \subset I$, which is equivalent to

$$I_f R := \left[0, \int_0^R f(\varphi^{-1}(s))ds\right] \subset I.$$

In Section 2, we present a lower and upper solution result for continuous problem (1.3) with $\lambda = 1$. In Section 3, we give some notations and fixed point reformulation of (1.3) with $\lambda = 1$ and prove all possible solutions and their first differences have a prior bounds, based on this, we calculate some topological degrees. Using the results of these two parts and the estimate of the first derivative of a concave function, in Section 4, we show that there is a $\Lambda > 0$ such that problem (1.3) has zero, at least one or at least two positive solutions when $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$, $\lambda > \Lambda$. Finally in Section 5, for the convenience of readers and integrity of the paper, we give the detailed derivation process of problem (1.3) with $\lambda = 1$.

The main result is as follows.

Theorem 1.1. *Assume that* $I_f R \subset I$ *and* $f'(t) \geq 0$, $H(t,r) < \frac{f'}{f}(t)$ *for all* $r \in [0, R]$, $t \in I_f R$ *and assume also that*

$$\begin{cases} \lim\limits_{t \to 0^+} \frac{N f'(t)}{\varphi(t)} = f_0, \\ \lim\limits_{t \to 0^+} \frac{N f(t) H(t,r)}{\varphi(t)} = H_0, \\ f_0 - H_0 = 0. \end{cases} \tag{A_{fH}}$$

Then there is a $\Lambda > \frac{2NM_0}{R^3}$ *such that problem* (1.3) *has zero, at least one or at least two positive solutions when* $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$, $\lambda > \Lambda$.

Notations: The space $C := C[0, R]$ will be endowed with the usual sup-norm $\|\cdot\|_\infty$ and $C^1 := C^1[0, R]$ will considered with the norm $\|u\| = \|u\|_\infty + \|u'\|_\infty$. $C^1_M := \{u \in C^1 : u'(0) = u(R) = 0\}$ is the closed subspace of C^1. For $u_0 \in C^1_M$, we set $B(u_0, \rho) := \{u \in C^1_M : \|u\| < \rho\}$ $(\rho > 0)$ and B_ρ is used to represent $B(0, \rho)$.

2 Lower and upper solutions

In this section, we develop the lower and upper solution method for the mixed boundary value problem

$$\begin{cases} -\left(r^{N-1}\phi(v')\right)' = N r^{N-1}\left[\dfrac{f'(\varphi^{-1}(v))}{\sqrt{1 - v'^2}} - f(\varphi^{-1}(v)) H(\varphi^{-1}(v), r)\right], & r \in (0, R), \\ |v'| < 1, \quad r \in (0, R), \\ v'(0) = v(R) = 0. \end{cases} \tag{2.1}$$

Definition 2.1. A lower solution α of (2.1) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < 1$, $r^{N-1}\phi(\alpha') \in C^1$, $I_f R \subset I$ and

$$-(r^{N-1}\phi(\alpha'))' \leq N r^{N-1}\left[\frac{f'(\varphi^{-1}(\alpha))}{\sqrt{1 - \alpha'^2}} - f(\varphi^{-1}(\alpha)) H(\varphi^{-1}(\alpha), r)\right], \quad r \in (0, R), \quad \alpha(R) \leq 0.$$

An upper solution β of (2.1) is a function $\beta \in C^1$ such that $\|\beta'\|_\infty < 1$, $r^{N-1}\phi(\beta') \in C^1$, $I_f R \subset I$ and

$$-(r^{N-1}\phi(\beta'))' \geq N r^{N-1}\left[\frac{f'(\varphi^{-1}(\beta))}{\sqrt{1 - \beta'^2}} - f(\varphi^{-1}(\beta)) H(\varphi^{-1}(\beta), r)\right], \quad r \in (0, R), \quad \beta(R) \geq 0.$$

Such a lower or an upper solution is called strict if the above inequalities are strict.

Theorem 2.2. *Assume that $I_f R \subset I$ and $f'(t) \geq 0$, $H(t,r) < \frac{f'}{f}(t)$ for all $r \in [0,R]$, $t \in I_f R$. If (2.1) has a lower solution α and an upper solution β such that $\alpha(r) \leq \beta(r)$ for all $r \in [0,R]$, then (2.1) has at least one solution v such that $\alpha(r) \leq v(r) \leq \beta(r)$ for all $r \in [0,R]$.*

Proof. Let $\gamma : [0,R] \times \mathbb{R} \to \mathbb{R}$ be the continuous function defined by

$$\gamma(r,v) = \begin{cases} \alpha(r), & \text{if } v < \alpha(r), \\ v, & \text{if } \alpha(r) \leq v \leq \beta(r), \\ \beta(r), & \text{if } v > \beta(r). \end{cases}$$

We consider the modified problem

$$\begin{cases} (r^{N-1}\phi(v'))' + Nr^{N-1}\Big[\dfrac{f'(\varphi^{-1}(\gamma(r,v)))}{\sqrt{1-v'^2}} \\ \qquad -H(\varphi^{-1}(\gamma(r,v)),r)f(\varphi^{-1}(\gamma(r,v))) - v + \gamma(r,v)\Big] = 0, \quad r \in (0,R), \\ |v'| < 1, \quad r \in (0,R), \\ v'(0) = 0 = v(R). \end{cases} \tag{2.2}$$

It follows from [2] that the problem (2.2) has at least one solution.

We show that if v is a solution (2.2), then $\alpha(r) \leq v(r) \leq \beta(r)$ for all $r \in [0,R]$. This will conclude the proof.

Suppose by contradiction that there is some $r_0 \in [0,R]$ such that

$$\max_{[0,R]}[\alpha - v] = \alpha(r_0) - v(r_0) > 0.$$

If $r_0 \in (0,R)$, then $\alpha'(r_0) = v'(r_0)$ and there are sequences $\{r_k\}$ in $(0,r_0)$ converging to r_0 such that $\alpha'(r_k) - v'(r_k) \geq 0$. Since ϕ is an increasing homeomorphism then we can have

$$r_k^{N-1}\phi(v'(r_k)) - r_0^{N-1}\phi(v'(r_0)) \leq r_k^{N-1}\phi(\alpha'(r_k)) - r_0^{N-1}\phi(\alpha'(r_0)),$$

which means

$$(r_0^{N-1}\phi(\alpha'(r_0)))' \leq (r_0^{N-1}\phi(v'(r_0))'.$$

Therefore, since α is a lower solution of (2.1) we have

$$\begin{aligned} &(r_0^{N-1}\phi(\alpha'(r_0)))' \\ &\quad \leq (r_0^{N-1}\phi(v'(r_0)))' \\ &\quad = Nr_0^{N-1}\left[-\frac{f'(\varphi^{-1}(\alpha(r_0)))}{\sqrt{1-(\alpha'(r_0))^2}} + H(\varphi^{-1}(\alpha(r_0)),r_0)f(\varphi^{-1}(\alpha(r_0))) + v(r_0) - \alpha(r_0)\right] \\ &\quad < Nr_0^{N-1}\left[-\frac{f'(\varphi^{-1}(\alpha(r_0)))}{\sqrt{1-(\alpha'(r_0))^2}} + H(\varphi^{-1}(\alpha(r_0)),r_0)f(\varphi^{-1}(\alpha(r_0)))\right] \\ &\quad \leq (r_0^{N-1}\phi(\alpha'(r_0)))', \end{aligned}$$

but this a contradiction.

If $\max_{[0,R]}[\alpha - v] = \alpha(R) - v(R) > 0$, then by definition of lower solutions, we obtain a contradiction again. If $\max_{[0,R]}[\alpha - v] = \alpha(0) - v(0) > 0$, then there exists $r_1 \in (0,R]$ such that $\alpha(r) - v(r) > 0$ for all $r \in [0,r_1]$ and $\alpha'(r_1) - v'(r_1) \leq 0$. It follows that

$$(r_1^{N-1}\phi(\alpha'(r_1)))' \leq (r_1^{N-1}\phi(v'(r_1)))'.$$

Note that $I_f R \subset I$ and $f'(t) \geq 0$ for all $t \in I_f R$. By using the fact and integrating (2.2) from 0 to r_1, we have that

$$\begin{aligned} r_1^{N-1}\phi(\alpha'(r_1)) &\leq r_1^{N-1}\phi(v'(r_1)) \\ &< N\int_0^{r_1} r^{N-1}\left[-\frac{f'(\varphi^{-1}(\alpha(r)))}{\sqrt{1-(v'(r))^2}} + H(\varphi^{-1}(\alpha(r)),r)f(\varphi^{-1}(\alpha(r)))\right]dr \\ &\leq N\int_0^{r_1} r^{N-1}\left[-\frac{f'(\varphi^{-1}(\alpha(r)))}{\sqrt{1-(\alpha'(r))^2}} + H(\varphi^{-1}(\alpha(r)),r)f(\varphi^{-1}(\alpha(r)))\right]dr \\ &\leq r_1^{N-1}\phi(\alpha'(r_1)). \end{aligned}$$

But this is a contradiction. Hence, $\alpha(r) \leq v(r)$ for all $r \in [0, R]$. Analogously, we can show that $v(r) \leq \beta(r)$ for all $r \in [0, R]$. □

Remark 2.3. The Theorem 2.2 still holds for $f(t) \equiv 1$.

3 Fixed point, a priori bounds and degree

In this section, we consider problems of type

$$\begin{cases} (r^{N-1}\phi(v'))' + r^{N-1}g(r, v, v') = 0, & r \in (0, R), \\ |v'| < 1, \quad r \in (0, R), \\ v'(0) = v(R) = 0, \end{cases} \tag{3.1}$$

where $N \geq 1$ is an integer, $R > 0$ is a constant, and we also assume that

(A_ϕ) $\phi : (-1, 1) \to \mathbb{R}$ is an odd, increasing homeomorphism;

(A_g) $g : [0, R] \times [0, \alpha) \times (-1, 1) \to [0, +\infty)$ is a continuous function with $0 < \alpha \leq +\infty$.

Recall, by a solution of (3.1) we mean a function $v \in C^1$ with $\|v'\|_\infty < 1$, such that $r^{N-1}\phi(v') \in C^1$ and (3.1) is satisfied.

Setting

$$\sigma(r) := 1/r^{N-1},$$

we introduce the linear operators

$$S : C \to C, \quad Sv(r) = \sigma(r)\int_0^r t^{N-1}v(t)\,dt \quad (r \in [0, R]), \quad Sv(0) = 0;$$

$$K : C \to C^1, \quad Kv(r) = \int_r^R v(t)\,dt \quad (r \in [0, R]).$$

It is easy to see the standard argument that K is bounded and that S is compact by the Arzelà–Ascoli theorem. This means that the nonlinear operator $K \circ \phi^{-1} \circ S : C \to C^1$ is compact. Moreover, for a given function $h \in C$, the problem

$$(r^{N-1}\phi(v'))' + r^{N-1}h(r) = 0, \quad r \in (0, R), \quad |v'| < 1, \quad v'(0) = v(R) = 0$$

has a unique solution

$$v = K \circ \phi^{-1} \circ S \circ h.$$

Next, let N_g be the Nemytskii operator associated with g, i.e.,

$$N_g : C \to C, \quad N_g = g(\cdot, v(\cdot), v'(\cdot)).$$

Noticing that N_g is continuous and maps a bounded set to a bounded set. So problem (3.1) has the following reformulation about fixed points.

Lemma 3.1. *A function $v \in C^1_M$ is a solution of problem (3.1) if and only if the compact nonlinear operator*

$$\mathcal{N}_g : C^1_M \to C^1_M, \quad \mathcal{N}_g = K \circ \phi^{-1} \circ S \circ N_g$$

has a fixed point, and furthermore the fixed point of $\mathcal{N}_g$ satisfies

$$\|v'\|_\infty < 1, \quad \|v\|_\infty < R \tag{3.2}$$

and

$$d_{LS}[I - \mathcal{N}_g, B_\rho, 0] = 1 \quad \textit{for all } \rho \geq (R+1).$$

Proof. Since the range of ϕ^{-1} is $(-1,1)$, the inequality (3.2) holds. Next, consider the compact homotopy

$$\mathcal{H} : [0,1] \times C^1_M, \quad \mathcal{H}(\tau, \cdot) = \tau \mathcal{N}_g$$

and

$$\mathcal{H}([0,1] \times C^1_M) \subset B_{(R+1)}.$$

Then, from the invariance under homotopy of the Leray–Schauder degree it follows that

$$d_{LS}[I - \mathcal{H}(0,\cdot), B_\rho, 0] = d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0] = d_{LS}[I - \mathcal{N}_g, B_\rho, 0] = 1,$$

for all $\rho \geq (R+1)$. □

In view of Theorem 2.2 and Remark 2.3, we have the following result.

Lemma 3.2. *Assume that (3.1) has a lower solution α and an upper solution β such that $\alpha(r) \leq \beta(r)$ for all $r \in [0,R]$, and let $\Omega_{\alpha,\beta} := \{v \in C^1_M : \alpha \leq v \leq \beta\}$. Assume also that (3.1) has an unique solution v_0 in $\Omega_{\alpha,\beta}$ and there exists $\rho_0 > 0$ such that $\overline{B}(v_0, \rho_0) \subset \Omega_{\alpha,\beta}$. Then*

$$d_{LS} = [I - \mathcal{N}_g, B(v_0, \rho), 0] = 1 \quad \textit{for all } 0 < \rho \leq \rho_0,$$

where $\mathcal{N}_g$ is the fixed point operator associated to (3.1).

Proof. Let $\mathcal{N}_g$ be the fixed point operator associated with (3.1). The proof of Theorem 2.2 shows that any fixed point v of $\mathcal{N}_g$ is contained in $\Omega_{\alpha,\beta}$, and this means that v_0 is the unique fixed of $\mathcal{N}_g$ and there exists $\rho_0 > 0$ such that $\overline{B}(v_0, \rho_0) \subset \Omega_{\alpha,\beta}$. From Lemma 3.1 and the excision property of the Leray–Schauder degree there is

$$d_{LS}[I - \mathcal{N}_g, B(v_0, \rho_0), 0] = 1,$$

which is

$$d_{LS}[I - \mathcal{N}_g, B(v_0, \rho), 0] = 1 \quad \text{for all } 0 < \rho \leq \rho_0.$$

□

Lemma 3.3. *Assume that (A_ϕ), (A_g) and*

(A_g') *$g(r, v, v') > 0$ for all $(r, v, v') \in (0, R] \times (0, \alpha) \times (-1, 1)$.*

Let v be a nontrivial solution of (3.1). Then $v > 0$ on $[0, R)$ and v is strictly decreasing.

Proof. Let's first integrate both sides of (3.1) from 0 to r, which is

$$v'(r) = -\phi^{-1}\left(\frac{1}{r^{N-1}}\int_0^r s^{N-1}g(s, v, v')ds\right). \tag{3.3}$$

Then integrate both sides of (3.3) from r to R to get

$$v(r) = \int_r^R \phi^{-1}\left(\frac{1}{t^{N-1}}\int_0^t s^{N-1}g(s, v, v')ds\right) dt. \tag{3.4}$$

So if $g(r, v, v') > 0$, we have $v > 0$ on $[0, R)$ and v is strictly decreasing. □

In the next lemma we assume that g is sublinear with respect to ϕ at zero.

Lemma 3.4. *Assume that conditions (A_ϕ), (A_g) and (A'_g) hold. Assume also that*

$$\lim_{s\to 0^+} \frac{g(r, s, s')}{\phi(s)} = 0 \quad \textit{uniformly for } r \times s' \in [0, R] \times (-1, 1) \tag{3.5}$$

and

$$\liminf_{s\to 0^+} \frac{\phi(\sigma s)}{\phi(s)} > 0 \quad \textit{for all } \sigma > 0. \tag{3.6}$$

Then there exists $\rho_0 > 0$ such that

$$d_{LS}[I - \mathcal{N}_g, B_\rho, 0] = 1 \quad \textit{for all } 0 < \rho \leq \rho_0.$$

Proof. Using (3.6) we can find $\varepsilon > 0$ such that

$$R\varepsilon/N < \liminf_{s\to 0} \frac{\phi(s/R)}{\phi(s)}. \tag{3.7}$$

Using (3.5) we can find $s_\varepsilon > 0$ such that

$$g(r, s, s') \leq \varepsilon\phi(s) \quad \text{for all } (r, s, s') \in [0, R] \times [0, s_\varepsilon] \times (-1, 1). \tag{3.8}$$

Next, we consider the compact homotopy

$$\mathcal{H} : [0, 1] \times C^1_M \to C^1_M, \quad \mathcal{H}(\tau, v) = \tau\mathcal{N}_g(v).$$

Let's we say have $\rho_0 > 0$ such that

$$v \neq \mathcal{H}(\tau, v) \quad \text{for all } (\tau, v) \in [0, 1] \times (\overline{B}_{\rho_0} \setminus \{0\}). \tag{3.9}$$

In fact, suppose there exists

$$v_k = \tau_k\mathcal{N}_g(v_k), \quad \tau_k \in [0, 1],$$

where $v_k \in C^1_M \setminus \{0\}$, $k \in \mathbb{N}$, $\|v_k\| \to 0$. From the previous lemma, v is strictly monotonically decreasing and strictly positive on $[0, R)$.

Asuming $\|v_k\| \leq s_\varepsilon$, $k \in \mathbb{N}$, we can see from (3.8)

$$g(r, v_k(r), v'_k(r)) \leq \varepsilon\phi(\|v_k\|_\infty) \quad \text{for all } r \in [0, R],\ k \in \mathbb{N}.$$

Then for any $k \in \mathbb{N}$, there is

$$\begin{aligned}\|v_k\|_\infty &\leq \int_0^R \phi^{-1}\left(\sigma(t)\int_0^t r^{N-1}g(r,v_k,v_k')dr\right)dt \\ &\leq R\phi^{-1}\left(\frac{\varepsilon R}{N}\phi(\|v_k\|_\infty)\right).\end{aligned}$$

That is, there is

$$\frac{\phi\left(\frac{\|v_k\|_\infty}{R}\right)}{\phi(\|v_k\|_\infty)} \leq \frac{\varepsilon R}{N}.$$

This contradicts (3.7) and so (3.9) is true. That is, for any $\rho \in (0, \rho_0]$, there is

$$d_{LS}[I - \mathcal{H}(1,\cdot), B_\rho, 0] = d_{LS}[I - \mathcal{N}_g, B_\rho, 0] = d_{LS}[I - \mathcal{H}(0,\cdot), B_\rho, 0] = d_{LS}[I, B_\rho, 0] = 1. \qquad \square$$

4 Proof of main result

First of all there is an important lemma before the main result of this paper.

Lemma 4.1. *Let $k \in (0,1)$, $\beta_0 \in (0, \frac{1-k}{8}R)$ be given. Let $I_{k,\beta_0} := [\frac{4\beta_0}{1-k}, R - \frac{4\beta_0}{1-k}]$. Then*

$$\frac{R}{2} \in I_{k,\beta_0}$$

and

$$|v'(s)| \leq 1-k, \quad \forall v \in \mathcal{A}, \quad \forall s \in I_{k,\beta_0},$$

where $\mathcal{A} := \{\, v \mid v$ is concave in $[0,R]$, $v'(0) < 1$, $v'(R) > -1$, $\|v\|_\infty \leq 4\beta_0\}$.

Proof. Let $a = 1-k$, $b = \frac{4\beta_0}{1-k}$, then

$$0 < a < 1, \quad b \in \left(0, \frac{R}{2}\right), \quad I := I_{k,\beta_0} = [b, R-b].$$

Since $v \in C^1[0,R]$, v is concave in $[0,R]$ and v' is decreasing. If there exists $s \in I$ such that $|v'(s)| > 1-k = a$, then $v'(s) > a$ or $v'(s) < -a$. If $v'(s) < -a$, then $\frac{v(s)-v(R)}{s-R} = v'(t)$, for some $t \in (s,R)$. So we have $\frac{v(s)}{s-R} \leq v'(s) < -a$. Therefore $v(s) > a(R-s) \geq ab = 4\beta_0 \geq \|v\|_\infty$. This is a contradiction. Analogously, we can get a contradiction for other case. $\square$

Proof of Theorem 1.1. Let us say

$$S_j := \{\lambda > 0 : (1.3) \text{ at least } j \text{ positive solutions}\}, \quad (j = 1,2).$$

1. *The existence of* Λ.

Let $\lambda > 0$ and v be a positive solution of (1.3). Firstly, using hypothesis (A_{fH}), we have: $\forall \varepsilon_0 > 0$, $\exists\, \delta_1$, for $|\varphi^{-1}(v) - 0| < \delta_1$, there can be $\left|\frac{Nf'(\varphi^{-1}(v))}{v} - f_0\right| < \varepsilon_0$. For the above ε_0, $\exists\, \delta_2$, when $|\varphi^{-1}(v) - 0| < \delta_2$, there is $\left|\frac{Nf(\varphi^{-1}(v))H(\varphi^{-1}(v),r)}{v} - H_0\right| < \varepsilon_0$.

Secondly, integrating (1.3) from 0 to $r \in (0, R]$ and using that v is a positive solution of (1.3) such that we obtain

$$\begin{aligned} -r^{N-1}\phi(v') &= \int_0^r \lambda t^{N-1}\left(\frac{Nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - Nf(\varphi^{-1}(v))H(\varphi^{-1}(v),t)\right)dt \\ &< \lambda \int_0^r t^{N-1}\left(\frac{f_0 v}{\sqrt{1-v'^2}} - H_0 v\right)dt \\ &= \lambda \int_0^r t^{N-1}\left(\frac{f_0 v}{\sqrt{1-v'^2}} - f_0 v\right)dt. \end{aligned}$$

Using Lemma 4.1, let $k = a_0$, $\beta_0 = \frac{(1-a_0)\eta}{8}R \in (0, \frac{1-a_0}{8}R)$, a_0 is the constant that satisfies the definition and $\eta \in (0,1)$ is the given constant, then there is $I = [\frac{\eta}{2}R, R - \frac{\eta}{2}R]$. Hence, $\|v\|_\infty \le \frac{(1-a_0)\eta}{2}R$, $|v'(s)| \le 1 - a_0$, for all $s \in I$.

Therefore,

$$\begin{aligned} -r^{N-1}\phi(v') &< \lambda \int_0^r t^{N-1}\left(\frac{f_0 v}{\sqrt{1-v'^2}} - f_0 v\right)dt \\ &\le \lambda \int_0^r t^{N-1} f_0 \frac{(1-a_0)\eta}{2} R\left(\frac{1}{\sqrt{1-(1-a_0)^2}} - 1\right)dt \\ &\le \lambda M R \int_0^r t^{N-1} dt \\ &= \frac{\lambda M R r^N}{N}, \end{aligned}$$

where $M = f_0 \frac{(1-a_0)\eta}{2}\left(\frac{1}{\sqrt{1-(1-a_0)^2}} - 1\right)$.

Therefore, there is

$$-v'(r) \le -\frac{v'(r)}{\sqrt{1-v'^2}} < \frac{\lambda M R r}{N}. \tag{4.1}$$

Integrating (4.1) from 0 to R we obtain

$$v(0) < \frac{\lambda M R^3}{2N}. \tag{4.2}$$

Next, using $v(0) > 0$, we obtain

$$\lambda > \frac{2NM_0}{R^3},$$

where $M_0 := v(0)/M$.

We know from [18] that the problem (1.3) has at least one positive solution for $\lambda > 0$. Specially, $S_1 \neq \emptyset$ and we can define

$$\Lambda = \Lambda(R) := \inf S_1.$$

Clearly, we have $\Lambda \ge \frac{2NM_0}{R^3}$. We claim that $\Lambda \in S_1$. Indeed, let $\lambda_k \subset S_1, \lambda_k \to \Lambda$ $(k \to \infty)$. Since $v_k \in C^1_M$, v_k is positive on $[0, R)$, then

$$v_k = K \circ \phi^{-1} \circ S \circ \left(\lambda_k\left(\frac{Nf'(\varphi^{-1}(v_k))}{\sqrt{1-v_k'^2}} - Nf(\varphi^{-1}(v_k))H(\varphi^{-1}(v_k), r)\right)\right).$$

Using (3.2) and the Arzelà–Ascoli theorem can have $v \in C$ and has a subsequence such that $\{v_k\} \to v$. So, it follows that $v \geq 0$ and

$$v = K \circ \phi^{-1} \circ S \circ \left(\Lambda \left(\frac{Nf'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - Nf(\varphi^{-1}(v))H(\varphi^{-1}(v), r) \right) \right).$$

With (4.2), we can see that there is a constant $c_1 > 0$ such that $v_k(0) > c_1$, $\forall k \in \mathbb{N}$. This ensures that $v(0) \geq c_1$, according to Lemma 3.3, has $v > 0$ on $[0, R)$. Hence, $\Lambda \in S_1$. Obviously, $\Lambda > \frac{2NM_0}{R^3}$.

Next, let $\lambda_0 > \Lambda$, where λ_0 is arbitrary. Here $\lambda_0 \in S_1$ is proved by Theorem 2.2. Let v_1 be a positive solution for (1.3) corresponding to $\lambda = \Lambda$. It is now easy to know that v_1 is a lower solution to problem (1.3) when $\lambda = \lambda_0$. Construct the upper solution, let $H > 0$, $\widetilde{R} > R$, while considering the problem

$$\left(r^{N-1} \frac{v'}{\sqrt{1-v'^2}} \right)' + r^{N-1} H = 0, \quad v'(0) = v(\widetilde{R}) = 0. \tag{4.3}$$

By integrating the above formula, we get

$$v(r) = \frac{N}{H} \left[\sqrt{1 + \frac{H^2}{N^2}\widetilde{R}^2} - \sqrt{1 + \frac{H^2}{N^2} r^2} \right].$$

For fixed $\lambda_2 > \lambda_0$, let v_2 is the solution of problem (4.3) corresponding to $H = \lambda_2 M \widetilde{R}$. By $v_2(R) > 0$ and

$$\lambda_0 \left(\frac{Nf'(\varphi^{-1}(v_2))}{\sqrt{1-v_2'^2}} - Nf(\varphi^{-1}(v_2))H(\varphi^{-1}(v_2), r) \right) \leq \lambda_2 M \widetilde{R}, \quad r \in [0, R].$$

Then we can see that v_2 is an upper solution of problem (1.3) when $\lambda = \lambda_0$, then

$$v_2(R) = N \left[\sqrt{\frac{1}{(\lambda_2 M \widetilde{R})^2} + \frac{\widetilde{R}^2}{N^2}} - \sqrt{\frac{1}{(\lambda_2 M \widetilde{R})^2} + \frac{R^2}{N^2}} \right].$$

Then there is $v_1(0) < v_2(R)$ when $\widetilde{R}$ is sufficiently large. Consider that v_1, v_2 is strictly decreasing, then there is $v_1 < v_2$ on $[0, R]$. Thus, from Theorem 2.2 we know that $\lambda_0 \in S_1$, therefore $S_1 \in [\Lambda, \infty]$.

2. *Multiplicity.*

Let $\lambda_0 > \Lambda$. Let us prove $\lambda_0 \in S_2$ by Lemma 3.1, 3.2, 3.4. Let v_1, v_2 be constructed as above. When $\lambda = \lambda_0$, let v_0 be a solution to problem (1.3) such that $v_1 \leq v_0 \leq v_2$, i.e., $v_0 \in \Omega_{v_1, v_2} := \{v_0 \in C^1_M : v_1 \leq v_0 \leq v_2\}$.

First, we claim that exists $\varepsilon > 0$ with $\overline{B}(v_0, \varepsilon) \subset \Omega_{v_1, v_2}$. For all $r \in [0, R]$, there is

$$\begin{aligned}
v_2(r) &= \int_r^{\widetilde{R}} \phi^{-1} \left(\sigma(t) \int_0^t s^{N-1} \lambda_2 M \widetilde{R} ds \right) dt \\
&> \int_r^R \phi^{-1} \left(\sigma(t) \int_0^t s^{N-1} \lambda_2 \left(\frac{Nf'(\varphi^{-1}(v_2))}{\sqrt{1-v_2'^2}} - Nf(\varphi^{-1}(v_2))H(\varphi^{-1}(v_2), s) \right) ds \right) dt \\
&\geq \int_r^R \phi^{-1} \left(\sigma(t) \int_0^t s^{N-1} \lambda_0 \left(\frac{Nf'(\varphi^{-1}(v_0))}{\sqrt{1-v_0'^2}} - Nf(\varphi^{-1}(v_0))H(\varphi^{-1}(v_0), s) \right) ds \right) dt \\
&= v_0(r).
\end{aligned}$$

Therefore, there exists $\varepsilon_2 > 0$ such that $v \le v_2$ for all $v \in \overline{B}(v_0, \varepsilon_2)$. Similarly on $[0, R/2]$ there is $v_1 < v_0$. Therefore $\varepsilon_1' > 0$ can be found such that

$$v \in C^1_M \text{ and } \|v - v_0\|_\infty \le \varepsilon_1' \Rightarrow v \ge v_1 \text{ on } [0, R/2]. \tag{4.4}$$

On the other hand, we have

$$-v_0' = \phi^{-1} \circ S \circ \lambda_0 \left(\frac{Nf'(\varphi^{-1}(v_0))}{\sqrt{1 - {v_0'}^2}} - Nf(\varphi^{-1}(v_0))H(\varphi^{-1}(v_0), r) \right)$$

and

$$-v_1' = \phi^{-1} \circ S \circ \Lambda \left(\frac{Nf'(\varphi^{-1}(v_1))}{\sqrt{1 - {v_1'}^2}} - Nf(\varphi^{-1}(v_1))H(\varphi^{-1}(v_1), r) \right),$$

yielding $v_0' < v_1'$ on $[R/2, R]$. So we can find a sufficiently small $\varepsilon_1 \in (0, \varepsilon_1')$ such that $v' < v_1'$ on $[R/2, R]$, where $v \in \overline{B}(v_0, \varepsilon_1)$. It follows from $v_0(R) = 0 = v(R)$ that for all $v \in \overline{B}(v_0, \varepsilon_1)$ has $v > v_1$ on [0,R]. Considering (4.4), we claim $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2\})$. Next, if the problem (1.3) has a second solution in Ω_{v_1, v_2}, then the proof of the multiplicity is completed.

If not, using Lemma 3.2 we get

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B(v_0, \rho), 0] = 1 \quad \text{for all } 0 < \rho \le \varepsilon,$$

where $\mathcal{N}_{\lambda_0}$ is the fixed point operator associated to (1.3) with $\lambda = \lambda_0$.

In addition, using Lemma 3.1 we have

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1 \quad \text{for all } \rho \ge (R + 1).$$

From Lemma 3.4 one has

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_\rho, 0] = 1 \quad \text{for all sufficiently small } \rho.$$

When ρ_1, ρ_2 is sufficiently small and $\rho_3 \ge R + 1$ such that $\overline{B}(v_0, \rho_1) \cap \overline{B}_{\rho_2} = \emptyset$ and $\overline{B}(v_0, \rho_1) \cup \overline{B}_{\rho_2} \subset B_{\rho_3}$. Then, from the additivity-excision property of the Leray–Schauder degree it follows that

$$d_{LS}[I - \mathcal{N}_{\lambda_0}, B_{\rho_3} \setminus [\overline{B}(v_0, \rho_1) \cup \overline{B}_{\rho_2}], 0] = -1,$$

which, together with the existence property of the Leray–Schauder degree, imply that $\mathcal{N}_{\lambda_0}$ has a fixed point $\widetilde{v}_0 \in B_{\rho_3} \backslash [\overline{B}(v_0, \rho_1) \bigcup \overline{B}_{\rho_2}]$. We infer that (1.3) has a second positive solution, and the proof is complete. □

Appendix: derivation process of problem (1.3)

To the best of our knowledge, problem (1.3) was first given in [20], but they did not given derivation process. For the convenience of readers and integrity of the paper, here we give the detailed derivation.

Without loss of generality, let us consider the radially symmetric spacelike solutions of the Dirichlet problem with the mean curvature operator in FLRW spacetime

$$\begin{cases} \operatorname{div}\left(\frac{\operatorname{grad} u}{f(u)\sqrt{f^2(u) - |\operatorname{grad} u|^2}} \right) + \frac{f'(u)}{\sqrt{f^2(u) - |\operatorname{grad} u|^2}} \left(N + \frac{|\operatorname{grad} u|^2}{f^2(u)} \right) = NH(u, |x|) & \text{in } B(R), \\ |\operatorname{grad} u| < f(u) & \text{in } B(R), \\ u = 0 & \text{on } \partial B(R), \end{cases} \tag{A.1}$$

where $B(R) = \{x \in \mathbb{R}^N : |x| < R\}$ and $N \geq 1$.

Step 1. If $N = 1$.

Then (A.1) reduces to

$$\begin{cases} \left(\dfrac{u'}{f(u)\sqrt{f^2(u) - u'^2}}\right)' + \dfrac{f'(u)}{\sqrt{f^2(u) - u'^2}}\left(1 + \dfrac{u'^2}{f^2(u)}\right) = H(u, |x|), \quad x \in (0, R), \\ |u'| < f(u), \quad x \in (0, R), \\ u'(0) = u(R) = 0. \end{cases} \tag{A.2}$$

In fact (A.2) can be converted to the following

$$\begin{cases} \left(\dfrac{1}{f(u)} \cdot \dfrac{u'}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}}\right)' + \dfrac{f'(u)(f^2(u) + u'^2)}{f^3(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} = H(u, |x|), \quad x \in (0, R), \\ |u'| < f(u), \quad x \in (0, R), \\ u'(0) = u(R) = 0. \end{cases} \tag{A.3}$$

Let $v(r) = \varphi(u(x))$ and $r = |x|$. Then

$$v'(r) = \varphi'(u)u'(x) = \frac{u'(x)}{f(u(x))}, \quad \left(\varphi(s) = \int_0^s \frac{dt}{f(t)}\right),$$

and accordingly,

$$u(x) = \varphi^{-1}(v(r)), \quad u'(x) = f(u(x))v'(r). \tag{A.4}$$

Since

$$\begin{aligned} &\left(\frac{1}{f(u)} \cdot \frac{u'}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}}\right)' + \frac{f'(u)(f^2(u) + u'^2)}{f^3(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} \\ &= \frac{-f'(u)u'}{f^2(u)} \cdot \frac{u'}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} + \frac{1}{f(u)} \cdot \left(\frac{u'}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}}\right)' \\ &\quad + \frac{f'(u)}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} + \frac{f'(u)u'^2}{f^3(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}} \\ &= \frac{1}{f(u)} \cdot \left(\frac{u'}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}}\right)' + \frac{f'(u)}{f(u)\sqrt{1 - \left(\frac{u'}{f(u)}\right)^2}}. \end{aligned} \tag{A.5}$$

Then, this fact together with (A.4), problem (A.3) can be converted to the following

$$\begin{cases} -\left(\dfrac{v'}{\sqrt{1 - v'^2}}\right)' = \dfrac{f'(\varphi^{-1}(v))}{\sqrt{1 - v'^2}} - f(\varphi^{-1}(v))H(\varphi^{-1}(v), r), \quad r \in (0, R), \\ |v'| < 1, \quad r \in (0, R), \\ v'(0) = v(R) = 0. \end{cases} \tag{A.6}$$

Step 2. If $N \geq 2$.

Given $u(x),\ x = (x_1, \ldots, x_N)$.

Let $v(r) = \varphi(u(x))$ and $r = |x| = \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}}$. Then

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\left(\sum_{i=1}^{N} x_i^2\right)^{-\frac{1}{2}} 2x_i = \frac{x_i}{r}. \tag{A.7}$$

$$\frac{\partial v}{\partial x_i} = v'(r)\frac{\partial r}{\partial x_i} = v'(r)\cdot\frac{x_i}{r} = \varphi'(u)\cdot\frac{\partial u}{\partial x_i} = \frac{1}{f(u)}\cdot\frac{\partial u}{\partial x_i}.$$

Hence

$$\frac{\partial u}{\partial x_i} = f(u)\cdot v'(r)\cdot\frac{x_i}{r}. \tag{A.8}$$

Since

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right),$$

then

$$|\operatorname{grad} u|^2 = \sum_{i=1}^{N}\left(\frac{\partial u}{\partial x_i}\right)^2 = \sum_{i=1}^{N}\left(f(u)\cdot v'(r)\cdot\frac{x_i}{r}\right)^2 = (f(u)v'(r))^2\sum_{i=1}^{N}\left(\frac{x_i}{r}\right)^2 = (f(u)v'(r))^2, \tag{A.9}$$

that is

$$\left(\frac{|\operatorname{grad} u|}{f(u)}\right)^2 = (v'(r))^2,$$

and accordingly, from this and (A.8), we have that

$$\begin{aligned}
&\operatorname{div}\left(\frac{\operatorname{grad} u}{f(u)\sqrt{f^2(u) - |\operatorname{grad} u|^2}}\right) + \frac{f'(u)}{\sqrt{f^2(u) - |\operatorname{grad} u|^2}}\left(N + \frac{|\operatorname{grad} u|^2}{f^2(u)}\right)\\
&= \operatorname{div}\left(\frac{1}{f(u)}\cdot\frac{\operatorname{grad} u}{f(u)\sqrt{1 - \left(\frac{|\operatorname{grad} u|}{f(u)}\right)^2}}\right) + \frac{f'(u)(Nf^2(u) + |\operatorname{grad} u|^2)}{f^3(u)\sqrt{1 - \left(\frac{|\operatorname{grad} u|}{f(u)}\right)^2}}\\
&= \operatorname{div}\left(\frac{1}{f(u)}\cdot\frac{\operatorname{grad} u}{f(u)\sqrt{1-(v'(r))^2}}\right) + \frac{f'(u)(Nf^2(u) + (f(u)v'(r))^2)}{f^3(u)\sqrt{1-(v'(r))^2}}\\
&= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{1}{f(u)}\cdot\frac{1}{f(u)\sqrt{1-(v'(r))^2}}\cdot f(u)\cdot v'(r)\cdot\frac{x_i}{r}\right)\\
&\quad + \frac{f'(u)(Nf^2(u) + (f(u)v'(r))^2)}{f^3(u)\sqrt{1-(v'(r))^2}}\\
&= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{1}{f(u)}\cdot\frac{v'(r)}{\sqrt{1-(v'(r))^2}}\cdot\frac{x_i}{r}\right) + \frac{f'(u)(Nf^2(u) + (f(u)v'(r))^2)}{f^3(u)\sqrt{1-(v'(r))^2}}.
\end{aligned} \tag{A.10}$$

From now on, let us fixed the notation $\phi(s) = \frac{s}{\sqrt{1-s^2}}$.

From (A.7), (A.8), it follows that

$$\begin{aligned}
&\frac{\partial}{\partial x_i}\left(\frac{1}{f(u)}\cdot\frac{v'(r)}{\sqrt{1-(v'(r))^2}}\cdot\frac{x_i}{r}\right)\\
&\quad=\frac{-f'(u)\cdot f(u)\cdot v'(r)\cdot\frac{x_i}{r}}{f^2(u)}\cdot\phi(v'(r))\cdot\frac{x_i}{r}\\
&\qquad+\frac{1}{f(u)}\left[\phi'(v'(r))\cdot\frac{x_i}{r}\cdot\frac{x_i}{r}+\phi(v'(r))\cdot\frac{r-x_i\cdot\frac{x_i}{r}}{r^2}\right]\\
&\quad=\frac{-f'(u)\cdot v'(r)\cdot\left(\frac{x_i}{r}\right)^2}{f(u)}\cdot\phi(v'(r))\\
&\qquad+\frac{1}{f(u)}\phi'(v'(r))\cdot\left(\frac{x_i}{r}\right)^2+\frac{1}{f(u)}\cdot\phi(v'(r))\cdot\frac{r^2-x_i^2}{r^3}.
\end{aligned}\tag{A.11}$$

Hence

$$\begin{aligned}
&\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{1}{f(u)}\cdot\frac{v'(r)}{\sqrt{1-(v'(r))^2}}\cdot\frac{x_i}{r}\right)\\
&\qquad=\frac{-f'(u)\cdot v'(r)}{f(u)}\cdot\phi(v'(r))+\frac{1}{f(u)}\phi'(v'(r))+\frac{1}{f(u)}\cdot\phi(v'(r))\cdot\frac{N-1}{r}.
\end{aligned}\tag{A.12}$$

From this and (A.10), we have that

$$\begin{aligned}
&\operatorname{div}\left(\frac{\operatorname{grad}u}{f(u)\sqrt{f^2(u)-|\operatorname{grad}u|^2}}\right)+\frac{f'(u)}{\sqrt{f^2(u)-|\operatorname{grad}u|^2}}\left(N+\frac{|\operatorname{grad}u|^2}{f^2(u)}\right)\\
&=\frac{-f'(u)\cdot v'(r)}{f(u)}\cdot\phi(v'(r))+\frac{1}{f(u)}\phi'(v'(r))+\frac{1}{f(u)}\cdot\phi(v'(r))\cdot\frac{N-1}{r}\\
&\quad+\frac{Nf'(u)}{f(u)\sqrt{1-(v'(r))^2}}+\frac{f'(u)v'(r)}{f(u)}\cdot\phi(v'(r))\\
&=\frac{1}{f(u)}\phi'(v'(r))+\frac{1}{f(u)}\cdot\phi(v'(r))\cdot\frac{N-1}{r}+\frac{Nf'(u)}{f(u)\sqrt{1-(v'(r))^2}}\\
&=NH(u,r).
\end{aligned}\tag{A.13}$$

Hence, we have

$$\phi'(v'(r))+\frac{N-1}{r}\phi(v'(r))=-\frac{Nf'(u)}{\sqrt{1-(v'(r))^2}}+Nf(u)H(u,r),$$

multiplying both sides of the equation by r^{N-1}, we get that

$$r^{N-1}\phi'(v'(r))+(N-1)r^{N-2}\phi(v'(r))=Nr^{N-1}\left[-\frac{f'(u)}{\sqrt{1-(v'(r))^2}}+f(u)H(u,r)\right],$$

that is

$$-(r^{N-1}\phi(v'(r)))'=Nr^{N-1}\left[\frac{f'(u)}{\sqrt{1-(v'(r))^2}}-f(u)H(u,r)\right].\tag{A.14}$$

From this and the fact

$$u(x)=\varphi^{-1}(v(r)),$$

problem (A.1) can be converted to

$$\begin{cases} -\left(r^{N-1}\phi(v')\right)' = Nr^{N-1}\left[\dfrac{f'(\varphi^{-1}(v))}{\sqrt{1-v'^2}} - f(\varphi^{-1}(v))H(\varphi^{-1}(v),r)\right], \quad r\in(0,R), \\ |v'|<1, \quad r\in(0,R), \\ v'(0)=v(R)=0. \end{cases} \tag{A.15}$$

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Global attractivity of a higher order nonlinear difference equation with decreasing terms

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Abstract. In the present paper, we further study the asymptotical behavior of the following higher order nonlinear difference equation

$$x(n+1) = ax(n) + bf(x(n)) + cf(x(n-k)), \qquad n = 0, 1, \dots$$

where a, b and c are constants with $0 < a < 1, 0 \leq b < 1, 0 \leq c < 1$ and $a + b + c = 1$, $f \in C[[0, \infty), [0, \infty)]$ with $f(x) > 0$ for $x > 0$, and k is a positive integer, which has been recently studied in: On global attractivity of a higher order difference equation and its applications [*Electron. J. Qual. Theory Diff. Equ.* **2022**, No. 2, 1–14 pp]. We obtain some new sufficient conditions for the global attractivity of positive solutions of the equation, and show the applications of these results to some population models..

Keywords: higher order nonlinear difference equation, positive equilibrium, global attractivity, population model.

2020 Mathematics Subject Classification: 39A10, 92D25.

1 Introduction

Consider the following higher order nonlinear difference equation

$$x(n+1) = ax(n) + bf(x(n)) + cf(x(n-k)), \qquad n = 0, 1, \dots, \tag{1.1}$$

where a, b and c are constants with $0 < a < 1, 0 \leq b < 1, 0 \leq c < 1$ and $a + b + c = 1$, $f \in C[[0, \infty), [0, \infty)]$ with $f(x) > 0$ for $x > 0$ and k is a positive integer. The case when the sum of the main coefficients of a higher order difference equation is equal to one is of a great interest and has been studied a lot see, e.g., [1,2,19–23] and the related references therein. One of the reasons is that such difference equations frequently model some processes in nature or society. Recently, asymptotic behavior of positive solutions of Eq. (1.1) has been studied in [1]. Among other results, the following one was presented therein.

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Theorem A. *Assume that $f(x)$ has a unique positive fixed point $\bar{x}$ and satisfies the negative feedback condition*

$$(x-\bar{x})(f(x)-x)<0, \qquad x>0,\ x\neq\bar{x}. \tag{1.2}$$

Suppose also $ax+bf(x)$ is increasing, and $f(x)$ is L-Lipschitz with

$$c\frac{1-a^{k+1}}{c+a^k b}L\leq 1. \tag{1.3}$$

Then every positive solution $\{x(n)\}$ of Eq. (1.1) converges to $\bar{x}$ as $n\to\infty$.

In addition, by using a different approach, a new result on the global attractivity of positive solutions of Eq. (1.1) was obtained in [2] for the special case that f is unimodal, that is, $f(x)=xg(x)$ where $g\in C[[0,\infty),[0,\infty)]$ is decreasing.

In the present paper, we are still interested in the study of global attractivity of positive solutions of Eq. (1.1), but for the case that f is decreasing, and furthermore for the case that f is an S-map, that is, $f:[0,\infty)\to[0,\infty)$ is three times differentiable with $(Sf)(x)<0$ and $f'(x)<0$ for $x>0$ where S is the Schwarzian derivative

$$(Sf)(x)=\frac{f'''(x)}{f'(x)}-\frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Clearly, if we let

$$x(-k),x(-k+1),\dots,x(0) \tag{1.4}$$

be $k+1$ given nonnegative numbers with $x(0)>0$, then Eq. (1.1) has a unique positive solution with initial condition (1.4).

In the next section, we establish two sufficient conditions on the global attractivity of positive solutions of Eq. (1.1) under the conditions that f is a decreasing function and f is an S-map, respectively. Our results can be applied to several difference equations derived from mathematical biology. We show these applications in Section 3.

In the following discussion,we always assume that f is decreasing. In addition, for the sake of convenience, we adopt the notation $\prod_{i=m}^{n}s(i)=1$ and $\sum_{i=m}^{n}s(i)=0$ whenever $\{s(n)\}$ is a real sequence and $m>n$.

2 Main results

Since f is decreasing, f has a unique positive fixed point $\bar{x}$ and satisfies the negative feedback condition (1.2). Hence by Lemma 2.1 in [1], every positive solution $\{x(n)\}$ of Eq. (1.1) is bounded and persistent.

In the following, we establish two sufficient conditions for every positive solution of Eq. (1.1) to converge to $\bar{x}$ as $n\to\infty$. By an argument similar to that in the proof of Theorem 2.2 in [1], we know that every nonoscillatory solution of Eq. (1.1) converges to $\bar{x}$. Hence we need to obtain conditions for every oscillatory solution of Eq. (1.1) to converge to $\bar{x}$ also.

The following lemma on the asymptotic behavior of oscillatory solutions of Eq. (1.1) is needed in the proof of our main results.

Lemma 2.1. *Assume that $ax+bf(x)$ is increasing and let $\{x(n)\}$ be a positive solution of Eq. (1.1) which oscillates about $\bar{x}$. Then for any nonnegative integer $m\geq 0$, there is a positive integer N_m such that*

$$u(2m)\leq x(n)\leq u(2m+1) \quad \textit{for } n\geq N_m \tag{2.1}$$

where $\{u(n)\}$ is defined by

$$\begin{cases} u(n) = c\frac{1-a^{k+1}}{c+a^k b} f(u(n-1)) + \frac{a^k(b+ac)}{c+a^k b}\bar{x}, \ n = 1,2,\dots, \\ u(0) = \frac{a^k(b+ac)}{c+a^k b}\bar{x}. \end{cases} \tag{2.2}$$

Proof. Let $y(n) = x(n) - \bar{x}$. Then $\{y(n)\}$ satisfies the equation

$$y(n+1) = ay(n) + b(f(y(n)+\bar{x}) - \bar{x}) + c(f(y(n-k)+\bar{x}) - \bar{x}) \tag{2.3}$$

and $\{y(n)\}$ oscillates about zero.

Let $y(i)$ and $y(j)$ be two consecutive members of the solution $\{y(n)\}$ such that

$$y(i) \geq 0, y(j+1) \geq 0 \quad \text{and} \quad y(n) < 0 \quad \text{for } i+1 \leq n \leq j. \tag{2.4}$$

and let

$$y(r) = \min\{y(i+1), y(i+2), \dots, y(j)\}.$$

Then by an argument similar to that in the proof of Theorem 2.2 in [1] (the increasing property of $ax + bf(x)$ is needed in the proof) we may show that

$$r - (i+1) \leq k \tag{2.5}$$

and

$$y(r) \geq \frac{1-a}{c+a^k b} a^r \sum_{n=i}^{r-1} \frac{c}{a^{n+1}} [f(y(n-k)+\bar{x}) - f(\bar{x})]. \tag{2.6}$$

Noting $f(y(n-k)+\bar{x}) \geq 0, f(\bar{x}) = \bar{x}$ and (2.5), we see that

$$y(r) \geq -\bar{x}\frac{1-a}{c+a^k b} a^r \sum_{n=i}^{r-1} \frac{c}{a^{n+1}} = -\bar{x}\frac{1-a}{c+a^k b} c \left(\frac{1-a^{r-i}}{1-a}\right) \geq -c\bar{x}\frac{1-a^{k+1}}{c+a^k b}$$

and so it follows that

$$y(n) \geq -c\bar{x}\frac{1-a^{k+1}}{c+a^k b}, \qquad i \leq n \leq j.$$

Since $y(i)$ and $y(j)$ are two arbitrary members of the solution with property (2.4), we see that there is a positive integer N_0' such that

$$y(n) \geq -c\bar{x}\frac{1-a^{k+1}}{c+a^k b} \stackrel{\text{def}}{=} z(0), \qquad n \geq N_0'. \tag{2.7}$$

Next, let $y(i)$ and $y(j)$ be two consecutive members of the solution $\{y(n)\}$ with $N_0' + k \leq i < j$ such that

$$y(i) \leq 0, \ y(j+1) \leq 0 \quad \text{and} \quad y(n) > 0 \quad \text{for } i+1 \leq n \leq j \tag{2.8}$$

and

$$y(t) = \max\{y(i+1), y(i+2), \dots, y(j)\}.$$

Then by a similar argument, we may show that

$$t - (i+1) \leq k \tag{2.9}$$

and

$$y(t) \le \frac{1-a}{c+a^k b} a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}} [f(y(n-k)+\bar{x}) - f(\bar{x})]. \tag{2.10}$$

Since

$$z(0)+\bar{x} = \left(1 - c\frac{1-a^{k+1}}{c+a^k b}\right)\bar{x} = \frac{a^k(b+ac)}{c+a^k b}\bar{x} > 0,$$

$f(z(0)+\bar{x})$ is well-defined. Since $z(0) < 0$ (see (2.7)) and f is decreasing, we see that

$$f(y(n-k)+\bar{x}) \le f(z(0)+\bar{x}) \quad \text{for } n \ge N_0' + k.$$

Hence, it follows from (2.9) and (2.10) that

$$y(t) \le \frac{1-a}{c+a^k b} a^t \sum_{n=i}^{t-1} \frac{c}{a^{n+1}} [f(z(0)+\bar{x}) - f(\bar{x})] \le c\frac{1-a^{k+1}}{c+a^k b}[f(z(0)+\bar{x}) - \bar{x}],$$

which yields

$$y(n) \le c\frac{1-a^{k+1}}{c+a^k b}[f(z(0)+\bar{x}) - \bar{x}], \qquad i \le n \le j.$$

Since $y(i)$ and $y(j)$ are two arbitrary members of the solution with property (2.8), we see that there is a positive integer $N_0 > N_0'$ such that

$$y(n) \le c\frac{1-a^{k+1}}{c+a^k b}[f(z(0)+\bar{x}) - \bar{x}] \stackrel{\text{def}}{=} z(1), \qquad n \ge N_0.$$

Then, by an easy induction, we see that for each $m \ge 0$, there is a positive integer N_m such that

$$z(2m) \le y(n) \le z(2m+1) \quad \text{for } n \ge N_m, \tag{2.11}$$

where $\{z(n)\}$ is defined by

$$\begin{cases} z(n) = c\frac{1-a^{k+1}}{c+a^k b}[f(z(n-1)+\bar{x}) - \bar{x}], & n = 1,2,\dots, \\ z(0) = -c\bar{x}\frac{1-a^{k+1}}{c+a^k b}. \end{cases} \tag{2.12}$$

Let $u(n) = z(n) + \bar{x}, n = 0,1,\dots$ Then (2.11) and (2.12) become (2.1) and (2.2), respectively. The proof is complete. □

Theorem 2.2. *Assume that $ax + bf(x)$ is increasing and*

$$\frac{c(1-a^{k+1})}{a^k(b+ac)\bar{x}}(xf(x))' > -1, \qquad x > 0. \tag{2.13}$$

Then every positive solution $\{x(n)\}$ of Eq. (1.1) tends to its positive equilibrium $\bar{x}$ as $n \to \infty$.

Proof. As indicated at the beginning of the section, every nonoscillatory solution of Eq. (1.1) converges to $\bar{x}$. Hence we only need to show that every oscillatory solution converges to $\bar{x}$ also. To this end, let $\{x(n)\}$ be an oscillatory solution of Eq. (1.1). Then by Lemma 2.1, $\{x(n)\}$ satisfies (2.1). Since $u(0) \le u(1)$, from (2.2) and the monotonicity of f it is not difficult to see that $\{u(2m)\}$ is increasing, $\{u(2m+1)\}$ is decreasing and $u(2m) \le \bar{x} \le u(2m+1)$, $m = 0,1,\dots$ Hence,

$$\lim_{m\to\infty} u(2m) = l \le \bar{x} \quad \text{and} \quad \lim_{m\to\infty} u(2m+1) = L \ge \bar{x}$$

exist, and l and L satisfy the equations

$$\begin{cases} l = c\frac{1-a^{k+1}}{c+a^k b} f(L) + \frac{a^k(b+ac)}{c+a^k b}\bar{x} \\ L = c\frac{1-a^{k+1}}{c+a^k b} f(l) + \frac{a^k(b+ac)}{c+a^k b}\bar{x}. \end{cases} \tag{2.14}$$

We now show that $l = L = \bar{x}$. To this end, let

$$g(x) = c\frac{1-a^{k+1}}{c+a^k b} x f(x) + \frac{a^k(b+ac)}{c+a^k b}\bar{x}x, \qquad x > 0$$

and observe that

$$g'(x) = c\frac{1-a^{k+1}}{c+a^k b}(xf(x))' + \frac{a^k(b+ac)}{c+a^k b}\bar{x}, \qquad x > 0.$$

In view of (2.13), we see that $g'(x) > 0$. However, it follows from (2.14) that $g(l) = g(L) = lL$. Hence $l = L = \bar{x}$ and so $\lim_{n\to\infty} u(n) = \bar{x}$. Then from (2.1) we see that $x(n) \to \bar{x}$ as $n \to \infty$. The proof is complete. □

For the proof of the next theorem, we need the following lemma which is extracted from [12].

Lemma 2.3. *Consider the following difference equation*

$$x(n+1) = h(x(n)), \qquad n = 0, 1, \ldots \tag{2.15}$$

where $h : [0,\infty) \to [0,\infty)$ is an S-map. Assume that $\bar{x}$ is the unique fixed point of h and $|h'(\bar{x})| \leq 1$. Then $\bar{x}$ is a global attractor of all solutions of Eq. (2.15).

Theorem 2.4. *Assume that $ax + bf(x)$ is increasing and f is an S-map with*

$$c\frac{1-a^{k+1}}{c+a^k b} f'(\bar{x}) \geq -1. \tag{2.16}$$

Then every positive solution $\{x(n)\}$ of Eq. (1.1) tends to its positive equilibrium $\bar{x}$ as $n \to \infty$.

Proof. We only need to show that every oscillatory solution of Eq. (1.1) converges to $\bar{x}$. Let $\{x(n)\}$ be an oscillatory solution. Then $\{x(n)\}$ satisfies (2.1). Hence, to show that $x(n) \to \bar{x}$ as $n \to \infty$ it suffices to show that $u(n) \to \bar{x}$ as $n \to \infty$. To this end, let

$$h(x) = c\frac{1-a^{k+1}}{c+a^k b} f(x) + \frac{a^k(b+ac)}{c+a^k b}\bar{x}.$$

Clearly, $h : [0,\infty) \to [0,\infty)$, $\bar{x}$ is the unique fixed point of h, $h'(x) = c\frac{1-a^{k+1}}{c+a^k b} f'(x) < 0$ and $(Sh)(x) = (Sf)(x) < 0$ for $x > 0$. Hence, h is an S-map. In addition, (2.16) yields $|h'(\bar{x})| \leq 1$. Therefore, all the conditions assumed in Lemma 2.3 are satisfied and so $u(n) \to \bar{x}$ as $n \to \infty$. Then it follows that $x(n) \to \bar{x}$ as $n \to \infty$. The proof is complete. □

Remark 2.5. By comparing Theorems 2.2 and 2.4 with Theorem A, we see that when f is a decreasing function, the condition (2.13) is different from the condition (1.3); while when f is an S-map, the condition (2.16) is better than the condition (1.3).

3 Applications

In this section, we apply our results obtained in the last section to some difference equations derived from mathematical biology.

Consider the following system of difference equations

$$\begin{cases} x(n+1) = (1-\epsilon)f(x(n)) + \epsilon y(n), \\ y(n+1) = (1-\epsilon)y(n) + \epsilon f(x(n)), \qquad n = 0,1,\ldots, \\ x(0) \geq 0,\ y(0) \geq 0,\ x(0)+y(0) > 0, \end{cases} \tag{3.1}$$

where $0 < \epsilon < 1$ is a positive constant and $f \in C[[0,\infty),[0,\infty)]$ with $f(x) > 0$ for $x > 0$. Sys. (3.1) is a population model proposed by Newman et al. [18] which assumes symmetric dispersal between active population $x(n)$ and refuge population $y(n)$. The chaotic behavior of positive solutions of Sys. (3.1) is studied in [18] by numerical simulations, whereas in [3] various properties of solutions of (3.1) are studied and several results on the asymptotic behavior of solutions of (3.1) are obtained. Recently, a sufficient condition on the global stability of positive solutions of (3.1) is obtained in [1].

Notice that Sys. (3.1) can be converted into the second order difference equation

$$x(n+1) = (1-\epsilon)x(n) + (1-\epsilon)f(x(n)) + (2\epsilon-1)f(x(n-1)), \qquad n = 0,1,\ldots \tag{3.2}$$

When f is decreasing and $\epsilon \geq 1/2$, Eq. (3.2) is in the form of (1.1) and f has a unique positive fixed point $\bar{x}$. Clearly, $\bar{x}$ is the unique positive equilibrium of Eq. (3.2) and $(\bar{x},\bar{x})$ is the unique positive equilibrium of Sys. (3.1).

By Theorems 2.2 and 2.4, we may have the following result on the global attractivity of positive solutions of Sys. (3.1).

Corollary 3.1. *Assume that $1/2 \leq \epsilon < 1$, f is decreasing and $x + f(x)$ is increasing. Suppose also that either $xf(x)$ is differentiable with*

$$\frac{(2\epsilon-1)(2-\epsilon)}{2(1-\epsilon)^2\bar{x}}(xf(x))' > -1, \qquad x > 0 \tag{3.3}$$

or f is an S-map with

$$(2-\epsilon)(2-1/\epsilon)f'(\bar{x}) \geq -1. \tag{3.4}$$

Then every positive solution $(x(n),y(n))$ of Sys. (3.1) tends to its positive equilibrium $(\bar{x},\bar{x})$ as $n \to \infty$.

Proof. As indicated above, Sys. (3.1) can be converted into (3.2) which is in the form of Eq. (1.1) with $a = b = 1-\epsilon, c = 2\epsilon - 1$ and $k = 1$. By the assumption, $ax + bf(x) = (1-\epsilon)(x + f(x))$ is increasing. In addition, noting

$$\frac{c(1-a^{k+1})}{a^k(b+ac)} = \frac{(2\epsilon-1)(2-\epsilon)}{2(1-\epsilon)^2} \tag{3.5}$$

and

$$c\frac{1-a^{k+1}}{c+a^kb} = (2-\epsilon)(2-1/\epsilon) \tag{3.6}$$

we see that when (3.5) or (3.6) holds, (2.13) or (2.16) holds respectively. Then by Theorems 2.2 and 2.4, every positive solution $\{x(n)\}$ of Eq. (3.2) converges to $\bar{x}$ as $n \to \infty$. Then from (3.1) we see that

$$\epsilon y(n) = x(n+1) - (1-\epsilon)f(x(n)) \to \bar{x} - (1-\epsilon)f(\bar{x}) \quad \text{as } n \to \infty,$$

which yields

$$y(n) \to \bar{x} \quad \text{as } n \to \infty.$$

Hence, it follows that every positive solution $(x(n), y(n))$ of Sys. (3.1) converges to $(\bar{x}, \bar{x})$. The proof is complete. □

Next, consider the following difference equation in the form

$$x(n+1) = \alpha x(n) + \beta g(x(n)) + \gamma g(x(n-k)), \qquad n = 0, 1, \dots, \tag{3.7}$$

where $0 < \alpha < 1, \beta \geq 0$ and $\gamma \geq 0$ with $\beta + \gamma > 0$ are constants, $g \in C[[0,\infty),[0,\infty)]$ and k is a positive integer, observe that it can be written as

$$x(n+1) = \alpha x(n) + \frac{\beta(1-\alpha)}{\beta+\gamma}\left[\frac{\beta+\gamma}{1-\alpha}g(x(n))\right] + \frac{\gamma(1-\alpha)}{\beta+\gamma}\left[\frac{\beta+\gamma}{1-\alpha}g(x(n-k))\right], \tag{3.8}$$

which is in the form of (1.1) with

$$a = \alpha, \qquad b = \frac{\beta(1-\alpha)}{\beta+\gamma}, \qquad c = \frac{\gamma(1-\alpha)}{\beta+\gamma} \quad \text{and} \quad f(x) = \frac{\beta+\gamma}{1-\alpha}g(x).$$

Assume that $\bar{x}$ is the unique positive fixed point of $f(x)$, that is, $\bar{x}$ is the only positive number satisfying

$$g(\bar{x}) = \frac{1-\alpha}{\beta+\gamma}\bar{x}.$$

Clearly $\bar{x}$ is the unique positive equilibrium of Eq. (3.7). Observing that

$$\frac{c(1-a^{k+1})}{a^k(b+ac)\bar{x}} = \frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}$$

and

$$c\frac{1-a^{k+1}}{c+a^k b}f'(\bar{x}) = \frac{\gamma(1-\alpha^{k+1})(\beta+\gamma)}{(1-\alpha)(\gamma+\alpha^k\beta)}g'(\bar{x}),$$

we see that the following corollary on the global attractivity of $\bar{x}$ is a direct consequence of Theorems 2.2 and 2.4.

Corollary 3.2. *Assume that g is decreasing and $\alpha x + \beta g(x)$ is increasing. Let $\bar{x}$ be the unique positive equilibrium of Eq. (3.7) and suppose that either $xg(x)$ is differentiable with*

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}(xg(x))' > -1 \tag{3.9}$$

or g is an S-map with

$$\frac{\gamma(1-\alpha^{k+1})(\beta+\gamma)}{(1-\alpha)(\gamma+\alpha^k\beta)}g'(\bar{x}) \geq -1. \tag{3.10}$$

Then every positive solution of Eq. (3.7) tends to $\bar{x}$ as $n \to \infty$.

When $\gamma = 0$, Eq. (3.7) reduces to

$$x(n+1) = \alpha x(n) + \beta g(x(n)), \qquad n = 0, 1, \dots \tag{3.11}$$

Clearly, (3.9) is automatically satisfied since the left side is 0. From Corollary 3.2 we know that when g is decreasing and $\alpha x + \beta g(x)$ is increasing, every positive solution $\{x(n)\}$ of Eq. (3.11) tends to its positive equilibrium $\bar{x}$ as $n \to \infty$ where $\bar{x}$ is the unique positive number satisfying $\bar{x} = \frac{\beta}{1-\alpha} g(\bar{x})$.

When $\beta = 0$, Eq. (3.7) reduces to

$$x(n+1) = \alpha x(n) + \gamma g(x(n-k)), \qquad n = 0, 1, \dots, \tag{3.12}$$

which includes several discrete models derived from mathematical biology. For instance, when $g(x) = \frac{1}{1+x^p}$ where p is a positive constant, Eq. (3.12) is a discrete analogue of a model that has been used to study blood cells production [13]; when $g(x) = e^{-qx}$ where q is a positive constant, Eq. (3.12) is a discrete version of a model of the survival of red blood cells in an animal [25]. Due to its theoretical interest and applications, asymptotic behavior of positive solutions of Eq. (3.12) and some related forms have been studied by numerous authors, see, for example, [1, 2, 4–11, 13–25] and the references cited therein. As a special case of Eq. (3.7), our results can be applied to Eq. (3.12) also.

In the following, we discuss the global attractivity of positive solutions of Eq. (3.7) when $g(x) = \frac{1}{1+x^p}$ and $g(x) = e^{-qx}$ where p and q are positive constants, respectively. When $g(x) = \frac{1}{1+x^p}$, Eq. (3.7) becomes

$$x(n+1) = \alpha x(n) + \frac{\beta}{1+x^p} + \frac{\gamma}{1+x^p(n-k)}, \qquad n = 0, 1, \dots \tag{3.13}$$

Clearly, g is decreasing and has a unique positive number $\bar{x}$ satisfying $g(\bar{x}) = \frac{1-\alpha}{\beta+\gamma}\bar{x}$ which is the only positive equilibrium of Eq. (3.13). When $\beta = 0$, $\alpha x + \beta g(x) = \alpha x$ is increasing; when $\beta > 0$ and $p \geq 1$, noting

$$g'(x) = \frac{-px^{p-1}}{(1+x^p)^2}$$

and

$$g''(x) = \frac{-px^{p-2}((p-1)-(p+1)x^p)}{(1+x^p)^3}$$

we see that $g'(x)$ takes minimum at $x^* = (\frac{p-1}{p+1})^{1/p}$ and

$$g'(x^*) = -\frac{1}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p}.$$

Hence, if $p \geq 1$ and

$$\frac{\beta}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p} \leq \alpha, \tag{3.14}$$

then

$$(\alpha x + \beta g(x))' \geq \alpha + \beta g'(x^*) = \alpha - \frac{\beta}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p} \geq 0$$

and so $\alpha x + \beta g(x)$ is increasing.

Next, observe that

$$(xg(x))' = \frac{1+(1-p)x^p}{(1+x^p)^2}$$

and

$$(xg(x))'' = \frac{px^{p-1}((p-1)x^p-(p+1))}{(1+x^p)^3}.$$

We see that when $p \le 1$, $(xg(x))' > 0$ and so (3.9) is true; when $p > 1$, $(xg(x))'$ has minimum at $x^* = \left(\frac{p+1}{p-1}\right)^{1/p}$. Hence in this case,

$$(xg(x))' \ge (xg(x))'|_{x=x^*} = -\frac{(p-1)^2}{4p}. \tag{3.15}$$

Clearly, if

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}\left(-\frac{(p-1)^2}{4p}\right) > -1,$$

that is,

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}\frac{(p-1)^2}{4p} < 1, \tag{3.16}$$

then by noting (3.15) we know that (3.9) is satisfied. Furthermore, by a simple calculation, we find that for $p > 1$,

$$(Sg)(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2 = \frac{1}{2}(1-p)(1+p)x^{-2} < 0, \qquad x > 0,$$

that is, g is an S-map. In addition, by noting

$$g'(\bar{x}) = -\frac{p\bar{x}^{p-1}}{(1+\bar{x}^p)^2} = -p\bar{x}^{p-1}g^2(\bar{x}) = -p\bar{x}^{p-1}\left(\frac{1-\alpha}{\beta+\gamma}\bar{x}\right)^2 = -p\left(\frac{1-\alpha}{\beta+\gamma}\right)^2\bar{x}^{p+1}$$

we see that if

$$\frac{\gamma(1-\alpha^{k+1})(\beta+\gamma)}{(1-\alpha)(\gamma+\alpha^k\beta)}\left(-p\left(\frac{1-\alpha}{\beta+\gamma}\right)^2\bar{x}^{p+1}\right) \ge -1,$$

that is,

$$\frac{\gamma(1-\alpha)(1-\alpha^{k+1})}{(\beta+\gamma)(\gamma+\alpha^k\beta)}p\bar{x}^{p+1} \le 1, \tag{3.17}$$

then (3.10) is satisfied. Hence, by Corollary 3.2, we have the following conclusion: every positive solution of Eq. (3.13) tends to its positive equilibrium $\bar{x}$ as $n \to \infty$ if one of the following holds

(i) $p \le 1$ and $\beta = 0$;

(ii) $p \ge 1$, (3.14) and (3.16) hold;

(iii) $p > 1$, (3.14) and (3.17) hold.

When $f(x) = \frac{1}{1+x^p}$, Sys. (3.1) becomes

$$\begin{cases} x(n+1) = \frac{1-\epsilon}{1+x^p(n)} + \epsilon y(n), \\ y(n+1) = (1-\epsilon)y(n) + \frac{\epsilon}{1+x^p(n)}, \qquad n = 0,1,\ldots, \\ x(0) \ge 0,\ y(0) \ge 0,\ x(0)+y(0) > 0, \end{cases} \tag{3.18}$$

and it can be converted into Eq. (3.13) with $\alpha = \beta = 1-\epsilon, \gamma = 2\epsilon - 1$ and $k = 1$. Since $\alpha = \beta$, (3.14) reduces to

$$\frac{1}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p} \leq 1. \tag{3.19}$$

Hence, when $p \geq 1$ and (3.19) holds, $x + f(x)$ is increasing. Note that $f(x) = g(x)$. From (3.15), (3.17) and the above discussion, we know that when $p \geq 1$,

$$(xf(x))' \geq -(p-1)^2/(4p) \tag{3.20}$$

and

$$f(\bar{x}) = -p\left(\frac{1-\alpha}{\beta+\gamma}\right)^2 \bar{x}^{p+1} = -p\bar{x}^{p+1}. \tag{3.21}$$

Clearly, (3.20) implies that if

$$\frac{(2\epsilon-1)(2-\epsilon)}{2(1-\epsilon)^2\bar{x}}\left(-\frac{(p-1)^2}{4p}\right) > -1,$$

that is,

$$\frac{(2\epsilon-1)(2-\epsilon)}{2(1-\epsilon)^2\bar{x}}\frac{(p-1)^2}{4p} < 1, \tag{3.22}$$

then (3.5) is satisfied, and (3.21) implies that if

$$(2-\epsilon)(2-1/\epsilon)(-p\bar{x}^{p+1}) \geq -1,$$

that is,

$$(2-\epsilon)(2-1/\epsilon)p\bar{x}^{p+1} \leq 1, \tag{3.23}$$

then (3.6) is satisfied. In addition, from the above discussion, we know that when $p > 1$, f is an S-map. Hence, by Corollary 3.1, we have the following conclusion: when $1/2 \leq \epsilon < 1$, every positive solution of Sys. (3.18) converges to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$ if either $p \geq 1$, (3.19) and (3.22) hold, or $p > 1$, (3.19) and (3.23) hold.

Example 3.3. Consider the equation

$$x(n+1) = (1/2)x(n) + (3/4)\frac{1}{1+x^2(n)} + (1/4)\frac{1}{1+x^2(n-3)}, \qquad n = 0, 1, \ldots, \tag{3.24}$$

which is in the form of Eq. (3.7) with $\alpha = 1/2, \beta = 3/4, \gamma = 1/4, k = 3$ and $g(x) = 1/(1+x^2)$. Note that $\bar{x} = 1$ is the unique positive equilibrium of Eq. (3.24). Since $p = 2$,

$$\frac{\beta}{4p}(p-1)^{1-1/p}(1+p)^{1+1/p} = (3/4)(1/8)3^{3/2} < 1/2 = \alpha,$$

that is, (3.14) is satisfied. In addition, observing that

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}\frac{(p-1)^2}{4p} = \frac{(1/4)(1-(1/2)^4)}{(1/2)^3((3/4)+(1/2)(1/4))}\cdot\frac{1}{8} = \frac{15}{56} < 1$$

we see that (3.16) is satisfied. Hence, from the above discussion, we know that every positive solution of Eq. (3.24) tends to its positive equilibrium $\bar{x} = 1$ as $n \to \infty$.

Example 3.4. Consider the system

$$\begin{cases} x(n+1) = \frac{7/15}{1+x^3(n)} + (8/15)y(n), \\ y(n+1) = (7/15)y(n) + \frac{8/15}{1+x^3(n)}, & n = 0,1,\dots, \\ x(0) \geq 0,\ y(0) \geq 0,\ x(0)+y(0) > 0, \end{cases} \tag{3.25}$$

which is in the form of Sys. (3.18) with $\epsilon = 8/15$ and $f(x) = 1/(1+x^3)$. Since $p = 3$,

$$\frac{1}{4p}(p-1)^{1-1/p}(p+1)^{1+1/p} = (1/12)2^{2/3}4^{4/3} < 1,$$

that is, (3.19) is satisfied. In addition, we know that f is an S-map. Sys. (3.25) has the unique positive equilibrium $(\bar{x}, \bar{x})$ where $\bar{x}$ is the unique positive fixed point of f. Observing $\bar{x}(1+\bar{x}^3) = 1$, we see that $\bar{x} < 1$. Then it follows that

$$(2-\epsilon)(2-1/\epsilon)p\bar{x}^{p+1} \leq (2-8/15)(2-15/8)3 = 11/20 < 1$$

and so (3.22) is satisfied. Hence, from the above discussion, we know that every positive solution of Sys. (3.25) tends to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$.

When $g(x) = e^{-qx}$, Eq. (3.7) becomes

$$x(n+1) = \alpha x(n) + \beta e^{-qx(n)} + \gamma e^{-qx(n-k)}, \qquad n = 0,1,\dots. \tag{3.26}$$

Since g is decreasing, there is a unique positive number $\bar{x}$ satisfying $g(\bar{x}) = \frac{1-\alpha}{\beta+\gamma}\bar{x}$. Clearly $\bar{x}$ is the only positive equilibrium of Eq. (3.26). Noting

$$(\alpha x + \beta g(x))' = (\alpha x + \beta e^{-qx})' = \alpha - q\beta e^{-qx}$$

we see that $\alpha x + \beta e^{-qx}$ is increasing when

$$\alpha \geq q\beta. \tag{3.27}$$

In addition, observing that

$$(xg(x))' = (1-qx)e^{-qx} \text{ and } (xg(x))'' = q(qx-2)e^{-qx}$$

we find that $(xg(x))'$ takes minimum when $x = 2/q$ and so

$$(xg(x))' \geq (xg(x))'|_{x=q/2} = -e^{-2}. \tag{3.28}$$

Hence, if

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}}(-e^{-2}) > -1,$$

that is,

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}} < e^2, \tag{3.29}$$

then (3.9) is satisfied. Furthermore, by a simple calculation, we find that

$$(Sg)(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2 = -(1/2)q^2 < 0, \qquad x > 0,$$

that is, g is an S-map. In addition, by noting

$$g'(\bar{x}) = -qe^{-q\bar{x}} = -q\frac{1-\alpha}{\beta+\gamma}\bar{x} \tag{3.30}$$

we see that if

$$\frac{\gamma(1-\alpha^{k+1})(\beta+\gamma)}{(1-\alpha)(\gamma+\alpha^k\beta)}\left(-q\frac{1-\alpha}{\beta+\gamma}\bar{x}\right) \geq -1$$

that is,

$$\frac{\gamma(1-\alpha^{k+1})}{\gamma+\alpha^k\beta}q\bar{x} \leq 1, \tag{3.31}$$

then (3.10) is satisfied. Hence, by Corollary 3.2, we have the following conclusion: if (3.27) holds and either (3.29) or (3.31) holds also, then every positive solution of Eq. (3.26) tends to its positive equilibrium as $n \to \infty$.

When $f(x) = e^{-qx}$, Sys. (3.1) is

$$\begin{cases} x(n+1) = (1-\epsilon)e^{-qx(n)} + \epsilon y(n), \\ y(n+1) = (1-\epsilon)y(n) + \epsilon e^{-qx(n)}, \qquad n = 0, 1, \ldots, \\ x(0) \geq 0,\ y(0) \geq 0,\ x(0)+y(0) > 0, \end{cases} \tag{3.32}$$

and it can be converted into Eq. (3.26) with $\alpha = \beta = 1-\epsilon, \gamma = 2\epsilon - 1$ and $k = 1$. Since $\alpha = \beta$, (3.27) reduces to $q \leq 1$. Noting $f(x) = g(x)$, from (3.28), (3.30) and the above discussion, we know that

$$(xf(x))' \geq -e^{-2} \tag{3.33}$$

and

$$f'(\bar{x}) = -qe^{-q\bar{x}} = -q\frac{1-\alpha}{\beta+\gamma}\bar{x} = -q\bar{x}. \tag{3.34}$$

Clearly, (3.33) implies that if

$$\frac{(2\epsilon-1)(2-\epsilon)}{2(1-\epsilon)^2\bar{x}}(-e^{-2}) > -1,$$

that is,

$$\frac{(2\epsilon-1)(2-\epsilon)}{2(1-\epsilon)^2\bar{x}} < e^2, \tag{3.35}$$

then (3.5) is satisfied, and if

$$(2-\epsilon)(2-1/\epsilon)(-q\bar{x}) \geq -1,$$

that is,

$$(2-\epsilon)(2-1/\epsilon)q\bar{x} \leq 1, \tag{3.36}$$

then (3.6) is satisfied. Hence, by Corollary 3.1, we have the following conclusion on the global attractivity of positive solutions of Sys. (3.32): if $q \leq 1$ and either (3.35) or (3.36) holds, then every positive solution of Sys. (3.32) tends to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$.

Example 3.5. Consider the equation

$$x(n+1) = (2/3)x(n) + (1/3)e^{-2x(n)} + (1/4)e^{-2x(n-3)}, \qquad n = 0, 1, \ldots \tag{3.37}$$

which is in the form of Eq. (3.26) with $\alpha = 2/3, \beta = 1/3, \gamma = 1/4, k = 3$ and $g(x) = e^{-2x}$. Noting $q = 2$, we see that (3.27) is satisfied. Let $\bar{x}$ be the unique positive equilibrium of Eq. (3.37). Then $\bar{x}$ satisfies $\bar{x}e^{2\bar{x}} = 7/4$. By noting $(1/2)e^{2(1/2)} < 7/4$, we see that $\bar{x} > 1/2$ and so it follows that

$$\frac{\gamma(1-\alpha^{k+1})}{\alpha^k(\beta+\alpha\gamma)\bar{x}} < \frac{(1/4)(1-(2/3)^4)}{(2/3)^3((1/3)+(2/3)(1/4))(1/2)} = \frac{65}{24} < e^2,$$

that is, (3.29) holds. Hence, from the above discussion, we know that every positive solution of Eq. (3.37) converges to its positive equilibrium $\bar{x}$ as $n \to \infty$.

Example 3.6. Consider the system

$$\begin{cases} x(n+1) = (2/5)e^{-(1/2)x(n)} + (3/5)y(n), \\ y(n+1) = (2/5)y(n) + (3/5)e^{-(1/2)x(n)}, \qquad n = 0, 1, \dots, \\ x(0) \geq 0,\ y(0) \geq 0,\ x(0) + y(0) > 0, \end{cases} \tag{3.38}$$

which is in the form of Sys. (3.32) with $\epsilon = 3/5$ and $f(x) = e^{-(1/2)x}$. Note that $q = 1/2 < 1$. Sys. (3.38) has the unique positive equilibrium $(\bar{x}, \bar{x})$ where $\bar{x}$ is the unique positive fixed point of f. Noting $\bar{x}e^{(1/2)\bar{x}} = 1$, we see that $\bar{x} < 1$. Then it follows that

$$(2-\epsilon)(2-1/\epsilon)q\bar{x} \leq (2-3/5)(2-5/3)(1/2) = 7/30 < 1$$

and so (3.36) is satisfied. Hence, from the above discussion, we know that every positive solution of Sys. (3.38) tends to its positive equilibrium $(\bar{x}, \bar{x})$ as $n \to \infty$.

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Existence of solutions for asymptotically periodic quasilinear Schrödinger equations with local nonlinearities

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Abstract. This paper is concerned with the existence of positive solutions for asymptotically periodic quasilinear Schrödinger equations. By using a Nehari-type constraint and Moser iteration, we get the existence results which is a complement to the ones in Chu and Liu [*Nonlinear Anal. Real World Appl.* **44**(2018), 118–127]. Moreover, we consider a new reformative asymptotic processes of the potential function and the nonlinearity term is only locally defined.

Keywords: quasilinear Schrödinger equation, L^∞-estimate, asymptotically periodic, Nehari manifold.

2020 Mathematics Subject Classification: 35A15, 35D30, 35B09.

1 Introduction and main results

We are concerned with the existence of solutions for the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \frac{u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}) = \lambda h(u), \qquad x \in \mathbb{R}^N, \tag{1.1}$$

which models the self-channeling of a high-power ultrashort laser in matter (see [2]).

The main mathematical difficulty with problem (1.1) is caused by the quasilinear term $\frac{u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2})$, the natural functional corresponding to problem (1.1) maybe not well defined for all $u \in H^1(\mathbb{R}^N)$. To overcome this difficulty, various arguments have been developed, such as a change of variables (see [1,4,5,11,15,17]) and a perturbation method (see [3]). Chu and Liu [1] proved that (1.1) has a positive solution by using the monotonicity trick and a priori estimate in the radial space. It is a little surprising that no condition is assumed on the nonlinear term $h(u)$ near infinity. For the periodic potential, there are references [4,5], they discussed the following equation

$$-\Delta u + V(x)u - [\Delta(1+u^2)^{\alpha/2}]\frac{\alpha u}{2(1+u^2)^{(2-\alpha)/2}} = h(x,u), \tag{1.2}$$

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where α is a parameter. Jalilian [4] considered equation (1.2) with $1.36 < \alpha \leq 2$ and proved that (1.2) had infinitely many geometrically distinct solutions. Then, Li [5] extended the results to $1 \leq \alpha \leq 2$ and proved the existence of a ground state solution for equation (1.2). Shen and Wang [11] studied the well potential and got the standing wave solutions for (1.1) with subcritical or critical growth by using Resonance Theorem and Hahn–Banach Theorem. For the steep potential well, one can see [15], the authors obtained the existence of a ground state solution by using the Mountain Pass Theorem, and considered the concentration behavior of the solution. In [17], the authors considered the constant potential and obtained the existence and multiplicity of radial and nonradial normalized solutions for problem (1.1) when h satisfies the well-known Berestycki–Lions condition. As far as we know, there are no results concerning problem (1.1) with the asymptotically periodic potential except [16].

However, the related semilinear equation with the asymptotically periodic condition has been extensively studied, see [6,8,13,18] and their references. We would like to point out that in reference [6,8], they discussed the asymptotically periodic potential and given reformative conditions which unify the asymptotic processes of V, h at infinity. The asymptotic processes is weaker than those in [13,18].

In the present paper, we borrow an idea from [1,6] to discuss problem (1.1) with the asymptotically periodic potential. Denote

$$\mathcal{F}_0 := \left\{ k(x) : \forall \epsilon > 0,\ \lim_{|y|\to\infty} \mathrm{meas}\{x \in B_1(y) : |k(x)| \geq \epsilon\} = 0 \right\}.$$

Then, we give some assumptions on the potential $V(x)$ and the nonlinear term $h(s)$.

(V) $0 \leq V(x) \leq V_0(x) \in L^\infty(\mathbb{R}^N)$, $V(x) - V_0(x) \in \mathcal{F}_0$, $\inf_{x\in\mathbb{R}^N} V_0(x) > 0$ and $V_0(x)$ satisfies $V_0(x+z) = V_0(x)$ for all $x \in \mathbb{R}^N$ and $z \in \mathbb{Z}^N$.

The function $h \in C(\mathbb{R}, \mathbb{R})$ satisfies

(h_1) there exist $p > 2$, $\delta \in (0,1)$ such that the function $s \mapsto \frac{h(s)}{s^{p-1}}$ is nondecreasing and $h(s) > 0$ on $(0, \delta]$.

(h_2) there exists $q \in (2, 2^*)$ such that $\liminf_{s\to 0^+} \frac{H(s)}{s^q} > 0$, where $H(s) = \int_0^s h(t)dt$ and $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent.

Now we state our main result.

Theorem 1.1. *Suppose that conditions (V) and (h_1), (h_2) are satisfied, then there is $\lambda_1 > 0$ such that problem (1.1) possesses a positive solution for $\lambda \geq \lambda_1$.*

Remark 1.2. (1) We emphasize that no condition is assumed on the nonlinear term $h(u)$ near infinity in Theorem 1.1. In all these previous works for problem (1.1), among other assumptions, the authors always assume that the nonlinear term $h(u)$ has growth conditions near infinity except [1]. However, Chu and Liu [1] investigated quasi-linear Schrödinger equations in the radial space. They had the compactness and got certain solutions easily. In our cases, we do not have compact embedding. Due to the lack of compact embedding, the existence of ground states of problem (1.1) becomes rather complicated. we borrow an idea from [6] to overcome this difficulty.

(2) Our results also can be seen as the extension of semilinear poroblem in [6] to the quasilinear one.

(3) For simplicity, we will abbreviate $\int_{\mathbb{R}^N} k(x)dx$ as $\int_{\mathbb{R}^N} k(x)$.

Notation: In this paper, we use the following notations.

- $H^1(\mathbb{R}^N)$ is the usual Hilbert space endowed with the norm

$$\|u\|_H^2 = \int_{\mathbb{R}^N} \Big(|\nabla u|^2 + u^2\Big).$$

- $L^s(\mathbb{R}^N)$ is the usual Banach space endowed with the norm

$$\|u\|_s^s = \int_{\mathbb{R}^N} |u|^s, \quad \forall s \in [1, +\infty).$$

- $\|u\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^N} |u(x)|$ denotes the usual norm in $L^\infty(\mathbb{R}^N)$.
- $E = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 < \infty\}$ is endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} \Big(|\nabla u|^2 + V(x)u^2\Big).$$

- $B_r(y) := \{x \in \mathbb{R}^N : |x - y| < r\}$.
- $C, C_1, C_2, \dots$ denote various positive (possibly different) constants.

2 Some preliminary results

We note that the solutions of problem (1.1) are the critical points of the functional

$$J_h(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[\left(1 + \frac{u^2}{2(1+u^2)}\right)|\nabla u|^2\right] + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 - \lambda \int_{\mathbb{R}^N} H(u).$$

Variational methods cannot be applied directly to find weak solutions of problem (1.1), since the natural associated functional $J_h(u)$ is not well defined in general in the space E. To overcome this difficulty, we borrow an idea from Shen and Wang [10].

Let $F(u) := \int_0^u f(t)dt$, where f is defined by

$$f(t) = \sqrt{1 + \frac{t^2}{2(1+t^2)}}. \tag{2.1}$$

After the change of variables $u = F^{-1}(v)$ from J, we get a new variational functional

$$I_h(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|F^{-1}(v)|^2) - \lambda \int_{\mathbb{R}^N} H(F^{-1}(v)).$$

Since f is a nondecreasing positive function, we obtain $|F^{-1}(v)| \le \frac{|v|}{f(0)} = |v|$. From this and the conditions of h, it is clear that I_h is well defined in E and $I_h \in C^1(E, \mathbb{R})$ (see [2, 10, 11] for details). Now, we give another equation

$$-\operatorname{div}\left[\left(1 + \frac{u^2}{2(1+u^2)}\right)\nabla u\right] + V(x)u + \frac{u}{2(1+u^2)^2}|\nabla u|^2 = \lambda h(u), \tag{2.2}$$

which is equivalent to (1.1). In fact, we only need to show that

$$-\text{div}\left[\left(1+\frac{u^2}{2(1+u^2)}\right)\nabla u\right]+\frac{u}{2(1+u^2)^2}|\nabla u|^2=-\Delta u-\frac{u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}).$$

By a direct calculation, we obtain

$$\begin{aligned}
&-\text{div}\left[\left(1+\frac{u^2}{2(1+u^2)}\right)\nabla u\right]+\frac{u}{2(1+u^2)^2}|\nabla u|^2\\
&=-\text{div}\nabla u-\frac{u^2}{2(1+u^2)}\text{div}\nabla u-\nabla u\cdot\nabla\frac{u^2}{2(1+u^2)}+\frac{u}{2(1+u^2)^2}|\nabla u|^2\\
&=-\Delta u-\frac{u^2}{2(1+u^2)}\Delta u-\frac{u}{2(1+u^2)^2}|\nabla u|^2\\
&=-\Delta u-\frac{u}{2\sqrt{1+u^2}}\left(\frac{u}{\sqrt{1+u^2}}\text{div}\nabla u+\nabla u\cdot\nabla\frac{u}{\sqrt{1+u^2}}\right)\\
&=-\Delta u-\frac{u}{2\sqrt{1+u^2}}\Delta(\sqrt{1+u^2}).
\end{aligned}$$

If u is a weak solution of problem (1.1), then it is also a weak solution of (2.2) and should satisfy

$$\int_{\mathbb{R}^N}\left[\left(1+\frac{u^2}{2(1+u^2)}\right)\nabla u\cdot\nabla\varphi+\frac{u}{2(1+u^2)^2}|\nabla u|^2\varphi+V(x)u\varphi-\lambda h(u)\varphi\right]=0, \tag{2.3}$$

for all $\varphi\in C_0^\infty(\mathbb{R}^N)$. Let $\varphi=\frac{\psi}{f(u)}$, then, it can be checked that (2.3) is equivalent to the following equality

$$\int_{\mathbb{R}^N}\left(\nabla v\cdot\nabla\psi+V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))}\psi-\lambda\frac{h(F^{-1}(v))}{f(F^{-1}(v))}\psi\right)=0. \tag{2.4}$$

Therefore, in order to find the solutions of problem (1.1), it suffices to study the existence of solutions of the following equation

$$-\triangle v+V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))}=\lambda\frac{h(F^{-1}(v))}{f(F^{-1}(v))},\qquad x\in\mathbb{R}^N. \tag{2.5}$$

Now, we summarize the properties of F^{-1}, f.

Lemma 2.1. *The functions F^{-1}, f satisfy the following properties:*

(1) $1\le f(t)\le\sqrt{\frac{3}{2}}$ *for all* $t\in\mathbb{R}$;

(2) $1\le\frac{F^{-1}(t)f(F^{-1}(t))}{t}\le 6-2\sqrt{6}$ *for all* $t\in\mathbb{R}, t\neq 0$;

(3) $\sqrt{\frac{2}{3}}|t|\le|F^{-1}(t)|\le|t|$ *for all* $t\in\mathbb{R}$;

(4) $\frac{F^{-1}(t)}{t}\to 1$ *as* $t\to 0$;

(5) $\frac{F^{-1}(t)}{t}\to\sqrt{\frac{2}{3}}$ *as* $t\to\infty$;

(6) $0\le\frac{f'(t)t}{f(t)}\le 5-2\sqrt{6}$ *for all* $t\in\mathbb{R}$;

(7) *The function* $\frac{t}{f(t)F(t)}$ *is strictly decreasing for all* $t \geq 0$*;*

(8) *The function* $\frac{t^\mu}{f(t)F(t)}$*,* $\mu \in (2, p)$ *is strictly increasing for all* $t \geq 0$*.*

Proof. The proof of the items (1)–(6) have been proved in [11], we only need to prove items (7)(8). Let $l_1(t) = \frac{t}{f(t)F(t)}$. Since $f(t)$ is strictly increasing in $(0, +\infty)$, one has

$$0 \leq F(t) = \int_0^t f(s)ds < tf(t). \tag{2.6}$$

Then using item (6) and (2.6), we obtain

$$l'(t) = \frac{F(t) - tf(t) - \frac{f'(t)t}{f(t)}F(t)}{f(t)F^2(t)} \leq \frac{F(t) - tf(t)}{f(t)F^2(t)} < 0.$$

The above inequality proves item (7).

Let $l_\mu(t) = \frac{t^\mu}{f(t)F(t)}$, $l_0(t) = [\mu - (5 - 2\sqrt{6})]F(t) - tf(t)$. It is following from item (6) that

$$l_0'(t) = (\mu F(t) - tf(t))' = f(t)\left(\mu - (6 - 2\sqrt{6}) - \frac{f'(t)t}{f(t)}\right) > 0.$$

We can get that $l_0(t)$ is strictly increasing in $(0, +\infty)$ and $l_0(t) > l_0(0) = 0$ for $t > 0$. Then, using item (6) again, we obtain

$$l_\mu'(t) = \frac{t^{\mu-1}}{f(t)F^2(t)}\left[\mu F(t) - tf(t) - \frac{f'(t)t}{f(t)}F(t)\right] \geq \frac{t^{\mu-1}}{f(t)F^2(t)}l_0(t) > 0.$$

The above inequality proves item (8). □

Lemma 2.2 ([8])**.** *Suppose that condition* (V) *is satisfied. Then, the norms* $\|\cdot\|_H$ *and* $\|\cdot\|$ *are equivalent in the space E and the embedding* $E \hookrightarrow L^\alpha(\mathbb{R}^N)$ *is continuous for any* $\alpha \in [2, 2^*]$*.*

Lemma 2.3 ([12])**.** *Let E be a real Banach space and* $I \in C^1(E, \mathbb{R})$*. Let S be a closed subset of E which disconnects E in distinct connected components* E_1*,* E_2*. Suppose further that* $I(0) = 0$ *and*

(1) $0 \in E_1$ *and there is* $\alpha > 0$ *such that* $I|_S \geq \alpha > 0$*.*

(2) *there is* $\rho > 0$*,* $e \in E_2$*,* $\|e\| > \rho$*, such that* $I(e) < 0$*.*

Then I possesses a sequence $\{u_n\} \subset E$ *satisfying*

$$I(u_n) \to c \geq \alpha, \qquad (1 + \|u_n\|)\|I'(u_n)\| \to 0, \tag{2.7}$$

where $c \geq \alpha > 0$ *given by*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \qquad \Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

We call the sequence $\{u_n\}$ that satisfies (2.7) ia a $(C)_c$ sequence of the functional I.

Lemma 2.4. *Assume that condition* (V) *holds. If* $\{u_n\}$ *is bounded in E and* $u_n \to 0$ *in* $L^\alpha_{\text{loc}}(\mathbb{R}^N)$ *for* $\alpha \in [2, 2^*)$*, one has*

$$A_{n1} := \int_{\mathbb{R}^N} (V(x) - V_0(x)) |F^{-1}(u_n)|^2 = o_n(1).$$

Proof. When $k(x) \in \mathcal{F}_0$, for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\int_{|k(x)|\geq \epsilon} u^2 \leq C_0 \int_{B_{R_\epsilon+1}(0)} u^2 + C_1 \epsilon^{2/N} \|u\|_H^2, \qquad \forall u \in E, \tag{2.8}$$

where C_0, C_1 are positive constants and independent on ϵ. Inequality (2.8) has already been proved in [8], we omit it here.

Let $k(x) := V(x) - V_0(x) \in \mathcal{F}_0$, then, $|k(x)| \leq 2|V_0(x)| \leq 2\|V_0\|_\infty$, by using Lemma 2.1-(3) and (2.8), we have

$$\begin{aligned} |A_{n1}| &\leq \int_{\mathbb{R}^N} |k(x)||F^{-1}(u_n)|^2 \leq \int_{\mathbb{R}^N} |k(x)u_n^2| \\ &= \int_{|k(x)|\geq\epsilon} |k(x)u_n^2| + \int_{|k(x)|<\epsilon} |k(x)u_n^2| \\ &\leq 2\|V_0\|_\infty \left[C_0 \int_{B_{R_\epsilon+1}(0)} u_n^2 + C_1 \epsilon^{\frac{2}{N}} \|u_n\|_H^2 \right] + \epsilon \int_{\mathbb{R}^N} |u_n|^2 \\ &= o_n(1) + C_2 \epsilon^{\frac{2}{N}} + C_3 \epsilon. \end{aligned}$$

Let $\epsilon \to 0$, Lemma 2.4 holds. □

Lemma 2.5. *Assume that condition (V) holds, $\{u_n\} \subset E$ is bounded, $|z_n| \to +\infty$. Then for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, one has*

$$B_{n1} := \int_{\mathbb{R}^N} (V(x) - V_0(x)) \frac{F^{-1}(u_n)}{f(F^{-1}(u_n))} \varphi(x - z_n) = o_n(1).$$

Proof. Since $\varphi \in C_0^\infty(\mathbb{R}^N)$, we get that

$$\int_{B_{R_\epsilon+1}(0)} |\varphi(x - z_n)|^2 = o_n(1). \tag{2.9}$$

Let $k(x) := V(x) - V_0(x) \in \mathcal{F}_0$, by using Lemma 2.1-(3), (2.8), (2.9) and the Hölder inequality, we have

$$\begin{aligned} |B_{n1}| &\leq \int_{|k|\geq\epsilon} \left| \frac{k(x)F^{-1}(u_n)}{f(F^{-1}(u_n))} \varphi(x - z_n) \right| + \int_{|k|<\epsilon} \left| \frac{k(x)F^{-1}(u_n)}{f(F^{-1}(u_n))} \varphi(x - z_n) \right| \\ &\leq 2\|V_0\|_\infty \int_{|k|\geq\epsilon} |u_n \varphi(x - z_n)| + \epsilon \int_{|k|<\epsilon} |u_n \varphi(x - z_n)| \\ &\leq 2\|V_0\|_\infty \|u_n\|_2 \left(\int_{|k|\geq\epsilon} |\varphi(x - z_n)|^2 \right)^{1/2} + \epsilon \|u_n\|_2 \|\varphi\|_2 \\ &\leq C_4 \left(C_0 \int_{B_{R_\epsilon+1}(0)} |\varphi(x - z_n)|^2 + C_1 \epsilon^{2/N} \|\varphi\|_H^2 \right)^{1/2} + C_5 \epsilon \\ &= o_n(1) + C_6 \epsilon^{1/N} + C_5 \epsilon. \end{aligned}$$

Let $\epsilon \to 0$, Lemma 2.5 is proved. □

3 Proof of Theorem 1.1

By assumptions (h_1) and (h_2), we get that $p \leq q$ and

$$|h(s)s| \leq C|s|^p, \qquad \forall |s| \leq \delta.$$

Then, we need to modify $h(u)$ to prove our main results. Set

$$g(s) := \begin{cases} 0, & s \le 0, \\ h(s), & 0 < s \le \delta, \\ C_1 s^{p-1}, & s > \delta. \end{cases}$$

We can fix $C_1 > 0$ such that $g \in C(\mathbb{R}, \mathbb{R}^+)$. According to the definition of g, and we can get the following lemma easily.

Lemma 3.1. *Suppose that (h_1) is satisfied. Then*

(1) $\lim_{s\to+\infty} \frac{G(s)}{s^2} = +\infty$, *where* $G(s) = \int_0^s g(t)dt$.

(2) *there exists $C > 0$ such that $|g(s)s| \le C|s|^p$ and $|G(s)| \le C|s|^p$ for all $s \in \mathbb{R}$.*

(3) *there exists $\mu \in (2, p)$ such that the function $s \mapsto \frac{g(s)}{s^{\mu-1}}$ is strictly increasing on $(0, +\infty)$.*

Let us consider the modified equation of problem (2.5) given by

$$-\triangle v + V(x)\frac{F^{-1}(v)}{f(F^{-1}(v))} = \lambda \frac{g(F^{-1}(v))}{f(F^{-1}(v))}, \qquad x \in \mathbb{R}^N. \tag{3.1}$$

We note that the solutions of problem (3.1) are the critical points of the functional

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|F^{-1}(v)|^2) - \lambda \int_{\mathbb{R}^N} G(F^{-1}(v)).$$

In order to prove our results, we need the periodic problem as follow

$$-\triangle v + V_0(x)\frac{F^{-1}(v)}{f(F^{-1}(v))} = \lambda \frac{g(F^{-1}(v))}{f(F^{-1}(v))}, \qquad x \in \mathbb{R}^N, \tag{3.2}$$

whose corresponding energy functional is denoted as

$$I_0(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla v|^2 + V_0(x)|F^{-1}(v)|^2\right] - \int_{\mathbb{R}^N} G(x, F^{-1}(v)).$$

Define

$$\mathcal{N} = \{u \in E : \langle I'(u), u\rangle = 0, u \ne 0\}, \qquad \mathcal{N}_0 = \{u \in E : \langle I_0'(u), u\rangle = 0, u \ne 0\},$$

$$c = \inf_{u\in\mathcal{N}} I(u), \qquad c_0 = \inf_{u\in\mathcal{N}_0} I_0(u).$$

Then we can deduce the following lemma.

Lemma 3.2. *Suppose that conditions (V) and $(h_1), (h_2)$ hold, then for each $u \in E$, $u \ne 0$, there is a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. Moreover, the maximum of $I(tu)$ for $t \ge 0$ is achieved at t_u.*

Proof. By Lemma 2.1-(3), Lemma 3.1-(2) and the Sobolev inequality, one has

$$\int_{\mathbb{R}^N} G(F^{-1}(tu)) \le C\int_{\mathbb{R}^N} |F^{-1}(tu)|^p \le Ct^p \int_{\mathbb{R}^N} u^p \le Ct^p\|u\|^p. \tag{3.3}$$

It follows from Lemma 2.1-(3), (3.3) and Lemma 2.2 that

$$\begin{aligned}\Psi(t) := I(tu) &= \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla(tu)|^2 + V(x)|F^{-1}(tu)|^2\right] - \lambda\int_{\mathbb{R}^N} G(F^{-1}(tu))\\ &\geq \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{t^2}{3}\int_{\mathbb{R}^N} V(x)u^2 - \lambda C t^p\|u\|^p\\ &\geq \frac{t^2}{3}\|u\|^2 - \lambda C t^p\|u\|^p.\end{aligned}$$

Therefore, we can get $\Psi(t) > 0$ whenever $t > 0$ is small enough.

Let $\Omega = \{x \in \mathbb{R}^N : u(x) > 0\}$, then thanks to Lemma 3.1-(1), Lemma 2.1-(3)(5) and the Fatou Lemma, we can deduce that

$$\limsup_{t\to\infty}\frac{\Psi(t)}{t^2} \leq \frac{1}{2}\|u\|^2 - \lambda\liminf_{t\to\infty}\int_\Omega \frac{G(F^{-1}(tu))}{|F^{-1}(tu)|^2}\cdot\frac{|F^{-1}(tu)|^2}{(tu)^2}\cdot u^2 = -\infty.$$

Hence, $\Psi(t) \to -\infty$ as $t \to \infty$ and Ψ has a positive maximum.

The condition $\Psi'(t) = 0$ is equivalent to

$$\int_{\mathbb{R}^N}|\nabla u|^2 = \int_{\mathbb{R}^N}\left[\frac{\lambda g(F^{-1}(tu))}{tuf(F^{-1}(tu))} - \frac{V(x)F^{-1}(tu)}{f(F^{-1}(tu))tu}\right]u^2.$$

Let

$$Z(s) := \frac{g(s)}{f(s)F(s)} - \frac{V(x)s}{f(s)F(s)}.$$

By Lemma 3.1-(3) and Lemma 2.1-(7)(8), $s \mapsto Z(s)$ is strictly increasing for $s > 0$, so there is a unique $t_u > 0$ such that $\Psi'(t_u) = 0$. The conclusion is true since $\Psi'(t) = t^{-1}\langle I'(tu), tu\rangle$. □

Lemma 3.3. *Suppose that (V) and (h_1), (h_2) hold. Then*

(i) there exists $\rho > 0$ such that $\|u\| \geq \rho$ for all $u \in \mathcal{N}$.

(ii) the functional I is bounded from below on $\mathcal{N}$ by a positive constant.

Proof. (i) For any $u \in \mathcal{N}$, By Lemma 3.1-(1)(2), Lemma 2.2-(1)(3) and the Sobolev inequality, we have

$$\begin{aligned}\frac{2}{3}\|u\|^2 &\leq \int_{\mathbb{R}^N}|\nabla u|^2 + \int_{\mathbb{R}^N} V(x)\frac{F^{-1}(u)}{f(F^{-1}(u))}u = \lambda\int_{\mathbb{R}^N}\frac{g(F^{-1}(u))}{f(F^{-1}(u))}u\\ &\leq \lambda C\int_{\mathbb{R}^N} u^p \leq \lambda C\|u\|^p.\end{aligned}$$

Hence, there exists $\rho > 0$ independent of u such that $\|u\| \geq \rho$.

(ii) It follows from (3.3) and Lemma 2.1-(3) that

$$\begin{aligned}I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|F^{-1}(u)|^2 - \lambda\int_{\mathbb{R}^N} G(F^{-1}(u))\\ &\geq \frac{1}{3}\|u\|^2 - \lambda C\|u\|^p.\end{aligned}$$

Since $p > 2$, there exists $\sigma > 0$ such that $I(u) \geq \frac{\sigma^2}{4} > 0$ for $\|u\| = \sigma > 0$. For any $v \in \mathcal{N}$, there exists $t_1 > 0$ such that $t_1\|v\| = \sigma$. By Lemma 3.1-(1)(2), we obtain

$$I(v) \geq I(t_1 v) \geq \frac{\sigma^2}{4}.$$

This completes the proof. □

Lemma 3.4. *Suppose that conditions (V) and $(h_1), (h_2)$ are satisfied. If $u \in \mathcal{N}$ and $I(u) = c$, then u is a ground state solution of problem* (3.1) *(see [8, 16]).*

It follows from [16] that the periodic problem (3.2) has a positive ground state solution u. From Lemma 3.2, there is a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. Moreover, the maximum of $I(tu)$ for $t \geq 0$ is achieved at t_u. Thanks to $V(x) \leq V_0(x)$, we obtain

$$c \leq I(t_u u) \leq I_0(t_u u) \leq I_0(u) = c_0, \tag{3.4}$$

hence $c \leq c_0$. Thanks to Lemma 3.3-(ii), we can also get $c > 0$.

As the argument in [14, Theorem 4.2], we obtain the following lemma due to Lemmas 3.1–3.3.

Lemma 3.5. *Suppose that (V) holds, h satisfies $(h_1), (h_2)$, then*

$$c = \inf_{u \in \mathcal{N}} I(u) = \inf_{u \in E} \max_{t > 0} I(tu) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, I(\gamma(t)) < 0\}$.

The above lemma is also valid for functional I_0.

Next, we will give the boundedness of the Cerami sequences.

Lemma 3.6. *Suppose that conditions (V) and $(h_1), (h_2)$ hold. Let $\{u_n\} \subset E$ be a $(C)_c$ sequence for the functional I. Then $\{u_n\}$ is bounded in E .*

Proof. Suppose by contradiction that $\{u_n\} \subset E$ be a sequence such that $\|u_n\| \to \infty$, $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$. Set $v_n := \frac{u_n}{\|u_n\|}$, then, there is a $v \in E$ such that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$.

If $v \neq 0$, let $\Omega_* = \{x \in \mathbb{R}^N : v(x) > 0\}$, then $\operatorname{meas} \Omega_* > 0$. For a.e. $x \in \Omega_*$, one has

$$u_n(x) \to +\infty \quad \text{as } \|u_n\| \to +\infty,$$

since $v_n(x) = \frac{u_n(x)}{\|u_n\|} \to v(x) > 0$ for a.e. $x \in \Omega_*$, from Lemma 2.1-(5) and the fact that $F^{-1}(t)$ is strictly increasing, we can deduce that for a.e. $x \in \Omega_*$,

$$F^{-1}(u_n) \to +\infty \quad \text{as } \|u_n\| \to +\infty.$$

It follows from Lemma 2.1-(3)(5) and Lemma 3.1-(1) that

$$\begin{aligned}
0 &= \limsup_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} \\
&\leq \limsup_{n\to\infty} \frac{\frac{1}{2}\|u_n\|^2 - \lambda \int_{\mathbb{R}^N} G(F^{-1}(u_n))}{\|u_n\|^2} \\
&= \frac{1}{2} - \lambda \liminf_{n\to\infty} \int_{\Omega_*} \left(\frac{G(F^{-1}(u_n))}{|F^{-1}(u_n)|^2} \cdot \frac{|F^{-1}(u_n)|^2}{u_n^2} \cdot v_n^2 \right) \\
&= -\infty.
\end{aligned}$$

A contradiction, thus $v = 0$. Define

$$\beta := \limsup_{n\to\infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} v_n^2 dx.$$

If $\beta = 0$, by the Lions lemma [14, Lemma 1.21], we get $v_n \to 0$ in $L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$. It follows from Lemma 3.1-(2) and Lemma 2.1-(3) that

$$\int_{\mathbb{R}^N} G(F^{-1}(tv_n)) \leq C \int_{\mathbb{R}^N} |F^{-1}(tv_n)|^p \leq Ct^p \int_{\mathbb{R}^N} |v_n|^p = o_n(1), \tag{3.5}$$

for any $t \geq 0$. Especially, set $t = 4\sqrt{c}$, we obtain

$$\int_{\mathbb{R}^N} G(F^{-1}(4\sqrt{c}v_n)) = o_n(1). \tag{3.6}$$

By Lemma 2.1-(4), one has $F^{-1}(4\sqrt{c}v_n) \to 4\sqrt{c}v_n$, since $4\sqrt{c}v_n \to 0$ a.e. in $\mathbb{R}^N$. Then, we can deduce that

$$\int_{\mathbb{R}^N} V(x) \left[(4\sqrt{c}v_n)^2 - [F^{-1}(4\sqrt{c}v_n)]^2 \right] = o_n(1). \tag{3.7}$$

Setting

$$k(x,s) = \lambda \frac{g(F^{-1}(s))}{f(F^{-1}(s))} - V(x) \frac{F^{-1}(s)}{f(F^{-1}(s))} + V(x)s,$$

and

$$K(x,s) := \int_0^s k(x,t)dt = \lambda G(F^{-1}(s)) - \frac{1}{2} V(x) |F^{-1}(s)|^2 + \frac{1}{2} V(x) s^2.$$

Then,

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)u^2] - \int_{\mathbb{R}^N} K(x,u). \tag{3.8}$$

Thanks to (3.6) and (3.7), we can obtain that

$$\begin{aligned} \int_{\mathbb{R}^N} K(x, 4\sqrt{c}v_n) = \lambda \int_{\mathbb{R}^N} G(F^{-1}(4\sqrt{c}v_n)) \\ + \frac{1}{2} \int_{\mathbb{R}^N} V(x) \left[(4\sqrt{c}v_n)^2 - [F^{-1}(4\sqrt{c}v_n)]^2 \right] = o_n(1). \end{aligned}$$

By the continuity of I, there exists $t_n \in [0,1]$ such that $I(t_n u_n) = \max_{0 \leq t \leq 1} I(tu_n)$. Since $\|u_n\| \to \infty$, we have $\frac{4\sqrt{c}}{\|u_n\|} \leq 1$ when n is large enough. Hence, one has

$$\begin{aligned} I(t_n u_n) + o_n(1) &\geq I\left(\frac{4\sqrt{c}}{\|u_n\|} u_n \right) + o_n(1) = I(4\sqrt{c}v_n) + o_n(1) \\ &= 8c\|v_n\|^2 - \int_{\mathbb{R}^N} K(x, 4\sqrt{c}v_n) + o_n(1) \\ &= 8c + o_n(1). \end{aligned}$$

Note that $I(u_n) \to c$, so $0 < t_n < 1$ and $\langle I'(t_n u_n), t_n u_n \rangle = 0$ when n is large enough. By Lemma 3.1-(3) and Lemma 2.1-(7)(8), the function

$$\frac{k(x,s)}{s} = \frac{\lambda g(F^{-1}(s))}{f(F^{-1}(s))s} - V(x) \frac{F^{-1}(s)}{f(F^{-1}(s))s} + V(x)$$

is strictly increasing for $s > 0$. Since $\{u_n\}$ is a Cerami sequence of I and the monotonicity of $\frac{k(x,s)}{s}$, we can conclude

$$
\begin{aligned}
c &= I(u_n) + o_n(1) \\
&= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o_n(1) \\
&= \int_{\mathbb{R}^N} \left(\frac{1}{2}k(x,u_n)u_n - K(x,u_n)\right) + o_n(1) \\
&\geq \int_{\mathbb{R}^N} \left(\frac{1}{2}k(x,t_nu_n)t_nu_n - K(x,t_nu_n)\right) + o_n(1) \\
&= I(t_nu_n) - \frac{1}{2}\langle I'(t_nu_n), t_nu_n\rangle + o_n(1) \\
&= I(t_nu_n) + o_n(1) \\
&\geq 8c + o_n(1),
\end{aligned}
$$

which is a contradiction for $c > 0$.

If $\beta > 0$, by the definition of β, there is $z_n \in \mathbb{R}^N$ such that

$$\frac{\beta}{2} < \int_{B_1(z_n)} v_n^2.$$

If z_n is bounded, there exists $R > 0$ such that

$$\frac{\beta}{2} < \int_{B_R(0)} v_n^2,$$

which is a contradiction with $v_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$.

If z_n is unbounded, up to a subsequence, $|z_n| \to \infty$. Let $w_n(x) := v_n(x+z_n) = \frac{u_n(x+z_n)}{\|u_n\|}$, we have

$$\frac{\beta}{2} < \int_{B_1(0)} w_n^2. \tag{3.9}$$

There is a function $w \in E$ such that $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. Moreover, by (3.9), one has $w(x) \neq 0$. Define $\Omega_{**} = \{x \in \mathbb{R}^N : w(x) > 0\}$, then $meas\Omega_{**} > 0$ and for a.e. $x \in \Omega_{**}$, we have

$$u_n(x+z_n) \to +\infty \quad \text{as } \|u_n\| \to +\infty.$$

Since $F^{-1}(t)$ is strictly increasing for $t \geq 0$, by Lemma 2.1-(5), we can conclude that for a.e. $x \in \Omega_{**}$,

$$F^{-1}(u_n(x+z_n)) \to +\infty \quad \text{as } \|u_n\| \to +\infty.$$

Then, from Lemma 3.1-(1) and Lemma 2.1-(5), one has

$$
\begin{aligned}
&\liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} G(F^{-1}(u_n))}{\|u_n\|^2} \\
&\quad = \liminf_{n\to\infty} \frac{\int_{\mathbb{R}^N} G(F^{-1}(u_n(x+z_n)))}{\|u_n\|^2} \\
&\quad \geq \liminf_{n\to\infty} \int_{\Omega_{**}} \frac{G(F^{-1}(u_n(x+z_n)))}{|F^{-1}(u_n(x+z_n))|^2} \frac{|F^{-1}(u_n(x+z_n))|^2}{(u_n(x+z_n))^2} w_n^2 \\
&\quad = +\infty.
\end{aligned}
$$

Combining the above inequality with Lemma 2.1-(3), we have

$$\begin{aligned} 0 &= \limsup_{n\to\infty} \frac{I(u_n)}{\|u_n\|^2} \\ &\le \frac{1}{2} - \lambda \liminf_{n\to\infty} \frac{1}{\|u_n\|^2} \int_{\mathbb{R}^N} G(F^{-1}(u_n)) \\ &= -\infty, \end{aligned}$$

this contradiction finished the proof. □

Lemma 3.7. *Suppose that conditions* (V) *and* $(h_1), (h_2)$ *hold. Then problem* (3.1) *has a positive ground state solution.*

Proof. It follows from Lemma 3.3-(ii) and (3.44) that

$$0 < c \le c_0.$$

If $c = c_0$, we can get from (3.4) that

$$c_0 = c \le I(t_u u) \le I_0(t_u u) \le I_0(u) = c_0.$$

Then $t_u u$ is a positive ground solution of problem (3.1).

If $0 < c < c_0$, we see that I satisfies the mountain pass geometry from the proof of Lemma 3.2. Then, we can get a Cerami sequence $\{u_n\}$ on level c due to Lemma 2.3. Applying Lemma 3.6, the $(C)_c$ sequence is bounded. Then, we may get, up to a subsequence, $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. By using the Lebesgue dominated convergence theorem, through the standard discussion, we can get that

$$0 = \langle I'(u_n), \phi\rangle + o_n(1) = \langle I'(u), \phi\rangle,$$

for any $\phi \in C_0^\infty(\mathbb{R}^N)$, i.e. u is a weak solution of problem (3.1).

(i) The case $u \ne 0$. Since u is a weak solution of problem (3.1), $I(u) \ge c$ and $u \in \mathcal{N}$. By (3.8), the monotonicity of $\frac{k(x,s)}{s}$ and the Fatou lemma, one has

$$\begin{aligned} c &= I(u_n) + o_n(1) \\ &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o_n(1) \\ &= \int_{\mathbb{R}^N} \left(\frac{1}{2}k(x,u_n)u_n - K(x,u_n)\right) + o_n(1) \\ &\ge \int_{\mathbb{R}^N} \left(\frac{1}{2}k(x,u)u - K(x,u)\right) + o_n(1) \\ &= I(u) - \frac{1}{2}\langle I'(u), u\rangle \\ &= I(u). \end{aligned}$$

Hence, $I(u) = c$ and $I'(u) = 0$, which implies that u is a ground state solution of problem (3.1). Moreover, we could deduce that u is a positive solution by applying the strongly maximum principle.

(ii) The case $u = 0$. Define

$$\beta := \limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_1(z)} u_n^2.$$

If $\beta = 0$, by the Lions lemma [14, Lemma 1.21], we get $u_n \to 0$ in $L^\alpha(\mathbb{R}^N)$ for $\alpha \in (2, 2^*)$. It is similar to the proof of (3.5), we can deduce

$$\int_{\mathbb{R}^N} G(F^{-1}(u_n)) \le o_n(1). \tag{3.10}$$

Combining (3.10) with Lemma 2.1-(3), we obtain

$$\begin{aligned} c &= I(u_n) + o_n(1) \\ &\le \frac{1}{2}\|u_n\|^2 - \lambda \int_{\mathbb{R}^N} G(F^{-1}(u_n)) = o_n(1). \end{aligned}$$

A contradiction, thus $\beta > 0$. By the definition of β, up to a subsequence, there exist $R > 0$ and $z_n \in \mathbb{Z}^N$ such that

$$\int_{B_R(0)} u_n^2(x + z_n) = \int_{B_R(z_n)} u_n^2(x) > \frac{\beta}{2}.$$

If z_n is bounded, there is $R' > 0$ such that

$$\int_{B_{R'}(0)} u_n^2 \ge \int_{B_R(z_n)} u_n^2 > \frac{\beta}{2},$$

which contradicts with $u_n \to u = 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$. Thus, z_n is unbounded, going if necessary to a subsequence, $|z_n| \to \infty$. Let $w_n(x) := u_n(x + z_n)$, then there exists a function $w \in E\backslash\{0\}$ such that $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. It follows from Lemma 2.5 that, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} 0 &= \langle I'(u_n), \varphi(x - z_n)\rangle + o_n(1) \\ &= \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla \varphi(x - z_n) + \int_{\mathbb{R}^N} V(x) \frac{F^{-1}(u_n)}{f(F^{-1}(u_n))} \varphi(x - z_n) \\ &\quad - \lambda \int_{\mathbb{R}^N} \frac{g(F^{-1}(u_n))}{f(F^{-1}(u_n))} \varphi(x - z_n) + o_n(1) \\ &= \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla \varphi(x - z_n) + \int_{\mathbb{R}^N} V_0(x) \frac{F^{-1}(u_n)}{f(F^{-1}(u_n))} \varphi(x - z_n) \\ &\quad - \lambda \int_{\mathbb{R}^N} \frac{g(F^{-1}(u_n))}{f(F^{-1}(u_n))} \varphi(x - z_n) + o_n(1) \\ &= \int_{\mathbb{R}^N} \nabla w_n \cdot \nabla \varphi + \int_{\mathbb{R}^N} V_0(x) \frac{F^{-1}(w_n)}{f(F^{-1}(w_n))} \varphi - \lambda \int_{\mathbb{R}^N} \frac{g(F^{-1}(w_n))}{f(F^{-1}(w_n))} \varphi + o_n(1) \\ &= \int_{\mathbb{R}^N} \nabla w \cdot \nabla \varphi + \int_{\mathbb{R}^N} V_0(x) \frac{F^{-1}(w)}{f(F^{-1}(w))} \varphi - \lambda \int_{\mathbb{R}^N} \frac{g(F^{-1}(w))}{f(F^{-1}(w))} \varphi \\ &= \langle I_0'(w), \varphi\rangle, \end{aligned}$$

i.e. w is a weak solution of the periodic problem (3.2).

On the one hand, it follows from Lemmas 2.4–2.5 that

$$\begin{aligned} c &= I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle + o_n(1) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} V(x)|F^{-1}(u_n)|^2 - \frac{1}{2} \int_{\mathbb{R}^N} V(x) \frac{F^{-1}(u_n)}{f(F^{-1}(u_n))} u_n \\ &\quad + \lambda \int_{\mathbb{R}^N} \frac{g(F^{-1}(u_n))}{2f(F^{-1}(u_n))} u_n - \lambda \int_{\mathbb{R}^N} G(F^{-1}(u_n)) + o_n(1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)|F^{-1}(u_n)|^2 - \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)\frac{F^{-1}(u_n)}{f(F^{-1}(u_n))}u_n \\
&\quad + \lambda\int_{\mathbb{R}^N} \frac{g(F^{-1}(u_n))}{2f(F^{-1}(u_n))}u_n - \int_{\mathbb{R}^N} G(F^{-1}(u_n)) + o_n(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)\left[|F^{-1}(w_n)|^2 - \frac{F^{-1}(w_n)}{f(F^{-1}(w_n))}w_n\right] \\
&\quad + \lambda\int_{\mathbb{R}^N} \left[\frac{g(F^{-1}(w_n))}{2f(F^{-1}(w_n))}w_n - G(F^{-1}(w_n))\right] + o_n(1) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} V_0(x)\left[|F^{-1}(w)|^2 - \frac{F^{-1}(w)}{f(F^{-1}(w))}w\right] \\
&\quad + \lambda\int_{\mathbb{R}^N} \left[\frac{g(F^{-1}(w))}{2f(F^{-1}(w))}w - G(F^{-1}(w))\right] \\
&= I_0(w) - \frac{1}{2}\langle I_0'(w), w\rangle \\
&= I_0(w) \geq c_0,
\end{aligned}
$$

which is a contradiction with $c \leq c_0$. Hence, the case $u = 0$ cannot happen, this completes the proof. □

Lemma 3.8. *Suppose that (V) and (h_1) hold. If u is a critical point of I, then $u \in L^\infty(\mathbb{R}^N)$. Moreover, there is a constant $C > 0$ independent of λ such that*

$$
\|u\|_\infty \leq C\lambda^{\frac{1}{2^*-p}}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{\frac{2^*-2}{2(2^*-p)}}.
$$

Proof. For all $k > 0$, we set

$$
u_k(x) = \begin{cases} u(x), & \text{if } |u(x)| \leq k, \\ \pm k, & \text{if } \pm u(x) > k. \end{cases}
$$

We use $\varphi_k = |u_k|^{2(\beta-1)}u$ with $\beta > 1$ as a test function and calculate $\langle I'(u), \varphi_k\rangle = 0$, namely,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |u_k|^{2(\beta-1)}|\nabla u|^2 + 2(\beta-1)\int_{\mathbb{R}^N} |u_k|^{2(\beta-2)}uu_k\nabla u\cdot\nabla u_k \\
&\qquad + \int_{\mathbb{R}^N} V(x)\frac{F^{-1}(u)}{f(F^{-1}(u))}|u_k|^{2(\beta-1)}u = \lambda\int_{\mathbb{R}^N} \frac{g(F^{-1}(u))}{f(F^{-1}(u))}|u_k|^{2(\beta-1)}u. \quad (3.11)
\end{aligned}
$$

According to the facts that $u^2|\nabla u_k|^2 \leq u_k^2|\nabla u|^2$, $\beta > 1$, and the Sobolev inequality, we obtain

$$
\begin{aligned}
&\beta^2\int_{\mathbb{R}^N} \left(|u_k|^{2(\beta-1)}|\nabla u|^2 + 2(\beta-1)|u_k|^{2(\beta-2)}uu_k\nabla u\cdot\nabla u_k\right) \\
&\quad \geq \int_{\mathbb{R}^N} |u_k|^{2(\beta-1)}|\nabla u|^2 + \int_{\mathbb{R}^N} (\beta-1)^2|u_k|^{2(\beta-2)}u^2|\nabla u_k|^2 \\
&\qquad + \int_{\mathbb{R}^N} 2(\beta-1)|u_k|^{2(\beta-2)}uu_k\nabla u\cdot\nabla u_k \\
&\quad = \int_{\mathbb{R}^N} \left|\nabla\left(|u_k|^{\beta-1}u\right)\right|^2 \\
&\quad \geq C\left(\int_{\mathbb{R}^N} \left||u_k|^{\beta-1}u\right|^{2^*}\right)^{\frac{2}{2^*}}, \quad (3.12)
\end{aligned}
$$

By Lemma 2.1-(1)(3) and Lemma 3.1-(2), we have

$$\int_{\mathbb{R}^N} \frac{g(F^{-1}(u))}{f(F^{-1}(u))}|u_k|^{2(\beta-1)}u \leq \int_{\mathbb{R}^N} g(F^{-1}(u))|u_k|^{2(\beta-1)}u \leq C\int_{\mathbb{R}^N} |u|^p|u_k|^{2(\beta-1)}. \tag{3.13}$$

Using Lemma 2.1-(1)(3) again, we can obtain

$$\int_{\mathbb{R}^N} V(x)\frac{F^{-1}(u)}{f(F^{-1}(u))}|u_k|^{2(\beta-1)}u \geq \frac{2}{3}\int_{\mathbb{R}^N} V(x)|u_k|^{2(\beta-1)}u^2 \geq 0. \tag{3.14}$$

By (3.11)–(3.14) and the Hölder inequality, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} \left||u_k|^{\beta-1}u\right|^{2^*}\right)^{\frac{2}{2^*}} \\
&\quad \leq C\beta^2\lambda\int_{\mathbb{R}^N}(|u|^{p-2}|u_k|^{2(\beta-1)}u^2) \\
&\quad \leq C\beta^2\lambda\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{\frac{p-2}{2^*}}\left(\int_{\mathbb{R}^N}\left||u_k|^{2(\beta-1)}u^2\right|^{\frac{2^*}{2^*-p+2}}\right)^{\frac{2^*-p+2}{2^*}}.
\end{aligned}$$

Then, let $k \to \infty$, we obtain

$$\|u\|_{\beta\cdot 2^*} \leq (C\beta^2\lambda)^{\frac{1}{2\beta}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{p-2}{4\beta}}\|u\|_{\frac{2\cdot 2^*\beta}{2^*-p+2}}. \tag{3.15}$$

Set

$$\beta_m = \left(\frac{2^*-p+2}{2}\right)^{m+1}, \qquad m = 0, 1, \ldots$$

Then we get

$$\frac{2\cdot 2^*\beta_m}{2^*-p+2} = 2^*\beta_{m-1}.$$

It follows from (3.15) that

$$\begin{aligned}
\|u\|_{\beta_m\cdot 2^*} &\leq (C\beta_m^2\lambda)^{\frac{1}{2\beta_m}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{p-2}{4\beta_m}}\|u\|_{\frac{2\cdot 2^*\beta_m}{2^*-p+2}} \\
&= (C\lambda)^{\frac{1}{2\beta_m}}\beta_m^{\frac{1}{\beta_m}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{p-2}{4\beta_m}}\|u\|_{\beta_{m-1}\cdot 2^*}.
\end{aligned}$$

According to the Moser iteration, we obtain

$$\|u\|_{\beta_m\cdot 2^*} \leq (C\lambda)^{\sum_{i=0}^m \frac{1}{2\beta_i}}\prod_{i=0}^m \beta_i^{\frac{1}{\beta_i}}\left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{p-2}{4}\sum_{i=0}^m\frac{1}{\beta_i}}\|u\|_{2^*}. \tag{3.16}$$

Since $\beta_0 = \left(\frac{2^*-p+2}{2}\right) > 1$ and $\beta_i = \beta_0^{i+1}$, we get

$$\sum_{i=0}^m \frac{1}{\beta_i} = \sum_{i=0}^m \frac{1}{\beta_0^{i+1}}, \qquad \prod_{i=0}^m \beta_i^{\frac{1}{\beta_i}} = \prod_{i=0}^m(\beta_0^{i+1})^{\frac{1}{\beta_0^{i+1}}} = (\beta_0)^{\sum_{i=0}^m\frac{i+1}{\beta_0^{i+1}}}.$$

We can see

$$\sum_{i=0}^\infty \frac{i+1}{\beta_0^{i+1}} = \beta^* < +\infty, \quad \sum_{i=0}^\infty \frac{1}{\beta_0^{i+1}} = \frac{2}{2^*-p}.$$

Then, letting $m \to \infty$ in (3.16), we obtain that $u \in L^\infty(\mathbb{R}^N)$ and

$$\begin{aligned}\|u\|_\infty &\le C\lambda^{\frac{1}{2^*-p}} \beta_0^{\beta_*} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^{\frac{p-2}{2(2^*-p)}} \|u\|_{2^*} \\ &\le C\lambda^{\frac{1}{2^*-p}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^{\frac{2^*-2}{2(2^*-p)}}. \end{aligned} \tag{3.17}$$

This lemma is proved. □

Proof of Theorem 1.1. According to Lemma 3.7, equation (3.1) has a ground state solution u and $u \in \mathcal{N}$. By (3.8), Lemma 3.3-(i) and the Sobolev embedding, we have

$$\begin{aligned} c &= I(u) - \frac{1}{\mu}\langle I'(u), u\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) + \int_{\mathbb{R}^N} [\frac{1}{\mu} k(x,u)u - K(x,u)] \\ &\ge \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u\|^2. \end{aligned} \tag{3.18}$$

We can choose $v \in E \cap L^\infty(\mathbb{R}^N)$ such that $\|v\|_\infty < 1$. By (h_2) and (3) of Lemma 2.1, there exists a positive constant C_1 independent of λ such that

$$G(F^{-1}(tv)) \ge C_1 |F^{-1}(tv)|^q \ge C|tv|^q, \quad t \in [0,1].$$

Meanwhile, there exists $\lambda_0 > 0$ such that $I(v) < 0$ for $\lambda \ge \lambda_0$. It follows from the definition of c, Lemma 3.1-(2) and Lemma 2.1-(3) that

$$\begin{aligned} c &\le \max_{t\in[0,1]} I(tv) \\ &\le \max_{t\in[0,1]} \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 dx + V(x)v^2) - \lambda \int_{\mathbb{R}^N} G(F^{-1}(tv)) \\ &\le \max_{t\in[0,1]} \frac{t^2}{2} \|v\|^2 - Ct^q \lambda \int_{\mathbb{R}^N} |v|^q \\ &\le C\lambda^{-\frac{2}{q-2}}. \end{aligned} \tag{3.19}$$

Combining (3.17), (3.18) with (3.19), one has

$$\|u\|_\infty \le C\lambda^{\frac{1}{2^*-p}} \|u\|^{\frac{2^*-2}{2^*-p}} \le C\lambda^{\frac{1}{2^*-p}} \lambda^{\frac{1}{2-q}\cdot\frac{2^*-2}{2^*-p}}.$$

Since $p, q \in (2, 2^*)$, there exists $\lambda_1 \ge \lambda_0$ such that

$$\|u\|_\infty \le C\lambda_1^{\frac{(2^*-q)}{(2^*-p)(2-q)}} \le \delta.$$

Therefore, by the definition of g, we can obtain that u is also a positive solution of equation (2.5) for $\lambda \ge \lambda_1$. This ends the proof. □

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Leighton–Wintner type oscillation criteria for second-order differential equations with $p(t)$-Laplacian

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Abstract. This paper deals with the oscillation problems for nonlinear differential equations of the form $(r(t)|x'|^{p(t)-2}x')' + c(t)f(x) = 0$ involving $p(t)$-Laplacian. The Leighton–Wintner type oscillation criteria are established without any conditions on the limit of $p(t)$. In addition, we discuss the applications to partial differential equations. Some examples are given to illustrate our results.

Keywords: oscillation, second-order differential equation, $p(t)$-Laplacian, quasilinear differential equation, Emden–Fowler differential equation, Riccati technique.

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1 Introduction

In this paper, we consider the second-order nonlinear differential equation

$$\Big(r(t)|x'|^{p(t)-2}x'\Big)' + c(t)f(x) = 0, \quad t \geq t_0 \in \mathbb{R}, \tag{1.1}$$

where $r(t) > 0$, $c(t)$, and $p(t) > 1$ are continuous functions, and $f(u)$ is a continuous function satisfying the condition $uf(u) > 0$ for $u \neq 0$.

A function $x(t)$ is said to be a *solution* of equation (1.1) defined on $[t_0, \tau) \subset \mathbb{R}$, if $x(t)$ and the quasiderivative $r(t)|x'(t)|^{p(t)-2}x'(t)$ are continuously differentiable and $x(t)$ satisfies equation (1.1) on $[t_0, \tau)$. A nontrivial solution $x(t)$ of equation (1.1) is said to be a *singular solution of the first kind*, if there exists a number $T_x > t_0$ such that $x(t) \equiv 0$ for $t \geq T_x$. It is said to be a *singular solution of the second kind* if $\tau < \infty$, which means that $x(t)$ is nonextendable to the right, i.e.,

$$\limsup_{t\to\tau-} \big(|x(t)| + |x'(t)|\big) = \infty$$

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holds. It is said to be a *proper* solution if $x(t)$ is nonsingular. Furthermore, a proper solution $x(t)$ of equation (1.1) can be divided into the following two types. It is called *oscillatory*, if there exists a sequence $\{t_n\}$ of $[t_0,\infty)$ such that $x(t_n) = 0$ and $t_n \to \infty$ as $n \to \infty$. Otherwise, it is called *nonoscillatory*.

A great deal of papers have been devoted to the oscillation problems for the quasilinear differential equation

$$\left(r(t)|x'|^{p-2}x'\right)' + c(t)|x|^{p-2}x = 0 \tag{1.2}$$

involving the classical p-Laplacian. It is easy to see that the constant multiple of a solution of equation (1.2) is also a solution, but the sum of solutions is not always a solution. In this point of view, equation (1.2) is known as a *half-linear* differential equation (see [1,8]). With this advantage, we can introduce the generalized trigonometric functions and Sturm's separation and comparison theorems as basic tools for p-Laplacian. Moreover, the global existence and uniqueness of solutions of equation (1.2) are guaranteed for initial-value problem, i.e., all nontrivial solutions of equation (1.2) are proper. For example, various results for the oscillation problems for equation (1.2) can be found in [1,7–9,14–17,22] and the references cited therein. Especially, the so-called Leighton–Wintner type oscillation criterion has been obtained as follows.

Theorem A ([1,8])**.** *Suppose that*

$$\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)^{1/(p-1)} dt = \infty \quad \textit{and} \quad \int_{t_0}^{\infty} c(t)\,dt = \infty.$$

Then, all nontrivial solutions of equation (1.2) *are oscillatory.*

The differential operator in equation (1.1) is called $p(t)$*-Laplacian*, which is a generalization of p-Laplacian. It is also known as the one-dimensional version of the partial differential operator $p(\mathbf{x})$-Laplacian, which appears in mathematical models of various research fields such as nonlinear elasticity theory, electrorheological fluids, and image processing (see [4,13,18]). For example, oscillation problems for quasilinear elliptic partial differential equations with $p(\mathbf{x})$-Laplacian are considered in [23–25]. In particular, sufficient conditions are obtained under which all radial solutions of the equation

$$\operatorname{div}\left(|\nabla u|^{p(\mathbf{x})-2}\nabla u\right) + \frac{1}{|\mathbf{x}|^{\theta(\mathbf{x})}}|u|^{q(\mathbf{x})-2}u = 0 \quad \text{in } \Omega$$

are oscillatory in [25] under certain conditions on the limits of p, θ, and q, where $\Omega = \{\mathbf{x} \in \mathbb{R}^N \mid |x| > r_0\}$ with the Euclidean norm. The proof is based on radialization technique with ordinary differential equation involving $p(t)$-Laplacian. In this way, there has been an increasing interest in the study of asymptotic behavior of solutions for ordinary differential equations involving $p(t)$-Laplacian. For instance, those results can be found in [3,5,6,10–12,19–21]. In [10], a kind of comparison theorem is proved to the oscillation problems for equation (1.1). In addition, the existence of proper solutions and singular solutions of equation (1.1) is treated in [3].

However, we point out that the solution space of the equation

$$\left(r(t)|x'|^{p(t)-2}x'\right)' + c(t)|x|^{p(t)-2}x = 0 \tag{1.3}$$

involving $p(t)$-Laplacian does not have homogeneity unlike equation (1.2). Hence, to the best of our knowledge, generalized trigonometric functions and Sturm's separation and comparison theorems are not obtained for equation (1.3). Hence, not a few results do not rule out the coexistence of oscillatory and nonoscillatory solutions. Moreover, the literature on $p(t)$-Laplacian often assumes certain conditions on the limit of $p(t)$. For example, the log-Hölder decay condition is assumed in [12, 25], i.e., there exist $p > 1$, and $M > 0$ such that

$$t^{|p-p(t)|} < M$$

for t sufficiently large. This implies that $p(t) \to p > 1$ as $t \to \infty$.

The purpose of this paper is to establish Leighton–Wintner type oscillation criteria for equation (1.1). This paper is organized as follows. In Section 2, we give two oscillation criteria. In Section 3, we deal with the existence of proper solutions. Finally, we consider an application to partial differential equations in Section 4.

2 Oscillation problem

In this section, we give Leighton–Wintner type oscillation criteria for equation (1.1).

Theorem 2.1. *Assume that $f(u)$ is a smooth function satisfying $f'(u) \geq 0$ for $u \in \mathbb{R}$. Suppose that for any $L > 0$,*

$$\int_{t_0}^{\infty} \left(\frac{L}{r(t)}\right)^{1/(p(t)-1)} dt = \infty \quad \text{and} \quad \int_{t_0}^{\infty} c(t)\, dt = \infty. \tag{2.1}$$

Then, all proper solutions of equation (1.1) are oscillatory.

Proof. Suppose, toward a contradiction, that equation (1.1) has a positive solution. That is to say, there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Let

$$w(t) = \frac{r(t)|x'(t)|^{p(t)-2}x'(t)}{f(x(t))}.$$

Then, we have

$$w'(t) = -c(t) - \frac{r(t)|x'(t)|^{p(t)}f'(x(t))}{(f(x(t)))^2}.$$

Integrating both sides of this equality from t_1 to $t \geq t_1$, we get

$$w(t) = w(t_1) - \int_{t_1}^{t} c(s)\, ds - \int_{t_1}^{t} \frac{r(s)|x'(s)|^{p(s)}f'(x(s))}{(f(x(s)))^2}\, ds.$$

From (2.1) and $f'(u) \geq 0$ ($u \in \mathbb{R}$), there exists $t_2 \geq t_1$ such that

$$\int_{t_2}^{t} c(s)\, ds \geq 0$$

and $w(t) < 0$ for $t \geq t_2$, which implies $x'(t) < 0$ for $t \geq t_2$.

Integrating both sides of equation (1.1) from t_2 to $t \geq t_2$, we get

$$\begin{aligned} -r(t)|x'(t)|^{p(t)-1} &= r(t)|x'(t)|^{p(t)-2}x'(t) \\ &= r(t_2)|x'(t_2)|^{p(t_2)-2}x'(t_2) - \int_{t_2}^{t} c(s)f(x(s))\, ds \\ &= r(t_2)|x'(t_2)|^{p(t_2)-2}x'(t_2) - f(x(t)) \int_{t_2}^{t} c(s)\, ds \\ &\quad + \int_{t_2}^{t} f'(x(s))x'(s) \int_{t_2}^{s} c(\tau)\, d\tau\, ds \\ &\leq r(t_2)|x'(t_2)|^{p(t_2)-2}x'(t_2) = -r(t_2)|x'(t_2)|^{p(t_2)-1}. \end{aligned}$$

Hence, we have

$$-x'(t) \geq \left(\frac{K}{r(t)}\right)^{\frac{1}{p(t)-1}}$$

for $t \geq t_2$, where $K = r(t_2)|x'(t_2)|^{p(t_2)-1} > 0$. Thus, by (2.1) we obtain

$$x(t) \leq x(t_2) - \int_{t_2}^{t} \left(\frac{K}{r(s)}\right)^{\frac{1}{p(s)-1}} ds \to -\infty$$

as $t \to \infty$, which is a contradiction to the positivity of $x(t)$. □

We also prove the following criterion.

Theorem 2.2. *Assume that $c(t) > 0$ for $t \geq t_0$ and there exists a smooth function $g(u)$ such that $ug(u) > 0$ ($u \neq 0$), $g'(u) \geq 0$, and $|f(u)| \geq |g(u)|$ ($u \in \mathbb{R}$). Suppose that (2.1) holds for any $L > 0$. Then, all proper solutions of equation (1.1) are oscillatory.*

Proof. Suppose, toward a contradiction, that equation (1.1) has a positive solution $x(t)$. That is to say, there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Hence, $x(t)$ satisfies

$$\left(r(t)|x'(t)|^{p(t)-2}x'(t)\right)' + C(t)g(x(t)) = 0, \quad t \geq t_1, \tag{2.2}$$

where $C(t) = c(t)f(x(t))/g(x(t))$. We note that $C(t)$ is continuous because $x(t) > 0$ for $t \geq t_1$ and $ug(u) > 0$ for $u \neq 0$. Since $|f(u)| \geq |g(u)|$ ($u \in \mathbb{R}$), we see that $C(t) \geq c(t)$, and therefore, we get

$$\int_{t_0}^{\infty} C(t)\,dt \geq \int_{t_0}^{\infty} c(t)\,dt = \infty.$$

Proceeding in the same manner as the proof of Theorem 2.1 with (2.2), we see that the assertion holds. □

Remark 2.3. Although the positivity of $c(t)$ is required, we don't need the monotonicity and the smoothness of $f(u)$ in Theorem 2.2.

We consider the special case that $f(u) = |u|^{\lambda-2}u$, where $\lambda > 1$ is a constant. Then, equation (1.1) becomes the equation

$$\left(r(t)|x'|^{p(t)-2}x'\right)' + c(t)|x|^{\lambda-2}x = 0. \tag{2.3}$$

In the rest of this paper, for simplicity, we focus on equation (2.3). By Theorem 2.1, we give the following corollary.

Corollary 2.4. *Suppose that (2.1) holds for any $L > 0$. Then, all proper solutions of equation (2.3) are oscillatory.*

3 Existence of proper solutions

In order to deal with the asymptotic behavior of solutions, we must pay attention to the existence of singular solutions. In fact, for example, when $p(t) \equiv p > 1$ and $r(t) \equiv 1$, equation (2.3) becomes the generalized Emden–Fowler type differential equation

$$\left(|x'|^{p-2}x'\right)' + c(t)|x|^{\lambda-2}x = 0. \tag{3.1}$$

It is known that if $p > \lambda$ (resp., $p < \lambda$) then equation (3.1) has a singular solution of the first (resp., second) kind for certain $c(t)$ (see [2, Theorem 4]).

In this section, we consider the existence of proper solutions of equation (2.3). According to [3], the following theorem is proved.

Theorem B ([3]). *Suppose that $p(t)$ and $(r(t))^{1/(p(t)-1)}$ are continuously differentiable, $p(t)$ is nondecreasing, and $c(t)$ is positive. Then, every nontrivial solutions of equation (2.3) is proper.*

Using Corollary 2.4 and Theorem B, we obtain the following corollary.

Corollary 3.1. *Assume that $p(t)$ and $(r(t))^{1/(p(t)-1)}$ are continuously differentiable, $p(t)$ is nondecreasing, and $c(t)$ is positive. Suppose that (2.1) holds for any $L > 0$. Then, all nontrivial solutions of equation (2.3) are oscillatory.*

We propose an example of Corollary 3.1.

Example 3.2. Let $t_0 = 1$, $r(t) \equiv 1$, $c(t) = 1/t$, and $p(t) = 3 - 1/t$. Then, equation (2.3) becomes

$$\left(|x'|^{3-1/t}x'\right)' + \frac{1}{t}|x|^{\lambda-2}x = 0. \tag{3.2}$$

From Corollary 3.1, all nontrivial solutions of equation (3.2) are oscillatory. Figure 3.1 indicates the solution is proper and oscillatory.

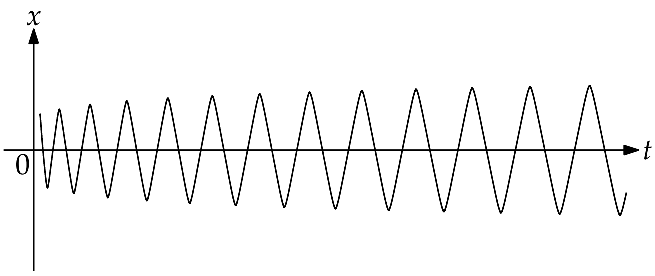

Figure 3.1: A solution $x(t)$ of equation (3.2) with $x(1) = 3$, $x'(1) = 0$, and $\lambda = 5$.

We next consider the case when $p(t)$ does not have monotonicity. For equation (2.3), the following propositions are derived from [3, Theorems 2.1, 2.2].

Proposition 3.3. *Suppose that $p(t) \le \lambda$ for $t \in [t_0, \infty)$. Then, equation (2.3) has no singular solutions of the first kind.*

Proposition 3.4. *Suppose that $p(t) \ge \lambda$ for $t \in [t_0, \infty)$. Then, equation (2.3) has no singular solutions of the second kind.*

In the case when $c(t)$ is negative, then the following result is given from Proposition 3.4.

Theorem 3.5. *Suppose that $p(t) \ge \lambda$ and $c(t) < 0$ for $t \in [t_0, \infty)$. Then, equation (2.3) has proper solutions.*

Proof. Let $x(t)$ be a solution of equation (2.3) satisfying the initial condition $x(t_0) > 0$ and $x'(t_0) > 0$. Since $c(t)$ is negative, we can find $T > t_0$ such that

$$\left(r(t)|x'(t)|^{p(t)-2}x'(t)\right)' = -c(t)|x(t)|^{\lambda-2}x(t) > 0$$

for $t \in [t_0, T)$, which implies that $r(t)|x'(t)|^{p(t)-2}x'(t)$ is positive increasing for $t \in [t_0, T)$. Hence, $x'(t)$ is positive for any $t \in [t_0, \infty)$, and therefore, $x(t)$ is a positive increasing solution. From Proposition 3.4, we see that $x(t)$ is proper. □

However, it is clear that (2.1) does not hold and equation (2.3) has no oscillatory solutions under the assumptions of Theorem 3.5.

In view of Propositions 3.3 and 3.4, we see that all nontrivial solutions of equation (2.3) are proper when $p(t) \equiv \lambda$. Otherwise, we cannot exclude the possibilities of the existence of singular solutions by using these propositions. To illustrate this problem, we introduce an example of Corollary 2.4 and Propositions 3.3, 3.4.

Example 3.6. Let $t_0 = 1$, $r(t) \equiv 1$, $c(t) = 1/t$, and $p(t) = \sin t + 5/2$. Then, equation (2.3) becomes

$$\left(|x'|^{\sin t + 1/2} x'\right)' + \frac{1}{t}|x|^{\lambda - 2}x = 0. \tag{3.3}$$

In the case of $\lambda \geq 7/2$, equation (3.3) has no singular solution of the first kind, as stated in Proposition 3.3. However, we cannot rule out the possibility that equation (3.3) has singular solutions of the second kind. In fact, keen spikes can be observed in Figure 3.2. On the other hand, when $1 < \lambda \leq 3/2$, equation (3.3) has no singular solution of the second kind according to Proposition 3.4. However, we cannot exclude the possibility that equation (3.3) has singular solutions of the first kind. We can identify the points in Figure 3.3 where the derivative of the solution is zero, even though they are not extrema. In the case of $3/2 < \lambda < 7/2$, there are possibilities that equation (3.3) has singular solutions of the first/second kind. In any cases, it can be derived from Corollary 2.4 that all proper solutions of equation (3.3) are oscillatory.

In the case of $p(t) \not\equiv \lambda$, the existence of proper solutions of equation (2.3) is proved by Theorem 4.1 in [3] under the additional assumption $\liminf_{t\to\infty} p(t) > 1$. However, in order to apply this result, the condition

$$\int_{t_0}^{\infty} |c(t)|\, dt < \infty$$

is also required, which is the opposite case of (2.1). It is an open problem if equation (2.3) has proper solutions under $p(t) \not\equiv \lambda$ and (2.1).

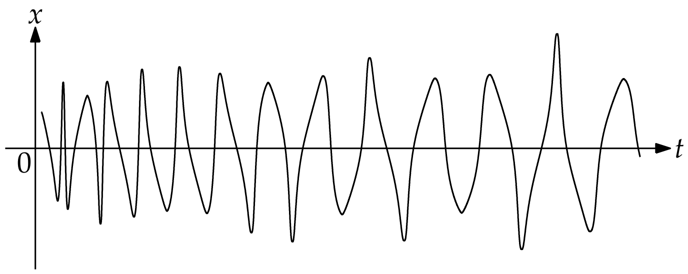

Figure 3.2: A solution $x(t)$ of equation (3.3) with $x(1) = 3$, $x'(1) = 0$, and $\lambda = 4$.

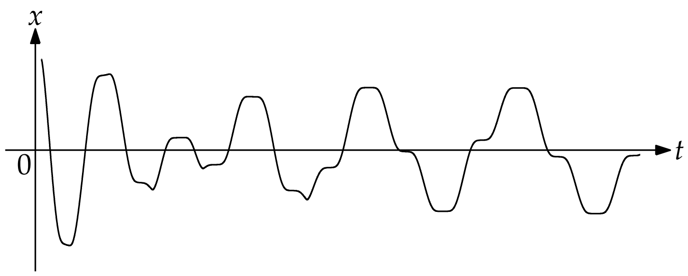

Figure 3.3: A solution $x(t)$ of equation (3.3) with $x(1) = 1$, $x'(1) = 0$, and $\lambda = 3/2$.

4 Applications

In this section, we propose an application to partial differential equations. Let us consider the quasilinear differential equation

$$\operatorname{div}\left(|\nabla u|^{p(\mathbf{x})-2}\nabla u\right)+F(\mathbf{x})|u|^{\lambda-2}u=0 \quad \text{in } \Omega, \tag{4.1}$$

where $\Omega=\{\mathbf{x}\in\mathbb{R}^N \mid |x|>r_0\}$. If u is a radially symmetric function, i.e., $u(\mathbf{x})=y(t)$, $t=|\mathbf{x}|$, we can write equation (4.1) as

$$\left(t^{N-1}|y'|^{p(t)-2}y'\right)'+t^{N-1}F(t)|y|^{\lambda-2}y=0 \quad \text{for } t>r_0. \tag{4.2}$$

We say that a radially symmetric solution $u(\mathbf{x})$ of (4.1) is oscillatory if it keeps neither positive nor negative, that is, the solution $y(t)$ of equation (4.2) corresponding to $u(\mathbf{x})$ is oscillatory. Using Corollary 2.4, we obtain the following theorem.

Theorem 4.1. *Suppose that for any $L>0$,*

$$\int_{t_0}^{\infty}\left(\frac{L}{t^{N-1}}\right)^{1/(p(t)-1)}dt=\infty \quad \textit{and} \quad \int_{t_0}^{\infty}t^{N-1}F(t)\,dt=\infty.$$

Then, all radially symmetric solutions of equation (4.1) are oscillatory.

Example 4.2. Let $N\in\mathbb{N}$, $F(t)=1/t^N$, and $p(t)=\sin t+N+3/2$. Then, equation (4.2) becomes

$$\left(t^{N-1}|y'|^{\sin t+N-1/2}y'\right)'+\frac{1}{t}|y|^{\lambda-2}y=0 \text{ for } t>r_0 \tag{4.3}$$

and it is easy to see that $\int_{r_0}^{\infty}t^{N-1}F(t)\,dt=\infty$. In addition, we have $1/(p(t)-1)\le 2/(2N-1)$. Hence, it is obvious that

$$\int_{r_0}^{\infty}\left(\frac{L}{t^{N-1}}\right)^{1/(p(t)-1)}dt=\infty$$

when $N=1$. In the case of $N\ge 2$, since $L/t^{N-1}\to 0$ as $t\to\infty$, we can find $r_1\ge r_0$ such that $L/t^{N-1}<1$. Hence, we have

$$\begin{aligned}\int_{r_1}^{t}\left(\frac{L}{s^{N-1}}\right)^{1/(p(s)-1)}ds &\ge \int_{r_1}^{t}\left(\frac{L}{s^{N-1}}\right)^{2/(2N-1)}ds = L^{2/(2N-1)}\int_{r_1}^{t}s^{-2(N-1)/(2N-1)}\,ds\\ &=(2N-1)L^{2/(2N-1)}\left(t^{1/(2N-1)}-r_1^{1/(2N-1)}\right)\to\infty\end{aligned}$$

as $t\to\infty$. From Theorem 4.1, all radially symmetric solutions of equation (4.1) are oscillatory.

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A variation of parameters formula for nonautonomous linear impulsive differential equations with piecewise constant arguments of generalized type

This paper is dedicated to the memory of Prof. Nicolás Yus Suárez

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Abstract. In this work, we give a variation of parameters formula for nonautonomous linear impulsive differential equations with piecewise constant arguments of generalized type. We cover several cases of differential equations with deviated arguments investigated before as particular cases. We also give some examples showing the applicability of our results.

Keywords: variation of parameters formula, piecewise constant argument, linear functional differential equations, DEPCAG, IDEPCAG.

2020 Mathematics Subject Classification: 34A36, 34A37, 34A38, 34K34, 34K45.

1 Introduction

Occasionally, natural phenomena must be modeled using differential equations that may have discontinuous solutions, such as a piecewise constant, or the impulsive effect must be present. Some examples of such modeling can be found in the works of S. Busenberg and K. Cooke [7] (where the authors modeled vertical transmission diseases) and L. Dai and M. C. Singh [12] (oscillatory motion of spring-mass systems subject to piecewise constant forces such $Ax([t])$ or $A\cos([t])$). The last work studied the motion of mechanisms modeled by

$$mx''(t) + kx_1 = A\sin\left(\omega\left[\frac{t}{T}\right]\right),$$

where $[\cdot]$ is the greatest integer function. (See [11]).

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In the 70's, A. Myshkis [15] studied differential equations with deviating arguments ($h(t) \leq t$, such as $h(t) = [t]$ or $h(t) = [t-1]$). The Ukrainian mathematician M. Akhmet generalized those systems, introducing differential equations of the form

$$y'(t) = f(t, y(t), y(\gamma(t))), \tag{1.1}$$

where $\gamma(t)$ is a *piecewise constant argument of generalized type*. In order to define such γ, let $(t_n)_{n\in\mathbb{Z}}$ and $(\zeta_n)_{n\in\mathbb{Z}}$ such that $t_n < t_{n+1}, \forall n \in \mathbb{Z}$ with $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to-\infty} t_n = -\infty$ and $\zeta_n \in [t_n, t_{n+1}]$. Then, $\gamma(t) = \zeta_n$, if $t \in I_n = [t_n, t_{n+1})$. I.e., $\gamma(t)$ is a step function. An elementary example of such functions is $\gamma(t) = [t]$ which is constant in every interval $[n, n+1[$ with $n \in \mathbb{Z}$ (see (1.3)).

If a piecewise constant argument is used, the interval I_n is decomposed into an advanced and delayed subintervals $I_n = I_n^+ \bigcup I_n^-$, where $I_n^+ = [t_n, \zeta_n]$ and $I_n^- = [\zeta_n, t_{n+1}]$. This class of differential equations is known as *Differential Equations with Piecewise Constant Argument of Generalized Type* (*DEPCAG*). They have continuous solutions, even though γ is discontinuous. If we assume continuity of the solutions of (1.1), integrating from t_n to t_{n+1}, we define a finite-difference equation, so we are in the presence of a hybrid dynamic (see [3,17]).

For example, taking $\gamma(t) = \left[\frac{t+l}{h}\right]h$ with $0 \leq l < h$, we have

$$\left[\frac{t+l}{h}\right]h = nh, \text{ when } t \in I_n = [nh-l, (n+1)h-l).$$

Then, we see that $\gamma(t) - t \geq 0 \Leftrightarrow t \leq nh$ and $\gamma(t) - t \leq 0 \Leftrightarrow t \geq nh$. Hence, we have

$$I_n^+ = [nh-l, nh], \quad I_n^- = [nh, (n+1)h-l].$$

Now, if an impulsive condition is defined at $\{t_n\}_{n\in\mathbb{Z}}$, we are in the presence of the *Impulsive differential equations with piecewise constant argument of generalized type* (*IDEPCAG*) (see [2]),

$$\begin{aligned} x'(t) &= f(t, x(t), x(\gamma(t))), \qquad t \neq t_n \\ \Delta x(t_n) &:= x(t_n) - x(t_n^-) = J_n(x(t_n^-)), \qquad t = t_n, \quad n \in \mathbb{N}, \end{aligned} \tag{1.2}$$

where $x(t_n^-) = \lim_{t\to t_n^-} x(t)$, and J_n is the impulsive operator (see [18]).

When the piecewise constant argument used in a differential equation is explicit, it will be called DEPCA (IDEPCA if it has impulses).

An elementary and illustrative example of IDEPCA

Consider the scalar IDEPCA

$$\begin{aligned} x'(t) &= (\alpha-1)x([t]), \qquad t \neq n \\ x(n) &= \beta x(n^-), \qquad t = n, \quad n \in \mathbb{N}. \end{aligned} \tag{1.3}$$

where $\alpha, \beta \in \mathbb{R}$, $\beta \neq 1$.

If $t \in [n, n+1)$ for some $n \in \mathbb{Z}$, equation (1.3) can be written as

$$x'(t) = (\alpha-1)x(n). \tag{1.4}$$

In the following, we will assume $t_0 = 0$. Now, integrating on $[n, n+1)$ from n to t we see that

$$x(t) = x(n)(1 + (\alpha-1)(t-n)). \tag{1.5}$$

Next, assuming continuity at $t = n+1$, we have

$$x((n+1)^-) = \alpha x(n).$$

Applying the impulsive condition to the last expression, we get the following *finite-difference equation*

$$x((n+1)) = (\alpha\beta)x(n).$$

Its solution is

$$x(n) = (\alpha\beta)^n x(0). \tag{1.6}$$

Finally, applying (1.6) in (1.5) we have

$$x(t) = (\alpha\beta)^{[t]} (1 + (\alpha - 1)(t - [t]))x(0). \tag{1.7}$$

Remark 1.1. From (1.7), we can conclude that the underlying dynamic is of mixed type. The discrete and the continuous parts of the system are dependent. For example, A stable continuous part (associated with the coefficient α) can be unstabilized by the discrete part (associated with the parameter β). See [18].

In the next table, we describe some of the behavior of the solutions of (1.7):

Behavior of solutions	**Condition**
$\lvert x(t)\rvert \xrightarrow{t\to\infty} 0$ exponentially.	$\lvert\alpha\beta\rvert < 1$ and $\alpha\beta \neq 0$.
$x(t)$ is constant.	$\alpha\beta = 0$ or $\alpha = \beta = 1$
$x(t)$ is oscillatory.	$\alpha\beta < 0$
$x(t)$ is piecewise constant.	$\alpha = 1$
$\lvert x(t)\rvert$ is piecewise constant and $x(t) \xrightarrow{t\to\infty} +\infty$.	$\alpha = 1$ and $\lvert\beta\rvert > 1$
$x(t)$ is piecewise constant and $x(t) \xrightarrow{t\to\infty} 0$.	$\alpha = 1$ and $0 < \beta < 1$
$\lvert x(t)\rvert \xrightarrow{t\to\infty} +\infty$ exponentially.	$\lvert\alpha\beta\rvert > 1$.

Table 1.1: Behavior of solutions of (1.7)

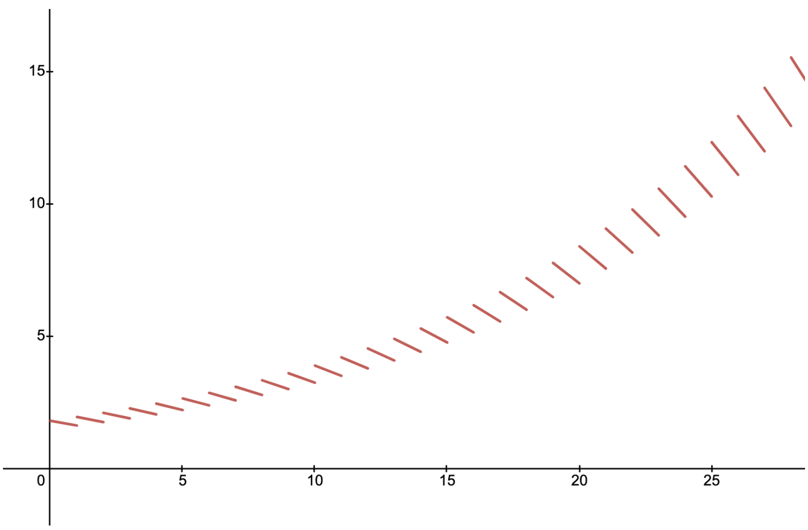

Figure 1.1: Solution of (1.3) with $\alpha = 0.9$, $\beta = 1.2$, $x_0 = 1.8$.

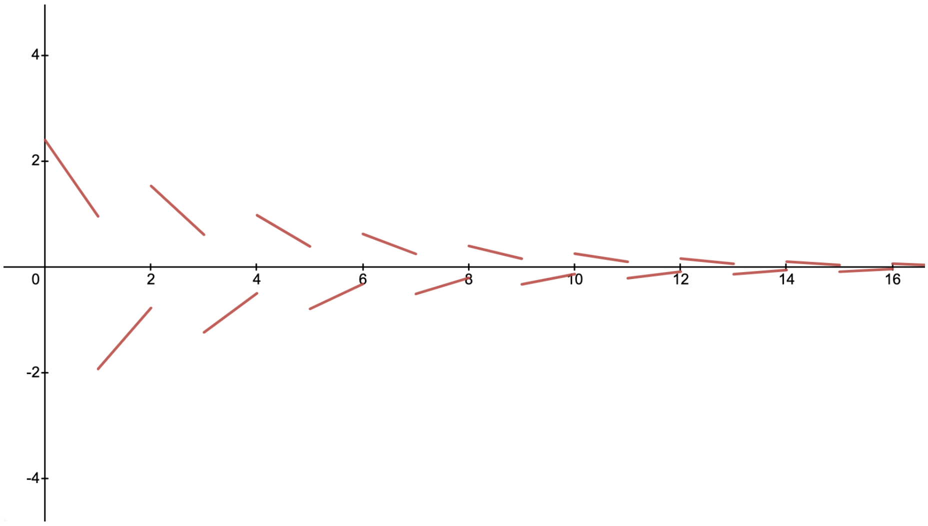

Figure 1.2: solution of (1.3) with $\alpha = 0.4$, $\beta = -2$, $x_0 = 2.4$.

1.1 Why study IDEPCAG?: impulses in action

Example 1.2. Let the following scalar linear DEPCA

$$x'(t) = a(t)(x(t) - x([t])), \qquad x(\tau) = x_0, \tag{1.8}$$

and the scalar linear IDEPCA

$$\begin{array}{ll} z'(t) = a(t)\left(z(t) - z([t])\right), & t \neq k \\ z(k) = c_k z(k^-), & t = k, \quad k \in \mathbb{Z}, \end{array} \tag{1.9}$$

where $a(t)$ is a continuous locally integrable function and $(c_k)_{k\in\mathbb{N}}$ a real sequence such that $c_k \notin \{0, 1\}$, for all $k \in \mathbb{N}$. As $\gamma(t) = [t]$, we have $t_k = k = \zeta_k = k$ if $t \in [k, k+1)$, $k \in \mathbb{Z}$.

The solution of (1.8) is $x(t) = x_0$, $\forall t \geq \tau$. I.e., all the solutions are constant (see [17]).

On the other hand, as we will see, the solution of (1.9) is

$$z(t) = \left(\prod_{j=k(\tau)+1}^{k(t)} c_j\right) z(\tau), \qquad t \geq \tau,$$

where $k(t) = k$ is the only integer such that $t \in [k, k+1]$.

Hence, all the solutions are nonconstant if $c_j \neq 1$ and $c_j \neq 0$, for all $j \geq k(\tau)$. This example shows the differences between DEPCA and IDEPCA systems. The discrete part of the system can greatly impact the whole dynamic, determining the qualitative properties of the solutions.

1.2 Fundamental matrices and variation of parameters formulas: an overview

1.2.1 The fundamental matrix of a DEPCA system

In [9], K. L. Cooke and J. Wiener were the first to obtain a fundamental matrix for a scalar *DEPCA*'s using the delayed piecewise constant arguments $\gamma(t) = [t]$, $\gamma(t) = [t-1]$, $\gamma(t) = [t-n]$ and $\gamma(t) = t - n[t]$. Also, they considered the very interesting scalar DEPCA

$$x'(t) = a(t)x(t) + \sum_{i=0}^{n} a_i(t)x([t-i]), \qquad a_n \neq 0,$$

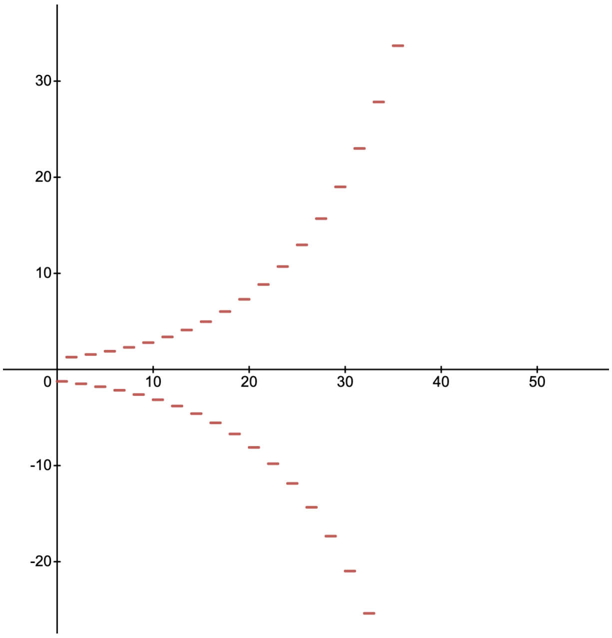

Figure 1.3: Solution of (1.9) with $c_k = -1.1$ and $z(0) = -1.2$

and

$$x'(t) = ax(t) + \sum_{i=1}^{n} a_i x(t - i[t]).$$

Also, in [19], S. M. Shah and J. Wiener studied the DEPCA

$$x'(t) = a(t)x(t) + \sum_{i=0}^{n} a_i(t)x([t+i]), \qquad a_n \neq 0, \quad n \geq 2.$$

Then, in [8], K. L. Cooke and J. Wiener studied the mixed-type piecewise constant argument $\gamma(t) = 2\left[\frac{t+1}{2}\right]$ and considered the DEPCA

$$z'(t) = az(t) + bz(2[(t+1)/2]).$$

Additionally, in [22], J. Wiener and A. R. Aftabizadeh considered the mixed-type piecewise constant argument $\gamma(t) = m\left[\frac{t+k}{m}\right]$ where $0 < k < m$, $k, m, n \in \mathbb{Z}^+$, and they studied the DEPCA

$$w'(t) = aw(t) + bw(m[(t+k)/m]).$$

1.2.2 Variation of parameters formula for a DEPCA

In [13] (1991), N. Jayasree and S. G. Deo were the first to consider the advanced and delayed parts of the solutions studying the equation

$$z'(t) = az(t) + bz(2[(t+1)/2]) + f(t),$$

obtaining a variation of parameters formula for this DEPCA, in terms of the homogeneous linear DEPCA associated:

$$\begin{aligned} z(t) = y(t) &+ \sum_{j=0}^{[(t+1)/2]-1} \lambda^{-1}(1) \int_{2j}^{2j+1} \Psi(t,2j)\phi(2j+1,s)f(s)ds \\ &- \sum_{j=1}^{[(t+1)/2]} \lambda^{-1}(1) \int_{2j}^{2j-1} \Psi(t,2j)\phi(2j-1,s)f(s)ds \\ &+ \int_{2[(t+1)/2]}^{t} \phi(t,s)f(s)ds, \end{aligned}$$

where

$$\lambda(t) = \exp(at)\left(1+a^{-1}b\right) - a^{-1}b,$$

ϕ and Ψ are the fundamental solutions of $x'(t) = ax(t)$ and $y'(t) = ay(t) + by(2[(t+1)/2])$ respectively.

In [14] (2001), Q. Meng and J. Yan obtained a variation of parameters formula for the differential equation

$$x'(t) + a(t)x(t) + b(t)x(g(t)) = f(t) \quad \text{for } t > 0,$$

where $a(t), b(t)$ and $f(t)$ are locally integrable functions on $[0,\infty)$, $g(t)$ is a piecewise constant function defined by $g(t) = np$ for $t \in [np-l,(n+1)p-l)$ with $n \in \mathbb{N}$ and p, l positive constants such that $p > l$. The authors studied the oscillation and asymptotic stability properties of the solutions.

In [1] (2008), M. Akhmet considered the DEPCAG for systems

$$z'(t) = A(t)z(t) + B(t)z(\gamma(t)) + F(t), \tag{1.10}$$

$$w'(t) = A(t)w(t) + B(t)w(\gamma(t)) + g\left(t, w(t), w(\gamma(t))\right), \tag{1.11}$$

where $A(t), B(t) \in C(\mathbb{R})$ are $n \times n$ real valued uniformly bounded on $\mathbb{R}$ matrices, $g(t,x,y) \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n)$ is an $n \times 1$ Lipschitz real valued function with $g(t,0,0) = 0$, $\gamma(t)$ is a piecewise constant argument of generalized type. The author found the following variation of parameters formula

$$\begin{aligned} w(t) = W(t,t_0)w_0 &+ W(t,t_0)\int_{t_0}^{\zeta_i} X(t_0,s)g(s,w(s),w(\gamma(s)))ds \\ &+ \sum_{k=i}^{j-1} W(t,t_{k+1}) \int_{\zeta_k}^{\zeta_{k+1}} X(t_{k+1},s)g(s,w(s),w(\gamma(s)))ds \\ &+ \int_{\zeta_j}^{t} X(t,s)g(s,w(s),w(\gamma(s)))ds, \end{aligned}$$

where $j = j(t)$ is the only $j \in \mathbb{Z}$ such that $t_{j(t)} \le t \le t_{j(t)+1}$, $t_k \le \zeta_k \le t_{k+1}$, $t_i \le t_0 \le t_{i+1}$, X is the fundamental matrix of

$$x'(t) = A(t)x(t),$$

and W is the fundamental matrix of the homogeneous linear DEPCAG

$$y'(t) = A(t)y(t) + B(t)y(\gamma(t)).$$

Later, in [17] (2011), M. Pinto gave a new DEPCAG variation of parameters formula. This time, the author considered the delayed and advanced intervals defined by the general piecewise constant argument

$$
\begin{aligned}
z(t) = W(t,t_0)z_0 + \underbrace{W(t,t_0)\int_{t_0}^{\zeta_i} X(t_0,s)g(s,z(s),z(\gamma(s)))ds}_{I_k^+} \\
+ \sum_{k=i+1}^{j} \underbrace{W(t,t_k)\int_{t_k}^{\zeta_k} X(t_k,s)g(s,z(s),z(\gamma(s)))ds}_{I_k^+} \\
+ \sum_{k=i}^{j-1} \underbrace{W(t,t_{k+1})\int_{\zeta_k}^{t_{k+1}} X(t_{k+1},s)g(s,z(s),z(\gamma(s)))ds}_{I_k^-} \\
+ \underbrace{\int_{\zeta_j}^{t} X(t,s)g(s,z(s),z(\gamma(s)))ds,}_{I_k^-}
\end{aligned}
$$

where $t_i \le t_0 \le t_{i+1}$ and $t_{j(t)} \le t \le t_{j(t)+1}$.

In the DEPCAG theory, decomposing the interval I_n into the advanced and delayed subintervals is critical. As we will see, it is necessary for the forward or backward continuation of solutions.

1.2.3 Variation of parameters formula for an IDEPCA: the impulsive effect applied

For the IDEPCA case, In [16] (2012), G. Oztepe and H. Bereketoglu studied the scalar IDEPCA

$$
\begin{aligned}
x'(t) &= a(t)(x(t) - x([t+1])) + f(t), \qquad x(0) = x_0, \quad t \neq n \in \mathbb{N} \\
\Delta x(n) &= d_n, \qquad t = n, \quad n \in \mathbb{N}.
\end{aligned} \tag{1.12}
$$

They proved the convergence of the solutions to a real constant when $t \to \infty$, and they showed the limit value in terms of x_0, using a suitable integral equation. They concluded the following expression for the solutions of (1.12)

$$
\begin{aligned}
x(t) = {} & \exp\left(\int_{[t]}^{t} a(u)du\right) x([t]) + \left(1 - \exp\left(\int_{[t]}^{t} a(u)du\right)\right) x([t+1]) \\
& + \int_{[t]}^{t} \exp\left(\int_{s}^{t} a(u)du\right) f(s)ds,
\end{aligned}
$$

where

$$
x([t]) = x_0 + \sum_{j=0}^{[t]-1} \left(\int_{j}^{j+1} \exp\left(-\int_{j}^{s} a(u)du\right) f(s)ds + \exp\left(-\int_{j}^{j+1} a(u)du\right) d_{j+1}\right).
$$

For the IDEPCA case, in [6] (2023), K-S. Chiu and I. Berna considered the following impulsive differential equation with a piecewise constant argument

$$
\begin{aligned}
y'(t) &= a(t)y(t) + b(t)y\left(p\left[\frac{t+l}{p}\right]\right), \quad y(\tau) = c_0, \qquad t \neq kp - l \\
\Delta y(kp - l) &= d_k y(kp - l^-), \qquad t = kp - l, \quad k \in \mathbb{Z},
\end{aligned} \tag{1.13}
$$

and

$$y'(t) = a(t)y(t) + b(t)y\left(p\left[\frac{t+l}{p}\right]\right) + f(t), \qquad y(\tau) = c_0, \quad t \neq kp - l$$
$$\Delta y(kp-l) = d_k y(kp - l^-), \qquad t = kp - l, \quad k \in \mathbb{Z}, \tag{1.14}$$

where $a(t) \neq 0$, $b(t)$ and $f(t)$ are real-valued continuous functions, $p < l$ and $d_k \in \mathbb{R} - \{1\}$. The authors obtained criteria for the existence and uniqueness, a variation of parameters formula, a Gronwall–Bellman inequality, stability and oscillation criteria for solutions for (1.13) and (1.14).

To our knowledge, there is no variation formula for impulsive differential equations with a generalized constant argument. As we have shown, some authors have studied just some particular cases before.

2 Aim of the work

We will get a variation of parameters formula associated with *IDEPCAG* system

$$\begin{array}{ll} x'(t) = A(t)x(t) + B(t)x(\gamma(t)) + F(t), & t \neq t_k \\ \Delta x|_{t=t_k} = C_k x(t_k^-) + D_k, & t = t_k, \end{array} \tag{2.1}$$

extending the particular case treated in [6] and the general results of the DEPCAG case studied in [17] to the IDEPCAG context.

3 Preliminaires

Let $\mathcal{PC}(X,Y)$ be the set of all functions $r : X \to Y$ which are continuous for $t \neq t_k$ and continuous from the left with discontinuities of the first kind at $t = t_k$. Similarly, let $\mathcal{PC}^1(X,Y)$ the set of functions $s : X \to Y$ such that $s' \in \mathcal{PC}(X,Y)$.

Definition 3.1 (DEPCAG solution)**.** A continuous function $x(t)$ is a solution of (1.1) if:

(i) $x'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception at the times t_k, $k \in \mathbb{Z}$, where the one side derivative exists.

(ii) $x(t)$ satisfies (1.1) on the intervals of the form (t_k, t_{k+1}), and it holds for the right derivative of $x(t)$ at t_k.

Definition 3.2 (IDEPCAG solution)**.** A piecewise continuous function $y(t)$ is a solution of (1.2) if:

(i) $y(t)$ is continuous on $I_k = [t_k, t_{k+1})$ with first kind discontinuities at t_k, $k \in \mathbb{Z}$, where $y'(t)$ exists at each $t \in \mathbb{R}$ with the possible exception at the times t_k, where lateral derivatives exist (i.e. $y(t) \in \mathcal{PC}^1([t_k, t_{k+1}), R^n)$).

(ii) The ordinary differential equation

$$y'(t) = f(t, y(t), y(\zeta_k))$$

holds on every interval I_k, where $\gamma(t) = \zeta_k$.

(iii) For $t = t_k$, the impulsive condition

$$\Delta y(t_k) = y(t_k) - y(t_k^-) = J_k(y(t_k^-))$$

holds. I.e., $y(t_k) = y(t_k^-) + J_k(y(t_k^-))$, where $y(t_k^-)$ denotes the left-hand limit of the function y at t_k.

Let the IDEPCAG system:

$$\begin{aligned} x'(t) &= f(t, x(t), x(\gamma(t))), \qquad && t \neq t_k \\ x(t_k) - x\left(t_k^-\right) &= J_k(x(t_k^-)), && t = t_k, \\ x(\tau) &= x_0, \end{aligned} \tag{3.1}$$

where $f \in C([\tau, \infty) \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $J_k \in C(\{t_k\}, \mathbb{R}^n)$ and $(\tau, x_0) \in \mathbb{R} \times \mathbb{R}^n$.

Let the following hypotheses hold:

(H1) Let $\eta_1, \eta_2 : \mathbb{R} \to [0, \infty)$ locally integrable functions and $\lambda_k \in \mathbb{R}^+$, $\forall k \in \mathbb{Z}$; such that

$$\begin{aligned} \|f(t, x_1, y_1) - f(t, x_2, y_2)\| &\leq \eta_1(t) \|x_1 - x_2\| + \eta_2(t) \|y_1 - y_2\|, \\ \left\|J_k(x_1(t_k^-)) - J_k(x_2(t_k^-))\right\| &\leq \lambda_k \left\|x_1\left(t_k^-\right) - x_2\left(t_k^-\right)\right\|. \end{aligned}$$

where $\|\cdot\|$ is some matricial norm.

(H2)
$$\overline{\nu} = \sup_{k \in \mathbb{Z}} \left(\int_{t_k}^{t_{k+1}} (\eta_1(s) + \eta_2(s))\, ds \right) < 1.$$

In the following, we mention some useful results: an integral equation associated with (2.1) and two Gronwall–Bellman type inequalities necessary to prove the uniqueness and stability of solutions.

3.1 An integral equation associated to (3.1)

Theorem 3.3 ([4, Lemma 4.2]). *a function $x(t) = x(t, \tau, x_0)$, $\tau \in \mathbb{R}^+$ is a solution of* (3.1) *on* $\mathbb{R}^+$ *if and only if satisfies:*

$$x(t) = x_0 + \int_\tau^t f(s, x(s), x(\gamma(s)))ds + \sum_{\tau < t_k \leq t} J_k\left(x\left(t_k^-\right)\right),$$

where

$$\begin{aligned} \int_\tau^t f(s, x(s), x(\gamma(t)))ds = \int_\tau^{t_1} f(s, x(s), x(\zeta_0))\, ds + \sum_{j=1}^{k(t)-1} \int_{t_j}^{t_{j+1}} f(s, x(s), x(\zeta_j))ds \\ + \int_{t_{k(t)}}^t f\left(s, x(s), x\left(\zeta_{k(t)}\right)\right) ds, \end{aligned}$$

3.2 First IDEPCAG Gronwall–Bellman type inequality

Lemma 3.4 ([20], [4, Lemma 4.3]). *Let I an interval and $u, \eta_1, \eta_2 : I \to [0, \infty)$ such that u is continuous (with possible exception at $\{t_k\}_{k \in \mathbb{N}}$), η_1, η_2 are continuous and locally integrable functions, $\eta : \{t_k\} \to [0, \infty)$ and $\gamma(t)$ a piecewise constant argument of generalized type such that $\gamma(t) = \zeta_k$, $\forall t \in I_k = [t_k, t_{k+1})$ with $t_k \leq \zeta_k \leq t_{k+1}$ $\forall k \in \mathbb{N}$. Assume that $\forall t \geq \tau$*

$$u(t) \leq u(\tau) + \int_\tau^t (\eta_1(s)u(s) + \eta_2(s)u(\gamma(s)))\, ds + \sum_{\tau < t_k \leq t} \eta(t_k)u(t_k^-)$$

and

$$\widehat{\vartheta}_k = \int_{t_k}^{\zeta_k} (\eta_1(s) + \eta_2(s))\,ds \le \widehat{\vartheta} := \sup_{k\in\mathbb{N}} \widehat{\vartheta}_k < 1. \tag{3.2}$$

hold. Then, for $t \ge \tau$, we have

$$u(t) \le \left(\prod_{\tau<t_k\le t}(1+\eta(t_k))\right)\exp\left(\int_\tau^t\left(\eta_1(s)+\frac{\eta_2(s)}{1-\widehat{\vartheta}}\right)ds\right)u(\tau), \tag{3.3}$$

$$u(\zeta_k) \le (1-\vartheta)u(t_k) \tag{3.4}$$

$$u(\gamma(t)) \le (1-\vartheta)^{-1}\left(\prod_{\tau<t_k\le t}(1+\eta_3(t_j))\right)\exp\left(\int_\tau^t\left(\eta_1(s)+\frac{\eta_2(s)}{1-\widehat{\vartheta}}\right)ds\right)u(\tau). \tag{3.5}$$

3.3 Second IDEPCAG Gronwall–Bellman type inequality

Lemma 3.5 ([5, 20])**.** *Let I an interval and $u, \eta_1, \eta_2 : I \to [0,\infty)$ such that u is continuous (with possible exception at $\{t_k\}_{k\in N}$), η_1, η_2 are continuous and locally integrable functions, $\eta : \{t_k\} \to [0,\infty)$ and $\gamma(t)$ a piecewise constant argument of generalized type such that $\gamma(t) = \zeta_k$, $\forall t \in I_k = [t_k, t_{k+1})$ with $t_k \le \zeta_k \le t_{k+1}$ $\forall k \in \mathbb{N}$. Assume that $\forall t \ge \tau$*

$$u(t) \le u(\tau) + \int_\tau^t (\eta_1(s)u(s) + \eta_2(s)u(\gamma(s)))ds + \sum_{\tau<t_k\le t}\eta(t_k)u(t_k^-) \tag{3.6}$$

and

$$\varrho_k = \int_{t_k}^{\zeta_k}\left(\eta_2(s)e^{\int_s^{\zeta_k}\eta_1(r)dr}\right)ds \le \varrho := \sup_{k\in\mathbb{N}}\varrho_k < 1. \tag{3.7}$$

Then, for $t \ge \tau$, we have

$$\begin{aligned} u(t) \le & \left(\prod_{\tau<t_k\le t}(1+\eta(t_k))\right) \\ & \cdot \exp\left(\frac{1}{1-\vartheta}\sum_{j=k(\tau)+1}^{k(t)}\int_{t_{j-1}}^{t_j}\eta_2(s)\exp\left(\int_{t_{j-1}}^{\zeta_{j-1}}\eta_1(r)dr\right)ds\right. \\ & \left. + \frac{1}{1-\vartheta}\int_{t_{k(t)}}^t \eta_2(s)\exp\left(\int_{t_{k(t)}}^{\zeta_{k(t)}}\eta_1(r)dr\right)ds + \int_\tau^t \eta_1(s)ds\right)u(\tau). \end{aligned} \tag{3.8}$$

3.4 Existence and uniqueness for (3.1)

Theorem 3.6 (Uniqueness [4, Theorem 4.5])**.** *Consider the I.V.P for (2.1) with $y(t,\tau,y(\tau))$. Let (H1)–(H2) hold. Then, there exists a unique solution y for (2.1) on $[\tau,\infty)$. Moreover, every solution is stable.*

Lemma 3.7 (Existence of solutions in $[\tau, t_k)$ [4, Lemma 4.6])**.** *Consider the I.V.P for (2.1) with $y(t,\tau,y(\tau))$. Let (H1)–(H2) hold. Then, for each $y_0 \in \mathbb{R}^n$ and $\zeta_k \in [t_{k-1}, t_k)$ there exists a solution $y(t) = y(t,\tau,y(\tau))$ of (2.1) on $[\tau, t_r)$ such that $y(\tau) = y_0$.*

Theorem 3.8 (Existence of solutions in $[\tau,\infty$ [4, Theorem 4.7])**.** *Let (H1)–(H2) hold. Then, for each $(\tau, y_0 \in \mathbb{R}_0^+ \times \mathbb{R}^n$, there exists $y(t) = y(t,\tau,y_0)$ for $t \ge \tau$, a unique solution for (2.1) such that $y(\tau) = y_0$.*

4 Variation of parameters formula for IDEPCAG

In this section, we will construct a variation of parameters formula for the IDEPCAG system

$$\begin{aligned} y'(t) &= A(t)y(t) + B(t)y(\gamma(t)) + F(t), && t \neq t_k \\ \Delta y|_{t=t_k} &= C_k y(t_k^-) + D_k, && t = t_k \end{aligned} \tag{4.1}$$

where $y \in \mathbb{R}^{n\times 1}, t \in \mathbb{R}, F(t) \in \mathbb{R}^{n\times 1}$ is a real valued continuous matrix, $A(t), B(t) \in \mathbb{R}^{n\times n}$ are real valued continuous locally integrable matrices, $C_k, D_k \in \mathbb{R}^{n\times n}$, $(I + C_k)$ invertible $\forall k \in \mathbb{Z}$, where $I_{n\times n} = I$ is the identity matrix and $\gamma(t)$ is a generalized piecewise constant argument. This time, we will consider the advanced and the delayed intervals in our approach.

First, we will find the *fundamental matrix* for the linear IDEPCAG

$$\begin{aligned} w'(t) &= A(t)w(t) + B(t)w(\gamma(t)), && t \neq t_k \\ \Delta w|_{t=t_k} &= C_k w(t_k^-), && t = t_k. \end{aligned} \tag{4.2}$$

Then, we will give the variation of parameters formula for (4.1).

Let $\Phi(t,s)$, $t, s \in \mathbb{R}$, with $\Phi(t,t) = I$ the transition (Cauchy) matrix of the ordinary system

$$x'(t) = A(t)x(t), \qquad t \in I_k = [t_k, t_{k+1}). \tag{4.3}$$

We will assume the following hypothesis:

(H3) Let

$$\rho_{k^+}(A) = \exp\left(\int_{t_k}^{\zeta_k} \|A(u)\|\, du\right), \qquad \rho_{k^-}(A) = \exp\left(\int_{\zeta_k}^{t_{k+1}} \|A(u)\|\, du\right),$$

$$\rho_k(A) = \rho_{k^+}(A)\cdot\rho_{k^-}(A), \qquad \nu_k^{\pm}(B) = \rho_k^{\pm}(A)\ln\rho_k^{\pm}(B),$$

and assume that

$$\rho(A) = \sup_{k\in\mathbb{Z}} \rho_k(A) < \infty, \qquad \nu^{\pm}(B) = \sup_{k\in\mathbb{Z}} \nu_k^{\pm}(B) < \infty.$$

Consider the following matrices

$$J(t,\tau) = I + \int_\tau^t \Phi(\tau,s)B(s)ds, \qquad E(t,\tau) = \Phi(t,\tau)J(t,\tau), \tag{4.4}$$

where

$$\nu_k^{\pm}(B) < \nu^{\pm}(B) < 1. \tag{4.5}$$

Remark 4.1. It is important to notice the following facts:

a) As a consequence of (H3), $J(t_k,\zeta_k)$ and $J(t_{k+1},\zeta_k)$ are invertible $\forall k \in \mathbb{Z}$, and

$$\left\|J^{-1}(t_k,\zeta_k)\right\| \leq \sum_{k=0}^{\infty}\left[\nu^+(B)\right]^k = \frac{1}{1-\nu^+(B)}, \qquad \|J(t_k,\zeta_k)\| \leq 1 + \nu^+(B), \tag{4.6}$$

$$\left\|J^{-1}(t_{k+1},\zeta_k)\right\| \leq \sum_{k=0}^{\infty}\left[\nu^-(B)\right]^k = \frac{1}{1-\nu^-(B)}, \qquad \|J(t_{k+1},\zeta_k)\| \leq 1 + \nu^-(B). \tag{4.7}$$

Additionally, setting $t_0 := \tau$, we will assume that $J^{-1}(\tau,\gamma(\tau))$ exists.

b) Also, due to (H3) and the Gronwall inequality, we have

$$|\Phi(t)| \leq \rho(A),$$

(see [17]).

4.1 The fundamental matrix of the linear homogeneous IDEPCAG

We adopt the following convention:

$$\overleftarrow{\prod_{k=j}^{j+p}} T_k = T_{j+p} \cdot T_{j+p-1} \cdot \ldots \cdot T_j.$$

Also, we will assume $\gamma(\tau) := \tau$ if $\gamma(\tau) < \tau$, where $k(\tau)$ is the only $k \in \mathbb{Z}$ such that $\tau \in I_{k(\tau)} = [t_{k(\tau)}, t_{k(\tau)+1})$. We will adopt the following notation:

$$\prod_{j=r+1}^{r} A_j = 1, \qquad \sum_{j=r+1}^{r} A_j = 0.$$

Let the system

$$\begin{array}{ll} w'(t) = A(t)w(t) + B(t)w(\gamma(t)), & t \neq t_k \\ w(t_k) = (I + C_k)\, w(t_k^-), & t = t_k \\ w_0 = w(\tau). & \end{array} \tag{4.8}$$

We will construct the fundamental matrix for system (4.8).
Let $t, \tau \in I_k = [t_k, t_{k+1})$ for some $k \in \mathbb{Z}$. In this interval, we are in the presence of the ordinary system

$$w'(t) = A(t)w(t) + B(t)w(\zeta_k).$$

So, the unique solution can be written as

$$w(t) = \Phi(t,\tau)w(\tau) + \int_\tau^t \Phi(t,s)B(s)w(\zeta_k)ds. \tag{4.9}$$

Keeping in mind $I_k^+ = [t_k, \zeta_k]$, evaluating the last expression at $t = \zeta_k$ we have

$$w(\zeta_k) = \Phi(\zeta_k,\tau)w(\tau) + \int_\tau^{\zeta_k} \Phi(\zeta_k,s)B(s)w(\zeta_k)ds. \tag{4.10}$$

Hence, we get

$$\left(I + \int_{\zeta_k}^{\tau} \Phi(\zeta_k,s)B(s)ds\right) w(\zeta_k) = \Phi(\zeta_k,\tau)w(\tau).$$

I.e.

$$w(\zeta_k) = J^{-1}(\tau,\zeta_k)\Phi(\zeta_k,\tau)w(\tau). \tag{4.11}$$

Then, by the definition of $E(t,\tau) = \Phi(t,\tau)J(t,\tau)$, we have

$$w(\zeta_k) = E^{-1}(\tau,\zeta_k)w(\tau). \tag{4.12}$$

Now, from (4.9) working on $I_k^- = [\zeta_k, t_{k+1})$, considering $\tau = \zeta_k$, we have

$$\begin{aligned} w(t) &= \Phi(t,\zeta_k)w(\zeta_k) + \int_{\zeta_k}^t \Phi(t,s)B(s)w(\zeta_k)ds \\ &= \Phi(t,\zeta_k)\left(I + \int_{\zeta_k}^t \Phi(\zeta_k,s)B(s)ds\right) w(\zeta_k). \end{aligned}$$

I.e.,

$$w(t) = E(t,\zeta_k)w(\zeta_k). \tag{4.13}$$

So, by (4.12), we can rewrite (4.13) as

$$w(t) = E(t,\zeta_k)E^{-1}(\tau,\zeta_k)w(\tau). \tag{4.14}$$

Then, setting

$$W(t,s) = E(t,\gamma(s))E^{-1}(s,\gamma(s)), \quad \text{if } t,s \in I_k = [t_k, t_{k+1}), \tag{4.15}$$

we have the solution for (4.8) for $t \in I_k$

$$w(t) = W(t,\tau)w(\tau). \tag{4.16}$$

Next, if we consider $\tau = t_k$ and assuming left side continuity of (4.16) at $t = t_{k+1}$, we have

$$w(t_{k+1}^-) = W(t_{k+1}, t_k)w(t_k).$$

Then, applying the impulsive condition to the last equation, we get

$$\begin{aligned} w(t_{k+1}) &= (I + C_{k+1})\, w(t_{k+1}^-) \\ &= (I + C_{k+1})\, W(t_{k+1}, t_k)w(t_k). \end{aligned}$$

This expression corresponds to a finite-difference equation. Then, by solving it, we get

$$w(t_{k(t)}) = \left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}} (I + C_{j+1})\, W(t_{j+1}, t_j) \right) w\left(t_{k(\tau)+1}\right). \tag{4.17}$$

Finally, by (4.16) and the impulsive condition, we have

$$w(t_{k(\tau)+1}) = (I + C_{k(\tau)+1})W(t_{k(\tau)+1}, \tau)w(\tau).$$

Hence, considering $\tau = t_k$ in (4.16) and applying (4.17) we get

$$\begin{aligned} w(t) &= W(t, t_{k(t)}) \left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}} (I + C_{j+1})\, W(t_{j+1}, t_j) \right) \left(I + C_{k(\tau)+1}\right) W(t_{k(\tau)+1}, \tau)w(\tau) \\ &= W(t,\tau)w(\tau), \quad \text{for } t \in I_{k(t)} \text{ and } \tau \in I_{k(\tau)}. \end{aligned} \tag{4.18}$$

The last equation is the solution of (4.8) on $[\tau, t)$.

We call the expression

$$W(t,\tau) = W(t, t_{k(t)}) \left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}} (I + C_{j+1})\, W(t_{j+1}, t_j) \right) \left(I + C_{k(\tau)+1}\right) W(t_{k(\tau)+1}, \tau) \tag{4.19}$$

the fundamental matrix for (4.8) for $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$.

Remark 4.2. We use the decomposition of $I_k = I_k^+ \cup I_k^-$ to define W. In fact, we can rewrite (4.19) in terms of the advanced and delayed parts using (4.15):

$$\begin{aligned} W(t,\tau) = E(t,\zeta_{k(t)})E^{-1}(t_{k(t)},\zeta_{k(t)}) &\left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}} (I + C_{j+1})\, E(t_{j+1},\zeta_j)E^{-1}(t_j,\zeta_j) \right) \\ &\cdot \left(I + C_{k(\tau)+1}\right) E(t_{k(\tau)+1}, \gamma(\tau))E^{-1}(\tau,\gamma(\tau)), \qquad \zeta_j = \gamma(t_j) \end{aligned}$$

for $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$.

Remark 4.3.

a) Considering $B(t) = 0$, we recover the classical fundamental matrix of the impulsive linear differential equation (see [18]).

b) If $C_k = 0, \forall k \in \mathbb{Z}$, we recover the DEPCAG case studied by M. Pinto in [17].

c) If we consider $\gamma(t) = p[\frac{t+l}{p}]$, with $p < l$, we recover the IDEPCA case studied by K.-S. Chiu in [6].

4.2 The variation of parameter formula for IDEPCAG

Let the IDEPCAG

$$\begin{aligned} &y'(t) = A(t)y(t) + B(t)y(\gamma(t)) + F(t), && t \neq t_k, \\ &y(t_k) = (I + C_k)y(t_k^-) + D_k, && t = t_k, \\ &y_0 = y(\tau). \end{aligned} \tag{4.20}$$

If $\tau, t \in I_k = [t_k, t_{k+1})$, then the unique solution of (4.20) is

$$y(t) = \Phi(t,\tau)y(\tau) + \int_\tau^t \Phi(t,s)B(s)y(\zeta_k)ds + \int_\tau^t \Phi(t,s)f(s)ds.$$

Then, if $\tau = \zeta_k$, we have

$$\begin{aligned} y(t) &= \Phi(t,\zeta_k)y(\zeta_k) + \int_{\zeta_k}^t \Phi(t,s)B(s)y(\zeta_k)ds + \int_{\zeta_k}^t \Phi(t,s)f(s)ds \\ &= \Phi(t,\zeta_k)\left(I + \int_{\zeta_k}^t \Phi(\zeta_k,s)B(s)ds\right)y(\zeta_k) + \int_{\zeta_k}^t \Phi(t,s)f(s)ds \\ &= \Phi(t,\zeta_k)J\,(t,\zeta_k)\,y(\zeta_k) + \int_{\zeta_k}^t \Phi(t,s)f(s)ds, \end{aligned}$$

i.e.

$$y(t) = E(t,\zeta_k)y(\zeta_k) + \int_{\zeta_k}^t \Phi(t,s)f(s)ds. \tag{4.21}$$

Now, if we consider $t = \tau$ in (4.21) we have

$$y(\tau) = E(\tau,\zeta_k)y(\zeta_k) + \int_{\zeta_k}^\tau \Phi(\tau,s)f(s)ds,$$

and, by $(H3)$, we get the following estimation for $y(\zeta_k)$

$$y(\zeta_k) = E^{-1}(\tau,\zeta_k)\left(y(\tau) + \int_\tau^{\zeta_k} \Phi(\tau,s)f(s)ds\right). \tag{4.22}$$

Then, applying (4.22) in (4.21) we obtain

$$y(t) = E(t,\zeta_k)E^{-1}(\tau,\zeta_k)\left(y(\tau) + \int_\tau^{\zeta_k} \Phi(\tau,s)f(s)ds\right) + \int_{\zeta_k}^t \Phi(t,s)f(s)ds,$$

i.e.,

$$y(t) = W(t,\tau)y(\tau) + \int_\tau^{\zeta_k} W(t,\tau)\Phi(\tau,s)f(s)ds + \int_{\zeta_k}^t \Phi(t,s)f(s)ds, \qquad \tau, t \in I_k. \tag{4.23}$$

Next, taking the left-side limit to the last expression, we have

$$y(t_{k+1}^-) = W(t_{k+1},\tau)\left(y(\tau) + \int_\tau^{\zeta_k} \Phi(\tau,s)f(s)ds\right) + \int_{\zeta_k}^{t_{k+1}} \Phi(t_{k+1},s)f(s)ds. \tag{4.24}$$

Applying the impulsive condition, we get

$$y(t_{k+1}) = (I + C_{k+1})\, y(t_{k+1}^-) + D_{k+1},$$

or

$$\begin{aligned} y(t_{k+1}) = &\ (I + C_{k+1})\, W(t_{k+1},\tau)\left(y(\tau) + \int_\tau^{\zeta_k} \Phi(\tau,s)f(s)ds\right) \\ &+ \int_{\zeta_k}^{t_{k+1}} (I + C_{k+1})\, \Phi(t_{k+1},s)f(s)ds + D_{k+1}. \end{aligned}$$

Therefore, considering $\tau = t_k$ in the last expression we have

$$\begin{aligned} y(t_{k+1}) = &\ (I + C_{k+1})\, W(t_{k+1},t_k)\left(y(t_k) + \int_{t_k}^{\zeta_k} \Phi(t_k,s)f(s)ds\right) \\ &+ \int_{\zeta_k}^{t_{k+1}} (I + C_{k+1})\, \Phi(t_{k+1},s)f(s)ds + D_{k+1}, \end{aligned}$$

or

$$y(t_{k+1}) = W_k\left(y(t_k) + \alpha_k^+\right) + \alpha_k^- + \beta_k,$$

which corresponds to a non-homogeneous linear difference equation, where

$$\begin{aligned} W_k &= (I + C_{k+1})W(t_{k+1},t_k), \\ \alpha_k^+ &= \int_{t_k}^{\zeta_k} \Phi(t_k,s)f(s)ds, \\ \alpha_k^- &= \int_{\zeta_k}^{t_{k+1}} (I + C_{k+1})\, \Phi(t_{k+1},s)f(s)ds, \\ \beta_k &= D_{k+1}. \end{aligned}$$

Recalling that

$$W(t_{k(t)},\tau) = \left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}} \left(I + C_{j+1}\right) W(t_{j+1},t_j)\right)\left(I + C_{k(\tau)+1}\right) W(t_{k(\tau)+1},\tau),$$

we get the discrete solution of (4.20):

$$\begin{aligned} y(t_{k(t)}) = &\left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}} \left(I + C_{j+1}\right) W(t_{j+1},t_j)\right)(I + C_{k(\tau)+1})W(t_{k(\tau)+1},\tau)y(\tau) \\ &+ \int_\tau^{\zeta_{k(\tau)}} W\left(t_{k(t)},\tau\right)\Phi(\tau,s)f(s)ds \\ &+ \sum_{r=k(\tau)+1}^{k(t)-1}\left(\overset{\leftarrow}{\prod_{j=r}^{k(t)-1}} \left(I + C_{j+1}\right) W(t_{j+1},t_j)\right)\int_{t_r}^{\zeta_r} \Phi(t_r,s)f(s)ds \\ &+ \sum_{r=k(\tau)}^{k(t)-1}\left(\overset{\leftarrow}{\prod_{j=r+1}^{k(t)-1}} (I + C_{j+1})W(t_{j+1},t_j)\right)\int_{\zeta_r}^{t_{r+1}} (I + C_{r+1})\Phi(t_{r+1},s)f(s)ds \\ &+ \sum_{r=k(\tau)}^{k(t)-1}\left(\overset{\leftarrow}{\prod_{j=r+1}^{k(t)-1}} (I + C_{j+1})W(t_{j+1},t_j)\right) D_{r+1}, \end{aligned}$$

or, written in terms of (4.19),

$$
\begin{aligned}
y(t_{k(t)}) &= W(t_{k(t)},\tau)y(\tau) + \int_{\tau}^{\zeta_{k(\tau)}} W\left(t_{k(t)},\tau\right)\Phi(\tau,s)f(s)ds \\
&+ \sum_{r=k(\tau)+1}^{k(t)-1}\int_{t_r}^{\zeta_r} W(t_{k(t)},t_r)\Phi(t_r,s)f(s)ds \\
&+ \sum_{r=k(\tau)}^{k(t)-1}\int_{\zeta_r}^{t_{r+1}} W(t_{k(t)},t_{r+1})(I+C_{r+1})\Phi(t_{r+1},s)f(s)ds \\
&+ \sum_{r=k(\tau)}^{k(t)-1} W(t_{k(t)},t_{r+1})D_{r+1}.
\end{aligned}
\tag{4.25}
$$

Now, considering $\tau = t_k$ in (4.23) we have

$$
y(t) = W(t,t_{k(t)})y(t_{k(t)}) + \int_{t_{k(t)}}^{\zeta_{k(t)}} W(t,t_{k(t)})\Phi(t_{k(t)},s)f(s)ds + \int_{\zeta_{k(t)}}^{t} \Phi(t,s)f(s)ds.
$$

Finally, replacing $y(t_{k(t)})$ by (4.25) and rewriting in terms of (4.19), we get the variation of parameters formula for IDEPCAG (4.20):

$$
\begin{aligned}
y(t) &= W(t,\tau)y(\tau) \\
&+ \int_{\tau}^{\zeta_{k(\tau)}} W(t,\tau)\Phi(\tau,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)}\int_{t_r}^{\zeta_r} W(t,t_r)\Phi(t_r,s)f(s)ds \\
&+ \sum_{r=k(\tau)}^{k(t)-1}\int_{\zeta_r}^{t_{r+1}} W(t,t_{r+1})\left(I+C_{r+1}\right)\Phi(t_{r+1},s)f(s)ds \\
&+ \int_{\zeta_{k(t)}}^{t} \Phi(t,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} W(t,t_r)D_r, \quad \text{for } t\in[\tau,t_{k(t)+1}),
\end{aligned}
\tag{4.26}
$$

where W is the fundamental matrix of (4.8).

4.2.1 Green type matrix for IDEPCAG

If we define the following Green matrix type for IDEPCAG:

$$
\widetilde{W}(t,s) = \begin{cases} W^{+}(t,s), & \text{if } t_r \le s \le \gamma(s) \\ W^{-}(t,s), & \text{if } \gamma(s) < s \le t_{r+1}, \end{cases}
\tag{4.27}
$$

where

$$
W^{+}(t,s) = W(t,t_r)\Phi(t_r,s), \quad \text{if } \ t_r \le s \le \gamma(s),\ s<t,
\tag{4.28}
$$

and

$$
W^{-}(t,s) = \begin{cases} W(t,t_{r+1})\left(I+C_{r+1}\right)\Phi(t_{r+1},s), & \text{if } \gamma(s) \le s < t_{r+1},\ t>s,\ t\le t_{k+1}, \\ \Phi(t,s), & \text{if } \gamma(t) < s \le t < t_{r+1}. \end{cases}
\tag{4.29}
$$

Hence, we can see that

$$\int_{\tau}^{t} W^{+}(t,s)f(s)ds = \int_{\tau}^{\zeta_{k(\tau)}} W(t,\tau)\Phi(\tau,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} \int_{t_r}^{\zeta_r} W(t,t_r)\Phi(t_r,s)f(s)ds,$$

$$\int_{\tau}^{t} W^{-}(t,s)f(s)ds = \sum_{r=k(\tau)}^{k(t)-1} \int_{\zeta_r}^{t_{r+1}} W(t,t_{r+1})\left(I+C_{r+1}\right)\Phi(t_{r+1},s)f(s)ds + \int_{\zeta_{k(t)}}^{t} \Phi(t,s)f(s)ds.$$

So, we have

$$\widetilde{W}(t,s) = W^{+}(t,s) + W^{-}(t,s).$$

In this way, (4.26) can be expressed as

$$y(t) = W(t,\tau)y(\tau) + \int_{\tau}^{t} \widetilde{W}(t,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} W(t,t_r)D_r. \tag{4.30}$$

4.3 Some special cases of (4.20)

In the following, we present some r cases for (4.20).

1. Let $\gamma^{-}(t) = t_k$ and $\gamma^{+}(t) = t_{k+1}$, for all $t \in I_k = [t_k, t_{k+1})$. I.e., we are considering the completely delayed and advanced general piecewise constant arguments. Then, taking in account Remark 4.2, the solution of (4.20) for both cases $y_{-}(t)$ and $y_{+}(t)$ respectively are:

$$y_{-}(t) = W_{-}(t,\tau)y(\tau) + \sum_{r=k(\tau)}^{k(t)-1} \int_{t_r}^{t_{r+1}} W_{-}(t,t_{r+1})\left(I+C_{r+1}\right)\Phi(t_{r+1},s)f(s)ds + \int_{t_{k(t)}}^{t} \Phi(t,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} W_{-}(t,t_r)D_r, \tag{4.31}$$

where

$$W_{-}(t,\tau) = E(t,t_{k(t)})\left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}} \left(I+C_{j+1}\right)E(t_{j+1},t_j)\right)\cdot\left(I+C_{k(\tau)+1}\right)E(t_{k(\tau)+1},\tau), \tag{4.32}$$

and

$$y_{+}(t) = W_{+}(t,\tau)y(\tau) + \int_{\tau}^{t_{k(\tau)+1}} W_{+}(t,\tau)\Phi(\tau,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} \int_{t_r}^{t_{r+1}} W_{+}(t,t_r)\Phi(t_r,s)f(s)ds - \int_{t}^{t_{k(t)+1}} \Phi(t,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} W_{+}(t,t_r)D_r, \tag{4.33}$$

where

$$W_+(t,\tau) = E(t,t_{k(t)+1})E^{-1}(t_{k(t)},t_{k(t)+1}) \left(\overleftarrow{\prod_{j=k(\tau)+1}^{k(t)-1}} \left(I + C_{j+1}\right) E^{-1}(t_j,t_{j+1}) \right) \cdot \left(I + C_{k(\tau)+1}\right) E^{-1}(\tau,t_{k(\tau)+1}),$$

for $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$, recalling that $\gamma(\tau) := \tau$ if $\gamma(\tau) < \tau$.

2. Let the IDEPCAG

$$\begin{aligned} w'(t) &= B(t)w(\gamma(t)), && t \neq t_k \\ w(t_k) &= (I + C_k)w(t_k^-), && t = t_k \\ w_0 &= w(\tau). \end{aligned} \tag{4.34}$$

We see that $\Phi(t,s) = I, E(t,s) = J(t,s)$ and $J(t,s) = I + \int_s^t B(u)du$, where I is the identity matrix. Hence the fundamental matrix for (4.34) is given by

$$W(t,\tau) = J(t,\zeta_{k(t)})J^{-1}(t_{k(t)},\zeta_{k(t)}) \left(\overleftarrow{\prod_{j=k(\tau)+1}^{k(t)-1}} \left(I + C_{j+1}\right) J(t_{j+1},\zeta_j)J^{-1}(t_j,\zeta_j) \right) \cdot \left(I + C_{k(\tau)+1}\right) J(t_{k(\tau)+1},\gamma(\tau))J^{-1}(\tau,\gamma(\tau)), \qquad \zeta_j = \gamma(t_j).$$

for $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$.

This case is very important because it is used for the approximation of solutions of differential equations considering $\gamma(t) = \left[\frac{t}{h}\right] h$, with $h > 0$ fixed.

3. Let the IDEPCAG

$$\begin{aligned} w'(t) &= Aw(t) + Bw(\gamma(t)), && t \neq t_k \\ w(t_k) &= (I + C)w(t_k^-), && t = t_k \\ w_0 &= w(\tau), \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} y'(t) &= Ay(t) + By(\gamma(t)) + f(t), && t \neq t_k \\ y(t_k) &= (I + C)y(t_k^-) + D_k, && t = t_k \\ y_0 &= y(\tau), \end{aligned} \tag{4.36}$$

where A^{-1} exist. By $(H3)$, we know that $J(t,\tau) = I + \int_\tau^t e^{A(\tau-s)}B\,ds$ is invertible, for $\tau, t \in I_k = [t_k, t_{k+1})$. Moreover, following [17], we see that

$$\begin{aligned} J(t,\tau) &= I + \int_\tau^t e^{A(\tau-s)}Bds \\ &= I + e^{A\tau}\left(\int_\tau^t (-A)e^{-As}ds\right)(-A^{-1})B \\ &= I + A^{-1}\left(I - e^{A(\tau-t)}\right)B. \end{aligned} \tag{4.37}$$

Then, as $E(t,\tau) = \Phi(t,\tau)J(t,\tau)$, we have

$$E(t,\tau) = e^{A(t-\tau)}\left(I + A^{-1}\left(I - e^{-A(t-\tau)}\right)B\right). \tag{4.38}$$

In light of the last calculations, we define

$$\begin{aligned}\widetilde{E}(t) &= e^{At}\left(I + A^{-1}\left(I - e^{-At}\right)B\right)\\ \eta_k^+ &= \zeta_k - t_k, \qquad \eta_k^- = t_{k+1} - \zeta_k, \quad k \in \mathbb{Z},\\ \eta(t) &= t - \gamma(t).\end{aligned}$$

Recalling that

$$\widehat{W}(t,s) = \widetilde{E}(t-\gamma(s))\widetilde{E}^{-1}(\eta(s)), \quad \text{if } t,s \in I_k = [t_k, t_{k+1}), \tag{4.39}$$

the solution of (4.35) is

$$w(t) = \widehat{W}(t,\tau)w(\tau),$$

where

$$\widehat{W}(t,\tau) = \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta_{k(t)}^+)\left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}}(I+C)\widetilde{E}(\eta_j^-)\widetilde{E}^{-1}(-\eta_j^+)\right) \tag{4.40}$$

$$\cdot (I+C)\widetilde{E}(\eta_{k(\tau)+1}^-)\widetilde{E}^{-1}(\eta(\tau)), \tag{4.41}$$

is the fundamental matrix for (4.35) with $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$.

The solution for (4.36) is given by

$$\begin{aligned}y(t) = {} & \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta_{k(t)}^+)\left(\overset{\leftarrow}{\prod_{j=k(\tau)+1}^{k(t)-1}}(I+C)\widetilde{E}(\eta_j^-)\widetilde{E}^{-1}(-\eta_j^+)\right)\\ & \cdot (I+C)\widetilde{E}(\eta_{k(\tau)+1}^-)\widetilde{E}^{-1}(\eta(\tau))\left(y(\tau) + \int_\tau^{\zeta_{k(\tau)}} e^{A(\tau-s)}f(s)ds\right)\\ & + \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta_{k(t)}^+)\\ & \cdot\left\{\sum_{r=k(\tau)+1}^{k(t)}\left(\overset{\leftarrow}{\prod_{j=r}^{k(t)-1}}(I+C)\widetilde{E}(\eta_j^-)\widetilde{E}^{-1}(-\eta_j^+)\right)\int_{t_r}^{\zeta_r} e^{A(t_r-s)}f(s)ds\right.\\ & + \sum_{r=k(\tau)}^{k(t)-1}\left(\overset{\leftarrow}{\prod_{j=r+1}^{k(t)}}(I+C)\widetilde{E}(\eta_j^-)\widetilde{E}^{-1}(-\eta_j^+)\right)\int_{\zeta_r}^{t_{r+1}}(I+C)e^{A(t_{r+1}-s)}f(s)ds\\ & \left.+ \sum_{r=k(\tau)}^{k(t)-1}\left(\overset{\leftarrow}{\prod_{j=r+1}^{k(t)}}(I+C)\widetilde{E}(\eta_j^-)\widetilde{E}^{-1}(-\eta_j^+)\right)D_r\right\}\\ & + \int_{\zeta_{k(t)}}^t e^{A(t-s)}f(s)ds.\end{aligned} \tag{4.42}$$

Also, if $\eta = \eta_k^+ = \eta_k^-$, $k \in \mathbb{Z}$, $\widehat{E} = (I+C)\widetilde{E}(\eta)\widetilde{E}^{-1}(-\eta)$, the solution of (4.35) is

$$w(t) = \widehat{W}(t,\tau)w(\tau),$$

where

$$\widehat{W}(t,\tau) = \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta_{k(t)}^+)\widehat{E}^{k(t)-k(\tau)-1}(I+C)\widetilde{E}(\eta)\widetilde{E}^{-1}(\eta(\tau)),$$

is the fundamental matrix for (4.35) with $t \in I_{k(t)}$ and $\tau \in I_{k(\tau)}$.
The solution for (4.36) is given by

$$\begin{aligned} y(t) &= \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta^{+}_{k(t)})\widehat{E}^{k(t)-k(\tau)-1}(I+C)\widetilde{E}(\eta^{-}_{k(\tau)+1})\widetilde{E}^{-1}(\eta(\tau)) \\ &\cdot \left(y(\tau) + \int_{\tau}^{\zeta_{k(\tau)}} e^{A(\tau-s)} f(s)ds\right) \\ &+ \widetilde{E}(\eta(t))\widetilde{E}^{-1}(-\eta^{+}_{k(t)}) \cdot \left\{ \sum_{r=k(\tau)+1}^{k(t)} \widehat{E}^{k(t)-r} \int_{t_r}^{\zeta_r} e^{A(t_r-s)} f(s)ds \right. \\ &+ \left. \sum_{r=k(\tau)}^{k(t)-1} \widehat{E}^{k(t)-r} \int_{\zeta_r}^{t_{r+1}} (I+C)e^{A(t_{r+1}-s)} f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} \widehat{E}^{k(t)-r} D_r \right\} \\ &+ \int_{\zeta_{k(t)}}^{t} e^{A(t-s)} f(s)ds. \end{aligned} \tag{4.43}$$

Remark 4.4.

1. We recover the variation of parameters concluded in [17] when $D_r = C_r = 0$.
2. Also, our result implies the variation of constant formulas given in section 1.2

5 Some examples of linear IDEPCAG systems

In [16], H. Bereketoglu and G. Oztepe studied the following linear IDEPCAG

$$\begin{aligned} &z'(t) = A(t)\left(z(t) - z(\gamma(t))\right) + f(t), && t \neq t_k \\ &z(t_k) = z(t_k^-) + D_k, && t = t_k. \\ &z(\tau) = z_0 \end{aligned} \tag{5.1}$$

where $\gamma(t)$ is some piecewise constant argument of generalized type, $A(t)$ is a continuous locally integrable matrix, $D : \mathbb{N} \to \mathbb{R}$ is such that $D_k \neq 0, \forall k \in \mathbb{N}$. The authors originally considered the cases $\gamma_1(t) = [t+1]$, and $\gamma_2(t) = [t-1]$. Hence, $t_k = k, \zeta_{1,k}k = k+1$ and $\zeta_{2,k} = k-1$, respectively.

Let $\Phi(t)$ be the fundamental matrix of the ordinary differential system

$$x'(t) = A(t)x(t). \tag{5.2}$$

It is well known that $\Phi^{-1}(t)$ is the fundamental matrix of the adjoint system associated with (5.2). So, it satisfies

$$\left(\Phi^{-1}\right)'(t) = -\Phi^{-1}(t)A(t).$$

Therefore, we have

$$\begin{aligned} J(t,t_k) &= I - \int_{t_k}^{t} \Phi(t_k,s)A(s)ds \\ &= I + \Phi(t_k)\left(\int_{t_k}^{t} -\Phi^{-1}(s)A(s)ds\right) \\ &= I + \Phi(t_k)\left(\Phi^{-1}(t) - \Phi^{-1}(t_k)\right) \\ &= \Phi(t_k,t) \\ &= \Phi^{-1}(t,t_k), \end{aligned}$$

$E(t,t_k) = \Phi(t,t_k)J(t,t_k) = \Phi(t,t_k)\Phi^{-1}(t,t_k) = I$, and, as a result of last estimations, for $t,t' \in I_k$, we have $W(t,t') = I$. Hence, the linear homogeneous IDEPCAG (is a DEPCAG because $C_k = 0$) system

$$\begin{aligned} w'(t) &= A(t)\left(w(t) - w(\gamma(t))\right), && t \neq t_k \\ w(t_k) &= w(t_k^-) && t = t_k. \\ w(\tau) &= w_0, \end{aligned} \tag{5.3}$$

has the constant solution $w(t) = w(\tau)$.

Finally, for the variation of parameters formula (4.26), the solution for (5.1) is

$$\begin{aligned} y(t) = y(\tau) &+ \int_{\tau}^{\zeta_{k(\tau)}} \Phi(\tau,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} \int_{t_r}^{\zeta_r} \Phi(t_r,s)f(s)ds \\ &+ \sum_{r=k(\tau)}^{k(t)-1} \int_{\zeta_r}^{t_{r+1}} \Phi(t_{r+1},s)f(s)ds + \int_{\zeta_{k(t)}}^{t} \Phi(t,s)f(s)ds + \sum_{r=k(\tau)+1}^{k(t)} D_r, \end{aligned}$$

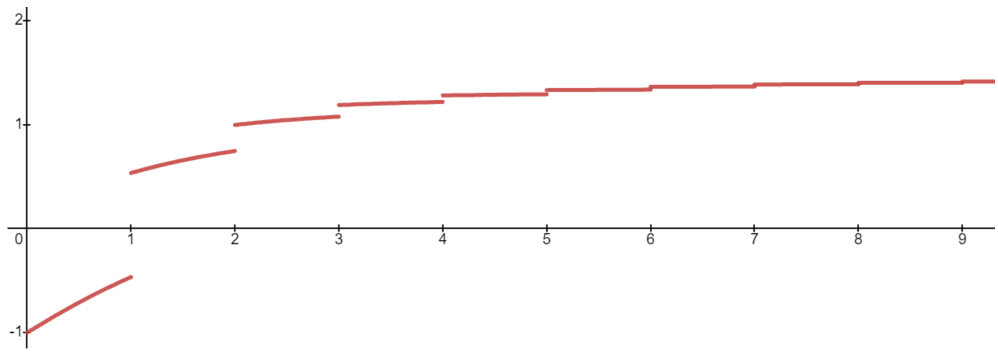

Figure 5.1: Solution of (5.1) with $\gamma(t) = [t] + 7/10$, $D_r = 1/r^2$, $A(t) = 1/(t+1)$, $f(t) = \exp(-t)$ and $y(0) = y_0 = -1$.

Remark 5.1. This is the IDEPCAG case for the well-known differential equation studied by K. L Cooke and J. A. Yorke in [10]. The authors investigated the following delay differential equation (DDE):

$$x'(t) = g(x(t)) - g(x(t-L)),$$

where $x(t)$ denotes the number of individuals in a population, the number of births is $g(x(t))$, and L is the constant life span of the individuals in the population. Then, the number of deaths $g(x(t-L))$. Since the difference $g(x(t)) - g(x(t-L)]$ means the change of the population. Therefore $x'(t)$ corresponds to the growth of the population at instant t.

In (5.3), we considered $g(x(t)) = A(t)x(t)$ and the constant delay in the Cooke–Yorke equation is regarded as a piecewise constant argument $\gamma(t)$. Notice that if D_r is summable and $f(t) = 0\ \forall t \geq \tau$, then the solution of (5.1) tends to the constant

$$y_\infty = y(\tau) + \sum_{t_r \geq t_{k(\tau)+1}} D_r, \quad \text{as } t \to \infty,$$

no matter what $\gamma(t)$ was used. For further about asymptotics in IDEPCAG, see [4].

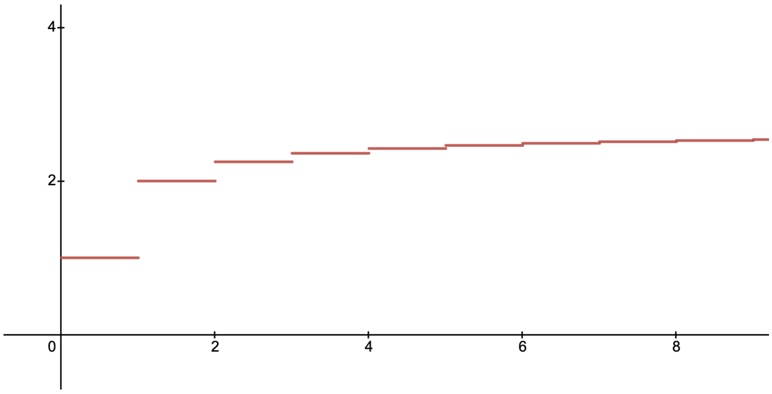

Figure 5.2: Solution of (5.1) with $D_k = 1/k^2,\ f(t) = 0$ and $z(0) = 1$.

Let us consider the following IDEPCA

$$\begin{aligned} &z'(t) = \sin(2\pi t) z\left(\left[\tfrac{t}{h}\right]h + \beta h\right) + 1, && t \neq kh, \quad k \in \mathbb{N}, \\ &z(kh) = \left(-\frac{1}{2}\right) z(kh^-) + \frac{1}{2}, && t = kh, \\ &z(0) = z_0, \end{aligned} \tag{5.4}$$

where $h > 0$, $0 \leq \beta \leq 1$.

It is easy to see that $t_k = kh,\ \zeta_k = (k+\beta)h,\ k \in \mathbb{N}$ and

$$I_k^+ = [kh, (k+\beta)h], \quad I_k^- = [(k+\beta)h, (k+1)h).$$

We see that

$$\begin{aligned} &\nu_k^+(\sin(2\pi t)) \leq \beta h < 1, \quad \text{if } h \text{ is small enough}, \\ &\nu_k^-(\sin(2\pi t)) \leq (1-\beta)h < 1, \quad \text{if } h \text{ is small enough}, \\ &E(t,\tau) = 1 + \int_\tau^t \sin(2\pi s) ds. \end{aligned}$$

The fundamental matrix of the homogeneous equation associated with (5.4) is

$$\begin{aligned} W(t,0) = &\left(1 + \int_{[t/h]h+\beta h}^{t} \sin(2\pi s)\, ds\right)\left(1 + \int_{[t/h]h+\beta h}^{[t/h]h} \sin(2\pi s)\, ds\right)^{(-1)} \\ &\cdot \left(-\frac{1}{2}\right)^{[t/h]} \left(\prod_{j=0}^{[t/h]-1} \left(1 + \int_{(j+\beta)h}^{(j+1)h} \sin(2\pi s)\, ds\right)\left(1 + \int_{(j+\beta)h}^{jh} \sin(2\pi s)\, ds\right)^{(-1)}\right). \end{aligned}$$

Hence, the solution of (5.4) is

$$\begin{aligned} z(t) = W(t,0)z_0 + \left(-\frac{1}{2}\right)(1-\beta)\, h \sum_{r=0}^{[t/h]-1} W(t,(r+1)\,h) + (t - ([t/h]h + \beta h)) \\ + W(t,0)\,\beta h + \beta h \sum_{r=1}^{[t/h]} W(t, rh) + \left(-\frac{1}{2}\right)\sum_{r=0}^{[t/h]-1} W(t,(r+1)\,h). \end{aligned}$$

The piecewise constant used in this example was introduced in [21] to study the approximation of solutions of differential equations (under some stability assumptions and taking $h \to 0$.)

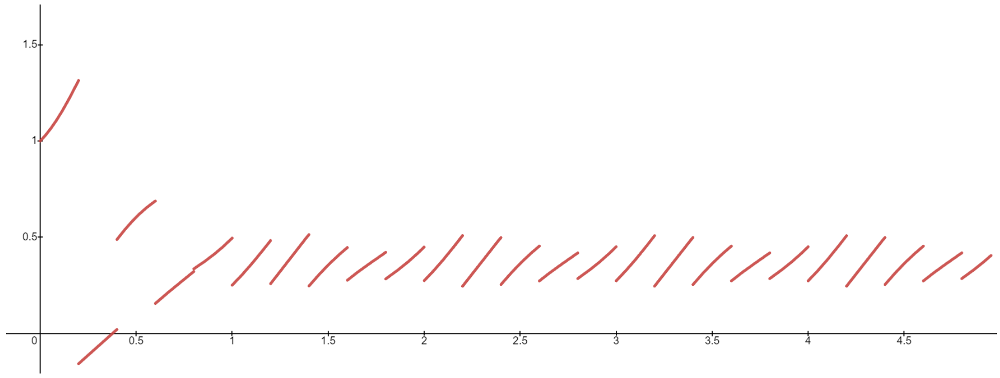

Figure 5.3: Solution of (5.4) with $h = \beta = 0, 2$.

6 Conclusions

In this work, we gave a variation of parameters formula for impulsive differential equations with piecewise constant arguments. We analyzed the constant coefficients case and gave several examples of formulas applied to some concrete piecewise constant arguments. We extended some cases treated before and showed the effect of the impulses in the dynamic.

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Multiple solutions for a fractional p-Kirchhoff system with singular nonlinearity

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Abstract. This paper examines a class of fractional p-Kirchhoff systems driven by a nonlocal integro-differential operator with singular nonlinearity. By making use of Nehari manifold techniques, the existence of two nontrivial solutions is established. Our results extend those in Xiang et al. [*Nonlinearity* **29**(2016), 3186–3205] for the corresponding subcritical case.

Keywords: p-Kirchhoff system, singular nonlinearity, Nehari manifold.

2020 Mathematics Subject Classification: 35A15, 35J60.

1 Introduction

We look for nontrivial solutions of the following fractional p-Kirchhoff system

$$\begin{cases} \left(\sum\limits_{i=1}^{k}[u_i]_{s,p}^p\right)^{\theta-1}(-\Delta)_p^s u_j(x)=\lambda_j|u_j|^{q-2}u_j+\sum\limits_{i\neq j}\beta_{ij}|u_i|^{1-m}|u_j|^{-m} & \text{in } \Omega,\\ u_j=0 & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.1}$$

where

$$[u_j]_{s,p}=\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u_j(x)-u_j(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}}, \qquad j=1,2,\dots,k,\ k\geq 2,$$

$\theta\geq 1$, $N>ps$ with $s\in(0,1)$, $0<m<1$, $0<2-2m<\theta p<q<p_s^*=\frac{Np}{N-sp}$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $\lambda_j>0$ is a parameter, $\beta_{ij}>0$ for all $1\leq i<j\leq k$, $\beta_{ij}=\beta_{ji}$ for $i\neq j$, $j=1,2,\dots,k$, and $(-\Delta)_p^s$ is the fractional p-Laplace operator which may be defined along any $v\in C_0^\infty(\mathbb{R}^N)$ as

$$(-\Delta)_p^s v=2\lim_{\delta\to 0^+}\int_{\mathbb{R}^N\setminus B_\delta(x)}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{N+ps}}dy \quad \text{for } x\in\mathbb{R}^N,$$

where $B_\delta(x)$ denotes the ball in $\mathbb{R}^N$ of radius δ centered at x. For more details on the fractional p-Laplacian, we can see [8] and the references therein.

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In [9], a steady-state Kirchhoff variational model in bounded regular domains of $\mathbb{R}^N$ was proposed by Fiscella and Valdinoci. In fact, problem (1.1) is a fractional version of Kirchhoff model. Specifically, Kirchhoff proposed the following model

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L} \|\nabla u\|^2_{L^2([0,L])} \right) \frac{\partial^2 u}{\partial x^2} = f(x,u), \tag{1.2}$$

where ρ, p_0, h, E, L are constants. As we all know, this model extends the classical D'Alambert wave equation. Set $M(y) = p_0/h + (E(2L))y$ with $y \geq 0$. If $M(0) = 0$, we call problem (1.2) degenerate, otherwise, it is called non-degenerate if $M(0) > 0$. For $M(0) = 0$, it has a very important physical significance, that is, the base tension of the string is equal to zero. Clearly, in this paper, we are concerned about the situation of degradation in the fractional p-Laplacian setting. We refer the interested reader to [2,6,12,13] for some related results.

In recent years, with the application of nonlocal operators in real life or engineering fields becoming more and more obvious, such as bridge survey, population model, image processing, etc., the fractional Laplacian operator has received extensive attention. Most recently, Sousa in [14] studied a class of fractional p-Laplacian differential operators with variable exponents. The author obtained the existence of a positive solution for the investigated fractional system of the Kirchhoff type by using the method of sub- and super- solutions, via technical assumptions on the nonlinearity. In [19] Zuo et al. considered a variational approach based on the scaling function method to solve optimization problems. Precisely, in [18] Zhao et al. studied a p-fractional Schrödinger–Kirchhoff equation with electromagnetic fields and the Hardy–Littlewood–Sobolev nonlinearity. They used the concentration-compactness principles and improved techniques to obtain Palais–Smile condition at level c. By variational methods, they obtained the existence and multiplicity of solutions. For more literature about the results for nonlocal fractional Laplacian operators and related nonlocal integro-differential equations, we can also refer to [1,7,17] and the references therein.

On the other hand, there are a lot of literature on the equation or system with singular nonlinearity. Consider the following semilinear problem

$$\begin{cases} (-\Delta)^s u = \lambda k(x) u^{-\gamma} + M u^q & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0 & \text{in } \Omega, \end{cases}$$

where $n > 2s$, $M \geq 0$, $0 < s < 1$, $\gamma > 0$, $\lambda > 0$, $1 < q < 2^*_s - 1$. The weights $k : \Omega \to \mathbb{R}$ are assumed to be nonnegative and (essentially) bounded. In [3], the authors studied the existence of distributional solutions for small λ using the uniform estimates of $\{u_n\}$ which are solutions of the regularized problems with singular term $u^{-\gamma}$ replaced by $(u + \frac{1}{m})^{-\gamma}$. This was extended for the p-fractional Laplace operator by Canino et al. in [5]. Assuming $0 < \gamma < 1$, Ghanmi and Saoudi [10] studied the existence of at least two solutions for singular equations with a positively homogeneous function by making use of variational methods. For fractional Laplacian system involving singular nonlinearity, the work [11] dealt with

$$\begin{cases} (-\Delta)^s u = \lambda a(x)|u|^{q-2}u + \frac{1-\alpha}{2-\alpha-\beta} c(x)|u|^{-\alpha}|v|^{1-\beta} & x \in \Omega, \\ (-\Delta)^s v = \mu b(x)|v|^{q-2}u + \frac{1-\alpha}{2-\alpha-\beta} c(x)|u|^{1-\alpha}|v|^{-\beta} & x \in \Omega, \\ u = v = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\lambda, \mu \in (0, \infty)$, $0 < \alpha, \beta < 1$, $N > 2s$, $1 < q < 2 < 2^*_s = \frac{2N}{N-2s}$, $s \in (0,1)$, and $a, b, c \in C(\overline{\Omega})$ are nonnegative functions. With the help of Nehari manifold, the authors obtained two nontrivial solutions to this system.

Inspired by above papers, the main purpose of this paper is to extend the following work [16]

$$\begin{cases} \left(\sum_{i=1}^{k}[u_i]_{s,p}^{p}\right)^{\theta-1}(-\Delta)_p^s u_j(x)=\lambda_j|u_j|^{q-2}u_j+\sum_{i\neq j}\beta_{ij}|u_i|^m|u_j|^{m-2}u_j & \text{in } \Omega,\\ u_j=0 & \text{in } \mathbb{R}^N\setminus\Omega. \end{cases} \tag{1.3}$$

In [16], when $1<q<\theta p<2m<p_s^*$, the authors obtained two distinct solutions to system (1.3). We try to study whether it is possible to get similar result when replacing $\sum_{i\neq j}\beta_{ij}|u_i|^{1-m}|u_j|^{-m}$ in the place of $\sum_{i\neq j}\beta_{ij}|u_i|^m|u_j|^{m-2}u_j$. The main difficulties in dealing with this problem come from the singular nonlinearity, i.e. $0<m<1$. To our best knowledge, our result for the fractional p-Kirchhoff system with singular nonlinearity is new.

Before describing main result, we recall some necessary definitions. For convenience, we denote by $|u|_r:=\|u\|_{L^r(\mathbb{R}^N)}$ the norm of Lebesgue space $L^r(\Omega)$ with $r\geq 1$. Define $W^{s,p}(\Omega)$ as a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in $W^{s,p}(\Omega)$ belongs to $L^p(\Omega)$ and

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy<\infty.$$

Equip $W^{s,p}(\Omega)$ with the norm

$$\|u\|_{W^{s,p}(\Omega)}=|u|_p+\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}}.$$

Obviously, $W^{s,p}(\Omega)$ is a Banach space. We shall consider the following closed linear subspace

$$W_0^{s,p}(\Omega)=\left\{u\in W^{s,p}(\Omega):u(x)=0 \text{ a.e. in } \mathbb{R}^N\setminus\Omega\right\}.$$

Moreover, we have that

$$\|u_j\|_{W_j}=\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u_j(x)-u_j(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}}$$

is an equivalent norm of $W_j=W_0^{s,p}(\Omega)$. It follows from the fractional Sobolev inequality that

$$S=\inf_{u_j\in W_j}\left(\frac{\|u_j\|_{W_j}}{|u_j|_{p_s^*}}\right)^p. \tag{1.4}$$

In this paper we will work in the reflexive Banach space $W=W_1\times\cdots\times W_k$ endowed with the norm

$$\|\mathbf{u}\|_W=\left(\|u_1\|_{W_1}^p+\cdots+\|u_k\|_{W_k}^p\right)^{\frac{1}{p}},\qquad \forall\mathbf{u}=(u_1,\ldots,u_k)\in W.$$

The variational functional of system (1.1) is

$$J(\mathbf{u})=\frac{1}{\theta p}\|\mathbf{u}\|_W^{\theta p}-\frac{1}{q}\sum_{j=1}^{k}\lambda_j|u_j|_q^q-\frac{1}{1-m}\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}|u_iu_j|_{1-m}^{1-m}, \tag{1.5}$$

for $\mathbf{u} = (u_1, \dots, u_k) \in W$. Note that $J \notin C^1(W, \mathbb{R})$, and classical variational methods are not applicable. Moreover, we say that function $\mathbf{u} = (u_1, \dots, u_k) \in W$ is a weak solution of system (1.1), if

$$\|\mathbf{u}\|_W^{(\theta-1)p} \sum_{j=1}^{k} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_j(x) - u_j(y)|^{p-2}(u_j(x) - u_j(y))(w_j(x) - w_j(y))}{|x-y|^{n+ps}} dxdy$$
$$= \sum_{j=1}^{k} \lambda_j \int_\Omega |u_j|^{q-2} u_j w_j dx + \sum_{j=1}^{k} \sum_{i \neq j} \beta_{ij} \int_\Omega |u_i|^{1-m} |u_j|^{-m} w_j dx,$$

for any $\mathbf{w} = (w_1, \dots, w_k) \in W$. It is easy to see that solutions of system (1.1) correspond to the critical points of J.

Set

$$\Lambda = \frac{\theta p - 2 + 2m}{q - \theta p} \left(\frac{q - 2 + 2m}{q - \theta p} \right)^{\frac{2-2m-q}{\theta p - 2 + 2m}} \left(\sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \right)^{\frac{\theta p - q}{\theta p - 2 + 2m}} |\Omega|^{\frac{(2m-2+q)(\theta p - p_s^*)}{p_s^*(\theta p - 2 + 2m)}} S^{\frac{\theta(2m-2+q)}{\theta p - 2 + 2m}},$$

$$\Lambda_0 = \left(\frac{\theta p}{2 - 2m} \right)^{\frac{\theta p - q}{\theta p - 2 + 2m}} \Lambda,$$

$$\Theta_\Lambda = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_k) \in (\mathbb{R}^+)^k : 0 < \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}} < \Lambda \right\}$$

and

$$\Theta_{\Lambda_0} = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_k) \in (\mathbb{R}^+)^k : 0 < \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}} < \Lambda_0 \right\}.$$

Obviously, $\Theta_{\Lambda_0} \subset \Theta_\Lambda$. Our main result is the following.

Theorem 1.1. *Suppose that $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \Theta_{\Lambda_0}$. Then system (1.1) has two distinct solutions.*

The remainder of this paper is organized as follows. In Section 2, we state some preliminary results. Section 3 is devoted to the proof of Theorem 1.1.

2 Preliminaries

In this section, we state some basic results. Define the constraint set (Nehari maniflod)

$$\mathcal{N} = \{\mathbf{u} \in W \backslash \{0\} : \langle J'(\mathbf{u}), \mathbf{u} \rangle = 0\}.$$

Thus, $\mathbf{u} \in \mathcal{N}$ if and only if

$$\|\mathbf{u}\|_W^{\theta p} = \sum_{j=1}^{k} \lambda_j \int_\Omega |u_j|^q dx + 2 \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx. \tag{2.1}$$

Fix $\mathbf{u} \in W$ and define the function of the form $K_\mathbf{u} : t \to J(t\mathbf{u})$ for $t > 0$. Such maps are famous fibering maps, which were discussed by Brown and Wu in [4]. Precisely,

$$K_\mathbf{u}(t) = J(t\mathbf{u}) = \frac{t^{\theta p}}{\theta p} \|\mathbf{u}\|_W^{\theta p} - \frac{t^q}{q} \sum_{j=1}^{k} \lambda_j \int_\Omega |u_j|^q dx - \frac{t^{2-2m}}{1-m} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx.$$

Therefore,

$$K'_{\mathbf{u}}(t) = t^{\theta p-1}\|\mathbf{u}\|_W^{\theta p} - t^{q-1}\sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q dx - 2t^{1-2m}\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^{1-m}dx$$

and

$$K''_{\mathbf{u}}(t) = (\theta p-1)t^{\theta p-2}\|\mathbf{u}\|_W^{\theta p} - (q-1)t^{q-2}\sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q dx$$
$$- 2(1-2m)t^{-2m}\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^{1-m}dx.$$

Lemma 2.1. *Let $\mathbf{u} \in W\backslash\{0\}$ and $t > 0$. Then $t\mathbf{u} \in \mathcal{N}$ if and only if $K'_{\mathbf{u}}(t) = 0$.*

Proof. Note that

$$tK'_{\mathbf{u}}(t) = \|t\mathbf{u}\|_W^{\theta p} - \sum_{j=1}^{k}\lambda_j\int_\Omega |tu_j|^q dx - 2\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |tu_itu_j|^{1-m}dx.$$

By (2.1), we can easily draw the conclusion of the lemma. □

Using methods similar to those used in [15], we split $\mathcal{N}$ into three sets. Accordingly, we define

$$\mathcal{N}^+ = \{t\mathbf{u} \in W : K'_{\mathbf{u}}(t) = 0, K''_{\mathbf{u}}(t) > 0\} = \{\mathbf{u} \in \mathcal{N} : K''_{\mathbf{u}}(1) > 0\};$$
$$\mathcal{N}^- = \{t\mathbf{u} \in W : K'_{\mathbf{u}}(t) = 0, K''_{\mathbf{u}}(t) < 0\} = \{\mathbf{u} \in \mathcal{N} : K''_{\mathbf{u}}(1) < 0\};$$
$$\mathcal{N}^0 = \{t\mathbf{u} \in W : K'_{\mathbf{u}}(t) = 0, K''_{\mathbf{u}}(t) = 0\} = \{\mathbf{u} \in \mathcal{N} : K''_{\mathbf{u}}(1) = 0\}.$$

In the next, we state some basic properties of submanifold.

Lemma 2.2. *Let $\mathbf{u}_0$ be a local minimizer for J such that $\mathbf{u}_0 \notin \mathcal{N}^0$. Then $\mathbf{u}_0$ is a critical point for J.*

Proof. Since $\mathbf{u}_0$ is a local minimizer of J on $\mathcal{N}$, it is a solution of the optimization problem

$$\text{minimize } J \text{ subject to } F(\mathbf{u}) = 0,$$

where

$$F(\mathbf{u}) = \|\mathbf{u}\|_W^{\theta p} - \sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q dx - 2\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^{1-m}dx.$$

Then, applying the theory of Lagrange multipliers, we can find a $\mu \in \mathbb{R}$ such that $J'(\mathbf{u}_0) = \mu F'(\mathbf{u}_0)$ which implies

$$0 = \langle J'(\mathbf{u}_0), \mathbf{u}_0\rangle = \mu\langle F'(\mathbf{u}_0), \mathbf{u}_0\rangle.$$

Further, from $\mathbf{u}_0 \in \mathcal{N}$ and $\mathbf{u}_0 \notin \mathcal{N}^0$ it is easy to know $\langle F'(\mathbf{u}_0), \mathbf{u}_0\rangle \neq 0$. So we obtain $\mu = 0$ and the proof is complete. □

Lemma 2.3. *The functional J is coercive and bounded below on $\mathcal{N}$.*

Proof. For any $\mathbf{u} \in \mathcal{N}$, by (1.4), (2.1), the Young and Hölder inequalities, we obtain

$$\begin{aligned}
J(\mathbf{u}) &= \left(\frac{1}{\theta p} - \frac{1}{q}\right) \|\mathbf{u}\|_W^{\theta p} - \left(\frac{1}{1-m} - \frac{2}{q}\right) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} |u_i u_j|_{1-m}^{1-m} \\
&\geq \left(\frac{1}{\theta p} - \frac{1}{q}\right) \|\mathbf{u}\|_W^{\theta p} - \frac{1}{2}\left(\frac{1}{1-m} - \frac{2}{q}\right) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \left(|u_i|_{2-2m}^{2-2m} + |u_j|_{2-2m}^{2-2m}\right) \\
&\geq \left(\frac{1}{\theta p} - \frac{1}{q}\right) \|\mathbf{u}\|_W^{\theta p} - \frac{1}{2}\left(\frac{1}{1-m} - \frac{2}{q}\right) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} |\Omega|^{\frac{p_s^*+2m-2}{p_s^*}} S^{\frac{2m-2}{p}} \|\mathbf{u}\|_W^{2-2m},
\end{aligned}$$

which together with $2-2m < \theta p$ yields that J is coercive and bounded below on $\mathcal{N}$. □

Set

$$I_{\mathbf{u}}(t) = t^{\theta p - q} \|\mathbf{u}\|_W^{\theta p} - 2t^{2-2m-q} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx.$$

Clearly, $t\mathbf{u} \in \mathcal{N}$ if and only if

$$I_{\mathbf{u}}(t) = \sum_{j=1}^{k} \lambda_j \int_\Omega |u_j|^q dx. \tag{2.2}$$

Moreover, $I_{\mathbf{u}}$ satisfies the following properties.

Lemma 2.4. *Suppose that* $\mathbf{u} \in W\backslash\{0\}$. *One has*

(i) the function $I_{\mathbf{u}}$ *possesses a unique maximum at*

$$t = t_{max} = \left(\frac{2(q-2+2m) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx}{(q-\theta p)\|\mathbf{u}\|_W^{\theta p}} \right)^{\frac{1}{\theta p - 2 + 2m}};$$

(ii) $I'_{\mathbf{u}}(t) > 0$ *for* $t \in (0, t_{max})$ *and* $I'_{\mathbf{u}}(t) < 0$ *for* $t \in (t_{max}, +\infty)$;

(iii) $\lim_{t\to 0^+} I_{\mathbf{u}}(t) = -\infty$, $\lim_{t\to+\infty} I_{\mathbf{u}}(t) = 0$.

Proof. Note that

$$I'_{\mathbf{u}}(t) = (\theta p - q) t^{\theta p - q - 1} \|\mathbf{u}\|_W^{\theta p} - 2(2-2m-q) t^{1-2m-q} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx.$$

Set $I'_{\mathbf{u}}(t) = 0$. Obviously, $I'_{\mathbf{u}}(t_{max}) = 0$ and $I''_{\mathbf{u}}(t_{max}) < 0$, with unique

$$t_{max} = \left(\frac{2(q-2+2m) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx}{(q-\theta p)\|\mathbf{u}\|_W^{\theta p}} \right)^{\frac{1}{\theta p - 2 + 2m}}.$$

Moreover, it is easy to see that (ii) and (iii) follow from the structure of $I_{\mathbf{u}}$. □

Lemma 2.5. *Suppose that* $t\mathbf{u} \in \mathcal{N}$. *Then* $t\mathbf{u} \in \mathcal{N}^+$ *or* $(\mathcal{N}^-)$ *if and only if* $I'_u(t) > 0$ *or* (< 0).

Proof. If $t\mathbf{u} \in \mathcal{N}$, by (2.1), we get

$$K''_{\mathbf{u}}(t) = (\theta p - q)t^{\theta p - 2}\|\mathbf{u}\|_W^{\theta p} - 2(2-2m-q)t^{-2m}\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_{\Omega}|u_i u_j|^{1-m}dx.$$

Note that

$$t^{1-q}I'_{\mathbf{u}}(t) = K''_{\mathbf{u}}(t),$$

which yields that $t\mathbf{u} \in \mathcal{N}^+$ or $(\mathcal{N}^-)$ if and only if $I'_u(t) > 0$ or (< 0). □

Lemma 2.6. *Suppose that* $\mathbf{u} \in W\backslash\{0\}$. *Then for* $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \Theta_\Lambda$, *there exist* $t^+, t^- > 0$ *such that* $t^+ < t_{max} < t^-$, $t^+\mathbf{u} \in \mathcal{N}^+$, $t^-\mathbf{u} \in \mathcal{N}^-$ *and*

$$J(t^+\mathbf{u}) = \inf_{0\le t\le t_{max}} J(t\mathbf{u}), \qquad J(t^-\mathbf{u}) = \sup_{t\ge 0} J(t\mathbf{u}).$$

Proof. By (1.4), the Young and Hölder inequalities, we have

$$\begin{aligned}
I_{\mathbf{u}}(t_{max}) &= \frac{\theta p - 2 + 2m}{q - \theta p}\left(\frac{q-2+2m}{q-\theta p}\right)^{\frac{2-2m-q}{\theta p - 2 + 2m}} \frac{\left(2\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}\int_\Omega |u_i u_j|^{1-m}dx\right)^{\frac{\theta p - q}{\theta p - 2 + 2m}}}{\left(\|\mathbf{u}\|_W^{\theta p}\right)^{\frac{2-2m-q}{\theta p - 2+2m}}} \\
&\ge \frac{\theta p - 2 + 2m}{q - \theta p}\left(\frac{q-2+2m}{q-\theta p}\right)^{\frac{2-2m-q}{\theta p - 2 + 2m}} \frac{\left(\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}|\Omega|^{\frac{p_s^*+2m-2}{p_s^*}}S^{\frac{2m-2}{p}}\|\mathbf{u}\|_W^{2-2m}\right)^{\frac{\theta p - q}{\theta p - 2 + 2m}}}{\left(\|\mathbf{u}\|_W^{\theta p}\right)^{\frac{2-2m-q}{\theta p - 2+2m}}} \\
&= \frac{\theta p - 2 + 2m}{q - \theta p}\left(\frac{q-2+2m}{q-\theta p}\right)^{\frac{2-2m-q}{\theta p - 2 + 2m}} \left(\sum_{j=1}^{k}\sum_{i<j}\beta_{ij}|\Omega|^{\frac{p_s^*+2m-2}{p_s^*}}S^{\frac{2m-2}{p}}\right)^{\frac{\theta p - q}{\theta p - 2 + 2m}} \|\mathbf{u}\|_W^{q}.
\end{aligned}$$

It follows from $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \Theta_\Lambda$ that

$$\begin{aligned}
0 &< \sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q dx \le |\Omega|^{\frac{p_s^*-q}{p_s^*}}\sum_{j=1}^{k}\lambda_j|u_j|_{p_s^*}^q \\
&\le |\Omega|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\sum_{j=1}^{k}\lambda_j\|u_j\|_{W_j}^q \le |\Omega|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\left(\sum_{j=1}^{k}\lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\left(\sum_{j=1}^{k}\|u_j\|_{W_j}^{\theta p}\right)^{\frac{q}{\theta p}} \\
&\le |\Omega|^{\frac{p_s^*-q}{p_s^*}}S^{-\frac{q}{p}}\left(\sum_{j=1}^{k}\lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\|\mathbf{u}\|_W^q < I_{\mathbf{u}}(t_{max}),
\end{aligned}$$

which implies that there exist $t^+, t^- > 0$ such that $t^+ < t_{max} < t^-$,

$$I_{\mathbf{u}}(t^+) = \sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q dx = I_{\mathbf{u}}(t^-),$$

$I'_{\mathbf{u}}(t^+) > 0$ and $I'_{\mathbf{u}}(t^-) < 0$. Then, by (2.2) and Lemma 2.5, we obtain $t^+\mathbf{u} \in \mathcal{N}^+$ and $t^-\mathbf{u} \in \mathcal{N}^-$. Combining Lemma 2.4 and

$$K'_{\mathbf{u}}(t) = t^{q-1}\left(I_{\mathbf{u}}(t) - \sum_{j=1}^{k}\lambda_j\int_\Omega |u_j|^q dx\right),$$

we get that $J(t\mathbf{u})$ is strictly decreasing on $(0,t^+)$, strictly increasing on (t^+,t^-) and strictly decreasing on $(t^-,+\infty)$. Hence

$$J(t^+\mathbf{u}) = \inf_{0\le t\le t_{max}} J(t\mathbf{u}), \qquad J(t^-\mathbf{u}) = \sup_{t\ge 0} J(t\mathbf{u}).$$

The proof is completed. □

Lemma 2.7. *Suppose that* $(\lambda_1,\lambda_2,\dots,\lambda_k)\in\Theta_\Lambda$. *Then* $\mathcal{N}^0=\emptyset$.

Proof. Arguing by contradiction, we assume that $\mathcal{N}^0\neq\emptyset$. Then for $\mathbf{u}\in\mathcal{N}^0$, we have by (2.1) that

$$\begin{aligned}0 = K''_{\mathbf{u}}(1) &= (\theta p - q)\|\mathbf{u}\|_W^{\theta p} - 2(2-2m-q)\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^{1-m}dx\\ &= (\theta p + 2m - 2)\|\mathbf{u}\|_W^{\theta p} - (q+2m-2)\sum_{j=1}^k \lambda_j\int_\Omega |u_j|^q dx.\end{aligned}$$

Hence, by (1.4), the Young and Hölder inequalities, we get

$$\begin{aligned}\|\mathbf{u}\|_W^{\theta p} &= \frac{2(2-2m-q)}{\theta p - q}\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega |u_iu_j|^{1-m}dx\\ &\le \frac{2-2m-q}{\theta p - q}\sum_{j=1}^k\sum_{i<j}\beta_{ij}|\Omega|^{\frac{p_s^*+2m-2}{p_s^*}} S^{\frac{2m-2}{p}}\|\mathbf{u}\|_W^{2-2m},\end{aligned}$$

which implies that

$$\|\mathbf{u}\|_W \le \left(\frac{2-2m-q}{\theta p - q}\sum_{j=1}^k\sum_{i<j}\beta_{ij}|\Omega|^{\frac{p_s^*+2m-2}{p_s^*}} S^{\frac{2m-2}{p}}\right)^{\frac{1}{\theta p - 2 + 2m}}. \tag{2.3}$$

Moreover, by (1.4) and the Hölder inequality, we get

$$\begin{aligned}\|\mathbf{u}\|_W^{\theta p} &= \frac{q+2m-2}{\theta p + 2m - 2}\sum_{j=1}^k \lambda_j\int_\Omega |u_j|^q dx\\ &\le \frac{q+2m-2}{\theta p + 2m - 2}|\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}}\left(\sum_{j=1}^k \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\|\mathbf{u}\|_W^q,\end{aligned}$$

which implies that

$$\|\mathbf{u}\|_W \ge \left(\frac{q+2m-2}{\theta p + 2m - 2}|\Omega|^{\frac{p_s^*-q}{p_s^*}} S^{-\frac{q}{p}}\left(\sum_{j=1}^k \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\right)^{\frac{1}{\theta p - q}}. \tag{2.4}$$

Combining (2.3) and (2.4), we obtain

$$\begin{aligned}&\left(\sum_{j=1}^k \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\\ &\quad\ge \frac{\theta p - 2 + 2m}{q - \theta p}\left(\frac{q-2+2m}{q-\theta p}\right)^{\frac{2-2m-q}{\theta p - 2 + 2m}}\left(\sum_{j=1}^k\sum_{i<j}\beta_{ij}\right)^{\frac{\theta p - q}{\theta p - 2 + 2m}}|\Omega|^{\frac{(2m-2+q)(\theta p - p_s^*)}{p_s^*(\theta p - 2 + 2m)}} S^{\frac{\theta(2m-2+q)}{\theta p - 2 + 2m}},\end{aligned}$$

which contradicts

$$0 < \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}} \right)^{\frac{\theta p - q}{\theta p}} < \Lambda.$$

This ends the proof. □

3 Proof of Theorem 1.1

By Lemmas 2.3 and 2.7, for $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Theta_\Lambda$, we obtain $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ and J is bounded from below on $\mathcal{N}^+$ and $\mathcal{N}^-$. Set

$$\alpha^+ = \inf_{\mathbf{u} \in \mathcal{N}^+} J(\mathbf{u}) \quad \text{and} \quad \alpha^- = \inf_{\mathbf{u} \in \mathcal{N}^-} J(\mathbf{u}).$$

Lemma 3.1. $\alpha^+ < 0$.

Proof. For $\mathbf{u} \in \mathcal{N}^+$, we have $K'_{\mathbf{u}}(1) = 0$ and $K''_{\mathbf{u}}(1) > 0$. Then

$$(\theta p - q)\|\mathbf{u}\|_W^{\theta p} > 2(2 - 2m - q) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |u_i u_j|^{1-m} dx.$$

This yields that

$$\begin{aligned} J(\mathbf{u}) &= \left(\frac{1}{\theta p} - \frac{1}{q}\right) \|\mathbf{u}\|_W^{\theta p} - \left(\frac{1}{1-m} - \frac{2}{q}\right) \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} |u_i u_j|_{1-m}^{1-m} \\ &\leq \left[\left(\frac{1}{\theta p} - \frac{1}{q}\right) - \left(\frac{1}{1-m} - \frac{2}{q}\right) \frac{q - \theta p}{2(q - 2 + 2m)} \right] \|\mathbf{u}\|_W^{\theta p} \\ &= \frac{(\theta p - q)(\theta p - 2 + 2m)}{(2 - 2m) q \theta p} \|\mathbf{u}\|_W^{\theta p} < 0, \end{aligned}$$

due to $0 < 2 - 2m < \theta p < q$. Therefore $\alpha^+ < 0$ follows from the definition α^+. □

Lemma 3.2. *The minimization problem*

$$\alpha^+ = \inf_{\mathbf{u} \in \mathcal{N}^+} J(\mathbf{u})$$

is achieved at a point $\mathbf{u}^+ \in \mathcal{N}^+$.

Proof. Let $\{\mathbf{u}_n\}$ be a minimizing sequence of the minimization problem, i.e. $\{\mathbf{u}_n\} \subset \mathcal{N}^+$ and $\lim_{n\to\infty} J(\mathbf{u}_n) = \alpha^+$. By Lemma 2.3, it is easy to see that $\{\mathbf{u}_n\}$ is bounded, we can find a $\mathbf{u}^+$ such that $\mathbf{u}_n \rightharpoonup \mathbf{u}^+$ weakly in W, $\mathbf{u}_n \to \mathbf{u}^+$ strongly in $L^r(\Omega)$, $1 \leq r < p_s^*$. Now, we prove

$$\lim_{n\to\infty} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |(\mathbf{u}_n)_i (\mathbf{u}_n)_j|^{1-m} dx = \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |(\mathbf{u}^+)_i (\mathbf{u}^+)_j|^{1-m} dx \tag{3.1}$$

and

$$\lim_{n\to\infty} \sum_{j=1}^{k} \lambda_j \int_\Omega |(\mathbf{u}_n)_j|^q dx = \sum_{j=1}^{k} \lambda_j \int_\Omega |(\mathbf{u}^+)_j|^q dx. \tag{3.2}$$

By the Vitali theorem, we claim that

$$\lim_{n\to\infty} \int_\Omega |(\mathbf{u}_n)_i (\mathbf{u}_n)_j|^{1-m} dx = \int_\Omega |(\mathbf{u}^+)_i (\mathbf{u}^+)_j|^{1-m} dx.$$

In fact, by the Young inequality, we have

$$\int_\Omega |(\mathbf{u}_n)_i(\mathbf{u}_n)_j|^{1-m}dx \le \frac{1}{2}\int_\Omega |(\mathbf{u}_n)_i|^{2-2m}dx + \frac{1}{2}\int_\Omega |(\mathbf{u}_n)_j|^{2-2m}dx.$$

By the Sobolev embedding theorem and boundedness of $\{(\mathbf{u}_n)_i\}$, we can find a constant $C > 0$ such that $|(\mathbf{u}_n)_i|_{p_s^*} \le C$. Moreover, it follows from the Hölder inequality that

$$\int_\Omega |(\mathbf{u}_n)_i|^{2-2m}dx \le |\Omega|^{\frac{p_s^*+2m-2}{p_s^*}} |(\mathbf{u}_n)_i|_{p_s^*}^{2-2m}. \tag{3.3}$$

From (3.3), for every $\epsilon > 0$, setting

$$\delta = \left(\frac{\epsilon}{C^{2-2m}}\right)^{\frac{p_s^*}{p_s^*+2m-2}},$$

when $A \subset \Omega$ with meas $A < \delta$, we obtain

$$\int_A |(\mathbf{u}_n)_i|^{2-2m}dx \le |A|^{\frac{p_s^*+2m-2}{p_s^*}} C^{2-2m} < \epsilon.$$

Similarly, $\int_A |(\mathbf{u}_n)_j|^{2-2m}dx < \epsilon$. This yields that

$$\left\{\int_\Omega |(\mathbf{u}_n)_i(\mathbf{u}_n)_j|^{1-m}dx, n \in \mathbb{N}\right\}$$

is equi-absolutely-continuous. Thus, our claim is true. This implies that (3.1) holds. On the other hand, for $1 \le j \le k$, it follows from the Hölder inequality and $\mathbf{u}_n \to \mathbf{u}^+$ strongly in $L^q(\Omega)$ that

$$\begin{aligned}\int_\Omega \big||(\mathbf{u}_n)_j|^q - |(\mathbf{u}^+)_j|^q\big|\,dx &= q\int_\Omega (|(\mathbf{u}^+)_j| + \tau(|(\mathbf{u}_n)_j - (\mathbf{u}^+)_j|))^{q-1}|(\mathbf{u}_n)_j - (\mathbf{u}^+)_j|dx \\ &\le q|(\mathbf{u}_n)_j + (\mathbf{u}^+)_j|_q^{q-1}|(\mathbf{u}_n)_j - (\mathbf{u}^+)_j|_q \\ &\le C|(\mathbf{u}_n)_j - (\mathbf{u}^+)_j|_q \to 0,\end{aligned}$$

as $n \to \infty$, where $\tau \in (0,1)$ and $C > 0$ denotes various constants. Therefore,

$$\lim_{n\to\infty}\int_\Omega \big||(\mathbf{u}_n)_j|^q - |(\mathbf{u}^+)_j|^q\big|\,dx = 0, \qquad \forall j \in \{1,2,,\dots k\},$$

which implies that (3.2) holds. Furthermore, we can prove that $\mathbf{u}_n \to \mathbf{u}^+$ strongly in W. Arguing by contradiction, we assume $\mathbf{u}_n \nrightarrow \mathbf{u}^+$ strongly in W. Then,

$$\|\mathbf{u}^+\|_W^{\theta p} < \lim_{n\to\infty}\inf \|\mathbf{u}_n\|_W^{\theta p}.$$

By Lemma 2.6, there exists $t^+ > 0$ such that $t^+\mathbf{u}^+ \in \mathcal{N}^+$. Then, for $\mathbf{u}_n \in \mathcal{N}^+$, one has

$$\begin{aligned}&\lim_{n\to\infty} K'_{\mathbf{u}_n}(t^+) \\ &= \lim_{n\to\infty}\left((t^+)^{\theta p-1}\|\mathbf{u}_n\|_W^{\theta p} - (t^+)^{q-1}\sum_{j=1}^k \lambda_j\int_\Omega |(\mathbf{u}_n)_j|^q dx - 2(t^+)^{1-2m}\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega |(\mathbf{u}_n)_i(\mathbf{u}_n)_j|^{1-m}dx\right) \\ &> (t^+)^{\theta p-1}\|\mathbf{u}^+\|_W^{\theta p} - (t^+)^{q-1}\sum_{j=1}^k \lambda_j\int_\Omega |(\mathbf{u}^+)_j|^q dx - 2(t^+)^{1-2m}\sum_{j=1}^k\sum_{i<j}\beta_{ij}\int_\Omega |(\mathbf{u}^+)_i(\mathbf{u}^+)_j|^{1-m}dx \\ &= K'_{\mathbf{u}^+}(t^+) = 0.\end{aligned}$$

This yields $K'_{\mathbf{u}_n}(t^+) > 0$ for large enough n. Note that $K'_{\mathbf{u}_n}(1) = 0$ for each n and $K'_{\mathbf{u}_n}(t) < 0$ for $t \in (0,1)$. It follows that $t^+ > 1$. Moreover, from that fact that $K_{\mathbf{u}^+}(t)$ is decreasing on $(0, t^+)$, we have

$$J(t^+\mathbf{u}^+) \le J(\mathbf{u}^+) < \lim_{n\to\infty} J(\mathbf{u}_n) = \alpha^+,$$

which contradicts the fact that $\alpha^+ = \inf_{\mathbf{u}\in\mathcal{N}^+} J(\mathbf{u})$. Thus, we conclude that $\mathbf{u}_n \to \mathbf{u}^+$ strongly in W. By $K'_{\mathbf{u}_n}(1) = 0$ and $K''_{\mathbf{u}_n}(1) > 0$, we get that $K'_{\mathbf{u}^+}(1) = 0$ and $K''_{\mathbf{u}^+}(1) \ge 0$. Note that $\mathcal{N}^0 = \emptyset$. Then $K''_{\mathbf{u}^+}(1) > 0$, which implies $\mathbf{u}^+ \in \mathcal{N}^+$. Above all, by $J(\mathbf{u}^+) = \inf_{\mathbf{u}\in\mathcal{N}^+} J(\mathbf{u}) < 0$, $\mathbf{u}^+$ is a minimizer of J on $\mathcal{N}^+$. The proof is completed. □

Lemma 3.3. *Suppose that $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Theta_{\Lambda_0}$. Then $\alpha^- > \alpha_0$ for some $\alpha_0 > 0$.*

Proof. For $\mathbf{u} \in \mathcal{N}^-$, we have $K'_{\mathbf{u}}(1) = 0$ and $K''_{\mathbf{u}}(1) < 0$. Then

$$(\theta p - 2 + 2m)\|\mathbf{u}\|_W^{\theta p} < (q - 2 + 2m)\sum_{j=1}^{k} \lambda_j \int_\Omega |u_j|^q dx.$$

By (1.4) and the Hölder inequality, we have

$$\sum_{j=1}^{k} \lambda_j \int_\Omega |u_j|^q dx \le |\Omega|^{\frac{p_s^* - q}{p_s^*}} S^{-\frac{q}{p}} \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}} \|\mathbf{u}\|_W^q.$$

Hence,

$$\|\mathbf{u}\|_W > \left[\frac{q - 2 + 2m}{\theta p - 2 + 2m} |\Omega|^{\frac{p_s^* - q}{p_s^*}} S^{-\frac{q}{p}} \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\right]^{\frac{1}{\theta p - q}}. \tag{3.4}$$

Then we obtain

$$\begin{aligned}
J(\mathbf{u}) &= \left(\frac{1}{\theta p} - \frac{1}{q}\right)\|\mathbf{u}\|_W^{\theta p} - \left(\frac{1}{1-m} - \frac{2}{q}\right)\sum_{j=1}^{k}\sum_{i<j} \beta_{ij} |u_i u_j|_{1-m}^{1-m} \\
&\ge \left(\frac{1}{\theta p} - \frac{1}{q}\right)\|\mathbf{u}\|_W^{\theta p} - \frac{1}{2}\left(\frac{1}{1-m} - \frac{2}{q}\right)\sum_{j=1}^{k}\sum_{i<j} \beta_{ij} |\Omega|^{\frac{p_s^* + 2m - 2}{p_s^*}} S^{\frac{2m-2}{p}} \|\mathbf{u}\|_W^{2-2m} \\
&= \|\mathbf{u}\|_W^{2-2m}\left[\left(\frac{1}{\theta p} - \frac{1}{q}\right)\|\mathbf{u}\|_W^{\theta p - 2 + 2m} - \frac{1}{2}\left(\frac{1}{1-m} - \frac{2}{q}\right)\sum_{j=1}^{k}\sum_{i<j} \beta_{ij} |\Omega|^{\frac{p_s^* + 2m - 2}{p_s^*}} S^{\frac{2m-2}{p}}\right] \\
&> \|\mathbf{u}\|_W^{2-2m}\Bigg\{\left(\frac{1}{\theta p} - \frac{1}{q}\right)\left[\frac{q - 2 + 2m}{\theta p - 2 + 2m} |\Omega|^{\frac{p_s^* - q}{p_s^*}} S^{-\frac{q}{p}} \left(\sum_{j=1}^{k} \lambda_j^{\frac{\theta p}{\theta p - q}}\right)^{\frac{\theta p - q}{\theta p}}\right]^{\frac{\theta p - 2 + 2m}{\theta p - q}} \\
&\quad - \frac{1}{2}\left(\frac{1}{1-m} - \frac{2}{q}\right)\sum_{j=1}^{k}\sum_{i<j} \beta_{ij} |\Omega|^{\frac{p_s^* + 2m - 2}{p_s^*}} S^{\frac{2m-2}{p}}\Bigg\} \ge \alpha_0 > 0,
\end{aligned}$$

thanks to $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Theta_{\Lambda_0}$ and (3.4). □

Lemma 3.4. *The minimization problem*

$$\alpha^- = \inf_{\mathbf{u}\in\mathcal{N}^-} J(\mathbf{u})$$

is achieved at a point $\mathbf{u}^- \in \mathcal{N}^-$.

Proof. Let $\{\mathbf{u}_n\}$ be a minimizing sequence of the minimization problem, i.e. $\{\mathbf{u}_n\} \subset \mathcal{N}^-$ and $\lim_{n\to\infty} J(\mathbf{u}_n) = \alpha^-$. By Lemma 2.3, it is easy to see that $\{\mathbf{u}_n\}$ is bounded, we can find a $\mathbf{u}^-$ such that $\mathbf{u}_n \rightharpoonup \mathbf{u}^-$ weakly in W, $\mathbf{u}_n \to \mathbf{u}^-$ strongly in $L^r(\Omega)$, $1 \le r < p_s^*$. Similar to Lemma 3.2, we have

$$\lim_{n\to\infty} \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |(\mathbf{u}_n)_i (\mathbf{u}_n)_j|^{1-m} dx = \sum_{j=1}^{k} \sum_{i<j} \beta_{ij} \int_\Omega |(\mathbf{u}^-)_i (\mathbf{u}^-)_j|^{1-m} dx$$

and

$$\lim_{n\to\infty} \sum_{j=1}^{k} \lambda_j \int_\Omega |(\mathbf{u}_n)_j|^q dx = \sum_{j=1}^{k} \lambda_j \int_\Omega |(\mathbf{u}^-)_j|^q dx.$$

Furthermore, we can prove that $\mathbf{u}_n \to \mathbf{u}^-$ strongly in W. Arguing by contradiction, we assume $\mathbf{u}_n \not\to \mathbf{u}^-$ strongly in W. Then,

$$\|\mathbf{u}^-\|_W^{\theta p} < \lim_{n\to\infty} \inf \|\mathbf{u}_n\|_W^{\theta p}.$$

By Lemma 2.6, there exists $t^- > 0$ such that $t^-\mathbf{u}^- \in \mathcal{N}^-$. Thus, since $\{\mathbf{u}_n\} \subset \mathcal{N}^-$ and $J(t\mathbf{u}_n) \le J(\mathbf{u}_n)$, for all $t > 0$ we have

$$J(t^-\mathbf{u}^-) < \lim_{n\to\infty} J(t^-\mathbf{u}_n) \le \lim_{n\to\infty} J(\mathbf{u}_n) = \alpha^-,$$

which contradicts the fact that $\alpha^- = \inf_{\mathbf{u}\in\mathcal{N}^-} J(\mathbf{u})$. Thus, we conclude that $\mathbf{u}_n \to \mathbf{u}^-$ strongly in W. By $K'_{\mathbf{u}_n}(1) = 0$ and $K''_{\mathbf{u}_n}(1) < 0$, we get that $K'_{\mathbf{u}^-}(1) = 0$ and $K''_{\mathbf{u}^-}(1) \le 0$. Note that $\mathcal{N}^0 = \emptyset$. Then $K''_{\mathbf{u}^+}(1) < 0$, which implies $\mathbf{u}^- \in \mathcal{N}^+$. Above all, by $J(\mathbf{u}^-) = \inf_{\mathbf{u}\in\mathcal{N}^-} J(\mathbf{u})$, $\mathbf{u}^-$ is a minimizer of J on $\mathcal{N}^-$. The proof is completed. □

Proof of Theorem 1.1. For all $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \Theta_{\Lambda_0}$, by Lemmas 3.2 and 3.4, we conclude that there exist $\mathbf{u}^+ \in \mathcal{N}^+$ and $\mathbf{u}^- \in \mathcal{N}^-$ satisfying $J(\mathbf{u}^+) = \inf_{\mathbf{u}\in\mathcal{N}^+} J(\mathbf{u})$ and $J(\mathbf{u}^-) = \inf_{\mathbf{u}\in\mathcal{N}^-} J(\mathbf{u})$. In view of Lemma 2.2, $\mathbf{u}^+$ and $\mathbf{u}^-$ are two solutions of system (1.1). Moreover, since $J(\mathbf{u}^+) = J(|\mathbf{u}^+|)$ and $|\mathbf{u}^+| \in \mathcal{N}^+$ and similarly $J(\mathbf{u}^-) = J(|\mathbf{u}^-|)$ and $|\mathbf{u}^-| \in \mathcal{N}^+$, so we may assume $\mathbf{u}^\pm \ge 0$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, two solutions of system (1.1) are distinct. And by Lemmas 3.1 and 3.3, we have $J(\mathbf{u}^+) < 0$ and $J(\mathbf{u}^-) > 0$. Hence we provided the existence of two nontrivial nonnegative solutions to our system (1.1). □

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Family of quadratic differential systems with invariant parabolas: a complete classification in the space $\mathbb{R}^{12}$

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Abstract. Consider the class **QS** of all non-degenerate planar quadratic differential systems and its subclass **QSP** of all its systems possessing an invariant parabola. This is an interesting family because on one side it is defined by an algebraic geometric property and on the other, it is a family where limit cycles occur. Note that each quadratic differential system can be identified with a point of $\mathbb{R}^{12}$ through its coefficients. In this paper, we provide necessary and sufficient conditions for a system in **QS** to have at least one invariant parabola. We give the global "bifurcation" diagram of the family **QS** which indicates where a parabola is present or absent and in case it is present, the diagram indicates how many parabolas there could be, their reciprocal position and what kind of singular points at infinity (simple or multiple) as well as their multiplicities are the points at infinity of the parabolas. The diagram is expressed in terms of affine invariant polynomials and it is done in the 12-dimensional space of parameters.

Keywords: quadratic vector fields, affine invariant polynomials, invariant algebraic curve, invariant parabola.

2020 Mathematics Subject Classification: 58K30, 34A26, 34C05, 34C40.

1 Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

where $P, Q \in \mathbb{R}[x,y]$, i.e. P, Q are polynomials in x, y over $\mathbb{R}$ and their associated vector fields

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

We call *degree* of a system (1.1) the integer $m = \max(\deg P, \deg Q)$. In particular we call *quadratic* a differential system (1.1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

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Quadratic systems appear in the modelling of many natural phenomena described in different branches of science, in biological and physical applications and applications of these systems became a subject of interest for the mathematicians. Many papers have been published about quadratic systems, see for example [19] for a bibliographical survey.

Here we always assume that the polynomials P and Q are coprime. Otherwise doing a rescaling of the time, systems (1.1) can be reduced to linear or constant systems. Quadratic systems under this assumption are called *non-degenerate quadratic systems*.

Let V be an open and dense subset of $\mathbb{R}^2$, we say that a nonconstant differentiable function $H : V \to \mathbb{R}$ is a first integral of a system (1.1) on V if $H(x(t), y(t))$ is constant for all of the values of t for which $(x(t), y(t))$ is a solution of this system contained in V. Obviously H is a first integral of systems (1.1) if and only if

$$X(H) = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0,$$

for all $(x, y) \in V$. When a system (1.1) has a first integral we say that this system is integrable.

The knowledge of the first integrals is of particular interest in planar differential systems because they allow us to draw their phase portraits.

On the other hand given $f \in \mathbb{C}[x, y]$ we say that the curve $f(x, y) = 0$ is an *invariant algebraic curve* of systems (1.1) if there exists $K \in \mathbb{C}[x, y]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. When $K = 0$, f is a polynomial first integral.

Let us suppose that $f(x, y) = 0$ is of degree n:

$$f(x, y) = c_{00} + c_{10}x + c_{01}y + \cdots + c_{n0}x^n + c_{n-1,1}x^{n-1}y + \cdots + c_{0n}y^n$$

with $\hat{c} = (c_{00}, \ldots, c_{0n}) \in \mathbb{C}^N$ where $N = (n+1)(n+2)/2$. We note that the equation $\lambda f(x, y) = 0$ where $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ yields the same locus of complex points in the plane as the locus induced by $f(x, y) = 0$. So, a curve of degree n defined by $\hat{c}$ can be identified with a point $[\hat{c}] = [c_{00} : c_{10} : \cdots : c_{0n}]$ in $P_{N-1}(\mathbb{C})$. We say that a sequence of degree n curves $f_i(x, y) = 0$ converges to a curve $f(x, y) = 0$ if and only if the sequence of points $[\hat{c}_i] = [c_{i00} : c_{i10} \cdots : c_{i0n}]$ converges to $[\hat{c}] = [c_{00} : c_{10} : \cdots : c_{0n}]$ in the topology of $P_{N-1}(\mathbb{C})$.

We observe that if we rescale the time $t' = \lambda t$ by a positive constant λ the geometry of the systems (1.1) does not change. So for our purposes we can identify a system (1.1) of degree m with a point in $[a_{10}, a_{10}, \ldots, b_{0m}]$ in $\mathbb{S}^{M-1}(\mathbb{R})$ with $M = (m+1)(m+2)$.

We compactify the space of all the polynomial differential systems of degree m on $\mathbb{S}^{M-1}$ with $M = (m+1)(m+2)$ by multiplying the coefficients of each systems with $1/\big(\sum(a_{ij}^2 + b_{ij}^2)\big)^{1/2}$.

Definition 1.1. We say that an invariant curve $\mathcal{L} : f(x, y) = 0$, $f \in \mathbb{C}[x, y]$ for a polynomial system (S) of degree m has *multiplicity* r if there exists a sequence of real polynomial systems (S_k) of degree m converging to (S) in the topology of $\mathbb{S}^{M-1}$, $M = (m+1)(m+2)$, such that each (S_k) has r distinct invariant curves $\mathcal{L}_{1,k} : f_{1,k}(x, y) = 0, \ldots, \mathcal{L}_{r,k} : f_{r,k}(x, y) = 0$ over $\mathbb{C}$, $\deg(f) = \deg(f_{i,k}) = n$, converging to $\mathcal{L}$ as $k \to \infty$, in the topology of $P_{N-1}(\mathbb{C})$, with $N = (n+1)(n+2)/2$ and this does not occur for $r+1$.

The motivation for studying the systems in the quadratic class is not only because of their usefulness in many applications but also for theoretical reasons, as discussed by Schlomiuk and Vulpe in the introduction of [20]. The study of non–degenerate quadratic systems could be done using normal forms and applying the invariant theory.

Systematic work on quadratic differential systems possessing an invariant conic began towards the end of the XX-th century and the beginning of this century. Quadratic systems having an invariant ellipse as a limit cycle were investigated by Y.-X. Qin [18]; the necessary and sufficient conditions on the coefficients of a quadratic system and also on the coefficients of a conic so as to have the conic as an invariant curve of the system were presented by Druzhkova [8]; Cairó and Llibre in [4] have investigated the Darboux integrability of the quadratic systems having invariant algebraic conics; Oliveira, Rezende and Vulpe [14] provided necessary and sufficient conditions for a system in **QS** to have at least one invariant hyperbola in terms of its coefficients and the necessary and sufficient affine invariant conditions for a system in **QS** so as to have the ellipse as an invariant curve of the system were presented by Oliveira, Rezende, Schlomiuk and Vulpe [16]. In [15] the authors classified the family of quadratic systems possessing an invariant hyperbola in terms of configurations of hyperbolas and presence or absence of invariant lines. This is an invariant classification, independent of specific normal forms. A similar classification in the case of an invariant ellipse is done in [13].

In this work we consider non-degenerate quadratic differential systems possessing an invariant parabola. We denote this family by **QSP**. Our goal in this paper is to obtain a characterization of systems in **QSP** in terms of invariant polynomials. Thus our equalities and inequalities in the bifurcation diagram splitting the parameter space into regions and subsets with distinct dynamics, will not be expressed in terms of coefficients of a fixed normal form or several such forms, coefficients which do not have any geometrical meaning and are rigidly tied to these normal forms. They will be expressed in terms of invariant polynomials which are very supple objects that can be easily be computed by a computer for any specific normal form and allowing us also to easily pass from one normal form to any other.

It is known that the coordinates of an infinite singular point p of a quadratic system (S) are defined by a linear factor in the factorization of the invariant polynomial $C_2(x,y) = yp_2(x,y) - xq_2(x,y)$ over $\mathbb{C}$. Here $p_2(x,y)$ and $q_2(x,y)$ are the corresponding quadratic homogeneous parts of (S). The multiplicity m of the singularity p has two components (see the concepts and notations introduced in [11]). If we denote them by (m^∞, m^f) (i.e. $m = m^\infty + m^f$) then m^∞ (respectively, m^f) is the maximum number of infinite (respectively, finite) singularities which can split from p, in small perturbations of the systems. In this case the number m^∞ coincides with the multiplicity of the linear factor of $C_2(x,y)$ which defines p.

Definition 1.2. By the direction of an invariant parabola of a quadratic system (S) we mean the direction of its axis of symmetry which intersects the invariant line $Z = 0$ at an infinite singular point p of (S) with the multiplicity (m^∞, m^f). We say that this direction of the invariant parabola is simple (respectively, double; triple) if $m^\infty = 1$ (respectively 2; 3). We denote this parabola by $\cup$ (respectively $\overset{2}{\cup}$; $\overset{3}{\cup}$). Moreover, if the infinite invariant line $Z = 0$ is filled up with singularities then we denote by $\overset{\infty}{\cup}$ the invariant parabola which is tangent to the line $Z = 0$ at a non-isolated singular point.

In order to distinguish the invariant parabolas that a quadratic system could have we use the following notations:

- $\cup$ for a simple invariant parabola;

- $\Cup$ for two simple invariant parabolas in the same direction (they could intersect);
- $\cup\subset$ for two simple invariant parabolas in different directions;
- $\boldsymbol{\cup}^2$ for one double invariant parabola;
- $\overset{2}{\cup}$ for one simple invariant parabola in double direction;
- $\overset{3}{\cup}$ for one simple invariant parabola in triple direction;
- $\overset{\infty}{\cup}$ for one simple invariant parabola when the line at infinity is filled up with singularities;
- $\overset{2}{\cup}\subset$ for two simple invariant parabola: one in a double direction and one in a simple direction;
- $\Cup\subset$ for three simple invariant parabolas: two in one direction and one in another direction;
- $\overset{2}{\cup}\overset{2}{\Cup}$ for three real invariant parabolas in the same double direction;
- $\overset{2}{\cup}\overset{2}{\Cup}{}^c$ for one real and two complex invariant parabolas in the same double direction;
- $\overset{2}{\cup}\overset{2}{\boldsymbol{\cup}}{}^2$ for one simple and one double real invariant parabolas in the same double direction;
- $\overset{2}{\boldsymbol{\cup}}{}^3$ for a triple real invariant parabola in a double direction;
- $\infty\overset{2}{\cup}$ for a 1-parameter family of invariant parabolas in the same double direction;
- $\infty\overset{3}{\cup}$ for a 1-parameter family of invariant parabolas in the same triple direction.

Our main results are stated in the following theorem.

Main Theorem. (A) *The condition $\chi_1 = \chi_2 = 0$ is necessary for a non-degenerate quadratic system to possess at least one invariant parabola.*

$(\boldsymbol{B})$ *Assume that for a non-degenerate quadratic system (S) the condition $\chi_1 = \chi_2 = 0$ holds. Then this system possesses at least one invariant parabola if and only if the corresponding conditions indicated below are satisfied, respectively. Furthermore in the case of the existence of an invariant parabola this systems could be brought via an affine transformation and time rescaling to one of the canonical forms presented below, correspondingly:*

$\boldsymbol{\alpha})$ *For $\eta > 0$ the system (S) could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$, $\boldsymbol{\cup}^2$, $\cup\subset$, $\Cup\subset$. Moreover (S) has one of the above sets of parabolas if and only if the corresponding conditions provided by the diagram given in Figure 1 are satisfied. Furthermore the system (S) with an invariant parabola could be brought via an affine transformation and time rescaling to the following canonical form*

$$\dot{x} = m + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + (g-1)xy + 2y^2 \qquad (S_{\boldsymbol{\alpha}})$$

possessing the invariant parabola $\Phi(x,y) = x^2 - y$.

$\boldsymbol{\beta}$) *For $\eta < 0$ the system (S) could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$, $\mathbf{U}^2$. Moreover (S) has one of the above sets of parabolas if and only if the corresponding conditions provided by the diagram given in Figure 1 are satisfied. Furthermore the system (S) with an invariant parabola could be brought via an affine transformation and time rescaling to the following canonical form*

$$\dot{x} = m + \frac{2n-1}{2}\,x - \frac{g}{2}\,y + gx^2 - xy, \quad \dot{y} = 2mx + 2ny - x^2 + gxy - 2y^2 \qquad (S_{\boldsymbol{\beta}})$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y$.

$\boldsymbol{\gamma}$) *For $\eta = 0$ and $\widetilde{M} \neq 0$ the system (S) could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$, $\mathbf{U}^2$, $\overset{2}{\cup}$, $\overset{2}{\cup}\subset$, $\overset{2}{\cup}\overset{2}{\Cup}$, $\overset{2}{\cup}\overset{2}{\Cup}{}^c$, $\overset{2}{\cup}\overset{2}{\mathbf{U}}{}^2$, $\overset{2}{\mathbf{U}}{}^3$, $\infty\overset{2}{\cup}$.*

Moreover (S) has one of the above sets of parabolas if and only if the corresponding conditions provided by the diagram given in Figure 1 are satisfied. Furthermore the system (S) with invariant parabola could be brought via an affine transformation and time rescaling to one of the following two normal forms:

$$\dot{x} = m + nx - gy/2 + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + gxy + 2y^2, \ g \in \{0, 1\} \qquad (S^1_{\boldsymbol{\gamma}})$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y$, or

$$\dot{x} = 2mx + 2ny + (h-1)xy, \quad \dot{y} = n - (h+1)x/2 + my + hy^2 \qquad (S^2_{\boldsymbol{\gamma}})$$

possessing the invariant parabola $\Phi(x, y) = y^2 - x$.

$\boldsymbol{\delta}$) *For $\eta = \widetilde{M} = 0$ and $C_2 \neq 0$ the system (S) could only possess one of the following sets of invariant parabolas: $\overset{3}{\cup}$, $\infty\overset{3}{\cup}$. Moreover (S) has one of the above sets of parabolas if and only if the corresponding conditions provided by the diagram given in Figure 1 (the branch $C_2 \neq 0$) are satisfied. Furthermore the system (S) with an invariant parabola could be brought via an affine transformation and time rescaling to the following canonical form*

$$\dot{x} = m + (2n-1)x/2 - gy/2 + gx^2, \quad \dot{y} = 2mx + 2ny - x^2 + gxy \qquad (S_{\boldsymbol{\delta}})$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y$.

$\boldsymbol{\theta}$) *For $\eta = \widetilde{M} = C_2 = 0$ the system (S) could only possess an invariant parabola $\overset{\infty}{\cup}$. Moreover (S) has this invariant parabola if and only if the corresponding conditions provided by the diagram given in Figure 1 (the branch $C_2 = 0$) are satisfied. Furthermore the system (S) with an invariant parabola could be brought via an affine transformation and time rescaling to the following canonical form*

$$\dot{x} = m + nx - y/2 + x^2, \quad \dot{y} = 2mx + 2ny + xy \qquad (S_{\boldsymbol{\theta}})$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y$.

The paper is organized as follows. In Section 2 we construct the invariant polynomials which are responsible for the existence of an invariant parabola and obtain the ten equations relating the coefficients of a quadratic system with those of an invariant parabola. In Section 3 we give the proof of the Main Theorem constructing the conditions for the existence of invariant parabolas as well as the corresponding canonical systems.

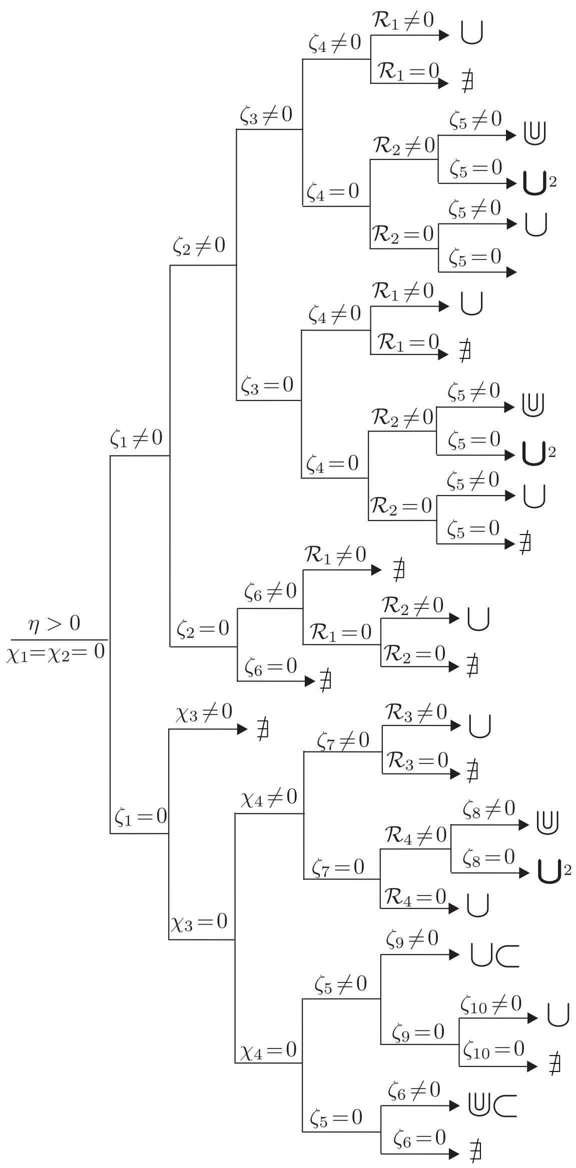

Figure 1.1: Quadratic systems with invariant parabolas: the case $\eta > 0$.

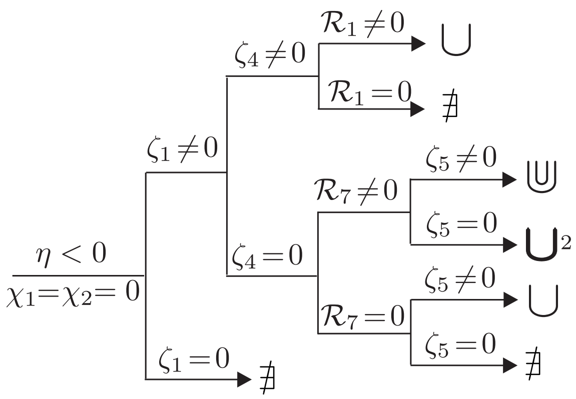

Figure 1.2: Quadratic systems with invariant parabolas: the case $\eta < 0$.

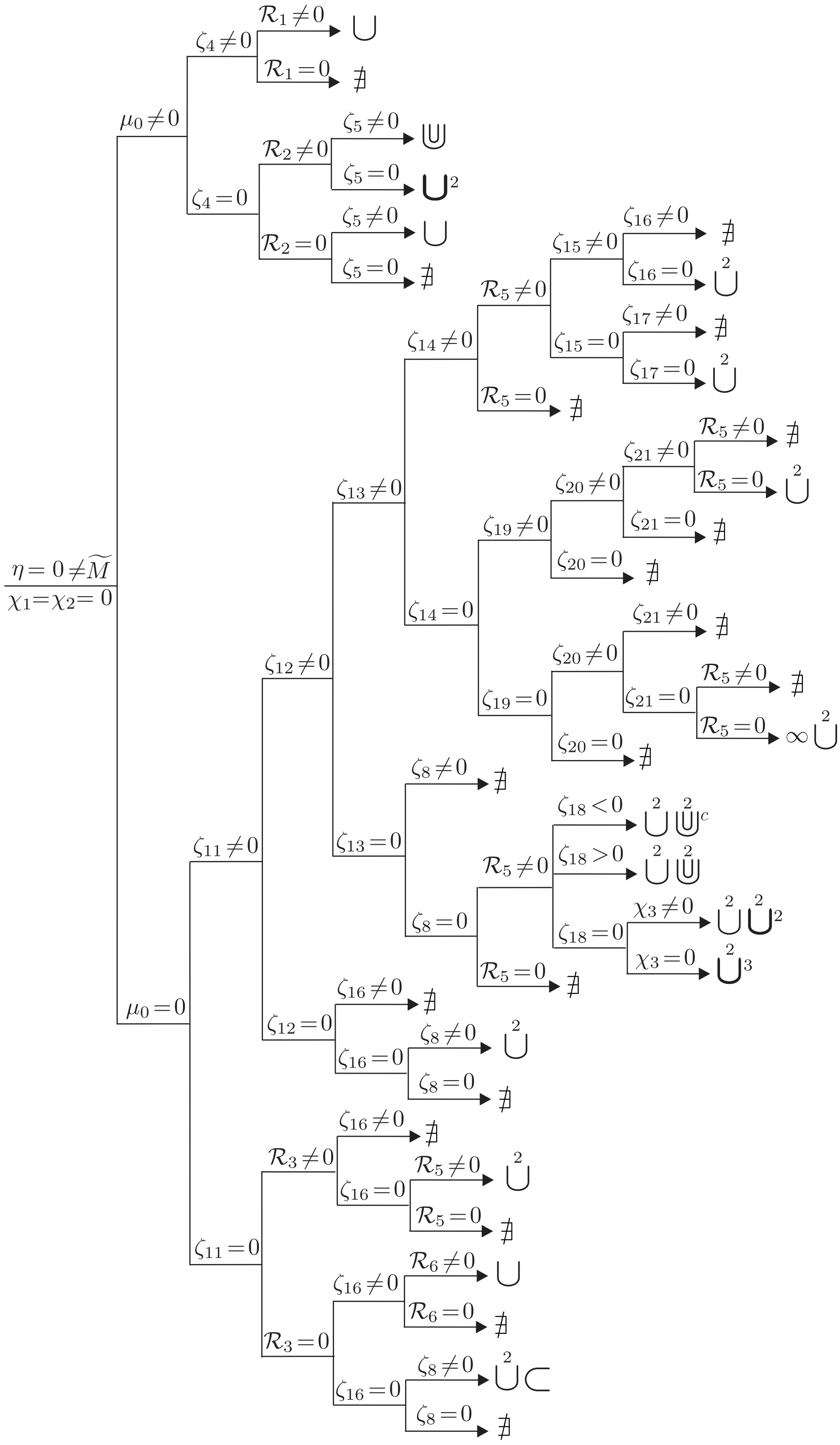

Figure 1.3: Quadratic systems with invariant parabolas: the case $\eta = 0 \neq \widetilde{M}$.

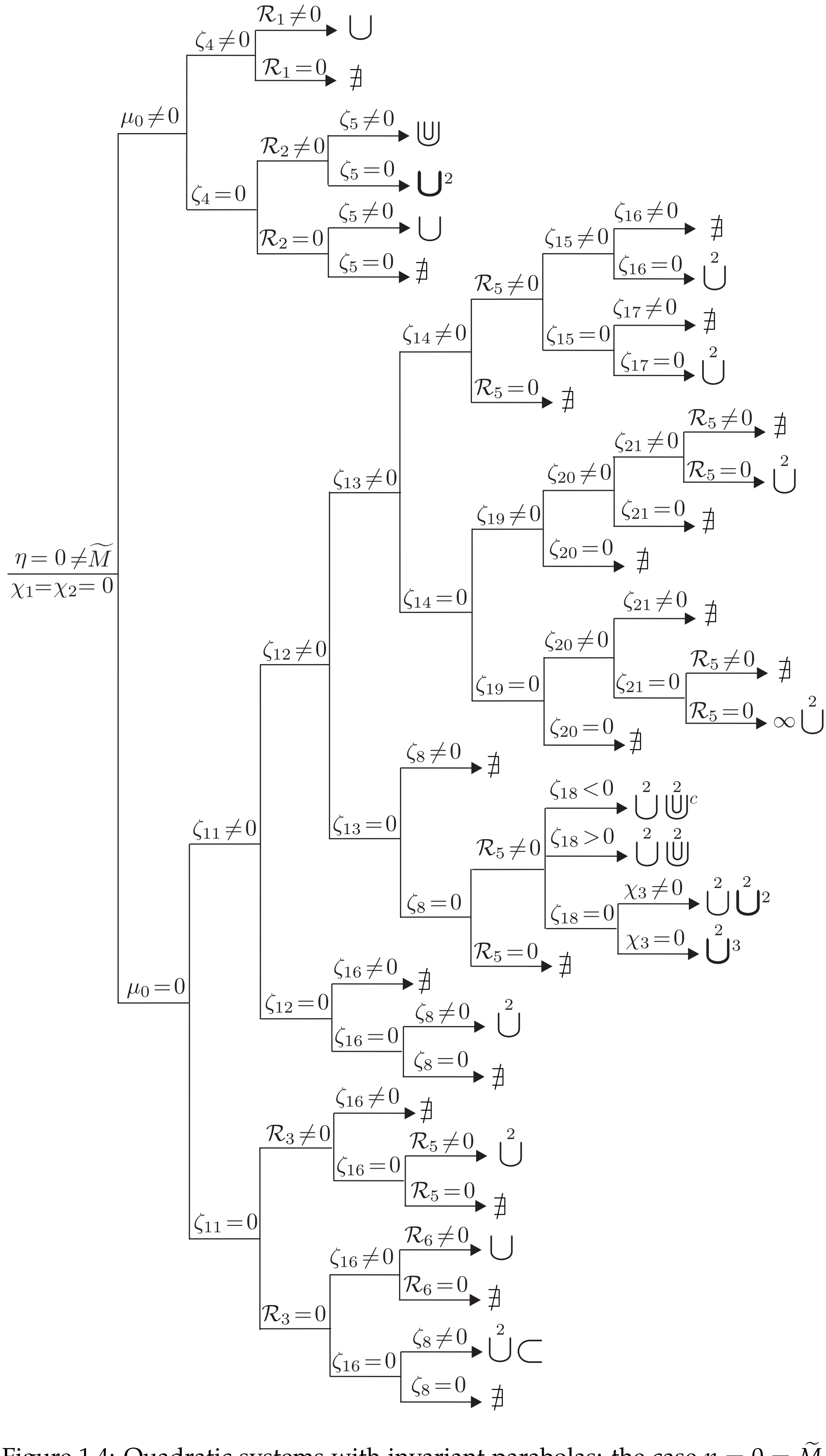

Figure 1.4: Quadratic systems with invariant parabolas: the case $\eta = 0 = \widetilde{M}$.

2 The construction of the invariant polynomials

Consider real quadratic systems of the form

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y), \end{aligned} \tag{2.1}$$

with homogeneous polynomials p_i and q_i $(i = 0, 1, 2)$ of degree i in x, y:

$$\begin{aligned} &p_0 = a_{00}, \quad && p_1(x,y) = a_{10}x + a_{01}y, \quad && p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ &q_0 = b_{00}, \quad && q_1(x,y) = b_{10}x + b_{01}y, \quad && q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

It is known that on the set of quadratic systems acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane (cf. [21]). For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **QS**. We can identify the set **QS** of systems (2.1) with a subset of $\mathbb{R}^{12}$ via the map **QS** $\longrightarrow \mathbb{R}^{12}$ which associates to each system (2.1) the 12-tuple $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ of its coefficients. We associate to this group action polynomials in x, y and parameters which behave well with respect to this action, the GL-comitants (GL-invariants), the T-comitants (affine invariants) and the CT-comitants. For their definitions as well as their detailed constructions we refer the reader to the paper [21] (see also [1]).

2.1 Main invariant polynomials associated with invariant parabolas

We single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (2.1):

$$\begin{aligned} C_i(\tilde{a},x,y) &= yp_i(x,y) - xq_i(x,y), \ (i = 0,1,2), \\ D_i(\tilde{a},x,y) &= \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y}, \ (i = 1,2). \end{aligned} \tag{2.2}$$

As it was shown in [23] these polynomials of degree one in the coefficients of systems (2.1) are GL-comitants of these systems. Let $f, g \in \mathbb{R}[\tilde{a}, x, y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The polynomial $(f,g)^{(k)} \in \mathbb{R}[\tilde{a}, x, y]$ is called *the transvectant of index k of (f,g)* (cf. [9,17])).

Proposition 2.1 (see [24]). *Any GL-comitant of systems (2.1) can be constructed from the elements (2.2) by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(*,*)^{(k)}$.*

Remark 2.2. We point out that the elements (2.2) generate the whole set of GL-comitants and hence also the set of affine comitants as well as the set of T-comitants.

We construct the following GL-comitants of the second degree with respect to the coefficients of the initial systems

$$\begin{aligned} &T_1=(C_0,C_1)^{(1)}, \quad T_2=(C_0,C_2)^{(1)}, \quad T_3=(C_0,D_2)^{(1)},\\ &T_4=(C_1,C_1)^{(2)}, \quad T_5=(C_1,C_2)^{(1)}, \quad T_6=(C_1,C_2)^{(2)},\\ &T_7=(C_1,D_2)^{(1)}, \quad T_8=(C_2,C_2)^{(2)}, \quad T_9=(C_2,D_2)^{(1)}. \end{aligned} \tag{2.3}$$

Using these GL-comitants as well as the polynomials (2.2) we construct the additional invariant polynomials. In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall define here a family of T-comitants expressed through C_i $(i=0,1,2)$ and D_j $(j=1,2)$:

$$\begin{aligned}
\hat{A} =\ & \left(C_1, T_8-2T_9+D_2^2\right)^{(2)}/144,\\
\widehat{D} =\ & \Big[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6-(C_1,T_5)^{(1)}+6D_1(C_1D_2-T_5)-9D_1^2C_2\Big]/36,\\
\widehat{E} =\ & \Big[D_1(2T_9-T_8)-3\,(C_1,T_9)^{(1)}-D_2(3T_7+D_1D_2)\Big]/72,\\
\widehat{F} =\ & [6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)+48C_0\,(D_2,T_9)^{(1)}-9D_2^2T_4+288D_1\widehat{E}\\
& -24\left(C_2,\widehat{D}\right)^{(2)}+120\left(D_2,\widehat{D}\right)^{(1)}-36C_1\,(D_2,T_7)^{(1)}+8D_1\,(D_2,T_5)^{(1)}\Big]/144,\\
\widehat{B} =\ & \Big\{16D_1\,(D_2,T_8)^{(1)}\,(3C_1D_1-2C_0D_2+4T_2)+32C_0\,(D_2,T_9)^{(1)}\,(3D_1D_2-5T_6+9T_7)\\
& +2\,(D_2,T_9)^{(1)}\left(27C_1T_4-18C_1D_1^2-32D_1T_2+32\,(C_0,T_5)^{(1)}\right)\\
& +6(D_2,T_7)^{(1)}[8C_0(T_8-12T_9)-12C_1(D_1D_2+T_7)+D_1(26C_2D_1+32T_5)+C_2(9T_4+96T_3)]\\
& +6\,(D_2,T_6)^{(1)}\,[32C_0T_9-C_1(12T_7+52D_1D_2)-32C_2D_1^2]+48D_2\,(D_2,T_1)^{(1)}\,(2D_2^2-T_8)\\
& -32D_1T_8\,(D_2,T_2)^{(1)}+9D_2^2T_4\,(T_6-2T_7)-16D_1\,(C_2,T_8)^{(1)}\,(D_1^2+4T_3)\\
& +12D_1\,(C_1,T_8)^{(2)}\,(C_1D_2-2C_2D_1)+6D_1D_2T_4\,(T_8-7D_2^2-42T_9)\\
& +12D_1\,(C_1,T_8)^{(1)}\,(T_7+2D_1D_2)+96D_2^2\Big[D_1\,(C_1,T_6)^{(1)}+D_2\,(C_0,T_6)^{(1)}\Big]\\
& -16D_1D_2T_3\,(2D_2^2+3T_8)-4D_1^3D_2\,(D_2^2+3T_8+6T_9)+6D_1^2D_2^2\,(7T_6+2T_7)\\
& -252D_1D_2T_4T_9\Big\}/(2^83^3),\\
\widehat{K} =\ & (T_8+4T_9+4D_2^2)/72, \quad \widehat{H}=(8T_9-T_8+2D_2^2)/72.
\end{aligned}$$

These polynomials in addition to (2.2) and (2.3) will serve as bricks in constructing affine invariant polynomials for systems (2.1).

The following 42 affine invariants $A_1,\ldots,A_{42}$ form the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [2] by constructing $A_1,\ldots,A_{42}$ using the above bricks.

$$\begin{aligned}
A_1 &= \hat{A}, & A_{22} &= \frac{1}{1152}[\![C_2,\widehat{D})^{(1)},D_2)^{(1)},D_2)^{(1)},D_2)^{(1)}D_2)^{(1)},\\
A_2 &= (C_2,\widehat{D})^{(3)}/12, & A_{23} &= [\![[\widehat{F},\widehat{H})^{(1)},\widehat{K})^{(2)}/8,\\
A_3 &= [\![C_2,D_2)^{(1)},D_2)^{(1)},D_2)^{(1)}/48, & A_{24} &= [\![C_2,\widehat{D})^{(2)},\widehat{K})^{(1)},\widehat{H})^{(2)}/32,\\
A_4 &= (\widehat{H},\widehat{H})^{(2)}, & A_{25} &= [\![\widehat{D},\widehat{D})^{(2)},\widehat{E})^{(2)}/16,\\
A_5 &= (\widehat{H},\widehat{K})^{(2)}/2, & A_{26} &= (\widehat{B},\widehat{D})^{(3)}/36,
\end{aligned}$$

$$
\begin{aligned}
A_6 &= (\widehat{E},\widehat{H})^{(2)}/2, & A_{27} &= [\![\widehat{B},D_2)^{(1)},\widehat{H})^{(2)}/24,\\
A_7 &= [\![[C_2,\widehat{E})^{(2)},D_2)^{(1)}/8, & A_{28} &= [\![C_2,\widehat{K})^{(2)},\widehat{D})^{(1)},\widehat{E})^{(2)}/16,\\
A_8 &= [\![\widehat{D},\widehat{H})^{(2)},D_2)^{(1)}/8, & A_{29} &= [\![\widehat{D},\widehat{F})^{(1)},\widehat{D})^{(3)}/96,\\
A_9 &= [\![\widehat{D},D_2)^{(1)},D_2)^{(1)},D_2)^{(1)}/48, & A_{30} &= [\![C_2,\widehat{D})^{(2)},\widehat{D})^{(1)},\widehat{D})^{(3)}/288,\\
A_{10} &= [\![\widehat{D},\widehat{K})^{(2)},D_2)^{(1)}/8, & A_{31} &= [\![\widehat{D},\widehat{D})^{(2)},\widehat{K})^{(1)},\widehat{H})^{(2)}/64,\\
A_{11} &= (\widehat{F},\widehat{K})^{(2)}/4, & A_{32} &= [\![\widehat{D},\widehat{D})^{(2)},D_2)^{(1)},\widehat{H})^{(1)},D_2)^{(1)}/64,\\
A_{12} &= (\widehat{F},\widehat{H})^{(2)}/4, & A_{33} &= [\![\widehat{D},D_2)^{(1)},\widehat{F})^{(1)},D_2)^{(1)},D_2)^{(1)}/128,\\
A_{13} &= [\![C_2,\widehat{H})^{(1)},\widehat{H})^{(2)},D_2)^{(1)}/24, & A_{34} &= [\![\widehat{D},\widehat{D})^{(2)},D_2)^{(1)},\widehat{K})^{(1)},D_2)^{(1)}/64,\\
A_{14} &= (\widehat{B},C_2)^{(3)}/36, & A_{35} &= [\![\widehat{D},\widehat{D})^{(2)},\widehat{E})^{(1)},D_2)^{(1)},D_2)^{(1)}/128,\\
A_{15} &= (\widehat{E},\widehat{F})^{(2)}/4, & A_{36} &= [\![\widehat{D},\widehat{E})^{(2)},\widehat{D})^{(1)},\widehat{H})^{(2)}/16,\\
A_{16} &= [\![\widehat{E},D_2)^{(1)},C_2)^{(1)},\widehat{K})^{(2)}/16, & A_{37} &= [\![\widehat{D},\widehat{D})^{(2)},\widehat{D})^{(1)},\widehat{D})^{(3)}/576,\\
A_{17} &= [\![\widehat{D},\widehat{D})^{(2)},D_2)^{(1)},D_2)^{(1)}/64, & A_{38} &= [\![C_2,\widehat{D})^{(2)},\widehat{D})^{(2)},\widehat{D})^{(1)},\widehat{H})^{(2)}/64,\\
A_{18} &= [\![\widehat{D},\widehat{F})^{(2)},D_2)^{(1)}/16, & A_{39} &= [\![\widehat{D},\widehat{D})^{(2)},\widehat{F})^{(1)},\widehat{H})^{(2)}/64,\\
A_{19} &= [\![\widehat{D},\widehat{D})^{(2)},\widehat{H})^{(2)}/16, & A_{40} &= [\![\widehat{D},\widehat{D})^{(2)},\widehat{F})^{(1)},\widehat{K})^{(2)}/64,\\
A_{20} &= [\![C_2,\widehat{D})^{(2)},\widehat{F})^{(2)}/16, & A_{41} &= [\![C_2,\widehat{D})^{(2)},\widehat{D})^{(2)},\widehat{F})^{(1)},D_2)^{(1)}/64,\\
A_{21} &= [\![\widehat{D},\widehat{D})^{(2)},\widehat{K})^{(2)}/16, & A_{42} &= [\![\widehat{D},\widehat{F})^{(2)},\widehat{F})^{(1)},D_2)^{(1)}/16.
\end{aligned}
$$

In the above list, the bracket "$[\![$" is used in order to avoid placing the otherwise necessary up to five parentheses "(".

Using the elements of the minimal polynomial basis given above we construct the affine invariant polynomials

$$
\begin{aligned}
\chi_1 &= 32A_3 + 45A_4 - 160A_5;\\
\chi_2 &= 32A_8(14A_8 - 48A_9 + 37A_{10} + 24A_{11})\\
&\quad + 16A_5(76A_{17} + 74A_{18} + 313A_{19} - 80A_{20} - 167A_{21})\\
&\quad + A_4(160A_2^2 + 368A_{18} - 3363A_{19} + 736A_{20} + 2109A_{21}) + 32(17A_{10}^2 + 27A_{10}A_{11} + 24A_{11}^2\\
&\quad - 48A_9A_{12} + 51A_{10}A_{12} + 24A_{11}A_{12} + 288A_6A_{14} - 96A_7A_{14});\\
\chi_3 &= 6520480A_{20}(407A_{18} - 2253A_{21}) + 24A_{18}(1057715458A_{19} + 5944853225A_{21})\\
&\quad + 28800A_{14}(1872476A_{25} - 122259A_{26}) + 144A_{12}(3620283092A_{29} - 1554910481A_{30})\\
&\quad + 1440A_{15}(107225339A_{25} - 19561440A_{26}) - 72A_{11}(8198511476A_{29} - 2965514443A_{30})\\
&\quad + 652048(4544A_{18}^2 + 125A_{20}^2 - 8955A_2A_{42}) - 9(264364688A_{19}^2 + 39417454842A_{19}A_{21}\\
&\quad - 54474141921A_{21}^2) + 3448898760A_{19}A_{20};\\
\chi_4 &= 62713A_{10}^2 + 45787A_{10}A_{11} - 157928A_{11}^2 + 81202A_{10}A_{12} + 353474A_{11}A_{12} - 145848A_{12}^2\\
&\quad + 64320A_7A_{15} + 28600A_5A_{17};\\
\zeta_1 &= 13A_4 - 24A_5;\\
\zeta_2 &= -A_4;\\
\zeta_3 &= 16A_5 - 17A_4;\\
\zeta_4 &= 9A_1A_4 - 7A_1A_5 - 2A_{16};
\end{aligned}
$$

$$\begin{aligned}
\zeta_5 &= 166A_8 + 384A_9 - 1120A_{10} - 512A_{11} - 62A_{12};\\
\zeta_6 &= -A_6;\\
\zeta_7 &= 40(71436A_7A_{20} - 640883A_7A_{21} + 1008622A_1A_{32}) + 12A_{12}(3585035A_{14} + 14919259A_{15})\\
&\quad - 5(8092193A_{10} + 15970731A_{11})A_{14} - (129780821A_{10} + 269944167A_{11})A_{15};\\
\zeta_8 &= A_2;\\
\zeta_9 &= 1040(2256A_7A_{15} + 143A_3A_{21}) - 264(162941A_{10} + 315202A_{11})A_{12}\\
&\quad + 3A_{11}(25887132A_{10} + 24385177A_{11}) + 20603609A_{10}^2 + 24896016A_{12}^2;\\
\zeta_{10} &= 250A_1^2 + 34A_{11} - 41A_{12};\\
\zeta_{11} &= D_2^2 + 28\widehat{H} - 32\widehat{K};\\
\zeta_{12} &= D_2^2 - 4\widehat{H} - 16\widehat{K};\\
\zeta_{13} &= D_2^2 - 18\widehat{K};\\
\zeta_{14} &= D_2^2 - 16\widehat{K};\\
\zeta_{15} &= \widehat{H};\\
\zeta_{16} &= A_2(24A_{18} - 42A_{17} - 1024A_{19} - 2A_{20} - 213A_{21}) + 5(420A_1A_{25} - 199A_{38}\\
&\quad - 225A_{39} + 60A_{40} + 8A_{41});\\
\zeta_{17} &= 3456(C_0,T_7)^{(1)}\big[(D_2,T_7)^{(1)}\big]^2 + 81D_1^3(C_1,T_8)^{(2)}(C_1,T_9)^{(2)} - 36D_1(D_2,T_7)^{(1)}\times\\
&\quad \times\big[8[\![T_8,C_2)^{(1)},C_1)^{(2)},C_0)^{(1)} + [\![C_1,T_5)^{(2)},36T_6 - 7D_1D_2)^{(1)}\big]\\
&\quad - 4[\![C_1,T_5)^{(2)},D_2)^{(1)}[\![C_1,T_5)^{(2)},T_6 + 309D_1D_2)^{(1)} + 70T_4(D_2,T_7)^{(1)}[\![C_1,T_5)^{(2)},D_2)^{(1)};\\
\zeta_{18} &= A_{37};\\
\zeta_{19} &= (C_2,\widetilde{D})^{(1)};\\
\zeta_{20} &= (C_2,\widetilde{D})^{(2)};\\
\zeta_{21} &= (C_2,\widetilde{E})^{(1)};\\
\zeta_{22} &= A_2(3A_2^2 - 4A18) + 72A_1(A25 + 2A26);\\
\zeta_{23} &= T_4;\\
\zeta_{24} &= 6C_2D_1^2 + 9C_2T_4 - 4D_1T_5;\\
\mathcal{R}_1 &= 531A_2A_4 - 1472A_2A_5 - 8352A_1A_6 + 320A_{22} - 3216A_{23} + 1488A_{24};\\
\mathcal{R}_2 &= 15A_{10} - 10A_8 - 6A_9;\\
\mathcal{R}_3 &= 4800(6650951968A_{14}A_{15} - 2382132830A_{14}^2 - 9860550485A_{15}^2) + 1600(4765089473A_{11}\\
&\quad - 7838161089A_{12})A_{20} + 640(15664652914A_{11} - 50944340271A_{12})A_{18}\\
&\quad - 6(20392663986679A_{10} + 34357804389813A_{11} - 739275727012A_{12})A_{21}\\
&\quad + 3(46944212550227A_{10} + 83455057317969A_{11} - 22899810934956A_{12})A_{19};\\
\mathcal{R}_4 &= 251A_1^2 + 25A_{12};\\
\mathcal{R}_5 &= [\![C_2,C_2)^{(2)},C_1)^{(2)};\\
\mathcal{R}_6 &= 851A_2A_{17} - 235A_{41} + 170A_{42};\\
\mathcal{R}_7 &= 62250A_1^2 + 8956A_9 - 46223A_{10} - 50129A_{11} + 14766A_{12}.
\end{aligned}$$

2.2 Preliminary results involving the use of polynomial invariants

Considering the GL-comitant $C_2(\tilde{a},x,y) = yp_2(\tilde{a},x,y) - xq_2(\tilde{a},x,y)$ as a cubic binary form of x and y we calculate

$$\eta(\tilde{a}) = \text{Discrim}[C_2,\xi], \quad \widetilde{M}(\tilde{a},x,y) = \text{Hessian}[C_2],$$

where $\xi = y/x$ or $\xi = x/y$. Following [23] (see also [21]) we have the next lemma.

Lemma 2.3. *The number of distinct roots (real and imaginary) of the polynomial $C_2(\tilde{a},x,y)$ is determined by the following conditions:*

(i) 3 *real if* $\eta > 0$*;*

(ii) 1 *real and* 2 *imaginary if* $\eta < 0$*;*

(iii) 2 *real (1 double) if* $\eta = 0$ *and* $\widetilde{M} \neq 0$*;*

(iv) 1 *real (triple) if* $\eta = \widetilde{M} = 0$ *and* $C_2 \neq 0$*;*

(v) ∞ *if* $\eta = \widetilde{M} = C_2 = 0$.

Moreover, for each one of these cases the quadratic systems (2.1) *can be brought via a linear transformation to one of the following canonical systems* $(\mathbf{S}_I)$–$(\mathbf{S}_{IV})$*:*

$$\begin{cases} \dot{x} = a + cx + dy + gx^2 + (h-1)xy, \\ \dot{y} = b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_I)$$

$$\begin{cases} \dot{x} = a + cx + dy + gx^2 + (h+1)xy, \\ \dot{y} = b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases} \dot{x} = a + cx + dy + gx^2 + hxy, \\ \dot{y} = b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases} \dot{x} = a + cx + dy + gx^2 + hxy, \\ \dot{y} = b + ex + fy - x^2 + gxy + hy^2, \end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases} \dot{x} = a + cx + dy + x^2, \\ \dot{y} = b + ex + fy + xy. \end{cases} \qquad (\mathbf{S}_V)$$

Some important necessary conditions for a quadratic system to possess invariant parabolas are provided by the next lemma.

Lemma 2.4. *If a quadratic system* (2.1) *possesses an invariant parabola then the conditions* $\chi_1 = \chi_2 = 0$ *hold.*

Proof. Assume that a quadratic system (2.1) possesses an invariant parabola. It is known that via an affine transformation this parabola could be brought to the canonical form $y = x^2$. Then as it was proved in [5] this quadratic system can be written in the form

$$\dot{x} = c(y - x^2) + (a + bx + gy) + ex, \quad \dot{y} = d(y - x^2) + 2x(a + bx + gy) + 2ey^2,$$

where a, b, c, d, g, h, e are real parameters. A straightforward calculation gives $\chi_1 = \chi_2 = 0$ for the above systems and this completes the proof of the lemma. □

Assume that a conic

$$\Phi(x,y) \equiv p + qx + ry + sx^2 + 2vxy + uy^2 = 0 \tag{2.4}$$

is an affine algebraic invariant curve for quadratic systems (2.1), which we rewrite in the form:

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + 2hxy + ky^2 \equiv P(x,y), \\ \frac{dy}{dt} &= b + ex + fy + lx^2 + 2mxy + ny^2 \equiv Q(x,y). \end{aligned} \tag{2.5}$$

Remark 2.5. Following [10] we construct the determinant

$$\Delta = \begin{vmatrix} s & v & q/2 \\ v & u & r/2 \\ q/2 & r/2 & p \end{vmatrix},$$

associated to the conic (2.4). By [10] this conic is irreducible (i.e. the polynomial Φ defining the conic is irreducible over $\mathbb{C}$) if and only if $\Delta \neq 0$.

According to definition of an invariant curve (see page 2) considering the cofactor $K = Ux + Vy + W \in \mathbb{R}[x,y]$ the following identity holds:

$$\frac{\partial \Phi}{\partial x} P(x,y) + \frac{\partial \Phi}{\partial y} Q(x,y) = \Phi(x,y)(Ux + Vy + W).$$

This identity yields a system of 10 equations for determining the 9 unknown parameters p, q, r, s, u, v, U, V, W:

$$\begin{aligned} Eq_1 &\equiv s(2a_{20} - U) + 2b_{20}v = 0, \\ Eq_2 &\equiv 2v(a_{20} + 2b_{11} - U) + s(4a_{11} - V) + 2b_{20}u = 0, \\ Eq_3 &\equiv 2v(2a_{11} + b_{02} - V) + u(4b_{11} - U) + 2a_{02}s = 0, \\ Eq_4 &\equiv u(2b_{02} - V) + 2a_{02}v = 0, \\ Eq_5 &\equiv q(a_{20} - U) + s(2a_{10} - W) + 2b_{10}v + b_{20}r = 0, \\ Eq_6 &\equiv r(2b_{11} - U) + q(2a_{11} - V) + 2v(a_{10} + b_{01} - W) + 2(a_{01}s + b_{10}u) = 0, \\ Eq_7 &\equiv r(b_{02} - V) + u(2b_{01} - W) + 2a_{01}v + a_{02}q = 0, \\ Eq_8 &\equiv q(a_{10} - W) + 2(a_{00}s + b_{00}v) + b_{10}r - pU = 0, \\ Eq_9 &\equiv r(b_{01} - W) + 2(b_{00}u + a_{00}v) + a_{01}q - pV = 0, \\ Eq_{10} &\equiv a_{00}q + b_{00}r - pW = 0. \end{aligned} \tag{2.6}$$

According to [6] (see also [3]) we have the next lemma.

Lemma 2.6. *Suppose that a polynomial system (1.1) of degree n has the invariant algebraic curve $f(x,y) = 0$ of degree m. Let P_n, Q_n and f_m be the homogeneous components of P, Q and f of degree n and m, respectively. Then the irreducible factors of f_m must be factors of $yP_n - xQn$.*

3 The proof of the Main Theorem

Assuming that a quadratic system (2.5) has an invariant parabola (2.4) by Lemma 2.4 we conclude that for this system the conditions $\chi_1 = \chi_2 = 0$ have to be fulfilled.

In what follows considering Lemma 2.3 we examine each one of the families of quadratic systems provided by this lemma.

3.1 Systems with three real infinite singularities

In this case according to Lemma 2.3 systems (2.5) via a linear transformation could be brought to the following family of systems

$$\frac{dx}{dt} = a + cx + dy + gx^2 + (h-1)xy, \qquad\qquad (3.1)$$
$$\frac{dy}{dt} = b + ex + fy + (g-1)xy + hy^2,$$

for which we have $C_2(x,y) = xy(x-y)$. Therefore the infinite singularities are located at the intersections of the lines $x = 0$, $y = 0$ and $x - y = 0$ with the line $Z = 0$ at infinity. So by Lemma 2.6 it is clear that if a parabola is invariant for these systems, then its homogeneous quadratic part has one of the following forms: $(i)\ kx^2$, $(ii)\ ky^2$, $(iii)\ k(x-y)^2$, where k is a real nonzero constant. Obviously we may assume $k = 1$ (otherwise instead of conic (2.4) we could consider $\Phi(x,y)/k = 0$).

According to Lemma 2.4 for the existence of an invariant parabola for a system (3.1) the condition $\chi_1 = 0$ is necessary. For the above systems we calculate

$$\chi_1 = 2(g-2)(h-2)(1+g+h) \qquad\qquad (3.2)$$

and therefore the condition $\chi_1 = 0$ is equivalent to $(g-2)(h-2)(1+g+h) = 0$.

On the other hand we have the following lemma.

Lemma 3.1. *Assume that a system (3.1) possesses an invariant parabola. Then its quadratic homogeneous part is of the form x^2 (respectively, y^2; $(x-y)^2$) only if the condition $h - 2 = 0$ (respectively, $g - 2 = 0$; $g + h + 1 = 0$) holds.*

Proof. Assume that a system (3.1) possesses an invariant parabola of the form $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic). Then considering equations (2.6) we obtain

$$s = 1,\ \ v = u = 0,\ \ Eq_2 = -2 + 2h - V = 0\ \Rightarrow\ V = 2(h-1).$$

Therefore we have $Eq_7 = (2-h)r = 0$ and since $r \neq 0$ this implies $h - 2 = 0$. So the statement of the lemma is true in this case.

If the system possesses an invariant parabola of the form $\Phi(x,y) = p + qx + ry + y^2$ with $q \neq 0$ then considering equations (2.6) we obtain

$$s = v = 0,\ \ u = 1,\ \ Eq_3 = -2 + 2g - U = 0\ \Rightarrow\ U = 2(g-1).$$

In this case we obtain $Eq_5 = (2-g)q = 0$ and due to $q \neq 0$ we get $g - 2 = 0$.

Assume now a system (3.1) possesses an invariant parabola of the form $\Phi(x,y) = p + qx + ry + (x-y)^2$ with $q + r \neq 0$. Then we have

$$s = 1,\ v = -1,\ u = 1,\ \ Eq_1 = 2g - U,\ Eq_4 = 2h - V$$

and therefore the equations $Eq_1 = 0$ and $Eq_4 = 0$ yield $U = 2g$ and $V = 2h$, respectively. Then we calculate $Eq_5 + Eq_6 + Eq_7 = -(1+g+h)(q+r) = 0$ and due to the condition $q + r \neq 0$ we get $1 + g + h = 0$ and this completes the proof of Lemma 3.1. □

Considering (3.2) it is clear that the condition $\chi_1 = 0$ implies either $h = 2$ or $g = 2$ or $h = -(1+g)$. On the other hand for systems (3.1) we have

$$\begin{aligned} \zeta_1 &= 2(g-2)(3+g) \quad \text{in the case } h = 2;\\ \zeta_1 &= 2(h-2)(3+h) \quad \text{in the case } g = 2;\\ \zeta_1 &= 2(g-2)(3+g) \quad \text{in the case } h = -(1+g) \end{aligned}$$

and therefore we conclude that if $\chi_1 = 0$ then the condition $\zeta_1 = 0$ imposes the vanishing of one more factor of the polynomial χ_1.

Remark 3.2. If $(h-2)(g-2)(1+g+h) = 0$ then without losing generality we may assume $h = 2$. Furthermore if two of these factors vanish simultaneously (i.e. $\zeta_1 = 0$) then we may assume $h = 2$ and $g = 2$.

Indeed assume $h - 2 \neq 0$ and suppose first that $g = 2$. We observe that the change

$$(x, y, a, b, c, d, e, f, g, h) \mapsto (y, x, b, a, e, d, f, c, h, g)$$

conserves systems (3.1) and hence the condition $g = 2$ is transformed into $h = 2$.

Admit now that the condition $1 + g + h = 0$ is fulfilled. Then applying to these systems the transformation

$$x_1 = x - y, \quad y_1 = -y$$

and arrive at the systems

$$\dot{x}_1 = a_1 + c_1x_1 + g_1x_1^2 + (h_1 - 1)x_1y_1, \quad \dot{y}_1 = b_1 + f_1y_1 + (g_1 - 1)x_1y_1 + h_1y_1^2$$

where (we are interested only in homogeneous quadratic parts)

$$g_1 = g, \quad h_1 = 1 - g - h, \Rightarrow \; g = g_1, \quad h = 1 - g_1 - h_1.$$

Therefore we obtain $1 + h_1 + g_1 = 1 + (1 - g - h) + g = 2 - h$ and hence via the above transformation the condition $1 + g + h = 0$ is reduced to the condition $h - 2 = 0$.

Assume now that two of the factors $(h-2)(g-2)(1+g+h)$ vanish. Then as it was shown above we may assume $h = 2$. In this case other two factors become $g - 2$ and $g + 3$. Supposing $h = 2$ and $g = -3$ systems (3.1) become

$$\dot{x} = a + cx + dy - 3x^2 + xy, \quad \dot{y} = b + ex + fy - 4xy + 2y^2,$$

which via the transformation $x_1 = x$, $y_1 = x - y$ can be brought to the systems

$$\dot{x}_1 = a_1 + c_1x_1 + d_1y_1 + 2x_1^2 + x_1y_1, \quad \dot{y}_1 = b_1 + e_1x_1 + f_1y_1 + x_1y_1 + 2y_1^2.$$

It remains to observe that these systems have the form (3.1) with $h = 2$ and $g = 2$ and we conclude that the statement of Remark 3.2 is valid.

Considering Lemma 3.1 and Remark 3.2 we conclude that for determining the conditions for the existence and the number of invariant parabolas for systems (3.1) it is sufficient to examine when the invariant parabolas have the forms $\Phi(x, y) = p + qx + ry + x^2$ and $\Phi(x, y) = p + qx + ry + y^2$. Moreover as it is mentioned above systems (3.1) could have invariant parabolas only in one direction (if $\chi_1 = 0$ and $\zeta_1 \neq 0$) and they could have invariant parabolas in two directions (if $\chi_1 = 0$ and $\zeta_1 = 0$). In what follows we examine each one of these two possibilities.

3.1.1 The possibility $\chi_1 = 0$ and $\zeta_1 \neq 0$

Then we may assume $h = 2$ and by Lemma 3.1 systems (3.1) could have invariant parabolas of the form $\Phi(x,y) = p + qx + ry + x^2$. Applying the translation $(x,y) \mapsto \big(x - d,\ y - c + 2dg\big)$ systems (3.1) can be brought to the systems

$$\dot{x} = a + gx^2 + xy, \quad \dot{y} = b + ex + fy + (g-1)xy + 2y^2. \tag{3.3}$$

Coefficient conditions for systems (3.3) to possess invariant parabolas

Lemma 3.3. *A system (3.3) possesses either one or two invariant parabolas or a double invariant parabola of the form $\Phi(x,y) = p + qx + ry + x^2$ $(r \neq 0)$ if and only if $\Omega_1 = 0$ and the corresponding set of conditions are satisfied, respectively:*

$(\mathcal{A}_1)$ $g(g+1) \neq 0,\ 2g+1 \neq 0,\ \mathcal{D}_1 \neq 0,\ \mathcal{G}_1 \neq 0 \Rightarrow \cup$;

$(\mathcal{A}_2)$ $g(g+1) \neq 0,\ 2g+1 \neq 0,\ \mathcal{D}_1 = 0,\ a \neq 0,\ \mathcal{F}_1 \neq 0 \Rightarrow \Cup$;

$(\mathcal{A}_3)$ $g(g+1) \neq 0,\ 2g+1 \neq 0,\ \mathcal{D}_1 = 0,\ a \neq 0,\ \mathcal{F}_1 = 0 \Rightarrow \cup^2$;

$(\mathcal{A}_4)$ $g(g+1) \neq 0,\ 2g+1 \neq 0,\ \mathcal{D}_1 = 0,\ a = 0,\ f \neq 0 \Rightarrow \cup$;

$(\mathcal{A}_5)$ $g = -1/2,\ \mathcal{D}_1 \neq 0,\ a \neq 0 \Rightarrow \cup$;

$(\mathcal{A}_6)$ $g = -1/2,\ \mathcal{D}_1 = 0,\ b \neq 0,\ e^2 - 2b \neq 0 \Rightarrow \Cup$;

$(\mathcal{A}_7)$ $g = -1/2,\ \mathcal{D}_1 = 0,\ b \neq 0,\ e^2 - 2b = 0 \Rightarrow \cup^2$;

$(\mathcal{A}_8)$ $g = -1/2,\ \mathcal{D}_1 = 0,\ b = 0,\ e \neq 0 \Rightarrow \cup$;

$(\mathcal{A}_9)$ $g = 0,\ b = a,\ e \neq 0,\ a \neq 0 \Rightarrow \cup$;

$(\mathcal{A}_{10})$ $g = -1,\ b = 0,\ e + f \neq 0,\ a \neq 0 \Rightarrow \cup$,

where

$$\begin{aligned}
\Omega_1 &= 2b^2(1+2g)^2 - b\big[4a(1+g)(1+2g)(1+3g) - (e-fg)(e+f+fg)\big] \\
&\quad + a(1+g)\big[2a(1+g)(1+3g)^2 - (e-2fg)(e+f+fg)\big]; \\
\mathcal{D}_1 &= e + f(g+1);\ \ \mathcal{G}_1 = a - b + 4ag - 2bg + 3ag^2;\ \ \mathcal{F}_1 = 8ag(1+g) - f^2(1+2g).
\end{aligned} \tag{3.4}$$

Proof. Considering the equations (2.6) and the form of invariant parabola $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$ we obtain

$$\begin{gathered}
s = 1,\ v = u = 0,\ U = 2g,\ V = 2,\ W = -gq, \\
Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0.
\end{gathered}$$

Calculating the remaining equations we obtain

$$\begin{aligned}
&Eq_6 = -q - r - gr, \qquad Eq_8 = 2a - 2gp + gq^2 + er, \\
&Eq_9 = -2p + fr + gqr, \qquad Eq_{10} = aq + gpq + br.
\end{aligned}$$

It is clear that the equations $Eq_6 = 0$ implies $q = -(1+g)r$ and then $Eq_9 = 0$ gives us $p = r(f - gr - g^2 r)/2$. Therefore calculations yield

$$\begin{aligned}
Eq_8 &= g(1+g)(1+2g)r^2 + (e - fg)r + 2a, \\
Eq_{10} &= r\big[g^2(1+g)^2 r^2 - fg(1+g)r - 2(a - b + ag)\big]/2 \equiv r\Psi/2
\end{aligned} \tag{3.5}$$

and since $r \neq 0$ the equation $Eq_{10} = 0$ is equivalent to $\Psi = 0$.

According to [12, Lemmas 11, 12] the equations $Eq_8 = 0$ and $\Psi = 0$ have a common solution of degree 2 with respect to the parameter r if and only if

$$Res_r^{(0)}(Eq_8, \Psi) = Res_r^{(1)}(Eq_8, \Psi) = 0$$

where $Res_r^{(1)}$ is the subresultant of order one and $Res_r^{(0)}$ is the subresultant of order zero which coincide with standard resultant (for detailed definition see [12], formula (19)). We calculate

$$\begin{aligned} Res_r^{(1)}(Eq_8, \Psi) &= -g^2(1+g)^2(e+f+fg) \equiv -g^2(1+g)^2\mathcal{D}_1, \\ Res_r^{(0)}(Eq_8, \Psi) &= 2g^2(g+1)^2\Omega_1. \end{aligned}$$

So we examine three possibilities: $g(g+1) \neq 0$, $g = 0$ and $g = -1$.

***1:* The possibility** $g(g+1) \neq 0$**.** Considering (3.4) we observe that the polynomial Ω_1 is quadratic with respect to the parameter b if $2g+1 \neq 0$. So we discuss two cases: $2g+1 \neq 0$ and $2g+1 = 0$.

***1.1:* The case** $2g+1 \neq 0$**.** We observe that due to the condition $g(g+1) \neq 0$ the subresultant of order one $Res_r^{(1)}(Eq_8, \Psi)$ vanishes if and only if $\mathcal{D}_1 = 0$. So we consider two subcases: $\mathcal{D}_1 \neq 0$ and $\mathcal{D}_1 = 0$.

1.1.1: *The subcase* $\mathcal{D}_1 \neq 0$. Then the invariant parabola exists if and only if $\Omega_1 = 0$ and therefore we have to examine the solutions of the equation $\Omega_1 = 0$. In this case we calculate

$$\text{Discrim}[\Omega_1, b] = -(e+f+fg)^2\big[8ag(1+g)(1+2g) - (e-fg)^2\big] \equiv -\mathcal{D}_1^2\mathcal{E}$$

and hence the equation $\Omega_1 = 0$ has real solutions with respect to the parameter b if and only if either $\mathcal{D}_1 = 0$ or $\mathcal{E} \leq 0$. However since the condition $\mathcal{D}_1 \neq 0$ holds it remains to examine the condition $\mathcal{E} \leq 0$.

In this case setting $\mathcal{E} = -w^2 \leq 0$ we calculate

$$a = \frac{(e-fg)^2 - w^2}{8g(g+1)(2g+1)} \tag{3.6}$$

and then we obtain:

$$\Omega_1 = \frac{B_+B_-}{32g^2(1+2g)^2},$$

where

$$B_\pm = 8bg(1+2g)^2 + (fg - e + \varepsilon w)\big[e(1+g) - fg(3+5g) + \varepsilon w(1+3g)\big], \quad \varepsilon = \pm 1.$$

Then the condition $\Omega_1 = 0$ gives us

$$b = \frac{(e - fg - \varepsilon w)}{8g(1+2g)^2}\big[e(1+g) - fg(3+5g) + \varepsilon w(1+3g)\big] \tag{3.7}$$

where $\varepsilon = 1$ if $B_+ = 0$ and $\varepsilon = -1$ if $B_- = 0$. In this case we obtain that the polynomials Eq_8 and $\Psi(e, f, g, r)$ have the common factor $\zeta = 2g(1+g)(1+2g)r + e - fg - \varepsilon w$ which is linear with respect to the parameter r. Setting $\zeta = 0$ we get

$$r = -\frac{e - fg - \varepsilon w}{2g(1+g)(1+2g)}$$

and we arrive at the family of systems

$$\begin{aligned} \dot{x} &= \frac{(e-fg)^2 - w^2}{8g(g+1)(2g+1)} + gx^2 + xy, \\ \dot{y} &= \frac{(e-fg-\varepsilon w)}{8g(1+2g)^2}\big[e(1+g) - fg(3+5g) + \varepsilon w(1+3g)\big] + ex + fy + (g-1)xy + 2y^2. \end{aligned} \tag{3.8}$$

This family of systems possess the following invariant parabola

$$\begin{aligned} \Phi(x,y) = &-\frac{(e-fg-\varepsilon w)(e+2f+3fg-\varepsilon w)}{8g(1+g)(1+2g)^2} \\ &+ \frac{e-fg-\varepsilon w}{2g(1+2g)}\,x - \frac{e-fg-\varepsilon w}{2g(1+g)(1+2g)}\,y + x^2. \end{aligned} \tag{3.9}$$

We observe that this conic is reducible if and only if $e - fg + \varepsilon w = 0$.

Considering (3.6) and (3.7) we get

$$w^2 = -8ag(1+g)(1+2g) + (e-fg)^2$$

and then we obtain

$$\begin{aligned} b = \frac{1}{8g(1+2g)^2}\big[(e-fg)(e+eg-3fg-5fg^2) - 2\varepsilon wg(e+f+fg) - (1+3g)w^2\big] \;\Rightarrow \\ 4b(1+2g)^2 - 4a(1+g)(1+2g)(1+3g) + (e+f+fg)(e-fg-\varepsilon w) = 0. \end{aligned}$$

Since $\mathcal{D}_1 = (e+f+fg) \neq 0$ we solve the last equation with respect to εw and we obtain

$$\varepsilon w = \frac{1}{e+f+fg}\big[4b(1+2g)^2 - 4a(1+g)(1+2g)(1+3g) + (e-fg)(e+f+fg)\big].$$

Then calculations yield

$$r = -\frac{e-fg-\varepsilon w}{2g(1+g)(1+2g)} = -\frac{2(a-b+4ag-2bg+3ag^2)}{g(1+g)(e+f+fg)} = -\frac{2\,\mathcal{G}_1}{g(1+g)(e+f+fg)} \neq 0.$$

This completes the proof of the statement $(\boldsymbol{A_1})$ of Lemma 3.3.

Since systems (3.8) are in the class defined by the condition $\eta > 0$, according to the statement $\boldsymbol{\alpha})$ of Main Theorem we have to prove that these systems could be brought via a transformation to the canonical form $(S_{\boldsymbol{\alpha}})$.

Indeed as $g(1+g)(1+2g) \neq 0$ we apply to the parabola (3.9) the translation

$$x = x_1 - \frac{e-fg-\varepsilon w}{4g(1+2g)}, \quad y = y_1 - \frac{e+3eg+fg(3+5g)-(1+3g)\varepsilon w}{8g(1+2g)} \tag{3.10}$$

and we get a simpler form of this parabola:

$$\widetilde{\Phi}(x_1,y_1) = x_1^2 - \frac{e-fg-\varepsilon w}{2g(1+g)(1+2g)}\,y_1.$$

On the other hand applying the same translation to the family of systems (3.8) we arrive at the systems

$$\begin{aligned}\dot{x}_1 =& \frac{(e-fg-\varepsilon w)\big[fg(1-g)(3+5g)+e(1+10g+13g^2)+(g-1)(1+3g)\varepsilon w\big]}{32g^2(1+g)(1+2g)^2}-\\ & \frac{fg(3+g)+e(1+7g)-(1+7g)\varepsilon w}{8g(1+2g)}x_1-\frac{e-fg-\varepsilon w}{4g(1+2g)}y_1+gx_1^2+x_1y_1,\\ \dot{y}_1 =& \frac{fg(1-g)(3+5g)+e(1+10g+13g^2)+(g-1)(1+3g)\varepsilon w}{8g(1+2g)}x_1-\\ & \frac{fg(3+g)+e(1+7g)-(1+7g)\varepsilon w}{4g(1+2g)}y_1+(g-1)x_1y_1+2y_1^2.\end{aligned} \tag{3.11}$$

Observation 3.4. We remark that simultaneously applying the same translation on systems (3.8) and on the corresponding invariant parabola (3.9) we arrive at systems (3.11). We point out that the linear parts of these systems together with the coefficients of the transformed parabola $\widetilde{\Phi}(x_1,y_1)$ suggest us the new notations for the simplification of the canonical forms.

Indeed due to the condition $g(1+g)(1+2g)(e-fg-\varepsilon w)\neq 0$ we could set the following new notations:

$$\begin{gathered}k=\frac{e-fg-\varepsilon w}{2g(1+g)(1+2g)},\quad n=-\frac{fg(3+g)+e(1+7g)-(1+7g)\varepsilon w}{8g(1+2g)},\\ m=\frac{fg(1-g)(3+5g)+e(1+10g+13g^2)+(g-1)(1+3g)\varepsilon w}{16g(1+2g)}\quad\Rightarrow\\ e=gk-g^3k+2m+n-gn,\quad f=-\frac{(1+g)(1+7g)k-4n}{2},\\ w=-\frac{g(1+g)(1+3g)k-2(2m+n+gn)}{2\varepsilon}\end{gathered}$$

where $k\neq 0$ due to $e-fg+\varepsilon w\neq 0$. Then we arrive at the following family of systems:

$$\begin{aligned}\dot{x}_1 &= km+nx_1-\frac{k}{2}(g+1)y_1+gx_1^2+xy,\\ \dot{y}_1 &= 2mx_1+2ny_1+(g-1)x_1y_1+2y_1^2.\end{aligned}$$

which possess the invariant parabola $\Phi(x_1,y_1)=x_1^2-ky_1,\quad k\neq 0$.

Remark 3.5. If $k\neq 0$ then due to a rescaling we may assume $k=1$ in the above systems as well as in the invariant parabola.

Indeed since $k\neq 0$ via the rescaling $(x_1,y_1,t_1)\mapsto(kx_1,ky_1,t/k)$ and setting $m/k=m$ and $n/k=n$ we may assume $k=1$ in the above systems. At the same time applying this rescaling to the above parabola we get $\Phi(x_1,y_1)=k^2(x_1^2-y_1)$ and we conclude that the parabola $\widetilde{\Phi}(x_1,y_1)=x_1^2-y_1$ also in invariant for the above systems.

Therefore due to this remark we get the canonical systems $(S_{\boldsymbol{\alpha}})$ provided by the statement $\boldsymbol{\alpha})$ of Main Theorem.

1.1.2: *The subcase* $\mathcal{D}_1=0$. Then $e=-f(1+g)$ and therefore we obtain:

$$\Omega_1=2\big[a(1+g)(1+3g)-b(1+2g)\big]^2=2\mathcal{G}_1^2$$

and since $2g+1 \neq 0$ the condition $\Omega_1 = 0$ implies

$$b = \frac{a(1+g)(1+3g)}{1+2g}.$$

Therefore we determine that in this case the polynomials Eq_8 and Eq_{10} have the following common factor

$$\tilde{\phi} = 2a - f(1+2g)r + g(1+g)(1+2g)r^2.$$

We observe that $\tilde{\phi}$ is quadratic in r with the discriminant

$$\text{Discrim}[\tilde{\phi}, r] = -(1+2g)\left[8ag(1+g) - f^2(1+2g)\right]$$

and setting this discriminant equal to be w^2 we obtain

$$a = \frac{f^2(1+2g)^2 - w^2}{8g(1+g)(1+2g)}. \tag{3.12}$$

Then we arrive at the following expressions for the polynomials Eq_8 and Eq_{10}:

$$Eq_8 = \frac{H_+ H_-}{4g(1+g)(1+2g)}, \quad Eq_{10} = \frac{rH_+ H_-}{8(1+2g)^2},$$

where

$$H_\pm = f(1+2g) - 2g(1+g)(1+2g)r \pm w.$$

Therefore the equations $Eq_8 = Eq_{10} = 0$ imply $H_+H_- = 0$. If $H_+ = 0$ we determine

$$r = \frac{f + 2fg + w}{2g(1+g)(1+2g)} \equiv r^+$$

and we obtain the parabola

$$\Phi_1(x,y) = \frac{f^2(1+2g)^2 - w^2}{8g(g+1)(2g+1)^2} - \frac{f+2fg+w}{2g(1+2g)}\,x + \frac{f+2fg+w}{2g(g+1)(2g+1)}\,y + x^2.$$

In the case $H_- = 0$ we obtain

$$r = \frac{f + 2fg - w}{2g(1+g)(1+2g)} \equiv r^-$$

and we get the parabola

$$\Phi_2(x,y) = \frac{f^2(1+2g)^2 - w^2}{8g(g+1)(2g+1)^2} - \frac{f+2fg-w}{2g(1+2g)}\,x + \frac{f+2fg-w}{2g(g+1)(2g+1)}\,y + x^2.$$

Both these parabolas are invariant for the following family of systems:

$$\begin{aligned} \dot{x} &= \frac{f^2(1+2g)^2 - w^2}{8g(1+g)(1+2g)} + gx^2 + xy, \\ \dot{y} &= \frac{(3g+1)\left[f^2(1+2g)^2 - w^2\right]}{8g(2g+1)^2} - f(g+1)x + fy + (g-1)xy + 2y^2). \end{aligned} \tag{3.13}$$

We observe that both parabolas $\Phi_i(x,y) = 0$ $(i = 1,2)$ exist (i.e. are not reducible) if and only if $r^+ r^- \neq 0$ and this is equivalent to

$$(f + 2fg + w)(f + 2fg - w) = f^2(1+2g)^2 - w^2 \neq 0$$

and considering (3.12) this is equivalent to $a \neq 0$.

On the other hand if only one of the factors vanishes we have $a = 0$ and

$$r^+ + r^- = (f + 2fg + w) + (f + 2fg - w) = 2f(1 + 2g) \neq 0$$

and due to $1 + 2g \neq 0$ we obtain that the above condition is equivalent to $f \neq 0$. Therefore for $a = 0$ and $f \neq 0$ we could have only one parabola.

We determine that in the case $w = 0$ we obtain $\Phi_1(x, y) = \Phi_2(x, y)$, i.e. the parabolas coalesce when w tends to zero and we obtain a double parabola. On the other hand considering (3.12) for $w = 0$ we obtain to

$$a - \frac{f^2(1 + 2g)}{8g(1 + g)} = \frac{8ag(1 + g) - f^2(1 + 2g)}{8g(1 + g)} = \frac{\mathcal{F}_1}{8g(1 + g)}$$

and we conclude that these invariant parabolas coalesce if and only if $\mathcal{F}_1 = 0$.

Thus we conclude that the statements $(\boldsymbol{A_2})$, $(\boldsymbol{A_3})$ and $(\boldsymbol{A_4})$ of Lemma 3.3 are proved.

Next we observe that the family of systems (3.13) is a subfamily of (3.8) defined by the condition $e = -f(1 + g)$ (i.e. $\mathcal{D}_1 = 0$). Moreover considering (3.9) for $e = -f(1 + g)$ we obtain:

$$\Phi(x, y) = \frac{f^2(1 + 2g)^2 - w^2}{8g(g + 1)(2g + 1)^2} - \frac{f + 2fg + \varepsilon w}{2g(1 + 2g)}\, x + \frac{f + 2fg + \varepsilon w}{2g(g + 1)(2g + 1)}\, y + x^2$$

and we observe that for $\varepsilon = 1$ (respectively $\varepsilon = -1$) the above parabola coincides with the invariant parabola $\Phi_1(x, y) = 0$ (respectively $\Phi_2(x, y) = 0$) of systems (3.13).

So taking the invariant parabola $\Phi_1(x, y) = 0$ (obtained for $e = -f(1 + g)$ and $\varepsilon = 1$) we could apply the same translation (3.10) in this particular case and we arrive at the subfamily of systems (3.11) defined by the conditions $e = -f(1 + g)$ and $\varepsilon = 1$ which possess the following invariant parabola

$$\widetilde{\Phi}_1(x_1, y_1) = x_1^2 + \frac{(f + 2fg + w)}{2g(1 + g)(1 + 2g)}\, y_1.$$

Since in the considered case we have only three free parameters, we set only two new parameters as follows:

$$k = -\frac{f + 2fg + w}{2g(1 + g)(1 + 2g)}, \quad n = \frac{f + 5fg + 6fg^2 + w + 7gw}{8g(1 + 2g)} \quad \Rightarrow$$
$$f = -\frac{k + 8gk + 7g^2k + 4n}{2}, \quad w = \frac{(1 + 2g)(k + 4gk + 3g^2k + 4n)}{2}.$$

In this case after an additional rescaling (to force $k = 1$, see Remark 3.5) we arrive at the subfamily of systems $(S_{\boldsymbol{\alpha}})$ defined by the condition

$$m = (1 + 3g)(1 + 4g + 3g^2 + 2n)/4.$$

***1.2:* The case** $2g + 1 = 0$**.** Then $g = -1/2$ and evaluating Ω_1 and $\mathcal{D}_1$ we obtain

$$\Omega_1 = \left[2b(2e + f)^2 + a(a - 4e^2 - 6ef - 2f^2)\right]/8 = 0, \quad \mathcal{D}_1 = (2e + f)/2. \tag{3.14}$$

So we discuss two subcases: $\mathcal{D}_1 \neq 0$ and $\mathcal{D}_1 = 0$.

1.2.1: *The subcase* $\mathcal{D}_1 \neq 0$. Then $2e+f \neq 0$ and then the condition $\Omega_1 = 0$ gives us

$$b = -\frac{a(a-4e^2-6ef-2f^2)}{2(2e+f)^2}.$$

In this case the polynomials Eq_8 and Eq_{10} have the common factor $4a+(2e+f)r$. Therefore the equations $Eq_8 = Eq_{10} = 0$ imply $r = -4a/(2e+f)$ and we arrive at the family of systems

$$\begin{aligned} \dot{x} &= a - x^2/2 + xy, \\ \dot{y} &= -\frac{a(a-4e^2-6ef-2f^2)}{2(2e+f)^2} + ex + fy - 3xy/2 + 2y^2, \end{aligned} \tag{3.15}$$

which possess the invariant parabola

$$\Phi(x,y) = \frac{2a(a-2ef-f^2)}{(2e+f)^2} + \frac{2a}{2e+f}\,x - \frac{4a}{2e+f}\,y + x^2. \tag{3.16}$$

Evidently this conic is irreducible if and only if $a \neq 0$. This completes the proof of the statement $(\boldsymbol{A_5})$ of Lemma 3.3.

Next we show that systems (3.15) could be brought via a transformation to the canonical form $(S_{\boldsymbol{\alpha}})$. Indeed since $2e+f \neq 0$ we apply to parabola (3.16) the translation

$$x = x_1 - \frac{a}{2e+f}, \quad y = y_1 + \frac{a-4ef-2f^2}{4(2e+f)}.$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1) = x_1^2 - \dfrac{4a}{2e+f}\,y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.15) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$\begin{aligned} k &= \frac{4a}{2e+f}, \quad n = -\frac{(-5a+4ef+2f^2)}{4(2e+f)}, \quad m = \frac{-3a+16e^2+20ef+6f^2}{16(2e+f)} \quad \Rightarrow \\ a &= -\frac{k(k-32m-8n)}{32}, \quad e = \frac{-3k+16m+12n}{8}, \quad f = \frac{5k-16n}{8}. \end{aligned}$$

Then after an additional rescaling (to force $k = 1$, see Remark 3.5) we arrive at the subfamily of systems $(S_{\boldsymbol{\alpha}})$ defined by the condition $g = -1/2$.

1.2.2: *The subcase* $\mathcal{D}_1 = 0$. This implies $f = -2e$ and considering (3.14) we have

$$\Omega_1 = a^2/8 = 0 \;\Rightarrow\; a = 0.$$

Therefore we obtain

$$Eq_8 = 0, \quad Eq_{10} = r(32b - 8er + r^2)/32 \equiv r\phi(b,e,r)/32 = 0.$$

Since $r \neq 0$ and $\text{Discrim}[\phi, r] = 64(e^2-2b)$ we must have $e^2 - 2b \geq 0$ and we set $e^2 - 2b = w^2 \geq 0$, i.e. $b = (e^2-w^2)/2$. Then we obtain

$$\phi = (4e - r - 4w)(4e - r + 4w) \equiv \varphi_1\varphi_2 = 0.$$

If $\varphi_1 = 0$ we obtain $r = 4(e-w) \neq 0$ and we obtain the parabola

$$\Phi_1'(x,y) = -2(e^2-w^2) - 2(e-w)x + 4(e-w)y + x^2 \tag{3.17}$$

which is invariant for the family of systems

$$\dot{x} = -x^2/2 + xy,$$
$$\dot{y} = (e^2 - w^2)/2 + ex - 2ey - 3xy/2 + 2y^2. \tag{3.18}$$

In the case $\varphi_2 = 0$ we obtain $r = 4(e + w) \neq 0$ and we obtain the parabola

$$\Phi_2'(x, y) = -2(e^2 - w^2) - 2(e + w)x + 4(e + w)y + x^2$$

which is invariant for the same family of systems (3.18).

We observe that both invariant parabolas exist only for $(e - w)(e + w) = e^2 - w^2 \neq 0$ and since $b = (e^2 - w^2)/2$ we obtain that the condition $b \neq 0$ must hold.

On the other hand we could have only one invariant parabola in the case when one of the factors vanishes, i.e. $(e - w)(e + w) = 0$. So we calculate

$$(e - w) + (e + w) = 2e$$

and we conclude that in the case $b = 0$ and $e \neq 0$ systems (3.18) possess only one invariant parabola.

We determine that in the case $w = 0$ we obtain $\Phi_1'(x, y) = \Phi_2'(x, y)$, i.e. the parabolas coalesce when w tends to zero and we obtain a double parabola. On the other hand considering the relation $e^2 - 2b = w^2$ for $w = 0$ we obtain $e^2 - 2b = 0$ and hence in the case $b \neq 0$ we have two distinct invariant parabolas if $e^2 - 2b \neq 0$ and one double invariant parabola if $e^2 - 2b = 0$.

Thus we conclude that the statements $(\boldsymbol{A_6})$, $(\boldsymbol{A_7})$ and $(\boldsymbol{A_8})$ of Lemma 3.3 are proved.

Next we show that systems (3.18) could be brought via a transformation to the canonical form $(S_{\boldsymbol{\alpha}})$. Indeed we apply to parabola (3.17) the translation

$$x = x_1 + e - w, \quad y = y_1 + (3e + w)/4$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 + 4(e - w)\, y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.18) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -4(e - w), \quad n = (-e + 5w)/4 \quad \Rightarrow$$
$$e = (-5k + 16n)/16, \quad w = (-k + 16n)/16.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems $(S_{\boldsymbol{\alpha}})$ defined by the conditions $g = -1/2$ and $m = (1 - 8n)/32$.

***2:* The possibility $g = 0$.** Then considering (3.5) we obtain

$$Eq_8 = 2a + er = 0, \quad Eq_{10} = (b - a)r = 0.$$

and since $r \neq 0$ we obtain $b = a$. We discuss two cases: $e \neq 0$ and $e = 0$.

***2.1:* The case $e \neq 0$.** Then we get $r = -2a/e$ and we arrive at the family of systems

$$\dot{x} = a + xy, \quad \dot{y} = a + ex + fy - xy + 2y^2, \tag{3.19}$$

which possess the invariant parabola

$$\Phi(x,y) = -\frac{af}{e} + \frac{2a}{e}x - \frac{2a}{e}y + x^2 \tag{3.20}$$

and clearly this conic is irreducible if and only if the condition $a \neq 0$ holds.

***2.2:* The case** $e = 0$**.** Then the equation $Eq_8 = 0$ gives us $a = 0$ and then the equation $Eq_{10} = br = 0$ implies $b = 0$. However in this case we get the degenerate system:

$$\dot{x} = xy, \quad \dot{y} = y(f - x + 2y).$$

So a system (3.3) with $g = 0$ possesses an invariant parabola if and only if the condition $ea \neq 0$ is satisfied. This completes the proof of the statement $(\boldsymbol{A_9})$ of Lemma 3.3.

Now we look for a transformation to brings systems (3.19) to the canonical form (S_{α}). For this we apply to parabola (3.20) the translation

$$x = x_1 - a/e, \quad y = y_1 - \frac{a+ef}{2e}$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 - \frac{2a}{e}y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.15) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = \frac{2a}{e}, \quad m = \frac{a + 2e^2 + ef}{4e}, \quad n = -\frac{a+ef}{2e} \quad \Rightarrow$$
$$a = k(2m+n)/2, \quad e = 2m+n, \quad f = -(k+4n)/2.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S_{α}) defined by the condition $g = 0$.

***3:* The possibility** $g = -1$**.** Then considering (3.5) we obtain

$$Eq_8 = 2a + (e+f)r = 0, \quad Eq_{10} = br = 0.$$

and since $r \neq 0$ we obtain $b = 0$. We discuss two cases: $e + f \neq 0$ and $e + f = 0$.

***3.1:* The case** $e + f \neq 0$**.** Then we get $r = -2a/(e+f)$ and we arrive at the family of systems

$$\dot{x} = a - x^2 + xy, \quad \dot{y} = ex + fy - 2xy + 2y^2, \tag{3.21}$$

which possess the invariant parabola

$$\Phi(x,y) = -\frac{af}{e+f} - \frac{2a}{e+f}y + x^2. \tag{3.22}$$

if $a(e+f) \neq 0$.

***3.2:* The case** $e + f = 0$**.** Then $f = -e$ and the equation $Eq_8 = 0$ gives us $a = 0$. Since $b = 0$ this leads to the degenerate system:

$$\dot{x} = -x(x-y), \quad \dot{y} = (e-2y)(x-y).$$

Thus we have proved that a system (3.3) with $g = -1$ possesses an invariant parabola if and only if the condition $a(e+f) \neq 0$ holds. This completes the proof of the statement $(\boldsymbol{A_{10}})$.

Next we show that systems (3.21) could be brought via a transformation to the canonical form (S_α). Indeed we apply to parabola (3.22) the translation

$$x = x_1, \quad y = y_1 - f/2$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 - \dfrac{2a}{e+f}\, y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.15) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = \frac{2a}{e+f}, \quad m = \frac{e+f}{2}, \quad n = -\frac{f}{2} \quad \Rightarrow$$
$$a = km, \quad e = 2(m+n), \quad f = -2n.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S_α) defined by the conditions $g = -1$.

Since all the cases are examined we deduce that Lemma 3.3 is proved. □

Invariant conditions: the case $\eta > 0$ and $\zeta_1 \neq 0$ Next we determine the affine invariant conditions for a system with $\eta > 0$ and $\zeta_1 \neq 0$ to possess an invariant parabola. According to Lemma 2.4 in this case the condition $\chi_1 = 0$ is necessary.

We prove the following theorem.

Theorem 3.6. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta > 0$, $\chi_1 = 0$ and $\zeta_1 \neq 0$ are satisfied. Then this system could possess invariant parabolas only in one direction. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$ and $\cup^2$. Moreover this system has one of the above sets of parabolas if and only if $\chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$(\mathcal{A}_1)$ $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 \neq 0, \mathcal{R}_1 \neq 0 \Rightarrow \cup$;
$(\mathcal{A}_2)$ $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 \neq 0 \Rightarrow \Cup$;
$(\mathcal{A}_3)$ $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 = 0 \Rightarrow \cup^2$;
$(\mathcal{A}_4)$ $\zeta_2 \neq 0, \zeta_3 \neq 0, \zeta_4 = 0, \mathcal{R}_2 = 0, \zeta_5 \neq 0 \Rightarrow \cup$;
$(\mathcal{A}_5)$ $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 \neq 0, \mathcal{R}_1 \neq 0 \Rightarrow \cup$;
$(\mathcal{A}_6)$ $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0\ \zeta_5 \neq 0 \Rightarrow \Cup$;
$(\mathcal{A}_7)$ $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 = 0, \mathcal{R}_2 \neq 0, \zeta_5 = 0 \Rightarrow \cup^2$;
$(\mathcal{A}_8)$ $\zeta_2 \neq 0, \zeta_3 = 0, \zeta_4 = 0, \mathcal{R}_2 = 0, \zeta_5 \neq 0 \Rightarrow \cup$;
$(\mathcal{A}_9)$ $\zeta_2 = 0, \ \zeta_6 \neq 0, \mathcal{R}_1 = 0, \mathcal{R}_2 \neq 0 \Rightarrow \cup$.

Proof. Assume that for an arbitrary non-degenerate quadratic system the condition $\eta > 0$ holds. Then according to Lemma 2.3 this system could be brought via a linear transformation to the family of systems (3.1). Forcing the condition $\chi_1 = 0$ to be fulfilled for these systems we get $(h-2)(g-2)(1+g+h) = 0$. Considering Remark 3.2 we may assume $h = 2$ and after an additional translation we arrive at the family of systems (3.3), i.e. at the systems

$$\dot{x} = a + gx^2 + xy, \quad \dot{y} = b + ex + fy + (g-1)xy + 2y^2. \tag{3.23}$$

For these systems we calculate

$$\zeta_1 = 2(g-2)(3+g), \quad \chi_2 = 384(g-2)(3+g)\Omega_1$$

and since $\zeta_1 \neq 0$ the condition $\chi_2 = 0$ is equivalent to $\Omega_1 = 0$.

Following the statements $(\mathcal{A}_1)$–$(\mathcal{A}_4)$ of the theorem for systems (3.23) we calculate:

$$\zeta_2 = 4g(g+1),\ \zeta_3 = 8(2g+1)^2,\ \zeta_4 = -(g-2)(3+g)\mathcal{D}_1/8,$$
$$\mathcal{R}_1 = 30(g-2)(3+g)(a-b+4ag-2bg+3ag^2) = 30(g-2)(3+g)\mathcal{G}_1.$$

We discuss two cases: $\zeta_2 \neq 0$ and $\zeta_2 = 0$.

***1:* The case $\zeta_2 \neq 0$.** Then $g(g+1) \neq 0$ and taking into account Lemma 3.3 we have to consider the condition $2g+1 \neq 0$ which is equivalent to $\zeta_3 \neq 0$.

***1.1:* The subcase $\zeta_3 \neq 0$.**

Then we have $2g+1 \neq 0$ and due to $\zeta_1 \neq 0$ (i.e. $(g-2)(3+g) \neq 0$) the condition $\zeta_4 \neq 0$ is equivalent to $\mathcal{D}_1 \neq 0$. So we examine two possibilities: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

1.1.1: *The possibility $\zeta_4 \neq 0$.* Then we have $\mathcal{D}_1 \neq 0$ and we observe that the condition $\mathcal{R}_1 \neq 0$ is equivalent to $\mathcal{G}_1 \neq 0$ since $\zeta_1 \neq 0$. Therefore all the conditions provided by the statement $(\boldsymbol{A_1})$ of Lemma 3.3 are satisfied and by this lemma systems (3.23) possess one invariant parabola.

1.1.2: *The possibility $\zeta_4 = 0$.* In this case due to $\zeta_1 \neq 0$ we obtain $\mathcal{D}_1 = 0$ and considering (3.4) we get:

$$\mathcal{D}_1 = e + f(1+g) = 0 \quad \Rightarrow \quad e = -f(1+g).$$

Then for systems (3.23) we obtain

$$\Omega_1 = 2\big[a(1+g)(1+3g) - b(1+2g)\big]^2$$

and due to the condition $1+2g \neq 0$ the condition $\Omega_1 = 0$ yields

$$b = \frac{a(1+g)(1+3g)}{1+2g}.$$

Therefore for systems (3.23) with the parameters e and b above determined we calculate

$$\zeta_5 = -\frac{19}{1+2g}(g-2)(3+g)\left[8ag(1+g) - f^2(1+2g)\right] = -\frac{19}{1+2g}(g-2)(3+g)\mathcal{F}_1.$$

So due to the condition $\zeta_1 \neq 0$ (i.e. $(g-2)(3+g) \neq 0$ we obtain that the condition $\mathcal{F}_1 = 0$ is equivalent to $\zeta_5 = 0$.

We determine that in the case under examination the condition $a \neq 0$ is equivalent to $\mathcal{R}_2 \neq 0$, which for systems (3.23) has the value

$$\mathcal{R}_2 = -\frac{a(g-2)(3+g)(8+27g+27g^2)}{4(1+2g)}.$$

Indeed first we observe that $\mathrm{Discrim}[8+27g+27g^2, g] = -135 < 0$ and secondly we have $(g-2)(3+g)(1+2g) \neq 0$ due to the condition $\zeta_1\zeta_3 \neq 0$. So considering Lemma 3.3 we conclude that systems (3.23) possess two parabolas if the conditions

$$\chi_1 = \chi_2 = 0,\ \zeta_1 \neq 0,\ \zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0$$

hold. Moreover by Lemma 3.3 these invariant parabolas are distinct if $\zeta_5 \neq 0$, i.e. $\mathcal{F}_1 \neq 0$ (see statement $(\boldsymbol{A_2})$) and they coalesce (obtaining a double parabola) if $\zeta_5 = 0$, i.e. $\mathcal{F}_1 = 0$ (see statement $(\boldsymbol{A_3})$).

Assume now that the condition $\mathcal{R}_2 = 0$ (i.e. $a = 0$) holds. Then for systems (3.23) with $\mathcal{D}_1 = 0$ and $\Omega_1 = 0$ we calculate

$$\zeta_5 = 19f^2(g-2)(3+g)$$

and since $(g-2)(3+g) \neq 0$ (due to $\zeta_1 \neq 0$) we conclude that the condition $f \neq 0$ is equivalent to $\zeta_5 \neq 0$. So we get the conditions provided by Lemma 3.3 (see statement $(\boldsymbol{A_4})$) and therefore we have one simple invariant parabola.

***1.2:* The subcase $\zeta_3 = 0$.** Then $1+2g = 0$, i.e. $g = -1/2$ and for systems (3.23) calculations yield:

$$\zeta_1 = -25/2,\ \zeta_2 = -1,\ \zeta_4 = 25(2e+f)/64 = 25\mathcal{D}_1/32,$$
$$\chi_2 = -300\big[2b(2e+f)^2 + a(a-4e^2-6ef-2f^2)\big].$$

So considering Lemma 3.3 (see statements $(\boldsymbol{A_5})$–$(\boldsymbol{A_8})$) we discuss two possibilities: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

1.2.1:* *The possibility $\zeta_4 \neq 0$. Then $2e+f \neq 0$ and the condition $\chi_2 = 0$ gives us

$$b = -\frac{a(a-4e^2-6ef-2f^2)}{2(2e+f)^2}.$$

So according to Lemma 3.3 (see statements $(\boldsymbol{A_5})$) systems (3.23) with $g = -1/2$ and the above given value of the parameter b possess one invariant parabola if in addition the condition $a \neq 0$ holds. It remains to observe that this condition is governed by the invariant polynomial $\mathcal{R}_1$ because for these systems we have $\mathcal{R}_1 = 375a/8$.

1.2.2:* *The possibility $\zeta_4 = 0$. Then we have $\mathcal{D}_1 = 0$ which implies $f = -2e$ and then we obtain $\chi_2 = -300a^2 = 0$, i.e. $a = 0$. As a result we arrive at the family of systems

$$\dot{x} = -x^2/2 + xy, \quad \dot{y} = b + ex - 2ey - 3xy/2 + 2y^2 \tag{3.24}$$

for which we calculate

$$\zeta_1 = -25/2,\ \zeta_2 = -1,\ \zeta_3 = \zeta_4 = \mathcal{R}_1 = 0,\ \ \mathcal{R}_2 = -125b/16,\ \zeta_5 = 475(2b-e^2).$$

So considering the statements $(\boldsymbol{A_6})$ and $(\boldsymbol{A_7})$ of Lemma 3.3 we deduce that in the case $\mathcal{R}_2 \neq 0$ (i.e. $b \neq 0$) systems (3.24) possess two distinct invariant parabolas if $\zeta_5 \neq 0$ and one double invariant parabola if $\zeta_5 = 0$.

Assuming $\mathcal{R}_2 = 0$ (i.e. $b = 0$) considering the value of the invariant polynomial ζ_5 given above we get $\zeta_5 = -475e^2$ and hence the condition $e \neq 0$ is equivalent to $\zeta_5 \neq 0$.

So we get the conditions provided by the statement $(\boldsymbol{A_8})$ of Lemma 3.3 and therefore systems (3.24) possess one simple invariant parabola.

***2:* The case $\zeta_2 = 0$.** Then we have $g(g+1) = 0$, i.e. either $g = 0$ or $g = -1$. We discuss each one of these possibilities.

2.1:* *The possibility $g = 0$. Then for for systems (3.23) we calculate

$$\chi_2 = -2304(a-b)(2a-2b-e^2-ef),\ \mathcal{R}_1 = -180(a-b).$$

According to the statement $(\boldsymbol{A_9})$ of Lemma 3.3 for the existence of invariant parabola the condition $b = a$ is necessary, i.e. we must have $\mathcal{R}_1 = 0$ and this implies $\chi_2 = 0$. Setting $b = a$ we obtain

$$\zeta_6 = e/2,\ \ \mathcal{R}_2 = 12a$$

and considering the statements (A_9) the condition $\zeta_6\mathcal{R}_2 \neq 0$ must be satisfied for the existence of an invariant parabola.

2.2: *The possibility* $g = -1$. In this case for systems (3.23) we obtain

$$\chi_2 = -2304b(2b + e^2 + ef), \; \mathcal{R}_1 = -180b.$$

Considering the statement (A_{10}) of Lemma 3.3 we deduce that for the existence of an invariant parabola the condition $b = 0$ is necessary, i.e. we must have $\mathcal{R}_1 = 0$ and this implies $\chi_2 = 0$. Setting $b = 0$ we calculate

$$\zeta_6 = -(e + f)/2, \;\; \mathcal{R}_2 = -12a$$

and therefore by the statements (A_{10}) the condition $\zeta_6\mathcal{R}_2 \neq 0$ must be satisfied for the existence of an invariant parabola.

We observe that in both cases $g = 0$ and $g = -1$ we have obtained the same invariant conditions $\mathcal{R}_1 = 0$ and $\zeta_6\mathcal{R}_2 \neq 0$. This completes the proof of the statement $(\boldsymbol{\mathcal{A}_9})$ of Theorem 3.6 as well as the proof of Theorem 3.6. □

3.1.2 The possibility $\chi_1 = \zeta_1 = 0$

Next we consider the case when systems (3.1) could possess invariant parabolas in two directions. Then two factors of χ_1 from (3.2) vanish. According to Remark 3.2 we could consider $h = 2 = g$ and due to the translation $(x, y) \mapsto (x - d, y - e)$ (forcing $d = e = 0$) we arrive at the family of systems

$$\dot{x} = a + cx + 2x^2 + xy, \quad \dot{y} = b + fy + xy + 2y^2. \tag{3.25}$$

Coefficient conditions for systems (3.25) to possess invariant parabolas. By Lemma 3.1 systems (3.25) could possess invariant parabolas either of the form $\Phi(x, y) = p + qx + ry + x^2$ $(r \neq 0)$ or of the form $\Phi(x, y) = p + qx + ry + y^2$ $(q \neq 0)$. We prove the following lemma.

Lemma 3.7. *A system (3.25) possesses either one or two invariant parabolas or a double invariant parabola of the indicated form if and only if the corresponding set of conditions are satisfied, respectively:*

$(\boldsymbol{B})$ $\Phi(x, y) = p + qx + ry + x^2 \quad \Leftrightarrow \quad \Omega_1' = 0$ *and either*

- $(\boldsymbol{B_1})$ $\mathcal{D}_1' \neq 0, \mathcal{G}_1' \neq 0 \Rightarrow$ *one invariant parabola; or*
- $(\boldsymbol{B_2})$ $\mathcal{D}_1' = 0, a \neq 0, \mathcal{F}_1' \neq 0 \Rightarrow$ *two invariant parabolas; or*
- $(\boldsymbol{B_3})$ $\mathcal{D}_1' = 0, a \neq 0, \mathcal{F}_1' = 0 \Rightarrow$ *one double invariant parabola; or*
- $(\boldsymbol{B_4})$ $\mathcal{D}_1' = 0, a = 0, c \neq 0 \Rightarrow$ *one invariant parabola.*

$(\boldsymbol{B'})$ $\Phi(x, y) = p + qx + ry + y^2 \quad \Leftrightarrow \quad \Omega_2' = 0$ *and either*

- $(\boldsymbol{B_1'})$ $\mathcal{D}_2' \neq 0, \mathcal{G}_2' \neq 0 \Rightarrow$ *one invariant parabola; or*
- $(\boldsymbol{B_2'})$ $\mathcal{D}_2' = 0, b \neq 0, \mathcal{F}_2' \neq 0 \Rightarrow$ *two invariant parabolas; or*
- $(\boldsymbol{B_3'})$ $\mathcal{D}_2' = 0, b \neq 0, \mathcal{F}_2' = 0 \Rightarrow$ *one double invariant parabola; or*
- $(\boldsymbol{B_4'})$ $\mathcal{D}_2' = 0, b = 0, f \neq 0 \Rightarrow$ *one invariant parabola.*

where

$$\begin{aligned}
&\Omega_1' = 50b^2 + b(109c^2 - 420a - 53cf - 6f^2) + 3(6a - c^2 + cf)(49a - 6c^2 - cf + 2f^2);\\
&\mathcal{D}_1' = 13c - 3f;\;\; \mathcal{G}_1' = 21a - 5b - 10c^2 + 5cf;\\
&\mathcal{F}_1' = 15a - 15b - 2c^2 + 2f^2;\\
&\Omega_2' = 50a^2 + a(109f^2 - 420b - 53cf - 6c^2) + 3(6b - f^2 + cf)(49b - 6f^2 - cf + 2c^2);\\
&\mathcal{D}_2' = 13f - 3c;\;\; \mathcal{G}_2' = 21b - 5a - 10f^2 + 5cf;\\
&\mathcal{F}_2' = 15b - 15a - 2f^2 + 2c^2.
\end{aligned} \tag{3.26}$$

Proof. Considering the equations (2.6) we examine each one of the statements of the above lemma.

(B) $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$. In this case we obtain

$$s = 1,\; v = u = 0,\; U = 4,\; V = 2,\; W = 2(c - q),$$
$$Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0.$$

Calculating the remaining equations we have

$$Eq_6 = -q - 3r,\;\; Eq_8 = 2a - 4p - cq + 2q^2,$$
$$Eq_9 = -2p - 2cr + fr + 2qr,\;\; Eq_{10} = aq - 2cp + 2pq + br.$$

It is clear that the equations $Eq_6 = 0$ implies $q = -3r$ whereas $Eq_9 = 0$ gives us $p = -r(2c - f + 6r)/2$. Therefore calculations yield

$$Eq_8 = 2a + (7c - 2f)r + 30r^2,$$
$$Eq_{10} = r\big[b - 3a + 2c^2 - cf + 3(4c - f)r + 18r^2\big] \equiv r\Psi'(a,b,c,f,r)$$

and since $r \neq 0$ the equation $Eq_{10} = 0$ is equivalent to $\Psi' = 0$.

According to [12, Lemmas 11,12] the equations $Eq_8 = 0$ and $\Psi' = 0$ have a common solution of degree 2 with respect to the parameter r if and only if

$$Res_r^{(0)}\,(Eq_8, \Psi') = Res_r^{(1)}\,(Eq_8, \Psi) = 0$$

where $Res_r^{(1)}$ is the subresultant of order one and $Res_r^{(0)}$ is the subresultant of order zero which coincide with standard resultant (for detailed definition see [12], formula (19)). We calculate

$$Res_r^{(1)}\,(Eq_8, \Psi) = 18(13c - 3f) \equiv 18\mathcal{D}_1', \quad Res_r^{(0)}\,(Eq_8, \Psi) \;\; = 18\Omega_1'.$$

So we examine two possibilities: $\mathcal{D}_1' \neq 0$ and $\mathcal{D}_1' = 0$.

1: **The possibility** $\mathcal{D}_1' \neq 0$**.** Therefore the equations $Eq_8 = 0$ and $\Psi' = 0$ could have a unique common solution with respect to the parameter r and for this it is necessary and sufficient $\Omega_1' = 0$. So we have to examine the solutions of the equation $\Omega_1' = 0$. In this case we calculate

$$\text{Discrim}[\Omega_1', b] = -(13c - 3f)^2(240a - 49c^2 + 28cf - 4f^2) \equiv -\mathcal{D}_1'^2\mathcal{E}'$$

and hence the equation $\Omega_1' = 0$ has real solutions with respect to the parameter b if and only if either $\mathcal{D}_1' = 0$ or $\mathcal{E}' \leq 0$. However since the condition $\mathcal{D}_1' \neq 0$ holds it remains to examine the condition $\mathcal{E}' \leq 0$. In this case setting $\mathcal{E}' = -w^2 \leq 0$ we calculate

$$a = \frac{(7c - 2f)^2 - w^2}{240} \tag{3.27}$$

and then we obtain $\Omega_1' = (E_+E_-)/3200$, where

$$E_\pm = 400b + 93c^2 - 16cf - 52f^2 + 4\varepsilon(13c - 3f)w + 7w^2, \quad \varepsilon = \pm 1.$$

Then the condition $\Omega_1' = 0$ gives us

$$b = -\frac{1}{400}(3c + 2f + \varepsilon w)(31c - 26f + 7\varepsilon w) \tag{3.28}$$

where $\varepsilon = 1$ if $E_+ = 0$ and $\varepsilon = -1$ if $E_- = 0$. In this case we obtain that the polynomials Eq_8 and $\Psi(c, f, r)$ have the common factor $\zeta = (7c - 2f + 60r - \varepsilon w)$ which is linear with respect to the parameter r. Setting $\zeta = 0$ we get

$$r = \frac{2f - 7c + \varepsilon w}{60}$$

and we arrive at the family of systems

$$\begin{aligned} \dot{x} &= \frac{(7c - 2f)^2 - w^2}{240} + cx + 2x^2 + xy, \\ \dot{y} &= -\frac{(3c + 2f + \varepsilon w)(31c - 26f + 7\varepsilon w)}{400} + fy + xy + 2y^2. \end{aligned} \tag{3.29}$$

This family of systems possess the following invariant parabola

$$\Phi(x, y) = \frac{(7c - 2f - \varepsilon w)(13c - 8f + \varepsilon w)}{1200} + \frac{(7c - 2f - \varepsilon w)}{20}x - \frac{(7c - 2f - \varepsilon w)}{60}y + x^2. \tag{3.30}$$

We observe that this conic is reducible if and only if $7c - 2f - \varepsilon w = 0$.

Considering (3.27) and (3.28) we get

$$w^2 = -240a + (7c - 2f)^2$$

and then we obtain

$$\begin{gathered} b = -\frac{1}{400}\big[(31c - 26f)(3c + 2f) + 4(13c - 3f)\varepsilon w + 7w^2\big] \;\Rightarrow \\ 100b - 420a + 109c^2 - 53cf - 6f^2 + (13c - 3f)\varepsilon w = 0. \end{gathered}$$

Since $\mathcal{D}_1' = 13c - 3f \neq 0$ we solve the last equation with respect to εw and we obtain

$$\varepsilon w = \frac{1}{13c - 3f}\big(420a - 100b - 109c^2 + 53cf + 6f^2\big).$$

Then calculations yield

$$r = \frac{(-7c + 2f + \varepsilon w)}{60} = \frac{21a - 5b - 10c^2 + 5cf}{3(13c - 3f)} = \frac{\mathcal{G}_1'}{3(13c - 3f)} \neq 0.$$

This completes the proof of the statement $(\boldsymbol{B_1})$ of Lemma 3.3.

Next we show that systems (3.29) could be brought via a transformation to the canonical form $(S_{\boldsymbol{\alpha}})$. Indeed we apply to parabola (3.30) the translation

$$x = x_1 + \frac{2f - 7c + \varepsilon w}{40}, \quad y = y_1 + \frac{31c - 26f + 7\varepsilon w}{80}.$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 + \dfrac{2f - 7c + \varepsilon w}{60} y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.29) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -\frac{2f - 7c + \varepsilon w}{60}, \quad m = -\frac{31c - 26f + 7\varepsilon w}{160}, \quad n = \frac{11c - 2f + 3\varepsilon w}{16} \quad \Rightarrow$$
$$c = 6k - 2m + n, \quad f = \frac{3k - 16m + 4n}{2}, \quad \varepsilon w = -21k + 2m + 3n.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems $(S_{\boldsymbol{\alpha}})$ defined by the conditions $g = 2$.

2: The possibility $\mathcal{D}_1' = 0$. Then $f = 13c/3$ and therefore we obtain:

$$\Omega_1' = 2(63a - 15b + 35c^2)^2/9$$

and hence the condition $\Omega_1' = 0$ implies

$$b = 7(9a + 5c^2)/15.$$

Therefore we determine that in this case the polynomials Eq_8 and Eq_{10} have the following common factor

$$\phi' = 6a - 5cr + 90r^2.$$

We observe that ϕ' is quadratic in r with the discriminant

$$\text{Discrim}[\phi', r] = -5(432a - 5c^2)$$

and setting this discriminant equal to be w^2 we obtain

$$a = \frac{25c^2 - w^2}{2160}. \tag{3.31}$$

Then we arrive at the following expressions for the polynomials Eq_8 and Eq_{10}:

$$Eq_{10} = \frac{3r}{5} Eq_8 = \frac{r(5c - 180r + w)(5c - 180r - w)}{1800} = \frac{rU_+U_-}{1800}.$$

Therefore the equations $Eq_8 = Eq_{10} = 0$ imply $U_+U_- = 0$. If $U_+ = 0$ we determine

$$r = \frac{5c + w}{180} \equiv r^+$$

and we obtain the parabola

$$\Phi_1'(x, y) = \frac{(65c - w)(5c + w)}{10800} - \frac{5c + w}{60} x + \frac{5c + w}{180} y + x^2. \tag{3.32}$$

In the case $U_- = 0$ we obtain

$$r = \frac{5c + w}{180} \equiv r^-$$

and we get the parabola

$$\Phi_2'(x, y) = \frac{(65c + w)(5c - w)}{10800} - \frac{5c - w}{60} x + \frac{5c - w}{180} y + x^2.$$

Both these parabolas are invariant for the following family of systems:

$$\dot{x} = \frac{25c^2 - w^2}{2160} + cx + 2x^2 + xy, \quad \dot{y} = \frac{7(1225c^2 - w^2)}{3600} + \frac{13c}{3}y + xy + 2y^2. \tag{3.33}$$

We observe that both parabolas $\Phi_i'(x,y) = 0$ $(i = 1,2)$ exist (i.e. are not reducible) if and only if $r^+r^- \neq 0$ and this is equivalent to

$$(5c + w)(5c - w) = 25c^2 - w^2$$

and considering (3.31) this is equivalent to $a \neq 0$.

On the other hand if only one of the factors vanishes we have $a = 0$ and

$$r^+ + r^- = (5c + w) + (5c - w) = 10c \neq 0.$$

Therefore for $a = 0$ and $c \neq 0$ we could have only one parabola.

We determine that in the case $w = 0$ we obtain $\Phi_1'(x,y) = \Phi_2'(x,y)$, i.e. the parabolas coalesce when w tends to zero and we obtain a double parabola. On the other hand considering (3.31) for $w = 0$ we obtain

$$a - \frac{25c^2}{2160} = \frac{432a - 5c^2}{432} = -\frac{9}{432}\mathcal{F}_1'$$

and we conclude that these invariant parabolas coalesce if and only if $\mathcal{F}_1' = 0$.

Thus we conclude that the statements $(\boldsymbol{B_2})$, $(\boldsymbol{B_3})$ and $(\boldsymbol{B_4})$ of Lemma 3.7 are proved.

Next we show that systems (3.33) could be brought via a transformation to the canonical form $(S_{\boldsymbol{\alpha}})$. Indeed we could apply to parabola (3.32) the translation

$$x = x_1 + \frac{5c + w}{120}, \quad y = y_1 - \frac{7(35c - w)}{240}$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 + \dfrac{5c + w}{180}y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.29) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -\frac{5c + w}{180}, \quad n = \frac{7c + 3w}{48} \quad \Rightarrow \quad c = -3(45k + 4n)/2, \quad w = 15(21k + 4n)/2.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems $(S_{\boldsymbol{\alpha}})$ defined by the conditions $g = 2$ and $m = 7(21k + 2n)/4$.

$(\boldsymbol{B'})$ $\Phi(x,y) = p + qx + ry + y^2$ with $q \neq 0$ otherwise we get a reducible conic. It is not too difficult to detect that this case can be brought to the case $(\boldsymbol{B})$ if we apply two changes: one in systems (3.25) and another in the formula of conic (2.4). More precisely the change

$$(x, y, a, b, c, f) \mapsto (y, x, b, a, f, c) \tag{3.34}$$

conserves systems (3.25) whereas the change

$$(x, y, p, q, r, s, v, u) \to (y, x, p, r, q, u, v, s)$$

conserves the conic (2.4). We observe that the second change transfers the parabola $\Phi(x,y) = p + qx + ry + x^2$ to the parabola $\Phi(x,y) = p + qx + ry + y^2$ and at the same time the first change transfers the conditions $(\boldsymbol{B_i})$, $i = 1,2,3,4$ from the statement $(\boldsymbol{B})$ of Lemma 3.7 to the conditions $(\boldsymbol{B_i'})$, $i = 1,2,3,4$ from the statement $(\boldsymbol{B'})$ of the same lemma, correspondingly. Since the conditions of the statement $(\boldsymbol{B})$ are proved, we conclude that the conditions of the statement $(\boldsymbol{B'})$ of Lemma 3.7 are also valid. This completes the proof of Lemma 3.7. □

We point out that Theorem 3.6 provides the necessary and sufficient conditions for the existence of invariant parabolas for an arbitrary quadratic systems with the conditions $\eta > 0$, $\chi_1 = 0$ and $\zeta_1 \neq 0$. As it was mentioned earlier (see page 16) the condition $\zeta_1 \neq 0$ does not allow this system to possess invariant parabolas in two directions.

Invariant conditions: the case $\eta > 0$ and $\zeta_1 = 0$ Next we consider the class of quadratic systems for which the conditions $\eta > 0$ and $\zeta_1 = 0$, which could possess invariant parabolas in two directions.

We prove the following theorem.

Theorem 3.8. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta > 0$ and $\chi_1 = \zeta_1 = 0$ are satisfied. Then this system could possess invariant parabolas in one or two directions. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$, $\cup^2$, $\cup\subset$ and $\Cup\subset$. Moreover this system has one of the above sets of invariant parabolas if and only if $\chi_3 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$(\mathcal{B}_1)$ $\chi_4 \neq 0,\ \zeta_7 \neq 0,\ \mathcal{R}_3 \neq 0 \Rightarrow \cup;$
$(\mathcal{B}_2)$ $\chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 \neq 0,\ \zeta_8 \neq 0 \Rightarrow \Cup;$
$(\mathcal{B}_3)$ $\chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 \neq 0,\ \zeta_8 = 0 \Rightarrow \cup^2;$
$(\mathcal{B}_4)$ $\chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 = 0 \Rightarrow \cup;$
$(\mathcal{B}_5)$ $\chi_4 = 0,\ \zeta_5 \neq 0,\ \zeta_9 \neq 0 \Rightarrow \cup\subset;$
$(\mathcal{B}_6)$ $\chi_4 = 0,\ \zeta_5 \neq 0,\ \zeta_9 = 0,\ \zeta_{10} \neq 0 \Rightarrow \cup;$
$(\mathcal{B}_7)$ $\chi_4 = 0,\ \zeta_5 = 0,\ \zeta_6 \neq 0 \Rightarrow \Cup\subset.$

Proof. As it was shown earlier (see page 29) if for a quadratic system with three real infinite singularities the conditions $\chi_1 = \zeta_1 = 0$ are satisfied, then via an affine transformation and time rescaling this system can be brought to the form (3.25). Thus in what follows we consider the family of quadratic systems

$$\dot{x} = a + cx + 2x^2 + xy, \quad \dot{y} = b + fy + xy + 2y^2. \tag{3.35}$$

Considering (3.26) for these systems we calculate

$$\chi_1 = \chi_2 = \zeta_1 = 0, \;\; \chi_3 = 2^4 3^4 5^3 83 \cdot 491\, \Omega_1' \Omega_2', \;\; \chi_4 = 123750(\Omega_1' + \Omega_2')$$

and therefore the condition $\chi_3 = 0$ yields $\Omega_1'\Omega_2' = 0$, i.e. one of the necessary conditions provided either by the statement $(\boldsymbol{B})$ of Lemma 3.7 or by the statement $(\boldsymbol{B'})$ of this lemma is satisfied. We discuss two cases: $\chi_4 \neq 0$ and $\chi_4 = 0$.

1: **The case $\chi_4 \neq 0$.** Then $\Omega_1' + \Omega_2' \neq 0$ and we conclude that only one of the polynomials Ω_1' or Ω_2' vanishes. Considering the change (3.34) we may assume without losing generality that the conditions $\Omega_1' = 0$ and $\Omega_2' \neq 0$ are fulfilled.

On the other hand for systems (3.35) we calculate

$$\zeta_7 = 105750\big(\mathcal{D}_2'\Omega_1' + \mathcal{D}_1'\Omega_2'\big), \quad \mathcal{R}_3 = 5134081342500\big(\mathcal{G}_2'\Omega_1' + \mathcal{G}_1'\Omega_2'\big).$$

Therefore since $\Omega_1' = 0$ and $\Omega_2' \neq 0$ we obtain that the condition $\mathcal{D}_1' = 0$ is equivalent to $\zeta_7 = 0$. Moreover in this case the condition $\mathcal{R}_3 \neq 0$ is equivalent to $\mathcal{G}_1' \neq 0$. So we discuss two subcases: $\zeta_7 \neq 0$ and $\zeta_7 = 0$.

1.1: **The subcase $\zeta_7 \neq 0$.** Then $\mathcal{D}_1' \neq 0$ and by Lemma 3.7 (see statement $(\boldsymbol{B_1})$) we deduce that systems (3.25) possess one invariant parabola if and only if $\mathcal{G}_1' \neq 0$. Due to $\Omega_1' = 0$ and

$\Omega'_2 \neq 0$ this condition is equivalent to $\mathcal{R}_3 \neq 0$ and we conclude that the statement $(\boldsymbol{\mathcal{B}_1})$ of Theorem 3.8 is proved.

***1.2:* The subcase $\zeta_7 = 0$.** This implies $\mathcal{D}'_1 = 13c - 3f = 0$, i.e. $f = 13c/3$ and then we get

$$\Omega'_1 = 2(63a - 15b + 35c^2)^2/9 = 0 \;\Rightarrow\; b = 7(9a + 5c^2)/15.$$

So we arrive at the family of systems

$$\dot{x} = a + cx + 2x^2 + xy, \quad \dot{y} = \frac{7}{15}(9a + 5c^2) + \frac{13c}{3}y + xy + 2y^2, \tag{3.36}$$

for which we calculate

$$\zeta_8 = -(432a - 5c^2)/9 = \mathcal{F}'_1, \quad \mathcal{R}_4 = 15600a.$$

We observe that the condition $\mathcal{R}_4 \neq 0$ is equivalent to $a \neq 0$ and therefore by Lemma 3.7 in the case $\mathcal{R}_4 \neq 0$ systems (3.36) possess two distinct parabolas in one direction (see statement $(\boldsymbol{B_2})$) if $\zeta_8 \neq 0$ and they possess one double invariant parabola (see statement $(\boldsymbol{B_3})$) if $\zeta_8 = 0$. This means that the statements $(\boldsymbol{\mathcal{B}_2})$ and $(\boldsymbol{\mathcal{B}_3})$ of Theorem 3.8 are proved.

Assume now that the condition $\mathcal{R}_4 = 0$ holds. Then $a = 0$ and for systems (3.36) we have $\chi_4 = 110000c^4 \neq 0$. Then according to the statement $(\boldsymbol{B_4})$ of Lemma 3.7 we conclude that these systems possess one invariant parabola and therefore the statements $(\boldsymbol{\mathcal{B}_4})$ of Theorem 3.8 is valid.

***2:* The case $\chi_4 = 0$.** Then we get $\Omega'_1 = \Omega'_2 = 0$ and since for systems (3.35) we have

$$\begin{aligned} \zeta_5 &= 25(13c - 3f)(13f - 3c)/4 = 25\mathcal{D}'_1\mathcal{D}'_2/4, \\ \zeta_9 &= -990000(21a - 5b - 10c^2 + 5cf)(5a - 21b - 5cf + 10f^2) = 990000\mathcal{G}'_1\mathcal{G}'_2, \\ \zeta_{10} &= 5(8a + 8b - 5c^2 + 5cf - 5f^2)/4 = 5(\mathcal{G}'_1 + \mathcal{G}'_2)/8. \end{aligned} \tag{3.37}$$

We examine two subcases: $\zeta_5 \neq 0$ and $\zeta_5 = 0$.

***2.1:* The subcase $\zeta_5 \neq 0$.** Then $\mathcal{D}'_1\mathcal{D}'_2 \neq 0$ and by Lemma 3.7 (see statements $(\boldsymbol{A'_1})$ and $(\boldsymbol{B'_1})$) we have one invariant parabola in the direction $x = 0$ if $\mathcal{G}'_1 \neq 0$ and one in the direction $y = 0$ if $\mathcal{G}'_2 \neq 0$. So considering (3.37) we examine two possibilities: $\zeta_9 \neq 0$ and $\zeta_9 = 0$.

2.1.1: *The possibility $\zeta_9 \neq 0$.* This implies $\mathcal{G}'_1\mathcal{G}'_2 \neq 0$ and by Lemma 3.7 in this case we have one invariant parabola in one direction and another invariant parabola in other direction. So the statement $(\boldsymbol{\mathcal{B}_5})$ of Theorem 3.8 is proved.

2.1.2: *The possibility $\zeta_9 = 0$.* Then we have $\mathcal{G}'_1\mathcal{G}'_2 = 0$, i.e. at least one of the factors vanishes. Considering (3.37) we conclude that both factors vanish if and only if $\zeta_{10} = 0$. In this case $\mathcal{G}'_1 = \mathcal{G}'_2 = 0$ and by Lemma 3.7 (see statements $(\boldsymbol{B_1})$ and $(\boldsymbol{B'_1})$) systems (3.25) could not possess any invariant parabolas.

On the other hand in the case $\zeta_{10} \neq 0$ we have $\mathcal{G}'_1 + \mathcal{G}'_2 \neq 0$ and since $\mathcal{G}'_1\mathcal{G}'_2 = 0$, by Lemma 3.7 we have one invariant parabola (either in direction $y = 0$ if $\mathcal{G}'_1 = 0$ or in direction $x = 0$ if $\mathcal{G}'_2 = 0$. This means that the statement $(\boldsymbol{\mathcal{B}_6})$ of Theorem 3.8 is valid.

***2.2:* The subcase $\zeta_5 = 0$.** In this case we get $\mathcal{D}'_1\mathcal{D}'_2 = 0$. On the other hand we obtain $\mathcal{D}'_1 + \mathcal{D}'_2 = 10(c + f)$ and hence both $\mathcal{D}'_1$ and $\mathcal{D}'_2$ vanish if and only if $c + f = 0$ and this condition is governed by the invariant polynomial $\zeta_6 = -(c + f)/2$. So we discuss two possibilities: $\zeta_6 \neq 0$ and $\zeta_6 = 0$.

2.2.1: *The possibility* $\zeta_6 \neq 0$. Then only one of the polynomials $\mathcal{D}'_1$ or $\mathcal{D}'_2$ vanishes and due to the change $(x, y, a, b, c, f) \mapsto (y, x, b, a, f, c)$ without losing generality we may assume that for systems (3.25) the condition $\mathcal{D}'_1 = 0$ holds. Considering (3.26) this condition implies $f = 13c/3$ and then we obtain

$$\Omega'_1 = 2(63a - 15b + 35c^2)^2/9 = 0 \Rightarrow b = 7(9a + 5c^2)/15.$$

Therefore we calculate

$$\Omega'_2 = 8(144a + 5c^2)(2704a + 5c^2)/225, \quad \zeta_6 = -8c/3 \neq 0$$

and the condition $\Omega'_2 = 0$ gives us either $a = -5c^2/144 \neq 0$ or $a = -5c^2/2704 \neq 0$ (due to $\zeta_6 \neq 0$). In this case we get either

$$\mathcal{F}'_1 = 20c^2/9 \neq 0, \quad \mathcal{G}'_2 = -120c^2 \neq 0$$

if $a = -5c^2/144$ or

$$\mathcal{F}'_1 = 980c^2/1521 \neq 0, \quad \mathcal{G}'_2 = -13720c^2/117 \neq 0$$

if $a = -5c^2/2704$. So considering the statements $(\boldsymbol{B_2})$ and $(\boldsymbol{B'_1})$ of Lemma 3.7 we conclude that systems (3.25) possess two distinct invariant parabolas in the direction $x = 0$ and one invariant parabola in the direction $y = 0$. This means that the statement $(\boldsymbol{\mathcal{B}_7})$ of Theorem 3.8 is valid.

2.2.2: *The possibility* $\zeta_6 = 0$. This condition implies $\mathcal{D}'_1 = \mathcal{D}'_2 = 0$ and considering (3.26) we obtain $c = f = 0$. Then we obtain

$$\Omega'_1 = 2(21a - 5b)^2, \quad \Omega'_2 = 2(5a - 21b)^2$$

and evidently the conditions $\Omega'_1 = \Omega'_2 = 0$ imply $a = b = 0$. Therefore we arrive at the following homogeneous system

$$\dot{x} = x(2x + y), \quad \dot{y} = y(x + 2y)$$

that could not possess any invariant parabola.

Since all the possibilities are examined we conclude that Theorem 3.8 is proved. □

3.2 Systems with one real and two complex infinite singularities

In this case according to Lemma 2.3 systems (2.5) could be brought via a linear transformation to the following family of systems

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + (h+1)xy, \\ \frac{dy}{dt} &= b + ex + fy - x^2 + gxy + hy^2. \end{aligned} \tag{3.38}$$

For these systems we calculate

$$C_2(x, y) = x(x^2 + y^2), \quad \chi_1 = -2(2 + h)\left[g^2 + (h-3)^2\right] \tag{3.39}$$

and by Lemma 2.6 we conclude that the above systems could have invariant parabolas only of the form $\Phi(x, y) = p + qx + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic).

On the other hand according to Lemma 2.4 for a system (3.38) to possess an invariant parabola the condition $\chi_1 = 0$ is necessary. Considering (3.39) this condition implies either $h = -2$ or $g = 0 = h - 3$. We claim that in the second case systems (3.38) could not possess any invariant parabola.

Indeed, assuming $g = 0$ and $h = 3$ and using a translation we may assume $c = d = 0$ and we arrive at the family of systems

$$\dot{x} = a + 4xy, \quad \dot{y} = b + ex + fy - x^2 + 3y^2. \tag{3.40}$$

Considering equations (2.6) and the form of the parabola $\Phi(x, y) = p + qx + ry + x^2$ with $r \neq 0$, for systems (3.40) we have

$$s = 1,\ v = u = 0,\ Eq_2 = 8 - V, Eq_7 = r(3 - V).$$

Evidently the conditions $Eq_2 = 0$ and $Eq_7 = 0$ imply $r = 0$, i.e. the conic $\Phi(x, y) = p + qx + ry + x^2$ with $r = 0$ is reducible and this completes the proof of our claim.

For systems (3.38) we calculate

$$\zeta_1 = -2\big[(h-3)(1+h)(2h-1) + g^2(3+2h)\big]$$

and clearly the conditions $g = 0$ and $h = 3$ imply $\zeta_1 = 0$. On the other hand for $h = -2$ we get $\zeta_1 = 2(25 + g^2) \neq 0$ and therefore the condition $h + 2 = 0$ is equivalent to $\chi_1 = 0$ and $\zeta_1 \neq 0$. So we have the next remark.

Remark 3.9. If a system (3.38) possesses an invariant parabola then the conditions $\chi_1 = 0$ and $\zeta_1 \neq 0$ are necessary.

According to this remark we assume that the conditions $\chi_1 = 0$ and $\zeta_1 \neq 0$ are fulfilled for systems (3.38). Then the condition $h = -2$ holds and due to a translation we may consider $c = d = 0$. So we arrive at the family of systems

$$\dot{x} = a + gx^2 - xy, \quad \dot{y} = b + ex + fy - x^2 + gxy - 2y^2. \tag{3.41}$$

3.2.1 Coefficient conditions for systems (3.41) to possess invariant parabolas

We prove the following lemma.

Lemma 3.10. *A system (3.41) possesses either one or two invariant parabolas or a double invariant parabola of the form $\Phi(x, y) = p + qx + ry + x^2$ if and only if $\widetilde{\Omega} = 0$ and the corresponding set of conditions are satisfied, respectively:*

(E_1) $\widetilde{\mathcal{D}} \neq 0, \widetilde{\mathcal{G}} \neq 0 \Rightarrow$ *one invariant parabola;*

(E_2) $\widetilde{\mathcal{D}} = 0, b \neq 0, \widetilde{\mathcal{F}} \neq 0 \Rightarrow$ *two invariant parabolas;*

(E_3) $\widetilde{\mathcal{D}} = 0, b \neq 0, \widetilde{\mathcal{F}} = 0 \Rightarrow$ *one double invariant parabola;*

(E_4) $\widetilde{\mathcal{D}} = 0, b = 0, f \neq 0 \Rightarrow$ *one invariant parabola,*

where

$$\begin{aligned}
&\widetilde{\Omega} = 2a^2(1+3g^2)^2 + a\big[8bg(1+3g^2) - (e-fg)(f+eg+2fg^2)\big] + b(8bg^2 + f^2g^2 - e^2),\\
&\widetilde{\mathcal{D}} = e - fg, \quad \widetilde{\mathcal{G}} = a + 2bg + 3ag^2,\\
&\widetilde{\mathcal{F}} = 608(b+ag)(25+g^2) + 25(49e^2 + 76f^2) - fg(850e + 299fg).
\end{aligned} \tag{3.42}$$

Proof. Considering equations (2.6) and the form of the parabola $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$ for systems (3.41) we obtain

$$s = 1,\ v = u = 0,\ U = 2g,\ V = -2,\ W = -gq - r,$$
$$Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0,\ \ Eq_6 = q - gr.$$

Therefore the condition $Eq_6 = 0$ gives us $q = gr$ and calculations yield:

$$Eq_9 = 2p + r(f + r + g^2 r) = 0 \ \Rightarrow \ p = -r(f + r + g^2 r)/2$$

and then we obtain

$$Eq_8 = 2a + (e + fg)r + 2g(1 + g^2)r^2,$$
$$Eq_{10} = \frac{r}{2}\big[2(b + ag) - f(1 + g^2)r - (1 + g^2)^2 r^2\big] \equiv \frac{r}{2}\widetilde{\Psi}(a, b, f, g, r).$$

Since $r \neq 0$ the equation $Eq_{10} = 0$ is equivalent to $\widetilde{\Psi} = 0$.

According to [12, Lemmas 11,12] the equations $Eq_8 = 0$ and $\widetilde{\Psi} = 0$ have a common solution of degree 2 with respect to the parameter r if and only if

$$Res_r^{(0)}\,(Eq_8, \widetilde{\Psi}) = Res_r^{(1)}\,(Eq_8, \widetilde{\Psi}) = 0$$

where $Res_r^{(1)}$ is the subresultant of order one and $Res_r^{(0)}$ is the subresultant of order zero which coincide with the standard resultant (for detailed definition see [12], formula (19)). We calculate

$$Res_r^{(1)}\,(Eq_8, \widetilde{\Psi}) = (1 + g^2)^2(e - fg) \equiv (1 + g^2)^2\widetilde{\mathcal{D}},$$
$$Res_r^{(0)}\,(Eq_8, \widetilde{\Psi}) = 2(1 + g^2)^2\widetilde{\Omega}.$$

We observe that the subresultant of order one $Res_r^{(1)}\,(Eq_8, \widetilde{\Psi})$ vanishes if and only if $\widetilde{\mathcal{D}} = 0$. So we consider two cases: $\widetilde{\mathcal{D}} \neq 0$ and $\widetilde{\mathcal{D}} = 0$.

***1:* The case $\widetilde{\mathcal{D}} \neq 0$.** Then the invariant parabola exists if and only if $\widetilde{\Omega} = 0$ and therefore we have to examine the solutions of the equation $\widetilde{\Omega} = 0$. We calculate

$$\mathrm{Discrim}[\widetilde{\Omega}, a] = (e - fg)^2\big[8b(1 + g^2)(1 + 3g^2) + (f + eg + 2fg^2)^2\big] \equiv \widetilde{\mathcal{D}}^2\widetilde{\mathcal{E}}$$

and hence the equation $\widetilde{\Omega} = 0$ has real solutions in the parameter a if and only if either $\widetilde{\mathcal{D}} = 0$ or $\widetilde{\mathcal{E}} \geq 0$. However since the condition $\widetilde{\mathcal{D}} \neq 0$ holds it remains to examine the condition $\widetilde{\mathcal{E}} \geq 0$.

In this case setting $\widetilde{\mathcal{E}} = w^2 \geq 0$ we calculate

$$b = -\frac{(f + eg + 2fg^2)^2 - w^2}{8(1 + g^2)(1 + 3g^2)} \tag{3.43}$$

and then we obtain

$$\widetilde{\Omega} = \frac{G_+ G-}{8(1 + g^2)^2(1 + 3g^2)^2},$$

where

$$G_\pm = 4a(1 + g^2)(1 + 3g^2)^2 - (f + eg + 2fg^2 + \varepsilon w)(e + 2eg^2 + fg^3 - \varepsilon gw),\ \ \varepsilon = \pm 1.$$

Then the condition $\widetilde{\Omega} = 0$ gives us

$$a = \frac{(f + eg + 2fg^2 + \varepsilon w)(e + 2eg^2 + fg^3 - \varepsilon wg)}{4(1 + g^2)(1 + 3g^2)^2}, \tag{3.44}$$

where $\varepsilon = 1$ if $G_+ = 0$ and $\varepsilon = -1$ if $G_- = 0$. In this case we obtain that the polynomials Eq_8 and $\widetilde{\Psi}$ have the common factor $\zeta = 2(1+g^2)(1+3g^2)r + f + eg + 2fg^2 + \varepsilon w$ which is linear with respect to the parameter r. Setting $\zeta = 0$ we get

$$r = -\frac{f + eg + 2fg^2 + \varepsilon w}{2(1+g^2)(1+3g^2)}$$

and we arrive at the family of systems

$$\begin{aligned} \dot{x} &= \frac{(f + eg + 2fg^2 + \varepsilon w)(e + 2eg^2 + fg^3 - g\varepsilon w)}{4(1+g^2)(1+3g^2)^2} + gx^2 - xy, \\ \dot{y} &= -\frac{(f + eg + 2fg^2)^2 - w^2}{8(1+g^2)(1+3g^2)} + ex + fy + gxy - 2y^2. \end{aligned} \tag{3.45}$$

This family of systems possess the following invariant parabola

$$\begin{aligned} \Phi(x,y) = &\frac{(f - eg + 4fg^2 - \varepsilon w)(f + eg + 2fg^2 + \varepsilon w)}{8(1+g^2)(1+3g^2)^2} - \frac{g(f + eg + 2fg^2 + \varepsilon w)}{2(1+g^2)(1+3g^2)^2}\,x \\ &- \frac{f + eg + 2fg^2 + \varepsilon w}{2(1+g^2)(1+3g^2)}\,y + x^2. \end{aligned} \tag{3.46}$$

We observe that this conic is reducible if and only if $f + eg + 2fg^2 + \varepsilon w = 0$.

Considering (3.43) we get

$$w^2 = 8b(1+g^2)(1+3g^2) + (f + eg + 2fg^2)^2$$

and then from (3.44) we obtain

$$\begin{aligned} a = \frac{1}{4(1+g^2)(1+3g^2)^2}\big[(f + eg + 2fg^2)(e + 2eg^2 + fg^3) + (e - fg)(1+g^2)\varepsilon w - gw^2\big] \;\Rightarrow \\ 8bg(1+3g^2) + 4a(1+3g^2)^2 - (e - fg)(f + eg + 2fg^2) - (e - fg)\varepsilon w = 0. \end{aligned}$$

Since $\widetilde{\mathcal{D}} = (e - fg) \neq 0$ we solve the last equation with respect to εw and we obtain

$$\varepsilon w = \frac{1}{e - fg}\big[4b(1+2g)^2 - 4a(1+g)(1+2g)(1+3g) + (e - fg)(e + f + fg)\big].$$

Then calculations yield

$$r = -\frac{f + eg + 2fg^2 + \varepsilon w}{2(1+g^2)(1+3g^2)} = -\frac{2(a + 2bg + 3ag^2)}{(e - fg)(1+g^2)} = -\frac{2\,\widetilde{\mathcal{G}}}{(e - fg)(1+g^2)} \neq 0.$$

This completes the proof of the statement $(\boldsymbol{E_1})$ of Lemma 3.10.

Next we show that systems (3.45) could be brought via a transformation to the canonical form $(S_{\boldsymbol{\beta}})$. Indeed we could apply to parabola (3.46) the translation

$$x = x_1 + \frac{g(f + eg + 2fg^2 + \varepsilon w)}{4(1+g^2)(1+3g^2)}, \quad y = y_1 + \frac{f(2 + 9g^2 + 6g^4) - (2 + 3g^2)(eg + \varepsilon w)}{8(1+g^2)(1+3g^2)},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 - \dfrac{f + eg + 2fg^2 + \varepsilon w}{2(1+g^2)(1+3g^2)}\,y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.45) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = \frac{f+eg+2fg^2+\varepsilon w}{2(1+g^2)(1+3g^2)}, \quad n = \frac{g(4e-fg+7eg^2+2fg^3)+(4+7g^2)\varepsilon w}{8(1+g^2)(1+3g^2)},$$
$$m = \frac{(2+3g^2)(4e-fg+7eg^2+2fg^3)-3g(2+g^2)\varepsilon w}{16(1+g^2)(1+3g^2)} \quad \Rightarrow$$
$$e = \frac{gk-2g^3k+4m+2gn}{2}, \quad f = \frac{4k+7g^2k-4n}{2}, \quad w = \frac{2n-2gm+3g^2n}{\varepsilon}.$$

Then after an additional rescaling (to force $k=1$) we arrive at the family of systems (S_{β}).

2: The case $\widetilde{\mathcal{D}}=0$. Then $e-fg=0$ and we have $e=fg$. Therefore we obtain:

$$\widetilde{\Omega} = 2(a+2bg+3ag^2)^2$$

and the condition $\widetilde{\Omega}=0$ implies

$$a = -\frac{2bg}{1+3g^2}.$$

Therefore we determine that in this case the polynomials Eq_8 and Eq_{10} have the following common factor

$$\tilde{\phi} = 2b - f(1+3g^2)r - (1+g^2)(1+3g^2)r^2.$$

We observe that $\tilde{\phi}$ is quadratic in r with the discriminant

$$\text{Discrim}[\tilde{\phi}, r] = (1+3g^2)(8b+f^2+8bg^2+3f^2g^2)$$

and clearly the condition $(8b+f^2+8bg^2+3f^2g^2) \geq 0$ must hold. Setting

$$8b+f^2+8bg^2+3f^2g^2 = (1+3g^2)w^2 \geq 0,$$

we obtain

$$b = -\frac{(1+3g^2)(f^2-w^2)}{8(1+g^2)}. \tag{3.47}$$

Then we arrive at the following expressions for the polynomials Eq_8 and Eq_{10}:

$$Eq_8 = \frac{gM_+M_-}{2(1+g^2)}, \quad Eq_{10} = -\frac{rM_+M_-}{8}, \quad M_{\pm} = f+2r+2g^2r \pm w.$$

Therefore the equations $Eq_8 = Eq_{10} = 0$ imply $M_+M_- = 0$.

If $M_+ = 0$ we determine

$$r = -\frac{f+w}{2(1+g^2)} \equiv r^+$$

and we obtain the parabola

$$\Phi_1(x,y) = \frac{(f-w)(f+w)}{8\,(g^2+1)} - \frac{g(f+w)}{2\,(g^2+1)}\,x - \frac{f+w}{2\,(g^2+1)}\,y + x^2. \tag{3.48}$$

In the case $M_- = 0$ we obtain

$$r = -\frac{f-w}{2(1+g^2)} \equiv r^-$$

and we get the parabola

$$\Phi_2(x,y) = \frac{(f-w)(f+w)}{8\,(g^2+1)} - \frac{g(f-w)}{2\,(g^2+1)}\,x - \frac{f-w}{2\,(g^2+1)}\,y + x^2.$$

Both these parabolas are invariant for the following family of systems:

$$\begin{aligned} \dot{x} &= \frac{g(f^2-w^2)}{4(1+g^2)} + gx^2 - xy, \\ \dot{y} &= -\frac{(1+3g^2)(f^2-w^2)}{8(1+g^2)} + fgx + fy + gxy - 2y^2. \end{aligned} \tag{3.49}$$

We observe that both parabolas $\Phi_i(x,y) = 0$ $(i = 1,2)$ exist (i.e. are not reducible) if and only if $r^+r^- \neq 0$ and this is equivalent to

$$(f-w)(f+w) = f^2 - w^2$$

and considering (3.47) this is equivalent to $b \neq 0$.

On the other hand if only one of the factors vanishes we have $b = 0$ and

$$r^+ + r^- = (f-w) + (f+w) = 2f \neq 0$$

i.e. $f \neq 0$. Therefore for $b = 0$ and $f \neq 0$ we could have only one parabola.

We determine that in the case $w = 0$ we obtain $\Phi_1(x,y) = \Phi_2(x,y)$, i.e. the parabolas coalesced when w tends to zero and we obtain a double parabola. On the other hand considering (3.47) for $w = 0$ we obtain to

$$b + \frac{f^2(1+3g^2)}{8(1+g^2)} = \frac{8b(1+g^2) + f^2(1+3g^2)}{8(1+g^2)} = \frac{(1+3g^2)}{608(25+g^2)(1+g^2)}\,\widetilde{\mathcal{F}}$$

and we conclude that these invariant parabolas coalesce if and only if $\widetilde{\mathcal{F}} = 0$. So the statements $(\boldsymbol{E_2})$–$(\boldsymbol{E_4})$ of Lemma 3.10 are valid.

As all the cases are examined we conclude that Lemma 3.10 is proved. □

Next we show that systems (3.49) could be brought via a transformation to the canonical form (S_{β}). Indeed we could apply to parabola (3.48) the translation

$$x = x_1 + \frac{g(f+w)}{4(1+g^2)}, \quad y = y_1 + \frac{f(2+g^2) - (2+3g^2)w}{8(1+g^2)}.$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1) = x_1^2 - \dfrac{f+w}{2(1+g^2)}\,y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.49) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = \frac{f+w}{2(1+g^2)}, \quad n = \frac{3fg^2 + 4w + 7g^2w}{8(1+g^2)} \quad \Rightarrow \quad f = \frac{4k + 7g^2k - 4n}{2}, \quad w = \frac{4n - 3g^2k}{2}.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S_{β}) defined by the conditions $m = 3g(1 + 3g^2 - 2n)/4$.

3.2.2 Invariant conditions: the case $\eta < 0$

Next using Lemma 3.10 we shall construct the equivalent affine invariant conditions for a system with $\eta < 0$ to possess an invariant parabola.

We prove the following theorem.

Theorem 3.11. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta < 0$, $\chi_1 = 0$ and $\zeta_1 \neq 0$ are satisfied. Then this system could possess invariant parabolas only in one (real) direction. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$ and $\bigcup^2$. Moreover this system has one of the above sets of invariant parabolas if and only if $\chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$(\mathcal{E}_1)$ $\zeta_4 \neq 0$, $\mathcal{R}_1 \neq 0$ $\Rightarrow \cup$;
$(\mathcal{E}_2)$ $\zeta_4 = 0$, $\mathcal{R}_7 \neq 0$, $\zeta_5 \neq 0$ $\Rightarrow \Cup$;
$(\mathcal{E}_3)$ $\zeta_4 = 0$, $\mathcal{R}_7 \neq 0$, $\zeta_5 = 0$ $\Rightarrow \bigcup^2$;
$(\mathcal{E}_4)$ $\zeta_4 = 0$, $\mathcal{R}_7 = 0$, $\zeta_5 \neq 0$ $\Rightarrow \cup$.

Proof. According to Remark 3.9 for a system (3.38) to possess an invariant parabola the conditions $\chi_1 = 0$ and $\zeta_1 \neq 0$ are necessary. As it was shown earlier (see page 37) if for a quadratic system with one real and two complex infinite singularities the conditions $\chi_1 = 0$ and $\zeta_1 \neq 0$ are satisfied, then via an affine transformation and time rescaling this system can be brought to the form (3.41). Thus in what follows we consider the family of quadratic systems

$$\dot{x} = a + gx^2 - xy, \quad \dot{y} = b + ex + fy - x^2 + gxy - 2y^2, \tag{3.50}$$

for which considering (3.42) we calculate.

$$\begin{aligned} &\chi_1 = 0, \quad \zeta_1 = 2(25+g^2), \quad \chi_2 = 384(25+g^2)\,\widetilde{\Omega}, \\ &\zeta_4 = -(25+g^2)\,\widetilde{\mathcal{D}}/8, \quad \mathcal{R}_1 = -30(25+g^2)\,\widetilde{\mathcal{G}}. \end{aligned} \tag{3.51}$$

Evidently the condition $\chi_2 = 0$ is equivalent to $\widetilde{\Omega} = 0$ and we consider two cases: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

1: **The case $\zeta_4 \neq 0$.** Then we have $\widetilde{\mathcal{D}} \neq 0$ and according to Lemma 3.10 in this case a quadratic system possesses an invariant parabola if and only if the condition $\widetilde{\mathcal{G}} \neq 0$ holds. According to (3.51) this condition is governed by the invariant polynomial $\mathcal{R}_1$. So we conclude that the statement $(\mathcal{E}_1)$ of Theorem 3.11 is valid.

2: **The case $\zeta_4 = 0$.** This implies $\widetilde{\mathcal{D}} = 0$ and considering (3.42) we get $e = fg$. Then for systems (3.50) we calculate

$$\chi_2 = 768(25+g^2)(a+2bg+3ag^2)^2 = 0 \;\Rightarrow\; a = -\frac{2bg}{1+3g^2}$$

and in this case we obtain:

$$\zeta_5 = \widetilde{\mathcal{F}}/4, \quad \mathcal{R}_7 = -64480b.$$

We examine two possibilities: $\mathcal{R}_7 \neq 0$ and $\mathcal{R}_7 = 0$.

2.1: **The possibility $\mathcal{R}_7 \neq 0$.** In this case we get $b \neq 0$. We observe that the condition $\zeta_5 = 0$ is equivalent to $\widetilde{\mathcal{F}} = 0$ and according to Lemma 3.10 due to $b \neq 0$ we get two invariant parabolas for $\zeta_5 \neq 0$ and one double invariant parabola if $\zeta_5 = 0$.

Thus the statements $(\mathcal{E}_2)$ and $(\mathcal{E}_3)$ of Theorem 3.11 are valid.

***2.2:* The possibility $\mathcal{R}_7 = 0$.** This implies $b = 0$ and for systems (3.50) with $e = fg$ we calculate

$$\chi_2 = 768a^2(25+g^2)(1+3g^2)^2, \quad \zeta_5 = 19(f^2+8ag)(25+g^2).$$

Therefore the condition $\chi_2 = 0$ gives us $a = 0$ and then we obtain $\zeta_5 = 19f^2(25+g^2)$. So the condition $f \neq 0$ is equivalent to $\zeta_5 \neq 0$ and considering the statement $(\boldsymbol{E_4})$ of Lemma 3.10 we conclude that the statement $(\mathcal{E}_4)$ of Theorem 3.11 is valid and this completes the proof of this theorem. □

3.3 Systems with two real distinct infinite singularities

In this case, according to Lemma 2.3, the conditions $\eta = 0$ and $\widetilde{M} \neq 0$ hold and systems (2.5) could be brought via a linear transformation and the additional change $(x, y, a, b, c, d, e, f, g, h) \mapsto (y, x, b, a, f, e, d, c, h, g)$ to the following family of systems

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + (h-1)xy, \\ \frac{dy}{dt} &= b + ex + fy + gxy + hy^2. \end{aligned} \tag{3.52}$$

For these systems we calculate

$$C_2(x,y) = -xy^2, \quad \chi_1 = 2g^2(h-2)$$

and by Lemma 2.6 we conclude that the above systems could have invariant parabolas either of the form $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic) or of the form $\Phi(x,y) = p + qx + ry + y^2$ with $q \neq 0$.

According to Lemma 2.4 for the existence of an invariant parabola for a system (3.52) the condition $\chi_1 = 0$ is necessary, i.e. $g(h-2) = 0$. We prove the following lemma.

Lemma 3.12. *Assume that a system (3.52) possesses an invariant parabola. Then its quadratic homogeneous part is of the form x^2 (respectively, y^2) only if the condition $h = 2$ (respectively, $g = 0$) holds.*

Proof. Assume that a system (3.52) possesses an invariant parabola of the form $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic). Then considering equations (2.6) we obtain

$$s = 1, \quad v = u = 0, \quad Eq_2 = -2 + 2h - V = 0 \ \Rightarrow \ V = 2(h-1).$$

Therefore we have $Eq_7 = -(h-2)r = 0$ and since $r \neq 0$ this implies $h = 2$. So the statement of the lemma is true in this case.

If the system possesses an invariant parabola of the form $\Phi(x,y) = p + qx + ry + y^2$ with $q \neq 0$ then considering equations (2.6) we obtain

$$s = v = 0, \quad u = 1, \quad Eq_3 = 2g - U = 0 \ \Rightarrow \ U = 2g.$$

In this case we obtain $Eq_5 = -gq = 0$ and due to $q \neq 0$ we get $g = 0$. This completes the proof of the lemma. □

Considering Lemma 3.12 we conclude that for determining the conditions for the existence and the number of invariant parabolas for systems (3.52) it is necessary and sufficient to examine the two possibilities: the existence of invariant parabolas of the form $\Phi(x,y) = p+qx+ry+x^2$ $(r \neq 0)$ and of the form $\Phi(x,y) = p+qx+ry+y^2$ $(q \neq 0)$. By Lemma 3.12 in the first case the condition $h=2$ holds whereas in the second we have $g=0$.

Taking into account that for systems (3.52) we have

$$\chi_1 = 2g^2(h-2), \quad \mu_0 = g^2 h$$

we conclude that the case $h-2=0$ is equivalent to $\chi_1 = 0$ and $\mu_0 \neq 0$ whereas the case $g=0$ is equivalent to $\chi_1 = \mu_0 = 0$. In what follows we examine each one of this two possibilities.

3.3.1 The possibility $\chi_1 = 0$ and $\mu_0 \neq 0$

Then we have $g \neq 0$ and $h=2$ and by Lemma 3.12 systems (3.52) could have invariant parabolas only of the form $\Phi(x,y) = p+qx+ry+x^2$ with $r \neq 0$. Applying the transformation $(x,y) \mapsto (x/g-d,\ y-c+2dg)$ we impose the conditions $g=1$ and $c=d=0$ to be fulfilled and we arrive at the family of systems

$$\dot{x} = a + x^2 + xy, \quad \dot{y} = b + ex + fy + xy + 2y^2. \tag{3.53}$$

Coefficient conditions for systems (3.53) to possess invariant parabolas. We prove the following lemma.

Lemma 3.13. *A system (3.53) possesses either one or two invariant parabolas or a double invariant parabola of the form $\Phi(x,y) = p+qx+ry+x^2$ $(r \neq 0)$ if and only if $\Upsilon_1 = 0$ and the corresponding set of conditions are satisfied, respectively:*

(H_1) $\mathfrak{D}_1 \neq 0,\ \mathfrak{G}_1 \neq 0 \Rightarrow \cup$;

(H_2) $\mathfrak{D}_1 = 0,\ a \neq 0,\ \mathfrak{F}_1 \neq 0 \Rightarrow \Cup$;

(H_3) $\mathfrak{D}_1 = 0,\ a \neq 0,\ \mathfrak{F}_1 = 0 \Rightarrow \cup^2$;

(H_4) $\mathfrak{D}_1 = 0,\ a = 0,\ e \neq 0 \Rightarrow \cup$.

where

$$\begin{aligned} &\Upsilon_1 = 8b^2 - b(24a - e^2 + f^2) + a(18a - e^2 + ef + 2f^2); \\ &\mathfrak{D}_1 = e + f; \quad \mathfrak{G}_1 = 3a - 2b; \quad \mathfrak{F}_1 = 4a - e^2. \end{aligned} \tag{3.54}$$

Proof. Considering the equations (2.6) and the form of invariant parabola $\Phi(x,y) = p+qx+ry+x^2$ with $r \neq 0$ for systems (3.53) we obtain

$$\begin{aligned} &s = 1,\ v = u = 0,\ U = 2,\ V = 2,\ W = -q, \\ &Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0. \end{aligned}$$

Then we have

$$Eq_6 = -q - r = 0, \ \ Eq_9 = -2p + fr + qr = 0 \ \Rightarrow \ q = -r,\ p = r(f-r)/2$$

and calculations yield

$$Eq_8 = 2a + (e-f)r + 2r^2, \ \ Eq_{10} = -\frac{r}{2}\big[2(a-b) + fr - r^2)\big] \equiv -\frac{r}{2}\Psi_1(a,b,f,r).$$

Since $r \neq 0$ the equation $Eq_{10} = 0$ is equivalent to $\Psi_1 = 0$.

According to [12, Lemmas 11,12] the equations $Eq_8 = 0$ and $\Psi_1 = 0$ have a common solution of degree 2 with respect to the parameter r if and only if

$$Res_r^{(0)}(Eq_8, \Psi_1) = Res_r^{(1)}(Eq_8, \Psi_1) = 0$$

where $Res_r^{(1)}$ is the subresultant of order one and $Res_r^{(0)}$ is the subresultant of order zero which coincide with the standard resultant (for detailed definition see [12], formula (19)). We calculate

$$Res_r^{(1)}(Eq_8, \Psi_1) = (e+f) \equiv \mathfrak{D}_1, \quad Res_r^{(0)}(Eq_8, \Psi_1) = 2\Upsilon_1.$$

So we consider two possibilities: $\mathfrak{D}_1 \neq 0$ and $\mathfrak{D}_1 = 0$.

1: **The possibility $\mathfrak{D}_1 \neq 0$.** Then the invariant parabola exists if and only if $\Upsilon_1 = 0$ and therefore we have to examine the solutions of the equation $\Upsilon_1 = 0$. In this case we calculate ??

$$\text{Discrim}[\Upsilon_1, b] = -(e+f)^2(16a - e^2 + 2ef - f^2) \equiv -\mathfrak{D}_1^2\mathcal{E}$$

and hence due to $\mathfrak{D}_1 \neq 0$ the equation $\Upsilon_1 = 0$ has real solutions with respect to the parameter b if and only if $\mathcal{E} \leq 0$. Then setting $\mathcal{E} = -w^2 \geq 0$ we calculate

$$a = \frac{(e-f)^2 - w^2}{16} \tag{3.55}$$

and then we obtain:

$$\Upsilon_1 = \frac{N_+N_-}{128},$$

where

$$N_\pm = 32b - e^2 + 6ef - 5f^2 + 2(e+f)\varepsilon w + 3w^2, \quad \varepsilon = \pm 1.$$

Then the condition $\Upsilon_1 = 0$ gives us

$$b = \frac{1}{32}(e - 5f - 3\varepsilon w)(e - f + \varepsilon w) \tag{3.56}$$

where $\varepsilon = 1$ if $N_+ = 0$ and $\varepsilon = -1$ if $N_- = 0$. In this case we obtain that the polynomials Eq_8 and $\Psi_1(e, f, g, r)$ have the common factor $\zeta = e - f + 4r + \varepsilon w$ which is linear with respect to the parameter r. Setting $\zeta = 0$ we get

$$r = -\frac{e - f + \varepsilon w}{4}$$

and we arrive at the family of systems

$$\begin{aligned} \dot{x} &= \frac{(e-f)^2 - w^2}{16} + x^2 + xy, \\ \dot{y} &= \frac{1}{32}(e - 5f - 3\varepsilon w)(e - f + \varepsilon w) + ex + fy + xy + 2y^2. \end{aligned} \tag{3.57}$$

This family of systems possess the following invariant parabola

$$\Phi(x, y) = -\frac{(e - f + \varepsilon w)(e + 3f + \varepsilon w)}{32} + \frac{e - f + \varepsilon w}{4}x - \frac{e - f + \varepsilon w}{4}y + x^2. \tag{3.58}$$

We observe that this conic is reducible if and only if $e - f + \varepsilon w = 0$.

Considering (3.55) we get

$$w^2 = -16a + (e-f)^2$$

and then we obtain

$$b = \frac{1}{32}\big[(e-5f)(e-f) - 2(e+f)\varepsilon w - 3w^2\big] \;\Rightarrow$$
$$16b - 24a + (e+f)(e-f+\varepsilon w) = 0.$$

Since $\mathfrak{D}_1 = e+f \neq 0$ we solve the last equation with respect to εw and we obtain

$$\varepsilon w = \frac{1}{e+f}\big(24a - 16b - e^2 + f^2\big).$$

Then calculations yield

$$r = -\frac{e-f+\varepsilon w}{4} = -\frac{2(3a-2b)}{e+f} = -\frac{2\,\mathfrak{G}_1}{e+f} \neq 0.$$

This completes the proof of the statement $(\boldsymbol{H_1})$ of Lemma 3.13.

Next we show that systems (3.57) could be brought via a transformation to the canonical form $(S^1_{\boldsymbol{\gamma}})$. Indeed we could apply to parabola (3.58) the translation

$$x = x_1 - \frac{e-f+\varepsilon w}{8}, \quad y = y_1 - \frac{3e+5f+3\varepsilon w}{16},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 - \dfrac{e-f+\varepsilon w}{4}\, y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.57) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = \frac{e-f+\varepsilon w}{4}, \quad n = -\frac{7e+f+7\varepsilon w}{16}, \quad m = \frac{13e-5f-3\varepsilon w}{32} \quad \Rightarrow$$
$$e = -k+2m-n, \quad f = -\frac{7k+4n}{2}, \quad w = \frac{3k-4m-2n}{2\varepsilon}.$$

Then after an additional rescaling (to force $k=1$) we arrive at the subfamily of systems $(S^1_{\boldsymbol{\gamma}})$ defined by the conditions $g=1$.

***2:* The possibility** $\mathfrak{D}_1 = 0$**.** Considering (3.54) we have $f = -e$ and then we calculate

$$\Upsilon_1 = 2(3a-2b)^2 = 0 \;\Rightarrow\; b = 3a/2.$$

Therefore we determine that in this case the polynomials Eq_8 and Eq_{10} have the following common factor

$$\phi_1 = a + er + r^2.$$

We observe that $\tilde{\phi}$ is quadratic in r with the discriminant Discrim$[\phi_1, r] = -4a + e^2$ and setting Discrim$[\phi_1, r] = w^2$ we obtain

$$a = \frac{e^2 - w^2}{4}. \tag{3.59}$$

Then we arrive at the following expressions for the polynomials Eq_8 and Eq_{10}:

$$Eq_8 = \frac{S_+S_-}{2}, \quad Eq_{10} = r\frac{S_+S_-}{8}, \quad S_\pm = (e + 2r \pm w).$$

Therefore the equations $Eq_8 = Eq_{10} = 0$ imply $S_+S_- = 0$. If $S_+ = 0$ we determine

$$r = -\frac{e+w}{2} \equiv r^+$$

and we obtain the parabola

$$\Phi_1(x,y) = \frac{e^2-w^2}{8} + \frac{e+w}{2}x - \frac{e+w}{2}y + x^2. \tag{3.60}$$

In the case $S_- = 0$ we obtain

$$r = -\frac{e-w}{2} \equiv r^-$$

and we get the parabola

$$\Phi_2(x,y) = \frac{e^2-w^2}{8} + \frac{e-w}{2}x - \frac{e-w}{2}y + x^2.$$

Both these parabolas are invariant for the following family of systems:

$$\dot{x} = \frac{e^2-w^2}{4} + x^2 + xy, \quad \dot{y} = \frac{3(e^2-w^2)}{8} + ex - ey + xy + 2y^2. \tag{3.61}$$

We observe that both parabolas $\Phi_i(x,y) = 0$ $(i = 1,2)$ exist (i.e. are not reducible) if and only if $r^+r^- \neq 0$ and this is equivalent to

$$(e+w)(e-w) = e^2 - w^2 \neq 0.$$

Considering (3.59) this is equivalent to $a \neq 0$.

On the other hand if only one of the factors vanishes we have $a = 0$ and

$$r^+ + r^- = (e+w) + (e-w) = 2e \neq 0$$

and we obtain that the above condition is equivalent to $e \neq 0$. Therefore for $a = 0$ and $e \neq 0$ we could have only one parabola and we have no parabolas for $a = e = 0$.

We determine that in the case $w = 0$ we obtain $\Phi_1(x,y) = \Phi_2(x,y)$, i.e. the parabolas coalesce when w tends to zero and we obtain a double parabola. On the other hand considering (3.59) for $w = 0$ we obtain

$$a - \frac{e^2-w^2}{4} = \frac{4a-e^2}{4} = \frac{\mathfrak{F}_1}{4}$$

and we conclude that these invariant parabolas coalesce if and only if $\mathfrak{F}_1 = 0$.

Thus we conclude that the statements $(\boldsymbol{H_2})$,$(\boldsymbol{H_3})$ and $(\boldsymbol{H_4})$ of Lemma 3.13 are proved. □

Next we show that systems (3.61) could be brought via a transformation to the canonical form (S^1_γ). Indeed we could apply to parabola (3.60) the translation

$$x = x_1 - (e+w)/4, \quad y = y_1 + (e-3w)/8,$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1) = x_1^2 - \dfrac{e+w}{2}y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.61) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = (e+w)/2, \;\; n = -(3e+7w)/8 \;\; \Rightarrow \;\; e = (7k+4n)/2, \;\; w = -(3k+4n)/2.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S^1_γ) defined by the conditions $g = 1$ and $m = 3(3+2n)/4$.

Invariant conditions: the case $\eta = 0$, $\widetilde{M} \neq 0$ and $\mu_0 \neq 0$. Next using Lemma 3.13 we shall construct the equivalent affine invariant conditions for a system with $\eta = 0$, $\widetilde{M} \neq 0$ and $\mu_0 \neq 0$ to possess an invariant parabola.

We prove the following theorem.

Theorem 3.14. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta = 0$, $\widetilde{M} \neq 0$, $\chi_1 = 0$ and $\mu_0 \neq 0$ are satisfied. Then this system could possess invariant parabolas only in one (simple) direction. More exactly it could only possess one of the following sets of invariant parabolas: ∪, ⩌ and $\bigcup^2$. Moreover this system has one of the above sets of invariant parabolas if and only if $\chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$(\mathcal{H}_1)$ $\zeta_4 \neq 0$, $\mathcal{R}_1 \neq 0$ $\Rightarrow$ ∪;

$(\mathcal{H}_2)$ $\zeta_4 = 0$, $\mathcal{R}_2 \neq 0$, $\zeta_5 \neq 0$ $\Rightarrow$ ⩌;

$(\mathcal{H}_3)$ $\zeta_4 = 0$, $\mathcal{R}_2 \neq 0$, $\zeta_5 = 0$ $\Rightarrow$ $\bigcup^2$;

$(\mathcal{H}_4)$ $\zeta_4 = 0$, $\mathcal{R}_2 = 0$, $\zeta_5 \neq 0$ $\Rightarrow$ ∪.

Proof. Assume that quadratic system the conditions $\eta = 0$ and $\widetilde{M} \neq 0$ are fulfilled. Then via a linear transformation this system can be brought to the canonical form (3.52). According to Lemma 2.4 for a system (3.52) to possess an invariant parabola the conditions $\chi_1 = \chi_2 = 0$ are necessary. Moreover it was shown earlier (see page 44) that a system (3.52) with $\chi_1 = 0$ and $\mu_0 \neq 0$ via an affine transformation ant time rescaling can be brought to the form (3.53). Thus in what follows we consider the family of quadratic systems

$$\dot{x} = a + x^2 + xy, \quad \dot{y} = b + ex + fy + xy + 2y^2. \tag{3.62}$$

Considering (3.54) for these systems we calculate.

$$\chi_1 = 0, \quad \chi_2 = 384\mathrm{Y}_1, \quad \zeta_4 = -\mathfrak{D}_1/8, \quad \mathcal{R}_1 = 30\mathfrak{G}_1. \tag{3.63}$$

Evidently the condition $\chi_2 = 0$ is equivalent to $\mathrm{Y}_1 = 0$ and we consider two cases: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

1: **The case $\zeta_4 \neq 0$.** Then we have $\mathfrak{D}_1 \neq 0$ and according to Lemma 3.13 in this case a quadratic system possesses an invariant parabola if and only if the condition $\mathfrak{G}_1 \neq 0$ holds. According to (3.63) this condition is governed by the invariant polynomial $\mathcal{R}_1$. So we conclude that the statement $(\mathcal{H}_1)$ of Theorem 3.14 is valid.

2: **The case $\zeta_4 = 0$.** This implies $\mathfrak{D}_1 = 0$ and considering (3.54) we get $f = -e$. Then for systems (3.62) we calculate

$$\chi_2 = 768(3a - 2b)^2 = 0 \;\Rightarrow\; b = 3a/2$$

and in this case we obtain:

$$\zeta_5 = -19(4a - e^2) = -19\mathfrak{F}_1, \quad \mathcal{R}_2 = -27a/2.$$

We examine two possibilities: $\mathcal{R}_2 \neq 0$ and $\mathcal{R}_2 = 0$.

2.1: **The subcase $\mathcal{R}_2 \neq 0$.** In this case we get $a \neq 0$. We observe that the condition $\zeta_5 = 0$ is equivalent to $\mathfrak{F}_1 = 0$ and according to Lemma 3.13 due to $b \neq 0$ (because $a \neq 0$) we get two invariant parabolas for $\zeta_5 \neq 0$ and one double invariant parabola if $\zeta_5 = 0$.

Thus the statements $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ of Theorem 3.14 are valid.

2.2: The subcase $\mathcal{R}_2 = 0$. This implies $a = 0$ and for systems (3.62) with $f = -e$ and $a = b = 0$ we calculate

$$\zeta_5 = 19e^2.$$

So the condition $e \neq 0$ is equivalent to $\zeta_5 \neq 0$ and considering Lemma 3.13 we conclude that the statement $(\mathcal{H}_4)$ of Theorem 3.14 is valid. This completes the proof of Theorem 3.14. □

3.3.2 The possibility $\chi_1 = \mu_0 = 0$

In this case considering the conditions $\chi_1 = 2g^2(h-2) = 0$ and $\mu_0 = g^2 h$ for systems (3.52) we obtain $g = 0$ and by Lemma 3.12 these systems could have invariant parabolas of the form $\Phi(x,y) = p + qx + ry + y^2$ with $q \neq 0$ and if in addition $h = 2$ then they could have invariant parabolas of the form $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$.

So we consider the family of systems

$$\dot{x} = a + cx + dy + (h-1)xy, \quad \dot{y} = b + ex + fy + hy^2. \tag{3.64}$$

Coefficient conditions for systems (3.64) to possess invariant parabolas. We prove the following lemma.

Lemma 3.15. *A system (3.64) could only possess one of the following sets of invariant parabolas:* $\cup$, $\overset{2}{\cup}$, $\overset{2}{\cup}\subset$, $\overset{2}{\cup}\overset{2}{\mathbb{U}}$, $\overset{2}{\cup}\overset{2}{\mathbb{U}}^c$, $\overset{2}{\cup}\overset{2}{\bigcup}^2$, $\overset{2}{\bigcup}^3$ *and* $\infty\overset{2}{\cup}$. *Moreover this system has one of the above sets of invariant parabolas if and only if the corresponding set of conditions are satisfied, respectively:*

- $(\boldsymbol{K_1})$ $h+1 \neq 0$, $3h+1 \neq 0$, $h \neq 0$, $h-2 \neq 0$, $\Upsilon_2 = 0$, $e \neq 0 \Rightarrow \overset{2}{\cup}$;
- $(\boldsymbol{K_2})$ $h = 2$, $\Upsilon_3 \neq 0$, $\Upsilon_2 = 0$, $e \neq 0 \Rightarrow \overset{2}{\cup}$;
- $(\boldsymbol{K_3})$ $h = 2$, $\Upsilon_3 = 0$, $\Upsilon_2 \neq 0$, $e(a-cd) \neq 0 \Rightarrow \cup$;
- $(\boldsymbol{K_4})$ $h = 2$, $\Upsilon_3 = 0$, $\Upsilon_2 = 0$, $e(4c-f) \neq 0 \Rightarrow \overset{2}{\cup}\subset$;
- $(\boldsymbol{K_5})$ $h = 0$, $\Upsilon_2 = 0$, $ef \neq 0 \Rightarrow \overset{2}{\cup}$;
- $(\boldsymbol{K_6})$ $h = -1/3$, $c = 2f$, $\mathfrak{D}_2 > 0$, $e \neq 0 \Rightarrow \overset{2}{\cup}\overset{2}{\mathbb{U}}$;
- $(\boldsymbol{K_7})$ $h = -1/3$, $c = 2f$, $\mathfrak{D}_2 < 0$, $e \neq 0 \Rightarrow \overset{2}{\cup}\overset{2}{\mathbb{U}}^c$;
- $(\boldsymbol{K_8})$ $h = -1/3$, $c = 2f$, $\mathfrak{D}_2 = 0$, $\mathfrak{F}_2 \neq 0$, $e \neq 0 \Rightarrow \overset{2}{\cup}\overset{2}{\bigcup}^2$;
- $(\boldsymbol{K_9})$ $h = -1/3$, $c = 2f$, $\mathfrak{D}_2 = 0$, $\mathfrak{F}_2 = 0$, $e \neq 0 \Rightarrow \overset{2}{\bigcup}^3$;
- $(\boldsymbol{K_{10}})$ $h = -1$, $e = 0$, $\mathfrak{G}_2 \neq 0$, $\mathfrak{H}_2 \neq 0$, $c - f \neq 0 \Rightarrow \overset{2}{\cup}$;
- $(\boldsymbol{K_{11}})$ $h = -1$, $e = 0$, $\mathfrak{G}_2 = 0$, $\mathfrak{H}_2 \neq 0$, $c - f = 0 \Rightarrow \infty\overset{2}{\cup}$,

where

$$\begin{aligned}
\Upsilon_2 &= aeh(1+3h)^3 - b(1+h)(1+3h)^2(f+2ch-fh) - (f+ch+fh)\big[2c^2h(1+h) \\
&\quad - cf(1+h)(5h-1) + de - 2f^2 + 6deh + 9deh^2 + 2f^2h^2\big]; \quad \Upsilon_3 = b + 2c^2 - de - cf; \\
\mathfrak{D}_2 &= 256b^3 + 576b^2(de+f^2) + 432b(de+f^2)^2 - 324a^2e^2 - 972ade^2f \\
&\quad + 27(de-2f^2)^2(4de+f^2); \quad \mathfrak{F}_2 = 4b + 3de + 3f^2; \quad \mathfrak{G}_2 = 2a + cd, \quad \mathfrak{H}_2 = 4b - c^2 + 2cf.
\end{aligned} \tag{3.65}$$

Proof. Considering the equations (2.6) and the form of invariant parabola $\Phi(x,y) = p + qx + ry + y^2$ with $q \neq 0$ we obtain

$$\begin{gathered} s = v = 0,\ u = 1,\ U = 0,\ V = 2h,\ W = 2f - hr, \\ Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0,\ Eq_6 = 2e - (h+1)q. \end{gathered} \tag{3.66}$$

So we have to consider two possibilities: $h + 1 \neq 0$ and $h + 1 = 0$.

1: **The possibility** $h + 1 \neq 0$**.** Then equation $Eq_6 = 0$ gives us $q = \dfrac{2e}{1+h}$ and since $q \neq 0$ we get $e \neq 0$. Therefore we calculate:

$$Eq_8 = \frac{e\big[2c - 4f + (1+3h)r\big]}{1+h}$$

and we have to examine two cases: $1 + 3h \neq 0$ and $1 + 3h = 0$.

1.1: **The case** $1 + 3h \neq 0$**.** Then due to $e \neq 0$ the equation $Eq_8 = 0$ implies $r = -\dfrac{2(c-2f)}{1+3h}$ and calculations yield:

$$\begin{aligned} Eq_9 &= \frac{2}{(1+h)(1+3h)^2}\big[de(1+3h)^2 + b(1+h)(1+3h)^2 + (c-2f)(1+h)(f+2ch-fh) \\ &\quad - h(1+h)(1+3h)^2 p\big], \\ Eq_{10} &= \frac{2}{(1+h)(1+3h)}\big[ae(1+3h) - b(c-2f)(1+h) - (1+h)(f+ch+fh)p\big]. \end{aligned} \tag{3.67}$$

We observe that both equations are linear with respect to parameter p and we calculate

$$Res_p(Eq_9, Eq_{10}) = -\frac{4}{(1+h)(1+3h)^3}\,\Upsilon_2 = 0.$$

Considering (3.65) we observe that Υ_2 is linear with respect to the parameter a with the coefficient $eh(1+3h)^3$ where $e(1+3h) \neq 0$. So we consider two subcases: $h \neq 0$ and $h = 0$.

1.1.1: *The subcase* $h \neq 0$. Then the condition $\Upsilon_2 = 0$ gives us

$$\begin{aligned} a = {} & \frac{1}{eh(1+3h)^3}\big[b(1+h)(1+3h)^2(f+2ch-fh) \\ & + (f+ch+fh)(de + cf - 2f^2 + 2c^2h + 6deh - 4cfh \\ & + 2c^2h^2 + 9deh^2 - 5cfh^2 + 2f^2h^2)\big] \equiv a' \end{aligned} \tag{3.68}$$

and then we calculate

$$Eq_9 = \frac{2}{(1+h)(1+3h)^2}\Psi(b,c,d,e,f,h), \quad Eq_{10} = \frac{2(f+ch+fh)}{h(1+h)(1+3h)^3}\Psi(b,c,d,e,f,h),$$

where

$$\Psi = b(1+h)(1+3h)^2 - h(1+h)(1+3h)^2 p + de(1+3h)^2 + (c-2f)(1+h)(f+2ch-fh).$$

Therefore the condition $Eq_9 = Eq_{10} = 0$ implies $\Psi = 0$ and we get

$$p = \frac{2}{h(1+h)(1+3h)^2}\big[b(1+h)(1+3h)^2 + de(1+3h)^2 + (c-2f)(1+h)(f+2ch-fh)\big] \equiv p'. \tag{3.69}$$

Thus we arrive at the family of systems

$$\dot{x} = a' + cx + dy + (h-1)xy, \quad \dot{y} \ = b + ex + fy + hy^2, \ \ e \neq 0, \tag{3.70}$$

where a' is given in (3.68). These systems possess the following invariant parabola:

$$\Phi = p' + \frac{2e}{h+1}x - \frac{2(c-2f)}{3h+1}y + y^2, \quad e \neq 0, \tag{3.71}$$

where p' is given in (3.69).

We recall that according to Lemma 3.12 in the case $h-2=0$ systems (3.64) could possess invariant parabolas of the form $\Phi(x,y) = p + qx + ry + x^2$. So we discuss two cases: $h-2 \neq 0$ and $h-2=0$.

1.1.1.1: *The possibility* $h-2 \neq 0$. Then by Lemma 3.12 systems (3.64) could not possess invariant parabolas in the second direction.

So we proved that in the case $(h+1)(3h+1)h(h-2)e \neq 0$ and $\Upsilon_2 = 0$ systems (3.64) possess an invariant parabola of the form $\Phi(x,y) = p + qx + ry + y^2$.

Thus the proof of the statement $(\boldsymbol{K_1})$ of Lemma 3.15 is completed.

Next we show that systems (3.70) could be brought via a transformation to the canonical form (S^2_{γ}). Indeed we could apply to parabola (3.71) the translation

$$x = x_1 - \frac{1}{2eh(1+3h)^2}\big[de(1+3h)^2 + b(1+h)(1+3h)^2 + (c-2f)(1+h)(f+ch+fh)\big],$$
$$y = y_1 + \frac{c-2f}{1+3h},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1) = y_1^2 + \dfrac{2e}{1+h}x_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.61) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -\frac{2e}{1+h}, \quad m = \frac{f+2ch-fh}{1+3h}, \quad n = \frac{h+1}{4eh(1+3h)^2}\big[de(1+3h)^2 -$$
$$b(h-1)(1+3h)^2 - (c-2f)(h-1)(f+ch+fh)\big] \quad \Rightarrow$$
$$c = \frac{f(h-1)+(1+3h)m}{2h}, \ d = \frac{(h-1)(f^2-m^2) - 4h(bh-b-2hkn)}{2h(1+h)k}, \ e = -\frac{(1+h)k}{2}.$$

Then after an additional rescaling (to force $k=1$) we arrive at the family of systems (S^2_{γ}).

1.1.1.2: *The possibility* $h-2=0$. In this case we examine the conditions for the existence of the invariant parabolas of the form $\Phi(x,y) = p + qx + ry + x^2$ $(r \neq 0)$.

Considering the equations (2.6) for systems (3.64) we obtain

$$s = 1, \ v = u = 0, \ U = 0, \ V = 2, \ W = 2c,$$
$$Eq_1 = Eq_2 = Eq_3 = Eq_4 = Eq_5 = Eq_7 = 0, \ Eq_6 = 2d - q.$$

Therefore the equation $Eq_6 = 0$ gives us $q = 2d$ and then calculations yield:

$$Eq_9 = 2d^2 - 2p - 2cr + fr = 0 \ \Rightarrow \ p = (2d^2 - 2cr + fr)/2.$$

In this case we obtain

$$Eq_8 = 2(a - cd) + er, \quad Eq_{10} = 2d(a - cd) + (b + 2c^2 - cf)r$$

and we claim that the condition $e \neq 0$ must hold in order to have an invariant parabola. Indeed suppose $e = 0$. Then $Eq_8 = 0$ gives us $a = cd$ and therefore due to $r \neq 0$ from $Eq_{10} = 0$ we get $b = c(f - 2c)$ and this leads to the following degenerate systems

$$\dot{x} = (d + x)(c + y), \; \dot{y} = -(2c - f - 2y)(c + y).$$

So $e \neq 0$ and we obtain

$$r = -2(a - cd))/e \neq 0.$$

In this case calculations yield

$$Eq_{10} = -2(a - cd)(b + 2c^2 - de - cf)/e = -2(a - cd)\Upsilon_3/e \tag{3.72}$$

and due to $a - cd \neq 0$ we obtain that $Eq_{10} = 0$ is equivalent to $\Upsilon_3 = 0$. Therefore we discuss two cases: $\Upsilon_3 \neq 0$ and $\Upsilon_3 = 0$.

a) *The case* $\Upsilon_3 \neq 0$. Then systems (3.64) could not possess invariant parabolas of the form $\Phi(x, y) = p + qx + ry + x^2$ ($r \neq 0$). However these systems could have invariant parabolas of the form $\Phi(x, y) = p + qx + ry + y^2$ ($q \neq 0$) and for this it is sufficient to force the conditions $\Upsilon_2 = 0$ and $e \neq 0$. Indeed the condition $h = 2$ implies $(h + 1)(3h + 1)h \neq 0$ and as it was shown above (see p. ***1.1.1:***) in the case $\Upsilon_2 = 0$ and $e \neq 0$ we arrive at the family of systems (3.70) possessing the invariant parabola (3.71) in this particular case with $h = 2$.

Thus we conclude that the statement $(\boldsymbol{K_2})$ of Lemma 3.15 is proved.

b) *The case* $\Upsilon_3 = 0$. Considering (3.65) the condition $\Upsilon_3 = 0$ gives us $b = -2c^2 + de + cf$. Then from (3.72) we get $Eq_{10} = 0$ and we arrive at the following systems

$$\dot{x} = a + cx + dy + xy, \quad \dot{y} = -2c^2 + de + cf + ex + fy + 2y^2, \; e \neq 0, \tag{3.73}$$

possessing the invariant parabola

$$\Phi = \frac{a(2c - f) - d(2c^2 - de - cf)}{e} + 2dx - \frac{2(a - cd)}{e}y + x^2, \quad e(a - cd) \neq 0. \tag{3.74}$$

Next we show that systems (3.73) could be brought via a transformation to the canonical form (S^1_γ). Indeed we could apply to parabola (3.74) the translation

$$x = x_1 - d, \quad y = y_1 + \frac{2c - f}{2}.$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 - \dfrac{2(a - cd)}{e}y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.73) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = \frac{2(a - cd)}{e}, \;\; m = \frac{e}{2}, \;\; n = \frac{4c - f}{2} \quad \Rightarrow$$
$$a = cd + km, \; e = 2m, \; f = 2(2c - n).$$

Then after an additional rescaling (to force $k=1$) we arrive at the subfamily of systems (S^1_{γ}) defined by the condition $g=0$.

Next we examine the possibility of the existence besides the parabola (3.74) another parabola of the form $\Phi(x,y)=p+qx+ry+y^2$ $(q\neq 0)$. As it was mentioned earlier the condition $h=2$ implies $(h+1)(3h+1)h\neq 0$ and according to the statement $(\boldsymbol{K_1})$ of Lemma 3.15 to have such a parabola the condition $\Upsilon_2=0$ is necessary. So we examine two possibilities: $\Upsilon_2\neq 0$ and $\Upsilon_2=0$.

b.1) *The possibility* $\Upsilon_2\neq 0$. In this case by the statement $(\boldsymbol{K_1})$ we could not have parabola of the form $\Phi(x,y)=p+qx+ry+y^2$ $(q\neq 0)$ and hence in the case under consideration we have a single parabola (3.74).

Thus we deduce that the conditions provided by the statement $(\boldsymbol{K_3})$ of Lemma 3.15 are valid.

b.2) *The possibility* $\Upsilon_2=0$. As it was shown above (see p. ***b)***) for $h=2$ and $\Upsilon_3=0$ systems (3.64) can be brought to the form (3.73). For these systems we calculate

$$\Upsilon_2=2(576c^3+343ae-343cde-432c^2f+108cf^2-9f^3).$$

So due to $e\neq 0$ we obtain

$$a=\frac{1}{343e}\big[c(343de-576c^2)+9f(48c^2-12cf+f^2)\big]$$

and we arrive at the family of systems

$$\begin{aligned}\dot{x}&=\frac{1}{343e}\big[c(343de-576c^2)+9f(48c^2-12cf+f^2)\big]+cx+dy+xy,\\ \dot{y}&=-2c^2+de+cf+ex+fy+2y^2,\quad e\neq 0,\end{aligned}\tag{3.75}$$

possessing the following two invariant parabolas:

$$\begin{aligned}\Phi_1&=d^2-\frac{9(2c-f)(4c-f)^3}{343e^2}+2dx+\frac{18(4c-f)^3}{343e^2}\,y+x^2,\quad e(4c-f)\neq 0;\\ \Phi_2&=\frac{1}{147}(60cf-141c^2+98de+3f^2)+\frac{2e}{3}\,x-\frac{2(c-2f)}{7}\,y+y^2,\quad e\neq 0.\end{aligned}\tag{3.76}$$

So the conditions provided by the statement $(\boldsymbol{K_4})$ of Lemma 3.15 are valid.

Next we show that systems (3.75) could be brought via a transformation to the canonical form (S^2_{γ}). Indeed we could apply to parabola $\Phi_2=0$ from (3.76) the translation

$$x=x_1+\frac{9(4c-f)^2-98de}{98e},\quad y=y_1+\frac{c-2f}{7}$$

which brings this parabola to the form $\widetilde{\Phi}(x_1,y_1)=y_1^2+\dfrac{2e}{3}\,x_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.75) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k=-\frac{2e}{3},\quad m=\frac{4c-f}{7}\quad\Rightarrow\quad c=\frac{f+7m}{4},\quad e=-\frac{3k}{2}.$$

Then after an additional rescaling (to force $k=1$) we arrive at the the subfamily of systems (S^2_{γ}) defined by the conditions $h=2$ and $n=-3m^2/2$.

1.1.2: *The subcase* $h = 0$. In this case considering (3.67) we obtain

$$Eq_9 = 2(b + de + cf - 2f^2), \quad Eq_{10} = 2(ae - bc + 2bf - fp)$$

and clearly the condition $Eq_9 = 0$ gives us $b = -de - cf + 2f^2$. We observe that the equation $Eq_{10} = 0$ is linear with respect to the parameter p with the coefficient f. So we discuss two possibilities: $f \neq 0$ and $f = 0$.

1.1.2.1: *The possibility* $f \neq 0$. Then the condition $Eq_{10} = 0$ implies

$$p = \left[ae + (c - 2f)(de + cf - 2f^2)\right]/f$$

and we arrive at the family of systems

$$\dot{x} = a + cx + dy - xy, \quad \dot{y} = -de - cf + 2f^2 + ex + fy, \;\; ef \neq 0, \tag{3.77}$$

possessing the following invariant parabola:

$$\Phi = \frac{1}{f}\left[ae + (c - 2f)(de + cf - 2f^2)\right] + 2ex - 2(c - 2f)y + y^2, \quad ef \neq 0. \tag{3.78}$$

On the other hand for $h = 0$ we have

$$\Upsilon_2 = -f(b + de + cf - 2f^2)$$

and since $f \neq 0$ we conclude that the condition $\Upsilon_2 = 0$ is equivalent to $b + de + cf - 2f^2 = 0$.

1.1.2.2: *The possibility* $f = 0$. Then considering (3.67) for $h = f = 0$ we obtain

$$Eq_9 = 2(b + de) = 0, \quad Eq_{10} = 2(ae - bc) = 0$$

and due to $e \neq 0$ (since $q \neq 0$) this implies $b = -de$ and $a = -cd$. However in this case we get the degenerate systems

$$\dot{x} = -(d - x)(c - y), \quad \dot{y} = -e(d - x).$$

Thus we have proved that for the existence of invariant parabola of systems (3.64) with $h = 0$ the conditions $\Upsilon_2 = 0$ and $ef \neq 0$ must hold and we deduce that the conditions provided by the statement $(\boldsymbol{K_5})$ of Lemma 3.15 are valid.

Next we show that systems (3.77) could be brought via a transformation to the canonical form (S^2_γ). Indeed we could apply to parabola (3.78) the translation

$$x = x_1 - \frac{a + cd - 2df}{2f}, \quad y = y_1 + c - 2f,$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = y_1^2 + 2e\,x_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.77) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -2e, \;\; m = f, \;\; n = \frac{a + cd}{4f} \quad \Rightarrow$$
$$a = -cd + 4mn, \;\; e = -k/2, \;\; f = m.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the the subfamily of systems (S^2_γ) defined by the conditions $h = 0$.

1.2: **The case** $1 + 3h = 0$**.** Then $h = -1/3$ and considering (2.6) and (3.66) we calculate $Eq_6 = 2/3(3e - q) = 0$ which implies $q = 3e \neq 0$. Therefore calculations yield:

$$Eq_8 = 3e(c - 2f), \quad Eq_9 = (6b + 9de + 2p - 3fr - r^2)/3$$

and since $e \neq 0$ the equation $Eq_8 = 0$ implies $c = 2f$ and from $Eq_9 = 0$ we obtain

$$p = \frac{1}{2}\big[r^2 + 3fr - 3(2b + 3de)\big].$$

Then we obtain

$$Eq_{10} = -\frac{1}{6}\big[r^3 + 9fr^2 - 3(4b + 3de - 6f^2)r - 18(ae + 2bf + 3def)\big] \equiv -\frac{1}{6}\Psi(r)$$

and we conclude that if r_0 is a solution of the equation $Eq_{10} = 0$ (i.e. $\Psi(r_0) = 0$) then systems

$$\dot{x} = a + 2fx + dy - 4xy/3, \quad \dot{y} = b + ex + fy - y^2/3, \quad e \neq 0 \tag{3.79}$$

possess the invariant parabola

$$\Phi_0(x, y) = (r_0^2 + 3fr_0 - 6b - 9de)/2 + 3ex + r_0 y + y^2. \tag{3.80}$$

On the other hand we calculate

$$\Psi'_r = 3(r^2 + 6fr - 4b - 3de + 6f^2), \quad \Psi''_r = 6(3f + r),$$
$$\text{Discrim}[\Psi, r] = 27\mathcal{D}_2, \quad \text{Res}_r(\Psi'_r, \Psi''_r) = -108\mathcal{F}_2,$$

and we conclude that systems (3.79) has the following invariant parabolas of the form (3.80):

- if $\mathcal{D}_2 > 0 \Rightarrow$ three real distinct invariant parabolas;
- if $\mathcal{D}_2 < 0 \Rightarrow$ one real and two complex invariant parabolas;
- if $\mathcal{D}_2 = 0$, $\mathcal{F}_2 \neq 0 \Rightarrow$ one simple and one double real invariant parabolas;
- if $\mathcal{D}_2 = 0$, $\mathcal{F}_2 = 0 \Rightarrow$ one triple real invariant parabolas.

So we conclude that the conditions provided by the statements $(\boldsymbol{K_6})$–$(\boldsymbol{K_9})$ of Lemma 3.15 are valid.

Next we show that systems (3.79) could be brought via a real transformation to the canonical form (S^2_γ). Indeed we could apply to parabola (3.80) with $r_0 \in \mathbb{R}$ the translation

$$x = x_1 + \frac{12b + 18de - 6fr_0 - r_0^2}{12e}, \quad y = y_1 - r_0/2,$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = y_1^2 + 3e\,x_1$.

On the other hand applying the same translation to systems (3.79) we arrive at the systems

$$\begin{aligned}
\dot{x}_1 &= -\frac{1}{8e}\Psi(r_0) + \frac{2}{3}(3f + r_0)x_1 - \frac{1}{9e}(12b + 9de - 6fr_0 - r_0^2)y_1 - \frac{4}{3}x_1y_1,\\
\dot{y}_1 &= \frac{1}{6}(12b + 9de - 6fr_0 - r_0^2) + ex_1 + \frac{1}{3}(3f + r_0)y_1 - \frac{1}{3}y_1^2.
\end{aligned}$$

We recall that $\Psi(r_0) = 0$ and we set the following new notations (suggested by the above parabola and the linear parts of the above systems):

$$k = -3e, \quad m = \frac{3f + r_0}{3}, \quad n = -\frac{12b + 9de - 6fr_0 - r_0^2}{18e} \quad \Rightarrow$$
$$b = \frac{3dk + 6kn + 6mr_0 - r_0^2}{12}, \; e = -\frac{k}{3}, \; f = \frac{3m - r_0}{3}.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S_{γ}^2) defined by the conditions $h = -1/3$.

***2:* The possibility $h + 1 = 0$.** Then $h = -1$ and considering (3.66) for $h = -1$ we have $Eq_6 = 2e = 0$, i.e. $e = 0$. Then taking into account (2.6) calculations yield:

$$Eq_8 = q(c - 2f - r), \quad Eq_9 = 2b + 2p + dq - fr - r^2$$

and since $q \neq 0$ the condition $Eq_8 = 0$ implies $c - 2f - r = 0$ and this gives us $r = c - 2f$. Then from $Eq_9 = 0$ we obtain $p = (c^2 - 2b - 3cf + 2f^2 - dq)/2$ and we obtain

$$Eq_{10} = \frac{1}{2}\big[(c - f)(4b - c^2 + 2cf) + (2a + cd)q\big]. \tag{3.81}$$

We observe that for systems (3.64) we have $\mathcal{G}_2 = 2a + cd$ and therefore we have to consider two cases: $\mathcal{G}_2 \neq 0$ and $\mathcal{G}_2 = 0$.

***2.1:* The case $\mathcal{G}_2 \neq 0$.** Then $2a + cd \neq 0$ and the equation $Eq_{10} = 0$ implies

$$q = -\frac{(c - f)(4b - c^2 + 2cf)}{2a + cd}$$

and we obtain the parabola

$$\begin{gathered}\Phi = \frac{-2ab + ac^2 - 3acf + 2af^2 + bcd - 2bdf}{2a + cd} - \frac{(c - f)(4b - c^2 + 2cf)}{2a + cd}x + (c - 2f)y + y^2, \\ (2a + cd)(c - f)\left(4b - c^2 + 2cf\right) \neq 0 \Leftrightarrow \mathfrak{G}_2\mathfrak{H}_2(c - f) \neq 0,\end{gathered} \tag{3.82}$$

which is invariant for the family of systems

$$\dot{x} = a + cx + dy - 2xy, \quad \dot{y} = b + fy - y^2. \tag{3.83}$$

So we conclude that the conditions provided by the statement $(\boldsymbol{K_{10}})$ of Lemma 3.15 are valid.

Next we show that systems (3.83) could be brought via a transformation to the canonical form (S_{γ}^2). Indeed we could apply to parabola (3.82) the translation

$$x = x_1 - \frac{2a - cd + 2df}{4(c - f)}, \quad y = y_1 - \frac{c - 2f}{2},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = y_1^2 - \dfrac{(c - f)(4b - c^2 + 2cf)}{2a + cd}x_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.83) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = \frac{(c - f)(4b - c^2 + 2cf)}{2a + cd}, \quad m = c - f, \quad n = \frac{2a + cd}{4(c - f)} \quad \Rightarrow$$
$$a = (4mn - cd)/2, \quad b = (2cm - c^2 + 4kn)/4, \; f = c - m.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the the subfamily of systems (S_{γ}^2) defined by the conditions $h = -1$.

2.2: **The case $\mathcal{G}_2 = 0$.** In this case $2a + cd = 0$, i.e. $a = -cd/2$ and considering (3.81) the equation $Eq_{10} = 0$ yields

$$(c - f)(4b - c^2 + 2cf) = 0 \quad \Rightarrow \quad (c - f)\mathfrak{H}_2 = 0.$$

If $\mathfrak{H}_2 = 0$ then we get $b = c(c - 2f)/4$ and this leads to degenerate systems:

$$\dot{x} = -(d - 2x)(c - 2y)/2, \quad \dot{y} = (c - 2y)(c - 2f + 2y)/4.$$

So the condition $\mathfrak{H}_2 \neq 0$ is necessary and then we have $c - f = 0$. So we get $f = c$ which leads to the family of systems

$$\dot{x} = -cd/2 + cx + dy - 2xy, \quad \dot{y} = b + cy - y^2 \tag{3.84}$$

possessing the following family of invariant parabolas:

$$\Phi = -(2b + dq)/2 + qx - cy + y^2, \quad q \in \mathbb{R}, \quad q \neq 0. \tag{3.85}$$

Next we show that systems (3.84) could be brought via a transformation to the canonical form (S^2_γ). Indeed we could apply to parabola (3.85) the translation

$$x = x_1 + \frac{4b + c^2 + 2dq}{4q}, \quad y = y_1 + \frac{c}{2},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = y_1^2 + q\,x_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.84) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -q, \quad m = c - f, \quad n = -\frac{4b + c^2}{4q} \quad \Rightarrow$$
$$b = (4kn - c^2)/4, \quad q = -k \neq 0.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S^2_γ) defined by the conditions $h = -1$ and $m = 0$. Moreover we observe that this subfamily of systems possess the following family of invariant parabolas:

$$\Phi = -n(1 + q) + qx + y^2, \quad q \in \mathbb{R}, \quad q \neq 0.$$

Evidently for $q = -1$ we get the parabola $y^2 - x = 0$.

Thus the the condition provided by the statement $(\boldsymbol{K_{11}})$ of Lemma 3.15 are valid and this completes the proof of the Lemma 3.15. □

Invariant conditions: the case $\eta = 0$, $\widetilde{M} \neq 0$ and $\mu_0 = 0$. Next we consider the class of quadratic systems for which the conditions $\eta = 0$, $\widetilde{M} \neq 0$, $\mu_0 = 0$, which by Lemma 3.12 could possess invariant parabolas in two directions.

We prove the following theorem.

Theorem 3.16. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta = 0$, $\widetilde{M} \neq 0$, $\chi_1 = \mu_0 = 0$ are satisfied. Then this system could possess invariant parabolas in two directions. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\overset{2}{\cup}$, $\overset{2}{\cup}\subset$, $\overset{2}{\cup}\overset{2}{\mathbb{U}}$, $\overset{2}{\cup}\overset{2}{\mathbb{U}}^c$, $\overset{2}{\cup}\overset{2}{\boldsymbol{\cup}}{}^2$, $\overset{2}{\boldsymbol{\cup}}{}^3$ and $\infty\overset{2}{\cup}$. Moreover this system has one of the above sets of invariant parabolas if and only if one of the following sets of conditions are satisfied, correspondingly:*

$$(\mathcal{K}_1)\ \zeta_{11}\zeta_{12}\zeta_{13}\zeta_{14}\neq 0,\ \mathcal{R}_5\neq 0,\ \begin{cases}\zeta_{15}\neq 0, \zeta_{16}=0\\ or\ \zeta_{15}=\zeta_{17}=0,\end{cases} \Rightarrow \overset{2}{\cup};$$

$$(\mathcal{K}_2)\ \zeta_{11}=0,\ \mathcal{R}_3\neq 0,\ \zeta_{16}=0,\ \mathcal{R}_5\neq 0 \Rightarrow \overset{2}{\cup};$$

$$(\mathcal{K}_3)\ \zeta_{11}=0,\ \mathcal{R}_3=0,\ \zeta_{16}\neq 0,\ \mathcal{R}_6\neq 0 \Rightarrow \cup;$$

$$(\mathcal{K}_4)\ \zeta_{11}=0,\ \mathcal{R}_3=0,\ \zeta_{16}=0,\ \zeta_8\neq 0 \Rightarrow \overset{2}{\cup}\subset;$$

$$(\mathcal{K}_5)\ \zeta_{12}=0,\ \zeta_{16}=0,\ \zeta_8\neq 0 \Rightarrow \overset{2}{\cup};$$

$$(\mathcal{K}_6)\ \zeta_{13}=0,\ \zeta_8=0,\ \zeta_{18}>0,\ \mathcal{R}_5\neq 0 \Rightarrow \overset{2}{\cup}\overset{2}{\Cup};$$

$$(\mathcal{K}_7)\ \zeta_{13}=0,\ \zeta_8=0,\ \zeta_{18}<0,\ \mathcal{R}_5\neq 0 \Rightarrow \overset{2}{\cup}\overset{2}{\Cup}^c;$$

$$(\mathcal{K}_8)\ \zeta_{13}=0,\ \zeta_8=0,\ \zeta_{18}=0,\ \mathcal{R}_5\neq 0,\ \chi_3\neq 0 \Rightarrow \overset{2}{\cup}\overset{2}{\cup}^2;$$

$$(\mathcal{K}_9)\ \zeta_{13}=0,\ \zeta_8=0,\ \zeta_{18}=0,\ \mathcal{R}_5\neq 0, \chi_3=0 \Rightarrow \overset{2}{\cup}^3;$$

$$(\mathcal{K}_{10})\ \zeta_{14}=0,\ \zeta_{19}\neq 0,\ \zeta_{20}\neq 0,\ \mathcal{R}_5=0,\ \zeta_{21}\neq 0 \Rightarrow \overset{2}{\cup};$$

$$(\mathcal{K}_{11})\ \zeta_{14}=0,\ \zeta_{19}=0,\ \zeta_{20}\neq 0,\ \mathcal{R}_5=0,\ \zeta_{21}=0 \Rightarrow \infty\overset{2}{\cup}.$$

Proof. Assume that quadratic system the conditions $\eta=0$ and $\widetilde{M}\neq 0$ are fulfilled. Then via a linear transformation this system can be brought to the canonical form (3.52). According to Lemma 2.4 for a system (3.52) to possess an invariant parabola the conditions $\chi_1=\chi_2=0$ are necessary. Moreover it was shown earlier (see page 49) that a system (3.52) with $\chi_1=\mu_0=0$ via an affine transformation and time rescaling can be brought to the form (3.64). Thus in what follows we consider the family of quadratic systems

$$\dot{x}=a+cx+dy+(h-1)xy,\quad \dot{y}=b+ex+fy+hy^2. \tag{3.86}$$

***1:* The statement $(\mathcal{K}_1)$.** Considering (3.65) for these systems we calculate

$$\begin{aligned}\chi_1=\chi_2=0,\ \ \zeta_{11}=-4(h-2)y^2,\ \ \zeta_{12}=4hy^2,\ \ \zeta_{13}=(1+3h)y^2,\ \ \zeta_{14}=(1+h)^2y^2,\\ \zeta_{15}=(h-1)^2y^2/4,\ \ \zeta_{16}=45e^3(h-1)^2\Upsilon_2/8,\ \ \mathcal{R}_5=32e.\end{aligned} \tag{3.87}$$

The condition $\zeta_{11}\neq 0$ implies $h-2\neq 0$ and according to Lemma 3.12 systems (3.64) could not possess invariant parabolas of the form $\Phi(x,y)=p+qx+ry+x^2$.

On the other hand evidently the condition $\zeta_{12}\zeta_{13}\zeta_{14}\neq 0$ is equivalent to $(1+h)(1+3h)h\neq 0$ and the condition $\mathcal{R}_5\neq 0$ is equivalent to $e\neq 0$. So considering the statement $(\boldsymbol{K_1})$ of Lemma 3.15 it remains to determine in invariant form the condition which is equivalent to $\Upsilon_2=0$. We consider two cases: $\zeta_{15}\neq 0$ and $\zeta_{15}=0$.

***1.1:* The case $\zeta_{15}\neq 0$.** Then $h-1\neq 0$ and due to $\mathcal{R}_5\neq 0$ (i.e. $e\neq 0$) we conclude that the condition $\Upsilon_2=0$ is equivalent to $\zeta_{16}=0$ and hence the statement $(\mathcal{K}_1)$ of Theorem 3.16 is valid in this case.

***1.2:* The case $\zeta_{15}=0$.** Then $h=1$ and we obtain

$$\Upsilon_2=-4(16bc+c^3-16ae+4cde+8def-4cf^2),\quad \zeta_{17}=13824e^2\Upsilon_2.$$

Therefore due to $e\neq 0$ we conclude that the condition $\Upsilon_2=0$ is equivalent to $\zeta_{17}=0$ and this completes the proof of the statement $(\mathcal{K}_1)$ of Theorem 3.16.

2: **The statements $(\mathcal{K}_2)$–$(\mathcal{K}_4)$.** For systems (3.86) the condition $\zeta_{11} = 0$ gives us $h = 2$. In this case we calculate

$$\mathcal{R}_3 = -503139971565000\, e^4(b + 2c^2 - de - cf) = -503139971565000\, e^4 Y_3,$$
$$\mathcal{R}_5 = 32e, \ \ \mathcal{R}_6 = 8925(a - cd)e^4/4, \ \ \zeta_{16} = 45e^3 Y_2/8, \ \ \zeta_8 = e(4c - f).$$

Therefore the condition $\mathcal{R}_5 \neq 0$ is equivalent to $e \neq 0$ and for $e \neq 0$ the condition $\mathcal{R}_3 = 0$ (respectively $\zeta_{16} = 0$) is equivalent with $Y_3 = 0$ (respectively $Y_2 = 0$). So in the case $\mathcal{R}_3 \neq 0$ we deduce that the conditions of the statements $(\mathcal{K}_2)$ provides the conditions of the statements$(\boldsymbol{K_2})$ of Lemma 3.15.

Assume now $\mathcal{R}_3 = 0$. We observe that the condition $\mathcal{R}_6 \neq 0$ (respectively $\zeta_8 \neq 0$) implies $e(a - cd) \neq 0$ (respectively $e(4c - f) \neq 0$), i.e. in both cases we have $e \neq 0$. Then the condition $\mathcal{R}_3 = 0$ is equivalent to Y_3 and the condition $\zeta_{16} = 0$ is equivalent to $Y_2 = 0$.

Thus considering Lemma 3.15 we conclude that in the case $\zeta_{16} \neq 0$ (respectively $\zeta_{16} = 0$) we get the conditions provided by the statement $(\boldsymbol{K_3})$ (respectively $(\boldsymbol{K_4})$) of this lemma. So the invariant conditions provided by the statements statements $(\mathcal{K}_2)$–$(\mathcal{K}_4)$ of Theorem 3.16 are valid.

3: **The statement $(\mathcal{K}_5)$.** Considering (3.87) the condition $\zeta_{12} = 0$ implies $h = 0$ and we calculate:

$$\zeta_{16} = 45e^3 Y_2/8, \ \ \zeta_8 = ef.$$

Clearly the condition $\zeta_8 \neq 0$ implies $ef \neq 0$ and then the condition $\zeta_{16} = 0$ is equivalent to $Y_2 = 0$. So we get the conditions provided by the statement $(\boldsymbol{K_5})$ of Lemma 3.15 and this implies the validity of the statement $(\mathcal{K}_5)$ of Theorem 3.16.

4: **The statements $(\mathcal{K}_6)$–$(\mathcal{K}_9)$.** Considering (3.87) we observe that the condition $\zeta_{13} = 0$ implies $h = -1/3$ and we calculate:

$$\mathcal{R}_5 = 32e, \ \ \zeta_8 = -2e(c - 2f)/3.$$

So the condition $\mathcal{R}_5 \neq 0$ implies $e \neq 0$ and then the condition $\zeta_8 = 0$ yields $c = 2f$. In this case we have

$$\zeta_{18} = 64/2187\, e^2\, \mathfrak{D}_2, \ \ \chi_3 = -\frac{15792269387776}{729}\, e^4\, \mathfrak{F}_2^2$$

and hence the condition $\zeta_{18} = 0$ is equivalent to $\mathfrak{D}_2 = 0$ and for $\zeta_{18} \neq 0$ we have $\mathrm{sign}(\zeta_{18}) = \mathrm{sign}(\mathfrak{D}_2)$. Moreover the condition $\chi_3 = 0$ is equivalent to $\mathfrak{F}_2 = 0$.

Thus we obtain that in the case $\mathcal{R}_5 \neq 0$, $\zeta_8 = 0$ and $\zeta_{18} > 0$ (respectively $\zeta_{18} < 0$) then we arrive at the conditions provided by the statement $(\mathcal{K}_6)$ (respectively $(\mathcal{K}_7)$) of Lemma 3.15.

In the case $\zeta_{18} = 0$ (i.e. $\mathfrak{D}_2 = 0$) we obtain the conditions provided by the statement $(\mathcal{K}_8)$ if $\chi_3 \neq 0$ and by the statement $(\mathcal{K}_9)$ if $\chi_3 = 0$ (i.e. $\mathfrak{F}_2 = 0$). This proves the validity of the statements $(\mathcal{K}_6)$–$(\mathcal{K}_9)$ of Theorem 3.16.

5: **The statements $(\mathcal{K}_{10})$, $(\mathcal{K}_{11})$.** From (3.87) we obtain that the condition $\zeta_{14} = 0$ implies $h = -1$ and then we have $\mathcal{R}_5 = 32e$. So the condition $\mathcal{R}_5 = 0$ implies $e = 0$ and we calculate

$$\zeta_{19} = 6(2a + cd)y^4 = 6\mathfrak{G}_2, \ \ \zeta_{20} = 8(4b - c^2 + 2cf)y^2 = 8\mathfrak{H}_2 y^2, \ \ \zeta_{21} = 2(c - f)y^3.$$

So we observe that for $\zeta_{19} \neq 0$, $\zeta_{20} \neq 0$ and $\zeta_{21} \neq 0$ we arrive at the conditions provided by the statement $(\boldsymbol{K_{10}})$ of Lemma 3.15. In the case $\zeta_{19} = 0$, $\zeta_{20} \neq 0$ and $\zeta_{21} = 0$ we get the conditions provided by the statement $(\boldsymbol{K_{11}})$ of the same lemma.

Thus we conclude that the statements $(\mathcal{K}_{10})$ and $(\mathcal{K}_{11})$ of Theorem 3.16 are valid and this completes the proof of this theorem. □

3.4 Systems with a unique infinite singular point which is real

In this case according to Lemma 2.3 systems (2.5) via a linear transformation could be brought to the following family of systems

$$\frac{dx}{dt} = a + cx + dy + gx^2 + hxy, \quad \frac{dy}{dt} = b + ex + fy - x^2 + gxy + hy^2. \tag{3.88}$$

For these systems we calculate

$$C_2(x,y) = x^3, \ \ \chi_1 = -2h^3$$

and by Lemma 2.6 we conclude that the above systems could have invariant parabolas only of the form $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic).

According to Lemma 2.4 for a system (3.88) to possess an invariant parabola the condition $\chi_1 = 0$ is necessary and this implies $h = 0$. Moreover we may assume $e = 0$ due to the translation $x \to x + e/2$, $y \to y$ and we arrive at the family of systems

$$\frac{dx}{dt} = a + cx + dy + gx^2, \quad \frac{dy}{dt} = b + fy - x^2 + gxy. \tag{3.89}$$

3.4.1 Coefficient conditions for systems (3.89) to possess invariant parabolas.

We prove the following lemma.

Lemma 3.17. *A system* (3.89) *could only possess one of the following sets of invariant parabolas:* $\overset{3}{\cup}$ *and* $\infty\overset{3}{\cup}$. *Moreover this system has one of the above sets of invariant parabolas if and only if the corresponding set of conditions are satisfied, respectively:*

$(\boldsymbol{L_1})$ $g \neq 0$, $\Upsilon_4 = 0$, $d \neq 0 \Rightarrow \overset{3}{\cup}$;

$(\boldsymbol{L_2})$ $g = 0$, $d = 0$, $c - f \neq 0$, $f(2c - f) \neq 0 \ \Rightarrow \overset{3}{\cup}$;

$(\boldsymbol{L_3})$ $g = 0$, $d = 0$, $f = c \neq 0 \ \Rightarrow \infty\overset{3}{\cup}$,

where

$$\Upsilon_4 = 27bdg^4 - 9ag^3(d - cg + 2fg) - (2d + cg - 2fg)(d - cg - fg)(2d - 2cg + fg). \tag{3.90}$$

Proof. Considering equations (2.6) and the form of the parabola $\Phi(x,y) = p + qr + ry + x^2$ with $r \neq 0$ (otherwise we get a reducible conic), for systems (3.89) we obtain

$$s = 1, \ v = u = 0, \ U = 2g, \ V = 0, \ W = 2c - gq - r, \ Eq_6 = 2d - gr$$

and clearly we have to discuss two possibilities: $g \neq 0$ and $g = 0$.

***1:* The possibility** $g \neq 0$. Then the equation $Eq_6 = 0$ yields $r = 2d/g \neq 0$ and we calculate:

$$Eq_8 = 2(a - gp) + q(2d - cg)/g + gq^2 = 0 \ \Rightarrow \ p = \frac{a}{g} - \frac{q(cg - 2d)}{2g^2} + \frac{q^2}{2}.$$

Then we obtain

$$Eq_9 = d(4d - 4cg + 2fg + 3g^2q)/g^2$$

and since $dg \neq 0$ the equation $Eq_8 = 0$ gives us

$$q = 2(2cg - 2d - fg)/(3g^2)$$

and finally we calculate the last equation $Eq_{10} = 0$:

$$\begin{aligned} Eq_{10} &= \frac{2}{27g^5}\big[27bdg^4 - 9ag^3(d - cg + 2fg) - (2d + cg - 2fg)(d - cg - fg)(2d - 2cg + fg)\big] \\ &= \frac{2}{27g^5}\Upsilon_4. \end{aligned}$$

Since $dg \neq 0$ the equation $Eq_{10} = 0$ gives us

$$b = \frac{1}{27dg^4}\big[9ag^3(d - cg + 2fg) + (2d + cg - 2fg)(d - cg - fg)(2d - 2cg + fg)\big] \equiv b_0$$

and we arrive at the family of systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b_0 + fy - x^2 + gxy \tag{3.91}$$

possessing the invariant parabola

$$\Phi(x,y) = \frac{9ag^3 - (2d + cg - 2fg)(2d - 2cg + fg)}{9g^4} - \frac{2(2d - 2cg + fg)}{3g^2}x + \frac{2d}{g}y + x^2. \tag{3.92}$$

This completes the proof of the statement $(\boldsymbol{L_1})$ of Lemma 3.17.

Next we show that systems (3.91) could be brought via a transformation to the canonical form (S_δ). Indeed we could apply to parabola (3.92) the translation

$$x = x_1 - \frac{2cg - 2d - fg}{3g^2}, \quad y = y_1 + \frac{8d^2 - 2d(5c - f)g + g^2(2c^2 + cf - f^2 - 9ag)}{18dg^3},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 + \dfrac{2d}{g}y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.91) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$\begin{gathered} k = -\frac{2d}{g},\ n = -\frac{-d + cg - 2fg}{3g},\ m = -\frac{16d^2 - 2d(7c - 5f)g - g^2(2c^2 + cf - f^2 - 9ag)}{36dg^2} \Rightarrow \\ a = \frac{4c^2 - 8ck - 5k^2 - 4n^2 + 4k(8gm + 3n)}{16g},\ d = -\frac{gk}{2},\ f = \frac{2c + k + 6n}{4}. \end{gathered}$$

Then after an additional rescaling (to force $k = 1$) we arrive at the family of systems (S_δ).

***2:* The possibility** $g = 0$. Then the equation $Eq_6 = 0$ yields $d = 0$ and we calculate:

$$Eq_9 = r(f - 2c + r)$$

and due to $r \neq 0$ we get $r = 2c - f \neq 0$. Then calculations yield:

$$Eq_8 = 2a + (c - f)q, \quad Eq_{10} = 2bc - bf - fp + aq$$

and we have to examine two cases: $c - f \neq 0$ and $c - f = 0$.

***2.1:* The case** $c - f \neq 0$**.** Then the equation $Eq_8 = 0$ gives us $q = -2a/(c-f)$ and then we obtain

$$Eq_{10} = \frac{2bc^2 - 2a^2 - 3bcf + bf^2}{c - f} - fp.$$

We claim that for the existence of an invariant parabola the condition $f \neq 0$ must hold. Indeed supposing $f = 0$ we obtain $Eq_{10} = 2(bc - a^2)/c$ and then the condition $Eq_{10} = 0$ implies $b = a^2/c$. However in this case we arrive at the degenerate systems

$$\dot{x} = a + cx, \quad \dot{y} = \frac{(a - cx)(a + cx)}{c^2}$$

and this completes the proof of our claim.

So we have $f \neq 0$ and then the condition $Eq_{10} = 0$ gives us

$$p = \frac{-2a^2 + 2bc^2 - 3bcf + bf^2}{f(c - f)}$$

and we arrive at the parabola

$$\Phi(x, y) = \frac{2bc^2 - 2a^2 - 3bcf + bf^2}{f(c - f)} - \frac{2a}{c - f}x + (2c - f)y + x^2, \;\; f(c - f)(2c - f) \neq 0, \quad (3.93)$$

which is invariant for the family of systems:

$$\dot{x} = a + cx, \quad \dot{y} = b + fy - x^2. \quad (3.94)$$

This completes the proof of the statement $(\boldsymbol{L_2})$ of Lemma 3.17.

Next we show that systems (3.94) could be brought via a transformation to the canonical form (S_δ). Indeed we could apply to parabola (3.93) the translation

$$x = x_1 + \frac{a}{c - f}, \quad y = y_1 - \frac{bc^2 - a^2 - 2bcf + bf^2}{(c - f)^2 f},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 + (2c - f)\,y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.95) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = f - 2c, \; m = -\frac{a}{c - f}, \; n = \frac{f}{2} \quad \Rightarrow \quad a = -\frac{m(k + 2n)}{2}, \; c = \frac{2n - k}{2}, \; f = 2n.$$

Then after an additional rescaling (to force $k = 1$) we arrive at the subfamily of systems (S_δ) defined by the condition $g = 0$.

***2.2:* The case** $c - f = 0$**.** Then we set $f = c$ and the equation $Eq_8 = 0$ gives us $a = 0$ and we obtain

$$Eq_{10} = c(b - p) = 0.$$

In this case $r = 2c - f = c \neq 0$ and hence the condition $Eq_{10} = 0$ implies $p = b$. Therefore we obtain the family of systems

$$\dot{x} = cx, \quad \dot{y} = b + cy - x^2, \quad (3.95)$$

which possess the family of the invariant parabolas depending on one parameter q.

$$\Phi(x, y) = b + qx + cy + x^2, \;\; c \neq 0. \quad (3.96)$$

This completes the proof of the Lemma 3.17. □

Next we show that systems (3.95) could be brought via a transformation to the canonical form (S_δ). Indeed we could apply to parabola (3.96) the translation

$$x = x_1 - \frac{q}{2}, \quad y = y_1 - \frac{4b - q^2}{4c},$$

which brings this parabola to the form $\widetilde{\Phi}(x_1, y_1) = x_1^2 + c\,y_1$.

On the other hand considering Observation 3.4 we apply the same translation to systems (3.95) and we set the following new notations (suggested by the above parabola and the linear parts of the transformed systems):

$$k = -c, \ m = \frac{q}{2} \ \Rightarrow \ c = -k, \ q = 2m.$$

Then after an additional rescaling (to force $k = 1$) arrive at the subfamily of systems (S_δ) defined by the conditions $g = 0$ and $n = -1/2$.

3.4.2 Invariant conditions: the case $\eta = \widetilde{M} = 0$ and $C_2 \neq 0$

Next, using Lemma 3.17 we shall construct the equivalent affine invariant conditions for a system with $\eta = \widetilde{M} = 0$ and $C_2 \neq 0$ to possess an invariant parabola.

We prove the following theorem.

Theorem 3.18. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta = \widetilde{M} = 0$, $\chi_1 = 0$ and $C_2 \neq 0$ are satisfied. Then this system could only possess one of the following sets of invariant parabolas: $\overset{3}{\cup}$ and $\infty\overset{3}{\cup}$. Moreover this system has one of the above sets of invariant parabolas if and only if one of the following sets of conditions are satisfied, correspondingly:*

$(\boldsymbol{\mathcal{L}_1})$ $\quad \zeta_{14} \neq 0, \zeta_{22} = 0, \mathcal{R}_2 \neq 0 \quad \Rightarrow \overset{3}{\cup}$;

$(\boldsymbol{\mathcal{L}_2})$ $\quad \zeta_{14} = 0, \zeta_{20} = 0, \zeta_{23} \neq 0, \zeta_{24} \neq 0 \quad \Rightarrow \overset{3}{\cup}$;

$(\boldsymbol{\mathcal{L}_3})$ $\quad \zeta_{14} = 0, \zeta_{20} = 0, \zeta_{23} = 0, \zeta_{24} \neq 0 \quad \Rightarrow \infty\overset{3}{\cup}$.

Proof. Assume that quadratic system the conditions $\widetilde{M} = 0$ and $C_2 \neq 0$ are fulfilled. Then via a linear transformation this system can be brought to the canonical form (3.88). According to Lemma 2.4 for a system (3.88) to possess an invariant parabola the conditions $\chi_1 = \chi_2 = 0$ are necessary. Moreover it was shown earlier (see page 60) that a system (3.88) with $\chi_1 = 0$ via an affine transformation ant time rescaling can be brought to the form (3.89). Thus in what follows we consider the family of quadratic systems

$$\frac{dx}{dt} = a + cx + dy + gx^2, \quad \frac{dy}{dt} = b + fy - x^2 + gxy. \tag{3.97}$$

1: **The statement $(\boldsymbol{\mathcal{L}_1})$.** Considering (3.90) for these systems we calculate

$$\chi_1 = \chi_2 = 0, \ \ \zeta_{14} = g^2x^2, \ \ \zeta_{22} = 9d^3g^3\mathrm{Y}_4, \ \ \mathcal{R}_2 = 9d^2g^4/4 \tag{3.98}$$

and clearly the condition $\zeta_{14} \neq 0$ is equivalent to $g \neq 0$ and in this case the condition $\mathcal{R}_2 \neq 0$ gives us $d \neq 0$. Therefore we conclude that for $\zeta_{14}\mathcal{R}_2 \neq 0$ the condition $\zeta_{22} = 0$ is equivalent to $\mathrm{Y}_4 = 0$.

2: **The statement $(\mathcal{L}_2)$.** From (3.98) evidently the condition $\zeta_{14} = 0$ implies $g = 0$ and then we calculate

$$\zeta_{23} = -2(c-f)^2, \quad \zeta_{20} = -12d^2x^2.$$

Therefore the condition $\zeta_{20} = 0$ is equivalent to $d = 0$ whereas $\zeta_{23} \neq 0$ implies $c - f \neq 0$. So for $\zeta_{20} = 0$ we get $d = 0$ and then we calculate

$$\zeta_{24} = 24f(2c-f)x^3$$

and hence the condition $\zeta_{24} \neq 0$ is equivalent to $f(2c-f) \neq 0$. Since the condition $\zeta_{23} \neq 0$ is equivalent to $c - f \neq 0$, considering Lemma 3.17 we conclude that the statement $(\mathcal{L}_2)$ of Theorem 3.18 is proved.

3: **The statement $(\mathcal{L}_3)$.** Since the condition $\zeta_{23} = -2(c-f)^2 = 0$ gives us $f = c$ for systems (3.97) with $g = d = 0$ and $f = c$ we calculate $\zeta_{24} = 24c^2x^3$ and clearly the condition $\zeta_{24} \neq 0$ is equivalent to $c \neq 0$. This completes the proof of Theorem 3.18. □

3.5 Systems with infinite line filled up with singularities

According to Lemma 2.3 in the case $C_2 = 0$ systems (2.5) via a linear transformation could be brought to the systems $(\mathbf{S}_V)$ for which in addition we may assume $e = f = 0$ due to a translation. So we consider the following family of quadratic systems

$$\frac{dx}{dt} = a + cx + dy + x^2, \quad \frac{dy}{dt} = b + xy. \tag{3.99}$$

We prove the following lemma.

Lemma 3.19. *A non-degenerate quadratic system (3.99) could only have invariant parabola of the form $\Phi(x,y) = p + qx + ry + x^2$ with $r \neq 0$. Moreover it possesses an invariant parabola of this form if and only if the following conditions hold:*

$$d \neq 0, \ \Upsilon_5 = 9ac - 2c^3 + 27bd = 0.$$

Proof. Suppose that these systems possess an invariant parabola

$$\Phi(x,y) \equiv p + qx + ry + sx^2 + 2vxy + uy^2 = 0$$

with $v^2 - su = 0$ and $u \neq 0$, i.e. its quadratic part is not of the form sx^2. Then clearly we may assume $u = 1$ and then we obtain $s = v^2$, i.e. we get the parabola

$$\Phi(x,y) \equiv p + qx + ry + (vx+y)^2 = 0, \tag{3.100}$$

for which the condition $q \neq rv$ must hold, otherwise we get a reducible conic.

Considering equations (2.6) and the form of the parabola (3.100) with $q \neq rv$, for systems (3.99) we obtain

$$s = v^2, \ u = 1, \ Eq_3 = 2 - U - 2vV, \ Eq_4 = -V$$

and evidently the equations $Eq_3 = 0$ and $Eq_4 = 0$ imply $V = 0$ and $U = 2$. Then calculations yield

$$Eq_5 = -q + 2cv^2 - v^2W = 0, \ \ Eq_6 = -r + 2cv + 2dv^2 - 2vW = 0, \ \ Eq_7 = 2dv - W = 0$$

and we get

$$W = 2dv, \;\; q = 2v^2(c - dv), \;\; r = 2v(c - dv) \;\Rightarrow\; q = rv.$$

So we obtain a reducible conic.

Thus we have proved that if systems (3.99) possess an invariant parabola then it is necessary of the form

$$\Phi(x, y) \equiv p + qx + ry + sx^2 = 0$$

with $s \neq 0$ and $r \neq 0$, otherwise we get a reducible conic. Then we may assume $s = 1$ and again, considering the ten equations (2.6) and the above parabola, for systems (3.99) we obtain

$$s = 1, \; u = v = 0, \;\; Eq_1 = 2 - U = 0, \;\; Eq_2 = -V = 0 \;\Rightarrow\; V = 0, \; U = 2$$

and then calculations yield:

$$Eq_5 = 2c - q - W = 0, \;\; Eq_6 = 2d - r = 0 \;\Rightarrow\; r = 2d \neq 0, \; W = 2c - q.$$

Therefore evaluating the remaining equations we obtain

$$Eq_8 = 2a - 2p - cq + q^2, \;\; Eq_9 = d(3q - 4c), \;\; Eq_{10} = 2bd - 2cp + aq + pq.$$

Since $d \neq 0$ (due to $r \neq 0$) the equation $Eq_9 = 0$ gives us $q = 4c/3$ and then from $Eq_8 = 0$ we get $p = (9a + 2c^2)/9$. In this case we obtain

$$Eq_{10} = \frac{2}{27}(9ac - 2c^3 + 27bd) = \frac{2}{27}\Upsilon_5 = 0.$$

Since $d \neq 0$ the condition $\Upsilon_5 = 0$ implies $b = \dfrac{c}{27d}(2c^2 - 9a)$ and we arrive at the systems

$$\dot{x} = a + cx + dy + x^2, \quad \dot{y} = \frac{c}{27d}(2c^2 - 9a) + xy \tag{3.101}$$

which possess the following invariant parabola:

$$\Phi(x, y) = \frac{1}{9}\left(9a + 2c^2\right) + \frac{4c}{3}x + 2dy + x^2, \;\; d \neq 0. \tag{3.102}$$

This complete the proof of the Lemma 3.19. □

Evaluating for systems (3.99) the invariant polynomials ζ_5 and ζ_{22} we obtain

$$\zeta_5 = -891d^2/4, \;\; \zeta_{22} = 9d^3(9ac - 2c^3 + 27bd) = 9d^3\Upsilon_5.$$

So the condition $d \neq 0$ is equivalent to $\zeta_5 \neq 0$ and in this case the condition $\Upsilon_5 = 0$ is equivalent to $\zeta_{22} = 0$. Therefore considering Lemma 3.19 we conclude that the following theorem is valid.

Theorem 3.20. *Assume that for a non-degenerate quadratic system the condition $C_2 = 0$ holds. Then this system possesses an invariant parabola (which is unique) if and only if the conditions $\zeta_5 \neq 0$ and $\zeta_{22} = 0$ hold.*

In order to determine a simpler canonical form of systems (3.101) we apply to these systems as well as to parabola (3.102) the translation

$$x = x_1 - \frac{2c}{3}, \quad y = y_1 - \frac{9a - 2c^2}{18d}.$$

Then we could set the following new notations:

$$k = -2d, \ m = -\frac{9a - 2c^2}{36d}, \ n = -\frac{c}{3} \ \Rightarrow$$
$$a = 2(km + n^2), \ c = -3n, \ d = -\frac{k}{2},$$

where $k \neq 0$ due to $d \neq 0$. Then we arrive at the family of systems

$$\dot{x}_1 = km + nx_1 - \frac{k}{2}y_1 + x_1^2, \quad \dot{y}_1 = 2mx_1 + 2ny_1 + x_1y_1,$$

which possess the invariant parabola

$$\Phi(x_1, y_1) = x_1^2 - ky_1, \quad k \neq 0.$$

Finally applying the rescaling $(x_1, y_1, t_1) \mapsto (kx, ky, t/k)$ we arrive at the systems

$$\dot{x} = m + nx - y/2 + x^2, \quad \dot{y} = 2mx + 2ny + xy,$$

which possess the invariant parabola $\Phi(x, y) = x^2 - y$.

As all the cases are investigated we conclude that the Main Theorem is proved.

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Schrödinger–Hardy system without Ambrosetti–Rambinowitz condition on Carnot groups

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Abstract. In this paper, we study the following Schrödinger–Hardy system

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{r(\xi)^2} u = F_u(\xi, u, v) & \text{in } \Omega, \\ -\Delta_{\mathbb{G}} v - \nu \frac{\psi^2}{r(\xi)^2} v = F_v(\xi, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain on Carnot groups $\mathbb{G}$, whose homogeneous dimension is $Q \geq 3$, $\Delta_{\mathbb{G}}$ denotes the sub-Laplacian operator on $\mathbb{G}$, μ and ν are real parameters, $r(\xi)$ is the natural gauge associated with fundamental solution of $-\Delta_{\mathbb{G}}$ on $\mathbb{G}$, ψ is the geometrical function defined as $\psi = |\nabla_{\mathbb{G}} r|$, and $\nabla_{\mathbb{G}}$ is the horizontal gradient associated with $\Delta_{\mathbb{G}}$. The difficulty is not only the nonlinearities F_u and F_v without Ambrosetti–Rabinowitz condition, but also the hardy terms and the structure on Carnot groups. We obtain the existence of nonnegative solution for this system by mountain pass theorem in a new framework.

Keywords: Schrödinger–Hardy system, without Ambrosetti–Rabinowitz condition, Carnot groups, mountain pass theorem.

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1 Introduction and main results

In this paper, we consider the following Schrödinger–Hardy system

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{r(\xi)^2} u = F_u(\xi, u, v) & \text{in } \Omega, \\ -\Delta_{\mathbb{G}} v - \nu \frac{\psi^2}{r(\xi)^2} v = F_v(\xi, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain on Carnot groups $\mathbb{G}$, whose homogeneous dimension is $Q \geq 3$, $\Delta_{\mathbb{G}}$ denotes the sub-Laplacian operator on $\mathbb{G}$, μ and ν are real parameters, $r(\xi)$ is

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the natural gauge associated with fundamental solution of $-\Delta_{\mathbb{G}}$ on $\mathbb{G}$, ψ is the geometrical function defined as $\psi = |\nabla_{\mathbb{G}} r|$, and $\nabla_{\mathbb{G}}$ is the horizontal gradient associated with $\Delta_{\mathbb{G}}$. The difficulty in this paper is not only the nonlinearities F_u and F_v without Ambrosetti–Rabinowitz condition, but also the hardy terms and the structure on Carnot groups.

In the context of stratified groups, the problem has been intensively studied in last decades, starting with the pioneering papers [21,22]. In particular, a number of literatures are related to Heisenberg group, such as [4,15,16,23,35,36] and references therein. Only few results concern the general Carnot setting. For related topics, see [2,3,11,31,37] and references therein.

We mention that Ferrara et al. [17] obtained the existence of a weak solution for the following problem

$$\begin{cases} -\Delta_{\mathbb{G}} u = \lambda f(\xi, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{G}$, $\lambda > 0$ is a real parameter, and f is a subcritical nonlinearity. For critical exponent subelliptic problem,

$$\begin{cases} -\Delta_{\mathbb{G}} u = |u|^{2^*-2}u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with $2^* = \frac{2Q}{Q-2}$. When $f = 0$, problem (1.2) does not admit any nonnegative non trivial solution on star-shaped domain, see [21, 22]. If Ω is a bounded domain of $\mathbb{G}$, Loiudice [27] established the existence of positive and sign changing solutions for $f = \lambda u$, extending the famous Brezis–Nirenberg results [8] to the subelliptic Carnot setting. Subsequently, by Nehari manifold and Ekeland variational principle, Loiudice [32] considered the general non-homogeneous problem (1.2) and proved the existence of at least two positive solutions, provided that non-homogeneous term f satisfies suitable assumptions.

Concerning the problem for sub-Laplacian operator involving critical Hardy–Sobolev nonlinearity

$$\begin{cases} -\Delta_{\mathbb{G}} u = \frac{\psi^{\alpha}}{r(\xi)^{\alpha}}|u|^{2^*(\alpha)-2}u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{G}$ is a bounded domain, $0 < \alpha < 2$, $2^*(\alpha) = \frac{2(Q-\alpha)}{Q-2}$ is the critical Sobolev–Hardy exponent, Loiudice [29] proved that if $\lambda = 0$, there is no nonnegative nontrivial solutions when Ω is a bounded star-shaped domain about the origin with respect to dilations of the group. Also, the existence of solution was established provided that $\lambda > 0$.

For more general nonlinearity with Hardy type potential, that is

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu\frac{\psi^2}{r(\xi)^2}u = f(\xi, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is an open subset of $\mathbb{G}$, $0 \in \Omega$, $0 \le \mu < (\frac{Q-2}{2})^2$, the function f satisfies $f(\xi, u) \le C(|u| + |u|^{2^*-1})$, $\forall(\xi, u) \in \Omega \times \mathbb{R}$ and $C > 0$ is a constant. By L^p regularity of solutions and Moser's iteration, Loiudice [30] showed that any positive solution of (1.3) has a stronger singularity as $\mu \to (\frac{Q-2}{2})^2$. When f is a purely critical nonlinearity, that is $f(u) = |u|^{2^*-2}u$, the behavior of solutions at origin shows the decay of solution at infinity by Kelvin transform on $\mathbb{R}^n$ in Euclidean setting. However, this technique fails in Carnot group, because there does not exist a suitable inversion with good conformal properties. We point out this technique is true for a special subclass of stratified groups, that is the Iwasawa-type groups $\mathbb{H}$. Loiudice [28] showed that if $u \in \mathcal{S}_0^1(\Omega)$ is a solution to

$$-\Delta_{\mathbb{H}} u - \mu \frac{\psi^2}{r(\xi)^2} u = |u|^{2^*-2} u \quad \text{in } \Omega,$$

there is $C_1 > 0$ such that

$$|u(\xi)| \le C_1 r(\xi)^{-\sqrt{\mu_{\mathbb{H}}} - \sqrt{\mu_{\mathbb{H}} - \mu}}, \quad \text{for } r(\xi) \text{ large.}$$

Moreover, if u is positive, there exists $C_2 > 0$ such that

$$|u(\xi)| \ge C_2 r(\xi)^{-\sqrt{\mu_{\mathbb{H}}} - \sqrt{\mu_{\mathbb{H}} - \mu}}, \quad \text{for } r(\xi) \text{ large,}$$

where $\mathcal{S}_0^1(\Omega)$ is the Folland–Stein space, defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{S}_0^1(\Omega)} = \Big(\int_\Omega |\nabla_{\mathbb{G}} u|^2 d\xi \Big)^{\frac{1}{2}},$$

and $\mu_{\mathbb{H}} = (\frac{Q-2}{2})^2$ is the best constant in Hardy inequality on Iwasawa-type groups,

$$\mu_{\mathbb{H}} \int_{\mathbb{H}} \psi^2 \frac{|u|^2}{r(\xi)^2} d\xi \le \int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^2 d\xi, \quad \forall u \in C_0^\infty(\mathbb{H}),$$

and it is never attained, some more details can be seen in [5, 10]. Moreover, this result was extended to the whole Carnot groups in [33] by using different methods, and Loiudice investigated the existence and nonexistence for subelliptic Brezis–Nirenberg type problem as follows

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \frac{\psi^2}{r(\xi)^2} u = u^{2^*-1} + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Concerning the results in the whole Carnot group, Zhang [39] considered the following equation

$$-\Delta_{\mathbb{G}} u = \lambda \frac{\psi^\alpha}{r(\xi)^\alpha} |u|^{2^*(\alpha)-2} u + \beta f(\xi) |u|^{p-2} u \quad \text{in } \mathbb{G},$$

where λ, $\beta > 0$ are parameters, $0 < \alpha \le 2$, Zhang proved the existence and multiplicity of solutions by variational methods and the theory of genus. Concerning multiple Hardy nonlinearities, Zhang [38] proved the attainability of best Sobolev–Hardy constant of

$$S_{\mu,\alpha} = \inf_{u \in \mathcal{S}^1(\mathbb{G}) \setminus \{0\}} \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 d\xi - \mu \int_{\mathbb{G}} \frac{\psi(\xi)^2}{r(\xi)^2} |u|^2 d\xi}{\Big(\int_{\mathbb{G}} \frac{\psi(\xi)^\alpha}{r(\xi)^\alpha} |u|^{2^*(\alpha)} d\xi \Big)^{\frac{2}{2^*(\alpha)}}}.$$

Moreover, as an application, by variational methods and local compactness of Palais–Smale sequences, Zhang obtained the existence of nontrivial weak solution to the following singularity sub-elliptic equation and system

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi(\xi)^2}{r(\xi)^2} u = \frac{\psi(\xi)^\alpha}{r(\xi)^\alpha} |u|^{2^*(\alpha)-2} u + \frac{\psi(\xi)^\beta}{r(\xi)^\beta} |u|^{2^*(\beta)-2} u \quad \text{in } \mathbb{G},$$

and

$$\begin{cases} -\Delta_{\mathbb{G}}u-\mu\frac{\psi(\xi)^2}{r(\xi)^2}u=\frac{\psi(\xi)^\alpha}{r(\xi)^\alpha}|u|^{2^*(\alpha)-2}u+\frac{\lambda\eta}{\eta+\theta}\frac{\psi(\xi)^\alpha}{r(\xi)^\alpha}|u|^{\eta-2}u|v|^\theta & \text{in } \mathbb{G}, \\ -\Delta_{\mathbb{G}}v-\mu\frac{\psi(\xi)^2}{r(\xi)^2}v=\frac{\psi(\xi)^\alpha}{r(\xi)^\alpha}|v|^{2^*(\alpha)-2}v+\frac{\lambda\eta}{\eta+\theta}\frac{\psi(\xi)^\alpha}{r(\xi)^\alpha}|u|^{\eta}|v|^{\theta-2}v & \text{in } \mathbb{G}, \end{cases}$$

where $0\le\alpha,\ \beta<2$ and $\eta,\ \theta>1$ with $\eta+\theta=2^*(\alpha)$, $\lambda>0$ is a parameter. Further, the problems with Hardy potential have been considered by [24] and [6,7,34,35] for Hardy nonlinearity in Heisenberg group. In particular, we mention that Bordoni and Pucci [6] first proved the existence of nontrivial nonnegative solutions of the Schrödinger system including multiple critical nonlinearities and Hardy potentials in Heisenberg groups.

In order to deal with (1.1), we introduce the Sobolev-type inequality: there exists a positive constant $C>0$ such that

$$\int_\Omega |u|^{2^*}d\xi\le C\Big(\int_\Omega|\nabla_{\mathbb{G}}u|^2d\xi\Big)^{\frac{2^*}{2}},\quad \forall u\in C_0^\infty(\Omega), \tag{1.4}$$

where 2^* is the critical exponent for $\Delta_{\mathbb{G}}$, the embedding $\mathcal{S}_0^1(\Omega)\hookrightarrow L^q(\Omega)$ is compact for $1\le q<2^*$ but only continuous for $q=2^*$, and the Hardy-type inequality is: for every $u\in C_0^\infty(\Omega)$, there holds

$$\Big(\frac{Q-2}{2}\Big)^2\Big(\int_\Omega\frac{\psi^2}{r(\xi)^2}|u|^2d\xi\Big)\le\int_\Omega|\nabla_{\mathbb{G}}u|^2d\xi, \tag{1.5}$$

where $\big(\frac{Q-2}{2}\big)^2$ is the optimal constant but never attained (see [12,20]). (1.5) is first proved by Garofalo and Lanconelli [20] in Heisenberg group, then, D'Ambrosio [12] extended this result to all Carnot groups. Moreover, the best Hardy constant $\mathcal{K}>0$ of (1.5) is given by

$$\mathcal{K}=\inf_{u\in\mathcal{S}_0^1(\Omega),u\neq0}\frac{\|u\|^2_{\mathcal{S}_0^1(\Omega)}}{\|u\|_\psi^2}\quad\text{with}\quad \|u\|_\psi^2=\int_\Omega\frac{\psi^2}{r(\xi)^2}|u|^2d\xi. \tag{1.6}$$

Now, let us define a suitable solution space $W=\mathcal{S}_0^1(\Omega)\times\mathcal{S}_0^1(\Omega)$, which is a separable, reflexive Banach space and endowed with the norm

$$\|(u,v)\|=\Big(\|u\|^2_{\mathcal{S}_0^1(\Omega)}+\|v\|^2_{\mathcal{S}_0^1(\Omega)}\Big)^{\frac12}, \tag{1.7}$$

we denote

$$\|(u,v)\|_p=\Big(\int_\Omega|(u,v)|^pd\xi\Big)^{\frac1p}=\Big(\int_\Omega|(u^2+v^2)^{\frac12}|^pd\xi\Big)^{\frac1p},$$

for $1\le p<\infty$, and let

$$\lambda^*=\inf_{(u,v)\in W\setminus\{(0,0)\}}\frac{\|(u,v)\|^2}{\|(u,v)\|_2^2}>0.$$

Throughout the paper, we assume that $F(\xi,u,v):\Omega\times\mathbb{R}^2\to\mathbb{R}$ is continuous, $F(\xi,0,0)=0$ in Ω, and it satisfies the following assumptions.

(f_1) The partial derivatives F_u, $F_v\in C(\Omega\times\mathbb{R}^2)$, $F(\xi,u,v)\ge0$ in $\Omega\times\mathbb{R}^2$. Moreover, for each $\xi\in\Omega$,

$$F_u(\xi,u,v)=0\begin{cases}\text{if } u\le0 \text{ and } v\in\mathbb{R},\\ \text{if } v\le0\text{ and } u\in\mathbb{R}.\end{cases}$$

(f_2) There exists $s \in (2, 2^*)$ and $\lambda \in [0, \lambda^*)$, then for each $\epsilon > 0$, there is a constant $C_\epsilon > 0$ such that

$$|F_w(\xi, w)| \leq (\lambda + \epsilon)|w| + C_\epsilon |w|^{s-1},$$

for every $(\xi, w) \in \Omega \times \mathbb{R}^2$, $w = (u, v)$, $|w| = \sqrt{u^2 + v^2}$, where $F_w = (F_u, F_v)$.

(f_3) $\lim\limits_{|w|\to\infty} \frac{2F(\xi,w)}{|w|^2} = \infty$, uniformly in Ω.

(f_4) For any $w = (u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $0 < \tau < 1$, there exists a nonnegative function $g \in L^1(\Omega)$ and a constant $C_F \geq 1$ such that $H(\xi, \tau w) \leq C_F H(\xi, w) + g(\xi)$, where $H(\xi, w) = F_w(\xi, w)w - 2F(\xi, w)$.

The main result can be stated as follows.

Theorem 1.1. *Assume that F satisfies (f_1)–(f_4). Then (1.1) has at least a nonnegative solution $(u, v) \in W$ for any μ, $\nu \in (-\infty, \mathcal{K})$ such that*

$$\Theta - \frac{2\lambda}{\lambda^*} > 0, \tag{1.8}$$

where $\lambda \in [0, \lambda^*)$, $\Theta = \min\left\{1 - \frac{\mu^+}{\mathcal{K}}, 1 - \frac{\nu^+}{\mathcal{K}}\right\}$, $\mu^+ = \max\{0, \mu\}$ and $\nu^+ = \max\{0, \nu\}$.

In this paper, the main difficulty is that the energy functional does not satisfy Palais–Smale condition since the nonlinearities F_u and F_v loss the Ambrosetti–Rabinowitz condition, see also [14, 25, 26]. It should be mentioned that the (f_4) plays an important role in proving the boundless of Palais–Smale sequence.

The rest of the paper is organized as follows. In Section 2, we recall the main notations and definitions related to the Carnot groups, and present some preparatory results. In Section 3, we prove that the energy functional satisfies the mountain pass geometry structures. In Section 4, we obtain the compactness theorem and prove the main result. Finally, we show two lemmas in Section 5.

2 The functional setting of Carnot groups

We briefly recall the definitions and notations related to the Carnot groups functional setting. For a complete treatment, we refer to [5, 18, 19].

2.1 The Carnot groups

A Carnot group is a homogeneous group, denoted as $\mathbb{G} = (\mathbb{R}^n, \circ, \mathfrak{F})$, whose Lie algebra $\mathfrak{g}$ is stratified, that is, $\mathfrak{g} = \bigoplus_{i=1}^r V_i$, where $r > 0$ is a integer number and called the *step* of $\mathbb{G}$, $\mathfrak{g}$ is the Lie algebra of left invariant vector fields on $\mathbb{G}$, V_i is a linear subspace of $\mathfrak{g}$, $i = 1, \ldots, r$, and satisfies

$$\begin{aligned}
&\dim V_i = n_i, \text{ for } i = 1, \ldots, r, \\
&[V_1, V_i] = V_{i+1}, \text{ for } 1 \leq i \leq r-1, \text{ and } [V_1, V_r] = \{0\}.
\end{aligned}$$

From these, we can see that $[V_1, V_i]$ stands for the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$, $Y \in V_i$.

In fact, $(\mathbb{R}^n, \circ)$ is a Lie group equipped with a family of group automorphisms (namely *dilatations*) $\mathfrak{F} := \{\delta_\eta\}_{\eta>0}$ such that, for every $\eta > 0$, the map

$$\delta_\eta : \prod_{i=1}^{r} \mathbb{R}^{n_i} \to \prod_{i=1}^{r} \mathbb{R}^{n_i},$$

shows that $\delta_\eta(\xi^{(1)}, \ldots, \xi^{(r)}) = (\eta\xi^{(1)}, \eta^2\xi^{(2)}, \ldots, \eta^r\xi^{(r)})$, where $\xi^{(i)} \in \mathbb{R}^{n_i}$, $i = 1, \ldots, r$, and $\sum_{i=1}^{r} n_i = n$. The structure $\mathbb{G} = (\mathbb{R}^n, \circ, \mathfrak{F})$ is called a *homogeneous group*, and $Q = \dim_h \mathbb{G} := \sum_{k=1}^{r} k n_k$ is called the *homogeneous dimension* of $\mathbb{G}$. In this paper, we pay attention to $\dim_h \mathbb{G} \geq 3$. In particular, $\mathbb{G}$ is the Euclidean space provided that $\dim_h \mathbb{G} \leq 3$, i.e. $\mathbb{G} = (\mathbb{R}^{\dim_h \mathbb{G}}, +)$.

Let $\{X_j\}_{j=1}^{n_1}$ be a basis of V_1, then the associated subelliptic operator $\Delta_{\mathbb{G}}$ is given by

$$\Delta_{\mathbb{G}} := \sum_{j=1}^{n_1} X_j^2,$$

which is the second order differential operator on $\mathbb{G}$. Here, n_1 is the dimension of the first step, moreover, the subelliptic gradient is $\nabla_{\mathbb{G}} := (X_1, X_2, \ldots, X_{n_1})$. As proved in [18], there exists a suitable homogeneous norm $r(\xi)$, called *gauge norm*, such that $\Gamma(\xi) = \frac{C}{r(\xi)^{Q-2}}$ is the fundamental solution of $-\Delta_{\mathbb{G}}$, where $C > 0$ is a constant. By definition, a homogeneous norm is any continuous function from $\mathbb{G}$ to $[0, +\infty)$ such that for $\eta > 0$, $\xi \in \mathbb{G}$, $r(\delta_\eta(\xi)) = \eta r(\xi)$, $r(\xi^{-1}) = r(\xi)$, $r(\xi) = 0$ if and only if $\xi = 0$.

2.2 Functional setting and preliminary results

In this subsection, we present some useful results and comments, (1.1) has a variational structure and the Euler–Lagrange functional $I_{\mu,\nu} : W \to \mathbb{R}$ is given by

$$I_{\mu,\nu}(u,v) = \frac{1}{2}\|u\|^2_{\mathcal{S}_0^1(\Omega)} + \frac{1}{2}\|v\|^2_{\mathcal{S}_0^1(\Omega)} - \frac{\mu}{2}\|u\|^2_\psi - \frac{\nu}{2}\|v\|^2_\psi - \int_\Omega F(\xi, u, v)d\xi,$$

for all $u, v \in W$. Indeed, $I_{\mu,\nu}$ is well defined and be of class $C^1(W)$ under the assumptions (f_1) and (f_2). A function $(u, v) \in W$ is a weak solution of (1.1) if holds

$$\langle u, \Phi\rangle + \langle v, \Psi\rangle - \mu\langle u, \Phi\rangle_\psi - \nu\langle v, \Psi\rangle_\psi = \int_\Omega \Big(F_u(\xi, u, v)\Phi + F_v(\xi, u, v)\Psi\Big)d\xi,$$

for every $(\Phi, \Psi) \in W$, where

$$\langle u, \Phi\rangle = \int_\Omega (\nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \Phi)d\xi, \qquad \langle v, \Psi\rangle = \int_\Omega (\nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} \Psi)d\xi,$$
$$\langle u, \Phi\rangle_\psi = \int_\Omega \frac{\psi^2}{r(\xi)^2} u\Phi d\xi, \qquad \langle v, \Psi\rangle_\psi = \int_\Omega \frac{\psi^2}{r(\xi)^2} v\Psi d\xi.$$

Moreover, for all $(u, v) \in W$, there holds

$$\langle I'_{\mu,\nu}(u, v), (\Phi, \Psi)\rangle = \langle u, \Phi\rangle + \langle v, \Psi\rangle - \mu\langle u, \Phi\rangle_\psi - \nu\langle v, \Psi\rangle_\psi - \int_\Omega \Big(F_u(\xi, u, v)\Phi + F_v(\xi, u, v)\Psi\Big)d\xi, \quad \text{for every } (\Phi, \Psi) \in W.$$

Therefore, the weak solutions of (1.1) are exactly the critical points of $I_{\mu,\nu}$.

Lemma 2.1. *The embedding* $W \hookrightarrow L^q(\Omega) \times L^q(\Omega)$ *is continuous for* $1 \le q \le 2^*$ *and* $\|(u,v)\|_q \le C_q'\|(u,v)\|$ *for all* $(u,v) \in W$ *and* $C_q' > 0$ *is a constant.*

Proof. From [19], we know that $\mathcal{S}_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le q \le 2^*$, thus, there is $C_q > 0$ such that

$$\|u\|_q \le C_q\|u\|_{\mathcal{S}_0^1(\Omega)} \quad \text{and} \quad \|v\|_q \le C_q\|v\|_{\mathcal{S}_0^1(\Omega)}.$$

Moreover, by (f_2), (1.7) and a fact $a+b \le \sqrt{2(a^2+b^2)}$ for each $a,b \in \mathbb{R}$, there holds

$$\begin{aligned}\|(u,v)\|_q &= \|\sqrt{u^2+v^2}\|_q \le \|\sqrt{(u+v)^2}\|_q \le \|u\|_q + \|v\|_q \\ &\le C_q(\|u\|_{\mathcal{S}_0^1(\Omega)} + \|v\|_{\mathcal{S}_0^1(\Omega)}) \le C_q\sqrt{2(\|u\|^2_{\mathcal{S}_0^1(\Omega)} + \|v\|^2_{\mathcal{S}_0^1(\Omega)})} = C_q'\|(u,v)\|.\end{aligned}$$

This proof is finished. □

Lemma 2.2 ([1])**.** *Let* $\{(u_k,v_k)\} \subset W$ *be such that* $(u_k,v_k) \rightharpoonup (u,v)$ *weakly in* W *as* $k \to \infty$*, then up to a subsequence,* $(u_k,v_k) \to (u,v)$ *a.e. in* Ω *as* $k \to \infty$*.*

Lemma 2.3. *Let* $\Omega \subset \mathbb{G}$ *be a smooth bounded domain, then, the embedding* $W \hookrightarrow L^q(\Omega) \times L^q(\Omega)$ *is compact when* $1 \le q < 2^*$*.*

Proof. From [19], it holds that $S_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $1 \le q < 2^*$, that is, if $\{u_k\}$ and $\{v_k\}$ are bounded sequences in $S_0^1(\Omega)$, then there exist $u, v \in W$ such that,

$$u_k \to u \quad \text{and} \quad v_k \to v \quad \text{in} \quad L^q(\Omega).$$

Hence, if $\{(u_k,v_k)\} \subset W$ be a bounded sequence, we have

$$\|(u_k,v_k) - (u,v)\|_q \le \|u_k - u\|_q + \|v_k - v\|_q \to 0.$$

It follows that $\{(u_k,v_k)\}$ strongly in $L^q(\Omega) \times L^q(\Omega)$. □

In the following, we recall the definition of Cerami sequence and Cerami condition.

Definition 2.4. Let $X = (X, \|\cdot\|)$ be a Banach space, X' denotes its dual space, the functional $I : X \to \mathbb{R}$ be of $C^1(X)$.

(i) Cerami sequence: A sequence $u_k \in X$ is called a Cerami sequence if for every $u_k \in X$, $I(u_k)$ is bounded and $(1 + \|u_k\|)\|I'(u_k)\|_{X'} \to 0$ as $k \to \infty$. In particular, $\|I'(u_k)\|_{X'} \to 0$ as $k \to \infty$.

(ii) Cerami condition: A functional I satisfies the Cerami condition if any Cerami sequence associated with I has a strongly convergent subsequence in X.

3 Mountain pass structure

In this section, the results concern the existence of Palais–Smale sequence for $I_{\mu,\nu}$.

Lemma 3.1 ([9])**.** *Let* E *be a real Banach space,* $\mathcal{I} \in C^1(E)$ *with* $\mathcal{I}(0) = 0$*. There are constants* $\rho, \tau > 0$ *and* $e \in E$ *with* $\|e\|_E > \rho$ *such that*

$$\inf_{\|u\|_E = \rho} \mathcal{I}(u) \ge \tau \quad \text{and} \quad \mathcal{I}(e) < 0.$$

Then there is a Cerami sequence $\{u_k\} \subset E$ *such that*

$$\mathcal{I}(u_k) \to c, \quad (1 + \|u_k\|_E)\|I'(u_k)\|_E \to 0,$$

where

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{I}(\gamma(t)) \geq \tau,$$

and

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

The number c is called mountain pass level. If the functional $\mathcal{I}$ *satisfies the Cerami condition at the minimax level c, then c is a critical value of* $\mathcal{I}$ *in E.*

We first show that the energy functional $\mathcal{I}_{\mu,\nu}$ satisfies the geometric structure required by Lemma 3.1.

Lemma 3.2. *Assume that* (f_2) *holds, then there exist* ζ, $\rho > 0$ *such that*

$$I_{\mu,\nu}(u,v) \geq \zeta, \quad \text{if } \|(u,v)\| = \rho.$$

Proof. Let us set $\chi = \min\{(1 - \frac{\mu^+}{\mathcal{K}}), (1 - \frac{\nu^+}{\mathcal{K}})\}$ and from (f_2), we have

$$\begin{aligned}
I_{\mu,\nu}(u,v) &= \frac{1}{2}\|u\|^2_{\mathcal{S}^1_0(\Omega)} + \frac{1}{2}\|v\|^2_{\mathcal{S}^1_0(\Omega)} - \frac{\mu}{2}\|u\|^2_{\psi} - \frac{\nu}{2}\|v\|^2_{\psi} - \int_\Omega F(\xi,u,v)d\xi \\
&\geq \frac{1}{2}\|u\|^2_{\mathcal{S}^1_0(\Omega)}\Big(1 - \frac{\mu^+}{\mathcal{K}}\Big) + \frac{1}{2}\|v\|^2_{\mathcal{S}^1_0(\Omega)}\Big(1 - \frac{\nu^+}{\mathcal{K}}\Big) \\
&\quad - \int_\Omega \Big(\frac{1}{2}(\lambda+\epsilon)|(u,v)|^2 + \frac{1}{s}C_\epsilon|(u,v)|^s\Big)d\xi \\
&\geq \frac{\chi}{2}(\|u\|^2_{\mathcal{S}^1_0(\Omega)} + \|v\|^2_{\mathcal{S}^1_0(\Omega)}) - \frac{1}{2}(\lambda+\epsilon)\|(u,v)\|^2_2 - \frac{1}{s}C_\epsilon\|(u,v)\|^s_s \\
&\geq \frac{\chi}{2}\|(u,v)\|^2 - \frac{1}{2\lambda^*}(\lambda+\epsilon)\|(u,v)\|^2 - \frac{1}{s}C_\epsilon C_s\|(u,v)\|^s \\
&= \frac{1}{2}\Big(\chi - \frac{\lambda+\epsilon}{\lambda^*} - \frac{2}{s}C_\epsilon C_s\|(u,v)\|^{s-2}\Big)\|(u,v)\|^2,
\end{aligned}$$

where $C_s > 0$ is a constant, $\mathcal{K}$ is given in (1.6) and $s \in (2, 2^*)$. Thus, if ρ is small enough such that

$$\chi - \frac{\lambda+\epsilon}{\lambda^*} - \frac{2}{s}C_\epsilon C_s\rho^{s-2} > 0,$$

it holds $I_{\mu,\nu}(u,v) \geq \frac{1}{2}(\chi - \frac{\lambda+\epsilon}{\lambda^*} - \frac{2}{s}C_\epsilon C_s\rho^{s-2})\rho^2 = \zeta > 0$ for all $(u,v) \in W$ with $\|(u,v)\| = \rho$. We obtain this lemma. □

Lemma 3.3. *Suppose that* (f_3) *holds, then there exists* $(\tilde{u},\tilde{v}) \in W$ *with* $\|(\tilde{u},\tilde{v})\| > \rho$ *such that* $I_{\mu,\nu}(\tilde{u},\tilde{v}) < 0$.

Proof. It suffices to prove that for a fixed $(u_0,v_0) \in W$, $I_{\mu,\nu}(tu_0,tv_0) \to -\infty$ as $t \to +\infty$. We assume that $(u,v) \in W$ with compact support $\mathbf{D_c}$. From (f_3), there are constants c_1, c_2, $\delta > 0$, such that for $|u|$, $|v| > \delta$, one has

$$F(\xi,u,v) \geq c_1|(u,v)|^2 \geq c_1|(u,v)|^2 - c_2, \quad \text{for } (u,v) \in W.$$

Now, choosing arbitrarily $(u_0, v_0) \in W$ with $u_0, v_0 > 0$, and $\|(u_0, v_0)\| = 1$, hence, for all $t > 0$, we set $\mu^- = \min\{0, \mu\}$ and $\nu^- = \min\{0, \nu\}$, then

$$\begin{aligned} I_{\mu,\nu}(tu_0, tv_0) &\leq \frac{t^2}{2}\Big(\|u_0\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_0\|^2_{\mathcal{S}^1_0(\Omega)} - \mu\|u_0\|^2_\psi - \nu\|v_0\|^2_\psi\Big) - \int_\Omega \Big(c_1|(tu_0, tv_0)|^2 - c_2\Big)d\xi \\ &\leq \frac{t^2}{2}\Big(\|u_0\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_0\|^2_{\mathcal{S}^1_0(\Omega)} + |\mu^-|\|u_0\|^2_\psi + |\nu^-|\|v_0\|^2_\psi - 2c_1\|(u_0, v_0)\|^2_2\Big) + c_2|\mathbf{D_c}|. \end{aligned}$$

If c_1 is large enough, there holds

$$0 < \|u_0\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_0\|^2_{\mathcal{S}^1_0(\Omega)} + |\mu^-|\|u_0\|^2_\psi + |\nu^-|\|v_0\|^2_\psi < 2c_1\|(u_0, v_0)\|^2_2.$$

Therefore, we have $I_{\mu,\nu}(tu_0, tu_0) \to -\infty$ as $t \to \infty$. Setting $(\tilde{u}, \tilde{v}) = (t_0u_0, t_0u_0) \in W$, such that $\|(\tilde{u}, \tilde{v})\| > \rho$ and $I_{\mu,\nu}(\tilde{u}, \tilde{v}) < 0$. We obtain this lemma. □

4 Cerami sequence and existence of solutions

4.1 Cerami sequence

In this section, we give an analysis of Cerami sequence and prove that $I_{\mu,\nu}$ satisfies Cerami condition.

Lemma 4.1. *Assume that (f_1)–(f_4) hold, then for each μ, $\nu \in (-\infty, \mathcal{K})$, any Cerami sequence of $I_{\mu,\nu}$ is bounded in W.*

Proof. Let $\{(u_k, v_k)\} \subset W$ be a Cerami sequence of $I_{\mu,\nu}$, then, there exists $\mathcal{L} > 0$ independent of k such that

$$|I_{\mu,\nu}(u_k, v_k)| \leq \mathcal{L} \quad \text{for all } k, \quad (1 + \|(u_k, v_k)\|)I'_{\mu,\nu}(u_k, v_k) \to 0 \quad \text{as } k \to \infty. \tag{4.1}$$

Thus, there is $\tau_k > 0$ and $\tau_k \to 0$ as $k \to \infty$, such that

$$|\langle I'_{\mu,\nu}(u_k, v_k), (\Phi, \Psi)\rangle| \leq \frac{\tau_k\|(\Phi, \Psi)\|}{1 + \|(u_k, v_k)\|}, \quad \forall(\Phi, \Psi) \in W. \tag{4.2}$$

Let us set $(\Phi, \Psi) = (u_k, v_k)$, then

$$\begin{aligned} &\Big|\langle u_k, u_k\rangle + \langle v_k, v_k\rangle - \mu\langle u_k, u_k\rangle_\psi - \nu\langle v_k, v_k\rangle_\psi - \int_\Omega \Big(F_u(\xi, u_k, v_k)u_k + F_v(\xi, u_k, v_k)v_k\Big)d\xi\Big| \\ &\quad = |\langle I'_{\mu,\nu}(u_k, v_k), (u_k, v_k)\rangle| \leq \frac{\tau_k\|(u_k, v_k)\|}{1 + \|(u_k, v_k)\|} \leq \tau_k \leq C, \end{aligned}$$

that is

$$\begin{aligned} &- \|u_k\|^2_{\mathcal{S}^1_0(\Omega)} - \|v_k\|^2_{\mathcal{S}^1_0(\Omega)} + \mu\|u_k\|^2_\psi + \nu\|v_k\|^2_\psi \\ &\qquad + \int_\Omega \Big(F_u(\xi, u_k, v_k)u_k + F_v(\xi, u_k, v_k)v_k\Big)d\xi \leq C. \end{aligned} \tag{4.3}$$

Now, we prove that (u_k, v_k) is bounded in W. Suppose, by contradiction, $\|(u_k, v_k)\| \to \infty$ as $k \to \infty$. We define a sequence as $(w_k, z_k) = \frac{(u_k, v_k)}{\|(u_k, v_k)\|}$, then, $\|(w_k, z_k)\| = 1$. By Lemmas 2.2 and 2.3, there exists $(w, z) \in W$ such that

$$\begin{aligned} &(w_k, z_k) \rightharpoonup (w, z) \quad \text{weakly in } W, \\ &(w_k, z_k) \to (w, z) \quad \text{strongly in } L^q(\Omega) \times L^q(\Omega) \quad \text{for } q \in [1, 2^*), \\ &(w_k, z_k) \to (w, z) \quad \text{a.e. in } \Omega. \end{aligned} \tag{4.4}$$

We divide the argument into several steps.

Step 1: We prove $w \geq 0$ and $z \geq 0$ *a.e.* in Ω. Let us set $w_k^- = \min\{0, w_k\}$ and $z_k^- = \min\{0, z_k\}$, then (w_k^-, z_k^-) is bounded because (w_k, z_k) in W is bounded. We choose $(\Phi, \Psi) = (w_k^-, z_k^-)$ in (4.2), it follows that

$$o(1) = \frac{|\langle I'_{\mu,\nu}(u_k, v_k), (w_k^-, z_k^-)\rangle|}{\|(u_k, v_k)\|} \quad \text{for } \|(u_k, v_k)\| \to \infty.$$

Therefore, from (f_1), the elementary inequality $|a^- - b^-|^2 \leq (a-b)(a^- - b^-)$, $(a, b \in \mathbb{R})$, (1.5) and a fact that $\mu, \nu < \mathcal{K}$, one has

$$\begin{aligned}
o(1) &= \frac{1}{\|(u_k, v_k)\|}\Big(\langle u_k, w_k^-\rangle + \langle v_k, z_k^-\rangle - \mu\langle u_k, w_k^-\rangle_\psi - \nu\langle v_k, z_k^-\rangle_\psi \\
&\quad - \int_\Omega \Big(F_u(\xi, u_k, v_k)w_k^- + F_v(\xi, u_k, v_k)z_k^-\Big)d\xi\Big) \\
&= \frac{1}{\|(u_k, v_k)\|^2}\Big(\langle u_k, u_k^-\rangle + \langle v_k, v_k^-\rangle - \mu\langle u_k, u_k^-\rangle_\psi - \nu\langle v_k, v_k^-\rangle_\psi\Big) \\
&\quad - \int_\Omega \frac{\Big(F_u(\xi, u_k, v_k)u_k^- + F_v(\xi, u_k, v_k)v_k^-\Big)}{\|(u_k, v_k)\|^2} d\xi \\
&= \frac{1}{\|(u_k, v_k)\|^2}\Big(\langle u_k, u_k^-\rangle + \langle v_k, v_k^-\rangle - \mu\langle u_k, u_k^-\rangle_\psi - \nu\langle v_k, v_k^-\rangle_\psi\Big) \\
&\geq \frac{1}{\|(u_k, v_k)\|^2}\Big(\|u_k^-\|^2_{\mathcal{S}_0^1(\Omega)} + \|v_k^-\|^2_{\mathcal{S}_0^1(\Omega)} - \mu\|u_k^-\|^2_\psi - \nu\|v_k^-\|^2_\psi\Big) \\
&\geq \Big(1 - \frac{\mu^+}{\mathcal{K}}\Big)\|w_k^-\|^2_{\mathcal{S}_0^1(\Omega)} + \Big(1 - \frac{\nu^+}{\mathcal{K}}\Big)\|z_k^-\|^2_{\mathcal{S}_0^1(\Omega)}.
\end{aligned}$$

It follows that

$$\|w_k^-\|_{\mathcal{S}_0^1(\Omega)} \to 0 \quad \text{and} \quad \|z_k^-\|_{\mathcal{S}_0^1(\Omega)} \to 0.$$

Hence, $(w_k^-, z_k^-) \to (0, 0)$ in W as $k \to \infty$, $(w_k^-, z_k^-) = (0, 0)$ *a.e* in Ω, by the definition of w_k^- and z_k^-, we get that $w \geq 0$ and $z \geq 0$ *a.e.* in Ω.

Step 2: We prove $(w, z) = (0, 0)$ *a.e* in Ω. Let us set $\mathbf{D}_+ = \{\xi \in \Omega : w > 0 \text{ or } z > 0\}$ and $\mathbf{D_0} = \{\xi \in \Omega : (w, z) = (0, 0)\}$. Assume that the Haar measure of $\mathbf{D}_+$ is positive. From the assumption that $\|(u_k, v_k)\| \to \infty$, we have

$$|(u_k, v_k)| = \|(u_k, v_k)\||(w_k, z_k)| \to \infty \quad a.e. \text{ in } \mathbf{D}_+.$$

Then, from (f_3), we get

$$\lim_{k\to\infty} \frac{F(\xi, u_k, v_k)}{\|(u_k, v_k)\|^2} = \lim_{k\to\infty} \frac{F(\xi, u_k, v_k)|(w_k, z_k)|^2}{|(u_k, v_k)|^2} = \infty \quad a.e. \text{ in } \mathbf{D}_+. \tag{4.5}$$

Moreover, by Fatou's lemma and (4.5), there holds

$$\begin{aligned}
\liminf_{k\to\infty} \int_\Omega \frac{F(\xi, u_k, v_k)}{\|(u_k, v_k)\|^2} d\xi &\geq \int_\Omega \liminf_{k\to\infty} \frac{F(\xi, u_k, v_k)}{\|(u_k, v_k)\|^2} d\xi \\
&= \int_\Omega \liminf_{k\to\infty} \frac{F(\xi, u_k, v_k)|(w_k, z_k)|^2}{|(u_k, v_k)|^2} d\xi = \infty \quad a.e. \text{ in } \mathbf{D}_+.
\end{aligned} \tag{4.6}$$

On the other hand, from (4.1), a fact that $\|u_k\|^2_{\mathcal{S}^1_0(\Omega)} \leq \|(u_k, v_k)\|^2$, $\|v_k\|^2_{\mathcal{S}^1_0(\Omega)} \leq \|(u_k, v_k)\|^2$ and (1.5), we get

$$\begin{aligned}\int_\Omega F(\xi, u_k, v_k)d\xi &\leq \frac{1}{2}\|u_k\|^2_{\mathcal{S}^1_0(\Omega)} + \frac{1}{2}\|v_k\|^2_{\mathcal{S}^1_0(\Omega)} - \frac{\mu}{2}\|u_k\|^2_\psi - \frac{\nu}{2}\|v_k\|^2_\psi + \mathcal{L} \\ &\leq \|(u_k, v_k)\|^2 + \frac{|\mu^-|}{2\mathcal{K}}\|(u_k, v_k)\|^2 + \frac{|\nu^-|}{2\mathcal{K}}\|(u_k, v_k)\|^2 + \mathcal{L} \quad \text{for } k \in \mathbb{N},\end{aligned}$$

where we have used a fact that $\|(u_k, v_k)\| \geq 1$ because the hypothesis $(u_k, v_k) \to \infty$. Hence

$$\limsup_{k\to\infty} \int_\Omega \frac{F(\xi, u_k, v_k)}{\|(u_k, v_k)\|^2} d\xi \leq 1 + \frac{|\mu^-|}{2\mathcal{K}} + \frac{|\nu^-|}{2\mathcal{K}} + \frac{Z}{\|(u_k, v_k)\|^2},$$

it contradicts with (4.6). Hence, the measure of $\mathbf{D}_+$ is zero, that is $(w, z) = (0, 0)$ *a.e* in Ω.

Step 3: We prove that $\{(u_k, v_k)\} \subset W$ is bounded. Choosing τ_k is the smallest value of $\tau \in [0, 1]$ such that $I_{\mu,\nu}(\tau_k u_k, \tau_k v_k) = \max_{0\leq\tau\leq1} I_{\mu,\nu}(\tau u_k, \tau v_k)$. For $\Lambda > 0$, we set $(W_k, Z_k) = \sqrt{2\Lambda}(w_k, z_k) = \sqrt{2\Lambda}\frac{(u_k, v_k)}{\|(u_k, v_k)\|}$, then, by (4.4) and Step 2, we obtain

$$\lim_{k\to\infty}(W_k, Z_k) = \lim_{k\to\infty}\sqrt{2\Lambda}(w_k, z_k) = \sqrt{2\Lambda}(w, z) = \sqrt{2\Lambda}(0, 0), \tag{4.7}$$

in $L^q(\Omega) \times L^q(\Omega)$ for $q \in [1, 2^*)$. By (f_1), (f_2), (4.7) and let $\epsilon = 1$, it holds

$$\begin{aligned}0 \leq \int_\Omega F(\xi, W_k, Z_k)d\xi &\leq \int_\Omega \Big((\lambda + 1)|(W_k, Z_k)| + C_1|(W_k, Z_k)|^s\Big)d\xi \\ &\leq (\lambda + 1)\|(W_k, Z_k)\|_1 + C_1\|(W_k, Z_k)\|^s_s \to 0, \quad \text{as } k \to \infty,\end{aligned}$$

for $s \in (2, 2^*)$, that is

$$\lim_{k\to\infty}\int_\Omega F(\xi, W_k, Z_k)d\xi = 0. \tag{4.8}$$

From $\|(u_k, v_k)\| \to \infty$, we assume that there is $k_0 \geq k$, such that $\frac{\sqrt{2\Lambda}}{\|(u_k, v_k)\|} \in (0, 1)$, then

$$\begin{aligned}I_{\mu,\nu}(\tau_k u_k, \tau_k v_k) &\geq I_{\mu,\nu}\left(\sqrt{2\Lambda}\frac{u_k}{\|(u_k, v_k)\|}, \sqrt{2\Lambda}\frac{v_k}{\|(u_k, v_k)\|}\right) \\ &\geq \Lambda\|w_k\|^2(1 - \frac{\mu^+}{\mathcal{K}}) + \|z_k\|^2(1 - \frac{\nu^+}{\mathcal{K}}) - \int_\Omega F(\xi, W_k, Z_k)d\xi \\ &\geq \Lambda\chi(\|w_k\|^2 + \|z_k\|^2) - \int_\Omega F(\xi, W_k, Z_k)d\xi \\ &\geq \frac{1}{2}\Lambda\chi - \int_\Omega F(\xi, W_k, Z_k)d\xi,\end{aligned}$$

where χ is defined in Lemma 3.2, $\|w_k\|^2_{\mathcal{S}^1_0(\Omega)} + \|z_k\|^2_{\mathcal{S}^1_0(\Omega)} = 1$ because $\|(w_k, z_k)\| = 1$. By (4.8), there is $k_1 \geq k_0$ such that $\int_\Omega F(\xi, W_k, Z_k)d\xi \leq \frac{1}{2}\Lambda\chi$ for $k \geq k_1$. It follows that

$$\lim_{k\to\infty} I_{\mu,\nu}(\tau_k u_k, \tau_k v_k) = \infty. \tag{4.9}$$

Since $0 < \tau_k < 1$, by (f_4), one has

$$\int_\Omega H(\xi, \tau_k u_k, \tau_k v_k)d\xi \leq C_F\int_\Omega H(\xi, u_k, v_k)d\xi + \int_\Omega g(\xi)d\xi. \tag{4.10}$$

From the facts that $I_{\mu,\nu}(0,0)=0$, $I_{\mu,\nu}(u_k,v_k)\to c\in\mathbb{R}$, (4.9), and $\tau_k\in(0,1)$, there holds

$$\begin{aligned}0=\tau_k\frac{d}{d\tau}I_{\mu,\nu}(\tau u_k,\tau v_k)\Big|_{\tau=\tau_k}&=\langle I'_{\mu,\nu}(\tau_k u_k,\tau_k v_k),(\tau_k u_k,\tau_k v_k)\rangle\\&=\|\tau_k u_k\|^2_{\mathcal{S}^1_0(\Omega)}+\|\tau_k v_k\|^2_{\mathcal{S}^1_0(\Omega)}-\mu\|\tau_k u_k\|^2_\psi-\nu\|\tau_k v_k\|^2_\psi\\&\quad-\int_\Omega\Big(F_u(\xi,\tau_k u_k,\tau_k v_k)\tau_k u_k+F_v(\xi,\tau_k u_k,\tau_k v_k)\tau_k v_k\Big)d\xi.\end{aligned}$$

By (f_4) and (4.10), it follows that

$$\begin{aligned}&\|\tau_k u_k\|^2_{\mathcal{S}^1_0(\Omega)}+\|\tau_k v_k\|^2_{\mathcal{S}^1_0(\Omega)}-\mu\|\tau_k u_k\|^2_\psi-\nu\|\tau_k v_k\|^2_\psi\\&=\int_\Omega\Big(F_u(\xi,\tau_k u_k,\tau_k v_k)\tau_k u_k+F_v(\xi,\tau_k u_k,\tau_k v_k)\tau_k v_k\Big)d\xi\\&=2\int_\Omega F(\xi,\tau_k u_k,\tau_k v_k)d\xi+\int_\Omega H(\xi,\tau_k u_k,\tau_k v_k)d\xi\\&\le 2\int_\Omega F(\xi,\tau_k u_k,\tau_k v_k)d\xi+C_F\int_\Omega H(\xi,u_k,v_k)d\xi+\int_\Omega g(\xi)d\xi.\end{aligned}\tag{4.11}$$

From (4.9) and (4.11), one has

$$\begin{aligned}2I_{\mu,\nu}(\tau_k u_k,\tau_k v_k)&=\|\tau_k u_k\|^2_{\mathcal{S}^1_0(\Omega)}+\|\tau_k v_k\|^2_{\mathcal{S}^1_0(\Omega)}-\mu\|\tau_k u_k\|^2_\psi-\nu\|\tau_k v_k\|^2_\psi\\&\quad-2\int_\Omega F(\xi,\tau_k u_k,\tau_k v_k)d\xi\\&\le C_F\int_\Omega H(\xi,u_k,v_k)d\xi+\int_\Omega g(\xi)d\xi\to\infty\quad\text{as } k\to\infty.\end{aligned}$$

Hence, we deduce that

$$\frac{1}{C_F}\Big(-C+\int_\Omega H(\xi,u_k,v_k)d\xi\Big)\to\infty\quad\text{as } k\to\infty.\tag{4.12}$$

On the other hand, by (4.1), (f_4) and (4.3), we have

$$\begin{aligned}\tilde{\mathcal{L}}&\ge 2I_{\mu,\nu}(u_k,v_k)\\&=\|u_k\|^2_{\mathcal{S}^1_0(\Omega)}+\|v_k\|^2_{\mathcal{S}^1_0(\Omega)}-\mu\|u_k\|^2_\psi-\nu\|v_k\|^2_\psi-2\int_\Omega F(\xi,u_k,v_k)d\xi\\&=\|u_k\|^2_{\mathcal{S}^1_0(\Omega)}+\|v_k\|^2_{\mathcal{S}^1_0(\Omega)}-\mu\|u_k\|^2_\psi-\nu\|v_k\|^2_\psi\\&\quad-\int_\Omega\Big(F_u(\xi,u_k,v_k)u_k+F_v(\xi,u_k,v_k)v_k\Big)d\xi+\int_\Omega H(\xi,u_k,v_k)d\xi\\&\ge -C+\int_\Omega H(\xi,u_k,v_k)d\xi,\end{aligned}\tag{4.13}$$

where $\tilde{\mathcal{L}}$ is a positive constant. Since $C_F\ge 1$ in (f_4) and by (4.13), we obtain

$$\frac{1}{C_F}\Big(-C+\int_\Omega H(\xi,u_k,v_k)d\xi\Big)\le -C+\int_\Omega H(\xi,u_k,v_k)d\xi\le\tilde{\mathcal{L}}.$$

This contradicts with (4.12), it follows that $\{(u_k,v_k)\}\subset W$ is a bounded Cerami sequence. We finish the proof of this lemma. □

In the following, we verify that $I_{\mu,\nu}$ satisfies the Cerami condition at level c.

Lemma 4.2. *Assume that* (f_2) *with* $\epsilon = 1$ *holds. Then, for all* μ, $\nu \in (-\infty, \mathcal{K})$, $I_{\mu,\nu}$ *satisfies the Cerami condition in* W.

Proof. Assume that $\{(u_k, v_k)\} \subset W$ is a Cerami sequence of $I_{\mu,\nu}$. Then, by Lemma 4.1, we know that $\{(u_k, v_k)\}$ is bounded. Then, up to subsequence, from (1.5), Lemmas 2.2 and 2.3, for $1 \le q < 2^*$, there exists $(u, v) \in W$ such that

$$
\begin{aligned}
(u_k, v_k) \rightharpoonup (u, v) \ \text{ in } W, \quad \|u_k - u\|_{\mathcal{S}_0^1(\Omega)} \to \bar{a}, \ \ \|v_k - v\|_{\mathcal{S}_0^1(\Omega)} \to \check{a}, \\
u_k \rightharpoonup u \ \text{ in } L^2(\Omega, \psi^2 r^{-2}), \ \ \|u_k - u\|_\psi \to \acute{a}, \\
v_k \rightharpoonup v \ \text{ in } L^2(\Omega, \psi^2 r^{-2}), \ \ \|v_k - v\|_\psi \to \grave{a}, \\
(u_k, v_k) \to (u, v) \ \text{ in } L^q(\Omega) \times L^q(\Omega), \quad (u_k, v_k) \to (u, v) \ \text{ a.e in } \Omega, \\
\nabla_{\mathbb{G}} u_k \rightharpoonup \nabla_{\mathbb{G}} u \ \text{ in } L^2(\Omega, \mathbb{R}^{2n}), \quad \nabla_{\mathbb{G}} v_k \rightharpoonup \nabla_{\mathbb{G}} v \ \text{ in } L^2(\Omega, \mathbb{R}^{2n}), \\
\nabla_{\mathbb{G}} u_k \rightharpoonup \vartheta \ \text{ in } L^2(\Omega, \mathbb{R}^{2n}), \quad \nabla_{\mathbb{G}} v_k \rightharpoonup \varsigma \ \text{ in } L^2(\Omega, \mathbb{R}^{2n}),
\end{aligned}
\tag{4.14}
$$

where $\vartheta, \varsigma \in L^2(\Omega, \mathbb{R}^{2n})$ are two vector field functions in Ω, and $\bar{a}$, $\acute{a}$, $\check{a}$, $\grave{a}$ are four nonnegative numbers.

From (4.14), we conclude that

$$
\int_\Omega \frac{\psi^2}{r(\xi)^2} u_k \Phi d\xi \to \int_\Omega \frac{\psi^2}{r(\xi)^2} u \Phi d\xi \quad \text{and} \quad \int_\Omega \frac{\psi^2}{r(\xi)^2} v_k \Psi d\xi \to \int_\Omega \frac{\psi^2}{r(\xi)^2} v \Psi d\xi, \tag{4.15}
$$

for $(\Phi, \Psi) \in W$. We choose $\epsilon = 1$ in (f_2), and by Hölder inequality, then

$$
\begin{aligned}
&\int_\Omega \Big| \Big(F_u(\xi, u_k, v_k) - F_u(\xi, u, v) \Big) (u_k - u) \\
&\qquad + \Big(F_v(\xi, u_k, v_k) - F_v(\xi, u, v) \Big) (v_k - v) \Big| d\xi \\
&= \int_\Omega \Big| F_w(\xi, w_k)(w_k - w) - F_w(\xi, w)(w_k - w) \Big| d\xi \\
&\le \int_\Omega \Big((\lambda + 1)(|w_k| + |w|)|w_k - w| + C_1(|w_k|^{s-1} + |w|^{s-1})|w_k - w| \Big) d\xi \\
&\le C_\lambda (\|w_k - w\|_2 + \|w_k - w\|_s) \to 0 \quad \text{as } k \to \infty,
\end{aligned}
\tag{4.16}
$$

where $C_\lambda > 0$ is a suitable constant. From (4.1), it holds that $I'_{\mu,\nu}(u_k, v_k) \to 0$ in W' as $k \to \infty$, then for every $(\Phi, \Psi) \in W$, we have

$$
\begin{aligned}
0 \leftarrow & \langle I'_{\mu,\nu}(u_k, v_k), (\Phi, \Psi) \rangle \\
= & \int_\Omega (\nabla_{\mathbb{G}} u_k, \nabla_{\mathbb{G}} \Phi) d\xi + \int_\Omega (\nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}} \Psi) d\xi - \mu \int_\Omega \frac{\psi^2}{r(\xi)^2} u_k \Phi d\xi - \nu \int_\Omega \frac{\psi^2}{r(\xi)^2} v_k \Psi d\xi \\
& - \int_\Omega \Big(F_u(\xi, u_k, v_k) \Phi + F_v(\xi, u_k, v_k) \Psi \Big) d\xi.
\end{aligned}
\tag{4.17}
$$

Subsequently, we prove that the (PS) sequence satisfies compactness condition by means of the Brézis–Lieb lemma.

From (4.17) and Lemma A.1 (it shows that (u_k, v_k) satisfies the Brézis–Lieb lemma's condition, see in the Appendix), one has

$$
\nabla_{\mathbb{G}} u_k \to \nabla_{\mathbb{G}} u \quad \text{and} \ \ \nabla_{\mathbb{G}} v_k \to \nabla_{\mathbb{G}} v \quad a.e. \text{ in } \ \Omega, \tag{4.18}
$$

and by (4.14), there holds $\nabla_{\mathbb{G}} u_k \rightharpoonup \vartheta$ and $\nabla_{\mathbb{G}} v_k \rightharpoonup \varsigma$ in $L^2(\Omega, \mathbb{R}^{2n})$. Hence, from Proposition A.7 in [1], we obtain $\nabla_{\mathbb{G}} u = \vartheta$ and $\nabla_{\mathbb{G}} v = \varsigma$ a.e. in Ω. It yields that $\nabla_{\mathbb{G}} u_k \rightharpoonup \nabla_{\mathbb{G}} u$ and $\nabla_{\mathbb{G}} v_k \rightharpoonup \nabla_{\mathbb{G}} v$ in $L^2(\Omega, \mathbb{R}^{2n})$, therefore, for any $(\Phi, \Psi) \in W$, one has

$$\int_\Omega (\nabla_{\mathbb{G}} u_k, \nabla_{\mathbb{G}} \Phi) d\xi \to \int_\Omega (\nabla_{\mathbb{G}} u, \nabla_{\mathbb{G}} \Phi) d\xi \quad \text{and} \quad \int_\Omega (\nabla_{\mathbb{G}} v_k, \nabla_{\mathbb{G}} \Phi) d\xi \to \int_\Omega (\nabla_{\mathbb{G}} v, \nabla_{\mathbb{G}} \Phi) d\xi.$$

It follows $\langle u_k, u\rangle \to \|u\|^2_{\mathcal{S}^1_0(\Omega)}$, $\langle u, u_k\rangle \to \|u\|^2_{\mathcal{S}^1_0(\Omega)}$ and $\langle v_k, v\rangle \to \|v\|^2_{\mathcal{S}^1_0(\Omega)}$, $\langle v, v_k\rangle \to \|v\|^2_{\mathcal{S}^1_0(\Omega)}$. Moreover, by (4.15) and (4.16), the weak limit $w = (u, v)$ is a critical point of $I_{\mu,\nu}$ in W. From (4.14) and (4.18), the Brézis–Lieb lemma holds that

$$\|u_k\|^2_{\mathcal{S}^1_0(\Omega)} = \|u_k - u\|^2_{\mathcal{S}^1_0(\Omega)} + \|u\|^2_{\mathcal{S}^1_0(\Omega)} + o(1), \quad \|v_k\|^2_{\mathcal{S}^1_0(\Omega)} = \|v_k - v\|^2_{\mathcal{S}^1_0(\Omega)} + \|v\|^2_{\mathcal{S}^1_0(\Omega)} + o(1),$$
$$\|u_k\|^2_\psi = \|u_k - u\|^2_\psi + \|u\|^2_\psi + o(1), \quad \|v_k\|^2_\psi = \|v_k - v\|^2_\psi + \|v\|^2_\psi + o(1).$$

Consequently, one has

$$\begin{aligned} o(1) &= \langle I'_{\mu,\nu}(w_k) - I'_{\mu,\nu}(w), w_k - w\rangle \\ &= \|u_k\|^2_{\mathcal{S}^1_0(\Omega)} + \|u\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_k\|^2_{\mathcal{S}^1_0(\Omega)} + \|v\|^2_{\mathcal{S}^1_0(\Omega)} - \langle u_k, u\rangle - \langle u, u_k\rangle - \langle v_k, v\rangle - \langle u, u_k\rangle \\ &\quad - \mu\Big(\|u_k\|^2_\psi + \|u\|^2_\psi - \langle u_k, u\rangle_\psi - \langle u, u_k\rangle_\psi\Big) \\ &\quad - \nu\Big(\|v_k\|^2_\psi + \|v\|^2_\psi - \langle v_k, v\rangle_\psi - \langle v v_k\rangle_\psi\Big) + o(1) \\ &= \|u_k\|^2_{\mathcal{S}^1_0(\Omega)} - \|u\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_k\|^2_{\mathcal{S}^1_0(\Omega)} - \|v\|^2_{\mathcal{S}^1_0(\Omega)} - \mu(\|u_k\|^2_\psi - \|u\|^2_\psi) \\ &\quad - \nu(\|v_k\|^2_\psi - \|v\|^2_\psi) + o(1) \\ &= \|u_k - u\|^2_{\mathcal{S}^1_0(\Omega)} + \|v_k - v\|^2_{\mathcal{S}^1_0(\Omega)} - \mu\|u_k - u\|^2_\psi - \nu\|v_k - v\|^2_\psi + o(1). \end{aligned}$$

From (4.14) and above equality, it follows that

$$\begin{aligned} \bar{a}^2 + \breve{a}^2 &= \lim_{k\to\infty} \|u_k - u\|^2_{\mathcal{S}^1_0(\Omega)} + \lim_{k\to\infty} \|v_k - v\|^2_{\mathcal{S}^1_0(\Omega)} \\ &= \mu \lim_{k\to\infty} \|u_k - u\|^2_\psi + \nu \lim_{k\to\infty} \|v_k - v\|^2_\psi \\ &= \mu \acute{a}^2 + \nu \grave{a}^2. \end{aligned} \tag{4.19}$$

Thus, when either $\mu^+ + \nu^+ = 0$ or $\acute{a} + \grave{a} = 0$, we get $(u_k, v_k) \to (u, v)$ in W as $k \to \infty$ and finish the proof about compactness condition for (PS) sequence of I_ϵ. In order to achieve this aim, we assume by contradiction, that is $\mu^+ + \nu^+ > 0$ and $\acute{a} + \grave{a} > 0$.

(1) If either $\mu^+ + \grave{a} = 0$ or $\nu^+ + \acute{a} = 0$, then either $\acute{a} > 0$ and $\bar{a} = 0$, or $\grave{a} > 0$ and $\breve{a} = 0$. However, all of cases are impossible because the nonnegative of norm in (4.14).

(2) If either $\mu^+ + \acute{a} = 0$ or $\nu^+ + \grave{a} = 0$, then either $\grave{a} > 0$, $\nu^+ > 0$ and $\breve{a}^2 \le \nu^+ \grave{a}^2 < \mathcal{K}\grave{a}^2 \le \breve{a}^2$, or $\acute{a} > 0$, $\mu^+ > 0$ and $\bar{a}^2 \le \mu^+ \acute{a}^2 < \mathcal{K}\acute{a}^2 \le \bar{a}^2$, it appears a contradiction.

(3) $\mu^+ > 0$, $\nu^+ > 0$, $\acute{a} > 0$ and $\grave{a} > 0$, from (4.19) and (1.5), we get

$$\bar{a}^2 + \breve{a}^2 = \mu\acute{a}^2 + \nu\grave{a}^2 < \mathcal{K}\acute{a}^2 + \mathcal{K}\grave{a}^2 \le \bar{a}^2 + \breve{a}^2,$$

and a contradiction arises. From above discussions, we get $\acute{a} + \grave{a} = 0$, that is $(u_k, v_k) \to (u, v)$ in W as $k \to \infty$, from (4.19), the proof of this lemma is finished. □

4.2 The existence of solution

In this part, we study the existence of nonnegative solution for (1.1).

Proof of Theorem 1.1. From Lemmas 3.2 and 3.3, we know that $I_{\mu,\nu}$ satisfies the mountain pass geometry structures. Moreover, the Cerami condition holds by Lemma 4.2. Therefore, for every $(\Phi, \Psi) \in W$, there exists $(u,v) \in W$, $(u,v) \neq (0,0)$ such that

$$\langle u, \Phi\rangle + \langle v, \Psi\rangle - \mu\langle u, \Phi\rangle_\psi - \nu\langle v, \Psi\rangle_\psi = \int_\Omega \Big(F_u(\xi,u,v)\Phi + F_v(\xi,u,v)\Psi\Big)d\xi.$$

Now, we prove that (u,v) is nonnegative. Let us set $\Phi = u^- = \min\{0,u\}$ and $\Psi = v^- = \min\{0,v\}$, then, from (f_1), (1.6) and (1.8), one has

$$\begin{aligned} 0 &= \int_\Omega \Big(F_u(\xi,u,v)u^- + F_v(\xi,u,v)v^-\Big)d\xi \\ &= \langle u, u^-\rangle + \langle v, v^-\rangle - \mu\langle u, u^-\rangle_\psi - \nu\langle v, v^-\rangle_\psi \\ &\geq (1 - \frac{\mu^+}{\mathcal{K}})\|u^-\|^2_{\mathcal{S}^1_0(\Omega)} + (1 - \frac{\nu^+}{\mathcal{K}})\|v^-\|^2_{\mathcal{S}^1_0(\Omega)} \geq 0. \end{aligned}$$

Thus, $u^- = 0$ and $v^- = 0$ a.e. in Ω, that is $u \geq 0$ and $v \geq 0$ a.e. in Ω, it shows that any solution of (1.1) is nonnegative. We finish the proof. □

A Appendix

In this section, we give a proof for the following lemma.

Lemma A.1. *Let (u_k, v_k) and (u,v) belongs to W and satisfying*

(i) $(u_k, v_k) \rightharpoonup (u,v)$ *in* W,

(ii) $(u_k, v_k) \to (u,v)$ *a.e. in* Ω,

(iii) $I'_{\mu,\nu}(u_k, v_k) \to 0$ *strongly in* W',

(iv) ϑ, $\varsigma \in \Omega$ *are two vector field functions with* ϑ, $\varsigma \in L^2(\Omega, \mathbb{R}^{2n})$ *such that* $\nabla_{\mathbb{G}} u_k \rightharpoonup \vartheta$ *and* $\nabla_{\mathbb{G}} v_k \rightharpoonup \varsigma$ *in* $L^2(\Omega, \mathbb{R}^{2n})$*, then, it holds*

$$\nabla_{\mathbb{G}} u_k \to \nabla_{\mathbb{G}} u \quad \text{and} \quad \nabla_{\mathbb{G}} v_k \to \nabla_{\mathbb{G}} v \quad \text{a.e. in } \Omega. \tag{A.1}$$

Proof. Let function $\beta_R \in C_0^\infty(\Omega)$ with $R > 0$, such that $0 \leq \beta_R \leq 1$ in Ω and $\beta_R \equiv 1$ in B_R. For every $z \in \mathbb{R}$, we define

$$\varrho_\epsilon(z) = \begin{cases} z, & \text{if } |z| < \epsilon, \\ \epsilon\dfrac{z}{|z|}, & \text{if } |z| \geq \epsilon. \end{cases}$$

We set $\phi_k = \beta_R \varrho_\epsilon \circ (u_k - u)$ and $\varphi_k = \beta_R \varrho_\epsilon \circ (v_k - v)$, thus, by Lemma 2.1, there holds ϕ_k,

$\varphi_k \in W^{1,2}(\Omega)$. Let $\Phi = \phi_k$ and $\Psi = \varphi_k$ in (4.17), then

$$\begin{aligned}
&\int_\Omega \beta_R\Big((\nabla_{\mathbb{G}}u_k - \nabla_{\mathbb{G}}u, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon\circ(u_k-u)\Big)\Big)d\xi \\
&\qquad + \int_\Omega \beta_R\Big(\nabla_{\mathbb{G}}v_k - \nabla_{\mathbb{G}}v, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon\circ(v_k-v)\Big)\Big)d\xi \\
&= -\int_\Omega \varrho_\epsilon\circ(u_k-u)(\nabla_{\mathbb{G}}u_k, \nabla_{\mathbb{G}}\beta_R)d\xi - \int_\Omega \beta_R\Big(\nabla_{\mathbb{G}}u, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon\circ(u_k-u)\Big)\Big)d\xi \\
&\qquad - \int_\Omega \varrho_\epsilon\circ(v_k-v)(\nabla_{\mathbb{G}}v_k, \nabla_{\mathbb{G}}\beta_R)d\xi - \int_\Omega \beta_R\Big(\nabla_{\mathbb{G}}v, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon\circ(v_k-v)\Big)\Big)d\xi \\
&\qquad + \langle I'_{\mu,\nu}(u_k,v_k),(\phi_k,\varphi_k)\rangle + \mu\int_\Omega \frac{\psi^2}{r(\xi)^2}u_k\phi_k d\xi + \nu\int_\Omega \frac{\psi^2}{r(\xi)^2}v_k\varphi_k d\xi \\
&\qquad + \int_\Omega \Big(F_u(\xi,u_k,v_k)\phi_k + F_v(\xi,u_k,v_k)\varphi_k\Big)d\xi. \qquad \text{(A.2)}
\end{aligned}$$

Now, we prove the each term in (A.2).

(1) We choose that $\tilde{\beta}_R$ be the support of β_R and contained in a suitable ball of Ω, since $|\varrho_\epsilon\circ(u_k-u)\nabla_{\mathbb{G}}\beta_R| \to 0$ in $L^2(\tilde{\beta}_R)$ and $|\varrho_\epsilon\circ(v_k-v)\nabla_{\mathbb{G}}\beta_R| \to 0$ in $L^2(\tilde{\beta}_R)$, and by (4.14), $\nabla_{\mathbb{G}}u_k \rightharpoonup \vartheta$ in $L^2(\Omega,\mathbb{R}^{2n})$, $\nabla_{\mathbb{G}}v_k \rightharpoonup \varsigma$ in $L^2(\Omega,\mathbb{R}^{2n})$, then

$$\int_\Omega \varrho_\epsilon\circ(u_k-u)(\nabla_{\mathbb{G}}u_k,\nabla_{\mathbb{G}}\beta_R)d\xi \to 0 \quad \text{and} \quad \int_\Omega \varrho_\epsilon\circ(v_k-v)(\nabla_{\mathbb{G}}v_k,\nabla_{\mathbb{G}}\beta_R)d\xi \to 0.$$

(2) Since $\nabla_{\mathbb{G}}\Big(\varrho_\epsilon\circ(u_k-u)\Big)$ in $L^2(\Omega,\mathbb{R}^{2n})$, $\nabla_{\mathbb{G}}\Big(\varrho_\epsilon\circ(v_k-v)\Big)$ in $L^2(\Omega,\mathbb{R}^{2n})$, $\nabla_{\mathbb{G}}u_k \in L^2(\Omega,\mathbb{R}^{2n})$, $\nabla_{\mathbb{G}}v_k \in L^2(\Omega,\mathbb{R}^{2n})$. From Lemma 2.1, $u_k \rightharpoonup u$ and $v_k \rightharpoonup v$ in W, one has

$$\int_\Omega \beta_R\Big(\nabla_{\mathbb{G}}u, \nabla_{\mathbb{G}}(\varrho_\epsilon\circ(u_k-u))\Big)d\xi \to 0 \quad \text{and} \quad \int_\Omega \beta_R\Big(\nabla_{\mathbb{G}}v, \nabla_{\mathbb{G}}(\varrho_\epsilon\circ(v_k-v))\Big)d\xi \to 0.$$

(3) From $I'_{\mu,\nu}(u_k,v_k) \to 0$ in W' and $(\varphi_k,\phi_k) \rightharpoonup 0$ in W as $k\to\infty$, we have

$$\langle I'_{\mu,\nu}(u_k,v_k),(\varphi_k,\phi_k)\rangle \rightharpoonup 0.$$

(4) For simplicity, we denote

$$M_k = \mu\frac{\psi^2}{r(\xi)^2}u_k + F_u(\xi,u_k,v_k), \quad N_k = \nu\frac{\psi^2}{r(\xi)^2}v_k + F_v(\xi,u_k,v_k), \qquad \text{(A.3)}$$

by $0 \le \beta_R \le 1$ in Ω, the definition of ϕ_k, φ_k and $\varrho_\epsilon(z)$, Lemma A.2, there holds

$$\begin{aligned}
\int_\Omega (M_k\varphi_k + N_k\phi_k)d\xi &\le \int_{\tilde{\beta}_R}\Big(|M_k|\cdot|\varrho_\epsilon\circ(u_k-u)| + |N_k|\cdot|\varrho_\epsilon\circ(v_k-v)|\Big)d\xi \\
&\le \epsilon\int_{\tilde{\beta}_R}(|M_k|+|N_k|)d\xi \le \epsilon C_R,
\end{aligned}$$

where $C_R > 0$ is a constant. Moreover

$$\begin{aligned}
&\beta_R\Big(\nabla_{\mathbb{G}}u_k - \nabla_{\mathbb{G}}u, \nabla_{\mathbb{G}}(\varrho_\epsilon\circ(u_k-u))\Big) \ge 0, \\
&\beta_R\Big(\nabla_{\mathbb{G}}v_k - \nabla_{\mathbb{G}}v, \nabla_{\mathbb{G}}(\varrho_\epsilon\circ(v_k-v))\Big) \ge 0 \quad \textit{a.e.} \text{ in } \Omega. \qquad \text{(A.4)}
\end{aligned}$$

Furthermore

$$\begin{aligned}
&\int_{B_R} \beta_R\Big(\nabla_{\mathbb{G}}u_k - \nabla_{\mathbb{G}}u, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (u_k - u)\Big)\Big)d\xi \\
&\qquad + \int_{B_R} \beta_R\Big(\nabla_{\mathbb{G}}v_k - \nabla_{\mathbb{G}}v, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (v_k - v)\Big)\Big)d\xi \\
&\quad \leq \int_{\Omega} \beta_R\Big(\nabla_{\mathbb{G}}u_k - \nabla_{\mathbb{G}}u, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (u_k - u)\Big)\Big)d\xi \\
&\qquad + \int_{\Omega} \beta_R\Big(\nabla_{\mathbb{G}}v_k - \nabla_{\mathbb{G}}v, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (v_k - v)\Big)\Big)d\xi.
\end{aligned}$$

From (1)–(4) and a fact that $\beta_R \equiv 1$ in B_R, then (A.2) becomes

$$\begin{aligned}
&\limsup_{k\to\infty} \Bigg[\int_{B_R} \Big(\nabla_{\mathbb{G}}u_k - \nabla_{\mathbb{G}}u, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (u_k - u)\Big)\Big)d\xi \\
&\qquad + \int_{B_R} \Big(\nabla_{\mathbb{G}}v_k - \nabla_{\mathbb{G}}v, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (v_k - v)\Big)\Big)d\xi\Bigg] \\
&\quad \leq \limsup_{k\to\infty} \Bigg[\int_{\Omega} \Big(\nabla_{\mathbb{G}}u_k - \nabla_{\mathbb{G}}u, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (u_k - u)\Big)\Big)d\xi \\
&\qquad + \int_{\Omega} \Big(\nabla_{\mathbb{G}}v_k - \nabla_{\mathbb{G}}v, \nabla_{\mathbb{G}}\Big(\varrho_\epsilon \circ (v_k - v)\Big)\Big)d\xi\Bigg] \leq \epsilon C_R.
\end{aligned} \tag{A.5}$$

Subsequently, let $g_k = g_{u,k} + g_{v,k}$ with $g_{u,k} = \big(\nabla_{\mathbb{G}}u_k - \nabla_{\mathbb{G}}u, \nabla_{\mathbb{G}}(u_k - u)\big)$ and $g_{v,k} = \big(\nabla_{\mathbb{G}}v_k - \nabla_{\mathbb{G}}v, \nabla_{\mathbb{G}}(v_k - v)\big)$. We will show that g_k is nonnegative and bounded in $L^1(\Omega)$.

Firstly, if we assume that g_k is negative, it appears a contradiction with (A.4), thus, g_k is nonnegative. Secondly, since $\nabla_{\mathbb{G}}u_k$ is bounded in $L^2(\Omega, \mathbb{R}^{2n})$, and by (4.14), we know that $\nabla_{\mathbb{G}}v_k$ is bounded in $L^2(\Omega, \mathbb{R}^{2n})$. Therefore

$$0 \leq \int_{\Omega} g_k(\xi)d\xi \leq \|\nabla_{\mathbb{G}}u_k - \nabla_{\mathbb{G}}u\|_2^2 + \|\nabla_{\mathbb{G}}v_k - \nabla_{\mathbb{G}}v\|_2^2 \leq C_0, \tag{A.6}$$

where C_0 is a suitable constant and independent of k.

We select $t \in (0,1)$ and divide the ball B_R into four parts,

$$\begin{aligned}
B^\epsilon_{u,k}(R) &= \{\xi \in B_R : |u_k(\xi) - u(\xi)| \leq \epsilon\}, &\quad \tilde{B}^\epsilon_{u,k}(R) &= B_R \setminus B^\epsilon_{u,k}(R), \\
B^\epsilon_{v,k}(R) &= \{\xi \in B_R : |v_k(\xi) - v(\xi)| \leq \epsilon\}, &\quad \tilde{B}^\epsilon_{v,k}(R) &= B_R \setminus B^\epsilon_{v,k}(R).
\end{aligned}$$

Since $\nabla_{\mathbb{G}}\big(\varrho_\epsilon \circ (u_k - u)\big) = \nabla_{\mathbb{G}}(u_k - u)$ in $B^\epsilon_{u,k}(R)$ and $\nabla_{\mathbb{G}}\big(\varrho_\epsilon \circ (v_k - v)\big) = \nabla_{\mathbb{G}}(v_k - v)$ in $B^\epsilon_{v,k}(R)$, and from (A.6), we get

$$\begin{aligned}
\int_{B_R} g_k^t d\xi &\leq \int_{B_R} g_{u,k}^t d\xi + \int_{B_R} g_{v,k}^t d\xi \\
&= \int_{B^\epsilon_{u,k}(R)} g_{u,k}^t d\xi + \int_{\tilde{B}^\epsilon_{u,k}(R)} g_{u,k}^t d\xi + \int_{B^\epsilon_{v,k}(R)} g_{v,k}^t d\xi + \int_{\tilde{B}^\epsilon_{v,k}(R)} g_{v,k}^t d\xi \\
&\leq \Big(\int_{B^\epsilon_{u,k}(R)} g_{u,k} d\xi\Big)^t |B^\epsilon_{u,k}(R)|^{1-t} + \Big(\int_{\tilde{B}^\epsilon_{u,k}(R)} g_k d\xi\Big)^t |\tilde{B}^\epsilon_{u,k}(R)|^{1-t} \\
&\quad + \Big(\int_{B^\epsilon_{v,k}(R)} g_{v,k} d\xi\Big)^t |B^\epsilon_{v,k}(R)|^{1-t} + \Big(\int_{\tilde{B}^\epsilon_{v,k}(R)} g_k d\xi\Big)^t |\tilde{B}^\epsilon_{v,k}(R)|^{1-t} \\
&\leq (\epsilon C_R)^t \Big(|B^\epsilon_{u,k}(R)|^{1-t} + |\tilde{B}^\epsilon_{v,k}(R)|^{1-t}\Big) + C_0^t \Big(|B^\epsilon_{u,k}(R)|^{1-t} + |\tilde{B}^\epsilon_{v,k}(R)|^{1-t}\Big).
\end{aligned}$$

Moreover, the definition of $B^{\epsilon}_{u,k}(R)$ and $B^{\epsilon}_{v,k}(R)$ follows that, $|\tilde{B}^{\epsilon}_{u,k}(R)|$ and $|\tilde{B}^{\epsilon}_{v,k}(R)|$ tends to 0 as k goes to ∞. Thus, $0 \le \limsup_{k\to\infty} \int_{B_R} g_k^t d\xi \le (\epsilon C_R)^t |B_R|^{1-t}$, which means that $g_k^t \to 0$ as $\epsilon \to 0$ in $L^1(B_R)$. Hence, $g_k \to 0$ a.e. in Ω for R is arbitrary, then, (A.1) is valid from Lemma 3 in [13]. □

Finally, the following result shows that the hardy term is bounded in W.

Lemma A.2. *Let $\{(u_k, v_k)\} \subset W$ be a bounded sequence and Ω_0 represent a compact set of Ω, M_k and N_k are given in* (A.3)*. Then there is a constant $C(\Omega_0) > 0$ such that*

$$\sup_k \int_{\Omega_0} (|M_k| + |N_k|) d\xi \le C(\Omega_0).$$

Proof. Since $\psi = |\psi| \le 1$ and the Jacobian determinant is r^4, thus, $\psi^2 r^{-2}$ be of class $L^1_{loc}(\Omega)$, by (1.5), one has

$$\int_{\Omega_0} \left(\left(\frac{\psi}{r}\right)^2 |u_k| + \left(\frac{\psi}{r}\right)^2 |v_k| \right) d\xi \le \left\| \frac{\psi}{r} \right\|_2 \sup_k \|u_k\|_\psi + \left\| \frac{\psi}{r} \right\|_2 \sup_k \|v_k\|_\psi = C_2(\Omega_0),$$

where $C_2(\Omega_0)$ is a positive constant depending on Ω_0. Moreover, from (f_2), it holds

$$\begin{aligned}
&\int_{\Omega_0} \Big| F_u(\xi, u_k, v_k) + F_v(\xi, u_k, v_k) \Big| d\xi \\
&\le \sqrt{2} \int_{\Omega_0} \left| \frac{\sqrt{H_u^2(\xi, u_k, v_k) + H_v^2(\xi, u_k, v_k)}}{2} \right| d\xi \\
&\le \sqrt{2} \int_{\Omega_0} \Big| (\lambda + 1) |(u_k, v_k)| + C_\epsilon |(u_k, v_k)|^{s-1} \Big| d\xi \\
&\le \sqrt{2} \Big((\lambda + 1) \sup_k \|(u_k, v_k)\|_{2^*} |\Omega_0|^{\frac{1}{t}} + C_\epsilon |\Omega_0|^{2^*-2+1} \sup_k \|(u_k, v_k)\|_{2^*}^{2-1} \Big) = C_3(\Omega_0),
\end{aligned}$$

where $t > 1$ and $t = \frac{2^*}{2^*-1}$ is the Lebesgue exponent for $s \in (2, 2^*)$. From above argument, we get $\sup_k \int_{\Omega_0} (|M_k| + |N_k|) d\xi \le C(\Omega_0)$, the proof of this lemma is completed. □

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Existence of solution for a generalized Schrödinger–Poisson system via bifurcation theory

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Abstract. In this paper, we study a generalized Schrödinger–Poisson system in a bounded domain of $\mathbb{R}^3$ and involving an asymptotically linear nonlinearity. We prove the existence of positive solutions using bifurcation theory.

Keywords: bifurcation theory, positive solutions, Schrödinger–Poisson system, topological degree.

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1 Introduction

This paper is concerned with the existence of solutions for the problem

$$\begin{cases} -\Delta u + \phi(x)u = \lambda f(u) \text{ in } \Omega, \\ -\Delta\phi(x) = g(u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ \phi > 0 \text{ in } \Omega, \\ u(x) = \phi(x) = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{P}$$

where $0 < \lambda$ is a parameter, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $f \in C^1([0,\infty),\mathbb{R})$ and $g \in C(\mathbb{R},[0,\infty))$.

When the function $g(t) = t^2$, this system represents the well known Schrödinger–Poisson (or Schrödinger–Maxwell) equations, that have been widely studied in the recent past. This equation appears in the mean field approach for the Hartree–Fock model and as a nonlinear Schrödinger equation that takes into account the electrostatic field generated by the wave, see [7, 10, 14, 15].

Recently, many authors have studied the existence, non-existence and multiplicity of solutions of the problem

$$\begin{cases} -\Delta u + \lambda\phi(x)u = z(u) \text{ in } \Omega, \\ -\Delta\phi(x) = u^2 \text{ in } \Omega, \\ u(x) = \phi(x) = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

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where z is a superlinear function; see for example [1,3,8,9,11,13] and the references therein. To prove their results they used the reduction argument and then employed variational methods. It is worth pointing out that in the proof of Theorem 2.1 of [13] the authors used the Leray–Schauder degree to prove the existence of a positive solution when the parameter λ is small enough. Also, in the references of the papers mentioned above the reader will find many works dealing with Schrödinger–Poisson systems where $\Omega = \mathbb{R}^3$.

Motivated by the papers above and Ambrosetti and Hess [4], we are interested in studying system (P) when f is asymptotically linear and g satisfies some suitable assumptions. Specifically, we introduce the following assumptions:

(F_1) $f \in C^1([0,\infty),\mathbb{R})$, $f(0) = 0$ and $m_0 = \lim_{t\to 0^+} \frac{f(t)}{t} > 0$ (namely $m_0 = f'_+(0)$);

(F_2) There exist $m_\infty > 0$, a function h and a constant C such that

$$f(t) = m_\infty t + h(t), \text{ where } h \in C^{0,1}(\mathbb{R}^+,\mathbb{R}) \text{ and } |h(t)| \le C, \forall t \in \mathbb{R}^+ (\mathbb{R}^+ = [0,\infty));$$

(G_1) $g(t) = t^{2p}$, where $0 < p < 2$;

(G_2) $g \in C(\mathbb{R},(0,\infty))$ and there exist the limit $\lim_{t\to\infty} g(t) = g(\infty)$ and a constant $c > 0$ such that $0 < g(t) < c$ for all $t \in \mathbb{R}$.

Some examples of functions satisfying the above assumptions are as follows.

Example 1.1.

(a) The function $f(t) = t - t^{10}, t \ge 0$, satisfies (F_1).

(b) The function $f(t) = t - arctg(t^2), t \ge 0$, satisfies (F_1) and (F_2).

(c) The function $f(t) = t, t \ge 0$, satisfies (F_1) and (F_2).

(e) The function $g(t) = \frac{t^2}{1+t^2} + 1, t \in \mathbb{R}$, satisfies (G_2).

(g) The function $g(t) = \frac{|t|}{1+t^2} + 1, t \in \mathbb{R}$, satisfies (G_2).

As we can see, the function f is allowed to change sign. Before stating our main results, we need some definitions and notations. First, we introduce the Banach space

$$X = C(\overline{\Omega},\mathbb{R})$$

endowed with the norm $\|u\| = \sup_{x\in\overline{\Omega}} |u(x)|$ for $u \in X$.

We say that $(\lambda, u, \phi_u) \in \mathbb{R} \times [(H_0^1(\Omega) \times H_0^1(\Omega)) \cap (X \times X)]$ is a solution of (P) if $u > 0$ in Ω, $\phi_u > 0$ in Ω and

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \phi_u(x) u \varphi dx = \lambda \int_\Omega f(u)\varphi dx, \tag{1.1}$$

$$\int_\Omega \nabla \phi_u \nabla \psi dx = \int_\Omega g(u)\psi dx, \tag{1.2}$$

for all $(\varphi,\psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$. When $u > 0$ in Ω, (u, ϕ_u) is a positive solution. Moreover, we say that (λ, u, ϕ_u) is a weak solution of (P) if $(u, \phi_u) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and it satisfies (1.1)–(1.2). It turns out that weak solutions are solutions provided f has subcritical growth (see Lemma 2.4).

A bifurcation point for (P) is a number $\lambda^* \in \mathbb{R}$ such that there exists a sequence $(\lambda_n, u_n, \phi_{u_n})$ $\in \mathbb{R} \times [(H_0^1(\Omega) \times H_0^1(\Omega)) \cap (X \times X)]$ satisfying the following properties:

(*i*) $\lambda_n \longrightarrow \lambda^*$;

(*ii*) $(\lambda_n, u_n, \phi_{u_n})$ is a solution of (P) with $u_n \neq 0$ and $\|u_n\| \longrightarrow 0$.

We say that $\lambda^* \in \mathbb{R}$ is a bifurcation point from infinity of (P) if there exists a sequence $(\lambda_n, u_n, \phi_{u_n}) \in \mathbb{R} \times [(H_0^1(\Omega) \times H_0^1(\Omega)) \cap (X \times X)]$ satisfying the following properties:

(*i*) $\lambda_n \longrightarrow \lambda^*$;

(*ii*) $(\lambda_n, u_n, \phi_{u_n})$ is a solution of (P) and $\|u_n\| \longrightarrow +\infty$.

It is well known that under the assumption (G_2) there exists a unique solution $\phi_\infty \in H_0^1(\Omega) \cap X$ of the problem

$$\begin{cases} -\Delta u = g(\infty) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Also, there exists a unique solution $\phi_0 \in H_0^1(\Omega) \cap X$ of the problem

$$\begin{cases} -\Delta u = g(0) \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\phi_0 > 0$ in Ω if $g(0) > 0$ holds.

Let us denote by $\lambda_1[\phi_\infty]$ and φ_∞ the first eigenvalue and the positive eigenfunction normalized by $\|\varphi_\infty\| = 1$, respectively, of the eigenvalue problem

$$-\Delta u + \phi_\infty(x) u = \lambda u \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega.$$

Similarly, let us denote by $\lambda_1[\phi_0]$ and φ_0 the first eigenvalue and the positive eigenfunction normalized by $\|\varphi_0\| = 1$, respectively, of the eigenvalue problem

$$-\Delta u + \phi_0(x) u = \lambda u \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega.$$

We observe that if $g(0) = 0$ then $\lambda_1[\phi_0]$ and φ_0 are the first eigenvalue and the positive eigenfunction, respectively, of $(-\Delta, H_0^1(\Omega))$.

Now we are ready to state our main results.

Theorem 1.2. *Suppose that* (F_1) *and* (G_1) *hold. Then* $\lambda_0 = \lambda_1[\phi_0]/m_0$ *is the unique bifurcation point of* (P)*. In addition, the continuum* Σ_0 *emanating from* $(\lambda_0, 0)$ *is unbounded. The same conclusion holds under the assumptions* (F_1) *and* (G_2)*.*

Theorem 1.3. *Assume that* (F_2) *and* (G_2) *hold. Then* $\lambda_\infty = \lambda_1[\phi_\infty]/m_\infty$ *is the unique bifurcation point from infinity of* (P)*. Moreover, there exists a subset* Σ_∞ *in* $\mathbb{R} \times X$ *of solutions of* (P) *such that* $\tilde{\Sigma}_\infty = \left\{(\lambda, z) : (\lambda, z/\|z\|^2) \in \Sigma_\infty\right\} \cup \left\{(\lambda_\infty, 0)\right\}$ *is connected and unbounded.*

After a bibliography review, we did not find any paper involving bifurcation theory and problems involving a generalized Schrödinger–Poisson system in a bounded domain as in the problem (P). Inspired by this fact, in the present paper we show that it is possible to apply the Leray–Schauder degree theory and the global bifurcation result due to Rabinowitz [12] to study the existence of solution for (P). To carry out this program, we first use the reduction argument (see [2]), which says that (P) is equivalent to a nonlocal problem (see problem (S)). After, we follow the same methodology as Ambrosetti and Hess [4]. However as we are working with a nonlocal problem it is necessary to do a careful study on some estimates and convergences involving the nonlocal term $\phi_u u$. Also, the calculation of Leray–Schauder degree of some maps involving the nonlocal term $\phi_u u$ must be justified (see Lemma 3.2). The reader is invited to verify that when $g(0) \neq 0$ the bifurcation points of Theorems 1.2 and 1.3 are different from those found in [4]. Moreover, under additional assumptions on f and g we will show that the bifurcation point found in our work is supercritical (the nontrivial solutions branch off on the right of λ_∞), while under the same assumption on f, the bifurcation point found in [4] is subcritical (the branching is on the left of bifurcation point).

Finally, we would like to point out that our results are new even in the case where $g(t) = t^2$ (that is, $p = 1$ in (G_1)), which is the case considered in the papers mentioned above and which allows us to apply variational methods. Indeed, in the papers mentioned above they did not study the existence of bifurcation points for problems of type (P). Also, they did not consider asymptotically linear nonlinearities as in our work. Thus, our work is the first to deal with the existence of bifurcation points and the continuum emanating from these points for Problem (P) with asymptotically linear nonlinearities even in the case when $p = 1$.

The paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 1.3. In section 5 we will show a result of multiplicity of solutions under additional assumptions on f and g.

Notation. Throughout this paper, we make use of the following notations:

- $L^p(\Omega)$, for $1 \leq p \leq \infty$, denotes the Lebesgue space with usual norm denoted by $|u|_p$.
- $H_0^1(\Omega)$ denotes the Sobolev space endowed with inner product

$$(u, v)_H = \int_\Omega \nabla u \nabla v, \ \forall u, v \in H_0^1(\Omega).$$

  The norm associated with this inner product will be denoted by $\| \ \|_H$.

- $W^{2,k}(\Omega)$ denotes the Sobolev space with norm $\|u\|_{W^{2,k}} = \left(\sum_{|\alpha| \leq 2} \|D^\alpha u\|_k^k\right)^{1/k}$.
- If u is a measurable function, we denote by u^- the negative part of u, which is given by $u^- = \max\{-u, 0\}$.
- The function $d(x, \partial\Omega)$ denotes the distance from a point $x \in \overline{\Omega}$ to the boundary $\partial\Omega$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of $\Omega \subset \mathbb{R}^N$.
- $\deg(I - \Psi, \mathcal{W}, 0)$ denotes the Leray–Schauder degree of $I - \Psi$ in $\mathcal{W}$ with respect to 0, where $\mathcal{W} \subset X$ is a bounded open set and $\Psi : \overline{\mathcal{W}} \longrightarrow X$ is a compact operator.
- $B_r(0) \subset X$ denotes the ball centered at $0 \in X$ with radius $r > 0$.
- $c, c_1, c_2, \ldots$ and $C, C_1, C_2, \ldots$ are possibly different positive constants which may change from line to line.

2 Preliminary results

Throughout this paper, unless it is explicitly stated, we will assume that (G_1) or (G_2) holds. In this section we will establish some results that we will need in the next sections.

For all $u \in L^{3/p}(\Omega)$ there exists a unique $\phi_u \in H_0^1(\Omega)$ which solves

$$-\Delta\phi = g(u(x)) \text{ in } \Omega,$$

and there holds

$$\phi_u(x) = \int_\Omega \frac{g(u(y))}{|x-y|}dy.$$

By L^p-theory one has $\phi_u \in W^{2,3/p}(\Omega)$, $0 < p < 2$, and so $\phi_u \in X$ (because $6/p > 3$). Since $g(u) \geq 0$, then by the maximum principle $\phi_u \geq 0$. Moreover, if $u \neq 0$ then $\phi_u > 0$ in Ω. Also, we have the following estimates.

Lemma 2.1. *For every $u \in L^{3/p}(\Omega)$ there holds*

$$\|\phi_u\| \leq C_2|g(u)|_{3/p},$$

for some constant $C_2 > 0$ independent of u. In particular, if $u \in X$, then

$$\|\phi_u\| \leq C\|g(u)\|, \tag{2.1}$$

for some constant $C > 0$ independent of u.

Proof. By L^p-theory one has $\phi_u \in W^{2,3/p}(\Omega)$ and

$$\|\phi_u\|_{W^{2,3/p}} \leq C_1|g(u)|_{3/p},$$

for some constant $C_1 > 0$, which depends only on Ω and p.

Combining this inequality with the embedding of $W^{2,3/p}(\Omega)$ into X we get

$$\|\phi_u\| \leq C_2|g(u)|_{3/p},$$

for some constant $C_2 > 0$, which depends only on Ω and p.

If in addition $u \in X$, then the inequality $|g(u)|_{3/p} \leq |\Omega|^{p/3}\|g(u)\|$ is valid, and therefore

$$\|\phi_u\| \leq C\|g(u)\|,$$

where $C = C_2(\Omega)|\Omega|^{p/3}$. This completes the proof of the lemma. □

We recall that a map $J : X \to X$ is bounded if it maps bounded sets onto bounded sets.

In order to apply Bifurcation Theory we will need the following lemma.

Lemma 2.2. *The map $\mathcal{J} : X \longrightarrow X$ defined by setting $\mathcal{J}(u) = \phi_u$ is continuous and bounded.*

Proof. Let $\{u_n\} \subset X$ be a sequence such that $u_n \to u$ in X. As $\phi_{u_n} - \phi_u \in H_0^1(\Omega)$ satisfies

$$-\Delta(\phi_{u_n} - \phi_u) = g(u_n) - g(u) \textit{ in } \Omega,$$

by elliptic regularity it follows that

$$\|\phi_{u_n} - \phi_u\| \leq C\|g(u_n) - g(u)\|,$$

for some constant $C > 0$ independent of u_n and u. Since $u_n \to u$ in X implies $g(u_n) \to g(u)$ in X, from the last inequality one deduces that $\phi_{u_n} \to \phi_u$ in X. This proves that the map $\mathcal{J}$ is continuous in X.

Finally, the boundedness of $\mathcal{J}$ follows from (2.1), and the proof is completed. □

Our next result establishes the positivity of weak solutions to a variational inequality.

Lemma 2.3. *Let $\phi \in X$ and suppose that $u \in H_0^1(\Omega)$ satisfies*

$$\begin{cases} -\Delta u + \phi(x)u \geq 0 \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega. \end{cases}$$

Then either $u \equiv 0$, or there exists $\epsilon > 0$ such that $u(x) \geq \epsilon d(x, \partial\Omega)$ in Ω.

Proof. Let $k = \|\phi\|$ and assume that $u \not\equiv 0$. In this case, we get

$$-\Delta u + ku \geq -\Delta u + \phi(x)u \geq 0 \text{ in } \Omega,$$

namely, u satisfies

$$\begin{cases} -\Delta u + ku \geq 0 \text{ in } \Omega, \\ u \gneqq 0 \text{ in } \Omega. \end{cases}$$

This allows us to apply Theorem 3 of Brezis–Nirenberg [6] to deduce that $u(x) \geq \epsilon d(x, \partial\Omega)$ in Ω, for some $\epsilon > 0$. This completes the proof. □

Now, we consider the nonlocal problem

$$\begin{cases} -\Delta u + \phi_u(x)u = z(u) \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega, \end{cases} \tag{Q}$$

under the following assumption on $z \in C(\mathbb{R}, \mathbb{R})$:

(H) $|z(t)| \leq c_1 + c_2|t|^q$, where $c_1, c_2 > 0$ are constants and $0 < q < 2^* - 1$.

Lemma 2.4. *Suppose that (H) holds. Then every $u \in H_0^1(\Omega)$ which is a weak solution of (Q) belongs to X.*

Proof. Indeed, $u \in H_0^1(\Omega)$ is a weak solution of the problem

$$-\Delta u = h(x, u) \text{ in } \Omega,$$

where $h(x,t) = z(t) - \phi_u(x)t$. From Lemma 2.2 and (H) one infers that

$$|h(x,t)| \leq c_3 + c_4|t|^q,$$

for all $x \in \Omega, t \in \mathbb{R}$ and some constants $c_3, c_4 > 0$. Thus, a standard bootstrap argument implies that $u \in X$. This completes the proof. □

3 Global bifurcation

The main goal of this section is to prove Theorem 1.2. To do this we need some definitions and auxiliary lemmas.

It is well known that Problem (P) is equivalent to the nonlocal problem

$$\begin{cases} -\Delta u + \phi_u(x)u = \lambda f(u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega. \end{cases} \tag{S}$$

We extend the function f to a continuous function $\tilde{f}$ defined on $\mathbb{R}$ in such a way that $\tilde{f}(t) = f(0)$ for all $t < 0$. Then, we can consider the nonlocal problem

$$\begin{cases} -\Delta u + \phi_u(x)u = \lambda \tilde{f}(u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u(x) = 0 \text{ on } \partial\Omega. \end{cases} \tag{$\tilde{S}$}$$

Now we prove the following result.

Lemma 3.1. *Assume that either* (F_1) *or* (F_2) *is satisfied. Then Problems* (S) *and* $(\tilde{S})$ *are equivalent.*

Proof. It is clear that if u is a solution of (S) then it is also a solution of $(\tilde{S})$. Now, we assume that u is a solution of $(\tilde{S})$. Taking u^- as test function in $(\tilde{S})$ we get

$$-\|u^-\|_H^2 - \int_\Omega \phi_u(x)(u^-)^2 = \int_\Omega \lambda f(0)u^-,$$

which implies $\|u^-\|_H = 0$, that is, $u \geq 0$ in Ω. Thus $\tilde{f}(u) = f(u)$ in Ω, and if either (F_1) or (F_2) is satisfied then

$$|\tilde{f}(t)| = |f(t)| \leq c_1|t|, \ \forall t \in [0, \|u\| + 1),$$

and for some constant $c_1 > 0$. Therefore, u satisfies

$$\begin{cases} -\Delta u + (\phi_u(x) + \lambda c_1)u \geq 0 \text{ in } \Omega, \\ u \gneqq 0 \text{ in } \Omega, \\ u(x) = 0 \ \text{ on } \partial\Omega, \end{cases}$$

and from Lemma 2.3 one infers that $u > 0$ in Ω. This completes the proof. □

Due to Lemma 3.1, the proof of Theorems 1.2 and 1.3 is reduced to proving the existence of the bifurcation points of Problem $(\tilde{S})$. To study Problem $(\tilde{S})$ we will transform it into a functional equation. From now on we will denote by K the Green operator of $-\Delta$ on $H_0^1(\Omega)$. It is well known that K is compact as a map from X in itself. From Lemma 2.2 it follows that the map $F_\lambda : X \to X$ given by

$$F_\lambda(u) = \lambda \tilde{f}(u) - \phi_u u$$

is continuous and bounded. As a consequence, the map $T : \mathbb{R} \times X \to X$ defined by $T(\lambda, u) = K(F_\lambda(u))$ is compact and Problem $(\tilde{S})$ is equivalent to the functional equation

$$\Phi(\lambda, u) = 0,$$

where $\Phi(\lambda, u) = u - T(\lambda, u)$ for $(\lambda, u) \in \mathbb{R} \times X$.

The first property of the map Φ that we highlight is the following.

Lemma 3.2. *For every* $\mu \in [0, 1]$ *the function* $u \equiv 0$ *is the unique solution of the problem*

$$\begin{cases} -\Delta u + \mu\phi_u(x)u = 0 \text{ in } \Omega, \\ u \in H_0^1(\Omega) \cap X. \end{cases} \tag{A}$$

In particular,

$$\deg(\Phi(0, \cdot), B_r(0), 0) = 1,$$

for all $r > 0$.

Proof. If u satisfies (A) then,

$$\|u\|_H^2 + \mu \int_\Omega \phi_u u^2 = 0,$$

and this implies that $\|u\|_H = 0$, namely that $u \equiv 0$.

Thus the homotopy $H(\mu, u) = u - \mu T(0, u), (\mu, u) \in [0, 1] \times X$, is admissible on the ball $B_r(0)$, for all $r > 0$. Using the homotopy invariance, it follows that

$$\deg(H(1, \cdot), B_r(0), 0) = \deg(I, B_r(0), 0) = 1,$$

and since $\Phi(0, \cdot) = H(1, \cdot)$, we get $\deg(\Phi(0, \cdot), B_r(0), 0) = 1$. □

Now, let us give the precise definition of bifurcation point of the functional equation $\Phi(\lambda, u) = 0$.

Definition 3.3. We say that λ_* is a bifurcation point of $\Phi(\lambda, u) = 0$ if there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times X$, with $u_n \neq 0$, such that $\lambda_n \longrightarrow \lambda_*$, $\|u_n\| \longrightarrow 0$ and $\Phi(\lambda_n, u_n) = 0$.

It turns out that the bifurcation points of $\Phi(\lambda, u) = 0$ are the bifurcation points of ($\tilde{S}$) (and therefore are also the bifurcation points of (P)).

Denoting by

$$\Sigma_\Phi = \{(\lambda, u) \in \mathbb{R} \times X : \Phi(\lambda, u) = 0,\ u \neq 0\},$$

and taking the closure $\overline{\Sigma}_\Phi$ of Σ_Φ, we see that λ_* is a bifurcation point of $\Phi(\lambda, u) = 0$ if and only if $(\lambda_*, 0) \in \overline{\Sigma}_\Phi$.

For each $\lambda \in \mathbb{R}$ fixed, the index of $\Phi_\lambda = \Phi(\lambda, \cdot)$ relative to 0, denoted by $i(\Phi_\lambda, 0)$, is defined by

$$i(\Phi_\lambda, 0) = \lim_{\epsilon \to 0} \deg(\Phi_\lambda, B_\epsilon(0), 0).$$

To prove Theorem 1.2 we have to prove the change of index of $\Phi(\lambda, \cdot)$ as λ crosses $\lambda = \lambda_0$. The proof is based on the following lemmas.

Lemma 3.4. *Let $\Lambda \subset \mathbb{R}^+$ be a compact interval with $\lambda_0 \notin \Lambda$. Then there exists $\epsilon > 0$ satisfying*

$$\Phi(\lambda, u) \neq 0, \qquad \forall \lambda \in \Lambda, \quad \forall 0 < \|u\| \leq \epsilon.$$

Proof. We argue by contradiction assuming that there exists a sequence $(\lambda_n, u_n) \in \Lambda \times X$ satisfying

$$\lambda_n \longrightarrow \lambda \neq \lambda_0, \qquad \|u_n\| \longrightarrow 0,$$
$$\Phi(\lambda_n, u_n) = 0, \qquad u_n > 0.$$

Now, we divide the equation $u_n = K(F_{\lambda_n}(u_n))$ by $\|u_n\|$ to get

$$v_n = K\left(\frac{F_{\lambda_n}(u_n)}{\|u_n\|}\right), \quad \text{where } v_n = \frac{u_n}{\|u_n\|}.$$

We claim that the sequence $\left\{\frac{F_{\lambda_n}(u_n)}{\|u_n\|}\right\}$ is bounded in $\Lambda \times X$. To prove this claim, let $\delta > 0$ such that $|f(t)| \leq (m_0 + 1)|t|$ for all $0 < t < \delta$ (the existence of δ is guaranteed by (F_1)). Since $\|u_n\| \longrightarrow 0$ there exists $n_0 \in \mathbb{N}$ such that $\|u_n\| < \delta$ for all $n > n_0$. From this and (2.1) we deduce that

$$\|F_{\lambda_n}(u_n)\| \leq C(\|u_n\| + \|g(u_n)\|\|u_n\|),$$

for all $n > n_0$ and for some constant $C > 0$ independent of n.

Therefore,

$$\|\frac{F_{\lambda_n}(u_n)}{\|u_n\|}\| \le C(1+\|g(u_n)\|) \le C(1+\max_{t\in[-\delta,\delta]}|g(t)|),$$

for $n > n_0$, which implies that the sequence $\{\frac{F_{\lambda_n}(u_n)}{\|u_n\|}\}$ is bounded in $\Lambda \times X$.

Since K is compact, from $v_n = K(\frac{F_{\lambda_n}(u_n)}{\|u_n\|})$ we deduce that, up to a subsequence, v_n strongly converges to some $v \in X$ with $\|v\| = 1$. Then, by Lemmas 2.1 and 2.2 and (F_1) one infers

$$\frac{F_{\lambda_n}(u_n)}{\|u_n\|} \longrightarrow (\lambda m_0 - \phi_0)v \text{ in } X,$$

and therefore

$$v = K((\lambda m_0 - \phi_0)v).$$

But this says that v is a solution of the problem

$$\begin{cases} -\Delta v + \phi_0(x)v = \lambda m_0 v \text{ in } \Omega, \\ v \ge 0 \ in \ \Omega, \end{cases}$$

and from Lemma 2.3 one infers that $v > 0$ in Ω. As a consequence v is an eigenfunction of norm one associated to λ.

Using φ_0 as a test function in this eigenvalue problem we obtain

$$\lambda_1[\phi_0]\int_\Omega v\varphi_0 = \int_\Omega \nabla v \nabla \varphi_0 dx + \int_\Omega \phi_0 v \varphi_0 dx = \lambda m_0 \int_\Omega v\varphi_0,$$

and we conclude that $\lambda_1[\phi_0] = \lambda m_0$, which is a contradiction and the proof is finished. □

As a consequence of the proof of Lemma 3.4 we obtain the following corollary.

Corollary 3.5. *The unique possible bifurcation point of solutions is* $\lambda = \lambda_0$.

Lemma 3.6. *If* $\lambda < \lambda_0$ *then* $i(\Phi_\lambda, 0) = 1$.

Proof. Fix any $\lambda < \lambda_0$ and take $\Lambda = [0, \lambda]$. For $t \in [0, 1]$, the parameter $t\lambda$ belongs to Λ and from Lemma 3.4 it follows that $\Phi(t\lambda, u) \neq 0$ for all $0 < \|u\| \le \epsilon$, where $\epsilon > 0$ is given by Lemma 3.4. Consider the homotopy $H(t, u) = \Phi(t\lambda, u)$. Using the homotopy invariance, we get

$$\deg(H(1,\cdot), B_\epsilon(0), 0) = \deg(H(0,\cdot), B_\epsilon(0), 0),$$

namely

$$i(\Phi_\lambda, 0) = \deg(\Phi_\lambda, B_\epsilon(0), 0) = \deg(\Phi_0, B_\epsilon(0), 0) = 1,$$

where we have used Lemma 3.2 in the last equality. This completes the proof. □

Lemma 3.7. *For every* $\lambda > \lambda_0$ *there exists* $\delta > 0$ *such that*

$$\Phi(\lambda, u) \neq \tau\varphi_1, \quad \forall 0 < \|u\| \le \delta, \quad \forall \tau \ge 0.$$

Proof. We fix $\lambda > \lambda_0$ and we assume, by contradiction, that there exist sequences $u_n \in X$ and $\tau_n \ge 0$ satisfying $u_n > 0$ in Ω, $\|u_n\| \longrightarrow 0$ and

$$\Phi(\lambda, u_n) = \tau_n \varphi_1,$$

or, equivalently,

$$u_n = K(F_\lambda(u_n)) + \tau_n \varphi_1.$$

Dividing this equation by $\|u_n\|$ one finds

$$v_n = K\left(\frac{F_\lambda(u_n)}{\|u_n\|}\right) + \varphi_1 \frac{\tau_n}{\|u_n\|}, \quad \text{where } v_n = \frac{u_n}{\|u_n\|}.$$

Arguing as in the proof of Lemma 3.4, we see that the sequence $\{\frac{F_\lambda(u_n)}{\|u_n\|}\}$ is bounded in X. Thus, using the compactness of K, we deduce that, up to a subsequence, $K(\frac{F_\lambda(u_n)}{\|u_n\|})$ is convergent and hence $\tau_n/\|u_n\|$ is bounded. Passing again to a subsequence, if necessary, we can assume that $\tau_n/\|u_n\| \longrightarrow \tau \geq 0$ and $u_n/\|u_n\| \longrightarrow v$ with $v \in X$ and $\|v\| = 1$. Arguing as we have done in the proof of Lemma 3.4, it is easy to see that v satisfies

$$\begin{cases} -\Delta v + \phi_0 v = \lambda m_0 v + \tau \lambda_1 \varphi_1 \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \\ \|v\| = 1. \end{cases}$$

Then, using φ_0 as a test function in this problem we obtain

$$\lambda_1[\phi_0] \int_\Omega v\varphi_0 = \lambda m_0 \int_\Omega v\varphi_0 + \int_\Omega \tau\lambda_1\varphi_1\varphi_0 \geq \lambda m_0 \int_\Omega v\varphi_0,$$

which implies that $\lambda_0 \geq \lambda$, a contradiction. The proof is finished. □

Lemma 3.8. *If $\lambda > \lambda_0$ then $i(\Phi_\lambda, 0) = 0$.*

Proof. If $\lambda > \lambda_0$ then, from Lemma 3.7, we derive that

$$\deg(\Phi_\lambda, B_\delta(0), 0) = \deg(\Phi_\lambda - \tau\varphi_1, B_\delta(0), 0), \quad \forall \tau > 0,$$

where $\delta > 0$ is given by Lemma 3.7.

But, again using Lemma 3.7, the problem

$$\begin{cases} -\Delta w + \phi_w(x) w = \lambda \tilde{f}(w) + \tau\lambda_1\varphi_1 \text{ in } \Omega, \\ w = 0 \text{ in } \partial\Omega, \end{cases}$$

has no nontrivial solution satisfying $0 < \|u\| \leq \delta$. Since, $w = 0$ is not a solution provided that $\tau > 0$, we deduce that

$$i(\Phi_\lambda, 0) = \deg(\Phi_\lambda, B_\delta(0), 0) = \deg(\Phi_\lambda - \tau\varphi_1, B_\delta(0), 0) = 0, \quad \forall \lambda > \lambda_0.$$

This completes the proof. □

Now, we are ready to prove Theorem 1.2.

Proof. (*of Theorem 1.2*) Assume that λ_0 is no bifurcation point. Then there exists $\epsilon > 0$ such that

$$\Phi_\lambda(u) \neq 0, \text{ for all } \lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon] \text{ and } 0 < \|u\| \leq \epsilon.$$

Thus, if we take

$$\lambda_0 - \epsilon < \tilde{\lambda} < \lambda_0 < \hat{\lambda} < \lambda_0 + \epsilon$$

one has

$$\deg(\Phi_{\tilde{\lambda}}, B_\epsilon(0), 0) = \deg(\Phi_{\hat{\lambda}}, B_\epsilon(0), 0),$$

and therefore,

$$i(\Phi_{\tilde{\lambda}}, 0) = i(\Phi_{\hat{\lambda}}, 0),$$

which contradicts Lemmas 3.6 and 3.8. Moreover, from Corollary 3.5 λ_0 is the unique bifurcation point for (P).

As a consequence, one can repeat the arguments carried out in the proof of the Global Bifurcation Theorem due to Rabinowitz [12] to show the existence of Σ_0. This completes the proof. □

4 Bifurcation from infinity

In this section we are going to prove Theorem 1.3. Hereafter we will assume that (F_2) and (G_2) hold. We start with the following definition.

Definition 4.1. We say that λ_∞ is a bifurcation point from infinity of $\Phi(\lambda, u) = 0$ if there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times X$ satisfying

$$\lambda_n \longrightarrow \lambda_\infty, \quad \|u_n\| \longrightarrow +\infty, \quad \Phi(\lambda_n, u_n) = 0.$$

It turns out that the bifurcation points from infinity of $\Phi(\lambda, u) = 0$ are the bifurcation points from infinity of ($\tilde{S}$) (and therefore are also the bifurcation points from infinity of (P)).

Following [4], if we make the Kelvin transform

$$z = \frac{u}{\|u\|^2}, \quad \text{with } u \neq 0,$$

we derive that

$$\Phi(\lambda, u) = 0, \;\; u \neq 0 \Leftrightarrow z - \|z\|^2 T\left(\lambda, \frac{z}{\|z\|^2}\right) = 0, \quad z \neq 0.$$

Thus we are led to define the map

$$\tilde{\Phi}(\lambda, z) = \begin{cases} z - \|z\|^2 T(\lambda, \frac{z}{\|z\|^2}), \text{ if } z \neq 0, \\ 0, \text{ if } z = 0. \end{cases}$$

Moreover, using Lemma 2.1 we find that

$$\|z\|^2 \|\phi_{z/\|z\|^2} \frac{z}{\|z\|^2}\| \leq C\|z\|,$$

for all $z \neq 0$ and some constant $C > 0$ independent of z. As a consequence we obtain

$$\lim_{z \to 0} \|z\|^2 \phi_{z/\|z\|^2} \frac{z}{\|z\|^2} = 0.$$

From this limit and assumption on f it readily follows that $\tilde{\Phi}$ is continuous. In particular, $\tilde{\Phi}$ is a compact perturbation of the identity and λ_∞ is a bifurcation point from infinity for $\Phi(\lambda, u) = 0$ if and only if λ_∞ is a bifurcation point for $\tilde{\Phi}(\lambda, z) = 0$. Moreover, arguing as in the proof of Lemma 3.2, we immediately deduce the following property:

$$\deg(\tilde{\Phi}(0, \cdot), B_\epsilon(0), 0) = 1, \text{ for all } \epsilon > 0.$$

The proof of Theorem 1.3 is based on the following lemmas.

Lemma 4.2. *Let $\Lambda \subset [0, \lambda_\infty)$ be any compact interval. Then*

(a) *there exists $r > 0$ such that $\Phi_\lambda(u) \neq 0$, for all $\lambda \in \Lambda$ and $\|u\| \geq r$,*

(b) *λ_∞ is the only possible bifurcation from infinity for $\Phi(\lambda, u) = 0$,*

(c) *$i(\tilde{\Phi}_\lambda, 0) = 1$ for all $\lambda < \lambda_\infty$.*

Proof. (a) We argue by contradiction assuming that there exists a sequence $(\lambda_n, u_n) \in \Lambda \times X$ satisfying

$$\lambda_n \longrightarrow \lambda \neq \lambda_\infty, \quad \|u_n\| \longrightarrow \infty,$$

$$\Phi(\lambda_n, u_n) = 0, \quad u_n > 0.$$

Setting $v_n = \|u_n\|^{-1} u_n$, we find

$$v_n = K\left(\lambda_n \frac{f(u_n)}{\|u_n\|} - \phi_{u_n} v_n\right).$$

By Lemma 2.1 we infer that there exists a constant $C > 0$ such that

$$\left\|\lambda_n \frac{f(u_n)}{\|u_n\|} - \phi_{u_n} v_n\right\| \leq \left[\left\|\lambda_n \left(m_\infty v_n + \frac{h(u_n)}{\|u_n\|}\right)\right\| + \|\phi_{u_n} v_n\|\right] \leq C, \text{ for all } n \in \mathbb{N}.$$

Since K is compact, from $v_n = K(\lambda_n \frac{f(u_n)}{\|u_n\|} - \phi_{u_n} v_n)$ we deduce that, up to a subsequence, v_n strongly converges to some $v \in X$ with $\|v\| = 1$. Note also that v_n converges weakly to v in $H_0^1(\Omega)$ and $v \geq 0$ in Ω. Moreover, there holds

$$\int_\Omega \nabla v_n \nabla \varphi dx + \int_\Omega \phi_{u_n} v_n \varphi dx = \int_\Omega \lambda_n \frac{f(u_n)}{\|u_n\|} \varphi dx, \quad \varphi \in H_0^1(\Omega). \tag{4.1}$$

On the other hand, the boundedness of g and the L^p-theory imply that, up to a subsequence, ϕ_{u_n} converges weakly in $H_0^1(\Omega)$ and strongly in X, to some $\phi \in H_0^1(\Omega) \cap X$. Thus, by the Lebesgue dominated convergence theorem we yield

$$\int_\Omega \nabla v \nabla \varphi dx + \int_\Omega \phi v \varphi dx = \int_\Omega \lambda m_\infty v \varphi dx, \quad \varphi \in H_0^1(\Omega), \tag{4.2}$$

which together with Lemma 2.3 implies that $v > 0$ in Ω. As a consequence we get that $u_n(x) = \|u_n\| v_n(x) \longrightarrow \infty$ for all $x \in \Omega$, and applying the Lebesgue dominated convergence theorem we found that

$$\int_\Omega \nabla \phi \nabla \varphi dx = \int_\Omega g(\infty) \varphi dx, \quad \varphi \in H_0^1(\Omega),$$

namely $\phi = \phi_\infty$.

Finally, using φ_∞ as a test function in (4.2) we obtain

$$\lambda_1[\phi_\infty] \int_\Omega v \varphi_\infty dx = \lambda m_\infty \int_\Omega v \varphi_\infty dx,$$

and we conclude that $\lambda_\infty = \lambda$, which is a contradiction. This contradiction proves (a).

Statement (b) follows immediately from (a). Regarding (c), fix any $\lambda < \lambda_\infty$ and take $\Lambda = [0, \lambda]$. For $t \in [0, 1]$, the parameter $t\lambda$ belongs to Λ and from (a) it follows that $u \neq T(t\lambda, u)$

for all $\|u\| \geq r$. This implies that $\tilde{\Phi}(t\lambda, z) \neq 0$ for all $0 < \|z\| \leq 1/r$. Consider the homotopy $H(t,z) = \tilde{\Phi}(t\lambda, z)$. Using the homotopy invariance, we get

$$\deg(\tilde{\Phi}_\lambda, B_{1/r}(0), 0) = \deg(\tilde{\Phi}_0, B_{1/r}(0), 0),$$

namely

$$i(\tilde{\Phi}_\lambda, 0) = \deg(\tilde{\Phi}_0, B_{1/r}(0), 0) = 1,$$

proving (c).

□

Lemma 4.3. *Let $\lambda > \lambda_\infty$. Then*

(a) *there exists $\epsilon > 0$ such that $\Phi_\lambda(u) \neq \tau\varphi_1$, for all $\tau \geq 0$ and $\|u\| \geq \epsilon$,*

(b) *$i(\tilde{\Phi}_\lambda, 0) = 0$ for all $\lambda > \lambda_\infty$.*

Proof. (a) We fix $\lambda > \lambda_\infty$ and we assume, by contradiction, that there exist $\tau_n \geq 0$ and $\|u_n\| \to \infty$ such that $\Phi_\lambda(u_n) = \tau_n\varphi_1$, namely

$$u_n - \tau_n\varphi_1 = K(\lambda f(u_n) - \phi_{u_n}u_n).$$

Setting $v_n = \|u_n\|^{-1}u_n$, we get

$$v_n - \tau_n\|u_n\|^{-1}\varphi_1 = K\left(\lambda\frac{f(u_n)}{\|u_n\|} - \phi_{u_n}v_n\right),$$

and arguing as in Lemma 4.2, one readily shows that the sequence $\left\{\frac{f(u_n)}{\|u_n\|} - \phi_{u_n}v_n\right\}$ is bounded in X. Thus, using the compactness of K, we deduce that, up to a subsequence,

$$K\left(\lambda\frac{f(u_n)}{\|u_n\|} - \phi_{u_n}v_n\right)$$

is convergent and hence $\tau_n/\|u_n\|$ is bounded. Passing again to a subsequence, if necessary, we can assume that $\tau_n/\|u_n\| \longrightarrow \tau \geq 0$ and $u_n/\|u_n\| \longrightarrow v$ with $v \in X$ and $\|v\| = 1$. Arguing as we have done in the proof of Lemma 4.2, we can deduce that $u_n(x) \to \infty$ for all $x \in \Omega$ and that v satisfies

$$\begin{cases} -\Delta v + \phi_\infty v = \lambda m_\infty v + \tau\lambda_1\varphi_1 \text{ in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \\ \|v\| = 1. \end{cases}$$

Therefore, using φ_∞ as a test function in this problem we obtain

$$\lambda_1[\phi_\infty]\int_\Omega v\varphi_\infty \geq \lambda m_\infty \int_\Omega v\varphi_\infty,$$

and we conclude that $\lambda_\infty \geq \lambda$, which is a contradiction. This proves (a).

(b) Take $\tau = t\|u\|^2$, with $t \in [0,1]$. By (a) it follows that $\Phi_\lambda(u) \neq t\|u\|^2\varphi_1$ for all $\|u\| \geq \epsilon$. This implies

$$\tilde{\Phi}_\lambda(z) \neq t\varphi_1, \quad \forall 0 < \|z\| \leq \frac{1}{\epsilon}, \quad \forall t \in [0,1]. \tag{4.3}$$

Using the homotopy $H(t,z) = \tilde{\Phi}_\lambda(z) - t\varphi_1$ on the ball $B_{1/\epsilon}(0)$ we find

$$i(\tilde{\Phi}_\lambda, 0) = \deg(\tilde{\Phi}_\lambda, B_{1/\epsilon}(0), 0) = \deg(\tilde{\Phi}_\lambda - \varphi_1, B_{1/\epsilon}(0), 0).$$

The latter degree is zero because (4.3), with $t = 1$, implies that $\tilde{\Phi}_\lambda(z) = \varphi_1$ has no solution on $B_{1/\epsilon}(0)$. This proves (b). □

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Arguing as in the proof of Theorem 1.2, the Lemmas 4.2 and 4.3 ensure that λ_∞ is the unique bifurcation point for the equation $\tilde{\Phi}(\lambda, z) = 0$, and that from $(\lambda_\infty, 0)$ emanates an unbounded continuum of solutions $\tilde{\Sigma}_\infty = \{(\lambda, z) : \tilde{\Phi}(\lambda, z) = 0\}$ in $\mathbb{R} \times X$. Moreover, $(\lambda, z) \in \tilde{\Sigma}_\infty, z \neq 0$, if and only if $(\lambda, z/\|z\|^2) \in \Sigma_\Phi = \{(\lambda, u) : \Phi(\lambda, u) = 0,\ u \neq 0\}$. We define $\Sigma_\infty = \{(\lambda, z/\|z\|^2) : (\lambda, z) \in \tilde{\Sigma}_\infty, z \neq 0\}$. Therefore, $\Sigma_\infty \subset \Sigma_\Phi$ and

$$\tilde{\Sigma}_\infty = \{(\lambda, z) : (\lambda, z/\|z\|^2) \in \Sigma_\infty\} \cup \{(\lambda_\infty, 0)\}$$

is connected and unbounded. This completes the proof. □

Remark 4.4. The reader can ask why we consider only assumption (G_2) in Theorem 1.3. To answer this question, we recall that in the proof of Lemmas 4.2 and 4.3 the boundedness of the sequence $\{\phi_{u_n} v_n\}$ plays a fundamental role. However, under the assumption (G_1), we have the inequality $\|\phi_{u_n}\| \leq C\|u_n\|^{2q}$, which does not ensure the boundedness of the sequence $\{\phi_{u_n} v_n\}$ as $\|u_n\| \longrightarrow \infty$.

5 Multiplicity of solutions

Throughout this section we will use the same notation as in the previous sections. In this section we will apply Theorems 1.2 and 1.3 to show a result of multiplicity of solutions for (P) under additional assumptions on f and g. Specifically, we introduce the following assumptions:

(F_3) $2^{-1} m_\infty t \leq f(t) \leq m_\infty t$ for all $t \geq 0$ and $f'_+(0) = m_\infty$;

(G_3) $g(\infty) = \lim_{t\to\infty} g(t) = \lim_{t\to 0} g(t)$ and $g(\infty) \leq g(t)$ for all $t \in \mathbb{R}$.

Assume that (G_3) is valid. For every $u \in H_0^1(\Omega)$ we have

$$-\Delta\phi_u = g(u) \geq g(\infty) = -\Delta\phi_\infty \text{ in } \Omega,$$

which implies

$$\phi_u \geq \phi_\infty \text{ in } \Omega. \tag{5.1}$$

Moreover, if we define

$$\tilde{g} = \sup_{t\geq 0} g(t) \quad \text{and} \quad -\Delta\phi_{\tilde{g}} = \tilde{g}, \phi_{\tilde{g}} \in H_0^1(\Omega),$$

we can show that $\phi_u \leq \phi_{\tilde{g}}$ in Ω (using the same argument as above) for all $u \in H_0^1(\Omega)$.

Let us denote by $\lambda_1[\phi_{\tilde{g}}]$ and $\varphi_{\tilde{g}}$ the first eigenvalue and the positive eigenfunction normalized by $\|\varphi_{\tilde{g}}\| = 1$, respectively, of the eigenvalue problem

$$-\Delta u + \phi_{\tilde{g}}(x) u = \lambda u \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega.$$

Let us point out that under the assumptions (F_1)–(F_3) and (G_2)–(G_3) one has $\lambda_0 = \lambda_\infty$.

Now we have the following lemma.

Lemma 5.1. *Suppose that* (F_3) *and* (G_3) *hold. If Problem* (S) *has a solution, then* $\lambda_0 \le \lambda \le \lambda_{\tilde{g}}$*, where* $\lambda_{\tilde{g}} = 2\lambda_1[\phi_{\tilde{g}}]/m_\infty$.

Proof. Indeed, if u is a solution of (S), then

$$\begin{aligned}\lambda_1[\phi_\infty]\int_\Omega u\varphi_\infty dx &= \int_\Omega \nabla u \nabla \varphi_\infty dx + \int_\Omega \phi_\infty u \varphi_\infty dx \\ &\le \int_\Omega \nabla u \nabla \varphi_\infty dx + \int_\Omega \phi_u u \varphi_\infty dx \quad \text{(by (5.1))} \\ &= \lambda \int_\Omega f(u)\varphi_\infty dx \\ &\le \lambda m_\infty \int_\Omega u\varphi_\infty dx \quad \text{(by } (F_3)).\end{aligned}$$

This now implies $\lambda \ge \lambda_0$.

Similarly,

$$\begin{aligned}2^{-1}m_\infty\lambda \int_\Omega u\varphi_{\tilde{g}} dx &\le \int_\Omega \lambda f(u)\varphi_{\tilde{g}} dx \\ &= \int_\Omega \nabla u \nabla \varphi_{\tilde{g}} dx + \int_\Omega \phi_u u \varphi_{\tilde{g}} dx \\ &\le \int_\Omega \nabla u \nabla \varphi_{\tilde{g}} dx + \int_\Omega \phi_{\tilde{g}} u \varphi_{\tilde{g}} dx \\ &= \lambda_1[\phi_{\tilde{g}}]\int_\Omega u\varphi_{\tilde{g}} dx,\end{aligned}$$

whence we infer that $\lambda_{\tilde{g}} \ge \lambda$. This proves the lemma. □

The main result of this section is the following theorem.

Theorem 5.2. *Assume that* (F_1)–(F_3) *and* (G_2)–(G_3) *hold. Then*

(*a*) $\Sigma_0 = \Sigma_\infty \cup \{(\lambda_\infty, 0)\}$,

(*b*) *there exists* $\epsilon > 0$ *such that Problem* (P) *has at least two solutions for* $\lambda_0 < \lambda < \lambda_0 + \epsilon$.

Proof. (*a*) First of all, let us remark that since $\tilde{\Sigma}_\infty$ is connected and $\tilde{\Sigma}_\infty \cap (\mathbb{R} \times \{0\}) = \{(\lambda_\infty, 0)\}$ then $\tilde{\Sigma}_\infty - \{(\lambda_\infty, 0)\}$ is connected. Now, the map $W : \tilde{\Sigma}_\infty - \{(\lambda_\infty, 0)\} \longrightarrow \Sigma_\Phi$ given by

$$W(\lambda, z) = (\lambda, z/\|z\|^2)$$

is continuous. Thus $W(\tilde{\Sigma} - \{(\lambda_*, 0)\}) = \Sigma_\infty$ is a connected subset of Σ_Φ. Using Lemma 5.1 and that λ_∞ is the unique bifurcation point of $\Phi(\lambda, u) = 0$ and the unique bifurcation point from infinity of $\Phi(\lambda, u) = 0$ we can see that $\overline{\Sigma_\infty} = \Sigma_\infty \cup \{(\lambda_\infty, 0)\}$ (which is a connected subset of Σ_Φ too).

Finally, we will show that $\Sigma_0 = \Sigma_\infty \cup \{(\lambda_\infty, 0)\}$. Clearly, $\Sigma_\infty \cup \{(\lambda_\infty, 0)\} \subset \Sigma_0$. We assume now that $(\lambda, u) \in \Sigma_0 - \{(\lambda_\infty, 0)\}$, namely, $u \ne 0$ and

$$u - T(\lambda, u) = 0.$$

Let us write $(\lambda, u) = (\lambda, z/\|z\|^2)$, where $z = u/\|u\|^2$. Thus, the last equality above can be rewritten as

$$\frac{z}{\|z\|^2} - T(\lambda, z/\|z\|^2) = 0,$$

which implies $(\lambda, u) = (\lambda, z/\|z\|^2) \in \Sigma_\infty$.

Then one finds:

$$\Sigma_0 - \{(\lambda_\infty, 0)\} \subset \Sigma_\infty$$

and as $\Sigma_\infty \subset \Sigma_0$ we conclude that $\Sigma_0 = \Sigma_\infty \cup \{(\lambda_\infty, 0)\}$. This proves (a).

(b) Let $u_\lambda \in \Sigma_0$ and $v_\lambda \in \Sigma_\infty$ be the solutions of (P) obtained in Theorems 1.2 and 1.3, respectively. By using the fact that $\|u_\lambda\| \to 0$ and $\|v_\lambda\| \to \infty$ as $\lambda \to \lambda_0$ and Lemma 5.1, we deduce that there exists $\epsilon > 0$ such that

$$\|u_\lambda\| < 1 < \|v_\lambda\| \quad \text{for } \lambda_0 < \lambda < \lambda_0 + \epsilon.$$

This allows us to conclude that $u_\lambda \neq v_\lambda$, and therefore u_λ and v_λ are two distinct solutions of (P) for $\lambda_0 < \lambda < \lambda_0 + \epsilon$. □

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Existence of nodal solutions to some nonlinear boundary value problems for ordinary differential equations of fourth order

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Abstract. In this paper, we study the existence of nodal solutions of some nonlinear boundary value problems for ordinary differential equations of fourth order with a spectral parameter in the boundary condition. To do this, we first study the global bifurcation of solutions from zero and infinity of the corresponding nonlinear eigenvalue problems in classes with a fixed oscillation count. Then, using these global bifurcation results, we prove the existence of solutions of the considered nonlinear boundary value problems with a fixed number of nodes.

Keywords: nonlinear problem, eigenvalue parameter, bifurcation point, nodal solution, component

2020 Mathematics Subject Classification: 34B24, 34C23, 34L15, 34L30, 47J10, 47J15.

1 Introduction

In this paper, we consider the existence of nodal solutions to the following nonlinear boundary value problem for ordinary differential equations of fourth order

$$\ell(y) \equiv (p(x)y''(x))'' - (q(x)y'(x))' = \chi r(x)h(y(x)),\ x \in (0,l), \tag{1.1}$$

$$y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \tag{1.2}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{1.3}$$

$$y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \tag{1.4}$$

$$(a\lambda + b)y(l) - (c\lambda + d)Ty(l) = 0, \tag{1.5}$$

where $Ty \equiv (py'')' - qy'$, p is a positive twice continuously differentiable function on $[0,l]$, q is a non-negative continuously differentiable function on $[0,l]$, χ is a positive number, $r(x)$ is

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a positive continuous function on $[0, l]$, α, β, γ, a, b, c, d are real constants such that $\alpha, \beta, \gamma \in [0, \pi/2]$ and $\sigma = bc - ad > 0$. The nonlinear term h has the form $f + g$, where f and g are real-valued continuous on $\mathbb{R}$ functions that satisfy the following conditions:

$$\underline{f}_0, \overline{f}_0, \underline{f}_\infty, \overline{f}_\infty \in \mathbb{R} \quad \text{with} \quad \underline{f}_0 \neq \overline{f}_0, \underline{f}_\infty \neq \overline{f}_\infty, \tag{1.6}$$

where

$$\underline{f}_0 = \liminf_{|s| \to 0} \frac{f(s)}{s}, \qquad \overline{f}_0 = \limsup_{|s| \to 0} \frac{f(s)}{s}, \tag{1.7}$$

$$\underline{f}_\infty = \liminf_{|s| \to +\infty} \frac{f(s)}{s}, \qquad \overline{f}_\infty = \limsup_{|s| \to +\infty} \frac{f(s)}{s}; \tag{1.8}$$

$$sg(s) > 0 \quad \text{for } s \in \mathbb{R} \setminus \{0\}; \tag{1.9}$$

there exist positive constants g_0, $g_\infty \in (0, +\infty)$ such that

$$g_0 = \lim_{|s| \to 0} \frac{g(s)}{s}, \qquad g_\infty = \lim_{|s| \to +\infty} \frac{g(s)}{s}. \tag{1.10}$$

The subject of this paper is to determine the interval of χ, in which there are solutions to problem (1.1)–(1.5) that have a fixed number of simple zeros in $(0, l)$.

It is well known that boundary value problems for ordinary differential equations arise in the study of many different processes of natural science, see [9,10,12,14,17] and the references therein. For example, problem (1.1)–(1.5) arises when studying of bending (of deformation) of a homogeneous rod, in the cross sections of which a longitudinal force acts and at the right end of which the mass is concentrated or on this end a tracking force acts.

Problems similar to (1.1)–(1.5) for ordinary differential equations of second and fourth orders have been considered before in, for example, [8, 11, 13, 16, 18–22, 26–28]. In [8, 11, 18–21, 26], the authors using the global bifurcation results of [1, 2, 7, 8, 11, 18, 23–25] show that there are nontrivial solutions of the considered nonlinear problems, which have the usual nodal properties (unfortunately, there are gaps in the proofs of the main assertions in [11, Theorems 2.2 and 3.1] and [18, Theorem 3.1]). Similar results were obtained in the paper [22] by analytical methods involving the Prüfer angular functions. Should be noted that in [13, 26, 27], problems with local and nonlocal boundary conditions are considered and the existence of positive solutions of these problems is established.

In the present paper, using the global bifurcation results from [1–4, 6] and removing the above gaps (see the proof of Steps 1–3 of Theorem 3.1), we prove the existence of two different solutions to problem (1.1)–(1.5) with a fixed number of nodal points.

The rest of this article is organized as follows. Section 2 provides, which we need in the future, known facts about the unilateral global bifurcation of solutions from zero and infinity of nonlinear eigenvalue problems for fourth-order ordinary differential equations. In Section 3, we determine an interval for a parameter χ, in which there are nodal solutions to problem (1.1)–(1.5). In this case, the proof of the main theorem, i.e. Theorem 3.1 consists of 4 steps. In Step 1, using (1.6), (1.7) and the first condition from (1.10), we find bifurcation intervals from zero and prove the existence of two families of unbounded components of the solution set of problem (1.1)–(1.5) bifurcating from these intervals and contained in classes with a fixed number of nodes. In Step 2, using (1.6), (1.8) and the second condition from (1.10), we find bifurcation intervals from infinity and prove the existence of two families of unbounded components of the set of solutions bifurcating from these intervals and contained in classes with a fixed number of nodes in the neighborhood of these intervals, which either intersect another bifurcation interval, or intersect the line of trivial solutions, or have unbounded projections onto the line of trivial solutions. In Step 3, it is established that the global components of solutions to problem (1.1)–(1.5) bifurcating from intervals infinity are also contained in the corresponding classes with a fixed number of nodes and coincide with the corresponding components of solutions bifurcating from intervals of the line of trivial solutions.

2 Preliminary

We consider the following linear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda r(x) y(x),\ x \in (0,l), \\ y \in (b.c.)_\lambda, \end{cases} . \tag{2.1}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $(b.c.)_\lambda$ is a set of functions satisfying the boundary conditions (1.2)–(1.5).

The spectral properties of (2.1) were studied in [15], where, in particular, it was shown that the spectrum of this problem is discrete and consists of an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ of real and simple eigenvalues. Moreover, if $c = 0$, then eigenfunction $y_k(x)$, $k \in \mathbb{N}$, corresponding to the eigenvalue λ_k has exactly $k-1$ simple zeros in $(0,1)$; if $c \neq 0$, then there exists $N \in \mathbb{N}$ such that the eigenfunction $y_k(x)$ corresponding to the eigenvalue λ_k has for $k \leq N$ exactly $k-1$ and for $k > N$ exactly $k-2$ simple zeros in $(0,l)$.

Remark 2.1. Throughout what follows we will assume that the coefficients of boundary conditions are chosen such that the first eigenvalue of problem (2.1) is positive.

Let E be a Banach space $C^3[0,l] \cap BC_0$ with the norm $\|y\|_3 = \sum_{s=0}^{3} \| y^{(s)} \|_\infty$, where $\|y\|_\infty = \max_{x\in[0,l]} |y\,(x)|$ and BC_0 is a set of functions which satisfy the boundary conditions (1.2)–(1.4).

From now on ν will denote an element of $\{+,-\}$ that is, either $\nu = +$ or $\nu = -$.

In a recent paper [4, §2, pp. 4–5], using the Prüfer type transformation for each $k \in \mathbb{N}$ and each ν, the authors constructed sets $\mathcal{S}_k^\nu$ of functions $y \in E$, which have the oscillatory properties of eigenfunctions of the linear problem (2.1) and their derivatives. Note that the sets $\mathcal{S}_k^+$, $\mathcal{S}_k^-$ and $\mathcal{S}_k = \mathcal{S}_k^+ \cup \mathcal{S}_k^-$ are pairwise disjoint open subsets of E. Moreover, it was shown in [1, Lemma 2.2] that if $y \in \partial\mathcal{S}_k^\nu\ (\partial\mathcal{S}_k)$, then y has at least one zero of multiplicity 4 in $(0,l)$.

To study the existence of solutions to problem (1.1)–(1.5) with a fixed number of nodes, consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(y) = \lambda r(x) y + \tilde{h}(x, y, y', y'', y''', \lambda),\ x \in (0,l), \\ y \in (b.c.)_\lambda. \end{cases} \tag{2.2}$$

Here $\tilde{h}$ has a representation $\tilde{f} + \tilde{g}$, where $\tilde{f}$ and $\tilde{g}$ are real-valued continuous functions on $[0,l] \times \mathbb{R}^5$ that satisfy the following conditions: there exist constants $\tilde{M} > 0$ and sufficiently small $\tau_0 > 0$ such that

$$\left| \frac{\tilde{f}(x,y,s,v,w,\lambda)}{y} \right| \leq \tilde{M},\ (x,y,s,v,w) \in [0,l] \times \mathbb{R}^4,\ 0 < |y| + |s| + |v| + |w|\ \leq \tau_0, \\ y \neq 0,\ \lambda \in \mathbb{R}; \tag{2.3}$$

$$\tilde{g}(x,y,s,v,w,\lambda) = o(|y|+|s|+|v|+|w|) \text{ as } |y|+|s|+|v|+|w| \to 0, \tag{2.4}$$

uniformly in $x \in [0,l]$ and $\lambda \in \Lambda$ for each bounded interval $\Lambda \subset \mathbb{R}$, or there exist constants $\tilde{\tilde{M}} > 0$ and sufficiently large $\varkappa_0 > 0$ such that

$$\left|\frac{\tilde{f}(x,y,s,v,w,\lambda)}{y}\right| \le \tilde{\tilde{M}},\ (x,y,s,v,w) \in [0,l] \times \mathbb{R}^4,\ |y|+|s|+|v|+|w| \ge \varkappa_0,\ y \neq 0,\ \lambda \in \mathbb{R}; \tag{2.5}$$

$$\tilde{g}(x,\,y,\,s,\,v,\,w,\,\lambda) = o(|y|+|s|+|v|+|w|) \text{ as } |y|+|s|+|v|+|w| \to \infty, \tag{2.6}$$

uniformly in $x \in [0,l]$ and $\lambda \in \Lambda$.

If conditions (2.3) and (2.4) are satisfied, then the bifurcation of nontrivial solutions of problem (2.2) from the line of trivial solutions $\mathbb{R} \times \{0\} = \{(\lambda,0) : \lambda \in \mathbb{R}\}$ is considered. In this case, the global bifurcation of nontrivial solutions of problem (2.2) is studied in [4], where the following results are obtained.

Lemma 2.2 ([4, Lemmas 3 and 4]). *Let conditions (2.3) and (2.4) be satisfied. Then for each $k \in \mathbb{N}$ and each ν the set of bifurcation points of (2.2) with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$ is nonempty and lies in $\tilde{I}_k \times \{0\}$, where $\tilde{I}_k = \left[\lambda_k - \frac{\tilde{M}}{r_0}, \lambda_k + \frac{\tilde{M}}{r_0}\right]$, $r_0 = \min_{x\in[0,l]} r(x)$.*

For each $k \in \mathbb{N}$ and each ν let $\tilde{D}_k^\nu$ be the union of all the components of the set of nontrivial solutions to problem (2.2) bifurcating from the points of the interval $\tilde{I}_k \times \{0\}$ with respect to $\mathbb{R} \times \mathcal{S}_k^\nu$. Moreover, let $D_k^\nu = \tilde{D}_k^\nu \cup (\tilde{I}_k \times \{0\})$. Note that D_k^ν is connected, but $\tilde{D}_k^\nu$ may not be connected in $\mathbb{R} \times E$.

Theorem 2.3 ([4, Theorem 3]). *Let conditions (2.3) and (2.4) be satisfied. Then for each $k \in \mathbb{N}$ and each ν the set $\tilde{D}_k^\nu$ is nonempty, lies in $\mathbb{R} \times S_k^\nu$ and is unbounded in $\mathbb{R} \times E$.*

In the case when conditions (2.5) and (2.6) are satisfied, then we consider the bifurcation of nontrivial solutions to problem (2.2) from infinity, or rather from the line $\mathbb{R} \times \{\infty\} = \{(\lambda,\infty) : \lambda \in \mathbb{R}\}$. Global bifurcation of nontrivial solutions of problem (2.2) from infinity with respect to the set $\mathbb{R} \times S_k^\nu$ was considered in [3] in the case of $\tilde{f} \equiv 0$. Using the results of [1,3] and [4] following the corresponding arguments in [6], we can obtain the following results.

Lemma 2.4. *Let conditions (2.5) and (2.6) be satisfied. Then for each $k \in \mathbb{N}$ and each ν the set of asymptotic bifurcation points of problem (2.2) with respect to the set $\mathbb{R} \times S_k^\nu$ is nonempty and lies in $\tilde{\tilde{I}}_k \times \{\infty\}$, where $\tilde{\tilde{I}}_k = \left[\lambda_k - \frac{\tilde{\tilde{M}}}{r_0}, \lambda_k + \frac{\tilde{\tilde{M}}}{r_0}\right]$.*

For each $k \in \mathbb{N}$ and each ν let $\tilde{\tilde{D}}_k^\nu$ be the union of all the components of the set of nontrivial solutions to problem (2.2) bifurcating from the points of the interval $\tilde{\tilde{I}}_k \times \{\infty\}$ with respect to the set $\mathbb{R} \times S_k^\nu$. Moreover, let $D_k^{\nu,*} = \tilde{\tilde{D}}_k^\nu \cup \left(\tilde{\tilde{I}}_k \times \{\infty\}\right)$ (in this case we add the points $\{(\lambda,\infty) : \lambda \in \mathbb{R}\}$ to our space $\mathbb{R} \times E$ and define an appropriate topology on the resulting set). Note that $D_k^{\nu,*}$ is connected.

Theorem 2.5. *For each $k \in \mathbb{N}$ and each ν the set $\tilde{\tilde{D}}_k^\nu$ is nonempty and for this set at least one of the following statements holds:*

(i) the set $\tilde{\tilde{D}}_k^\nu$ meets $\tilde{\tilde{I}}_{k'} \times \{\infty\}$ with respect to $\mathbb{R} \times \mathcal{S}_{k'}^{\nu'}$ for some $(k',\nu') \neq (k,\nu)$;

(ii) the set $\tilde{\tilde{D}}_k^\nu$ meets $\mathbb{R} \times \{0\}$ for some $\lambda \in \mathbb{R}$;

(iii) the projection of $\tilde{\tilde{D}}_k^\nu$ on $\mathbb{R} \times \{0\}$ is unbounded.

In addition, if cases (ii) and (iii) are not satisfied for the union $\tilde{\tilde{D}}_k = \tilde{\tilde{D}}_k^+ \cup \tilde{\tilde{D}}_k^-$, then case (i) is satisfied for it with $k' \neq k$.

3 Existence of solutions to problem (1.1)–(1.5) with fixed oscillation count

In this section we will determine the interval of χ, in which there exist nodal solutions of problem (1.1)–(1.5).

Theorem 3.1. *Let $g_0 > -\underline{f}_0$, $g_\infty > -\underline{f}_\infty$, and for some $k \in \mathbb{N}$ one of the following conditions is satisfied:*

$$\frac{\lambda_k}{g_0 + \underline{f}_0} < \chi < \frac{\lambda_k}{g_\infty + \overline{f}_\infty}; \qquad \frac{\lambda_k}{g_\infty + \underline{f}_\infty} < \chi < \frac{\lambda_k}{g_0 + \overline{f}_0}.$$

Then there are two solutions $\tilde{y}_k^+$ and $\tilde{y}_k^-$ of problem (1.1)–(1.5) such that $\tilde{y}_k^+ \in \mathcal{S}_k^+$ and $\tilde{y}_k^- \in \mathcal{S}_k^-$, i.e., $\tilde{y}_k^+$ has either $k-1$ or $k-2$ simple zeros in $(0, l)$ and is positive near $x = 0$, and $\tilde{y}_k^-$ has either $k-1$ or $k-2$ simple zeros in $(0, l)$ and is negative near $x = 0$.

Proof. To prove the theorem, consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda \chi r(x) g(y(x)) + \chi r(x) f(y(x)), \ x \in (0, l), \\ y \in (b.c.)_\lambda, \end{cases} \tag{3.1}$$

where $\lambda \in \mathbb{R}$ is an eigenvalue parameter.

Step 1. It follows from the first condition of (1.10) that the function $g(s)$, $s \in \mathbb{R}$, can be represented in the following form

$$g(s) = s g_0 + \rho(s), \tag{3.2}$$

where $\rho(s)$ is a real-valued continuous functions on $\mathbb{R}$ that satisfies the condition

$$\lim_{|s| \to 0} \frac{\rho(s)}{s} = 0. \tag{3.3}$$

Let $\zeta(u) = \max_{|s| \in [0, u]} |\rho(s)|$. It is obvious that the function $\zeta(u)$ is nondecreasing on $[0, +\infty)$.

It follows from (3.3) that for any sufficiently small $\varepsilon > 0$ one can find a sufficiently small $\delta_\varepsilon > 0$ such that for any $s \in \mathbb{R}$ satisfying condition $|s| < \delta_\varepsilon$ we have $|\rho(s)| < \varepsilon |s|$. Then we have

$$\frac{\zeta(u)}{u} < \varepsilon \quad \text{for any } u \in (0, \delta_\varepsilon). \tag{3.4}$$

Since the function $\zeta(u)$ is nondecreasing on $[0, +\infty)$ for any $x \in [0, l]$ we get

$$\frac{|\rho(y(x))|}{\|y\|_3} \leq \frac{\zeta(\|y\|_\infty)}{\|y\|_3} \leq \frac{\zeta(\|y\|_3)}{\|y\|_3}. \tag{3.5}$$

Let $y \in E$ such that $\|y\|_3 < \delta_\varepsilon$. Then by (3.4) we have

$$\frac{\zeta(\|y\|_3)}{\|y\|_3} < \varepsilon,$$

and consequently, for any $x \in [0, l]$ we get

$$\frac{|\rho(y(x))|}{\|y\|_3} < \varepsilon \quad \text{for any } x \in [0, l], \tag{3.6}$$

in view of (3.5). Therefore, it follows from (3.6) that

$$\|\rho(y)\|_\infty = o(\|y\|_3) \quad \text{as } \|y\|_3 \to 0. \tag{3.7}$$

Considering (3.2), the problem (3.1) can be written in the following equivalent form

$$\begin{cases} \ell(y)(x) = \lambda\chi r(x) g_0 y(x) + \chi r(x) f(y(x)) + \lambda\chi r(x)\rho(y(x)),\ x \in (0,l), \\ y \in (b.c.)_\lambda. \end{cases} \tag{3.8}$$

Let $\delta_0 > 0$ be a sufficiently small number. Then it follows from (1.6) and (1.7) that there exists sufficiently small $\sigma_0 \in (0, \tau_0)$ such that

$$\underline{f}_0 - \frac{g_0\delta_0}{2} < \frac{f(s)}{s} < \overline{f}_0 + \frac{g_0\delta_0}{2} \quad \text{for any } s \in \mathbb{R},\ 0 < |s| < \sigma_0. \tag{3.9}$$

Relation (3.9) implies that

$$\left|\frac{f(s)}{s}\right| \le \tilde{M}_0 \quad \text{for any } s \in \mathbb{R},\ \ 0 < |s| < \sigma_0, \tag{3.10}$$

where $\tilde{M}_0 = \max\left\{|\underline{f}_0 - \frac{g_0\delta_0}{2}|, |\overline{f}_0 + \frac{g_0\delta_0}{2}|\right\} > 0$. Then by (3.7) (see also (3.6)) and (3.10) it follows from Lemma 2.2 that for each $k \in \mathbb{N}$ and each ν the set of bifurcation points of (3.8) (or (3.1)) with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$ is nonempty. If $(\lambda^*, 0)$ is a bifurcation point of problem (3.8) with respect to $\mathbb{R} \times \mathcal{S}_k^\nu$, then there exists a sequence $\{(\lambda_n^*, y_n^*)\}_{n=1}^\infty \subset \mathbb{R} \times \mathcal{S}_k^\nu$ such that

$$\begin{cases} \ell(y_n^*)(x) = \lambda_n^*\chi r(x) g_0 y_n^*(x) + \chi r(x) f(y_n^*(x)) + \lambda_n^*\chi r(x)\rho(y_n^*(x)),\ x \in (0,l), \\ y_n^* \in (b.c.)_{\lambda_n^*}, \end{cases} \tag{3.11}$$

and

$$(\lambda_n^*, y_n^*) \to (\lambda^*, 0) \quad \text{in } \mathbb{R} \times E \text{ as } n \to \infty. \tag{3.12}$$

Let

$$\varphi_n^*(x) = \begin{cases} -\frac{f(\tilde{y}_n^*(x))}{\tilde{y}_n^*(x)} & \text{if } \tilde{y}_n^*(x) \neq 0, \\ 0 & \text{if } \tilde{y}_n^*(x) = 0. \end{cases} \tag{3.13}$$

Then by (3.13) it follows from (3.11) that for each $n \in \mathbb{N}$ the pair (λ_n^*, y_n^*) is a solution of the following linearizable problem

$$\begin{cases} \frac{1}{\chi r(x) g_0}\,\ell(y)(x) + \frac{1}{g_0}\,\varphi_n^*(x) y(x) = \lambda y(x) + \frac{1}{g_0}\,\lambda\rho(y(x)),\ x \in (0,l), \\ y \in (b.c.)_\lambda. \end{cases} \tag{3.14}$$

In view of (3.12) we can choose $n \in \mathbb{N}$ so large that

$$-\left(\frac{\overline{f}_0}{g_0} + \frac{\delta_0}{2}\right) < \frac{1}{g_0}\varphi_n^*(x) < -\left(\frac{\underline{f}_0}{g_0} - \frac{\delta_0}{2}\right) \quad \text{for any } x \in [0,l], \tag{3.15}$$

in view of (3.9) and (3.13).

It follows from [5, Remark 4.2 and Theorem 4.3] that for each fixed $n \in \mathbb{N}$ the eigenvalues of the linear eigenvalue problem

$$\begin{cases} \frac{1}{\chi r(x) g_0}\,\ell(y)(x) + \frac{1}{g_0}\,\varphi_n^*(x) y(x) = \lambda y(x),\ x \in (0,l), \\ y \in (b.c.)_\lambda, \end{cases} \tag{3.16}$$

are real and simple, and form an infinitely increasing sequence $\{\lambda^*_{k,n}\}_{k=1}^{\infty}$; moreover, the eigenfunction $y^*_{k,n}(x)$, $k \in \mathbb{N}$, corresponding to the eigenvalue $\lambda^*_{k,n}$ lies in $\mathcal{S}_k$.

In view of relation (3.15), by following the arguments in Lemmas 5.1 and 5.3 of [1] we get

$$\tilde{\lambda}_k - \frac{\overline{f}_0}{g_0} - \frac{\delta_0}{2} \leq \lambda^*_{k,n} \leq \tilde{\lambda}_k - \frac{\underline{f}_0}{g_0} + \frac{\delta_0}{2}, \tag{3.17}$$

where $\tilde{\lambda}_k$, $k \in \mathbb{N}$, is a kth eigenvalue of the linear eigenvalue problem

$$\begin{cases} \frac{1}{\chi r(x) g_0}\, \ell(y)(x) = \lambda y(x), \; x \in (0,l), \\ y \in (b.c.)_\lambda. \end{cases} \tag{3.18}$$

Since $(\lambda^*_{k,n}, 0)$ is a unique bifurcation point of problem (3.14) with respect to $\mathbb{R} \times \mathcal{S}_k^\nu$ by (3.12) we can again choose $n \in \mathbb{N}$ so large that

$$\lambda^*_{k,n} - \frac{\delta_0}{2} < \lambda^*_n < \lambda^*_{k,n} + \frac{\delta_0}{2}\,. \tag{3.19}$$

Then it follows from (3.17) and (3.19) that

$$\tilde{\lambda}_k - \frac{\overline{f}_0}{g_0} - \delta_0 < \lambda^*_n < \tilde{\lambda}_k - \frac{\underline{f}_0}{g_0} + \delta_0\,, \tag{3.20}$$

whence, with regard to (3.12), we obtain

$$\tilde{\lambda}_k - \frac{\overline{f}_0}{g_0} - \delta_0 \leq \lambda^* \leq \tilde{\lambda}_k - \frac{\underline{f}_0}{g_0} + \delta_0\,. \tag{3.21}$$

As can be seen from (3.18) that $\lambda_k = \chi g_0 \tilde{\lambda}_k$ for each $k \in \mathbb{N}$. Consequently, it follows from (3.21) that

$$\frac{\lambda_k}{\chi g_0} - \frac{\overline{f}_0}{g_0} - \delta_0 \leq \lambda^* \leq \frac{\lambda_k}{\chi g_0} - \frac{\underline{f}_0}{g_0} + \delta_0\,. \tag{3.22}$$

Since δ_0 is arbitrary small enough, it follows from (3.22) that

$$\frac{\lambda_k}{\chi g_0} - \frac{\overline{f}_0}{g_0} \leq \lambda^* \leq \frac{\lambda_k}{\chi g_0} - \frac{\underline{f}_0}{g_0}\,. \tag{3.23}$$

Thus, (3.23) shows that the bifurcation points of problem (3.1) (or (3.8)) with respect to the set $\mathbb{R} \times \mathcal{S}_k^\nu$ are contained in the interval $I_k^0 \times \{0\}$, where

$$I_k^0 = \left[\frac{\lambda_k}{\chi g_0} - \frac{\overline{f}_0}{g_0}, \frac{\lambda_k}{\chi g_0} - \frac{\underline{f}_0}{g_0}\right].$$

Then, by Theorem 2.3, for each $k \in \mathbb{N}$ and each ν there exists a component $D^\nu_{k,0}$ of the set of solutions of problem (3.1), which contains $I_k^0 \times \{0\}$, lies in $(\mathbb{R} \times \mathcal{S}_k^\nu) \cup (I_k^0 \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.

Step 2. By the second condition in (1.10) we can represent the function $g(s)$, $s \in \mathbb{R}$, as follows:

$$g(s) = s g_\infty + \varrho(s), \tag{3.24}$$

where

$$\lim_{|s|\to+\infty}\frac{\varrho(s)}{s}=0. \tag{3.25}$$

Let $\varsigma(u)=\max_{|s|\in[0,u]}|\varrho(s)|$. It is obvious that the function $\varsigma(u)$ is nondecreasing on $[0,+\infty)$.

In view of (3.25), for any sufficiently small $\varepsilon>0$ there exists a sufficiently large $\Delta_\varepsilon>0$ such that

$$|\varrho(s)|<\frac{1}{2}\,\varepsilon|s| \quad \text{for any } s\in\mathbb{R},\ |s|>\Delta_\varepsilon. \tag{3.26}$$

Let $u\in[\Delta_\varepsilon,\infty)$ be arbitrary. Then we have

$$\varsigma(u)=\max\left\{\max_{|s|\in[0,\Delta_\varepsilon]}|\varrho(s)|,\ \max_{|s|\in[\Delta_\varepsilon,u]}|\varrho(s)|\right\}. \tag{3.27}$$

Let $K_\varepsilon=\max_{|s|\in[0,\Delta_\varepsilon]}|\varrho(s)|$. We will choose $\Delta_{1,\varepsilon}>\Delta_\varepsilon$ so large that $\frac{K_\varepsilon}{\Delta_{1,\varepsilon}}<\frac{1}{2}\,\varepsilon$. Now let $u>\Delta_{1,\varepsilon}$. Then by (3.26) it follows from (3.27) that

$$\begin{aligned}\frac{\varsigma(u)}{u}&=\frac{\max\{K_\varepsilon,\max_{|s|\in[\Delta_\varepsilon,u]}|\varrho(s)|\}}{u}\le\frac{\max\{K_\varepsilon,\frac{1}{2}\,\varepsilon\,u\}}{u}\\&=\max\left\{\frac{K_\varepsilon}{u},\frac{1}{2}\,\varepsilon\right\}\le\max\left\{\frac{K_\varepsilon}{\Delta_{1,\varepsilon}},\frac{1}{2}\,\varepsilon\right\}\le\frac{1}{2}\,\varepsilon<\varepsilon.\end{aligned} \tag{3.28}$$

Since the function $\varsigma(u)$ is nondecreasing on $[0,+\infty)$ for any $x\in[0,l]$ we have

$$\frac{|\varrho(y(x))|}{\|y\|_3}\le\frac{\varsigma(\|y\|_\infty)}{\|y\|_3}\le\frac{\varsigma(\|y\|_3)}{\|y\|_3}. \tag{3.29}$$

If $\|y\|_3>\Delta_{1,\varepsilon}$, then by (3.28) it follows from (3.29) that

$$\frac{|\varrho(y(x))|}{\|y\|_3}<\varepsilon \quad \text{for any } x\in[0,l],$$

which shows that

$$\|\varrho(y)\|_\infty=o(\|y\|_3) \quad \text{as } \|y\|_3\to\infty. \tag{3.30}$$

Taking into account (3.24), we can rewrite the problem (3.1) in the following equivalent form

$$\begin{cases}\ell(y)(x)=\lambda\chi r(x)g_\infty y(x)+\chi r(x)f(y(x))+\lambda\chi r(x)\varrho(y(x)),\ x\in(0,l),\\ y\in(b.c.)_\lambda.\end{cases} \tag{3.31}$$

Using [1, Lemma 5.1], Lemma 2.4, relations (1.6), (1.8), (3.30) and following the above arguments, we can show that if $(\tilde\lambda^*,\infty)$ is an asymptotic bifurcation point of problem (3.1) (or (3.31)), then

$$\tilde\lambda^*\in I_k^\infty=\left[\frac{\lambda_k}{\chi g_\infty}-\frac{\overline{f}_\infty}{g_\infty},\frac{\lambda_k}{\chi g_\infty}-\frac{\underline{f}_\infty}{g_\infty}\right].$$

Hence it follows from Theorem 2.5 that for each $k\in\mathbb{N}$ and each ν there exists a component $D_{k,\infty}^\nu$ of the set of solutions to problem (3.1) containing $I_k^\infty\times\{\infty\}$ and for which at least one of the following statements holds:

(i) the set $D_{k,\infty}^\nu$ meets $I_{k'}^\infty\times\{\infty\}$ with respect to $\mathbb{R}\times\mathcal{S}_{k'}^{\nu'}$ for some $(k',\nu')\neq(k,\nu)$;

(ii) the set $D^{\nu}_{k,\infty}$ meets $\mathbb{R} \times \{0\}$ for some $\lambda \in \mathbb{R}$;

(iii) the projection of $D^{\nu}_{k,\infty}$ on $\mathbb{R} \times \{0\}$ is unbounded.

Step 3. By following the arguments in Theorem 3.3 of [25] we can show that for each $k \in \mathbb{N}$ and each ν, $D^{\nu}_{k,\infty} \backslash (I^{\infty}_k \times \{\infty\}) \subset \mathbb{R} \times \mathcal{S}^{\nu}_k$, and consequently, alternative (i) above for $D^{\nu}_{k,\infty}$ cannot hold. Moreover, if $D^{\nu}_{k,\infty}$ meets $\mathbb{R} \times \{0\}$ for some $\lambda \in \mathbb{R}$, then $\lambda \in I^0_k$. Similarly, if $D^{\nu}_{k,0}$ meets $\mathbb{R} \times \{\infty\}$ for some $\lambda \in \mathbb{R}$, then $\lambda \in I^{\infty}_k$. Hence we conclude that if $D^{\nu}_{k,\infty}$ has a bounded projection on $\mathbb{R} \times \{0\}$, then $D^{+}_{k,0} = D^{+}_{k,\infty}$ and $D^{-}_{k,0} = D^{-}_{k,\infty}$.

Now we show that for each $k \in \mathbb{N}$ and each ν the set $D^{\nu}_{k,\infty}$ has a bounded projection on $\mathbb{R} \times \{0\}$. Indeed, otherwise there exists a sequence $\{(\overline{\lambda}_n, \overline{y}_n)\}_{n=1}^{\infty} \subset (D^{\nu}_{k,\infty} \backslash \mathcal{Q}_{k,\infty}) \subset (\mathbb{R} \times \mathcal{S}^{\nu}_k)$ such that

$$\lim_{n\to\infty} \overline{\lambda}_n = \pm\infty, \tag{3.32}$$

where $\mathcal{Q}_{k,\infty}$ is a some neighborhood of $I^{\infty}_k \times \{\infty\}$.

By (1.6)–(1.10) there exists a positive constants κ_0, κ_1 and κ_2 such that

$$\kappa_0 \leq \frac{g(s)}{s} \leq \kappa_1 \quad \text{and} \quad \left|\frac{f(s)}{s}\right| \leq \kappa_2 \quad \text{for any } s \in \mathbb{R},\ s \neq 0. \tag{3.33}$$

We define the functions $\overline{\varphi}_n(x)$ and $\overline{\phi}_n(x)$, $x \in [0, l]$, as follows:

$$\overline{\varphi}_n(x) = \begin{cases} \frac{g(\overline{y}_n(x))}{\overline{y}_n(x)} & \text{if } \overline{y}_n(x) \neq 0, \\ 0 & \text{if } \overline{y}_n(x) = 0, \end{cases} \qquad \overline{\phi}_n(x) = \begin{cases} -\frac{f(\overline{y}_n(x))}{\overline{y}_n(x)} & \text{if } \overline{y}_n(x) \neq 0, \\ 0 & \text{if } \overline{y}_n(x) = 0. \end{cases} \tag{3.34}$$

Since $\overline{y}_n \in \mathcal{S}^{\nu}_k$ by (3.34) it follows from (3.1) that $\overline{\lambda}_n$ for each $n \in \mathbb{N}$ is kth eigenvalue of the following linear eigenvalue problem

$$\begin{cases} \ell(y)(x) + \chi r(x)\overline{\phi}_n(x)\, y(x) = \lambda \chi r(x)\overline{\varphi}_n(x)\, y(x)),\ x \in (0, l), \\ y \in (b.c.)_{\lambda}. \end{cases} \tag{3.35}$$

By (3.33) from (3.34) we get

$$\kappa_0 \leq \overline{\varphi}_n(x) \leq \kappa_1 \quad \text{and} \quad |\overline{\phi}_n(x)| \leq \kappa_2 \quad \text{for any } x \in [0, l]. \tag{3.36}$$

It is known (see [1, 4]) that problem (3.35) reduces to the spectral problem for the self-adjoint operator in the Hilbert space $H = L_2(0, l) \oplus \mathbb{C}$ with corresponding scalar product. In view of (3.36), by the maximum-minimum property of eigenvalues (see [1, 2]) we obtain that the eigenvalues of problem (3.35) are uniformly bounded from below with respect to $n \in \mathbb{N}$. Consequently, the relation

$$\lim_{n\to\infty} \overline{\lambda}_n = -\infty$$

is not possible. Should be noted that the relation

$$\lim_{n\to\infty} \overline{\lambda}_n = +\infty,$$

is also impossible, since for a sufficiently large n, by [5, Theorem 4.3], the number of zeros of the function $\overline{y}_n$ will be large enough, which contradicts the condition $\overline{y}_n \in \mathcal{S}^{\nu}_k$.

Therefore, for any $k \in \mathbb{N}$ we have

$$D^{+}_{k,0} = D^{+}_{k,\infty} \quad \text{and} \quad D^{-}_{k,0} = D^{-}_{k,\infty}. \tag{3.37}$$

Step 4. It is obvious that any solution to problem (3.1) of the form $(1, y)$ gives a solution y to problem (1.1)–(1.5). In order for problem (1.1)–(1.5) to have a solution y which is contained in $\mathcal{S}_k^\nu$ for some $k \in \mathbb{N}$, by (3.37) it is sufficient that on the real axis $\mathbb{R}$ the interval I_k^0 lies to the left of 1 and the interval I_k^∞ lies to the right of 1, or the interval I_k^0 lies to the right of 1, and the interval I_k^∞ lies to the left of 1.

Let the conditions $g_0 > -\underline{f}_0$ and $g_\infty > -\underline{f}_\infty$ be satisfied. Hence we have $g_\infty > -\overline{f}_\infty$. If the condition $\frac{\lambda_k}{g_0+\underline{f}_0} < \chi < \frac{\lambda_k}{g_\infty+\overline{f}_\infty}$ is satisfied, then we get

$$\frac{\lambda_k}{\chi g_0} - \frac{\underline{f}_0}{g_0} < \frac{\lambda_k}{\frac{\lambda_k}{g_0+\underline{f}_0} g_0} - \frac{\underline{f}_0}{g_0} = \frac{\lambda_k(g_0+\underline{f}_0)}{\lambda_k g_0} - \frac{\lambda_k \underline{f}_0}{\lambda_k g_0} = 1$$

and

$$\frac{\lambda_k}{\chi g_\infty} - \frac{\overline{f}_\infty}{g_\infty} > \frac{\lambda_k}{\frac{\lambda_k}{g_\infty+\overline{f}_\infty} g_\infty} - \frac{\overline{f}_\infty}{g_\infty} = \frac{\lambda_k(g_\infty+\overline{f}_\infty)}{\lambda_k g_\infty} - \frac{\lambda_k \overline{f}_\infty}{\lambda_k g_\infty} = 1.$$

The case in which $\frac{\lambda_k}{g_\infty+\underline{f}_\infty} < \chi < \frac{\lambda_k}{g_0+\overline{f}_0}$ can be considered in a similar way. The proof of this theorem is complete. □

Step 4 of the proof of Theorem 3.1 makes it possible to obtain other conditions for the existence of solutions to problem (1.1)–(1.5) contained in the sets $\mathcal{S}_k^+$ and $\mathcal{S}_k^-$ for some $k \in \mathbb{N}$.

Theorem 3.2. *Let $g_0 > -\underline{f}_0$, $-\overline{f}_\infty < g_\infty \leq -\underline{f}_\infty$, and for some $k \in \mathbb{N}$ the following condition is satisfied:*

$$\frac{\lambda_k}{g_0+\underline{f}_0} < \chi < \frac{\lambda_k}{g_\infty+\overline{f}_\infty}.$$

Then the statement of Theorem 3.1 holds.

Theorem 3.3. *Let $g_0 > -\underline{f}_0$, $g_\infty \leq -\overline{f}_\infty$, and for some $k \in \mathbb{N}$ the following condition is satisfied:*

$$\chi > \frac{\lambda_k}{g_0+\underline{f}_0}.$$

Then the statement of Theorem 3.1 holds.

Theorem 3.4. *Let $-\overline{f}_0 < g_0 \leq -\underline{f}_0$, $g_\infty > -\underline{f}_\infty$, and for some $k \in \mathbb{N}$ the following condition is satisfied:*

$$\frac{\lambda_k}{g_\infty+\underline{f}_\infty} < \chi < \frac{\lambda_k}{g_0+\overline{f}_0}.$$

Then the statement of Theorem 3.1 holds.

Theorem 3.5. *Let $g_0 \leq -\overline{f}_0$, $g_\infty > -\underline{f}_\infty$, and for some $k \in \mathbb{N}$ the following condition is satisfied:*

$$\chi > \frac{\lambda_k}{g_\infty+\underline{f}_\infty}.$$

Then the statement of Theorem 3.1 holds.

The proofs of these theorems are similar to that of Step 4 of Theorem 3.1.

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Fully nonlinear degenerate equations with applications to Grad equations

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Abstract. We consider a class of degenerate elliptic fully nonlinear equations with applications to Grad equations:

$$\begin{cases} |Du|^{\gamma}\mathcal{M}^{+}_{\lambda,\Lambda}\big(D^2u(x)\big) = f\big(|u \geq u(x)|\big) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $\gamma \geq 1$ is a constant, Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary. We prove the existence of a $W^{2,p}$-viscosity solution to the above equation, which degenerates when the gradient of the solution vanishes.

Keywords: fully nonlinear degenerate elliptic equations, viscosity solution, Pucci's extremal operator, Dirichlet boundary value problem.

2020 Mathematics Subject Classification: 35J25, 35J60, 35J70, 35D40.

1 Introduction

We investigate the following degenerate problem:

$$\begin{cases} |Du|^{\gamma}\mathcal{M}^{+}_{\lambda,\Lambda}\big(D^2u(x)\big) = f\big(|u \geq u(x)|\big) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\gamma \geq 1$ is a constant, Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary, $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$, $f : [0, |\Omega|] \longrightarrow \mathbb{R}$ is a non-decreasing, non-negative continuous function and $u : \Omega \longrightarrow \mathbb{R}$. Here, $\mathcal{M}^{+}_{\lambda,\Lambda}$ is the Pucci's extremal operator. In our setting, by $u \geq u(x)$, we mean,

$$\{\omega \in \Omega : u(\omega) \geq u(x)\}$$

called the superlevel sets of u. We establish the existence of a $W^{2,p}$-viscosity solution (also known as L^p-viscosity solution) to (1.1). It is worth mentioning that the notion of $W^{2,p}$-viscosity solution was defined by Caffarelli et al. [7]. In the case when $\gamma = 0$ in (1.1), the existence of a $W^{2,p}$-viscosity solution is proven by L. Caffarelli and I. Tomasetti [8].

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The pioneer contribution in this direction was due to H. Grad [12], who introduced such equations, which appear in plasma physics, called "Grad equations". In their seminal work, Grad examined the following equation in three-dimension:

$$\Delta\Psi = F(V, \Psi, \Psi', \Psi''),$$

where the right hand side (R.H.S.) represents a second-order differential operator acting on $\Psi(V)$ for a surface defined by $\Psi =$ constant. Here, $\Psi'(V)$ represents the derivative with respect to volume and $\Psi(V)$ stands for the inverse function to $V(\Psi)$, denoting the volume enclosed by Ψ. Furthermore, they pointed out the potential for simplifying plasma equations by introducing u^* defined as:

$$u^*(t) := \inf\Big\{s : |u < s| \geq t\Big\}.$$

These equations, also known as Queer Differential Equations in the literature, have a wide range of applications across various fields. One notable application is their appearance in plasma modeling, specifically in analyzing plasma confined within toroidal containers. We refer to [12] for the details. Moreover, these equations exhibit connections in financial mathematics, see [23]. R. Temam [22] pioneered the investigation of problems akin to (1.1) concerning the Laplacian, a direction extensively examined by several researchers. They notably established the existence of solutions to equations having the model structure:

$$\Delta u = g\big(|u < u(x)|, u(x)\big) + f(x),$$

by exploiting the properties of directional derivatives of u^*. For further insights into this topic, we refer the interested readers to the works of J. Mossino and R. Temam [17], as well as those by P. Laurence and E. Stredulinsky [15,16], along with the related references therein.

In all the aforementioned research works, the problem was studied using variational methods. However, in a recent work, L. Caffarelli and I. Tomasetti [8] studied the equation similar to J. Mossino and R. Temam [17] for fully nonlinear uniformly elliptic operators using the viscosity approach. Specifically, they addressed the following problem:

$$\begin{cases} F\big(D^2u(x)\big) = f\big(|u \geq u(x)|\big) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where F represents a convex, uniformly elliptic operator. They established the existence of a $W^{2,p}$-viscosity solution u to this problem, satisfying the following estimate:

$$\|u\|_{W^{2,p}(\Omega)} \leq C\big[\|u\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f\big(|u \geq u(x)|\big)\|_{p,\Omega}\big].$$

For further insights into the existence and qualitative questions pertaining to extremal Pucci's equations, we refer to [10,11,18,20,21,24–26] .

Concurrently, equations involving gradient degenerate fully nonlinear elliptic operators have been widely investigated over the past decade. Pioneering works in this direction are attributed to I. Birindelli and F. Demengel. They proved several important results for these operators in a series of papers. These contributions involve comparison principle and Liouville-type results [3], regularity and uniqueness of eigenvalues and eigenfunctions [4,5], $C^{1,\alpha}$ regularity in the radial case [6]. Furthermore, the equations of the form:

$$|Du|^\gamma F(D^2u) = f \quad \text{in } B_1, \tag{1.2}$$

when $\gamma \geq 0$ is a constant and $f \in L^\infty(B_1, \mathbb{R})$, were investigated by C. Imbert and L. Silvestre [13]. In particular, they established the interior $C^{1,\alpha}$ regularity of solutions for equations of the form (1.2). One may also see [19] for insights into variable exponent degenerate mixed fully nonlinear local and nonlocal equations.

Motivated by the above works and recently by the work of L. Caffarelli and I. Tomasetti [8], it is natural to ask the following question:

Question: *Do we have the existence of a viscosity solution to* (1.1)*?*

The aim of this paper is to answer this question affirmatively. The crucial difference to our problem from [8] is due to the fact that $|Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2u)$ degenerates along the set of critical points,

$$\mathcal{C} := \{x : Du(x) = 0\}.$$

The problem is challenging due to the following reasons:

(C1) The R.H.S. of (1.1) is a function of measure of superlevel sets. This makes the problem nonlocal.

(C2) The L.H.S. of (1.1) is degenerate. The fundamental theory of L^p-viscosity solutions does not work directly here since it requires the uniform ellipticity of the operator. Also, when $f \in C(\Omega)$, the problem can be discussed in the C-viscosity sense but in the case of discontinuous data, when $f \in L^p(\Omega)$, the problem needs to be treated in the L^p-viscosity sense. We point out that this situation occurs while approximating the R.H.S. of (1.1).

We use the L^p-viscosity solution approach for Monge–Ampère equation as in [1, 8] to (1.1). To handle the above mentioned challenges, we first consider the following approximate problem:

$$\begin{cases} |Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}\big(D^2u(x)\big) + \varepsilon\Delta u = f\big(|u \geq u(x)|\big) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

for $\varepsilon > 0$. Further, using the approximations in the R.H.S. of the equation and exploiting the results available for uniform elliptic operators, for instance, Theorem 2.5 and Theorem 2.7 (see next), we establish the existence of a viscosity solution to the approximate problem (1.3). This yields the existence of a viscosity solution to (1.1). More precisely, using the idea of Amadori et al. [1], we first get the existence of a $W^{2,p}$-viscosity solution to the approximate problem (1.3) by invoking Theorem 2.1 [8]. We recall that the estimate established in [8] is not adequate to claim the uniform bound on the $W^{2,p}$-viscosity solution of (1.3). To show the existence of a solution to the original problem (1.1), we seek the uniform bound on the solutions of (1.3), which is crucial in approaching $\varepsilon \longrightarrow 0^+$. We invoke the Alexandroff–Bakelman–Pucci (ABP) estimates from Caffarelli et al. [7] to sort this issue. These estimates play a crucial role in obtaining uniform bounds on the $W^{2,p}$-viscosity solutions to (1.3).

Throughout the paper, we consider Ω to be a bounded $C^{1,1}$ domain in $\mathbb{R}^N$, $N \geq 2$.

The main result of this paper is the following:

Theorem 1.1. *Let $\gamma \geq 1$ be a constant. Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain. Let $f \in C\big([0, |\Omega|], \mathbb{R}\big)$ be a non-decreasing, non-negative function and $g \in W^{2,p}(\Omega) \cap C(\overline{\Omega})$, $p > N$. Consider the problem*

$$\begin{cases} |Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}\big(D^2u(x)\big) = f\big(|u \geq u(x)|\big) & \textit{in } \Omega \\ u = g & \textit{on } \partial\Omega. \end{cases} \tag{1.4}$$

Then, there exists a $W^{2,p}$-viscosity solution of (1.4). Moreover, the solution satisfies the following estimate:

$$\|u\|_{W^{2,p}(\Omega)} \le C\Big(\|u\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f(|u \ge u(x)|)\|_{p,\Omega}\Big),$$

where $C > 0$ is a constant.

Remark 1.2. By Sobolev's embedding theorem we have that the solution is $C^{1,\alpha}(\overline{\Omega})$ regular for any $0 < \alpha < 1$.

The organization of the paper is as follows. In Section 2, we recall the basic definitions and several key results used in the ensuing sections of the paper. Section 3 is devoted to the proof of our main result. Here, we sketch the plan of our proof:

(i) Perturb the left-hand side (L.H.S.), i.e., the operator $|Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2u)$ by adding $\varepsilon\Delta u$, for $\varepsilon > 0$. (This makes the problem uniformly elliptic.)

(ii) Fix a Lipschitz function v in the R.H.S. of (1.3).

(iii) Construct a sequence of L^p-functions converging to R.H.S. (for fixed Lipschitz function v) and obtain a sequence of solutions.

(iv) Obtain the existence of solution to equation pertaining $|Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \varepsilon\Delta u$ for fixed Lipschitz function v in the R.H.S.

(v) Use Theorem 2.1 [8] (an application of Schaefer fixed point theorem) to show the existence of a solution to (1.3).

(vi) Establish the existence of a $W^{2,p}$-viscosity solution to (1.4).

2 Preliminaries

We recall that a continuous mapping $F : S^N \longrightarrow \mathbb{R}$ is *uniformly elliptic* if:
For any $A \in \mathcal{S}^N$, where $\mathcal{S}^N$ is the set of all $N \times N$ real symmetric matrices, there exist two positive constants $\Lambda \ge \lambda > 0$ s.t.

$$\lambda\|B\| \le F(A+B) - F(A) \le N\Lambda\|B\| \quad \text{for all positive semi-definite } B \in \mathcal{S}^N,$$

and $\|B\|$ is the largest eigenvalue of B. Here, we have the usual partial ordering: $A \le B$ in $\mathcal{S}^N$ means that $\langle A\xi, \xi\rangle \le \langle B\xi, \xi\rangle$ for any $\xi \in \mathbb{R}^N$. In other words, $B - A$ is positive semidefinite.

Let $S \in \mathcal{S}^N$, then for the given two parameters $\Lambda \ge \lambda > 0$, Pucci's maximal operator is defined as follows:

$$\mathcal{M}^+_{\lambda,\Lambda}(S) := \Lambda \sum_{e_i \ge 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

where $\{e_i\}_{i=1}^N$ are the eigenvalues of S. This operator is uniformly elliptic and subadditive, that is

$$\mathcal{M}^+_{\lambda,\Lambda}(A+B) \le \mathcal{M}^+_{\lambda,\Lambda}(A) + \mathcal{M}^+_{\lambda,\Lambda}(B),$$

for $A, B \in \mathcal{S}^N$. Clearly, for $\lambda = \Lambda = 1$, $\mathcal{M}^+_{\lambda,\Lambda} \equiv \Delta$, the classical Laplace operator.

Next, we recall the notion of a viscosity solution. M. G. Crandall and P.-L. Lions [9] were the first to introduce the concept of a viscosity solution. Now, we recall the definition of *continuous viscosity solution* to the following equation:

$$|Du|^\gamma F(D^2u(x)) = f \quad \text{in } \Omega, \tag{2.1}$$

for $f \in C(\Omega)$.

Definition 2.1 ([3])**.** Let $u : \overline{\Omega} \longrightarrow \mathbb{R}$ be an upper semicontinuous (USC) function in Ω. Then, u is called a *viscosity subsolution* of (2.1) if

$$|D\varphi(x)|^{\gamma} F\big(D^2\varphi(x)\big) \geq f(x),$$

whenever $\varphi \in C^2(\Omega)$ and $x \in \Omega$ is a local maximizer of $u - \varphi$ with $D\varphi \neq \mathbf{0} \in \mathbb{R}^N$.

Definition 2.2 ([3])**.** Let $u : \overline{\Omega} \longrightarrow \mathbb{R}$ be a lower semicontinuous (LSC) function in Ω. Then, u is called a *viscosity supersolution* of (2.1) if

$$|D\psi(x)|^{\gamma} F\big(D^2\psi(x)\big) \leq f(x),$$

whenever $\psi \in C^2(\Omega)$ and $x \in \Omega$ is a local minimizer of $u - \psi$ with $D\psi \neq \mathbf{0} \in \mathbb{R}^N$.

Definition 2.3 ([3])**.** A continuous function u is said to be a *viscosity solution* to (2.1) if it is a supersolution as well as subsolution to (2.1).

Let $h \in L^p(\Omega)$, $g \in W^{2,p}(\Omega) \cap C(\overline{\Omega})$ for $p > N$. Let us consider the problem

$$\begin{cases} |Du|^{\gamma} \mathcal{M}^{+}_{\lambda,\Lambda}(D^2 u) = h & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

We mention that the classical definition of $W^{2,p}$-viscosity solution can not be applied for (2.2), due to the lack of uniform ellipticity. Consider the problem:

$$\begin{cases} |Du|^{\gamma} \mathcal{M}^{+}_{\lambda,\Lambda}\big(D^2 u\big) + \varepsilon \Delta u = h & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

for $p \in \mathbb{R}^N$. Motivated by Caffarelli et al. [7] and Ishii et al. [14], we define the L^p-viscosity subsolution (supersolution) to (2.3) as follows.

Definition 2.4. Let u be an USC (respectively, LSC) function on $\overline{\Omega}$. We say that u is an L^p-viscosity subsolution (respectively, supersolution) to (2.3) if $u \leq g$ (resp., $u \geq g$) on $\partial\Omega$ and for all $\phi \in W^{2,p}(\Omega)$,

$$\operatorname{ess}\liminf_{x \longrightarrow y} \big(|D\phi(x)|^{\gamma} \mathcal{M}^{+}_{\lambda,\Lambda}\big(D^2\phi(x)\big) + \varepsilon\Delta\phi(x) - h(x)\big) \geq 0$$

$$\left(\text{resp., } \operatorname{ess}\limsup_{x \longrightarrow y} \big(|D\phi(x)|^{\gamma} \mathcal{M}^{+}_{\lambda,\Lambda}\big(D^2\phi(x)\big) + \varepsilon\Delta\phi(x) - h(x)\big) \leq 0\right),$$

for $y \in \Omega$, the point of local maxima (respectively, minima) to $u - \phi$.

We say that any continuous function u is an L^p-viscosity solution to (2.3) if it is both L^p-viscosity subsolution and supersolution to (2.3). Now, we state a result concerning the existence and uniqueness of $W^{2,p}$-viscosity solution to the operator F under certain hypotheses. The following result is due to N. Winter [27].

Theorem 2.5 ([27, Theorem 4.6]). *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^N$. Let $F(p,M)$ be a uniformly elliptic operator and convex in M-variable. Also, let $F(0,0) \equiv 0$ in Ω, $f \in L^p(\Omega)$ and $g \in W^{2,p}(\Omega)$ for $p > N$. Then, there exists a unique $W^{2,p}$-viscosity solution to*

$$\begin{cases} F(Du, D^2u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Moreover, $u \in W^{2,p}(\Omega)$ and

$$\|u\|_{W^{2,p}(\Omega)} \leq C\Big(\|u\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f\|_{p,\Omega}\Big),$$

for some positive constant C.

Theorem 2.6 ([2, Theorem 1.1]). *Let Ω be a bounded domain with C^2-boundary. Let $\gamma \geq 0$ and F be a uniformly elliptic operator and $f \in C(\overline{\Omega})$, $g \in C^{1,\beta}(\partial\Omega)$ for some $\beta \in (0,1)$. Then, any viscosity solution u of*

$$\begin{cases} |Du|^\gamma F(D^2u) = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

is in $C^{1,\alpha}$ for some $\alpha = \alpha(\lambda, \Lambda, \|f\|_{\infty,\Omega}, N, \Omega, , \beta)$. Moreover, u satisfies the following estimate

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\left(\|g\|_{C^{1,\beta}(\partial\Omega)} + \|u\|_{\infty,\Omega} + \|f\|_{\infty,\Omega}^{\frac{1}{1+\gamma}}\right),$$

for some positive constant $C = C(\alpha)$.

The following result plays an important role in Step 5 of the proof of our main result.

Theorem 2.7 ([7, Theorem 3.8]). *Let F_i, F be uniformly elliptic and $p > N$. Let $f, f_i \in L^p(\Omega)$. Let $u_i \in C(\Omega)$ be $W^{2,p}$-viscosity subsolutions (supersolutions) to*

$$F_i(D^2u_i) = f_i \quad \text{in } \Omega,$$

for $i = 1, 2, \ldots$ Assume that $u_i \longrightarrow u$ locally uniformly in Ω. Also, assume that if for each $B(x_0, r) \subset \Omega$ and $g \in W^{2,p}(B(x_0,r))$, we have

$$\|\big(F_i(D^2u_i) - f_i(x) - F(D^2(u)) + f(x)\big)^+\|_{p,B(x_0,r)} \longrightarrow 0,$$
$$\Big(\|\big(F_i(D^2u_i) - f_i(x) - F(D^2(u)) + f(x)\big)^-\|_{p,B(x_0,r)} \longrightarrow 0\Big).$$

Then, u is a $W^{2,p}$-viscosity subsolution (supersolution) to

$$F(D^2u) = f \quad \text{in } \Omega.$$

3 Proof of our main result

Proof of Theorem 1.1. The original problem is

$$\begin{cases} |Du|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2u(x)) = f(|u \geq u(x)|) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Step 1: Perturbing the L.H.S. by adding $\varepsilon\Delta u$. Consider the approximate problem:

$$\begin{cases} |Du|^{\gamma}\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) + \varepsilon\Delta u = f(|u \geq u(x)|) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

for $\varepsilon > 0$. Since, $Gu := |Du|^{\gamma}\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) + \varepsilon\Delta u$ is uniformly elliptic, so by Theorem 2.1 [8], we immediately have the existence of a $W^{2,p}$-viscosity solution (say u_ε) to (3.2) satisfying the following estimate:

$$\|u_\varepsilon\|_{W^{2,p}(\Omega)} \leq C\Big(\|u_\varepsilon\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|f(|u_\varepsilon \geq u_\varepsilon(x)|)\|_{p,\Omega}\Big).$$

By the above estimate, one can not directly claim the uniform bound on u_ε, which is crucial in order to pass the limit $\varepsilon \longrightarrow 0$ to establish the existence of $W^{2,p}$-viscosity solution to (3.1). To overcome this difficulty, we further approximate problem (3.2).

Step 2: Freeze a Lipschitz function v for the R.H.S.. Next, following the arguments similar to [8], we fix a Lipschitz function v in Ω, and consider $h_v(x) := f(|v \geq v(x)|)$ and reduce to the following problem:

$$\begin{cases} |Du|^{\gamma}\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) + \varepsilon\Delta u = h_v & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Step 3: Building a sequence of functions in R.H.S. We consider a sequence of functions $\{h_v^i\}_{i=1}^{\infty}$ defined as

$$h_v^i(x) := f\Big(i\int_0^{\frac{1}{i}} |v \geq v(x) - t|dt\Big).$$

We approximate the function

$$h_v(x)\ (= f(|v \geq v(x)|))$$

in the R.H.S. of (3.3) by the sequence of functions $\{h_v^i\}_{i=1}^{\infty}$. Hence, we have the following approximate problem:

$$\begin{cases} |Du|^{\gamma}\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) + \varepsilon\Delta u = h_v^i & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

for $i \geq 1$. Since $\{h_v^i\} \in L^p(\Omega)$. For each i, by Theorem 2.5, we have the existence of a unique $W^{2,p}$-viscosity solution to (3.4).

Lemma 3.1. *There exists a unique $W^{2,p}$-viscosity solution to* (3.4). *Moreover, it satisfies the following estimate:*

$$\|u_\varepsilon^i\|_{W^{2,p}(\Omega)} \leq C\Big(\max_{\partial\Omega} g + \|g\|_{W^{2,p}(\Omega)} + f(|\Omega|)|\Omega|^{\frac{1}{p}}\Big).$$

Proof. By Theorem 2.5, we have the existence of a unique $W^{2,p}$-viscosity solution u_ε^i to (3.4) satisfying the following estimate:

$$\|u_\varepsilon^i\|_{W^{2,p}(\Omega)} \leq C\Big(\|u_\varepsilon^i\|_{\infty,\Omega} + \|g\|_{W^{2,p}(\Omega)} + \|h_v^i\|_{p,\Omega}\Big),$$

Also, it is easy to observe that

$$\|h_v^i\|_{\infty,\Omega} \leq f(|\Omega|), \tag{3.5}$$

and

$$\begin{aligned} \|h_v^i\|_{p,\Omega} &= \left(\int_\Omega |h_v^i(x)|^p dx \right)^{\frac{1}{p}} \\ &\le \|h_v^i\|_{\infty,\Omega} |\Omega|^{\frac{1}{p}} \\ &\le f(|\Omega|)|\Omega|^{\frac{1}{p}}, \end{aligned} \tag{3.6}$$

for each $i \ge 1$. Thus the sequence of functions h_v^i is uniformly bounded. Now, by ABP estimates established in [7], we have

$$\sup_\Omega u_\varepsilon^i \le \sup_{\partial\Omega} u_\varepsilon^i + C\|h_v^i\|_{p,\Omega},$$

and similarly for the $\inf_\Omega u_\varepsilon^i$. For more details, see Proposition 3.3 [7]. Using this along with the estimates (3.5) and (3.6), we have the following:

$$\|u_\varepsilon^i\|_{W^{2,p}(\Omega)} \le \widetilde{C},$$

where $\widetilde{C}$ is a positive constant independent of i and ε. □

Step 4: Establish the existence of solution to (3.3)**.** It further gives that $\{u_\varepsilon^i\}$ is uniformly bounded in $W^{2,p}(\Omega)$ (with respect to i). Now, by reflexivity of $W^{2,p}(\Omega)$, u_ε^i converges weakly in $W^{2,p}(\Omega)$. Moreover, since $p > N/2$. Using the similar arguments as above, we have the existence of a subsequence such that $u_\varepsilon^i \longrightarrow u_{\varepsilon,v}$ in the Lipschitz norm. As a consequence of Theorem 2.7, $u_{\varepsilon,v}$ is a $W^{2,p}$-viscosity solution to (3.3). Moreover, $u_{\varepsilon,v}$ satisfies the following estimate:

$$\|u_{\varepsilon,v}\|_{W^{2,p}(\Omega)} \le C\left(\max_{\partial\Omega} |g| + \|g\|_{W^{2,p}(\Omega)} + \|f(|v \ge v(x)|)\|_{p,\Omega} \right).$$

Step 5: Establish the existence of solution to (3.2)**.** Further, using Theorem 2.1 [8] (an application of Schaefer fixed point theorem), we have the existence of a $W^{2,p}$-viscosity solution to (3.2) for each $0 < \varepsilon < 1$, say u_ε (a Lipschitz fixed point). Moreover, u_ε satisfies the following estimate:

$$\|u_\varepsilon\|_{W^{2,p}(\Omega)} \le C\left(\max_{\partial\Omega} |g| + \|g\|_{W^{2,p}(\Omega)} + \|f(|u_\varepsilon \ge u_\varepsilon(x)|)\|_{p,\Omega} \right). \tag{3.7}$$

Step 6: Establish the existence of solution to (3.1) **on** $\varepsilon \longrightarrow 0$**.** Since u_ε is uniformly bounded in $W^{2,p}(\Omega)$ (with respect to ε) so we have that along some subsequence, u_ε converges weakly in $W^{2,p}(\Omega)$. Moreover, by the Rellich–Kondrasov theorem, along some subsequence $u_\varepsilon \longrightarrow u$ in $C(\overline{\Omega})$ (since $p > N$) to a Lipschitz function u. We further claim that u is an L^p-viscosity solution to (3.1). We use the idea of [1]. We just check the supersolution part. Further, one can check for the subsolution part using the similar arguments. Let, if possible, assume that u is not an L^p-viscosity supersolution to (3.1). Then by definition, there exists a point $x_0 \in \Omega$ and a function $\phi \in W^{2,p}(\Omega)$ with $D\phi \ne 0$ such that $u - \phi$ has local minimum at x_0 and

$$|D\phi|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) - f(|u \ge u(x_0)|) \ge \alpha \text{ a.e. in some ball } B(x_0, r), \tag{3.8}$$

for some constant $\alpha > 0$. In other words, $u - \phi$ restricted to $\overline{B(x_0, r)}$ has a global strict minima at x_0. Next, using the above information, we get a contradiction by constructing a function $\phi_\varepsilon = \phi - \psi_\varepsilon$ corresponding to u_ε such that

$$|D\phi_\varepsilon|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi_\varepsilon) + \varepsilon\Delta\phi_\varepsilon - f(|u \ge u(x_0)|) \ge \alpha \quad \text{a.e. in } B(x_0, r) \tag{3.9}$$

for small enough $\varepsilon > 0$ and

$$\phi_\varepsilon \longrightarrow \phi \quad \text{uniformly.}$$

Now, since u_ε is an L^p-viscosity solution to (3.2) so (3.9) implies that $u_\varepsilon - \phi_\varepsilon$ can not attain minimum in the ball $B(x_0, r)$. However, since $u_\varepsilon - \phi_\varepsilon$ is continuous and $\overline{B(x_0, r)}$ is compact. Therefore, $u_\varepsilon - \phi_\varepsilon$ attains minimum in $\overline{B(x_0, r)}$. Let it be x_ε. It gives that $x_\varepsilon \longrightarrow x_0$ along some subsequence. It further implies that $x_\varepsilon \in B(x_0, r)$ for small enough ε, which is a contradiction. Thus such a function ϕ constructed in (3.8) does not exist, which proves our claim that u is an L^p-supersolution to (3.1). Similarly, one can check the subsolution part.

Next, we show that u is the limit function of the sequence of functions u_ε as $\varepsilon \longrightarrow 0$. Let if possible, ε_i and $\widetilde{\varepsilon}_i$ be two sequences approaching 0 with u and $\widetilde{u}$ being the corresponding limit functions to the sequences, respectively. Up to subsequences, we may assume that

$$\cdots \leq \widetilde{\varepsilon}_{i+1} \leq \varepsilon_i \leq \widetilde{\varepsilon}_i \leq \varepsilon_{i-1} \leq \cdots .$$

Our aim is to show that $w = u_{\varepsilon_i} - u_{\widetilde{\varepsilon}_{i+1}} \leq 0$. If we show that

$$|Dw|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2 w) + \varepsilon \Delta w \geq 0 \quad \text{in } \Omega \text{ (in } C\text{-viscosity sense)},$$

we are done. As by comparison principle, we would immediately get $w \leq 0$. Therefore, $u_{\varepsilon_i} \leq u_{\widetilde{\varepsilon}_{i+1}}$. Thus, in order to show that

$$w = u_{\varepsilon_i} - u_{\widetilde{\varepsilon}_{i+1}} \leq 0,$$

we only need to show that

$$|Dw|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2 w) + \varepsilon_i \Delta w \geq 0.$$

As shown above, it immediately gives $w \leq 0$. Let us assume the contrary, i.e., there exists some point $x_0 \in \Omega$ such that for some $\varphi \in C^2(\Omega)$, $w - \varphi$ attains local maxima at x_0, i.e., there exists a ball $B(x_0, r)$ such that

$$|D\varphi|^\gamma \mathcal{M}^+_{\lambda,\Lambda}(D^2 \varphi) + \varepsilon_i \Delta \varphi \leq -\alpha \quad \text{in } B(x_0, r),$$

for some $\alpha > 0$ and $w - \varphi = \left(u_{\varepsilon_i} - u_{\widetilde{\varepsilon}_{i+1}}\right) - \varphi = u_{\varepsilon_i} - \left(u_{\widetilde{\varepsilon}_{i+1}} + \varphi\right)$ has a global strict maximum at x_0 in $B(x_0, r)$. Now, consider a function

$$\Psi := \varphi + u_{\widetilde{\varepsilon}_{i+1}}.$$

Clearly, $\Psi \in W^{2,p}(\Omega)$ and touches u_{ε_i} from above at x_0. Also, consider a test function, Φ for

$u_{\widetilde{\varepsilon}_{i+1}}$ touching from below with $|D\Phi(x_0)|$ sufficiently larger than $|D\varphi(x_0)|$. We have

$$
\begin{aligned}
&|D\Psi(x_0)|^{\gamma}\mathcal{M}^{+}_{\lambda,\Lambda}\big(D^2\Psi(x_0)\big)+\varepsilon_i\Delta\Psi(x_0)-f\big(|u\geq u(x_0)|\big)+\alpha\\
&\quad\leq |D\Psi(x_0)|^{\gamma}\big(\mathcal{M}^{+}_{\lambda,\Lambda}\big(D^2\varphi(x_0)\big)+\mathcal{M}^{+}_{\lambda,\Lambda}\big(D^2\Phi(x_0)\big)\big)+\varepsilon_i\Delta\varphi(x_0)\\
&\qquad+\varepsilon_i\Delta\Phi(x_0)-f\big(|u\geq u(x_0)|\big)+\alpha\\
&\quad= |D\Psi(x_0)|^{\gamma}\mathcal{M}^{+}_{\lambda,\Lambda}\big(D^2\varphi(x_0)\big)+\varepsilon_i\Delta\varphi(x_0)+|D\Psi(x_0)|^{\gamma}\mathcal{M}^{+}_{\lambda,\Lambda}\big(D^2\Phi(x_0)\big)\\
&\qquad+\varepsilon_i\Delta\Phi(x_0)-f\big(|u\geq u(x_0)|\big)+\alpha\\
&\quad= |D\varphi(x_0)+D\Phi(x_0)|^{\gamma}\mathcal{M}^{+}_{\lambda,\Lambda}\big(D^2\varphi(x_0)\big)+\varepsilon_i\Delta\varphi(x_0)+\alpha\\
&\qquad+|D\varphi(x_0)+D\Phi(x_0)|^{\gamma}\mathcal{M}^{+}_{\lambda,\Lambda}(D^2\Phi(x_0))+\varepsilon_i\Delta\Phi(x_0)-f\big(|u\geq u(x_0)|\big)\\
&\quad\leq \frac{|D\varphi(x_0)+D\Phi(x_0)|^{\gamma}}{|D\varphi(x_0)|^{\gamma}}\big(-\alpha-\varepsilon_i\Delta\varphi(x_0)\big)+\varepsilon_i\Delta\varphi(x_0)+\alpha\\
&\qquad+\frac{|D\varphi(x_0)+D\Phi(x_0)|^{\gamma}}{|D\Phi(x_0)|^{\gamma}}\big(f\big(|u\geq u(x_0)|\big)-\varepsilon_i\Delta\Phi(x_0)\big)+\varepsilon_i\Delta\Phi(x_0)\\
&\qquad-f\big(|u\geq u(x_0)|\big)\\
&\quad\leq \frac{|D\varphi(x_0)+D\Phi(x_0)|^{\gamma}}{|D\varphi(x_0)|^{\gamma}}\big(-\alpha-\varepsilon_i\Delta\varphi(x_0)\big)+\varepsilon_i\Delta\varphi(x_0)+\alpha\\
&\qquad+2^{\gamma-1}\frac{|D\varphi(x_0)|^{\gamma}+|D\Phi(x_0)|^{\gamma}}{|D\Phi(x_0)|^{\gamma}}\big(f\big(|u\geq u(x_0)|\big)-\varepsilon_i\Delta\Phi(x_0)\big)+\varepsilon_i\Delta\Phi(x_0)\\
&\qquad-f\big(|u\geq u(x_0)|\big)\\
&\quad\leq \big(-\alpha-\varepsilon_i\Delta\varphi(x_0)\big)\left(\frac{|D\varphi(x_0)+D\Phi(x_0)|^{\gamma}}{|D\varphi(x_0)|^{\gamma}}-1\right)\\
&\qquad+\big(f\big(|u\geq u(x_0)|\big)-\varepsilon_i\Delta\Phi(x_0)\big)\left(2^{\gamma-1}\frac{\big(|D\varphi(x_0)|^{\gamma}+|D\Phi(x_0)|^{\gamma}\big)}{|D\Phi(x_0)|^{\gamma}}-1\right),
\end{aligned}
\tag{3.10}
$$

for all large enough $i\in\mathbb{N}$. Note that in the second last step we used the fact that for any positive real numbers a,b and $r\geq 1$, we have

$$|a+b|^r\leq 2^{r-1}\big(|a|^r+|b|^r\big).$$

Further, by the choice of test function Φ made before (3.10), we have

$$\mathcal{M}^{+}_{\lambda,\Lambda}(D^2\Psi)+\varepsilon_i\Delta\Psi-f\big(|u\geq u(x_0)|\big)\leq-\alpha<0,$$

which contradicts the fact that u_{ε_i} is an L^p-viscosity solution to (3.2). Thus we have that $u_{\varepsilon_i}\leq u_{\widetilde{\varepsilon}_{i+1}}$. Letting $i\longrightarrow\infty$, we get $u\leq\widetilde{u}$. Also, following the similar arguments, one can show that $u_{\widetilde{\varepsilon}_{i+1}}\leq u_{\varepsilon_i}$. Thus, we have $\widetilde{u}\leq u$ and hence $u=\widetilde{u}$.
Therefore, we have the existence of a $W^{2,p}$-viscosity solution, u to (3.1). Moreover, by (3.7), we have the following estimate:

$$\|u\|_{W^{2,p}(\Omega)}\leq C\Big(\|u\|_{\infty,\Omega}+\|g\|_{W^{2,p}(\Omega)}+\|f(|u\geq u(x)|)\|_{p,\Omega}\Big).\qquad\square$$

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Heteroclinic solutions in singularly perturbed discontinuous differential equations: a non-generic case

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Abstract. We derive Melnikov type conditions for the persistence of heteroclinic solutions in perturbed slowly varying discontinuous differential equations. Opposite to [*J. Differential Equations* **400**(2024), 314–375] we assume that the unperturbed (frozen) equation has a parametric system of heteroclinic solutions and extend a result in [*SIAM J. Math. Anal.* **18**(1987), 612–629] and [*SIAM J. Math. Anal.* **19**(1988), 1254–1255] to higher dimensional non-Hamiltonian discontinuous singularly perturbed differential equations.

Keywords: discontinuous differential equations, heteroclinic solutions, Melnikov conditions, persistence.

2020 Mathematics Subject Classification: Primary 34C37, Secondary 34C23, 34D15, 37G20.

1 Introduction

Let $h(x,y), f_i(x,y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, N+1$, be C^r-functions, $r \geq 2$, bounded on $\mathbb{R}^n \times \mathbb{R}^m$ together with their derivatives, and $c_1 < c_2 < \dots < c_N < c_{N+1}$ be real numbers.

In this paper we study the problem of existence of continuous, piecewise smooth, bounded solutions of a singularly perturbed equation like

$$\begin{aligned} \dot{x} &= f(x,y), \\ \dot{y} &= \varepsilon g(x,y,\varepsilon) \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ and

$$f(x,y) := \begin{cases} f_i(x,y) & \text{if } c_{i-1} < h(x,y) < c_i, \\ & i = 1, \dots, N \\ f_{N+1}(x,y) & \text{if } h(x,y) > c_N \end{cases} \tag{1.2}$$

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where we take for notational simplicity $c_0 = -\infty$. It is assumed that for all $y \in \mathbb{R}^m$, the frozen system

$$\dot{x} = f(x, y) \tag{1.3}$$

has hyperbolic fixed points $x = w_\pm(y)$ with an associated piecewise C^r, heteroclinic solution $u(t, y)$ intersecting transversally the manifolds $\mathcal{S}_i(y) = \{x \mid h(x, y) = c_i\}$. We intend to give a Melnikov like condition guaranteeing that the perturbed system (1.1) has a solution $(x(t, \varepsilon), y(t, \varepsilon))$ such that $\sup_{t\in\mathbb{R}} |x(t, \varepsilon) - u(t, y(t, \varepsilon))| \to 0$ as $\varepsilon \to 0$. This paper has been motivated by [10, 11] where the authors considered a perturbation of a smooth, Hamiltonian, three-dimensional system. The main result of our paper (Theorem 6.2) concerns higher dimension, discontinuous and not necessarily Hamiltonian systems. Moreover the approach in [10, 11] is basically geometrical, while in this paper it is based on Lyapunov–Schmidt reduction.

This paper is a continuation of series of our works [3–6] on the study of existence of bounded solutions for slowly varying discontinuous differential equations. Papers [3–5] deal with the persistence of periodic solutions in case of existence of either a single or a family of periodic solutions for the frozen system (1.3). Next, in [6] generic conditions have been given for persistence of an isolated homoclinic-heteroclinic solution for the frozen system. Thus it is a natural step to study the case when the frozen system possesses a parametric system of bounded-homoclinic-heteroclinic solutions, which is the purpose of this paper.

To prove Theorem 6.2 we use a general result in [6] concerning the characterization of bounded solutions on both the positive and the negative line for the perturbed equation, Then, in [6], this result is used, jointly with a Lyapunov–Schmidt reduction, to write down a bifurcation equation which is the scalar product of certain vectors with the difference at $t = 0$ of the value of these solutions. Now, in [6] the case is considered where this function has a simple zero at $\varepsilon = 0$, while in this paper it is identically zero at $\varepsilon = 0$. This fact makes a big difference and indeed the Melnikov functions obtained in the two cases are quite different.

We now briefly sketch the content of this paper. For the reader convenience and also for the completeness of this paper, we recall necessary results from [6] in Sections 2–5. Namely, Section 2 provides basic assumptions and defines the piecewise smooth heteroclinic solution of the unperturbed system. Section 3 recalls the definition of exponential dichotomy and extends this notion to discontinuous, piecewise linear, systems with jumps at some points; moreover some results concerning existence of bounded solutions on either $t \geq 0$ and $t \leq 0$ are extended to these systems. In Section 4 we construct families of bounded solutions and describes them in terms of some parameters. These solutions are continuous and piecewise smooth and give the bounded solutions we look for, when they assume the same value at $t = 0$. Section 5 defines the discontinuous variational equation.

Our main results are proved in Section 6 where we obtain a Melnikov-type condition assuring that the bifurcation function has a manifold of solutions. Motivated by [8], Section 7, is devoted to the construction of an example of application of the main result of this paper. Although the equation is three-dimensional and Hamiltonian, the vector field is discontinuous and then the results in [8, 10, 11] do not apply.

Finally, in Section 8 we show that the Melnikov function given here extends to the heteroclinic case with finitely many discontinuity points, the Melnikov function given in [5] for the periodic case with two discontinuity points.

In the whole paper we will use the following notation. Given a vector v or a matrix A with v^T, A^T we denote the transpose of v, A.

2 Notation and basic assumptions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $c_1 < \ldots < c_{N+1}$ be real numbers and $h : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a C^r-functions, $r \geq 2$, with bounded derivatives. For $\ell = 1, \ldots, N+1$, we set

$$\Omega_\ell = \{(x,y) \in \Omega \times \mathbb{R}^m \mid c_{i-1} \leq h(x,y) < c_i\},$$

where we set for simplicity, $c_0 = -\infty$. Then let $f_\ell : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ be C^r-functions, bounded together with their derivatives in $\Omega \times \mathbb{R}^m$.

First we give the definition of solutions of equation

$$\dot{x} = f_i(x,y), \quad (x,y) \in \Omega_i,\ i = 1, \ldots, N+1 \tag{2.1}$$

we are considering in this paper.

Definition 2.1. A continuous, piecewise smooth function $u(t,y)$ is a solution of equation (2.1) on $t \geq 0$ intersecting transversally the sets $\mathcal{S}_i(y) = \{x \in \Omega \mid h(x,y) = c_i\}$, $i = 1, \ldots, N$, if there exist $\eta > 0$ and C^r-functions bounded together with their derivatives $0 < t_1(y) < \ldots < t_N(y)$ such that the following conditions hold for $1 \leq i \leq N$ (note that we set $t_0(y) = 0$)

$a_1)$ $\dot{u}(t,y) = f_i(u(t,y),y)$ for $t_{i-1}(y) < t < t_i(y)$ and $\dot{u}(t,y) = f_{N+1}(u(t,y),y)$ for $t > t_N(y)$;

$a_2)$ $h(u(t_i(y),y),y) = c_i$, and $h_x(u(t_i(y),y),y)\dot{u}(t_i(y)^\pm,y) > 2\eta$;

$a_3)$ $c_{i-1} < h(u(t,y),y) < c_i$, for $t_{i-1}(y) < t < t_i(y)$ and $h(u(t,y),y) > c_N$, for $t > t_N(y)$.

Similarly, a continuous, piecewise smooth function $u(t,y)$ is a solution of equation (2.1) on $t \leq 0$ intersecting transversally the sets $\mathcal{S}_i(y)$, if there exist $\eta > 0$ and C^r-functions bounded together with their derivatives $t_{-N}(y) < \ldots < t_{-1}(y) < 0$ such that the following conditions hold for any $1 \leq i \leq N$:

$a_1')$ $\dot{u}(t,y) = f_i(u(t,y),y)$ for $t_{-i}(y) < t < t_{-i+1}(y)$ and $\dot{u}(t,y) = f_{N+1}(u(t,y),y)$ for $t < t_{-N}(y)$;

$a_2')$ $h(u(t_{-i}(y),y),y) = c_i$, and $h_x(u(t_{-i}(y),y),y)\dot{u}(t_{-i}(y)^\pm,y) < -2\eta$;

$a_3')$ $c_{i-1} < h(u(t,y),y) < c_i$, for $t_{-i}(y) < t < t_{-i+1}(y)$ and $h(u(t,y),y) > c_N$, for $t < t_{-N}(y)$.

In this paper we assume that a continuous, piecewise smooth solution $u(t,y)$ of equation (2.1) exist, for $t \in \mathbb{R}$, such that the following conditions hold.

$A_1)$ $w_0(y) := u(0,y)$ and its derivatives are bounded functions on $\mathbb{R}^m$ and $w_0(y)$ belongs to an open and bounded subset $B \subset \mathbb{R}^n$ such that $\overline{B} \times \mathbb{R}^m \subset \Omega_1$.

$A_2)$ There exist smooth and bounded functions $w_\pm(y)$ and $\mu_0 > 0$, such that

$$f_{N+1}(w_\pm(y),y) = 0,$$
$$h(w_\pm(y),y) - c_N > \mu_0,$$

for any $y \in \mathbb{R}^m$ and

$$\lim_{t \to \pm\infty} u(t,y) - w_\pm(y) = 0$$

uniformly with respect to $y \in \mathbb{R}^m$.

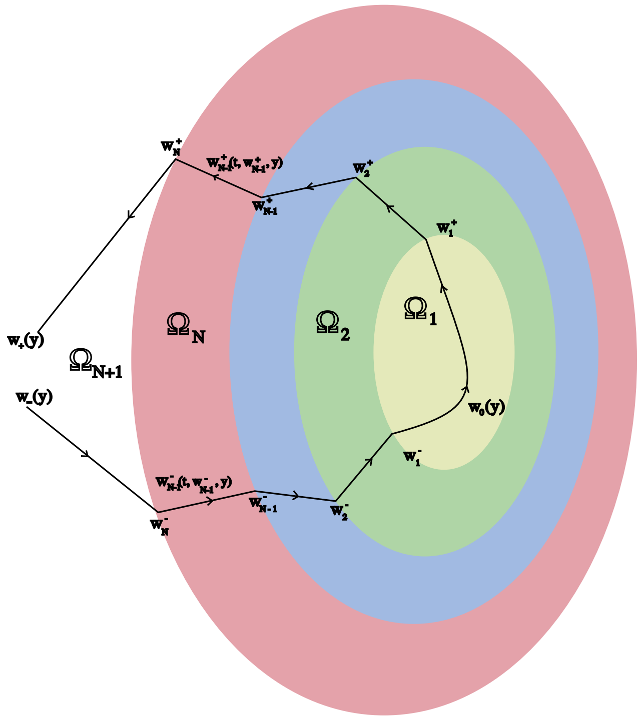

Figure 2.1: The piecewise C^1 bounded solution of (1.3). For simplicity we write $w_j^\pm$ instead of $w_j^\pm(y)$.

A_3) For any $y \in \mathbb{R}^m$, $f_{N+1,x}(w_\pm(y), y)$ have k eigenvalues with negative real parts and $n-k$ eigenvalues with positive real parts, counted with multiplicities and there exists $\delta_0 > 0$ such that all these eigenvalues satisfy

$$|\operatorname{Re}\lambda(y)| > \delta_0.$$

We set

$$t_0(y) = 0, \qquad \forall y \in \mathbb{R}^m.$$

So, we are considering solutions of (2.1) which are contained in $C \times \mathbb{R}^n \subset \Omega \times \mathbb{R}^m$, where C is a compact subset of Ω. Then we may and will assume that $\Omega = \mathbb{R}^n$.

Remark 2.2. i) As in [6], all results in this paper can be easily generalised to the case where the solutions exit transversally Ω_i and enter into either Ω_{i+1} or Ω_{i-1} transversally. We can formalize all of this as follows: there exists $(j_0, \dots, j_M)$ such that given j_i then j_{i+1} is either $j_i - 1$ or $j_i + 1$ and for $t_i(y) < t < t_{i+1}(y)$ we have

$$c_{j_i - 1} < h(u(t,y), y) < c_{j_i}.$$

Moreover

$$|h_x(u(t_i(y), y), y) f_i(u(t_i(y), y), y)| > 2\eta.$$

for any $i = 1, \dots, N$. A similar generalization can be made for $t \le 0$ and all other assumption will be changed accordingly.

ii) From a_2) and a_2') it follows that, for $i = 1, \dots, N$:

$$\frac{\partial}{\partial t} h(u(t,y),y)_{|t=t_i(y)} \geq 0,$$
$$\frac{\partial}{\partial t} h(u(t,y),y)_{|t=0} \geq 0$$

that is

$$h_x(u(t_i(y),y),y) f_{i-1}(u(t_i(y),y),y) \geq 0,$$
$$h_x(u(t_i(y),y),y) f_i(u(t_i(y),y),y) \geq 0.$$

Similarly

$$h_x(u(t_{-i}(y),y),y) f_i(u(t_{-i}(y),y),y) \leq 0;$$
$$h_x(u(t_{-i}(y),y),y) f_{i+1}(u(t_{-i}(y),y),y) \leq 0.$$

So a_2) and a_2') are a kind of transversality assumption on $u(t,y)$.

Let $w_0^{\pm}(y) = u(0,y)$ and set, for $i = 1, \dots, N$:

$$w_i^{\pm}(y) = u(t_{\pm i}(y),y) \in \mathcal{S}_i(y). \tag{2.2}$$

The following result has been proved in [6]

Lemma 2.3. *$w_i^{\pm}(y)$ are C^r-functions bounded together with their derivatives. Moreover $u(t,y)$ and its derivatives with respect to y are bounded uniformly with respect to y, on both $t \geq t_N^+(y)$ and $t \leq t_N^-(y)$.*

Let $i = 1, \dots, N+1$. For $t \geq 0$, let $u_i^+(t,y)$ be the solution of $\dot{x} = f_i(x,y)$ such that $u_i^+(t_{i-1}(y),y) = w_{i-1}^+(y)$. Similarly, let $u_i^-(t,y)$ be the solution of $\dot{x} = f_i(x,y)$ such that $u_i^-(t_{1-i}(y),y) = w_{i-1}^-(y)$. Note that $u_i^{\pm}(t,y)$ is defined for $t \in \mathbb{R}$ and

$$u(t,y) = \begin{cases} u_i^-(t,y) & \text{for } t_{-i}(y) \leq t \leq t_{1-i}(y),\, i = 1, \dots N+1, \\ u_i^+(t,y) & \text{for } t_{i-1}(y) \leq t \leq t_i(y),\, i = 1, \dots, N+1 \end{cases} \tag{2.3}$$

where, for simplicity, we set $t_{-N-1}(y) = -\infty$ and $t_{N+1}(y) = \infty$. Note that

$$u_i^+(t_i(y),y) = u(t_i(y),y) = w_i^+(y) = u_{i+1}^+(t_i(y),y)$$

and similarly,

$$u_i^-(t_{-i}(y),y) = u(t_{-i}(y),y) = w_i^-(y) = u_{i+1}^-(t_{-i}(y),y).$$

3 Exponential dichotomy for piecewise discontinuous systems

A basic tool in this paper is the notion of exponential dichotomy, whose definition we recall here. Let J be either $[a,\infty)$, $(-\infty,a]$, or $\mathbb{R}$ and $A(t)$, $t \in J$, be a $n \times n$ continuous matrix. We say that the linear system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n \tag{3.1}$$

has an exponential dichotomy on J if there exist a projection $P : \mathbb{R}^n \to \mathbb{R}^n$ and constants $\delta > 0$ and $K \geq 1$ such that the fundamental matrix $X(t)$ of (3.1) satisfying $X(a) = \mathbb{I}$, when $J = [a, \infty), (-\infty, a]$, or $X(0) = \mathbb{I}$ when $J = \mathbb{R}$, satisfies

$$|X(t)PX(s)^{-1}| \leq Ke^{-\delta(t-s)}, \quad \text{for } s \leq t, s, t \in J,$$
$$|X(s)(\mathbb{I} - P)X(t)^{-1}| \leq Ke^{-\delta(t-s)}, \quad \text{for } s \leq t, s, t \in J.$$

K and δ are called the constant and the exponent of the exponential dichotomy.

In [6] the notion of exponential dichotomy has been extended to systems with discontinuities.

Let $t_0 < t_1 < \ldots < t_N$ be real numbers, $B_1, \ldots, B_N$ be invertible $n \times n$ matrices and $\mathcal{A}(t)$, $t \geq t_0$ be a piecewise continuous matrix with possible discontinuity jumps at $t = t_1, \ldots, t_N$, that is

$$\mathcal{A}(t) = \begin{cases} A_i(t) & \text{if } t_{i-1} \leq t < t_i, \\ & i = 1, \ldots, N \\ A_{N+1}(t) & \text{if } t \geq t_N \end{cases} \tag{3.2}$$

where $A_i(t)$ is continuous for $t_{i-1} \leq t \leq t_i$, $A_{N+1}(t)$ is continuous for $t \geq t_N$. Note that $\mathcal{A}(t)$ is continuous for $t \geq t_0$, $t \neq t_i$, $i \neq 1, \ldots, N$ and right-continuous at $t = t_i$, $i = 1, \ldots, N$ with possible jumps at $t = t_i$, $i = 1, \ldots, N$ given by the matrix $A_{i+1}(t_i) - A_i(t_i)$.

For $t \geq t_0$ the fundamental matrix of the linear, discontinuous, system

$$\begin{aligned} \dot{x} &= \mathcal{A}(t)x, \\ x(t_i^+) &= B_i x(t_i^-), \qquad i = 1, \ldots, N \end{aligned} \tag{3.3}$$

is defined as

$$X_+(t) = \begin{cases} U_1(t) & \text{if } 0 \leq t < t_1, \\ U_{i+1}(t)U_{i+1}(t_i)^{-1}B_i X_+(t_i^-) & \text{if } t_i \leq t < t_{i+1}, \\ & i = 1 \ldots, N-1 \\ U_{N+1}(t)U_{N+1}(t_N)^{-1}B_N X_+(t_N^-) & \text{if } t \geq t_N, \end{cases}$$

where $U_i(t)$ is the fundamental matrix of the linear systems

$$\dot{x} = A_i(t)x$$

on $\mathbb{R}$, that is $\dot{U}_i(t) = A_i(t)U_i(t)$, $t \in \mathbb{R}$, and $U_i(0) = \mathbb{I}$.

Similarly, if $t_{-N} < \ldots t_{-1} < t_0$ and

$$\mathcal{A}(t) = \begin{cases} A_{N+1}(t) & \text{if } t \leq t_{-N}, \\ A_i(t) & \text{if } t_{-i} < t \leq t_{-i+1}, \\ & i = 1, \ldots, N \end{cases} \tag{3.4}$$

where $A_i(t)$ is continuous for $t_{-i-1} \leq t \leq t_{-i}$ and $A_{N+1}(t)$ is continuous for $t \leq t_{-N}$, the fundamental matrix, for $t \leq t_0$, of the linear, discontinuous, system

$$\begin{aligned} \dot{x} &= \mathcal{A}(t)x, \\ x(t_{-i}^+) &= B_i x(t_{-i}^-) \end{aligned} \tag{3.5}$$

is

$$X_-(t) = \begin{cases} U_1(t) & \text{if } t_{-1} < t \leq 0, \\ U_{i+1}(t)U_{i+1}(t_{-i})^{-1}B_i^{-1}X_-(t_{-i}^+) & \text{if } t_{-i-1} < t \leq t_{-i}, \\ & i = 1, \ldots, N-1 \\ U_{N+1}(t)U_{N+1}(t_{-N})^{-1}B_N^{-1}X_+(t_{-N}^+) & \text{if } t \leq t_{-N}. \end{cases}$$

Note that, on $t \le t_0$, $\mathcal{A}(t)$ is continuous for $t \le t_0$, $t \ne t_{-i}$, $i \ne 1, \dots, N$ and left-continuous at $t = t_{-i}$, $i = 1, \dots, N$ with possible jumps at $t = t_{-i}$, $i = 1, \dots, N$ given by the matrix $A_i(t_{-i}) - A_{-i+1}(t_{-i})$.

Remark 3.1. As a matter of facts, for $t \ge t_0$, we will consider

$$\mathcal{A}(t) = \begin{cases} A_i(t) & \text{if } t_{i-1} \le t \le t_i, \\ & \ i = 1, \dots, N \\ A_{N+1}(t) & \text{if } t \ge t_N \end{cases}$$

and similarly for $t \le t_0$. This may cause a duplicate definition of $\mathcal{A}(t)$ at $t = t_i$, however it will be always clear which one among the functions $A_i(t)$ will be taken into account at that point.

Without loss of generality we may and will assume that $t_0 = 0$.

Note that $X_+(t)$ is continuous for $t \ne t_1, \dots, t_N$ and right-continuous at $t = t_1, \dots, t_N$ and $X_-(t)$ is continuous for $t \ne t_{-1}, \dots, t_{-N}$ and left-continuous at $t = t_{-1}, \dots, t_{-N}$.

It is clear that $\dot{X}_\pm(t) = \mathcal{A}(t)X_\pm(t)$, for any $\pm t \ge 0$, $t \ne t_{\pm 1}, \dots, t_{\pm N}$, $X_\pm(0) = \mathbb{I}$, the identity matrix, and

$$\begin{aligned} X_+(t_i^+) &= B_i X_+(t_i^-), \\ X_-(t_{-i}^+) &= B_i X_-(t_{-i}^-) \end{aligned} \tag{3.6}$$

for any $i = 1, \dots, N$. Actually we can write

$$X_+(t_i) = B_i X_+(t_i^-), \quad X_-(t_{-i}) = B_i^{-1} X_-(t_{-i}^+)$$

since $X_+(t)$ is right-continuous and $X_-(t)$ is left-continuous.

Remark 3.2. Let $\tau \ge 0$ be a fixed number. For $t \ge 0$, $x(t) = X_+(t)X_+(\tau)^{-1}\tilde{x}$ is the right-continuous solution of

$$\begin{cases} \dot{x} = \mathcal{A}(t)x, & \text{for } t \ge 0, t \ne t_1, \dots, t_N \\ x(t_i^+) = B_i x(t_i^-) \\ x(\tau^+) = \tilde{x}. \end{cases} \tag{3.7}$$

Indeed, it is obvious that $\dot{x}(t) = \mathcal{A}(t)x(t)$ for $t \ge 0$, $t \ne t_1, \dots, t_N$ and that $x(t_i^+) = B_i x(t_i^-)$, since $X_+(t_i^+) = B_i X_+(t_i^-)$. Moreover, for any $\tau \ge 0$ we have $x(\tau^+) = X_+(\tau^+)X_+(\tau)^{-1}\tilde{x} = X_+(\tau)X_+(\tau)^{-1}\tilde{x} = \tilde{x}$, since $X_+(t)$ is right-continuous at any $t \ge 0$.

Similarly, for $t \le 0$ and any fixed $\tau \le 0$, $x(t) = X_-(t)X_-(\tau)^{-1}\tilde{x}$ is the left-continuous solution of

$$\begin{cases} \dot{x} = \mathcal{A}(t)x, & \text{for } t \le 0, t \ne t_{-1}, \dots, t_{-N} \\ x(t_{-i}^-) = B_i^{-1} x(t_{-i}^+) \\ x(\tau^-) = \tilde{x}. \end{cases} \tag{3.8}$$

The following results have been proved in [6]:

Lemma 3.3. *Suppose that the linear system*

$$\dot{x} = A_{N+1}(t)x$$

has an exponential dichotomy on $t \ge t_N$ (resp. $t \le t_{-N}$) with constant K, exponent δ and projection $\mathcal{P}_+$ (resp. $\mathcal{P}_-$ when $t \le t_{-N}$). Then, the linear system (3.3) *(resp.* (3.5)*) with $\mathcal{A}(t)$ as in* (3.2) *(resp.*

(3.4)) *has an exponential dichotomy on* $\mathbb{R}_+$, *(resp.* $\mathbb{R}_-$*) with the same exponent* δ, *constant* $\tilde{K} \geq K$ *and projection*

$$\begin{aligned}\tilde{\mathcal{P}}_+ &= X_+(t_N^+)^{-1}\mathcal{P}_+X_+(t_N^+),\\ \tilde{\mathcal{P}}_- &= X_-(t_{-N}^-)^{-1}\mathcal{P}_-X_-(t_{-N}^-).\end{aligned} \tag{3.9}$$

Lemma 3.4. *Let* $\mathcal{A}(t)$ *be either as in* (3.2) *or* (3.4)*. Suppose that the condition of Lemma 3.3 holds and let* $\tilde{\mathcal{P}}_\pm$ *be as in* (3.9)*. Then* $\xi_+ \in \mathcal{R}\tilde{\mathcal{P}}_+$ *if and only if the solution of the discontinuous system* (3.3) *such that* $x(0) = \xi_+$ *is bounded for* $t \geq 0$*. Similarly,* $\xi_- \in \mathcal{N}\tilde{\mathcal{P}}_-$ *if and only if the solution of the discontinuous system* (3.5) *such that* $x(0) = \xi_-$ *is bounded for* $t \leq 0$*.*

Lemma 3.5. *Let* B_i, $i = 1,\dots,N$, *be invertible* $n \times n$ *matrices and* $k(t)$ *be a bounded integrable function for* $t \geq 0$, *(resp.* $t \leq 0$*). Suppose the condition of Lemma 3.3 hold and set*

$$\begin{aligned}\tilde{\mathcal{P}}_+^\tau &= X_+(\tau)\tilde{\mathcal{P}}_+X_+(\tau)^{-1},\\ \tilde{\mathcal{P}}_-^\tau &= X_-(-\tau)\tilde{\mathcal{P}}_-X_-(-\tau)^{-1}\end{aligned}$$

where $\tilde{\mathcal{P}}_\pm$ *is as in* (3.9) *and* $0 \leq \tau \in \mathbb{R}$ *is a fixed number. Then, for any* $\xi_+ \in \mathcal{R}\tilde{\mathcal{P}}_+^\tau$ *(resp.* $\xi_- \in \mathcal{N}\tilde{\mathcal{P}}_-^\tau$*) the linear inhomogeneous system*

$$\begin{aligned}\dot{x} &= \mathcal{A}(t)x + k(t),\\ x(t_i^+) &= B_i x(t_i^-), \quad i = 1,\dots,N\\ \tilde{\mathcal{P}}_+^\tau x(\tau) &= \xi_+\end{aligned} \tag{3.10}$$

with $t \geq 0$, *[resp.*

$$\begin{aligned}\dot{x} &= \mathcal{A}(t)x + k(t),\\ x(t_i^-) &= B_i^{-1}x(t_i^+),\\ (\mathbb{I} - \tilde{\mathcal{P}}_-^\tau)x(-\tau) &= \xi_-\end{aligned}$$

when $t \leq 0$*] has the unique right-continuous, [resp. left-continuous when* $t \leq 0$*] bounded solution*

$$\begin{aligned}x(t) = X_+(t)\tilde{\mathcal{P}}_+X_+(\tau)^{-1}\xi_+ &+ \int_\tau^t X_+(t)\tilde{\mathcal{P}}_+X_+(s)^{-1}k(s)ds\\ &- \int_t^\infty X_+(t)(\mathbb{I} - \tilde{\mathcal{P}}_+)X_+(s)^{-1}k(s)ds\end{aligned} \tag{3.11}$$

[resp.

$$\begin{aligned}x(t) = X_-(t)(\mathbb{I} - \tilde{\mathcal{P}}_-)X_-(-\tau)^{-1}\xi_- &+ \int_{-\infty}^t X_-(t)\tilde{\mathcal{P}}_-X_-(s)^{-1}k(s)ds\\ &- \int_t^{-\tau} X_-(t)(\mathbb{I} - \tilde{\mathcal{P}}_-)X_-(s)^{-1}k(s)ds\end{aligned} \tag{3.12}$$

if $t \leq 0$*]. Moreover such a solution satisfies*

$$\sup_{t\geq\tau}|x(t)| \leq K[|\xi_+| + 2\delta^{-1}\sup_{t\geq 0}|k(t)|] \tag{3.13}$$

if $t \geq 0$ *[resp.*

$$\sup_{t\leq-\tau}|x(t)| \leq K[|\xi_-| + 2\delta^{-1}\sup_{t\leq 0}|k(t)|] \tag{3.14}$$

if $t \leq 0$*].*

4 Bounded solutions on the half lines

From A_3) we know that the number of the eigenvalues of $f_{N+1,x}(w_\pm(y),y)$ with negative (and then also positive) real parts, counted with multiplicities, is independent of $y \in \mathbb{R}^m$. Moreover it also follows that all eigenvalues are bounded functions of $y \in \mathbb{R}^m$. Indeed, since $f_{N+1,x}(w_\pm(y),y)$ is bounded, the matrix $\mathbb{I} - \lambda^{-1} f_{N+1,x}(w_\pm(y),y)$ is invertible for $|\lambda| > R$, sufficiently large and independent of y. Hence all eigenvalues have to satisfy $|\lambda| \le R$.

Let δ_0 be any positive number strictly less than $\min\{|\operatorname{Re}\lambda(y)|\}$, where $\lambda(y)$ are the eigenvalues of $f_{N+1,x}(w_\pm(y),y)$. According to [7] the system $\dot{x} = f_{N+1,x}(w_\pm(y),y)x$ has an exponential dichotomy on $\mathbb{R}$ with exponent δ_0 and spectral projection (of rank k)

$$\begin{aligned} P^0_\pm(y) &= \frac{1}{2\pi i}\int_\Gamma (z\mathbb{I} - f_{N+1,x}(w_\pm(y),y))^{-1} dz \\ &= \sum_{Re\lambda(y)<0} \operatorname{Res}((z\mathbb{I} - f_{N+1,x}(w_\pm(y),y))^{-1}, z = \lambda(y)) \end{aligned}$$

where $\operatorname{Res}(F(z), z = z_0)$ is the residual of the meromorphic function $F(z)$ at z_0 and Γ is a closed curve that contains in its interior all eigenvalues of $f_{N+1,x}(w_\pm(y),y)$ with negative real parts, but none of those with positive real parts. Hence $|P^0(y)| \le M$, for any $y \in \mathbb{R}^m$ and some $M \ge 1$.

Now, recalling (2.3), from A_2) and the boundedness of $t_N(y)$, it follows immediately that

$$\lim_{t\to\pm\infty} u^\pm_{N+1}(t,y) = w_\pm(y)$$

uniformly with respect to $y \in \mathbb{R}^m$.

Let $T_+ > \sup_{y\in\mathbb{R}^m} t_N(y)$, $T_- < \inf_{y\in\mathbb{R}^m} t_{-N}(y)$ and take $0 < \delta < \delta_0$. From the roughness of exponential dichotomies (cfr. [7, Proposition 2, p. 34]) the linear systems

$$\dot{x} = f_{N+1,x}(u^+_{N+1}(t+T_+,y),y)x \tag{4.1}$$

and

$$\dot{x} = f_{N+1,x}(u^-_{N+1}(t+T_-,y),y)x \tag{4.2}$$

have an exponential dichotomy on $\mathbb{R}_+$, $\mathbb{R}_-$ resp., uniformly with respect to $y \in \mathbb{R}^m$, with projections $P_+(y)$, resp. $P_-(y)$, of rank k, constant K and exponent δ. Moreover, according to [8, Proposition 2.3], it can be assumed that, for $|y - y_0|$ sufficiently small it results: $\mathcal{N}P_+(y) = \mathcal{N}P_+(y_0)$, $\mathcal{R}P_-(y) = \mathcal{R}P_-(y_0)$ and in this case the projections are smooth with respect to y. Note that, $\mathcal{N}P_+(y) = \mathcal{N}P_+(y_0)$ and $\mathcal{R}P_-(y) = \mathcal{R}P_-(y_0)$ are equivalent to

$$\begin{aligned} P_+(y) &= P_+(y)P_+(y_0), \qquad & P_+(y_0) &= P_+(y_0)P_+(y) \\ P_-(y) &= P_-(y_0)P_-(y), \qquad & P_-(y_0) &= P_-(y)P_-(y_0). \end{aligned} \tag{4.3}$$

Let $U^\pm_i(t,y)$ be the fundamental matrix of

$$\dot{x} = f_{i,x}(u^\pm_i(t,y),y)x$$

in $\mathbb{R}_\pm$ resp., that is

$$\begin{aligned} &\dot{U}^\pm_i(t,y) = f_{i,x}(u^\pm_i(t,y),y)U^\pm_i(t,y), \quad \pm t \ge 0, \\ &U^\pm_i(0,y) = \mathbb{I}. \end{aligned}$$

As in [6] we see that

Lemma 4.1. *For any $\tau \in \mathbb{R}$ the linear system*

$$\dot{x} = f_{N+1,x}(u^+_{N+1}(t,y),y)x, \tag{4.4}$$

resp.

$$\dot{x} = f_{N+1,x}(u^-_{N+1}(t,y),y)x, \tag{4.5}$$

has an exponential dichotomy on $t \geq \tau$, resp. $t \leq \tau$, with exponent δ, constant $\tilde{K}$ independent on y and projections

$$Q_+(y) = U^+_{N+1}(\tau,y)U^+_{N+1}(T_+,y)^{-1}P_+(y)U^+_{N+1}(T_+,y)U^+_{N+1}(\tau,y)^{-1}$$
$$Q_-(y) = U^-_{N+1}(\tau,y)U^-_{N+1}(T_-,y)^{-1}P_-(y)U^-_{N+1}(T_-,y)U^-_{N+1}(\tau,y)^{-1}.$$

In particular, if $\tau = T_+$, resp. $\tau = T_-$, then $Q_+(y) = P_+(y)$, resp. $Q_-(y) = P_-(y)$, and $\tilde{K} = K$.

Finally, the following result holds (see [6, Theorems 4.3, 4.5]).

Theorem 4.2. *There exist $\rho > 0$, bounded C^r-functions*

$$t^*_{-N}(\xi_-,\alpha,\varepsilon) < \ldots < t^*_{-1}(\xi_-,\alpha,\varepsilon) < t^*_0(\xi_-,\alpha,\varepsilon) = 0 < t^*_1(\xi_+,\alpha,\varepsilon) < \ldots < t^*_N(\xi_+,\alpha,\varepsilon)$$

such that, for all $i = 1,\ldots,N$,

$$\lim_{(\xi_+,\varepsilon)\to 0} |t^*_i(\xi_+,\alpha,\varepsilon) - t_i(\alpha)| = 0,$$
$$\lim_{(\xi_-,\varepsilon)\to 0} |t^*_{-i}(\xi_-,\alpha,\varepsilon) - t_{-i}(\alpha)| = 0$$

uniformly with respect to $\alpha \in \mathbb{R}^m$, and continuous, piecewise C^r, solutions of (1.1)

$$(x_\pm(t,\xi_\pm,\alpha,\varepsilon), y_\pm(t,\xi_\pm,\alpha,\varepsilon))$$

defined for $t \geq 0$ and $t \leq 0$ resp., and such that

$$\begin{aligned}
c_{i-1} < h(x_+(t,\xi_+,\alpha,\varepsilon),y_+(t,\xi_+,\alpha,\varepsilon)) < c_i, &\quad \text{for } t^*_{i-1}(\xi_+,\alpha,\varepsilon) < t < t^*_i(\xi_+,\alpha,\varepsilon),\\
h(x_+(t,\xi_+,\alpha,\varepsilon),y_+(t,\xi_+,\alpha,\varepsilon)) > c_N, &\quad \text{for } t > t^*_N(\xi_+,\alpha,\varepsilon),\\
c_{i-1} < h(x_-(t,\xi_-,\alpha,\varepsilon),y_-(t,\xi_-,\alpha,\varepsilon)) < c_i, &\quad \text{for } t^*_{-i}(\xi_-,\alpha,\varepsilon) < t < t^*_{-i+1}(\xi_-,\alpha,\varepsilon),\\
h(x_-(t,\xi_-,\alpha,\varepsilon),y_-(t,\xi_-,\alpha,\varepsilon)) > c_N, &\quad \text{for } t < t^*_{-N}(\xi_-,\alpha,\varepsilon),
\end{aligned}$$

$$\begin{aligned}
h(x_+(t^*_i(\xi_+,\alpha,\varepsilon),\xi_+,\alpha,\varepsilon),y_+(t^*_i(\xi_+,\alpha,\varepsilon),\xi_+,\alpha,\varepsilon)) &= c_i,\\
h(x_-(t^*_{-i}(\xi_-,\alpha,\varepsilon),\xi_-,\alpha,\varepsilon),y_-(t^*_{-i}(\xi_-,\alpha,\varepsilon),\xi_-,\alpha,\varepsilon)) &= c_i,\\
\frac{\partial}{\partial t}h(x_+(t,\xi_+,\alpha,\varepsilon),y_+(t,\xi_+,\alpha,\varepsilon))_{|t=t^*_i(\xi_+,\alpha,\varepsilon)} &> \eta,\\
\frac{\partial}{\partial t}h(x_-(t,\xi_-,\alpha,\varepsilon),y_-(t,\xi_-,\alpha,\varepsilon))_{|t=t^*_{-i}(\xi_-,\alpha,\varepsilon)} &< -\eta,\\
y_\pm(T_\pm,\xi_\pm,\alpha,\varepsilon) &= \alpha,\\
P_+(\alpha)[x(T_+) - u(T_+,\alpha)] &= \xi_+,\\
(\mathbb{I} - P_-(\alpha))[x(T_-) - u(T_-,\alpha)] &= \xi_-
\end{aligned}$$

where $c_0 = -\infty$. Moreover

$$\begin{aligned} \sup_{t\geq 0} |x_+(t,\xi_+,\alpha,\varepsilon) - u(t,y_+(t,\xi_+,\alpha,\varepsilon))| < \rho, \\ \sup_{t\leq 0} |x_-(t,\xi_-,\alpha,\varepsilon) - u(t,y_-(t,\xi_-,\alpha,\varepsilon))| < \rho \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \sup_{t\geq 0} |x_+(t,\xi_+,\alpha,\varepsilon) - u(t,y_+(t,\xi_+,\alpha,\varepsilon))| \to 0 \quad \textit{as } |\xi_+| + |\varepsilon| \to 0, \\ \sup_{t\leq 0} |x_-(t,\xi_-,\alpha,\varepsilon) - u(t,y_-(t,\xi_-,\alpha,\varepsilon))| \to 0 \quad \textit{as } |\xi_-| + |\varepsilon| \to 0 \end{aligned} \tag{4.7}$$

uniformly with respect to α as well as

$$\lim_{\varepsilon\to 0} y_\pm(0,\xi_\pm,\alpha,\varepsilon) = \alpha$$

uniformly with respect to $(\xi_\pm,\alpha)$.

Remark 4.3. According to Theorem 4.2 we have

$$\begin{aligned} h(x_+(t_i^*(\xi_+,\alpha,\varepsilon),\xi_+,\alpha,\varepsilon),y_+(t_i^*(\xi_+,\alpha,\varepsilon),\xi_+,\alpha,\varepsilon)) = c_i, \\ h(x_-(t_{-i}^*(\xi_-,\alpha,\varepsilon),\xi_-,\alpha,\varepsilon),y_-(t_{-i}^*(\xi_-,\alpha,\varepsilon),\xi_-,\alpha,\varepsilon)) = c_i. \end{aligned} \tag{4.8}$$

Differentiating (4.8) with respect to ξ_+, ξ_-, at $\varepsilon = 0$ we obtain a formula for the derivatives

$$\frac{\partial t_i^*}{\partial \xi_+}(\xi_+,\alpha,0), \quad \frac{\partial t_{-i}^*}{\partial \xi_-}(\xi_-,\alpha,0), \quad i = 1,\ldots,N.$$

However we have to distinguish when $t \to t_i^*(\xi_+,\alpha,0)^+$ or $t \to t_i^*(\xi_+,\alpha,0)^-$ (resp. $t \to t_{-i}^*(\xi_-,\alpha,0)^+$ or $t \to t_{-i}^*(\xi_-,\alpha,0)^-$). For example if $t \to t_i^*(\xi_+,\alpha,0)^+$, $x_+(t,\xi_+,\alpha,0)$ is the solution of $\dot{x} = f_{i+1}(x,\alpha)$ and then, differentiating (4.8) with respect to ξ_+, we get, with $t_i^* = t_i^*(\xi_+,\alpha,0)$:

$$h_x(x_+(t_i^*,\xi_+,\alpha,0),\alpha)[f_{i+1}(x_+(t_i^*,\xi_+,\alpha,0),\alpha)\frac{\partial t_i^*}{\partial \xi_+}(\xi_+,\alpha,0) + x_{+,\xi_+}(t_i^{*+},\xi_+,\alpha,0)] = 0.$$

Vice versa, when $t \to t_i^*(\xi_+,\alpha,0)^-$, $x_+(t,\xi_+,\alpha,0)$ is the solution of $\dot{x} = f_i(x,\alpha)$ and then

$$h_x(x_+(t_i^*,\xi_+,\alpha,0),\alpha)[f_i(x_+(t_i^*,\xi_+,\alpha,0),\alpha)\frac{\partial t_i^*}{\partial \xi_+}(\xi_+,\alpha,0) + x_{+,\xi_+}(t_i^{*-},\xi_+,\alpha,0)] = 0.$$

Similarly we get, with $t_{-i}^* = t_{-i}^*(\xi_-,\alpha,0)$:

$$h_x(x_-(t_{-i}^*,\xi_-,\alpha,0),\alpha)[f_{i+1}(x_-(t_{-i}^*,\xi_-,\alpha,0),\alpha)\frac{\partial t_{-i}^*}{\partial \xi_-}(\xi_-,\alpha,0) + x_{-,\xi_-}(t_{-i}^{*-},\xi_-,\alpha,0)] = 0$$

and

$$h_x(x_-(t_{-i}^*,\xi_-,\alpha,0),\alpha)[f_i(x_+(t_{-i}^*,\xi_-,\alpha,0),\alpha)\frac{\partial t_{-i}^*}{\partial \xi_-}(\xi_-,\alpha,0) + x_{-,\xi_-}(t_{-i}^{*+},\xi_-,\alpha,0)] = 0.$$

We will use this remark in the next section.

5 The discontinuous variational equation

For any fixed $\alpha \in \mathbb{R}^m$ and $\ell = \pm 1, \dots, \pm N$ we define linear operators $B_\ell(\alpha) : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$B_\ell(\alpha)x = x - \frac{h_x(u(t_\ell(\alpha), \alpha), \alpha)x}{h_x(u(t_\ell(\alpha), \alpha), \alpha)\dot{u}(t_\ell(\alpha)^-, \alpha)}[\dot{u}(t_\ell(\alpha)^-, \alpha), \alpha) - \dot{u}(t_\ell(\alpha)^+, \alpha), \alpha)]. \quad (5.1)$$

The following result has been proved in [6, Propositioon 5.1, 5.2].

Proposition 5.1. *For any $\alpha \in \mathbb{R}^m$, $x \mapsto B_\ell(\alpha)x$ are invertible linear maps. Moreover $x_{+,\xi_+}(t, 0, \alpha, 0)$ is a solution of*

$$\begin{aligned} \dot{x} = A(t,\alpha)x &:= \begin{cases} f_{i,x}(u(t,\alpha),\alpha)x & \text{if } t_{i-1}(\alpha) \le t < t_i(\alpha), \\ & \quad i = 1,\dots,N \\ f_{N+1,x}(u(t,\alpha),\alpha)x & \text{if } t \ge t_N(\alpha), \end{cases} \\ x(t_i(\alpha)^+) &= B_i(\alpha)x(t_i(\alpha)^-), \quad i = 1,\dots,N \end{aligned} \quad (5.2)$$

which is C^1 for $t \neq t_i(\alpha)$, bounded for $t \ge 0$ and can be assumed to be right-continuous at $t = t_i(\alpha)$. Similarly $x_{-,\xi_-}(t, 0, \alpha, 0)$ is a solution of

$$\begin{aligned} \dot{x} = A(t,\alpha)x &:= \begin{cases} f_{i,x}(u(t,\alpha),\alpha)x & \text{if } t_{-i}(\alpha) < t \le t_{-i+1}(\alpha), \\ & \quad i = 1,\dots,N \\ f_{N+1,x}(u(t,\alpha),\alpha)x & \text{if } t \le t_{-N}(\alpha) \end{cases} \\ x(t_{-i}(\alpha)^+) &= B_{-i}(\alpha)x(t_{-i}(\alpha)^-), \quad i = 1,\dots,N \end{aligned} \quad (5.3)$$

which is C^1 for $t \neq t_{-i}(\alpha)$,bounded for $t \le 0$ and can be assumed to be left-continuous at $t = t_{-i}(\alpha)$. Finally, for $t \ge 0$, resp. $t \le 0$, the function

$$\dot{u}(t,\alpha) = \begin{cases} \dot{u}_i^+(t,\alpha) & \text{for } t_{i-1}(\alpha) \le t < t_i(\alpha), \\ & \quad i = 1,\dots,N \\ \dot{u}_{N+1}^+(t,\alpha) & \text{for } t \ge T_N(\alpha) \end{cases}$$

resp.

$$\dot{u}(t,\alpha) = \begin{cases} \dot{u}_i^-(t,\alpha) & \text{for } t_{-i}(\alpha) < t \le t_{-i+1}(\alpha), \\ & \quad i = 1,\dots,N \\ \dot{u}_{N+1}^-(t,\alpha) & \text{for } t \le T_{-N}(\alpha) \end{cases}$$

is a solution of (5.2) (resp. (5.3)) bounded on $t \ge 0$ (resp, $t \le 0$).

6 Main result

First we recall that $P_+(y)$ is the projections of the exponential dichotomy on $t \ge 0$, of the linear system (4.1) with constant K and exponent δ. Then, from Lemma 4.1, we see that (4.4) has an exponential dichotomy on $t \ge t_N(y)$ with exponent δ and projection

$$U_{N+1}^+(t_N(y), y)U_{N+1}^+(T_+, y)^{-1}P_+(y)U_{N+1}^+(T_+, y)U_{N+1}^+(t_N(y), y)^{-1}.$$

Similarly, the linear system (4.5) has an exponential dichotomy on $t \le t_{-N}(y)$ with exponent δ and projection

$$U_{N+1}^-(t_{-N}(y), y)U_{N+1}^-(T_-, y)^{-1}P_-(y)U_{N+1}^-(T_-, y)U_{N+1}^-(t_{-N}(y), y)^{-1}.$$

From Lemma 3.3–3.4 we obtain the following

Proposition 6.1. *For any $\alpha \in \mathbb{R}^m$, the discontinuous linear system (5.2), resp. (5.3), has an exponential dichotomy on $\mathbb{R}_+$, resp. $\mathbb{R}_-$, with projections $Q_+(\alpha)$, resp. $Q_-(\alpha)$, given by*

$$\begin{aligned} Q_+(\alpha) &= X_+(t_N(\alpha)^+,\alpha)^{-1}U^+_{N+1}(t_N(\alpha),\alpha)U^+_{N+1}(T_+,\alpha)^{-1} \\ &\quad \cdot P_+(\alpha)U^+_{N+1}(T_+,\alpha)U^+_{N+1}(t_N(\alpha),\alpha)^{-1}X_+(t_N(\alpha)^+,\alpha) \\ Q_-(\alpha) &= X_-(t_{-N}(\alpha)^-,\alpha)^{-1}U^-_{N+1}(t_{-N}(\alpha),\alpha)U^-_{N+1}(T_-,\alpha)^{-1} \\ &\quad \cdot P_-(\alpha)U^-_{N+1}(T_-,\alpha)U^-_{N+1}(t_{-N}(\alpha),\alpha)^{-1}X_-(t_{-N}(\alpha)^-,\alpha) \end{aligned}$$

where

$$\begin{aligned} X_+(t_N^+(\alpha),\alpha) &= B_N(\alpha)U_N^+(t_N(\alpha))U_N^+(t_{N-1}(\alpha),\alpha)^{-1}\dots B_1(\alpha)U_1^+(t_1(\alpha),\alpha) \\ X_-(t_{-N}(\alpha)^-,\alpha) &= \ B_{-N}(\alpha)^{-1}U_N^-(t_{-N}(\alpha))U_N^-(t_{-N+1}(\alpha),\alpha)^{-1}\dots B_{-1}(\alpha)^{-1}U_1^-(t_{-1}(\alpha),\alpha). \end{aligned}$$

Moreover $\mathcal{R}Q_+(\alpha)$ (resp. $\mathcal{N}Q_-(\alpha)$) is the space of initial conditions of solutions of (5.2), resp. (5.3), right-continuous, when $t \geq 0$ (resp. left-continuous, when $t \leq 0$) and bounded on $\mathbb{R}_+$, (resp, on $\mathbb{R}_-$).

We assume the following condition holds:

A_5) For any $\alpha \in \mathbb{R}^m$, $\dim[\mathcal{R}Q_+(\alpha) \cap \mathcal{N}Q_-(\alpha)] = d \leq m$.

From Proposition 5.1 we know that $\dot{u}(0,\alpha) \in \mathcal{R}Q_+(\alpha) \cap \mathcal{N}Q_-(\alpha)$ so

$$1 \leq \dim[\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)]^\perp = d.$$

Next, from A_3) it follows that $\dim \mathcal{R}Q_+(\alpha) = k$ and $\dim \mathcal{N}Q_-(\alpha) = n-k$, hence $d \leq \min\{k, n-k\}$.

Let $\psi_1(\alpha), \dots, \psi_d(\alpha) \in \mathbb{R}^n$ be such that

$$[\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)]^\perp = \operatorname{span}\{\psi_1(\alpha), \dots, \psi_d(\alpha)\}.$$

Without loss of generality we assume that $\{\psi_1(\alpha), \dots, \psi_d(\alpha)\}$ is an orthonormal set.

The purpose of this section is to prove the following

Theorem 6.2. *Suppose that A_1)–A_5) hold. Suppose further that there exists $\alpha_0 \in \mathbb{R}^m$ such that the vector function*

$$M(\alpha) := \left(\int_{-\infty}^{\infty} \psi_j(\alpha)^T G(t,\alpha) u_y(t,\alpha) g(u(t,\alpha),\alpha,0) dt \right)_{j=1,\dots,d}$$

where

$$G(t,\alpha) = \begin{cases} Q_-(\alpha)X_-(t,\alpha)^{-1} & \text{if } t \leq 0, \\ (\mathbb{I} - Q_+(\alpha))X_+(t,\alpha)^{-1} & \text{if } t \geq 0 \end{cases}$$

satisfies $M(\alpha_0) = 0$ and $\operatorname{rank} M'(\alpha_0) = d$. Then there exists $\rho > 0$ and $\varepsilon_0 > 0$ such that for $0 \leq \varepsilon \leq \varepsilon_0$ system (1.1) has a $(m-d)$-dimensional manifold of continuous, piecewise C^r solutions $(x(t), y(t))$ such that

$$\sup_{t\in\mathbb{R}} |x(t) - u(t, y(t)) < \rho,$$

$$\sup_{t\in\mathbb{R}} |x(t) - u(t, y(t))| \to 0$$

as $\varepsilon \to 0$.

Proof. Arguing as in [6, Theorem 6.2] we know that

$$\begin{aligned} x_+(t,\xi_+,\alpha,\varepsilon) = u(t,y_+(t,\xi_+,\alpha,\varepsilon)) + X_+(t,\alpha)\tilde{\xi}_+ \\ + \int_0^t X_+(t,\alpha)Q_+(\alpha)X_+(s,\alpha)^{-1}b_+(s)ds \\ s - \int_t^\infty X_+(t,\alpha)(\mathbb{I} - Q_+(\alpha))X_+(s,\alpha)^{-1}b_+(s)ds \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} b_+(t) &= b_+(t,\xi_+,\alpha,\varepsilon) \\ &:= f(x_+(t,\xi_+,\alpha,\varepsilon),y_+(t,\xi_+,\alpha,\varepsilon)) - f(u(t,y_+(t,\xi_+,\alpha,\varepsilon)),y_+(t,\xi_+,\alpha,\varepsilon)) \\ &\quad - \mathcal{A}(t,\alpha)[x_+(t,\xi_+,\alpha,\varepsilon) - u(t,y_+(t,\xi_+,\alpha,\varepsilon))] \\ &\quad - \varepsilon u_y(t,y_+(t,\xi_+,\alpha,\varepsilon))g(x_+(t,\xi_+,\alpha,\varepsilon),y_+(t,\xi_+,\alpha,\varepsilon),\varepsilon) \end{aligned}$$

and

$$\tilde{\xi}_+ = Q_+(\alpha)[x_+(0,\xi_+,\alpha,\varepsilon) - u(0,y_+(0,\xi_+,\alpha,\varepsilon))] \in \mathcal{R}Q_+(\alpha).$$

Moreover, for ε sufficiently small, the map $(\xi_+,\alpha) \mapsto (\tilde{\xi}_+, y_+(0,\xi_+,\alpha,\varepsilon))$ from $\mathcal{R}P_+(\alpha) \times \mathbb{R}^m$ into $\mathcal{R}Q_+(\alpha) \times \mathbb{R}^m$ is linearly invertible.

Similarly, for $|\alpha_- - y_0|$ sufficiently small we have

$$\begin{aligned} x_-(t,\xi_-,\alpha_-,\varepsilon) = u(t,y_-(t,\xi_-,\alpha_-,\varepsilon)) + X_-(t,\alpha_-)\tilde{\xi}_- \\ + \int_{-\infty}^t X_-(t,\alpha_-)Q_-(\alpha_-)X_-(s,\alpha_-)^{-1}b_-(s)ds \\ - \int_t^0 X_-(t,\alpha_-)(\mathbb{I} - Q_-(\alpha_-))X_-(s,\alpha_-)^{-1}b_-(s)ds \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} b_-(t) &= b_-(t,\xi_-,\alpha_-,\varepsilon) \\ &:= f(x_-(t,\xi_-,\alpha_-,\varepsilon),y_-(t,\xi_-,\alpha_-,\varepsilon)) - f(u(t,y_-(t,\xi_-,\alpha_-,\varepsilon)),y_-(t,\xi_-,\alpha_-,\varepsilon)) \\ &\quad - \mathcal{A}(t,\alpha_-)[x_-(t,\xi_-,\alpha_-,\varepsilon) - u(t,y_-(t,\xi_-,\alpha_-,\varepsilon))] \\ &\quad - \varepsilon u_y(t,y_-(t,\xi_-,\alpha_-,\varepsilon))g(x_-(t,\xi_-,\alpha_-,\varepsilon),y_-(t,\xi_-,\alpha_-,\varepsilon),\varepsilon) \end{aligned}$$

and

$$\tilde{\xi}_- = [\mathbb{I} - Q_-(\alpha_-)][x_-(0,\xi_-,\alpha_-,\varepsilon) - u(0,y_-(0,\xi_+,\alpha_-,\varepsilon))] \in \mathcal{N}Q_-(\alpha_-).$$

Moreover, for ε sufficiently small, the map $(\xi_-,\alpha_-) \mapsto (\tilde{\xi}_-, y_-(0,\xi_-,\alpha_-,\varepsilon))$ from $\mathcal{N}P_-(\alpha_-) \times \mathbb{R}^m$ into $\mathcal{N}Q_-(\alpha_-) \times \mathbb{R}^m$ is linearly invertible.

From (6.1)-(6.2) we get, for $|\alpha - y_0| + |\alpha_- - y_0|$ sufficiently small

$$\begin{aligned} x_+(0,\xi_+,\alpha,\varepsilon) - x_-(0,\xi_-,\alpha_-,\varepsilon) \\ = u(0,y_+(0,\xi_+,\alpha,\varepsilon)) - u(0,y_-(0,\xi_-,\alpha_-,\varepsilon)) + \tilde{\xi}_+ - \tilde{\xi}_- \\ - \int_0^\infty (\mathbb{I} - Q_+(\alpha))X_+(s,\alpha)^{-1}b_+(s)ds - \int_{-\infty}^0 Q_-(\alpha_-)X_-(s,\alpha_-)^{-1}b_-(s)ds. \end{aligned} \tag{6.3}$$

Then the system

$$\begin{cases} x_+(0,\xi_+,\alpha,\varepsilon) = x_-(0,\xi_-,\alpha_-,\varepsilon), \\ y_+(0,\xi_+,\alpha,\varepsilon) = y_-(0,\xi_-,\alpha_-,\varepsilon) \end{cases}$$

is equivalent to

$$\begin{cases} \tilde{\xi}_+ - \tilde{\xi}_- = k(\xi_+, \xi_-, \alpha, \alpha_-, \varepsilon), \\ y_-(0, \xi_-, \alpha_-, \varepsilon) - y_+(0, \xi_+, \alpha, \varepsilon) = 0 \end{cases} \tag{6.4}$$

where

$$k(\xi_+, \xi_-, \alpha, \alpha_-, \varepsilon) = \int_0^\infty (\mathbb{I} - Q_+(\alpha))X_+(s,\alpha)^{-1} b_+(s)ds + \int_{-\infty}^0 Q_-(\alpha_-)X_-(s,\alpha_-)^{-1}b_-(s)ds.$$

Differentiating $b_+(t) = b_+(t, \xi_+, \alpha, \varepsilon)$ with respect to ξ_+ at $\xi_+ = 0$, $\varepsilon = 0$ and also using $x_+(t, 0, \alpha, 0) = u(t, \alpha)$, $y_+(t, 0, \alpha, 0) = \alpha$, we see that, for $t_{i-1}(\alpha) < t < t_i(\alpha)$, we have

$$\tfrac{\partial b_+}{\partial \xi_+}(t, 0, \alpha, 0) = [f_{i,x}(u(t,\alpha), \alpha) - A(t,\alpha)]x_{+,\xi_+}(t, 0, \alpha, 0) = 0$$

and for $t > t_N(\alpha)$:

$$\tfrac{\partial b_+}{\partial \xi_+}(t, 0, \alpha, 0) = [f_{N+1,x}(u(t,\alpha), \alpha) - A(t,\alpha)]x_{+,\xi_+}(t, 0, \alpha, 0) = 0.$$

Then

$$\frac{\partial}{\partial \xi_+}\left[\int_0^\infty (\mathbb{I} - Q_+(\alpha))X_+(s,\alpha)^{-1}b_+(s)ds\right]_{\xi_+=0,\varepsilon=0} = 0.$$

Similarly we get, for $|\alpha_- - y_0|$ sufficiently small,

$$\frac{\partial}{\partial \xi_-}\left[\int_{-\infty}^0 Q_-(\alpha_-)X_-(s,\alpha_-)^{-1}b_-(s)ds\right]_{\xi_-=0,\varepsilon=0} = 0.$$

As a consequence (6.4) reads:

$$\begin{aligned} \tilde{\xi}_+ - \tilde{\xi}_- &= R_1(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon), \\ \alpha_- - \alpha &= R_2(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon) \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} R_1(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon) &= k(\xi_+, \xi_-, \alpha, \alpha_-, \varepsilon) \\ \tilde{\xi}_+ &= Q_+(\alpha)[x_+(0, \xi_+, \alpha, \varepsilon) - u(0, y_+(0, \xi_+, \alpha, \varepsilon))] \\ \tilde{\xi}_- &= [\mathbb{I} - Q_-(\alpha_-)][x_-(0, \xi_-, \alpha_-, \varepsilon) - u(0, y_-(0, \xi_-, \alpha_-, \varepsilon))]. \end{aligned}$$

Note that, being $(\xi_+, \xi_-) \mapsto (\tilde{\xi}_+, \tilde{\xi}_-)$ linearly invertible, we derive: $R_1(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon) = O(|\tilde{\xi}_+|^2 + |\tilde{\xi}_-|^2 + |\varepsilon|)$ and $R_2(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon) = O(|\varepsilon|)$, uniformly with respect to (α, α_-).

Now, as $\tilde{\xi}_- \in \mathcal{N}Q_-(\alpha_-)$, we have

$$(\mathbb{I} - Q_-(\alpha))\tilde{\xi}_- = \tilde{\xi}_- - (Q_-(\alpha) - Q_-(\alpha_-))\tilde{\xi}_-$$

and hence

$$\frac{1}{2}|\tilde{\xi}_-| \le |(\mathbb{I} - Q_-(\alpha))\tilde{\xi}_-| \le 2|\tilde{\xi}_-|$$

provided $|\alpha_- - y_0|$ and $|\alpha - y_0|$ are sufficiently small. Hence the map $\tilde{\xi}_- \mapsto (\mathbb{I} - Q_-(\alpha))\tilde{\xi}_-$ from $\mathcal{N}Q_-(\alpha_-)$ into $\mathcal{N}Q_-(\alpha)$ is linearly invertible. Then, setting

$$\bar{\xi}_+ = \tilde{\xi}_+, \quad \bar{\xi}_- = (\mathbb{I} - Q_-(\alpha))\tilde{\xi}_-, \tag{6.6}$$

(6.5) can be written as

$$\begin{aligned} \bar{\xi}_+ - \bar{\xi}_- &= \bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon), \\ \alpha_- - \alpha &= \bar{R}_2(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) \end{aligned} \tag{6.7}$$

with

$$\begin{aligned}(\bar{\xi}_+, \bar{\xi}_-) &\in \mathcal{R}Q_+(\alpha) \times \mathcal{N}Q_-(\alpha),\\ \bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) &= R_1(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon),\\ \bar{R}_2(\bar{\xi}_+, \bar{\xi}_-, \alpha_+, \alpha_-, \varepsilon) &= R_2(\tilde{\xi}_+, \tilde{\xi}_-, \alpha, \alpha_-, \varepsilon)\end{aligned}$$

where $(\tilde{\xi}_+, \tilde{\xi}_-)$ is the point corresponding to $(\bar{\xi}_+, \bar{\xi}_-)$ through (6.6). Note that it holds $\bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) = O(|\bar{\xi}_+|^2 + |\bar{\xi}_-|^2 + |\varepsilon|)$, $\bar{R}_2(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) = O(|\varepsilon|)$ uniformly with respect to (α, α_-).

Let α, α_- be such that $|\alpha - y_0|$ and $|\alpha_- - y_0|$ are sufficiently small. The map $(\bar{\xi}_+, \bar{\xi}_-) \mapsto \bar{\xi}_+ - \bar{\xi}_-$ is a linear map from $\mathcal{R}Q_+(\alpha) \times \mathcal{N}Q_-(\alpha)$ into $\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)$ whose kernel is $\mathcal{R}Q_+(\alpha) \cap \mathcal{N}Q_-(\alpha)$ which, by assumption A_5), is d-dimensional.

Let $W(\alpha) \subset \mathcal{R}Q_+(\alpha)$ be a complement of $\mathcal{R}Q_+(\alpha) \cap \mathcal{N}Q_-(\alpha)$ in $\mathcal{R}Q_+(\alpha)$, so that

$$\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha) = W(\alpha) \oplus \mathcal{N}Q_-(\alpha).$$

Note that $\dim W(\alpha) = k - d$ and

$$\mathbb{R}^n = [\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)] \oplus \operatorname{span}\{\psi_1(\alpha), \ldots, \psi_d(\alpha)\}.$$

Next, let $Q(\alpha) : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection such that $\mathcal{R}Q(\alpha) = \mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)$ and $\mathcal{N}Q(\alpha) = \operatorname{span}\{\psi_1(\alpha), \ldots, \psi_d(\alpha)\}$. Since

$$(\mathbb{I} - Q(\alpha))x \in \mathcal{N}Q(\alpha) = \operatorname{span}\{\psi_1(\alpha), \ldots, \psi_d(\alpha)\}$$

and $(\psi_1(\alpha), \ldots, \psi_d(\alpha))$ is orthonormal we get

$$\begin{aligned}(\mathbb{I} - Q(\alpha))x &= \sum_{j-1}^{d} \langle \psi_j(\alpha), (\mathbb{I} - Q(\alpha))x \rangle \psi_j(\alpha)\\ &= \sum_{j-1}^{d} \langle (\mathbb{I} - Q(\alpha))\psi_j(\alpha), x \rangle \psi_j(\alpha) = \sum_{j-1}^{d} (\psi_j(\alpha)^T x)\psi_j(\alpha).\end{aligned}$$

Hence we replace (6.7) with

$$\begin{aligned}\bar{\xi}_+ - \bar{\xi}_- &= Q(\alpha)\bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon),\\ \alpha - \alpha_- &= \bar{R}_2(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon),\\ \psi_j^T(\alpha)\bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) &= 0.\end{aligned} \tag{6.8}$$

We solve (6.8) for $(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-) \in W(\alpha) \times \mathcal{N}Q_-(\alpha) \times \mathbb{R}^m \times \mathbb{R}^m$ in terms of ε.

Since $\dim[\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)] = n - d$, we see that for any fixed ε

$$\begin{aligned}\bar{\xi}_+ - \bar{\xi}_- &= Q(\alpha)\bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon)],\\ \alpha - \alpha_- &= \bar{R}_2(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon)\end{aligned} \tag{6.9}$$

is essentially a system of $n - d + m$ equations in the $n - d + 2m$ variables $(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-)$ such that, when $\varepsilon = 0$, has the solution

$$(\bar{\xi}_+, \bar{\xi}_-) = (0, 0), \qquad \alpha_- = \alpha.$$

The Jacobian matrix at this point is

$$J = \begin{pmatrix} L & 0 & 0 \\ 0 & \mathbb{I}_{\mathbb{R}^m} & -\mathbb{I}_{\mathbb{R}^m} \end{pmatrix}$$

where $L : W \times \mathcal{N}Q_-(\alpha) \to W \oplus \mathcal{N}Q_-(\alpha)$ is the invertible linear map given by $L(\bar{\xi}_+, \bar{\xi}_-) = \bar{\xi}_+ - \bar{\xi}_-$. We have

$$\operatorname{rank} J = n - d + m$$

hence, for $\varepsilon \neq 0$ and sufficiently small (6.9) has a m-dimensional C^r-manifold of solutions

$$\bar{\xi}_+ = \bar{\xi}_+(\alpha, \varepsilon), \quad \bar{\xi}_- = \bar{\xi}_-(\alpha, \varepsilon), \quad \alpha_- = \alpha_-(\alpha, \varepsilon)$$

where

$$\begin{aligned} |\bar{\xi}_\pm(\alpha, \varepsilon)| &= O(|\varepsilon|), \\ |\alpha_-(\alpha, \varepsilon) - \alpha| &= O(|\varepsilon|) \end{aligned} \tag{6.10}$$

uniformly with respect to α. Plugging this solution in the third equation in (6.8) we obtain the system of equations

$$\psi_j^T(\alpha) \bar{R}_1(\bar{\xi}_+(\alpha, \varepsilon), \bar{\xi}_-(\alpha, \varepsilon), \alpha, \alpha_-(\alpha, \varepsilon), \varepsilon) = 0, \quad j = 1, \ldots, d.$$

As $\bar{R}_1(0, 0, \alpha, \alpha, 0) = 0$ we see that this equation is identically satisfied when $\varepsilon = 0$, so we replace it with

$$\mathcal{M}(\alpha, \varepsilon) = 0$$

where $\mathcal{M}(\alpha, \varepsilon)$ is the C^{r-1}-function:

$$\mathcal{M}(\alpha, \varepsilon) = \begin{cases} \varepsilon^{-1} \left(\psi_j^T(\alpha) [\bar{R}_1(\bar{\xi}_+(\alpha, \varepsilon), \bar{\xi}_-(\alpha, \varepsilon), \alpha, \alpha_-(\alpha, \varepsilon), \varepsilon) \right)_{j=1,\ldots,d} & \text{for } \varepsilon \neq 0, \\ \left[\left(\frac{\partial}{\partial \varepsilon} \psi_j^T(\alpha) \bar{R}_1(\bar{\xi}_+(\alpha, \varepsilon), \bar{\xi}_-(\alpha, \varepsilon), \alpha, \alpha_-(\alpha, \varepsilon), \varepsilon) \right)_{j=1,\ldots,d} \right]_{|\varepsilon=0} & \text{for } \varepsilon = 0. \end{cases}$$

We have already observed that $\bar{R}_1(\bar{\xi}_+, \bar{\xi}_-, \alpha, \alpha_-, \varepsilon) = O(|\bar{\xi}_+|^2 + |\bar{\xi}_-|^2 + |\varepsilon|)$ uniformly with respect to (α, α_-), then

$$\mathcal{M}(\alpha, 0) = \left(\psi_j^T(\alpha) \bar{R}_{1,\varepsilon}(0, 0, \alpha, \alpha, 0)] \right)_{j=1,\ldots,d}.$$

We now compute $\bar{R}_{1,\varepsilon}(0, 0, \alpha, \alpha, 0)$. Since the map $(\tilde{\xi}_+, \tilde{\xi}_-) \mapsto (\bar{\xi}_+, \bar{\xi}_-)$ where $\bar{\xi}_+ = \tilde{\xi}_+$ and $\bar{\xi}_- = (\mathbb{I} - Q(\alpha))\tilde{\xi}_-$ is a linear isomorphism we see that

$$\bar{R}_1(0, 0, \alpha, \alpha, \varepsilon) = k(0, 0, \alpha, \alpha, \varepsilon)$$

and hence

$$\begin{aligned} \bar{R}_{1\varepsilon}(0, 0, \alpha, \alpha, 0) &= k_\varepsilon(0, 0, \alpha, \alpha, 0) \\ &= \int_{-\infty}^{0} Q_-(\alpha) X_-(t, \alpha)^{-1} \frac{\partial b_-}{\partial \varepsilon}(t, 0, 0, \alpha, \alpha, 0) dt \\ &\quad + \int_{0}^{\infty} (\mathbb{I} - Q_+(\alpha)) X_+(t, \alpha)^{-1} \frac{\partial b_+}{\partial \varepsilon}(t, 0, 0, \alpha, \alpha, 0) dt \end{aligned}$$

that is

$$\mathcal{M}(\alpha,0) = \left(\int_{-\infty}^{\infty} \psi_j^T(t,\alpha) b_\varepsilon(t,0,0,\alpha,\alpha,0)dt\right)_{j=1,\dots,d}$$

where

$$\psi_j^T(t,\alpha) = \begin{cases} \psi_j^T(\alpha) Q_-(\alpha) X_-(t,\alpha)^{-1} & \text{if } t<0, \\ \psi_j^T(\alpha)(\mathbb{I} - Q_+(\alpha)) X_+(t,\alpha)^{-1} & \text{if } t \geq 0 \end{cases} \tag{6.11}$$

and

$$b_\varepsilon(t,0,0,\alpha,\alpha,0) = \begin{cases} \frac{\partial b_-}{\partial \varepsilon}(t,0,0,\alpha,\alpha,0) & \text{if } t<0, \\ \frac{\partial b_+}{\partial \varepsilon}(t,0,0,\alpha,\alpha,0) & \text{if } t \geq 0. \end{cases}$$

Now, it is easy to check that

$$\begin{cases} \frac{\partial b_-}{\partial \varepsilon}(t,0,0,\alpha,\alpha,0) & \text{if } t<0, \\ \frac{\partial b_+}{\partial \varepsilon}(t,0,0,\alpha,\alpha,0) & \text{if } t \geq 0 \end{cases} = -u_y(t,\alpha) g(u(t,\alpha),\alpha,0).$$

Hence

$$\mathcal{M}(\alpha,0) = -\left(\int_{-\infty}^{\infty} \psi_j^T(t,\alpha) u_y(t,\alpha) g(u(t,\alpha),\alpha,0)dt\right)_{j=1,\dots,d} = -M(\alpha). \tag{6.12}$$

The thesis follows now from the Implicit Function Theorem. □

Remark 6.3. i) The orthonormal basis $(\psi_1(\alpha),\dots,\psi_d(\alpha))$ of $[\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)]^\perp$ can be replaced by any independent set $(\tilde\psi_1(\alpha),\dots,\tilde\psi_d(\alpha))$ such that

$$\mathbb{R}^n = [\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)] \oplus \operatorname{span}\{\tilde\psi_1(\alpha),\dots,\tilde\psi_d(\alpha)\}.$$

Indeed, let $\langle\cdot,\cdot\rangle$ be a scalar product on $\mathbb{R}^n$ such that

$$[\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)]^\perp = \operatorname{span}\{\tilde\psi_1(\alpha),\dots,\tilde\psi_d(\alpha)\}$$

and let $(\psi_1(\alpha),\dots,\psi_d(\alpha))$ be an orthonormal basis of $\operatorname{span}\{\tilde\psi_1(\alpha),\dots,\tilde\psi_d(\alpha)\}$. Then a smooth, invertible $d\times d$ matrix $N(\alpha)$ exists such that

$$(\tilde\psi_1(\alpha)\dots\tilde\psi_d(\alpha)) = (\psi_1(\alpha)\dots\psi_d(\alpha))N(\alpha).$$

Set

$$\tilde M(\alpha) = \left[\int_{-\infty}^{\infty} \tilde\psi_j^T(t,\alpha) u_y(t,\alpha) g(u(t,\alpha),\alpha,0)dt\right]_{j=1,\dots,d}.$$

We have

$$\begin{aligned} &[\tilde\psi_j(\alpha)^T u_y(t,\alpha) g(u(t,\alpha),\alpha,0)]_{j=1,\dots,d} \\ &\quad = (\tilde\psi_1(\alpha)\dots\tilde\psi_d(\alpha))^T [u_y(t,\alpha) g(u(t,\alpha),\alpha,0)] \\ &\quad = N(\alpha)^T (\psi_1(\alpha)\dots\psi_d(\alpha))^T u_y(t,\alpha) g(u(t,\alpha),\alpha,0) \\ &\quad = N(\alpha)^T [\psi_j(\alpha)^T [u_y(t,\alpha) g(u(t,\alpha),\alpha,0)]_{j=1,\dots,d} \end{aligned}$$

that is

$$\tilde M(\alpha) = N(\alpha)^T M(\alpha).$$

Now, assuming that $M(\alpha_0) = 0$ and $\operatorname{rank} M'(\alpha_0) = d$, we see that $\tilde M(\alpha_0) = N(\alpha_0)^T M(\alpha_0) = 0$ and

$$\tilde M'(\alpha_0) = N(\alpha_0)^T M'(\alpha_0).$$

So $\tilde{M}(\alpha_0) = 0$ and $\operatorname{rank} \tilde{M}'(\alpha_0) = d$. The vice versa is proved in the same way using the equality

$$M(\alpha) = [N(\alpha)^T]^{-1}\tilde{M}(\alpha).$$

ii) The adjoint system to (5.2) and (5.3) is given by [1]

$$\begin{aligned} \dot{w} &= -A^T(t,\alpha)w \quad \text{if } t \geq 0, \\ w(t_i(\alpha)^+) &= (B_i^*(\alpha)^T)^{-1}w(t_i(\alpha)^-), \\ w(t_{-i}(\alpha)^+) &= (B_{*,i}(\alpha)^T)^{-1}w(t_{-i}(\alpha)^-). \end{aligned} \tag{6.13}$$

It is easy to check that, if $\psi(\alpha) \in [\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)]^\perp$, the function $\psi(t,\alpha)$ defined in (6.11) is a bounded solution of (6.13). We prove that if

$$\operatorname{span}\{\psi_1(\alpha), \dots, \psi_d(\alpha)\} = [\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha)]^\perp$$

then $\{\psi_1(t,\alpha), \dots, \psi_d(t,\alpha)\}$ is a basis for the space of the bounded solutions of (6.13). Indeed, the fundamental matrix of (6.13) on $t \geq 0$ is $[X_+(t,\alpha))^T]^{-1}$, and the fundamental matrix of (6.13) on $t \leq 0$ is $[X_-(t,\alpha))^T]^{-1}$. As a consequence (6.13) has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projections $(\mathbb{I} - Q_+^T)$ and $(\mathbb{I} - Q_-^T)$ respectively. So, the space of bounded solutions of (6.13), C^1 for $t \neq t_{\pm i}(\alpha)$, are those whose initial condition belongs to

$$\mathcal{R}(\mathbb{I} - Q_+^T(\alpha)) \cap \mathcal{N}(\mathbb{I} - Q_-^T(\alpha)) = (\mathcal{R}Q_+v)^\perp \cap (\mathcal{N}Q_-(\alpha))^\perp = (\mathcal{R}Q_+(\alpha) + \mathcal{N}Q_-(\alpha))^\perp.$$

Then the dimension of the space of solutions of (6.13), bounded on $\mathbb{R}$, is d and vectors $\{\psi_1(t,\alpha), \dots, \psi_d(t,\alpha)\}$ span this space.

As in [5,9] we see that if $x(t,\alpha)$ and $\psi(t,\alpha)$ are bounded solutions on $\mathbb{R}$ of (5.2)–(5.3) and (6.13) resp., both continuous for $t \neq t_{\pm i}(\alpha)$ then $\psi(t,\alpha)^T x(t,\alpha)$ is constant on $\mathbb{R}$.

7 An example

The simplest case of application of Theorem 6.2 is when $d = 1$ that is when

$$\mathcal{R}Q_+(\alpha) \cap \mathcal{N}Q_-(\alpha) = \operatorname{span}\{\dot{u}(0,\alpha)\}.$$

This condition is trivially satisfied when $n = 2$ since in this case $k = n - k = 1$. Moreover, when $n = 2$, we also have $\dim \mathcal{R}Q_+(\alpha) = \dim \mathcal{N}Q_-(\alpha) = 1$ and hence

$$\mathcal{R}Q_+(\alpha) = \mathcal{N}Q_-(\alpha) = \operatorname{span}\{\dot{u}(0,\alpha)\}. \tag{7.1}$$

In this section we consider examples of applications of Theorem 6.2 with $n = 2$, $m = 1$ and $d = 1$. Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The following result that has been proved in [6]:

Proposition 7.1. *Consider the system in $\mathbb{R}^3$:*

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2, y), \\ \dot{x}_2 &= F_2(x_1, x_2, y), \\ \dot{y} &= \varepsilon g(x_1, x_2, y). \end{aligned} \tag{7.2}$$

and suppose that for any $\alpha \in \mathbb{R}$, the unperturbed equation

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2, \alpha), \\ \dot{x}_2 &= F_2(x_1, x_2, \alpha) \end{aligned} \tag{7.3}$$

has a solution $u(t,\alpha)$ satisfying assumptions $A_1) - A_5)$. Let

$$A(t,\alpha) = [a_{jk}(t,\alpha)]_{1\le j,k\le 2} := [F_{j,x_k}(u_1(t), u_2(t), \alpha)]_{1\le j,k\le 2},$$

$B_i(\alpha)$ as in (5.1) and

$$v(t,\alpha) := e^{-\int_0^t a_{11}(s,\alpha)+a_{22}(s,\alpha)ds} J\dot{u}(t,y_0) = e^{-\int_0^t a_{11}(s,\alpha)+a_{22}(s,\alpha)ds} \begin{pmatrix} -\dot{u}_2(t,\alpha) \\ \dot{u}_1(t,\alpha) \end{pmatrix}. \tag{7.4}$$

Then the space of bounded solution of the adjoint variational system are of the form

$$\psi(t,\alpha) = \begin{cases} \mu_{-N}(\alpha)v(t,\alpha) & \text{for } t \le t_{-N}(\alpha), \\ \mu_{-i}(\alpha)v(t,\alpha), & \text{for } t_{-i-1}(\alpha) < t \le t_{-i}(\alpha), \\ \mu_i(\alpha)v(t,\alpha), & \text{for } t_i(\alpha) \le t < t_{i+1}(\alpha), \\ \mu_N(\alpha)v(t,\alpha) & \text{for } t \ge t_N(\alpha) \end{cases}$$

where $\mu_{-N}(\alpha) \neq 0$ is arbitrary and, for any $i = 1, \ldots, N$,

$$\begin{aligned} \mu_{-i+1}(\alpha)v(t_{-i}(\alpha)^+,\alpha) &= \mu_{-i}(\alpha)[B_{-i}(\alpha)^T]^{-1} v(t_{-i}(\alpha)^-,\alpha), \\ \mu_i(\alpha)v(t_i(\alpha)^+,\alpha) &= \mu_{i-1}(\alpha)[B_i(\alpha)^T]^{-1} v(t_i(\alpha)^-,\alpha). \end{aligned} \tag{7.5}$$

Remark 7.2. i) From (7.5) we have

$$\begin{aligned} \mu_{-i+1}(\alpha)J\dot{u}(t_{-i}(\alpha)^+,\alpha)) &= \mu_{-i}(\alpha)[B_{-i}^T(\alpha)]^{-1} J\dot{u}(t_{-i}(\alpha)^-,\alpha)), \\ \mu_i(\alpha)J\dot{u}(t_i(\alpha)^+,\alpha)) &= \mu_{i-1}(\alpha)[B_i^T(\alpha)]^{-1} J\dot{u}(t_i(\alpha)^-,\alpha)) \end{aligned}$$

and then

$$\begin{aligned} \mu_i(\alpha)\|\dot{u}(t_i(\alpha)^+,\alpha))\|^2 &= \mu_i(\alpha)\langle J\dot{u}(t_i(\alpha)^+,\alpha)), J\dot{u}(t_i(\alpha)^+,\alpha))\rangle, \\ &= \mu_{i-1}(\alpha)\langle [B_i^T(\alpha)]^{-1} J\dot{u}(t_i(\alpha)^-,\alpha)), J\dot{u}(t_i(\alpha)^+,\alpha))\rangle \end{aligned}$$

and similarly

$$\begin{aligned} \mu_{-i+1}(\alpha)\|\dot{u}(t_{-i}(\alpha)^+,\alpha))\|^2 &= \mu_{-i}(\alpha)\langle J\dot{u}(t_{-i}(\alpha)^+,\alpha)), J\dot{u}(t_{-i}(\alpha)^+,\alpha))\rangle, \\ &= \mu_{-i}(\alpha)\langle [B_{-i}^T(\alpha)]^{-1} J\dot{u}(t_{-i}(\alpha)^-,\alpha)), J\dot{u}(t_{-i}(\alpha)^+,\alpha))\rangle. \end{aligned}$$

Hence all $\mu_i(\alpha)$'s can be computed in terms of $\dot{u}(t_i(\alpha)^\pm,\alpha))$.

ii) Since $\mu_{-N}(\alpha) \neq 0$ and all $B_i(\alpha)$, $B_{-i}(\alpha)$ are invertible, we see that $\mu_{\pm i}(\alpha) \neq 0$ for all $i = 0, \ldots, N$.

The next Proposition, proved in [6], states that in some circumstances all $\mu_i(\alpha)$'s are equal. This case is of particular interest, since then we can take $\psi(t,\alpha) = v(t,\alpha)$ and the Melnikov condition reads

$$\Delta(\alpha_0) = 0, \quad \operatorname{rank} \Delta'(\alpha_0) = d$$

where

$$\Delta(\alpha) := \int_{-\infty}^{\infty} e^{-\int_0^t a_{11}(s,\alpha)+a_{22}(s,\alpha)ds} \begin{pmatrix} -\dot{u}_2(t,\alpha) \\ \dot{u}_1(t,\alpha) \end{pmatrix}^T \begin{pmatrix} u_{1,\alpha}(t,\alpha) \\ u_{2,\alpha}(t,\alpha) \end{pmatrix} g(u(t,\alpha),\alpha,0)dt.$$

If, moreover, $a_{11}(t,\alpha) + a_{22}(t,\alpha) = 0$ we have

$$\Delta(\alpha) = \int_{-\infty}^{\infty} [\dot{u}_1(t,\alpha)u_{2,\alpha}(t,\alpha) - \dot{u}_2(t,\alpha)u_{1,\alpha}(t,\alpha)] g(u(t,\alpha),\alpha,0)dt. \tag{7.6}$$

Proposition 7.3. *Equations* (7.5) *are satisfied with* $\mu_i = 1$, $i = -N, \ldots, N$, *if and only if there exist* $\nu_i^{\pm}(\alpha)$, $i = 1, \ldots, N$, *such that*

$$J[f_{i+1}(u(t_{\pm i}(\alpha), \alpha), \alpha) - f_i(u(t_{\pm i}(\alpha), \alpha), \alpha)] = \nu_i(\alpha)^{\pm} h_x(u(t_{\pm i}(\alpha), \alpha), \alpha)^T. \tag{7.7}$$

For example, suppose

$$h(x, y) = x_k$$

where either $k = 1$ or $k = 2$. Recalling that

$$f_i(x, y) = \begin{pmatrix} F_1^i(x_1, x_2, y) \\ F_2^i(x_1, x_2, y) \end{pmatrix}$$

we get, omitting arguments (that can be either $(u(t_{-i}), y_0)$ or $(u(t_i), y_0)$) for simplicity:

$$J[f_{i+1} - f_i] = \begin{pmatrix} F_2^i - F_2^{i+1} \\ F_1^{i+1} - F_1^i \end{pmatrix}$$

and then (7.7) holds if and only if

$$F_k^i(u(t_{\pm i}(\alpha)), \alpha) = F_k^{i+1}(u(t_{\pm i}(\alpha)), \alpha) \tag{7.8}$$

for all $i = 1, \ldots, N$.

As a concrete example we consider the following two dimensional equation (see [8]):

$$\ddot{x} = \lambda p(t) \Phi(x)$$

where $\lambda \gg 1$ is a large parameter, $p(t) > 0$ is a positive, C^2, periodic function, and $\Phi(x)$ is a piecewise C^2 function such that

$$\Phi(x) = \begin{cases} \Phi_-(x) & \text{if } x < \frac{1}{2}, \\ \Phi_+(x) & \text{if } x > \frac{1}{2}. \end{cases}$$

Then $h(x_1, x_2) = x_1$ and the discontinuity manifold is $\mathcal{S} = \{x_1 = \frac{1}{2}\}$.

Taking $\lambda = \varepsilon^{-2}$ and changing the time scale $t \mapsto \varepsilon^{-1} t$, the equation reads

$$\begin{aligned} \ddot{x} &= p(y) \Phi(x), \\ \dot{y} &= \varepsilon \end{aligned} \tag{7.9}$$

or

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= p(y) \Phi(x_1), \\ \dot{y} &= \varepsilon. \end{aligned}$$

We assume that $x = 0$, $\dot{x} = 0$ is a hyperbolic fixed point of equation $\ddot{x} = \Phi(x)$ with an associated solution $(u(t), \dot{u}(t))$, homoclinic to $(0, 0)$ and such that

$$\begin{aligned} 0 < u(t) < \frac{1}{2} &\quad \text{for } t < t_- \text{ or } t > t_+, \\ u(t) > \frac{1}{2} &\quad \text{for } t_- < t < t_+, \\ u(t_+) = u(t_-) = \frac{1}{2}, & \\ \dot{u}(t_\pm) \neq 0. & \end{aligned} \tag{7.10}$$

Then (7.9) has the family of homoclinic solutions

$$(u(t,\alpha),\dot{u}(t,\alpha)) = (u(t\sqrt{p(\alpha)}), \sqrt{p(\alpha)}\dot{u}(t\sqrt{p(\alpha)}))$$

that satisfy assumptions A_1)–A_4). Note that, according to assumption we have

$$t_\pm(\alpha) = \frac{t_\pm}{\sqrt{p(\alpha)}}$$

and

$$\left|\begin{pmatrix} u(t,\alpha) \\ \dot{u}(t,\alpha) \end{pmatrix}\right| \leq \sqrt{1+p(\alpha)}e^{-\delta t\sqrt{p(\alpha)}} \leq \sqrt{1+p_{\max}}e^{-\delta t\sqrt{p_{\min}}}$$

where $0 < p_{\min} := \min\{p(\alpha)\} < \max\{p(\alpha)\} := p_{\max}$.

As $F_1(x_1,x_2,y) = x_2$ is continuous and $h(x_1,x_2,y) = x_1$ we see that (7.8) is certainly satisfied. Then

$$\begin{aligned}\Delta(\alpha) = \int_{-\infty}^{\infty} \sqrt{p(\alpha)}\dot{u}(t\sqrt{p(\alpha)}) \left[\frac{p'(\alpha)}{2\sqrt{p(\alpha)}}\dot{u}(t\sqrt{p(\alpha)}) + \sqrt{p(\alpha)}\ddot{u}(t\sqrt{p(\alpha)})\frac{tp'(\alpha)}{2\sqrt{p(\alpha)})}\right] \\ - p(\alpha)\ddot{u}(t\sqrt{p(\alpha)}\dot{u}(t\sqrt{p(\alpha)})\frac{tp'(\alpha)}{2\sqrt{p(\alpha)})}dt = \frac{1}{2}p'(\alpha)\int_{-\infty}^{\infty} \dot{u}(t\sqrt{p(\alpha)})^2 dt \\ = \frac{p'(\alpha)}{2\sqrt{p(\alpha)}}\int_{-\infty}^{\infty} \dot{u}(t)^2 dt.\end{aligned}$$

As a consequence $\Delta(\alpha)$ has a simple zero at $\alpha = \alpha_0$ if and only if $p(\alpha)$ has a non degenerate critical point at $\alpha = \alpha_0$. From Theorem 6.2 we conclude with the following

Proposition 7.4. *Let $\Phi_\pm(x)$ be C^2-functions and suppose that*

$$\ddot{x} = \Phi(x) := \begin{cases} \Phi_-(x) & \text{if } 0 < x < \frac{1}{2}, \\ \Phi_+(x) & \text{if } x > \frac{1}{2} \end{cases}$$

has the hyperbolic fixed point $(u,\dot{u}) = (0,0)$ together with a homoclinic orbit such that (7.10) holds. Then, if $p(\alpha)$ is a periodic C^2-functions having a non-degenerate maximum (or minimum) at $\alpha = \alpha_0$ then there exists $\lambda_0 \gg 1$ and a unique, C^1, $\alpha(\lambda)$ such that $\lim_{\lambda\to\infty}\alpha(\lambda) = \alpha_0$ and for $\lambda > \lambda_0$ the perturbed equation

$$\ddot{x} = \lambda p(t)\Phi(x)$$

has a solution $(x(t,\lambda),\dot{x}(t,\lambda))$ such that

$$\sup_{t\in\mathbb{R}} |x(t,\lambda) - u(t, t\lambda^{-\frac{1}{2}} + \alpha(\lambda))| \to 0,$$
$$\sup_{t\in\mathbb{R}} |\dot{x}(t,\lambda) - \dot{u}(t, t\lambda^{-\frac{1}{2}} + \alpha(\lambda)))| \to 0$$

as $\lambda \to \infty$.

As a concrete example we take

$$\Phi_\pm(x) = \mp x, \quad p(y) = 2 + \sin y$$

so that system (7.9) reads

$$\begin{aligned} \ddot{x} &= (2+\sin(y))x, && x < \frac{1}{2}, \\ \ddot{x} &= -(2+\sin(y))x, && x > \frac{1}{2}, \\ \dot{y} &= \varepsilon. \end{aligned} \tag{7.11}$$

The homoclinic solution of the frozen system ($\varepsilon = 0$) is

$$u(t) = \begin{cases} \frac{e^{\frac{\pi}{4}}}{2} e^t & \text{if } t \le -\frac{\pi}{4}, \\ \frac{1}{\sqrt{2}} \cos t & \text{if } -\frac{\pi}{4} \le t \le \frac{\pi}{4}, \\ \frac{e^{\frac{\pi}{4}}}{2} e^{-t} & \text{if } t \ge \frac{\pi}{4}. \end{cases}$$

Solving $p'(\alpha) = \cos\alpha = 0$, we get $\alpha_1 = \frac{\pi}{2}$ and $\alpha_2 = \frac{3\pi}{2}$ with $p''(\alpha_{1,2}) = -\sin\alpha_{1,2} = \mp 1 \neq 0$. Now we add some numerical figures of solutions of (7.11) near

$$v(t) = u\left(t\sqrt{2+\sin(\alpha_1+\varepsilon t)}\right)$$

and

$$w(t) = u\left(t\sqrt{2+\sin(\alpha_2+\varepsilon t)}\right)$$

for ε small, say $\varepsilon = 0.1$ and for

$$y(0) \sim \alpha_{1,2}. \tag{7.12}$$

Note

$$v(0) = w(0) = u(0) = \frac{1}{\sqrt{2}}, \qquad \dot{v}(0) = \dot{w}(0) = \dot{u}(0) = 0.$$

Here we draw some pictures of the solutions of equation (7.11) where we take $y(0) = \frac{\pi}{2} \pm 0.05$. In all these pictures we take $\varepsilon = 0.1$. Figures 7.2–7.5 in the paper show the curves of $(t, x(t))$

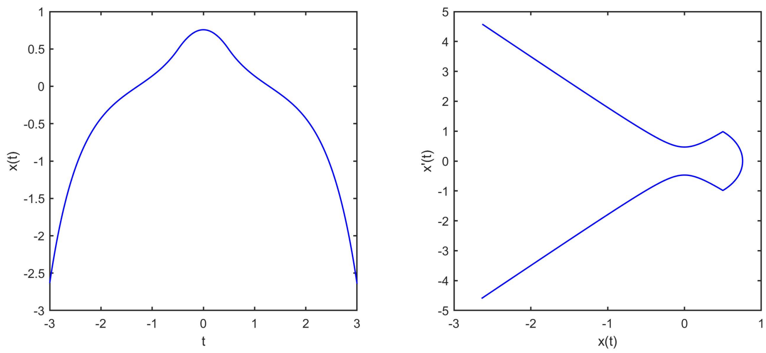

Figure 7.1: The plot of $(v(t), \dot{v}(t))$ for $t \in (-10, 10)$.

and $(x(t), x'(t))$ corresponding to $\varepsilon = 0.1$.

In the case of keeping initial conditions unchanged but taking $\varepsilon = 0.01$, we have the following Figures 7.6–7.9.

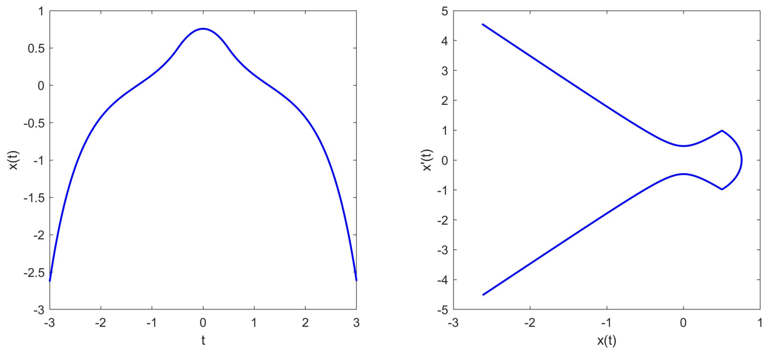

Figure 7.2: The plot of $(t, x(t))$ and $(x(t), \dot{x}(t))$ for $t = [-3, 3]$ with $y(0) = \frac{\pi}{2} + 0.05$, $x(0) = \frac{1}{\sqrt{2}} + 0.05$, $\dot{x}(0) = 0$. Note that the solution escapes very quickly from a neighbourhood of the fixed point $x = \dot{x} = 0$ as $t \to \pm\infty$.

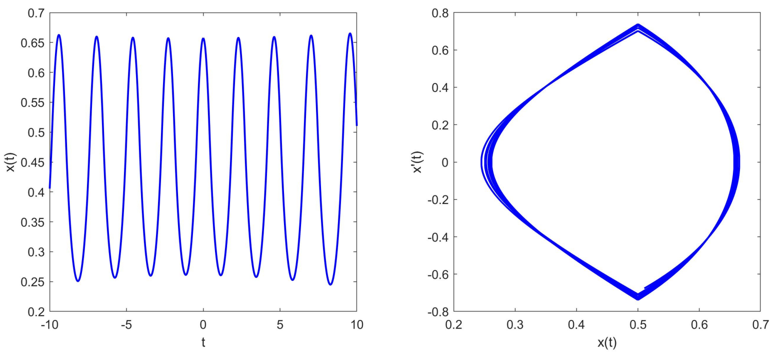

Figure 7.3: The plot of $(t, x(t))$ and $(x(t), \dot{x}(t))$ for $t = [-10, 10]$ with $y(0) = \frac{\pi}{2} + 0.05$, $x(0) = \frac{1}{\sqrt{2}} - 0.05$, $\dot{x}(0) = 0$. Here the solution looks like a periodic solution in the bounded domain $0.2 \leq x \leq 0.7$, $-0.8 \leq \dot{x} \leq 0.8$.

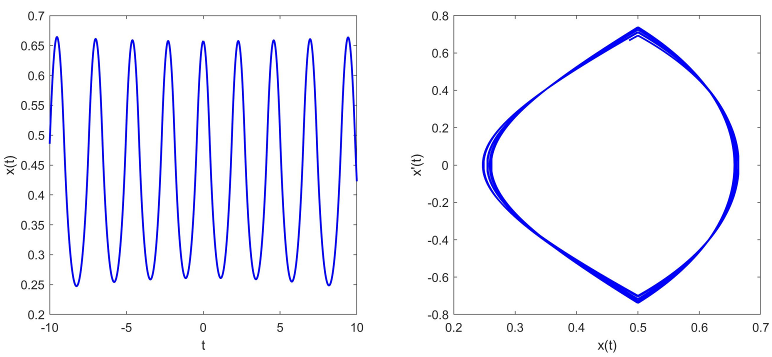

Figure 7.4: The plot of $(t, x(t))$ and $(x(t), \dot{x}(t))$ for $t = [-10, 10]$ with $y(0) = \frac{\pi}{2} - 0.05$, $x(0) = \frac{1}{\sqrt{2}} - 0.05$, $\dot{x}(0) = 0$. Also in this case the solution looks like a periodic solution in the bounded domain $0.2 \leq x \leq 0.7$, $-0.8 \leq \dot{x} \leq 0.8$.

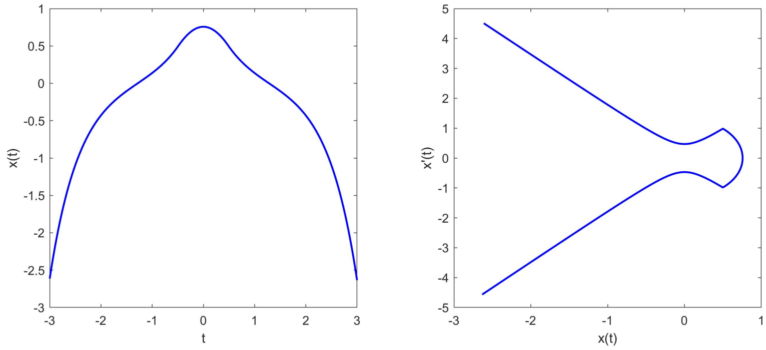

Figure 7.5: The plot of $(t, x(t))$ and $(x(t), \dot{x}(t))$ for $t = [-3, 3]$ with $y(0) = \frac{\pi}{2} - 0.05$, $x(0) = \frac{1}{\sqrt{2}} + 0.05$, $\dot{x}(0) = 0$. For these initial values the solution escapes very quickly from a neighbourhood of the fixed point $x = \dot{x} = 0$ as $t \to \pm\infty$.

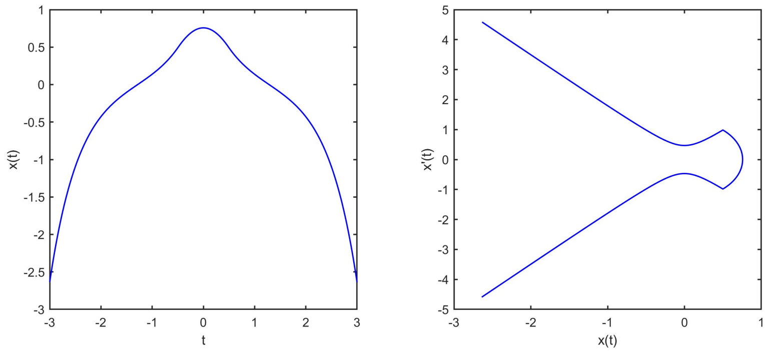

Figure 7.6: Corresponding to the case of $\varepsilon = 0.01$ in Figure 7.2.

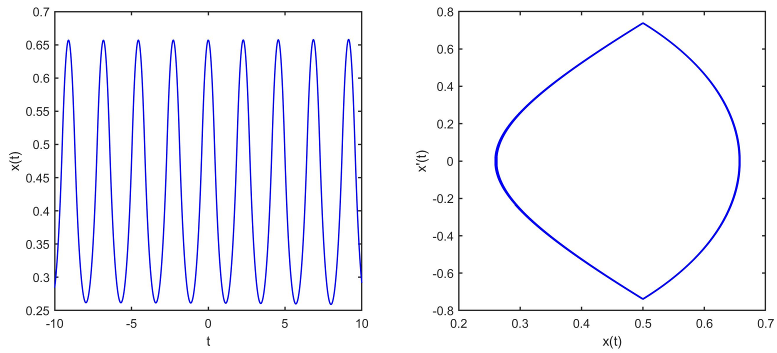

Figure 7.7: Corresponding to the case of $\varepsilon = 0.01$ in Figure 7.3.

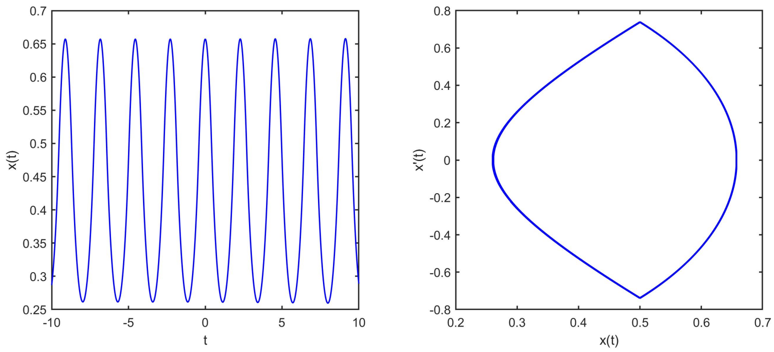

Figure 7.8: Corresponding to the case of $\varepsilon = 0.01$ in Figure 7.4.

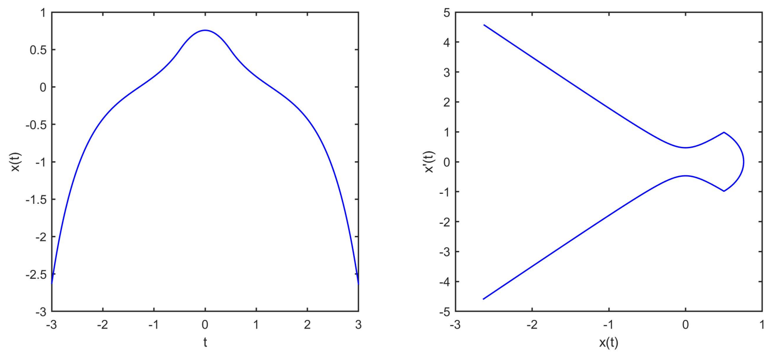

Figure 7.9: Corresponding to the case of $\varepsilon = 0.01$ in Figure 7.5.

8 Concluding remark

According to the results in [4], with the correction given in [5], the Melnikov function in the periodic case, with two discontinuity points and a family of periodic solutions $u(t,\alpha)$ of the unperturbed equation, is

$$\int_{-T(\alpha)/2}^{T(\alpha)/2} \psi_j(t,\alpha)^T f_y(u(t,\alpha),\alpha) y_\varepsilon(t,0,\alpha,0)\,dt \\ + \psi_j(t_*(\alpha)^+,\alpha)^T \frac{h_{y,*} y_{\varepsilon,*}}{h_{x,*} f_{+,*}} (f_{-,*} - f_{+,*}) + \psi_j(t^*(\alpha)^+,\alpha)^T \frac{h_y^* y_\varepsilon^*}{h_x^* f_-^*} (f_+^* - f_-^*).$$

In the following we prove that $-M(\alpha)$ extends the above expression to the heteroclinic case (i.e. with ∞ replacing $T(\alpha)$) with several discontinuity points.

Differentiating $\dot u(t,y) = f(u(t,y),y)$ with respect to y we see that, for $t \neq t_{\pm i}(\alpha)$, $i = 1,\dots,N$:

$$\dot u_y(t,\alpha) = A(t,\alpha) u_y(t,\alpha) + f_y(u(t,\alpha),\alpha)$$

and then

$$\frac{d}{dt}(u_y(t,\alpha) y_\varepsilon(t,0,\alpha,0)) = \dot u_y(t,\alpha) y_\varepsilon(t,0,\alpha,0) + u_y(t,\alpha) \dot y_\varepsilon(t,0,\alpha,0) \\ = A(t,\alpha) u_y(t,\alpha) y_\varepsilon(t,0,\alpha,0) + f_y(u(t,\alpha),\alpha) y_\varepsilon(t,0,\alpha,0) + u_y(t,\alpha) g(u(t,\alpha),\alpha).$$

So

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi_j^T(t,\alpha)u_y(t,\alpha)g(u(t,\alpha),\alpha,0)dt \\
&\quad = \int_{-\infty}^{\infty} \psi_j^T(t,\alpha)\left[\frac{d}{dt}(u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)) - A(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)\right. \\
&\qquad \left. - f_y(u(t,\alpha),\alpha)y_\varepsilon(t,0,\alpha,0)\right]dt \\
&\quad = \int_{-\infty}^{\infty} \psi_j^T(t,\alpha)\frac{d}{dt}(u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)) + \dot{\psi}_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0) \\
&\qquad - \psi_j^T(t,\alpha)f_y(u(t,\alpha),\alpha)y_\varepsilon(t,0,\alpha,0)dt \\
&\quad = \int_{-\infty}^{\infty} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)] - \psi_j^T(t,\alpha)f_y(u(t,\alpha),\alpha)y_\varepsilon(t,0,\alpha,0)dt.
\end{aligned}$$

Then the j-th component of $-M(\alpha)$, say $-M_j(\alpha)$, is

$$\begin{aligned}
-M_j(\alpha) = &\int_{-\infty}^{\infty} \psi_j^T(t,\alpha)f_y(u(t,\alpha),\alpha)y_\varepsilon(t,0,\alpha,0)dt \\
&- \int_{-\infty}^{\infty} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt.
\end{aligned}$$

Using the continuity of $y_\varepsilon(t,\xi,\alpha,\eta)$ we get:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt \\
&\quad = \int_{-\infty}^{t_{-N}(\alpha)} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt \\
&\qquad + \sum_{i=-N}^{N-1} \int_{t_i(\alpha)}^{t_{i+1}(\alpha)} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt \\
&\qquad + \int_{t_N(\alpha)}^{\infty} \frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt \\
&\quad = \psi_j^T(t_{-N}(\alpha)^-,\alpha)u_y(t_{-N}(\alpha)^-,\alpha)y_\varepsilon(t_{-N}(\alpha),0,\alpha,0) \\
&\qquad + \sum_{i=-N}^{N-1} \Big[\psi_j^T(t_{i+1}(\alpha)^-,\alpha)u_y(t_{i+1}(\alpha)^-,\alpha)y_\varepsilon(t_{i+1}(\alpha),0,\alpha,0) \\
&\qquad - \psi_j^T(t_i(\alpha)^+,\alpha)u_y(t_i(\alpha)^+,\alpha)y_\varepsilon(t_i(\alpha),0,\alpha,0)\Big] \\
&\qquad - \psi_j^T(t_N(\alpha)^+,\alpha)u_y(t_N(\alpha)^+,\alpha)y_\varepsilon(t_N(\alpha),0,\alpha,0) \\
&\quad = \psi_j^T(t_{-N}(\alpha)^-,\alpha)u_y(t_{-N}(\alpha)^-,\alpha)y_\varepsilon(t_{-N}(\alpha),0,\alpha,0) \\
&\qquad + \sum_{i=-N+1}^{N} \psi_j^T(t_i(\alpha)^-,\alpha)u_y(t_i(\alpha)^-,\alpha)y_\varepsilon(t_i(\alpha),0,\alpha,0) \\
&\qquad - \sum_{i=-N}^{N-1} \psi_j^T(t_i(\alpha)^+,\alpha)u_y(t_i(\alpha)^+,\alpha)y_\varepsilon(t_i(\alpha),0,\alpha,0) \\
&\qquad - \psi_j^T(t_N(\alpha)^+,\alpha)u_y(t_N(\alpha)^+,\alpha)y_\varepsilon(t_N(\alpha),0,\alpha,0)
\end{aligned}$$

$$
\begin{aligned}
&= \psi_j^T(t_{-N}(\alpha)^-,\alpha)u_y(t_{-N}(\alpha)^-,\alpha)y_\varepsilon(t_{-N}(\alpha),0,\alpha,0)\\
&\quad + \sum_{i=-N+1}^{N-1}[\psi_j^T(t_i(\alpha)^-,\alpha)u_y(t_i(\alpha)^-,\alpha) - \psi_j^T(t_i(\alpha)^+,\alpha)u_y(t_i(\alpha)^+,\alpha)]y_\varepsilon(t_i(\alpha),0,\alpha,0)\\
&\quad + \psi_j^T(t_N(\alpha)^-,\alpha)u_y(t_N(\alpha)^-,\alpha)y_\varepsilon(t_N(\alpha),0,\alpha,0)\\
&\quad - \psi_j^T(t_{-N}(\alpha)^+,\alpha)u_y(t_{-N}(\alpha)^+,\alpha)y_\varepsilon(t_{-N}(\alpha),0,\alpha,0)\\
&\quad - \psi_j^T(t_N(\alpha)^+,\alpha)u_y(t_N(\alpha)^+,\alpha)y_\varepsilon(t_N(\alpha),0,\alpha,0)\\
&= \sum_{i=-N}^{N}[\psi_j^T(t_i(\alpha)^-,\alpha)u_y(t_i(\alpha)^-,\alpha) - \psi_j^T(t_i(\alpha)^+,\alpha)u_y(t_i(\alpha)^+,\alpha)]y_\varepsilon(t_i(\alpha),0,\alpha,0)\\
&= \sum_{i=-N,i\neq 0}^{N}[\psi_j^T(t_i(\alpha)^-,\alpha)u_y(t_i(\alpha)^-,\alpha) - \psi_j^T(t_i(\alpha)^+,\alpha)u_y(t_i(\alpha)^+,\alpha)]y_\varepsilon(t_i(\alpha),0,\alpha,0).
\end{aligned}
$$

The last equality follows from the fact that $\psi_j(t,\alpha)$ and $u_y(t,\alpha)$ are continuous at $t = t_0(\alpha) = 0$.

Next, from (3.6)–(6.11) we see that, for any $\ell = \pm 1,\dots,\pm N$, we have

$$\psi_j^T(t_\ell(\alpha)^-,\alpha) = \psi_j^T(t_\ell(\alpha)^+,\alpha)B_\ell(\alpha)$$

Hence:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\frac{d}{dt}[\psi_j^T(t,\alpha)u_y(t,\alpha)y_\varepsilon(t,0,\alpha,0)]dt\\
&\qquad \sum_{\ell=-N,\ell\neq 0}^{N}\psi_j^T(t_\ell(\alpha)^+,\alpha)[B_\ell(\alpha)u_y(t_\ell(\alpha)^-,\alpha) - u_y(t_\ell(\alpha)^+,\alpha)]y_\varepsilon(t_\ell(\alpha),0,\alpha,0).
\end{aligned}
$$

From (5.1) we obtain

$$
\begin{aligned}
&B_\ell(\alpha)u_y(t_\ell(\alpha)^-,\alpha) - u_y(t_\ell(\alpha)^-,\alpha)\\
&\qquad = -\frac{h_x(u(t_\ell(\alpha),\alpha),\alpha)u_y(t_\ell(\alpha)^-,\alpha)}{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot u(t_\ell(\alpha)^-,\alpha),\alpha)}(\dot u(t_\ell(\alpha)^-,\alpha) - \dot u(t_\ell(\alpha)^+,\alpha)).
\end{aligned}
$$

Differentiating $h(u(t_\ell(\alpha),\alpha),\alpha) = c_{|\ell|}$ with respect to α we get

$$
\begin{aligned}
&h_x(u(t_\ell(\alpha),\alpha),\alpha)u_y(t_\ell(\alpha)^-,\alpha)\\
&\qquad = -h_x(u(t_\ell(\alpha),\alpha),y)\dot u(t_\ell(\alpha)^-,\alpha)t'_\ell(\alpha) - h_y(u(t_\ell(\alpha),\alpha),\alpha)
\end{aligned}
$$

and then

$$
\begin{aligned}
&B_\ell(\alpha)u_y(t_\ell(\alpha)^-,\alpha) - u_y(t_\ell(\alpha)^-,\alpha)\\
&\quad = \frac{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot u(t_\ell(\alpha)^-,y)t'_\ell(\alpha) + h_y(u(t_\ell(\alpha),\alpha),\alpha)}{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot u(t_\ell(\alpha)^-,\alpha)}(\dot u(t_\ell(\alpha)^-,\alpha) - \dot u(t_\ell(\alpha)^+,\alpha))\\
&\quad = \left[t'_\ell(\alpha) + \frac{h_y(u(t_\ell(\alpha),\alpha),\alpha)}{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot u(t_\ell(\alpha)^-,\alpha)}\right](\dot u(t_\ell(\alpha)^-,\alpha) - \dot u(t_\ell(\alpha)^+,\alpha)).
\end{aligned}
$$

So

$$
\begin{aligned}
&B_\ell(\alpha)u_y(t_\ell(\alpha)^-,\alpha) - u_y(t_\ell(\alpha)^+,\alpha)\\
&\qquad = u_y(t_\ell(\alpha)^-,\alpha) - u_y(t_\ell(\alpha)^+,\alpha) + t'_\ell(\alpha)[\dot u(t_\ell(\alpha)^-,\alpha) - \dot u(t_\ell(\alpha)^+,\alpha)]\\
&\qquad\quad + \frac{h_y(u(t_\ell(\alpha),\alpha),\alpha)}{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot u(t_\ell(\alpha)^-,\alpha)}(\dot u(t_\ell(\alpha)^-,\alpha) - \dot u(t_\ell(\alpha)^+,\alpha)).
\end{aligned}
\tag{8.1}
$$

Next, for $t_\ell(\alpha) < t < t_{\ell+1}(\alpha)$ we have:

$$u(t,\alpha) = u(t_\ell(\alpha)^-,\alpha) + \int_{t_\ell(\alpha)}^{t} \dot{u}(s,\alpha)ds.$$

Hence

$$u_y(t_\ell(\alpha)^+,\alpha) = \dot{u}(t_\ell(\alpha)^-,\alpha)t'_\ell(\alpha) + u_y(t_\ell(\alpha)^-,\alpha) - \dot{u}(t_\ell(\alpha)^+,\alpha)t'_\ell(\alpha)$$

and then

$$u_y(t_\ell(\alpha)^-,\alpha) - u_y(t_\ell(\alpha)^+,\alpha) = [\dot{u}(t_\ell(\alpha)^+,\alpha) - \dot{u}(t_\ell(\alpha)^-,\alpha)]t'_\ell(\alpha). \tag{8.2}$$

Plugging (8.2) into (8.1) we finally obtain:

$$\begin{aligned}&[B_\ell(\alpha)u_y(t_\ell(\alpha)^-,\alpha) - u_y(t_\ell(\alpha)^+,\alpha)]y_\varepsilon(t_\ell(\alpha),0,\alpha,0)\\ &\quad = \frac{h_y(u(t_\ell(\alpha),\alpha),\alpha)y_\varepsilon(t_\ell(\alpha),0,\alpha,0)}{h_x(u(t_\ell(\alpha),\alpha),\alpha)\dot{u}(t_\ell(\alpha)^-,\alpha)}(\dot{u}(t_\ell(\alpha)^-,\alpha) - \dot{u}(t_\ell(\alpha)^+,\alpha))\end{aligned}$$

Putting everything together we finally get:

$$\begin{aligned}-M_j(\alpha) = &\int_{-\infty}^{\infty} \psi_j^T(t,\alpha)f_y(u(t,\alpha),\alpha)y_\varepsilon(t,0,\alpha,0)dt\\ &+ \sum_{i=1}^{N} \psi_j^T(t_{-i}(\alpha)^+,\alpha)\frac{h_y(u(t_{-i}(\alpha),\alpha),\alpha)y_\varepsilon(t_{-i}(\alpha),0,\alpha,0)}{h_x(u(t_{-i}(\alpha),\alpha),\alpha)f_{i+1}(u(t_{-i}(\alpha),\alpha),\alpha)}\\ &\cdot (f_i(u(t_{-i}(\alpha),\alpha),\alpha) - f_{i+1}(u(t_{-i}(\alpha),\alpha),\alpha))\\ &+ \sum_{i=1}^{N} \psi_j^T(t_i(\alpha)^+,\alpha)\frac{h_y(u(t_i(\alpha),\alpha),\alpha)y_\varepsilon(t_i(\alpha),0,\alpha,0)}{h_x(u(t_i(\alpha),\alpha),\alpha)f_i(u(t_i(\alpha),\alpha),\alpha)}\\ &\cdot (f_{i+1}(u(t_i(\alpha),\alpha),\alpha) - f_i(u(t_i(\alpha),\alpha),\alpha)).\end{aligned} \tag{8.3}$$

This completes the proof that $-M(\alpha)$ extends the Melnikov function for the periodic case with two discontinuity points to the heteroclinic case with a finite number of discontinuity points.

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A remark on energy balance of the Navier–Stokes equations

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Abstract. In this short note, we provide a self-contained proof for the criterion $\nabla u \in L^{\frac{5}{2}}(0,T;L^2(\mathbb{R}^3))$ to Navier–Stokes equations energy balance, which improves some recent results on this problem.

Keywords: Navier–Stokes equations, energy equality, distributional solutions.

2020 Mathematics Subject Classification: 35Q35, 76D03, 35B65.

1 Introduction

We are interested in the energy balance of distributional solutions to Navier–Stokes equations

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta u + \nabla p = 0, & x\in\mathbb{R}^3, t>0 \\ \nabla\cdot u = 0, & x\in\mathbb{R}^3, t>0 \\ u|_{t=0} = u_0(x), & x\in\mathbb{R}^3. \end{cases} \tag{1.1}$$

It is well known since the work of Leray [8] and Hopf [6], that for any $u_0 \in L^2_\sigma(\mathbb{R}^3)$ one can construct a global weak solutions to (1.1), namely, a function u that, for each $T>0$, is in the class

$$u \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3)) \tag{1.2}$$

and solves (1.1) in a distributional sense. Here, $L^2_\sigma(\mathbb{R}^3)$ is the subspace of $L^2(\mathbb{R}^3)$ of divergence-free vector functions. In addition, such a u satisfies the so-called energy inequality:

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(\tau)\|^2_{L^2} d\tau \le \|u_0\|^2_{L^2}, \qquad \forall\, t\ge 0. \tag{1.3}$$

Much about the solutions of the Navier–Stokes equation is unknown, including uniqueness and regularity. The main barrier is the fact that the energy equality, which states that for any smooth solution u, it obeys the following energy balance:

$$\|u(t)\|^2_{L^2} + 2\int_0^t \|\nabla u(\tau)\|^2_{L^2} d\tau = \|u_0\|^2_{L^2}, \qquad \forall\, t\ge 0. \tag{1.4}$$

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A natural question immediately arises: does any Leray–Hopf weak (distributional) solution of the Navier–Stokes equations automatically satisfy the energy balance (1.4)? To date this question remains open, and only conditional results are available.

Energy equality is clearly a prerequisite for regularity, and can be a first step in proving conditional regularity results [2,3,11,12]. Lions [9] and Ladyzhenskaya [7] proved independently that a Leray–Hopf weak solution satisfy the (global) energy equality (1.4) under the additional assumption $u \in L^4L^4$. Shinbrot [13] generalized the Lions–Ladyzhenskaya condition to

$$u \in L^r(0,T;L^s(\mathbb{R}^d)) \quad \text{with} \quad \frac{2}{r}+\frac{2}{s} \leq 1, s \geq 4. \tag{1.5}$$

Recently, Yu [14] given a new proof to the Shinbrot energy conservation criterion. In addition, Berselli and Chiodaroli in [1] prove some new energy balance criteria in terms of the gradient of the velocity. Specially, they showed that

$$\nabla u \in L^{\frac{5}{2}}(0,T;L^2(\Omega)) \tag{1.6}$$

can ensure the energy identity.

From the PDEs point of view, it is significant to study the motion of distributional (very weak) solutions of fluid equations, see Definition 1.1. In this regard, there is not any available regularity on velocity field u, apart the solution being in $L^2_{\text{loc}}\left(\mathbb{R}^3 \times [0,T)\right)$. Recently, The famous mathematician Giovanni P. Galdi [4,5] systematically studied the relation between very weak and Leray–Hopf solutions to Navier–Stokes equations, and he first proved that if distributional solution in $L^4(0,T;L^4(\mathbb{R}^3))$, and with initial data u_0 in $L^2(\mathbb{R}^3)$, then energy equality (1.4) holds true, in particular, he emphasized that the requirement (1.2) is entirely redundant. The key observation is the use of the duality argument and the above conditions to improve the regularity of the solution (i.e., $L^\infty\left(0,T;L^2\left(\mathbb{R}^3\right)\right) \cap L^2\left(0,T;H^1\left(\mathbb{R}^3\right)\right)$).

As everyone knows, in fluid mechanics, the gradient of velocity(∇u) is an important physical quantity. Objective of this note is to prove that control the gradient of velocity, i.e., $\nabla u \in L^{\frac{5}{2}}(L^2(\mathbb{R}^3))$ along with the (necessary) condition $u_0 \in L^2_\sigma(\mathbb{R}^3)$ can ensure the energy balance. More precisely, setting

$$\mathcal{D}_T := \{\varphi \in C_0^\infty(\mathbb{R}^3 \times [0,T)) : \ \operatorname{div} \varphi = 0\}.$$

Definition 1.1 (Distributional solution)**.** Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, $T > 0$. The function $u \in L^2_{\text{loc}}\left(\mathbb{R}^3 \times [0,T)\right)$ is a distributional solution to the Navier–Stokes equations (1.1) if

1. for any $\Phi \in \mathcal{D}_T$, we have

$$\int_0^T \int_{\mathbb{R}^3} u \cdot (\partial_t \Phi + \Delta \Phi + u \cdot \nabla \Phi) dx dt = -\int_{\mathbb{R}^3} u(x,0) \cdot \Phi(x,0) dx;$$

2. for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, it holds that

$$\int_{\mathbb{R}^3} u \cdot \nabla \varphi dx = 0,$$

for a.e. $t \in (0,T)$.

We will show the following.

Theorem 1.2. *Suppose that* $u \in L^2_{\text{loc}}\left(\mathbb{R}^3 \times [0,T)\right)$ *be a distributional solution in the sense of Definition 1.1 to the Navier–Stokes (1.1). If*

$$\nabla u \in L^{\frac{5}{2}}\left(0,T;L^2(\mathbb{R}^3)\right),$$

then

$$\int_{\mathbb{R}^3} |u(t,x)|^2 dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u(x,\tau)|^2 dx d\tau = \int_{\mathbb{R}^3} |u_0|^2 \, dx$$

for any $t \in [0,T]$.

Remark 1.3. From a purely mathematical perspective, it seems that a new strategy for studying the energy balance of distributional solutions based on gradient of velocity, which may be applied to other incompressible fluid equations.

Remark 1.4. Note that this result is supercritical with respect to the Prodi–Serrin scaling since $\frac{3}{p} + \frac{2}{q} = \frac{3}{2} + \frac{4}{5} > 2$, showing that the energy balance holds even if one does not expect the full regularity of solutions to hold.

Remark 1.5. Berselli and Chiodaroli [1] obtained energy equality via $\nabla u \in L^{\frac{5}{2}}\left(0,T;L^2(\Omega)\right)$, however, we want to emphasize is that the finite energy ($u \in L^\infty L^2 \cap L^2 H^1$) plays a key role in their proof.

2 Proof of Theorem 1.2

This section is devoted to proof of Theorem 1.2. For the sake of simplicity, we will proceed as if the solution is differentiable in time. The extra arguments needed to mollify in time are straightforward.

Let $\eta : \mathbb{R}^3 \to \mathbb{R}$ be a standard mollifier, i.e. $\eta(x) = Ce^{\frac{1}{|x|^2-1}}$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| \geqslant 1$, where constant $C > 0$ selected such that $\int_{\mathbb{R}^3} \eta(x)\mathrm{d}x = 1$. For any $\varepsilon > 0$, we define the rescaled mollifier $\eta_\varepsilon(x) = \varepsilon^{-3}\eta\left(\frac{x}{\varepsilon}\right)$. For any function $f \in L^1_{\text{loc}}(\mathbb{R}^3)$, its mollified version is defined as

$$f^\varepsilon(x) = (f * \eta_\varepsilon)(x) = \int_{\mathbb{R}^3} \eta_\varepsilon(x-y)f(y)\mathrm{d}y.$$

If $f \in W^{1,p}(\mathbb{R}^3)$, the following local approximation is well known

$$f^\varepsilon(x) \to f \quad \text{in } W^{1,p}_{\text{loc}}(\mathbb{R}^3) \quad \forall p \in [1,\infty).$$

The crucial ingredient to prove Theorem 1.2 is the following lemmas.

Lemma 2.1 ([10])**.** *Let* ∂ *be a partial derivative in one direction. Let* $f, \partial f \in L^p\left(\mathbb{R}^+ \times \mathbb{R}^3\right)$, $g \in L^q\left(\mathbb{R}^+ \times \mathbb{R}^d\right)$ *with* $1 \leq p,q \leq \infty$, *and* $\frac{1}{p} + \frac{1}{q} \leq 1$. *Then, we have*

$$\|\partial(fg) * \eta_\varepsilon - \partial\left(f(g*\eta_\varepsilon)\right)\|_{L^r(\mathbb{R}^+\times\mathbb{R}^3)} \leq C\|\partial f\|_{L^p\left(\mathbb{R}^+\times\mathbb{R}^d\right)}\|g\|_{L^q(\mathbb{R}^+\times\mathbb{R}^3)}$$

for some constant $C > 0$ *independent of* ε, f *and* g, *and with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *In addition,*

$$\partial(fg) * \eta_\varepsilon - \partial\left(f(g*\eta_\varepsilon)\right) \to 0 \quad \text{in } L^r\left(\mathbb{R}^+ \times \mathbb{R}^3\right)$$

as $\varepsilon \to 0$, *if* $r < \infty$.

Lemma 2.2. *Let $u_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and let u be a distributional solution in the sense of Definition 1.1 to the Navier–Stokes equations (1.1) and satisfies*

$$\nabla u \in L^{\frac{5}{2}}\left(0, T; L^2(\mathbb{R}^3)\right),$$

then we have

$$\sup_{t\geq 0} \|u^\varepsilon(\cdot,t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx d\tau \leq K, \quad \forall\, t \in [0,T],$$

where K is a constant depending only on $\|u_0\|_{L^2}$ and $\int_0^T \|\nabla u\|_{L^2}^{\frac{5}{2}} dt$.

Remark 2.3. Lemma 2.2 shows distributional solution u falls into the class of Leray–Hopf weak solutions provided that $\nabla u \in L^{\frac{5}{2}}\left(0, T; L^2(\mathbb{R}^3)\right)$.

Proof of Lemma 2.2. Multiplying $(1.1)_1$ by $(u^\varepsilon)^\varepsilon$, then integrating over $(0,T) \times \mathbb{R}^3$, we infer that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u^\varepsilon|^2 dx + \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx = -\int_{\mathbb{R}^3} \operatorname{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon dx. \tag{2.1}$$

Indeed, taking advantage of the interpolation inequality, Hölder's inequality and Young's inequality, we know that

$$\begin{aligned}
\left| -\int_{\mathbb{R}^3} \operatorname{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon dx \right| &\leq C\|(u\cdot\nabla u)\|_{L^{\frac{3}{2}}} \|u^\varepsilon\|_{L^3} \\
&\leq C\|u\|_{L^6}\|\nabla u\|_{L^2}\|u^\varepsilon\|_{L^3} \\
&\leq C\|\nabla u\|_{L^2}^2\|u^\varepsilon\|_{L^3} \\
&\leq C\|\nabla u\|_{L^2}^2\|u^\varepsilon\|_{L^2}^{\frac{1}{2}}\|u^\varepsilon\|_{L^6}^{\frac{1}{2}} \\
&\leq C\|\nabla u\|_{L^2}^{\frac{5}{2}}\left(\|u^\varepsilon\|_{L^2}^2 + 1\right).
\end{aligned} \tag{2.2}$$

Then substituting estimates (2.2) into (2.1), we arrive at

$$\frac{d}{dt}\int_{\mathbb{R}^3} |u^\varepsilon|^2 dx + \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx \leq C\|\nabla u\|_{L^2}^{\frac{5}{2}}\left(\|u^\varepsilon\|_{L^2}^2 + 1\right). \tag{2.3}$$

Applying Gronwall's inequality to see that

$$\begin{aligned}
\sup_{t\geq 0} \|u^\varepsilon(\cdot,t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx d\tau &\leq \|u_0\|_{L^2}^2 \exp C \int_0^t \|\nabla u\|_{L^2}^{\frac{5}{2}} ds \\
&\leq K,
\end{aligned} \tag{2.4}$$

for all $t \in [0,T]$, where K is a constant depending only on initial data u_0 and $\int_0^T \|\nabla u\|_{L^2}^{\frac{5}{2}} dt$. Let $\varepsilon \to 0$ in (2.4), one has

$$\sup_{t\geq 0} \|u(\cdot,t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx d\tau \leq K. \tag{2.5}$$

Then we complete the proof of Lemma 2.2. □

Proof of Theorem 1.2. With Lemma 2.1 and Lemma 2.2 in hand, we are ready to prove our main result. First, we appeal to $u \in L^\infty\left(L^2\right) \cap L^{\frac{5}{2}}(H^1)$, by interpolation inequality we show that

$$\|u\|_{L^4} \leq \|u\|_{L^2}^{\frac{1}{4}}\|u\|_{L^6}^{\frac{3}{4}},$$

By integration in $(0,T)$ one easily proves that the estimate

$$\int_0^T \|u\|_{L^4}^{\frac{10}{3}} dt \le \int_0^T \|u\|_{L^6}^{\frac{5}{2}} \|u\|_{L^2}^{\frac{5}{6}} dt \le \int_0^T \|\nabla u\|_{L^2}^{\frac{5}{2}} \|u\|_{L^2}^{\frac{5}{6}} dt \le C. \tag{2.6}$$

Next, modifying the momentum equation $(1.1)_1$ and taking the inner-product with u^ε, thus we have

$$\int_{\mathbb{R}^3} u^\varepsilon \left(\partial_t u + u \cdot \nabla u - \Delta u + \nabla p\right)^\varepsilon dx = 0. \tag{2.7}$$

This yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u^\varepsilon|^2 dx + \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx = -\int_{\mathbb{R}^3} \operatorname{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon dx. \tag{2.8}$$

Clearly,

$$\int_{\mathbb{R}^3} |u^\varepsilon|^2 dx - \int_{\mathbb{R}^3} |u_0^\varepsilon|^2 dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u^\varepsilon|^2 dx d\tau = -2\int_0^t \int_{\mathbb{R}^3} \operatorname{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon dx d\tau. \tag{2.9}$$

Notice that the incompressible condition $\operatorname{div} u = 0$ ensures

$$-2\int_0^t \int_{\mathbb{R}^3} \operatorname{div}(u \otimes u^\varepsilon) \cdot u^\varepsilon dx d\tau = 0,$$

by using Hölder's equality and Lemma 2.1, one has

$$\begin{aligned}
&-2\int_0^t \int_{\mathbb{R}^3} \operatorname{div}(u \otimes u)^\varepsilon \cdot u^\varepsilon - \operatorname{div}(u \otimes u^\varepsilon) \cdot u^\varepsilon dx d\tau \\
&= 2\int_0^t \int_{\mathbb{R}^3} [(u \otimes u)^\varepsilon - (u \otimes u^\varepsilon)] \cdot \nabla u^\varepsilon dx d\tau \\
&\le 2\int_0^t \int_{\mathbb{R}^3} \left(|(u \otimes u)^\varepsilon - u \otimes u| + |u \otimes u - u \otimes u^\varepsilon|\right) |\nabla u^\varepsilon| \, dx d\tau \\
&\le C\|\nabla u\|_{L^{\frac{5}{2}}(0,T;L^2(\mathbb{R}^3))} \Big(\|(u \otimes u)^\varepsilon - u \otimes u\|_{L^{\frac{5}{3}}(0,T;L^2(\mathbb{R}^3))} \\
&\quad + \|u\|_{L^{\frac{10}{3}}(L^4(\mathbb{R}^3))} \|u - u^\varepsilon\|_{L^{\frac{10}{3}}(0,T;L^4(\mathbb{R}^3))} \Big).
\end{aligned} \tag{2.10}$$

Thanks to (2.6) and standard properties of mollifier, we know that the right hand side of (2.10) becomes zero as $\varepsilon \to 0$, which completes the proof of this case.

Finally, letting ε go to zero in (2.9), and using the facts (2.10), what we have proved is that in the limit

$$\int_{\mathbb{R}^3} |u(t,x)|^2 dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla u(x,\tau)|^2 dx d\tau = \int_{\mathbb{R}^3} |u_0|^2 \, dx.$$

This completes the proof of Theorem 1.2. □

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Tightening Poincaré–Bendixson theory after counting separately the fixed points on the boundary and interior of a planar region

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Abstract. This paper tightens the classical Poincaré–Bendixson theory for a positively invariant, simply-connected compact set $\mathcal{M}$ in a continuously differentiable planar vector field by further characterizing for any point $p \in \mathcal{M}$, the composition of the limit sets $\omega(p)$ and $\alpha(p)$ after counting separately the fixed points on $\mathcal{M}$'s boundary and interior. In particular, when $\mathcal{M}$ contains finitely many boundary but no interior fixed points, $\omega(p)$ contains only a single fixed point, and when $\mathcal{M}$ may have infinitely many boundary but no interior fixed points, $\omega(p)$ can, in addition, be a continuum of fixed points. When $\mathcal{M}$ contains only one interior and finitely many boundary fixed points, $\omega(p)$ or $\alpha(p)$ contains exclusively a fixed point, a closed orbit or the union of the interior fixed point and homoclinic orbits joining it to itself. When $\mathcal{M}$ contains in general a finite number of fixed points and neither $\omega(p)$ nor $\alpha(p)$ is a closed orbit or contains just a fixed point, at least one of $\omega(p)$ and $\alpha(p)$ excludes all boundary fixed points and consists only of a number of the interior fixed points and orbits connecting them.

Keywords: Poincaré–Bendixson theory, planar vector field, limit set.

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1 Introduction

Determining the asymptotic behavior of general continuous vector fields, even qualitatively, is still a daunting task. In the nineteenth century, Poincaré studied this problem for planar systems by focusing on the global behavior of the systems' trajectories without integrating the corresponding differential equations [7,13]. The analysis was later completed by Bendixson [2]. The related classical results are commonly referred to as the Poincaré–Bendixson theorem [2,7,9–11,14–17]. Consider the vector field

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^2 \tag{1.1}$$

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where f is $\mathbf{C}^1$ on an open set $\mathcal{U}$ in $\mathbb{R}^2$. A point $x^* \in \mathbb{R}^2$ is a "fixed point" of the vector field if $f(x^*) = 0$. Denote the omega and alpha limit sets of a point p by $\omega(p)$ and $\alpha(p)$, respectively.

Theorem 1.1 (Poincaré–Bendixson theorem [23, Theorem 9.0.6], [12, Theorem 1.8])**.** *For the vector field (1.1), let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant complex for the vector field containing a finite number of fixed points. For any $p \in \mathcal{M}$, one of the following holds:*

1. *$\omega(p)$ is a fixed point;*
2. *$\omega(p)$ is a closed orbit;*
3. *$\omega(p)$ consists of a finite number of fixed points $p_1, \ldots, p_n$ and orbits γ with $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$, where p_i and p_j are not necessarily different. Moreover, for two distinct fixed points p_i and p_j, there exists at most one orbit γ such that $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$.*

From this theorem, although possibilities such as strange attractors and chaotic orbits can be easily ruled out, the third case in the theorem still gives rise to sometimes a large number of possible limiting behaviors. For example, when $\mathcal{M}$ contains just four fixed points on its boundary, there can be more than 300 different compositions of $\omega(p)$ even under the simplifying assumption that there is at most one homoclinic orbit at each fixed point. Some existing results have tried to reduce the possible scenarios; in [1, Theorem 68], [18, Theorem 3] the third case has been stated more precisely by stipulating that the trajectories γ must be the continuations of one another, and in [19, Section 3.7, Theorem 3] the number of homoclinic orbits at each fixed point is limited by one when the vector field is "relatively prime analytic". However, then for the example just mentioned, $\omega(p)$ can still have more than 50 different compositions. This example shows that if one is interested in categorizing all possible asymptotic behaviors of a planar system qualitatively, a greatly needed task in fields such as mathematical biology [4], one may still encounter difficulty even with the help of the existing most tightened form of Poincaré–Bendixson theorem.

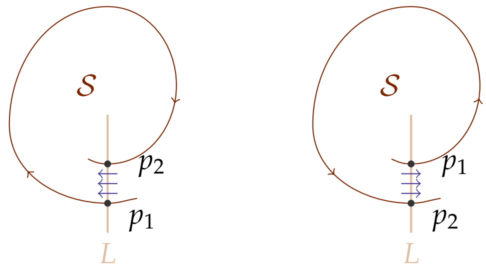
The aim of this paper is to reduce the number of possible compositions of the limit sets of a vector field when knowing the number of fixed points on the boundary and in the interior of a given positively invariant, simply-connected compact set $\mathcal{M}$.

Notations: Let $\phi(t, x)$ denote the flow generated by the vector field (1.1), which is the solution of (1.1) passing through x at time t. For a point $p \in \mathbb{R}^2$, let $\mathcal{O}(p)$ denote the *orbit* of p defined by $\mathcal{O}(p) = \{x \in \mathbb{R}^2 \mid x = \phi(t, p), t \in \mathbb{R}\}$, and $\mathcal{O}_+(p)$ denote the *positive semi-orbit* of p, defined by $\mathcal{O}_+(p) = \{x \in \mathbb{R}^2 \mid x = \phi(t, p), t \geq 0\}$ [23]. Correspondingly, for $p_1, p_2 \in \mathcal{O}_+(p)$, define the segment semi-orbit $\mathcal{O}_+(p)$ *from* p_1 *to* p_2 as $\mathcal{O}_+(p_2) - \mathcal{O}_+(p_1)$. A *homoclinic orbit* is a trajectory that joins a fixed point to itself. For a set $\mathcal{M}$, denote its *interior* by Int $\mathcal{M}$, its *boundary* by $\partial\mathcal{M}$, and its closure by $\overline{\mathcal{M}}$.

2 Main results

We first review some basic relevant results. The following lemma is applicable to higher dimensional spaces, but we restrict it here to the plane.

Lemma 2.1 ([23, Proposition 8.1.3], [3, Theorem 3-3.6])**.** *For the vector field (1.1), let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant compact set. Then for any point $p \in \mathcal{M}$, it holds that $\omega(p)$ is nonempty, connected, and compact.*



(a) (b)

Figure 2.1: The two possible cases for the positive semi-orbit $\mathcal{O}_+(p)$ in the proof of Theorem 2.4.

A continuous connected arc in the plane is said to be *transverse to the vector field*, if the vector field has no fixed points on the arc and nowhere becomes tangent to the arc [11]. By a *transversal* we refer to a closed line segment $\mathcal{L}$ that is transverse to the vector field. Due to the continuity of the vector field, clearly one can construct a transversal through any non-fixed point. The following lemma illustrates how the flow through a point p approaches a transversal through a non-fixed omega limit point $q \in \omega(p)$ when it exists.

Lemma 2.2 ([8, reformulation of Lemma 1.26])**.** *For the vector field* (1.1)*, consider a point $p \in \mathcal{U}$ such that $\mathcal{O}(p) \subset \mathcal{U}$. Let $q \in \omega(p)$ be a non-fixed point of* (1.1) *and let $\mathcal{L}$ be a transversal through q. Then there exists a sequence $\{t_i\} \to \infty$, such that $\{\phi(t_i, p)\} \in \mathcal{L}$ and $\{\phi(t_i, p)\} \to q$.*

The following result guarantees the existence of a fixed point inside a closed orbit [3,6,9,23]:

Lemma 2.3 ([23, Corollary 6.0.2])**.** *Enclosed by any closed orbit of* (1.1) *in $\mathcal{U}$, there must be at least one fixed point.*

Now we are ready to present the main results of the paper.

2.1 $\mathcal{M}$ has no interior fixed point

Theorem 2.4 (No interior fixed points, positively invariant vector field)**.** *For the vector field* (1.1)*, consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that contains a finite number of fixed points, all on $\partial\mathcal{M}$. Then for any $p \in \mathcal{M}$, $\omega(p)$ is a fixed point on $\partial\mathcal{M}$.*

Proof. From Theorem 1.1, it suffices to prove that $\omega(p)$ contains only fixed points since then only situation 1 is possible and the corresponding fixed point can only be on $\partial\mathcal{M}$ as Int $\mathcal{M}$ contains no fixed points. We prove this by contradiction, so assume on the contrary that there is a non-fixed point $q \in \omega(p)$. Then one can construct a transversal $\mathcal{L}$ through q, and from Lemma 2.2, we know that $\mathcal{O}_+(p)$ intersects $\mathcal{L}$ for infinitely many times and such intersection points are in $\mathcal{M}$ since $\mathcal{O}_+(p) \subset \mathcal{M}$. So one can pick two consecutive intersection points p_1 and p_2 such that the line segment p_1p_2 lies in $\mathcal{M}$. Should p_1 and p_2 coincide, $\omega(p)$ would be a closed orbit, lying in $\mathcal{M}$, but encircling no fixed point as all the fixed points are on $\partial\mathcal{M}$. This cannot happen in view of Lemma 2.3, and thus, p_1 and p_2 must be distinct.

As illustrated by Fig. 2.1, we construct the simply-connected compact set $\mathcal{S}$ whose boundary is formed by the segment semi-orbit $\mathcal{O}_+(p)$ from p_1 to p_2 and the line segment p_1p_2. Since

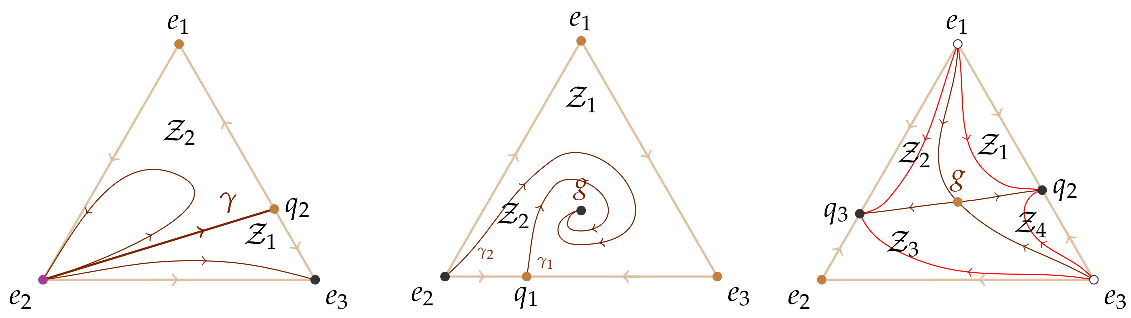


Figure 2.2: Phase portrait examples for an invariant compact set Δ. **(a)** e_1 and q_2 are hyperbolic saddle, e_3 is a hyperbolic stable and e_2 is a center fixed point. The stable invariant manifold of q_2 divides Δ into $\mathcal{Z}_1$ and $\mathcal{Z}_2$. Theorem 2.5 and the local stability results imply that for each $z \in \text{Int } \mathcal{Z}_1$, $\alpha(z) = e_2$ and $\omega(z) = e_3$, and for each $z \in \text{Int } \mathcal{Z}_2$, $\alpha(z) = \omega(z) = e_2$. **(b)** e_1, e_3 and q_1 are hyperbolic saddle, e_2 is a hyperbolic unstable, and g is a hyperbolic stable fixed point. Because of Theorem 2.7, the local stability results and the fact that no limit cycle exists, $\omega(p) = \{g\}$ for all $p \in \text{int}(\Delta)$. Hence, the unique out-going trajectory from q_1, denoted by γ_1, converges to g. The rest of the orbits in $\text{int}(\Delta)$ start from e_2 and end at g. This is because any out-going trajectory from e_2, e.g., γ_2, together with γ_1 divide the simplex into the zones $\mathcal{Z}_1$ and $\mathcal{Z}_2$, each of which satisfy the condition of $\mathcal{M}$ in Theorem 2.7. Hence, every trajectory in $\text{Int } \mathcal{Z}_i, i = 1, 2$, starts from e_2 and end at g. **(c)** e_2 and g are hyperbolic saddle, e_1 and e_3 are hyperbolic unstable and q_3 and q_2 are hyperbolic stable fixed points. The trajectories γ_1 and γ_2 lie on the unstable invariant manifold of g. Because of Theorem 2.7 and the local stability results, the unstable invariant manifold of g is confined to q_2 and q_3 and the stable invariant manifold of g is confined to e_1 and e_3. This results in the four zones $\mathcal{Z}_1, \ldots, \mathcal{Z}_4$. In view of Theorem 2.4, $\forall z \in \text{Int } \mathcal{Z}_1$, $\alpha(z) = e_1$ and $\omega(z) = q_2$, $\forall z \in \text{Int } \mathcal{Z}_2$, $\alpha(z) = e_1$ and $\omega(z) = q_3$, $\forall z \in \text{Int } \mathcal{Z}_3$, $\alpha(z) = e_3$ and $\omega(z) = q_3$, and $\forall z \in \text{Int } \mathcal{Z}_4$, $\alpha(z) = e_3$ and $\omega(z) = q_2$.

$\mathcal{O}_+(p)$ always intersects $\mathcal{L}$ from the same side to the other, the orientation of the p_1-to-p_2 semi-orbit with respect to the line segment p_1p_2 must be one of the two cases shown in Fig. 2.1. From the definition of $\mathcal{L}$, the vector field at any point on p_1p_2 intersects p_1p_2 from the same side of the line, and thus $\mathcal{S}$ is either positively invariant as shown in Fig. 2.1.(a) or negatively invariant as shown in Fig. 2.1.(b).

Since the boundary p_1-to-p_2 semi-orbit and p_1p_2 both lie in $\mathcal{M}$, we know that $\mathcal{S} \subseteq \mathcal{M}$. Hence, $\text{Int } \mathcal{S} \subseteq \text{Int } \mathcal{M}$ and contains no fixed point. Moreover, neither $\mathcal{O}_+(p)$ nor $\mathcal{L}$ contains any fixed point, so $\partial\mathcal{S}$ does not contain any fixed point. Therefore, $\mathcal{S}$ contains no fixed point. Consequently, if $\mathcal{S}$ is positively invariant, applying Theorem 1.1, we know that for any point $s \in \mathcal{S}$, $\omega(s)$ can only be a closed orbit confined in $\mathcal{S}$. But this contradicts Lemma 2.3. If on the other hand, $\mathcal{S}$ is negatively invariant, we apply the same argument after inverting the direction of the vector field and again reach the same contradiction. So the proof is complete. □

In term of the example given in the introduction, Theorem 2.4 implies that $\omega(p)$ in the example can only be one of the fixed points, so at most four possibilities. If in addition to being positively invariant, $\mathcal{M}$ is also negatively invariant, i.e., $\mathcal{M}$ is invariant, then Theorem 2.4 can get even more strengthened.

Theorem 2.5 (No interior fixed points, invariant vector field)**.** *For the vector field* (1.1)*, consider an invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that contains a finite number of fixed points, all on $\partial\mathcal{M}$. Then for any $p \in \mathcal{M}$, both $\omega(p)$ and $\alpha(p)$ are fixed points, not necessarily different, on $\partial\mathcal{M}$.*

Proof. Theorem 2.4 implies that for any $p \in \mathcal{M}$, $\omega(p)$ contains only a single fixed point on $\partial\mathcal{M}$. The same holds for $\alpha(p)$ after reversing the direction of the vector field since $\mathcal{M}$ is also negatively invariant. This completes the proof. □

Fig. 2.2 demonstrates an example from planar *replicator dynamics* [20–22], where the triangle $e_1e_2e_3$, known as a *face*, is invariant. Part (a) corresponds to Theorem 2.5. The reader may refer to [4, 5] for all 49 possible qualitatively different phase portraits of the dynamics.

2.2 $\mathcal{M}$ has no interior, but infinitely many boundary fixed points

We obtain the following theorem that is the counterpart of Theorem 2.4 when the vector field may have infinitely many fixed points on $\partial\mathcal{M}$.

Theorem 2.6. *For the vector field* (1.1)*, consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that has no interior fixed point, but may contain an infinite number of fixed points on $\partial\mathcal{M}$. Then for any $p \in \mathcal{M}$, one of the following two holds:*

1. *$\omega(p)$ is a fixed point on $\partial\mathcal{M}$;*
2. *$\omega(p)$ is a continuum of fixed points on $\partial\mathcal{M}$.*

Proof. Following similar steps as those in the proof for Theorem 2.4, one can construct the simply-connected compact set $\mathcal{S}$ as illustrated in Fig. 2.1. Using similar arguments for $\mathcal{S}$ as those in the proof for Theorem 2.4, after applying Theorem 6.1 in [7], which is the extension of Poincaré–Bendixson theorem to the case when there are infinitely many fixed points, one knows that $\omega(p)$ does not contain any fixed point. On the other hand, $\omega(p)$ has to be connected in view of Lemma 2.1, so it can only be a connected subset of the fixed points in $\mathcal{M}$, which is either a fixed point or a continuum of fixed points on $\partial\mathcal{M}$. □

2.3 $\mathcal{M}$ has exactly one interior fixed point

Now we present the counterpart of Theorem 2.4 discussing the case when $\mathcal{M}$ contains exactly one interior and finitely many boundary fixed points.

Theorem 2.7 (One interior fixed point)**.** *For the vector field* (1.1)*, consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ that contains exactly one interior fixed point x^* and a finite number of fixed points on its boundary. Then for any $p \in \mathcal{M}$, at least one of the following holds:*

1. *$\omega(p)$ is a fixed point, a closed orbit encircling x^* or the union of $\{x^*\}$ and a (possibly union of) homoclinic orbit(s) joining x^* to itself;*
2. *$\alpha(p)$ is $\{x^*\}$, a closed orbit encircling x^* or the union of $\{x^*\}$ and a (possibly union of) homoclinic orbit(s) joining x^* to itself.*

Proof. We investigate all possibilities for $\omega(p)$ and show that each results in one of the cases of the theorem. Should $\omega(p)$ be a singleton fixed point or a closed orbit that has to encircle g according to Lemma 2.3, we arrive at Part 1. of the theorem. So consider the situation when $\omega(p)$ is neither. It then follows Theorem 1.1 that $\omega(p)$ contains non-fixed points; we pick one such point q and construct a transversal $\mathcal{L}$ through q. From Lemma 2.2, we know that $\mathcal{O}_+(p)$ intersects $\mathcal{L}$ for infinitely many times. Consider two consecutive intersections p_1 and p_2 which have to be distinctive since $\omega(p)$ is not a closed orbit. We construct the simply-connected compact set $\mathcal{S}$ whose boundary is formed by the semi-orbit $\mathcal{O}_+(p)$ from p_1 to p_2 and the line segment p_1p_2. Similar to the proof of Theorem 2.4, one can show that:

(i) $\mathcal{S}$ is in the form of one of the two cases shown in Fig. 2.1,

(ii) $\mathcal{S}$ is positively invariant in Case (a) of the figure and negatively invariant in Case (b), and

(iii) $x^* \in \operatorname{Int} \mathcal{S}$ is the only fixed point in $\mathcal{S}$.

If $\mathcal{S}$ is positively invariant, $\mathcal{O}_+(p) \cap \operatorname{Int} \mathcal{S} \neq \emptyset$, implying the existence of some $s_p \in \mathcal{O}_+(p) \cap \operatorname{Int} \mathcal{S}$. Consequently, $\omega(s_p) = \omega(p)$. Then, applying Theorem 1.1, we know that $\omega(s_p)$ consists of a number of fixed points in $\mathcal{S}$ and the orbits connecting them. However, since x^* is the only fixed point in Int $\mathcal{M}$, such orbits can only connect x^* to itself. So $\omega(s_p)$ is the union of $\{x^*\}$ and a (possibly union of) homoclinic orbit(s) joining x^* to itself, so is $\omega(p)$. So in this case Part 1 of the theorem holds.

Otherwise, if $\mathcal{S}$ is negatively invariant, then there exists a point $s_p \in \mathcal{O}_-(p) \cap \operatorname{Int} \mathcal{S}$ where $\mathcal{O}_-(p)$ is the same as $\mathcal{O}_+(p)$, but when time is reversed. Consequently, after reversing the direction of the vector field, one can check the three cases in Theorem 1.1 as $\omega(s_p)$ lead to the three cases in Part 2 of the theorem respectively. □

Theorem 2.7 is indeed restricting the third case of Theorem 1.1, for at least one of the ω or α limit sets. Note that if, in addition, x^* is hyperbolic and the vector field contains no closed orbits, then for any point $p \in \mathcal{M}$, either $\omega(p)$ is a fixed point or $\alpha(p) = \{x^*\}$. See Fig. 2.2.(b) and (c) for two examples. We highlight that the first case in Theorem 2.7 may not cover all possibilities for $\omega(p)$ (see Fig. 2.3); however, then the second case of the Theorem will be in force, determining the structure of $\alpha(p)$.

It is also worth mentioning that some cases in Part 1 and Part 2 of Theorem 2.7 never take place at the same time. For example, it is impossible to have both $\omega(p)$ and $\alpha(p)$ being the union of $\{x^*\}$ and a homoclinic orbit joining x^* to itself. We exclude such cases for general positively invariant compact regions as follows. A point is *periodic* if it is on a closed orbit.

Proposition 2.8. *Let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant compact set under the vector field* (1.1)*. For any non-periodic point $p \in \mathcal{M}$, if $\omega(p) = \alpha(p)$, then the limit sets contain only fixed points.*

Either Lemma 9.0.2 in [23] or the results on the characterization of non-periodic orbits in [6] can be used for the proof, which we skip here. In case $\mathcal{M}$ contains finitely many fixed points, we can sharpen the result of Proposition 2.8 by using Proposition 8.1.3 in [23].

Corollary 2.9. *For the vector field* (1.1)*, let $\mathcal{M} \subset \mathcal{U}$ be a positively invariant compact set containing a finite number of fixed points. Then for any non-periodic point $p \in \mathcal{M}$, if $\omega(p) = \alpha(p)$, then the limit sets exclusively contain a single fixed point.*

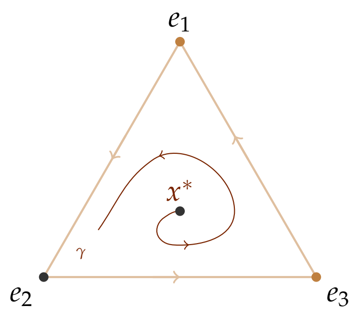


Figure 2.3: Phase portrait example for an invariant compact set $\mathcal{M}$ defined by the triangle $e_1e_2e_3$, where e_1, e_2 and e_3 are fixed points. There is exactly one interior fixed point, g. For every point p in the interior of $\mathcal{M}$, the ω-limit set of p equals $\partial\mathcal{M}$, that is the union of the fixed points e_1, e_2 and e_3 and the heteroclinic orbits connecting them to each other. This is not covered by the first case of Theorem 2.7. However, $\alpha(p) = \{x^*\}$, which is satisfied by the second case of Theorem 2.7.

2.4 $\mathcal{M}$ has finitely many interior fixed points

Following the previous subsection of having one interior fixed point in the positively invariant compact set $\mathcal{M}$, we now extend the result to the more general case of having finitely many interior fixed points in $\mathcal{M}$.

Theorem 2.10 (Finitely many interior fixed points)**.** *For the vector field* (1.1)*, consider a positively invariant, simply-connected compact set $\mathcal{M} \subset \mathcal{U}$ containing a finite number of fixed points. Then for any point $p \in \mathcal{M}$, at least one of the following holds:*

1. *$\omega(p)$ is a fixed point, a closed orbit encircling at least one interior fixed point or the union of some interior fixed points together with the orbits connecting them;*

2. *$\alpha(p)$ is an interior fixed point, a closed orbit encircling at least one interior fixed point or the union of some interior fixed points together with the orbits connecting them.*

Proof. The proof is similar to that for Theorem 2.7 and we omit it here. □

Compared to the classical form of Poincaré–Bendixson Theorem 1.1, what Theorem 2.10 has further clarified is the role of the interior fixed points of $\mathcal{M}$ play to influence the topological structure of the limit sets. For example, as an immediate result of Theorem 2.10, if the third case of Theorem 1.1 takes place for p, then $\omega(p)$ and $\alpha(p)$ cannot be free of interior fixed points at the same time; in other words, unless $\omega(p)$ is simply a fixed point or a closed orbit, some interior fixed points must be in either $\omega(p)$ or $\alpha(p)$. Another implication of Theorem 2.10 is the exclusion of the boundary fixed points from one of $\omega(p)$ and $\alpha(p)$. From Theorem 2.10, if $\omega(p)$ is not simply a fixed point, then at least one of $\omega(p)$ or $\alpha(p)$ does not contain any boundary fixed point. In a sense, this implies that the interior fixed points are more important for determining the structures of the limit sets. Finally, we note that Corollary 2.9 can also be utilized here to rule out some trivial cases when $\omega(p)$ and $\alpha(p)$ are the same.

At the end of this section, we present the following version of Theorem 2.10 without requiring $\mathcal{M}$ to be simply connected.

Theorem 2.11. *For the vector field* (1.1)*, consider a positively invariant, compact set* $\mathcal{M} \subset \mathcal{U}$ *that contains a finite number of fixed points. Then for any* $p \in \mathcal{M}$*, at least one of the following holds:*

1. $\omega(p)$ *is a fixed point, a closed orbit or the union of some interior fixed points with the orbits connecting them;*

2. $\alpha(p)$ *is one of the interior fixed points, a closed orbit or the union of some interior fixed points with the orbits connecting them.*

Proof. The proof is similar to that of Theorem 2.7. The difference is that if $\omega(p)$ or $\alpha(p)$ is a closed orbit, it may encircle areas that do not belong to $\mathcal{M}$. □

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On the existence of patterns in reaction-diffusion problems with Dirichlet boundary conditions

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Abstract. Consider a general reaction-diffusion problem, $u_t = \Delta u + f(x, u, u_x)$, on a revolution surface or in an n-dimensional ball with Dirichlet boundary conditions. In this work, we provide conditions related to the geometry of the domain and the spatial heterogeneities of the problem that ensure the existence or not of a non-constant stationary stable solution. Several applications are presented, particularly with regard to the Allen–Cahn, Fisher–KPP and sine-Gordon equations.

Keywords: patterns, Dirichlet boundary conditions, surface of revolution.

2020 Mathematics Subject Classification: 35B35, 35B36.

1 Introduction

In this work we consider the following problem

$$\begin{cases} u_t = \Delta u + f(x, u, u_x), & (t, x) \in \mathbb{R}^+ \times \Omega, \\ u(t, x) = B, & (t, x) \in \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{1.1}$$

where f is a C^1 function, $B \in \mathbb{R}$ and Ω is a surface of revolution in $\mathbb{R}^3$ or is an n-dimensional ball. We say that U is a *stationary solution* of (1.1) if U is a solution of (1.1) independent of temporal variable t, that is

$$\begin{cases} \Delta U + f(x, U, U_x) = 0, & x \in \Omega, \\ U(x) = B, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

A stationary solution U of (1.1) is called *stable* if for every $\eta > 0$ there exists $\delta > 0$ such that for every solution v to (1.1) satisfying $\|v(0, \cdot) - U(\cdot)\|_{L^\infty} < \delta$ it holds that $\|v(t, \cdot) - U(\cdot)\|_{L^\infty} < \eta$, for all $t > 0$. Finally, if U is a non-constant stable stationary solution of (1.1), then U is commonly referred to as the *spatial pattern* or simply *pattern*.

The study of reaction-diffusion equations has been a central focus in the field of mathematical modeling for several decades. These systems have wide-ranging applications in various scientific disciplines, including chemistry, biology, physics, and ecology. One of the intriguing phenomena that often arises in reaction-diffusion systems is the spontaneous formation of spatial patterns. These patterns, which can take on diverse shapes and structures, emerge as a consequence of the interplay between the underlying reaction kinetics and the diffusion of the interacting species [9,20].

In this work, we are interested in investigating the role of spatial heterogeneities as well as the domain geometry concerning the existence or not of patterns. The literature on this subject is extensive, mainly when Neumann boundary conditions are considered. The difficulty in obtaining results with Dirichlet boundary conditions leads to a reduced number of studies. Here, we cite [8,11,21], where the authors achieve results for one-dimensional problems, and [13] for problems in n-dimensional balls. Some results on surfaces of revolution can be found in [19].

Our proposal to study the problem on surfaces of revolution is motivated by the recent interest of the scientific community in these domains [3,4,14,16,18,19], and success is primarily attributed to the well-established symmetry properties of stable solutions in this domain (similar phenomena are observed in the case of balls in $\mathbb{R}^n$). Such symmetry leads us to one-dimensional problems, and this is crucial for obtaining the results.

The proposed ideas can be applied in various situations. In this study, we provide several examples involving the Allen–Cahn, Fisher–KPP, and sine-Gordon problems. These choices were made given the significant relevance of these models. However, as evident, the potential applications extend to many other cases, including reaction-convection-diffusion problems.

The work is divided as follows. In Section 2, we present preliminary results related to the existence and uniqueness of solutions for one-dimensional nonlinear second-order problems. In this section, we illustrate how to obtain results regarding the existence or not of patterns in one-dimensional problems, underscoring the significance of this section in its own right. In Section 3, we present the main results for the problem on surfaces of revolution, whereas Section 4 is dedicated to the sine-Gordon problem in an n-dimensional ball. Finally, in Section 5, we provide some concluding remarks.

2 Preliminaries and some general one-dimensional results

In this section, we will present three general results on the existence and uniqueness of solutions for certain elliptic problems in a interval. In this case, for the sake of simplicity, we will replace the notation u_x with u'. Results of this type are commonly understood when $f(x,u,u')$ satisfies a specific uniform Lipschitz condition, assuming the interval length for the variable u where the problem occurs is sufficiently small. However, in numerous scenarios, it is necessary to extend this result to include functions $f(x,u,u')$ that are Lipschitz not for all u but solely for u within a bounded interval. This is what is accomplished in the first theorem below, which considers Dirichlet conditions at the boundary.

Additionally, it is crucial to highlight that instead of the typical Lipschitz condition, we presume a set of one-sided conditions which, while not more restrictive, proves to be considerably more practical. Further elaboration on this matter can be found in [1,2,6].

The subsequent results in this section are related to a function $g \in C^1$ such that

$$g(s,0,0) = 0 \tag{2.1}$$

and

$$G_1(u-v, u'-v') \le g(s,u,u') - g(s,v,v') \le G_2(u-v, u'-v') \tag{2.2}$$

where

$$G_2(u,u') = \begin{cases} M_2u' + K_2u, & u \geq 0,\ u' \geq 0, \\ M_1u' + K_2u, & u \geq 0,\ u' \leq 0, \\ M_1u' + K_1u, & u \leq 0,\ u' \leq 0, \\ M_2u' + K_1u, & u \leq 0,\ u' \geq 0, \end{cases} \tag{2.3}$$

$$G_1(u,u') = \begin{cases} M_1u' + K_1u, & u \geq 0,\ u' \geq 0, \\ M_2u' + K_1u, & u \geq 0,\ u' \leq 0, \\ M_2u' + K_2u, & u \leq 0,\ u' \leq 0, \\ M_1u' + K_2u, & u \leq 0,\ u' \geq 0, \end{cases} \tag{2.4}$$

and $M_i, K_i \in \mathbb{R}$ ($i = 1,2$) are constant.

The next three theorems are crucial to all the results of this work.

Theorem 2.1 (Theorem 1 in [1]). *For $(s,u,u') \in [0,L] \times J \times \mathbb{R}$, where J is a closed interval in $\mathbb{R}$, let $g(s,u,u')$ be a continuous function satisfying (2.1) and (2.2). If the two problems ($i = 1,2$)*

$$\begin{cases} u_i''(s) + G_i(u_i(s), u_i'(s)) = 0, & s \in (a,b), \\ u_i(a) = A', \quad u_i(b) = B', \end{cases} \tag{2.5}$$

have unique solutions on every sub-interval $[a,b]$ of $[0,L]$ for arbitrary A', B', and if for $a = 0$, $b = L$, $A' = A$, $B' = B$ the ranges of u_i ($i = 1,2$) are subsets of J, then the problem

$$\begin{cases} u''(s) + g(s,u(s),u'(s)) = 0, \\ u(0) = A, \quad u(L) = B, \end{cases} \tag{2.6}$$

has a unique solution $u(s)$, which remains in J and it satisfies

$$u_1(s) \leq u(s) \leq u_2(s),$$

where u_1 and u_2 are solutions of (2.5) with G_1 and G_2, respectively, and $a = 0$, $b = L$, $A' = A$, $B' = B$.

Before stating the next theorem, we define

$$\alpha(M,K) = \begin{cases} \dfrac{2}{\sqrt{4K - M^2}} \cos^{-1}\left(\dfrac{M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 > 0, \\ \dfrac{2}{\sqrt{M^2 - 4K}} \cosh^{-1}\left(\dfrac{M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 < 0,\ M > 0,\ K > 0, \\ \dfrac{2}{M}, & \text{if } 4K - M^2 = 0,\ M > 0, \\ +\infty, & \text{otherwise} \end{cases} \tag{2.7}$$

and

$$\beta(M,K) = \begin{cases} \dfrac{2}{\sqrt{4K - M^2}} \cos^{-1}\left(\dfrac{-M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 > 0, \\ \dfrac{2}{\sqrt{M^2 - 4K}} \cosh^{-1}\left(\dfrac{-M}{2\sqrt{K}}\right), & \text{if } 4K - M^2 < 0,\ M < 0,\ K > 0, \\ \dfrac{-2}{M}, & \text{if } 4K - M^2 = 0,\ M < 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.8}$$

The next theorem is fundamental for verifying the existence and uniqueness of solution to problems in (2.5) in the theorem above.

Theorem 2.2 (Theorem 1 in [2]). *Let $G(y, y')$ be a continuous real valued function satisfying $G(0, 0) = 0$ and (2.2) (assuming $g(s, y, y') = G(y, y')$) with G_1 and G_2 defined in (2.4) and (2.3). If*

$$L < \alpha(M_2, K_2) + \beta(M_1, K_2),$$

then the boundary value problem

$$\begin{cases} u''(s) + G(u(s), u'(s)) = 0, & s \in (0, L), \\ u(0) = A', \quad u(L) = B' \end{cases} \tag{2.9}$$

has a unique solution for every pair of real numbers A', B'.

Below, we present a new result regarding existence and uniqueness, specifically for mixed boundary conditions. In particular, we will use this theorem to investigate the sine-Gordon problem in an n-dimensional ball.

Theorem 2.3 (Theorem 1 in [6]). *Let $g(s, y, y')$ be a continuous real valued function satisfying (2.2) with G_1 and G_2 defined in (2.4) and (2.3). If*

$$L < \beta(M_1, K_2),$$

then the mixed boundary value problem

$$\begin{cases} u''(s) + g(s, u(s), u'(s)) = 0, & s \in (0, L), \\ u'(0) = A, \quad u(L) = B \end{cases} \tag{2.10}$$

has a unique solution for every pair of real numbers A, B.

With the aforementioned theorems, it is not difficult to derive results regarding the non-existence of patterns in one-dimensional problems and zero Dirichlet boundary conditions. Although this is not the main objective of this work, we present a simple example below.

Example 2.4. Consider the following problem

$$\begin{cases} u_t = u_{xx} + \rho(x)u(1-u), & (t, x) \in \mathbb{R}^+ \times (0, L), \\ u(0) = 0, \quad u(L) = 0, \end{cases} \tag{2.11}$$

where ρ is a continuous function with sign-changing or not. Note that this includes the important Fisher–KPP equation which will be further elucidated in the subsequent section.

In this case we consider $J = [0, 1]$ and then, $g(x, u) = \rho(x)u(1-u)$ satisfies (2.1) and (2.2) with G_1 and G_2 given by (2.4) and (2.3) if

$$M_1 = M_2 = 0, \quad K_1 = -\sup_{x \in [0,L]} |\rho(x)| \text{ and } K_2 = \sup_{x \in [0,L]} |\rho(x)|.$$

Now, in order to use Theorem 2.1 we have to analyse

$$\begin{cases} z'' + G_i(z) = 0, & (0, L), \\ z(0) = A', \quad z(L) = B' \end{cases} \tag{2.12}$$

with $i = 1,2$. We can use Theorem 2.2 to conclude that if $L < \alpha(M_2, K_2) + \beta(M_1, K_2)$, that is

$$\frac{2}{\sqrt{4K_2 - M_2^2}} \cos^{-1}\left(\frac{M_2}{2\sqrt{K_2}}\right) + \frac{2}{\sqrt{4K_2 - M_1^2}} \cos^{-1}\left(\frac{-M_1}{2\sqrt{K_2}}\right) = \frac{\pi}{2\sqrt{K_2}} + \frac{\pi}{2\sqrt{K_2}} = \frac{\pi}{\sqrt{K_2}} > L$$

or

$$K_2 < (\pi/L)^2,$$

then (2.12) has a unique solution (for $i = 1,2$) for any $A', B' \in \mathbb{R}$. In particular, it is easy to see that if $A' = B' = 0$, then $z \equiv 0 \in J = [0,1]$ is the unique solution. Finally, Theorem 2.1 yields that, in these conditions, $u \equiv 0$ is the unique stationary solution of (2.11), and thus, (2.11) does not admit patterns.

Remark 2.5. Note that the interval J ($= [0,1]$ in the above example) is associated with the range of variation of u and with the inequalities in (2.2). Evidently, its choice affects the values of M_i and K_i, and consequently the application of the results of existence and uniqueness of solution to determine whether patterns emerge or not.

Below, we present a simple example of pattern existence for a problem with mixed boundary conditions. In this case, the chosen nonlinearity is related to the sine-Gordon equation.

Example 2.6. Consider the following problem with mixed boundary conditions

$$\begin{cases} u_t = (e^{5x} u_x)_x + (x+6)\sin(u), & (t,x) \in \mathbb{R}^+ \times (0,1), \\ u_x(t,0) = 1/2, \quad u(t,1) = 1/4. \end{cases} \tag{2.13}$$

This problem can be written as

$$\begin{cases} \dfrac{u_t}{e^{5x}} = u_{xx} + 5u_x + \dfrac{(x+6)}{e^{5x}} \sin(u), & (t,x) \in \mathbb{R}^+ \times (0,1), \\ u_x(t,0) = 1/2, \quad u(t,1) = 1/4, \end{cases} \tag{2.14}$$

and the corresponding stationary problem is

$$\begin{cases} u_{xx} + 5u_x + \dfrac{(x+6)}{e^{5x}} \sin(u) = 0, & x \in (0,1), \\ u_x(0) = 1/2, \quad u(1) = 1/4. \end{cases} \tag{2.15}$$

We note that $h(x, u, u_x) = 5u_x + \frac{(x+6)}{e^{5x}} \sin(u)$ satisfies (2.2) with G_1 and G_2 defined in (2.4) and (2.3) with

$$M_1 = M_2 = 5, \quad K_1 = -6, \quad K_2 = 6.$$

A simple analysis of (2.8) shows that

$$\beta(M_1, K_2) = \infty.$$

From Theorem 2.3, it follows that (2.15) has a unique solution U. Now note that $E : \{u \in H^1(0,1); u(1) = 1/4\} \to \mathbb{R}$ defined by

$$E[u] = \int_0^1 \frac{e^{5x}}{2}(u_x)^2 - F(u,x)dx + \frac{u(0)}{2},$$

where $F(u,x) = (x+6)\int_0^u \sin(\sigma)d\sigma$ is the energy functional associated with (2.15), its critical points are solutions to (2.15). Now, we state that E serves as a strict Lyapunov functional for (2.13) i.e., except at stationary states, $E[u(t,\cdot)]$ is strictly decreasing on orbits. To verify this, we take a solution u of (2.13) and a function $v \in H^1(0,1)$ such that $v(1) = 0$. Then

$$u_t = (e^{5x}u_x)_x + (x+6)\sin(u)$$

and we can multiply this equation by v, integrate on $(0,1)$ and use integration by parts to achieve

$$\begin{aligned}\int_0^1 vu_t dx &= \int_0^1 ve^{5x}u_{xx}dx + \int_0^1 v5e^{5x}u_x dx + \int_0^1 v(x+6)\sin(u)dx \\ &= \int_0^1 v5e^{5x}u_x dx + v(1)e^5 u_x(t,1) - v(0)u_x(t,0) - \int_0^1 (ve^{5x})_x u_x dx \\ &\quad + \int_0^1 v(x+6)\sin(u)dx \\ &= \int_0^1 v5e^{5x}u_x dx - \frac{v(0)}{2} - \int_0^1 v_x e^{5x}u_x dx - \int_0^1 v5e^{5x}u_x dx \\ &\quad + \int_0^1 v(x+6)\sin(u)dx.\end{aligned}$$

We can cancel the first and fourth term of the last equality to obtain

$$\int_0^1 vu_t dx = -\frac{v(0)}{2} - \int_0^1 v_x e^{5x}u_x dx + \int_0^1 v(x+6)\sin(u)dx. \tag{2.16}$$

Now observe that if u is a solution of (2.13), then $u_t(t,\cdot) \in H^1(0,1)$, $u(t,1) = 1/4$ for all t, and thus $u_t(t,1) = 0$. If we differentiate $E[u(t,\cdot)]$ with respect to t, we obtain

$$\frac{d}{dt}E[u(t,\cdot)] = \int_0^1 e^{5x}u_x u_{tx}dx - \int_0^1 (x+6)\sin(u)u_t dx + \frac{u_t(t,0)}{2}. \tag{2.17}$$

We can compare (2.16) and (2.17) to get

$$\frac{d}{dt}E[u(t,\cdot)] = -\int_0^1 (u_t)^2 dx.$$

Therefore, we have a system with a gradient structure, and then the bounded trajectories of (2.13) approach the set of stationary solutions (for the reader's convenience, we cite [7] for topics related to the dynamics of (2.13) and [15, Chapter 2] for results related to the existence and boundedness of solutions of (2.13)). Since U is non-constant and the only stationary solution of the problem, we conclude that U is a pattern as defined above.

3 Surfaces of revolution

Considering a smooth curve C in $\mathbb{R}^3$ parameterized by $(\psi(s), 0, \chi(s))$, where $s \in [l_1, l_2]$ ($[0,1] \subset (l_1, l_2)$), with $\psi(l_1) = \psi(l_2) = 0$, we can generate a borderless surface of revolution $\mathcal{M}$. This surface can be parametrized by

$$x = (\psi(s)\cos(\theta), \psi(s)\sin(\theta), \chi(s)), \quad (s,\theta) \in [l_1, l_2] \times [0, 2\pi). \tag{3.1}$$

Let $\mathcal{M}$ be the surface of revolution parametrized by (3.1). We also assume that $\psi, \chi \in C^2$, $\psi > 0$ in (l_1, l_2), $(\psi_s)^2 + (\chi_s)^2 = 1$ and $\chi_s(s) \geq 0$ in $[l_1, l_2]$. Moreover, $\psi_s(l_1) = -\psi_s(l_2) = 1$, and as stated above, we assume $\psi(l_1) = \psi(l_2) = 0$.

By setting $x^1 = s$ and $x^2 = \theta$, we can conclude that the surface of revolution $\mathcal{M}$, with the above parametrization, is a 2-dimensional Riemannian manifold with the metric

$$g = ds^2 + \psi^2(s)d\theta^2. \tag{3.2}$$

$\mathcal{M}$ has no boundary, and we always assume that $\mathcal{M}$ and the Riemannian metric g on it are smooth (see [5], for instance). The area element on $\mathcal{M}$ is given by $d\sigma = \psi d\theta ds$, and the gradient of u with respect to the metric g is given by

$$\nabla_g u = \left(\partial_s u, \frac{1}{\psi^2}\partial_\theta u\right).$$

The Laplace–Beltrami operator Δ_g on $\mathcal{M}$ can be expressed as

$$\Delta_g u = u_{ss} + \frac{\psi_s}{\psi}u_s + \frac{1}{\psi^2}u_{\theta\theta}. \tag{3.3}$$

We consider $\mathcal{S} \subset \mathcal{M}$ as a surface of revolution with a boundary parameterized by

$$x = (\psi(s)\cos(\theta), \psi(s)\sin(\theta), \chi(s)), \quad (s, \theta) \in [0,1] \times [0, 2\pi). \tag{3.4}$$

Hence, $\partial\mathcal{S} = \mathcal{C}_0 \cup \mathcal{C}_1$, where $\mathcal{C}_0$ and $\mathcal{C}_1$ are two circles parameterized by ($\theta \in [0, 2\pi)$)

$$(\psi(0)\cos(\theta), \psi(0)\sin(\theta), \chi(0))$$

and

$$(\psi(1)\cos(\theta), \psi(1)\sin(\theta), \chi(1)),$$

respectively.

Theorem 3.1. *Consider the following problem on a surface $\mathcal{S}$ as defined above*

$$\begin{cases} u_t = \Delta_g u + h(x, u), & (t, x) \in \mathbb{R}^+ \times \mathcal{S}, \\ u(t, x) = 0, & (t, x) \in \mathbb{R}^+ \times \partial\mathcal{S} = \mathbb{R}^+ \times (\mathcal{C}_0 \cup \mathcal{C}_1), \end{cases} \tag{3.5}$$

where h is a function of class C^1 and $h(\cdot, \eta)$ is independent of angular variation. Suppose that

(a) $\tilde{h}(s, u, u_s) := \frac{\psi_s}{\psi}u_s + h(s, u)$ satisfies (2.1) and (2.2) with G_i ($i = 1, 2$) given by (2.3), (2.4) for $(s, u, u_s) \in [0, 1] \times J \times \mathbb{R}$ where $J \subset \mathbb{R}$ is a closed interval containing 0;

(b) $\alpha(M_2, K_2) + \beta(M_1, K_2) > 1$ where α and β are numbers defined in (2.7) and (2.8).

Then problem (3.5) does not admit patterns.

Proof. First, we observe that stable stationary solutions of (3.5) must be independent of angular variation. This is a well-known result that can be seen in [3, 14]. Thus, due to (3.3), we can conclude that if u is a stable stationary solution of (3.5), then u satisfies:

$$\begin{cases} u_{ss} + \dfrac{\psi_s}{\psi}u_s + h(s, u) = 0, & s \in [0, 1], \\ u(0) = u(1) = 0. \end{cases} \tag{3.6}$$

Our goal now is to prove that problem (3.6) has a unique solution $u \equiv 0$. To achieve this, we use Theorem 2.1.

According to hypothesis (a), $\tilde{h}(s,u,u_s) = \frac{\psi_s}{\psi}u_s + h(s,u)$ satisfies (2.1) and (2.2) with G_i given by (2.3) and (2.4), and it is the first part of Theorem 2.1. Note that, by hypothesis (b), we can use Theorem 2.2 twice (with $L = 1$ and $G = G_1$ and again with $G = G_2$) to conclude that the two problems ($i = 1,2$)

$$\begin{cases} u_i'' + G_i(u_i(s), u_i'(s)) = 0, \\ u_i(a) = A', \quad u_i(b) = B' \end{cases} \tag{3.7}$$

have unique solutions on every sub-interval $[a,b] \subset [0,1]$ for arbitrary A', B'.

Finally, the problems ($i = 1,2$)

$$\begin{cases} z'' + G_i(z,z') = 0, \quad s \in [0,1], \\ z(0) = z(1) = 0, \end{cases} \tag{3.8}$$

has $z = 0 \in J$ as solution, which allows us to utilize Theorem 2.1 and conclude that $u \equiv 0$ is the unique solution of (3.6). Hence, it follows that problem (3.5) does not admit the existence of patterns, and the theorem is proved. □

3.1 The Allen–Cahn problem

The aim now is to apply Theorem 3.1 to some relevant cases commonly found in the literature. In this subsection, we address the nonlinearity of Allen–Cahn. In this case, we consider problem (3.5) with $h(x,u) = u - u^3$, i.e.

$$\begin{cases} u_t = \Delta_g u + u - u^3, \quad (t,x) \in \mathbb{R}^+ \times \mathcal{S}, \\ u(t,x) = 0, \quad (t,x) \in \mathbb{R}^+ \times \partial\mathcal{S} = \mathbb{R}^+ \times (\mathcal{C}_0 \cup \mathcal{C}_1). \end{cases} \tag{3.9}$$

As we know, a stable solution of (3.9) must satisfy

$$\begin{cases} u_{ss} + \dfrac{\psi_s}{\psi}u_s + u - u^3 = 0, \quad s \in [0,1], \\ u(0) = u(1) = 0. \end{cases} \tag{3.10}$$

A simple computation shows that $\tilde{h}(s,u,u_s) = \frac{\psi_s}{\psi}u_s + u - u^3$ satisfies (2.2) if we consider, for example: $J = [0,1]$, G_1 and G_2 given by (2.4) and (2.3), respectively, with

$$M_1 = \inf_{s\in[0,1]} \left\{ \frac{\psi_s}{\psi} \right\}, \quad M_2 = \sup_{s\in[0,1]} \left\{ \frac{\psi_s}{\psi} \right\}, \quad K_1 = -2 \text{ and } K_2 = 1. \tag{3.11}$$

Hence, if we assume ψ such that

$$\alpha(M_2, K_2) + \beta(M_1, K_2) > 1,$$

we have the hypothesis (b) and we can use Lemma 2.2 to ensure that the problems ($i = 1,2$)

$$\begin{cases} z'' + G_i(z,z') = 0, \quad s \in [0,1], \\ z(0) = z(1) = 0, \end{cases} \tag{3.12}$$

have unique solutions $u_i \equiv 0 \in [0,1]$ ($i = 1,2$).

Remark 3.2. The conditions (a) and (b) of Theorem 3.1 involve the geometry of the domain (represented by the function ψ) along with the reaction term of the problem. In particular, $\frac{\psi'}{\psi}$ (see also (3.11)) represents the geodesic curvature of the parallel circles $s =$ constant on $\mathcal{S}$.

Example 3.3. Consider the problem (3.9) where $\mathcal{S}_1$ is a finite straight cylinder, that is, $\psi(s) = 1$ ($\chi(s) = s + 1$) for all $s \in [0, 1]$. In this case, $M_1 = M_2 = 0$, $K_1 = -2$ and $K_2 = 1$. Thus, it easy to see that

$$\alpha(0,1) + \beta(0,1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi > 1$$

and we can conclude that there are no patterns for this case.

Similarly, if $\psi(s) = s^2/4 + 1/2$ and $\chi(s) = \frac{s}{4}\sqrt{4 - s^2} + \arcsin(s/2)$ for $s \in [0, 1]$, then $\mathcal{S}_2$ resembles a frustum of a hyperboloid (see figure below) and we have $M_1 = 0$, $M_2 = 2/3$, $K_1 = -2$ and $K_2 = 1$. It follows that

$$\alpha(2/3, 1) + \beta(0, 1) > 1$$

and again there are no patterns for the problem (3.9) in this case.

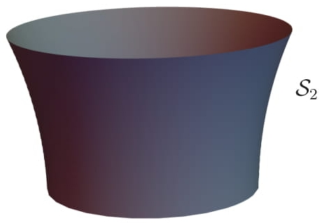

Figure 3.1: Surface of revolution $\mathcal{S}_2$

Our results can also be applied to spatially heterogeneous problems. For instance, we can consider $a \in C^1(\mathcal{S})$ as a positive diffusivity coefficient and $b \in C^1(\mathcal{S})$ as a positive reaction coefficient multiplying $u - u^3$. In this case, the problem becomes:

$$\begin{cases} u_t = \operatorname{div}_g(a(x)\nabla u) + b(x)(u - u^3), & (t, x) \in \mathbb{R}^+ \times \mathcal{S}, \\ u(t, x) = 0, & (t, x) \in \mathbb{R}^+ \times \partial\mathcal{S} = \mathbb{R}^+ \times (\mathcal{C}_0 \cup \mathcal{C}_1). \end{cases} \tag{3.13}$$

If we assume the functions a and b are independent of angular variation, then we have

$$\begin{cases} u_t = a u_{ss} + \dfrac{(a\psi)_s}{\psi} u_s + \dfrac{a}{\psi^2} u_{\theta\theta} + b(u - u^3), & (t, s, \theta) \in \mathbb{R}^+ \times [0, 1] \times [0, 2\pi), \\ u(t, 0, \theta) = u(t, 1, \theta) = 0, & (t, \theta) \in \mathbb{R}^+ \times [0, 2\pi), \end{cases} \tag{3.14}$$

and the stable solutions satisfy

$$\begin{cases} u_{ss} + \tilde{h}(s, u, u_s) = 0, & s \in [0, 1], \\ u(0) = u(1) = 0, \end{cases} \tag{3.15}$$

where $\tilde{h}(s, u, u_s) = \frac{(a(s)\psi(s))_s}{a(s)\psi(s)} u_s(s) + \frac{b(s)}{a(s)}(u(s) - u^3(s))$.

In this case, $\tilde{h}$ satisfies (2.1) and satisfies (2.2) if we consider

$$M_1 = \inf_{s\in[0,1]} \left\{ \frac{(a\psi)_s}{a\psi} \right\}, \tag{3.16}$$

$$M_2 = \sup_{s\in[0,1]} \left\{ \frac{(a\psi)_s}{a\psi} \right\}, \tag{3.17}$$

$$K_1 = \inf_{s\in[0,1]} \left\{ \frac{-2b(s)}{a(s)} \right\} \tag{3.18}$$

and

$$K_2 = \sup_{s\in[0,1]} \left\{ \frac{b(s)}{a(s)} \right\}. \tag{3.19}$$

Now, if we proceed as before, we find that if a, b, ψ are taken such that

$$\alpha(M_2, K_2) + \beta(M_1, K_2) > 1$$

occurs, we can also conclude the non-existence of patterns for this spatially heterogeneous problem.

3.2 The Fisher–KPP problem

A similar analysis can also be conducted for the Fisher–KPP problem. In this case, considering $a, b \in C^1(\mathcal{S})$, we have

$$\begin{cases} u_t = \mathrm{div}_g(a(x)\nabla u) + b(x)(u-u^2), & (t,x) \in \mathbb{R}^+ \times \mathcal{S}, \\ u(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \partial\mathcal{S} = \mathbb{R}^+ \times (\mathcal{C}_0 \cup \mathcal{C}_1). \end{cases} \tag{3.20}$$

Similar to the previous case, several instability results can be derived from the relationship between functions a and b, and the geometry of $\mathcal{S}$ represented by the function ψ. However, now, we will demonstrate with examples how we can utilize the ideas developed here to obtain results of the existence of patterns for problems with non-zero Dirichlet boundary conditions. In this case, we once again make use of the symmetry of stable solutions, and as usual (see [3,14,18]), we analyze the existence of stable solution to the problem

$$\begin{cases} u_t = au_{ss} + \dfrac{(a(s)\psi)_s}{\psi} u_s + b(s)(u-u^2), & (t,s) \in \mathbb{R}^+ \times [0,1], \\ u(t,0) = A, & t \in \mathbb{R}^+, \\ u(t,1) = B, & t \in \mathbb{R}^+. \end{cases} \tag{3.21}$$

Example 3.4. Consider $a \equiv b \equiv 1$ and $\mathcal{S}$ again a finite straight cylinder ($\psi(s) = 1$ and $\chi(s) = s+1$ for all $s \in [0,1]$). Then we consider the following Fisher–KPP problem with non-zero Dirichlet boundary conditions

$$\begin{cases} u_t = \Delta_g u + (u-u^2), & (t,x) \in \mathbb{R}^+ \times \mathcal{S}, \\ u(t,x) = 1/3, & (t,x) \in \mathbb{R}^+ \times \mathcal{C}_0, \\ u(t,x) = 1/2, & (t,x) \in \mathbb{R}^+ \times \mathcal{C}_1. \end{cases} \tag{3.22}$$

In this case, we have to analyze the following problem

$$\begin{cases} u_{ss} + (u-u^2) = 0, & s \in [0,1], \\ u(0) = 1/3, \quad u(1) = 1/2. \end{cases} \tag{3.23}$$

It is not difficult to see that $h(u) = u - u^2$ satisfies (2.1) and (2.2) with $J = [0,1]$ and

$$M_1 = M_2 = 0, \quad K_1 = -1, \quad K_2 = 1.$$

Moreover,

$$\begin{aligned}\alpha(M_2, K_2) + \beta(M_1, K_2) &= \frac{2}{\sqrt{4K_2 - M_2^2}} \cos^{-1}\left(\frac{M_2}{2\sqrt{K_2}}\right) + \frac{2}{\sqrt{4K_2 - M_1^2}} \cos^{-1}\left(\frac{-M_1}{2\sqrt{K_2}}\right) \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi > 1.\end{aligned}$$

Hence, we can use Lemma 2.2 to ensure that

$$\begin{cases} z'' + z = 0, & s \in (0,1), \\ z(a) = A', & z(b) = B' \end{cases} \tag{3.24}$$

and

$$\begin{cases} z'' - z = 0, & s \in (0,1), \\ z(a) = A', & z(b) = B' \end{cases} \tag{3.25}$$

have unique solutions on every sub-interval $[a,b]$ of $[0,1]$ for arbitrary A', B'. Finally, in order to apply Theorem 2.1 we have to consider the problems (3.24) and (3.25) with $a = 0$, $b = 1$, $A' = 1/3$ e $B' = 1/2$. After a few calculations, it's not hard to see that

$$z_1(s) = \frac{2\cos(s) - 2\cot(1)\sin(s) + 3\csc(1)\sin(s)}{6}$$

and

$$z_2(s) = \frac{e^{-s}(-3e + 2e^2 - 2e^{2s} + 3e^{(1+2s)})}{6(e^2 - 1)}$$

are solutions of (3.24) and (3.25) respectively (with $a = 0$, $b = 1$, $A' = 1/3$ e $B' = 1/2$), and both solutions have range contained in $[0,1]$. According to Theorem 2.1, problem (3.23) has a unique solution U. Now we proceed as in the Example 2.6. The energy functional $E : \{u \in H^1(\mathcal{S}); u(x) = 1/3 \text{ for } x \in \mathcal{C}_0,\ u(x) = 1/2 \text{ for } x \in \mathcal{C}_1\} \to \mathbb{R}$ associated with the problem (3.23) is defined by

$$E[u] = \int_{\mathcal{S}} \frac{1}{2}|\nabla_g u|^2 + F(u)dx$$

where $F(u) = \int_0^u s - s^2 ds$. It is routine to verify that (3.22) is a gradient system (see Example 2.6), so we can conclude that U is a pattern.

Once again, depending on computational capacity, one can contemplate more general problems involving heterogeneities, different boundary values, and alternative surfaces.

4 Sine-Gordon equation in an n-dimensional ball

This section is dedicated to studying the sine-Gordon equation. In this equation, we have $f(u) = \sin(u)$ and since f is globally bounded, we can, in this case, analyze the problem in an n-dimensional ball $\mathcal{B}$ centered at the origin with a radius equal to 1. Hence, we consider the following problem

$$\begin{cases} u_t = \mathrm{div}(a(x)\nabla u) + b(x)\sin(u), & (t,x) \in \mathbb{R}^+ \times \mathcal{B}, \\ u(t,x) = B, & (t,x) \in \mathbb{R}^+ \times \partial\mathcal{B}, \end{cases} \tag{4.1}$$

where a and b are functions of class C^1 with radial symmetry, and $B \in \mathbb{R}$.

It is well-known that stable solutions of (4.1) are radially symmetric; thus, if u is a stable solution it satisfies (for simplicity, we consider $n = 2$)

$$\begin{cases} u_{rr} + \dfrac{(a(r)r)_r}{a(r)r} u_r + \dfrac{b(r)}{a(r)} \sin(u) = 0, & r \in (0,1), \\ u_r(0) = 0, \quad u(1) = B. \end{cases} \tag{4.2}$$

We can state the following theorem.

Theorem 4.1. *Consider* (4.1) *and suppose that*

$$h(r,u,u_r) := \frac{(a(r)r)_r}{a(r)r} u_r + \frac{b(r)}{a(r)} \sin(u)$$

satisfies (2.2) *with* G_1 *and* G_2 *defined in* (2.4) *and* (2.3)*, respectively, and* $\beta(M_1, K_2) > 1$*. Then, if* $B = 2k\pi$ *(*$k \in \mathbb{Z}$*)* (4.1) *does not admit patterns.*

Proof. A direct application of Theorem 2.3 gives us that the problem (4.2) with $B = 2k\pi$ ($k \in \mathbb{Z}$) has $u \equiv 2k\pi$ as its unique solution. Therefore, (4.1) does not admit patterns. □

Example 4.2. If a and b are taken such that $a(r) = e^{5r}/r$ and $b(r) = (r+6)e^{5r}/r$ then $h(r,u,u_r) = 5u_r + (r+6)\sin(u)$. Hence h satisfies (2.2) with

$$M_1 = M_2 = 5, \quad K_1 = -6, \quad K_2 = 6.$$

It follows that $\beta(M_1, K_2) = \infty$ e therefore (4.1) does not admit patterns if $B = 2k\pi$ ($k \in \mathbb{Z}$).

5 Concluding remarks

In this paper, we present a straightforward and efficient approach to studying pattern formation in problems with Dirichlet boundary conditions. The symmetry of the domains under consideration, along with the well-known properties of stable solutions, enabled us to leverage results on the existence, uniqueness, and stability of solutions in one-dimensional problems to achieve our objectives. Below, we provide some concluding remarks that complement the ideas discussed thus far.

(i) Evidently, the problem of sine-Gordon could be considered on revolution surfaces as before, and then results of existence or non-existence of patterns would also be generated for this case. On the other hand, the absence of a result like Theorem 2.1 for problems with mixed boundary conditions prevents us from considering the nonlinearities of Allen–Cahn and Fisher–KPP in an n-dimensional ball.

(ii) The examples presented in this work serve the purpose of illustrating how one can apply the developed theory. In this regard, the parameters (surfaces and heterogeneities) were chosen in a way to simplify the computations. Particularly, in Example 4.2, the choice of the diffusion coefficient $a(r) = e^{5r}/r$ made the problem more straightforward and allowed us to use Theorem 4.1.

(iii) Similarly, other equations can be considered beyond those highlighted in this work (namely Allen–Cahn, Fisher–KPP and sine-Gordon equations). For instance, with a nonlinearity of the form $f(u,x) = u(u-\theta(x))(1-u)$, where $0 < \theta(x) < 1$, which is related to the *Fife–Greenlee equation* [10], or the *perturbed sine-Gordon equation* where $f(u) = \sin(u) - g(u)$ [17], or even in problems with advection terms, that is, in reaction–convection–diffusion problems, see [12] and references therein.

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Well-posedness and controllability of a nonlinear system for surface waves

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Abstract. In this paper we study the well-posedness for the periodic Cauchy problem and the internal controllability of a one-dimensional system that describes the propagation of long water waves with small amplitude in the presence of surface tension. The well-posedness is proved by using the Fourier transform restriction method and the controllability is proved by using the moment method.

Keywords: nonlinear system, water waves, well-posedness, Bourgain spaces, spectral analysis, internal control.

2020 Mathematics Subject Classification: 25E15, 93B05, 35Q35.

1 Introduction

In the present work we consider the periodic Cauchy problem and the internal controllability of the following one-dimensional system

$$\begin{cases} \eta_t + \partial_x^2 \Phi - \partial_x^4 \Phi + \partial_x (\eta \partial_x \Phi) = 0, \\ \Phi_t + \eta - \partial_x^2 \eta + \dfrac{1}{2} (\partial_x \Phi)^2 = 0, \end{cases} \tag{1.1}$$

which is a rescaled version of the system derived in [14] from the evolution of long water waves with small amplitude in the presence of surface tension, where $\Phi = \Phi(x,t)$ represents the nondimensional velocity potential on the bottom $z = 0$ and the variable $\eta = \eta(x,t)$ corresponds the free surface elevation.

As happens in water wave models, there is a Hamiltonian type structure which is clever to characterize the space for the study of the Cauchy problem. In our particular system (1.1), the Hamiltonian functional $\mathcal{H} = \mathcal{H}(t)$ is defined as

$$\mathcal{H}\begin{pmatrix} \eta \\ \Phi \end{pmatrix} = \frac{1}{2} \int_{\mathbb{R}} \left(\eta^2 + (\partial_x \eta)^2 + (\partial_x \Phi)^2 + (\partial_x^2 \Phi)^2 + \eta (\partial_x \Phi)^2 \right) dx,$$

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and the Hamiltonian type structure is given by

$$\begin{pmatrix} \eta_t \\ \Phi_t \end{pmatrix} = \mathcal{J}\mathcal{H}' \begin{pmatrix} \eta \\ \Phi \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We see directly that the functional $\mathcal{H}$ is well defined when

$$\eta(\cdot,t), \Phi_x(\cdot,t) \in H^1(\mathbb{R}),$$

for t in some interval. These conditions already characterize the natural space for the study of solutions of the system (1.1). Certainly, J. Quintero and A. Montes in [16] showed for the model (1.1) the existence of solitary wave solutions which propagate with speed of wave $\theta > 0$, i.e. solutions of the form

$$\eta(x,t) = u(x-\theta t), \quad \Phi(x,t) = v(x-\theta t),$$

in the energy space $H^1(\mathbb{R}) \times \mathcal{V}^2(\mathbb{R})$, where $H^1(\mathbb{R})$ is the usual Sobolev space of order 1 and the space $\mathcal{V}^2(\mathbb{R})$ is defined with respect to the norm given by

$$\|w\|^2_{\mathcal{V}^2(\mathbb{R})} = \|w'\|^2_{H^1(\mathbb{R})} = \int_{\mathbb{R}} \left((w')^2 + (w'')^2\right) dx.$$

They also showed, using the estimates of the Kato's commutator, the local well-posedness for the Cauchy problem associated to the system (1.1) in the Sobolev type space $H^s(\mathbb{R}) \times \mathcal{V}^{s+1}(\mathbb{R})$, with $s > 3/2$, where $H^s(\mathbb{R})$ is the usual Sobolev space of order s defined as the completion of the Schwartz class with respect to the norm

$$\|w\|_{H^s(\mathbb{R})} = \| \,(1+|\xi|)^s\, \widehat{w}(\xi)\|_{L^2_\xi}$$

and $\mathcal{V}^{s+1}(\mathbb{R})$ denotes the completion of the Schwartz class with respect to the norm

$$\|w\|_{\mathcal{V}^{s+1}(\mathbb{R})} = \| \,(1+|\xi|)^s\, |\xi|\widehat{w}(\xi)\|_{L^2_\xi},$$

where $\widehat{w}$ is the Fourier transform of w in the space variable x and ξ is the variable in the frequency space related to the variable x. Using Bourgain type spaces, in work [13] the authors showed that the Cauchy problem associated to the system (1.1) is locally well-posedness in the space $H^s(\mathbb{R}) \times \mathcal{V}^{s+1}(\mathbb{R})$ for $s \geq 0$.

On the case of the periodic domain $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ (the one-dimensional torus), it was proved in [15] the local well-posedness of the Cauchy problem associated to system (1.1) in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$, for $s > 3/2$, where the periodic Sobolev space $H^s(\mathbb{T})$ is defined by

$$H^s(\mathbb{T}) = \left\{ w = \sum_{k\in\mathbb{Z}} w_k e^{ikx} \;\; : \;\; \sum_{k\in\mathbb{Z}} (1+|k|^2)^s |w_k|^2 \, < \, +\infty \right\}$$

and the space $\mathcal{V}^{s+1}(\mathbb{T})$ is defined by the norm

$$\|w\|_{\mathcal{V}^{s+1}(\mathbb{T})} = \left[\sum_{k\in\mathbb{Z}} (1+|k|^2)^s |k|^2 |w_k|^2 \right]^{1/2},$$

where $w_k = \widehat{w}(k)$ denotes the k-Fourier coefficient with respect to the spatial variable x. In this paper, we prove that the Cauchy problem associated to system (1.1) with the initial condition

$$\eta(x,0) = \eta_0(x), \quad \Phi(x,0) = \Phi_0(x) \tag{1.2}$$

is locally well-posed in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ for $s \geq 0$. Hence we improve the result found in [15].

To study the Cauchy problem (1.1)–(1.2) we use its integral equivalent formulation,

$$(\eta(t), \Phi(t)) = S(t)(\eta_0, \Phi_0) - \int_0^t S(t-t')\Big(\partial_x(\eta \partial_x \Phi), \frac{1}{2}(\partial_x \Phi)^2\Big)(t')dt', \tag{1.3}$$

where $S(t)(\eta_0, \Phi_0)$ is the solution of the linear problem, that is

$$S(t)(\eta_0, \Phi_0) = (S_1(t)(\eta_0, \Phi_0), S_2(t)(\eta_0, \Phi_0))$$

with

$$S_1(t)(\eta_0, \Phi_0) = \sum_{k \in \mathbb{Z}} e^{ikx} \left[\cos(\phi(k)t)\widehat{\eta}_0(k) + |k| \sin(\phi(k)t)\widehat{\Phi}_0(k)\right],$$

$$S_2(t)(\eta_0, \Phi_0) = \sum_{k \in \mathbb{Z}} e^{ikx} \left[-\frac{\sin(\phi(k)t)\widehat{\eta}_0(k)}{|k|} + \cos(\phi(k)t)\widehat{\Phi}_0(k)\right],$$

and the function ϕ defined as

$$\phi(k) = |k|^3 + |k|$$

is the Fourier symbol associated to the spacial linear part of the system (1.1).

The method of proof will be the application of the contraction mapping principle in a suitable Banach function space $C\left([0,T] : H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})\right) \cap \mathcal{Z}^s$, where the appropriated space-time weight norm for $\mathcal{Z}^s$ is determined by the knowledge of certain estimates for the solutions of the linear part. This method, introduced by J. Bourgain in [2]-[3] and simplified by C. Kenig, G. Ponce and L. Vega in [8]-[9], not only benefits of the above mentioned space-time estimates, but also exploits structural properties of the nonlinearity.

As usual when dealing with dispersive models in Bourgain spaces, we slightly modify the terms in the right-hand of (1.3) by means of a cut off function. In the following, let $\psi \in C_0^\infty(\mathbb{R})$ with support in $(-2, 2)$, such that $0 \leq \psi \leq 1$, and $\psi \equiv 1$ in $[-1, 1]$. Thus, for $0 < T < 1$ we consider the following modified version of (1.3),

$$(\eta(t), \Phi(t)) = \psi(t)S(t)(\eta_0, \Phi_0) - \psi(t) \int_0^t S(t-t')\psi(t')\Big(\partial_x(\eta \partial_x \Phi), \frac{1}{2}\left(\partial_x \Phi\right)^2\Big)(t')dt'. \tag{1.4}$$

We will show the existence of a solution of the integral problem (1.4) using the Banach fixed point theorem and appropriate linear and nonlinear estimates.

The second part of this paper is concerned with the internal control problem for the system (1.1) on the periodic domain $\mathbb{T}$: choose an appropriate internal control function

$$F = F(x,t) = (f_1(x,t), f_2(x,t))$$

to guide the model

$$\begin{cases} \eta_t + \partial_x^2 \Phi - \partial_x^4 \Phi + \partial_x(\eta \partial_x \Phi) = f_1, & x \in \mathbb{T},\ t \geq 0, \\ \Phi_t + \eta - \partial_x^2 \eta + \frac{1}{2}\left(\partial_x \Phi\right)^2 = f_2, & x \in \mathbb{T},\ t \geq 0, \end{cases} \tag{1.5}$$

during a time interval $[0, T]$, from a given initial state to another preassigned terminal state, in an appropriate function space of system states.

During the last years, there have been many contributions to the internal controllability for different dispersive wave models. For instance, in the case of the Korteweg–de Vries equation

D. Russell and B. Zhang in [18] showed that for $T > 0$ and functions $u_0, u_T \in H^s(\mathbb{T})$, $s \geq 0$, one can always find a control f so that the Cauchy problem

$$u_t + uu_x + u_{xxx} = f, \quad u(x,0) = u_0(x),$$

has a solution $u \in C\left([0,T] : H^s(\mathbb{T})\right)$ satisfying

$$u(x,T) = u_T(x), \quad x \in \mathbb{T},$$

when the initial and terminal states are sufficiently small. A similar result was proved by B. Zhang in [20] for the Boussinesq model,

$$u_{tt} - u_{xx} + \left(u^2 + u_{xx}\right)_{xx} = f, \quad u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x),$$

with the condition

$$u(x,T) = u_T(x), \quad u_t(x,T) = v_T(x),$$

in the space $H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ with $s \geq 2$. In the work [5], E. Cerpa and I. Rivas showed controllability for the Boussinesq equation in low regularity, this is, in the space $H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ with $s > -\frac{1}{4}$.

For the Benjamin–Bona–Mahony equation, L. Rosier and B. Zhang in [17] proved that

$$u_t + u_x - u_{xxt} + uu_x = a(x+ct)h(x,t),$$

with a moving distributed control is controllable in $H^s(\mathbb{T})$ for any $s \geq 1$ in (sufficiently) large time. The control time is chosen in such a way that the support of the control, which is moving at the constant velocity c, can visit all the domain $\mathbb{T}$.

C. Laurent, F. Linares and L. Rosier in [11] and F. Linares and L. Rosier in [12] considered the control problem for the Benjamin–Ono equation,

$$u_t + \mathcal{H}(u_{xx}) + uu_x = f, \quad u(x,0) = u_0(x), \quad u(x,T) = u_T(x).$$

In the latter work, authors proved a controllability result in $L^2(\mathbb{T})$ that allows to prove the global controllability in large time.

Our main result in Theorem 5.4 gives a positive answer to the internal controllability for the system (1.5) in a local sense. We will show that for $T > 0$ and initial an terminal states

$$(\eta_0, \Phi_0), \quad (\eta_T, \Phi_T) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T}), \quad s \geq 0,$$

sufficiently small, there exists a control function $F = (f_1, f_2)$ such that the Cauchy problem associated to the system (1.5) with the initial condition

$$\eta(x,0) = \eta_0(x), \quad \Phi_0(x,0) = \Phi_0(x), \quad x \in \mathbb{T}, \tag{1.6}$$

has a solution $(\eta, \Phi) \in C\left([0,T] : H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})\right)$ satisfying the condition

$$\eta(x,T) = \eta_T(x), \quad \Phi(x,T) = \Phi_T(x), \quad x \in \mathbb{T}.$$

Following the same approach used in the case of the KdV equation and Boussinesq equation, we restrict our attention to a control of the form

$$F(x,t) = (f_1(x,t), f_2(x,t)) = (\rho_1 h_1(x,t), \rho_2 h_2(x,t)),$$

with ρ_i being a smooth function defined on $\mathbb{T}$. Thus

$$H(x,t) = (h_1(x,t), h_2(x,t))$$

is a new control input. Here, the function ρ_i can have a support strictly contained on the torus; thus, it can represent a localization of the control $h_i(x,t)$, which would be only able to act on a part of the domain. First, we perform a spectral analysis for the operator

$$M = \begin{pmatrix} 0 & -(I-\partial_x^2)\partial_x^2 \\ -(I-\partial_x^2) & 0 \end{pmatrix},$$

defined in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$. Using that the k-Fourier symbol for the operator M is given by

$$M_k = \begin{pmatrix} 0 & (1+k^2)k^2 \\ -(1+k^2) & 0 \end{pmatrix},$$

we prove for M the existence of a discrete spectral decomposition since the eigenvectors form a Riesz basis of the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$. Next, using this spectral analysis and the moment method we establish that the linear system associated with (1.5),

$$\begin{cases} \eta_t + \partial_x^2\Phi - \partial_x^4\Phi = f_1, \\ \Phi_t + \eta - \partial_x^2\eta = f_2, \end{cases} \tag{1.7}$$

is exactly controllable in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$, with the conditions

$$\eta(0) = \eta_0, \quad \eta(T) = \eta_T, \quad \Phi(0) = \Phi_0, \quad \Phi(T) = \Phi_T. \tag{1.8}$$

Finally, the nonlinear problem is treated as a perturbation by fixed point theory.

The paper is organized as follows. In Section 2, first we define the Bourgain spaces related to our problem and next we establish all the linear estimates needed to proved the result of well-posedness. In Section 3 we estimate the bilinear forms $\partial_x(\eta\partial_x\Phi)$ and $(\partial_x\Phi)(\partial_x\Phi_1)$ associated to the nonlinear part of the system. The Section 4 will be dedicated to establish the result of local well-posedness, via a standard fixed point argument. In Section 5.1, we perform the spectral analysis for the operator M defined in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$, for $s \geq 0$. In Section 5.2, by solving a moment problem we found the characterization of the internal control $F = (f_1, f_2)$ for the linear problem (1.7)–(1.8). In Section 5.3, we prove the exact controllability result for the nonlinear problem, by imposing smallness of the initial and terminal states. The proof of this result is mainly based on the linear controllability and the Banach Fixed Point Theorem.

2 Bourgain spaces and linear estimates

We star with the definition of the Bourgain type spaces. We consider the space $\mathcal{Y}$ of functions w such that

(i) $w : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, (ii) $w(x,\cdot) \in \mathcal{S}(\mathbb{R})$ for all $x \in \mathbb{T}$,

(iii) $x \to w(x,\cdot) \in C^\infty(\mathbb{R})$, (iv) $\widehat{w}(0,t) = 0$ for all $t \in \mathbb{R}$,

Definition 2.1. For $s, \beta \in \mathbb{R}$ we define the Bourgain spaces $X^{s,\beta}$ to be the completion of the space $\mathcal{Y}$ with respect to the norm

$$\|w\|_{X^{s,\beta}} = \|\langle k\rangle^s \langle|\tau| - \phi(k)\rangle^\beta \widetilde{w}\|_{\ell^2_k L^2_\tau},$$

where $\langle a\rangle = 1 + |a|$; $\tilde{w}$ denotes the time-space Fourier transform of w,

$$\widetilde{w}(k,\tau) = \frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{T}} e^{-ixk-it\tau} w(x,t)dxdt;$$

and the function ϕ is defined as

$$\phi(k) = |k|^3 + |k|.$$

The spaces $Y^{s+1,\beta}$ to be the completion of the Schwartz class $\mathcal{S}_{per,2\pi} = \mathcal{S}(\mathbb{T}\times\mathbb{R})$ with respect to the norm

$$\|w\|_{Y^{s+1,\beta}} = \||k|\langle|\tau| - \phi(k)\rangle^\beta \langle k\rangle^s \widetilde{w}\|_{\ell^2_k L^2_\tau}.$$

We similarly introduce the spaces Z^s, W^{s+1}, $s \in \mathbb{R}$, with the norms

$$\|w\|_{Z^s} = \|w\|_{X^{s,-1/2}} + \left\|\frac{\langle k\rangle^s \widetilde{w}(k,\tau)}{\langle|\tau| - \phi(k)\rangle}\right\|_{\ell^2_k L^1_\tau},$$

and

$$\|w\|_{W^{s+1}} = \|w\|_{Y^{s+1,-1/2}} + \left\|\frac{|k|\langle k\rangle^s \widetilde{w}(k,\tau)}{\langle|\tau| - \phi(k)\rangle}\right\|_{\ell^2_k L^1_\tau}.$$

Also, we consider the spaces $U^s, V^{s+1}, s \in \mathbb{R}$, where U^s denotes the completion of the Schwartz class $\mathcal{S}_{per,2\pi}$ with respect to the norm

$$\|w\|_{U^s} = \|w\|_{X^{s,1/2}} + \|\langle k\rangle^s \widetilde{w}(k,\tau)\|_{\ell^2_k L^1_\tau}$$

and V^{s+1} denotes the completion of the Schwartz class $\mathcal{S}_{per,2\pi}$ with respect to the norm

$$\|w\|_{V^{s+1}} = \|w\|_{Y^{s+1,1/2}} + \||k|\langle k\rangle^s \widetilde{w}(k,\tau)\|_{\ell^2_k L^1_\tau}.$$

For $T > 0$ we denote by U^s_T the space space of the restrictions to the interval $[0,T]$ of the elements $w \in U^s$ with norm defined by

$$\|\eta\|_{U^s_T} = \inf_{w\in U^s}\{\|w\|_{U^s} : \eta(t) = w(t) \quad \text{on } [0,T]\}.$$

and by V^{s+1}_T the space space of the restrictions to the interval $[0,T]$ of the elements $w \in V^{s+1}$ with norm defined by

$$\|\Phi\|_{V^{s+1}_T} = \inf_{w\in V^{s+1}}\{\|w\|_{V^{s+1}} : \Phi(t) = w(t) \quad \text{on } [0,T]\}.$$

Next we look at some basic results.

Lemma 2.2. *Let $s \in \mathbb{R}$, then there exists $C > 0$ such that*

(i) $\|\psi\eta\|_{X^{s,-1/2}} \le C\|\eta\|_{X^{s,-1/2}}$,

(ii) $\|\psi\Phi\|_{Y^{s+1,-1/2}} \le C\|\Phi\|_{Y^{s+1,-1/2}}$.

Proof. We will use the notation $\widehat{w}^{(t)}$ for the Fourier transform of w in the time variable t. Note that

$$\widetilde{\psi\eta}(k,\tau) = \frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{T}} e^{-ixk}e^{-it\tau}\left(\int_{\mathbb{R}} e^{it\lambda}\widehat{\psi}^{(t)}(\lambda)d\lambda\right)\eta(x,t)dxdt$$
$$= \int_{\mathbb{R}} \widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda$$

and also

$$\|\psi\eta\|^2_{X^{s,-1/2}} = \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle|\tau|-\phi(k)\rangle^{-1}\left|\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\right|^2 d\tau.$$

Moreover

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{-\infty}^{0}\langle|\tau|-\phi(k)\rangle^{-1}\left|\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\right|^2 d\tau\\
&\quad\le \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle\tau+\phi(k)\rangle^{-1}\left|\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\right|^2 d\tau\\
&\quad= \left\|\langle\tau+\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\right\|^2_{\ell^2_kL^2_\tau}\\
&\quad\le \int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\lambda)|^2\|\langle\tau+\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\widetilde{\eta}(k,\tau-\lambda)\|^2_{\ell^2_kL^2_\tau}d\lambda
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{0}^{+\infty}\langle|\tau|-\phi(k)\rangle^{-1}\left|\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\right|^2 d\tau\\
&\quad\le \left\|\langle\tau-\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\int_{\mathbb{R}}\widehat{\psi}^{(t)}(\lambda)\widetilde{\eta}(k,\tau-\lambda)d\lambda\right\|^2_{\ell^2_kL^2_\tau}\\
&\quad\le \int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\lambda)|^2\|\langle\tau-\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\widetilde{\eta}(k,\tau-\lambda)\|^2_{\ell^2_kL^2_\tau}d\lambda.
\end{aligned}$$

Next, using the inequality

$$\big||\tau|-\phi(k)\big| \le \min\{|\tau-\phi(k)|,|\tau+\phi(k)|\},$$

we have for all $\lambda\in\mathbb{R}$, $\langle|\tau|-\phi(k)\rangle \le \langle\tau\pm\phi(k)\rangle \le \langle\tau+\lambda\pm\phi(k)\rangle\langle\lambda\rangle$, and then

$$\begin{aligned}
&\int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\lambda)|^2\|\langle\tau\pm\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\widetilde{\eta}(k,\tau-\lambda)\|^2_{\ell^2_kL^2_\tau}d\lambda\\
&\quad= \int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\lambda)|^2\Big(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle\tau+\lambda\pm\phi(k)\rangle^{-1}|\widetilde{\eta}(k,\tau)|^2d\tau\Big)d\lambda\\
&\quad\le \int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\lambda)|^2\Big(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle\lambda\rangle\langle|\tau|-\phi(k)\rangle^{-1}|\widetilde{\eta}(k,\tau)|^2d\tau\Big)d\lambda\\
&\quad= \|\langle|\tau|-\phi(k)\rangle^{-1/2}\langle k\rangle^{s}\widetilde{\eta}\|^2_{\ell^2_kL^2_\tau}\int_{\mathbb{R}}\langle\lambda\rangle|\widehat{\psi}^{(t)}(\lambda)|^2d\lambda\\
&\quad\le C\|\eta\|^2_{X^{s,-1/2}}.
\end{aligned}$$

Thus, we conclude that

$$\|\psi\eta\|_{X^{s,-1/2}} \le C\|\eta\|_{X^{s,-1/2}}.$$

In a similar fashion we have that

$$\int_{\mathbb{R}} |\widehat{\psi}(\lambda)^{(t)}|^2 \|\langle \tau \pm \phi(k)\rangle^{-1/2}|k|\langle k\rangle^s \widetilde{\Phi}(k,\tau-\lambda)\|^2_{\ell^2_k L^2_\tau} d\lambda \\ \leq \|\langle |\tau| - \phi(k)\rangle^{-1/2}|k|\langle k\rangle^s \widetilde{\Phi}\|^2_{\ell^2_k L^2_\tau} \int_{\mathbb{R}} \langle\lambda\rangle |\widehat{\psi}^{(t)}(\lambda)|^2 d\lambda \leq C\|\Phi\|^2_{Y^{s+1,-1/2}}.$$

Therefore

$$\|\psi\Phi\|_{Y^{s+1,-1/2}} \leq C\|\Phi\|_{Y^{s+1,-1/2}}. \qquad \square$$

Similarly, we have also the following lemma.

Lemma 2.3. *Let $s \in \mathbb{R}$, then there exists $C > 0$ such that*

(i) $\left\| \frac{\langle k\rangle^s \widetilde{\psi\eta}(k,\tau)}{\langle |\tau| - \phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau} \leq C \left\| \frac{\langle k\rangle^s \widetilde{\eta}(k,\tau)}{\langle |\tau| - \phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau}$,

(ii) $\left\| \frac{|k|\langle k\rangle^s \widetilde{\psi\Phi}(k,\tau)}{\langle |\tau| - \phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau} \leq C \left\| \frac{|k|\langle k\rangle^s \widetilde{\Phi}(k,\tau)}{\langle |\tau| - \phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau}$.

In the following lemmas we establish estimations related with the semigroup $S(t)$.

Lemma 2.4. *Let $s \in \mathbb{R}$, then exists $C_1 > 0$ such that*

$$\|\psi(t)S_1(t)(\eta_0,\Phi_0)\|_{U^s} \leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})},$$

$$\|\psi(t)S_2(t)(\eta_0,\Phi_0)\|_{V^{s+1}} \leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}.$$

Proof. We see that

$$\left[\psi(t)\sum_{k\in\mathbb{Z}} e^{ikx}e^{\pm i\phi(k)t}\widehat{\eta}_0(k)\right]^{\sim}(k,\tau) = \widehat{\eta}_0(k)\widehat{\psi}^{(t)}(\tau \mp \phi(k)).$$

So that

$$\left\|\psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i\phi(k)t}\widehat{\eta}_0(k)\right\|^2_{X^{s,1/2}} = \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\eta}_0(k)|^2\int_{\mathbb{R}}\langle |\tau| - \phi(k)\rangle|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|^2 d\tau \\ \leq \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\eta}_0(k)|^2\int_{\mathbb{R}}\langle\tau\rangle|\widehat{\psi}^{(t)}(\tau)|^2 d\tau \leq C\|\eta_0\|^2_{H^s(\mathbb{T})}.$$

Also, we note that

$$\left\|\langle k\rangle^s\left[\psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i\phi(k)t}\widehat{\eta}_0(k)\right]^{\sim}\right\|^2_{\ell^2_k L^1_\tau} = \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{\eta}_0(k)|^2\left(\int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|d\tau\right)^2 \\ \leq C\|\eta_0\|^2_{H^s(\mathbb{T})}.$$

In a similar fashion,

$$\left\|\psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i\phi(k)t}|k|\widehat{\Phi}_0(k)\right\|^2_{X^{s,1/2}} \\ = \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|k|^2|\widehat{\Phi}_0(k)|^2\int_{\mathbb{R}}\langle |\tau| - \phi(k)\rangle|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|^2 d\tau \leq C\|\Phi_0\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}$$

and

$$\Big\| \langle k\rangle^s \Big[\psi(t) \sum_{k\in\mathbb{Z}} e^{ixk} e^{\pm i\phi(k)t} |k| \widehat{\Phi}_0(k)\Big]^{\sim} \Big\|^2_{\ell^2_k L^1_\tau} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |k|^2 |\widehat{\Phi}_0(k)|^2 \left(\int_{\mathbb{R}} |\widehat{\psi}^{(t)}(\tau\mp\phi(k))| d\tau\right)^2$$
$$\leq C\|\Phi_0\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}.$$

Thus, from the previous estimates we obtain that

$$\|\psi(t)S_1(t)(\eta_0,\Phi_0)\|_{U^s} \leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}.$$

Similarly we have that

$$\Big\|\psi(t) \sum_{k\in\mathbb{Z}} \frac{e^{ikx} e^{\pm i\phi(k)t}\widehat{\eta}_0(k)}{|k|}\Big\|_{Y^{s+1,1/2}} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |\widehat{\eta}_0(k)|^2 \int_{\mathbb{R}} \langle |\tau| - \phi(k)\rangle |\widehat{\psi}^{(t)}(\tau\mp\phi(k))|^2 d\tau$$
$$\leq C\|\eta_0\|^2_{H^s(\mathbb{T})}$$

and

$$\Big\| |k|\langle k\rangle^s \Big[\psi(t) \sum_{k\in\mathbb{Z}} \frac{e^{ikx} e^{\pm i\phi(k)t}\widehat{\eta}_0(k)}{|k|}\Big]^{\sim} \Big\|_{\ell^2_k L^1_\tau} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |\widehat{\eta}_0(k)|^2 \Big(\int_{\mathbb{R}} |\widehat{\psi}^{(t)}(\tau\mp\phi(k))| d\tau\Big)^2$$
$$\leq C\|\eta_0\|^2_{H^s(\mathbb{T})}.$$

Also, we have that

$$\Big\|\psi(t) \sum_{k\in\mathbb{Z}} e^{ikx} e^{\pm i\phi(k)t}\widehat{\Phi}_0(k)\Big\|_{Y^{s+1,1/2}} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |k|^2 |\widehat{\Phi}_0(k)|^2 \int_{\mathbb{R}} \langle |\tau| - \phi(k)\rangle |\widehat{\psi}^{(t)}(\tau\mp\phi(k))|^2 d\tau$$
$$\leq C\|\Phi_0\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}$$

and

$$\Big\| |k|\langle k\rangle^s \Big[\psi(t) \sum_{k\in\mathbb{Z}} e^{ikx} e^{\pm i\phi(k)t}\widehat{\Phi}_0(k)\Big]^{\sim} \Big\|_{\ell^2_k L^1_\tau} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |k|^2 |\widehat{\Phi}_0(k)|^2 \Big(\int_{\mathbb{R}} |\widehat{\psi}^{(t)}(\tau\mp\phi(k))| d\tau\Big)^2$$
$$\leq C\|\Phi_0\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}.$$

Then we conclude that

$$\|\psi(t)S_2(t)(\eta_0,\Phi_0)\|_{V^{s+1}} \leq C_1\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}. \qquad \square$$

Lemma 2.5. *Let $s\in\mathbb{R}$, then there exists $C_2>0$ such that*

(i) $\Big\|\psi(t)\int_0^t f(t')dt'\Big\|_{H_t^{1/2}} \leq C_2\left(\|f\|_{H_t^{-1/2}} + \|\langle\tau\rangle^{-1}\widehat{f}^{(t)}\|_{L^1_\tau}\right).$

(ii) $\Big\|\psi(t)\int_0^t S_1(t-t')(\eta,\Phi)(t')\,dt'\Big\|_{U^s} \leq C_2\left(\|\eta\|_{Z^s} + \|\Phi\|_{W^{s+1}}\right),$

(iii) $\Big\|\psi(t)\int_0^t S_2(t-t')(\eta,\Phi)(t')\,dt'\Big\|_{V^{s+1}} \leq C_2\left(\|\eta\|_{Z^s} + \|\Phi\|_{W^{s+1}}\right).$

Proof. For the inequality (i) see Remark 3.13 of [4]. To prove the inequality (ii), first we note that

$$\begin{aligned}&\Big(\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\Big)^{\wedge}(k,t)\\ &\quad=\psi(t)\int_0^t e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\\ &\quad=e^{\pm i\phi(k)t}\psi(t)\int_0^t e^{\mp i\phi(k)t'}\widehat{\eta}(k,t')dt'=e^{\pm i\phi(k)t}\widehat{w}(k,t),\end{aligned}$$

where $w(x,t)=\psi(t)\int_0^t e^{\mp i\phi(k)t'}\eta(x,t')dt'$. Then we obtain that

$$\Big[\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\Big]^{\sim}(k,\tau)=\widetilde{w}(k,\tau\mp\phi(k)).$$

Using the fact that

$$\max\{\big||\tau+\phi(k)|-\phi(k)\big|,\big||\tau-\phi(k)|-\phi(k)\big|\}\le|\tau|$$

we have that

$$\begin{aligned}\Big\|\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\Big\|^2_{X^{s,1/2}}&=\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle|\tau\pm\phi(k)|-\phi(k)\rangle|\widetilde{w}(k,\tau)|^2d\tau\\ &\le\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle\tau\rangle|\widetilde{w}(k,\tau)|^2d\tau=\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\|\widehat{w}\|^2_{H_t^{1/2}}.\end{aligned}$$

By using part (i) we have that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\|\widehat{w}\|^2_{H_t^{1/2}}&=\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big\|\psi(t)\int_0^t e^{\mp i\phi(k)t'}\widehat{\eta}(k,t')dt'\Big\|^2_{H_t^{1/2}}\\ &\le C\Big(\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big\|e^{\mp i\phi(k)t}\widehat{\eta}(k,t)\Big\|^2_{H_t^{-1/2}}\\ &\qquad+\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big\|\langle\tau\rangle^{-1}\Big[e^{\mp i\phi(k)t}\widehat{\eta}(k,t)\Big]^{\wedge(t)}\Big\|^2_{L^1_\tau}\Big)\\ &\le C\Big[\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\int_{\mathbb{R}}\langle|\tau|-\phi(k)\rangle^{-1}|\widetilde{\eta}(k,\tau)|^2d\tau\\ &\qquad+\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big(\int_{\mathbb{R}}\langle|\tau|-\phi(k)\rangle^{-1}|\widetilde{\eta}(k,\tau)|d\tau\Big)^2\Big].\end{aligned}$$

Thus

$$\Big\|\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\Big\|^2_{X^{s,1/2}}\le C\|\eta\|^2_{Z^s}.$$

Let ϱ a smooth cutoff function in the time variable, supported in $A=[-1,1]$. Then

$$\begin{aligned}\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'&=\psi(t)\sum_{k\in\mathbb{Z}}e^{ixk}\int_{\mathbb{R}}e^{\pm it\phi(k)}\widetilde{\eta}(k,\tau)\Big(\int_0^t e^{it'(\tau\mp\phi(k))}dt'\Big)d\tau\\ &=\psi(t)\sum_{k\in\mathbb{Z}}e^{ixk}\int_{\mathbb{R}}\frac{e^{i\tau t}-e^{\pm it\phi(k)}}{i(\tau\mp\phi(k))}\widetilde{\eta}(k,\tau)d\tau=S_1+S_2-S_3,\end{aligned}$$

where

$$S_1 = \psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}\int_{\mathbb{R}} \frac{e^{i\tau t}-e^{\pm it\phi(k)}}{i(\tau\mp\phi(k))}\varrho(\tau\mp\phi(k))\widetilde{\eta}(k,\tau)d\tau,$$
$$S_2 = \psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}\int_{\mathbb{R}} \frac{[1-\varrho(\tau\mp\phi(k))]}{i(\tau\mp\phi(k))}\widetilde{\eta}(k,\tau)e^{i\tau t}d\tau,$$
$$S_3 = \psi(t)\sum_{k\in\mathbb{Z}} e^{i(xk\pm t\phi(k))}\int_{\mathbb{R}} \frac{[1-\varrho(\tau\mp\phi(k))]}{i(\tau\mp\phi(k))}\widetilde{\eta}(k,\tau)d\tau.$$

Now,

$$\begin{aligned} S_1 &= \psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}\int_{\mathbb{R}} \frac{e^{\pm it\phi(k)}\left(e^{it(\tau\mp\phi(k))}-1\right)}{i(\tau\mp\phi(k))}\varrho(\tau\mp\phi(k))\widetilde{\eta}(k,\tau)d\tau \\ &= \psi(t)\sum_{k\in\mathbb{Z}} e^{i(xk\pm\phi(k))}\int_{\mathbb{R}}\sum_{n\geq 1}\frac{t^n[i(\tau\mp\phi(k))]^{n-1}}{n!}\varrho(\tau\mp\phi(k))\widetilde{\eta}(k,\tau)d\tau \\ &= \psi(t)\sum_{n\geq 1}\frac{t^n i^{n-1}}{n!}\Big(\sum_{k\in\mathbb{Z}} e^{i(xk\pm t\phi(k))}\int_{\mathbb{R}}(\tau\mp\phi(k))^{n-1}\varrho(\tau\mp\phi(k))\widetilde{\eta}(k,\tau)d\tau\Big). \end{aligned}$$

Thus, using the notation

$$f_n(k) = \int_{\mathbb{R}} i^{n-1}(\tau\mp\phi(k))^{n-1}\varrho(\tau\mp\phi(k))\widetilde{\eta}(k,\tau)d\tau, \quad \omega_n(t)=\psi(t)t^n,$$

we see that

$$\begin{aligned} \widetilde{S_1}(k,\tau) &= \Big[\sum_{n\geq 1}\frac{\omega_n(t)}{n!}\Big(\sum_{k\in\mathbb{Z}} e^{i(xk\pm t\phi(k))}f_n(k)\Big)\Big]^{\sim}(k,\tau) \\ &= \sum_{n\geq 1}\frac{1}{n!}f_n(k)\widehat{\omega}_n^{(t)}(\tau\mp\phi(k)). \end{aligned}$$

Therefore

$$\begin{aligned} \|\langle k\rangle^s\widetilde{S}_1\|^2_{\ell^2_kL^1_\tau} &\leq C\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big(\|\chi_A(\tau\mp\phi(k))\widetilde{\eta}(k,\tau)\|_{L^1_\tau}\sum_{n\geq 1}\frac{1}{n!}\int_{\mathbb{R}}|\widehat{\omega}_n^{(t)}(\tau\mp\phi(k))|d\tau\Big)^2 \\ &\leq C\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big(\int_{\mathbb{R}}\langle\tau\mp\phi(k)\rangle^{-1}|\widetilde{\eta}(k,\tau)|d\tau\Big)^2 = C\|\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1}\widetilde{\eta}\|^2_{\ell^2_kL^1_\tau}. \end{aligned}$$

Now, if we use the notation

$$g(k,\tau) = [i(\tau\mp\phi(k))]^{-1}[1-\varrho(\tau\mp\phi(k))]\widetilde{\eta}(k,\tau)$$

then

$$\widetilde{S}_2(k,\tau) = \frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{T}} e^{-ixk}e^{-it\tau}\psi(t)g^{\sim^{-1}}(x,t)dxdt = \widehat{\psi}^{(t)}(\tau)*g(k,\tau).$$

So that, from Young's inequality,

$$\begin{aligned} \|\langle k\rangle^s\widetilde{S}_2\|^2_{\ell^2_kL^1_\tau} &\leq \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\|\widehat{\psi}^{(t)}(\tau)\|^2_{L^1_\tau}\|g(k,\tau)\|^2_{L^1_\tau} \\ &\leq C\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\left(\int_{\mathbb{R}}\langle\tau\mp\phi(k)\rangle^{-1}|\chi_B(\tau\mp\phi(k))\widetilde{\eta}(k,\tau)|d\tau\right)^2 \\ &\leq C\|\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1}\widetilde{\eta}\|^2_{\ell^2_kL^1_\tau}, \end{aligned}$$

where $B = \{\tau : |\tau| \geq 1\}$. Next, let

$$\widehat{h}(k) = \int_{\mathbb{R}} [i(\tau \mp \phi(k))]^{-1}[1 - \varrho(\tau \mp \phi(k))]\widetilde{\eta}(k,\tau)d\tau.$$

Then

$$\begin{aligned}\|\langle k\rangle^s \widetilde{S}_3\|^2_{\ell^2_k L^1_\tau} &= \|\langle k\rangle^s \widehat{h}(k)\widehat{\psi}^{(t)}(\tau \mp \phi(k))\|^2_{\ell^2_k L^1_\tau} \\ &\leq C\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}\Big(\int_{\mathbb{R}} \langle \tau \mp \phi(k)\rangle^{-1}|\chi_B(\tau \mp \phi(k))\widetilde{\eta}(k,\tau)|d\tau\Big)^2 \\ &\leq C\|\langle k\rangle^s \langle |\tau| - \phi(k)\rangle^{-1}\widetilde{\eta}\|^2_{\ell^2_k L^1_\tau}.\end{aligned}$$

Hence, from the previous estimates we conclude that

$$\Big\|\langle k\rangle^s \Big[\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}\widehat{\eta}(k,t')dt'\Big]^{\sim}\Big\|_{\ell^2_k L^1_\tau} \leq C\|\langle k\rangle^s\langle|\tau| - \phi(k)\rangle^{-1}\widetilde{\eta}\|_{\ell^2_k L^1_\tau}.$$

In what follows we will use similar arguments. First

$$\Big[\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}|k|\widehat{\Phi}(k,t')dt'\Big]^{\sim}(k,\tau) = \widetilde{v}(k,\tau \mp \phi(k)).$$

where $v(x,t) = \psi(t)\int_0^t e^{\mp i\phi(k)t'}|k|\Phi(x,t')dt'$. Then we obtain that

$$\begin{aligned}\Big\|\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}|k|\widehat{\Phi}(k,t')dt'\Big\|^2_{X^{s,1/2}} &\leq \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\|\widehat{v}\|^2_{H^{1/2}_t} \\ &\leq C\Big[\sum_{k\in\mathbb{Z}} |k|^2\langle k\rangle^{2s}\int_{\mathbb{R}} \langle \tau \mp \phi(k)\rangle^{-1}|\widetilde{\Phi}(k,\tau)|^2 d\tau \\ &\quad + \sum_{k\in\mathbb{Z}} |k|^2\langle k\rangle^{2s}\Big(\int_{\mathbb{R}}\langle \tau \mp \phi(k)\rangle^{-1}|\widetilde{\Phi}(k,\tau)|d\tau\Big)^2\Big] \\ &\leq C\|\Phi\|^2_{W^{s+1}}.\end{aligned}$$

Now,

$$\psi(t)\int_0^t \sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}|k|\widehat{\Phi}(k,t')dt' = S_4 + S_5 - S_6,$$

where

$$\begin{aligned}S_4 &= \psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}\int_{\mathbb{R}} \frac{e^{i\tau t} - e^{\pm it\phi(k)}}{i(\tau \mp \phi(k))}\varrho(\tau \mp \phi(k))|k|\widetilde{\Phi}(k,\tau)d\tau, \\ S_5 &= \psi(t)\sum_{k\in\mathbb{Z}} e^{ixk}\int_{\mathbb{R}} \frac{[1 - \varrho(\tau \mp \phi(k))]}{i(\tau \mp \phi(k))}|k|\widetilde{\Phi}(k,\tau)e^{i\tau t}d\tau, \\ S_6 &= \psi(t)\sum_{k\in\mathbb{Z}} e^{i(xk \pm t\phi(k))}\int_{\mathbb{R}} \frac{[1 - \varrho(\tau \mp \phi(k))]}{i(\tau \mp \phi(k))}|k|\widetilde{\Phi}(k,\tau)d\tau.\end{aligned}$$

We note that

$$S_4 = \psi(t)\sum_{n\geq 1}\frac{t^n i^{n-1}}{n!}\Big(\sum_{k\in\mathbb{Z}} e^{i(xk \pm t\phi(k))}\int_{\mathbb{R}} (\tau \mp \phi(k))^{n-1}\varrho(\tau \mp \phi(k))|k|\widetilde{\Phi}(k,\tau)d\tau\Big).$$

Thus, if we use the notation

$$\zeta_n(k) = \int_{\mathbb{R}} i^{n-1}(\tau \mp \phi(k))^{n-1}\varrho(\tau \mp \phi(k))|k|\widetilde{\Phi}(k,\tau)d\tau, \quad \omega_n(t) = \psi(t)t^n,$$

we obtain that

$$\begin{aligned}\widetilde{S}_4(k,\tau) &= \Big[\sum_{n\geq 1}\frac{\omega_n(t)}{n!}\Big(\sum_{k\in\mathbb{Z}} e^{i(xk\pm t\phi(k))}\zeta_n(k)\Big)\Big]^{\sim}(k,\tau) \\ &= \sum_{n\geq 1}\frac{1}{n!}\zeta_n(k)\widehat{\omega}_n^{(t)}(\tau \mp \phi(k)).\end{aligned}$$

Therefore

$$\begin{aligned}\|\langle k\rangle^s \widetilde{S}_4\|^2_{\ell^2_k L^1_\tau} &\leq \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\Big(\sum_{n\geq 1}\frac{1}{n!}|\zeta_n(k)|\int_{\mathbb{R}}|\widehat{\omega}_n^{(t)}(\tau\mp\phi(k))|d\tau\Big)^2 \\ &\leq C\sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\|\chi_A(\tau\mp\phi(k))|k|\widetilde{\Phi}(k,\tau)\|^2_{L^1_\tau} \\ &= C\||k|\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1}\widetilde{\Phi}\|^2_{\ell^2_k L^1_\tau}.\end{aligned}$$

Using the notation $g_1(k,\tau) = [i(\tau\mp\phi(k))]^{-1}[1-\varrho(\tau\mp\phi(k))]|k|\widetilde{\Phi}(k,\tau)$ we see that

$$\widetilde{S}_5(k,\tau) = \Big[\psi(t)\sum_{k\in\mathbb{Z}} e^{ikx}\int_{\mathbb{R}} e^{it\tau}g_1(k,\tau)d\tau\Big]^{\sim}(k,\tau) = \widehat{\psi}^{(t)}(\tau) * g_1(k,\tau).$$

Hence

$$\begin{aligned}\|\langle k\rangle^s\widetilde{S}_5\|^2_{\ell^2_k L^1_\tau} &\leq \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}\|\widehat{\psi}^{(t)}(\tau)\|^2_{L^1_\tau}\|g_1(k,\tau)\|^2_{L^1_\tau} \\ &\leq C\||k|\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1}\widetilde{\Phi}\|^2_{\ell^2_k L^1_\tau}.\end{aligned}$$

Now, let $\widehat{h}_1(k) = \int_{\mathbb{R}}[i(\tau\mp\phi(k))]^{-1}[1-\varrho(\tau\mp\phi(k))]|k|\widetilde{\Phi}(k,\tau)d\tau$. Then

$$\begin{aligned}\|\langle k\rangle^s\widetilde{S}_6\|^2_{\ell^2_k L^1_\tau} &= \sum_{k\in\mathbb{Z}}\langle k\rangle^{2s}|\widehat{h}_1(k)|^2\left(\int_{\mathbb{R}}|\widehat{\psi}^{(t)}(\tau\mp\phi(k))|d\tau\right)^2 \\ &\leq C\||k|\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1}\widetilde{\Phi}\|^2_{\ell^2_k L^1_\tau}.\end{aligned}$$

Consequently, from the previous estimates we have that

$$\Big\|\langle k\rangle^s\Big[\psi(t)\int_0^t\sum_{k\in\mathbb{Z}} e^{ixk}e^{\pm i(t-t')\phi(k)}|k|\widehat{\Phi}(k,t')dt'\Big]^{\sim}\Big\|_{\ell^2_k L^1_\tau} \leq C\||k|\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1}\widetilde{\Phi}\|_{\ell^2_k L^1_\tau}.$$

Therefore, we conclude that

$$\Big\|\psi(t)\int_0^t S_1(t-t')(\eta,\Phi)(t')\,dt'\Big\|_{U^s} \leq C_2\Big(\|\eta\|_{Z^s} + \|\Phi\|_{W^{s+1}}\Big).$$

Similarly we obtain the other inequality in (iii). □

In the following lemma we show the continuous embedding of the space $U^s \times V^{s+1}$ in the class $C(\mathbb{R} : H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T}))$ for $s \in \mathbb{R}$.

Lemma 2.6. *Let $s \in \mathbb{R}$, then there exists $C > 0$ such that*

$$\|(\eta, \Phi)\|_{C(\mathbb{R} : H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T}))} \leq C\|(\eta, \Phi)\|_{U^s \times V^{s+1}}.$$

Proof. First we prove that $U^s \subseteq L^\infty(\mathbb{R} : H^s(\mathbb{T}))$. Since

$$\|\eta(t)\|_{H^s(\mathbb{T})} \leq \Big(\sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \Big(\int_{\mathbb{R}} |\widetilde{\eta}(k, \tau)| d\tau \Big)^2 \Big)^{1/2} \leq \|\eta\|_{U^s},$$

we have that $\|\eta\|_{L^\infty(\mathbb{R} : H^s(\mathbb{T}))} \leq \|\eta\|_{U^s}$. Now,

$$\|\eta(t) - \eta(t')\|^2_{H^s(\mathbb{T})} \leq \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} \Big(\int_{\mathbb{R}} |e^{it\tau} - e^{it'\tau}| |\widetilde{\eta}(k, \tau)| d\tau \Big)^2.$$

Then, using the Dominated Convergence Theorem,

$$\|\eta(t) - \eta(t')\|_{H^s(\mathbb{T})} \to 0, \quad t \to t'.$$

Thus $\eta \in C(\mathbb{R} : H^s(\mathbb{T}))$ and moreover $\|\eta\|_{C(\mathbb{R} : H^s(\mathbb{T}))} \leq C\|\eta\|_{U^s}$. Finally,

$$\|\Phi(t)\|_{\mathcal{V}^{s+1}(\mathbb{T})} = \Big(\sum_{k \in \mathbb{Z}} |k|^2 \langle k \rangle^{2s} \Big| \int_{\mathbb{R}} e^{it\tau} \widetilde{\Phi}(k, \tau) d\tau \Big|^2 \Big)^{1/2} \leq \|\Phi\|_{V^{s+1}}.$$

Hence, as in the previous case, $\|\Phi\|_{C(\mathbb{R} : \mathcal{V}^{s+1}(\mathbb{T}))} \leq C\|\Phi\|_{V^{s+1}}$ and then

$$\|(\eta, \Phi)\|_{C(\mathbb{R} : H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T}))} \leq C\|(\eta, \Phi)\|_{U^s \times V^{s+1}}.$$

□

3 Bilinear estimates

Before proceed to the proof of the bilinear estimates, we state some elementary calculus inequalities that will be useful later, and whose proofs can be seen, respectively, in Lemma 5.3 of [10], Lemma 2.5 of [19], and Lemma 4.2 in [6].

Lemma 3.1. *If $\mu > 1/2$ and $\nu = \nu(k, \tau) > 0$, then*

$$\sup_{(k,\tau) \in \mathbb{Z} \times \mathbb{R}} \sum_{k_1 \in \mathbb{Z}} \frac{1}{(\nu + |k_1^2 + \alpha_1 k_1 + \alpha_2|)^\mu} < +\infty,$$

where $\alpha_1 = \alpha_1(k, \tau)$ and $\alpha_2 = \alpha_2(k, \tau)$.

Lemma 3.2. *If $\mu > 1/3$ and $\nu = \nu(k, \tau) > 0$, then*

$$\sup_{(k,\tau) \in \mathbb{Z} \times \mathbb{R}} \sum_{k_1 \in \mathbb{Z}} \frac{1}{(\nu^3 + |k_1^3 + \alpha_1 k_1^2 + \alpha_2 k_1 + \alpha_3|)^\mu} < +\infty,$$

where $\alpha_1 = \alpha_1(k, \tau), \alpha_2 = \alpha_2(k, \tau)$ and $\alpha_3 = \alpha_3(k, \tau)$.

Lemma 3.3. *For $p, q > 0$ and $r = \min\{p, q, p + q - 1\}$ with $p + q > 1$, we have that*

$$\int_{\mathbb{R}} \frac{dx}{\langle x - \lambda \rangle^p \langle x - \mu \rangle^q} \leq \frac{C}{\langle \lambda - \mu \rangle^r}. \tag{3.1}$$

The following nonlinear estimates constitute an important result for this work. We prove these estimates using a method originally due to Bourgain (see [2,3]) and considerably improved by Kenig, Ponce and Vega (see [8,9]).

Lemma 3.4. *Let $s \geq 0$, then exists $C_3 > 0$ such that*

(i) $\|\partial_x(\eta\partial_x\Phi)\|_{X^{s,-1/2}} \leq C_3\|\eta\|_{X^{s,1/2}}\|\Phi\|_{Y^{s+1,1/2}}$,

(ii) $\|(\partial_x\Phi)(\partial_x\Phi_1)\|_{Y^{s+1,-1/2}} \leq C_3\|\Phi\|_{Y^{s+1,1/2}}\|\Phi_1\|_{Y^{s+1,1/2}}$.

Proof. First we note that

$$\begin{aligned}
&\|\partial_x(\eta\partial_x\Phi)\|_{X^{s,-1/2}} \\
&\quad= \|\langle|\tau|-\phi(k)\rangle^{-1/2}k\langle k\rangle^s(\widetilde{\eta}*\widetilde{\partial_x\Phi})(k,\tau)\|_{\ell^2_kL^2_\tau} \\
&\quad= \sup_{\|h\|_{\ell^2_kL^2_\tau}=1}\left|\sum_{k,k_1\in\mathbb{Z}}\int_{\mathbb{R}^2} k\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1/2}\widetilde{\eta}(k-k_1,\tau-\tau_1)k_1\widetilde{\Phi}(k_1,\tau_1)h(k,\tau)\,d\tau d\tau_1\right|.
\end{aligned}$$

Thus, by letting

$$f(k,\tau) = \langle|\tau|-\phi(k)\rangle^{1/2}\langle k\rangle^s\widetilde{\eta}(k,\tau), \quad g(k,\tau) = \langle|\tau|-\phi(k)\rangle^{1/2}\langle k\rangle^s k\widetilde{\Phi}(k,\tau),$$

we have that (i) is equivalent to

$$|J(f,g,h)| \leq C\|f\|_{\ell^2_kL^2_\tau}\|g\|_{\ell^2_kL^2_\tau}\|h\|_{\ell^2_kL^2_\tau}, \tag{3.2}$$

where

$$J(f,g,h) = \sum_{k,k_1\in\mathbb{Z}}\int_{\mathbb{R}^2}\frac{k\langle k\rangle^s}{\langle k_1\rangle^s\langle k-k_1\rangle^s}\frac{f(k-k_1,\tau-\tau_1)g(k_1,\tau_1)h(k,\tau)d\tau d\tau_1}{\langle|\tau|-\phi(k)\rangle^{1/2}\langle|\tau_1|-\phi(k_1)\rangle^{1/2}\langle|\tau-\tau_1|-\phi(k-k_1)\rangle^{1/2}}.$$

For to perform the inequality (3.2), we analyse all possible cases for the sign of τ, τ_1 and $\tau-\tau_1$. To do this we split $\mathbb{Z}^2\times\mathbb{R}^2$ into the following regions

$$\begin{aligned}
\Gamma_1 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1<0, \tau-\tau_1<0\},\\
\Gamma_2 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1\geq 0, \tau-\tau_1<0, \tau\geq 0\},\\
\Gamma_3 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1\geq 0, \tau-\tau_1<0, \tau<0\},\\
\Gamma_4 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1<0, \tau-\tau_1\geq 0, \tau\geq 0\},\\
\Gamma_5 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1<0, \tau-\tau_1\geq 0, \tau<0\},\\
\Gamma_6 &= \{(k,k_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : \tau_1\geq 0, \tau-\tau_1\geq 0\}.
\end{aligned}$$

We note that $\tau_1<0$ and $\tau-\tau_1<0$ implies $\tau<0$, and $\tau_1\geq 0$ and $\tau-\tau_1\geq 0$ implies $\tau\geq 0$. Then the cases $\tau_1<0$, $\tau-\tau_1<0$, $\tau\geq 0$ and $\tau_1\geq 0$, $\tau-\tau_1\geq 0$, $\tau<0$ cannot occur. Now, since

$$1+|k| \leq (1+|k_1|)(1+|k-k_1|),$$

then for $s\geq 0$ we see that

$$\frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{2s}\langle k-k_1\rangle^{2s}} \leq 1. \tag{3.3}$$

So, we will prove the inequality (3.2) with $Z(f,g,h)$ instead of $J(f,g,h)$, where

$$Z(f,g,h) = \sum_{k,k_1\in\mathbb{Z}} \int_{\mathbb{R}^2} \frac{kf(k_2,\tau_2)g(k_1,\tau_1)h(k,\tau)\,d\tau d\tau_1}{\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}$$

with $k_2 = k - k_1, \tau_2 = \tau - \tau_1$ and $\sigma, \sigma_1, \sigma_2$ belonging to one of the following cases

$$\begin{aligned}
&(C_1)\,\sigma = \tau + |k|^3 + |k|,\ \sigma_1 = \tau_1 + |k_1|^3 + |k_1|,\ \sigma_2 = \tau_2 + |k_2|^3 + |k_2|,\\
&(C_2)\,\sigma = \tau - |k|^3 - |k|,\ \sigma_1 = \tau_1 - |k_1|^3 - |k_1|,\ \sigma_2 = \tau_2 + |k_2|^3 + |k_2|,\\
&(C_3)\,\sigma = \tau + |k|^3 + |k|,\ \sigma_1 = \tau_1 - |k_1|^3 - |k_1|,\ \sigma_2 = \tau_2 + |k_2|^3 + |k_2|,\\
&(C_4)\,\sigma = \tau - |k|^3 - |k|,\ \sigma_1 = \tau_1 + |k_1|^3 + |k_1|,\ \sigma_2 = \tau_2 - |k_2|^3 - |k_2|,\\
&(C_5)\,\sigma = \tau + |k|^3 + |k|,\ \sigma_1 = \tau_1 + |k_1|^3 + |k_1|,\ \sigma_2 = \tau_2 - |k_2|^3 - |k_2|,\\
&(C_6)\,\sigma = \tau - |k|^3 - |k|,\ \sigma_1 = \tau_1 - |k_1|^3 - |k_1|,\ \sigma_2 = \tau_2 - |k_2|^3 - |k_2|.
\end{aligned}$$

By hypotheses we have that $\widehat{\eta}(0,t) = 0$, for all $t \in \mathbb{R}$. Thus, if $k = k_1$ then $f(k_2,\tau_2) = 0$. Similarly if $k_1 = 0$ then $g(k_1,\tau_1) = 0$. Then, we will estimate $Z(f,g,h)$ when $k \neq 0$, $k_1 \neq 0$ and $k - k_1 \neq 0$.

By symmetry it is sufficient to estimate $Z(f,g,h)$ into the following set

$$R = \{(k,k_1,\tau,\tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 \,:\, |\sigma_2| \leq |\sigma_1|\}.$$

Now, we write $Z(f,g,h)$ as the sum $S_1 + S_2$, where

$$S_j = \sum_k \sum_{k_1} \iint_{R_j} \frac{kf(k_2,\tau_2)g(k_1,\tau_1)h(k,\tau)\chi_{R_j}\,d\tau d\tau_1}{\langle\sigma\rangle^{1/2}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}, \quad j = 1,2,$$

and the sets R_1, R_2 are defined by

$$R_1 = \{(k,k_1,\tau,\tau_1) \in R \,:\, |\sigma_1| \leq |\sigma|\}, \quad R_2 = \{(k,k_1,\tau,\tau_1) \in R \,:\, |\sigma| \leq |\sigma_1|\}.$$

We first consider $\sigma, \sigma_1, \sigma_2$ as the case (C_1). We will use the notations $\sum_k F_1(k)$, $\int F_2(x)dx$ to indicate that the sum or the integral are calculated, respectively, at some subset of $\mathbb{Z}$ or $\mathbb{R}$. Using the Cauchy–Schwarz inequality,

$$|S_1|^2 \leq \|h\|^2_{\ell^2_k L^2_\tau} \sum_k \int \left(\sum_{k_1} \int |f(k_2,\tau_2)g(k_1,\tau_1)|^2 d\tau_1 \right) \left(\sum_{k_1} \int \frac{\chi^2_{R_1}|k|^2 d\tau_1}{\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle} \right) d\tau.$$

We will prove that the expression

$$\sum_{k_1} \int \frac{\chi^2_{R_1}|k|^2 d\tau_1}{\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle} = \frac{|k|^2}{\langle\sigma\rangle} \sum_{k_1} \int \frac{\chi^2_{R_1} d\tau_1}{\langle\sigma_1\rangle\langle\sigma_2\rangle}$$

is bounded. But, by using inequality (3.1) in Lemma 3.3 we have that

$$\int_{\mathbb{R}} \frac{d\tau_1}{\langle\tau_1 + |k_1|^3 + |k_1|\rangle\langle\tau - \tau_1 + |k_2|^3 + |k_2|\rangle} \leq \frac{C}{\langle\tau + |k_1|^3 + |k_1| + |k_2|^3 + |k_2|\rangle}.$$

Then we will prove that there exists $C > 0$ such that

$$\frac{|k|^2}{\langle\tau + |k|^3 + |k|\rangle} \sum_{k_1} \frac{1}{\langle\tau + |k_1|^3 + |k_1| + |k - k_1|^3 + |k - k_1|\rangle} \leq C, \quad \text{on } R_1.$$

Since for $k \neq 0$, $k_1 \neq 0$ and $k \neq k_1$, $|k| = |k_1 + (k-k_1)| \leq |k_1| + |k-k_1| \leq 2|k_1(k-k_1)|$. Then

$$\frac{k^2}{2} \leq |kk_1(k-k_1)|. \tag{3.4}$$

Moreover, we observe the relation

$$\begin{aligned} \tau + |k|^3 + |k| - \left[\tau_1 + |k_1|^3 + |k_1| + \tau_2 + |k_2|^3 + |k_2|\right] \\ = |k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|. \end{aligned} \tag{3.5}$$

If $(k, k_1, \tau, \tau_1) \in R_1$ then

$$|\tau_2 + |k_2|^3 + |k_2|| \leq |\tau_1 + |k_1|^3 + |k_1|| \leq |\tau + |k|^3 + |k||.$$

Hence, using the triangle inequality in (3.5) we conclude that

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| \leq 3|\tau + |k|^3 + |k||. \tag{3.6}$$

Assume $k_1 > 0$ and $k - k_1 > 0$. Then $k > k_1 > 0$ and, using Lemma 3.1, we obtain that

$$\begin{aligned} \sum_{k_1} \frac{1}{\langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1| \rangle} &\leq \sum_{k_1 \in \mathbb{Z}} \frac{1}{\langle \tau + k^3 + k - 3k^2k_1 + 3kk_1^2 \rangle} \\ &\leq \sup_{(k,\tau) \in \mathbb{Z}^* \times \mathbb{R}} \sum_{k_1 \in \mathbb{Z}} \frac{1}{\left(\frac{1}{3|k|} + \left| k_1^2 - kk_1 + \frac{\tau}{3k} + \frac{k^2}{3} + \frac{1}{3} \right| \right)} \\ &\leq C, \end{aligned}$$

where $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$. Moreover,

$$|k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2| = k^3 + k - k_1^3 - k_1 - k_2^3 - k_2 = 3kk_1(k-k_1) > 0.$$

So, from (3.6) and inequality (3.4) we see that

$$|\tau + k^3 + k| \geq |kk_1(k-k_1)| \geq \frac{|k|^2}{2}.$$

Thus

$$\frac{|k|^2}{\langle \tau + |k|^3 + |k| \rangle} \leq C.$$

Assume $k_1 < 0$ and $k - k_1 < 0$. Then $k < k_1 < 0$ and, using Lemma 3.1, we see that

$$\begin{aligned} \sum_{k_1} \frac{1}{\langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1| \rangle} &\leq \sum_{k_1 \in \mathbb{Z}} \frac{1}{\langle \tau - k^3 - k + 3k^2k_1 - 3kk_1^2 \rangle} \\ &\leq \sup_{(k,\tau) \in \mathbb{Z}^* \times \mathbb{R}} \sum_{k_1 \in \mathbb{Z}} \frac{1}{\left(\frac{1}{3|k|} + \left| k_1^2 - kk_1 - \frac{\tau}{3k} + \frac{k^2}{3} + \frac{1}{3} \right| \right)} \\ &\leq C. \end{aligned}$$

Moreover

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = \big| -k^3 - k + k_1^3 + k_1 + k_2^3 + k_2 \big| = 3|kk_1(k-k_1)|.$$

Hence from (3.6) and (3.4) we obtain that

$$|\tau - k^3 - k| \geq |kk_1(k-k_1)| \geq \frac{|k|^2}{2} \quad \text{and} \quad \frac{|k|^2}{\langle \tau - k^3 - k \rangle} \leq C.$$

Assume $k_1 > 0$ and $k - k_1 < 0$. Using Lemma 3.2,

$$\begin{aligned}
&\sum_{k_1} \frac{1}{\langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1| \rangle} \\
&\quad \leq \sum_{k_1 \in \mathbb{Z}} \frac{1}{\langle \tau + 2k_1^3 + 2k_1 - k^3 - k + 3k^2k_1 - 3kk_1^2 \rangle} \\
&\quad \leq \sup_{(k,\tau)\in\mathbb{Z}\times\mathbb{R}} \sum_{k_1 \in \mathbb{Z}} \frac{1}{\left(\frac{1}{2} + \left|k_1^3 - \frac{3}{2}kk_1^2 + k_1 + \frac{3}{2}k^2k_1 + \frac{\tau}{2} - \frac{k^3}{2} - \frac{k}{2}\right|\right)} \leq C.
\end{aligned}$$

Moreover, if $k > 0$ then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k-k_1| \left[\left(k_1 - \frac{k}{4}\right)^2 + \frac{15k^2}{16} + 1\right] \geq \frac{15}{8}k^2.$$

If $k < 0$,

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k_1| \left[\left(k_1 - \frac{3k}{4}\right)^2 + \frac{15k^2}{16} + 1\right] \geq \frac{15k^2}{8}.$$

Thus, from inequality (3.6) we have that

$$|\tau + |k|^3 + |k|| \geq \frac{5k^2}{8} \quad \text{and} \quad \frac{|k|^2}{\langle \tau + |k|^3 + |k| \rangle} \leq C.$$

Assume $k_1 < 0$ and $k - k_1 > 0$. Using Lemma 3.2,

$$\begin{aligned}
&\sum_{k_1} \frac{1}{\langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1| \rangle} \\
&\quad \leq \sup_{(k,\tau)\in\mathbb{Z}\times\mathbb{R}} \sum_{k_1 \in \mathbb{Z}} \frac{1}{\left(\frac{1}{2} + \left|k_1^3 - \frac{3}{2}kk_1^2 + k_1 + \frac{3}{2}k^2k_1 - \frac{\tau}{2} - \frac{k^3}{2} - \frac{k}{2}\right|\right)} \leq C.
\end{aligned}$$

Now, if $k > 0$ then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k_1| \left[\left(k_1 - \frac{3k}{4}\right)^2 + \frac{15k^2}{16} + 1\right] \geq \frac{15k^2}{8}.$$

and if $k < 0$ then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| = 2|k-k_1| \left[\left(k_1 - \frac{k}{4}\right)^2 + \frac{15k^2}{16} + 1\right] \geq \frac{15}{8}k^2.$$

Thus, by using (3.6),

$$|\tau + |k|^3 + |k|| \geq \frac{5k^2}{8} \quad \text{and} \quad \frac{|k|^2}{\langle \tau + |k|^3 + |k| \rangle} \leq C.$$

Consequently, from the previous estimates there exists $C > 0$ such that

$$\frac{|k|^2}{\langle\sigma\rangle}\sum_{k_1}\int\frac{d\tau_1}{\langle\sigma_1\rangle\langle\sigma_2\rangle}\leq C,\quad \text{on}\quad R_1.$$

Therefore

$$\begin{aligned}|S_1|^2 &\leq C\|h\|^2_{\ell^2_k L^2_\tau}\sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}}|g(k_1,\tau_1)|^2\Big(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|f(k_2,\tau_2)|^2 d\tau\Big)d\tau_1\\ &\leq C\|f\|^2_{\ell^2_k L^2_\tau}\|g\|^2_{\ell^2_k L^2_\tau}\|h\|^2_{\ell^2_k L^2_\tau}.\end{aligned}$$

In a similar fashion,

$$|S_2|^2\leq\|g\|^2_{\ell^2_k L^2_\tau}\sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}}\left(\sum_k\int|f(k_2,\tau_2)h(k,\tau)|^2 d\tau\right)\left(\sum_k\int\frac{\chi^2_{R_2}|k|^2 d\tau}{\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle}\right)d\tau_1.$$

We will prove that the expression

$$\sum_k\int\frac{\chi^2_{R_2}|k|^2 d\tau}{\langle\sigma\rangle\langle\sigma_1\rangle\langle\sigma_2\rangle}=\frac{1}{\langle\sigma_1\rangle}\sum_k\int\frac{\chi^2_{R_2}|k|^2 d\tau}{\langle\sigma\rangle\langle\sigma_2\rangle}$$

is bounded. Using inequality (3.1) in Lemma 3.3 we have that

$$\int_{\mathbb{R}}\frac{d\tau}{\langle\tau+|k|^3+|k|\rangle\langle\tau-\tau_1+|k_2|^3+|k_2|\rangle}\leq\frac{C}{\langle\tau_1+|k|^3+|k|-|k_2|^3-|k_2|\rangle}.$$

Thus, we will show that there exists $C > 0$ such that

$$\frac{1}{\langle\tau_1+|k_1|^3+|k_1|\rangle}\sum_k\frac{|k|^2}{\langle\tau_1+|k|^3+|k|-|k-k_1|^3-|k-k_1|\rangle}\leq C,\quad \text{on } R_2.$$

We note that if $(k,k_1,\tau,\tau_1)\in R_2$,

$$|\tau+|k|^3+|k||\leq|\tau_1+|k_1|^3+|k_1||,\qquad |\tau_2+|k_2|^3+|k_2||\leq|\tau_1+|k_1|^3+|k_1||.$$

So, using the triangle inequality in (3.5) we see that

$$\big||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\big|\leq 3|\tau_1+|k_1|^3+|k_1||. \tag{3.7}$$

Assume $k > 0$ and $k - k_1 > 0$. Thus

$$\sum_k\frac{|k|^2}{\langle\tau_1+|k|^3+|k|-|k_2|^3-|k_2|\rangle}\leq\sum_{k\in\mathbb{Z}}\frac{|k|^2}{\langle\tau_1+k_1^3+k_1+3k^2k_1-3kk_1^2\rangle}=:J_1.$$

Moreover, if $k_1 > 0$ then, using inequality (3.4),

$$\big||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\big|=3|kk_1(k-k_1)|\geq\frac{3k^2}{2}$$

and if $k_1 < 0$ then

$$\big||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\big|=2|k_1|\left[\left(k_1-\frac{3k}{4}\right)^2+\frac{15k^2}{16}+1\right]\geq\frac{15k^2}{8}.$$

Hence, from (3.7) there exists $C>0$ such that

$$|\tau_1+|k_1|^3+|k_1||\geq Ck^2$$

and consequently, using Lemma 3.1, we obtain

$$\frac{1}{\langle\tau_1+|k_1|^3+|k_1|\rangle}J_1\leq\ C\sup_{(k_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}}\sum_{k\in\mathbb{Z}}\frac{1}{\left(\frac{1}{3|k_1|}+\left|k^2-kk_1+\frac{\tau_1}{3k_1}+\frac{k_1^2}{3}+\frac{1}{3}\right|\right)}\leq C.$$

Assume $k<0$ and $k-k_1<0$. Thus

$$\sum_k\frac{|k|^2}{\langle\tau_1+|k|^3+|k|-|k-k_1|^3-|k-k_1|\rangle}\leq\sum_{k\in\mathbb{Z}}\frac{|k|^2}{\langle\tau_1-k_1^3-k_1-3k^2k_1+3kk_1^2\rangle}=:J_2.$$

Moreover, if $k_1>0$ then

$$\left||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\right|=2|k_1|\left[\left(k_1-\frac{3k}{4}\right)^2+\frac{15k^2}{16}+1\right]\geq\frac{15k^2}{8}.$$

If $k_1<0$ then, using (3.4),

$$\left||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\right|=3|kk_1(k-k_1)|\geq\frac{3k^2}{2}.$$

Thus, from (3.7) there exists $C>0$ such that

$$|\tau_1+|k_1|^3+|k_1||\geq Ck^2$$

and so

$$\frac{1}{\langle\tau_1+k_1^3+k_1\rangle}J_2\leq\ C\sup_{(k_1,\tau_1)\in\mathbb{Z}^*\times\mathbb{R}}\sum_{k\in\mathbb{Z}}\frac{1}{\langle\tau_1-k_1^3-k_1-3k^2k_1+3kk_1^2\rangle}\leq C.$$

Assume $k>0$ and $k-k_1<0$. Then $k_1>k>0$ and

$$\sum_k\frac{|k|^2}{\langle\tau_1+|k|^3+|k|-|k-k_1|^3-|k-k_1|\rangle}\leq\sum_{k\in\mathbb{Z}}\frac{|k|^2}{\langle\tau_1+2k^3+2k-3k^2k_1+3kk_1^2-k_1^3-k_1\rangle}=:J_3.$$

Moreover, we see that

$$\left||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\right|\geq\frac{15k^2}{8}\quad\text{and}\quad|\tau_1+k_1^3+k_1|\geq\frac{5k^2}{8}.$$

Consequently

$$\begin{aligned}\frac{1}{\langle\tau_1+k_1^3+k_1\rangle}J_3&\leq C\sup_{(k_1,\tau_1)\in\mathbb{Z}\times\mathbb{R}}\sum_{k\in\mathbb{Z}}\frac{1}{\langle\tau_1+2k^3+2k-3k^2k_1+3kk_1^2-k_1^3-k_1\rangle}\\&\leq\sup_{(k_1,\tau_1)\in\mathbb{Z}\times\mathbb{R}}\sum_{k\in\mathbb{Z}}\frac{1}{\left(\frac{1}{2}+\left|k^3-\frac{3}{2}k^2k_1+k+\frac{3}{2}kk_1^2+\frac{\tau_1}{2}-\frac{k_1^3}{2}-\frac{k_1}{2}\right|\right)}\leq C.\end{aligned}$$

Assume $k<0$ and $k-k_1>0$. Then

$$\sum_k \frac{|k|^2}{\langle \tau_1+|k|^3+|k|-|k-k_1|^3-|k-k_1|\rangle} \leq \sum_{k\in\mathbb{Z}} \frac{|k|^2}{\langle \tau_1-2k^3-2k+k_1^3+k_1+3k^2k_1-3kk_1^2\rangle} =: J_4.$$

Also we see that

$$\big||k|^3+|k|-|k_1|^3-|k_1|-|k_2|^3-|k_2|\big| \geq \frac{15k^2}{8} \quad \text{and} \quad |\tau_1+k_1^3+k_1| \geq \frac{5k^2}{8}.$$

Thus, using (3.7),

$$|\tau_1+k_1^3+k_1| \geq \frac{5k^2}{8}.$$

So that

$$\begin{aligned}\frac{1}{\langle \tau_1+k_1^3+k_1\rangle} J_4 &\leq C \sup_{(k_1,\tau_1)\in\mathbb{Z}\times\mathbb{R}} \sum_{k\in\mathbb{Z}} \frac{1}{\langle \tau_1-2k^3-2k+3k^2k_1-3kk_1^2+k_1^3+k_1\rangle} \\ &\leq C \sup_{(k_1,\tau_1)\in\mathbb{Z}\times\mathbb{R}} \sum_{k\in\mathbb{Z}} \frac{1}{\left(\frac{1}{2}+\left|k^3-\frac{3}{2}k^2k_1+k+\frac{3}{2}kk_1^2-\frac{\tau_1}{2}-\frac{k_1^3}{2}-\frac{k_1}{2}\right|\right)} \leq C.\end{aligned}$$

Hence, from previous estimates we see that there exists $C>0$ such that

$$\frac{1}{\langle \sigma_1\rangle}\sum_k \int \frac{|k|^2 d\tau}{\langle \sigma\rangle\langle \sigma_2\rangle} \leq C, \quad \text{on } R_2.$$

Therefore

$$|S_2|^2 \leq C\|f\|^2_{\ell^2_k L^2_\tau}\|g\|^2_{\ell^2_k L^2_\tau}\|h\|^2_{\ell^2_k L^2_\tau}.$$

The proof of the others is similar to case (C_1). Finally, note that

$$\begin{aligned}&\|(\partial_x\Phi)(\partial_x\Phi_1)\|_{Y^{s+1,-1/2}} \\ &= \sup_{\|h\|_{\ell^2_k L^2_\tau}=1}\left|\sum_{k,k_1\in\mathbb{Z}}\int_{\mathbb{R}^2} k\langle k\rangle^s\langle|\tau|-\phi(k)\rangle^{-1/2}(k-k_1)\widetilde{\Phi}(k-k_1,\tau-\tau_1)k_1\widetilde{\Phi}_1(k_1,\tau_1)h(k,\tau)\,d\tau d\tau_1\right|.\end{aligned}$$

Then, by letting

$$f(k,\tau)=\langle|\tau|-\phi(k)\rangle^{1/2}\langle k\rangle^s k\widetilde{\Phi}(k,\tau), \quad f_1(k,\tau)=\langle|\tau|-\phi(k)\rangle^{1/2}\langle k\rangle^s k\widetilde{\Phi}_1(k,\tau)$$

we have that (ii) is equivalent to

$$|K(f,f_1,h)| \leq C\|f\|_{\ell^2_k L^2_\tau}\|f_1\|_{\ell^2_k L^2_\tau}\|h\|_{\ell^2_k L^2_\tau}, \tag{3.8}$$

where

$$K(f,f_1,h)=\sum_{k,k_1\in\mathbb{Z}}\int_{\mathbb{R}^2}\frac{k\langle k\rangle^s}{\langle k_1\rangle^s\langle k-k_1\rangle^s}\frac{f(k-k_1,\tau-\tau_1)f_1(k_1,\tau_1)h(k,\tau)\,d\tau d\tau_1}{\langle|\tau|-\phi(k)\rangle^{1/2}\langle|\tau_1|-\phi(k_1)\rangle^{1/2}\langle|\tau-\tau_1|-\phi(k-k_1)\rangle^{1/2}}.$$

The proof of (3.8) is analogous to the proof of (3.2). □

The proof of the following estimates is analogous to the proof of Lemma 3.4.

Lemma 3.5. *Let $s \geq 0$, then there exists $C_4 > 0$ such that*

(i) $\left\| \frac{\langle k\rangle^s [\partial_x(\eta\partial_x\Phi)]^{\sim}(k,\tau)}{\langle|\tau|-\phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau} \leq C_4 \|\eta\|_{X^{s,1/2}} \|\Phi\|_{Y^{s+1,1/2}}$,

(ii) $\left\| \frac{|k|\langle k\rangle^s [(\partial_x\Phi)(\partial_x\Phi_1)]^{\sim}(k,\tau)}{\langle|\tau|-\phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau} \leq C_4 \|\Phi\|_{Y^{s+1,1/2}} \|\Phi_1\|_{Y^{s+1,1/2}}$.

Proof. First, notice that

$$\begin{aligned}
&\left\| \frac{\langle k\rangle^s [\partial_x(\eta\partial_x\Phi)]^{\sim}(k,\tau)}{\langle|\tau|-\phi(k)\rangle} \right\|_{\ell^2_k L^1_\tau} \\
&\quad = \left\| \frac{k\langle k\rangle^s}{\langle|\tau|-\phi(k)\rangle} \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}} \frac{f(k-k_1,\tau-\tau_1)g(k_1,\tau_1)d\tau_1}{\langle k_1\rangle^s\langle k-k_1\rangle^s \langle|\tau_1|-\phi(k_1)\rangle^{1/2}\langle|\tau-\tau_1|-\phi(k-k_1)\rangle^{1/2}} \right\|_{\ell^2_k L^1_\tau} \\
&\quad =: J(f,g),
\end{aligned}$$

where

$$f(k,\tau) = \langle|\tau|-\phi(k)\rangle^{1/2}\langle k\rangle^s \widetilde{\eta}(k,\tau), \quad g(k,\tau) = \langle|\tau|-\phi(k)\rangle^{1/2}\langle k\rangle^s k\widetilde{\Phi}(k,\tau).$$

In view of inequality (3.3) we will prove inequality in (i) with $Z(f,g)$ instead of $J(f,g)$ where

$$Z(f,g) = \left\| \frac{k}{\langle\sigma\rangle} \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}} \frac{f(k_2,\tau_2)g(k_1,\tau_1)d\tau_1}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \right\|_{\ell^2_k L^1_\tau}.$$

More exactly, we will study the expression

$$Z_j(f,g) = \left\| \frac{k}{\langle\sigma\rangle} \sum_{k_1} \int \frac{f(k_2,\tau_2)g(k_1,\tau_1)\chi_{R_j} d\tau_1}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \right\|_{\ell^2_k L^1_\tau}, \quad j=1,2,$$

with $k_2 = k-k_1, \tau_2 = \tau-\tau_1$; σ,σ_1,σ_2 belonging to one of the cases (C_1)-(C_6); and the sets R R_1, R_2 are defined by

$$R = \{(k,k_1,\tau,\tau_1) \in \mathbb{Z}^2\times\mathbb{R}^2 : |\sigma_2| \leq |\sigma_1|\},$$

$$R_1 = \{(k,k_1,\tau,\tau_1) \in R : |\sigma_1| \leq |\sigma|\} \quad \text{and} \quad R_2 = \{(k,k_1,\tau,\tau_1) \in R : |\sigma| \leq |\sigma_1|\}.$$

Using a duality argument we see that

$$Z_1(f,g) = \sup_{\|h\|_{\ell^2_k}=1} \left| \sum_k \sum_{k_1} \int\int \frac{kf(k_2,\tau_2)g(k_1,\tau_1)h(k)\chi_{R_1} d\tau d\tau_1}{\langle\sigma\rangle\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \right|.$$

Now, consider σ,σ_1,σ_2 as in the case (C_1) and note that

$$\begin{aligned}
&\left[\sum_k \sum_{k_1} \int\int \frac{|kf(k_2,\tau_2)g(k_1,\tau_1)h(k)|\chi_{R_1} d\tau d\tau_1}{\langle\sigma\rangle\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}} \right]^2 \\
&\quad \leq \|g\|^2_{\ell^2_k L^2_\tau} \sum_{k_1} \left\| \sum_{k\in\mathbb{Z}} |h(k)|^2 \int_{\mathbb{R}} |f(k_2,\tau_2)|^2 d\tau \right\|_{L^\infty_{\tau_1}} \sum_k \int\int \frac{\chi^2_{R_1}|k|^2 d\tau d\tau_1}{\langle\sigma\rangle^2\langle\sigma_1\rangle\langle\sigma_2\rangle}.
\end{aligned}$$

Then we will prove that the expression

$$\sum_k |k|^2 \int \frac{1}{\langle\sigma\rangle^2}\Big(\int \frac{\chi^2_{R_1} d\tau_1}{\langle\sigma_1\rangle\langle\sigma_2\rangle}\Big) d\tau$$

is bounded. In fact, if $(k, k_1, \tau, \tau_1) \in R_1$,

$$|\tau_2 + |k_2|^3 + |k_2|| \leq |\tau_1 + |k_1|^3 + |k_1|| \leq |\tau + |k|^3 + |k||. \tag{3.9}$$

Hence, using inequality (3.1) in Lemma 3.3, we have for $0 < r < 1/4$ that

$$\begin{aligned}
&\frac{1}{\langle\tau + |k|^3 + |k|\rangle^2} \int \frac{d\tau_1}{\langle\tau + |k_1|^3 + |k_1|\rangle\langle\tau_2 + |k_2|^3 + |k_2|\rangle} \\
&\quad\leq \frac{1}{\langle\tau + |k|^3 + |k|\rangle^{2(1-r)}} \int_{\mathbb{R}} \frac{d\tau_1}{\langle\tau + |k_1|^3 + |k_1|\rangle^{1+r}\langle\tau_2 + |k_2|^3 + |k_2|\rangle^{1+r}} \\
&\quad\leq \frac{C}{\langle\tau + |k|^3 + |k|\rangle^{2(1-r)}\langle\tau + |k_1|^3 + |k_1| + |k_2|^3 + |k_2|\rangle^{1+r}}.
\end{aligned}$$

So, for $0 < r < 1/4$, we will prove that there exists $C > 0$ such that

$$\sum_k |k|^2 \int \frac{d\tau}{\langle\tau + |k|^3 + |k|\rangle^{2(1-r)}\langle\tau + |k_1|^3 + |k_1| + |k_2|^3 + |k_2|\rangle^{1+r}} \leq C, \quad \text{on } R_1.$$

The importance of the choice of r will be noted later. We have the relation

$$\begin{aligned}
\tau + |k|^3 + |k| - \big[\tau_1 + |k_1|^3 + |k_1| + \tau_2 + |k_2|^3 + |k_2|\big]& \\
= |k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|.&
\end{aligned} \tag{3.10}$$

Using the triangle inequality in (3.10) and inequality (3.9) we obtain that

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2|\big| \leq 3|\tau + |k|^3 + |k||. \tag{3.11}$$

Assume $k_1 > 0$ and $k - k_1 > 0$. Then

$$|k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 + |k_2| = 3kk_1(k - k_1) > 0.$$

Thus, from (3.4) and (3.10) we see that

$$|\tau + k^3 + k| \geq |kk_1(k - k_1)| \geq \frac{|k|^2}{2},$$

and consequently, for $0 < r < 1/4$, we have that

$$\begin{aligned}
&\sum_k |k|^2 \int \frac{d\tau}{\langle\tau + |k|^3 + |k|\rangle^{2(1-r)}\langle\tau + |k_1|^3 + |k_1| + |k - k_1|^3 + |k - k_1|\rangle^{1+r}} \\
&\quad= \sum_k |k|^2 \int \frac{d\tau}{\langle\tau + k^3 + k\rangle^{2(1-r)}\langle\tau + k_1^3 + k_1 + (k - k_1)^3 + (k - k_1)\rangle^{1+r}} \\
&\quad\leq C \sum_{k\in\mathbb{Z}^*} \frac{1}{|k|^{2-4r}} \int_{\mathbb{R}} \frac{d\tau}{\langle\tau + k^3 + k - 3k^2k_1 + 3kk_1^2\rangle^{1+r}}.
\end{aligned}$$

Assume $k_1 < 0$ and $k - k_1 < 0$. Then

$$\big||k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 + |k_2|\big| = 3|kk_1(k - k_1)|.$$

So, using (3.4) and (3.11) we see that

$$|\tau - k^3 - k| \geq |kk_1(k-k_1)| \geq \frac{|k|^2}{2}.$$

Hence, for $0 < r < 1/4$,

$$\sum_k |k|^2 \int \frac{d\tau}{\langle \tau + |k|^3 + |k| \rangle^{2(1-r)} \langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1| \rangle^{1+r}}$$
$$\leq C \sum_{k \in \mathbb{Z}^*} \frac{1}{|k|^{2-4r}} \int_{\mathbb{R}} \frac{d\tau}{\langle \tau - k^3 + 3k^2 k_1 - 3kk_1^2 - k \rangle^{1+r}}.$$

Assume $k_1 > 0$ and $k - k_1 < 0$. Hence, if $k > 0$ then

$$\big| |k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2| \big| = 2|k-k_1| \Big[\Big(k_1 - \frac{k}{4}\Big)^2 + \frac{15k^2}{16} + 1 \Big] \geq \frac{15}{8} k^2$$

and if $k < 0$ then

$$\big| |k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2| \big| = 2|k_1| \Big[\Big(k_1 - \frac{3k}{4}\Big)^2 + \frac{15k^2}{16} + 1 \Big] \geq \frac{15}{8} k^2.$$

So, from inequality (3.11) we conclude that

$$|\tau + |k|^3 + |k|| \geq \frac{5}{8} k^2,$$

and

$$\sum_k |k|^2 \int \frac{d\tau}{\langle \tau + |k|^3 + |k| \rangle^{2(1-r)} \langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1| \rangle^{1+r}}$$
$$\leq C \sum_{k \in \mathbb{Z}^*} \frac{1}{|k|^{2-4r}} \int_{\mathbb{R}} \frac{d\tau}{\langle \tau + 2k_1^3 + 2k_1 - k^3 + 3k^2 k_1 - 3kk_1^2 - k \rangle^{1+r}}.$$

Assume $k_1 < 0$ and $k - k_1 > 0$. Hence, if $k > 0$ then

$$\big| |k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2| \big| = 2|k_1| \Big[\Big(k_1 - \frac{3k}{4}\Big)^2 + \frac{15k^2}{16} + 1 \Big] \geq \frac{15k^2}{8}$$

and if $k < 0$ then

$$\big| |k|^3 + |k| - |k_1|^3 - |k_1| - |k_2|^3 - |k_2| \big| = 2|k-k_1| \Big[\Big(k_1 - \frac{k}{4}\Big)^2 + \frac{15k^2}{16} + 1 \Big] \geq \frac{15}{8} k^2.$$

Thus, using (3.11),

$$|\tau + |k|^3 + |k|| \geq \frac{5}{8} k^2$$

and so

$$\sum_k |k|^2 \int \frac{d\tau}{\langle \tau + |k|^3 + |k| \rangle^{2(1-r)} \langle \tau + |k_1|^3 + |k_1| + |k-k_1|^3 + |k-k_1| \rangle^{1+r}}$$
$$\leq C \sum_{k \in \mathbb{Z}^*} \frac{1}{|k|^{2-4r}} \int_{\mathbb{R}} \frac{d\tau}{\langle \tau - 2k_1^3 - 2k_1 + k^3 - 3k^2 k_1 + 3kk_1^2 + k \rangle^{1+r}}.$$

Therefore, for any $h \in \ell^2_k$ we have that

$$\begin{aligned}|Z_1|^2 &\leq C\|g\|^2_{\ell^2_k L^2_\tau}\Big(\sum_{k\in\mathbb{Z}}|h(k)|^2\sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}}|f(k_2,\tau)|^2 d\tau\Big)\\ &\leq C\|f\|^2_{\ell^2_k L^2_\tau}\|g\|^2_{\ell^2_k L^2_\tau}\|h\|^2_{\ell^2_k}.\end{aligned}$$

Next, choosing $1/2 < r < 3/4$ we see that

$$\begin{aligned}|Z_2| &\leq C\Big\|\frac{|k|}{\langle\sigma\rangle^{1-r}}\sum_{k_1}\int\frac{|f(k_2,\tau_2)g(k_1,\tau_1)|\chi_{R_2}d\tau_1}{\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\Big\|_{\ell^2_k L^2_\tau}\\ &= C\sup_{\|h\|_{\ell^2_k L^2_\tau}=1}\Big|\sum_k\sum_{k_1}\int\int\frac{|kf(k_2,\tau_2)g(k_1,\tau_1)|h(k,\tau)\chi_{R_2}d\tau d\tau_1}{\langle\sigma\rangle^{1-r}\langle\sigma_1\rangle^{1/2}\langle\sigma_2\rangle^{1/2}}\Big|.\end{aligned}$$

As before, for $1/2 < r < 3/4$ it is possible to prove that there exists $C > 0$ such that

$$\frac{1}{\langle\tau_1+|k_1|^3+|k_1|\rangle}\sum_k\frac{|k|^2}{\langle\tau_1+|k|^3+|k|-|k-k_1|^3-|k-k_1|\rangle^{2(1-r)}} \leq C, \quad \text{on } R_2.$$

and, by using Lemma 3.3, we have that

$$\int_{\mathbb{R}}\frac{d\tau}{\langle\tau+|k|^3+|k|\rangle^{2(1-r)}\langle\tau_2+|k_2|^3+|k_2|\rangle} \leq \frac{C}{\langle\tau_1+|k|^3+|k|-|k_2|^3-|k_2|\rangle^{2(1-r)}},$$

then the expression

$$\frac{1}{\langle\sigma_1\rangle}\sum_k|k|^2\int\frac{\chi^2_{R_2}d\tau}{\langle\sigma\rangle^{2(1-r)}\langle\sigma_2\rangle}$$

is bounded. Therefore, for any $h \in \ell^2_k L^2_\tau$ we have that

$$\begin{aligned}|Z_2|^2 &\leq C\|g\|^2_{\ell^2_k L^2_\tau}\sum_{k_1}\int\Big(\sum_k\int|f(k_2,\tau_2)h(k,\tau)|^2 d\tau\Big)\Big(\sum_k\int\frac{\chi^2_{R_2}|k|^2 d\tau}{\langle\sigma\rangle^{2(1-r)}\langle\sigma_1\rangle\langle\sigma_2\rangle}\Big)d\tau_1\\ &\leq C\|f\|^2_{\ell^2_k L^2_\tau}\|g\|^2_{\ell^2_k L^2_\tau}\|h\|^2_{\ell^2_k L^2_\tau}.\end{aligned}$$

In a similar way we have the rest of the proof. □

As a direct consequence of previous lemmas we have the following corollary.

Corollary 3.6. *Let $s \geq 0$, then there exists $C_5 > 0$ such that*

(i) $\|\psi\partial_x(\eta\partial_x\Phi)\|_{Z^s} \leq C_5\|\eta\|_{X^{s,1/2}}\|\Phi\|_{Y^{s+1,1/2}}$,

(ii) $\|\psi(\partial_x\eta)(\partial_x\Phi_1)\|_{W^{s+1}} \leq C_5\|\Phi\|_{Y^{s+1,1/2}}\|\Phi_1\|_{Y^{s+1,1/2}}$.

4 Well-posedness

In this section we establish the local well-posedness for the model (1.1) in the space $U^s \times V^s$.

Theorem 4.1. *Let $s \geq 0$, then for all $(\eta_0, \Phi_0) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ we have that there exist a time $T = T(\|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})}) > 0$ and a unique solution (η, Φ) of the Cauchy problem (1.1)-(1.2) such that*

$$\eta \in C([0,T] : H^s(\mathbb{T})) \cap U^s_T \quad \text{and} \quad \Phi \in C([0,T] : \mathcal{V}^{s+1}(\mathbb{T})) \cap V^{s+1}_T.$$

Moreover, for all $0 < T' < T$ there exists a neighborhood $\mathbb{V}$ of (η_0, Φ_0) in $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ such that the map data-solution is Lipschitz from $\mathbb{V}$ in the class

$$C([0,T'] : H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})) \cap (U^s_T \times V^{s+1}_T).$$

Proof. For $(\eta_0, \Phi_0) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ we consider the operator $\Gamma = (\Gamma_1, \Gamma_2)$ where

$$\Gamma_1(\eta, \Phi)(t) = \psi(t) S_1(t)(\eta_0, \Phi_0) - \psi(t) \int_0^t S_1(t-t')\psi(t') \Big(\partial_x(\eta \partial_x \Phi), \frac{1}{2}(\partial_x \Phi)^2\Big)(t')dt'$$

and

$$\Gamma_2(\eta, \Phi)(t) = \psi(t) S_2(t)(\eta_0, \Phi_0) - \psi(t) \int_0^t S_2(t-t')\psi(t') \Big(\partial_x(\eta \partial_x \Phi), \frac{1}{2}(\partial_x \Phi)^2\Big)(t')dt'.$$

Let Z_M the closed ball of radius M centered at the origin in $U^s \times V^{s+1}$,

$$Z_M = \{(\eta, \Phi) \in U^s \times V^{s+1} : \|(\eta, \Phi)\|_{U^s \times V^{s+1}} \leq M\}.$$

We will show that the correspondence $(\eta, \Phi) \mapsto \Gamma(\eta, \Phi)$ maps Z_M into itself and defines a contraction if M is well chosen. In fact, using Lemma 2.4, Lemma 2.5, and Corollary 3.6 we have that

$$\begin{aligned}
\|\Gamma_1(\eta, \Phi)\|_{U^s} &\leq C_1 \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} + C_2 \Big(\|\psi \partial_x(\eta \partial_x \Phi)\|_{Z^s} + \|\psi(\partial_x \Phi)^2\|_{W^{s+1}}\Big) \\
&\leq C_1 \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} + C_2 C_5 \Big(\|\eta\|_{X^{s,1/2}} \|\Phi\|_{Y^{s+1,1/2}} + \|\Phi\|^2_{Y^{s+1,1/2}}\Big) \\
&\leq C_1 \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} + C_2 C_5 \|(\eta, \Phi)\|^2_{U^s \times V^{s+1}}
\end{aligned}$$

and also that

$$\begin{aligned}
\|\Gamma_2(\eta, \Phi)\|_{V^{s+1}} &\leq C_1 \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} + C_2 \Big(\|\psi \partial_x(\eta \partial_x \Phi)\|_{Z^s} + \|\psi(\partial_x \Phi)^2\|_{W^{s+1}}\Big) \\
&\leq C_1 \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} + C_2 C_5 \|(\eta, \Phi)\|^2_{U^s \times V^{s+1}},
\end{aligned}$$

so that

$$\|\Gamma(\eta, \Phi)\|_{U^s \times V^{s+1}} \leq C_1 \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} + C_2 C_5 \|(\eta, \Phi)\|^2_{U^s \times V^{s+1}}. \tag{4.1}$$

Choosing $M = 2C_1 \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})}$ such that

$$K_1 = 4C_1 C_2 C_5 \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} < 1,$$

we obtain that

$$\begin{aligned}
\|\Gamma(\eta, \Phi)\|_{U^s \times V^{s+1}} &\leq C_1 \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} (1 + 4C_1 C_2 C_5 \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})}) \\
&\leq 2C_1 \|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} = M
\end{aligned}$$

and that Γ maps Z_M to itself. Now, let us prove that Γ is a contraction. In fact, if (η,Φ), $(\eta_1,\Phi_1)\in Z_M$, using Lemma 2.5 and Corollary 3.6 we have that

$$\begin{aligned}
&\|\Gamma_1(\eta,\Phi)-\Gamma_1(\eta_1,\Phi_1)\|_{U^s}\\
&\quad\le C_2\Big(\|\psi\partial_x(\eta\partial_x\Phi-\eta_1\partial_x\Phi_1)\|_{Z^s}+\|\psi((\partial_x\Phi)^2-(\partial_x\Phi_1)^2)\|_{W^{s+1}}\Big)\\
&\quad\le C_2\Big(\|\partial_x(\eta\partial_x(\Phi-\Phi_1))\|_{Z^s}+\|\partial_x((\eta-\eta_1)\partial_x\Phi_1)\|_{Z^s}\\
&\qquad+\|\partial_x(\Phi-\Phi_1)\partial_x(\Phi+\Phi_1)\|_{W^{s+1}}\Big)\\
&\quad\le C_2C_5\Big(\|\eta\|_{X^{s,1/2}}\|\Phi-\Phi_1\|_{Y^{s+1,1/2}}+\|\eta-\eta_1\|_{X^{s,1/2}}\|\Phi_1\|_{Y^{s+1,1/2}}\\
&\qquad+\|\Phi-\Phi_1\|_{Y^{s+1,1/2}}\|\Phi+\Phi_1\|_{Y^{s+1,1/2}}\Big)\\
&\quad\le C_2C_5\|(\eta,\Phi)-(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\left(\|(\eta,\Phi)\|_{U^s\times V^{s+1}}+\|(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\right).
\end{aligned}$$

In a similar fashion we see that

$$\begin{aligned}
&\|\Gamma_2(\eta,\Phi)-\Gamma_2(\eta_1,\Phi_1)\|_{V^{s+1}}\\
&\quad\le C_2\Big(\|\psi\partial_x(\eta\partial_x\Phi-\eta_1\partial_x\Phi_1)\|_{Z^s}+\|\psi((\partial_x\Phi)^2-(\partial_x\Phi_1)^2)\|_{W^{s+1}}\Big)\\
&\quad\le C_2C_5\|(\eta,\Phi)-(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\left(\|(\eta,\Phi)\|_{U^s\times V^{s+1}}+\|(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\right).
\end{aligned}$$

Then, we conclude

$$\begin{aligned}
&\|\Gamma(\eta,\Phi)-\Gamma(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\\
&\quad\le C_2C_5\|(\eta,\Phi)-(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\left(\|(\eta,\Phi)\|_{U^s\times V^{s+1}}+\|(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\right).
\end{aligned}\tag{4.2}$$

So, if (4) holds we obtain that

$$\|\Gamma(\eta,\Phi)-\Gamma(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}\le K_1\|(\eta,\Phi)-(\eta_1,\Phi_1)\|_{U^s\times V^{s+1}}$$

and then Γ is a contraction in Z_M. Thus, the contraction mapping principle guarantees the existence of a unique fixed point (η,Φ) of Γ in Z_M, which is solution of the truncated integral problem (1.4). Now, if (η_1,Φ_1) is a restriction of (η,Φ) on $[0,T]$, then using Lemma 2.6 we have that

$$\eta_1\in C([0,T]\,:\,H^s(\mathbb{T}))\cap U^s,\quad \Phi_1\in C([0,T]\,:\,\mathcal{V}^{s+1}(\mathbb{T}))\cap V^{s+1}$$

and (η_1,Φ_1) is a solution of the integral problem (1.3) on $[0,T]$.

By the fixed point argument used we have the uniqueness of the solution of the truncated integral problem (1.4) in the set Z_M. We will use an argument as in [1] to obtain the uniqueness of the integral problem (1.3) in the space $U^s_T\times V^{s+1}_T$.

Let $T>0$ and $(\eta,\Phi)\in U^s\times V^{s+1}$ be the solution of the truncated integral problem (1.4) obtained above and $(\eta_1,\Phi_1)\in U^s_T\times V^{s+1}_T$ a solution of the integral problem (1.3) with the same initial data $(\eta_0,\Phi_0)\in H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})$. Fix an extension $(\eta_2,\Phi_2)\in U^s\times V^{s+1}$ of (η_1,Φ_1), then, for some $T^*<T<1$ to be fixed later, we have that

$$\eta_2(t)=\psi(t)S_1(t)(\eta_0,\Phi_0)-\psi(t)\int_0^t S_1(t-t')\psi(t')\Big(\partial_x(\eta_2\partial_x\Phi_2),\frac{1}{2}(\partial_x\Phi_1)^2\Big)(t')dt'$$

and

$$\Phi_2(t)=\psi(t)S_2(t)(\eta_0,\Phi_0)-\psi(t)\int_0^t S_2(t-t')\psi(t')\Big(\partial_x(\eta_2\partial_x\Phi_2),\frac{1}{2}(\partial_x\Phi_1)^2\Big)(t')dt',$$

for all $t \in [0, T^*]$.

Now, by definition of $U^s_{T^*} \times V^{s+1}_{T^*}$ we have that for any $\epsilon > 0$, there exists $(\omega, \vartheta) \in U^s \times V^{s+1}$ such that for all $t \in [0, T^*]$,

$$\omega(t) = \eta(t) - \eta_2(t), \quad \vartheta(t) = \Phi(t) - \Phi_2(t)$$

and

$$\|\omega\|_{U^s} \leq \|\eta - \eta_2\|_{U^s_{T^*}} + \epsilon, \quad \|\vartheta\|_{V^{s+1}} \leq \|\Phi - \Phi_2\|_{V^{s+1}_{T^*}} + \epsilon. \tag{4.3}$$

We define

$$\omega_1(t) = -\psi(t) \int_0^t S_1(t-t')\psi(t')\Big(\partial_x(\eta\partial_x\vartheta) + \partial_x(\omega\partial_x\Phi_2), \frac{1}{2}\partial_x\vartheta\partial_x(\Phi+\Phi_2)\Big)(t')dt',$$

$$\vartheta_1(t) = -\psi(t) \int_0^t S_2(t-t')\psi(t')\Big(\partial_x(\eta\partial_x\vartheta) + \partial_x(\omega\partial_x\Phi_2), \frac{1}{2}\partial_x\vartheta\partial_x(\Phi+\Phi_2)\Big)(t')dt'.$$

Then we have that $\omega_1(t) = \eta(t) - \eta_2(t)$ and $\vartheta_1(t) = \Phi(t) - \Phi_2(t)$ for all $t \in [0, T^*]$. Thus, from Lemma 2.5 and Corollary 3.6 we obtain that

$$\begin{aligned}\|\eta - \eta_2\|_{U^s_{T^*}} &\leq \|\omega_1\|_{U^s} \\ &\leq C_2C_5\|(\omega,\vartheta)\|_{U^s\times V^{s+1}}\Big(\|(\eta,\Phi)\|_{U^s\times V^{s+1}} + \|(\eta_2,\Phi_2)\|_{U^s\times V^{s+1}}\Big) \\ &\leq 2C_2C_5N\|(\omega,\vartheta)\|_{U^s\times V^{s+1}}\end{aligned} \tag{4.4}$$

where we assume that

$$\max\{\|(\eta,\Phi)\|_{U^s\times V^{s+1}}, \ \|(\eta_2,\Phi_2)\|_{U^s\times V^{s+1}}\} \leq N.$$

In a similar fashion we have that

$$\begin{aligned}\|\Phi - \Phi_2\|_{V^{s+1}_{T^*}} \leq \|\vartheta_1\|_{V^{s+1,1/2}} &\leq C_2C_5\|(\omega,\vartheta)\|_{U^s\times V^{s+1}}\Big(\|(\eta,\Phi)\|_{U^s\times V^{s+1}} + \|(\eta_1,\Phi_2)\|_{U^s\times V^{s+1}}\Big) \\ &\leq 2C_2C_5N\|(\omega,\vartheta)\|_{U^s\times V^{s+1}}.\end{aligned} \tag{4.5}$$

If $4C_2C_5N \leq 1/2$, then we obtain, using (4.3), (4.4) and (4.5), that

$$\begin{aligned}\|\eta - \eta_2\|_{U^s_{T^*}} + \|\Phi - \Phi_2\|_{V^{s+1}_{T^*}} &\leq 4C_2C_5N\|(\omega,\vartheta)\|_{U^s\times V^{s+1}} \\ &\leq \frac{1}{2}\Big(\|\eta - \eta_2\|_{U^s_{T^*}} + \epsilon + \|\Phi - \Phi_2\|_{V^{s+1}_{T^*}} + \epsilon\Big).\end{aligned}$$

So, we see that

$$\|\eta - \eta_2\|_{U^s_{T^*}} + \|\Phi - \Phi_2\|_{V^{s+1}_{T^*}} \leq 2\epsilon.$$

Therefore $\eta = \eta_2$ and $\Phi = \Phi_2$ on $[0, T^*]$. Now, since the argument does not depend on the initial data, we can iterate this process a finite number of times to extend the uniqueness result in the whole existence interval $[0, T]$.

Combining an identical argument to the one used in the existence proof with Lemma 2.6, one can easily show that the map data-solution is locally Lipschitz. □

5 Internal controllability

5.1 Spectral analysis

In this section we perform the spectral analysis for the operator

$$M = \begin{pmatrix} 0 & -(I-\partial_x^2)\partial_x^2 \\ -(I-\partial_x^2) & 0 \end{pmatrix},$$

defined in the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$. The result in this analysis will be the basis to transfer the internal controllability of the associated linear system to the nonlinear system. Let us define

$$E_{1,k} = \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}, \quad E_{2,k} = \begin{pmatrix} 0 \\ \frac{1}{k}e^{ikx} \end{pmatrix},$$

for $k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. If we set

$$M_k = \begin{pmatrix} 0 & (1+k^2)k^2 \\ -(1+k^2) & 0 \end{pmatrix}, \qquad \Sigma_k = \begin{pmatrix} 0 & (1+k^2)k \\ -(1+k^2)k & 0 \end{pmatrix}, \quad k \in \mathbb{Z}^*$$

then we see directly that

$$M_k(E_{1,k}, E_{2,k}) = (E_{1,k}, E_{2,k})\Sigma_k, \quad k \in \mathbb{Z}^*.$$

Moreover, we have that the eigenvalues for Σ_k are

$$\lambda_{1,k} = i\sqrt{(1+k^2)^2k^2}, \quad \lambda_{2,k} = -i\sqrt{(1+k^2)^2k^2}, \quad k \in \mathbb{Z}^*,$$

with corresponding eigenvectors

$$\tilde{e}_{1,k} = \begin{pmatrix} 1 \\ \frac{\lambda_{1,k}}{(1+k^2)k} \end{pmatrix}, \quad \tilde{e}_{2,k} = \begin{pmatrix} 1 \\ \frac{\lambda_{2,k}}{(1+k^2)k} \end{pmatrix}, \quad k \in \mathbb{Z}^*.$$

Thus, we have that

$$\begin{aligned} M(E_{1,k}, E_{2,k})(\tilde{e}_{1,k}, \tilde{e}_{2,k}) &= (E_{1,k}, E_{2,k})\Sigma_k(\tilde{e}_{1,k}, \tilde{e}_{2,k}) \\ &= (\lambda_{1,k}(E_{1,k}, E_{2,k})\tilde{e}_{1,k}, \lambda_{2,k}(E_{1,k}, E_{2,k})\tilde{e}_{2,k}),\ k \in \mathbb{Z}^*, \end{aligned}$$

meaning that $\lambda_{1,k}$ and $\lambda_{2,k}$ are the eigenvalues for the operator M with corresponding eigenvectors

$$\eta_{j,k} = (E_{1,k}, E_{2,k})\tilde{e}_{j,k}, \quad j = 1, 2, \ \ k \in \mathbb{Z},$$

where

$$\lambda_{1,0} = \lambda_{2,0} = 0, \quad \tilde{e}_{1,0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{e}_{2,0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E_{1,0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad E_{2,0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

On the other hand, we see that

$$\lim_{k\to\pm\infty} \frac{\lambda_{1,k}}{(1+k^2)k} = \pm i, \quad \lim_{k\to\pm\infty} \frac{\lambda_{2,k}}{(1+k^2)k} = \mp i.$$

Then

$$\lim_{k\to\infty} (\tilde{e}_{1,k}, \tilde{e}_{2,k}) = \begin{pmatrix} 1 & 1 \\ \pm i & \mp i \end{pmatrix} \quad \text{and} \quad \lim_{k\to\infty} \det(\tilde{e}_{1,k}, \tilde{e}_{2,k}) = \mp 2i \neq 0.$$

In other words, $\{\nu_{1,0}, \nu_{2,0}, \nu_{1,k}, \nu_{2,k} \ : \ k \in \mathbb{Z}^*\}$ forms a Riesz basis for $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ with

$$\nu_{j,k} = \frac{\eta_{j,k}}{\|\eta_{j,k}\|_{H^s \times \mathcal{V}^{s+1}}}.$$

Moreover, we also have for $j = 1, 2$ that $\nu_{j,k} = \vec{b}_{j,k} e^{ikx}$ with

$$0 < B_1 \leq \|\vec{b}_{j,k}\| \leq B_2, \quad k \in \mathbb{Z}, \quad s \geq 0. \tag{5.1}$$

From the above discussion we have the following result.

Theorem 5.1. *Let λ_k and $\phi_{j,k}$, $j = 1, 2$ be given by*

$$\lambda_k = i \ \mathrm{sign}(k) \sqrt{(1+k^2)^2 k^2}, \quad k \in \mathbb{Z},$$

$$\phi_{1,k} = \begin{cases} \nu_{1,k}, & k = 0, 1, 2, 3, \ldots \\ \nu_{2,k}, & k = -1, -2, -3, \ldots, \end{cases} \qquad \phi_{2,k} = \begin{cases} \nu_{1,-k}, & k = 1, 2, 3, \ldots \\ \nu_{2,-k}, & k = 0, -1, -2, -3, \ldots \end{cases}$$

then we have that

- *(i) The spectrum of the operator M is $\sigma(M) = \{\lambda_k \ : \ k \in \mathbb{Z}\}$, in which each λ_k is a double eigenvalue with eigenvectors $\phi_{1,k}$ and $\phi_{2,k}$.*
- *(ii) The set $\{\phi_{1,k}, \phi_{2,k} \ : \ k \in \mathbb{Z}\}$ forms an orthonormal basis for the space $H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ such that any $(\eta, \Phi) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ has the following Fourier expansion*

$$(\eta, \Phi) = \sum_{k \in \mathbb{Z}} \left(\alpha_{1,k}\phi_{1,k} + \alpha_{2,k}\phi_{2,k} \right), \quad \alpha_{j,k} = \left\langle (\eta, \Phi), \phi_{j,k} \right\rangle_Q, \quad j = 1, 2, \ k \in \mathbb{Z},$$

where $Q = L^2(\mathbb{T}) \times L^2(\mathbb{T})$.

5.2 Linear controllability

In this section we consider the internal control problem for the linear system

$$\begin{cases} \eta_t + \partial_x^2 \Phi - \partial_x^4 \Phi = f_1, \\ \qquad \Phi_t + \eta - \partial_x^2 \eta = f_2, \end{cases} \tag{5.2}$$

with the initial condition

$$\eta(x, 0) = \eta_0(x), \quad \Phi(x, 0) = \Phi_0(x). \tag{5.3}$$

Theorem 5.2. *Suppose that $\rho = (\rho_1, \rho_2)$ is a non-zero smooth function defined on $\mathbb{T}$. Let $s \geq 0$ and $T > 0$, then for any (η_0, Φ_0), $(\eta_T, \Phi_T) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ there exists a function $H = (h_1, h_2) \in L^2\left(0, T; H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})\right)$ such that if*

$$F = (f_1(x,t), f_2(x,t)) = (\rho_1 h_1(x,t), \rho_2(x) h_2(x,t))$$

we have that the problem (5.2)–(5.3) has a unique solution

$$(\eta, \Phi) \in C\left([0, T] \ : \ H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T}) \right)$$

satisfying

$$\eta(x, T) = \eta_T(x), \quad \Phi(x, T) = \Phi_T(x).$$

Moreover, there exists $C = C(T) > 0$ such that

$$\|H\|_{L^2(0,T;H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T}))} \leq C \left(\|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} + \|(\eta_T, \Phi_T)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} \right).$$

Proof. For sake of simplicity in the proof we will consider $\rho_2 = \Phi_0 = \Phi_T = 0$. For any function $h = h(x,t)$, we define the control operator L by

$$(Lh)(x,t) = \rho_1(x)h(x,t).$$

If $f_1 = Lh$ and $f_2 = 0$, then we rewrite the problem (5.2)–(5.3) as the following first order linear

$$\begin{pmatrix}\eta\\ \Phi\end{pmatrix}_t = M\begin{pmatrix}\eta\\ \Phi\end{pmatrix} + Bh, \tag{5.4}$$

with the initial condition

$$\eta(x,0) = \eta_0(x), \quad \Phi(x,0) = 0 \tag{5.5}$$

where

$$Bh = \begin{pmatrix}Lh\\ 0\end{pmatrix}.$$

In this case for $h \in L^2(0,T;H^s(\mathbb{T}))$, the solution of the linear problem (5.4)–(5.5) is given by

$$(\eta(t),\Phi(t)) = S(t)\,(\eta_0,0) + \int_0^t S(t-\tau)Bh(\tau)\,d\tau.$$

Now, using the spectral analysis on the operator M we have that

$$\begin{aligned}(\eta(t),\Phi(t)) = &\sum_{n\in\mathbb{Z}} e^{\lambda_n t}(\alpha_{1,n}\phi_{1,n} + \alpha_{2,n}\phi_{2,n})\\ &+ \sum_{n\in\mathbb{Z}}\int_0^t e^{\lambda_n(t-\tau)}\,(\beta_{1,n}(\tau)\phi_{1,n} + \beta_{2,n}(\tau)\phi_{2,n})\,d\tau,\end{aligned} \tag{5.6}$$

where $\alpha_{j,n}$ and $\beta_{j,n}$, for $j = 1,2$, $n \in \mathbb{Z}$, are given by

$$\alpha_{j,n} = \left\langle(\eta_0,\Phi_0),\phi_{j,n}\right\rangle_Q, \quad \beta_{j,n}(t) = \left\langle Bh,\phi_{j,n}\right\rangle_Q. \tag{5.7}$$

We verify easily that L is a self-adjoint operator in $L^2(\mathbb{T})$ such that

$$\langle Bh,(\eta,\Phi)\rangle_Q = \langle(Lh,0),(\eta,\Phi)\rangle_Q = \langle Lh,\eta\rangle_{L^2(\mathbb{T})} = \langle h,L\eta\rangle_{L^2(\mathbb{T})},$$

then we have that

$$\alpha_{j,n} = \left\langle \eta_0,\phi_{j,n}^{(1)}\right\rangle_{L^2(\mathbb{T})}, \quad \beta_{j,n}(t) = \left\langle h(\cdot,t),L(\phi_{j,n}^{(1)})\right\rangle_{L^2(\mathbb{T})},$$

where $\phi^{(m)}$ denoting the m component of ϕ.

The internal control problem consists of finding a $h \in L^2(0,T;H^s(\mathbb{T}))$ such that

$$\eta(x,T) = \eta_T(x), \quad \Phi(x,T) = 0.$$

Then, let η_0 and η_T be having the following decompositions

$$\eta_0 = \sum_{n\in\mathbb{Z}}\left(\alpha_{1,n}\phi_{1,n}^{(1)} + \alpha_{2,n}\phi_{2,n}^{(1)}\right), \quad \eta_T = \sum_{n\in\mathbb{Z}}\left(\gamma_{1,n}\phi_{1,n}^{(1)} + \gamma_{2,n}\phi_{2,n}^{(1)}\right),$$

where $\gamma_{j,n} = \langle\eta_T,\phi_{j,n}^{(1)}\rangle_{L^2(\mathbb{T})}$. But, we know that

$$\begin{aligned}(\eta(x,T),\Phi(x,T)) = &\sum_{n\in\mathbb{Z}} e^{\lambda_n T}\,(\alpha_{1,n}\phi_{1,n} + \alpha_{2,n}\phi_{2,n})\\ &+ \sum_{n\in\mathbb{Z}}\int_0^T e^{\lambda_n(T-\tau)}\,(\beta_{1,n}(\tau)\phi_{1,n} + \beta_{2,n}(\tau)\phi_{2,n})\,d\tau.\end{aligned}$$

So, in each node, we have for $j = 1, 2$ and $n \in \mathbb{Z}$ that

$$\alpha_{j,n} + \int_0^T e^{-\lambda_n \tau} \beta_{j,n}(\tau)\, d\tau = \gamma_{j,n} e^{-\lambda_n T}.$$

Now, from [7] we have that $\mathcal{P} = \{e^{\lambda_k t} : k \in \mathbb{Z}\}$ is a Riesz basis for its closed span $\mathcal{P}_T = \overline{\text{gen}}\,\mathcal{P}$ generated in $L^2(0, T)$, with a unique dual Riesz basis given by $\mathcal{L} = \{q_k : k \in \mathbb{Z}\}$ satisfying that

$$\int_0^T q_l(t) \overline{e^{\lambda_k t}}\, dt = \delta_l^k, \quad l, k \in \mathbb{Z}.$$

We assume that f_1 has the form $f_1 = Lh$ with h given by the expansion

$$h(x, t) = \sum_{l \in \mathbb{Z}} q_l(t) \left(c_{1,l} L(\phi_{1,l}^{(1)}) + c_{2,l} L(\phi_{2,l}^{(1)}) \right), \tag{5.8}$$

where the coefficients $c_{1,l}$ and $c_{2,l}$ are to be determined so that, among other things, the series (5.8) is appropriately convergent. In this case, for $j = 1, 2$ and $n \in \mathbb{Z}$ we have that

$$\begin{aligned}
\int_0^T e^{-\lambda_n \tau} \beta_{j,n}(\tau)\, d\tau &= \int_0^T e^{-\lambda_n \tau} \left\langle h(\cdot, \tau), L(\phi_{j,n}^{(1)}) \right\rangle_{L^2(\mathbb{T})} d\tau \\
&= \sum_{k \in \mathbb{Z}} \int_0^T e^{-\lambda_n \tau} \Big(\int_{\mathbb{T}} h(y, \tau) e^{iky}\, dy \Big) \overline{\left(L(\phi_{j,n}^{(1)}) \right)_k} d\tau \\
&= \sum_{k \in \mathbb{Z}} \overline{\left(L(\phi_{j,n}^{(1)}) \right)_k} \int_{\mathbb{T}} \left(\sum_{l \in \mathbb{Z}} \int_0^T q_l(\tau) \overline{e^{\lambda_n \tau}} d\tau \left(c_{1,l} L(\phi_{1,l}^{(1)}) + c_{2,l} L(\phi_{2,l}^{(1)}) \right) \right) e^{iky}\, dy \\
&= \sum_{k \in \mathbb{Z}} \overline{\left(L(\phi_{j,n}^{(1)}) \right)_k} \int_{\mathbb{T}} \left(c_{1,n} L(\phi_{1,n}^{(1)}) + c_{2,n} L(\phi_{2,n}^{(1)}) \right) e^{iky}\, dy \\
&= c_{1,n} \left\langle L(\phi_{1,n}^{(1)}), L(\phi_{j,n}^{(1)}) \right\rangle_{L^2(\mathbb{T})} + c_{2,n} \left\langle L(\phi_{2,n}^{(1)}), L(\phi_{j,n}^{(1)}) \right\rangle_{L^2(\mathbb{T})},
\end{aligned}$$

where $\left(L(\phi_{j,n}^{(1)}) \right)_k = \left(\widehat{L(\phi_{j,n}^{(1)})} \right)(k)$. Now, using the computations above, for $n \in \mathbb{Z}$, we have that $c_{1,n}$ and $c_{2,n}$ must satisfy the linear system

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} c_{1,n} \\ c_{2,n} \end{pmatrix} = \begin{pmatrix} -\alpha_{1,n} + \gamma_{1,n} e^{-\lambda_n T} \\ -\alpha_{2,n} + \gamma_{2,n} e^{-\lambda_n T} \end{pmatrix},$$

where $a_{jl} = \langle L(\phi_{j,n}^{(1)}), L(\phi_{l,n}^{(1)}) \rangle_{L^2(\mathbb{T})}$. Using the fact that $L(\phi_{1,n}^{(1)})$ and $L(\phi_{2,n}^{(1)})$ are linear independent, we obtain that

$$\Delta_n = \det \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \|L(\phi_{1,n}^{(1)})\|_{L^2(\mathbb{T})}^2 \|L(\phi_{2,n}^{(1)})\|_{L^2(\mathbb{T})}^2 - \left| \langle L(\phi_{1,n}^{(1)}), L(\phi_{2,n}^{(1)}) \rangle_{L^2(\mathbb{T})} \right|^2 \neq 0.$$

Moreover, using that $\nu_{j,n}^{(1)} = b_{j,n}^{(1)} e^{inx}$ and the estimate (5.1), we have that

$$\|L(\phi_{j,n}^{(1)})\|_{L^2(\mathbb{T})}^2 \sim |b_{j,n}^{(1)}|^2 \geq C > 0. \tag{5.9}$$

In addition (see the estimations by B. Zhang for the Boussinesq equation in [20]), it is not hard to prove that

$$\left\langle L(\phi_{1,n}^{(1)}), L(\phi_{2,n}^{(1)}) \right\rangle \to 0, \quad n \to \infty.$$

Hence there exists $\epsilon > 0$ such that $|\Delta_n| > \epsilon$. So, we conclude that $c_{1,n}$ and $c_{2,n}$ are uniquely determine by

$$c_{1,n} = \frac{\begin{vmatrix} -\alpha_{1,n} + \gamma_{1,n}e^{-\lambda_n T} & a_{21} \\ -\alpha_{2,n} + \gamma_{2,n}e^{-\lambda_n T} & a_{22} \end{vmatrix}}{\Delta_n}, \quad c_{2,n} = \frac{\begin{vmatrix} a_{11} & -\alpha_{1,n} + \gamma_{1,n}e^{-\lambda_n T} \\ a_{12} & -\alpha_{2,n} + \gamma_{2,n}e^{-\lambda_n T} \end{vmatrix}}{\Delta_n}. \tag{5.10}$$

It remains to show that h defined by (5.8)–(5.10) belongs to the space $L^2(0,T;H^s(\mathbb{T}))$ provided that $\eta_0, \eta_T \in H^s(\mathbb{T})$. To this end, let us write

$$L(\phi_{j,l}^{(1)}) = \sum_{k\in\mathbb{Z}} a_{j,lk}e^{ikx}, \quad a_{j,lk} = \left(L(\phi_{j,l}^{(1)})\right)_k, \quad l,k \in \mathbb{Z},\ j = 1,2.$$

Thus

$$h(x,t) = h_1(x,t) + h_2(x,t),$$

where

$$h_j(x,t) = \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_{j,l}a_{j,lk}q_l(t)e^{ikx}, \quad j = 1,2.$$

From this, we conclude that

$$\begin{aligned}
&\|h_j\|^2_{L^2(0,T;H^s(\mathbb{T}))} \\
&= \int_0^T \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} |\left(h_j(\cdot,t)\right)_k|^2\,dt = \int_0^T \sum_{k\in\mathbb{Z}} (1+|k|)^{2s}\Big|\sum_{l\in\mathbb{Z}} a_{j,lk}c_{j,l}q_l(t)\Big|^2 dt \\
&= \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} \int_0^T \Big|\sum_{l\in\mathbb{Z}} a_{j,lk}c_{j,l}q_l(t)\Big|^2 dt \le C \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} \sum_{l\in\mathbb{Z}} |c_{j,l}|^2|a_{j,lk}|^2 \\
&= C\sum_{l\in\mathbb{Z}} |c_{j,l}|^2 \sum_{k\in\mathbb{Z}} (1+|k|)^{2s}|a_{j,lk}|^2,
\end{aligned}$$

where the constant $C > 0$ comes from the Riesz basis property of $\mathcal{L}$ in $\mathcal{P}_T$. Now, using (5.1), if $\rho_1 = \sum_{m\in\mathbb{Z}} \rho_m^1 e^{imx}$ we have that there exists $C > 0$ such that

$$\begin{aligned}
|a_{j,lk}| &= \left|\left\langle L(\phi_{j,l}^{(1)}),\, e^{ikx}\right\rangle_{L^2(\mathbb{T})}\right| = \left|\left\langle \rho_1\phi_{j,l}^{(1)},\, e^{ikx}\right\rangle_{L^2(\mathbb{T})}\right| \\
&= \left|\sum_{m\in\mathbb{Z}} \rho_m^1 \left\langle \phi_{j,l}^{(1)}e^{imx},\, e^{ikx}\right\rangle_{L^2(\mathbb{T})}\right| = \left|\sum_{m\in\mathbb{Z}} \rho_m^1 \left\langle b_{j,l}^{(1)}e^{ilx}e^{imx},\, e^{ikx}\right\rangle_{L^2(\mathbb{T})}\right| \\
&\le C\Big|\int_{\mathbb{T}} e^{-ix(k-l)} \sum_{m\in\mathbb{Z}} \rho_m^1 e^{imx}\,dx\Big| \le C\left|\rho_{k-l}^1\right|.
\end{aligned}$$

Then we see that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} (1+|k|)^{2s}|a_{j,lk}|^2 &\le C\sum_{k\in\mathbb{Z}} (1+|k|)^{2s}|\rho_{k-l}^1|^2 \le C\sum_{k\in\mathbb{Z}} (1+|k+l|)^{2s}|\rho_k^1|^2 \\
&\le C(1+|l|)^{2s}\sum_{k\in\mathbb{Z}} (1+|k|)^{2s}|\rho_k^1|^2 = (1+|l|)^{2s}\,\|\rho_1\|^2_{H^s(\mathbb{T})}.
\end{aligned}$$

On the other hand, we can to see that

$$\begin{aligned}
|c_{1,l}|^2 &\le C\left(|a_{22}|^2 + |a_{21}|^2\right)\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right) \\
&= C\left(\|L(\phi_{2,l}^{(1)})\|^4_{L^2(\mathbb{T})} + \left|\left\langle L(\phi_{2,l}^{(1)}), L(\phi_{1,l}^{(1)})\right\rangle_{L^2(\mathbb{T})}\right|^2\right)\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right) \\
&\le C\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
|c_{2,l}|^2 &\leq C\Big(\|L(\phi_{1,l}^{(1)})\|_{L^2(\mathbb{T})}^4 + \Big|\left\langle L(\phi_{1,l}^{(1)}), L(\phi_{2,l}^{(1)})\right\rangle_{L^2(\mathbb{T})}\Big|^2\Big) \times \left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right) \\
&\leq C\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right).
\end{aligned}
$$

From this, we conclude that

$$
\begin{aligned}
\|h\|_{L^2(0,T;H^s(\mathbb{T}))}^2 &\leq C\|\rho_1\|_{H^s(\mathbb{T})}^2 \left(\sum_{l\in\mathbb{Z}} (1+|l|)^{2s}\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{1,l}|^2\right)\right) \\
&\leq C\|\rho_1\|_{H^s(\mathbb{T})}^2 \left(\|\eta_0\|_{H^s(\mathbb{T})}^2 + \|\eta_T\|_{H^s(\mathbb{T})}^2\right).
\end{aligned}
$$

Now we consider $\rho_1 = \eta_0 = \eta_T = 0$. Since the proof is similar to the previous case, we only present some ideas. The solution of the linear problem

$$
\begin{pmatrix}\eta\\ \Phi\end{pmatrix}_t = M\begin{pmatrix}\eta\\ \Phi\end{pmatrix} + Bh, \quad (\eta(x,0),\Phi(x,0)) = (0,\Phi_0(x))
$$

with

$$
(Bh)(x,t) = \begin{pmatrix}0\\ (Lh)(x,t)\end{pmatrix} = \begin{pmatrix}0\\ \rho_2(x)h(x,t)\end{pmatrix}
$$

is given by

$$
(\eta(t),\Phi(t)) = \sum_{n\in\mathbb{Z}} e^{\lambda_n t}\left(\alpha_{1,n}\phi_{1,n} + \alpha_{2,n}\phi_{2,n}\right) + \sum_{n\in\mathbb{Z}}\int_0^t e^{\lambda_n(t-\tau)}\left(\beta_{1,n}(\tau)\phi_{1,n} + \beta_{2,n}(\tau)\phi_{2,n}\right)d\tau,
$$

where $\alpha_{j,n}$ and $\beta_{j,n}$ for $j = 1,2$, $n \in \mathbb{Z}$ are given by

$$
\alpha_{j,n} = \left\langle \Phi_0, \phi_{j,n}^{(2)}\right\rangle_{L^2(\mathbb{T})}, \quad \beta_{j,n}(t) = \left\langle h(\cdot,t), L(\phi_{j,n}^{(2)})\right\rangle_{L^2(\mathbb{T})}.
$$

Then, using the decompositions

$$
\Phi_0 = \sum_{n\in\mathbb{Z}}\left(\alpha_{1,n}\phi_{1,n}^{(2)} + \alpha_{2,n}\phi_{2,n}^{(2)}\right), \quad \Phi_T = \sum_{n\in\mathbb{Z}}\left(\gamma_{1,n}\phi_{1,n}^{(2)} + \gamma_{2,n}\phi_{2,n}^{(2)}\right),
$$

where $\gamma_{j,n} = \langle \Phi_T, \phi_{j,n}^{(2)}\rangle_{L^2(\mathbb{T})}$, then we have for $j = 1,2$ and $n \in \mathbb{Z}$ that

$$
\alpha_{j,n} + \int_0^T e^{-\lambda_n\tau}\beta_{j,n}(\tau)\,d\tau = \gamma_{j,n}e^{-\lambda_n T}.
$$

Now, we take the control h to have the form

$$
h(x,t) = \sum_{l\in\mathbb{Z}} q_l(t)\left(c_{1,l}L(\phi_{1,l}^{(2)}) + c_{2,l}L(\phi_{2,l}^{(2)})\right)
$$

with

$$
\int_0^T q_l(t)\overline{e^{\lambda_k t}}\,dt = \delta_l^k, \quad l,k \in \mathbb{Z}.
$$

Then, for $j = 1, 2$ and $n \in \mathbb{Z}$ we have that

$$\begin{aligned}\int_0^T e^{-\lambda_n \tau}\beta_{j,n}(\tau)\, d\tau &= \sum_{k\in\mathbb{Z}} \overline{\left(L\phi_{j,n}^{(2)}\right)_k} \int_0^T e^{-\lambda_n\tau}\Big(\int_{\mathbb{T}} h(y,\tau)e^{iky}\, dy\Big)\, d\tau \\ &= \sum_{k\in\mathbb{Z}} \overline{\left(L(\phi_{j,n}^{(2)})\right)_k} \int_{\mathbb{T}} \left(c_{1,n}L(\phi_{1,n}^{(2)}) + c_{2,n}L(\phi_{2,n}^{(2)})\right) e^{iky}\, dy \\ &= c_{1,n}\left\langle L(\phi_{1,n}^{(2)}), L(\phi_{j,n}^{(2)})\right\rangle_{L^2(\mathbb{T})} + c_{2,n}\left\langle L(\phi_{2,n}^{(2)}), L(\phi_{j,n}^{(2)})\right\rangle_{L^2(\mathbb{T})}.\end{aligned}$$

Thus, the coefficients $c_{1,n}$ and $c_{2,n}$ are uniquely determine by

$$c_{1,n} = \frac{\begin{vmatrix} -\alpha_{1,n} + \gamma_{1,n}e^{-\lambda_n T} & a_{21} \\ -\alpha_{2,n} + \gamma_{2,n}e^{-\lambda_n T} & a_{22}\end{vmatrix}}{\Delta_n}, \quad c_{2,n} = \frac{\begin{vmatrix} a_{11} & -\alpha_{1,n} + \gamma_{1,n}e^{-\lambda_n T} \\ a_{12} & -\alpha_{2,n} + \gamma_{2,n}e^{-\lambda_n T}\end{vmatrix}}{\Delta_n},$$

where

$$a_{jl} = \left\langle L(\phi_{j,n}^{(2)}), L(\phi_{l,n}^{(2)})\right\rangle_{L^2(\mathbb{T})}$$

and

$$\Delta_n = \|L(\phi_{1,n}^{(2)})\|^2_{L^2(\mathbb{T})}\|L(\phi_{2,n}^{(2)})\|^2_{L^2(\mathbb{T})} - \left|\langle L(\phi_{1,n}^{(2)}), L(\phi_{2,n}^{(2)})\rangle_{L^2(\mathbb{T})}\right|^2.$$

Finally, we write

$$\rho_2 = \sum_{k\in\mathbb{Z}} \rho_k^2 e^{ikx},\ L(\phi_{j,l}^{(2)}) = \sum_{k\in\mathbb{Z}} a_{j,lk}e^{ikx},\ a_{j,lk} = \left(L(\phi_{j,l}^{(2)})\right)_k,\ l,k\in\mathbb{Z},\ j = 1,2.$$

Then

$$h(x,t) = \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_{1,l}a_{1,lk}q_l(t)e^{ikx} + \sum_{l\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_{2,l}a_{2,lk}q_l(t)e^{ikx}.$$

Hence, we see that

$$\begin{aligned}\|h\|^2_{L^2(0,T;\mathcal{V}^{s+1}(\mathbb{T}))} &= \sum_{j=1}^{2}\sum_{k\in\mathbb{Z}} |k|^2(1+|k|)^{2s}\int_0^T \Big|\sum_{l\in\mathbb{Z}} a_{j,lk}c_{j,l}q_l(t)\Big|^2 dt \\ &\leq C\sum_{j=1}^{2}\sum_{l\in\mathbb{Z}} |c_{j,l}|^2 \sum_{k\in\mathbb{Z}} |k|^2(1+|k|)^{2s}|a_{j,lk}|^2.\end{aligned}$$

The fact $\left|a_{j,lk}\right| \leq C\left|\rho^2_{k-l}\right|$ implies

$$\sum_{k\in\mathbb{Z}} |k|^2(1+|k|)^{2s}|a_{j,lk}|^2 \leq C\sum_{k\in\mathbb{Z}} |k+l|^2(1+|k+l|)^{2s}|\rho_k^2|^2 \leq C|l|^2\left(1+|l|\right)^{2s}\|\rho_2\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}.$$

Then, using

$$|c_{j,l}|^2 \leq C\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{2,l}|^2\right),$$

we conclude that

$$\begin{aligned}\|h\|^2_{L^2(0,T;\mathcal{V}^{s+1}(\mathbb{T}))} &\leq C\|\rho_2\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}\left(\sum_{l\in\mathbb{Z}} |l|^2(1+|l|)^{2s}\left(|\alpha_{1,l}|^2 + |\alpha_{2,l}|^2 + |\gamma_{1,l}|^2 + |\gamma_{1,l}|^2\right)\right) \\ &\leq C\|\rho_2\|^2_{H^s(\mathbb{T})}\left(\|\Phi_0\|^2_{\mathcal{V}^{s+1}(\mathbb{T})} + \|\Phi_T\|^2_{\mathcal{V}^{s+1}(\mathbb{T})}\right).\end{aligned}$$

□

Remark 5.3. If $T > 0$ and $(\eta_0, \Phi_0), (\eta_T, \Phi_T) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ with $s \geq 0$, the Theorem 5.2 implies that there is F such that

$$S(T)(\eta_0, \Phi_0) + \int_0^T S(T-\tau)F(\tau)\, d\tau = (\eta_T, \Phi_T).$$

Moreover, we have that there exists $C = C(T) > 0$ such that

$$\|F\|_{L^1(0,T;H^s \times \mathcal{V}^{s+1})} \leq C \left(\|(\eta_0, \Phi_0)\|_{H^s \times \mathcal{V}^{s+1}} + \|(\eta_T, \Phi_T)\|_{H^s \times \mathcal{V}^{s+1}} \right).$$

5.3 Nonlinear controllability

Now we turn to the nonlinear system

$$\begin{cases} \eta_t + \partial_x^2 \Phi - \partial_x^4 \Phi + \partial_x (\eta \partial_x \Phi) = f_1, \\ \Phi_t + \eta - \partial_x^2 \eta + \frac{1}{2} (\partial_x \Phi)^2 = f_2, \end{cases} \tag{5.11}$$

with the initial condition

$$\eta(x,0) = \eta_0(x), \quad \Phi(x,0) = \Phi_0(x), \tag{5.12}$$

Theorem 5.4. *Let $s \geq 0$ and $T > 0$ be given, then there exists $\delta > 0$ such that for any (η_0, Φ_0), $(\eta_T, \Phi_T) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$ satisfying*

$$\|(\eta_0, \Phi_0)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})}, \quad \|(\eta_T, \Phi_T)\|_{H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})} < \delta,$$

there exists a control function $F = (f_1, f_2) \in L^1(0, T; H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T}))$ such that the solution $(\eta, \Phi) \in C\left([0,T] : H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})\right) \cap U_T^s \times V_T^{s+1}$ of the problem (5.11)–(5.12) satisfies

$$\eta(x,T) = \eta_T(x), \quad \Phi(x,T) = \Phi_T(x).$$

Proof. We rewrite the Cauchy problem (5.11)–(5.12) in its equivalent form:

$$\begin{aligned} (\eta(t), \Phi(t)) = S(t)(\eta_0, \Phi_0) &+ \int_0^t S(t-t')F(t')\, dt' \\ &- \int_0^t S(t-t')\Big(\partial_x(\eta\partial_x\Phi), \frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt'. \end{aligned} \tag{5.13}$$

Now, for any $(\eta, \Phi) = (\eta(x,t), \Phi(x,t))$ we define

$$w((\eta, \Phi), T) = \int_0^T S(T-t')\Big(\partial_x(\eta\partial_x\Phi), \frac{1}{2}(\partial_x\Phi)^2\Big)(t')\, dt'$$

According to Theorem 5.2, for given $(\eta_0, \Phi_0), (\eta_T, \Phi_T) \in H^s(\mathbb{T}) \times \mathcal{V}^{s+1}(\mathbb{T})$, if one chooses

$$F = F_{(\eta, \Phi)}$$

such that

$$S(T)(\eta_0, \Phi_0) + \int_0^T S(T-t')F_{(\eta,\Phi)}(t')\, dt' = (\eta_T, \Phi_T) + w((\eta, \Phi), T)$$

in the equation (5.13), then

$$(\eta(t),\Phi(t)) = S(t)(\eta_0,\Phi_0) + \int_0^t S(t-t')F_{(\eta,\Phi)}(t')\,dt' - \int_0^t S(t-t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt', \tag{5.14}$$

with $(\eta(0),\Phi(0)) = (\eta_0,\Phi_0)$ and

$$\begin{aligned}(\eta(T),\Phi(T)) &= S(T)(\eta_0,\Phi_0) + \int_0^T S(T-t')F_{(\eta,\Phi)}(t')dt' \\ &\quad - \int_0^T S(T-t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt' \\ &= (\eta_T,\Phi_T) + w((\eta,\Phi),T) - w((\eta,\Phi),T) = (\eta_T,\Phi_T).\end{aligned}$$

This suggests that we consider the map

$$\begin{aligned}\Gamma(\eta,\Phi) &= S(t)(\eta_0,\Phi_0) + \int_0^t S(t-t')F_{(\eta,\Phi)}(t')dt' \\ &\quad - \int_0^t S(t-t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt'.\end{aligned}$$

If the map Γ is shown to be a contraction in an appropriate space, then its fixed point (η,Φ) is a solution of (5.11)-(5.12) and satisfies $(\eta(x,T),\Phi(x,T)) = (\eta_T(x),\Phi_T(x))$. We show this is the case in the space $U^s\times V^{s+1}$.

As in the case of the KdV equation in [18], we modify the map $\Gamma=(\Gamma_1,\Gamma_2)$ as follow:

$$\begin{aligned}\Gamma_1(\eta,\Phi)(t) &= \psi_1(t)S_1(t)(\eta_0,\Phi_0) + \psi_1(t)\int_0^t S_1(t-t')\psi_2(t')F_{(\eta,\Phi)}(t')dt' \\ &\quad - \psi_1(t)\int_0^t S_1(t-t')\psi_2(t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt',\end{aligned}$$

and

$$\begin{aligned}\Gamma_2(\eta,\Phi)(t) &= \psi_1(t)S_2(t)(\eta_0,\Phi_0) + \psi_1(t)\int_0^t S_2(t-t')\psi_2(t')F_{(\eta,\Phi)}(t')dt' \\ &\quad - \psi_1(t)\int_0^t S_2(t-t')\psi_2(t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')dt',\end{aligned}$$

where ψ_1 is a smooth function with its support inside the interval $(T-1,T+1)$ and $\psi_1(t)=1$ for $t\in[-T,T]$, and ψ_2 is a nonnegative smooth function with $supp\,\psi_2\subset(-T-1,T+1)$ satisfying $\psi_2(t)=1$ for any t in the support of ψ_1.

As in Theorem 4.1, let Z_M be the closed ball of radius M centered at the origin in $U^s\times V^{s+1}$. Using Remark 5.3 and slight modifications of results in Lemmas 2.2–2.6 we have that

$$\begin{aligned}\|w((\eta,\Phi),T)\|_{H^s\times\mathcal{V}^{s+1}} &= \Big\|\int_0^T S(T-t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')\,dt'\Big\|_{H^s\times\mathcal{V}^{s+1}} \\ &\le \sup_{t\in\mathbb{R}}\Big\|\psi_1(t)\int_0^t S(t-t')\psi_2(t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')\,dt'\Big\|_{H^s\times\mathcal{V}^{s+1}} \\ &\le C\Big\|\psi_1(t)\int_0^t S(t-t')\psi_2(t')\Big(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\Big)(t')\,dt'\Big\|_{U^s\times V^{s+1}} \\ &\le C\Big(\|\partial_x(\eta\partial_x\Phi)\|_{Z^s} + \|(\partial_x\Phi)^2\|_{W^{s+1}}\Big) \\ &\le C\left(\|\eta\|_{X^{s,1/2}}\|\Phi\|_{Y^{s+1,1/2}} + \|\Phi\|^2_{Y^{s+1,1/2}}\right)\end{aligned}$$

and also that

$$\begin{aligned}\|\Gamma_1(\eta,\Phi)\|_{U^s} &\leq C(T)\left(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|F_{(\eta,\Phi)}\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}\right)\\ &\quad + C_2C_5\left(\|\eta\|_{X^{s,1/2}}\|\Phi\|_{Y^{s+1,1/2}}+\|\Phi\|^2_{Y^{s+1,1/2}}\right)\\ &\leq C(T)\big(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta,\Phi)\|^2_{U^s\times V^{s+1}}\big)\end{aligned}$$

and

$$\|\Gamma_2(\eta,\Phi)\|_{V^{s+1}} \leq C(T)\big(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta,\Phi)\|^2_{U^s\times V^{s+1}}\big).$$

So that

$$\|\Gamma(\eta,\Phi)\|_{U^s\times V^{s+1}} \leq C(T)\big(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta,\Phi)\|^2_{U^s\times V^{s+1}}\big).$$

Choosing $\delta > 0$ and

$$M = 2C(T)\left(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}\right)$$

in such a way that

$$2C^2(T)M < 1, \quad 2C(T)\delta \leq M,$$

then we conclude for any $(\eta,\Phi)\in Z_M$ that

$$\begin{aligned}\|\Gamma(\eta,\Phi)\|_{U^s\times V^{s+1}} &\leq C(T)\left(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}\right)(1+4C^2(T)M)\\ &\leq 2C(T)\left(\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}\right)\\ &\leq 2C(T)\delta \leq M\end{aligned}$$

provided that $\|(\eta_0,\Phi_0)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})}+\|(\eta_T,\Phi_T)\|_{H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})} < \delta$. Now, using the same of computations, we have that Γ is a contraction on Z_M, and so, the Banach Fixed Point Theorem guaranties the existence fixed point (η,Φ) of Γ in Z_M. This fixed point (η,Φ) is a unique solution of the integral equation

$$\begin{aligned}(\eta(t),\Phi(t)) &= \psi_1(t)S(t)(\eta_0,\Phi_0)+\psi_1(t)\int_0^t S(t-t')\psi_2(t')F_{(\eta,\Phi)}(t')dt'\\ &\quad -\psi_1(t)\int_0^t S(t-t')\psi_2(t')\left(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\right)(t')dt'.\end{aligned}$$

In particular for $t\in[0,T]$,

$$\begin{aligned}(\eta(t),\Phi(t)) &= S(t)(\eta_0,\Phi_0)+\int_0^t S(t-t')(t')F_{(\eta,\Phi)}(t')dt'\\ &\quad -\int_0^t S(t-t')(t')\left(\partial_x(\eta\partial_x\Phi),\frac{1}{2}(\partial_x\Phi)^2\right)(t')dt'.\end{aligned}$$

That is to say, $(\eta,\Phi)\in C\left([0,T]:H^s(\mathbb{T})\times\mathcal{V}^{s+1}(\mathbb{T})\right)$ solves

$$\begin{cases}\eta_t+\partial_x^2\Phi-\partial_x^4\Phi+\partial_x(\eta\partial_x\Phi)=f_1,\\ \Phi_t+\eta-\partial_x^2\eta+\frac{1}{2}(\partial_x\Phi)^2=f_2,\end{cases}$$

with the conditions

$$\eta(x,0)=\eta_0(x),\quad \Phi(x,0)=\Phi_0(x),\quad \eta(x,T)=\eta_T(x),\quad \Phi(x,T)=\Phi_T(x) \qquad \square$$

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An existence result for (p, q)-Laplacian BVP with falling zeros

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Abstract. We show the existence of a positive solution to the (p, q)-Laplacian problem

$$\begin{cases} -\Delta_p u - a\Delta_q u = \lambda f(u) - h(x) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

for λ large, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega, a$ is a nonnegative constant, $h \in L^\infty(\Omega)$, $p > q > 1$, and f satisfies $f(0) = f(r) = 0$ with $f > 0$ on $(0, r)$ for some $r > 0$.

Keywords: positive solutions, (p, q) Laplacian, semipositone.

2020 Mathematics Subject Classification: 34J60, 35J62.

1 Introduction

Consider the (p, q) Laplacian problem

$$\begin{cases} -\Delta_p u - a\Delta_q u = \lambda f(u) - h(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $p > q > 1$, $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$, $f : [0, \infty) \to \mathbb{R}$, $h : \Omega \to \mathbb{R}$, a is a nonnegative constant, and λ is a positive parameter.

In contrast to the p-Laplacian, the (p, q) -Laplacian is not homogenous and occurs in applied areas such as chemical reactions and quantum physics (see e.g. [2, 6]) and has been studied extensively in recent years. The existence of a positive solution to (1.1) for λ large when f is p-sublinear at ∞ was studied in [1]. We are interested here in the case when f has falling zeroes and are motivated by a result in [9, Theorem 1.1], where the existence of a positive solution to (1.1) was established for λ large when $a = 0$ (the p-Laplacian equation), $h \equiv \varepsilon$ is small, and f satisfies the following condition:

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(H) There exists a constant $r > 0$ such that $f : [0, r] \to \mathbb{R}$ is continuous with $f(0) = f(r) = 0$ and $f > 0$ on $(0, r)$.

This result extended a previous work in [4] where $p = 2$ and $f(u) = u - u^3$. Note that under the assumption (H), the function $g(u) = \lambda f(u) - \varepsilon$ has at least two zeroes for λ large as $g(0) = g(r) < 0$ and $g(r/2) = \lambda f(r/2) - \varepsilon > 0$ for λ large. The purpose of this note is to extend the result in [9] to the general (p, q)-Laplacian. In fact, we show that for any $h \in L^\infty(\Omega)$, (1.1) has a positive solution provided that λ is large enough. This extension is nontrivial since the lack of homogeneity of the operator makes it difficult to create a positive subsolution.

Our main result is

Theorem 1.1. *Let (H) hold and $c_0 > 0$. Suppose $h \in L^\infty(\Omega)$ with $0 \le h \le c_0$ in Ω. Then there exists a constant $\lambda_0 > 0$ depending on c_0 such that (1.1) has a positive solution for $\lambda > \lambda_0$.*

We shall denote by $\|\cdot\|_p, |\cdot|_1$, and $|\cdot|_{1,\nu}$ the norms in $L^p(\Omega), C^1(\bar{\Omega})$, and $C^{1,\nu}(\bar{\Omega})$ respectively.

Lemma 1.2. *Let $f \in L^\infty(\Omega)$ with $\|f\|_\infty \le M$. Then the problem*

$$\begin{cases} -\Delta_p u - a\Delta_q u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has a unique solution $u \in C^{1,\nu}(\bar{\Omega})$ for some $\nu \in (0, 1)$. Furthermore $|u|_{1,\nu} \le C$, where $C > 0$ is a constant depending on M (but not on a and f).

Proof. Let $E = W_0^{1,p}(\Omega)$ with norm $\|u\| = \left(\int_\Omega |\nabla u|^p\right)^{1/p}$. Define

$$\langle Au, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v + a\int_\Omega |\nabla u|^{q-2}\nabla u \cdot \nabla v$$

and

$$F(v) = \int_\Omega fv$$

for $u, v \in E$. Then it is easily seen that $A : E \to E^*$ is continuous with

$$\frac{\langle Au, u\rangle}{\|u\|} \ge \|u\|^{p-1} \to \infty \quad \text{as } \|u\| \to \infty$$

and

$$\langle Au - Av, u - v\rangle \ge \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \cdot \nabla u - \nabla v\right) > 0 \quad \text{for } u \ne v.$$

Hence by the Minty–Browder Theorem (see [3]), there exists a unique $u \in E$ such that $Au = F$ in E^* i.e. u is the unique weak solution of (1.2). To show that $u \in C^{1,\nu}(\bar{\Omega})$ for some $\nu \in (0, 1)$, we need Lieberman's regularity result in [8]. By the weak comparison principle [10, Theorem 10.1], $|u| \le \tilde{u}$ in Ω, where $\tilde{u}$ satisfies

$$\begin{cases} -\Delta_p \tilde{u} - a\Delta_q \tilde{u} = M & \text{in } B(0, R), \\ \tilde{u} = 0 & \text{on } \partial B(0, R), \end{cases}$$

where $R > 0$ is such that $\Omega \subset B(0,R)$ and $B(0,R)$ denotes the open ball centered at 0 with radius R in $\mathbb{R}^n$. Note that $\tilde{u}$ is unique, radial, and

$$\tilde{u}(x) = \int_{|x|}^{R} \phi^{-1}\left(\frac{Ms}{n}\right) ds \le \int_0^R \left(\frac{Ms}{n}\right)^{\frac{1}{p-1}} ds = \left(\frac{M}{n}\right)^{\frac{1}{p-1}} R^{\frac{p}{p-1}} \equiv M_0 \qquad \forall x \in B(0,R),$$

where $\phi(t) = |t|^{p-2}t + a|t|^{q-2}t$.

Next, let $w \in C^{1,\nu}(\bar{\Omega})$ satisfy $\Delta w = f$ in $\Omega, w = 0$ on $\partial\Omega$. Then the equation in (1.2) becomes

$$\operatorname{div} A(x,u,\nabla u) = 0 \quad \text{in } \Omega,$$

where $A(x,z,\mu) = |\mu|^{p-2}\mu + a|\mu|^{q-2}\mu + \nabla w(x)$. Since $A(x,z,\mu)$ satisfies assumptions (1.10a)–(1.10d) in [8, p. 320] and $|u| \le M_0$ in Ω, it follows from the remark after Theorem 1.7 in [8] that $u \in C^{1,\nu}(\bar{\Omega})$ for some $\nu \in (0,1)$ and $|u|_{1,\nu} \le C$, where C depends on M. □

Lemma 1.3. *Let $f, g \in L^\infty(\Omega)$ and $u, v \in W_0^{1,p}(\Omega)$ satisfy*

$$\begin{cases} -\Delta_p u - a\Delta_q u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta_p v - a\Delta_q v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $|u - v|_1 \to 0$ as $\|f - g\|_1 \to 0$.

Proof. By Lemma 1.2, $u, v \in C^{1,\nu}(\bar{\Omega})$ for some $\nu \in (0,1)$ and $|u|_{1,\nu}, |v|_{1,\nu} \le C$, where C depends on an upper bound of $\|f\|_\infty, \|g\|_\infty$.

Multiplying the equation

$$-(\Delta_p u - \Delta_p v) - a(\Delta_q u - \Delta_q v) = f - g \quad \text{in } \Omega$$

by $u - v$ and integrating, we get

$$\begin{aligned} \int_\Omega |\nabla(u-v)|^p + a\int_\Omega |\nabla(u-v)|^q &= \int_\Omega (f-g)(u-v) \\ &\le 2C\|f-g\|_1 \to 0 \end{aligned}$$

as $\|f - g\|_1 \to 0$. From this and the interpolation inequality [7, Corollary 1.3],

$$|w|_1 \le c|w|_{1,\beta}^{1-\theta} \|w\|_{W^{1,p}}^{\theta} \qquad \forall w \in C^{1,\beta}(\bar{\Omega})$$

for some $c > 0$ and $\theta \in (0,1)$, we obtain $|u - v|_1 \to 0$ as $\|f - g\|_1 \to 0$, which completes the proof. □

Lemma 1.4. *Let $m > 0$ and u_m be the solution of*

$$\begin{cases} -\Delta_p u - a\Delta_q u = m & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

(i) $\|u_m\|_\infty \to \infty$ *as* $m \to \infty$.

(ii) $\|u_m\|_\infty \to 0$ *as* $m \to 0$.

Proof. *(i)* A calculation shows that $u_m = m^{\frac{1}{p-1}} v_m$, where v_m satisfies

$$\begin{cases} -\Delta_p v_m - a m^{\frac{q-p}{p-1}} \Delta_q v_m = 1 & \text{in } \Omega, \\ v_m = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Suppose $\|u_m\|_\infty \not\longrightarrow \infty$ as $m \to \infty$. Then by going to a subsequence if necessary, we can assume that $\|u_m\|_\infty \le M \ \forall m > 0$ for some $M > 0$.

This implies $|v_m| \le M m^{-\frac{1}{p-1}} \le M$ in Ω for $m > 1$. By Lemma 1.2, $|v_m|_{1,\nu} \le C$, where $C > 0$ is independent of m. Hence there exists $v_0 \in C^1(\bar{\Omega})$ and a subsequence of (v_m), which we still denote by (v_m), such that $v_m \to v_0$ in $C^1(\bar{\Omega})$. Since

$$\int_\Omega |\nabla v_m|^{p-2} \nabla v_m \cdot \nabla \psi + a m^{\frac{q-p}{p-1}} \int_\Omega |\nabla v_m|^{q-2} \nabla v_m \cdot \nabla \psi = \int_\Omega \psi \qquad \forall \psi \in W_0^{1,p}(\Omega),$$

it follows by letting $m \to \infty$ that

$$\int_\Omega |\nabla v_0|^{p-2} \nabla v_0 \cdot \nabla \psi = \int_\Omega \psi \qquad \forall \psi \in W_0^{1,p}(\Omega),$$

i.e v_0 satisfies $-\Delta_p v_0 = 1$ in Ω, $v_0 = 0$ on $\partial\Omega$. Consequently,

$$\|u_m\|_\infty = m^{\frac{1}{p-1}} \|v_m\|_\infty \to \infty \qquad \text{as } m \to \infty$$

a contradiction which proves (i).

(ii) Using Lemma 1.3 with $f = m$ and $g = 0$, we obtain the result. □

Proof of Theorem 1.1. Let u_m be defined by Lemma 1.4. By Lemma 1.3, the map $m \mapsto \|u_m\|_\infty$ is continuous. This, together with Lemma 1.4, implies the existence of an $m > 0$ such that $\|u_m\|_\infty = r$. By [10, Corollary 8.4], $u_m > 0$ in Ω and $\frac{\partial u_m}{\partial n} < 0$ on $\partial\Omega$, where n denotes the outward unit normal on $\partial\Omega$. Let $0 < \alpha < \beta < r$ and $z_{\alpha,\beta} \in C^{1,\beta}(\bar{\Omega})$ be the solution of

$$-\Delta_p z - a\Delta_q z = \begin{cases} m & \text{if } u_m \in [\alpha, \beta], \\ -c_0 & \text{otherwise} \end{cases} \equiv h_{\alpha,\beta}, \qquad z = 0 \quad \text{on } \partial\Omega.$$

Note that the existence of $z_{\alpha,\beta}$ follows from Lemma 1.2. Since $-\Delta_p u_m - a\Delta_q u_m = m$ in Ω and

$$\|h_{\alpha,\beta} - m\|_1 = (m + c_0)|B| \to 0$$

as $\alpha \to 0$ and $\beta \to r$, where $|B|$ denotes the Lebesgue measure of

$$B = \{x : u_m(x) < \alpha\} \cup \{x : \beta < u_m(x) \le r\},$$

it follows from Lemma 1.3 that $|z_{\alpha,\beta} - u_m|_1 \to 0$ as $\alpha \to 0$ and $\beta \to r$. Hence there exist α, β such that $z_{\alpha,\beta} \equiv z_0$ such that

$$\frac{u_m}{2} \le z_0 \le u_m \quad \text{in } \Omega. \tag{1.4}$$

Note that the right side inequality in (1.4) follows from the weak comparison principle in [10, Theorem 10.1]. In particular, $\frac{\alpha}{2} \le z_0 \le \beta$ when $u_m \in [\alpha, \beta]$, which implies $f(z_0) \ge \inf_{[\alpha/2,\beta]} f \equiv \gamma > 0$ and therefore

$$-\Delta_p z_0 - a\Delta_q z_0 = m \le \lambda\gamma - c_0 \le \lambda f(z_0) - h(x) \tag{1.5}$$

for $u_m \in [\alpha, \beta]$ and $\lambda > \frac{m+c_0}{\gamma}$. For such λ and $u_m \notin [\alpha, \beta]$,

$$-\Delta_p z_0 - a\Delta_q z_0 = -c_0 \le -h(x) \le \lambda f(z_0) - h(x) \tag{1.6}$$

since $f(z_0) \ge 0$ in view of (1.4). Combining (1.5) and (1.6), we see that z_0 is a subsolution of (1.1). Clearly, $z_1 \equiv r$ is a supersolution of (1.1) with $z_0 \le z_1$ in Ω. Hence (1.1) has a solution z with $z_0 \le z \le z_1$ in Ω by [5, Corollary 1], which completes the proof. □

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Infinitely many solutions for an anisotropic differential inclusion on unbounded domains

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Abstract. We consider the differential inclusion problem given by

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) + V(x)|u(x)|^{p_N^o-2}u \in a(x)\partial F(x,u),$$

in $\mathbb{R}^N$. The problem deals with the anisotropic $p(x)$-Laplacian operator where p_i are Lipschitz continuous functions $2 \le p_i(x) < N$ for all $x \in \mathbb{R}^N$ and $i \in \{1,\dots,N\}$. Assume $p_N^o(x) = \max_{1\le i\le N} p_i(x)$, $a \in L^1_+(\mathbb{R}^N) \cap L^{\frac{N}{p_N^o(x)-1}}(\mathbb{R}^N)$, $F(x,t)$ is locally Lipschitz in the t-variable integrand and $\partial F(x,t)$ is the subdifferential with respect to the t-variable in the sense of Clarke. By establishing the existence of infinitely many solutions, we achieve a first result within the anisotropic framework.

Keywords: anisotropic $p(x)$-Laplacian, differential inclusion problem, locally Lipschitz function, infinitely many solutions, variational method.

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1 Introduction

The Wulff shape, also known as the equilibrium crystal shape, is closely associated with a convex hypersurface in $\mathbb{R}^N$ based on a given norm. Wulff [36] introduced a variational problem concerning an anisotropic geometric functional in the context of crystal growth. He conjectured, without providing a proof, that the Wulff shape, among closed convex hypersurfaces with constant enclosed volume, minimizes the anisotropic surface energy. This seminal work by Wulff has spurred significant research in the field of phase transitions, particularly in scenarios involving anisotropic and nonhomogeneous media. Recent studies have considered the existence of solutions for anisotropic problems, (see [4, 7, 12–16, 29–34, 37] and the related literature for more details).

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In 2019, Ge and Rădulescu [19] proved the existence of infinitely many solutions to the differential inclusion problem involving the $p(x)$-Laplacian

$$-\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u \in a(x)\partial F(x,u), \quad \text{in } \mathbb{R}^N$$

via a combination the variational principle for locally Lipschitz functions with the properties of the generalized Lebesgue Sobolev space.

Here, with the inspiration of [19], we investigate the existence of infinitely many solutions of a differential inclusion problem involving the anisotropic $p(x)$-Laplacian

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right) + V(x)|u(x)|^{p_N^o(x)-2}u \in a(x)\partial F(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where p_i are Lipschitz continuous functions such that $2 \le p_i(x) < N$ for all $x \in \mathbb{R}^N$ and $i \in \{1,\dots,N\}$, $p_N^o(x) = \max_{1\le i\le N} p_i(x)$, $a \in L^1_+(\mathbb{R}^N) \cap L^{\frac{N}{p_N^o(x)-1}}(\mathbb{R}^N)$, $F(x,t)$ is locally Lipschitz in the t-variable integrand and $\partial F(x,t)$ is the subdifferential with respect to the t-variable in the sense of Clarke [5]. Notice that $L^1_+(\mathbb{R}^N) := \{\eta \in L^1(\mathbb{R}^N) : \eta(x) > 0 \text{ for all } x \in \mathbb{R}^N\}$ and

$$C_+(\mathbb{R}^N) := \left\{h \in C(\mathbb{R}^N) : h(x) > 1 \text{ for all } x \in \mathbb{R}^N\right\}.$$

For any $h \in C_+(\mathbb{R}^N)$, we will denote

$$h^- := \inf_{x\in\mathbb{R}^N} h(x) \quad \text{and} \quad h^+ := \sup_{x\in\mathbb{R}^N} h(x).$$

Also we set the order $h_1 \ll h_2$ the fact that $\inf_{x\in\mathbb{R}^N}(h_2(x) - h_1(x)) > 0$.

For the potential function V, we make the following assumptions:

(V_1) $V \in C(\mathbb{R}^N)$, $0 < V^-$.

(V_2) There exists $r > 0$ such that for any $b > 0$,

$$\lim_{|y|\to\infty} \mu\left(\left\{x \in \mathbb{R}^N : V(x) \le b\right\} \cap B_r(y)\right) = 0,$$

where μ is the Lebesgue measure on $\mathbb{R}^N$.

For the nonlinearity F, suppose the function $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is such that $F(x,0) = 0$ a.e. on $\mathbb{R}^N$, and

(f_0) F is a Carathéodory function, that is, for all $t \in \mathbb{R}$, the mapping $x \mapsto F(x,t)$ is measurable and, for almost all $x \in \mathbb{R}^N$, the function $t \mapsto F(x,t)$ is locally Lipschitz.

(f_1) for almost all $x \in \mathbb{R}^N$, all $t \in \mathbb{R}$ and all $w \in \partial F(x,t)$, we have

$$|w| \le c\left(1 + |t|^{p_N^o(x)-1}\right).$$

(f_2) there exist $\delta > 0$ and $\subseteq \in L^\infty_-(\mathbb{R}^N)$ such that, for almost all $x \in \mathbb{R}^N$, we have

$$\sup_{0<|t|<\delta} F(x,t) \le \subseteq(x) < 0,$$

where $L^\infty_-(\mathbb{R}^N) = \{\eta \in L^\infty(\mathbb{R}^N) : \eta(x) < 0 \text{ for all } x \in \mathbb{R}^N\}$.

(f_3) There exists $q \in C_+(\mathbb{R}^N)$ with

$$p_N^{o\,+} < q^- \le q(x) < \frac{Np_N^o(x) - p_N^o(x)(p_N^o(x)-1)}{N - p_N^o(x)} < \kappa(x) \quad \text{for all } x \in \mathbb{R}^N$$

where $\kappa(x) = p_N^{o\,*}(x) = \frac{Np_N^o(x)}{N-p_N^o(x)}$, such that for almost all $x \in \mathbb{R}^N$ we have

$$\liminf_{|t|\to\infty} \frac{F(x,t)}{|t|^{q(x)}} > 0.$$

(f_4) $F(x,-t) = F(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Now we state the main result of this article.

Theorem 1.1. *Suppose that* (f_0)*–*(f_4)*,* (V_1) *and* (V_2) *hold. Then problem* (1.1) *has infinitely many nontrivial solutions.*

The unique aspect of this paper lies in its capacity to extend the conclusions of Ge and Rădulescu [19] to a more generalized, anisotropic setting.

The subsequent sections of this paper are structured as follows. In Section 2, we will revisit the definitions and various properties associated with variable exponent Sobolev spaces. Moving on to Section 3, we will establish the existence of an infinite number of nontrivial solutions for the problem (1.1), with credit given to the essential ideas derived from [19].

2 Preliminaries

We recall some preliminary results on the theory of variable exponent Sobolev space. For more details see [8,10,11,17,21,23,24,26–28], where additional information and specifics can be found.

For $p(x) \in C_+(\mathbb{R}^N)$, we define the variable exponent Lebesgue space

$$L^{p(x)} := \left\{ u : u \text{ is measurable and } \int_{\mathbb{R}^N} |u(x)|^{p(x)} dx < +\infty \right\}$$

with the norm

$$|u|_{L^{p(x)}(\mathbb{R}^N)} := |u|_{p(x)} := \inf \left\{ \Lambda > 0 : \int_{\mathbb{R}^N} |\frac{u(x)}{\Lambda}|^{p(x)} dx \le 1 \right\},$$

and we define the variable exponent Sobolev space

$$W^{1,p(x)}(\mathbb{R}^N) := \left\{ u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N) \right\},$$

with the norm $|u|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}$. We recall that spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.

Here, we introduce a natural generalization of the variable exponent Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^N)$ that will enable us to study problem (1.1) with sufficient accuracy (see [21,22]).

Let us denote by $\overrightarrow{p} : \mathbb{R}^N \to \mathbb{R}^N$ the vectorial function

$$\overrightarrow{p}(x) = (p_1(x), \dots, p_N(x)).$$

Also assume $1 < p_1(x), \ldots, p_N(x) < \infty$.

One can define $\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$ by

$$\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N) := \left\{ u \in L^{p_N^o(\cdot)}(\mathbb{R}^N) : \partial_{x_i} u \in L^{p_i(x)}(\mathbb{R}^N), i = 1, \ldots, N \right\}.$$

This is a reflexive Banach space with respect to the norm

$$\|u\|_{1,\overrightarrow{p}(\cdot)} := \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(\cdot)}(\mathbb{R}^N)}.$$

This space continuously embedded in $L^{p_N^o(\cdot)}(\mathbb{R}^N)$, where $p_N^o = \max_{1\le i\le N} p_i(x)$. We introduce $\overrightarrow{P}_+, \overrightarrow{P}_- \in \mathbb{R}^N$ by

$$\overrightarrow{P}_+ := (p_1^+, \ldots, p_N^+), \qquad \overrightarrow{P}_- := (p_1^-, \ldots, p_N^-)$$

and $P_+^+, P_-^+, P_-^- \in \mathbb{R}^+$ by

$$P_+^+ = \max\{p_1^+, \ldots, p_N^+\}, \qquad P_-^+ = \max\{p_1^-, \ldots, p_N^-\}, \qquad P_-^- = \min\{p_1^-, \ldots, p_N^-\}. \tag{2.1}$$

Throughout this paper we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1. \tag{2.2}$$

We define

$$p^*(x) = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i(x)} - 1}, \qquad p_\infty = \max\{P_-^+, p^{*-}\}, \tag{2.3}$$

where $p^{*-} = \inf_{x\in\mathbb{R}^N} p^*(x)$.

Define $J : \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N) \to \mathbb{R}$ by

$$J(u) := \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} |\partial_{x_i} u(x)|^{p_i(x)} dx + \int_{\mathbb{R}^N} \frac{1}{p_N^o(x)} V(x) |u(x)|^{p_N^o(x)} dx$$

for all $u \in \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$. Then $J \in C^1(\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N, \mathbb{R})$. If we denote

$$A = J' : \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N) \to (\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N))^*,$$

then

$$\begin{aligned}\langle A(u), v\rangle = &\sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} |\partial_{x_i} u(x)|^{p_i(x)-2} \partial_{x_i} u(x) \partial_{x_i} v(x) \\ &+ \int_{\mathbb{R}^N} \frac{1}{p_N^o(x)} V(x) |u(x)|^{p_N^o(x)-2} uv dx\end{aligned}$$

for all $u, v \in \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between $(\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N))^*$ and $\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$.

Definition 2.1. A mapping $f : X \to X^*$ is said to be of type $(S)_+$, if $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty} \langle f(u_n), u_n - u\rangle \le 0$ implies $u_n \to u$ in X.

Similar to [3] we have the following proposition.

Proposition 2.2. *Suppose $G := \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$ and A is as above. Then $A : G \to G^*$ is*

(1) convex, bounded and strictly monotone operator;

(2) a mapping of type $(S)_+$;

(3) a homeomorphism.

Denote by $L^{q(x)}(\mathbb{R}^N)$ the conjugate space of $L^{p_N^o(x)}(\mathbb{R}^N)$ with $\frac{1}{p_N^o(x)} + \frac{1}{q(x)} = 1$. Then the Hölder type inequality

$$\int_{\mathbb{R}^N} |uv| dx \leq \left(\frac{1}{p_N^{o\,-}} + \frac{1}{q^-} \right) |u|_{L^{p_N^o(x)}(\mathbb{R}^N)} |v|_{L^{q(x)}(\mathbb{R}^N)}$$

holds for all $u \in L^{p_N^o(x)}(\mathbb{R}^N)$ and $v \in L^{q(x)}(\mathbb{R}^N)$. Furthermore, if we define the mapping $\rho : L^{p_N^o(x)}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\rho(u) = \int_{\mathbb{R}^N} |u|^{p_N^o(x)} dx,$$

then the following relations hold:

$$|u|_{p_N^o(x)} = \mu \text{ for } u \neq 0 \Longleftrightarrow \rho\left(\frac{u}{\mu}\right) = 1, \tag{2.4}$$

$$|u|_{p_N^o(x)} < 1 (= 1, > 1) \Longleftrightarrow \rho(u) < 1 (= 1, > 1), \tag{2.5}$$

$$|u|_{p_N^o(x)} > 1 \Longrightarrow |u|_{p_N^o(x)}^{p_N^{o\,-}} \leq \rho(u) |u|_{p_N^o(x)}^{p_N^{o\,+}}, \tag{2.6}$$

$$|u|_{p_N^o(x)} < 1 \Longrightarrow |u|_{p_N^o(x)}^{p_N^{o\,+}} \leq \rho(u) |u|_{p_N^o(x)}^{p_N^{o\,-}}. \tag{2.7}$$

Proposition 2.3 ([9]). *Let $p_N^o(x), q(x)$ be measurable functions such that $p_N^o(x) \in L^\infty(\mathbb{R}^N)$ and $1 \leq p_N^o(x) q(x) \leq \infty$ almost everywhere in $\mathbb{R}^N$. Let $u \in L^{q(x)}(\mathbb{R}^N)$, $u \neq 0$. Then*

$$|u|_{p_N^o(x)q(x)} \geq 1 \Longrightarrow |u|_{p_N^o(x)q(x)}^{p_N^{o\,-}} \leq ||u|^{p_N^o(x)}|_{q(x)} \leq |u|_{p_N^o(x)q(x)}^{p_N^{o\,+}},$$

$$|u|_{p_N^o(x)q(x)} \leq 1 \Longrightarrow |u|_{p_N^o(x)q(x)}^{p_N^{o\,+}} \leq ||u|^{p_N^o(x)}|_{q(x)} \leq |u|_{p_N^o(x)q(x)}^{p_N^{o\,-}}.$$

In particular, if $p_N^o(x) = p_N^o$ is a constant, then $||u|^{p_N^o(x)}|_{q(x)} = |u|_{p_N^o q(x)}^{p_N^o}$.

Next we consider the case that V satisfies (V_1) and (V_2). We equip the linear subspace

$$E = \left\{ u \in \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N) : \sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_{x_i} u|^{p_i(x)} dx + \int_{\mathbb{R}^N} V(x) |u|^{p_N^o(x)} dx < +\infty \right\}$$

with the norm

$$\|u\|_E = \inf \left\{ \lambda > 0 : \sum_{i=1}^N \int_{\mathbb{R}^N} |\frac{\partial_{x_i} u}{\lambda}|^{p_i(x)} dx + \int_{\mathbb{R}^N} V(x) |\frac{u}{\lambda}|^{p_N^o(x)} dx < +\infty \right\}.$$

Then $(E, \|\cdot\|_E)$ is continuously embedded into $\mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$ as a closed subspace. Therefore, $(E, \|\cdot\|_E)$ is also a separable reflexive Banach space. It is easy to see that with the norm $\|\cdot\|_E$, Proposition 2.2 remains valid, that is, the following properties hold true.

Proposition 2.4. *Set* $I(u) = \sum_{i=1}^{N} \int_{\mathbb{R}^N} |\partial_{x_i} u|^{p_i(x)} dx + \int_{\mathbb{R}^N} V(x)|u|^{p_N^o(x)} dx$. *If* $u(x) \in \mathcal{D}^{1,\overrightarrow{p}(\cdot)}(\mathbb{R}^N)$, *then*

(i) *for* $u \neq 0$, $\|u\|_E = \lambda$ *if and only if* $I(\frac{u}{\lambda}) = 1$,

(ii) $\|u\|_E < 1 (= 1, > 1)$ *if and only if* $I(u) < 1 (= 1, > 1)$,

(iii) $\|u\|_E > 1$ *implies* $\|u\|_E^{{p_N^o}^-} \leq I(u) \leq \|u\|_E^{{p_N^o}^+}$,

(iv) $\|u\|_E < 1$ *implies* $\|u\|_E^{{p_N^o}^+} \leq I(u) \leq \|u\|_E^{{p_N^o}^-}$.

Here we recall the following theorem from [6, Theorem 2.4.] which implies there exits an embedding from $\mathcal{D}^{1,\overrightarrow{p}}(\mathbb{R}^N)$ into $L^{\overline{p}^\star}(\mathbb{R}^N)$, where $\overline{p}^\star := \frac{N\overline{p}}{N-\overline{p}}$. This is a particular case of the result of Troisi [35].

Theorem 2.5. *Assume* $p_i \geq 1$ *for* $i = 1, \dots, N$. *Then there exists a continuous embedding* $\mathcal{D}^{1,\overrightarrow{p}}(\mathbb{R}^N)$ *into* $L^{\overline{p}^\star}(\mathbb{R}^N)$, *i.e.* $\mathcal{D}^{1,\overrightarrow{p}}(\mathbb{R}^N) \hookrightarrow L^{\overline{p}^\star}(\mathbb{R}^N)$.

This implies that we have a continuous embedding $E \hookrightarrow L^{p_N^o(x)}(\mathbb{R}^N)$.

Now by a similar argument as [1, Theorem 3.2] which is in the Heisenberg group setting (or by combining [25, Theorem 2.1] and [2, Lemma 4.4] which are in the Euclidean setting) we have the following proposition.

Proposition 2.6. *Assume* $\kappa(x) = {p_N^o}^*(x) = \frac{Np_N^o(x)}{N-p_N^o(x)}$. *If* V *satisfies* (V_1) *and* (V_2), *then*

(i) *we have a compact embedding* $E \hookrightarrow L^{p_N^o(x)}(\mathbb{R}^N)$,

(ii) *for any measurable function* $q : \mathbb{R}^N \to \mathbb{R}$ *with* $p_N^o(x) < q(x) \ll \kappa(x)$, *we have a compact embedding* $E \hookrightarrow L^{q(x)}(\mathbb{R}^N)$.

Let $(Y, \|\cdot\|)$ be a real Banach space and Y^* its topological dual. A function $\phi : Y \to \mathbb{R}$ is called locally Lipschitz if each point $u \in Y$ possesses a neighborhood N_u such that $|f(u_1) - f(u_2)| \leq \|u_1 - u_2\|$, for all $u_1, u_2 \in N_u$, for a constant $L > 0$ depending on N_u. The generalized directional derivative of ϕ at the point $u \in Y$ in the direction $h \in Y$ is

$$\phi^o(u,h) = \liminf_{\substack{w \to u \\ t \to 0^+}} \frac{\phi(w+th) - \phi(w)}{t}.$$

The generalized gradient of ϕ at $u \in Y$ is defined by

$$\partial\phi(u) = \{u^* \in Y^* : \langle u^*, h\rangle \leq \phi^o(u,h) \text{ for all } h \in Y\},$$

which is a nonempty, convex and w^*-compact subset of Y, where $\langle\cdot,\cdot\rangle$ is the duality pairing between Y^* and Y. We say that $u \in Y$ is a critical point of ϕ if $0 \in \partial\phi(u)$, (see [3] for further details).

Definition 2.7. Let Y be a real Banach space, and $\phi : Y \to \mathbb{R}$ is a locally Lipschitz function. We say that ϕ satisfies the nonsmooth (PS_c) condition if any sequence $\{u_n\} \subset Y$ such that $\phi(u_n) \to c$ and $m(u_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence, where $m(u_n) = \inf\{\|u^*\|_{Y^*} : u^* \in \partial\phi(u_n)\}$. If this property holds at every level $c \in \mathbb{R}$, then we simply say that ϕ satisfies the nonsmooth (PS) condition.

We recall the following lemma [18, Theorem 2.1.7].

Lemma 2.8. *Assume that X is an infinite-dimensional Banach space, and let $\phi : X \to \mathbb{R}$ be a locally Lipschitz function that satisfies the nonsmooth (PS_c) condition for every $c > 0$. Assume $\phi(u) = \phi(-u)$ for all $u \in X$ and $\phi(0) = 0$. Suppose $X = X_1 \bigoplus X_2$, where X_1 is finite-dimensional, and assume the following conditions:*

(a) $\alpha > 0, \delta > 0$ such that $\|u\| = \delta$ with $u \in X_2$ implies $\phi(u) \geq \alpha$.

(b) For any finite-dimensional subspace $W \subset X_1$, there is $R = R(W)$ such that $\phi(u) \leq 0$ for $u \in W$ with $\|u\| \geq R$.

Then ϕ possesses an unbounded sequence of critical values.

Before ending this section we can define the weighted variable exponent Lebesgue space $L^{q(x)}_{a(x)}(\mathbb{R}^N)$ (see [20]) as follows:

Assume $a \in L^1_+(\mathbb{R}^N)$, then $a(x)$ is a measurable, nonnegative real-valued function for $x \in \mathbb{R}^N$. Define

$$L^{q(x)}_{a(x)}(\mathbb{R}^N) = \left\{ u \text{ is measurable and } \int_{\mathbb{R}^N} a(x)|u(x)|^{q(x)}dx < +\infty \right\}$$

with the norm

$$|u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} = \inf\left\{ \sigma > 0 : \int_{\mathbb{R}^N} a(x)|\frac{u}{\sigma}|^{q(x)}dx \leq 1 \right\}.$$

Then $L^{q(x)}_{a(x)}(\mathbb{R}^N)$ is a Banach space.

Remark 2.9. The embedding $E \hookrightarrow W^{1,p^o_N(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}_{a(x)}(\mathbb{R}^N)$ is continuous.

Set $h(x) = q(x)\frac{N}{N-p^o_N(x)-1}$, where $q(x)$ is mentioned in (f_3). Then ${p^o_N}^+ < h^-$ and $p^o_N(x) < h(x) < \kappa(x)$ for all $x \in \mathbb{R}^N$, where $\kappa(x) = {p^o_N}^*(x) = \frac{Np^o_N(x)}{N-p^o_N(x)}$. Hence, by Proposition 2.6, there is a continuous embedding $E \hookrightarrow L^{h(x)}(\mathbb{R}^N)$. Thus, for $u \in E$, we have $|u(x)|^{q(x)} \in L^{\frac{N}{N-p^o_N(x)+1}}(\mathbb{R}^N)$. By the Hölder inequality,

$$\int_{\mathbb{R}^N} a(x)|u|^{q(x)}dx \leq 2|a|_{L^{\frac{N}{p^o_N(x)-1}}(\mathbb{R}^N)} ||u|^{q(x)}|_{L^{\frac{N}{N-p^o_N(x)+1}}(\mathbb{R}^N)} < +\infty. \tag{2.8}$$

It follows that $u \in L^{q(x)}_{a(x)}(\mathbb{R}^N)$, and hence the embedding $E \hookrightarrow W^{1,p^o_N(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}_{a(x)}(\mathbb{R}^N)$ is continuous.

3 Infinitely many nontrivial solutions

Here, we introduce the energy functional $\phi : E \to \mathbb{R}$ associated with problem (1.1) by

$$\phi(u) = \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}dx + \int_{\mathbb{R}^N} \frac{1}{p^o_N(x)} V(x)|u|^{p^o_N(x)}dx - \int_{\mathbb{R}^N} F(x,u)dx.$$

From the hypotheses on F, it is standard to check that ϕ is locally Lipschitz on E and $\partial\phi(u) \subset A(u) - \partial F(x,u)$ for all $u \in E$ (see [3]).

Definition 3.1. $u \in E$ is called a solution of (1.1) to which there corresponds a mapping $x \in \mathbb{R}^N \to w(x)$ with $w(x) \in \partial F(x,u)$ for almost every $x \in \mathbb{R}^N$ having the property $x \to w(x)h(x) \in L^1(\mathbb{R}^N)$ for every $h \in E$, and

$$\sum_{i=1}^{N} \int_{\mathbb{R}^N} |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} h dx + \int_{\mathbb{R}^N} V(x)|u|^{p_N^o(x)-2} uh\, dx = \int_{\mathbb{R}^N} w(x)h(x)dx.$$

Notice that weak solutions of problem (1.1) are exactly the critical points of the functional ϕ.

Lemma 3.2. *Assume that all conditions of Theorem 1.1 are satisfied. Then the energy functional ϕ satisfies the nonsmooth (PS) condition in E.*

Proof. Suppose that $\{u_n\} \subset E$ is a $(\mathrm{PS_c})$ sequence for ϕ, that is $\phi(u_n) \to c$ and $m(u_n) \to 0$ as $n \to \infty$, which shows that

$$c = \phi(u_n) + o(1), \qquad m(u_n) = o(1), \tag{3.1}$$

where $o(1) \to 0$ as $n \to +\infty$.

We claim that the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded. Suppose that this is not the case. By passing to a subsequence if necessary, we may assume that $\|u_n\|_E \to +\infty$ as $n \to \infty$. Without loss of generality, we assume $\|u_n\|_E \geq 1$. Let $u_n^* \partial\phi(u_n)$ be such that $m(u_n) = \|u_n^*\|_{E^*}$, $n \in N$. We have $u_n^* = A(u_n) - w_n$, $w_n(x) \in \partial F(x, u_n(x))$, $w_n \in L^{p'(x)}(\mathbb{R}^N)$, where $\frac{1}{p_N^o(x)} + \frac{1}{p'(x)} = 1$. By (3.1), there is a constant $M_1 > 0$ such that

$$|\phi(u_n)| \leq M_1, \quad \text{for all } n \geq 1, \tag{3.2}$$

and there is a constant $C > 0$ such that

$$\begin{aligned} C\|u_n\|_E &\geq \langle u_n^*, u_n \rangle \\ &= \langle A(u_n), u_n \rangle - \int_{\mathbb{R}^N} a(x) w_n u_n dx \\ &= \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} dx + \int_{\mathbb{R}^N} \frac{1}{p_N^o(x)} V(x)|u|^{p_N^o(x)} dx - \int_{\mathbb{R}^N} a(x) w_n u_n dx. \end{aligned} \tag{3.3}$$

Then by (3.2) and (3.3), we have

$$M_1 p_N^{o\,-} + C\|u_n\|_E \geq \int_{\mathbb{R}^N} a(x)(p_N^{o\,-} F(x,u_n) - w_n u_n) dx, \tag{3.4}$$

where $p_N^{o\,-}$ is given by (2.1).

Next we estimate (3.4). By virtue of (f_3), there exists $c_1 > 0$ and $M_2 > 0$ such that, for almost all $x \in \mathbb{R}^N$ and all $|t| \geq M_2$, we have $F(x,t) \geq c_1 |t|^{q(x)}$. On the other hand, from (f_1), for almost all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$ such that $|t| < M_2$, we have $|F(x,t)| \leq C$, where $C = C(M_2) > 0$. Therefore, for almost all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$ the above two inequalities imply

$$F(x,t) \geq c_1 |t|^{q(x)} - c_2, \quad \text{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{3.5}$$

where $c_2 = C + \max\{M_2^{q^-}, M_2^{q^+}\} c_1$. Using (f_1) again, we deduce another estimate:

$$|wt| \leq c(|t| + |t|^{p_N^o(x)}) \leq 2c(1 + |t|^{p_N^o(x)}). \tag{3.6}$$

From (3.5) and (3.6), we have

$$p_N^{o\,-}F(x,t) - wt \geq p_N^{o\,-}c_1|t|^{q(x)} - p_N^{o\,-}c_2 - 2c(1+|t|^{p_N^o(x)}). \tag{3.7}$$

By (3.4) and (3.7), we get

$$\begin{aligned} &p_N^{o\,-}c_1\int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}dx - 2c\int_{\mathbb{R}^N} a(x)|u_n|^{p_N^o(x)}dx \\ &\leq M_1 p_N^{o\,-} + C\|u_n\|_E + (p_N^{o\,-}c_2 + 2c)|a|_1. \end{aligned} \tag{3.8}$$

Note that $q^- > p_N^{o\,+}$. Then, applying Young's inequality with ϵ, we get

$$\begin{aligned} |u|^{p_N^o(x)} = 1\times|u|^{p_N^o(x)} &= \epsilon^{-\frac{p_N^o(x)}{q(x)}} \times \epsilon^{\frac{p_N^o(x)}{q(x)}}|u|^{p_N^o(x)} \\ &\leq (\epsilon^{-\frac{p_N^o(x)}{q(x)}})^{\frac{q(x)}{q(x)-p_N^o(x)}} + \epsilon||u(x)|^{p_N^o(x)}|^{\frac{q(x)}{p_N^o(x)}} \\ &= \epsilon^{-\frac{p_N^o(x)}{q(x)-p_N^o(x)}} + \epsilon|u(x)|^{q(x)} \\ &\leq \epsilon^{-\frac{p_N^{o\,+}}{q^- - p_N^{o\,+}}} + \epsilon|u(x)|^{q(x)}. \end{aligned} \tag{3.9}$$

Hence, by (3.8) and (3.9), we obtain

$$(p_N^{o\,-}c_1 - 2c\epsilon)\int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}dx \leq M_1 p_N^{o\,-} + C\|u_n\|_E + (p_N^{o\,-}c_2+2c)|a|_1 + 2c\epsilon^{\frac{p_N^{o\,+}}{q^- - p_N^{o\,+}}}|a|_1.$$

Then, choosing ϵ_0 small enough such that $0 < \epsilon < \frac{p_N^{o\,-}c_1}{2c}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}dx \leq &\frac{M_1 p_N^{o\,-} + (p_N^{o\,-}c_2+2c)|a|_1 + 2c\epsilon^{\frac{p_N^{o\,+}}{q^- - p_N^{o\,+}}}|a|_1}{(p_N^{o\,-}c_1 - 2c\epsilon)} \\ &+ \frac{C}{(p_N^{o\,-}c_1 - 2c\epsilon)}\|u_n\|_E, \end{aligned} \tag{3.10}$$

for all $n \geq 1$.

On the other hand, using (f_1) again, we deduce another estimate:

$$|F(x,u_n)| \leq 2c(1+|u_n|^{p(x)}). \tag{3.11}$$

Hence we obtain from (3.2), (3.11) and $p_N^{o\,+} < q(x)$ that

$$\begin{aligned} \frac{1}{P_+^+}\|u_n\|_E^{p_N^{o\,-}} &\leq \sum_{i=1}^N \int_{\mathbb{R}^N} \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}dx + \int_{\mathbb{R}^N} V(x)|u_n|^{p_N^o(x)}dx \\ &= \phi(u_n) + \int_{\mathbb{R}^N} a(x)F(x,u)dx \\ &\leq M_1 + 2c|a|_1 + 2c\int_{\mathbb{R}^N} a(x)|u_n|^{p_N^o(x)}dx \\ &\leq M_1 + 2c|a|_1 + 2c\int_{\mathbb{R}^N} a(x)(1+|u_n|^{q(x)})dx \\ &= M_1 + 4c|a|_1 + 2c\int_{\mathbb{R}^N} a(x)|u_n|^{q(x)}dx. \end{aligned} \tag{3.12}$$

Therefore, combining (3.10) and (3.12), the boundedness of $\{u_n\}_{n=1}^{\infty}$ immediately follows, that is, there is constant $C > 0$ such that $\|u_n\|_E \le C$. Thus, passing to a subsequence if necessary, we assume that $u_n \rightharpoonup u_0$ in E, so it follows from (3.1) that

$$\langle A(u_n), u_n - u_0\rangle - \int_{\mathbb{R}^N} a(x) w_n (u_n - u_0) dx \le \epsilon_n, \tag{3.13}$$

with $\epsilon_n \downarrow 0$, $w_n(x)\partial F(x, u_n(x))$.

Next we prove that $\int_{\mathbb{R}^N} a(x) w_n(u_n - u_0)dx$ as $n \to +\infty$. Clearly, by hypothesis (f_1), we have

$$\int_{\mathbb{R}^N} a(x)|w_n|\;|u_n - u_0|dx \le c\int_{\mathbb{R}^N} a(x)|u_n - u_0|dx + \int_{\mathbb{R}^N} a(x)|u_n|^{p_N^o(x)-1}|u_n - u_0|dx. \tag{3.14}$$

On the other hand, using Hölder's inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x)|u_n|^{p_N^o(x)-1}|u_n - u_0|dx \\
&\le 3|a|_{L^{\frac{N}{p_N^o(x)-1}}(\mathbb{R}^N)} \left||u_n|^{p_N^o(x)-1}\right|_{L^{\frac{\kappa(x)}{p_N^o(x)-1}}(\mathbb{R}^N)} |u_n - u_0|_{L^{p_N^o(x)}(\mathbb{R}^N)} \\
&\le 3|a|_{L^{\frac{p_N^o(x)}{p_N^o(x)-r(x)}}(\mathbb{R}^N)} |u_n - u_0|_{L^{\frac{\kappa(x)}{p_N^o(x)}}(\mathbb{R}^N)} \left||u_n|^{p_N^o(x)-1}\right|_{L^{\frac{\kappa(x)}{p_N^o(x)-1}}(\mathbb{R}^N)}.
\end{aligned} \tag{3.15}$$

where $\kappa(x) = {p_N^o}^*(x)$. We claim that

$$\left||u_n|^{p_N^o(x)-1}\right|_{L^{\frac{\kappa(x)}{p_N^o(x)-1}}(\mathbb{R}^N)} \le |u_n|_{\kappa(x)}^{{p_N^o}^+-1} + 2. \tag{3.16}$$

Indeed, we have that

$$\text{if } |u_n|_{\kappa(x)} \ge 1, \text{ then } \left||u_n|^{p_N^o(x)-1}\right|_{L^{\frac{\kappa(x)}{p_N^o(x)-1}}(\mathbb{R}^N)} \le |u_n|_{\kappa(x)}^{{p_N^o}^+-1}. \tag{I}$$

This is seen as follows: According to (2.4), to prove (I), this is equivalent to proving that $|u_n|_{\kappa(x)} \ge 1$ implies

$$\int_{\mathbb{R}^N} \frac{|u_n(x)|^{(p_N^o(x)-1)\frac{\kappa(x)}{p_N^o(x)-1}}}{|u_n(x)|_{\kappa(x)}^{({p_N^o}^+-1)\frac{\kappa(x)}{p_N^o(x)-1}}} dx = \int_{\mathbb{R}^N} \frac{|u_n(x)|^{\kappa(x)}}{|u_n(x)|_{\kappa(x)}^{({p_N^o}^+-1)\frac{\kappa(x)}{p_N^o(x)-1}}} dx \le 1.$$

This inequality is justified as follows: since $|u_n|_{p_N^o(x)} \ge 1$ and

$$\begin{aligned}
({p_N^o}^+ - 1)\frac{\kappa(x)}{p_N^o(x)-1} - \kappa(x) &= {p_N^o}^+\frac{\kappa(x)}{p_N^o(x)-1} - \left(\kappa(x) + \frac{\kappa(x)}{p_N^o(x)-1}\right) \\
&= {p_N^o}^+\frac{\kappa(x)}{p_N^o(x)-1} - p_N^o(x)\frac{\kappa(x)}{p_N^o(x)-1} \\
&= \frac{\kappa(x)}{p_N^o(x)-1}({p_N^o}^+ - p_N^o(x)) \\
&\ge 0,
\end{aligned}$$

we infer that

$$\frac{|u_n(x)|^{\kappa(x)}}{|u_n|_{\kappa(x)}^{({p_N^o}^+-1)\frac{\kappa(x)}{p_N^o(x)-1}}} = \frac{|u_n(x)|^{\kappa(x)}}{|u_n|_{\kappa(x)}^{\kappa(x)}} \frac{1}{|u_n|_{\kappa(x)}^{({p_N^o}^+-1)\frac{\kappa(x)}{p_N^o(x)-1}-\kappa(x)}} \le \frac{|u_n(x)|^{\kappa(x)}}{|u_n|_{\kappa(x)}^{\kappa(x)}},$$

which implies

$$\int_{\mathbb{R}^N} \frac{|u_n(x)|^{(p_N^o(x)-1)p'(x)}}{|u_n|_{p_N^o(x)}^{(p_N^{o\,+}-1)p'(x)}} dx \le \int_{\mathbb{R}^N} \frac{|u_n(x)|^{\kappa(x)}}{|u_n|_{\kappa(x)}^{\kappa(x)}} dx = 1,$$

and the proof of (I) is complete.

$$\text{If } |u_n|_{\kappa(x)} < 1, \text{ then } ||u_n|^{p_N^o(x)-1}|_{\frac{\kappa(x)}{p_N^o(x)-1}} < 2. \tag{II}$$

Indeed, by $|u_n|_{\kappa(x)} < \int_{\mathbb{R}^N} |u_n|^{\kappa(x)} dx + 1$ and (2.7), we obtain

$$\begin{aligned} ||u_n|^{p_N^o(x)-1}|_{\frac{\kappa(x)}{p_N^o(x)-1}} &< \int_{\mathbb{R}^N} |u_n|^{(p_N^o(x)-1)\frac{\kappa(x)}{p_N^o(x)-1}} dx + 1 \\ &= \int_{\mathbb{R}^N} |u_n|^{\kappa(x)} dx + 1 < 1 + 1 = 2. \end{aligned}$$

Clearly, (3.6) is a consequence of (I) and (II).

Notice that the inclusion $E \hookrightarrow L^{\kappa(x)}(\mathbb{R}^N)$ is continuous, and hence there exists $C_1 > 0$ such that

$$|u_n|_{\kappa(x)} \le C_1 \|u_n\|_E \le CC_1. \tag{3.17}$$

By Proposition 2.6, the embedding $E \hookrightarrow L^{p_N^o(x)}(\mathbb{R}^N)$ is compact, and $u_n \rightharpoonup u$ in E implies $u_n \to u$ in $L^{p_N^o(x)}(\mathbb{R}^N)$. Hence using (3.15), (3.16) and (3.17), we have

$$\int_{\mathbb{R}^N} a(x)|u_n|_{p_N^o(x)-1}|u_n - u_0| dx \to 0 \quad \text{as } n \to +\infty. \tag{3.18}$$

Choose $\theta(x) = \frac{p_N^o(x)}{p_N^o(x)-1}$. Then $\theta \in C_+(\mathbb{R}^N), 1 < \theta(x) < \frac{N}{p_N^o(x)-1}$ for all $x \in \mathbb{R}^N$, and there exists $\lambda : \mathbb{R}^N \to (0,1)$ such that

$$\frac{1}{\theta(x)} = \frac{\lambda(x)}{1} + \frac{1-\lambda(x)}{\frac{N}{p_N^o(x)-1}} \quad \text{a.e. } x \in \mathbb{R}^N.$$

Then, for $x \in \mathbb{R}^N$, we have

$$s(x) = \frac{1}{\theta(x)\lambda(x)} > 1, \qquad t(x) = \frac{N}{\theta(x)(p(x)-1)(1-\lambda(x))} > 1.$$

Using $a \in L^1_+(\mathbb{R}^N) \cap L^{\frac{N}{p_N^o(x)-1}}(\mathbb{R}^N)$, we deduce

$$\begin{aligned} \int_{\mathbb{R}^N} |a|^{\theta(x)} dx &= \int_{\mathbb{R}^N} |a|^{\frac{1}{s(x)}} |a|^{\frac{\frac{N}{p_N^o(x)-1}}{t(x)}} dx \\ &\le 2\left[\left(\int_{\mathbb{R}^N} |a| dx\right)^{\frac{1}{s^+}} + \left(\int_{\mathbb{R}^N} |a| dx\right)^{\frac{1}{s^-}}\right] \\ &\quad \times \left[\left(\int_{\mathbb{R}^N} |a|^{\frac{N}{p(x)-1}} dx\right)^{\frac{1}{t^+}} + \left(\int_{\mathbb{R}^N} |a|^{\frac{N}{p(x)-1}} dx\right)^{\frac{1}{t^-}}\right]. \end{aligned} \tag{3.19}$$

This implies $a \in L^{\frac{p_N^o(x)}{p_N^o(x)-1}}(\mathbb{R}^N)$. Hence

$$\int_{\mathbb{R}^N} a(x)|u_n - u_0| dx \le 2|a|_{\frac{p(x)}{p(x)-1}} |u_n - u_0|_{p(x)} \to 0 \quad \text{as } n \to +\infty.$$

Combining (3.13), (3.14), (3.18) and (3.19), we get $\lim_{n\to+\infty}\langle A(u_n), u_n - u\rangle = 0$. By Proposition 2.2 (2), we get $u_n \to u_0$ in E. This proves that $\phi(u)$ satisfies the nonsmooth (PS) condition on E. □

Lemma 3.3. *Assume that all conditions of Theorem 1.1 are satisfied. Then there exist $\alpha > 0$ and $\nu > 0$ such that, for any $u \in E$ with $\|u\|_E = \nu$, we have $\phi(u) \geq \alpha$.*

Proof. Firstly, choose $q \in C_+(\mathbb{R}^N)$ (q is mentioned in (f_3)). Then

$$1 < \frac{\kappa(x)}{\kappa(x)-q(x)} = \frac{Np_N^o(x)}{Np_N^o(x) - q(x)(N-p_N^o(x))} < \frac{N}{p_N^o(x)-1}, \qquad x \in \mathbb{R}^N. \tag{3.20}$$

By Proposition 2.6, the embedding $E \hookrightarrow L^{\kappa(x)}(\mathbb{R}^N)$ is continuous, and there is constant $c_5 > 0$ such that

$$|u|_{\kappa(x)} \leq c_5\|u\|_E \quad \text{for all } u \in E. \tag{3.21}$$

Now choose $\gamma > 0$ such that $\gamma < \min\{1, \frac{1}{c_5}\}$. Then, for such a fixed γ, we have

$$|u|_{\kappa(x)} \leq 1 \quad \text{for all } u \in E \text{ with } \|u\|_E = \gamma. \tag{3.22}$$

Moreover, by virtue of hypothesis (f_2), we obtain

$$F(x,t) \leq \vartheta(x), \tag{3.23}$$

for any $x \in \mathbb{R}^N$ and $0 < |t| < \delta$.

On the other hand, for all $x \in \mathbb{R}^N$ and all $|t| \geq \delta$, (f_1) implies

$$|F(x,t)| \leq c_6|t|^{p(x)}, \tag{3.24}$$

where $c_6 = (1+\frac{1}{\tau})c$ and $\tau = \min\{|\delta|^{p_N^{o\,+}}, |\delta|^{p_N^{o\,-}}\}$.

From (3.23) and (3.24), for all $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$, we have $F(x,t) \leq \vartheta(x) + c_7|t|^{p(x)}$, where $c_7 = c_6 + \frac{|\vartheta|_\infty}{\tau}$.
Thus, for all $u \in E$ with $\|u\|_E = \gamma$, we have

$$\begin{aligned}
\phi(u) &= \sum_{i=1}^N \int_{\mathbb{R}^N} \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)}dx + \int_{\mathbb{R}^N} \frac{1}{p_N^o(x)}V(x)|u|^{p_N^o(x)}dx - \int_{\mathbb{R}^N} a(x)F(x,u)dx \\
&\geq \frac{1}{p_N^{o\,+}}\|u\|_E^{p_N^{o\,+}} - c_7\int_{\mathbb{R}^N} a(x)|u|^{p_N^o(x)}dx - \int_{\mathbb{R}^N} a(x)\vartheta(x)dx \\
&\geq \frac{1}{p_N^{o\,+}}\|u\|_E^{p_N^{o\,+}} - c_7\int_{\mathbb{R}^N} a(x)|u|^{p_N^o(x)}dx + \int_{\mathbb{R}^N} a(x)\vartheta(x)dx.
\end{aligned} \tag{3.25}$$

Applying Young's inequality with ε, we get

$$\begin{aligned}
|u|^{p_N^o(x)} &= 1 \times |u|^{p_N^o(x)} \\
&\leq \varepsilon \times 1^{\frac{q(x)}{q(x)-p_N^o(x)}} + \varepsilon^{-\frac{q(x)-p_N^o(x)}{p_N^o(x)}} \left||u|^{p_N^o(x)}\right|^{\frac{q(x)}{p_N^o(x)}} \\
&= \varepsilon + \varepsilon^{-\frac{q(x)-p_N^o(x)}{p_N^o(x)}}|u|^{q(x)} \\
&\leq \varepsilon + \varepsilon^{-\frac{q^+-p_N^{o\,-}}{p_N^{o\,-}}}|u|^{q(x)}.
\end{aligned} \tag{3.26}$$

So, returning to (3.25) and using (3.26), for all $u \in E$ with $\|u\|_E = \gamma$, we obtain

$$\begin{aligned}\phi(u) \geq & \frac{1}{p_N^{o\,+}}\|u\|^{p_N^{o\,+}} - \varepsilon^{-\frac{q^+ - p_N^{o\,-}}{p_N^{o\,-}}} c_7 \int_{\mathbb{R}^N} a(x)|u(x)|^{q(x)}dx \\ & - c_7\varepsilon \int_{\mathbb{R}^N} a(x)dx - \int_{\mathbb{R}^N} a(x)\nu(x)dx.\end{aligned} \tag{3.27}$$

Since $\nu \in L^\infty(\mathbb{R}^N)$, there exists some $c_8 > 0$ such that $-\nu(x) > c_8$. We can choose and ε_0 small enough such that $c_8|a|_1 - \varepsilon_0 c_7 |a|_1 > 0$, and then (3.27) immediately implies

$$\phi(u) \geq \frac{1}{p_N^{o\,+}}\|u\|_E^{p_N^{o\,+}} - \varepsilon^{-\frac{q^+ - p_N^{o\,-}}{p_N^{o\,-}}} c_7 \int_{\mathbb{R}^N} a(x)|u(x)|^{q(x)}dx. \tag{3.28}$$

Similarly to the proof of (3.19), and combining inequality (3.20), we have $a(x) \in L^{\frac{\kappa(x)}{\kappa(x)-q(x)}}(\mathbb{R}^N)$. Using Proposition 2.3, (3.21) and (3.22), for all $u \in E$ with $\|u\|_E = \gamma$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} a(x)|u|^{q(x)}dx \leq & |a|_{L^{\frac{\kappa(x)}{\kappa(x)-q(x)}}(\mathbb{R}^N)} \, ||u|^{q(x)}|_{L^{\frac{\kappa(x)}{q(x)}}(\mathbb{R}^N)} \\ \leq & |a|_{L^{\frac{\kappa(x)}{\kappa(x)-q(x)}}(\mathbb{R}^N)} \, |u|^{q^-}_{\kappa(x)} \\ \leq & |a|_{L^{\frac{\kappa(x)}{\kappa(x)-q(x)}}(\mathbb{R}^N)} \, c_5^{q^-} \|u\|_E^{q^-}.\end{aligned} \tag{3.29}$$

Using (3.29) in (3.28), we see that, for any $u \in E$ with $\|u\|_E = \gamma$, we have

$$\phi(u) \geq \frac{1}{p_N^{o\,+}}\|u\|_E^{p_N^{o\,+}} - \varepsilon_0^{-\frac{q^+ - p_N^{o\,-}}{p_N^{o\,-}}} c_5^{q^-} |a|_{L^{\frac{\kappa(x)}{\kappa(x)-q(x)}}(\mathbb{R}^N)} \|u\|_E^{q^-}.$$

which implies that there exist $\alpha > 0$ and $\nu > 0$ such that $\phi(u) \geq \alpha$ for any $u \in E$ with $\|u\|_E = \nu$. □

Lemma 3.4. *Assume that all conditions of Theorem 1.1 are satisfied. Then $\phi(u) \to -\infty$ as $\|u\|_E \to +\infty$ for all $u \in \mathcal{F}$, where $\mathcal{F}$ is an arbitrary finite-dimensional subspace of E.*

Proof. By virtue of hypothesis (f_3), we can find $M_4 > 0$ such that

$$F(x,t) \geq c_9|t|^{q(x)} \quad \text{for all } x \in \mathbb{R}^N,\ |t| \geq M_4. \tag{3.30}$$

In addition, from hypothesis (f_1), for almost all $x \in \mathbb{R}^N$ and $|t| < M_4$, we have

$$|F(x,t)| \geq c_3, \tag{3.31}$$

where $c_3 = (1 + M_4^{p_N^{o\,+}} + M_4^{p_N^{o\,-}})c$. Thus, using 3.30 and 3.31, we obtain

$$F(x,t) \geq c_9|t|^{q(x)} \quad \text{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{3.32}$$

where $c_4 = c_3 + c_9 \max\{M_4^{p_N^{o\,+}}, M_4^{p_N^{o\,-}}\}$. Moreover, similar to (2.6) and (2.7), we get

$$\begin{aligned}|u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} > 1 \Longrightarrow |u|^{q^-}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} a(x)|u|^{q(x)}dx \leq |u|^{q^+}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} \\ |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} < 1 \Longrightarrow |u|^{q^+}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} a(x)|u|^{q(x)}dx \leq |u|^{q^-}_{L^{q(x)}_{a(x)}(\mathbb{R}^N)}.\end{aligned} \tag{3.33}$$

Because W is a finite-dimensional subspace of E, all norms are equivalent, so we can find $0 < C = C(\mathcal{F}) < 1$ such that

$$C\|u\|_E \le |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} \le \frac{1}{C}\|u\|_E \quad \text{for all } u \in \mathcal{F}. \tag{3.34}$$

Taking into account (3.32), (3.33) and (3.34), for every $u \in \mathcal{F}$ with $\|u\|_E > 1$ and $|u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} > 1$, we have

$$\begin{aligned}
\phi(u) &= \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} dx + \int_{\mathbb{R}^N} \frac{1}{p_N^o(x)} V(x)|u|^{p_N^o(x)} dx - \int_{\mathbb{R}^N} a(x)F(x,u)dx \\
&\le \frac{1}{p_N^{o\,-}} \|u\|^{p_N^{o\,+}} - c_9 \int_{\mathbb{R}^N} a(x)|u|^{q(x)} dx + c_4 \int_{\mathbb{R}^N} a(x)dx \\
&\le \begin{cases} \frac{1}{p_N^{o\,-}} \|u\|^{p_N^{o\,+}} + c_4|a|_1 - c_9 C^{q^-} \|u\|_E^{q^-} & \text{if } |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} > 1, \\ \frac{1}{p_N^{o\,-}} \|u\|^{p_N^{o\,+}} + c_4|a|_1 - c_9 C^{q^+} \|u\|_E^{q^+} & \text{if } |u|_{L^{q(x)}_{a(x)}(\mathbb{R}^N)} < 1, \end{cases}
\end{aligned} \tag{3.35}$$

Because of $q^+ \ge q^- > p_N^{o\,+}$, we see that $\phi(u) \to -\infty$ as $\|u\|_E \to +\infty$. □

Here is the proof of Theorem 1.1.

Proof. It is obvious that ϕ is even and $\phi(0) = 0$. Besides, Lemmas 3.2, 3.3 and 3.4 permit the application of Lemma 2.8 with $X = E, X_1 = \mathcal{F}$ (see Lemma 3.4) and $X_2 = E \oplus \mathcal{F}$ (see Lemma 3.3). Therefore, we obtain that the functional ϕ has an unbounded sequence of critical values, so problem (1.1) possesses infinitely many nontrivial solutions. □

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Nonexistence results of solutions for some fractional p-Laplacian equations in $\mathbb{R}^N$

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Abstract. In the present paper, we study the nonexistence of nontrivial weak solutions to a class of fractional p-Laplacian equation in two cases which are $sp > N$ and $sp < N$. In each of these cases, by using fractional Laplacian theory and inequality techniques, we obtain concrete range of parameter for which nontrivial weak solution of the problem does not exist. Our work complements the known nonexistence results in this direction.

Keywords: fractional p-Laplacian equation, nonexistence, weak solution.

2020 Mathematics Subject Classification: 35A01, 35J60, 35R11, 46E35.

1 Introduction

In this paper, we investigate the following fractional p-Laplacian equation of the type

$$\begin{cases} (-\Delta)_p^s u + \lambda V(x)|u|^{q-2}u = m(x)|u|^{r-2}u, & \text{in } \Omega, \\ u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $s \in (0,1)$, p, q, r are positive numbers satisfying $1 < p < r < q < \infty$ or $1 < q < r < p < \infty$, $m, V \in L^1(\Omega)$ are positive functions and λ is a positive parameter.

The fractional p-Laplacian operator is defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \, dy, \quad x \in \mathbb{R}^N,$$

where $B_\epsilon(x) = \{y \in \mathbb{R}^N : |x - y| < \epsilon\}$.

In recent years, many papers have been devoted to the study of the fractional p-Laplacian equations due to their interesting applications, such as game theory, image processing, optimization and so on (see [3–5]). In particular, the existence, nonexistence, multiplicity and

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some other properties of solutions to the following type of fractional p-Laplacian equation where $sp < N$

$$\begin{cases}(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u), & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega,\end{cases} \tag{1.2}$$

have been widely studied by many scholars (see [1,2,6,8,9,12–16] and the references therein). For instance, Goyal and Sreenadh [6] obtained some results on the existence and nonexistence of solutions for the following equation with respect to the parameter λ

$$\begin{cases}(-\Delta)_p^s u - \lambda V(x)|u|^{p-2}u = m(x)|u|^{r-2}u, & \text{in } \Omega,\\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega,\end{cases} \tag{1.3}$$

where $sp < N$ and $1 < r < p$ or $p < r < p_s^* = \frac{Np}{N-sp}$.

Wu and Chen [15] studied the following equation

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = |u|^{r-2}u + \lambda |u|^{q-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

for the case $sp < N$ and $1 < q < p < r$. They deduced some existence results of nontrivial solution for some range of λ.

However, as far as we know, in the case $sp > N$, there have been rarely any existence or nonexistence results for problem (1.2). Inspired by the above mentioned papers, our purpose is to establish some results on the nonexistence of nontrivial weak solution for the problem (1.1) in both cases $sp > N$ and $sp < N$ under the assumptions $1 < p < r < q < \infty$ or $1 < q < r < p < \infty$. More precisely, we aim to obtain concrete range of parameter for which nontrivial weak solution of the problem does not exist in the case $sp > N$ and the case $sp < N$, respectively.

The rest of our paper is organized as follows. In Section 2, we will introduce some necessary lemmas and properties, which will be used in the sequel. In Section 3, we derive somewhat sharp nonexistence conditions of nontrivial solutions for (1.1) in both cases: $sp > N$ and $sp < N$.

2 Preliminaries

To state our results, we introduce some notations. Let $s \in (0,1)$ and $1 < p < \infty$ be real numbers. The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as follows:

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : \ [u]_{s,p} < \infty\}$$

equipped with the norm

$$\|u\|_{s,p} := \left(\|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{s,p}\right)^{1/p},$$

where

$$[u]_{s,p} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\right)^{1/p}$$

is the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We shall work on the space

$$W_0^{s,p}(\Omega) := \left\{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\},$$

which can be equivalently renormed by $[u]_{s,p}$.

Lemma 2.1 ([10])**.** *Let $\Omega \subset \mathbb{R}^N$ be bounded and open, $sp > N$ and $s \in (0,1)$. Then there is a constant $C_M > 0$ such that for all $u \in W_0^{s,p}(\Omega)$,*

$$|u(x) - u(y)| \leq C_M |x-y|^{\beta} [u]_{s,p}, \qquad x, y \in \mathbb{R}^N,$$

where $\beta = \frac{sp-N}{p}$.

Lemma 2.2 ([4])**.** *Let $\Omega \subset \mathbb{R}^N$ be bounded and open, $s \in (0,1)$, $1 < p < \infty$ with $sp < N$. Then, there exists a constant $C_H > 0$ such that*

$$\|u\|^p_{L^{p_s^*}(\mathbb{R}^N)} \leq C_H [u]^p_{s,p}, \qquad u \in W_0^{s,p}(\Omega),$$

where $p_s^* = \frac{Np}{N-sp}$.

Lemma 2.3 ([4])**.** *Let $\Omega \subset \mathbb{R}^N$ be an extension domain for $W^{s,p}$ with no external cusps and let $p \in [1, +\infty)$, $s \in (0,1)$ be such that $sp > N$. Then, there exists $C > 0$, depending on N, s, p and Ω, such that*

$$\|u\|_{C^{0,\alpha}(\Omega)} \leq C \Big(\|u\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \Big)^{\frac{1}{p}},$$

for any $u \in L^p(\Omega)$, with $\alpha = (sp - N)/p$.

Lemma 2.4 ([7])**.** *Let $s \in (0,1)$ and $1 < p < \infty$ be such that $sp < N$. Assume that $\Omega \subset \mathbb{R}^N$ is a (bounded) uniform domain with a (locally) (s,p)-uniformly fat boundary. Then Ω admits an (s,p)-Hardy inequality, that is, there is a constant $C_K > 0$ such that*

$$\int_\Omega \frac{|u(x)|^p}{d(x, \partial\Omega)^{sp}} dx \leq C_K [u]^p_{s,p}, \qquad u \in W_0^{s,p}(\Omega),$$

where $d(x, \partial\Omega) = \inf\{|x-y| : y \in \partial\Omega\}$.

Lemma 2.5 ([11])**.** *Let $M > 0, L > 0, p > 0$, $q > 0$ and $r > 0$ be given. If*

(i) $1 < p < r < q$;

or

(ii) $1 < q < r < p$,

then for each $x \geq 0$,

$$Mx^r - Lx^q \leq \frac{M(q-r)}{q-p} \left(\frac{(r-p)M}{(q-p)L} \right)^{\frac{r-p}{q-r}} x^p$$

holds.

Definition 2.6. We say that $u \in W_0^{s,p}(\Omega)$ is a weak solution of (1.1) if

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}} dxdy + \lambda \int_\Omega V(x)|u(x)|^{q-2}u(x)v(x)dx = \int_\Omega m(x)|u(x)|^{r-2}u(x)v(x)dx, \quad (2.1)$$

for all $v \in W_0^{s,p}(\Omega)$.

3 Main results

In this section, we suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain satisfying the regularities required by the fractional Sobolev inequalities given by Lemmas 2.1–2.4.

3.1 The case $sp > N$

Theorem 3.1. *Suppose that $sp > N$ and $m\left(\frac{m}{V}\right)^{\frac{r-p}{q-r}} \in L^1(\Omega)$. If*

$$\lambda > \frac{r-p}{q-p}\left(C_M^p R_\Omega^{sp-N}\frac{q-r}{q-p}\right)^{\frac{q-r}{r-p}}\left[\int_\Omega m(x)\left(\frac{m(x)}{V(x)}\right)^{\frac{r-p}{q-r}}dx\right]^{\frac{q-r}{r-p}}, \tag{3.1}$$

then problem (1.1) has no nontrivial weak solution $u \in W_0^{s,p}(\Omega)$, where C_M is given in Lemma 2.1 and $R_\Omega = \max\{d(x,\partial\Omega) : x \in \Omega\}$.

Proof. Suppose on the contrary that problem (1.1) has a nontrivial weak solution $u \in W_0^{s,p}(\Omega)$. Taking $v = u$ in (2.1) and from Lemma 2.5, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy \\
&\quad= \int_\Omega\left[m(x)|u(x)|^{r-2}u(x) - \lambda V(x)|u(x)|^{q-2}u(x)\right]u(x)dx \\
&\quad\le \int_\Omega\left[m(x)|u(x)|^r - \lambda V(x)|u(x)|^q\right]dx \\
&\quad\le \int_\Omega \frac{q-r}{q-p}\left(\frac{r-p}{q-p}\right)^{\frac{r-p}{q-r}} m(x)\left(\frac{m(x)}{\lambda V(x)}\right)^{\frac{r-p}{q-r}}|u(x)|^p dx,
\end{aligned}$$

i.e.,

$$[u]_{s,p}^p \le \int_\Omega \frac{q-r}{q-p}\left(\frac{r-p}{q-p}\right)^{\frac{r-p}{q-r}} m(x)\left(\frac{m(x)}{\lambda V(x)}\right)^{\frac{r-p}{q-r}}|u(x)|^p dx. \tag{3.2}$$

By $sp > N$ and Lemma 2.3, we get u is continuous in $\mathbb{R}^N$, in particular in $\overline{\Omega}$. Then there is some $\xi \in \Omega$ such that

$$|u(\xi)| = \max\left\{|u(x)| : x \in \mathbb{R}^N\right\} > 0.$$

From Lemma 2.1, there is a constant C_M such that

$$|u(x)-u(y)| \le C_M|x-y|^{\frac{sp-N}{p}}[u]_{s,p}, \qquad x,y \in \mathbb{R}^N.$$

Taking $x = \xi$ in the above inequality, we obtain

$$|u(\xi)| \le C_M|\xi-y|^{\frac{sp-N}{p}}[u]_{s,p}, \qquad y \in \partial\Omega,$$

i.e.,

$$|u(\xi)| \le C_M R_\Omega^{\frac{sp-N}{p}}[u]_{s,p}. \tag{3.3}$$

Combining (3.2) with (3.3), we obtain

$$\begin{aligned}
|u(\xi)| &\le C_M R_\Omega^{\frac{sp-N}{p}}\left(\int_\Omega \frac{q-r}{q-p}\left(\frac{r-p}{q-p}\right)^{\frac{r-p}{q-r}} m(x)\left(\frac{m(x)}{\lambda V(x)}\right)^{\frac{r-p}{q-r}}|u(x)|^p dx\right)^{\frac{1}{p}} \\
&\le C_M R_\Omega^{\frac{sp-N}{p}}\left(\int_\Omega \frac{q-r}{q-p}\left(\frac{r-p}{q-p}\right)^{\frac{r-p}{q-r}} m(x)\left(\frac{m(x)}{\lambda V(x)}\right)^{\frac{r-p}{q-r}}dx\right)^{\frac{1}{p}}|u(\xi)|,
\end{aligned}$$

which yields

$$1 \leq C_M R_\Omega^{\frac{sp-N}{p}} \left(\int_\Omega \frac{q-r}{q-p} \left(\frac{r-p}{q-p} \right)^{\frac{r-p}{q-r}} m(x) \left(\frac{m(x)}{\lambda V(x)} \right)^{\frac{r-p}{q-r}} dx \right)^{\frac{1}{p}}.$$

Thus

$$\lambda^{\frac{p-r}{q-r}} \frac{q-r}{q-p} \left(\frac{r-p}{q-p} \right)^{\frac{r-p}{q-r}} \int_\Omega m(x) \left(\frac{m(x)}{V(x)} \right)^{\frac{r-p}{q-r}} dx \geq \frac{1}{C_M^p R_\Omega^{sp-N}},$$

which implies that

$$\lambda^{\frac{p-r}{q-r}} \geq \frac{1}{C_M^p R_\Omega^{sp-N} \frac{q-r}{q-p} \left(\frac{r-p}{q-p} \right)^{\frac{r-p}{q-r}} \int_\Omega m(x) \left(\frac{m(x)}{V(x)} \right)^{\frac{r-p}{q-r}} dx}.$$

Hence, from $\frac{p-r}{q-r} < 0$ we obtain

$$\lambda \leq \frac{r-p}{q-p} \Big(C_M^p R_\Omega^{sp-N} \frac{q-r}{q-p} \Big)^{\frac{q-r}{r-p}} \Big[\int_\Omega m(x) \Big(\frac{m(x)}{V(x)} \Big)^{\frac{r-p}{q-r}} dx \Big]^{\frac{q-r}{r-p}}, \tag{3.4}$$

which contradicts to (3.1). This completes the proof. □

3.2 The case $sp < N$

Theorem 3.2. *Suppose that* $sp < N$, $m\left(\frac{m}{V}\right)^{\frac{r-p}{q-r}} \in L^\mu(\Omega)$ *and* $\frac{N}{sp} < \mu < \infty$. *Assume that*

$$\lambda > \frac{r-p}{q-p} \left(C_K^{1-\frac{N}{\mu sp}} C_H^{\frac{N}{\mu sp}} R_\Omega^{sp-\frac{N}{\mu}} \frac{q-r}{q-p} \right)^{\frac{q-r}{r-p}} \left[\int_\Omega m(x) \left(\frac{m(x)}{V(x)} \right)^{\frac{r-p}{q-r}} dx \right]^{\frac{q-r}{r-p}}, \tag{3.5}$$

then problem (1.1) *has no nontrivial weak solution* $u \in W_0^{s,p}(\Omega)$, *where* C_H *and* C_K *are given in Lemmas* 2.2 *and* 2.4, *and* $R_\Omega = \max\{d(x, \partial\Omega) : x \in \Omega\}$.

Proof. Suppose on the contrary that problem (1.1) has a nontrivial weak solution $u \in W_0^{s,p}(\Omega)$. From the proof of Theorem 3.1, we have (3.2) holds. Let $\eta = \frac{1}{\mu-1}(\mu - \frac{N}{sp})$, $\theta = \eta p + (1-\eta)p_s^*$ where $p_s^* = \frac{Np}{N-sp}$. By a straightforward computation, we have $0 < \eta < 1$, $\theta = p\nu$, where $\frac{1}{\mu} + \frac{1}{\nu} = 1$. On the other hand, we get

$$\frac{1}{R_\Omega^{\eta sp}} \int_\Omega |u(x)|^\theta dx \leq \int_\Omega \frac{|u(x)|^\theta}{d(x, \partial\Omega)^{\eta sp}} dx, \tag{3.6}$$

and by Hölder's inequality, Lemma 2.2, Lemma 2.4 and (3.2), we obtain

$$
\begin{aligned}
&\int_\Omega \frac{|u(x)|^\theta}{d(x,\partial\Omega)^{\eta sp}}dx \\
&= \int_\Omega \frac{|u(x)|^{\eta p}|u(x)|^{(1-\eta)p_s^*}}{d(x,\partial\Omega)^{\eta sp}}dx \\
&\leq \left[\int_\Omega \frac{|u(x)|^p}{d(x,\partial\Omega)^{sp}}dx\right]^\eta \left[\int_\Omega |u(x)|^{p_s^*}dx\right]^{1-\eta} \\
&\leq C_K^\eta [u]_{s,p}^{p\eta} C_H^{\frac{(1-\eta)p_s^*}{p}} [u]_{s,p}^{(1-\eta)p_s^*} \\
&= C[u]_{s,p}^{p\eta+(1-\eta)p_s^*} \\
&= C[u]_{s,p}^{p\frac{p\eta+(1-\eta)p_s^*}{p}} \\
&= C[u]_{s,p}^{p\frac{\theta}{p}} \\
&\leq C\left(\int_\Omega \frac{q-r}{q-p}\left(\frac{r-p}{q-p}\right)^{\frac{r-p}{q-r}} m(x)\left(\frac{m(x)}{\lambda V(x)}\right)^{\frac{r-p}{q-r}} |u(x)|^p dx\right)^{\frac{\theta}{p}} \\
&= C\left(\int_\Omega \frac{q-r}{q-p}\left(\frac{r-p}{q-p}\right)^{\frac{r-p}{q-r}} m(x)\left(\frac{m(x)}{\lambda V(x)}\right)^{\frac{r-p}{q-r}} |u(x)|^p dx\right)^{\nu} \\
&\leq C\left(\int_\Omega \left[\frac{q-r}{q-p}\left(\frac{r-p}{q-p}\right)^{\frac{r-p}{q-r}} m(x)\left(\frac{m(x)}{\lambda V(x)}\right)^{\frac{r-p}{q-r}}\right]^\mu dx\right)^{\frac{\nu}{\mu}} \int_\Omega |u(x)|^\theta dx, \qquad (3.7)
\end{aligned}
$$

where $C = C_K^\eta C_H^{\frac{(1-\eta)p_s^*}{p}}$. Thus, by (11) and (12), we have

$$
\frac{1}{R_\Omega^{\eta sp}} \leq C\left(\int_\Omega \left[\frac{q-r}{q-p}\left(\frac{r-p}{q-p}\right)^{\frac{r-p}{q-r}} m(x)\left(\frac{m(x)}{\lambda V(x)}\right)^{\frac{r-p}{q-r}}\right]^\mu dx\right)^{\frac{\nu}{\mu}}.
$$

Accordingly,

$$
\int_\Omega \left[\frac{q-r}{q-p}\left(\frac{r-p}{q-p}\right)^{\frac{r-p}{q-r}} m(x)\left(\frac{m(x)}{\lambda V(x)}\right)^{\frac{r-p}{q-r}}\right]^\mu dx \geq \frac{1}{C^{\frac{\mu}{\nu}} R_\Omega^{\mu sp-N}}. \qquad (3.8)
$$

Therefore,

$$
\lambda^{\mu\frac{p-r}{q-r}} \left[\frac{q-r}{q-p}\left(\frac{r-p}{q-p}\right)^{\frac{r-p}{q-r}}\right]^\mu \left[\int_\Omega m(x)\left(\frac{m(x)}{V(x)}\right)^{\frac{r-p}{q-r}}\right]^\mu \geq \frac{1}{C^{\frac{\mu}{\nu}} R_\Omega^{\mu sp-N}}. \qquad (3.9)
$$

Hence, from $\frac{p-r}{q-r} < 0$ we obtain

$$
\lambda \leq \frac{r-p}{q-p}\left(C^{\frac{1}{\nu}} R_\Omega^{sp-\frac{N}{\mu}} \frac{q-r}{q-p}\right)^{\frac{q-r}{r-p}} \left[\int_\Omega m(x)\left(\frac{m(x)}{V(x)}\right)^{\frac{r-p}{q-r}} dx\right]^{\frac{q-r}{r-p}}. \qquad (3.10)
$$

Combining the definition of C and the inequality (3.10), we have

$$
\lambda \leq \frac{r-p}{q-p}\left(C_K^{1-\frac{N}{\mu sp}} C_H^{\frac{N}{\mu sp}} R_\Omega^{sp-\frac{N}{\mu}} \frac{q-r}{q-p}\right)^{\frac{q-r}{r-p}} \left[\int_\Omega m(x)\left(\frac{m(x)}{V(x)}\right)^{\frac{r-p}{q-r}} dx\right]^{\frac{q-r}{r-p}},
$$

which contradicts to (3.5). This completes the proof. □

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Existence results for singular nonlinear BVPs in the critical regime

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Abstract. We study the existence of solutions for a class of boundary value problems on the half line, associated to a third order ordinary differential equation of the type

$$\left(\Phi(k(t,u'(t))u''(t))\right)'(t) = f\left(t,u(t),u'(t),u''(t)\right), \quad \text{a.a. } t \in \mathbb{R}_0^+.$$

The prototype for the operator Φ is the Φ-*Laplacian*; the function k is assumed to be continuous and it may vanish in a subset of zero Lebesgue measure, so that the problem can be *singular*; finally, f is a Carathéodory function satisfying a weak growth condition of *Winter–Nagumo* type. The approach we follow is based on fixed point techniques combined with the upper and lower solutions method.

Keywords: boundary value problems on unbounded domains, heteroclinic solutions, Φ-Laplacian operator, singular equations, Wintner–Nagumo condition, third order ODEs.

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1 Introduction

In this paper we are concerned with the existence of solutions to boundary value problems (BVPs) on the half line, associated to strongly nonlinear third order ordinary differential equations, of the type

$$\begin{cases} \left(\Phi\left(k(\cdot,u'(\cdot))u''(\cdot)\right)\right)'(t) = f\left(t,u(t),u'(t),u''(t)\right), & \text{a.a. } t \in \mathbb{R}_0^+, \\ u(0) = u_0, \;\; u'(0) = \nu_1, \;\; u'(+\infty) = \nu_2. \end{cases} \tag{$\mathcal{P}$}$$

Here, the operator $\Phi : \mathbb{R} \to \mathbb{R}$ is the so-called Φ-*Laplacian* and $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ is a Carathéodory function satisfying a weak growth condition of *Winter–Nagumo* type. Moreover, the function $k : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}_0^+$ is continuous and it may vanish in a subset of zero Lebesgue measure, so that problem ($\mathcal{P}$) is possibly *singular*. Finally, $u_0, \nu_1, \nu_2 \in \mathbb{R}$ are fixed real numbers.

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According to the existing literature, Φ-Laplacian type equations involve a strictly increasing homeomorphism

$$\Phi : (-a, a) \to (-b, b), \quad \text{with } 0 < a, b \le +\infty,$$

such that $\Phi(0) = 0$. When $a = b = +\infty$ the main prototype for the Φ-Laplacian is the classical *r-Laplacian* $\Phi(s) = |s|^{r-2}s$, with $r > 1$. When $a = +\infty$ and $b < +\infty$, the map Φ is usually called *non-surjective* or *bounded* Φ-Laplacian and its main prototype is the mean curvature operator

$$\Phi(s) = \frac{s}{\sqrt{1+s^2}}, \quad s \in \mathbb{R},$$

cf. [4, 21]. When $a < +\infty$, the Φ-Laplacian is said to be *singular* and in this case the main prototype is the relativistic operator

$$\Phi(s) = \frac{s}{\sqrt{1-s^2}}, \quad s \in (-1, 1),$$

see [5–7, 15, 28]. Further details on *Φ-Laplacians* BVPs can be found also in [18, 22].

One of the main reasons to study problem ($\mathcal{P}$) is the great amount of applications of the Φ-Laplace operator in different fields of physics and applied mathematics, such as non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity, theory of capillary surfaces, see e.g. [20, 25], and, more recently, the modeling of glaciology, see for instance [12, 23, 29]. Moreover, problems like ($\mathcal{P}$) find many applications in fluid dynamics as generalizations of the Blasius problem modeling the flat plate problem in boundary layer theory for viscous fluids, cf. [16].

Due to the wide class of their applications, as well as for a more theoretical interest, several papers have been devoted to Φ-Laplacian type equations. Many contributions concern BVPs associated to a second order counterpart of ($\mathcal{P}$) involving equations of the type

$$\left(\Phi(k(\cdot, u(\cdot))u'(\cdot))\right)'(t) = f\left(t, u(t), u'(t)\right), \tag{1.1}$$

both in bounded and unbounded domains, under various assumptions for Φ and f, alongside with different types of boundary conditions, see [8,14,27]. We also mention [30] for third-order BVPs and [26] for higher-order BVPs in the half-line.

Recently, the multiplicity of solutions to differential equations with Φ-Laplacian has been largely investigated under periodic, Dirichlet or Neumann boundary conditions. In [17, 19] second order differential equations posed in bounded domains are considered by means of the fixed point index theory, while in [3] the authors find positive unbounded solutions for singular second-order BVPs set on the half-line.

We prove the solvability of ($\mathcal{P}$) assuming that f may have *critical* rate of decay -1 at infinity, that is $f(t, \cdot, \cdot, \cdot) \sim 1/t$ as $t \to +\infty$, cf. assumption (H_3) and Remark 3.7. Together with this assumption, we require a suitable form of the so-called *Nagumo–Wintner* condition on f, cf. (3.2) in assumption (H_2). The *Nagumo–Wintner* condition allows us to obtain a priori estimates on the derivatives of any solution u to ($\mathcal{P}$) on compact intervals of $\mathbb{R}$, see Lemma 3.11. We stress the fact that the version of the *Nagumo–Wintner* condition most frequently used in the literature for second order BVPs (see e.g. [10, 15]) is not useful in our case since it would not provide the desired estimates on the *higher* order derivative of the solution. Hence, a suitable adaptation of this condition turns out to be necessary in this context.

Our main result is Theorem 3.9 in which we prove the existence of a weak solution for problem ($\mathcal{P}$), in a sense that will be specified later. The proof is based on a fixed point technique, combined with the method of lower and upper solutions, named, respectively α and β; see assumption (H_1). More precisely, we start by proving the existence of a solution u_n to the auxiliary boundary value problem

$$\begin{cases} (\Phi(k(\cdot,u'(\cdot))u''(\cdot)))'(t) = f(t,u(t),u'(t),u''(t)), & \text{a.a. } t \in I_n, \\ u(0) = u_0,\ u'(0) = \alpha'(0),\ u'(n) = \beta'(n), \end{cases} \tag{$\mathcal{P}_n$}$$

where $I_n = [0,n]$, with $n \in \mathbb{N}$ sufficiently large. Then, by means of the a priori estimates provided by the *Nagumo–Wintner* condition, we show that a suitable sequence $(x_n)_n$, constructed by extending the functions u_n to the entire half-line, somehow converges to a solution of $(\mathcal{P})$.

Despite assumptions (H_1)–(H_3) seem rather technical, they are fulfilled by a wide class of functions, as we shall prove in Section 4. In particular, they hold for BVPs of the following type

$$\begin{cases} \left(\Phi\left(k(\cdot,u'(\cdot))u''(\cdot)\right)\right)'(t) = f_1\left(t,u(t),u'(t)\right) f_2(u''(t)), & \text{a.a. } t \in \mathbb{R}_0^+, \\ u(0) = u_0,\ \ u'(0) = \nu_1,\ \ u'(+\infty) = \nu_2, \end{cases}$$

where f_1 and f_2 satisfies suitable growth conditions and *either*

$$k(t,y) = k_1(t)k_2(y), \qquad \text{with } k_1 \geq 0,\ k_2 > 0 \text{ in } [\nu_1,\nu_2], \quad \textit{or}$$
$$k(t,y) = k_1(t) + k_2(y), \qquad \text{with } k_1,k_2 \geq 0.$$

The *critical* rate for f, as well as the possibility of dealing with *singular* equations, have been yet considered in [11], where the authors prove the existence of heteroclinic solutions for BVPs associated to (1.1). A similar framework, with $k = k(t)$, can be found in the recent work [2], where BVPs associated to third order differential equations are studied in a compact domain, and in [1], where the author proves existence results for integro-differential BVPs in a non-critical regime and in the half-line. In this context, Theorem 3.9 throws a further light on the subject treating third order equations and it extends Theorem 3.3 of [2] since the function k is more general and solutions are obtained in the half-line. In particular, in [2] only solutions on compact domains are considered and the function k only depends on t. Concerning unbounded domains, a first contribution for the existence is contained in [1], where the sub-critical regime for the asymptotic behavior of the right-hand side f is investigated together with some non-existence results. The present work aims at providing a careful study of the critical regime for the asymptotic behavior of f at the same time generalizing the choice of the function k, which can also depend on the function v and not only on t.

By performing the change of variable $v(t) = u'(t)$, our results apply to integro-differential BVPs of the type

$$\begin{cases} \left(\Phi(k(\cdot,v(\cdot))v'(\cdot))\right)'(t) = f\left(t, \int_0^t v(s)\,ds, v(t), v'(t)\right), & \text{a.a. } t \in \mathbb{R}_0^+, \\ v(0) = \nu_1,\ \ v(+\infty) = \nu_2. \end{cases}$$

Hence, when $\nu_1 \neq \nu_2$, our analysis leads to the existence of heteroclinic solutions for such BVPs. These solutions are relevant in the study of biological, physical and chemical models since they represent a phase transition process in which the system evolves from an unstable equilibrium to a stable one; see [24,27,31] and the references therein.

Finally, we highlight that a straightforward adaptation of Theorem 3.9 to problem $(\mathcal{P})$ with $k = k(t,u(t),u'(t))$ directly follows in $\mathbb{R}_0^+$ considering an additional monotonicity assumption on the second variable for $k = k(t,x,y)$ and it will be object of a forthcoming paper.

The paper is organized as follows. In Section 2 we present some preliminary facts; in particular, Theorem 2.2 is a very general result on the solvability of Φ-Laplacian BVPs in compact intervals. Section 3 is devoted to our main result Theorem 3.9. Finally, in Section 4 we present a class of examples of functions Φ, k and f satisfying the assumptions used throughout the paper.

2 Preliminary results

In this section, we present an existence result for very general BVPs in compact real intervals, that is Theorem 2.2. The proof of Theorem 2.2 is based on the forthcoming lemma. Even if the proof of Lemma 2.1 somehow follows the proof of analogous results, cf. [2, Lemma 2.1] and [10, Lemma 2.6], for the sake of clarity we prefer to show it completely.

Lemma 2.1. *Let $T > 0$ be a fixed real number and denote by $I = [0, T] \subseteq \mathbb{R}$ and let $p > 1$ be fixed. Let $F : W^{2,p}(I) \to L^1(I)$, $v \mapsto F_v \in L^1(I)$, be a continuous operator for which there exists $\Theta \in L^1(I)$ such that*

$$|F_v(t)| \leq \Theta(t) \quad \text{for all } v \in W^{2,p}(I) \text{ and a.a. } t \in I. \tag{2.1}$$

Let $K : W^{2,p}(I) \subseteq C(I; \mathbb{R}) \to C(I; \mathbb{R})$, $v \mapsto K_v \in C(I; \mathbb{R})$, be continuous with respect to the uniform topology of $C(I; \mathbb{R})$ and suppose that there exist $k_1, k_2 \in C(I; \mathbb{R})$ satisfying

$$k_1, k_2 > 0 \quad \text{a.e. in } I \quad \text{and} \quad \frac{1}{k_1}, \frac{1}{k_2} \in L^p(I), \tag{2.2}$$

such that

$$k_1(t) \leq K_v(t) \leq k_2(t) \quad \text{for all } v \in W^{2,p}(I) \text{ and a.a. } t \in I. \tag{2.3}$$

Finally, let $\Psi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing homeomorphism. Then for all $v \in W^{2,p}(I)$ and for all $\delta_1, \delta_2 \in \mathbb{R}$ there exists a unique $\xi_v \in \mathbb{R}$ such that

$$\int_a^b \frac{1}{K_v(t)} \Psi^{-1} (\xi_v + \mathscr{F}_v(t))\, dt = \delta_2 - \delta_1, \tag{2.4}$$

where

$$\mathscr{F} : W^{2,p}(I) \to C(I; \mathbb{R}), \quad v \mapsto \mathscr{F}_v(t) = \int_a^t F_v(s) ds, \quad t \in I.$$

Furthermore, there exists $c_0 > 0$, independent on v, such that:

$$|\xi_v| \leq c_0 \quad \text{for all } v \in W^{2,p}(I). \tag{2.5}$$

Proof. First, we observe that the operator $\mathscr{F}$ is well defined being $\mathscr{F}_v$ continuous in I for all $v \in W^{2,p}(I)$ by (2.1). Moreover, $\mathscr{F}$ is continuous from $W^{2,p}(I)$ in $C(I, \mathbb{R})$. Indeed, F is continuous from $W^{2,p}(I)$ in $L^1(I)$ by assumption and

$$\sup_{t \in I} |\mathscr{F}_u(t) - \mathscr{F}_v(t)| \leq \|F_u - F_v\|_{L^1(I)} \quad \text{for all } u, v \in W^{2,p}(I). \tag{2.6}$$

Furthermore

$$\sup_{t \in I} |\mathscr{F}_v(t)| \leq \|\Theta\|_{L^1(I)} \quad \text{for all } v \in W^{2,p}(I). \tag{2.7}$$

Now, we fix $v \in W^{2,p}(I)$ and define

$$\mathfrak{F}_v : \mathbb{R} \to \mathbb{R}, \quad \xi \mapsto \mathfrak{F}_v(\xi) = \int_a^b \frac{1}{K_v(t)} \Psi^{-1} (\xi + \mathscr{F}_v(t))\, dt.$$

Note that also $\mathfrak{F}_v$ is continuous in I. Indeed, $\mathscr{F}_v$ is continuous on I as noted above, the function Ψ^{-1} is continuous by assumption and so, by Lebesgue's Dominated Convergence Theorem, we can infer that $\mathfrak{F}_v \in C(\mathbb{R};\mathbb{R})$.

Moreover $\mathfrak{F}_v$ is strictly increasing in $\mathbb{R}$ since $K_v \geq 0$ and Ψ^{-1} is strictly increasing by assumption. Finally

$$\Psi^{-1}\left(\xi - \|\Theta\|_{L^1(I)}\right)\int_a^b \frac{1}{K_v(t)}dt \leq \mathfrak{F}_v(\xi) \leq \Psi^{-1}\left(\xi + \|\Theta\|_{L^1(I)}\right)\int_a^b \frac{1}{K_v(t)}dt$$

which implies that $\lim_{\xi\to\pm\infty}\mathfrak{F}_v(\xi) = \pm\infty$. Then, by Bolzano's Theorem, there exists a unique $\xi_v \in \mathbb{R}$ such that $\mathfrak{F}_v(\xi_v) = \delta_2 - \delta_1$ for any choice of $\delta_1, \delta_2 \in \mathbb{R}$.

In order to prove (2.5), note that, by the Mean Value Theorem, there exists $t_v \in I$ such that

$$\mathfrak{F}_v(\xi_v) = \int_a^b \frac{1}{K_v(t)}\Psi^{-1}\left(\xi_v + \mathscr{F}_v(t)\right)dt = \delta_2 - \delta_1 = \Psi^{-1}\left(\xi_v + \mathscr{F}_v(t_v)\right)\int_a^b \frac{1}{K_v(t)}dt$$

and so

$$\xi_v + \mathscr{F}_v(t_v) = \Psi\left((\delta_2 - \delta_1)\left(\int_a^b \frac{1}{K_v(t)}dt\right)^{-1}\right). \tag{2.8}$$

Now observe that, by (2.3)

$$(\delta_2 - \delta_1)\left(\int_a^b \frac{1}{K_v(t)}dt\right)^{-1} \leq |\delta_2 - \delta_1|\left(\int_a^b \frac{1}{k_2(t)}dt\right)^{-1} =: C. \tag{2.9}$$

Hence, denoted by $\hat{\Psi} = \max_{[-C,C]}|\Psi|$ and recalling that Ψ is strictly increasing, relations (2.8)–(2.9) give

$$\begin{aligned}|\xi_v| &\leq |\xi_v + \mathscr{F}_v(t_v)| + |\mathscr{F}_v(t_v)| = \left|\Psi\left((\delta_2 - \delta_1)\left(\int_a^b \frac{1}{K_v(t)}dt\right)^{-1}\right)\right| + |\mathscr{F}_v(t_v)| \\ &\leq |\Psi(C)| + |\mathscr{F}_v(t_v)| \leq \hat{\Psi} + \|\Theta\|_{L^1(I)}.\end{aligned}$$

Choosing $c_0 = \hat{\Psi} + \|\Theta\|_{L^1(I)}$ the proof is complete. □

Theorem 2.2. *Let $T > 0$, $p > 1$ and the operators F, K and Ψ be as in in Lemma 2.1.*

Then, for all $v_0, \omega_1, \omega_2 \in \mathbb{R}$ there exists a solution v of the problem

$$\begin{cases}\left(\Psi \circ K_v v''\right)'(t) = F_v(t), \quad \text{a.a. } t \in I, \\ v(0) = v_0,\ v'(0) = \omega_1,\ v'(T) = \omega_2,\end{cases} \tag{$\mathcal{PA}$}$$

that is a function $v \in W^{2,p}(I)$ such that

- $t \mapsto (\Psi \circ K_v v'')(t) \in W^{1,p}(I)$;
- $\left(\Psi \circ K_v v''\right)'(t) = F_v(t), \quad$ *a.a.* $t \in I$;
- $v(0) = v_0,\ v'(0) = \omega_1,\ v'(T) = \omega_2.$

Proof. The proof is an adaptation of [2, Theorem 2.2] and [11, Theorem 3.1]. First, we observe that, since the operators F, K and Ψ satisfy all the assumptions of Lemma 2.1, for all $v \in$

$W^{2,p}(I)$ and all $\delta_1, \delta_2 \in \mathbb{R}$, there exists a unique $\xi_v \in \mathbb{R}$ such that (2.4) and (2.5) still hold when $\delta_1 = w_1$ and $\delta_2 = w_2$. Set

$$\mathcal{W}_0 = \left\{v \in W^{2,p}(I) : v(0) = v_0\right\}$$

and define the operator $G : \mathcal{W}_0 \to \mathcal{W}_0$, with $v \mapsto G_v$, as follows

$$G_v(t) = v_0 + \omega_1 t + \int_0^t \int_0^s g_v(\tau) d\tau ds,$$

where

$$g_v(t) = \frac{1}{K_v(t)} \Psi^{-1}\left(\xi_v + \mathscr{F}_v(t)\right), \quad t \in I.$$

It is easy to see that G is well defined and that the solutions to ($\mathcal{PA}$) correspond to the fixed points of G. Finally, following [2] and [11], it is possible to show that G is bounded, continuous and compact, so that, by Schauder's Fixed Point Theorem, we get the existence of a function v which is a fixed point of G in I, that is a solution to the problem ($\mathcal{PA}$). □

3 Functional setting and main result

Throughout the paper we assume the following structural assumptions on Φ, k and f.

(A_1) $\Phi : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing homeomorphism such that* $\Phi(0) = 0$ *with*

$$\liminf_{s \to 0^+} \frac{\Phi(s)}{s^\rho} > 0 \quad \text{for some } \rho > 0.$$

(A_2) $k : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ *is a continuous function such that*

- $k(t,y) > 0$ *for a.a.* $(t,y) \in \mathbb{R}_0^+ \times \mathbb{R}$;
- $t \mapsto 1/k(t,y) \in L^p_{loc}(\mathbb{R}_0^+)$ *for all* $y \in \mathbb{R}$, *for some* $p > 1$.

(A_3) $f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}$ is a *Carathéodory function, that is*

- $t \mapsto f(t,x,y,z)$ *is measurable for all* $(x,y,z) \in \mathbb{R}^3$;
- $(x,y,z) \mapsto f(t,x,y,z)$ *is continuous for a.a.* $t \in \mathbb{R}_0^+$,

which is also decreasing with respect to the second variable, that is

$$f(t,x_1,y,z) \geq f(t,x_2,y,z) \quad \text{for a.a. } t \in \mathbb{R}_0^+,$$

for every $x_1, x_2, y, z \in \mathbb{R}$ *such that* $x_1 \leq x_2$.

Moreover, we shall refer to the equation in ($\mathcal{P}$) as (ODE), that is

$$\left(\Phi \circ \mathcal{K}_u\right)'(t) = f(t,u(t),u'(t),u''(t)), \quad \text{for a.a. } t \in \mathbb{R}_0^+, \tag{ODE}$$

where, for simplicity, we denote

$$\mathcal{K}_u(t) = k(t,u'(t))u''(t) \quad \text{with } u \in W^{2,p}_{loc}(\mathbb{R}_0^+) \text{ and } t \in \mathbb{R}_0^+.$$

Definition 3.1. A function $u \in C^1(\mathbb{R}_0^+;\mathbb{R})$ is said to be a (weak) solution of $(\mathcal{P})$ if

- $u \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ and $t \mapsto (\Phi \circ \mathcal{K}_u)(t) \in W^{1,1}_{loc}(\mathbb{R}_0^+)$;
- $(\Phi \circ \mathcal{K}_u)'(t) = f(t, u(t), u'(t), u''(t))$ for a.a. $t \in \mathbb{R}_0^+$;
- $u(0) = u_0,\ u'(0) = \nu_1,\ u'(+\infty) = \nu_2$.

Remark 3.2. Since we allow the function k to vanish in a set having null measure, equation (ODE) can become singular. In this context, we search for solutions no more belonging to $C^2(\mathbb{R}_0^+)$, but to $W^{2,p}_{loc}(\mathbb{R}_0^+) \cap C^1(\mathbb{R}_0^+)$. The choice of this solution space is fairly natural if we consider that the map $t \mapsto 1/k(t,y)$ is assumed to be in $L^p_{loc}(\mathbb{R}_0^+)$ for all $y \in \mathbb{R}$.

Remark 3.3. Since $u \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ is a solution of $(\mathcal{P})$, and, in particular, the map $t \mapsto (\Phi \circ \mathcal{K}_u)(t)$ is in $W^{1,1}_{loc}(\mathbb{R}_0^+)$, and Φ is a homeomorphism, then $\mathcal{K}_u$ can be considered continuous in $\mathbb{R}_0^+$ (see [9, Remark 2.1]).

Definition 3.4. A function $\alpha \in C^1(\mathbb{R}_0^+;\mathbb{R})$ is said to be a (weak) lower solution of (ODE) if

- $\alpha \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ and $t \mapsto (\Phi \circ \mathcal{K}_\alpha)(t) \in W^{1,1}_{loc}(\mathbb{R}_0^+)$;
- $(\Phi \circ \mathcal{K}_\alpha)'(t) \geq f(t, \alpha(t), \alpha'(t), \alpha''(t))$ for a.a. $t \in \mathbb{R}_0^+$.

Definition 3.5. A function $\beta \in C^1(\mathbb{R}_0^+;\mathbb{R})$ is said to be a (weak) upper solution of (ODE) if

- $\beta \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ and $t \mapsto (\Phi \circ \mathcal{K}_\beta)(t) \in W^{1,1}_{loc}(\mathbb{R}_0^+)$;
- $(\Phi \circ \mathcal{K}_\beta)'(t) \leq f(t, \beta(t), \beta'(t), \beta''(t))$ for a.a. $t \in \mathbb{R}_0^+$.

Finally, we say that *a pair (α, β) of lower and upper solutions of* (ODE) *is ordered if*

$$\alpha'(t) \leq \beta'(t) \quad \text{for all } t \in \mathbb{R}_0^+.$$

Besides the structural assumptions (A_1)–(A_3) introduced before, we also consider some further natural requirements, including a suitable form of the so-called *Nagumo–Wintner* growth condition on f, see (3.2) below, which allows us to obtain a priori estimates on the derivatives of the solutions of $(\mathcal{P})$ on any compact interval of $\mathbb{R}_0^+$.

From now on, let $T_0 > 0$ be a fixed positive number and denote by $J = [0, T_0]$. Finally, assume the following conditions.

There exists an ordered pair (α, β) of lower and upper solutions to (ODE) *such that $\alpha(0) = \beta(0) = u_0$, $\alpha'(0) = \nu_1$, β' is increasing in $(T_0, +\infty)$ and $\lim_{t\to+\infty} \beta'(t) = \nu_2$, satisfying the following assumptions.*

(H_1) *Denoting by*

$$k_*(t) = \min\{k(t,y) : y \in [\alpha'(t), \beta'(t)]\}, \quad k^*(t) = \max\{k(t,y) : y \in [\alpha'(t), \beta'(t)]\},$$

assume that $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$.

(H_2) *There exist a constant $H > 0$, a non-negative function $\ell \in L^1(J)$, a non-negative function $\mu \in L^q(J)$, for some $1 < q \leq +\infty$, and a measurable function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

$$\frac{1}{\psi} \in L^1_{loc}(\mathbb{R}^+) \quad \textit{and} \quad \int^\infty \frac{1}{\psi(t)} dt = +\infty, \tag{3.1}$$

such that

$$|f(t,x,y,z)| \leq \psi(|\Phi(k(t,y)z)|)\left(\ell(t)+\mu(t)|z|^{\frac{q-1}{q}}\right) \qquad (3.2)$$

for a.a. $t \in J$ and all $x,y,z \in \mathbb{R}$ such that $x \in [\alpha(t),\beta(t)]$, $y \in [\alpha'(t),\beta'(t)]$ and $|z| \geq H$.

(H_3) *For any $L > 0$ there exist a non-negative function $\eta_L \in L^1(\mathbb{R}_0^+)$ and a continuous function $K_L \in W^{1,1}_{loc}(\mathbb{R}_0^+)$, with K_L null in $[0,T_0]$ and strictly increasing in $[T_0,+\infty)$, satisfying*

$$\int_{T_0}^{\infty} \frac{1}{k_*(t)} e^{-\frac{K_L(t)}{\rho}} dt < +\infty, \qquad (3.3)$$

such that

- (i) *$|f(t,x,y,z)| \geq K_L'(t)|\Phi(k(t,y)z)|$ for a.a. $t \in [T_0,+\infty)$, all $x \in [\alpha(t),\beta(t)]$, all $y \in [\alpha'(t),\beta'(t)]$ and all $z \in \mathbb{R}$ with $|z| \leq \mathcal{N}_L(t)/k(t,y)$;*
- (ii) *$|f(t,x,y,z)| \leq \eta_L(t)$ for a.a. $t \in \mathbb{R}_0^+$, all $x \in [\alpha(t),\beta(t)]$, all $y \in [\alpha'(t),\beta'(t)]$ and all $z \in \mathbb{R}$ with $|z| \leq \hat{\gamma}_L(t)$;*
- (iii) *$f(t,x,y,z) \leq 0$ for a.a. $t \in [T_0,+\infty)$, all $x \in [\alpha(t),\beta(t)]$, all $y \in [\alpha'(t),\beta'(t)]$ and all $z \in \mathbb{R}$ with $|z| \leq \hat{\gamma}_L(t)$;*

where

$$\gamma_L(t) = \frac{\mathcal{N}_L(t)}{k_*(t)} \quad \text{and} \quad \hat{\gamma}_L(t) = \gamma_L(t) + |\alpha''(t)| + |\beta''(t)| \quad \text{a.a. } t \in \mathbb{R}_0^+,$$
$$\mathcal{N}_L(t) = \Phi^{-1}\left\{\Phi(L)e^{-K_L(t)}\right\} \quad t \in \mathbb{R}_0^+.$$

Remark 3.6. Note that, since $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$ by (H_1), also $1/k^* \in L^p_{loc}(\mathbb{R}_0^+)$.

Moreover, by definition of k_*, we have

$$\frac{\mathcal{N}_L(t)}{k(t,y)} \leq \frac{\mathcal{N}_L(t)}{k_*(t)} = \gamma_L(t) \leq \hat{\gamma}_L(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+ \text{ and all } y \in [\alpha'(t),\beta'(t)],$$

so that, on account of (H_3)–(iii), we can rewrite (H_3)–(i) as follows

$$f(t,x,y,z) \leq -K_L'(t)|\Phi(k(t,y)z)|$$

for a.a. $t \in [T_0,+\infty)$, all $x \in [\alpha(t),\beta(t)]$, all $y \in [\alpha'(t),\beta'(t)]$ and all $z \in \mathbb{R}$ such that $|z| \leq \mathcal{N}_L(t)/k(t,y)$.

Remark 3.7. Despite their technicality, assumptions (H_1)–(H_3) are fulfilled in several remarkable cases, as it we be clear from the examples presented in Section 4.

In particular, the request (3.3) in (H_3) is compatible with the *critical* nature of the problem connected with the growth of f at infinity; see Section 4 and [11] for further details.

Remark 3.8. It is worth noting that $\mathcal{N}_L$ is continuous so that $\gamma_L = \mathcal{N}_L/k_* \in L^p_{loc}(\mathbb{R}_0^+)$, since $1/k_*$ in $L^p_{loc}(\mathbb{R}_0^+)$ by (A_2). Moreover, $\mathcal{N}_L$ is strictly positive by definition, being $\Phi(L) > 0$. Furthermore, recalling the monotonicity of K_L and the fact that Φ is a strictly increasing homeomorphism, we infer that $\mathcal{N}_L$ is strictly decreasing in $[T_0,+\infty)$. In particular, gathering the definition of $\mathcal{N}_L$ and the monotonicity of Φ, we deduce that $\mathcal{N}_L < L$ in $(T_0,+\infty)$ and $\mathcal{N}_L(t) = L$ for all $t \in J$.

Moreover, using the liminf condition in (A_1) we deduce that

$$\limsup_{\xi \to 0^+} \frac{\Phi^{-1}(\xi)}{\xi^{1/\rho}} < +\infty.$$

Consequently, combining the above considerations with (3.3) and the fact that $1/k_* \in L^p_{loc}(\mathbb{R}^+_0)$, we obtain that $\gamma_L = \mathcal{N}_L/k_* \in L^1(\mathbb{R}^+_0)$. Finally, since $\alpha, \beta \in W^{2,p}_{loc}(\mathbb{R}^+_0)$, we also have that $\hat{\gamma}_L = \gamma_L + |\alpha''| + |\beta''| \in L^1_{loc}(\mathbb{R}^+_0)$; see [9, Remark 3.7] and [11, Remark 1].

We are now ready to state our main result.

Theorem 3.9. *Assume (A_1)–(A_3) and (H_1)–(H_3). Then, problem $(\mathcal{P})$ admits at least a weak solution $u \in W^{2,p}_{loc}(\mathbb{R}^+_0)$ satisfying*

$$\alpha \leq u \leq \beta \quad \text{and} \quad \alpha' \leq u' \leq \beta' \ \text{ a.e. in } \mathbb{R}^+_0.$$

The proof of Theorem 3.9 is divided into two steps.

Step 1. Solvability on compact sets. Let $n \in \mathbb{N}$ be such that $n > T_0$ and consider problem $(\mathcal{P}_n)$. By solution to $(\mathcal{P}_n)$ we mean a function $u_n \in W^{2,p}(I_n)$ such that

- $t \mapsto \big(\Phi \circ \mathcal{K}_{u_n}\big)(t) \in W^{1,1}(I_n)$;
- $\big(\Phi \circ \mathcal{K}_{u_n}\big)'(t) = f(t, u_n(t), u_n'(t), u_n''(t))$ for a.a. $t \in I_n$;
- $u_n(0) = u_0,\ u_n'(0) = \alpha'(0),\ u_n'(n) = \beta(n)$.

Step 2. A limit argument. Once the existence on compact sets I_n is established, we construct a new sequence of functions $(x_n)_n$ by extending the functions u_n to $\mathbb{R}^+_0$ and we prove that the limit function of $(x_n)_n$ is a solution of $(\mathcal{P})$.

3.1 Solvability on compact sets

In order to prove the existence of solutions for $(\mathcal{P}_n)$ we first consider and intermediate *truncated problem*. Following [8], see also [2], for any pair of functions $\xi, \zeta \in L^1(I_n)$, satisfying the relation $\xi \leq \zeta$ a.e. in I_n, we define the truncation operator

$$\mathcal{T}^{\xi,\zeta} : L^1(I_n) \to L^1(I_n), \quad \eta \mapsto \mathcal{T}^{\xi,\zeta}_\eta,$$
$$\mathcal{T}^{\xi,\zeta}_\eta(t) = \max\{\xi(t), \min\{\eta(t), \zeta(t)\}\}, \quad t \in I_n.$$

Observe that, by definition,

$$T^{\xi,\zeta}_\eta(t) \in [\xi(t), \zeta(t)] \quad \text{for all } \eta \in L^1(I_n) \text{ and for all } t \in I_n. \tag{3.4}$$

Moreover, by [8, Lemma A.1] we know that

$(\mathcal{T}_1)$ $|\mathcal{T}^{\xi,\zeta}_{\eta_1}(t) - \mathcal{T}^{\xi,\zeta}_{\eta_2}(t)| \leq |\eta_1(t) - \eta_2(t)|$ *for all $\eta_1, \eta_2 \in L^1(I_n)$ and all $t \in I_n$;*

$(\mathcal{T}_2)$ *if $\xi, \zeta \in W^{1,1}(I_n)$, then $\mathcal{T}^{\xi,\zeta}(W^{1,1}(I_n)) \subseteq W^{1,1}(I_n)$;*

$(\mathcal{T}_3)$ *if $\xi, \zeta \in W^{1,1}(I_n)$, then $\mathcal{T}^{\xi,\zeta}$ is continuous from $W^{1,1}(I_n)$ into itself.*

Now, for all $u \in W^{2,p}(I_n)$ and for all $t \in I_n$ we denote

$$\mathcal{D}_{u'}(t) = \mathcal{T}^{-\hat{\gamma}_L,\hat{\gamma}_L}_{\left(\mathcal{T}^{\alpha',\beta'}_{u'}\right)'}(t),$$

where $\hat{\gamma}_L$ derives from assumption (H_3), and observe that this definition is well posed whenever $u \in W^{2,p}_{loc}(\mathbb{R}^+_0)$ by $(\mathcal{T}_2)$. Moreover, by property (3.4) it results that

$$|\mathcal{D}_{u'}(t)| \leq \hat{\gamma}_L(t) \quad \text{for all } t \in I_n. \tag{3.5}$$

Finally, we set

$$W_0 = \left\{u \in W^{2,p}(I_n) : \ u(0) = u_0\right\}$$

and consider the operator

$$\mathcal{F} : W_0 \to L^1(I_n), \quad u \mapsto \mathcal{F}_u,$$

defined as

$$\mathcal{F}_u(t) = f\left(t, \mathcal{T}^{\alpha,\beta}_u(t), \mathcal{T}^{\alpha',\beta'}_{u'}(t), \mathcal{D}_{u'}(t)\right) + \arctan\left(u'(t) - \mathcal{T}^{\alpha',\beta'}_{u'}(t)\right), \quad t \in I_n. \tag{3.6}$$

Then, we are in a position to introduce the truncated problem

$$\begin{cases} \left(\Phi \circ \mathcal{K}^{\mathcal{T}}_u u''\right)'(t) = \mathcal{F}_u(t), & \text{a.a. } t \in I_n, \\ u(0) = u_0, \ u'(0) = \alpha'(0), \ u'(n) = \beta'(n), \end{cases} \tag{$\mathcal{PT}_n$}$$

where

$$\mathcal{K}^{\mathcal{T}} : W^{2,p}(I_n) \to C(I_n; \mathbb{R}), \quad u \mapsto \mathcal{K}^{\mathcal{T}}_u(t) = k\left(t, \mathcal{T}^{\alpha',\beta'}_{u'}(t)\right). \tag{3.7}$$

We are going to prove that $(\mathcal{PT}_n)$ admits at least a weak solution, that is a function $u_n \in W_0$ such that

- $t \mapsto \left(\Phi \circ \mathcal{K}^{\mathcal{T}}_{u_n} u''_n\right)(t) \in W^{1,1}(I_n)$;
- $\left(\Phi \circ \mathcal{K}^{\mathcal{T}}_{u_n} u''_n\right)'(t) = \mathcal{F}_{u_n}(t)$ for a.a. $t \in I_n$;
- $u_n(0) = u_0, \ u'_n(0) = \alpha'(0), \ u'_n(n) = \beta(n)$.

To this aim, in the next result we show that $(\mathcal{PT}_n)$ can be framed into the functional setting of Theorem 2.2 and therefore it admits at least one solution. This is an intermediate step between the solvability of $(\mathcal{P}_n)$ and the solvability of $(\mathcal{P})$.

Theorem 3.10. ***Existence for*** $(\mathcal{PT}_n)$. *Assume* (A_1)–(A_3) *and* (H_1)–(H_3). *Then, problem* $(\mathcal{PT}_n)$ *admits at least a weak solution.*

Proof. We want to show that the operators $\mathcal{F}$ and $\mathcal{K}^{\mathcal{T}}$ defined in (3.6) and (3.7), respectively, satisfy the assumptions of Theorem 2.2.

First note that, since $\alpha'(t) \leq \beta'(t)$ for all $t \in \mathbb{R}^+_0$ and $\alpha(0) = \beta(0) = u_0$, we also have $\alpha(t) \leq \beta(t)$ for all $t \in \mathbb{R}^+_0$. Now, by (3.4), for all $t \in I_n$ and all $u \in W_0$ we have

$$\mathcal{T}^{\alpha,\beta}_u(t) \in [\alpha(t), \beta(t)] \quad \text{and} \quad \mathcal{T}^{\alpha',\beta'}_{u'}(t) \in [\alpha'(t), \beta'(t)],$$

so that, by assumption (H_3)–(ii) and (3.5), we get

$$\begin{aligned}|\mathcal{F}_u(t)| &= \left| f\left(t, \mathcal{T}_u^{\alpha,\beta}(t), \mathcal{T}_{u'}^{\alpha',\beta'}(t), \mathcal{D}_{u'}(t)\right) + \arctan\left(u'(t) - \mathcal{T}_{u'}^{\alpha',\beta'}(t)\right)\right| \\ &\le \left| f\left(t, \mathcal{T}_u^{\alpha,\beta}(t), \mathcal{T}_{u'}^{\alpha',\beta'}(t), \mathcal{D}_{u'}(t)\right)\right| + \frac{\pi}{2} \\ &\le \eta_L(t) + \frac{\pi}{2},\end{aligned}$$

for all $u \in W_0$ and all $t \in I_n$. Hence $\mathcal{F}$ satisfies assumption (2.1) of Lemma 2.1 with $\Theta = \eta_L + \pi/2$.

Now we shall prove that $\mathcal{F}_u$ is continuous from $W_0 \subseteq W^{2,p}(I_n)$ into $L^1(I_n)$. To this aim, let $(u_i)_i \subseteq W_0$ be a sequence in W_0 such that $u_i \to u$ in W_0. Fix a subsequence of $(\mathcal{F}u_i)_i$, still denoted by $(\mathcal{F}u_i)_i$ for simplicity. Since $u_i \to u$ in W_0 we also have

$$u_i \longrightarrow u \text{ in } W^{2,p}(I_n), \qquad u_i' \longrightarrow u' \text{ in } W^{1,p}(I_n), \qquad u_i'' \longrightarrow u'' \text{ in } L^p(I_n),$$

so that, by property $(\mathcal{T}_3)$, it follows that

$$\mathcal{T}_{u_i}^{\alpha,\beta} \longrightarrow \mathcal{T}_u^{\alpha,\beta} \quad \text{and} \quad \mathcal{T}_{u_i'}^{\alpha',\beta'} \longrightarrow \mathcal{T}_{u'}^{\alpha',\beta'} \quad \text{in } W^{1,1}(I_n),$$

and in turn

$$\left(\mathcal{T}_{u_i'}^{\alpha',\beta'}\right)' \longrightarrow \left(\mathcal{T}_{u'}^{\alpha',\beta'}\right)' \quad \text{in } L^1(I_n).$$

Hence, by Theorem 4.9 of [13], for a.a. $t \in I_n$ we have

$$\mathcal{T}_{u_i}^{\alpha,\beta}(t) \to \mathcal{T}_u^{\alpha,\beta}(t) \quad \text{and} \quad \left(\mathcal{T}_{u_i'}^{\alpha',\beta'}\right)'(t) \longrightarrow \left(\mathcal{T}_{u'}^{\alpha',\beta'}\right)'(t), \tag{3.8}$$

possibly up to a further subsequence. Moreover, using $(\mathcal{T}_1)$, for a.a. $t \in I_n$ we find

$$|\mathcal{D}_{u_i'}(t) - \mathcal{D}_{u'}(t)| = \left| \mathcal{T}^{-\hat{\gamma}_L,\hat{\gamma}_L}_{\left(\mathcal{T}_{u_i'}^{\alpha',\beta'}\right)'}(t) - \mathcal{T}^{-\hat{\gamma}_L,\hat{\gamma}_L}_{\left(\mathcal{T}_{u'}^{\alpha',\beta'}\right)'}(t)\right| \le \left|\mathcal{T}_{u_i'}^{\alpha',\beta'}(t) - \mathcal{T}_{u'}^{\alpha',\beta'}(t)\right| \longrightarrow 0,$$

that is

$$\mathcal{D}_{u_i'}(t) \longrightarrow \mathcal{D}_{u'}(t) \quad \text{for a.a. } t \in I_n. \tag{3.9}$$

Combining (3.8)–(3.9), with the fact that f is a Carathéodory function, we get

$$\begin{aligned}\mathcal{F}_{u_i}(t) &= f\left(t, \mathcal{T}_{u_i}^{\alpha,\beta}(t), \mathcal{T}_{u_i'}^{\alpha',\beta'}(t), \mathcal{D}_{u_i'}(t)\right) + \arctan\left(u_i'(t) - \mathcal{T}_{u_i'}^{\alpha',\beta'}(t)\right) \\ &\longrightarrow f\left(t, \mathcal{T}_u^{\alpha,\beta}(t), \mathcal{T}_{u'}^{\alpha',\beta'}(t), \mathcal{D}_{u'}(t)\right) + \arctan\left(u'(t) - \mathcal{T}_{u'}^{\alpha',\beta'}(t)\right) \\ &= \mathcal{F}_u(t) \quad \text{for a.a. } t \in I_n.\end{aligned}$$

Therefore we get the continuity of $\mathcal{F}$ by the Lebesgue Dominated Convergence Theorem. Now, putting

$$\alpha_n = \min_{t \in I_n} \alpha(t), \quad \alpha_n' = \min_{t \in I_n} \alpha'(t), \quad \beta_n = \min_{t \in I_n} \beta(t), \quad \beta_n' = \min_{t \in I_n} \beta'(t),$$

it follows that for all $t \in I_n$

$$\alpha_n \le \alpha(t) \le \mathcal{T}_u^{\alpha,\beta}(t) \le \beta(t) \le \beta_n \quad \text{and} \quad \alpha_n' \le \alpha'(t) \le \mathcal{T}_{u'}^{\alpha',\beta'}(t) \le \beta'(t) \le \beta_n'.$$

Since $\mathcal{T}$ is a continuous operator in $C(I_n;\mathbb{R})$, the uniform continuity of k in $I_n \times [\alpha'_n, \beta'_n]$ implies that $\mathcal{K}^{\mathcal{T}}$ is continuous with respect to the uniform topology of $C(I_n;\mathbb{R})$. Moreover, if $u \in W^{2,p}(I_n)$, then for all $t \in I_n$

$$\mathcal{T}_u^{\alpha,\beta}(t) \in [\alpha(t), \beta(t)] \subseteq [\alpha_n, \beta_n] \quad \text{and} \quad \mathcal{T}_{u'}^{\alpha',\beta'}(t) \in [\alpha'(t), \beta'(t)] \subseteq [\alpha'_n, \beta'_n]$$

so that

$$0 < k_*(t) \le \mathcal{K}_u^{\mathcal{T}}(t) = k\left(t, \mathcal{T}_{u'}^{\alpha',\beta'}(t)\right) \le k^*(t),$$

where the functions k_* and k^* have been introduced in (H_1). Thus we have that $\mathcal{K}^{\mathcal{T}}$ satisfies the assumptions of Lemma 2.1 with $k_1 = k_*$ and $k_2 = k^*$.

Finally, taking $\omega_1 = \alpha'(0)$, $\omega_2 = \beta'(n)$, problem ($\mathcal{PT}_n$) becomes ($\mathcal{PA}$). Therefore ($\mathcal{PT}_n$) admits a solution by Theorem 2.2. □

Now, in order to prove the existence of a solution for $(\mathcal{P}_n)$, we consider some useful properties characterizing every solution of $(\mathcal{PT}_n)$ that will be proved in the next lemma. To this aim, we define

$$M = \max_{t\in J} \beta'(t) - \min_{t\in J} \alpha'(t). \tag{3.10}$$

Note that the constant M is well defined, being α, $\beta \in C^1(\mathbb{R}_0^+;\mathbb{R})$. Moreover, since Φ is a continuous and strictly increasing function with $\Phi(0) = 0$, there exists $N \in \mathbb{R}^+$ such that

$$\Phi(N) > 0, \quad \Phi(-N) < 0 \quad \text{and} \quad N > \max\left\{H, \frac{M}{2T_0}\right\} \|k^*\|_{L^\infty(J)}, \tag{3.11}$$

where H is the positive constant introduced in assumption (H_2).

Finally, take $L > N$ such that

$$\min\left\{\int_{\Phi(N)}^{\Phi(L)} \frac{1}{\psi(t)}dt, \int_{-\Phi(-N)}^{-\Phi-(L)} \frac{1}{\psi(t)}dt\right\} > \|\ell\|_{L^1(J)} + \|\mu\|_{L^q(J)} \cdot M^{\frac{q-1}{q}}, \tag{3.12}$$

which is possible by virtue of (3.1).

Lemma 3.11. *Assume (A_1)–(A_3) and (H_1)–(H_3). Let $u_n \in W^{2,p}(I_n)$ be a solution of $(\mathcal{PT}_n)$. Then the following properties hold:*

(i) $\alpha(t) \le u_n(t) \le \beta(t)$ *and* $\alpha'(t) \le u'_n(t) \le \beta'(t)$ *for all* $t \in I_n$;

(ii) $\min_{t\in J} |\mathcal{K}_{u_n}(t)| \le N$;

(iii) $|\mathcal{K}_{u_n}(t)| < L$ *for all* $t \in J$;

(iv) $\mathcal{K}_{u_n}$ *is decreasing in* $[T_0, n]$;

(v) $\mathcal{K}_{u_n} \ge 0$ *in* $[T_0, n]$;

(vi) *if there exists* $t_1 \in [T_0, n]$ *such that* $\mathcal{K}_{u_n}(t_1) = 0$, *then* $\mathcal{K}_{u_n}(t) = 0$ *for all* $t \in [t_1, n]$;

(vii) $|\mathcal{K}_{u_n}(t)| \le \mathcal{N}_L(t)$ *for all* $t \in I_n$;

(viii) $\mathcal{D}_{u'_n}(t) = u''_n(t) \le \gamma_L(t) \le \hat{\gamma}_L(t)$ *for all* $t \in I_n$.

Proof. The proof is similar to other results already known for second order BVPs, so that we present it here for the sake of clarity and completeness.

Let $u_n \in W^{2,p}(I_n)$ be a solution of $(\mathcal{PT}_n)$.

Claim (i). First we show that

$$\alpha'(t) \leq u_n'(t) \leq \beta'(t) \quad \text{for all } t \in I_n. \tag{3.13}$$

Let us start with the first inequality in (3.13). Assume by contradiction that there exists $\bar{t} \in I_n$ such that $u_n'(\bar{t}) < \alpha'(\bar{t})$ and let us define

$$z(t) = u_n'(t) - \alpha'(t), \qquad t \in I_n.$$

Since u_n solves $(\mathcal{PT}_n)$ we find that

$$z(0) = u_n'(0) - \alpha'(0) = 0, \qquad z(n) = u_n'(n) - \alpha'(n) = \beta'(n) - \alpha'(n) \geq 0 \quad \text{and} \quad z(\bar{t}) < 0.$$

Thus, by the continuity of z and the compactness of I_n, there exists $\hat{t} \in I_n$ such that

$$z(\hat{t}) = \min_{t \in I_n} z(t) < 0.$$

Therefore we can find $t_1 \in [0, \hat{t})$ and $t_2 \in (\hat{t}, n]$ such that

$$z(t_1) = z(t_2) = 0 \quad \text{and} \quad z(t) < 0 \quad \text{for all } t \in (t_1, t_2). \tag{3.14}$$

Thus, by the definition of the truncating operator $\mathcal{T}$ and the fact that $u_n'(t) < \alpha'(t)$ for all $t \in (t_1, t_2)$, it follows

$$\mathcal{T}_{u_n'}^{\alpha', \beta'}(t) = \alpha'(t) \quad \text{for all } t \in (t_1, t_2), \tag{3.15}$$

and consequently

$$\mathcal{D}_{u_n'}(t) = \mathcal{T}^{-\hat{\gamma}_L, \hat{\gamma}_L}_{\left(\mathcal{T}_{u_n'}^{\alpha', \beta'}\right)'}(t) = \mathcal{T}_{\alpha''}^{-\hat{\gamma}_L, \hat{\gamma}_L}(t) = \alpha''(t) \quad \text{for all } t \in (t_1, t_2), \tag{3.16}$$

the last equality in (3.16) being true since $|\alpha''(t)| \leq \hat{\gamma}_L(t)$ for all $t \in (t_1, t_2)$ by the definition of $\hat{\gamma}_L$. Now, recalling that u_n is a weak solution of $(\mathcal{PT}_n)$ and α is a lower solution of (ODE), by (3.6) and (3.14)–(3.16), we infer

$$\begin{aligned}(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})'(t) = \mathcal{F}_{u_n}(t) &= f\left(t, \mathcal{T}_{u_n}^{\alpha,\beta}(t), \mathcal{T}_{u_n'}^{\alpha',\beta'}(t), \mathcal{D}_{u_n'}(t)\right) + \arctan\left(u_n'(t) - \mathcal{T}_{u_n'}^{\alpha',\beta'}(t)\right) \\ &= f\left(t, \mathcal{T}_{u_n}^{\alpha,\beta}(t), \alpha'(t), \alpha''(t)\right) + \arctan\left(u_n'(t) - \alpha'(t)\right) \\ &< f(t, \mathcal{T}_{u_n}^{\alpha,\beta}(t), \alpha'(t), \alpha''(t)) \leq f(t, \alpha(t), \alpha'(t), \alpha''(t)) \leq (\Phi \circ \mathcal{K}_\alpha)'(t),\end{aligned}$$

for all $t \in (t_1, t_2)$, that is

$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})'(t) < (\Phi \circ \mathcal{K}_\alpha)'(t) \quad \text{for all } t \in (t_1, t_2). \tag{3.17}$$

Now, we set

$$Z_1 = \{t \in (t_1, \hat{t}) : z'(t) < 0\} \quad \text{and} \quad Z_2 = \{t \in (t_1, \hat{t}) : z'(t) > 0\}.$$

Note that both Z_1 and Z_2 have positive Lebesgue measure so that, recalling that k is positive a.e. in $\mathbb{R}_0^+ \times \mathbb{R}$, we can find $t_1^* \in Z_1$ and $t_2^* \in Z_2$ such that

$$k\left(t_1^*, \mathcal{T}_{u_n'}^{\alpha',\beta'}(t_1^*)\right) > 0 \quad \text{and} \quad z'(t_1^*) < 0 \tag{3.18}$$

and
$$k\left(t_2^*, \mathcal{T}_{u_n'}^{\alpha',\beta'}(t_2^*)\right) > 0 \quad \text{and} \quad z'(t_2^*) > 0. \tag{3.19}$$
Integrating (3.17) in $[t_1^*, \hat{t}]$ we get
$$\int_{t_1^*}^{\hat{t}} \left(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}}\right)'(t)dt \le \int_{t_1^*}^{\hat{t}} \left(\Phi \circ \mathcal{K}_\alpha\right)'(t)dt$$
which is equivalent to
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(\hat{t}) - (\Phi \circ \mathcal{K}_\alpha)(\hat{t}) \le (\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(t_1^*) - (\Phi \circ \mathcal{K}_\alpha)(t_1^*). \tag{3.20}$$
Now observe that
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(t_1^*) - (\Phi \circ \mathcal{K}_\alpha)(t_1^*) < 0. \tag{3.21}$$
Indeed,
$$\begin{aligned}(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(t_1^*) - (\Phi \circ \mathcal{K}_\alpha)(t_1^*) &= \Phi\left(k\left(t_1^*, \mathcal{T}_{u_n'}^{\alpha',\beta'}(t_1^*)\right) u_n''(t_1^*)\right) - \Phi\left(k\left(t_1^*, \alpha'(t_1^*)\right)\alpha''(t_1^*)\right)\\ &= \Phi\left(k\left(t_1^*, \alpha'(t_1^*)\right) u_n''(t_1^*)\right) - \Phi\left(k\left(t_1^*, \alpha'(t_1^*)\right)\alpha''(t_1^*)\right) < 0,\end{aligned}$$
since $\mathcal{T}_{u_n}^{\alpha,\beta}(t_1^*) = \alpha(t_1^*)$, $u_n''(t_1^*) < \alpha''(t_1^*)$ by (3.18) and Φ is strictly increasing.

Thus, combining (3.20) and (3.21), we find
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(\hat{t}) - (\Phi \circ \mathcal{K}_\alpha)(\hat{t}) < 0. \tag{3.22}$$
Similarly, integrating (3.17) in $[\hat{t}, t_2^*]$, we get
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(\hat{t}) - (\Phi \circ \mathcal{K}_\alpha)(\hat{t}) \ge (\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(t_2^*) - (\Phi \circ \mathcal{K}_\alpha)(t_2^*).$$
Additionally,
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(t_2^*) - (\Phi \circ \mathcal{K}_\alpha)(t_2^*) > 0,$$
since $T_{u_n}^{\alpha,\beta}(t_2^*) = \alpha(t_2^*)$, $u_n''(t_2^*) > \alpha''(t_2^*)$ by (3.19) and Φ is strictly increasing. Therefore
$$(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}})(\hat{t}) - (\Phi \circ \mathcal{K}_\alpha)(\hat{t}) > 0. \tag{3.23}$$
Now, (3.22) and (3.23) provide a contradiction. Therefore $\alpha'(t) \le u_n'(t)$ for a.a. $t \in I_n$.

By adapting the previous argument, one is also able to prove $u_n'(t) \le \beta'(t)$ for a.a. $t \in I_n$ finally obtaining $\alpha'(t) \le u_n'(t) \le \beta'(t)$ for a.a. $t \in I_n$.

Eventually, we get the thesis just by integrating (3.13) and recalling that $u_n(0) = \alpha(0) = \beta(0)$ by assumption.

Claim (ii). Suppose, by contradiction, that $\mathcal{K}_{u_n}(t) > N$ for a.a. $t \in J$. Note that
$$u_n''(t) = \frac{\mathcal{K}_{u_n}(t)}{k(t, u_n'(t))} > \frac{N}{k(t, u_n'(t))} > 0$$
for a.a. $t \in J$. Thus, applying (i) we find
$$\begin{aligned} NT_0 = \int_0^{T_0} N dt < \int_0^{T_0} \mathcal{K}_{u_n}(t)dt &= \int_0^{T_0} k\left(t, u_n'(t)\right) u_n''(t)dt\\ &\le \|k^*\|_{L^\infty(J)} \int_0^{T_0} u_n''(t)dt = \|k^*\|_{L^\infty(J)} \left[u_n'(T_0) - u_n'(0)\right]\\ &\le \|k^*\|_{L^\infty(J)} \left[\beta'(T_0) - \alpha'(0)\right] \le M\|k^*\|_{L^\infty(J)} < NT_0,\end{aligned}$$

by the choice of N in (3.11), which is a contradiction.

Similarly, we would get a contradiction assuming that $\mathcal{K}_{u_n}(t) < -N$ for a.a. $t \in J$.

Claim (iii). Suppose, by contradiction, that there exists $\bar{t} \in J$ such that $|\mathcal{K}_{u_n}(\bar{t})| \geq L$. It follows that either $\mathcal{K}_{u_n}(\bar{t}) \geq L$ or $\mathcal{K}_{u_n}(\bar{t}) \leq -L$.

Let us assume that $\mathcal{K}_{u_n}(\bar{t}) \geq L$. By (ii) we know that

$$\min_{t\in J} \mathcal{K}_{u_n}(t) \leq \min_{t\in J} |\mathcal{K}_{u_n}(t)| \leq N < L$$

and so there exists $\hat{t} \in J$ such that $\mathcal{K}_{u_n}(\hat{t}) = \min_{t\in J} \mathcal{K}_{u_n}(t) \leq N$. By the continuity of $\mathcal{K}_{u_n}$ we can find $t_1, t_2 \in J$, with $t_1 \leq t_2$, such that $\mathcal{K}_{u_n}(t_1) = N$, $\mathcal{K}_{u_n}(t_2) = L$ and

$$N < \mathcal{K}_{u_n}(t) < L \quad \text{for all } t \in (t_1, t_2). \tag{3.24}$$

Consequently, recalling the definition of $\mathcal{K}_{u_n}$ and the fact that $k > 0$ a.e. in J, we find

$$\frac{N}{k(t, u_n'(t))} < u_n''(t) < \frac{L}{k(t, u_n'(t))} \quad \text{for all } t \in (t_1, t_2).$$

Hence, taking into account Remark 3.8, for a.a. $t \in (t_1, t_2)$

$$0 < H < \frac{N}{\|k^*\|_{L^\infty(J)}} \leq \frac{N}{k(t, u_n'(t))} < u_n''(t) < \frac{L}{k(t, u_n'(t))} \leq \frac{L}{k_*(t)} = \frac{\mathcal{N}_L(t)}{k_*(t)} = \gamma_L(t) \leq \hat{\gamma}_L(t),$$

which implies, in particular, that

$$\mathcal{D}_{u_n'(t)} = u_n''(t) \quad \text{for a.a. } t \in (t_1, t_2).$$

Therefore, recalling that u_n solves ($\mathcal{PT}_n$), using (H_2) and the fact that $\Phi \circ \mathcal{K}_{u_n}(t) > 0$ by (3.24), the monotonicity of Φ and the choice of N, for a.a. $t \in (t_1, t_2)$ it results

$$\begin{aligned}
\left|\left(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}}\right)'(t)\right| &= \left|\left(\Phi\left(k\left(t, \mathcal{T}_{u_n'}^{\alpha',\beta'}(t)\right) u_n''(t)\right)\right)'(t)\right| \\
&= \left|f\left(t, \mathcal{T}_{u_n}^{\alpha,\beta}(t), \mathcal{T}_{u_n'}^{\alpha',\beta'}(t), \mathcal{D}_{u_n'}(t)\right) + \arctan\left(u_n'(t) - \mathcal{T}_{u_n'}^{\alpha',\beta'}(t)\right)\right| \\
&= \left|f\left(t, u_n(t), u_n'(t), u_n''(t)\right)\right| \\
&\leq \psi(|\Phi \circ \mathcal{K}_{u_n}(t)|)\left(\ell(t) + \mu(t)|u_n''(t)|^{\frac{q-1}{q}}\right) \\
&= \psi(\Phi \circ \mathcal{K}_{u_n}(t))\left(\ell(t) + \mu(t)|u_n''(t)|^{\frac{q-1}{q}}\right).
\end{aligned}$$

Hence, by Hölder's inequality, we get

$$\begin{aligned}
\int_{\Phi(N)}^{\Phi(L)} \frac{1}{\psi(\tau)} d\tau = \int_{\Phi(\mathcal{K}_{u_n}(t_1))}^{\Phi(\mathcal{K}_{u_n}(t_2))} \frac{1}{\psi(\tau)} d\tau &= \int_{t_1}^{t_2} \frac{(\Phi \circ \mathcal{K}_{u_n})'(t)}{\psi(\Phi \circ \mathcal{K}_{u_n}(t))} dt \\
&\leq \int_{t_1}^{t_2} \left[\ell(t) + \mu(t)(u_n''(t))^{\frac{q-1}{q}}\right] dt \\
&\leq \|\ell\|_{L^1(J)} + \|\mu\|_{L^q(J)} \left(\int_{t_1}^{t_2} u_n''(t) dt\right)^{\frac{q-1}{q}} \\
&\leq \|\ell\|_{L^1(J)} + \|\mu\|_{L^q(J)} \left[\beta'(t_2) - \alpha'(t_1)\right]^{\frac{q-1}{q}} \\
&\leq \|\ell\|_{L^1(J)} + \|\mu\|_{L^q(J)} M^{\frac{q-1}{q}}.
\end{aligned}$$

This relation contradicts the choice of L in (3.12). Similarly, if we suppose $\mathcal{K}_{u_n}(\bar{t}) \leq -L$ we reach a contradiction again.

Claim (iv). Since u_n is a solution to ($\mathcal{PT}_n$), using (i), we find that

$$(\Phi \circ \mathcal{K}_{u_n})'(t) = f\left(t, u_n(t), u_n'(t), \mathcal{D}_{u_n'}(t)\right) \quad \text{for a.a. } t \in I_n.$$

On the other hand, by (i) and (H_3)–(iii), we get

$$f\left(t, u_n(t), u_n'(t), \mathcal{D}_{u_n'}(t)\right) \leq 0 \quad \text{for a.a. } t \in [T_0, n],$$

being $|D_{u_n'}(t)| \leq \hat{\gamma}_L(t)$ by (3.5). Hence

$$(\Phi \circ \mathcal{K}_{u_n})'(t) \leq 0 \quad \text{for a.a. } t \in [T_0, n]$$

and the claim follows since Φ is a strictly increasing homeomorphism.

Claim (v). By contradiction, suppose that there exists $t_1 \in [T_0, n]$ such that $\mathcal{K}_{u_n}(t_1) < 0$. Then, from (iv) we have that

$$\mathcal{K}_{u_n}(t) \leq \mathcal{K}_{u_n}(t_1) < 0 \quad \text{for all } t \in [t_1, n].$$

Hence, considering the definition of $\mathcal{K}_{u_n}$ we deduce

$$u_n''(t) = \frac{\mathcal{K}_{u_n}(t)}{k(t, u_n'(t))} < 0 \quad \text{for a.a. } t \in [t_1, n].$$

Now, we recall that u_n solves ($\mathcal{PT}_n$), and so, using (i) and assumption (H_1), we get

$$\beta'(n) = u_n'(n) = u_n'(t_1) + \int_{t_1}^{n} u_n''(\tau)d\tau < u_n'(t_1) \leq \beta'(t_1) \leq \beta'(n),$$

since β' is increasing in $(T_0, +\infty)$, which is a contradiction.

Claim (vi). The statement directly follows from (iv) and (v).

Claim (vii). Recalling that $\mathcal{N}_L = L$ in J, by virtue of (iii) and (v), it is sufficient to prove that

$$0 \leq \mathcal{K}_{u_n}(t) \leq \mathcal{N}_L(t) \qquad \forall t \in I_n \setminus J.$$

Put

$$t^* = \sup\left\{t \geq T_0 : \; \mathcal{K}_{u_n}(s) < \mathcal{N}_L(s) \qquad \forall\, s \in [T_0, t)\right\}.$$

Note that t^* is well defined, since $\mathcal{K}_{u_n}(T_0) < L = \mathcal{N}_L(T_0)$, and $t^* > T_0$.

We want to prove that $t^* > n$. Proceed by contradiction and suppose that $t^* \leq n$. This implies that

$$\mathcal{K}_{u_n}(t) > 0 \quad \text{for all } t \in [T_0, t^*].$$

Indeed, if there exists $\bar{t} \in [T_0, t^*]$ such that $\mathcal{K}_{u_n}(\bar{t}) = 0$, we would get $\mathcal{K}_{u_n}(t) = 0 < \mathcal{N}_L(t)$ for all $t \in [\bar{t}, n]$ by (vi). On the other hand, by definition of t^*, it results $\mathcal{K}_{u_n}(t) < \mathcal{N}_L(t)$ for all $t \in [T_0, t^*)$. Consequently, since $\bar{t} \leq t^*$, we would find $\mathcal{K}_{u_n}(t) < \mathcal{N}_L(t)$ for all $t \in [T_0, n]$ and this relation contradicts the maximality of t^*. Hence

$$0 < \mathcal{K}_{u_n}(t) = k(t, u_n'(t))u_n''(t) < \mathcal{N}_L(t) \quad \text{for a.a. } t \in [T_0, t^*),$$

and so

$$0 < u_n''(t) < \frac{\mathcal{N}_L(t)}{k(t, u_n'(t))} \leq \frac{\mathcal{N}_L(t)}{k_*(t)} = \gamma_L(t) \leq \hat{\gamma}_L(t) \quad \text{for a.a. } t \in [T_0, t^*).$$

Now, recalling that u_n'' is a solution to ($\mathcal{PT}_n$), assumptions (H_3)–(i) and (H_3)–(iii) give

$$(\Phi \circ \mathcal{K}_{u_n})'(t) = f\left(t, u_n(t), u_n'(t), u_n''(t)\right) \le -K_L'(t)\Phi \circ \mathcal{K}_{u_n}(t) \quad \text{for a.a. } t \in [T_0, t^*).$$

Recalling that $\mathcal{K}_{u_n} > 0$ a.e. in $[T_0, t^*)$ and that Φ is strictly increasing with $\Phi(0) = 0$, we infer

$$\frac{(\Phi \circ \mathcal{K}_{u_n})'(t)}{\Phi \circ \mathcal{K}_{u_n}(t)} \le -K_L'(t) \quad \text{for a.a. } t \in [T_0, t^*).$$

Integrating both sides of the previous estimate in $[T_0, t^*)$, we get

$$\log(\Phi \circ \mathcal{K}_{u_n})(t^*) - \log(\Phi \circ \mathcal{K}_{u_n})(T_0) \le -K_L(t^*),$$

since $K_L(T_0) = 0$. Now, by (iii) we know that $\mathcal{K}_{u_n}(T_0) < L$; therefore,

$$\log \frac{\Phi \circ \mathcal{K}_{u_n}(t^*)}{\Phi(L)} < -K_L(t^*),$$

being Φ strictly increasing, and in turn

$$\mathcal{K}_{u_n}(t^*) < \Phi^{-1}\left(\Phi(L)e^{-K_L(t^*)}\right) = \mathcal{N}_L(t^*).$$

This contradicts the maximality of t^*. Hence we conclude that $t^* > n$. This fact assures that $0 \le \mathcal{K}_{u_n}(t) < \mathcal{N}_L(t)$ for all $t \in [T_0, n]$.

Claim $(viii)$. Using (i) and (vii) we find

$$|u_n''(t)| = \frac{|k(t, u_n'(t))u_n''(t)|}{k(t, u_n'(t))} = \frac{|\mathcal{K}_{u_n}(t)|}{k(t, u_n'(t))} \le \frac{\mathcal{N}_L(t)}{k_*(t)} = \gamma_L(t) \le \hat{\gamma}_L(t) \quad \text{for a.a. } t \in I_n,$$

so that, by the definition of $\mathcal{D}$, we get the claim. □

Theorem 3.12. *Assume* (A_1)–(A_3) *and* (H_1)–(H_3). *Then, if* $u_n \in W^{2,p}(I_n)$ *is a solution of the truncated problem* ($\mathcal{PT}_n$)*, then it is also a solution to* ($\mathcal{P}_n$).

Proof. Let $u_n \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ be a solution of ($\mathcal{PT}_n$). Then, by Lemma 3.11–(i) and $(viii)$, we find that for a.a. $t \in I_n$

$$\begin{aligned}(\Phi \circ \mathcal{K}_{u_n})'(t) &= \left(\Phi\left(k(t, u_n'(t))u_n''(t)\right)\right)'(t) = \left(\Phi\left(k\left(t, \mathcal{T}_{u_n'}^{\alpha',\beta'}\right)u_n''(t)\right)\right)'(t) \\ &= \left(\Phi \circ \mathcal{K}_{u_n}^{\mathcal{T}}\right)'(t) = \mathcal{F}_{u_n}(t) \\ &= f\left(t, \mathcal{T}_{u_n}^{\alpha,\beta}(t), \mathcal{T}_{u_n'}^{\alpha',\beta'}(t), \mathcal{D}_{u_n'}(t)\right) + \arctan\left(u_n'(t) - \mathcal{T}_{u_n'}^{\alpha',\beta'}(t)\right) \\ &= f(t, u_n(t), u_n'(t), u_n''(t)),\end{aligned}$$

that is u_n solves the equation in ($\mathcal{P}_n$). Moreover $u_n(0) = u_0$, $u_n'(0) = \alpha'(0)$ and $u_n'(n) = \beta'(n)$. Hence u_n is a weak solution to ($\mathcal{P}_n$). □

3.2 A limit argument

In order to complete the proof of Theorem 3.9 we consider a sequence $(u_n)_n$ of solutions to ($\mathcal{PT}_n$). By Theorem 3.12, every u_n is also a solution to ($\mathcal{P}_n$). We shall find a solution to ($\mathcal{P}$) via a limit argument.

To this aim, for any $n > T_0$ define

$$x_n : \mathbb{R}_0^+ \to \mathbb{R}, \qquad x_n(t) = \begin{cases} u_n(t), & t \in I_n, \\ u_n(n) + \beta'(n)(t-n), & t > n, \end{cases}$$

so that

$$x_n'(t) = \begin{cases} u_n'(t), & t \in I_n, \\ \beta'(n), & t > n. \end{cases}$$

Moreover, for all $t \in \mathbb{R}_0^+$ put

$$z_n(t) = x_n''(t) = \begin{cases} u_n''(t), & t \in I_n, \\ 0, & t > n; \end{cases} \qquad \Phi_n(t) = \begin{cases} (\Phi \circ \mathcal{K}_{u_n})'(t), & t \in I_n, \\ 0, & t > n. \end{cases}$$

By Lemma 3.11–($viii$) we have

$$|z_n(t)| = |x_n''(t)| = \begin{cases} |u_n''(t)|, & \text{a.a. } t \in I_n, \\ 0, & t > n, \end{cases} \leq \gamma_L(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+. \tag{3.25}$$

Now, since u_n is a solution to ($\mathcal{P}_n$), by Lemma 3.11–(i) and ($viii$), together with (H_3)–(ii), we also find

$$|\Phi_n(t)| = \begin{cases} |f(t, u_n(t), u_n'(t), u_n''(t))|, & \text{a.a. } t \in I_n, \\ 0, & t > n, \end{cases} \leq \eta_L(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+. \tag{3.26}$$

Moreover, since $\eta_L \in L^1(\mathbb{R}_0^+)$ by assumption and also $\gamma_L \in L^1(\mathbb{R}_0^+)$, see Remark 3.8, both $(z_n)_n$ and $(\Phi_n)_n$ are equi-integrable in $\mathbb{R}_0^+$ so that, by the Dunford–Pettis Theorem, there exist $z, \hat{\Phi} \in L^1(\mathbb{R}_0^+)$ such that

$$z_n \rightharpoonup z \quad \text{and} \quad \Phi_n \rightharpoonup \hat{\Phi} \quad \text{in } L^1(\mathbb{R}_0^+) \quad \text{as } n \to +\infty, \tag{3.27}$$

up to subsequences. Consequently, for all $s \in \mathbb{R}_0^+$

$$\int_0^s z_n(\tau)d\tau \longrightarrow \int_0^s z(\tau)d\tau \quad \text{and} \quad \int_0^s \Phi_n(\tau)d\tau \longrightarrow \int_0^s \hat{\Phi}(\tau)d\tau \quad \text{as } n \to +\infty. \tag{3.28}$$

On the other hand, by Lemma 3.11–(i) and (iii), the sequences $(u_n'(0))_n$ and $(\mathcal{K}_{u_n}(0))_n$ are bounded in $\mathbb{R}$ and so there exists $\mathcal{K}_0 \in \mathbb{R}$ such that

$$u_n'(0) = x_n'(0) \longrightarrow \alpha'(0) = \nu_1 \quad \text{and} \quad \mathcal{K}_{u_n}(0) \longrightarrow \mathcal{K}_0 \quad \text{as } n \to +\infty, \tag{3.29}$$

up to subsequences. Now, let us define

$$x(t) = u_0 + \nu_1 t + \int_0^t \int_0^s z(\tau)d\tau ds, \qquad t \in \mathbb{R}_0^+.$$

We want to show that x is a solution to ($\mathcal{P}$). Clearly, $x(0) = u_0$. Moreover, for all $t \in \mathbb{R}_0^+$

$$x'(t) = \nu_1 + \int_0^t z(s)ds, \quad \text{with} \quad x'(0) = \alpha'(0) = \nu_1, \quad \text{and} \quad x''(t) = z(t).$$

By (3.28)–(3.29) we have

$$x_n'(t) = x_n'(0) + \int_0^t z_n(s)ds \longrightarrow x'(t) \quad \text{for all } t \in \mathbb{R}_0^+. \tag{3.30}$$

Furthermore, for all $s \in \mathbb{R}_0^+$

$$\left| \int_0^s z_n(\tau)d\tau \right| \leq \int_0^s |z_n(\tau)|d\tau \leq \|\gamma_L\|_{L^1(\mathbb{R}_0^+)},$$

so that, by (3.28)

$$\int_0^t \int_0^s z_n(\tau)d\tau ds \longrightarrow \int_0^t \int_0^s z(\tau)d\tau ds \quad \text{for all } t \in \mathbb{R}_0^+,$$

and in turn, using also (3.29), it results

$$x_n(t) = u_0 + u_n'(0)t + \int_0^t \int_0^s z_n(\tau)d\tau ds \longrightarrow x(t) \quad \text{for all } t \in \mathbb{R}_0^+. \tag{3.31}$$

Moreover, recalling that $u_n'(t) = x_n'(t)$ and $u_n''(t) = x_n''(t)$ for a.a. $t \in I_n$, we find

$$\Phi(k(t, x_n'(t))x_n''(t)) = \Phi \circ \mathcal{K}_{u_n}(t) = \Phi \circ \mathcal{K}_{u_n}(0) + \int_0^t \Phi_n(s)ds,$$

that is

$$x_n''(t) = \frac{1}{k(t, x_n'(t))} \Phi^{-1} \left(\Phi \circ \mathcal{K}_{u_n}(0) + \int_0^t \Phi_n(s)ds \right) \quad \text{for a.a. } t \in I_n.$$

Hence, recalling that k and Φ^{-1} are continuous, and using (3.28)–(3.30), we get

$$z_n(t) = x_n''(t) \longrightarrow \frac{1}{k(t, x'(t))} \mathcal{U}(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+, \tag{3.32}$$

where

$$\mathcal{U}(t) = \begin{cases} \Phi^{-1} \left(\Phi(\mathcal{K}_0) + \int_0^t \hat{\Phi}(s)ds \right), & t \in I_n, \\ 0, & t > n. \end{cases}$$

Observe that $\mathcal{U} \in C(\mathbb{R}_0^+; \mathbb{R})$, $\Phi \circ \mathcal{U} \in AC(\mathbb{R}_0^+; \mathbb{R})$ and $(\Phi \circ \mathcal{U})' = \hat{\Phi} \in L^1(\mathbb{R}_0^+)$. Now, by (3.25) and (3.32) we obtain

$$z_n = x_n'' \longrightarrow \frac{\mathcal{U}(\cdot)}{k(\cdot, x')} \quad \text{in } L^1(\mathbb{R}),$$

so that, by (3.27), we get

$$z(t) = \frac{\mathcal{U}(t)}{k(t, x'(t))} \quad \text{for a.a. } t \in \mathbb{R}_0^+, \tag{3.33}$$

which implies

$$x_n''(t) = z_n(t) \longrightarrow z(t) = x''(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+. \tag{3.34}$$

Combining (3.30), (3.31) and (3.34) with the fact that f is Carathéodory, we infer

$$f(t, x_n(t), x_n'(t), x_n''(t)) \longrightarrow f(t, x(t), x'(t), x''(t)) \quad \text{for a.a. } t \in \mathbb{R}_0^+. \tag{3.35}$$

Now, fix $t \in \mathbb{R}_0^+$; clearly there exists $\bar{n} > T_0$ such that $t \in I_n$ for all $n \geq \bar{n}$. Hence, recalling that for all $n > T_0$ the function u_n solves ($\mathcal{P}_n$), for any fixed $t \in \mathbb{R}_0^+$ we have

$$\Phi_n(t) = (\Phi \circ \mathcal{K}_{u_n})'(t) = f(t, u_n(t), u_n'(t), u_n''(t)) = f(t, x_n(t), x_n'(t), x_n''(t)) \quad \text{for all } n \geq \bar{n}.$$

Consequently, from (3.35) we obtain

$$\Phi_n(t) \longrightarrow f(t, x(t), x'(t), x''(t)) \quad \text{for a.a. } t \in \mathbb{R}_0^+,$$

and, by (3.26)

$$\Phi_n \longrightarrow f(\cdot, x, x', x'') \quad \text{in } L^1(\mathbb{R}_0^+).$$

Hence, by (3.33) we deduce that

$$(\Phi \circ \mathcal{K}_x)'(t) = (\Phi \circ \mathcal{U})'(t) = \hat{\Phi}(t) = f(t, x(t), x'(t), x''(t)) \quad \text{for a.a. } t \in \mathbb{R}_0^+,$$

that is x is a solution to (ODE).

Now, by Lemma 3.11–(i), we have

$$\alpha(t) \leq x_n(t) \leq \beta(t) \quad \text{and} \quad \alpha'(t) \leq x_n'(t) \leq \beta'(t) \quad \text{for a.a. } t \in I_n \text{ and all } n > T_0,$$

so that, by (3.30)–(3.31)

$$\alpha(t) \leq x(t) \leq \beta(t) \quad \text{and} \quad \alpha'(t) \leq x'(t) \leq \beta'(t) \quad \text{for a.a. } t \in \mathbb{R}_0^+. \tag{3.36}$$

Finally, by (3.25) and (3.34) we infer that $x_n'' \longrightarrow x''$ in $L^1(\mathbb{R}_0^+)$ and, recalling (3.29), we find

$$\sup_{t \in \mathbb{R}_0^+} |x_n'(t) - x'(t)| \leq |x_n'(0) - \nu_1| + \|x_n'' - x''\|_{L^1(\mathbb{R}_0^+)} \to 0,$$

so that $x_n' \to x'$ uniformly in $\mathbb{R}_0^+$. In particular,

$$\lim_{t \to +\infty} x'(t) = \lim_{n \to +\infty} \left(\lim_{t \to +\infty} x_n'(t) \right) = \lim_{n \to +\infty} \beta'(n) = \nu_2.$$

Concerning the regularity of x, we first observe that $x \in C^1(\mathbb{R}_0^+; \mathbb{R})$, being $x_n \in C^1(\mathbb{R}_0^+; \mathbb{R})$, and so $x \in L^p_{loc}(\mathbb{R}_0^+)$. Moreover, $\mathcal{U}$ is locally bounded in $\mathbb{R}_0^+$, since it is continuous, and $1/k(\cdot, x') \in L^p_{loc}(\mathbb{R}_0^+)$ by virtue of (3.36), being $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$ by assumption. Therefore $x'' = \mathcal{U}/k(\cdot, x') \in L^p_{loc}(\mathbb{R}_0^+)$. Furthermore also $x' \in L^p_{loc}(\mathbb{R}_0^+)$, being

$$\int_a^b |x'(t)|^p dt \leq 2^{p-1}|b-a| \left(|\nu_1|^p + \|x''\|_{L^1([a,b])} \right) < +\infty \quad \text{for all } a, b \in \mathbb{R}_0^+.$$

Hence $x \in W^{2,p}_{loc}(\mathbb{R}_0^+)$.

Finally, $\Phi \circ \mathcal{K}_x = \Phi \circ \mathcal{U} \in AC(\mathbb{R}_0^+; \mathbb{R})$ so that $\Phi \circ \mathcal{K}_x \in L^1_{loc}(\mathbb{R}_0^+)$ and $(\Phi \circ \mathcal{K}_x)' = \hat{\Phi} \in L^1(\mathbb{R}_0^+)$. Therefore $\Phi \circ \mathcal{K}_x \in W^{1,1}_{loc}(\mathbb{R}_0^+)$.

In conclusion, $x \in W^{2,p}_{loc}(\mathbb{R}_0^+)$ is a solution to ($\mathcal{P}$) and the proof is complete.

4 Examples

In this section we present a class of examples of functions Φ, k and f satisfying conditions (A_1)–(A_3) and (H_1)–(H_3).

Let $u_0, \nu_1, \nu_2 \in \mathbb{R}$ be such that $\nu_1 < \nu_2$, and let's consider the following BVP

$$\begin{cases} \left(\Phi\left(k(t,u'(\cdot))u''(\cdot)\right)\right)'(t) = f_1\left(t,u(t),u'(t)\right) f_2(u''(t)), & \text{a.a. } t \in \mathbb{R}_0^+, \\ u(0) = u_0, \;\; u'(0) = \nu_1, \;\; u'(+\infty) = \nu_2, \end{cases} \tag{4.1}$$

where the functions Φ, k, f_1 and f_2 fulfill the assumptions listed below.

(I) $\Phi : \mathbb{R} \to \mathbb{R}$ is an *odd* strictly increasing homeomorphism, with $\Phi(0) = 0$, and there exists $\rho > 0$ such that

$$\liminf_{s\to 0^+} \frac{\Phi(s)}{s^\rho} > 0. \tag{4.2}$$

(II) $k : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is continuous, strictly positive a.e. in $\mathbb{R}_0^+ \times \mathbb{R}$ and bounded in $\mathbb{R}_0^+ \times [\nu_1, \nu_2]$.

Moreover, if we denote by

$$k_* = \min_{y\in[\nu_1,\nu_2]} k(t,y) \quad \text{and} \quad k^* = \max_{y\in[\nu_1,\nu_2]} k(t,y),$$

we suppose that there exist $p > 1$ and $\sigma > 0$ such that

(II)$_1$ $t \mapsto 1/k(t,y) \in L^p_{loc}(\mathbb{R}_0^+)$ for all $y \in \mathbb{R}$ and $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$;

(II)$_2$ $\displaystyle\int_1^\infty \frac{1}{t^\sigma k_*^p(t)} dt < +\infty.$

(III) $f_1 : \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, decreasing with respect to the x variable, and there exists $T_0 > 0$ for which the following properties hold:

(III)$_1$ there exists $\tilde{f}_1 \in L^\infty_{loc}(\mathbb{R}_0^+)$ such that

$$|f_1(t,x,y)| \le \tilde{f}_1(t)$$

for a.a. $t \in [0, T_0]$, all $x \in [u_0 + t\nu_1, u_0 + t\nu_2]$ and all $y \in [\nu_1, \nu_2]$;

(III)$_2$ there exist c_1, $c_2 > 0$ and $\delta \ge -1$ such that

$$c_1 t^{-1} \le |f_1(t,x,y)| \le c_2 t^\delta$$

for a.a. $t \ge T_0$, all $x \in [u_0 + t\nu_1, u_0 + t\nu_2]$ and all $y \in [\nu_1, \nu_2]$;

(III)$_3$ $f_1(t,x,y) \le 0$ for all $t \ge T_0$, all $x \in [u_0 + t\nu_1, u_0 + t\nu_2]$ and all $y \in [\nu_1, \nu_2]$.

(IV) $f_2 \in C(\mathbb{R};\mathbb{R})$ and it verifies:

(IV)$_1$ $f_2(z) > 0$ for $z > 0$ and $f_2(0) = 0$;

(IV)$_2$ there exist $z^* > 0$, two real constants d_1, $d_2 > 0$ and a number $\gamma \le 1$ such that

$$d_1|\Phi(z)| \le f_2(z) \le d_2|\Phi(z)|^\gamma \quad \text{for all } z \in \mathbb{R} \text{ with } |z| < z^*;$$

(IV)$_3$ there exist $H > 0$ and $d_3 > 0$ such that if $z \in R$ and $|z| \geq H$ then

$$f_2(z) \leq d_3|z|^{\frac{q-1}{q}} \quad \text{for some } 1 < q \leq +\infty;$$

(IV)$_4$ f_2 is homogeneous of degree $d > 0$ in $\mathbb{R}$, with $d \leq p$, that is

$$f_2(tz) = t^d f_2(z) \quad \text{for all } t > 0 \text{ and } z \in \mathbb{R}.$$

Finally, put

$$K_M = \max_{t\in\mathbb{R}_0^+} k^*(t) = \sup\{k(t,y) : (t,y) \in \mathbb{R}_0^+ \times [\nu_1, \nu_2]\}$$

and suppose that

$$\frac{c_1 d_1}{K_M^d} \geq \sigma\rho \quad \text{and} \quad \gamma\frac{c_1 d_1}{K_M^d} \geq \sigma + \delta. \tag{4.3}$$

Remark 4.1. When $\delta = -1$ in (III)$_2$ we address the critical case.

Our aim is to prove that, in the present setting, all the hypotheses of Theorem 3.9 are satisfied. As a consequence, there exists a solution $u \in C^1(\mathbb{R}_0^+; \mathbb{R}) \cap W^{2,p}_{loc}(\mathbb{R}_0^+)$ of (4.1).

Remark 4.2. Before proceeding, we highlight, for future reference, a few consequences of the above assumptions (I)–(IV), that will also be employed in the sequel.

(A) For every $\nu \in (-\infty, p]$ it results

$$\int_1^\infty \frac{1}{t^\sigma k_*^\nu(t)} dt < +\infty.$$

Indeed, since $k(t,y) \geq k_*(t)$ for all for all $(t,y) \in \mathbb{R}_0^+ \times [\nu_1,\nu_2]$ and k is bounded in $\mathbb{R}_0^+ \times [\nu_1,\nu_2]$, it follows that k_* is bounded in $\mathbb{R}_0^+$. Therefore, recalling that the map $t \mapsto t^{-\sigma}/k_*^p(t)$ is integrable in $\{t \geq 1\}$ by (II)$_2$, for any $\nu \in (-\infty, p]$ we have

$$\int_1^\infty \frac{1}{t^\sigma k_*^\nu(t)} dt \leq \sup_{\mathbb{R}_0^+} k_*^{p-\nu} \int_1^\infty \frac{1}{t^\sigma k_*^p(t)} dt < +\infty.$$

(B) For all $\zeta > 0$ it is $\max_{|z|\leq\zeta} |\Phi(z)| = \Phi(\zeta)$. Indeed, if $z \in \mathbb{R}$ is such that $|z| \leq \zeta$, since Φ is odd and strictly increasing, we get $-\Phi(\zeta) = \Phi(-\zeta) \leq \Phi(z) \leq \Phi(\zeta)$.

(C) Combining (III)$_2$ with (III)$_3$ we have that

$$f_1(t,x,y) \leq -c_1 t^{-1} < 0$$

for a.a. $t \geq T_0$, all $x \in [u_0 + t\nu_1, u_0 + t\nu_2]$, all $y \in [\nu_1, \nu_2]$.

Obviously assumptions (A_1)–(A_3) are verified by virtue of (I)–(IV), with

$$f : \mathbb{R}_0^+ \times \mathbb{R}^3 \to \mathbb{R}, \quad f(t,x,y,z) = f_1(t,x,y) f_2(z).$$

Now, we are going to prove that also assumptions (H_1)–(H_3) are verified. To this aim, let T_0 be the positive number introduced in (III), and define

$$\alpha(t) = u_0 + t\nu_1 \quad \text{and} \quad \beta(t) = u_0 + t\nu_2, \quad t \in \mathbb{R}_0^+.$$

Clearly $\alpha, \beta \in C^\infty(\mathbb{R}_0^+;\mathbb{R}) \subset W^{2,p}_{loc}(\mathbb{R}_0^+)$, with $p > 1$ introduced in (II), and they are, respectively, a lower and a upper solution to (ODE), since $f_2(0) = 0$ by (IV)$_1$ and $\Phi(0) = 0$ by (I). Finally, $\alpha(0) = \beta(0) = u_0$ and $\alpha'(t) = \nu_1 < \nu_2 = \beta'(t)$ for all $t \in \mathbb{R}_0^+$, so that the pair (α, β) is ordered in $\mathbb{R}_0^+$.

Hypothesis (H_1)**.** Obviously β' is increasing in $(T_0, +\infty)$, being $\beta''(t) = 0$ for all $t \in \mathbb{R}_0^+$, and $\lim_{t\to+\infty} \beta'(t) = \nu_2$.

Hypothesis (H_2)**.** Let $H > 0$, $d_3 > 0$ and $q \in (1, +\infty]$ be as in (IV)$_3$. Combining (IV)$_1$ with (III)$_1$ and (IV)$_3$, we obtain

$$|f(t,x,y,z)| = |f_1(t,x,y)|f_2(z) \le d_3 \tilde{f}_1(t)|z|^{\frac{q-1}{q}}$$

for a.a. $t \in [0, T_0]$, all $x \in [\alpha(t), \beta(t)]$, all $y \in [\nu_1, \nu_2]$ and all $z \in \mathbb{R}$ with $|z| \ge H$. Hence assumption (H_2) is verified with

$$\psi \equiv 1, \quad \ell \equiv 0, \quad \mu(t) = d_3 \tilde{f}_1(t).$$

Observe that $\mu = d_3 \tilde{f}_1 \in L^q([0, T_0])$, being $\tilde{f}_1 \in L^\infty_{loc}(\mathbb{R}_0^+)$ by (III)$_1$.

Hypothesis (H_3)**.** Define the map $K_0 : \mathbb{R}_0^+ \to \mathbb{R}$ as follows

$$K_0(t) = \begin{cases} 0, & 0 \le t \le T_0, \\ \displaystyle\int_{T_0}^t f_0(s)ds, & t > T_0, \end{cases}$$

where

$$f_0(s) = \min\left\{|f_1(s,x,y)| : (x,y) \in [\alpha(t), \beta(t)] \times [\nu_1, \nu_2]\right\}.$$

Observe that

- f_0 is well defined in $\mathbb{R}_0^+$ since f_1 is Carathéodory by assumption (III);
- $f_0 \in L^1_{loc}(\mathbb{R}_0^+)$, being $\tilde{f}_1 \in L^\infty_{loc}(\mathbb{R}_0^+)$ by (III)$_1$ and

$$f_0(t) \le |f_1(t,x,y)| \le \tilde{f}_1(t) \quad \text{for all } t \in \mathbb{R}_0^+, \text{ all } x \in [\alpha(t), \beta(t)] \text{ and all } y \in [\nu_1, \nu_2].$$

Hence $K_0 \in AC(\mathbb{R}_0^+;\mathbb{R}) \cap W^{1,1}_{loc}(\mathbb{R}_0^+)$. Furthermore, K_0 is strictly increasing in $[T_0, +\infty)$, and

$$K_0(t) \ge c_1 \int_{T_0}^t \frac{ds}{s} = c_1 \log \frac{t}{T_0} \quad \text{for all } t \ge T_0, \tag{4.4}$$

by (III)$_2$ and (III)$_3$; see also Remark 4.2–(C).

Now, let $L > 0$ be fixed arbitrarily and put

$$m(L) = \min_{z^* \le |z| \le L} f_2(z), \quad M(L) = \max_{z^* \le |z| \le L} |\Phi(z)|, \quad c(L) = \min\left\{\frac{d_1}{K_M^d}, \frac{m(L)}{M(L)K_M^d}\right\} > 0.$$

Define the function $K_L : \mathbb{R}_0^+ \to \mathbb{R}$ as follows

$$H_L(t) = \Phi^{-1}\left(\Phi(L)e^{-c(L)K_0(t)}\right).$$

By (4.4) it follows that $K_0(t) \to +\infty$ as $t \to +\infty$, so that $H_L(t) \to 0$ as $t \to +\infty$. Therefore it is possible to find $t_L > T_0$ such that

$$H_L(t) \le z^* \quad \text{for all } t \ge t_L. \tag{4.5}$$

We then claim that the function $K_L(t) : \mathbb{R}_0^+ \to \mathbb{R}$ defined by

$$K_L(t) = \begin{cases} c(L)K_0(t), & 0 \le t \le t_L, \\ c(L)K_0(t_L) + \dfrac{d_1}{K_M^d} \displaystyle\int_{t_L}^t f_0(s)ds, & t > t_L, \end{cases}$$

satisfies all the properties in assumption (H_3). Observe that

- K_L is continuous in $\mathbb{R}_0^+$, being K_0 continuous in $\mathbb{R}_0^+$;
- $K_L \equiv K_0 \equiv 0$ in $[0, T_0]$ and K_L is strictly increasing in $[T_0, +\infty)$ since the same holds for K_0 and $t_L > T_0$; see Remark 4.2–(C).

Moreover, $K_L \in W^{1,1}_{loc}(\mathbb{R}_0^+)$, since the same is true for K_0, and for a.a. $t \in R_0^+$, all $x \in [\alpha(t), \beta(t)]$ and all $y \in [\nu_1, \nu_2]$, it is

$$K_L'(t) = \begin{cases} 0, & 0 \le t \le T_0 \\ c(L)f_0(t), & T_0 < t \le t_L \\ \dfrac{d_1}{K_M^d} f_0(t), & t > t_L \end{cases} \le \begin{cases} 0, & 0 \le t \le T_0, \\ c(L)|f_1(t,x,y)|, & T_0 < t \le t_L, \\ \dfrac{d_1}{K_M^d}|f_1(t,x,y)|, & t > t_L. \end{cases} \tag{4.6}$$

Now, we observe that $K_L(t) \ge c(L)K_0(t)$ for all $t \in \mathbb{R}_0^+$ so that

$$\mathcal{N}_L(t) = \Phi^{-1}\left(\Phi(L)e^{-K_L(t)}\right) \le H_L(t) \quad \text{for all } t \in \mathbb{R}_0^+,$$

and in turn, by (4.5), it follows that

$$\mathcal{N}_L(t) \le z^* \quad \text{for all } t \ge t_L. \tag{4.7}$$

Consequently, using $(IV)_4$ and $(IV)_2$, by (4.6), we obtain

$$\begin{aligned} |f(t,x,y,z)| &= \frac{|f_1(t,x,y)|}{k^d(t,y)} f_2(k(t,y)z) \ge \frac{d_1}{k^d(t,y)}|f_1(t,x,y)| \cdot |\Phi(k(t,y)z)| \\ &\ge \frac{d_1}{K_M^d}|f_1(t,x,y)| \cdot |\Phi(k(t,y)z)| \ge K_L'(t)|\Phi(k(t,y)z)|, \end{aligned} \tag{4.8}$$

for a.a. $t > t_L$, $x \in [\alpha(t), \beta(t)]$, $y \in [\nu_1, \nu_2]$ and $z \in \mathbb{R}$ such that $|k(t,y)z| \le \mathcal{N}_L(t)$.

On the other hand, by definition of $\mathcal{N}_L$ it is $\mathcal{N}_L(t) \le L$ for all $t \in \mathbb{R}_0^+$. Therefore, using again $(IV)_4$ and $(IV)_2$ and (4.6), it results

$$\begin{aligned} |f(t,x,y,z)| &= \frac{|f_1(t,x,y)|}{k^d(t,y)} f_2(k(t,y)z) \ge \frac{1}{K_M^d}|f_1(t,x,y)|f_2(k(t,y)z) \\ &\ge \begin{cases} \dfrac{d_1}{K_M^d}|f_1(t,x,y)| \cdot |\Phi(k(t,y)z)|, & |k(t,y)z| \le z^*, \\ \dfrac{m_L}{M_L K_M^d}|f_1(t,x,y)| \cdot |\Phi(k(t,y)z)|, & z^* \le |k(t,y)z| \le \mathcal{N}_L(t), \end{cases} \\ &\ge c(L)|f_1(t,x,y)| \cdot |\Phi(k(t,y)z)| \\ &\ge K_L'(t)|\Phi(k(t,y)z)|, \end{aligned} \tag{4.9}$$

a.a. $t \in [T_0, t_L]$, $x \in [\alpha(t), \beta(t)]$, $y \in [\nu_1, \nu_2]$ and $z \in \mathbb{R}$ such that $|k(t,y)z| \le \mathcal{N}_L(t)$.

Combining (4.8) with (4.9), it follows that

$$|f(t,x,y,z)| \geq K_L'(t)|\Phi(k(t,y)z)|$$

for a.a. $t \geq T_0$, all $x \in [\alpha(t), \beta(t)]$, all $y \in [\nu_1, \nu_2]$ and all $z \in \mathbb{R}$ such that $|z| \leq \mathcal{N}_L(t)/k(t,y)$, that is condition (H_3)–(i) is satisfied.

Now we are going to prove the validity of (3.3). To this aim, observe that, since K_L is continuous in $\mathbb{R}_0^+$ and $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$ by (II), it results

$$\int_{T_0}^{t_L} \frac{1}{k_*(t)} e^{-\frac{K_L(t)}{\rho}}\, dt < +\infty. \tag{4.10}$$

On the other hand, using the definition of K_L and (III)$_2$, we get

$$\begin{aligned}
\int_{t_L}^{\infty} \frac{1}{k_*(t)} e^{-\frac{K_L(t)}{\rho}}\, dt &= e^{-\frac{c(L)K_0(t_L)}{\rho}} \int_{t_L}^{\infty} \frac{1}{k_*(t)} e^{-\frac{d_1}{\rho K_M^d}\int_{t_L}^{t} f_0(s)ds}\, dt \\
&\leq e^{-\frac{c(L)K_0(t_L)}{\rho}} \int_{t_L}^{\infty} \frac{1}{k_*(t)} \left(\frac{t_L}{t}\right)^{\frac{c_1 d_1}{\rho K_M^d}} dt \\
&= t_L^{\frac{c_1 d_1}{\rho K_M^d}} e^{-\frac{c(L)K_0(t_L)}{\rho}} \int_{t_L}^{\infty} \frac{1}{k_*(t)} \left(\frac{1}{t}\right)^{\frac{c_1 d_1}{\rho K_M^d}} dt.
\end{aligned} \tag{4.11}$$

Now, recalling that the map $t \mapsto t^{-\sigma}/k_*(t)$ is integrable in $\{t \geq 1\}$, see Remark 4.2–(A), it follows that also the map $t \mapsto t^{-\frac{c_1 d_1}{\rho K_M^d}}/k_*(t)$ is integrable in $\{t \geq 1\}$, since

$$\int_{t_L}^{\infty} \frac{1}{k_*(t)} \left(\frac{1}{t}\right)^{\frac{c_1 d_1}{\rho K_M^d}} dt \leq \int_{t_L}^{\infty} \frac{1}{k_*(t)} \left(\frac{1}{t}\right)^{\sigma} dt < +\infty,$$

being $\sigma \leq c_1 d_1/\rho K_M^d$ by (4.3), which implies

$$\int_{t_L}^{\infty} \frac{1}{k_*(t)} \left(\frac{1}{t}\right)^{\frac{c_1 d_1}{\rho K_M^d}} dt < +\infty. \tag{4.12}$$

Combining (4.11) with (4.12) we get (3.3).

Now, we want to prove the existence of a non-negative function $\eta_L \in L^1(\mathbb{R}_0^+)$ satisfying (H_3)–(ii) (note that $\gamma_L = \hat{\gamma}_L$ in $\mathbb{R}_0^+$, being $\alpha'' \equiv \beta'' \equiv 0$ in $\mathbb{R}_0^+$). By (4.7), using also (IV)$_4$, (III)$_2$ and (IV)$_2$, see also Remark 4.2–(C), for a.a. $t \in \mathbb{R}_0^+$, all $x \in [\alpha(t), \beta(t)]$, all $y \in [\nu_1, \nu_2]$ and all $z \in \mathbb{R}$ such that $|k_*(t)z| \leq \mathcal{N}_L(t)$ we find

$$\begin{aligned}
|f(t,x,y,z)| &= \frac{|f_1(t,x,y)|}{k_*^d(t)} f_2(k_*(t)z) \\
&\leq \begin{cases} c_2 d_2 \dfrac{t^\delta}{k_*^d(t)} |\Phi(k_*(t)z)|^\gamma \leq c_2 d_2 \dfrac{t^\delta}{k_*^d(t)} \left[\Phi(\mathcal{N}_L(t))\right]^\gamma, & t > t_L, \\[2ex] \max\limits_{[0,L]} f_2 \cdot \dfrac{\tilde{f}_1(t)}{k_*^d(t)}, & 0 \leq t \leq t_L, \end{cases} \\
&=: \eta_L(t).
\end{aligned}$$

Finally, we claim that $\eta_L \in L^1(\mathbb{R}_0^+)$. First observe that, since $d \le p$, $\tilde{f}_1 \in L^\infty_{loc}(\mathbb{R}_0^+)$ and $1/k_* \in L^p_{loc}(\mathbb{R}_0^+)$, we get

$$\begin{aligned}\int_0^{t_l} \eta_L(t)dt &\le \max_{[0,L]} f_2 \cdot \|\tilde{f}_1\|_{L^\infty([0,t_L])} \int_0^{t_L} \frac{1}{k_*^d(t)} dt \\ &\le \max_{[0,L]} f_2 \cdot \|\tilde{f}_1\|_{L^\infty([0,t_L])} \sup_{\mathbb{R}_0^+} k_*^{p-d} \int_0^{t_L} \frac{1}{k_*^p(t)} dt \\ &< +\infty.\end{aligned} \tag{4.13}$$

On the other hand, by (III)$_2$, from the definition of η_L, it is

$$\begin{aligned}\int_{t_L}^\infty \eta_L(t)dt &= c_2 d_2 \int_{t_L}^\infty \frac{t^\delta}{k_*^d(t)} \left[\Phi(\mathcal{N}_L(t))\right]^\gamma dt \\ &= c_2 d_2 \Phi(L)^\gamma \int_{t_L}^\infty \frac{t^\delta}{k_*^d(t)} e^{-\gamma K_L(t)} dt \\ &= c_2 d_2 \Phi(L)^\gamma e^{-\gamma c(L) K_0(t_L)} \int_{t_L}^\infty \frac{t^\delta}{k_*^d(t)} e^{-\gamma \frac{d_1}{K_M^d} \int_{t_L}^t f_0(s)ds} dt \\ &\le c_3 \int_{t_L}^\infty \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt,\end{aligned} \tag{4.14}$$

with $c_3 > 0$ appropriate constant.

Now, observe that, if $t_L \ge 1$, recalling that $\gamma \frac{c_1 d_1}{K_M^d} - \delta \ge \sigma$ by (4.3), we get

$$\int_{t_L}^\infty \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt \le \int_1^\infty \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt \le \int_1^\infty \frac{1}{k_*^d(t) t^\sigma} dt < +\infty,$$

since $d \le p$ by assumption, see Remark 4.2–(A); on the other hand, if $t_L < 1$, then

$$\begin{aligned}\int_{t_L}^\infty \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt &= \int_{t_L}^1 \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt + \int_1^\infty \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt \\ &\le \int_{t_L}^1 \frac{1}{k_*^d(t)} \left(\frac{1}{t}\right)^{\gamma \frac{c_1 d_1}{K_M^d} - \delta} dt + \int_1^\infty \frac{1}{k_*^d(t) t^\sigma} dt \\ &< +\infty,\end{aligned}$$

since the map $t \mapsto k_*^{-d}(t) t^{\delta - \gamma \frac{c_1 d_1}{K_M^d}}$ is bounded in $[t_L, 1]$ and, as before, $\int_1^\infty \frac{1}{k_*^d(t) t^\sigma} dt < +\infty$, being $d \le p$. Therefore, from (4.14), we infer that

$$\int_{t_L}^\infty \eta_L(t) dt < +\infty. \tag{4.15}$$

Combining (4.13) with (4.15) we get the claim.

Gathering together all these facts, we are entitled to apply Theorem 3.9 to the BVP (4.1), getting at least a solution $u \in C^1(\mathbb{R}_0^+) \cap W^{2,p}_{loc}(\mathbb{R}_0^+)$ such that

$$u_0 + t\nu_1 \le u(t) \le u_0 + t\nu_2 \quad \text{and} \quad \nu_1 \le u'(t) \le \nu_2 \quad \text{for a.a. } t \in \mathbb{R}_0^+.$$

Remark 4.3. It is worth noting that, in the particular case when the homeomorphism Φ in (I) is also homogeneous of degree $r \in (0, p]$, the growth assumption (IV)$_3$ can be replaced with the following one:

(IV)$'_3$ there exists $H > 0$ and $d_3 > 0$ such that, if $z \in \mathbb{R}$ and $|z| \geq H$, then

$$f_2(z) \leq d_3 |\Phi(z)|^{\overline{\alpha}} \quad \text{for some } \overline{\alpha} \leq 1. \tag{4.16}$$

Indeed, if (4.16) holds, it follows

$$|f_1(t,x,y)| f_2(z) \leq d_3 |f_1(t,x,y)| \cdot |\Phi(z)|^{\overline{\alpha}} = d_3 \frac{|f_1(t,x,y)|}{k(t,y)^{r\overline{\alpha}}} |\Phi(k(t,y)z)|^{\overline{\alpha}}$$
$$\leq d_3 \frac{\tilde{f}_1(t)}{k_*^{r\overline{\alpha}}(t)} |\Phi(k(t,y)z)|^{\overline{\alpha}},$$

for a.a. $t \in [0, T_0]$, all $x \in [u_0 + tv_1, u_0 + tv_2]$, all $y \in [v_1, v_2]$ and all $z \in \mathbb{R}$ such that $|z| \geq H$.

As a consequence, hypothesis (H_2) in Theorem 3.9 is fulfilled with

$$\psi(s) = s^{\overline{\alpha}}, \quad \ell(t) = d_3 \frac{\tilde{f}_1(t)}{k_*^{\overline{\alpha} r}(t)}, \quad \mu(t) \equiv 0.$$

Note that, since $\overline{\alpha} \leq 1$, the function ψ satisfies (H_2)–(i); furthermore, since $\tilde{f}_1 \in L^\infty_{loc}(\mathbb{R}^+_0)$, the function $1/k_* \in L^p_{loc}(\mathbb{R}^+_0)$ and $\overline{\alpha} r \leq r \leq p$, we infer that $\ell \in L^1([0, T_0])$.

We conclude this section by presenting some concrete examples of BVPs of the form (4.1), satisfying assumptions (I)–(IV) introduced above.

Example 4.4. Let us consider the following boundary value problem

$$\begin{cases} \left(\Phi \left(e^{-u'(t)^2} \min \left\{ \sqrt{t}, \dfrac{1}{t^2} \right\} u'(t) \right) \right)' = f(t, u, u'(t), u''(t)), \quad \text{a.a. } t \in \mathbb{R}^+_0, \\ u(0) = 0, \quad u'(0) = 0, \quad u'(+\infty) = 1, \end{cases} \tag{4.17}$$

with $m > 1$ to be fixed later, $\theta \in (0, 1)$ and

$$\Phi(z) = z + \sin z, \quad f(t,x,y,z) = -m[\arctan(x^3 + y^2) + \pi] \cdot |z|^\theta \frac{t}{1+t^2}.$$

Obviously, problem (4.17) takes the form (4.1) with

- $\Phi : \mathbb{R} \to \mathbb{R}, \quad \Phi(z) = z + \sin z;$
- $k : \mathbb{R}^+_0 \times \mathbb{R} \to \mathbb{R}, \quad k(t,y) = e^{-y^2} \min\{\sqrt{t}, 1/t^2\};$
- $f_1 : \mathbb{R}^+_0 \times \mathbb{R}^2 \to \mathbb{R}, \quad f_1(t,x,y) = -[\arctan(x^3 + y^2) + \pi] \dfrac{mt}{1+t^2};$
- $f_2 : \mathbb{R} \to \mathbb{R}, \quad f_2(z) = |z|^\theta.$

We claim that the functions Φ, k, f_1 and f_2 satisfy all the assumptions (I)–(IV) introduced in this section, with suitable constants fulfilling (4.3).

Assumption (I) It is straightforward to recognize that Φ is an odd strictly increasing homeomorphism. Moreover, since

$$\lim_{z \to 0^+} \frac{\Phi(z)}{z} = \lim_{z \to 0^+} \frac{z + \sin z}{z} = 2,$$

condition (4.2) is satisfied with $\rho = 1$.

Assumption (II) Clearly k is continuous, strictly positive a.e. in $\mathbb{R}_0^+ \times \mathbb{R}$ and bounded in $\mathbb{R}_0^+ \times [0,1]$. Moreover, it is very easy to see that the map $t \mapsto 1/k(t,y) \in L^p_{loc}(\mathbb{R}_0^+)$ for all $y \in \mathbb{R}$, for every fixed $p \in (1,2)$, and the same is true for $1/k_*(t) = e/\min\{\sqrt{t}, t^2\}$. Moreover,

$$\int_1^\infty \frac{1}{t^\sigma k_*^p(t)}dt = e^p \int_1^\infty \frac{1}{t^{\sigma-2p}}dt < +\infty, \quad \text{for every } \sigma > 2p+1.$$

Assumption (III) f_1 is a Carathéodory function in $\mathbb{R}_0^+ \times \mathbb{R}^2$, being continuous in the same set, and it is also decreasing with respect to x in $\mathbb{R}_0^+ \times \mathbb{R}^2$.

Moreover, for all $t \in \mathbb{R}_0^+$, all $x \in [0,t]$ and all $y \in [0,1]$ it is

$$\begin{aligned}|f_1(t,x,y)| &= \left[\arctan(x^3+y^2)+\pi\right]\frac{mt}{1+t^2}\\ &\le \left[\arctan(1+t^3)+\pi\right]\frac{mt}{1+t^2} =: \tilde{f}_1(t) \in C(\mathbb{R}_0^+;\mathbb{R}) \subseteq L^\infty_{loc}(\mathbb{R}_0^+)\end{aligned}$$

and also

$$\frac{\pi}{2}\cdot\frac{mt^2}{1+t^2} = \left(\pi - \frac{\pi}{2}\right)\frac{mt^2}{1+t^2} \le t|f_1(t,x,y)| \le t\tilde{f}_1(t) = \left[\arctan(1+t^3)+\pi\right]\frac{mt^2}{1+t^2}. \tag{4.18}$$

Now, since

$$\lim_{t\to+\infty}\frac{\pi}{2}\cdot\frac{mt^2}{1+t^2} = m\frac{\pi}{2} \quad \text{and} \quad \lim_{t\to+\infty}\left[\arctan(1+t^3)+\pi\right]\frac{mt^2}{1+t^2} = \frac{3}{2}m\pi,$$

from (4.18) we deduce that for any $\varepsilon \in (0, m\pi/2)$ there exists $T_0 = T_0(\varepsilon) > 0$ such that

$$\left(m\frac{\pi}{2} - \varepsilon\right)\frac{1}{t} \le |f_1(t,x,y)| \le \left(\frac{3}{2}m\pi + \varepsilon\right)\frac{1}{t}$$

for all $t \ge T_0$, all $x \in [0,t]$ and all $y \in [0,1]$, so that $(\mathrm{III})_2$ holds with

$$c_1 = m\frac{\pi}{2} - \varepsilon, \quad c_2 = \frac{3}{2}m\pi + \varepsilon, \quad \delta = -1. \tag{4.19}$$

Note that $c_1 = m\frac{\pi}{2} - \varepsilon > 0$. Finally, condition $(\mathrm{III})_3$ is trivially satisfied.

Assumption (IV) Clearly f_2 is continuous in $\mathbb{R}$, being $\theta > 0$, and $(\mathrm{IV})_1$ trivially holds. Moreover,

$$\lim_{z\to 0}\frac{f_2(z)}{|\Phi(z)|^\theta} = \lim_{z\to 0}\frac{|z|^\theta}{|z+\sin z|^\theta} = \frac{1}{2^\theta},$$

so that for any $\varepsilon \in (0, 1/2^\theta)$ there exists $z^* = z^*(\varepsilon) > 0$ such that

$$\left(\frac{1}{2^\theta} - \varepsilon\right)|\Phi(z)| \le f_2(z) \le \left(\frac{1}{2^\theta} + \varepsilon\right)|\Phi(z)|^\theta \quad \text{for all } z \in \mathbb{R} : \ |z| < z^*.$$

Hence condition $(\mathrm{IV})_2$ is satisfied with

$$d_1 = \frac{1}{2^\theta} - \varepsilon, \quad d_2 = \frac{1}{2^\theta} + \varepsilon, \quad \gamma = \theta. \tag{4.20}$$

Furthermore, condition $(\mathrm{IV})_3$ holds with any constant $H > 0$, any number $d_3 \ge 1$ and $q = 1/(1-\theta) > 1$. Finally f_2 is homogeneous of degree $d = \theta < 1 < p$.

Now, we claim that, for any fixed $p \in (1,2)$, it is possible to choose

$$m > 1 \quad \text{and} \quad 0 < \varepsilon < \frac{1}{2^\theta} < m\frac{\pi}{2}$$

in such a way that (4.3) holds. To prove the claim, first note that

$$K_M = \sup_{\mathbb{R}_0^+ \times [0,1]} e^{-y^2} \min\{\sqrt{t}, 1/t^2\} = 1.$$

Now, since

$$\lim_{\varepsilon \to 0^+} (m - \varepsilon) \cdot \left(\frac{1}{2^\theta} - \varepsilon\right) = \frac{m}{2^\theta},$$

we can choose $m > 1$ and $\varepsilon \in (0, 1/2^\theta)$ satisfying

$$(m - \varepsilon) \cdot \left(\frac{1}{2^\theta} - \varepsilon\right) > \max\left\{1 + 2p, \frac{2p}{\theta}\right\}. \tag{4.21}$$

Consequently, by (4.19)–(4.20) and recalling that $\rho = 1$, one gets

$$\frac{c_1 d_1}{K_M^d} = (m - \varepsilon) \cdot \left(\frac{1}{2^\theta} - \varepsilon\right) > 1 + 2p = (1 + 2p)\rho.$$

On the other hand, using again (4.19)–(4.20) and recalling that $\delta = -1$, we see that

$$\gamma \frac{c_1 d_1}{K_M^d} = \theta(m - \varepsilon) \cdot \left(\frac{1}{2^\theta} - \varepsilon\right) > 2p = (1 + 2p) + \delta.$$

From this, since σ can be chosen arbitrarily close to $1 + 2p$, we conclude that (4.3) is satisfied.

We are then entitled to apply Theorem 3.9 which ensures the existence of a solution $u \in C^1(\mathbb{R}_0^+; \mathbb{R}) \cap W_{loc}^{2,p}(\mathbb{R}_0^+)$ of (4.17).

Remark 4.5. A further example which covers the critical case can be constructed following Example 4.4, but choosing $\nu_1 = 1 < \nu_2$ and substituting the function f_1 with the following one

$$f_1(t, x, y) = m\left[e^{-(x+y)} - 1\right] t \sin\left(\frac{1}{t^2 + 1}\right).$$

Indeed, $f_1 \in C(\mathbb{R}_0^+ \times \mathbb{R}^2)$, it is non positive whenever $x + y \geq 0$ and decreasing with respect to x in the whole of $\mathbb{R}_0^+ \times \mathbb{R}^2$.

Furthermore, for all $t \geq 1$, all $x \in [t, t\nu_2]$ and all $y \in [1, \nu_2]$ we have

$$\frac{m}{2} t \sin\left(\frac{1}{t^2 + 1}\right) \leq m\left[1 - e^{-(x+y)}\right] t \sin\left(\frac{1}{t^2 + 1}\right) = |f_1(t, x, y)| \leq mt \sin\left(\frac{1}{t^2 + 1}\right) =: \tilde{f}_1(t),$$

with $\tilde{f}_1 \in C(\mathbb{R}_0^+)$, which implies

$$\frac{m}{2} t^2 \sin\left(\frac{1}{t^2 + 1}\right) \leq t|f_1(t, x, y)| \leq mt^2 \sin\left(\frac{1}{t^2 + 1}\right).$$

Therefore, for any $\varepsilon \in (0, m/2)$ there exists $T_0 = T_0(\varepsilon)$ that can be chosen greater or equal than 1, such that, for all $t \geq T_0$, all $x \in [t, t\nu_2]$ and all $y \in [1, \nu_2]$, it is

$$\left(\frac{m}{2} - \varepsilon\right) \frac{1}{t} \leq |f_1(t, x, y)| \leq (m - \varepsilon)\frac{1}{t},$$

so that condition (III) holds with

$$T_0 = 1, \quad c_1 = \frac{m}{2} - \varepsilon, \quad c_2 = m - \varepsilon, \quad \delta = -1.$$

Finally, we conclude by choosing $m > 1$ and $\varepsilon \in \left(0, 1/2^\theta\right)$ satisfying

$$\left(\frac{m}{2} - \varepsilon\right) \cdot \left(\frac{1}{2^\theta} - \varepsilon\right) > \max\left\{1 + 2p, \frac{2p}{\theta}\right\}$$

in place of (4.21).

Example 4.6. Let $\overline{p} \in (1, +\infty)$ be fixed and let $r, \theta \in \mathbb{R}$ be such that

$$1 < r < \overline{p} + 1 \quad \text{and} \quad 0 < \theta < r - 1.$$

Let us consider the following boundary value problem

$$\begin{cases} \left(\Phi\left(\frac{|\sin t|^{1/\overline{p}} + |\sin^2 u'(t)|}{2} u''(t)\right)\right)' = f(t, u(t), u'(t), u''(t)), & \text{a.a. } t \in \mathbb{R}_0^+, \\ u(0) = 0, \ u'(0) = 1, \ u'(+\infty) = \nu_2, \end{cases} \tag{4.22}$$

with $\nu_2 > 1$, $m > 1$ to be fixed later and

$$\Phi(z) = \Phi_r(z) = |z|^{r-2}z \quad \text{and} \quad f(t, x, y, z) = -\frac{m + t|\cos t|}{1 + t}\left(x + |\cos y|\right)^3 |z|^\theta.$$

Problem (4.22) takes the form (4.1) with

- $\Phi : \mathbb{R} \to \mathbb{R}$ is the *r-Laplacian*;
- $k : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}, \quad k(t, y) = \frac{|\sin t|^{1/\overline{p}} + |\sin^2 y|}{2}$;
- $f_1 : \mathbb{R}_0^+ \times \mathbb{R}^2 \to \mathbb{R}, \quad f_1(t, x, y) = -\frac{m + t|\cos t|}{1 + t}(x + |\cos y|)^3$;
- $f_2 : \mathbb{R} \to \mathbb{R}, \quad f_2(z) = |z|^\theta$.

We shall see that the functions Φ, k, f_1 and f_2 satisfy all the assumptions (I)–(IV) introduced in this section, with suitable constants fulfilling (4.3).

Assumption (I) Clearly the *r-Laplacian* is an odd strictly increasing homeomorphism, for which condition (4.2) is satisfied with $\rho = r - 1$, being $\lim_{z \to 0^+} \frac{\Phi(z)}{z^{r-1}} = 1$.

Assumption (II) The function k is continuous and bounded in $\mathbb{R}_0^+ \times \mathbb{R}$, and it is strictly positive a.e. in $\mathbb{R}_0^+ \times \mathbb{R}$.

Moreover, it is very easy to check that the map $t \mapsto 1/k(t, y) \in L^p_{loc}(\mathbb{R}_0^+)$ for all $y \in \mathbb{R}$, for every fixed $p \in (1, \overline{p})$, and the same is true for $1/k_*(t) = 2/|\sin t|^{1/\overline{p}}$, so that (II)$_1$ holds.

Now, choose $p > 1$ in such a way that

$$p \in (\max\{1, r - 1\}, \overline{p}).$$

For any $\sigma > 1$ it results

$$\begin{aligned} \int_{2\pi}^{\infty} \frac{1}{t^\sigma k_*^p(t)} dt &= 2^p \int_{2\pi}^{\infty} \frac{1}{t^\sigma |\sin t|^{p/\overline{p}}} dt = 2^p \sum_{n=1}^{\infty} \int_{2n\pi}^{2(n+1)\pi} \frac{1}{t^\sigma |\sin t|^{p/\overline{p}}} dt \\ &= 2^p \sum_{n=1}^{+\infty} \int_0^{2\pi} \frac{1}{(t + 2n\pi)^\sigma |\sin t|^{p/\overline{p}}} dt \le 2^p \sum_{n=1}^{+\infty} \int_0^{2\pi} \frac{1}{(2n\pi)^\sigma |\sin t|^{p/\overline{p}}} dt \\ &= 2^p \sum_{n=1}^{+\infty} \frac{1}{(2n\pi)^\sigma} \int_0^{2\pi} \frac{1}{|\sin t|^{p/\overline{p}}} dt < +\infty, \end{aligned}$$

and, consequently, assumption (II)$_2$ is fulfilled.

Assumption (III) f_1 is a Carathéodory function in $\mathbb{R}_0^+ \times \mathbb{R}^2$, being continuous in the same set, and it is also decreasing with respect to x in $\mathbb{R}_0^+ \times \mathbb{R}^2$.

Now, take $T_0 = 1$. For all $t \geq T_0$, all $x \in [t, t\nu_2]$ and all $y \in \mathbb{R}$ it results $f_1(t,x,y) \leq 0$ and

$$|f_1(t,x,y)| = \frac{m + t|\cos t|}{1+t}(x + |\cos y|)^3 \leq 4\frac{(m+t|\cos t|)}{1+t}(x^3 + |\cos y|^3)$$
$$\leq \frac{4(m+t)(t^3\nu_2^3 + 1)}{1+t} =: \tilde{f}_1(t),$$

with $\tilde{f}_1 \in C(\mathbb{R}_0^+;\mathbb{R}) \subseteq L^\infty_{loc}(\mathbb{R}_0^+)$. Furthermore, for all $t \geq 1$, all $x \in [t, t\nu_2]$ and all $y \in \mathbb{R}$

$$|f_1(t,x,y)| \leq \tilde{f}_1(t) \leq 4(m+1)(\nu_2^3 + 1)t^3. \tag{4.23}$$

On the other hand, for all $t \geq 1$, all $x \in [t, t\nu_2]$ and all $y \in \mathbb{R}$, it is also true that

$$|f_1(t,x,y)| \geq \frac{m + t|\cos t|}{1+t}x^3 \geq \frac{m}{1+t} \geq \frac{m}{2}\cdot\frac{1}{t}. \tag{4.24}$$

Hence, combining (4.23) with (4.24) we see that (III)$_2$ holds with

$$T_0 = 1, \quad c_1 = \frac{m}{2}, \quad c_2 = 2(m+1)(1+\nu_2^2), \quad \delta = 3. \tag{4.25}$$

Assumption (IV) Clearly $f_2 > 0$ in $\mathbb{R}^+$ and $f_2(0) = 0$. Moreover, if $z \in \mathbb{R}$ and $|z| < 1$, then

$$|\Phi(z)| = |z|^{r-1} < |z|^\theta = f_2(z) = |z|^{\gamma(r-1)} = |\Phi(z)|^\gamma,$$

provided that $\gamma = \theta/(r-1) < 1$. Therefore, condition (IV)$_2$ is satisfied with $z^* = 1$ and

$$d_1 = d_2 = 1. \tag{4.26}$$

Now, observe that, since Φ is homogeneous of degree $r - 1 < p$ and

$$f_2(z) = |z|^\theta \leq |z|^{r-1} = |\Phi(z)| \quad \text{for all } z \in \mathbb{R} \text{ with } |z| \geq 1,$$

we find that assumption (IV)$_3'$ in Remark 4.3 holds with

$$H = 1, \quad d_3 = 1 \quad \overline{\alpha} = 1.$$

Finally, f_2 is homogeneous of degree $d = \theta$.

Now, in order to prove the validity of (4.3), take $m > 1$ satisfying

$$m > 2(r-1)\max\left\{1, \frac{4}{\theta}\right\}.$$

By definition

$$K_M = \sup_{\mathbb{R}_0^+ \times [1,\nu_2]} \frac{|\sin t|^{1/\overline{p}} + \sin^2 y}{2} \leq 1,$$

so that, by (4.25)–(4.26), and recalling that $\rho = r - 1$, it is

$$\frac{c_1 d_1}{K_M^d} \geq c_1 d_1 = \frac{m}{2} > r - 1 = \rho.$$

On the other hand, using again (4.25)–(4.26), we find

$$\gamma\frac{c_1 d_1}{K_M^d} \geq \gamma c_1 d_1 = \frac{\theta}{r-1}\cdot\frac{m}{2} > 4 = 1 + \delta.$$

Choosing σ arbitrarily close to 1 we obtain (4.3).

In conclusion, by Theorem 3.9 there exists a solution $u \in C^1(\mathbb{R}_0^+;\mathbb{R}) \cap W^{2,p}_{loc}(\mathbb{R}_0^+)$ of (4.22).

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On the logarithmic fractional Schrödinger–Poisson system with saddle-like potential

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Abstract. In this paper, we use variational methods to prove the existence of a positive solution for the following class of logarithmic fractional Schrödinger–Poisson system:

$$\begin{cases} \epsilon^{2s}\,(-\Delta)^s\,u + V(x)u - \phi(x)u = u\log u^2 & \text{in } \mathbb{R}^3, \\ \epsilon^{2t}\,(-\Delta)^t\,\phi = |u|^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\epsilon > 0$, $s,t \in (0,1)$, $(-\Delta)^\alpha$ is the fractional Laplacian and V is a saddle-like potential.

Keywords: fractional Schrödinger–Poisson system, logarithmic nonlinearity, variational methods.

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1 Introduction and main result

In this article, we consider the following fractional Schrödinger–Poisson system:

$$\begin{cases} \epsilon^{2s}\,(-\Delta)^s\,u + V(x)u - \phi(x)u = u\log u^2 & \text{in } \mathbb{R}^3, \\ \epsilon^{2t}\,(-\Delta)^t\,\phi = |u|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\epsilon > 0$ is a small parameter, $s,t \in (0,1)$ and $(-\Delta)^\alpha$, with $\alpha \in \{s,t\}$, is the fractional Laplacian operator which may be defined for any $u : \mathbb{R}^3 \to \mathbb{R}$ belonging to the Schwartz class by

$$(-\Delta)^\alpha u(x) = C(3,\alpha)P.V.\int_{\mathbb{R}^3} \frac{u(x)-u(y)}{|x-y|^{3+2\alpha}}\mathrm{d}y \quad (x \in \mathbb{R}^3),$$

where P.V. stands for the Cauchy principal value and $C(3,\alpha)$ is a normalizing constant; see Di Nezza–Palatucci–Valdinoci [13]. In recent years, there has been a surge of interest in studying partial differential equations involving nonlocal fractional Laplace operators. This type of nonlocal operator comes up naturally in the real world in many different applications, such as

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phase transitions, game theory, finance, image processing, Lévy processes, and optimization. For more details and applications, we refer the interested reader to the works of Applebaum [6], Bahrouni–Rădulescu–Winkert [7], Caffarelli–Silvestre [9], Di Nezza–Palatucci–Valdinoci [13], Molica Bisci–Rădulescu–Servadei [21], Pucci–Xiang–Zhang [23, 24] and their references.

In the fractional scenario, there are many results for the fractional Schrödinger–Poisson system. Teng [29] studied the existence of ground state solutions for the fractional Schrödinger–Poisson system with the critical Sobolev exponent. Yang–Yu–Zhao [31] were concerned with the existence and concentration behavior of ground state solutions for the fractional Schrödinger–Poisson system with critical nonlinearity. Ambrosio [5] used penalization techniques and Ljusternik–Schnirelmann theory to deal with the multiplicity and concentration of positive solutions for a fractional Schrödinger–Poisson type system with critical growth. Meng–Zhang–He [20] dealt with the existence of a positive and a sign-changing least energy solution for a class of fractional Schrödinger–Poisson system with critical growth and vanishing potentials. Finally, other interesting results in this direction can be found in the papers of Chen–Li–Peng [10], Ji [15], Murcia–Siciliano [22], Qu–He [25] and the references therein.

The case where potential V has a saddle-like geometry was considered in del Pino–Felmer–Miyagaki [12], essentially they assumed the potential V is bounded and $V \in C^2(\mathbb{R}^3)$, which verifies the following conditions:

Fix two subspaces $X, Y \subset \mathbb{R}^3$ such that $\mathbb{R}^3 = X \oplus Y$, then fix $c_0, c_1 > 0$ such that

$$c_0 = \inf_{z\in\mathbb{R}^3} V(z) > 0 \quad \text{and} \quad c_1 = \sup_{x\in X} V(x),$$

satisfying the following geometric condition

(V_1) There exists a number $\lambda \in (0,1)$, such that

$$c_0 = \inf_{R>0} \sup_{x\in\partial B_R(0)\cap X} V(x) < \inf_{y\in Y_\lambda} V(y).$$

where Y_λ is the cone about Y given by

$$Y_\lambda = \left\{ z \in \mathbb{R}^3 : |z \cdot y| > \lambda |z||y|, \text{ for some } y \in Y \right\}.$$

In addition to the above hypotheses, they imposed the conditions below:

(V_2) The functions $V, \frac{\partial V}{\partial x_i}, \frac{\partial^2 V}{\partial x_i \partial x_j}$ are bounded in $\mathbb{R}^3$, for all $i, j \in \{1,2,3\}$;

(V_3) V satisfies the Palais–Smale condition, that is, if $(x_n) \subset \mathbb{R}^3$, such that $(V(x_n))$ is limited and $\nabla V(x_n) \to 0$, then (x_n) possesses a convergent subsequence in $\mathbb{R}^3$.

Using the above conditions on V, and supposing that

$$c_1 < 2^{\frac{2(p-1)}{N+2-p(N-2)}} c_0,$$

the authors studied the existence of positive solutions for the following Schrödinger equation:

$$-\epsilon^2 \Delta u + V(z)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

where $p \in (2, 2^*)$ if $N \geq 3$ and $p \in (2, +\infty)$ if $N = 1, 2$, for $\epsilon > 0$ small enough. After, Alves [1] showed the existence of a positive solution for the following elliptic equation with exponential critical growth in $\mathbb{R}^2$:

$$-\epsilon^2 \Delta u + V(z)u = f(u) \quad \text{in } \mathbb{R}^2,$$

Alves and Miyagaki [4] considered the following nonlinear fractional elliptic equation with critical growth in $\mathbb{R}^N$:

$$\epsilon^{2s} (-\Delta)^s u + V(z)u = \lambda |u|^{q-2} u + |u|^{2_s^* - 2} u \quad \text{in } \mathbb{R}^N,$$

where $\lambda > 0$ is a positive parameter, $q \in (2, 2_s^*)$. Recently, under the same assumptions on the potential V, Alves and Ji [3] used the variational method to prove the existence of a positive solution for the following logarithmic Schrödinger equation:

$$-\epsilon^2 \Delta u + V(z)u = u \log u^2 \quad \text{in } \mathbb{R}^N.$$

Motivated by the above papers, in this work we consider the correlation result of the fractional Schrödinger–Poisson system. Now, we state the main result.

Theorem 1.1. *Suppose that V satisfies (V_1)–(V_3). If $V(0) > c_0$ and $c_1 < c_0 + 1$, then there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$, the system (1.1) has a positive solution.*

Remark 1.2. Noting that in this paper we consider the nonlocal term ϕ with negative coefficient. We want to point out that if we deal with the positive nonlocal term, i.e.,

$$\begin{cases} \epsilon^{2s} (-\Delta)^s u + V(x)u + \phi(x)u = u \log u^2 & \text{in } \mathbb{R}^3, \\ \epsilon^{2t} (-\Delta)^t \phi = |u|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

it is not easy to obtain the boundedness of Palais–Smale sequence (u_n). In fact, by the logarithmic Sobolev inequality, the key point is to prove the boundedness of (u_n), where the negative coefficient plays an important role, see Lemma 3.7. In contrast, if we study (1.2), the inequality may not necessarily hold true, then the boundedness of (u_n) fails to obtain. However, we believe system (1.2) is an interesting problem, we shall consider it further in our future work.

The paper is organized as follows. In Section 2, we recall some lemmas which we will use in the paper. In Section 3, we show some estimates and prove a technical result. In Section 4, we apply the deformation lemma to provide the proof of Theorem 1.1.

2 Preliminaries

If $A \subset \mathbb{R}^3$, we denote by $|u|_{L^q(A)}$ the $L^q(A)$-norm of a function $u : \mathbb{R}^3 \to \mathbb{R}$, and by $|u|_q$ its $L^q(\mathbb{R}^3)$-norm. Let us define $D^{s,2}(\mathbb{R}^3)$ as the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to

$$[u]^2 = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \mathrm{d}x\mathrm{d}y.$$

Then, we consider the fractional Sobolev space

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2 \left(\mathbb{R}^3\right) : [u] < \infty \right\},$$

endowed with the norm

$$\|u\|^2 = [u]^2 + |u|_2^2.$$

Now, we recall the following main embeddings for the fractional Sobolev spaces, see Di Nezza–Palatucci–Valdinoci [13].

Lemma 2.1. *Let $s \in (0,1)$. Then $H^s(\mathbb{R}^3)$ is continuously embedded in $L^p(\mathbb{R}^3)$ for any $p \in [2, 2_s^*]$ and compactly in $L^p_{loc}(\mathbb{R}^3)$ for any $p \in [1, 2_s^*)$ with $2_s^* = \frac{6}{3-2s}$.*

We also recall a version of the well-known concentration-compactness principle, see Felmer–Quaas–Tan [14].

Lemma 2.2. *If (u_n) is a bounded sequence in $H^s(\mathbb{R}^3)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \,\mathrm{d}x = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for all $r \in (2, 2_s^*)$.

By Lemma 2.1, we have

$$H^s(\mathbb{R}^3) \subset L^{\frac{12}{3+2t}}(\mathbb{R}^3). \tag{2.1}$$

For any fixed $u \in H^s(\mathbb{R}^3)$, $L_u : D^{t,2}(\mathbb{R}^3) \to \mathbb{R}$ be the functional given by

$$L_u(v) = \int_{\mathbb{R}^3} u^2 v \mathrm{d}x,$$

which is continuous in view of the Hölder inequality and (2.1). Indeed

$$|L_u(v)| \leq \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}} \mathrm{d}x\right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |v|^{2_t^*} \mathrm{d}x\right)^{\frac{1}{2_t^*}} \leq C\|u\|^2 \|v\|_{D^{t,2}},$$

where

$$\|v\|^2_{D^{t,2}} = \iint_{\mathbb{R}^6} \frac{|v(x) - v(y)|^2}{|x-y|^{3+2t}} \mathrm{d}x\mathrm{d}y.$$

Then, by the Lax–Milgram Theorem there is a unique $\phi_u^t \in D^{t,2}(\mathbb{R}^3)$, such that $\langle \phi_u^t, v\rangle$ for each $v \in D^{t,2}(\mathbb{R}^3)$, where $\langle \cdot, \cdot \rangle$ is the inner product on $D^{t,2}(\mathbb{R}^3)$. Thus, we obtain the t-Riesz formula

$$\phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} \mathrm{d}y, \quad \text{where } c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(3-2t)}{\Gamma(t)},$$

is the only weak solution of the problem

$$(-\Delta)^t \phi_u^t = u^2 \quad \text{in } \mathbb{R}^3.$$

Then, we state the following useful properties whose proofs can be found in Liu–Zhang [19] and Teng [29]:

Lemma 2.3. *For all $u \in H^s(\mathbb{R}^3)$, then the following properties hold:*

(1) $\|\phi_u^t\|_{D^{t,2}} \leq C|u|^2_{\frac{12}{3+2t}} \leq C\|u\|^2$ *and* $\int_{\mathbb{R}^3} \phi_u^t u^2 \mathrm{d}x \leq C_t |u|^4_{\frac{12}{3+2t}}$. *Moreover* $\phi_u^t : H^s(\mathbb{R}^3) \to D^{t,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets;*

(2) $\phi_u^t \geq 0$ *in* $\mathbb{R}^3$*;*

(3) *if* $y \in \mathbb{R}^3$ *and* $\bar{u}(x) = u(x+y)$, *then* $\phi^t_{\bar{u}}(x) = \phi^t_u(x+y)$ *and* $\int_{\mathbb{R}^3} \phi^t_{\bar{u}} \bar{u}^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi^t_u u^2 \mathrm{d}x$;

(4) $\phi^t_{ru} = r^2 \phi^t_u$ *for all* $r \in \mathbb{R}$;

(5) *if* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi^t_{u_n} \rightharpoonup \phi^t_u$ *in* $D^{t,2}(\mathbb{R}^3)$;

(6) *if* $u_n \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$, *then* $\int_{\mathbb{R}^3} \phi^t_{u_n} u^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi^t_{(u_n-u)} (u_n-u)^2 \mathrm{d}x + \int_{\mathbb{R}^3} \phi^t_u u^2 \mathrm{d}x + o_n(1)$;

(7) *if* $u_n \to u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi^t_{u_n} \to \phi^t_u$ *in* $D^{t,2}(\mathbb{R}^3)$ *and* $\int_{\mathbb{R}^3} \phi^t_{u_n} u^2 \mathrm{d}x \to \int_{\mathbb{R}^3} \phi^t_u u^2 \mathrm{d}x$.

In order to study system (1.1), we use the change of variable $x \to \epsilon x$, and the system (1.1) is equivalent to the easier handle system

$$\begin{cases} (-\Delta)^s u + V(\epsilon x)u - \phi(\epsilon x)u = u \log u^2 & \text{in } \mathbb{R}^3, \\ (-\Delta)^t \phi = |u|^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.2}$$

Substituting $\phi^t = \phi^t_u$ into system (2.2), we can rewrite (2.2) as a single equation

$$(-\Delta)^s u + V(\epsilon x)u - \phi^t_u u = u \log u^2 \quad \text{in } \mathbb{R}^3. \tag{2.3}$$

We shall use the variational method to study the problem (2.3). Note that, a weak solution of (2.3) in $H^s(\mathbb{R}^3)$ is a critical point of the associated energy functional

$$\mathcal{I}_\epsilon(u) := \frac{1}{2}\|u\|^2_\epsilon - \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_u |u|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u^2 \log u^2 \mathrm{d}x,$$

defined for all $u \in \mathcal{H}_\epsilon$ where

$$\mathcal{H}_\epsilon := \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\epsilon x) u^2 \mathrm{d}x < \infty \right\}$$

is endowed with the norm

$$\|u\|^2_\epsilon := [u]^2 + \int_{\mathbb{R}^3} (V(\epsilon x) + 1)\, u^2 \mathrm{d}x.$$

Obviously, $\mathcal{H}_\epsilon$ is a Hilbert space with inner product

$$(u,v)_\epsilon = \iint_{\mathbb{R}^6} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} (V(\epsilon x) + 1)\, uv \mathrm{d}x.$$

Definition 2.4. A solution of the problem (2.3) is a function $u \in H^s(\mathbb{R}^3)$ such that $u^2 \log u^2 \in L^1(\mathbb{R}^3)$ and

$$\iint_{\mathbb{R}^6} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V(\epsilon x) uv \mathrm{d}x \\ - \int_{\mathbb{R}^3} \phi^t_u uv \mathrm{d}x = \int_{\mathbb{R}^3} uv \log u^2 \mathrm{d}x, \quad \forall u, v \in C^\infty_0(\mathbb{R}^3).$$

Due to the lack of smoothness of $\mathcal{I}_\epsilon$, we shall use the approach explored in Ji–Szulkin [16] and Squassina–Szulkin [26]. Let us decompose it into a sum of a C^1 functional plus a convex lower semicontinuous functional, respectively. For $\delta > 0$, let us define the following functions:

$$F_1(\xi) = \begin{cases} 0, & \text{if } \xi = 0, \\ -\frac{1}{2}\xi^2 \log \xi^2 & \text{if } 0 < |\xi| < \delta, \\ -\frac{1}{2}\xi^2 \left(\log \delta^2 + 3\right) + 2\delta|\xi| - \frac{1}{2}\delta^2, & \text{if } |\xi| \geq \delta \end{cases}$$

and

$$F_2(\xi) = \begin{cases} 0, & \text{if } |\xi| < \delta, \\ \frac{1}{2}\xi^2 \log\left(\xi^2/\delta^2\right) + 2\delta|\xi| - \frac{3}{2}\xi^2 - \frac{1}{2}\delta^2, & \text{if } |\xi| \geq \delta. \end{cases}$$

Then,

$$F_2(\xi) - F_1(\xi) = \frac{1}{2}\xi^2 \log \xi^2, \quad \forall \xi \in \mathbb{R},$$

and the functional $\mathcal{I}_\epsilon : \mathcal{H}_\epsilon \to (-\infty, +\infty]$ may be rewritten as

$$\mathcal{I}_\epsilon(u) = \Phi_\epsilon(u) + \Psi(u), \quad u \in \mathcal{H}_\epsilon, \tag{2.4}$$

where

$$\Phi_\epsilon(u) = \frac{1}{2}\|u\|_\epsilon^2 - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t |u|^2 \mathrm{d}x - \int_{\mathbb{R}^3} F_2(u)\mathrm{d}x,$$

and

$$\Psi(u) = \int_{\mathbb{R}^3} F_1(u)\mathrm{d}x,$$

As proven in Ji–Szulkin [16] and Squassina–Szulkin [26], $F_1, F_2 \in C^1(\mathbb{R}, \mathbb{R})$. If $\delta > 0$ is small enough, F_1 is convex, even,

$$F_1(\xi) \geq 0 \quad \text{and} \quad F_1'(\xi)\xi \geq 0, \quad \forall \xi \in \mathbb{R}.$$

For each fixed $p \in (2, 2_s^*)$, there exists $C > 0$ such that

$$|F_2'(\xi)| \leq C|\xi|^{p-1}, \quad \forall \xi \in \mathbb{R}.$$

If potential V in (2.3) is replaced by a constant $A > -1$, we have the following problem

$$(-\Delta)^s u + Au - \phi_u^t u = u \log u^2 \quad \text{in } \mathbb{R}^3. \tag{2.5}$$

And the corresponding energy functional associated to (2.5) will be denoted by $\mathcal{I}_A : \mathcal{H}_\epsilon \to (-\infty, +\infty]$ and defined as

$$\mathcal{I}_A(u) = \frac{1}{2}[u]^2 + \frac{1}{2}\int_{\mathbb{R}^3} (A+1)u^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t |u|^2 \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} u^2 \log u^2 \mathrm{d}x.$$

Moreover, let us denote by $m(A)$ the mountain pas level associated with $\mathcal{I}_A$, which possesses the following characterizations

$$m(A) = \inf_{u \in \mathcal{H}_\epsilon \setminus \{0\}} \left\{ \max_{t \geq 0} \mathcal{I}_A(tu) \right\} = \inf_{u \in \mathcal{M}_A} \mathcal{I}_A(u),$$

where $\mathcal{M}_A$ is the Nehari Manifold associated with $\mathcal{I}_A$, given by

$$\mathcal{M}_A = \left\{ u \in \mathcal{H}_\epsilon \setminus \{0\} : \mathcal{I}_A'(u)u = 0 \right\}.$$

3 Technical results

In the section, we recall some definitions that can be found in Szulkin [28].

Definition 3.1. Let E be a Banach space, E' be the dual space of E and $\langle \cdot, \cdot \rangle$ be the duality paring between E' and E. Let $J : E \to \mathbb{R}$ be a functional of the form $J(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1(E, \mathbb{R})$ and Ψ is convex and lower semicontinuous. Let us list some definitions:

(i) The sub-differential $\partial J(u)$ of the functional J at a point $u \in E$ is the following set

$$\left\{w \in E' : \left\langle \Phi'(u), v-u \right\rangle + \Psi(v) - \Psi(u) \geq \langle w, v-u \rangle, \forall v \in E \right\}; \tag{3.1}$$

(ii) A critical point of J is a point $u \in E$ such that $J(u) < +\infty$ and $0 \in \partial J(u)$, i.e.

$$\left\langle \Phi'(u), v-u \right\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in E; \tag{3.2}$$

(iii) A Palais–Smale sequence at level d for J is a sequence $(u_n) \subset E$ such that $J(u_n) \to d$ and there exists a numerical sequence $\tau_n \to 0^+$ with

$$\left\langle \Phi'(u_n), v-u_n \right\rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n \|v-u_n\|, \quad \forall v \in E;$$

(iv) The functional J satisfies the Palais–Smale condition at level d ($(\mathrm{PS})_d$ condition, for short) if all Palais–Smale sequences at level d have a convergent subsequence;

(v) The effective domain of J is the set $D(J) = \{u \in E : J(u) < +\infty\}$.

In what follows, for each $u \in D(\mathcal{I}_\epsilon)$, we set the functional $\mathcal{I}'_\epsilon(u) : \mathcal{H}_{\epsilon,c} \to \mathbb{R}$ given by

$$\left\langle \mathcal{I}'_\epsilon(u), z \right\rangle = \left\langle \Phi'_\epsilon(u), z \right\rangle - \int F'_1(u) z \mathrm{d}x, \quad \forall z \in \mathcal{H}_{\epsilon,c},$$

where

$$\mathcal{H}_{\epsilon,c} = \{u \in \mathcal{H}_\epsilon : u \text{ has compact support } \},$$

and define

$$\|\mathcal{I}'_\epsilon(u)\| = \sup\left\{\left\langle \mathcal{I}'_\epsilon(u), z \right\rangle : z \in \mathcal{H}_{\epsilon,c} \text{ and } \|z\|_\epsilon \leq 1\right\}.$$

If $\|\mathcal{I}'_\epsilon(u)\|$ is finite, then $\mathcal{I}'_\epsilon(u)$ may be extended to a bounded operator in $\mathcal{H}_\epsilon$, and so, it can be seen as an element of $\mathcal{H}'_\epsilon$.

Lemma 3.2. *Let $\mathcal{I}_\epsilon$ satisfy (2.4), then:*

(i) *If $u \in D(\mathcal{I}_\epsilon)$ is a critical point of $\mathcal{I}_\epsilon$. Then, the following hold:*

$$\left\langle \Phi'_\epsilon(u), v-u \right\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in \mathcal{H}_\epsilon;$$

(ii) *For each $u \in D(\mathcal{I}_\epsilon)$ such that $\|\mathcal{I}'_\epsilon(u)\| < +\infty$, we have $\partial \mathcal{I}_\epsilon(u) \neq \emptyset$, that is, there exists $w \in \mathcal{H}'_\epsilon$, which is denoted by $w = \mathcal{I}'_\epsilon(u)$, such that*

$$\left\langle \Phi'_\epsilon(u), v-u \right\rangle + \int_{\mathbb{R}^3} F_1(v) \mathrm{d}x - \int_{\mathbb{R}^3} F_1(u) \mathrm{d}x \geq \langle w, v-u \rangle, \quad \forall v \in \mathcal{H}_\epsilon;$$

(iii) *If a function $u \in D(\mathcal{I}_\epsilon)$ is a critical point of $\mathcal{I}_\epsilon$, then u is a solution of (2.3);*

(iv) *If $(u_n) \subset \mathcal{H}_\epsilon$ is a Palais–Smale sequence, then*

$$\left\langle \mathcal{I}'_\epsilon(u_n), z \right\rangle = o_n(1) \|z\|_\epsilon, \quad \forall z \in \mathcal{H}_{\epsilon,c};$$

(v) *If Ω is a bounded domain with regular boundary, then Ψ (and hence $\mathcal{I}_\epsilon$) is of class C^1 in $H^s(\Omega)$. More precisely, the functional*

$$\Psi(u) = \int_\Omega F_1(u) \mathrm{d}x, \quad \forall u \in H^s(\Omega)$$

belongs to $C^1(H^s(\Omega), \mathbb{R})$.

Proof. (i) follows from (3.2). (ii) can be obtained arguing as in the proof of Squassina–Szulkin [27] and recalling that $C_c^\infty(\mathbb{R}^3)$ is dense in $\mathcal{H}_\epsilon$. (iii) and (iv) follow the same lines of the proofs of Ji–Szulkin [16]. To verify (v), since $|F_1'(\tau)| \le C(1+|\tau|^{q-1})$ with $q \in (2, 2_s^*)$, it is enough to proceed as in the proof of Willem [30]. □

As a consequence of the above proprieties, we have the following result.

Lemma 3.3. *If $u \in D(\mathcal{I}_\epsilon)$ and $\|\mathcal{I}_\epsilon'(u)\| < +\infty$, then $F_1'(u)u \in L^1(\mathbb{R}^3)$.*

Proof. Let $\varpi \in C_c^\infty(\mathbb{R}^3)$ be such that $0 \le \varpi \le 1$ in $\mathbb{R}^3$, $\varpi(x) = 1$ for $|x| \le 1$ and $\varpi(x) = 0$ for $|x| \ge 2$. For $R > 0$ and $u \in D(\mathcal{I}_\epsilon)$, let $\varpi_R(x) = \varpi(\frac{x}{R})$ and $u_R(x) = \varpi_R(x)u(x)$. Let us prove that

$$\lim_{R\to\infty} \|u_R - u\|_\epsilon = 0. \tag{3.3}$$

Clearly, $u_R \to u$ in $L^2(\mathbb{R}^3)$. On the other hand,

$$\begin{aligned}[u_R - u]^2 &\le 2\left[\iint_{\mathbb{R}^6} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}}|u(x)|^2 \mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}}|\varpi_R(x) - 1|^2 \mathrm{d}x\mathrm{d}y\right] \\ &= 2\left[\mathcal{A}_R + \mathcal{B}_R\right].\end{aligned}$$

Since

$$\begin{aligned}\mathcal{A}_R &= \iint_{\mathbb{R}^6} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}}|u(x)|^2 \mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^3} |u(x)|^2 \left(\int_{|x-y|>R} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x + \int_{|x-y|\le R} \frac{|\varpi_R(x) - \varpi_R(y)|^2}{|x-y|^{3+2s}}\mathrm{d}x\right)\mathrm{d}y \\ &\le \int_{\mathbb{R}^3} |u(x)|^2 \left(\int_{|x-y|>R} \frac{4\|\varpi\|^2_{L^\infty(\mathbb{R}^3)}}{|x-y|^{3+2s}}\mathrm{d}x + R^{-2}\int_{|x-y|\le R} \frac{\|\nabla\varpi\|^2_{L^\infty(\mathbb{R}^3)}}{|x-y|^{3+2s}}\mathrm{d}x\right)\mathrm{d}y \\ &\le C\int_{\mathbb{R}^3} |u(x)|^2 \mathrm{d}y \left(\int_R^\infty \frac{1}{r^{2s+1}}\mathrm{d}r + R^{-2}\int_0^R \frac{1}{r^{2s-1}}\mathrm{d}r\right) \\ &\le \frac{C}{R^{2s}},\end{aligned}$$

it follows that $0 \le \mathcal{A}_R \to 0$. Moreover, $\mathcal{B}_R \to 0$ by the dominated convergence theorem. Then, (3.3) holds.

From Lemma 3.2-(ii),

$$\langle \Phi_\epsilon'(u), u_R\rangle + \int_{\mathbb{R}^3} F_1'(u_R)u_R \mathrm{d}x = \langle w, u_R\rangle, \quad \forall w \in \mathcal{H}_\epsilon'. \tag{3.4}$$

Then, combining (3.3), (3.4) with Lemma 3.2-(v), we can see that $\int_{\mathbb{R}^3} F_1'(u)u_R \mathrm{d}x \le C$ for large $R > 0$. From $u_R \to u$ a.e. in $\mathbb{R}^3$ as $R \to \infty$ and Fatou's lemma, we derive that

$$\int_{\mathbb{R}^3} F_1'(u)u\mathrm{d}x \le \liminf_{R\to\infty} \int_{\mathbb{R}^3} F_1'(u)u_R \mathrm{d}x \le \liminf_{R\to\infty} \int_{\mathbb{R}^3} F_1'(u)\varpi_R \mathrm{d}x \le C$$

The proof has been completed. □

An immediate consequence of the last lemma is the following.

Corollary 3.4. *For each $u \in D(\mathcal{I}_\epsilon)\backslash\{0\}$ with $\|\mathcal{I}'_\epsilon(u)\| < +\infty$, we have that*

$$\mathcal{I}'_\epsilon(u)u = [u]^2 + \int_{\mathbb{R}^3} V(\epsilon x)u^2 \mathrm{d}x - \int_{\mathbb{R}^3} \phi_u^t u^2 \mathrm{d}x - \int_{\mathbb{R}^3} u^2 \log u^2 \mathrm{d}x,$$

and

$$\mathcal{I}_\epsilon(u) - \frac{1}{2}\mathcal{I}'_\epsilon(u)u = \frac{1}{2}\int_{\mathbb{R}^3} u^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^t u^2 \mathrm{d}x.$$

Corollary 3.5. *If $(u_n) \subset \mathcal{H}_\epsilon$ is a* (PS) *sequence for $\mathcal{I}_\epsilon$, then $\mathcal{I}'_\epsilon(u_n)u_n = o_n(1)\|u_n\|_\epsilon$. If (u_n) is bounded, we have*

$$\begin{aligned}\mathcal{I}_\epsilon(u_n) &= \mathcal{I}_\epsilon(u_n) - \frac{1}{2}\mathcal{I}'_\epsilon(u_n)u_n + o_n(1)\|u_n\|_\epsilon \\ &= \frac{1}{2}\int_{\mathbb{R}^3} u_n^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 \mathrm{d}x + o_n(1)\|u_n\|_\epsilon, \forall n \in \mathbb{N}.\end{aligned}$$

Corollary 3.6. *If $u \in \mathcal{H}_\epsilon$ is a critical point of $\mathcal{I}_\epsilon$ and $v \in \mathcal{H}_\epsilon$ verifies $F_1'(u)v \in L^1(\mathbb{R}^3)$, then $\mathcal{I}'_\epsilon(u)v = 0$.*

Now, we will prove some results that will be useful in the proof of Theorem 1.1.

Lemma 3.7. *For any $\epsilon > 0$, all* (PS) *sequences of $\mathcal{I}_\epsilon$ are bounded in $\mathcal{H}_\epsilon$.*

Proof. Let (u_n) be a $(\mathrm{PS})_d$ sequence. By Corollary 3.5, one concludes

$$\begin{aligned}\int_{\mathbb{R}^3} u_n^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{u_n}^t u_n^2 \mathrm{d}x &= 2\mathcal{I}_\epsilon(u_n) - \mathcal{I}'_\epsilon(u_n)u_n \\ &= 2d + o_n(1) + o_n(1)\|u_n\|_\epsilon \\ &\leq C + o_n(1)\|u_n\|_\epsilon,\end{aligned}$$

for some $C > 0$. Consequently,

$$\|u_n\|_\epsilon^2 \leq C + o_n(1)\|u_n\|_\epsilon. \tag{3.5}$$

Let us employ the following logarithmic Sobolev inequality found in Lieb–Loss [17],

$$\int_{\mathbb{R}^3} u^2 \log u^2 \mathrm{d}x \leq \frac{a^2}{\pi}|\nabla u|_2^2 + \left(\log|u|_2^2 - 3(1+\log a)\right)|u|_2^2, \tag{3.6}$$

for all $a > 0$. Fixing $\frac{a^2}{\pi} = \frac{1}{4}$ and $\xi \in (0,1)$, the inequalities (3.5) and (3.6) yield that

$$\begin{aligned}\int_{\mathbb{R}^3} u_n^2 \log u_n^2 \mathrm{d}x &\leq \frac{1}{4}|\nabla u_n|_2^2 + C\left(\log|u_n|_2^2 + 1\right)|u_n|_2^2 \\ &\leq \frac{1}{4}|\nabla u_n|_2^2 + C_1\left(1+\|u_n\|_\epsilon\right)^{1+\xi}.\end{aligned} \tag{3.7}$$

Then by (3.7), we have that

$$\begin{aligned}d + o_n(1) &= \mathcal{I}_\epsilon(u_n) - \frac{1}{4}\mathcal{I}'_\epsilon(u_n)u_n \\ &\geq \frac{1}{4}\|u_n\|_\epsilon^2 - \frac{1}{4}\int_{\mathbb{R}^3} u_n^2 \log u_n^2 \mathrm{d}x \\ &\geq C\left(\|u_n\|_\epsilon^2 - (1+\|u_n\|_\epsilon)^{1+\xi}\right),\end{aligned}$$

which shows that the sequence (u_n) is bounded. □

Lemma 3.8. *Suppose that V satisfies (V_1)–(V_3). For each $\sigma > 0$, there exists $\epsilon_0 = \epsilon_0(\sigma) > 0$ such that, if (u_n) is a $(\mathrm{PS})_c$ sequence for $\mathcal{I}_\epsilon$ with $c \in (m(c_0) + \sigma/2, 2m(c_0) - \sigma)$ and $\epsilon \in (0, \epsilon_0)$, then (u_n) has a weak limit $u_0 \neq 0$.*

Proof. We shall prove the lemma arguing by contradiction, by supposing that there exists $\sigma > 0$, a sequence $\epsilon_n \to 0$ and $(u_m^n) \subset \mathcal{H}_\epsilon$ such that

$$\lim_{m\to+\infty} \mathcal{I}_{\epsilon_n}(u_m^n) = c_n \quad \text{and} \quad \lim_{m\to+\infty} \|\mathcal{I}'_{\epsilon_n}(u_m^n)\| = 0,$$

with $u_m^n \rightharpoonup 0$, as $m \to +\infty$.

Claim I: There exists $\delta > 0$, such that

$$\liminf_{m\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_m^n|^2 \,\mathrm{d}x \geq \delta, \quad \forall n \in \mathbb{N}.$$

Indeed, if the Claim does not hold, there is $(n_j) \subset \mathbb{N}$ satisfying

$$\liminf_{m\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} \left|u_m^{n_j}\right|^2 \mathrm{d}x \leq \frac{1}{j}, \quad \forall j \in \mathbb{N}.$$

Then, for each $j \in \mathbb{N}$, there is m_j large enough such that

$$\sup_{y\in\mathbb{R}^3} \int_{B_R(y)} |u_{m_j}^{n_j}|^2 \mathrm{d}x \leq \frac{2}{j}, \quad |\mathcal{I}_{\epsilon_n}(u_{n_m}^{n_j}) - c_{n_j}| \leq \frac{1}{j}, \quad \text{and} \quad \left\|\mathcal{I}'_{\epsilon_n}(u_{m_j}^{n_j})\right\| \leq \frac{1}{j}, \quad \forall j \in \mathbb{N}. \tag{3.8}$$

Setting $w_j = u_{m_j}^{n_j}$, it shows that (w_j) is a bounded sequence, and by Lions [18],

$$\limsup_{j\to+\infty} |w_j|_p = 0, \quad \forall p \in (2, 2_s^*).$$

Then, we can see

$$\limsup_{j\to+\infty} \int_{\mathbb{R}^3} F_2'(w_j)\, w_j \mathrm{d}x = 0 \quad \text{and} \quad \limsup_{j\to+\infty} \int_{\mathbb{R}^3} \phi_{w_j}^t w_j^2 \mathrm{d}x = 0.$$

On the other hand, it follows from (3.8) that

$$\begin{aligned} \|w_j\|_{\epsilon_{n_j}}^2 + \int_{\mathbb{R}^3} F_1'(w_j) w_j \mathrm{d}x &= \mathcal{I}'_{\epsilon_{n_j}}(w_j) w_j + \int_{\mathbb{R}^3} F_2'(w_j) w_j \mathrm{d}x + \int_{\mathbb{R}^3} \phi_{w_j}^t w_j^2 \mathrm{d}x \\ &\leq o_j(1) \|w_j\|_{\epsilon_{n_j}}, \end{aligned}$$

where it follows that

$$\limsup_{j\to+\infty} \|w_j\|_{\epsilon_{n_j}}^2 = 0 \quad \text{and} \quad \limsup_{j\to+\infty} \int_{\mathbb{R}^3} F_1'(w_j) w_j \mathrm{d}x = 0.$$

Combining this fact with convexity of F_1, we can see that

$$\limsup_{j\to+\infty} \int_{\mathbb{R}^3} F_1(w_j) \mathrm{d}x = 0.$$

The above analysis imply that $\mathcal{I}_{\epsilon_{n_j}}(w_j) \to 0$ as $j \to +\infty$, and so, $c_{n_j} \to 0$ as $j \to +\infty$, which is contradictory because $c_{n_j} > m(c_0) + \sigma/2$ for all $j \in \mathbb{N}$. This proves the Claim I.

For each $n \in \mathbb{N}$, there exists $(z_m^n) \subset \mathbb{R}^3$ such that

$$\int_{B_R(z_m^n)} |u_m^n|^2 \,\mathrm{d}x \geq \frac{\delta}{2}, \quad \forall n \in \mathbb{N}.$$

Since $u_m^n \rightharpoonup 0$ as $m \to +\infty$, we have that $|z_m^n| \to +\infty$ as $m \to +\infty$. From the above study, for each $n \in \mathbb{N}$, we fix $m_n \in \mathbb{N}$ large enough satisfying

$$\int_{B_R(z_m^n)} |u_m^n|^2 \mathrm{d}x \geq \frac{\delta}{2}, \quad |\epsilon_n z_{m_n}^n| \geq n, \quad \|\mathcal{I}'_{\epsilon_n}(u_{m_n}^n)\|_\epsilon \leq \frac{1}{n} \quad \text{and} \quad |\mathcal{I}_{\epsilon_n}(u_{m_n}^n) - c_n| \leq \frac{1}{n}.$$

In what follows, we denote by (z_n) and (u_n) the sequences $(z_{m_n}^n)$ and $(u_{m_n}^n)$ respectively. Then,

$$\int_{B_R(z_n)} |u_n|^2 \mathrm{d}x \geq \frac{\delta}{2}, \quad |\epsilon_n z_n| \geq n, \quad \|\mathcal{I}'_{\epsilon_n}(u_n)\|_\epsilon \leq \frac{1}{n} \quad \text{and} \quad |\mathcal{I}_{\epsilon_n}(u_n) - c_n| \leq \frac{1}{n}.$$

The boundedness of (u_n) follows by standard arguments. Then, for some subsequence, there exists $u \in \mathcal{H}_\epsilon$ such that

$$u_n \rightharpoonup u \quad \text{in } \mathcal{H}_\epsilon.$$

Considering $\omega_n = u_n(\cdot + z_n)$, we have that (ω_n) is bounded in $\mathcal{H}_\epsilon$. Thus, there exists $\omega \in \mathcal{H}_\epsilon$ such that

$$\omega_n \rightharpoonup \omega \quad \text{in } \mathcal{H}_\epsilon,$$

and

$$\int_{B_R(0)} |\omega|^2 \mathrm{d}x = \liminf_{n\to+\infty} \int_{B_R(0)} |\omega_n|^2 \,\mathrm{d}x = \liminf_{n\to+\infty} \int_{B_R(z_n)} |u_n|^2 \mathrm{d}x \geq \frac{\delta}{2},$$

which implies that $\omega \neq 0$.

Now, for each $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have the equality below

$$\begin{aligned} &\iint_{\mathbb{R}^6} \frac{(\omega_n(x) - \omega_n(y))(\varphi(x) - \varphi(y))}{|x-y|^{3+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V(\epsilon_n z_n + \epsilon_n x)\, \omega_n \varphi \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \phi_{\omega_n}^t \omega_n \varphi \mathrm{d}x - \int_{\mathbb{R}^3} \omega_n \varphi \log \omega_n^2 \mathrm{d}x = o_n(1)\|\varphi\|_\epsilon, \end{aligned} \tag{3.9}$$

showing that ω is a nontrivial solution of the problem

$$(-\Delta)^s u + \alpha_1 u - \phi_u^t u = u \log u^2 \quad \text{in } \mathbb{R}^3, \tag{3.10}$$

where

$$\alpha_1 = \lim_{n\to+\infty} V(\epsilon_n z_n).$$

From Cabré–Sire [8], Caffarelli–Silvestre [9] and d'Avenia–Montefusco–Squassina [11], we can see that $\omega \in C^2(\mathbb{R}^3) \cap \mathcal{H}_\epsilon$.

For each $k \in \mathbb{N}$, there is $\varphi_k \in C_0^\infty(\mathbb{R}^3)$ such that

$$\|\varphi_k - \omega\|_\epsilon \to 0 \quad \text{as } k \to +\infty,$$

that is,

$$\|\varphi_k - \omega\|_\epsilon = o_k(1).$$

Using $\frac{\partial\varphi_k}{\partial x_i}$ as a test function of (3.9), we have

$$\iint_{\mathbb{R}^6} \frac{(\omega_n(x)-\omega_n(y))(\frac{\partial\varphi_k}{\partial x_i}(x)-\frac{\partial\varphi_k}{\partial x_i}(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} V(\epsilon_n z_n+\epsilon_n x)\,\omega_n\frac{\partial\varphi_k}{\partial x_i}\mathrm{d}x$$
$$-\int_{\mathbb{R}^3}\phi^t_{\omega_n}\omega_n\frac{\partial\varphi_k}{\partial x_i}\mathrm{d}x - \int_{\mathbb{R}^3}\omega_n\frac{\partial\varphi_k}{\partial x_i}\log\omega_n^2\mathrm{d}x = o_n(1).$$

Observing that

$$\iint_{\mathbb{R}^6} \frac{(\omega_n(x)-\omega_n(y))(\frac{\partial\varphi_k}{\partial x_i}(x)-\frac{\partial\varphi_k}{\partial x_i}(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y$$
$$= \iint_{\mathbb{R}^6} \frac{(\omega(x)-\omega(y))(\frac{\partial\varphi_k}{\partial x_i}(x)-\frac{\partial\varphi_k}{\partial x_i}(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y + o_n(1),$$

$$\int_{\mathbb{R}^3}\phi^t_{\omega_n}\omega_n\frac{\partial\varphi_k}{\partial x_i}\mathrm{d}x = \int_{\mathbb{R}^3}\phi^t_{\omega_n}\omega\frac{\partial\varphi_k}{\partial x_i}\mathrm{d}x + o_n(1),$$

and

$$\int_{\mathbb{R}^3}\omega_n\frac{\partial\varphi_k}{\partial x_i}\log\omega_n^2\mathrm{d}x = \int_{\mathbb{R}^3}\omega\frac{\partial\varphi_k}{\partial x_i}\log\omega^2\mathrm{d}x + o_n(1).$$

Gathering the above limit with (3.10), we derive that

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^3}\left(V(\epsilon_n z_n+\epsilon_n x)-V(\epsilon_n z_n)\right)\omega_n\frac{\partial\varphi_k}{\partial x_i}\mathrm{d}x\right| = 0.$$

As φ_k has compact support, the above limit gives

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^3}\left(V(\epsilon_n z_n+\epsilon_n x)-V(\epsilon_n z_n)\right)\omega\frac{\partial\varphi_k}{\partial x_i}\mathrm{d}x\right| = 0.$$

Recalling that $\frac{\partial\omega}{\partial x_i}\in L^2(\mathbb{R}^3)$, we have that $(\frac{\partial\varphi_k}{\partial x_i})$ is bounded in $L^2(\mathbb{R}^3)$. Then,

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^3}\left(V(\epsilon_n z_n+\epsilon_n x)-V(\epsilon_n z_n)\right)\varphi_k\frac{\partial\varphi_k}{\partial x_i}\mathrm{d}x\right| = o_k(1),$$

and so,

$$\limsup_{n\to+\infty}\left|\frac{1}{2}\int_{\mathbb{R}^3}\left(V(\epsilon_n z_n+\epsilon_n x)-V(\epsilon_n z_n)\right)\frac{\partial\left(\varphi_k^2\right)}{\partial x_i}\mathrm{d}x\right| = o_k(1).$$

Using Green's Theorem together with the fact that φ_k has compact support, we get the limit below

$$\limsup_{n\to+\infty}\left|\int_{\mathbb{R}^3}\frac{\partial V}{\partial x_i}(\epsilon_n z_n+\epsilon_n x)\,\varphi_k^2\mathrm{d}x\right| = o_k(1),$$

which combined with (V_2) loads to

$$\limsup_{n\to+\infty}\left|\frac{\partial V}{\partial x_i}(\epsilon_n z_n)\int_{\mathbb{R}^3}|\varphi_k|^2\mathrm{d}x\right| = o_k(1).$$

As

$$\int_{\mathbb{R}^3}|\varphi_k|^2\,\mathrm{d}x \to \int_{\mathbb{R}^3}|\omega|^2\mathrm{d}x > 0 \quad \text{as } k\to+\infty,$$

it shows that

$$\limsup_{n\to+\infty}\left|\frac{\partial V}{\partial x_i}(\epsilon_n z_n)\right| = o_k(1), \quad \forall i \in \{1,2,3\}.$$

Since k is arbitrary, we obtain

$$\nabla V(\epsilon_n z_n) \to 0 \quad \text{and} \quad V(\epsilon_n z_n) \to \alpha_1,$$

as $n \to \infty$. Therefore, $(\epsilon_n z_n)$ is a $(\mathrm{PS})_{\alpha_1}$ sequence for V, which is a contradiction, because by hypotheses V satisfies the (PS) condition and $(\epsilon_n z_n)$ does not have any convergent subsequence in $\mathbb{R}^3$. Thus, the proof is completed. □

Hereafter, we denote by $\mathcal{N}_\epsilon$ the Nehari manifold associated with $\mathcal{I}_\epsilon$, that is,

$$\mathcal{N}_\epsilon = \{u \in \mathcal{H}_\epsilon \backslash \{0\} : \mathcal{I}'_\epsilon(u)u = 0\}.$$

The next lemma will be crucial in our study to show an important estimate.

Lemma 3.9. *Let $\epsilon_n \to 0$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$ such that $\mathcal{I}_{\epsilon_n}(u_n) \to m(c_0)$. Then, there are $(z_n) \subset \mathbb{R}^3$ with $|z_n| \to +\infty$ and $u_1 \in \mathcal{H}_\epsilon \backslash \{0\}$ such that*

$$u_n(\cdot + z_n) \to u_1 \quad \text{in } \mathcal{H}_\epsilon.$$

Moreover,

$$\liminf_{n\to+\infty} |\epsilon_n z_n| > 0.$$

Proof. Since $u_n \in \mathcal{N}_{\epsilon_n}$, we can see that $\mathcal{I}'_{c_0}(u_n)\, u_n \le 0$ and $\mathcal{I}_{c_0}(u) \le \mathcal{I}_{\epsilon_n}(u)$ for all $u \in \mathcal{H}_\epsilon$ and $n \in \mathbb{N}$. Then, there exists $\tau_n \in (0,1]$ such that

$$(\tau_n u_n) \subset \mathcal{N}_{c_0} \quad \text{and} \quad \mathcal{I}_{c_0}(\tau_n u_n) \to m(c_0).$$

Since (τ_n) is bounded, by Alves–de Morais Filho [2], there exist $(z_n) \subset \mathbb{R}^3$, $u_1 \in \mathcal{H}_\epsilon \backslash \{0\}$, and a subsequence of (u_n), still denote by (u_n), verifying

$$u_n(\cdot + z_n) \to u_1 \quad \text{in } \mathcal{H}_\epsilon.$$

Claim II:

$$\liminf_{n\to+\infty} |\epsilon_n z_n| > 0.$$

Indeed, since $u_n \in \mathcal{N}_{\epsilon_n}$ for all $n \in N$, the function $u_n^1 = u_n(\cdot + z_n)$ must verify

$$\begin{aligned}[t] [u_n^1]^2 + \int_{\mathbb{R}^3} V(\epsilon_n z_n + \epsilon_n x)\left|u_n^1\right|^2 \mathrm{d}x - \int_{\mathbb{R}^3} \phi^t_{u_n^1}|u_n^1|^2 \mathrm{d}x + \int_{\mathbb{R}^3} F_1'\left(u_n^1\right) u_n^1 \mathrm{d}x \\ = \int_{\mathbb{R}^3} F_2'\left(u_n^1\right) u_n^1 \mathrm{d}x. \end{aligned} \tag{3.11}$$

Since F_1 is convex, even and $F_1(\tau) \ge F_1(0) = 0$ for all $\tau \in \mathbb{R}$, we can derive that $0 \le F_1(\tau) \le F_1'(\tau)\tau$ for all $\tau \in \mathbb{R}$. Supposing by contradiction that for some subsequence

$$\lim_{n\to+\infty} \epsilon_n z_n = 0.$$

Taking the limit of $n \to +\infty$ in (3.11), we have

$$[u_1]^2 + \int_{\mathbb{R}^3} V(0)\,|u_1|^2 \mathrm{d}x - \int_{\mathbb{R}^3} \phi^t_{u_1}|u_1|^2 \mathrm{d}x + \int_{\mathbb{R}^3} F_1'(u_1)\, u_1 \mathrm{d}x \le \int_{\mathbb{R}^3} F_2'(u_1)\, u_1 \mathrm{d}x.$$

Then, there is $\tau_1 \in (0,1]$ such that $\tau_1 u_1 \in \mathcal{M}_{V(0)}$. Thus, since $V(0) > c_0$, we derive that

$$\mathcal{I}_{V(0)}(\tau_1 u_1) \geq m(V(0)) > m(c_0) > 0. \tag{3.12}$$

On the other hand,

$$\mathcal{I}_{\epsilon_n}(u_n) \to \mathcal{I}_{V(0)}(\tau_1 u_1),$$

which leads to

$$m(c_0) \geq \mathcal{I}_{V(0)}(\tau_1 u_1). \tag{3.13}$$

From (3.12) and (3.13), we can find a contradiction, which finishes the proof. □

4 A special minimax level

To prove Theorem 1.1, we shall consider a special minimax level. To do that, we begin fixing the barycenter function by

$$\beta(u) = \frac{\int_{\mathbb{R}^3} \frac{x}{|x|} |u|^2 \mathrm{d}x}{\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x}, \quad \forall u \in \mathcal{H}_\epsilon \backslash \{0\}.$$

For each $z \in \mathbb{R}^N$ and $\epsilon > 0$, let us define the function

$$\varphi_{\epsilon,z}(x) = \tau_{\epsilon,z} u_0 \left(x - \frac{z}{\epsilon}\right),$$

where $\tau_{\epsilon,z} > 0$ is such that $\varphi_{\epsilon,z} \in \mathcal{N}_\epsilon$ and u_0 is a radial positive ground state solution for $\mathcal{I}_{c_0}$, that is,

$$\mathcal{I}_{c_0}(u_0) = m(c_0) \quad \text{and} \quad \mathcal{I}'_{c_0}(u_0) = 0.$$

In what follows, we set $\Upsilon_\epsilon(z) = \varphi_{\epsilon,z}$ for all $z \in \mathbb{R}^3$.

Lemma 4.1. *The function $\Upsilon_\epsilon : \mathbb{R}^3 \to \mathcal{N}_\epsilon$ is a continuous function.*

Proof. Let $(z_n) \subset \mathbb{R}^3$ and $z \in \mathbb{R}^3$ with $z_n \to z$ in $\mathbb{R}^3$. We must prove that

$$\Upsilon_\epsilon(z_n) \to \Upsilon_\epsilon(z) \quad \text{in } \mathcal{H}_\epsilon.$$

Here, the main point is to prove that

$$\tau_{\epsilon,z_n} \to \tau_{\epsilon,z} \quad \text{in } \mathbb{R}.$$

By definition of τ_{ϵ,z_n} and $\tau_{\epsilon,z}$, they are the unique numbers that satisfy

$$\mathcal{I}_\epsilon\left(\tau_{\epsilon,z_n} u_0\left(\cdot - \frac{z_n}{\epsilon}\right)\right) = \frac{1}{2}\int_{\mathbb{R}^3} \left|\tau_{\epsilon,z_n} u_0\left(x - \frac{z_n}{\epsilon}\right)\right|^2 \mathrm{d}x,$$

and

$$\mathcal{I}_\epsilon\left(\tau_{\epsilon,z} u_0\left(\cdot - \frac{z}{\epsilon}\right)\right) = \frac{1}{2}\int_{\mathbb{R}^3} \left|\tau_{\epsilon,z} u_0\left(x - \frac{z}{\epsilon}\right)\right|^2 \mathrm{d}x,$$

that is,

$$\begin{aligned} &\frac{1}{2}[\tau_{\epsilon,z_n} u_0]^2 + \frac{1}{2}\int_{\mathbb{R}^3} (V(\epsilon x + z_n) + 1) |\tau_{\epsilon,z_n} u_0|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{\tau_{\epsilon,z_n} u_0} |\tau_{\epsilon,z_n} u_0|^2 \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} F_1(\tau_{\epsilon,z_n} u_0)\,\mathrm{d}x - \int_{\mathbb{R}^3} F_2(\tau_{\epsilon,z_n} u_0)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^3} |\tau_{\epsilon,z_n} u_0|^2 \mathrm{d}x, \end{aligned} \tag{4.1}$$

and

$$\frac{1}{2}[\tau_{\epsilon,z}u_0]^2 + \frac{1}{2}\int_{\mathbb{R}^3} \left(V\left(\epsilon x + z\right) + 1\right)\left|\tau_{\epsilon,z}u_0\right|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{\tau_{\epsilon,z}u_0}|\tau_{\epsilon,z}u_0|^2\mathrm{d}x$$
$$+ \int_{\mathbb{R}^3} F_1\left(\tau_{\epsilon,z}u_0\right)\mathrm{d}x - \int_{\mathbb{R}^3} F_2\left(\tau_{\epsilon,z}u_0\right)\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^3} \left|\tau_{\epsilon,z}u_0\right|^2 \mathrm{d}x.$$

A simple calculation gives that (τ_{ϵ,z_n}) is bounded, thus for some subsequence, we can assume that $\tau_{\epsilon,z_n} \to \tau_*$. Since F_1 is increasing in $[0,+\infty)$ and $F_1(\zeta u_0) \in L^1(\mathbb{R}^3)$ for all $\zeta > 0$, taking the limit of $n \to +\infty$ in (4.1) and using the Lebesgue Theorem, we have

$$\frac{1}{2}[\tau_* u_0]^2 + \frac{1}{2}\int_{\mathbb{R}^3} \left(V\left(\epsilon x + z\right) + 1\right)\left|\tau_* u_0\right|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{\tau_* u_0}|\tau_* u_0|^2\mathrm{d}x$$
$$+ \int_{\mathbb{R}^3} F_1\left(\tau_* u_0\right)\mathrm{d}x - \int_{\mathbb{R}^3} F_2\left(\tau_* u_0\right)\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^3} \left|\tau_* u_0\right|^2 \mathrm{d}x,$$

By uniqueness of $\tau_{\epsilon,z}$, it shows that $\tau_{\epsilon,z} = \tau_*$, and so, $\tau_{\epsilon,z_n} \to \tau_{\epsilon,z}$. Then, since

$$u_0\left(\cdot - \frac{z_n}{\epsilon}\right) \to u_0\left(\cdot - \frac{z}{\epsilon}\right) \quad \text{in } \mathcal{H}_\epsilon,$$

the proof is completed. □

We establish several properties involving β and Υ_ϵ.

Lemma 4.2. *For each $r > 0$, we have*

$$\lim_{\epsilon\to 0}\left(\sup\left\{\left|\beta\left(\Upsilon_\epsilon(z)\right) - \frac{z}{|z|}\right| : |z| \geq r\right\}\right) = 0.$$

Proof. It is enough to show that for any $(z_n) \subset \mathbb{R}^3$ with $|z_n| \geq r$ and $\epsilon_n \to 0$, we have that

$$\left|\beta\left(\Upsilon_{\epsilon_n}(z_n)\right) - \frac{z_n}{|z_n|}\right| \to 0 \quad \text{as } n \to +\infty.$$

By change of variables,

$$\left|\beta\left(\Upsilon_{\epsilon_n}(z_n)\right) - \frac{z_n}{|z_n|}\right| = \frac{\int_{\mathbb{R}^3}\left|\frac{\epsilon_n x + z_n}{|\epsilon_n x + z_n|} - \frac{z_n}{|z_n|}\right| |u_0(x)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^3}|u_0(x)|^2\,\mathrm{d}x}.$$

Since for each $x \in \mathbb{R}^3$, we have

$$\left|\frac{\epsilon_n x + z_n}{|\epsilon_n x + z_n|} - \frac{z_n}{|z_n|}\right| \to 0 \quad \text{as } n \to +\infty,$$

by the Lebesgue Dominated Convergence Theorem, we get that

$$\int_{\mathbb{R}^3}\left|\frac{\epsilon_n x + z_n}{|\epsilon_n x + z_n|} - \frac{z_n}{|z_n|}\right| |u_0(x)|^2\,\mathrm{d}x \to 0 \quad \text{as } n \to +\infty,$$

which completes the proof. □

As a by-product of the arguments explored in the proof of the last lemma, we have

Corollary 4.3. *Fixed $r > 0$, there is $\epsilon_0 > 0$ such that*

$$\left(\beta\left(\Upsilon_\epsilon(z)\right), z\right) > 0, \quad \forall |z| \geq r \text{ and } \epsilon \in (0,\epsilon_0).$$

Proof. By Lemma 4.2, for fixed $r > 0$, there is $\epsilon_0 > 0$ such that

$$\left|\beta\left(\Upsilon_\epsilon(z)\right) - \frac{z}{|z|}\right| < \frac{1}{2}, \quad \forall |z| \geq r \text{ and } \epsilon \in (0, \epsilon_0).$$

On the other hand,

$$\begin{aligned}(\beta\left(\Upsilon_\epsilon(z)\right), z) &= \left(\beta\left(\Upsilon_\epsilon(z)\right) - \frac{z}{|z|}, z\right) + \left(\frac{z}{|z|}, z\right) \\ &= \left(\beta\left(\Upsilon_\epsilon(z)\right) - \frac{z}{|z|}, z\right) + |z|, \quad \forall z \in \mathbb{R}^3 \backslash \{0\}.\end{aligned}$$

Therefore, for $|z| \geq r$, we have

$$(\beta\left(\Upsilon_\epsilon(z)\right), z) \geq |z| \left(1 - \left|\beta\left(\Upsilon_\epsilon(z)\right) - \frac{z}{|z|}\right|\right) > \frac{|z|}{2} \geq \frac{r}{2} > 0,$$

showing the corollary. □

In the sequel, we define the set

$$\mathcal{B}_\epsilon = \{u \in \mathcal{N}_\epsilon : \beta(u) \in Y\}.$$

Note that $\mathcal{B}_\epsilon \neq \emptyset$, since $\beta\left(\varphi_{\epsilon,0}\right) = 0 \in Y$, for all $\epsilon > 0$. Associated with the above set, let us consider the real number D_ϵ given by

$$D_\epsilon = \inf_{u \in \mathcal{B}_\epsilon} \mathcal{I}_\epsilon(u).$$

The next lemma establishes an important relation between the levels D_ϵ and $m(c_0)$.

Lemma 4.4. *The following conclusions are valid:*

(a) *There are ϵ_0 and $\sigma > 0$ such that*

$$D_\epsilon \geq m\left(c_0\right) + \sigma, \quad \forall \epsilon \in (0, \epsilon_0).$$

(b)

$$\limsup_{\epsilon \to 0} \left\{\sup_{x \in X} \mathcal{I}_\epsilon(\Upsilon_\epsilon(x))\right\} < 2m\left(c_0\right) - \sigma.$$

Proof. (a) By the definition of D_ϵ, we can see

$$D_\epsilon \geq m(c_0), \quad \forall \epsilon > 0.$$

Supposing by contradiction that the lemma does not hold, there is $\epsilon_n \to 0$ satisfying

$$D_{\epsilon_n} \to m(c_0).$$

Hence, there exists $u_n \in \mathcal{N}_{\epsilon_n}$ with $\beta(u_n) \in Y$ such that

$$\mathcal{I}_{\epsilon_n}(u_n) \to m(c_0).$$

Thereby, by Lemma 3.9, there are $u_1 \in \mathcal{H}_\epsilon \backslash \{0\}$ and a sequence $(z_n) \subset \mathbb{R}^3$ with

$$\liminf_{n \to +\infty} |\epsilon_n z_n| > 0$$

verifying

$$u_n(\cdot + z_n) \to u_1 \quad \text{in } \mathcal{H}_\epsilon,$$

that is

$$u_n = u_1(\cdot - z_n) + \omega_n \quad \text{with } \omega_n \to 0 \text{ in } \mathcal{H}_\epsilon.$$

From the definition of β,

$$\beta(u_1(\cdot - z_n)) = \frac{\int_{\mathbb{R}^3} \frac{\epsilon_n x + \epsilon_n z_n}{|\epsilon_n x + \epsilon_n z_n|} |u_1|^2 \, dx}{\int_{\mathbb{R}^3} |u_1|^2 \, dx}.$$

Repeating the same arguments explored in the proof of Lemma 4.2, we know that

$$\beta(u_1(\cdot - z_n)) = \frac{z_n}{|z_n|} + o_n(1),$$

and so,

$$\beta(u_n) = \beta(u_1(\cdot - z_n)) + o_n(1) = \frac{z_n}{|z_n|} + o_n(1).$$

Since $\beta(u_n) \in Y$, we conclude that $\frac{z_n}{|z_n|} \in Y_\lambda$ for n large enough. Consequently, $z_n \in Y_\lambda$ for n large enough, implying that

$$\liminf_{n\to\infty} V(\epsilon_n z_n) > c_0.$$

If $A = \liminf_{n\to\infty} V(\epsilon_n z_n)$, the last inequality and the Fatou's lemma show that

$$m(c_0) = \liminf_{n\to\infty} \mathcal{I}_{\epsilon_n}(u_n) \geq \liminf_{n\to\infty} \mathcal{I}_{\epsilon_n}(\tau u_n) \geq \mathcal{I}_A(\tau u_1) \geq m(A) > m(c_0),$$

which is a contradiction, recalling that there exists $\tau \in (0,1]$ such that $\mathcal{I}'_A(\tau u_1)\tau u_1 = 0$ and $u_1 \neq 0$.

(b) By $V(0) > c_0$, $c_1 < c_0 + 1$ and the fact that u_0 is a ground state solution associated with $\mathcal{I}_{c_0}$, we infer that

$$\begin{aligned}
\limsup_{\epsilon\to 0} \left\{ \sup_{x\in X} \mathcal{I}_\epsilon(\Upsilon_\epsilon(x)) \right\} &\leq \frac{1}{2}[u_o]^2 + \frac{1}{2}\int_{\mathbb{R}^3} (c_1+1)|u_0|^2 \, dx \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^3} \phi^t_{u_0} |u_0|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} u_0^2 \log u_0^2 dx \\
&= \mathcal{I}_{c_0}(u_0) + \frac{(c_1 - c_0)}{2}\int_{\mathbb{R}^3} |u_0|^2 \, dx \\
&= \mathcal{I}_{c_0}(u_0) + (c_1 - c_0)\mathcal{I}_{c_0}(u_0) \\
&= (1 + c_1 - c_0)\, m(c_0) \\
&< 2m(c_0),
\end{aligned}$$

which completes the proof. □

Now, we are ready to show the minimax level. We fix $\epsilon \in (0, \epsilon_0)$ and the following sets

$$\mathcal{I}_\epsilon^d = \{u \in \mathcal{H}_\epsilon : \mathcal{I}_\epsilon(u) \leq d\}, \quad Q = \bar{B}_R(0) \cap X \quad \text{and} \quad \partial Q = \partial \bar{B}_R(0) \cap X.$$

By the above notations, we define the class of the functions

$$\Gamma = \{h \in C(Q, K_r) : h(x) = \Upsilon_\epsilon(x), \quad \forall x \in \partial Q\},$$

where $r > 0, K = \Upsilon_\epsilon(Q)$ and $K_r = \{u \in \mathcal{H}_\epsilon : \text{dist}(u, K) < r\}$. Note that $\Gamma \neq \emptyset$, since Lemma 4.1 ensures that $\Upsilon_\epsilon \in \Gamma$. Then, we set

$$\Omega_r = \{u \in K_r : \beta(u) \in Y\},$$

which is not empty because $\Upsilon_\epsilon(0) = \varphi_{\epsilon,0} \in K_r$ for all $r > 0$. Here we have used the fact that $\Upsilon_\epsilon(0) \in \Upsilon_\epsilon(Q)$ and $\beta(\Upsilon_\epsilon(0)) = 0 \in Y$.

Lemma 4.5. *There exists $r_0 > 0$ such that*

$$\Theta_r = \inf_{u \in \Omega_r} \mathcal{I}_\epsilon(u) > m(c_0) + \sigma/2, \quad \forall r \in (0, r_0).$$

Moreover, there is $R > 0$ such that

$$\mathcal{I}_\epsilon(\Upsilon_\epsilon(x)) \leq \frac{1}{2}\left(m(c_0) + \Theta_r\right), \quad \forall x \in \partial B_R(0) \cap X.$$

Proof. Assume by contradiction that the lemma does not hold. Then, there exist $r_n \to 0$ and $u_n \in \Omega_{r_n}$ such that $\mathcal{I}_\epsilon(u_n) \leq m(c_0) + \sigma/2$. By definition of Ω_{r_n}, there exists $v_n \in K$ such that $\|u_n - v_n\| \leq r_n$. Since K is compact, there are a subsequence of (v_n), still denoted by itself, and $v \in K$ such that $v_n \to v$ in $\mathcal{H}_\epsilon$, then $u_n \to v$ in $\mathcal{H}_\epsilon$ and $\beta(v) \in Y$, from where it follows that $v \in \mathcal{B}_\epsilon$, then by Lemma 4.4-$(a)$, $\mathcal{I}_\epsilon(v) \geq m(c_0) + \sigma$. On the other hand, since $\mathcal{I}_\epsilon$ is lower semicontinuous, we have

$$\liminf_{n \to +\infty} \mathcal{I}_\epsilon(u_n) \geq \mathcal{I}_\epsilon(v),$$

which is a contradiction.

By (V_1), given $\delta > 0$, there exist $\epsilon_0 > 0$ and $R > 0$ such that

$$\sup\{\mathcal{I}_\epsilon(\Upsilon_\epsilon(x)) : x \in \partial B_R(0) \cap X\} \leq m(c_0) + \delta, \quad \forall \epsilon \in (0, \epsilon_0).$$

Fixing $\delta = \frac{\sigma}{4}$, where σ was given in Lemma 4.4-(a), we derive

$$\sup\{\mathcal{I}_\epsilon(\Upsilon_\epsilon(x)) : x \in \partial B_R(0) \cap X\} \leq \frac{1}{2}\left(2m(c_0) + \frac{\sigma}{2}\right) < \frac{1}{2}\left(m(c_0) + \Theta_r\right), \quad \forall \epsilon \in (0, \epsilon_0),$$

which completes the proof. □

Lemma 4.6. *If $h \in \Gamma$, then $h(Q) \cap \Omega_r \neq \emptyset$ for all $r \in (0, r_0)$.*

Proof. It is enough to show that for all $h \in \Gamma$, there exists $x_* \in Q$ such that

$$\beta\left(h\left(x_*\right)\right) \in Y.$$

For each $h \in \Gamma$, we set the function $g : Q \to \mathbb{R}^3$ given by

$$g(x) = \beta(h(x)) \quad \forall x \in Q,$$

and the homotopy $\mathcal{F} : [0,1] \times Q \to X$ as

$$\mathcal{F}(\tau, x) = \tau P_X(g(x)) + (1 - \tau)x,$$

where P_X is the projection onto $X = \left\{(x, 0) : x \in \mathbb{R}^3\right\}$. By Corollary 4.3, fixed $R > 0$ and $\epsilon > 0$ small enough, one has

$$(\mathcal{F}(\tau, x), x) > 0, \quad \forall (\tau, x) \in [0,1] \times \partial Q.$$

Applying the homotopy invariance property of the topological degree, we have

$$d(g, Q, 0) = 1,$$

which implies that there is $x_* \in Q$ such that $\beta\left(h\left(x_*\right)\right) = 0$. □

Now, we define the minimax value

$$C_\epsilon = \inf_{h\in\Gamma}\sup_{x\in Q}\mathcal{I}_\epsilon(h(x)).$$

By Lemmas 4.5 and 4.6,

$$C_\epsilon \geq \Theta_r = \inf_{u\in\Omega_r}\mathcal{I}_\epsilon(u) \geq m(c_0) + \sigma/2, \tag{4.2}$$

for ϵ is small enough. On the other hand,

$$C_\epsilon \leq \sup_{x\in Q}\mathcal{I}_\epsilon(\Upsilon_\epsilon(x)).$$

Then, by Lemma 4.4-(b), if ϵ is small enough,

$$C_\epsilon \leq \sup_{x\in Q}\mathcal{I}_\epsilon(\Upsilon_\epsilon(x)) < 2m\left(c_0\right) - \sigma. \tag{4.3}$$

From (4.2) and (4.3), there is ϵ_0 such that

$$C_\epsilon \in \left(m(c_0) + \sigma/2, 2m(c_0) - \sigma\right), \quad \forall \epsilon \in (0, \epsilon_0).$$

Proof of Theorem 1.1. Before proving Theorem 1.1, we first propose the following claim.

Claim III: For a given $\tau > 0$ small enough, there exists $u_\tau \in E$ such that

$$\Phi_\epsilon'\left(u_\tau\right)\cdot\left(v-u_\tau\right)+\Psi(v)-\Psi\left(u_\tau\right) \geq -3\tau\left\|v-u_\tau\right\|_\epsilon, \quad \forall v \in E,$$

and

$$\mathcal{I}_\epsilon(u_\tau) \in \left[C_\epsilon - \tau, C_\epsilon + \tau\right].$$

In fact, to prove the claim, we follow the ideas explored in Alves–de Morais Filho [2] and Szulkin [28]. Have this in mind, by Lemma 4.5, we can fix $\tau > 0$ small enough such that

$$C_\epsilon - \tau/2 > \frac{1}{2}\left(m(c_0) + \Theta_r\right),$$

and we set

$$\Gamma_1 = \left\{h \in C(Q, K_r) : h|_{\partial Q} \approx \Upsilon_\epsilon|_{\partial Q} \text{ in } \mathcal{I}_\epsilon^{C_\epsilon - \tau/4}, \sup_{x\in\partial Q}\mathcal{I}_\epsilon(h(x)) \leq C_\epsilon - \tau/2\right\},$$

where $\approx$ denotes the homotopy relation and the number

$$C^* = \inf_{h\in\Gamma_1}\sup_{x\in Q}\mathcal{I}_\epsilon(h(x)).$$

Arguing as in Szulkin [28], we have that $C^* = C_\epsilon$, and so, it is enough to prove that Claim III holds for C^* instead C_ϵ. In order to show this, firstly let us fix $\tau > 0$ small enough and $h \in \Gamma_1$ such that

$$\Pi(h) \leq C^* + \tau \quad \text{and} \quad \Pi(g) - \Pi(h) \geq -\tau d(g, h), \quad \forall g \in \Gamma_1, \tag{4.4}$$

where

$$\Pi(g) = \sup_{x\in Q}\mathcal{I}_\epsilon(g(x)), \quad \forall g \in \Gamma_1,$$

and

$$d(g,h)=\sup_{x\in Q}\|g(x)-h(x)\|.$$

Supposing by contradiction that Claim III does not hold and arguing as in Alves–de Morais Filho [2], we can apply Proposition 2.3 of Szulkin [28] with $A=h(Q)$ to find a closed subset W containing A in it interior and a deformation $\alpha_s : W\to\mathcal{H}_\epsilon$ having the following properties:

$$\begin{cases} \|u-\alpha_s(u)\|_\epsilon\le s, & \forall u\in W \text{ and } s\approx 0^+,\\ \mathcal{I}_\epsilon(\alpha_s(u))-\mathcal{I}_\epsilon(u)\le 2s, & \forall u\in W,\\ \mathcal{I}_\epsilon(\alpha_s(u))-\mathcal{I}_\epsilon(u)\le -2\tau s, & \forall u\in W \text{ with } \mathcal{I}_\epsilon(u)\ge C^*-\tau, \end{cases} \tag{4.5}$$

and

$$\sup_{u\in A}\mathcal{I}_\epsilon(\alpha_s(u))-\sup_{u\in A}\mathcal{I}_\epsilon(u)\le -2\tau s. \tag{4.6}$$

It is easy to see that $g=\alpha_s\circ h\in\Gamma_1$, for s small enough. However, by(4.4), (4.5) and (4.6), we have

$$-\tau s\le -\tau d(g,h)\le \Pi(g)-\Pi(h)\le -2\tau s,$$

which is a contradiction. This contradiction shows that Claim III is true.

From Claim III, there exists a $(\mathrm{PS})_{C_\epsilon}$ sequence for $\mathcal{I}_\epsilon$, which will denoted by (u_n). By Lemma 3.8, we can assume that $u_n\rightharpoonup u_\epsilon$ for some $u_\epsilon\in\mathcal{H}_\epsilon\backslash\{0\}$. On the other hand, it follows from Lemma 3.2 that for each $v\in C_0^\infty(\mathbb{R}^3)$, there holds the limit $\langle\mathcal{I}_\epsilon'(u_n),v\rangle=o_n(1)\|v\|_\epsilon$, from where it shows that $\langle\mathcal{I}_\epsilon'(u_\epsilon),v\rangle=0$, or equivalently,

$$\begin{aligned}&\iint_{\mathbb{R}^6}\frac{(u_\epsilon(x)-u_\epsilon(y))(v(x)-v(y))}{|x-y|^{3+2s}}\mathrm{d}x\mathrm{d}y+\int_{\mathbb{R}^3}V(\epsilon x)u_\epsilon\cdot v\mathrm{d}x-\int_{\mathbb{R}^3}\phi_{u_\epsilon}^t u_\epsilon v\mathrm{d}x\\ &\quad=\int u_\epsilon v\log u_\epsilon^2\mathrm{d}x,\quad \forall v\in C_0^\infty(\mathbb{R}^3).\end{aligned}$$

Moreover, a similar computation also gives that

$$[u_\epsilon]^2+\int_{\mathbb{R}^3}V(\epsilon x)|u_\epsilon|^2\mathrm{d}x-\int_{\mathbb{R}^3}\phi_{u_\epsilon}^t|u_\epsilon|^2\mathrm{d}x+\int F_1'(u_\epsilon)u_\epsilon\mathrm{d}x\le\int F_2'(u_\epsilon)u_\epsilon\mathrm{d}x,$$

which implies that $u^2\log u^2\in L^1(\mathbb{R}^3)$. This proves that u_ϵ is a critical point of $\mathcal{I}_\epsilon$ with $\phi=\phi_{u_\epsilon}$ for ϵ small enough. Finally, the last inequality together with Fatou's Lemma implies that

$$\mathcal{I}_\epsilon(u_\epsilon)\le C_\epsilon<2m(c_0).$$

By Squassina–Szulkin [26], local estimates and standard bootstrap arguments show that $u_\epsilon\in C^2(\mathbb{R}^3,\mathbb{R})$. Moreover, by the Maximum Principle, we have that

$$u_\epsilon(x)>0 \text{ for } x\in\mathbb{R}.$$

For each $\epsilon>0$ small enough, let u_ϵ denote the positive solution obtained above. Setting $v_\epsilon=u_\epsilon(\frac{x}{\epsilon})$, then it shows that $(v_\epsilon,\phi_{v_\epsilon})$ gives rise to a pair of solutions of (1.1). □

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Energy conservation and conditional regularity for the incompressible Navier–Stokes–Maxwell system

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Abstract. In this paper, we study a hydrodynamic system modeling the evolution of a plasma subject to a self-induced electromagnetic Lorentz force in incompressible viscous fluids. The system consists of the Navier–Stokes equations coupled with a Maxwell equation. In the three dimensional case, we show that every weak solution verifies the energy equality for the incompressible Navier–Stokes–Maxwell equations with damping. We also establish some non-explosion criteria in terms of the velocity and magnetic of local strong solution for standard Navier–Stokes–Maxwell system.

Keywords: Navier–Stokes–Maxwell system, damping term, energy conservation, non-explosion criteria.

2020 Mathematics Subject Classification: 35Q35, 76D03.

1 Introduction

In this paper, we are interested in studying the unconditional energy conservation for the weak solutions of Navier–Stokes–Maxwell (NSM for short) equations with damping (or the tamed NSM equations)

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nu |u|^{\alpha-1} u = -\nabla P + j \times B, & \operatorname{div} u = 0, \\ \frac{1}{c}\partial_t E - \nabla \times B = -j, & j = \sigma(cE + u \times B), \\ \frac{1}{c}\partial_t B + \nabla \times E + \nu |B|^{\beta-1} B = 0, & \operatorname{div} B = 0, \end{cases} \tag{1.1}$$

with initial data

$$u(0,x) = u_0, \quad E(0,x) = E_0, \quad B(0,x) = B_0 \tag{1.2}$$

for $(t,x) \in \mathbb{R}^+ \times \Omega$, where Ω is a periodic domain $\mathbb{T}^3$ or whole space $\mathbb{R}^3$. Here $c > 0$ denotes the speed of light, μ and ν denote respectively the positive viscosity and damping coefficients of the fluid, and $\sigma > 0$ is the electrical conductivity. In the above system (1.1),

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$u = (u_1, u_2, u_3) = u(t,x)$ stands for the velocity field of the (incompressible) fluid, while $E = (E_1, E_2, E_3) = E(t,x)$ and $B = (B_1, B_2, B_3) = B(t,x)$ are the electric and magnetic fields, respectively. The scalar function $P = P(t,x)$ is the pressure and is also an unknown. Observe, though, that the electric current $j = j(t,x)$ is not an unknown, for it is fully determined by (u, E, B) through Ohm's law. The exponent α, β can be greater than or equal to 1.

The standard Navier–Stokes–Maxwell system (i.e., $\alpha = \beta = 1$) describes the evolution of a plasma (i.e., a charged fluid) subject to a self-induced electromagnetic Lorentz force $j \times B$.

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + j \times B, & \operatorname{div} u = 0, \\ \frac{1}{c}\partial_t E - \nabla \times B = -j, & j = \sigma(cE + u \times B), \\ \frac{1}{c}\partial_t B + \nabla \times E = 0, & \operatorname{div} B = 0. \end{cases} \tag{1.3}$$

Mathematically, NSM equations is a coupled system, constituted by the parabolic nature of the Navier–Stokes equations from fluid dynamics and the hyperbolic of the Maxwell equation from electromagnetism. Moreover, it can be derived from the Vlasov–Maxwell–Boltzmann system [3]. In the 2D case, Masmoudi [14] prove global existence of regular solutions to the Maxwell–Navier–Stokes system (1.3), and also provide an exponential growth estimate for the H^s norm of the solution when the time goes to infinity. In the 3D case, Ibrahim and Keraani [7] showed the existence of global small mild solutions of NSM system (1.3). Recently, Arsenio and Gallagher [1] have made important and new progress in this direction, they established that global existence of solutions to the 3D system (1.3) holds whenever the initial data tum (u_0, E_0, B_0) is chosen in the natural energy space L^2, while the electromagnetic field (E_0, B_0) alone lies in $\dot{H}^s$, for some given $s \in \left[\frac{1}{2}, \frac{3}{2}\right)$, and is sufficiently small when compared to some non-linear function of the initial energy

$$\mathcal{E}_0 := \frac{1}{2}\left(\|u_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2\right).$$

Kang and Lee [8] showed that the maximal existence time of the local strong solution T^* is finite if and only if

$$\int_0^{T^*} \|u\|_{L^\infty}^2 + \|B\|_{L^\infty}^{\frac{8}{3}} \, \mathrm{d}t = \infty.$$

This was improved by Fan-Zhou [4] to be

$$\int_0^{T^*} \|u\|_{L^\infty}^2 + \|B\|_{L^\infty}^2 \, \mathrm{d}t = \infty.$$

Thereafter, Ma, Jiang and Zhu [12] proved the following three regularity criteria:

- $u \in L^2\left(0, T; L^\infty\left(\mathbb{R}^3\right)\right)$ and $\nabla u \in L^2\left(0, T; L^3\left(\mathbb{R}^3\right)\right)$;
- $u \in L^p\left(0, T; L^q\left(\mathbb{R}^3\right)\right)$, $\frac{2}{p} + \frac{3}{q} = 1, 3 < q \leq \infty$ and $\nabla B \in L^2\left(0, T; L^3\left(\mathbb{R}^3\right)\right)$;
- $\nabla u \in L^p\left(0, T; L^q\left(\mathbb{R}^3\right)\right)$, $\frac{2}{p} + \frac{3}{q} = 2, \frac{3}{2} < q \leqslant \infty$ and $\nabla B \in L^2\left(0, T; L^3\left(\mathbb{R}^3\right)\right)$.

More recently, this result has been improved by Zhang, Pan and iu [20], who proved that if

$$u \in L^{\frac{2}{1-r}}\left(0, T; \dot{B}^{-r}_{\infty,\infty}\left(\mathbb{R}^3\right)\right), \qquad -1 < r < 1$$

and

$$\nabla B \in L^p\left(0, T; L^q\left(\mathbb{R}^3\right)\right), \qquad \frac{2}{p} + \frac{3}{q} = 2 \quad \text{with } 2 \leq q \leq 3,$$

then the strong solution to the NSM system (1.3) can be smoothly extended beyond T. In addition, the energy balance of distributional solutions/Leray–Hopf weak solutions for the NSM system was obtained in [13,19]. In particular, for a Leray–Hopf weak solution (u, E, B) satisfies

$$u \in L^q\left(0, T; L^p(\Omega)\right), \qquad \frac{1}{q} + \frac{1}{p} \leq \frac{1}{2}, \qquad p \geq 4, \tag{1.4}$$

and

$$B \in L^r\left(0, T; L^s(\Omega)\right), \qquad \frac{1}{r} + \frac{1}{s} \leq \frac{1}{2}, \qquad s \geq 4. \tag{1.5}$$

Then it keeps energy balance.

The damping comes from the resistance to the motion of the flow. It describes various physical phenomena such as porous media flow, drag or friction effects, and some dissipative mechanisms [2]. Recently, Liu, Sun and Xin [11] considered the following 3D NSM system with damping:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + |u|^{\alpha-1} u = -\nabla P + j \times B, & \operatorname{div} u = 0, \\ \partial_t E - \nabla \times B + |E| E = -j, & j = \sigma(E + u \times B), \\ \partial_t B + \nabla \times E + |B|^4 B = 0, & \operatorname{div} B = 0, \end{cases} \tag{1.6}$$

and proved the existence and uniqueness of strong solutions for system (1.6) provided that $\alpha \geq 3$.

When $E = B \equiv 0$, system (1.1) reduces to the Navier–Stokes system with damping(or the tamed Navier–Stokes equations). When viscosity and damping coefficients μ and ν equal to one, Cai and Jiu [2] first established the global existence of strong solutions provided that $\alpha \geq \frac{7}{2}$. Later, it was improved to the $\alpha \geq 3$ by Zhou [21]. Very recently, Hajduk and Robinson in [6] give a simple proof of the existence of global-in-time smooth solutions for tamed Navier–Stokes equations on a 3D periodic domain, for values of the absorption exponent α larger than 3. Furthermore, they prove that global, regular solutions exist also for the critical value of exponent $\alpha = 3$, provided that the coefficients satisfy the relation $4\mu\nu \geq 1$. Additionally, they showed that in the critical case every Leray-Hopf weak solution verifies the energy identity:

$$\|u(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u(s)\|_{L^2}^2 \, \mathrm{d}s + 2\nu \int_0^t \|u(s)\|_{L^4}^4 \, \mathrm{d}s = \|u(0)\|_{L^2}^2, \qquad 0 < t < T.$$

To the best of our knowledge, the validity of the energy equality is not to date verified for the NSM equations with damping (1.1) for the range of exponent values $\alpha, \beta \in [1, 3]$. For larger values of the exponent α, β with lower order damping term $|E|E$, it was already shown that the tamed NSM equations (1.6) enjoy existence of global-in-time strong solutions (see proof for whole spaces $\mathbb{R}^3$ in [11]) and hence the energy equality is satisfied. A natural question immediately arises: **does any Leray–Hopf weak solution of the tamed NSM equations** (1.1) **automatically satisfy the energy balance?** In this work, we will try to answer this question, and show the energy balance to the critical case $\alpha = \beta = 3$. In addition, for the standard NSM system (1.3) (i.e., $\alpha = \beta = 1$ in (1.1)), we will show some blow-up criteria under scaling invariant conditions on gradient of the velocity and the magnetic in some different Banach spaces, including the homogeneous Sobolev space, the weighted L^∞-space and the Morrey space, etc.

In a same fashion with [6], we first state the definition of the Leray–Hopf weak solutions to system (1.1).

Definition 1.1 (Leray–Hopf weak solution)**.** Let $(u_0, E_0, B_0) \in L^2(\Omega)$ with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$, $T > 0$. The function (u, E, B) is said to be a Leray–Hopf weak solution to tamed NSM system(1.1) if

1. $$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\dot{H}^1((\Omega))) \cap L^{\alpha+1}(0,T;L^{\alpha+1}(\Omega))$$
$$B \in L^\infty(0,T;L^2(\Omega)) \cap L^{\beta+1}(0,T;L^{\beta+1}(\Omega))$$
$$E \in L^\infty(0,T;L^2(\Omega)),\ j \in L^2(0,T;L^2(\Omega));$$

2. for any smooth test function $\varphi \in C_c^\infty(\Omega \times [0,T))$ and $\nabla \cdot \varphi = 0$, holds that
$$-\int_0^T\int_\Omega u \cdot \partial_t \varphi \mathrm{d}x\mathrm{d}t + \mu\int_0^T\int_\Omega \nabla u : \nabla\varphi \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega [u \cdot \nabla u - j \times B + \nu|u|^{\alpha-1}u] \cdot \varphi \mathrm{d}x\mathrm{d}t$$
$$= \int_\Omega u_0 \cdot \varphi(x,0)\mathrm{d}x,$$
$$-\frac{1}{c}\int_0^T\int_\Omega E \cdot \partial_t\varphi \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega B\nabla \times \varphi \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega j \cdot \varphi \mathrm{d}x\mathrm{d}t = \int_\Omega E_0 \cdot \varphi(x,0)\mathrm{d}x,$$
and
$$-\frac{1}{c}\int_0^T\int_\Omega B \cdot \partial_t\varphi \mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega E\nabla \times \varphi \mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \nu|B|^{\beta-1}B \cdot \varphi \mathrm{d}x\mathrm{d}t = \int_\Omega B_0 \cdot \varphi(x,0)\mathrm{d}x,$$

3. for any $\Phi \in C_c^\infty(\mathbb{R}^d)$, it holds that
$$\int_\Omega u \cdot \nabla\Phi \mathrm{d}x = \int_\Omega B \cdot \nabla\Phi \mathrm{d}x = 0$$
a.e. $t \in (0,T)$;

4. (u, E, B) satisfies the energy inequality
$$\frac{1}{2}(\|u(\cdot,t)\|_{L^2}^2 + \|E(\cdot,t)\|_{L^2}^2 + \|B(\cdot,t)\|_{L^2}^2) + \int_0^T (\mu\|\nabla u\|_{L^2}^2 + \frac{1}{\sigma}\|j\|_{L^2}^2)dt$$
$$+\nu\int_0^T \|u\|_{L^{\alpha+1}}^{\alpha+1} + c\|B\|_{L^{\beta+1}}^{\beta+1}\mathrm{d}t \le \frac{1}{2}\int_\Omega(|u_0|^2 + |E_0|^2 + |B_0|^2)\mathrm{d}x,$$
for all $t \in [0,T]$.

We now make the observation that Leray–Hopf weak solutions of the tamed NSM system (1.1) with $\alpha = \beta = 3$ by Definition 1.1 satisfy the condition (1.4)–(1.5). This suggests that the energy balance holds for all weak solutions of this problem, and we will prove this in the following unconditional energy balance theorem.

Theorem 1.2. *Let $\alpha = \beta = 3$ in tamed NSM system (1.1), then every Leray–Hopf weak solution (u, E, B) with $(u_0, E_0, B_0) \in L^2(\Omega)$ satisfies the energy balance:*
$$\frac{1}{2}(\|u(\cdot,t)\|_{L^2}^2 + \|E(\cdot,t)\|_{L^2}^2 + \|B(\cdot,t)\|_{L^2}^2) + \int_0^T (\mu\|\nabla u\|_{L^2}^2 + \frac{1}{\sigma}\|j\|_{L^2}^2)dt$$
$$+\nu\int_0^T \|u\|_{L^4}^4 + c\|B\|_{L^4}^4\mathrm{d}t = \frac{1}{2}(\|u_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2), \tag{1.7}$$
for all $t \in [0,T]$.

Remark 1.3. In [5], the authors approximate functions defined on smooth bounded domains by elements of the eigenspaces of the Laplacian or the Stokes operator in such a way that the approximations are bounded and converge in both Sobolev and Lebesgue spaces. As a direct application, they prove that all weak solutions of the tamed NS equations posed on a bounded domain in $\mathbb{R}^3$ satisfy the energy equality. One may establish similar result to tamed NSM equations (1.1) in smooth bounded domain by the the method developed in [4].

As mentioned in the introduction, we will give several new non-explosion criteria in terms of the velocity and magnetic for standard NSM system on the framework of different Banach spaces. (Definitions of various Banach spaces can be found in Section 3.)

Theorem 1.4. *Let $(u_0, E_0, B_0) \in H^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$. Assume that (u, E, B) be the local strong solution of the standard NSM system (1.3). If*

$$S, \nabla B \in L^2\left(0, T; \mathcal{X}\left(\mathbb{R}^3\right)\right), \tag{1.8}$$

where $\mathcal{X}$ is one of the Banach spaces:

- $\mathcal{X} = \dot{H}^{1/2}\left(\mathbb{R}^3\right)$ *(the homogeneous Sobolev space),*
- $\mathcal{X} = L^3\left(\mathbb{R}^3\right)$ *(the Lebesgue space),*
- $\mathcal{X} = \left\{f \in L^\infty_{loc}\left(\mathbb{R}^3\right) : \|f\| = \sup_{x\in\mathcal{X}} |x||f(x)| < \infty\right\}$ *(the weighted L^∞-space),*
- $\mathcal{X} = \mathcal{PM}^2\left(\mathbb{R}^3\right)$ *(the Le Jan–Sznitman space),*
- $\mathcal{X} = L^{3,\infty}\left(\mathbb{R}^3\right)$ *(the Marcinkiewicz space),*
- $\mathcal{X} = \dot{M}^3_p\left(\mathbb{R}^3\right)$ *for each $2 < p \leqslant 3$ (the Morrey space).*

Then the local strong solution can be smoothly extended beyond T. Here, $S = \nabla_{sym}(u)_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$.

Remark 1.5. The result in Theorem 1.4 is more weaker condition to that in [12] or [20]. On the one hand, the deformation tensor S in theorem 1.4 can be replaced ∇u or $\nabla \times u$, on the other hand, we see that the framework of several Banach spaces in theorem 1.4 is more flexible and larger than that of Lebesgue spaces.

2 Proof of Theorem 1.2

The goal of this section is to prove Theorem 1.2. The main idea is to use a Leray–Hopf weak solution as a test function. We cannot do this directly since it is not sufficiently regular in space or time. Therefore, for the sake of simplicity, we will proceed as if the solution is differentiable in time. The extra arguments needed to mollify in time are straightforward. To this end we recall here some standard facts of the theory of mollification and introduce a crucial lemma. The key lemma is as follows which was proved by Lions in [10].

Let us to define $\eta : \mathbb{R}^d \to \mathbb{R}$ be a standard mollifier, i.e. $\eta(x) = Ce^{\frac{1}{|x|^2-1}}$ for $|x| < 1$ and $\eta(x) = 0$ for $|x| \geqslant 1$, where constant $C > 0$ selected such that $\int_{\mathbb{R}^d} \eta(x)\mathrm{d}x = 1$. For any $\varepsilon > 0$, we define the rescaled mollifier $\eta_\varepsilon(x) = \varepsilon^{-d}\eta\left(\frac{x}{\varepsilon}\right)$. For any function $f \in L^1_{\mathrm{loc}}(\Omega)$, its mollified version is defined as

$$f^\varepsilon(x) = (f * \eta_\varepsilon)(x) = \int_\Omega \eta_\varepsilon(x-y)f(y)\mathrm{d}y.$$

If $f \in W^{1,p}(\Omega)$, the following local approximation is well known

$$f^\varepsilon(x) \to f \quad \text{in } W^{1,p}_{loc}(\Omega) \quad \forall p \in [1,\infty).$$

Lemma 2.1. *Let ∂ be a partial derivative in one direction. Let $f, \partial f \in L^p(\mathbb{R}^+ \times \Omega)$, $g \in L^q(\mathbb{R}^+ \times \Omega)$ with $1 \le p,q \le \infty$, and $\frac{1}{p}+\frac{1}{q} \le 1$. Then, we have*

$$\|\partial(fg) * \eta_\varepsilon - \partial\left(f(g*\eta_\varepsilon)\right)\|_{L^r(\mathbb{R}^+\times\Omega)} \le C\|\partial f\|_{L^p(\mathbb{R}^+\times\Omega)}\|g\|_{L^q(\mathbb{R}^+\times\Omega)}$$

for some constant $C>0$ independent of ε, f and g, and with $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. In addition,

$$\partial(fg)*\eta_\varepsilon - \partial\left(f(g*\eta_\varepsilon)\right) \to 0 \quad \text{in } L^r\left(\mathbb{R}^+\times\Omega\right)$$

as $\varepsilon \to 0$, if $r<\infty$.

Proof of Theorem 1.2. First, testing system $(1.1)_{1,2,3}$ by $(u^\varepsilon)^\varepsilon$, $(E^\varepsilon)^\varepsilon$ and $(B^\varepsilon)^\varepsilon$, respectively, we infer that

$$\begin{cases} \int_\Omega u^\varepsilon\left(\partial_t u + u\cdot\nabla u - \mu\Delta u + \nu|u|^2u + \nabla P - j\times B\right)^\varepsilon \mathrm{d}x = 0, \\ \int_\Omega E^\varepsilon\left(\partial_t E - c\nabla\times B + cj\right)^\varepsilon \mathrm{d}x = 0, \\ \int_\Omega B^\varepsilon\left(\partial_t B + c\nabla\times E + c\nu|B|^2B\right)^\varepsilon \mathrm{d}x = 0. \end{cases} \tag{2.1}$$

Moreover, it yields that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega\left(|u^\varepsilon|^2+|E^\varepsilon|^2+|B^\varepsilon|^2\right)\mathrm{d}x + \mu\int_\Omega|\nabla u^\varepsilon|^2\mathrm{d}x \\ &\quad + \nu\int_\Omega\left((|u|^2u)^\varepsilon\cdot u^\varepsilon + c(|B|^2B)^\varepsilon\cdot B^\varepsilon\right)dx \\ &= -\int_\Omega \mathrm{div}(u\otimes u)^\varepsilon\cdot u^\varepsilon\mathrm{d}x + \int_\Omega (j\times B)^\varepsilon\cdot u^\varepsilon\mathrm{d}x + c\int_\Omega(\nabla\times B)^\varepsilon\cdot E^\varepsilon\mathrm{d}x \\ &\quad - c\int_\Omega j^\varepsilon\cdot E^\varepsilon\mathrm{d}x - c\int_\Omega(\nabla\times E)^\varepsilon\cdot B^\varepsilon\mathrm{d}x. \end{aligned} \tag{2.2}$$

Clearly,

$$\begin{aligned} &\int_\Omega\left(|u^\varepsilon|^2+|E^\varepsilon|^2+|B^\varepsilon|^2\right)dx - \int_\Omega\left(|u_0^\varepsilon|^2+|E_0^\varepsilon|^2+|B_0^\varepsilon|^2\right)dxdt + 2\mu\int_0^T\int_\Omega|\nabla u^\varepsilon|^2\mathrm{d}x\mathrm{d}t \\ &\quad + 2\nu\int_0^T\int_\Omega\left((|u|^2u)^\varepsilon\cdot u^\varepsilon + c(|B|^2B)^\varepsilon\cdot B^\varepsilon\right)\mathrm{d}x\mathrm{d}t \\ &= -2\int_0^T\int_\Omega \mathrm{div}(u\otimes u)^\varepsilon\cdot u^\varepsilon\mathrm{d}x\mathrm{d}t + 2\int_0^T\int_\Omega(j\times B)^\varepsilon\cdot u^\varepsilon\mathrm{d}x\mathrm{d}t \\ &\quad + 2c\int_0^T\int_\Omega(\nabla\times B)^\varepsilon\cdot E^\varepsilon\mathrm{d}x\mathrm{d}t - 2c\int_0^T\int_\Omega j^\varepsilon\cdot E^\varepsilon\mathrm{d}x\mathrm{d}t \\ &\quad - 2c\int_0^T\int_\Omega(\nabla\times E)^\varepsilon\cdot B^\varepsilon\mathrm{d}x\mathrm{d}t \\ &= I_1+I_2+I_3+I_4+I_5. \end{aligned} \tag{2.3}$$

We want to pass to the limit in (2.3) as $\varepsilon \to 0$. To this end, using Hölder's inequality, we observe the following estimates for the nonlinear terms:

$$\begin{aligned}
&\left|\int_0^T\int_\Omega (|u|^2u)^\varepsilon\cdot u^\varepsilon \mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega |u|^2u\cdot u\mathrm{d}x\mathrm{d}t\right|\\
&\quad\le\left|\int_0^T\int_\Omega \left((|u|^2u)^\varepsilon-|u|^2u\right)\cdot u^\varepsilon+|u|^2u(u^\varepsilon-u)\mathrm{d}x\mathrm{d}t\right|\\
&\quad\le\|u^\varepsilon\|_{L^4(0,T;L^4(\Omega))}\left\|\left((|u|^2u)^\varepsilon-|u|^2u\right)\right\|_{L^{\frac43}\left(0,T;L^{\frac43}(\Omega)\right)}\\
&\qquad+\|u^\varepsilon-u\|_{L^4(0,T;L^4(\Omega))}\left\||u|^2u\right\|_{L^{\frac43}\left(0,T;L^{\frac43}(\Omega)\right)}\to 0\quad\text{ as }\varepsilon\to 0.
\end{aligned}\tag{2.4}$$

Similarly,

$$\left|\int_0^T\int_\Omega (|B|^2B)^\varepsilon\cdot B^\varepsilon \mathrm{d}x\mathrm{d}t-\int_0^T\int_\Omega |B|^2B\cdot u\mathrm{d}x\mathrm{d}t\right|\to 0\quad\text{ as }\varepsilon\to 0.\tag{2.5}$$

In addition, since

$$\begin{aligned}
\operatorname{div}(u\otimes u)^\varepsilon&=[\operatorname{div}(u\otimes u)^\varepsilon-\operatorname{div}(u\otimes u^\varepsilon)]+[\operatorname{div}(u\otimes u^\varepsilon)-\operatorname{div}(u^\varepsilon\otimes u^\varepsilon)]+\operatorname{div}(u^\varepsilon\otimes u^\varepsilon)\\
&=I_{11}+I_{12}+I_{13},
\end{aligned}$$

and so

$$I_1=-2\int_0^T\int_\Omega (I_{11}+I_{12}+I_{13})\cdot u^\varepsilon\mathrm{d}x\mathrm{d}t.\tag{2.6}$$

From Lemma 2.1, one obtains

$$\|I_{11}\|_{L^{\frac43}\left(0,T;L^{\frac43}(\Omega)\right)}\le C\|u^\varepsilon\|_{L^4(0,T;L^4(\Omega))}\|\nabla u^\varepsilon\|_{L^2(0,T;L^2(\Omega))}$$

and it converges to zero in $L^{\frac43}\left(0,T;L^{\frac43}(\Omega)\right)$ as ε tends to zero. Thus, as ε goes to zero, it follows that

$$\begin{aligned}
\left|-2\int_0^T\int_\Omega I_{11}u^\varepsilon\mathrm{d}x\mathrm{d}t\right|&=\left|\int_0^T\int_\Omega [\operatorname{div}(u\otimes u)^\varepsilon-\operatorname{div}(u\otimes u^\varepsilon)]\cdot u^\varepsilon\mathrm{d}x\mathrm{d}t\right|\\
&\le\|I_{11}\|_{L^{\frac43}\left(0,T;L^{\frac43}(\Omega)\right)}\|u^\varepsilon\|_{L^4(0,T;L^4(\Omega))}\\
&\to 0.
\end{aligned}\tag{2.7}$$

Moreover, we have

$$\begin{aligned}
\left|-2\int_0^T\int_\Omega I_{12}u^\varepsilon\mathrm{d}x\mathrm{d}t\right|&=\left|\int_0^T\int_\Omega [\operatorname{div}(u\otimes u^\varepsilon)-\operatorname{div}(u^\varepsilon\otimes u^\varepsilon)]\cdot u^\varepsilon dxdt\right|\\
&=\left|\int_0^T\int_\Omega (u\otimes u^\varepsilon-u^\varepsilon\otimes u^\varepsilon)\cdot\nabla u^\varepsilon\mathrm{d}x\mathrm{d}t\right|\\
&\le\|u-u^\varepsilon\|_{L^4(0,T;L^4(\Omega))}\|u^\varepsilon\|_{L^4(0,T;L^4(\Omega))}\|\nabla u^\varepsilon\|_{L^2(0,T;L^2(\Omega))}\\
&\to 0\quad\text{ as }\varepsilon\to 0.
\end{aligned}\tag{2.8}$$

Since $\operatorname{div} u^\varepsilon=0$, one has

$$-2\int_0^T\int_\Omega I_{13}\cdot u^\varepsilon\mathrm{d}x=0.\tag{2.9}$$

Combining (2.6)–(2.9), we know that

$$I_1\to 0,\quad\text{ as }\varepsilon\to 0.\tag{2.10}$$

For the term I_2, we claim that

$$I_2 = 2\int_0^T \int_\Omega (j\times B)^\varepsilon \cdot u^\varepsilon \mathrm{d}x\mathrm{d}t \to 2\int_0^T \int_\Omega (j\times B)\cdot u \mathrm{d}x\mathrm{d}t, \quad \text{as } \varepsilon \to 0. \tag{2.11}$$

Indeed,

$$\begin{aligned}
&2\int_0^T \int_\Omega (j\times B)^\varepsilon \cdot u^\varepsilon - (j\times B)\cdot u \mathrm{d}x\mathrm{d}t \\
&\quad = 2\int_0^T \int_\Omega (j\times B)^\varepsilon \cdot u^\varepsilon - (j\times B)\cdot u^\varepsilon + (j\times B)\cdot u^\varepsilon - (j\times B)\cdot u \mathrm{d}x\mathrm{d}t \\
&\quad = 2\int_0^T \int_\Omega \left[(j\times B)^\varepsilon - (j\times B)\right]\cdot u^\varepsilon + (j\times B)\cdot (u^\varepsilon - u)\,\mathrm{d}x\mathrm{d}t \\
&\quad \le 2\left\|(j\times B)^\varepsilon - (j\times B)\right\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)} \|u^\varepsilon\|_{L^4(0,T;L^4(\Omega))} \\
&\qquad + 2\left\|j\times B\right\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)} \|u^\varepsilon - u\|_{L^4(0,T;L^4(\Omega))} \\
&\quad \to 0, \quad \text{as } \varepsilon \to 0,
\end{aligned} \tag{2.12}$$

where we used fact that

$$\|j\times B\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)} \le \|j\|_{L^2(0,T;L^2(\Omega))}\|B\|_{L^4(0,T;L^4(\Omega))} \le C.$$

By using the same trick, we find that

$$I_4 = -2c\int_0^T \int_\Omega j^\varepsilon \cdot E^\varepsilon \mathrm{d}x\mathrm{d}t \to -2c\int_0^T \int_\Omega j\cdot E \mathrm{d}x\mathrm{d}t, \quad \text{as } \varepsilon \to 0. \tag{2.13}$$

After integration by part, the term I_3 can be dominated as

$$\begin{aligned}
I_3 &= 2c\int_0^T \int_\Omega (\nabla\times B)^\varepsilon \cdot E^\varepsilon \mathrm{d}x\mathrm{d}t \\
&= 2c\int_0^T \int_\Omega \left(\epsilon_{ijk}\partial_j B_k\right)^\varepsilon \cdot E_i^\varepsilon \mathrm{d}x\mathrm{d}t \\
&= -2c\int_0^T \int_\Omega \epsilon_{ijk} B_k^\varepsilon \cdot \partial_j E_i^\varepsilon \mathrm{d}x\mathrm{d}t \\
&= 2c\int_0^T \int_\Omega \epsilon_{kji} B_k^\varepsilon \cdot \partial_j E_i^\varepsilon \mathrm{d}x\mathrm{d}t \\
&= 2c\int_0^T \int_\Omega B^\varepsilon \cdot (\nabla\times E)^\varepsilon \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{2.14}$$

So, it follows that

$$I_3 + I_5 = 0. \tag{2.15}$$

Letting ε goes to zero in (2.3), and using the facts (2.4)–(2.5) and (2.10)–(2.15), we infer that

$$\begin{aligned}
&(\|u(\cdot,t)\|_{L^2}^2 + \|E(\cdot,t)\|_{L^2}^2 + \|B(\cdot,t)\|_{L^2}^2) + 2\int_0^T \left(\mu\|\nabla u\|_{L^2}^2 + \frac{1}{\sigma}\|j\|_{L^2}^2\right) dt \\
&\quad + 2\nu\int_0^T \|u\|_{L^4}^4 + c\|B\|_{L^4}^4 dt = (\|u_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2),
\end{aligned} \tag{2.16}$$

where we have used the facts that

$$2\int_0^T\int_\Omega (j\times B)\cdot u\mathrm{d}x\mathrm{d}t-2c\int_0^T\int_\Omega j\cdot E\mathrm{d}x\mathrm{d}t=-2\int_\Omega (u\times B)\cdot j\mathrm{d}x-2c\int_\Omega E\cdot j\mathrm{d}x$$

and

$$j=\sigma(cE+u\times B).$$

The energy equality (1.7) follows easily from (2.16). □

3 Proof of Theorem 1.4

In this section, we first introduce several definitions used throughout proof of theorem 1.4 and also recall the well-known results for our analysis (see e.g. [9,16,17] and [18]). And then show the derivation details of theorem 1.4.

Definition 3.1. The weighted L^∞-space is defined by

$$\text{Weighted } L^\infty=\left\{f\in L^\infty_{loc}\left(\mathbb{R}^3\right):\|f\|_{\mathcal{X}}=\sup_{x\in\mathcal{X}}|x||f(x)|<\infty\right\}.$$

Definition 3.2. The Le Jan–Sznitman space is defined by

$$\mathcal{PM}^2=\left\{v\in\mathcal{S}'\left(\mathbb{R}^3\right):\widehat{v}\in L^1_{loc}\left(\mathbb{R}^3\right),\quad \|v\|_{\mathcal{PM}^2}=\operatorname*{ess\,sup}_{\xi\in\mathbb{R}^3}|\xi|^2|\widehat{v}(\xi)|<\infty\right\}.$$

Definition 3.3. Let $1<p\le q<\infty$, the homogeneous Morrey spaces are defined as

$$\dot{M}^q_p\left(\mathbb{R}^3\right)=\left\{f\in L^q_{loc}\left(\mathbb{R}^3\right):\|f\|_{\dot{M}^q_p}=\sup_{R>0}\sup_{x\in\mathbb{R}^3}R^{3\left(\frac{1}{p}-\frac{1}{q}\right)}\left(\int_{B_R(x)}|f(y)|^p\,\mathrm{d}y\right)^{\frac{1}{p}}<\infty\right\}.$$

Lemma 3.4 ([15])**.** *For all $-\frac{3}{2}<\alpha<\frac{3}{2}$ and for all u divergence free in the sense that $\xi\cdot\hat{u}(\xi)=0$ almost everywhere,*

$$\|S\|^2_{\dot{H}^\alpha}=\|A\|^2_{\dot{H}^\alpha}=\frac{1}{2}\|\omega\|^2_{\dot{H}^\alpha}=\frac{1}{2}\|\nabla\otimes u\|^2_{\dot{H}^\alpha},\tag{3.1}$$

where symmetric part $S=S_{ij}=\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)$, which we refer to as the strain tensor, anti-symmetric part $A=A_{ij}=\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}\right)$, $\omega=\nabla\times u$.

Lemma 3.5 (Hardy-type inequalities [9])**.** *There exists a constant $K>0$ such that the following inequality holds true*

$$\left|\int_{\mathbb{R}^3}W\cdot(g\cdot\nabla)h\,\mathrm{d}x\right|\leqslant K\|W\|_{\mathcal{X}_\sigma}\|\nabla g\|_{L^2}\|\nabla h\|_{L^2},$$

for all vector fields $g,h\in\dot{H}^1\left(\mathbb{R}^3\right)$ and all $W\in\mathcal{X}_\sigma$, where $\mathcal{X}_\sigma$ (the subspace of $\mathcal{X}$ of divergence-free vector functions) is one of the Banach spaces

- *$\mathcal{X}_\sigma=\dot{H}^{1/2}_\sigma\left(\mathbb{R}^3\right)$ (the homogeneous Sobolev space),*
- *$\mathcal{X}_\sigma=L^3_\sigma\left(\mathbb{R}^3\right)$ (the Lebesgue space),*

- $\mathcal{X}_\sigma = \left\{ f \in L^\infty_{loc}\left(\mathbb{R}^3\right) : \|f\| = \sup_{x\in\mathcal{X}_\sigma} |x| |f(x)| < \infty \right\}$ *(the weighted L^∞-space),*
- $\mathcal{X}_\sigma = \mathcal{PM}^2\left(\mathbb{R}^3\right)$ *(the Le Jan–Sznitman space),*
- $\mathcal{X}_\sigma = L^{3,\infty}_\sigma\left(\mathbb{R}^3\right)$ *(the Marcinkiewicz space),*
- $\mathcal{X}_\sigma = \dot{M}^3_p\left(\mathbb{R}^3\right)$ *for each* $2 < p \leqslant 3$ *(the Morrey space).*

The following proposition is an ad hoc variant of the preceding lemma 3.5. It will be more applicable to proving theorem 1.4. This proposition is inspired by the details of the proof from lemma 3.5.

Proposition 3.6. *There exists a constant $C > 0$ such that the following inequality holds true*

$$\left| \int_{\mathbb{R}^3} Wgh \, \mathrm{d}x \right| \leqslant C \|W\|_{\mathcal{X}} \|\nabla g\|_{L^2} \|h\|_{L^2},$$

for all vector fields $g \in \dot{H}^1\left(\mathbb{R}^3\right)$, $h \in L^2\left(\mathbb{R}^3\right)$ and all $W \in \mathcal{X}$, where $\mathcal{X}$ is one of the Banach spaces

- $\mathcal{X} = \dot{H}^{1/2}\left(\mathbb{R}^3\right)$ *(the homogeneous Sobolev space),*
- $\mathcal{X} = L^3\left(\mathbb{R}^3\right)$ *(the Lebesgue space),*
- $\mathcal{X} = \{ f \in L^\infty_{loc}\left(\mathbb{R}^3\right) : \|f\| = \sup_{x\in\mathcal{X}} |x| |f(x)| < \infty \}$ *(the weighted L^∞-space),*
- $\mathcal{X} = \mathcal{PM}^2\left(\mathbb{R}^3\right)$ *(the Le Jan–Sznitman space),*
- $\mathcal{X} = L^{3,\infty}\left(\mathbb{R}^3\right)$ *(the Marcinkiewicz space),*
- $\mathcal{X} = \dot{M}^3_p\left(\mathbb{R}^3\right)$ *for each* $2 < p \leqslant 3$ *(the Morrey space).*

Proof of Theorem 1.4. We now shall prove Theorem 1.4, we only need to establish a priori estimates. Without loss of generality, we may assume $c = \sigma = 1$ in system (1.3).

Step 1: Energy estimates. Testing $(1.3)_{1,2,3}$ by u, E, B respectively, and adding up the results, we have the well-known energy equality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|(u,E,B)\|^2_{L^2} + \|\nabla u\|^2_{L^2} \\
&\quad = \int_{\mathbb{R}^3} (j\times B)\cdot u \mathrm{d}x + \int_{\mathbb{R}^3} \operatorname{curl} B \cdot E \mathrm{d}x - \int_{\mathbb{R}^3} j\cdot E \mathrm{d}x - \int_{\mathbb{R}^3} \operatorname{curl} E\cdot B \mathrm{d}x \\
&\quad = -\int_{\mathbb{R}^3} (u\times B)\cdot j \mathrm{d}x - \int_{\mathbb{R}^3} E\cdot j \mathrm{d}x \\
&\quad = -\|j\|^2_{L^2},
\end{aligned}$$

where we have used Ohm's law: $j = (E + u\times B)$. Which gives

$$\|(u,E,B)(t)\|^2_{L^2} + 2\int_0^T \|(\nabla u, j)(t)\|^2_{L^2} \mathrm{d}t = \|(u_0,E_0,B_0)\|^2_{L^2}, \quad \forall t \geq 0.$$

Step 2: $\dot{H}^1$ estimates. First, taking $\nabla\times$ on the first equation of (1.3), we get

$$\partial_t \omega + (u\cdot\nabla)\omega - \Delta\omega - S\omega = \nabla\times(j\times B), \tag{3.2}$$

the vortex stretching term $S\omega$ is often written $(\omega \cdot \nabla)u$, that is to say $S\omega = (\omega \cdot \nabla)u$. Indeed, the symmetric part S is given by

$$S_{ij} = \nabla_{sym}(u)_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$$

and the anti-symmetric part A is given by

$$A_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right).$$

Naturally, $\nabla u = S + A$. In addition, we know that in three spatial dimensions the anti-symmetric matrix A can be represented as a vector. Here, we write as:

$$A = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}.$$

Then we have $A\omega = 0$.

Next, multiplying (3.2) by ω and integrating by parts over $\mathbb{R}^3$, one has

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 &= \int_{\mathbb{R}^3} S\omega \cdot \omega dx + \int_{\mathbb{R}^3} \mathrm{curl}\,(j \times B) \cdot \omega dx \\ &= \int_{\mathbb{R}^3} S\omega \cdot \omega dx - \int_{\mathbb{R}^3} (j \times B) \cdot \Delta u dx. \end{aligned} \tag{3.3}$$

Testing $(1.3)_{2,3}$ by $-\Delta E, -\Delta B$ in $L^2(\mathbb{R}^3)$ respectively, and putting together, we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla E, \nabla B)\|_{L^2}^2 &= -\int_{\mathbb{R}^3} \mathrm{curl}\, B \cdot \Delta E \mathrm{d}x + \int_{\mathbb{R}^3} j \cdot \Delta E \mathrm{d}x + \int_{\mathbb{R}^3} \mathrm{curl}\, E \cdot \Delta B \mathrm{d}x \\ &= -\int_{\mathbb{R}^3} \partial_i j \cdot \partial_i E \mathrm{d}x. \end{aligned} \tag{3.4}$$

Since

$$\begin{aligned} -\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i j \cdot \partial_i E \mathrm{d}x &= -\sum_{i=1}^{3}\int_{\mathbb{R}^3} \partial_i j \cdot \partial_i (j - u \times B) \mathrm{d}x \\ &= -\|\nabla j\|_{L^2}^2 + \int_{\mathbb{R}^3} \partial_i (u \times B) \cdot \partial_i j \mathrm{d}x. \end{aligned} \tag{3.5}$$

From the equalities in (3.3)–(3.5), it follows that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\omega, \nabla E, \nabla B)\|_{L^2}^2 + \|(\nabla\omega, \nabla j)\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} S\omega \cdot \omega \mathrm{d}x + \int_{\mathbb{R}^3} \partial_i (j \times B) \cdot \partial_i u \mathrm{d}x + \int_{\mathbb{R}^3} \partial_i (u \times B) \cdot \partial_i j \mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} S\omega \cdot \omega \mathrm{d}x + \int_{\mathbb{R}^3} |j||\nabla u||\nabla B| \mathrm{d}x + \int_{\mathbb{R}^3} |u||\nabla B||\nabla j| \mathrm{d}x \\ &= I + J + K, \end{aligned} \tag{3.6}$$

where we have used the following cancellation property

$$\int_{\mathbb{R}^3} \partial_i j \times B \cdot \partial_i u \mathrm{d}x + \int_{\mathbb{R}^3} \partial_i j \cdot \partial_i u \times B \mathrm{d}x = 0.$$

Let (1.8) hold true. We use proposition 3.6 and lemma 3.4 to bound I as follows

$$\begin{aligned} I &\leq C\|S\|_{\mathcal{X}}\|\omega\|_{L^2}\|\nabla\omega\|_{L^2} \\ &\leq C\,\|S\|_{\mathcal{X}}^2\,\|\omega\|_{L^2}^2+\epsilon\|\nabla\omega\|_{L^2}^2, \end{aligned} \tag{3.7}$$

Similarly, we bound $J+K$ as follows

$$J+K \leq 2\epsilon\,\|\nabla j\|_{L^2}^2 + C\,\|\nabla B\|_{\mathcal{X}}^2\,\|\nabla u\|_{L^2}^2. \tag{3.8}$$

Where $\mathcal{X}$ is one of the Banach spaces:

- $\mathcal{X}=\dot{H}^{1/2}\left(\mathbb{R}^3\right)$ (the homogeneous Sobolev space),
- $\mathcal{X}=L^3\left(\mathbb{R}^3\right)$ (the Lebesgue space),
- $\mathcal{X}=\left\{f\in L^\infty_{loc}\left(\mathbb{R}^3\right):\|f\|=\sup_{x\in\mathcal{X}}|x\|f(x)|<\infty\right\}$ (the weighted L^∞-space),
- $\mathcal{X}=\mathcal{PM}^2\left(\mathbb{R}^3\right)$ (the Le Jan–Sznitman space),
- $\mathcal{X}=L^{3,\infty}\left(\mathbb{R}^3\right)$ (the Marcinkiewicz space),
- $\mathcal{X}=\dot{M}^3_p\left(\mathbb{R}^3\right)$ for each $2<p\leqslant 3$ (the Morrey space).

Applying the entropy identity (i.e., Lemma 3.4) and collecting (3.7) and (3.8) into (3.6), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\omega,\nabla E,\nabla B)\|_{L^2}^2+\|(\nabla\omega,\nabla j)\|_{L^2}^2 \leq C\left(\|S\|_{\mathcal{X}}^2+\|\nabla B\|_{\mathcal{X}}^2\right)\|\omega\|_{L^2}^2. \tag{3.9}$$

Using the Gronwall inequality, we conclude that

$$\sup_{0\leq t\leq T}\int|\nabla u|^2+|\nabla E|^2+|\nabla B|^2\mathrm{d}x\leq C$$

and

$$\int_0^T\|\Delta u\|_{L^2}^2+\|\nabla j\|_{L^2}^2\mathrm{d}t\leq C.$$

Step 3: $\dot{H}^2$ estimates. Applying Δ to $(1.3)_{1,2,3}$ and multiplying by Δu, ΔE and ΔB, respectively, and add the result equations, one has

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\,(\Delta u,\Delta B,\Delta E)\,\|_{L^2}^2+\|\,(\nabla\Delta u,\Delta j)\,\|_{L^2}^2 \\ &\quad=\int_{\mathbb{R}^3}-\Delta(u\cdot\nabla u)\cdot\Delta u+\Delta(j\times B)\cdot\Delta u+\Delta j\cdot\Delta(u\times B)\mathrm{d}x \\ &\quad=C\left(\|\nabla u\|_{L^3}^2+1\right)\|\Delta u\|_{L^2}^2+\epsilon\|\nabla\Delta u\|_{L^2}^2 \\ &\qquad+\int_{\mathbb{R}^3}\partial_{kk}j\times B\cdot\Delta u+\partial_k j\times\partial_k B\cdot\Delta u+j\times\partial_{kk}B\cdot\Delta u\mathrm{d}x \\ &\qquad+\int_{\mathbb{R}^3}\Delta j\cdot\partial_{kk}u\times B+\Delta j\cdot\partial_k u\times\partial_k B+\Delta j\cdot u\times\partial_{kk}B\mathrm{d}x \\ &\quad=C\left(\|\nabla u\|_{L^3}^2+1\right)\|\Delta u\|_{L^2}^2+\epsilon\|\nabla\Delta u\|_{L^2}^2+\underbrace{\int_{\mathbb{R}^3}\partial_k j\times\partial_k B\cdot\Delta u+j\times\partial_{kk}B\cdot\Delta u\mathrm{d}x}_{I} \\ &\qquad+\underbrace{\int_{\mathbb{R}^3}\Delta j\cdot\partial_k u\times\partial_k B+\Delta j\cdot u\times\partial_{kk}B\mathrm{d}x}_{J}, \end{aligned} \tag{3.10}$$

where used the following cancellation property:

$$\int_{\mathbb{R}^3} \partial_{kk} j \times B \cdot \Delta u \mathrm{d}x + \int_{\mathbb{R}^3} \Delta j \cdot \partial_{kk} u \times B \mathrm{d}x = 0.$$

I can be estimated as

$$\begin{aligned} I &\leq C\|\Delta u\|_{L^2}\|\nabla B\|_{L^3}\|\nabla j\|_{L^6} + C\|j\|_{L^3}\|\Delta B\|_{L^2}\|\Delta u\|_{L^6} \\ &\leq C\|\Delta u\|_{L^2}^2\left(\|\nabla B\|_{L^2}^2 + \|\Delta B\|_{L^2}^2\right) + C\left(\|j\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right)\|\Delta B\|_{L^2}^2 + \epsilon\|\Delta j\|_{L^2} + \epsilon\|\nabla\Delta u\|_{L^2}^2 \end{aligned} \tag{3.11}$$

Similarly,

$$J \leq C\left(\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right)\|\Delta B\|_{L^2}^2 + C\left(\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2\right)\|\Delta B\|_{L^2}^2 + 2\epsilon\|\Delta j\|_{L^2}^2 \tag{3.12}$$

Putting (3.11) and (3.12) into (3.10), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|(\Delta u, \Delta B, \Delta E)\|_{L^2}^2 + \|(\nabla\Delta u, \Delta j)\|_{L^2}^2 \\ &\qquad \leq C\left(\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|j\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right)\|(\Delta u, \Delta B, \Delta E)\|_{L^2}^2, \end{aligned} \tag{3.13}$$

and by Gronwall's lemma, we get that

$$\begin{aligned} u, B, E &\in L^\infty\left(0, T, \dot{H}^2\right); \\ \nabla u, j &\in L^2\left(0, T, \dot{H}^2\right). \end{aligned}$$

This completes the proof of Theorem 1.4. □

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Oscillation criteria for perturbed half-linear differential equations

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Abstract. Oscillatory properties of perturbed half-linear differential equations are investigated. We make use of the modified Riccati technique. A certain linear differential equation associated with the modified Riccati equation plays an important part. Improved oscillation criteria for a perturbed half-linear Riemann–Weber differential equation can be obtained.

Keywords: half-linear differential equation, oscillation criteria, Riemann–Weber differential equation, principal solution.

2020 Mathematics Subject Classification: 34C10.

1 Introduction

In this paper we consider the second order half-linear ordinary differential equation

$$(p(t)\Phi_\alpha(x'))' + q(t)\Phi_\alpha(x) = 0, \quad t \geq t_0, \tag{1.1}$$

where $\Phi_\alpha(x) = |x|^\alpha \operatorname{sgn} x$ with $\alpha > 0$, $p(t)$ and $q(t)$ are real-valued continuous functions on $[t_0, \infty)$, and $p(t) > 0$ for $t \geq t_0$. If $\alpha = 1$, then (1.1) reduces to the linear equation

$$(p(t)x')' + q(t)x = 0, \quad t \geq t_0. \tag{1.2}$$

The half-linear equation (1.1) can be seen as a natural generalization of the linear equation (1.2).

For a solution $x(t)$ of (1.1), the vector function

$$(x(t), y(t)) = (x(t), p(t)\Phi_\alpha(x'(t)))$$

is a solution of the two-dimensional nonlinear system

$$x' = p(t)^{-1/\alpha}\Phi_{1/\alpha}(y), \quad y' = -q(t)\Phi_\alpha(x). \tag{1.3}$$

Conversely, for a solution $(x(t), y(t))$ of (1.3), the first component $x(t)$ is a solution of (1.1). The system of the type (1.3) was considered by Mirzov [11]. Using the result of Mirzov

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[11, Lemma 2.1], we see that all local solutions of (1.1) can be continued to t_0 and ∞, and so all solutions of (1.1) exist on the entire interval $[t_0, \infty)$. Analogues of Sturm's comparison theorem and Sturm's separation theorem remain valid for (1.1) (Mirzov [11, Theorem 1.1]). Hence, if the equation (1.1) has a nonoscillatory solution, then any other nontrivial solution is also nonoscillatory. If the equation (1.1) has an oscillatory solution, then any other nontrivial solution is also oscillatory. Clearly, if $x(t)$ is a solution of (1.1), then so is $-x(t)$. Therefore we can suppose without loss of generality that a nonoscillatory solution of (1.1) is eventually positive.

In the last three decades, many results have been obtained in the theory of oscillatory and asymptotic behavior of solutions of half-linear differential equations. It is known that basic results for the second order linear equations can be generalized to the second order half-linear equations. The important works are summarized in the book of Došlý and Řehák [8]. For the recent results to half-linear equations we refer the reader to, for example, [1–7,9,10,12–16]. The present paper is strongly motivated by oscillatory and nonoscilaltory results in [2–4,6,7,9].

For the equation (1.1), it is sometimes assumed that

$$\int_{t_0}^{\infty} p(s)^{-1/\alpha} ds = \lim_{t\to\infty} \int_{t_0}^{t} p(s)^{-1/\alpha} ds = \infty \tag{1.4}$$

and

$$\begin{cases} \displaystyle\int_{t_0}^{\infty} q(s)ds = \lim_{t\to\infty} \int_{t_0}^{t} q(s)ds \quad \text{is convergent, and} \\ \displaystyle\int_{t}^{\infty} q(s)ds \geq 0, \not\equiv 0 \quad \text{on} \quad [t_0^+, \infty) \quad \text{for any} \quad t_0^+ \geq t_0. \end{cases} \tag{1.5}$$

We point out that for any nonoscillatory solution $x(t)$ of (1.1) with (1.4) and (1.5) the derivative $x'(t)$ does not vanish eventually. More precisely,

Proposition 1.1. *Consider the equation* (1.1) *under the conditions* (1.4) *and* (1.5)*. Let $x(t)$ be a nonoscillatory solution of* (1.1) *such that $x(t) > 0$ for $t \geq T$ $(\geq t_0)$. Then, $x'(t) > 0$ for $t \geq T$.*

The above fact is easily deduced from the generalized Riccati integral equation associated with (1.1). See Lemma 2.3 in the next section.

Together with the equation (1.1), we consider the equation of the same type

$$(p(t)\Phi_\alpha(x'))' + q_0(t)\Phi_\alpha(x) = 0, \quad t \geq t_0, \tag{1.6}$$

where $q_0(t)$ is a real-valued continuous function on $[t_0, \infty)$. The equation (1.1) is regarded as a perturbation of the equation (1.6). In this paper it will be assumed that (1.6) has a nonoscillatory solution $x = x_0(t)$ such that

$$x_0(t) > 0, \quad x_0'(t) > 0 \quad \text{for } t \geq T \tag{1.7}$$

and

$$\int_T^{\infty} \frac{1}{p(t)x_0(t)^2 x_0'(t)^{\alpha-1}} dt = \infty. \tag{1.8}$$

The condition (1.8) is closely related to an integral characterization of the principal solution of (1.6). For the concept of the principal solution, see Došlý and Řehák [8, Section 4.2]. The following result is known.

Theorem 1.2 (Došlý and Elbert [5] and Došlá and Došlý [1, Proposition 2]). *Suppose that $x = x_0(t)$ is a nonoscillatory solution of (1.6) satisfying (1.7).*

(i) *Let $0 < \alpha \le 1$. If (1.8) is satisfied, then $x_0(t)$ is the principal solution of (1.6).*

(ii) *Let $\alpha \ge 1$. If $x_0(t)$ is the principal solution of (1.6), then (1.8) holds.*

(iii) *Let $\alpha \ge 1$, and suppose that the conditions (1.4) and*

$$\int_{t_0}^{\infty} q_0(s)ds \quad \text{exists and} \quad \int_{t}^{\infty} q_0(s)ds \ge 0, \not\equiv 0 \quad \text{eventually} \tag{1.9}$$

are satisfied. Then, $x_0(t)$ is the principal solution of (1.6) if and only if (1.8) holds.

Note that the part (iii) of Theorem 1.2 is stated in [5, Theorem 3.3] and [8, Theorem 4.2.8] without the condition $\alpha \ge 1$. The part (iii) of Theorem 1.2 may fail to hold for $0 < \alpha < 1$ (see [1, Example 1]).

As an important oscillatory result the following theorem is known.

Theorem 1.3 (Došlý and Lomtatidze [7, Theorem 1]). *Suppose that the equation (1.6) is nonoscillatory and let $x = x_0(t)$ be the principal solution of (1.6) satisfying $x_0(t) > 0$ for $t \ge T$. If*

$$\int_{T}^{\infty} x_0(t)^{\alpha+1} \left[q(t) - q_0(t)\right] dt = \infty,$$

then the equation (1.1) is oscillatory.

Now let us consider the case where the equation (1.6) has a nonoscillatory solution $x = x_0(t)$ satisfying (1.7), (1.8) and

$$\int_{T}^{\infty} x_0(t)^{\alpha+1} \left[q(t) - q_0(t)\right] dt \quad \text{is convergent.} \tag{1.10}$$

It is not assumed that $x = x_0(t)$ is principal. Then we set

$$P(t) = p(t)x_0(t)^2 x_0'(t)^{\alpha-1} \quad \text{and} \quad Q(t) = x_0(t)^{\alpha+1}\left[q(t) - q_0(t)\right]. \tag{1.11}$$

Note that (1.7) implies $P(t) > 0$ $(t \ge T)$.

The condition

$$\liminf_{t\to\infty} p(t)x_0(t)x_0'(t)^{\alpha} > 0 \tag{1.12}$$

also plays an important part. The following nonoscillatory result has been showed by Došlý and Fišnarová.

Theorem 1.4 (Došlý and Fišnarová [6, Theorem 3]). *Suppose that the equation (1.6) has a nonoscillatory solution $x = x_0(t)$ satisfying (1.7), (1.8) and (1.12). Suppose moreover that (1.10) holds. If there exists $\varepsilon > 0$ such that the linear equation*

$$(P(t)x')' + (1+\varepsilon)\frac{\alpha+1}{2\alpha}Q(t)x = 0 \tag{1.13}$$

is nonoscillatory, then the equation (1.1) is nonoscillatory.

The following corollary is obtained by applying the classical Hille–Nehari nonoscillation criterion to the linear equation (1.13).

Corollary 1.5 (Došlý and Fišnarová [6, Corollary 1 (i)]). *Suppose that* (1.6) *has a nonoscillatory solution* $x = x_0(t)$ *satisfying* (1.7), (1.8) *and* (1.12). *Suppose moreover that* (1.10) *holds. If*

$$-\frac{3\alpha}{2(\alpha+1)} < \liminf_{t\to\infty} \left(\int_T^t \frac{1}{P(s)} ds\right) \left(\int_t^\infty Q(s) ds\right) \\ \leq \limsup_{t\to\infty} \left(\int_T^t \frac{1}{P(s)} ds\right) \left(\int_t^\infty Q(s) ds\right) < \frac{\alpha}{2(\alpha+1)},$$

then the equation (1.1) *is nonoscillatory.*

In this paper the following theorem will be proved.

Theorem 1.6. *Suppose that* $p(t)$ *and* $q(t)$ *satisfy* (1.4) *and* (1.5), *respectively. Suppose that the equation* (1.6) *has a nonoscillatory solution* $x = x_0(t)$ *satisfying* (1.7), (1.8) *and* (1.12). *Suppose moreover that* (1.10) *holds. If there exists a number* ε *with* $0 < \varepsilon < 1$ *such that the linear equation*

$$(P(t)x')' + (1-\varepsilon)\frac{\alpha+1}{2\alpha} Q(t)x = 0 \tag{1.14}$$

is oscillatory, then the equation (1.1) *is oscillatory.*

Theorem 1.6 was proved in [3, Theorem 1] under the restricted condition that

$$\lim_{t\to\infty} p(t)x_0(t)x_0'(t)^\alpha \quad \text{exists and is a positive finite value.}$$

Theorem 1.6 gives a partial extension of Theorem 4 in [6]. Applying the classical Hille–Nehari oscillation criterion to the linear equation (1.14), we have the following corollary.

Corollary 1.7. *Suppose that* $p(t)$ *and* $q(t)$ *satisfy* (1.4) *and* (1.5), *respectively. Suppose that* (1.6) *has a nonoscillatory solution* $x = x_0(t)$ *satisfying* (1.7), (1.8) *and* (1.12). *Suppose moreover that* (1.10) *holds. If*

$$\liminf_{t\to\infty} \left(\int_T^t \frac{1}{P(s)} ds\right) \left(\int_t^\infty Q(s) ds\right) > \frac{\alpha}{2(\alpha+1)}, \tag{1.15}$$

then the equation (1.1) *is oscillatory.*

Corollary 1.7 is a new result, while it is similar to Corollary 1 (ii) in [6].

Now, let

$$E(\alpha) = \frac{1}{\alpha+1}\left(\frac{\alpha}{\alpha+1}\right)^\alpha, \quad \mu(\alpha) = \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right)^\alpha, \tag{1.16}$$

and

$$\log_0 t = t, \quad \log_k t = \log(\log_{k-1} t), \quad \operatorname{Log}_k t = \prod_{j=1}^k \log_j t \quad (k = 1, 2, 3, \dots).$$

Then, consider the half-linear equation

$$(\Phi_\alpha(x'))' + \left(\frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}} \sum_{j=1}^n \frac{1}{(\operatorname{Log}_j t)^2} + c(t)\right) \Phi_\alpha(x) = 0, \tag{1.17}$$

where $c(t)$ is a continuous function on an interval $[t_0, \infty)$ with sufficiently large t_0. The equation (1.17) is regarded as a perturbation of the half-linear Riemann–Weber (sometimes also called Euler–Weber) differential equation

$$(\Phi_\alpha(x'))' + \left(\frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}} \sum_{j=1}^n \frac{1}{(\operatorname{Log}_j t)^2}\right) \Phi_\alpha(x) = 0. \tag{1.18}$$

It is known that (1.18) is nonoscillatory. Moreover, the asymptotic forms of (nonoscillatory) solutions of (1.18) are investigated by Elbert and Schneider [9, Corollary 1]. In this paper we pay attention to the fact that (1.18) has a nonoscillatory solution $x(t)$ such that

$$x(t) \sim t^{\alpha/(\alpha+1)}(\mathrm{Log}_n\, t)^{1/(\alpha+1)} \quad (t \to \infty). \tag{1.19}$$

We can prove the following theorem.

Theorem 1.8. *If*

$$\int_{t_0}^{\infty} t^{\alpha}(\mathrm{Log}_n\, t)c(t)dt = \infty, \tag{1.20}$$

then (1.17) *is oscillatory.*

The case $n = 1$ in Theorem 1.8 was obtained by Došlý [2, Corollary 1]. Theorem C in Elbert and Schneider [9] can be regarded as the case $n = 0$ in Theorem 1.8.

Next, consider the case where

$$\int_{t_0}^{\infty} t^{\alpha}(\mathrm{Log}_n\, t)c(t)dt \quad \text{is convergent.} \tag{1.21}$$

The following theorem is known.

Theorem 1.9 (Došlý [4, Theorem 3.3 (i)])**.** *Consider the equation* (1.17) *under the condition* (1.21)*. If*

$$\begin{aligned} -3\mu(\alpha) &< \liminf_{t\to\infty}\,(\log_{n+1} t) \int_t^{\infty} s^{\alpha}(\mathrm{Log}_n\, s)c(s)ds \\ &\le \limsup_{t\to\infty}\,(\log_{n+1} t) \int_t^{\infty} s^{\alpha}(\mathrm{Log}_n\, s)c(s)ds < \mu(\alpha), \end{aligned}$$

then (1.17) *is nonoscillatory.*

In the present paper the following theorem will be proved.

Theorem 1.10. *Consider the equation* (1.17) *under the condition* (1.21)*. If*

$$\liminf_{t\to\infty}\,(\log_{n+1} t) \int_t^{\infty} s^{\alpha}(\mathrm{Log}_n\, s)c(s)ds > \mu(\alpha), \tag{1.22}$$

then (1.17) *is oscillatory.*

Theorem 1.10 gives an improvement of Theorem 3.3 (ii) in [4].

Theorem 5 in [9] can be regarded as the case $n = 0$ in Theorems 1.9 and 1.10. Note that Theorem 5 in [9] is restricted to the case $n = 0$ and

$$\int_t^{\infty} s^{\alpha}c(s)ds \ge 0 \quad \text{for all large } t.$$

In the next section we state several basic (non)oscillatory results for the half-linear differential equation (1.1). The proofs are contained in the book of Došlý and Řehák [8]. For the proof of Theorem 1.6 we need some estimates for the function $F(u,v)$ which appears in the modified Riccati equation associated with (1.1). In Section 3 we state and prove the estimates for $F(u,v)$. The proof of Theorem 1.6 is given in Section 4. The proofs of Theorems 1.8 and 1.10 are presented in Section 5.

2 Basic results

For the convenience of the reader we summarize basic (non)oscillatory results for the half-linear differential equation (1.1). As usual, we use the asterisk notation

$$\xi^{\alpha *} = \Phi_\alpha(\xi) = |\xi|^\alpha \operatorname{sgn} \xi, \quad \xi \in \mathbb{R}, \quad \alpha > 0.$$

Then it is easy to see that, for $\xi, \eta \in \mathbb{R}$ and $\alpha, \alpha_1, \alpha_2 > 0$,

- $(\xi\eta)^{\alpha *} = \xi^{\alpha *}\eta^{\alpha *}, \quad (-\xi)^{\alpha *} = -\xi^{\alpha *};$
- $(\xi^{\alpha_1 *})^{\alpha_2 *} = \xi^{(\alpha_1\alpha_2)*}, \quad (\xi^{\alpha *})^{(1/\alpha)*} = \xi, \quad \left(\xi^{(1/\alpha)*}\right)^{\alpha *} = \xi;$
- $\xi^{\alpha *} \le \eta^{\alpha *}$ if and only if $\xi \le \eta$; $\xi^{\alpha *} < \eta^{\alpha *}$ if and only if $\xi < \eta$;
  $\xi^{\alpha *} = \eta^{\alpha *}$ if and only if $\xi = \eta$.

With this asterisk notation, the equation (1.1) is rewritten as

$$(p(t)(x')^{\alpha *})' + q(t)x^{\alpha *} = 0, \quad t \ge t_0.$$

Lemma 2.1. *The equation (1.1) is nonoscillatory if and only if there is a continuously differentiable function $y(t)$ which satisfies the generalized Riccati differential inequality*

$$y' + q(t) + \alpha p(t)^{-1/\alpha}|y|^{(\alpha+1)/\alpha} \le 0$$

on an interval $[T, \infty)$, $T \ge t_0$.

In what follows we consider the equation (1.1) under the condition (1.4). The next theorem is a half-linear extension of the classical Wintner oscillation criterion for (1.2).

Lemma 2.2. *Suppose that (1.4) holds. If*

$$\int_{t_0}^{\infty} q(t)dt = \infty,$$

then (1.1) is oscillatory.

As a next step we consider the case where (1.4) holds and

$$\int_{t_0}^{\infty} q(t)dt \quad \text{is convergent.} \tag{2.1}$$

Lemma 2.3. *Consider the equation (1.1) under the conditions (1.4) and (2.1). Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(t) > 0$ for $t \ge T$ $(\ge t_0)$. Then*

$$p(t)\left(\frac{x'(t)}{x(t)}\right)^{\alpha *} = \int_t^{\infty} q(s)ds + \alpha \int_t^{\infty} p(s)\left|\frac{x'(s)}{x(s)}\right|^{\alpha+1} ds, \quad t \ge T.$$

If the additional condition

$$\int_t^{\infty} q(s)ds \ge 0, \not\equiv 0 \quad \text{on} \quad [T^+, \infty) \quad \text{for any} \quad T^+ \ge T$$

is satisfied, then $x'(t) > 0$ for $t \ge T$.

The following lemma is the Hille–Nehari type (non)oscillation criteria for the equation (1.1).

Lemma 2.4. *Consider the equation* (1.1) *under the conditions* (1.4) *and* (2.1). *Let* $E(\alpha)$ *be the constant defined by the former part of* (1.16).

(i) *The equation* (1.1) *is nonoscillatory provided*

$$\begin{aligned} -(2\alpha+1)E(\alpha) &< \liminf_{t\to\infty}\left(\int_{t_0}^{t} p(s)^{-1/\alpha}ds\right)^{\alpha}\left(\int_{t}^{\infty} q(s)ds\right) \\ &\le \limsup_{t\to\infty}\left(\int_{t_0}^{t} p(s)^{-1/\alpha}ds\right)^{\alpha}\left(\int_{t}^{\infty} q(s)ds\right) < E(\alpha). \end{aligned}$$

(ii) *The equation* (1.1) *is oscillatory provided*

$$\liminf_{t\to\infty}\left(\int_{t_0}^{t} p(s)^{-1/\alpha}ds\right)^{\alpha}\left(\int_{t}^{\infty} q(s)ds\right) > E(\alpha).$$

The results mentioned here are half-linear extensions of the classical results for the linear equation (1.2). For the proofs, see [8].

3 Lemmas

It is known that the function

$$F(u,v) = |u+v|^{(\alpha+1)/\alpha} - |v|^{(\alpha+1)/\alpha} - \frac{\alpha+1}{\alpha}v^{(1/\alpha)*}u, \quad u,v\in\mathbb{R}, \tag{3.1}$$

plays a crucial role in the study of the oscillation and nonoscillation of (1.1).

Lemma 3.1 (see, e.g., Došlý and Fišnarová [6, Lemma 4])**.** *Let* $x = x(t)$ *and* $x = x_0(t)$ *be nonoscillatory solutions of* (1.1) *and* (1.6), *respectively. Suppose that* $x(t) > 0$ *and* $x_0(t) > 0$ *for* $t \ge T$ $(\ge t_0)$. *Then the function*

$$u(t) = p(t)x_0(t)^{\alpha+1}\left[\left(\frac{x'(t)}{x(t)}\right)^{\alpha*} - \left(\frac{x_0'(t)}{x_0(t)}\right)^{\alpha*}\right], \quad t \ge T, \tag{3.2}$$

is a solution of the modified Riccati differential equation

$$\begin{aligned} &u'(t) + x_0(t)^{\alpha+1}[q(t) - q_0(t)] \\ &\quad + \alpha p(t)^{-1/\alpha}x_0(t)^{-(\alpha+1)/\alpha}F(u(t), p(t)x_0(t)x_0'(t)^{\alpha*}) = 0, \quad t \ge T, \end{aligned} \tag{3.3}$$

where $F(u,v)$ *is defined by* (3.1).

Lemma 3.2. *Let* $F(u,v)$ *be the function which is defined by* (3.1).

(i) $F(u,v) \ge 0$ *for all* $u,v \in \mathbb{R}$; $F(u,v) = 0$ *if and only if* $u = 0$.

(ii) *Let* $k > 0$ *be a constant. Then there are constants* $L_1(k) > 0$ *and* $L_2(k) > 0$ *such that*

$$L_1(k)|v|^{(1/\alpha)-1}u^2 \le F(u,v) \le L_2(k)|v|^{(1/\alpha)-1}u^2 \tag{3.4}$$

for $v > 0$ *and* $-v < u \le kv$.

(iii) *Let* k_1 *and* k_2 *be constants satisfying* $0 < k_1 < k_2$. *Then there is a constant* $L(k_1, k_2) > 0$ *such that* $F(u, v)$ *can be expressed in the following form*

$$F(u,v) = \frac{\alpha+1}{2\alpha^2}|v|^{(1/\alpha)-1}u^2(1+R(u,v))$$

with

$$|R(u,v)| \le \frac{|\alpha-1|}{3\alpha}L(k_1,k_2)|u|$$

for $v > 0$ *and* $|u| \le k_1 < k_2 \le v$.

Proof. It is obvious that $F(0, v) = 0$. Differentiating the function $F(u, v)$ with respect to u, we obtain

$$\begin{aligned} F_u(u,v) &= \frac{\alpha+1}{\alpha}(u+v)^{(1/\alpha)*} - \frac{\alpha+1}{\alpha}v^{(1/\alpha)*}, \quad u,v \in \mathbb{R}, \\ F_{uu}(u,v) &= \frac{\alpha+1}{\alpha^2}|u+v|^{(1/\alpha)-1}, \quad u > -v, \\ F_{uuu}(u,v) &= \frac{(\alpha+1)(-\alpha+1)}{\alpha^3}(u+v)^{[(1/\alpha)-2]*}, \quad u > -v. \end{aligned}$$

Then, $F_u(0, v) = 0$ ($v \in \mathbb{R}$) and $F_{uu}(0, v) = [(\alpha + 1)/\alpha^2]|v|^{(1/\alpha)-1}$ ($v > 0$).

(i) Let $v \in \mathbb{R}$ be fixed. It is seen that $F_u(u, v) < 0$ for $u < 0$, $F_u(u, v) > 0$ for $u > 0$ and $F_u(0, v) = 0$. This means that $F(u, v)$ is strictly decreasing on $(-\infty, 0)$ and $F(u, v)$ is strictly increasing on $(0, \infty)$. Then, since $F(0, v) = 0$, it is clear that $F(u, v) \ge 0$ for $u \in \mathbb{R}$ and $F(u, v) = 0$ if and only if $u = 0$.

(ii) Let $v > 0$ and $-v < u \le kv$. By Taylor's theorem with integral remainder we have

$$F(u,v) = F(0,v) + F_u(0,v)u + \int_0^u (u-s)F_{uu}(s,v)ds.$$

Hence

$$\begin{aligned} F(u,v) &= \frac{\alpha+1}{\alpha^2}\int_0^u (u-s)|s+v|^{(1/\alpha)-1}ds \\ &= \frac{\alpha+1}{\alpha^2}\int_0^1 (u-u\sigma)|u\sigma+v|^{(1/\alpha)-1}ud\sigma \\ &= \frac{\alpha+1}{\alpha^2}|v|^{(1/\alpha)-1}u^2\int_0^1 (1-\sigma)\left|\frac{u}{v}\sigma+1\right|^{(1/\alpha)-1}d\sigma. \end{aligned}$$

Then, noting

$$0 \le -\sigma+1 \le \frac{u}{v}\sigma+1 \le k\sigma+1 \quad (0 \le \sigma \le 1),$$

we find that, for the case $0 < \alpha \le 1$,

$$\begin{aligned} \frac{\alpha}{\alpha+1} = \int_0^1 (1-\sigma)^{1/\alpha}d\sigma &\le \int_0^1 (1-\sigma)\left|\frac{u}{v}\sigma+1\right|^{(1/\alpha)-1}d\sigma \\ &\le \int_0^1 (1-\sigma)(k\sigma+1)^{(1/\alpha)-1}d\sigma; \end{aligned}$$

and, for the case $\alpha > 1$,

$$\begin{aligned} \int_0^1 (1-\sigma)(k\sigma+1)^{(1/\alpha)-1}d\sigma &\le \int_0^1 (1-\sigma)\left|\frac{u}{v}\sigma+1\right|^{(1/\alpha)-1}d\sigma \\ &\le \int_0^1 (1-\sigma)^{1/\alpha}d\sigma = \frac{\alpha}{\alpha+1}. \end{aligned}$$

This shows that (3.4) holds with positive constants $L_1(k)$ and $L_2(k)$ such that, for the case $0 < \alpha \le 1$,

$$L_1(k) = \frac{1}{\alpha} \quad \text{and} \quad L_2(k) = \frac{\alpha+1}{\alpha^2}\int_0^1 (1-\sigma)(k\sigma+1)^{(1/\alpha)-1}d\sigma;$$

and, for the case $\alpha > 1$,

$$L_1(k) = \frac{\alpha+1}{\alpha^2}\int_0^1 (1-\sigma)(k\sigma+1)^{(1/\alpha)-1}d\sigma \quad \text{and} \quad L_2(k) = \frac{1}{\alpha}.$$

(iii) Let $v > 0$ and $|u| \le k_1 < k_2 \le v$. By Taylor's theorem there is θ such that $0 < \theta < 1$ and

$$F(u,v) = F(0,v) + F_u(0,v)u + \frac{F_{uu}(0,v)}{2!}u^2 + \frac{F_{uuu}(\theta u,v)}{3!}u^3.$$

Hence

$$\begin{aligned} F(u,v) &= \frac{\alpha+1}{2\alpha^2}|v|^{(1/\alpha)-1}u^2 + \frac{(\alpha+1)(-\alpha+1)}{6\alpha^3}(\theta u+v)^{[(1/\alpha)-2]*}u^3 \\ &= \frac{\alpha+1}{2\alpha^2}|v|^{(1/\alpha)-1}u^2\left[1 + \frac{-\alpha+1}{3\alpha}|v|^{-(1/\alpha)+1}(\theta u+v)^{[(1/\alpha)-2]*}u\right]. \end{aligned}$$

Notice here that

$$\theta u + v \ge -|u| + v \ge -k_1 + k_2 > 0 \quad (0 < \theta < 1)$$

and

$$0 < -\frac{k_1}{k_2} + 1 \le \frac{u}{v}\theta + 1 = \frac{\theta u + v}{v} \le \frac{k_1}{k_2} + 1 \quad (0 < \theta < 1).$$

Put

$$R(u,v) = \frac{-\alpha+1}{3\alpha}|v|^{-(1/\alpha)+1}(\theta u+v)^{[(1/\alpha)-2]*}u.$$

Then it is easy to see that

$$|R(u,v)| \le \frac{|\alpha-1|}{3\alpha}\left|\frac{\theta u+v}{v}\right|^{(1/\alpha)-1}|\theta u+v|^{-1}|u| \le \frac{|\alpha-1|}{3\alpha}L(k_1,k_2)|u|,$$

where $L(k_1,k_2)$ is given by

$$L(k_1,k_2) = \begin{cases} \left(1+\dfrac{k_1}{k_2}\right)^{(1/\alpha)-1}(k_2-k_1)^{-1} & (0 < \alpha \le 1), \\ \left(1-\dfrac{k_1}{k_2}\right)^{(1/\alpha)-1}(k_2-k_1)^{-1} & (\alpha > 1). \end{cases}$$

This proves the part (iii) of Lemma 3.2. □

4 Proofs of the results

Proof of Theorem 1.6. Suppose that there is $\varepsilon \in (0,1)$ such that (1.14) is oscillatory. Assume, by contradiction, that the equation (1.1) has a nonoscillatory solution $x(t)$. We may suppose that $x(t) > 0$ for $t \ge T$. Then, we define the function $u(t)$ by (3.2). By Lemma 3.1, $u(t)$ satisfies (3.3). Integrating (3.3) from T to t, we obtain

$$\begin{aligned} &u(t) - u(T) + \int_T^t x_0(s)^{\alpha+1}[q(s)-q_0(s)]ds \\ &\quad + \alpha\int_T^t p(s)^{-1/\alpha}x_0(s)^{-(\alpha+1)/\alpha}F(u(s),p(s)x_0(s)x_0'(s)^{\alpha})ds = 0 \end{aligned} \tag{4.1}$$

for $t \geq T$. Since the integrand of the last integral in the left-hand side of (4.1) is nonnegative for $t \geq T$ (see Lemma 3.2 (i)), we have either

$$\int_T^\infty p(s)^{-1/\alpha} x_0(s)^{-(\alpha+1)/\alpha} F(u(s), p(s)x_0(s)x_0'(s)^\alpha)ds = \infty \tag{4.2}$$

or

$$\int_T^\infty p(s)^{-1/\alpha} x_0(s)^{-(\alpha+1)/\alpha} F(u(s), p(s)x_0(s)x_0'(s)^\alpha)ds < \infty. \tag{4.3}$$

Suppose first that (4.2) holds. Since (1.10) is assumed to hold, it follows from (4.1) that $u(t) \to -\infty$ as $t \to \infty$. We may suppose that $u(t) < 0$ for $t \geq T$. By Lemma 2.3 we have $x'(t) > 0$ for $t \geq T$. Hence, by (3.2), we get

$$-p(t)x_0(t)x_0'(t)^\alpha = -p(t)x_0(t)^{\alpha+1}\left(\frac{x_0'(t)}{x_0(t)}\right)^{\alpha*} < u(t) \leq p(t)x_0(t)x_0'(t)^\alpha, \quad t \geq T.$$

Applying Lemma 3.2 (ii) to the case $k = 1$, $u = u(t)$ and $v = p(t)x_0(t)x_0'(t)^\alpha$, we find that there are constants $L_1 = L_1(1) > 0$ and $L_2 = L_2(1) > 0$ such that

$$\begin{aligned} &L_1 p(t)^{-1} x_0(t)^{-2} x_0'(t)^{-\alpha+1} u(t)^2 \\ &\quad \leq p(t)^{-1/\alpha} x_0(t)^{-(\alpha+1)/\alpha} F(u(t), p(t)x_0(t)x_0'(t)^\alpha) \\ &\quad \leq L_2 p(t)^{-1} x_0(t)^{-2} x_0'(t)^{-\alpha+1} u(t)^2, \quad t \geq T. \end{aligned}$$

Therefore, (3.3) yields

$$u'(t) + x_0(t)^{\alpha+1}\left[q(t) - q_0(t)\right] + \alpha L_1 p(t)^{-1} x_0(t)^{-2} x_0'(t)^{-\alpha+1} u(t)^2 \leq 0$$

for $t \geq T$, and (4.2) gives

$$\int_T^\infty p(t)^{-1} x_0(t)^{-2} x_0'(t)^{-\alpha+1} u(t)^2 dt = \infty.$$

Thus we obtain

$$u'(t) + Q(t) + \alpha L_1 P(t)^{-1} u(t)^2 \leq 0, \quad t \geq T, \tag{4.4}$$

and

$$\int_T^\infty P(t)^{-1} u(t)^2 dt = \infty. \tag{4.5}$$

Here the functions $P(t)$ and $Q(t)$ are given by (1.11).

Put

$$\varphi(t) = \int_T^t P(s)^{-1} ds, \quad t \geq T. \tag{4.6}$$

It follows from (4.4) that

$$\begin{aligned} &\int_T^t (\varphi(t) - \varphi(s))^2 u'(s)ds + \int_T^t (\varphi(t) - \varphi(s))^2 Q(s)ds \\ &\quad + \alpha L_1 \int_T^t (\varphi(t) - \varphi(s))^2 P(s)^{-1} u(s)^2 ds \leq 0, \quad t \geq T. \end{aligned} \tag{4.7}$$

Denote by $I(t)$ the first term of the left-hand side of (4.7). Then, it is seen that

$$I(t) = -u(T)\varphi(t)^2 + 2\int_T^t (\varphi(t) - \varphi(s))P(s)^{-1} u(s)ds, \quad t \geq T,$$

and so, by the Cauchy–Schwarz inequality, we find that

$$|I(t)| \le |u(T)|\varphi(t)^2 + 2\left(\int_T^t P(s)^{-1}ds\right)^{1/2}\left(\int_T^t (\varphi(t)-\varphi(s))^2 P(s)^{-1}u(s)^2 ds\right)^{1/2}$$

for $t \ge T$. Therefore, (4.6) and (4.7) yield

$$\begin{aligned}\frac{1}{\varphi(t)^2}\int_T^t(\varphi(t)-\varphi(s))^2Q(s)ds &\le |u(T)| \\ &+ \frac{2}{\varphi(t)^{1/2}}\left(\frac{1}{\varphi(t)^2}\int_T^t(\varphi(t)-\varphi(s))^2P(s)^{-1}u(s)^2ds\right)^{1/2} \\ &- \alpha L_1\left(\frac{1}{\varphi(t)^2}\int_T^t(\varphi(t)-\varphi(s))^2P(s)^{-1}u(s)^2ds\right), \quad t > T.\end{aligned} \tag{4.8}$$

It follows from (1.8) and (4.6) that

$$\lim_{t\to\infty}\varphi(t) = \int_T^\infty P(s)^{-1}ds = \int_T^\infty p(s)^{-1}x_0(s)^{-2}x_0'(s)^{-\alpha+1}ds = \infty,$$

and, by L'Hospital's rule and (4.5), we get

$$\lim_{t\to\infty}\frac{1}{\varphi(t)^2}\int_T^t(\varphi(t)-\varphi(s))^2P(s)^{-1}u(s)^2ds = \int_T^\infty P(s)^{-1}u(s)^2ds = \infty.$$

Let β be a constant such that $0 < \beta < \alpha L_1$. Then it is easy to check that (4.8) yields

$$\frac{1}{\varphi(t)^2}\int_T^t(\varphi(t)-\varphi(s))^2Q(s)ds \le -\frac{\beta}{\varphi(t)^2}\int_T^t(\varphi(t)-\varphi(s))^2P(s)^{-1}u(s)^2ds$$

for all large t, and consequently,

$$\lim_{t\to\infty}\frac{1}{\varphi(t)^2}\int_T^t(\varphi(t)-\varphi(s))^2Q(s)ds = -\infty.$$

On the other hand, the condition (1.10), i.e., the condition

$$\lim_{t\to\infty}\int_T^t Q(s)ds = \int_T^\infty Q(s)ds \quad \text{is convergent}$$

implies

$$\lim_{t\to\infty}\frac{1}{\varphi(t)^2}\int_T^t(\varphi(t)-\varphi(s))^2Q(s)ds = \int_T^\infty Q(s)ds \in \mathbb{R}.$$

This is a contradiction. Therefore, (4.2) does not occur.

Next suppose that (4.3) holds. Using (1.10), (4.1) and (4.3), we see that $\lim_{t\to\infty}u(t)$ exists and is finite. Put $\lim_{t\to\infty}u(t) = \ell$ $(\in \mathbb{R})$. Integrating the equality (3.3) from t to τ $(T \le t \le \tau)$ and letting $\tau \to \infty$, we obtain

$$\begin{aligned}u(t) = \ell &+ \int_t^\infty x_0(s)^{\alpha+1}[q(s)-q_0(s)]ds \\ &+ \alpha\int_t^\infty p(s)^{-1/\alpha}x_0(s)^{-(\alpha+1)/\alpha}F(u(s), p(s)x_0(s)x_0'(s)^\alpha)ds\end{aligned}$$

for $t \ge T$. By Lemma 2.3 we have $x'(t) > 0$ for $t \ge T$. Hence, by (1.7), (3.2) and (1.12), there is a positive constant k such that

$$-p(t)x_0(t)x_0'(t)^\alpha < u(t) \le kp(t)x_0(t)x_0'(t)^\alpha \tag{4.9}$$

for all large t. We may suppose that (4.9) is valid for $t \geq T$. Applying Lemma 3.2 (ii) to the case $u = u(t)$ and $v = p(t)x_0(t)x_0'(t)^\alpha$, we find that there is a constant $L_1(k) > 0$ such that

$$\begin{aligned} &L_1(k)p(t)^{-1}x_0(t)^{-2}x_0'(t)^{-\alpha+1}u(t)^2 \\ &\quad \leq p(t)^{-1/\alpha}x_0(t)^{-(\alpha+1)/\alpha}F(u(t), p(t)x_0(t)x_0'(t)^\alpha), \quad t \geq T. \end{aligned}$$

Hence, (4.3) gives

$$\int_T^\infty p(t)^{-1}x_0(t)^{-2}x_0'(t)^{-\alpha+1}u(t)^2 dt < \infty.$$

If $\lim_{t\to\infty} u(t) = \ell \neq 0$, then the above fact contradicts the condition (1.8). Therefore we see that $\ell = 0$.

Since

$$\lim_{t\to\infty} u(t) = \ell = 0, \tag{4.10}$$

we find from (1.12) that there are positive constants k_1 and k_2 such that

$$|u(t)| \leq k_1 < k_2 \leq p(t)x_0(t)x_0'(t)^\alpha \tag{4.11}$$

for all large t. We may suppose that (4.11) holds for $t \geq T$. Then, applying Lemma 3.2 (iii) to the case $u = u(t)$ and $v = p(t)x_0(t)x_0'(t)^\alpha$, we deduce that $F(u(t), p(t)x_0(t)x_0'(t)^\alpha)$ is expressed as

$$F(u(t), p(t)x_0(t)x_0'(t)^\alpha) = \frac{\alpha+1}{2\alpha^2}|p(t)x_0(t)x_0'(t)^\alpha|^{(1/\alpha)-1}u(t)^2(1+R(t)) \tag{4.12}$$

with

$$|R(t)| \leq \frac{|\alpha-1|}{3\alpha}L(k_1,k_2)|u(t)| \tag{4.13}$$

for $t \geq T$. Here, $L(k_1,k_2)$ is the constant appearing in Lemma 3.2 (iii). Then, (4.12) gives

$$\begin{aligned} &p(t)^{-1/\alpha}x_0(t)^{-(\alpha+1)/\alpha}F(u(t), p(t)x_0(t)x_0'(t)^\alpha) \\ &\quad = \frac{\alpha+1}{2\alpha^2}p(t)^{-1}x_0(t)^{-2}x_0'(t)^{-\alpha+1}u(t)^2(1+R(t)), \quad t \geq T. \end{aligned} \tag{4.14}$$

By (4.10) and (4.13), we have $\lim_{t\to\infty} R(t) = 0$, and so

$$R(t) \geq -\varepsilon \quad \text{for all large } t, \tag{4.15}$$

where $\varepsilon \in (0,1)$ is the number in the statement of Theorem 1.6. Then, by (3.3), (4.14) and (4.15), we find that

$$u'(t) + Q(t) + (1-\varepsilon)\frac{\alpha+1}{2\alpha}P(t)^{-1}u(t)^2 \leq 0 \quad \text{for all large } t.$$

Therefore the function

$$y(t) = (1-\varepsilon)\frac{\alpha+1}{2\alpha}u(t) \quad \text{with } 0 < \varepsilon < 1$$

satisfies

$$y'(t) + (1-\varepsilon)\frac{\alpha+1}{2\alpha}Q(t) + P(t)^{-1}y(t)^2 \leq 0 \quad \text{for all large } t.$$

Hence, Lemma 2.1 of the case $\alpha = 1$ implies that the linear equation (1.14) is nonoscillatory. This is a contradiction to the assumption that (1.14) is oscillatory. Therefore, (4.3) also does not occur. Consequently the equation (1.1) is oscillatory. The proof of Theorem 1.6 is complete. □

Proof of Corollary 1.7. Corollary 1.7 is a simple combination of Theorem 1.6 and Lemma 2.4 (ii) with $\alpha = 1$. □

5 Proofs of the results (continued)

In this section we prove Theorem 1.8 and Theorem 1.10. It is known that the half-linear Riemann–Weber differential equation (1.18) has a nonoscillatory solution $x(t)$ satisfying (1.19) (see Elbert and Schneider [9, Corollary 1]). Put

$$x_0(t) = t^{\alpha/(\alpha+1)}(\mathrm{Log}_n\, t)^{1/(\alpha+1)}, \quad t \geq T, \tag{5.1}$$

where T is taken sufficiently large such that t and $\mathrm{Log}_j\, t$ $(j = 1, 2, \ldots, n)$ are positive for $t \geq T$. It is trivial that $x = x_0(t)$ is a positive solution of the equation

$$(\Phi_\alpha(x'))' - \frac{(\Phi_\alpha(x_0'(t)))'}{\Phi_\alpha(x_0(t))}\Phi_\alpha(x) = 0$$

on $[T, \infty)$. We define the function $c_0(t)$ by

$$c_0(t) = -\frac{(\Phi_\alpha(x_0'(t)))'}{\Phi_\alpha(x_0(t))} - \left(\frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}}\sum_{j=1}^{n}\frac{1}{(\mathrm{Log}_j\, t)^2}\right). \tag{5.2}$$

Then the function $x_0(t)$ is a positive solution of

$$(\Phi_\alpha(x'))' + \left(\frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}}\sum_{j=1}^{n}\frac{1}{(\mathrm{Log}_j\, t)^2} + c_0(t)\right)\Phi_\alpha(x) = 0. \tag{5.3}$$

In the equations (1.1) and (1.6), let $p(t) \equiv 1$ and

$$q(t) = \frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}}\sum_{j=1}^{n}\frac{1}{(\mathrm{Log}_j\, t)^2} + c(t) \tag{5.4}$$

and

$$q_0(t) = \frac{\alpha E(\alpha)}{t^{\alpha+1}} + \frac{\mu(\alpha)}{t^{\alpha+1}}\sum_{j=1}^{n}\frac{1}{(\mathrm{Log}_j\, t)^2} + c_0(t). \tag{5.5}$$

Then, the equations (1.1) and (1.6) become (1.17) and (5.3), respectively. The key idea of the proofs of Theorems 1.8 and 1.10 is to use the equation (5.3), not the equation (1.18).

From the calculation in [4] we see that

$$x_0'(t) = \frac{\alpha}{\alpha+1}t^{-1/(\alpha+1)}(\mathrm{Log}_n\, t)^{1/(\alpha+1)}\left(1 + \frac{1}{\alpha}\sum_{j=1}^{n}\frac{1}{\mathrm{Log}_j\, t}\right), \tag{5.6}$$

and

$$\begin{aligned}(\Phi_\alpha(x_0'(t)))' = &- t^{-(2\alpha+1)/(\alpha+1)}(\mathrm{Log}_n\, t)^{\alpha/(\alpha+1)} \\ &\times \left(\alpha E(\alpha) + \mu(\alpha)\sum_{j=1}^{n}\frac{1}{(\mathrm{Log}_j\, t)^2} + O\left(\frac{1}{(\log t)^3}\right)\right)\end{aligned}$$

as $t \to \infty$. Therefore, the function $c_0(t)$ defined by (5.2) satisfies

$$c_0(t) = O\left(\frac{1}{t^{\alpha+1}(\log t)^3}\right) \quad (t \to \infty). \tag{5.7}$$

By (5.7) it is clear that

$$\int_T^\infty c_0(s)ds \quad \text{is convergent}$$

and

$$\lim_{t\to\infty} t^\alpha \int_t^\infty c_0(s)ds = 0.$$

Therefore, in the present case, we find that

$$\int_T^\infty q_0(s)ds \quad \text{is convergent}$$

and

$$\int_t^\infty q_0(s)ds = \frac{E(\alpha)}{t^\alpha} + \mu(\alpha)\sum_{j=1}^n \int_t^\infty \frac{1}{s^{\alpha+1}(\mathrm{Log}_j\, s)^2}ds + \int_t^\infty c_0(s)ds.$$

Then it is easy to see that

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q_0(s)ds = E(\alpha) > 0.$$

Consequently, the condition (1.9) is satisfied.

By (5.1) and (5.6), the condition (1.7) is satisfied. Furthermore it is easily checked that

$$x_0(t)^{-2}x_0'(t)^{-\alpha+1} \sim \left(\frac{\alpha}{\alpha+1}\right)^{-\alpha+1} \frac{1}{t\,\mathrm{Log}_n\, t} \quad (t\to\infty).$$

Note here that

$$\frac{d}{dt}\log_{n+1} t = \frac{1}{t\,\mathrm{Log}_n\, t},$$

and so

$$\int_T^t \frac{1}{s\,\mathrm{Log}_n\, s}ds = \log_{n+1} t - \log_{n+1} T.$$

This implies

$$\int_T^t x_0(s)^{-2}x_0'(s)^{-\alpha+1}ds \sim \left(\frac{\alpha}{\alpha+1}\right)^{-\alpha+1} \log_{n+1} t \quad (t\to\infty), \tag{5.8}$$

which yields (1.8) with $p(t) \equiv 1$.

We have

$$\int_T^t x_0(s)^{\alpha+1}\left[q(s)-q_0(s)\right]ds = \int_T^t s^\alpha(\mathrm{Log}_n\, s)\left[c(s)-c_0(s)\right]ds.$$

Došlý [4] showed that

$$\int_T^\infty \frac{\mathrm{Log}_n\, s}{s(\log s)^3}ds < \infty$$

and

$$\lim_{t\to\infty}(\log_{n+1} t)\int_t^\infty \frac{\mathrm{Log}_n\, s}{s(\log s)^3}ds = 0.$$

Therefore we deduce from (5.7) that

$$\int_T^\infty s^\alpha(\mathrm{Log}_n\, s)c_0(s)ds \quad \text{is convergent} \tag{5.9}$$

and

$$\lim_{t\to\infty}(\log_{n+1} t)\int_t^\infty s^\alpha(\mathrm{Log}_n\, s)c_0(s)ds = 0. \tag{5.10}$$

We are now ready to prove Theorems 1.8 and 1.10.

Proof of Theorem 1.8. We apply Theorem 1.3 with $p(t) \equiv 1$ to the equations (1.17) and (5.3). Let $q(t)$ and $q_0(t)$ be the functions defined by (5.4) and (5.5), respectively. Since (1.8) $(p(t) \equiv 1)$ and (1.9) are satisfied, the function $x_0(t)$ which is defined by (5.1) is the principal solution of (5.3). In fact, this can be derived from a direct application of Theorem 1.2 with $p(t) \equiv 1$. For the case $0 < \alpha \le 1$, use the part (i), and for the case $\alpha \ge 1$ the part (iii). If (1.20) is satisfied, then (5.9) gives

$$\int_T^\infty x_0(s)^{\alpha+1}\,[q(s) - q_0(s)]\,ds = \int_T^\infty s^\alpha (\operatorname{Log}_n s)\,[c(s) - c_0(s)]\,ds = \infty.$$

Therefore it follows from Theorem 1.3 with $p(t) \equiv 1$ that if (1.20) is satisfied, then the equation (1.17) is oscillatory. The proof of Theorem 1.8 is complete. □

Proof of Theorem 1.10. We apply Corollary 1.7 with $p(t) \equiv 1$ to the equations (1.17) and (5.3). Let $q(t)$ and $q_0(t)$ be the functions defined by (5.4) and (5.5), respectively. We first show that if (1.21) holds, then (1.5) is satisfied. To see this, note that

$$\frac{d}{dt}\operatorname{Log}_n t = \frac{\operatorname{Log}_n t}{t}\left(\sum_{j=1}^n \frac{1}{\operatorname{Log}_j t}\right)$$

and

$$\begin{aligned}\int_T^t c(s)ds &= \int_T^t \frac{1}{s^\alpha \operatorname{Log}_n s} s^\alpha (\operatorname{Log}_n s) c(s) ds \\ &= -\frac{1}{t^\alpha \operatorname{Log}_n t}\int_t^\infty r^\alpha (\operatorname{Log}_n r) c(r) dr + \frac{1}{T^\alpha \operatorname{Log}_n T}\int_T^\infty r^\alpha (\operatorname{Log}_n r) c(r) dr \\ &\quad - \int_T^t \left[\frac{\alpha}{s^{\alpha+1}\operatorname{Log}_n s} + \frac{1}{s^{\alpha+1}\operatorname{Log}_n s}\left(\sum_{j=1}^n \frac{1}{\operatorname{Log}_j s}\right)\right]\left(\int_s^\infty r^\alpha (\operatorname{Log}_n r) c(r) dr\right) ds.\end{aligned}$$

Therefore we deduce that

$$\lim_{t\to\infty}\int_T^t c(s)ds \quad \text{exists and is finite}$$

and

$$\begin{aligned}\int_t^\infty c(s)ds = \frac{1}{t^\alpha \operatorname{Log}_n t}\int_t^\infty r^\alpha (\operatorname{Log}_n r) c(r) dr \\ - \int_t^\infty \left[\frac{\alpha}{s^{\alpha+1}\operatorname{Log}_n s} + \frac{1}{s^{\alpha+1}\operatorname{Log}_n s}\left(\sum_{j=1}^n \frac{1}{\operatorname{Log}_j s}\right)\right]\left(\int_s^\infty r^\alpha (\operatorname{Log}_n r) c(r) dr\right) ds\end{aligned}$$

for $t \ge T$. Then it is easy to find that

$$\lim_{t\to\infty} t^\alpha \int_t^\infty c(s)ds = 0.$$

Since

$$\int_t^\infty q(s)ds = \frac{E(\alpha)}{t^\alpha} + \mu(\alpha)\sum_{j=1}^n \int_t^\infty \frac{1}{s^{\alpha+1}(\operatorname{Log}_j s)^2} ds + \int_t^\infty c(s)ds,$$

we obtain

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)ds = E(\alpha)\ (>0).$$

Thus we see that the condition (1.5) is satisfied.

As mentioned before, the conditions (1.7) and (1.8) with $p(t) \equiv 1$ are satisfied. By (5.1) and (5.6), we have

$$x_0(t)x_0'(t)^\alpha = \left(\frac{\alpha}{\alpha+1}\right)^\alpha (\mathrm{Log}_n\, t)\left(1 + \frac{1}{\alpha}\sum_{j=1}^{n}\frac{1}{\mathrm{Log}_j\, t}\right)^\alpha,$$

and so $\lim_{t\to\infty} x_0(t)x_0'(t)^\alpha = \infty$. Hence the condition (1.12) with $p(t) \equiv 1$ is also satisfied. By the definition of $P(t)$ and $Q(t)$ and the properties (5.8) and (5.10), we have

$$\begin{aligned}
&\left(\int_T^t \frac{1}{P(s)}ds\right)\left(\int_t^\infty Q(s)ds\right)\\
&= \left(\int_T^t x_0(s)^{-2}x_0'(s)^{-\alpha+1}ds\right)\left(\int_t^\infty x_0(s)^{\alpha+1}[q(s)-q_0(s)]ds\right)\\
&= \varepsilon_1(t)\left(\frac{\alpha}{\alpha+1}\right)^{-\alpha+1}(\log_{n+1} t)\left(\int_t^\infty s^\alpha(\mathrm{Log}_n\, s)[c(s)-c_0(s)]ds\right)\\
&= \varepsilon_1(t)\left(\frac{\alpha}{\alpha+1}\right)^{-\alpha+1}(\log_{n+1} t)\left(\int_t^\infty s^\alpha(\mathrm{Log}_n\, s)c(s)ds\right) + \varepsilon_2(t),
\end{aligned}$$

where $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are functions such that

$$\lim_{t\to\infty}\varepsilon_1(t) = 1 \quad \text{and} \quad \lim_{t\to\infty}\varepsilon_2(t) = 0,$$

respectively. Then it is easy to see that (1.22) implies (1.15). Thus, by Corollary 1.7, we can conclude that if (1.22) holds, then the equation (1.17) is oscillatory. The proof of Theorem 1.10 is complete. □

Finally we present an equation whose oscillation follows from Theorem 1.10 and does not follow from Theorem 3.3 (ii) in [4].

Example 5.1. Consider the equation (1.17) of the case

$$c(t) = \mu(\alpha)\frac{k + \sin(2k\log_{n+2} t) - 2k\cos(2k\log_{n+2} t)}{t^{\alpha+1}(\mathrm{Log}_{n+1}\, t)^2}, \tag{5.11}$$

where k is a constant satisfying $k > 2$. In this case it can be shown without difficulty that the condition (1.21) is satisfied and

$$\int_t^\infty s^\alpha(\mathrm{Log}_n\, s)c(s)ds = \mu(\alpha)\frac{k + \sin(2k\log_{n+2} t)}{\log_{n+1} t}.$$

Therefore we have

$$\liminf_{t\to\infty}(\log_{n+1} t)\int_t^\infty s^\alpha(\mathrm{Log}_n\, s)c(s)ds = \mu(\alpha)(k-1) > \mu(\alpha).$$

Hence, by Theorem 1.10, we can conclude that, for any $\alpha > 0$, the equation (1.17) with (5.11) is oscillatory.

Theorem 3.3 (ii) in [4] requires the condition that there is a constant γ satisfying

$$t^{\alpha+1}(\log t)^3 c(t) \geq \gamma > \frac{2(\alpha+1)(\alpha-1)}{3\alpha^2}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$

for all large t. If $\alpha \geq 1$, this condition leads to $c(t) > 0$ for all large t. There exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $\lim t_i = \infty$ $(i \to \infty)$ and

$$k \log_{n+2} t_i = \pi i \quad \text{for all large } i.$$

Then, for the function $c(t)$ given by (5.11),

$$c(t_i) = \mu(\alpha) \frac{-k}{t_i^{\alpha+1} (\mathrm{Log}_{n+1} t_i)^2} < 0 \quad \text{for all large } i.$$

Therefore, if $\alpha \geq 1$, then we cannot apply Theorem 3.3 (ii) in [4] to the present case.

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Note on oscillation of neutral differential equations with multiple delays

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Abstract. This note is a reaction on a recently published sufficient condition for oscillation of all solutions of a neutral delay differential equation. It is shown by a counterexample that the result is not correct and the problem is explained in details. Several, more general, classes of neutral differential equations with time-dependent discrete, distributed as well as mixed delays are considered. New sufficient conditions for oscillation of all their solutions are proved. Applications are given for illustration.

Keywords: oscillatory solution, discrete delay, distributed delay, mixed delay.

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1 Introduction

Oscillation theory for first order linear neutral differential equations with delay,

$$[x(t) + P(t)\,x(t-\tau)]' + Q(t)\,x(t-\sigma) = 0, \quad t \geq t_0,$$

has attracted researchers' interest for decades (see, e.g., [1–3] and references therein).

Recently, a sufficient condition was proved in [5] for oscillation of all solutions of the neutral delay differential equation

$$[x(t) - x(\tau(t))]' + Q(t)\,x(\sigma(t)) = 0, \quad t \geq t_0 \tag{1.1}$$

for some $t_0 \in \mathbb{R}$, where $Q \in C([t_0,\infty),\mathbb{R}_+)$, $\mathbb{R}_+ = [0,\infty)$, and $\tau, \sigma \in \mathcal{T}_{t_0}$ with

$$\mathcal{T}_\xi = \left\{ f \in C([\xi,\infty),\mathbb{R}) \;\middle|\; \begin{array}{l} f \text{ is strictly increasing;} \\ f(t) < t \;\forall t \geq \xi;\ \lim_{t\to\infty} f(t) = \infty \end{array} \right\}. \tag{1.2}$$

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However, as we shall show in this paper, the proof is not correct and the statement does not hold. It is worth to mention that the limit property of functions from $\mathcal{T}_\xi$ was not explicitly assumed in [5], but it was applied in the proof.

In this paper, we prove a similar statement by a cautious use of a mathematical induction. We give a short remark explaining the problem of the original proof from [5]. Next, we generalize the result for the case of a convex combination of multiple discrete delays. We also consider neutral differential equations with distributed and mixed delays, and we prove analogous statements. More precisely, in addition to equation (1.1), the following equations are investigated in this paper:

$$\left[x(t) - \sum_{i=1}^{n} \lambda_i \, x(\tau_i(t))\right]' + \sum_{j=1}^{m} Q_j(t) \, x(\sigma_j(t)) = 0, \quad t \geq t_0, \tag{1.3}$$

$$\left[x(t) - \frac{\int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s)x(s)\, ds}{\int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s)\, ds}\right]' + Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s)x(s)\, ds = 0, \quad t \geq t_0, \tag{1.4}$$

and

$$\begin{aligned}\left[x(t) - \left(\sum_{i=1}^{n_1} \lambda_i(t)x(\tau_i(t)) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s)x(s)\, ds\right)\right]' \\ + \sum_{j=1}^{m_1} Q_j(t)x(\sigma_j(t)) + \sum_{j=1}^{m_2} S_j(t) \int_{\underline{\sigma}_j(t)}^{\overline{\sigma}_j(t)} R_j(s)x(s)\, ds = 0, \quad t \geq t_0 \end{aligned} \tag{1.5}$$

with appropriate parameters (see Theorems 3.7, 3.9, and 3.11 below).

The paper is organized as follows. In the next section, we conclude preliminary results and introduce an auxiliary function. Section 3 is devoted to the main results of this paper – sufficient conditions for oscillation of all solutions of various classes of neutral differential equations with delays. In the final section, applications to equations with particular types of delays, and concrete examples are given for illustration.

Throughout the paper, we denote by $\mathbb{N}$ ($\mathbb{N}_0$) the set of all positive (nonnegative) integers.

2 Preliminaries

In this section, we introduce some notation and prove auxiliary results.

Let us fix $\xi \in \mathbb{R}$ and consider $\tau \in \mathcal{T}_\xi$. Analogously to the iterations of function τ: $\tau^k = \tau \circ \tau^{k-1}$, $k \in \mathbb{N}$, $\tau^0 = id$, we denote $\tau^{-k} = \tau^{-1} \circ \tau^{-(k-1)}$, $k \in \mathbb{N}$ the iterations of the inverse function $\tau^{-1}\colon [\tau(\xi), \infty) \to [\xi, \infty)$. Then the following result holds.

Lemma 2.1. *Let $\xi \in \mathbb{R}$ and $\tau \in \mathcal{T}_\xi$. For any $\zeta \in [\tau(\xi), \infty)$, the sequence $\{\tau^{-k}(\zeta)\}_{k=1}^{\infty}$ is strictly increasing to ∞.*

Proof. Let $\zeta \in [\tau(\xi), \infty)$ be arbitrary and fixed. Then, $\tau(\zeta) < \zeta$ implies $\zeta < \tau^{-1}(\zeta)$, which yields $\tau^{-1}(\zeta) < \tau^{-2}(\zeta)$, etc. So, by induction, one can see that $\{\tau^{-k}(\zeta)\}_{k=1}^{\infty}$ is a strictly increasing sequence. Now, suppose by contrary that $\lim_{k\to\infty} \tau^{-k}(\zeta) = C < \infty$. Then,

$$C = \lim_{k\to\infty} \tau^{-k}(\zeta) = \lim_{k\to\infty} \tau^{-1}(\tau^{-(k-1)}(\zeta)) = \tau^{-1}\left(\lim_{k\to\infty} \tau^{-(k-1)}(\zeta)\right) = \tau^{-1}(C)$$

is a contradiction, and the proof is complete. □

For any $\zeta \in [\tau(\xi), \infty)$, we define a function $N^\tau_\zeta \colon [\zeta, \infty) \to \mathbb{N}$ such that for any $t \in [\zeta, \infty)$, $N^\tau_\zeta(t)$ satisfies

$$\tau^{-(N^\tau_\zeta(t)-1)}(\zeta) \leq t < \tau^{-N^\tau_\zeta(t)}(\zeta). \tag{2.1}$$

Due to Lemma 2.1, function N^τ_ζ is well defined. Note that

$$N^\tau_\zeta([\tau^{-k}(\zeta), \tau^{-(k+1)}(\zeta))) = k+1 \tag{2.2}$$

for each $k \in \mathbb{N}_0$. Then, it is easy to see that N^τ_ζ in nondecreasing on $[\zeta, \infty)$ and unbounded from above. Another important property of N^τ_ζ is proved in the next lemma.

Lemma 2.2. *Let $\xi \in \mathbb{R}$, $\tau \in \mathcal{T}_\xi$, $\alpha_1 \in [\xi, \infty)$, $\alpha_2 = \tau^{-k}(\alpha_1)$ for some $k \in \mathbb{N}_0$. Then*

$$N^\tau_{\alpha_1}(t) = N^\tau_{\alpha_2}(t) + k, \quad t \geq \alpha_2.$$

Proof. For any $t \geq \alpha_2$,

$$\begin{aligned} \tau^{-(N^\tau_{\alpha_2}(t)+k-1)}(\alpha_1) &= \tau^{-(N^\tau_{\alpha_2}(t)+k-1)}(\tau^k(\alpha_2)) \\ &= \tau^{-(N^\tau_{\alpha_2}(t)-1)}(\alpha_2) \leq t < \tau^{-N^\tau_{\alpha_2}(t)}(\alpha_2) \\ &= \tau^{-N^\tau_{\alpha_2}(t)}(\tau^{-k}(\alpha_1)) = \tau^{-(N^\tau_{\alpha_2}(t)+k)}(\alpha_1). \end{aligned}$$

But we know that for any $t \geq \alpha_2$ (even for any $t \geq \alpha_1$), there is a unique $\kappa \in \mathbb{N}$ satisfying $\tau^{-(\kappa-1)}(\alpha_1) \leq t < \tau^{-\kappa}(\alpha_1)$, and it is given by $\kappa = N^\tau_{\alpha_1}(t)$. Therefore, $N^\tau_{\alpha_1}(t) = N^\tau_{\alpha_2}(t) + k$. □

We will investigate solutions of equation (1.1) in the sense of the following definitions.

Definition 2.3. Let $t_0 \in \mathbb{R}$ and $Q \in C([t_0, \infty), \mathbb{R}_+)$, $\tau, \sigma \in \mathcal{T}_{t_0}$, $\varphi \in C([\min\{\tau(t_0), \sigma(t_0)\}, t_0], \mathbb{R})$ be given functions. We say that

$$x \in C([\min\{\tau(t_0), \sigma(t_0)\}, \infty), \mathbb{R})$$

is a solution of equation (1.1) along with the initial condition

$$x(t) = \varphi(t), \quad t \in [\min\{\tau(t_0), \sigma(t_0)\}, t_0] \tag{2.3}$$

if $x(t) - x(\tau(t))$ is continuously differentiable for all $t \in [t_0, \infty)$ and x satisfies (1.1), (2.3).

In the rest of the paper, we often omit initial condition (2.3). So, x is a solution of (1.1) if there exists a suitable function φ such that x solves initial value problem (1.1), (2.3).

Definition 2.4. Let $t_0 \in \mathbb{R}$ and $Q \in C([t_0, \infty), \mathbb{R}_+)$, $\tau, \sigma \in \mathcal{T}_{t_0}$ be given functions. Solution x of (1.1) is called eventually positive (eventually negative) if there is $T > t_0$ such that $x(t) > 0$ ($x(t) < 0$) for all $t \geq T$. In this case, x is called nonoscillatory. Otherwise, we say that x oscillates or that it is oscillatory.

In other neutral differential equations studied in the paper, their solutions are understood in an analogous sense.

Finally, in this section, we present an auxiliary lemma.

Lemma 2.5. *Let $A \geq B \geq 0$ and $\alpha > 1$. Then*

$$(A-B)^{\frac{1}{\alpha}} \geq A^{\frac{1}{\alpha}} - B^{\frac{1}{\alpha}}.$$

Proof. If $A = 0$, the statement is obvious. Now, let $A \neq 0$ and consider the function $f(x) = (1-x)^{\frac{1}{\alpha}} - (1 - x^{\frac{1}{\alpha}})$ for $x \in [0,1]$. Then $f(0) = 0 = f(1)$. The derivative,

$$f'(x) = \frac{1}{\alpha}\left(x^{\frac{1-\alpha}{\alpha}} - (1-x)^{\frac{1-\alpha}{\alpha}}\right)$$

vanishes if and only if $x = \frac{1}{2}$. Since

$$f\left(\frac{1}{2}\right) = \frac{2 - 2^{\frac{1}{\alpha}}}{2^{\frac{1}{\alpha}}} > 0,$$

we get that $f(x) \geq 0$ for all $x \in [0,1]$. In particular, $f(\frac{B}{A}) \geq 0$ which proves the statement. □

3 Main results

Here, we recall the result from [5] and provide a counterexample to show that it does not hold. Next, by correcting the wrong proof from [5], we prove a new sufficient condition for oscillation of all solutions of equation (1.1). Then, we give a generalization to multiple discrete delays. In Subsection 3.2, an analogous problem is studied for neutral differential equations with distributed and mixed delays.

3.1 Discrete delays

In [5], the next result was stated (we use the quotation marks to warn readers that the result is not correct):

"Theorem" 3.1. *Let $t_0 > 0$ and $Q \in C([t_0,\infty), \mathbb{R}_+)$, $\tau, \sigma \in \mathcal{T}_{t_0}$ be given functions. If*

$$\int_{t_0}^{\infty} Q(s)\, ds = \infty \tag{3.1}$$

or

$$\int_{t_0}^{\infty} s\, Q(s)\, ds = \infty, \tag{3.2}$$

then every solution of equation (1.1) oscillates.

It will be shown in the proof of Theorem 3.3 below (and it was correctly proved in [5]) that inequality (3.1) is indeed a sufficient condition for oscillation of all solutions of (1.1). In the next example, we illustrate that if (3.1) does not hold, inequality (3.2) does not guarantee the oscillation of all solutions of (1.1).

Example 3.2. Let us consider the following equation

$$\left[x(t) - x\left(\frac{t}{2}\right)\right]' + \frac{1}{t^2}\, x\left(\frac{t}{2 - t\ln(2 - e^{\frac{1}{t}})}\right) = 0, \quad t \geq t_0 \tag{3.3}$$

for some $t_0 > \frac{1}{\ln 2}$.

Since $0 < 2 - e^{\frac{1}{t_0}} \leq 2 - e^{\frac{1}{t}} < 1$, we have $\ln(2 - e^{\frac{1}{t}}) < 0$ $\forall t \geq t_0$. So,

$$\sigma(t) = \frac{t}{2 - t\ln(2 - e^{\frac{1}{t}})} < t, \quad t \geq t_0.$$

Furthermore, from

$$\sigma'(t) = \frac{4 - e^{\frac{1}{t}}}{\left(2 - e^{\frac{1}{t}}\right)\left(2 - t\ln(2 - e^{\frac{1}{t}})\right)^2},$$

one can see that $\sigma'(t) > 0$ for all $t \geq t_0$. It is easy to verify that $\sigma(t) \xrightarrow{t\to\infty} \infty$. Thus, $\sigma \in \mathcal{T}_{t_0}$. Clearly, $\tau \in \mathcal{T}_{t_0}$ for $\tau(t) = \frac{t}{2}$. Moreover,

$$\int_{t_0}^{\infty} \frac{ds}{s^2} = \frac{1}{t_0} < \infty, \qquad \int_{t_0}^{\infty} \frac{ds}{s} = \infty,$$

i.e., condition (3.1) is not satisfied, but (3.2) holds. Hence, by "Theorem" 3.1, every solution of equation (3.3) oscillates. However, a positive function $e^{-\frac{1}{t}}$ solves this equation. Indeed, for $x(t) = e^{-\frac{1}{t}}$, the left-hand side of (3.3) reads as

$$\frac{e^{-\frac{1}{t}}}{t^2} - \frac{2\,e^{-\frac{2}{t}}}{t^2} + \frac{1}{t^2}\,e^{-\frac{2 - t\ln(2 - e^{\frac{1}{t}})}{t}} = \frac{1}{t^2}\left[e^{-\frac{1}{t}} - 2\,e^{-\frac{2}{t}} + e^{-\frac{2}{t}}(2 - e^{\frac{1}{t}})\right] = 0.$$

Next, we present our result for equation (1.1).

Theorem 3.3. *Let $t_0 \in \mathbb{R}$ and $Q \in C([t_0,\infty),\mathbb{R}_+)$, $\tau,\sigma \in \mathcal{T}_{t_0}$ be given functions. If condition (3.1) is satisfied or*

$$\int_{\sigma^{-1}(t_0)}^{\infty} (N_{t_0}^{\tau}(\sigma(s)))^{\frac{1}{p}}\, Q(s)\, ds = \infty \tag{3.4$_p$}$$

for some $p > 1$, where $N_{t_0}^{\tau}(t)$ is given by (2.1), then every solution of equation (1.1) oscillates.

Note that in the label of condition $(3.4)_p$, we use the parameter $p > 1$ as the lower index.

Proof. One can easily see that condition (3.1) as well as condition $(3.4)_p$ implies that Q does not vanish for all t sufficiently large, i.e.,

$$\forall t \geq t_0\ \exists T \geq t : Q(T) > 0.$$

Without any loss of generality, we suppose in contrary that x is an eventually positive solution of (1.1). Since $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$, there is $t_1 \geq t_0$ such that $x(t)$, $x(\tau(t))$ and $x(\sigma(t))$ are positive for all $t \geq t_1$. For $z(t) = x(t) - x(\tau(t))$, equation (1.1) gives $z'(t) \leq 0$ $\forall t \geq t_1$. Moreover, from the nonvanishing property of Q, we know that for any $t \geq t_1$ there is $T \geq t$ such that $z'(T) < 0$. Hence, $z(t)$ can not vanish for all sufficiently large t, but it is either eventually negative or eventually positive.

If z is eventually negative, then, since it is nonincreasing, there exist $t_2 \geq t_1$ and $\mu > 0$ such that $z(t) \leq -\mu$ for all $t \geq t_2$. Equivalently, we have

$$x(t) \leq x(\tau(t)) - \mu, \quad t \geq t_2.$$

In particular,

$$x(\tau^{-k}(t_2)) \leq x(\tau^{-(k-1)}(t_2)) - \mu \leq \cdots \leq x(t_2) - k\mu$$

for each $k \in \mathbb{N}$. A contradiction with the eventual positivity of x follows, since the right side tends to $-\infty$ as $k \to \infty$.

Hence, z is eventually positive, i.e., there is $t_2 \geq t_1$ such that $z(t) > 0$ $\forall t \geq t_2$. This means that

$$x(t) > x(\tau(t)) > 0, \quad t \geq t_2. \tag{3.5}$$

Therefore,

$$x(t) \geq \min_{s \in [\tau(t_2), t_2]} x(s) =: \omega > 0, \quad t \geq t_2. \tag{3.6}$$

From equation (1.1), we obtain

$$0 = z'(t) + Q(t)x(\sigma(t)) \geq z'(t) + \omega Q(t)$$

for all $t \geq t_3$ for some $t_3 \geq \sigma^{-1}(t_2)$. This gives

$$z(t) \leq z(t_3) - \omega \int_{t_3}^{t} Q(s)\, ds, \quad t \geq t_3.$$

So, if condition (3.1) is satisfied, we get $\lim_{t\to\infty} z(t) = -\infty$ which is a contradiction, and x is oscillatory.

Now, assume that

$$\int_{t_0}^{\infty} Q(s)\, ds < \infty \tag{3.7}$$

and that condition $(3.4)_p$ is satisfied for some $p > 1$. Let us take $t_4 \geq \tau^{-1}(t_2)$ such that $t_4 = \tau^{-\kappa}(t_0)$ for some $\kappa \in \mathbb{N}$.

From

$$x(t) = z(t) + x(\tau(t)), \quad t \geq t_4, \tag{3.8}$$

we get

$$x(t) = z(t) + z(\tau(t)) + \cdots + z(\tau^{(N-1)}(t)) + x(\tau^N(t))$$

for any $t \in [\tau^{-(N-1)}(t_4), \tau^{-N}(t_4))$, $N \in \mathbb{N}$. Since z is nonincreasing and $\tau \in \mathcal{T}_{t_0}$, this identity implies

$$x(t) \geq Nz(t) + x(\tau^N(t)), \quad t \in [\tau^{-(N-1)}(t_4), \tau^{-N}(t_4)),\ N \in \mathbb{N}$$

or, equivalently,

$$x(t) \geq N^{\tau}_{t_4}(t)z(t) + x(\tau^{N^{\tau}_{t_4}(t)}(t)), \quad t \geq t_4$$

(see (2.2)). Note that $\tau^{N^{\tau}_{t_4}(t)}(t) \in [\tau(t_4), t_4) \subset [t_2, \infty)$ for any $t \geq t_4$. Hence, by (3.6),

$$x(t) \geq N^{\tau}_{t_4}(t)z(t) + \omega, \quad t \geq t_4.$$

Next, using the Young inequality,

$$\frac{A^p}{p} + \frac{B^q}{q} \geq AB$$

for $A, B > 0$ and $q = \frac{p}{p-1}$, we derive

$$x(t) \geq \frac{\left(\left(pN^{\tau}_{t_4}(t)z(t)\right)^{\frac{1}{p}}\right)^p}{p} + \frac{\left((q\omega)^{\frac{1}{q}}\right)^q}{q} \geq \left(pN^{\tau}_{t_4}(t)z(t)\right)^{\frac{1}{p}} (q\omega)^{\frac{1}{q}}$$

for all $t \geq t_4$. Let us denote $\omega_1 := p^{\frac{1}{p}}(q\omega)^{\frac{1}{q}} > 0$ and take $t_5 = \sigma^{-1}(t_4)$. Then, (1.1) implies

$$\begin{aligned} z'(t) = -Q(t)x(\sigma(t)) &\leq -\omega_1 Q(t) \left(N^{\tau}_{t_4}(\sigma(t))z(\sigma(t))\right)^{\frac{1}{p}} \\ &\leq -\omega_1 Q(t) \left(N^{\tau}_{t_4}(\sigma(t))z(t)\right)^{\frac{1}{p}}, \quad t \geq t_5 \end{aligned}$$

since z is nonincreasing. Dividing by $z^{\frac{1}{p}}(t)$ and integrating over $[t_5, t]$ yields

$$\int_{t_5}^{t} \frac{z'(s)\,ds}{z^{\frac{1}{p}}(s)} = qz^{\frac{1}{q}}(t) - qz^{\frac{1}{q}}(t_5) \leq -\omega_1 \int_{t_5}^{t} \left(N_{t_4}^{\tau}(\sigma(s))\right)^{\frac{1}{p}} Q(s)\,ds$$

for all $t \geq t_5$. Now, it only remains to prove that

$$\int_{t_5}^{\infty} \left(N_{t_4}^{\tau}(\sigma(s))\right)^{\frac{1}{p}} Q(s)\,ds = \infty. \tag{3.9}$$

Consequently, we get a contradiction with the eventual positivity of z, implying that x is oscillatory.

Using Lemmas 2.2, 2.5, we derive

$$\left(N_{t_4}^{\tau}(t)\right)^{\frac{1}{p}} = \left(N_{t_0}^{\tau}(t) - \kappa\right)^{\frac{1}{p}} \geq \left(N_{t_0}^{\tau}(t)\right)^{\frac{1}{p}} - \kappa^{\frac{1}{p}}$$

for all $t \geq t_4$. Therefore, assumptions $(3.4)_p$, (3.7) imply for $t \geq t_5$,

$$\begin{aligned}\int_{t_5}^{t} \left(N_{t_4}^{\tau}(\sigma(s))\right)^{\frac{1}{p}} Q(s)\,ds &\geq \int_{t_5}^{t} \left(N_{t_0}^{\tau}(\sigma(s))\right)^{\frac{1}{p}} Q(s)\,ds - \kappa^{\frac{1}{p}} \int_{t_5}^{t} Q(s)\,ds \\ &= \int_{\sigma^{-1}(t_0)}^{t} \left(N_{t_0}^{\tau}(\sigma(s))\right)^{\frac{1}{p}} Q(s)\,ds - \kappa^{\frac{1}{p}} \int_{t_0}^{t} Q(s)\,ds + C \xrightarrow{t\to\infty} \infty\end{aligned}$$

with an appropriate constant $C \in \mathbb{R}$. □

Remark 3.4. Condition (3.1) was proved in [5], but in the proof of Theorem 3.3, we emphasize were the missing assumption was needed. Namely, to get the existence of t_1.

Remark 3.5. The original proof of "Theorem" 3.1 from [5] contains the following issues:

1. Constant $\tau = \inf_{t \geq t_3}(t - \tau(t))$ was introduced and used as positive. However, the case $\tau(t) \nearrow t$ as $t \to \infty$ was not considered.

2. For fixed t, the value $x(\tau^{N(t)}(t))$ was used, where $N(t) = \left\lfloor \frac{t-t_3}{\tau} \right\rfloor^2$, τ is defined in the previous point of this remark, and $\lfloor \cdot \rfloor$ is the greatest integer function (or the floor function). This can be a problem if $N(t)$ is so large that $\tau^{N(t)}(t) < \tau(t_2)$, because then one can not use the estimation

$$x(\tau^{N(t)-1}(t)) > x(\tau^{N(t)}(t)).$$

   Similarly, we use estimation (3.5), but, in our case, $N_{t_4}^{\tau}(t)$ is bounded for any fixed $t \geq t_4$ (as it does not depend on the infimum).

3. The proof from [5] does not work even if τ is far from zero (e.g., constant delay). The problem is in the power 2 in the definition of $N(t)$ (see the previous point). Because then one can not iterate expansion (3.8) $N(t)$-times, due to $\tau^{N(t)}(t) < \tau(t_4)$.

Remark 3.6. Since $N_{t_0}^{\tau}(t) \in \mathbb{N}$ and $Q(t) \geq 0$ for all $t \geq t_0$, inequality $k^{\frac{1}{p_1}} \leq k^{\frac{1}{p_2}}$ for each $k \in \mathbb{N}$ and all $1 \leq p_2 \leq p_1$ gives that, $(3.4)_{p_1}$ implies $(3.4)_{p_2}$ for any $1 < p_2 \leq p_1$. Similarly, (3.1) implies $(3.4)_p$ for all $p > 1$.

Now, we generalize Theorem 3.3 to the case of multiple delays.

Theorem 3.7. *Let $t_0 \in \mathbb{R}$, $n, m \in \mathbb{N}$, $\lambda_i > 0$ for $i = 1, 2, \dots, n$ be such that $\sum_{i=1}^{n} \lambda_i = 1$, and $Q_j \in C([t_0, \infty), \mathbb{R}_+)$, $\tau_i, \sigma_j \in \mathcal{T}_{t_0}$ for $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ be given functions. If there exists $j_0 \in \{1, 2, \dots, m\}$ such that*

$$\int_{t_0}^{\infty} Q_{j_0}(s)\, ds = \infty \tag{3.10}$$

or

$$\int_{\sigma_{j_0}^{-1}(t_0)}^{\infty} (N_{t_0}^{\underline{\tau}}(\sigma_{j_0}(s)))^{\frac{1}{p}} Q_{j_0}(s)\, ds = \infty \tag*{(3.11)$_p$}$$

for some $p > 1$, where $\underline{\tau} = \min_{i=1,2,\dots,n} \tau_i$ and $N_{t_0}^{\underline{\tau}}(t)$ is given by (2.1), then every solution of equation (1.3) oscillates.

Proof. In this proof, we skip some details that are the same as in the proof of Theorem 3.3.

As in the proof of Theorem 3.3, each one of conditions (3.10), (3.11)$_p$ implies that

$$\forall t \geq t_0 \; \exists T \geq t : Q_{j_0}(T) > 0.$$

Suppose that x is an eventually positive solution of (1.3). Then, there is $t_1 \geq t_0$ such that $x(t)$, $x(\tau_i(t))$, $x(\sigma_j(t))$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$ are positive for all $t \geq t_1$. From equation (1.3), we get $z'(t) \leq 0$ $\forall t \geq t_1$ for $z(t) = x(t) - \sum_{i=1}^{n} \lambda_i x(\tau_i(t))$. Again, z can be only eventually negative or eventually positive.

If z is eventually negative, then there exist $t_2 \geq t_1$ and $\mu > 0$ such that $z(t) \leq -\mu$ for all $t \geq t_2$, i.e.,

$$\begin{aligned} x(t) &\leq -\mu + \sum_{i=1}^{n} \lambda_i x(\tau_i(t)) \leq -\mu + \max_{i=1,2,\dots,n} x(\tau_i(t)) \\ &\leq -\mu + \max_{s \in I(t)} x(s) \end{aligned} \tag{3.12}$$

for all $t \geq t_2$, where $I(t) = [\underline{\tau}(t), \overline{\tau}(t)]$, $\overline{\tau} = \max_{i=1,2,\dots,n} \tau_i$. Note that $\underline{\tau}, \overline{\tau} \in \mathcal{T}_{t_0}$. Denote

$$\begin{aligned} I_\ell &:= \left[\underline{\tau}^{-(\ell-1)}(t_2), \underline{\tau}^{-\ell}(t_2)\right), \quad \ell \in \mathbb{N}_0, \\ I_\ell^k &:= I_\ell \cap \left[\overline{\tau}^{-(k-1)}(\underline{\tau}^{-(\ell-1)}(t_2)), \overline{\tau}^{-k}(\underline{\tau}^{-(\ell-1)}(t_2))\right), \quad \ell \in \mathbb{N},\ k = 1, 2, \dots, K(\ell), \end{aligned} \tag{3.13}$$

where $K(\ell)$ is the largest $k \in \mathbb{N}$ for which $I_\ell^k \neq \emptyset$. Notice that by (2.2), $t \in I_\ell$ for $\ell \in \mathbb{N}_0$ if and only if $N_{t_2}^{\underline{\tau}}(t) = \ell$. Now, if $t \in I_\ell^1$ for $\ell \in \mathbb{N}$, then $\underline{\tau}(t) \in \underline{\tau}(I_\ell) = I_{\ell-1}$ and

$$\begin{aligned} \overline{\tau}(t) &\in \overline{\tau}\left(\left[\underline{\tau}^{-(\ell-1)}(t_2), \overline{\tau}^{-1}(\underline{\tau}^{-(\ell-1)}(t_2))\right)\right) \\ &= \left[\overline{\tau}(\underline{\tau}^{-(\ell-1)}(t_2)), \underline{\tau}^{-(\ell-1)}(t_2)\right) \subset I_{\ell-1}. \end{aligned}$$

Similarly, if $t \in I_\ell^k$ for $\ell \in \mathbb{N}$, $k \in \{2, 3, \dots, K(\ell)\}$, then $\underline{\tau}(t) \in \underline{\tau}(I_\ell) = I_{\ell-1}$ and

$$\begin{aligned} \overline{\tau}(t) &\in \overline{\tau}\left(\left[\overline{\tau}^{-(k-1)}(\underline{\tau}^{-(\ell-1)}(t_2)), \overline{\tau}^{-k}(\underline{\tau}^{-(\ell-1)}(t_2))\right)\right) \\ &= \left[\overline{\tau}^{-(k-2)}(\underline{\tau}^{-(\ell-1)}(t_2)), \overline{\tau}^{-(k-1)}(\underline{\tau}^{-(\ell-1)}(t_2))\right) \subset I_\ell^{k-1}. \end{aligned}$$

Using the above inclusions, we are able to work more precisely with $I(t)$ for particular values of t.

Now, we use the mathematical induction with respect to the intervals I_1^1, I_1^2, $\dots$, $I_1^{K(1)}$, $I_2^1, \dots$ to prove an estimation of $x(t)$ for all $t \geq t_2$. Let us denote $\Omega := \sup_{s \in I_0} x(s) =$

$\max_{s\in\overline{I_0}} x(s)$ and $C_\ell^k := \operatorname{conv}\{I_{\ell-1}, I_\ell^k\}$, the convex hull of the corresponding sets for $\ell \in \mathbb{N}$, $k \in \{1,2,\dots,K(\ell)\}$. For $I(I_\ell^k) = \bigcup_{t\in I_\ell^k}[\underline{\tau}(t), \overline{\tau}(t)]$, we get $I(I_\ell^1) \subset \overline{I_{\ell-1}}$ for each $\ell \in \mathbb{N}$. So, if $t \in I_1^1$, then by (3.12),

$$x(t) \le -\mu + \max_{s\in\overline{I(I_1^1)}} x(s) = -\mu + \max_{s\in\overline{I_0}} x(s) = -\mu + \Omega = -N_{t_2}^{\tau}(t)\mu + \Omega.$$

Furthermore, $I(I_\ell^k) \subset \overline{\operatorname{conv}\{I_{\ell-1}, I_\ell^{k-1}\}} = \overline{C_\ell^{k-1}}$ for $l \in \mathbb{N}$, $k \in \{2,3,\dots,K(\ell)\}$. Let us suppose that $\ell \in \mathbb{N}$, $k \in \{1,2,\dots,K(\ell)\}$ are fixed and

$$x(t) \le -N_{t_2}^{\tau}(t)\mu + \Omega$$

for all $t \in \operatorname{conv}\{I_0, I_\ell^k\}$ (due to the continuity of x, this estimation is valid for all $t \in \overline{\operatorname{conv}\{I_0, I_\ell^k\}}$). Now, if $k < K(\ell)$, for $t \in I_\ell^{k+1}$ we have $I(t) \subset I(I_\ell^{k+1}) \subset \overline{C_\ell^k}$. Hence, by (3.12),

$$\begin{aligned} x(t) &\le -\mu + \max_{s\in\overline{C_\ell^k}} x(s) \le -\mu + \max_{s\in\overline{C_\ell^k}}(-N_{t_2}^{\tau}(s)\mu + \Omega) \\ &= -\mu - (\ell-1)\mu + \Omega = -\ell\mu + \Omega = -N_{t_2}^{\tau}(t)\mu + \Omega. \end{aligned}$$

On the other side, if $k = K(\ell)$, for $t \in I_{\ell+1}^1$ we obtain

$$\begin{aligned} x(t) &\le -\mu + \max_{s\in\overline{I_\ell}} x(s) \le -\mu + \max_{s\in\overline{I_\ell}}(-N_{t_2}^{\tau}(s)\mu + \Omega) \\ &= -\mu - \ell\mu + \Omega = -(\ell+1)\mu + \Omega = -N_{t_2}^{\tau}(t)\mu + \Omega. \end{aligned}$$

So, we have proved that

$$x(t) \le -N_{t_2}^{\tau}(t)\mu + \Omega, \quad t \ge t_2.$$

Using $N_{t_2}^{\tau}(t) \xrightarrow{t\to\infty} \infty$, for $t \to \infty$ we obtain a contradiction with x being eventually positive.

Therefore, z is eventually positive, i.e., there is $t_2 \ge t_1$ such that

$$x(t) > \sum_{i=1}^{n} \lambda_i x(\tau_i(t)) \ge \min_{i=1,2,\dots,n} x(\tau_i(t)) \ge \min_{s\in I(t)} x(s) \ge \min_{s\in[\underline{\tau}(t),t]} x(s)$$

for all $t \ge t_2$. In this part of the proof, we adapt the notation from the previous part with this new value of t_2. So, we have

$$x(t) \ge \min_{s\in\overline{I_0}} x(s) =: \omega > 0, \quad t \ge t_2. \tag{3.14}$$

Consequently, from equation (1.3), we get

$$0 = z'(t) + \sum_{j=1}^{m} Q_j(t)x(\sigma_j(t)) \ge z'(t) + \omega \sum_{j=1}^{m} Q_j(t)$$

for all $t \ge t_3$ for some $t_3 \ge \underline{\sigma}^{-1}(t_2)$, $\underline{\sigma} = \min_{j=1,2,\dots,m} \sigma_j \in \mathcal{T}_{t_0}$. Integrating over $[t_3, t]$ gives

$$z(t) \le z(t_3) - \omega \sum_{j=1}^{m} \int_{t_3}^{t} Q_j(s)\, ds \le z(t_3) - \omega \int_{t_3}^{t} Q_{j_0}(s)\, ds, \quad t \ge t_3.$$

Assuming condition (3.10), this estimation results in a contradiction with eventual positivity of z for $t \to \infty$, which implies that x is oscillatory.

Now, assume that

$$\int_{t_0}^{\infty} Q_j(s)\,ds < \infty$$

for each $j = 1, 2, \ldots, m$, and that condition $(3.11)_p$ is satisfied for some $p > 1$. Then

$$x(t) = z(t) + \sum_{i=1}^{n} \lambda_i x(\tau_i(t)) \geq z(t) + \min_{i=1,2,\ldots,n} x(\tau_i(t)) \geq z(t) + \min_{s\in I(t)} x(s)$$

for all $t \geq t_4$, where $t_4 \geq \underline{\tau}^{-1}(t_2)$ is such that $t_4 = \underline{\tau}^{-\kappa}(t_0)$ for some $\kappa \in \mathbb{N}$. Let us fix arbitrary $T \geq t_4$. Then, due to $z'(t) \leq 0$ for all $t \geq t_4$,

$$x(t) \geq z(T) + \min_{s\in I(t)} x(s), \quad t \in [t_4, T].$$

Using induction as for (3.12), one can now show that

$$x(t) \geq N_{t_4}^{\underline{\tau}}(t) z(T) + \omega, \quad t \in [t_4, T].$$

In particular, this estimation is valid for $t = T$. Since $T \geq t_4$ was arbitrary, we have

$$x(t) \geq N_{t_4}^{\underline{\tau}}(t) z(t) + \omega, \quad t \geq t_4.$$

Applying Young inequality with $p > 1$ such that $(3.11)_p$ holds yields

$$x(t) \geq \left(p N_{t_4}^{\underline{\tau}}(t) z(t)\right)^{\frac{1}{p}} (q\omega)^{\frac{1}{q}}$$

for all $t \geq t_4$. Denoting $\omega_1 := p^{\frac{1}{p}} (q\omega)^{\frac{1}{q}} > 0$, we have

$$\begin{aligned} z'(t) &= -\sum_{j=1}^{m} Q_j(t) x(\sigma_j(t)) \leq -\omega_1 \sum_{j=1}^{m} Q_j(t) \left(N_{t_4}^{\underline{\tau}}(\sigma_j(t)) z(\sigma_j(t))\right)^{\frac{1}{p}} \\ &\leq -\omega_1 \sum_{j=1}^{m} Q_j(t) \left(N_{t_4}^{\underline{\tau}}(\sigma_j(t)) z(t)\right)^{\frac{1}{p}}, \quad t \geq t_5, \end{aligned}$$

where $t_5 = \underline{\sigma}^{-1}(t_4)$. Dividing by $z^{\frac{1}{p}}(t)$ and integrating over $[t_5, t]$ yields

$$q z^{\frac{1}{q}}(t) - q z^{\frac{1}{q}}(t_5) \leq -\omega_1 \int_{t_5}^{t} \left(N_{t_4}^{\underline{\tau}}(\sigma_{j_0}(s))\right)^{\frac{1}{p}} Q_{j_0}(s)\,ds, \quad t \geq t_5.$$

Now, the proof is finished as the proof of Theorem 3.3. □

Remark 3.8. Note that conditions (3.10) and $(3.11)_p$ are equivalent to

$$\sum_{j=1}^{m} \int_{t_0}^{\infty} Q_j(s)\,ds = \infty$$

and

$$\sum_{j=1}^{m} \int_{\sigma_j^{-1}(t_0)}^{\infty} \left(N_{t_0}^{\underline{\tau}}(\sigma_j(s))\right)^{\frac{1}{p}} Q_j(s)\,ds = \infty,$$

respectively.

3.2 Distributed delays

Here, we consider neutral differential equations with distributed and mixed delays.

Theorem 3.9. *Let $t_0 \in \mathbb{R}$, $\underline{\tau}, \overline{\tau}, \underline{\sigma}, \overline{\sigma} \in \mathcal{T}_{t_0}$ satisfy $\underline{\tau}(t) < \overline{\tau}(t)$ and $\underline{\sigma}(t) \leq \overline{\sigma}(t)$ for all $t \geq t_0$, $\lambda \in C([\underline{\tau}(t_0), \infty), (0, \infty))$, $Q \in C([t_0, \infty), \mathbb{R}_+)$, $R \in C([\underline{\sigma}(t_0), \infty), \mathbb{R}_+)$. If*

$$\int_{t_0}^{\infty} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} R(r)\, dr\, ds = \infty \tag{3.15}$$

or

$$\int_{\underline{\sigma}^{-1}(t_0)}^{\infty} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} (N_{t_0}^{\underline{\tau}}(r))^{\frac{1}{p}} R(r)\, dr\, ds = \infty \qquad (3.16)_p$$

for some $p > 1$, where $N_{t_0}^{\underline{\tau}}(t)$ is given by (2.1), then every solution of equation (1.4) oscillates.

Proof. Again, the proof is similar to the proofs of Theorems 3.3, 3.7, so we only provide some key points. For brevity, we also denote

$$\Lambda(t) = \left(\int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s)\, ds \right)^{-1}, \quad t \geq t_0.$$

As before, each one of conditions (3.15), $(3.16)_p$ implies that

$$\forall t \geq t_0 \quad \exists T \geq t: \quad Q(T) \int_{\underline{\sigma}(T)}^{\overline{\sigma}(T)} R(s)\, ds > 0.$$

We suppose that x is an eventually positive solution of (1.4). Then there exists $t_1 \geq t_0$ such that $x(t)$, $x(\underline{\tau}(t))$, $x(\underline{\sigma}(t))$ are positive for all $t \geq t_1$. Hence, by equation (1.4), $z'(t) \leq 0\ \forall t \geq t_1$ for $z(t) = x(t) - \Lambda(t) \int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s) x(s)\, ds$. We know that z is either eventually negative or eventually positive.

If z is eventually negative, there are $t_2 \geq t_1$ and $\mu > 0$ such that

$$x(t) \leq -\mu + \Lambda(t) \int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s) x(s)\, ds \leq -\mu + \max_{s \in I(t)} x(s) \tag{3.17}$$

for all $t \geq t_2$, where $I(t) = [\underline{\tau}(t), \overline{\tau}(t)]$. Analogously to inequality (3.12), one can show by induction that estimation (3.17) implies

$$x(t) \leq -N_{t_2}^{\underline{\tau}}(t)\mu + \Omega, \quad t \geq t_2,$$

where $\Omega = \max_{s \in \overline{I_0}} x(s)$ using the notation from the proof of Theorem 3.7. Using $N_{t_2}^{\underline{\tau}}(t) \xrightarrow{t \to \infty} \infty$, a contradiction is obtained for $t \to \infty$ with x being eventually positive.

Therefore, z is eventually positive. So, there is $t_2 \geq t_1$ such that

$$x(t) > \Lambda(t) \int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s) x(s)\, ds \geq \min_{s \in I(t)} x(s)$$

for all $t \geq t_2$. Adapting the notation (3.13) for I_ℓ, I_ℓ^k, estimation (3.14) follows. Next, from equation (1.4), we get

$$0 = z'(t) + Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s) x(s)\, ds \geq z'(t) + \omega Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s)\, ds$$

for all $t \geq t_3$ for some $t_3 \geq \underline{\sigma}^{-1}(t_2)$. Integration over $[t_3, t]$ results in

$$z(t) \leq z(t_3) - \omega \int_{t_3}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} R(r)\, dr\, ds, \quad t \geq t_3.$$

Assuming condition (3.15), we get a contradiction with eventual positivity of z, since the right side of the latter inequality tends to $-\infty$ as $t \to \infty$.

Now, assume that

$$\int_{t_0}^{\infty} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} R(r)\, dr\, ds < \infty \tag{3.18}$$

and that condition $(3.16)_p$ is satisfied for some $p > 1$. Then

$$x(t) = z(t) + \Lambda(t) \int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s) x(s)\, ds \geq z(t) + \min_{s \in I(t)} x(s)$$

for all $t \geq t_4$, where $t_4 \geq \underline{\tau}^{-1}(t_2)$ is such that $t_4 = \underline{\tau}^{-\kappa}(t_0)$ for some $\kappa \in \mathbb{N}$. As in the proof of Theorem 3.7, it can be shown that

$$x(t) \geq N_{t_4}^{\underline{\tau}}(t) z(t) + \omega, \quad t \geq t_4,$$

and Young inequality implies

$$x(t) \geq \left(p N_{t_4}^{\underline{\tau}}(t) z(t) \right)^{\frac{1}{p}} (q\omega)^{\frac{1}{q}}$$

for all $t \geq t_4$. Denoting $\omega_1 := p^{\frac{1}{p}} (q\omega)^{\frac{1}{q}} > 0$, from equation (1.4) we derive

$$\begin{aligned} z'(t) &= -Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s) x(s)\, ds \leq -\omega_1 Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s) \left(N_{t_4}^{\underline{\tau}}(s) z(s) \right)^{\frac{1}{p}} ds \\ &\leq -\omega_1 Q(t) \int_{\underline{\sigma}(t)}^{\overline{\sigma}(t)} R(s) \left(N_{t_4}^{\underline{\tau}}(s) \right)^{\frac{1}{p}} ds\, z(t)^{\frac{1}{p}}, \quad t \geq t_5, \end{aligned}$$

where $t_5 = \underline{\sigma}^{-1}(t_4)$. Dividing by $z^{\frac{1}{p}}(t)$ and integrating over $[t_5, t]$ gives

$$q z^{\frac{1}{q}}(t) - q z^{\frac{1}{q}}(t_5) \leq -\omega_1 \int_{t_5}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} \left(N_{t_4}^{\underline{\tau}}(r) \right)^{\frac{1}{p}} R(r)\, dr\, ds, \quad t \geq t_5.$$

Now, it only remains to show that

$$\int_{t_5}^{\infty} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} \left(N_{t_4}^{\underline{\tau}}(r) \right)^{\frac{1}{p}} R(r)\, dr\, ds = \infty$$

to obtain a contradiction with eventual positivity of z, implying that x is oscillatory. Using Lemmas 2.2, 2.5 (see the proof of Theorem 3.3), we obtain

$$\begin{aligned} &\int_{t_5}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} \left(N_{t_4}^{\underline{\tau}}(r) \right)^{\frac{1}{p}} R(r)\, dr\, ds \\ &\quad \geq \int_{t_5}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} \left(N_{t_0}^{\underline{\tau}}(r) \right)^{\frac{1}{p}} R(r)\, dr\, ds - \kappa^{\frac{1}{p}} \int_{t_5}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} R(r)\, dr\, ds \\ &\quad = \int_{\underline{\sigma}^{-1}(t_0)}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} \left(N_{t_0}^{\underline{\tau}}(r) \right)^{\frac{1}{p}} R(r)\, dr\, ds - \kappa^{\frac{1}{p}} \int_{t_0}^{t} Q(s) \int_{\underline{\sigma}(s)}^{\overline{\sigma}(s)} R(r)\, dr\, ds + C \end{aligned}$$

for an appropriate constant $C \in \mathbb{R}$. Note that, by conditions $(3.16)_p$ and (3.18), the right side tends to ∞ as $t \to \infty$. This completes the proof. □

Remark 3.10. Condition $\lambda \in C([\underline{\tau}(t_0),\infty),(0,\infty))$ in Theorem 3.9 can be weakened to $\lambda \in C([\underline{\tau}(t_0),\infty),\mathbb{R}_+)$ satisfying $\int_{\underline{\tau}(t)}^{\overline{\tau}(t)} \lambda(s)\,ds > 0$ for all $t \geq t_0$.

Finally, we present a result for neutral differential equations with mixed delays and time-dependent coefficients.

Theorem 3.11. *Let $t_0 \in \mathbb{R}$, $n_{1,2}, m_{1,2} \in \mathbb{N}_0$ be such that $n_1 + n_2 \geq 1$, $m_1 + m_2 \geq 1$. Moreover, let the following assumptions be fulfilled:*

1. *$\lambda_i \in C([t_0,\infty),\mathbb{R}_+)$ and $\tau_i \in \mathcal{T}_{t_0}$ for each $i = 1,2,\ldots,n_1$,*
2. *$\vartheta_i \in C([\underline{\tau}_i(t_0),\infty),\mathbb{R}_+)$ and $\underline{\tau}_i, \overline{\tau}_i \in \mathcal{T}_{t_0}$ are such that $\underline{\tau}_i(t) \leq \overline{\tau}_i(t)$ for all $t \geq t_0$ and for each $i = 1,2,\ldots,n_2$,*
3. *$Q_j \in C([t_0,\infty),\mathbb{R}_+)$ and $\sigma_j \in \mathcal{T}_{t_0}$ for each $j = 1,2,\ldots,m_1$,*
4. *$S_j \in C([t_0,\infty),\mathbb{R}_+)$, $R_j \in C([\underline{\sigma}_j(t_0),\infty),\mathbb{R}_+)$, and $\underline{\sigma}_j, \overline{\sigma}_j \in \mathcal{T}_{t_0}$ are such that $\underline{\sigma}_j(t) \leq \overline{\sigma}_j(t)$ for all $t \geq t_0$ and each $j = 1,2,\ldots,m_2$,*
5. *for all $t \geq t_0$,*
$$\sum_{i=1}^{n_1} \lambda_i(t) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s)\,ds = 1.$$

If

$$\sum_{j=1}^{m_1} \int_{t_0}^{\infty} Q_j(s)\,ds + \sum_{j=1}^{m_2} \int_{t_0}^{\infty} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} R_j(r)\,dr\,ds = \infty \tag{3.19}$$

or

$$\begin{aligned}\sum_{j=1}^{m_1} &\int_{\sigma_j^{-1}(t_0)}^{\infty} (N_{t_0}^{\underline{\tau}}(\sigma_j(s)))^{\frac{1}{p}}\, Q_j(s)\,ds \\ &+ \sum_{j=1}^{m_2} \int_{\underline{\sigma}_j^{-1}(t_0)}^{\infty} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} (N_{t_0}^{\underline{\tau}}(r))^{\frac{1}{p}}\, R_j(r)\,dr\,ds = \infty\end{aligned} \tag{3.20$_p$}$$

for some $p > 1$, where $\underline{\tau} = \min\{\min_{i=1,2,\ldots,n_1} \tau_i, \min_{i=1,2,\ldots,n_2} \underline{\tau}_i\}$ and $N_{t_0}^{\underline{\tau}}(t)$ is given by (2.1), then every solution of equation (1.5) oscillates.

Proof. Each one of conditions (3.19), $(3.20)_p$ implies that

$$\begin{aligned}\forall t \geq t_0 \quad \exists T \geq t: \quad &Q_j(T) > 0 \quad \text{for some } j \in \{1,2,\ldots,m_1\} \\ \text{or} \quad S_j(T) \int_{\underline{\sigma}_j(T)}^{\overline{\sigma}_j(T)} &R_j(s)\,ds > 0 \quad \text{for some } j \in \{1,2,\ldots,m_2\}.\end{aligned} \tag{3.21}$$

Let us denote

$$\overline{\tau} := \max\left\{\max_{i=1,2,\ldots,n_1} \tau_i, \max_{i=1,2,\ldots,n_2} \overline{\tau}_i\right\}, \qquad \underline{\sigma} := \min\left\{\min_{j=1,2,\ldots,m_1} \sigma_j, \min_{j=1,2,\ldots,m_2} \underline{\sigma}_j\right\}.$$

Note that $\underline{\tau}, \overline{\tau}, \underline{\sigma} \in \mathcal{T}_{t_0}$. Let us assume without any loss of generality that x is an eventually positive solution of (1.5). Take $t_1 \geq t_0$ such that $x(t)$, $x(\underline{\tau}(t))$ and $x(\underline{\sigma}(t))$ are positive for all $t \geq t_1$. Then, by (1.5), $z'(t) \leq 0$ $\forall t \geq t_1$ for

$$z(t) = x(t) - \left(\sum_{i=1}^{n_1} \lambda_i(t) x(\tau_i(t)) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s) x(s)\,ds\right). \tag{3.22}$$

Due to (3.21), z is either eventually negative or eventually positive.

If z is eventually negative, there are $t_2 \geq t_1$ and $\mu > 0$ such that

$$\begin{aligned} x(t) &\leq -\mu + \sum_{i=1}^{n_1} \lambda_i(t) x(\tau_i(t)) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s) x(s)\, ds \\ &\leq -\mu + \left(\sum_{i=1}^{n_1} \lambda_i(t) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s)\, ds \right) \max_{s \in I(t)} x(s) = -\mu + \max_{s \in I(t)} x(s) \end{aligned}$$

for all $t \geq t_2$, where $I(t) = [\underline{\tau}(t), \overline{\tau}(t)]$. As for (3.12), one can use mathematical induction to show that

$$x(t) \leq -N_{t_2}^{\underline{\tau}}(t)\mu + \Omega, \quad t \geq t_2,$$

where $\Omega = \max_{s \in \overline{I_0}} x(s)$ using the notation from the proof of Theorem 3.7. Consequently, $N_{t_2}^{\underline{\tau}}(t) \xrightarrow{t \to \infty} \infty$ yields a contradiction for $t \to \infty$ with x being eventually positive.

Hence, z is eventually positive. Take $t_2 \geq t_1$ such that

$$\begin{aligned} x(t) &> \sum_{i=1}^{n_1} \lambda_i(t) x(\tau_i(t)) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s) x(s)\, ds \\ &\geq \left(\sum_{i=1}^{n_1} \lambda_i(t) + \sum_{i=1}^{n_2} \int_{\underline{\tau}_i(t)}^{\overline{\tau}_i(t)} \vartheta_i(s)\, ds \right) \min_{s \in I(t)} x(s) \\ &= \min_{s \in I(t)} x(s) \geq \min_{s \in [\underline{\tau}(t), t)} x(s) \geq \min_{s \in \overline{I_0}} x(s) =: \omega \end{aligned}$$

for all $t \geq t_2$, where we used the notation from the proof of Theorem 3.7, again. As a consequence, equation (1.5) implies

$$0 \geq z'(t) + \omega \left(\sum_{j=1}^{m_1} Q_j(t) + \sum_{j=1}^{m_2} S_j(t) \int_{\underline{\sigma}_j(t)}^{\overline{\sigma}_j(t)} R_j(s)\, ds \right)$$

for all $t \geq t_3$ for some $t_3 \geq \underline{\sigma}^{-1}(t_2)$. Integrating the latter inequality over $[t_3, t]$ gives

$$z(t) \leq z(t_3) - \omega \left(\sum_{j=1}^{m_1} \int_{t_3}^{t} Q_j(s)\, ds + \sum_{j=1}^{m_2} \int_{t_3}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} R_j(r)\, dr\, ds \right)$$

for all $t \geq t_3$. This results in a contradiction with the eventual positivity of z for $t \to \infty$ if (3.19) holds. So, x is oscillatory.

Now suppose that

$$\sum_{j=1}^{m_1} \int_{t_0}^{\infty} Q_j(s)\, ds + \sum_{j=1}^{m_2} \int_{t_0}^{\infty} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} R_j(r)\, dr\, ds < \infty \tag{3.23}$$

and that condition $(3.20)_p$ is fulfilled for some $p > 1$. Then, by (3.22) and assumption (5), we get

$$x(t) \geq z(t) + \min_{s \in I(t)} x(s), \quad t \geq t_4,$$

where $t_4 \geq \underline{\tau}^{-1}(t_2)$ is such that $t_4 = \underline{\tau}^{-\kappa}(t_0)$ for some $\kappa \in \mathbb{N}$. By induction as in the proof of Theorem 3.7, we derive

$$x(t) \geq N_{t_4}^{\underline{\tau}}(t) z(t) + \omega, \quad t \geq t_4.$$

Denoting $\omega_1 := p^{\frac{1}{p}}(q\omega)^{\frac{1}{q}} > 0$, Young's inequality yields

$$x(t) \geq \omega_1 \left(N^{\tau}_{t_4}(t)z(t)\right)^{\frac{1}{p}}, \quad t \geq t_4.$$

Then, equation (1.5) gives

$$\begin{aligned} z'(t) &\leq -\omega_1 \left(\sum_{j=1}^{m_1} Q_j(t) \left(N^{\tau}_{t_4}(\sigma_j(t))z(\sigma_j(t))\right)^{\frac{1}{p}} + \sum_{j=1}^{m_2} S_j(t) \int_{\underline{\sigma}_j(t)}^{\overline{\sigma}_j(t)} R_j(s) \left(N^{\tau}_{t_4}(s)z(s)\right)^{\frac{1}{p}} ds \right) \\ &\leq -\omega_1 z(t)^{\frac{1}{p}} \left(\sum_{j=1}^{m_1} Q_j(t)(N^{\tau}_{t_4}(\sigma_j(t)))^{\frac{1}{p}} + \sum_{j=1}^{m_2} S_j(t) \int_{\underline{\sigma}_j(t)}^{\overline{\sigma}_j(t)} R_j(s) \left(N^{\tau}_{t_4}(s)\right)^{\frac{1}{p}} ds \right) \end{aligned}$$

for all $t \geq t_5 = \underline{\sigma}^{-1}(t_4)$. Dividing by $z^{\frac{1}{p}}(t)$ and integrating over $[t_5, t]$ results in

$$\begin{aligned} &qz^{\frac{1}{q}}(t) - qz^{\frac{1}{q}}(t_5) \\ &\quad \leq -\omega_1 \left(\sum_{j=1}^{m_1} \int_{t_5}^{t} (N^{\tau}_{t_4}(\sigma_j(s)))^{\frac{1}{p}} Q_j(s)\, ds + \sum_{j=1}^{m_2} \int_{t_5}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} (N^{\tau}_{t_4}(r))^{\frac{1}{p}} R_j(r)\, dr\, ds \right), \quad t \geq t_5. \end{aligned}$$

If the right side tends to $-\infty$ as $t \to \infty$, we get a contradiction with the eventual positivity of z, which implies that x is oscillatory. To see this, we use Lemmas 2.2, 2.5 to estimate

$$\begin{aligned} &\sum_{j=1}^{m_1} \int_{t_5}^{t} (N^{\tau}_{t_4}(\sigma_j(s)))^{\frac{1}{p}} Q_j(s)\, ds + \sum_{j=1}^{m_2} \int_{t_5}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} (N^{\tau}_{t_4}(r))^{\frac{1}{p}} R_j(r)\, dr\, ds \\ &\geq \sum_{j=1}^{m_1} \left(\int_{t_5}^{t} (N^{\tau}_{t_0}(\sigma_j(s)))^{\frac{1}{p}} Q_j(s)\, ds - \kappa^{\frac{1}{p}} \int_{t_5}^{t} Q_j(s)\, ds \right) \\ &\quad + \sum_{j=1}^{m_2} \left(\int_{t_5}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} (N^{\tau}_{t_0}(r))^{\frac{1}{p}} R_j(r)\, dr\, ds - \kappa^{\frac{1}{p}} \int_{t_5}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} R_j(r)\, dr\, ds \right) \\ &= \sum_{j=1}^{m_1} \left(\int_{\sigma_j^{-1}(t_0)}^{t} (N^{\tau}_{t_0}(\sigma_j(s)))^{\frac{1}{p}} Q_j(s)\, ds - \kappa^{\frac{1}{p}} \int_{t_0}^{t} Q_j(s)\, ds \right) \\ &\quad + \sum_{j=1}^{m_2} \left(\int_{\underline{\sigma}_j^{-1}(t_0)}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} (N^{\tau}_{t_0}(r))^{\frac{1}{p}} R_j(r)\, dr\, ds - \kappa^{\frac{1}{p}} \int_{t_0}^{t} S_j(s) \int_{\underline{\sigma}_j(s)}^{\overline{\sigma}_j(s)} R_j(r)\, dr\, ds \right) + C \end{aligned}$$

for an appropriate constant $C \in \mathbb{R}$. Condition $(3.20)_p$ and inequality (3.23) imply that the right-hand side tends to ∞ as $t \to \infty$. This completes the proof. □

4 Applications

In this section, we apply the results of Section 3 to concrete neutral differential equations.

4.1 Discrete delays

First, let us consider the neutral differential equation with one constant and one variable delay,

$$[x(t) - x(t-\alpha)]' + Q(t)x(\sigma(t)) = 0, \quad t \geq t_0 \tag{4.1}$$

for some $t_0 \in \mathbb{R}$, $\alpha > 0$, $\sigma \in \mathcal{T}_{t_0}$, and $Q \in C([t_0, \infty), \mathbb{R}_+)$. Then $\tau^k(t) = t - k\alpha$ for $k \in \mathbb{Z}$. Now, inequality (2.1) has the form

$$\zeta + (N_\zeta^\tau(t) - 1)\alpha \le t < \zeta + N_\zeta^\tau(t)\alpha.$$

Therefrom, we derive

$$\frac{t-\zeta}{\alpha} < N_\zeta^\tau(t) \le \frac{t-\zeta}{\alpha} + 1$$

that gives

$$N_\zeta^\tau(t) = \left\lfloor \frac{t-\zeta}{\alpha} \right\rfloor + 1.$$

Since we are interested in the convergence of the integral on the left side of $(3.4)_p$ in a neighborhood of ∞, it is enough to assume that $s \ge \tilde{t}_0$, where $\tilde{t}_0 \ge \sigma^{-1}(t_0)$ is such that $\sigma(\tilde{t}_0) > 0$. Then, dividing the inequality

$$\frac{\sigma(s) - t_0}{\alpha} < N_{t_0}^\tau(\sigma(s)) \le \frac{\sigma(s) - t_0}{\alpha} + 1$$

by $\sigma(s)/\alpha$ and taking the limit $s \to \infty$, we obtain

$$\lim_{s\to\infty} \frac{\alpha}{\sigma(s)} N_{t_0}^\tau(\sigma(s)) = 1.$$

Therefore, condition $(3.4)_p$ holds if and only if

$$\int_{\tilde{t}_0}^{\infty} (\sigma(s))^{\frac{1}{p}} Q(s)\, ds = \infty. \tag{4.2$_p$}$$

Using Theorem 3.3, one can easily prove the following result.

Proposition 4.1. *Let $t_0 \in \mathbb{R}$, $\alpha > 0$, $\sigma \in \mathcal{T}_{t_0}$, $Q \in C([t_0, \infty), \mathbb{R}_+)$, and $\tilde{t}_0 \ge \sigma^{-1}(t_0)$ be such that $\sigma(\tilde{t}_0) > 0$. Every solution of equation (4.1) oscillates if condition (3.1) or $(4.2)_p$ for some $p > 1$ is satisfied.*

In a particular case of equation (4.1) when $\sigma(t) = t - \beta$, $\beta > 0$, this statement can be simplified. From the inequality

$$\frac{s - \beta - t_0}{\alpha} < N_{t_0}^\tau(\sigma(s)) \le \frac{s - \beta - t_0}{\alpha} + 1$$

for $s \ge \tilde{t}_0$, where $\tilde{t}_0 \ge t_0 + \beta$ is positive, we get

$$\lim_{s\to\infty} \frac{\alpha}{s} N_{t_0}^\tau(\sigma(s)) = 1.$$

Hence, condition $(3.4)_p$ is equivalent to

$$\int_{\tilde{t}_0}^{\infty} s^{\frac{1}{p}} Q(s)\, ds = \infty. \tag{4.3$_p$}$$

Proposition 4.2. *Let $t_0 \in \mathbb{R}$, $\alpha, \beta > 0$, $Q \in C([t_0, \infty), \mathbb{R}_+)$, and $\tilde{t}_0 \ge t_0 + \beta$ be positive. Every solution of the equation*

$$[x(t) - x(t-\alpha)]' + Q(t)x(t-\beta) = 0, \quad t \ge t_0 \tag{4.4}$$

oscillates if condition (3.1) or $(4.3)_p$ for some $p > 1$ is satisfied.

Remark 4.3. Note that condition (3.1) or $(4.3)_p$ for some $p > 1$ implies

$$\int_{t_0}^{\infty} sQ(s)\,ds = \infty$$

which, by [6], means that equation (4.4) does not have a bounded positive solution.

Remark 4.4. For $Q(t) = t^{-\alpha}$, $1 < \alpha$, equation (4.4) reads as

$$[x(t) - x(t-\alpha)]' + t^{-\alpha}x(t-\beta) = 0, \quad t \geq t_0.$$

This is known [6] to have a bounded positive solution if $\alpha > 2$, since

$$\int_{t_0}^{\infty} s^{1-\alpha}\,ds < \infty.$$

To see that for $1 < \alpha < 2$ every solution is oscillatory, one can verify that

$$\int_{t_0}^{\infty} Q(s) \exp\left\{\frac{1}{\tau}\int_{t_0}^{s} rQ(r)\,dr\right\} ds = \infty$$

with $Q(t) = t^{-\alpha}$ from [4], or take $p = \frac{1}{\alpha-1} > 1$ in $(4.3)_p$ to get

$$\int_{\tilde{t}_0}^{\infty} s^{\frac{1}{p}} Q(s)\,ds = \int_{\tilde{t}_0}^{\infty} s^{\frac{1}{p}-\alpha}\,ds = \int_{\tilde{t}_0}^{\infty} s^{-1}\,ds = \infty.$$

The case $\alpha = 2$ still remains to be unanswered, despite of the fact that in [5, Corollary] the equation is stated to be oscillatory. At least for the variable delays, we proved that the equation has a positive solution (see Example 3.2).

4.2 Distributed delays

Example 4.5. Let us consider the following equation

$$\left[x(t) - \frac{2}{\pi}\int_{t-\pi}^{t-\frac{\pi}{2}} x(s)\,ds\right]' + \frac{2}{\pi(\sin 2\sigma - \sin\sigma)}\int_{t-2\sigma}^{t-\sigma} x(s)\,ds = 0, \quad t \geq t_0 \tag{4.5}$$

for some $t_0 \in \mathbb{R}$, where $\sigma = \frac{1}{3}\left(\pi - 2\arctan\frac{2}{\pi+2}\right) \doteq 0.79988 > 0$.

This equation is of the form (1.4) with $\lambda(t) \equiv 1$, $R(t) \equiv 1$, $Q(t) \equiv \frac{2}{\pi(\sin 2\sigma - \sin\sigma)} \doteq 2.25506 > 0$, $\underline{\tau}(t) = t - \pi$, $\overline{\tau}(t) = t - \frac{\pi}{2}$, $\underline{\sigma}(t) = t - 2\sigma$, and $\overline{\sigma}(t) = t - \sigma$. It is easy to see that condition (3.15) is satisfied. Thus, by Theorem 3.9, every solution of equation (4.5) oscillates. One of such solutions is $x(t) = \sin t$. Indeed, for this function, the left-hand side of (4.5) is equal to

$$\begin{aligned}\left[\sin(t) + \frac{2}{\pi}(\cos t + \sin t)\right]' &+ \frac{2(\cos(t-2\sigma) - \cos(t-\sigma))}{\pi(\sin 2\sigma - \sin\sigma)} \\ &= \left(1 + \frac{2}{\pi}\right)\cos(t) - \frac{2}{\pi}\sin t + \frac{2\cos t}{\pi}\frac{\cos 2\sigma - \cos\sigma}{\sin 2\sigma - \sin\sigma} + \frac{2}{\pi}\sin t. \end{aligned}\tag{4.6}$$

Noting that

$$\begin{aligned}\frac{\cos 2\sigma - \cos\sigma}{\sin 2\sigma - \sin\sigma} &= -\tan\frac{3}{2}\sigma = -\tan\left(\frac{\pi}{2} - \arctan\frac{2}{\pi+2}\right) \\ &= -\cot\left(\arctan\frac{2}{\pi+2}\right) = -\frac{\pi+2}{2} = -1 - \frac{\pi}{2}\end{aligned}$$

makes the right side of (4.6) vanish.

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Dynamics analysis of a diffusive prey-taxis system with memory and maturation delays

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Abstract. In this paper, a diffusive predator-prey system considering prey-taxis term with memory and maturation delays under Neumann boundary conditions is investigated. Firstly, the existence and stability of equilibria, especially the existence, uniqueness and stability of the positive equilibrium, are studied. Secondly, we prove that: (i) there is no spatially homogeneous steady state bifurcation as the eigenvalue of the negative Laplace operator is zero; (ii) as this system is only with memory delay τ_1, the the spatially nonhomogeneous Hopf bifurcation appears; (iii) when the model is only with maturation delay τ_2, the system has spatially homogeneous and nonhomogeneous periodic solutions; (iv) for the case of two delays, the system has rich dynamics, for example, stability switches, whose curves have four forms. Finally, some numerical simulations are produced to verify and support the theoretical results.

Keywords: diffusive system, fear effect, prey-taxis, memory delay, maturation delay.

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1 Introduction

The predator-prey dynamics is of great significance for developing the mathematical ecology and has been investigated by many scholars [5, 9, 10, 15, 18, 19]. That the prey population is also affected by the fear of predators not only the direct killing has been found [18]. On the basis of the experiment of Zanette [18], Wang et al. originally introduced the fear effect into the predator-prey model. The results showed that the incorporation of fear effect into the predator-prey model with Holling-II functional response can affect the stability of equilibrium [15]. With further research, for the various biological factors, Holling-II functional response of the predator-prey model with fear effect is modified differently, such as Allee effect [5], Leslie–Gower term [9] and prey refuge [19].

It is well known that in the spatial predator-prey model, predator and prey are usually considered to move randomly and are modeled by the reaction-diffusion equation. However, species also move towards certain directions due to the attraction or repulsion of some

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chemical signals, which is commonly called chemotactic movement [16]. In biology, predator population tends to move to the area where the density of the prey population is higher, which is termed prey-taxis [6,16]. This phenomenon was first noticed in a regional experiment about individual ladybugs and aphids by Karevia and Odell [7]. They derived a predator-prey model by considering prey-taxis as biased random walks, which is as follows

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \Delta u - \nabla \cdot (u\rho(u,v)\nabla v) + G_1(u,v), \\ \dfrac{\partial v(x,t)}{\partial t} = D\Delta v + G_2(u,v), \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ represent separately the density of the prey and predator at time $t > 0$ and space location x; $-\nabla \cdot (u\rho(u,v)\nabla v)$ stands for the prey-taxis term, and $\rho(u,v)$ is a coefficient that may rely on $u(x,t)$ or $v(x,t)$ and D represents the diffusion rate; $G_1(u,v)$ and $G_2(u,v)$ describe the functional response functions.

Considering that the ability to perceive danger is also related to the memory of animals, Fagan pointed out that it is vital to incorporate spatial memory into models of animal movements [3]. Namely, combining the reaction-diffusion equation with the memory delay term, form the spatial memory model or the memory-based diffusion system, which has attracted many researchers [1, 12, 13]. For example, Aly [1] studied bifurcations of a memory-based diffusive predator-prey system. Shi et al. [12] showed the wellposedness of the memory-based diffusive system; Song [13] investigated Hopf bifurcation caused by memory delay for a memory-based diffusive system. Recently, some scholars have considered to combine memory delay with fear effect into diffusion system [2, 17]. For example, Debnath et al. [2] explored the role of memory and fear effect on prey-predator dynamics. Yang et al. [17] considered memory delay and fear effect into a predator-prey model with diffusion. They proved that the fear effect has both the stabilizing and the destabilizing effect on the coexisting equilibrium under different conditions.

After the predator gets its food, it does not immediately respond to a change in the number of population, but requires a period of digestion or pregnancy. Namely, there is a time delay to allow the predator to reach maturity. Therefore, it is necessary to introduce the maturation or digestion delay into the model. For example, Liu et al. [8] introduced the digestion delay into a predator-prey model with fear effect. They showed that the occurrence of stability switches and Hopf bifurcations as the digestion delay passes through a series of critical values. Shi et al. [11] studied a model incorporating memory-based diffusion and maturation delay. They proved that the proper association of two delay mechanisms can cause the appearance of the spatially inhomogeneous time-periodic patterns. Wang et al. [14] investigated the model collecting the spatial memory, maturation effect, prey-taxis and fear effect, which is as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + \alpha\nabla(u\nabla v_{\tau_1}) + \dfrac{r_0 u}{1+kv_{\tau_2}} - du - au^2 - \dfrac{puv}{1+cu}, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial v}{\partial t} = d_2\Delta v - mv^2 + \dfrac{quv}{1+cu}, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0, \\ u(x,t) = u_0(x) \geq 0, v(x,t) = v_0(x) \geq 0, \ x \in \Omega, \ t \in (-\max\{\tau_1,\tau_2\}, 0], \end{cases} \tag{1.2}$$

where the meanings of $u(x,t)$, $v(x,t)$ are the same to those of model (1.1); $v_{\tau_1} = v(x, t-\tau_1)$,

$v_{\tau_2} = v(x, t - \tau_2)$; τ_1 is the memory delay; τ_2 represents the maturation delay; d_1 and d_2 are self-diffusion coefficients; r_0 and d stand for separately prey's growth rate and natural death rate without considering fear cost; a and m are on behalf of the death rates for prey's and predator's intra-special competition, respectively; $\frac{1}{1+kv}$ denotes the fear factor; $\alpha\nabla(u\nabla v_{\tau_1})$ stands for the prey-taxis term; $\frac{uv}{1+cu}$ is the Holling-II functional response; Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$; $\partial\nu$ is the outer flux; Δ and ∇ are Laplace and gradient operator defined in Ω. All of the parameters are positive. They showed that the model can exhibit rich dynamics, such as Turing instability, Hopf bifurcation and spatially nonhomogeneous (homogeneous) periodic distribution. They considered the spatial memory, pregnancy effect and fear effect for prey into a diffusive prey-taxis model with Holling-II functional response function. Motivated by this, we interest the system that the spatial memory and maturation effect in predator and fear effect in prey are incorporated in a diffusive prey-taxis model with the modified Leslie–Gower term and are curious about what dynamic behaviors for this complex model occur. In this paper, we aim to study the diffusive prey-taxis system considering fear effect with memory and maturation delays as follows,

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1\Delta u + \dfrac{r_0 u}{1+av} - du - cu^2 - \dfrac{puv}{u+kv}, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = d_2\Delta v - \chi\nabla(v\nabla u_{\tau_1}) + sv\left(1 - \dfrac{qv}{u_{\tau_2}+m}\right), \ x \in \Omega, \ t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu} = \dfrac{\partial v(x,t)}{\partial \nu} = 0, \ x \in \partial\Omega, \ t > 0, \\ u(x,t) = u_0(x) \geq 0, v(x,t) = v_0(x) \geq 0, \ x \in \Omega, \ t \in (-\max\{\tau_1,\tau_2\}, 0], \end{cases} \tag{1.3}$$

where $u_{\tau_1} = u(x, t - \tau_1)$; $u_{\tau_2} = u(x, t - \tau_2)$; $\frac{1}{1+av}$ is the fear factor; c represents the birth rate of prey; $-\chi\nabla(v\nabla u)$ stands for prey-taxis term; χ is prey-taxis coefficient; $\chi > 0$ (< 0) is called attractive (repulsive) prey-taxis; s is the intrinsic growth rate of predator; $\frac{puv}{u+kv}$ is the Holling-II functional response; $\frac{qv}{u+m}$ is the modified Leslie–Gower term. Keep the meanings and qualities of other parameters and functions be the same to system (1.2).

The remainder of this paper is structured as follows. In Sect. 2, we not only discuss the number and stability of equilibria, but also give the conditions for the existence and stability of the unique positive equilibrium. In Sect. 3, we analyze the existence of the spatially homogeneous and nonhomogeneous steady states and Hopf bifurcation, and exhibit the dynamics of the model with the cases of $\tau_1 > 0$, $\tau_2 = 0$; $\tau_1 = 0$, $\tau_2 > 0$; $\tau_1 > 0$, $\tau_2 > 0$. At the end, numerical simulations are given to substantiate the theoretical findings.

2 The existence and stability of equilibria

First, we discuss the existence and stability of the equilibria for the following ordinary differential equation of system (1.3)

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \dfrac{r_0 u}{1+av} - du - cu^2 - \dfrac{puv}{u+kv}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = sv\left(1 - \dfrac{qv}{u+m}\right). \end{cases} \tag{2.1}$$

Clearly, system (2.1) always has the trivial equilibrium $(0,\ 0)$ and a semi-trivial equilibrium $e_{01}(0, \frac{m}{q})$; as $r_0 > d$, the semi-trivial equilibrium $e_{10}(\frac{r_0-d}{c}, 0)$ exists; as

$$r_0 > r_0^* \triangleq (dk+p)(q+am)/kq, \tag{2.2}$$

system (2.1) has the unique positive solution $e_2(\bar{u}, \bar{v})$ with $\bar{v} = \frac{\bar{u}+m}{q}$, and $\bar{u}$ is the positive root of the equation

$$\frac{r_0 q}{a\bar{u} + am + q} - d - c\bar{u} - \frac{p(\bar{u}+m)}{(q+k)\bar{u} + km} = 0.$$

For each nonnegative equilibrium $e_2(\bar{u}, \bar{v})$, the Jacobi matrix can be expressed by

$$J_{(u,v)} := \begin{pmatrix} \dfrac{r_0}{1+av} - d - 2cu - \dfrac{pkv^2}{(u+kv)^2} & -\dfrac{ar_0 u}{(1+av)^2} - \dfrac{pu^2}{(u+kv)^2} \\ \dfrac{qsv^2}{(u+m)^2} & s - \dfrac{2qsv}{u+m} \end{pmatrix}. \tag{2.3}$$

For $(0,0)$ and $e_{10}(\frac{r_0-d}{c}, 0)$, they are always unstable because $\lambda_2 = s > 0$; for $e_{01}(0, \frac{m}{q})$, the eigenvalues of the Jacobi matrix are $\lambda_1 = \frac{r_0 q}{q+am} - d - \frac{p}{k}$, $\lambda_2 = -s < 0$, then e_{01} is locally asymptotically stable (unstable) if $r_0 < r_0^*$($r_0 > r_0^*$); the corresponding characteristic equation at the positive equilibrium $e_2\,(\bar{u}, \bar{v})$ is

$$\lambda^2 - (A_{11} - s)\,\lambda + \left(-sA_{11} - \frac{s}{q}A_{12}\right) = 0,$$

where

$$A_{11} = \frac{r_0}{1+a\bar{v}} - d - 2c\bar{u} - \frac{pk\bar{v}^2}{(\bar{u}+k\bar{v})^2},$$
$$A_{12} = -\frac{p\bar{u}}{(\bar{u}+k\bar{v})^2} - \frac{ar_0\bar{u}}{(1+a\bar{v})^2}.$$

Hence, if

$$A_{11} - s < 0, \qquad qA_{11} + A_{12} < 0, \tag{2.4}$$

then the positive equilibrium $e_2\,(\bar{u}, \bar{v})$ is locally asymptotically stable.

Summarizing the above works, we have the following theorem.

Theorem 2.1. *Model (2.1) always has an unstable trivial equilibrium $(0,0)$; if $r_0 > d$, then system (2.1) has a saddle $e_{10}(\frac{r_0-d}{c}, 0)$; the semi-trivial equilibrium $e_{01}(0, \frac{m}{q})$ is locally asymptotically stable when $r_0 < (dk+p)(q+am)/kq$ and unstable when $r_0 > (dk+p)(q+am)/kq$; suppose that condition (2.2) holds, then model (2.1) has the unique positive equilibrium $e_2\,(\bar{u}, \bar{v})$, and it is locally asymptotically stable as (2.4) are satisfied.*

Remark 2.2. Notice that (2.2) is not only the condition for the existence of positive equilibrium e_2, but also the change situation of the stability of equilibrium e_{01}. In other words, the appearance of the positive equilibrium e_2 leads to the instability of the boundary equilibrium e_{01}. Moreover, the condition for the existence of the equilibrium e_{10} is also contained in (2.2).

3 Stability analysis

In this section, we are going to analyse the stability of the positive equilibrium on one-dimension $\Omega = (0, \ell\pi)$. The linearized system of model (1.3) at $e_2(\bar{u}, \bar{v})$ is

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) + A_{11} u(x,t) + A_{12} v(x,t), & x \in \Omega, t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = d_2 \Delta v(x,t) - \chi \bar{v} \Delta u_{\tau_1} + \dfrac{s}{q} u_{\tau_2} - s v(x,t), & x \in \Omega, t > 0, \\ \dfrac{\partial u(x,t)}{\partial v} = 0, \dfrac{\partial v(x,t)}{\partial v} = 0, & x \in \partial\Omega, t > 0, \\ u(x,t) = u_0(x) \geq 0, v(x,t) = v_0(x) \geq 0, & x \in \Omega, t \in (-\max\{\tau_1, \tau_2\}, 0]. \end{cases} \tag{3.1}$$

The characteristic equation of model (3.1) at $e_2(\bar{u}, \bar{v})$ is

$$\Delta_n := \lambda^2 + A_n \lambda + B_n + C_n e^{-\lambda \tau_1} + D_n e^{-\lambda \tau_2} = 0, \qquad n \in \mathbb{N}, \tag{3.2}$$

where

$$\begin{aligned} A_n &= (d_1 + d_2) \frac{n^2}{l^2} - (A_{11} - s) > 0, & B_n &= \left(d_1 \frac{n^2}{l^2} - A_{11} \right) \left(d_2 \frac{n^2}{l^2} + s \right), \\ C_n &= -A_{12} \chi \bar{v} \frac{n^2}{l^2} > 0, & D_n &= -\frac{s}{q} A_{12} > 0. \end{aligned} \tag{3.3}$$

First, we discuss the existence of the steady states of model (1.3). Assume that $\lambda = 0$, then the characteristic equation (3.2) becomes

$$B_n + C_n + D_n = 0. \tag{3.4}$$

Note that there is no delay in equation (3.4), which is equivalent to, $\tau_1 = \tau_2 = 0$.

3.1 Steady states

If $n = 0$, one can deduce that $C_0 = 0$, then equation (3.4) can be rewritten as

$$B_0 + D_0 = -s A_{11} - \frac{s}{q} A_{12} = 0,$$

which is a contradiction with condition (2.4), therefore, there is no spatially homogeneous steady state bifurcation.

As $n \neq 0$, equation (3.4) becomes

$$d_1 d_2 \frac{n^4}{\ell^4} - (A_{11} d_2 - s d_1 + \chi A_{12} \bar{v}) \frac{n^2}{\ell^2} - \left(s A_{11} + \frac{s}{q} A_{12} \right) = 0. \tag{3.5}$$

Regard χ as a function of n^2, if

$$A_{11} d_2 - s d_1 > 0, \quad q (A_{11} d_2 + s d_1)^2 + 4 d_1 d_2 s A_{12} > 0, \tag{3.6}$$

then there is

$$\chi(n^2) = \frac{d_1 d_2 n^4 - (A_{11} d_2 - s d_1)\, l^2 n^2 - \left(s A_{11} + \frac{s}{q} A_{12}\right) l^4}{A_{12} \bar{v} l^2 n^2} > 0.$$

Taking the derivative of $\chi(n^2)$, there is

$$\chi'(n^2) = \frac{d_1 d_2 n^4 + \left(s A_{11} + \frac{s}{q} A_{12}\right) l^4}{A_{12} \bar{v} l^2 n^4}.$$

If $n < n_T$, one can deduce that $\chi'(n^2) > 0$, then $\chi(n^2)$ is increasing with n^2; if $n > n_T$, then $\chi'(n^2) < 0$, and $\chi(n^2)$ is decreasing with n^2 , where

$$n_T^2 = \ell^2 \sqrt{\frac{-(q s A_{11} + s A_{12})}{q d_1 d_2}}.$$

In order to ensure n^* is a positive integer, let

$$n^* = \begin{cases} [n_T], & \text{if } \chi([n_T]^2) > \chi\left(([n_T]+1)^2\right), \\ [n_T]+1, & \text{if } \chi([n_T]^2) < \chi\left(([n_T]+1)^2\right), \end{cases}$$

and $\chi^* = \chi(n^*)$, if $\chi > \chi^*$, then $B_n + C_n + D_n > 0$; if $\chi < \chi^*$, then there is $n \in \mathbb{N}_+$ satisfying $B_n + C_n + D_n = 0$.

To sum up, we have the following theorem.

Theorem 3.1. *Suppose that conditions (2.2), (2.4) and (3.6) hold. Let $n = 0$, then $e_2(\bar{u}, \bar{v})$ is always stable and there is no spatially homogeneous steady state bifurcation. Let $n \in \mathbb{N}_+$, if $\chi > \chi^*$, then $e_2(\bar{u}, \bar{v})$ is asymptotically stable; if $\chi < \chi^*$, then the spatially homogeneous steady state occurs.*

Remark 3.2. According to Theorem 3.1, the stability of positive equilibrium $e_2(\bar{u}, \bar{v})$ is affected by χ for the predator-prey system without delay ($\tau_1 = \tau_2 = 0$). That is to say, fast memory diffusion ($\chi > \chi^*$) remains the stability of the system, while slow memory diffusion ($\chi < \chi^*$) causes the system to be unstable. Moreover, if $A_{11} d_2 - s d_1 < 0$, $\chi(n^2) < 0$ for each $n \in \mathbb{N}_+$, then $e_2(\bar{u}, \bar{v})$ is asymptotically stable for $\chi > 0$. That is, for a sufficiently large self-diffusion d_1, there is no spatially homogeneous steady state bifurcation.

3.2 Hopf bifurcations

In this subsection, we always assume $\chi > \chi^*$ to analyze the Hopf bifurcation of model (1.3).

Let $\lambda = i\omega (\omega > 0)$, then the characteristic equation (3.2) becomes

$$-\omega^2 + B_n + C_n \cos(\omega\tau_1) + D_n \cos(\omega\tau_2) + i\big(A_n \omega - C_n \sin(\omega\tau_1) - D_n \sin(\omega\tau_2)\big) = 0. \quad (3.7)$$

As $n = 0$, equation (3.7) becomes

$$-\omega^2 + B_0 + D_0 \cos(\omega\tau_2) + i\left(A_0 \omega - D_0 \sin(\omega\tau_2)\right) = 0, \quad (3.8)$$

which only contains pregnancy delay τ_2. By equation (3.8), we have

$$\sin(\omega\tau_2) = \frac{A_0\omega}{D_0} > 0, \qquad \cos(\omega\tau_2) = \frac{\omega^2 - B_0}{D_0},$$

and

$$\omega^4 + \left(A_0^2 - 2B_0\right)\omega^2 + B_0^2 - D_0^2 = 0. \tag{3.9}$$

By condition (2.4), one can obtain $B_0 + D_0 > 0$, $A_0^2 - 2B_0 > 0$, $\left(A_0^2 - 2B_0\right)^2 - 4(B_0^2 - D_0^2) > 0$. So we consider equation (3.9) from several different cases.

1) $qA_{11} < A_{12}$, there are no positive real roots of equation (3.9), so the system is always stable;

2) $qA_{11} > A_{12}$, for this case, equation (3.9) has only one positive root satisfying

$$\omega_{2,0}^{+} = \sqrt{\frac{-q\left(A_{11}^2 + s^2\right) + \sqrt{q^2\left(A_{11}^2 - s^2\right)^2 + 4s^2A_{12}^2}}{2q}}, \tag{3.10}$$

and the transversality condition

$$\left(\frac{\mathrm{d}\Re(\lambda)}{\mathrm{d}\tau_2}\right)^{-1}\Bigg|_{\tau_2=\tau_{2,0}^{j+}} = \frac{\sqrt{q^4(1+4sA_{11})(s-A_{11})^2 + 4q^2s^2A_{12}^2}}{s^2A_{12}^2} > 0, \tag{3.11}$$

where $\tau_{2,0}^{j+}$ is a sequence as follows

$$\tau_{2,0}^{j+} = \frac{1}{\omega_{2,0}^{+}}\left(\arccos\frac{q\omega_{2,0}^{+2} + qsA_{11}}{-sA_{12}} + 2j\pi\right), \qquad j = 0, 1, 2, \ldots \tag{3.12}$$

The sequence $\{\tau_{2,0}^{j+}\}_{j=0}^{\infty}$ is an increasing sequence for j, thus $\tau_2^* = \tau_{2,0}^{0+} = \min_{j\in\mathbb{N}} \tau_{2,0}^{j+}$. If $0 < \tau_2 < \tau_2^*$, all the real parts of the roots of equation (3.2) are negative; if $\tau_2 = \tau_{2,0}^{j+}$, equation (3.2) has a pair of pure imaginary roots; if $\tau_2 > \tau_2^*$, at least, one of the roots of equation (3.2) is positive.

Theorem 3.3. *Suppose that conditions (2.2) and (2.4) hold. When $n = 0$, for $\tau_1 = 0$, we have the following statements.*

(1) If $qA_{11} < A_{12}$ or $qA_{11} > A_{12}$, $\tau_2 < \tau_2^*$, then e_2 is asymptotically stable;

(2) if $qA_{11} > A_{12}$, $\tau_2 > \tau_2^*$, then e_2 is unstable;

(3) if $qA_{11} > A_{12}$, $\tau_2 = \tau_{2,0}^{j+}$ $(j = 0, 1, 2, \ldots)$, then the spatially homogeneous Hopf bifurcation occurs.

Remark 3.4. From the above discussion, we can see that model (1.3) does not undergo the spatially homogeneous Hopf bifurcation as it only has the spatial memory delay τ_1, and for this case, memory diffusion χ has no effect on the stability of the positive equilibrium $(\bar{u}, \bar{v})$ of system (1.3).

Next, we assume $n \neq 0$, and consider the two cases (1) $\tau_1 > 0$, $\tau_2 = 0$; or $\tau_1 = 0$, $\tau_2 > 0$; (2) $\tau_1 > 0$, $\tau_2 > 0$.

Case 1. $\tau_1 = 0$, $\tau_2 > 0$ or $\tau_1 > 0$, $\tau_2 = 0$.

When $\tau_1 = 0$, $\tau_2 > 0$, equation (3.7) can be simplified as

$$-\omega^2 + B_n + C_n + D_n \cos(\omega\tau_2) + i\left(A_n\omega - D_n \sin(\omega\tau_2)\right) = 0. \tag{3.13}$$

Solving equation (3.13), we obtain

$$\sin(\omega\tau_2) = \frac{A_n\omega}{D_n} > 0, \qquad \cos(\omega\tau_2) = \frac{\omega^2 - (B_n + C_n)}{D_n},$$

and

$$\omega^4 + \left(A_n^2 - 2(B_n + C_n)\right)\omega^2 + (B_n + C_n)^2 - D_n^2 = 0, \tag{3.14}$$

where for $\forall n \in \mathbb{N}_+$, $(B_n + C_n)^2 - D_n^2 \neq 0$.

1). There exists at least a $n \in \mathbb{N}_+$ satisfying the condition

$$|B_n + C_n| < D_n, \tag{3.15}$$

or

$$|B_n + C_n| > D_n, \quad A_n^2 = 2\left((B_n + C_n) - \sqrt{(B_n + C_n)^2 - D_n^2}\right), \tag{3.16}$$

such that equation (3.14) has a unique positive root

$$\omega_{2,n}^{+} = \sqrt{\frac{-\left(A_n^2 - 2(B_n + D_n)\right) + \sqrt{A_n^4 - 4(B_n + C_n)A_n^2 + 4D_n^2}}{2}},$$

and the corresponding τ_2 is

$$\tau_{2,n}^{j+} = \frac{1}{\omega_{2,n}^{+}}\left(\arccos\frac{\omega_{2,n}^{+2} - (B_n + C_n)}{D_n} + 2j\pi\right), \qquad j = 0, 1, 2, \ldots$$

For each fixed n, $\{\tau_{2,n}^{j+}\}_{j=0}^{\infty}$ is an increasing sequence with the variable j. Denote

$$\tau_2^* := \tau_{2,n_c}^{0+} = \min_{n\in\mathbb{N}_+}\left\{\tau_{2,n}^{0+}\right\},$$

and τ_2^* is the minimum value of the sequence of $\left\{\tau_{2,n}^{j+}\right\}_{j=0}^{\infty}$, $n \in \mathbb{N}_+$. The transversality condition is

$$\left(\frac{\mathrm{d}\Re(\lambda)}{\mathrm{d}\tau_2}\right)^{-1}\Bigg|_{\tau_2=\tau_{2,0}^{j+}} = \frac{2\left(\omega_{2,0}^{+2} - B_0\right) + A_0^2}{\left(\omega_{2,0}^{+2} - B_0\right)^2 + A_0^2\omega_{2,0}^{+2}} > 0. \tag{3.17}$$

That is, if $\tau_2 < \tau_2^*$, then all the real parts of the roots of equation (3.2) are negative; if $\tau_2 = \tau_2^*$, then equation (3.2) has a pair of pure imaginary roots; if $\tau_2 > \tau_2^*$, then there is at least a root of equation (3.2) that has positive real part.

2). There exists a $n \in \mathbb{N}_+$ satisfying

$$|B_n + C_n| > D_n, \quad A_n^2 < 2\left((B_n + C_n) - \sqrt{(B_n + C_n)^2 - D_n^2}\right), \tag{3.18}$$

such that, equation (3.13) has two positive roots,

$$\omega_{2,n}^{\pm} = \sqrt{\frac{-\left(A_n^2 - 2\left(B_n + C_n\right)\right) \pm \sqrt{A_n^4 - 4\left(B_n + C_n\right) A_n^2 + 4D_n^2}}{2}},$$

and the corresponding τ_2 are

$$\tau_{2,n}^{j\pm} = \frac{1}{\omega_{2,n}^{\pm}} \left(\arccos \frac{\omega_{2,n}^{\pm 2} - (B_n + C_n)}{D_n} + 2j\pi \right), \qquad j = 0, 1, 2, \ldots$$

The transversality condition is

$$\left(\frac{\mathrm{d}\Re(\lambda)}{\mathrm{d}\tau_2} \right)^{-1} \Bigg|_{\tau_2 = \tau_{2,n}^{j+}} = \frac{\sqrt{A_n^4 - 4\left(B_n + C_n\right) A_n^2 + 4D_n^2}}{\left(\omega_{2,n}^{+2} - B_n\right)^2 + A_n^2 \omega_{2,n}^{+2}} > 0,$$

$$\left(\frac{\mathrm{d}\Re(\lambda)}{\mathrm{d}\tau_2} \right)^{-1} \Bigg|_{\tau_2 = \tau_{2,n}^{j-}} = -\frac{\sqrt{A_n^4 - 4\left(B_n + C_n\right) A_n^2 + 4D_n^2}}{\left(\omega_{2,n}^{-2} - B_n\right)^2 + A_n^2 \omega_{2,n}^{-2}} < 0.$$

For each fixed $n \in \mathbb{N}_+$, $\{\tau_{2,n}^{j+}\}_{j=0}^{\infty}$, $\{\tau_{2,n}^{j-}\}_{j=0}^{\infty}$ are increasing sequences with j, and $\tau_{2,n}^{j+} < \tau_{2,n}^{j-}$ due to $\omega_{2,n}^{j+} > \omega_{2,n}^{j-}$. Reorder $\{\tau_{2,n}^{j+}\}_{j=0}^{\infty}$, $\{\tau_{2,n}^{j-}\}_{j=0}^{\infty}$ as increasing subsequences and denote as $\{\tau_2^{S+}\}_{S=1}^{\infty}$, $\{\tau_2^{S-}\}_{S=1}^{\infty}$, respectively, and $\tau_2^* = \tau_2^{0+}$ is the minimum value. There exists a $K_2 \in \mathbb{N}$, such that all the real parts of the roots of model (3.13) are negative for

$$\tau_2 \in (0, \tau_2^{0+}) \bigcup \left(\tau_2^{0-}, \tau_2^{1+}\right) \bigcup \cdots \bigcup \left(\tau_2^{(K_2-1)-}, \tau_2^{K_2+}\right);$$

at least one root of model (3.13) has positive real part for

$$\tau_2 \in (\tau_2^{0+}, \tau_2^{0-}) \bigcup \left(\tau_2^{1+}, \tau_2^{1-}\right) \bigcup \cdots \bigcup \left(\tau_2^{K_2+}, \infty\right).$$

3). For each $n \in \mathbb{N}_+$ satisfying

$$|B_n + C_n| > D_n, \quad A_n^2 > 2\left((B_n + C_n) - \sqrt{(B_n + C_n)^2 - D_n^2} \right), \tag{3.19}$$

(3.13) has no positive root.

Theorem 3.5. *Suppose that (2.2), (2.4) and $\chi < \chi^*$ hold. When $\tau_1 = 0$, for $n \neq 0$, we have the results.*

1. The positive equilibrium $e_2(\bar{u}, \bar{v})$ is asymptotically stable if one of the following conditions is satisfied:

(1) $\exists n \in \mathbb{N}_+$, (3.15) or (3.16), $\tau_2 < \tau_2^*$;

(2) $\exists n \in \mathbb{N}_+$, (3.18), $\tau_2 \in \left(0, \tau_2^{0+}\right) \cup \cdots \cup \left(\tau_2^{(K_2-1)-}, \tau_2^{K_2+}\right)$, $K_2 \geq 0$;

(3) $\forall n \in \mathbb{N}_+$, (3.19).

2. The positive equilibrium $e_2(\bar{u}, \bar{v})$ is unstable if one of the following conditions holds:

(1) $\exists n \in \mathbb{N}_+$, (3.15) *or* (3.16) *,* $\tau_2 > \tau_2^*$*;*

(2) $\exists n \in \mathbb{N}_+$, (3.18) *holds,* $\tau_2 \in \left(\tau_2^{0+}, \tau_2^{0-}\right) \cup \cdots \cup \left(\tau_2^{K_2+}, +\infty\right)$*,* $K_2 \geq 0$.

3. *System (1.3) undergoes the spatially nonhomogeneous Hopf bifurcation if one of the following conditions is met:*

(1) $\exists n \in \mathbb{N}_+$, (3.15) *or* (3.16) *holds,* $\tau_2 = \tau_{2,n}^{j+}$ $(j = 0, 1, 2, \ldots)$*;*

(2) $\exists n \in \mathbb{N}_+$, (3.18) *holds,* $\tau_2 = \tau_2^{S\pm}$ $(S = 0, 1, 2, \ldots)$*.*

When $\tau_1 > 0$ and $\tau_2 = 0$, the discussion process is the same as above. Denote

$$\omega_{1,n}^{\pm} = \sqrt{\frac{-\left(A_n^2 - 2\left(B_n + D_n\right)\right) \pm \sqrt{A_n^4 - 4\left(B_n + D_n\right) A_n^2 + 4C_n^2}}{2}}, \tag{3.20}$$

and the corresponding delay τ_1 are

$$\tau_{1,n}^{j\pm} = \frac{1}{\omega_{1,n}^{\pm}} \left(\arccos \frac{\omega_{1,n}^{\pm 2} - \left(B_n + D_n\right)}{C_n} + 2j\pi \right), \qquad j = 0, 1, 2, \ldots \tag{3.21}$$

Let

$$\tau_1^* := \tau_1^{0+} = \min_{n \in \mathbb{N}_+} \left\{ \tau_{1,n}^{0+} \right\}. \tag{3.22}$$

We have the following statements.

Theorem 3.6. *Suppose that* (2.2), (2.4) *and* $\chi < \chi^*$ *hold. When* $\tau_2 = 0$*, for* $n \neq 0$*, the following statements hold.*

1. *The positive equilibrium* $(\bar{u}, \bar{v})$ *is asymptotically stable if the parameters satisfy one of the following conditions:*

(1) $\exists n \in \mathbb{N}_+$, $|B_n + D_n| < C_n$ *or* $|B_n + D_n| > C_n$, $A_n^2 = 2\left(\left(B_n + D_n\right) - \sqrt{\left(B_n + D_n\right)^2 - C_n^2}\right)$ *hold,* $\tau_1 < \tau_1^*$*;*

(2) $\exists n \in \mathbb{N}_+$, $|B_n + D_n| > C_n$, $A_n^2 < 2\left(\left(B_n + D_n\right) - \sqrt{\left(B_n + D_n\right)^2 - C_n^2}\right)$ *hold,* $\tau_1 \in \left(0, \tau_1^{0+}\right) \cup \cdots \cup \left(\tau_1^{(K_1-1)-}, \tau_2^{K_1+}\right)$*,* $K_1 \geq 0$*;*

(3) $\forall n \in \mathbb{N}_+$, $|B_n + D_n| > C_n$, $A_n^2 > 2\left(\left(B_n + D_n\right) - \sqrt{\left(B_n + D_n\right)^2 - C_n^2}\right)$ *hold.*

2. *The positive equilibrium* $(\bar{u}, \bar{v})$ *is unstable if the parameters meet one of the following conditions:*

(1) $\exists n \in \mathbb{N}_+$, $|B_n + D_n| < C_n$ *or* $|B_n + D_n| > C_n$, $A_n^2 = 2\left(\left(B_n + D_n\right) - \sqrt{\left(B_n + D_n\right)^2 - C_n^2}\right)$ *hold,* $\tau_1 > \tau_1^*$*;*

(2) $\exists n \in \mathbb{N}_+$, $|B_n + D_n| > C_n$, $A_n^2 < 2\left(\left(B_n + D_n\right) - \sqrt{\left(B_n + D_n\right)^2 - C_n^2}\right)$ *hold,* $\tau_1 \in \left(\tau_1^{0+}, \tau_1^{0-}\right) \cup \cdots \cup \left(\tau_1^{K_1+}, +\infty\right)$*,* $K_1 \geq 0$*;*

3. System (1.3) undergoes the spatially nonhomogeneous Hopf bifurcation if the parameters fulfill one of the following conditions:

(1) $\exists n \in \mathbb{N}_+$, $|B_n + D_n| < C_n$ *or* $|B_n + D_n| > C_n$, $A_n^2 = 2\Big((B_n + D_n) - \sqrt{(B_n + D_n)^2 - C_n^2}\Big)$ *hold,* $\tau_1 = \tau_{1,n}^{j+}$ $(j = 0, 1, 2, \dots)$;

(2) $\exists n \in \mathbb{N}_+$, $|B_n + D_n| > C_n$, $A_n^2 < 2\Big((B_n + D_n) - \sqrt{(B_n + D_n)^2 - C_n^2}\Big)$ *hold,* $\tau_1 = \tau_{1,n}^{S+}$ $(S = 0, 1, 2, \dots)$.

Case 2. $\tau_1, \tau_2 > 0$

We rewrite (3.2) with $\tau_1 > 0$ and $\tau_2 > 0$ as

$$D_n(\lambda, \tau_1, \tau_2) := P_{0,n}(\lambda) + P_{1,n}(\lambda)e^{-\lambda\tau_1} + P_{2,n}(\lambda)e^{-\lambda\tau_2} = 0, \tag{3.23}$$

where

$$P_{0,n}(\lambda) = \lambda^2 + A_n\lambda + B_n, \qquad P_{1,n}(\lambda) = C_n, \qquad P_{2,n}(\lambda) = D_n, \tag{3.24}$$

A_n, B_n, C_n and D_n are defined in (3.3). $\forall n \in \mathbb{N}_+$, $P_{S,n}(\lambda)$ $(S = 0, 1, 2)$ satisfy

(I) $\deg P_{0,n}(\lambda) \geq \max\{\deg P_{1,n}(\lambda), \deg P_{2,n}(\lambda)\}$;

(II) $P_{0,n}(0) + P_{1,n}(0) + P_{2,n}(0) = B_n + C_n + D_n \neq 0$;

(III) $P_{0,n}(\lambda)$, $P_{1,n}(\lambda)$, $P_{2,n}(\lambda)$ has no common zeros;

(IV) $\lim_{\lambda\to\infty}\left(\left|\frac{P_{1,n}(\lambda)}{P_{0,n}(\lambda)}\right| + \left|\frac{P_{2,n}(\lambda)}{P_{0,n}(\lambda)}\right|\right) < 1$.

Notice that $\lambda = 0$ is the solution of (3.23), thus we assume that the root of (3.23) is $\lambda = i\omega$ $(\omega > 0)$, and for $\forall\omega > 0$, $P_{j,n}(i\omega) \neq 0$ $(j = 0, 1, 2)$. According to [4], $\lambda = i\omega$ $(\omega > 0)$ is a solution of (3.23) if and only if Ω_n is nonempty. Ω_n is defined as,

$$\Omega_n = \left\{\omega \in \mathbb{R}_+ : |P_{1,n}(i\omega)| + |P_{2,n}(i\omega)| \geq |P_{0,n}(i\omega)|, \big||P_{1,n}(i\omega)| - |P_{2,n}(i\omega)|\big| \leq |P_{0,n}(i\omega)|\right\}. \tag{3.25}$$

If Ω_n is nonempty, then we denote the delays (τ_1, τ_2) satisfying (3.25) as

$$\begin{aligned}
\tau_{1,n,K_1}^{\pm}(\omega) &= \frac{\angle\arg\dfrac{P_{1,n}(i\omega)}{P_{0,n}(i\omega)} + (2K_1 - 1)\pi \pm \theta_{1,n}(\omega)}{\omega}, \quad K_1 = K_{1,n}^{\pm}, K_{1,n}^{\pm} + 1, K_{1,n}^{\pm} + 2, \dots, \\
\tau_{2,n,K_2}^{\pm}(\omega) &= \frac{\angle\arg\dfrac{P_{2,n}(i\omega)}{P_{0,n}(i\omega)} + (2K_2 - 1)\pi \mp \theta_{2,n}(\omega)}{\omega}, \quad K_2 = K_{2,n}^{\pm}, K_{2,n}^{\pm} + 1, K_{2,n}^{\pm} + 2, \dots,
\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}
\theta_{1,n}(\omega) &= \arccos\left(\frac{|P_{0,n}(i\omega)|^2 + |P_{1,n}(i\omega)|^2 - |P_{2,n}(i\omega)|^2}{2|P_{0,n}(i\omega)||P_{1,n}(i\omega)|}\right), \\
\theta_{2,n}(\omega) &= \arccos\left(\frac{|P_{0,n}(i\omega)|^2 - |P_{1,n}(i\omega)|^2 + |P_{2,n}(i\omega)|^2}{2|P_{0,n}(i\omega)||P_{2,n}(i\omega)|}\right).
\end{aligned} \tag{3.27}$$

$K_{1,n}^{\pm}$ and $K_{2,n}^{\pm}$ is the smallest integers to ensure $\tau_{1,n,K_1}^{\pm}$, $\tau_{2,n,K_2}^{\pm}$ are positive. Furthermore, the mode-n stability switching curves (3.23) are

$$\mathcal{T}_n = \bigcup_{K=1}^{N} \left\{ \bigcup_{K_1=-\infty}^{+\infty} \bigcup_{K_2=-\infty}^{+\infty} \left(\mathcal{T}^{+K}_{n,K_1,K_2}, \mathcal{T}^{-K}_{n,K_1,K_2} \right) \bigcap \mathbb{R}^2_+ \right\},$$

where

$$\mathcal{T}^{\pm K}_{n,K_1,K_2} = \left\{ \left(\tau^{\pm}_{1,n,K_1}(\omega), \tau^{\mp}_{2,n,K_2}(\omega) \right) : \omega \in \Omega_n \right\}.$$

By [4, Proposition 4.5], we have the following conclusion about $\mathcal{T}_n$ and Ω_n.

Theorem 3.7. *The mode-n stability switching curves $\mathcal{T}_n$ and the crossing set Ω_n have the following structures with $\forall n \in \mathbb{N}_+$,*

(1) $\mathcal{T}_n$ is a series of spiral-like curves

(1a) for $\Omega_n = [\omega^r_{2,n}, \omega^r_{1,n}]$, if $|B_n| < |C_n - D_n|$;

(1b) for $\Omega_n = [\omega^l_{1,n}, \omega^l_{2,n}] \bigcup [\omega^r_{2,n}, \omega^r_{1,n}]$, if

$$C_n + D_n < |B_n|, \quad A_n^2 < 2\left(B_n - \sqrt{B_n^2 - (C_n - D_n)^2}\right);$$

(2) $\mathcal{T}_n$ contains a series of open ended curves and a series of spiral-like curves for $\Omega_n = (0, \omega^l_{2,n}] \bigcup [\omega^r_{2,n}, \omega^r_{1,n}]$, if

$$|C_n - D_n| < |B_n| < C_n + D_n, \quad A_n^2 < 2\left(B_n - \sqrt{B_n^2 - (C_n - D_n)^2}\right);$$

(3) $\mathcal{T}_n$ is a series of open ended curves for $\Omega_n = \left(0, \omega^r_{1,n}\right]$, if

$$|C_n - D_n| < |B_n| < C_n + D_n, \quad A_n^2 > 2\left(B_n - \sqrt{B_n^2 - (C_n - D_n)^2}\right);$$

(4) $\mathcal{T}_n$ is a series of closed curves for $\Omega_n = [\omega^l_{1,n}, \omega^r_{1,n}]$, if

$$C_n + D_n < |B_n|, \quad 2\left(B_n - \sqrt{B_n^2 - (C_n - D_n)^2}\right) < A_n^2 < 2\left(B_n - \sqrt{B_n^2 - (C_n + D_n)^2}\right),$$

where

$$\omega^l_{1,n} = \sqrt{\frac{-\left(A_n^2 - 2B_n\right) - \sqrt{\Delta_1}}{2}}, \qquad \omega^r_{1,n} = \sqrt{\frac{-\left(A_n^2 - 2B_n\right) + \sqrt{\Delta_1}}{2}},$$

$$\omega^l_{2,n} = \sqrt{\frac{-\left(A_n^2 - 2B_n\right) - \sqrt{\Delta_2}}{2}}, \qquad \omega^r_{2,n} = \sqrt{\frac{-\left(A_n^2 - 2B_n\right) + \sqrt{\Delta_2}}{2}},$$

and

$$\Delta_1 = A_n^4 - 4B_n A_n^2 + 4\left(C_n + D_n\right)^2,$$
$$\Delta_2 = A_n^4 - 4B_n A_n^2 + 4\left(C_n - D_n\right)^2.$$

Proof. By (3.25) and (3.24), $|P_{1,n}(i\omega)| + |P_{2,n}(i\omega)| = |P_{0,n}(i\omega)|$ can be rewritten as

$$\omega^4 + \left(A_n^2 - 2B_n\right)\omega^2 + B_n^2 - \left(C_n + D_n\right)^2 = 0. \tag{3.28}$$

We have the cases:

- when $C_n + D_n < |B_n|$, if $A_n^2 > 2\left(B_n - \sqrt{B_n^2 - (C_n + D_n)^2}\right)$, for $\forall\omega > 0$, then $|P_{1,n}(i\omega)| + |P_{2,n}(i\omega)| < |P_{0,n}(i\omega)|$, thus $\Omega_n = \emptyset$; if $A_n^2 < 2\left(B_n - \sqrt{B_n^2 - (C_n + D_n)^2}\right)$, for $\omega \in \left[\omega_{1,n}^l, \omega_{1,n}^r\right]$, then $|P_{1,n}(i\omega)| + |P_{2,n}(i\omega)| \geq |P_{0,n}(i\omega)|$;
- when $C_n + D_n > |B_n|$, $|P_{1,n}(i\omega)| + |P_{2,n}(i\omega)| \geq |P_{0,n}(i\omega)|$ for $\omega \in \left(0, \omega_{1,n}^r\right]$.

Similarly, $\left||P_{1,n}(i\omega)| - |P_{2,n}(i\omega)|\right| = |P_{0,n}(i\omega)|$ can expressed as

$$\omega^4 + \left(A_n^2 - 2B_n\right)\omega^2 + B_n^2 - \left(C_n - D_n\right)^2 = 0. \tag{3.29}$$

For the same discussion, we also have:

- when $|C_n - D_n| < |B_n|$, $\Omega_n = (0, \infty)$ for $A_n^2 < 2\left(B_n - \sqrt{B_n^2 - (C_n - D)^2}\right)$; $\Omega_n = \left(0, \omega_{2,n}^l\right] \cup \left[\omega_{2,n}^r, +\infty\right)$ for $A_n^2 > 2\left(B_n - \sqrt{B_n^2 - (C_n - D)^2}\right)$;
- when $|C_n - D_n| > |B_n|$, $\Omega_n = \left[\omega_{2,n}^r, +\infty\right)$.

Particularly, $\omega_{1,n}^l < \omega_{2,n}^l$, $\omega_{2,n}^r < \omega_{1,n}^r$ due to $\Delta_1 > \Delta_2$. □

Remark 3.8. In addition, $\Omega_n = \emptyset$ for

$$C_n + D_n < |B_n|, A_n^2 > 2\left(B_n - \sqrt{B_n^2 - (C_n - D_n)^2}\right),$$

the conditions are continue holding for $n \to \infty$.

Let $\lambda = \sigma + i\omega$, and view τ_1, τ_2 as functions $\tau_1(\sigma, \omega)$, $\tau_2(\sigma, \omega)$. Calculating from (3.23), we have

$$\begin{aligned}
\frac{P_{1,n}(i\omega)}{P_{0,n}(i\omega)}e^{-i\omega\tau_1} = & \frac{C_n}{\omega^4 + (A_n^2 - 2B_n)\omega^2 + B_n^2}\left(\left(-\omega^2 + B_n\right)\cos(\omega\tau_1) - A_n\omega\sin(\omega\tau_1)\right) \\
& + \frac{-C_n i}{\omega^4 + (A_n^2 - 2B_n)\omega^2 + B_n^2}\left(\left(-\omega^2 + B_n\right)\sin(\omega\tau_1) + A_n\omega\cos(\omega\tau_1)\right), \\
\frac{P_{2,n}(i\omega)}{P_{0,n}(i\omega)}e^{-i\omega\tau_2} = & \frac{D_n i}{\omega^4 + (A_n^2 - 2B_n)\omega^2 + B_n^2}\left(\left(-\omega^2 + B_n\right)\cos(\omega\tau_2) - A_n\omega\sin(\omega\tau_2)\right) \\
& + \frac{-D_n i}{\omega^4 + (A_n^2 - 2B_n)\omega^2 + B_n^2}\left(\left(-\omega^2 + B_n\right)\sin(\omega\tau_2) + A_n\omega\cos(\omega\tau_2)\right),
\end{aligned}$$

and

$$\begin{aligned}
R_1 &= \mathrm{Re}\left(\frac{P_{1,n}(i\omega)}{P_{0,n}(i\omega)}e^{-i\omega\tau_1}\right), \qquad I_1 = \mathrm{Im}\left(\frac{P_{1,n}(i\omega)}{P_{0,n}(i\omega)}e^{-i\omega\tau_1}\right), \\
R_2 &= \mathrm{Re}\left(\frac{P_{2,n}(i\omega)}{P_{0,n}(i\omega)}e^{-i\omega\tau_2}\right), \qquad I_2 = \mathrm{Im}\left(\frac{P_{2,n}(i\omega)}{P_{0,n}(i\omega)}e^{-i\omega\tau_2}\right).
\end{aligned}$$

Then

$$R_2 I_1 - R_1 I_2 = \frac{-C_n D_n}{\omega^4 + (A_n^2 - 2B_n)\,\omega^2 + B_n^2} \sin\left(\omega\left(\tau_1 - \tau_2\right)\right). \tag{3.30}$$

The sign of $R_2 I_1 - R_1 I_2$ is determined by $\sin\left(\omega\left(\tau_1 - \tau_2\right)\right)$ because $-C_n D_n < 0$, $\omega^4 + \left(A_n^2 - 2B_n\right)\omega^2 + B_n^2 > 0$, for $\forall \omega > 0$.

From [4, Proposition 6.1], we have the following lemma.

Lemma 3.9. *Let $\omega \in \Omega_n$, $(\tau_1, \tau_2) \in \mathcal{T}_n$ such that $i\omega$ is a simple root of (3.23). A pair of conjugate complex roots cross the imaginary axis to the right (left) for $\sin\left(\omega\left(\tau_1 - \tau_2\right)\right) < 0$ (> 0) as (τ_1, τ_2) moves from the region on the right to the left of $\mathcal{T}_n$.*

If the following conditions hold:

(1) when $n = 0$, $|B_0| > D_0$, $A_0^2 < 2\left(B_0 - \sqrt{B_0^2 - D_0^2}\right)$ or $|B_0| < D_0$;

(2) when $n \in \mathbb{N}_+$, $|B_n| > C_n + D_n$, $A_n^2 < 2\left(B_n - \sqrt{B_n^2 - (C_n + D_n)^2}\right)$ or $|B_n| < C_n + D_n$;

then there exists (τ_1^0, τ_2^0) such that (3.23) has the pure imaginary root $i\omega^0$. Moreover, when $\omega^0\left(\tau_1^0 - \tau_2^0\right) \neq k\pi$ ($k \in \mathbb{Z}$), there is a neighborhood U_1 of (τ_1^0, τ_2^0), the following results hold.

Theorem 3.10. *Denote that U_2 is the stable region enclosed by the stability curves $\mathcal{T}_n$ and $\tau_1 - \tau_2$, but not contain $\mathcal{T}_n$, then*

(1) when $(\tau_1, \tau_2) \in U_1 \cap U_2$, the positive equilibrium $(\bar{u}, \bar{v})$ is asymptotically stable;

(2) when $(\tau_1, \tau_2) \in U_1 \backslash \bar{U}_2$, the positive equilibrium $(\bar{u}, \bar{v})$ is unstable;

(3) when $(\tau_1, \tau_2) \in \mathcal{T}_n$, system (1.3) undergoes the spatially nonhomogeneous Hopf bifurcation at $(\bar{u}, \bar{v})$.

4 Numerical simulations

In this section, we give some numerical simulations to support the findings of this paper.

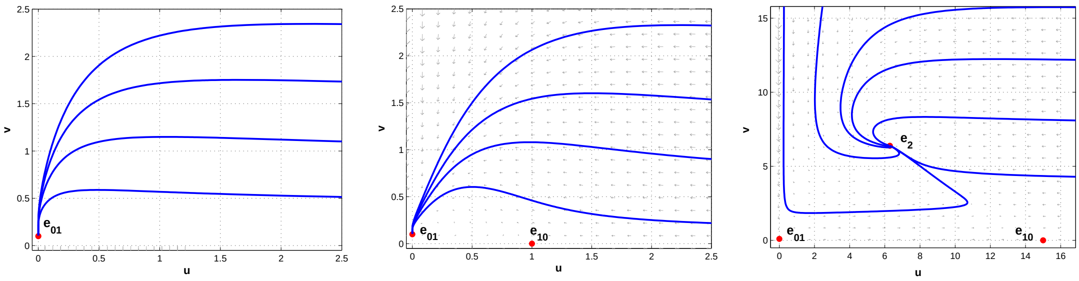
The parameters are chosen as $a = 0.1$, $c = 0.1$, $p = 0.1$, $k = 0.1$, $s = 0.1$, $m = 0.1$, $q = 1$, $d = 0.5$. Model (1.3) has only one non-negative stable solution $e_{01} = (0, 0.1)$ for $r_0 = 0.1 < d$ (see Fig. 4.1(a)); when $d < r_0 = 0.6 < (dk + p)(q + am)/kq$, model (1.3) has non-negative stable solution $e_{01} = (0, 0.1)$ and unstable solution $e_{10} = (1, 0)$ (see Fig. 4.1(b)); for $r_0 = 2 > (dk + p)(q + am)/kq$, model (1.3) has only one positive solution e_2, and non-negative solutions e_{01} and e_{10} are unstable (see Fig. 4.1(c)).

Taking the parameter values

$$r_0 = 80, \quad a = 2, \quad c = 3, \quad d = 2, \quad p = 18, \quad k = 1, \quad m = 1, \quad q = 1,$$

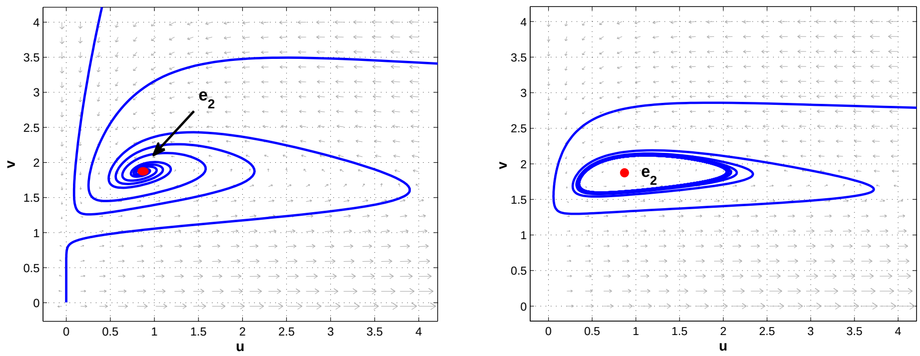
there are $(\bar{u}, \bar{v}) = (0.8678, 1.8678)$, $a_{11} = 1.292$, $a_{12} = -8.27$. According to condition (2.4), for $qa_{11} + a_{12} < 0$, the stability of $(\bar{u}, \bar{v}) = (0.8678, 1.8678)$ is determined by s. The positive equilibrium $(\bar{u}, \bar{v})$ is locally asymptotically stable when $s = 3 > 1.292$ (see Fig. 4.2(a)) and unstable when $s = 1 < 1.292$ (see Fig. 4.2(b)).

Assuming $d_1 = 0.01$, $d_2 = 10$, $\ell = 1$, $s = 1.5$, other parameters are the same as those of Fig. 4.2. Now $(\bar{u}, \bar{v}) = (0.8678, 1.8678)$, $A_{11} = 1.2952$, condition (3.6) is satisfied, $\chi^* = 0.7043$



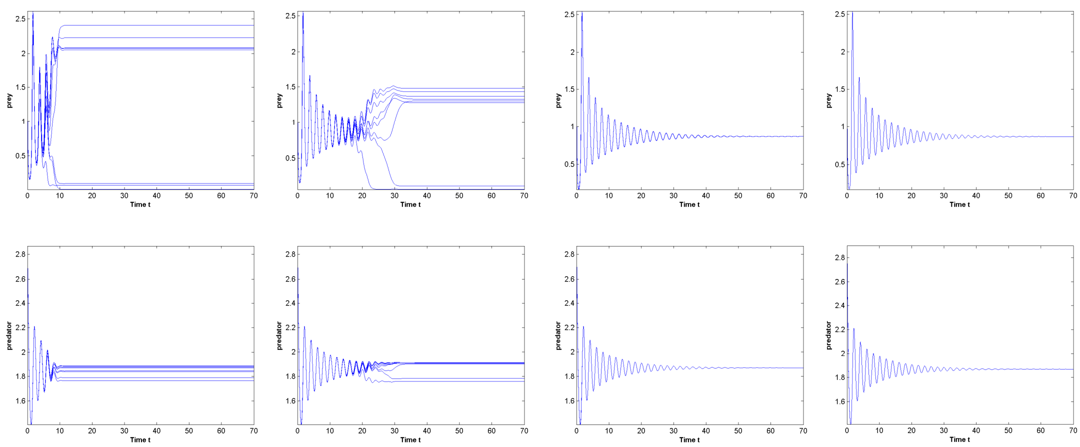
(a) $r_0 = 0.1$ (b) $r_0 = 0.6$ (c) $r_0 = 2$

Figure 4.1: The change of the number and stability of equilibrium points with the parameter r_0.



(a) $s = 3$ (b) $s = 1$

Figure 4.2: The relationship of the stability of $(\bar{u}, \bar{v})$ and s.

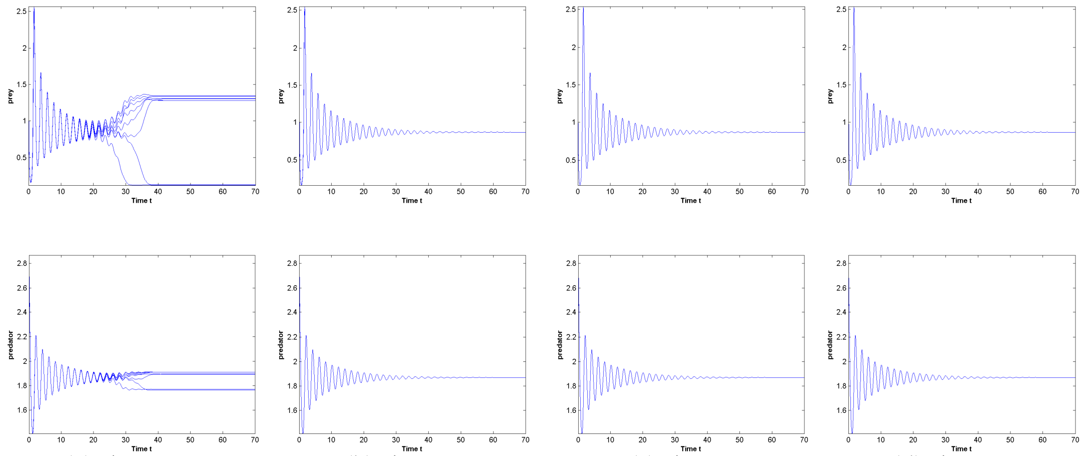


(a) $\chi = 0.01$ (b) $\chi = 0.5$ (c) $\chi = 1$ (d) $\chi = 10$

Figure 4.3: The first and second lines represent the populations of prey and predator, the stability of model (1.3) is controlled by χ, $(\bar{u}, \bar{v})$ is unstable for $\chi < \chi^*$ as (**a**) $\chi = 0.01$, (**b**) $\chi = 0.5$ and locally stable for $\chi > \chi^*$ as (**c**) $\chi = 1$, (**d**) $\chi = 10$.

for $n^* = 10$. Fig. 4.3 verifies Theorem 3.1, for $n \in \mathbb{N}_+$, $(\bar{u}, \bar{v})$ is asymptotically stable for $\chi > \chi^*$; when $\chi < \chi^*$, notice that $(\bar{u}, \bar{v})$ is unstable, Turing instability occurs (see Fig. 4.3(a)–(d)).

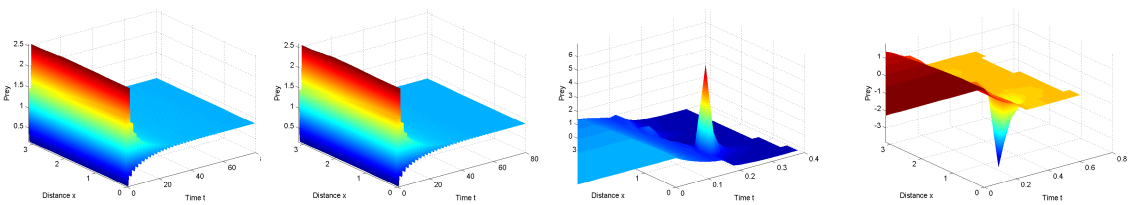
In addition, for the given χ, the stability of model (1.3) also affected by d_1. From Fig. 4.4, the spatiotemporal diagram of the prey is displayed in the figures of the first line and that of predator is showed in the figures of the second line for $\chi = 0.5$. Fig. 4.4 shows that, as the self-diffusion d_1 is big enough, there is no spatially or non-spatially homogeneous steady state bifurcation for the system.



(a) $d_1 = 0.02$ (b) $d_1 = 0.2$ (c) $d_1 = 2$ (d) $d_1 = 20$

Figure 4.4: The spatiotemporal diagram of the system. The value of d_1 is set as (**a**) $d_1 = 0.02$, (**b**) $d_1 = 0.2$, (**c**) $d_1 = 2$ and (**d**) $d_1 = 20$.

Next, we illustrate the influence of delay τ_1 and τ_2. For the delay τ_1, taking the parameter $\chi = 1 > \chi^*$, others are the same as those of Fig.4.3, then there exists $n \in \mathbb{N}_+$ satisfied $|B_n + D_n| < C_n$, and the critical values are $\omega_{1,0}^{3+} = 1.0828$ and $\tau_1^* = 0.7194$ for $n = 3$, so the positive equilibrium $(\bar{u}, \bar{v})$ is asymptotically stable for $\tau_1 < \tau_1^*$ (see Fig. 4.5(a)–(b)), and unstable for $\tau_1 > \tau_1^*$ (see Fig. 4.5(c)–(d)).

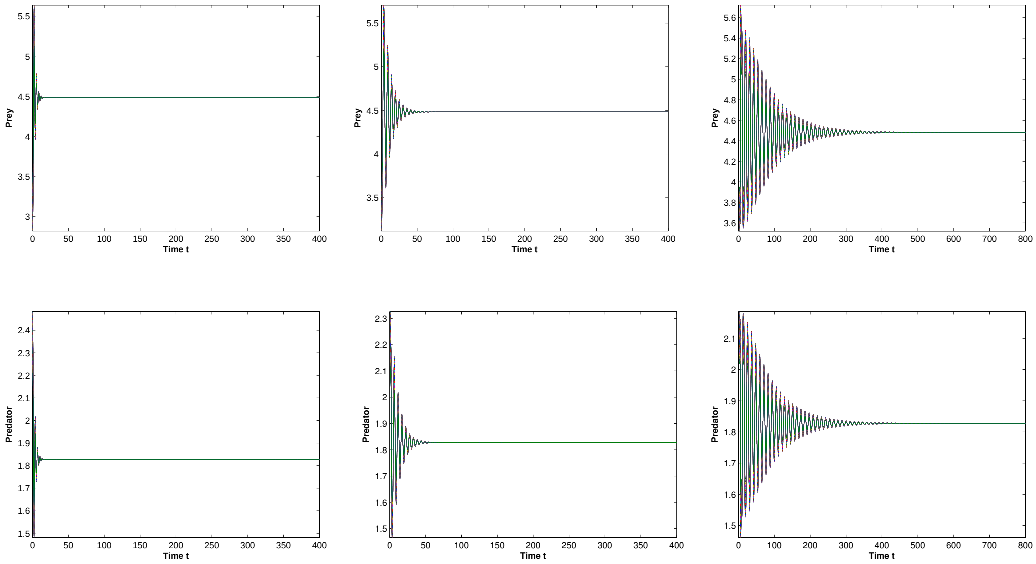


(a) $\tau_1 = 0.1$ (b) $\tau_1 = 0.2$ (c) $\tau_1 = 0.8$ (d) $\tau_1 = 1$

Figure 4.5: When $\tau_1 < \tau_1^*$, the system is always stable and unstable when $\tau_1 > \tau_1^*$, for the fixed value $\chi > \chi^*$.

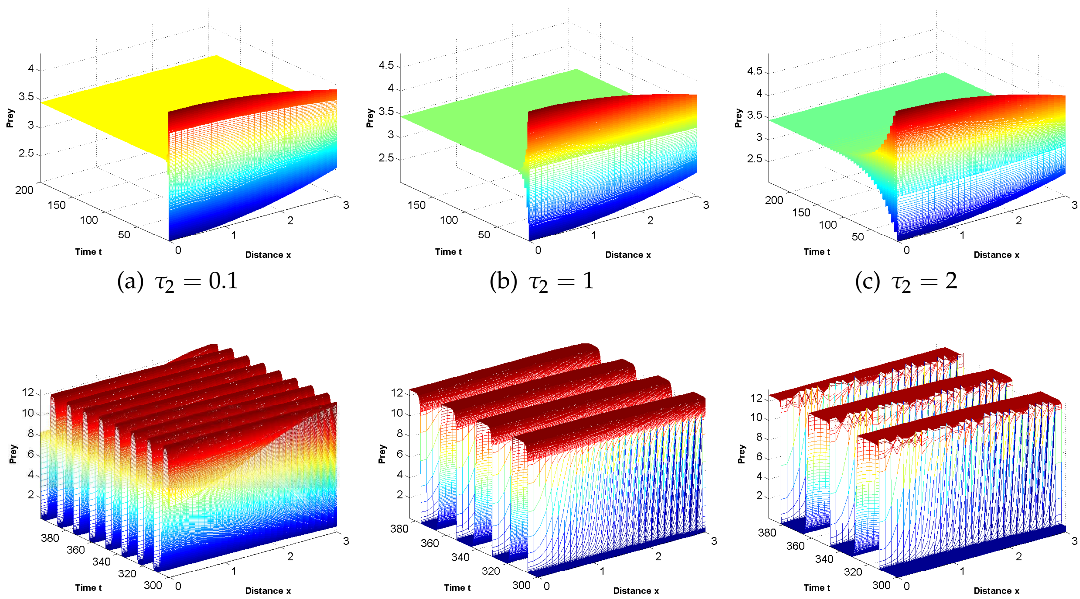
For delay τ_2 on the spatial distribution when $n = 0$, taking the parameter $q = 3$, model (1.3) has the unique positive equilibrium $(\bar{u}, \bar{v}) = (4.4826, 1.8275)$, the system is stable for $qA_{11} < A_{12}$ (see Fig. 4.6(a)–(c)).

While $q = 2$, $(\bar{u}, \bar{v}) = (3.4458, 2.2229)$, $qA_{11} > A_{12}$, we obtain the critical values $\omega_{2,0}^{0+} = 0.6008$ and $\tau_2^* = \tau_{2,0}^{0+} = 2.5366$. Taking $\tau_2 = 0.1, 1, 2 < \tau_2^*$ (see Fig. 4.7(a)–(c)), and $\tau_2 = 5, 10, 15 > \tau_2^*$ (see Fig. 4.7(d)–(f)) to verify the results of Theorem 3.3, the interval of the oscillation period becomes longer with the increasing of τ_2.



(a) $q = 3$, $\tau_2 = 1$ (b) $q = 3$, $\tau_2 = 2$ (c) $q = 3$, $\tau_2 = 5$

Figure 4.6: Spatiotemporal diagram of model (1.3). The system is always stable when $q = 3$ for any τ_2, due to $qA_{11} < A_{12}$.



(a) $\tau_2 = 0.1$ (b) $\tau_2 = 1$ (c) $\tau_2 = 2$

(d) $\tau_2 = 5$ (e) $\tau_2 = 10$ (f) $\tau_2 = 15$

Figure 4.7: Spatiotemporal diagram of prey for model (1.3). The first line shows that the system is always stable for $\tau_2 < \tau_2^*$, and the second line shows that Hopf bifurcation occurs for $\tau_2 > \tau_2^*$.

For the model with two delay ($\tau_1, \tau_2 \neq 0$), when $s = 3$, $\ell = 1$ and other parameters are the same as those of Fig. 2, the positive equilibrium is $(0.8678, 1.8678)$. For $n = 1$, the crossing set is $\Omega_1 = (0, 3.9051]$, satisfying $|C_1 - D_1| < |B_1| < C_1 + D_1$, $A_1^2 > 2\big(B_1 - \sqrt{B_1^2 - (C_1 - D_1)^2}\big)$, then the stability switching curves are a series of open ended curves, so Theorem 3.7(3) is verified (see Fig. 4.8(a)); for $n = 3$, $\Omega_1 = [2.4861, 3.0613]$, satisfying $|B_2| < |C_2 - D_2|$, the stability switching curves are a series of spiral-like curves, and Theorem 3.7(1a) is verified (see Fig. 4.8(b)). Others can be got similarly.

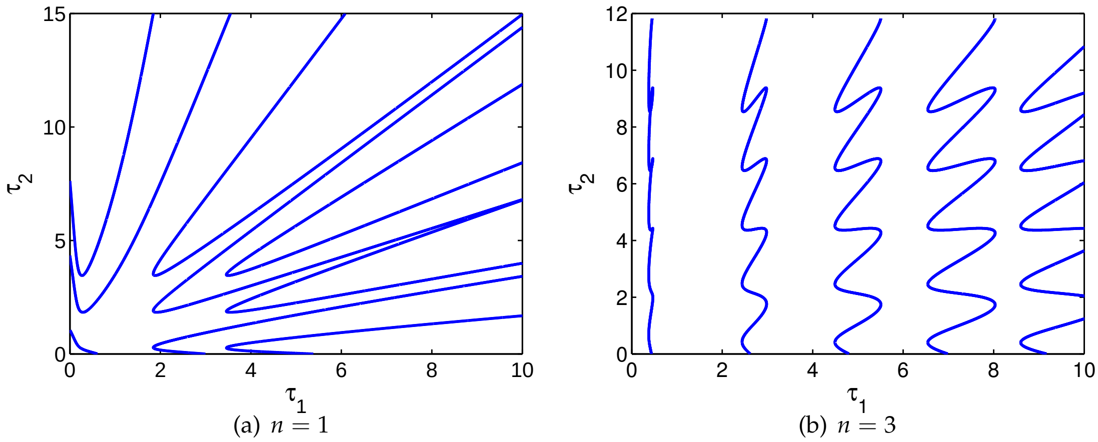


Figure 4.8: The stability switching curves for $n = 1, 2$. (a) open ended curves for $n = 1$; (b) spiral-like curves for $n = 2$.

5 Conclusion

In the paper, we propose a diffusive predator-prey system with two delays. We introduce the modified Leslie–Gower term and fear effect to the system, and consider the stability of the model with the memory delay τ_1 and maturation delay τ_2, obtaining the following results.

(1) System (1.3) always has the semi-trivial equilibrium e_{01}, which is stable for $r_0 < (dk + p)(q + am)/kq$; when $r_0 > (dk + p)(q + am)/kq$, the equilibrium e_{01} loses stable; when $r_0 > d$, there exists semi-trivial equilibrium e_{10}, which is always unstable. Meanwhile, the model has the unique positive equilibrium e_2, which is locally asymptotically stable for $a_{11} - s < 0$, $qa_{11} + a_{12} < 0$. The number and stability of model (1.3) are determined by r_0.

(2) For $n = 0$, as conditions (2.2) and (2.4) hold, there is no spatially homogeneous steady state bifurcation. When $n \neq 0$ and condition (3.6) is satisfied, for $\chi > \chi^*$, $(\bar{u}, \bar{v})$ is asymptotically stable; for $\chi < \chi^*$, the spatially homogeneous steady state bifurcation occurs at $(\bar{u}, \bar{v})$. Therefore, Turing instability appears. From the condition $A_{11}d_2 - sd_1 < 0$, one can conclude that slow prey-taxis and fast self-diffusion would cause Turing patterns to occur.

(3) System (1.3) exists the spatially nonhomogeneous Hopf bifurcation at $(\bar{u}, \bar{v})$ for fast memory delay when it only has delay τ_1; model (1.3) undergoes the spatially homogeneous and nonhomogeneous Hopf bifurcation at $(\bar{u}, \bar{v})$ for fast maturation delay when it only has delay τ_2. Specially, there is no spatially homogeneous Hopf bifurcation for any delay τ_2 ($\tau_1 = 0$) when q is big enough. For the model with two delay ($\tau_1, \tau_2 \neq 0$), the structures of mode-n stability switching curves $\mathcal{T}_n$ and the crossing set Ω_n are shown as in Theorem 3.7, and the dynamical behavior are much richer than one delay.

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Interval of the existence of positive solutions for a boundary value problem for system of three second-order differential equations

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Abstract. The aim of this paper is to estimate an interval of the existence of positive solutions for a boundary value problem for the system of three nonlinear second-order ordinary differential equations. Krasnosel'skiĭ–Precup fixed point theorem is used to determine this interval theoretically. For Dirichlet boundary conditions theoretical result is compared with result obtained numerically.

Keywords: boundary value problem, system of second-order ODEs, interval of the existence of solutions, positive solutions, Krasnosel'skiĭ–Precup fixed point theorem.

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1 Introduction

Systems of three second-order ordinary differential equations emerge naturally from the application of Newton's laws in modeling three body interaction: each equation represents the acceleration of a body in response to the forces exerted by the other two bodies. Such systems have a vital role in modeling problems of mechanics and oscillations.

In this paper, we investigate the interval of the existence of (strictly) positive solutions, i.e. we determine real positive τ for which at least one positive solution exists, for the following system of nonlinear second-order differential equations

$$x_i''(t) + f_i(t, x_1(t), x_2(t), x_3(t)) = 0, \quad t \in (0, \tau),\ i = 1, 2, 3, \tag{1.1}$$

coupled with nonlocal boundary conditions

$$x_i(0) = \varphi_i[x_i] + a_i, \quad x_i(\tau) = \psi_i[x_i] + b_i, \tag{1.2}$$

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where $a_i, b_i \geq 0$, $f_i : [0,\tau] \times [0,+\infty)^3 \to [0,+\infty)$ are continuous, $\varphi_i[x] = \int_0^\tau x(t)\mathrm{d}\Phi_i(t)$ and $\psi_i[x] = \int_0^\tau x(t)\mathrm{d}\Psi_i(t)$ are linear functionals defined via Riemann–Stieltjes integrals, where $\Phi_i, \Psi_i : [0,\tau] \to \mathbb{R}$ are functions of bounded variation.

We write $\varphi_i[\mathrm{Id}]$ and $\varphi_i[\tau]$ to denote φ_i applied to the identity function and constant function with value τ, respectively. The notation $|A|$ denotes the determinant of a square matrix A. Throughout the paper, we assume that

(A1) $0 \leq \varphi_i[\mathrm{Id}]$, $0 \leq \varphi_i[\tau - \mathrm{Id}] \leq \tau$ and $0 \leq \psi_i[\tau - \mathrm{Id}]$, $0 \leq \psi_i[\mathrm{Id}] \leq \tau$,

(A2) $0 < D_i = \begin{vmatrix} \tau - \varphi_i[\tau - \mathrm{Id}] & -\varphi_i[\mathrm{Id}] \\ -\psi_i[\tau - \mathrm{Id}] & \tau - \psi_i[\mathrm{Id}] \end{vmatrix}$,

are valid for every $i = 1,2,3$.

By a positive solution of problem (1.1), (1.2) we understand $(x_1, x_2, x_3) \in \left(C^2[0,\tau]\right)^3$, which satisfies system of differential equations (1.1), boundary conditions (1.2) and positive *coexistence* condition, i.e. $x_i(t) > 0$ for all $t \in (0,\tau)$ and every $i = 1,2,3$.

We do not assume $\varphi_i[x_i] \geq 0$ and $\psi_i[x_i] \geq 0$ for all $x_i \geq 0$, but we allow $\mathrm{d}\Phi_i$ and $\mathrm{d}\Psi_i$ to be signed measures. For details on signed measure and Riemann–Stieltjes integrals we refer reader, for instance, to [17–19]. But we require $\varphi_i[x_i] \geq 0$ and $\psi_i[x_i] \geq 0$ for corresponding component of the positive solution (x_1, x_2, x_3).

The term "coexistence" was introduced by Lan [11] in context of fixed points in product Banach spaces. Coexistence fixed point denotes a fixed point with all the components different from zero. The common approach to obtain solutions of operator equation is to seek the fixed points. The best-known fixed point theorems for positive solutions are Krasnosel'skiĭ's fixed point theorem in cones [10] and its generalizations, for instance, Krasnosel'skiĭ–Benjamin fixed point theorem [1], where conditions are weakened, and Guo–Krasnosel'skiĭ fixed point theorem [3], where considered region is more general. But, as it was mentioned in [12,13,15], these theorems cannot guarantee the coexistence fixed point. Motivated by this, Precup [12, 13] established (2-dimensional) vector version of Krasnosel'skiĭ's fixed point theorem, which allows to localize fixed point in the component-wise manner. Recently, Rodríguez-López [15] showed an alternative proof via fixed point index theory. As it was pointed out in [15], the result by Precup remains valid for n-dimensions. For multiplicity result of positive solutions by vector version of Krasnosel'skiĭ's fixed point theorem we refer reader to [8,14].

Generalized version of problem (1.1), (1.2) with $\tau = 1$ and $i = 1,2$, was studied by Henderson and Luca [5,6]. In [5] was considered problem (in our notations)

$$(a_i(t)x_i(t)')' - b_i(t)x_i(t) + \lambda_i p_i(t) f_i(t, x_1(t), x_2(t)) = 0, \quad t \in (0,1),\ i = 1,2, \tag{1.3}$$

$$\alpha_i x_i(0) - \beta_i a(0)x_i'(0) = \varphi_i[x_i], \quad \gamma_i x_i(1) + \delta_i a(1)x_i'(1) = \psi_i[x_i], \tag{1.4}$$

and sufficient conditions on λ_i and f_i were given such that non-negative solutions of problem (1.3), (1.4) exist. The result was based on Guo–Krasnosel'ski fixed point theorem. In [6] by applying fixed point index theory results on existence and multiplicity of positive solutions were obtained for the slightly modified problem (1.3), (1.4): the functions f_i depended on only one unknown $x_{j\neq i}$, i.e. $f_i(t, x_{j\neq i}(t))$.

In this paper, we apply two methods that allow us to obtain an interval of the existence of positive solutions for the problem (1.1), (1.2). First we find τ by solving system of inequalities, which is based on Green's functions of problem (1.1), (1.2) and behavior of functions f_i. To prove that for these τ there exist positive solutions we apply vector version of Krasnosel'skiĭ's fixed point theorem, or Krasnosel'skiĭ–Precup fixed point theorem. Let us recall this result here. A nonempty closed convex subset $K \subset X$ of normed space $(X, \|\cdot\|)$ is called a cone if $\lambda x \in K$ for every $x \in K$ and for all $\lambda \geq 0$, and $K \cap (-K) = 0$.

Theorem 1.1 (Krasnosel'skiĭ–Precup, [12,15]). *Let $(X, \|\cdot\|)$ be a normed space, $K_1, \ldots, K_n$ cones in X, $K = K_1 \times \cdots \times K_n$, $r = (r_1, \ldots, r_n)$, $R = (R_1, \ldots, R_n)$, with $0 < r_i < R_i$ for $i \in \{1, \ldots, n\}$, and*

$$\overline{K}_{r,R} = \{x = (x_1, \ldots, x_n) \in K : \forall\, i \in \{1, \ldots, n\} \quad r_i \leq \|x_i\| \leq R_i\}.$$

Assume that $T = (T_1, .., T_n) : \overline{K}_{r,R} \to K$ is a completely continuous map and for each $i \in \{1, .., n\}$ there exists $\mathrm{h}_i \in K_i \setminus \{0\}$ such that one of the following conditions is satisfied in $\overline{K}_{r,R}$:

(i) $T_i x + \mu \mathrm{h}_i \neq x_i$ *if* $\|x_i\| = r_i$ *and* $\mu > 0$, *and* $T_i x \neq \lambda x_i$ *if* $\|x_i\| = R_i$ *and* $\lambda > 1$;

(ii) $T_i x \neq \lambda x_i$ *if* $\|x_i\| = r_i$ *and* $\lambda > 1$, *and* $T_i x + \mu \mathrm{h}_i \neq x_i$ *if* $\|x_i\| = R_i$ *and* $\mu > 0$.

Then T has at least one fixed point $x \in K$ with $r_i \leq \|x_i\| \leq R_i$, $i \in \{1, \ldots, n\}$.

Conditions (i) and (ii) are called compression type and expansion type condition, respectively.

To satisfy compression and expansion type conditions various authors considered asymptotic behavior of f/x at zero and infinity. This approach is widely used in case of one differential equation or systems in which all f_i depend on only one unknown $x_{j \neq i}$ (see, for instance, [2,6,7,9,18,19]). The idea is to use limits

$$\lim_{x \to 0} \sup_t \frac{f(t,x)}{x}, \qquad \lim_{x \to \infty} \sup_t \frac{f(t,x)}{x}$$
$$\lim_{x \to 0} \inf_t \frac{f(t,x)}{x}, \qquad \lim_{x \to \infty} \inf_t \frac{f(t,x)}{x}.$$

In [9] the case where the above limits were zero or infinity was studied. In [2,7] the limits were allowed to be small or large enough, in a sense that necessary inequalities hold. In the case of systems of differential equations in which f_i depend on all unknowns many authors require additional assumptions on f_i to construct similar limits. For instance, f_i is monotone with respect to x_j, see [12,13], or bounded with respect to x_j, see [4].

If we let $\varphi_i \equiv 0$ and $\psi_i \equiv 0$, then boundary conditions (1.2) become Dirichlet boundary conditions. For such problem we compare the theoretical result with the result based on built-in functions of program *Mathematica* [20]. The numerical result is obtained by shooting method: we consider the initial value problem for system of differential equations and determine τ.

The outline of the rest of the paper is as follows. In Section 2, we rewrite boundary value problem (1.1), (1.2) as an equivalent system of integral equations by constructing the Greens functions and show the estimations of Greens functions. We prove the existence of positive solutions by applying Krasnosel'skiĭ–Precup fixed point theorem in Section 3 and formulate main result of this article in Theorem 3.7. Finally, in Section 4, we compare theoretical and numerical results for problem (1.1) with the boundary conditions $x_i(0) = a_i$, $x_i(\tau) = b_i$.

2 Construction and estimation of Green's functions

Standard approach is to rewrite problem (1.1), (1.2) as an equivalent system of integral equations via corresponding Green's functions. Results of this section are well-known and for details we refer reader to [16,18,19].

The Green's function G_0 corresponding to problem $x''(t)+h(t)=0$, $x(1)=0=x(\tau)$, is given by

$$G_0(t,s)=\frac{1}{\tau}\begin{cases} s(\tau-t), & 0\le s\le t\le\tau,\\ t(\tau-s), & 0\le t\le s\le\tau.\end{cases} \tag{2.1}$$

We denote $\mathcal{G}_{\varphi_i}(s)=\int_0^\tau G_0(t,s)\mathrm{d}\Phi_i(t)$, $\mathcal{G}_{\psi_i}(s)=\int_0^\tau G_0(t,s)\mathrm{d}\Psi_i(t)$ and in addition to (A1) and (A2) we assume

(A3) $\mathcal{G}_{\varphi_i}(s)\ge 0$ and $\mathcal{G}_{\psi_i}(s)\ge 0$ for all $s\in[0,\tau]$ and every $i=1,2,3$.

Recall that D_i is given by (A2) and $|A|$ denotes the determinant of a square matrix A.

Proposition 2.1. *A triple (x_1,x_2,x_3) is a solution of boundary value problem (1.1), (1.2) if and only if (x_1,x_2,x_3) is a solution of the system of integral equations*

$$x_i(t)=\int_0^\tau G_i(t,s)f_i(s,x_1(s),x_2(s),x_3(s))\mathrm{d}s+g_i(t),\quad t\in[0,\tau],\ i=1,2,3, \tag{2.2}$$

where

$$G_i(t,s)=\frac{1}{D_i}\begin{vmatrix}\tau-t & \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}]\\ t & -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}]\\ G_0(t,s) & -\mathcal{G}_{\varphi_i}(s) & -\mathcal{G}_{\psi_i}(s)\end{vmatrix} \tag{2.3}$$

and

$$g_i(t)=\frac{1}{D_i}\begin{vmatrix}\tau-t & \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}]\\ t & -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}]\\ 0 & -a_i & -b_i\end{vmatrix}. \tag{2.4}$$

Proof. Let (x_1,x_2,x_3) be a solution of boundary value problem (1.1), (1.2). For every $i=1,2,3$, integrating (1.1) twice from 0 to t and applying boundary conditions (1.2), we get

$$x_i(t)=\int_0^\tau G_0(t,s)f_i(s,x_1(s),x_2(s),x_3(s))\mathrm{d}s+\frac{t}{\tau}(b_i+\psi_i[x_i])+\frac{\tau-t}{\tau}(a_i+\varphi_i[x_i]). \tag{2.5}$$

Let us denote $(Fx_i)(t)=\int_0^\tau G_0(t,s)f_i(s,x_1(s),x_2(s),x_3(s))\mathrm{d}s$. Applying φ_i and ψ_i to (2.5), we get

$$\begin{aligned}\varphi_i[x_i](\tau-\varphi_i[\tau-\mathrm{Id}])-\varphi_i[\mathrm{Id}]\psi_i[x_i]&=\tau\,\varphi_i[Fx_i]+b_i\varphi_i[\mathrm{Id}]+a_i\varphi_i[\tau-\mathrm{Id}],\\ \psi_i[x_i](\tau-\psi_i[\mathrm{Id}])-\psi_i[\tau-\mathrm{Id}]\varphi_i[x_i]&=\tau\,\psi_i[Fx_i]+b_i\psi_i[\mathrm{Id}]+a_i\psi_i[\tau-\mathrm{Id}].\end{aligned} \tag{2.6}$$

We rewrite (2.6) in matrix form

$$\begin{pmatrix}\tau-\varphi_i[\tau-\mathrm{Id}] & -\varphi_i[\mathrm{Id}]\\ -\psi_i[\tau-\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}]\end{pmatrix}\begin{pmatrix}\varphi_i[x_i]\\ \psi_i[x_i]\end{pmatrix}=\begin{pmatrix}\tau\,\varphi_i[Fx_i]\\ \tau\,\psi_i[Fx_i]\end{pmatrix}+\begin{pmatrix}\varphi_i[\tau-\mathrm{Id}] & \varphi_i[\mathrm{Id}]\\ \psi_i[\tau-\mathrm{Id}] & \psi_i[\mathrm{Id}]\end{pmatrix}\begin{pmatrix}a_i\\ b_i\end{pmatrix}.$$

By assumption (A2), $D_i>0$ and it follows

$$\begin{aligned}\begin{pmatrix}\varphi_i[x_i]\\ \psi_i[x_i]\end{pmatrix}&=\frac{1}{D_i}\begin{pmatrix}\tau-\psi_i[\mathrm{Id}] & \varphi_i[\mathrm{Id}]\\ \psi_i[\tau-\mathrm{Id}] & \tau-\varphi_i[\tau-\mathrm{Id}]\end{pmatrix}\begin{pmatrix}\tau\,\varphi_i[Fx_i]\\ \tau\,\psi_i[Fx_i]\end{pmatrix}\\ &+\frac{1}{D_i}\begin{pmatrix}\tau-\psi_i[\mathrm{Id}] & \varphi_i[\mathrm{Id}]\\ \psi_i[\tau-\mathrm{Id}] & \tau-\varphi_i[\tau-\mathrm{Id}]\end{pmatrix}\begin{pmatrix}\varphi_i[\tau-\mathrm{Id}] & \varphi_i[\mathrm{Id}]\\ \psi_i[\tau-\mathrm{Id}] & \psi_i[\mathrm{Id}]\end{pmatrix}\begin{pmatrix}a_i\\ b_i\end{pmatrix}.\end{aligned} \tag{2.7}$$

Substituting $\varphi_i[x_i]$ and $\psi_i[x_i]$ from (2.7) in (2.5), we get

$$\begin{aligned} x_i(t) &= (Fx_i)(t) - \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ -\varphi_i[Fx_i] & -\psi_i[Fx_i] \end{vmatrix} + \frac{\tau-t}{D_i}\begin{vmatrix} -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -\varphi_i[Fx_i] & -\psi_i[Fx_i] \end{vmatrix} \\ &\quad - \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ -a_i & -b_i \end{vmatrix} + \frac{\tau-t}{D_i}\begin{vmatrix} -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -a_i & -b_i \end{vmatrix} \\ &= \frac{1}{D_i}\begin{vmatrix} \tau-t & \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ t & -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ (Fx_i)(t) & -\varphi_i[Fx_i] & -\psi_i[Fx_i] \end{vmatrix} + \frac{1}{D_i}\begin{vmatrix} \tau-t & \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ t & -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ 0 & -a_i & -b_i \end{vmatrix} \\ &= \int_0^\tau G_i(t,s) f_i(s, x_1(s), x_2(s), x_3(s))\mathrm{d}s + g_i(t), \end{aligned}$$

where G_i is given by (2.3) and g_i is given by (2.4).

Now, let (x_1, x_2, x_3) satisfy system of integral equations (2.2). It follows that each x_i also satisfies (2.5). By differentiating (2.5) twice, it is easy to see that (x_1, x_2, x_3) satisfies (1.1), (1.2) and $(x_1, x_2, x_3) \in \left(C^2[0,\tau]\right)^3$. □

Remark 2.2. Note that $G_i \geq 0$ and $g_i \geq 0$ for every $i = 1, 2, 3$. Indeed, expansion of (2.3) and (2.4) along the first column is

$$\begin{aligned} G_i(t,s) &= \frac{\tau-t}{D_i}\begin{vmatrix} -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -\mathcal{G}_{\varphi_i}(s) & -\mathcal{G}_{\varphi_i}(s) \end{vmatrix} - \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ -\mathcal{G}_{\varphi_i}(s) & -\mathcal{G}_{\varphi_i}(s) \end{vmatrix} + G_0(t,s) \\ &= \frac{\tau-t}{D_i}\begin{vmatrix} \varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -\mathcal{G}_{\varphi_i}(s) & \mathcal{G}_{\varphi_i}(s) \end{vmatrix} + \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & \psi_i[\tau-\mathrm{Id}] \\ -\mathcal{G}_{\varphi_i}(s) & \mathcal{G}_{\varphi_i}(s) \end{vmatrix} + G_0(t,s) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} g_i(t) &= \frac{\tau-t}{D_i}\begin{vmatrix} -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -a_i & -b_i \end{vmatrix} - \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ -a_i & -b_i \end{vmatrix} \\ &= \frac{\tau-t}{D_i}\begin{vmatrix} \varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ -a_i & b_i \end{vmatrix} + \frac{t}{D_i}\begin{vmatrix} \tau-\varphi_i[\tau-\mathrm{Id}] & \psi_i[\tau-\mathrm{Id}] \\ -a_i & b_i \end{vmatrix}. \end{aligned} \tag{2.9}$$

By assumptions (A1)–(A3), and $a_i, b_i \geq 0$, and fact that G_0, given by (2.1), is non-negative, determinants in last parts of (2.8) and (2.9) are non-negative for all $(t,s) \in [0,\tau] \times [0,\tau]$ and $t \in [0,\tau]$, respectively.

Let $m(t) = \min\{t/\tau,\ 1 - t/\tau\}$. It is known that Green's function G_0 satisfies

$$m(t)G_0(s,s) \leq G_0(t,s) \leq G_0(s,s), \quad (t,s) \in [0,\tau] \times [0,\tau].$$

Proposition 2.3. *Green's function G_i, given by (2.3), satisfies*

$$m(t)H_i(s) \leq G_i(t,s) \leq H_i(s), \quad (t,s) \in [0,\tau] \times [0,\tau],$$

where

$$H_i(s) = \frac{1}{D_i}\begin{vmatrix} \tau & \tau-\varphi_i[\tau-\mathrm{Id}] & -\psi_i[\tau-\mathrm{Id}] \\ \tau & -\varphi_i[\mathrm{Id}] & \tau-\psi_i[\mathrm{Id}] \\ G_0(s,s) & -\mathcal{G}_{\varphi_i}(s) & -\mathcal{G}_{\psi_i}(s) \end{vmatrix}.$$

Proof. Expansion of $G_i(t,s)$ along the first column is given by (2.8). We replace $\tau - t$ with τ in first determinant, t with τ in second determinant, $G_0(t,s)$ with $G_0(s,s)$ in third determinant and get $H_i(s)$. Therefore, $H_i \geq 0$ by the same argument as $G_i \geq 0$.

We get inequality $G_i(t,s) \leq H_i(s)$ by estimating $\tau - t \leq \tau$, $t \leq \tau$ and $G_0(t,s) \leq G_0(s,s)$.

It is clear that $1 - t/\tau \geq m(t)$ and $t/\tau \geq m(t)$ for all $t \in [0,\tau]$. We get inequality $G_i(t,s) \geq m(t)H_i(s)$ by estimating $\tau - t \geq m(t)\tau$, $t \geq m(t)\tau$ and $G_0(t,s) \geq m(t)G_0(s,s)$. □

Observe that if $a_i = b_i = 0$, then $g_i \equiv 0$. By (2.9), it is easy to see that $g_i(t)$ is a polynomial with degree at most one. Hence g_i is concave. Concavity of g_i implies

$$g_i(t) \geq m(t)\, g_i(t_0), \quad (t,t_0) \in [0,\tau] \times [0,\tau]. \tag{2.10}$$

For every $c \in (0,\tau/2)$ inequality $\tau c \leq m(t)$ holds for $t \in [c,\tau - c]$. As it was mentioned in [18, 19], for Green's function G_0 optimal constant is $c = \tau/4$. Optimal in a sense that $\inf\{\int_c^{\tau-c} G_0(t,s)\mathrm{d}s : t \in [c,\tau-c]\}$ is maximal.

3 Theoretical result on the existence of a positive solution

Consider Banach space $C[0,\tau]$ endowed with the norm $\|x\| = \max\{|x(t)| : t \in [0,\tau]\}$. We define cone k_i by

$$k_i = \left\{ u \in C[0,\tau] : u(t) \geq 0 \text{ for } t \in [0,\tau],\ \min_{t\in[\tau/4,3\tau/4]} u(t) \geq \frac{1}{4}\|u\|,\ \varphi_i[u] \geq 0,\ \psi_i[u] \geq 0 \right\}.$$

Let $K = k_1 \times k_2 \times k_3$, $x = (x_1,x_2,x_3)$ and $T = (T_1,T_2,T_3) : K \to (C[0,\tau])^3$ be an operator defined by

$$(T_i x)(t) = \int_0^\tau G_i(t,s) f_i(s,x_1(s),x_2(s),x_3(s))\mathrm{d}s + g_i(t), \tag{3.1}$$

where G_i is given by (2.3) and g_i is given by (2.4).

Observe that T is a completely continuous operator. Indeed, g_i is obviously completely continuous and $T_i x - g_i$ is completely continuous by application of Arzelà–Ascoli theorem. Boundary value problem (1.1), (1.2) has a non-negative solution if and only if operator T has a fixed point in K. To prove that maximal value of each x_i is positive, and hence the solution is positive, we apply Krasnosel'skiĭ–Precup fixed point theorem (Theorem 1.1). Now, we show that T maps K into itself.

Proposition 3.1. *Operator T, given by (3.1), satisfies $T(K) \subset K$.*

Proof. It is obvious that $T_i x \geq 0$ for each $i = 1,2,3$.

Let $T_i x$ achieve maximum value at point t_0, i.e. $(T_i x)(t_0) = \|T_i x\|$. By Proposition 2.3 and (2.10), for every $t \in [\tau/4, 3\tau/4]$ we have

$$\begin{aligned}
(T_i x)(t) &= \int_0^\tau G_i(t,s) f_i(s,x_1(s),x_2(s),x_3(s))\mathrm{d}s + g_i(t) \\
&\geq m(t) \int_0^\tau H_i(s) f_i(s,x_1(s),x_2(s),x_3(s))\mathrm{d}s + m(t) g_i(t_0) \\
&\geq \frac{1}{4}\Big(\int_0^\tau G_i(t_0,s) f_i(s,x_1(s),x_2(s),x_3(s))\mathrm{d}s + g_i(t_0) \Big) \\
&= \frac{1}{4}\|T_i x\|.
\end{aligned}$$

Next, consider

$$\varphi_i[Tx_i] = \int_0^\tau \left(\int_0^\tau G_i(t,s) \mathrm{d}\Phi_i(t) \right) f_i(s, x_1(s), x_2(s), x_3(s)) \mathrm{d}s + \varphi_i[g_i].$$

By (A1)–(A3), we get $\int_0^\tau G_i(t,s)\mathrm{d}\Phi_i(t) \geq 0$ and $\varphi_i[g_i] \geq 0$. Hence $\varphi_i[Tx_i] \geq 0$. Similarly $\psi_i[Tx_i] \geq 0$. Therefore, $T(K) \subset K$. □

Now, we briefly describe the main result. First, we show that if certain conditions on f_i hold, then T_i satisfies compression type condition (i) or expansion type condition (ii) of Krasnosel'skiĭ–Precup fixed point theorem (Theorem 1.1). Then we choose r and R such that each T_i satisfies either condition (i) or (ii) for all $x \in \overline{K}_{r,R}$. Finally, we conclude that at least one positive solution of problem (1.1), (1.2) exists.

Let us introduce notations

$$A_i = \inf_{t \in [\tau/4, 3\tau/4]} \int_{\tau/4}^{3\tau/4} G_i(t,s) \mathrm{d}s, \quad B_i = \sup_{t \in [0,\tau]} \int_0^\tau G_i(t,s) \mathrm{d}s.$$

To prove the following Lemma 3.2 (and Proposition 3.6) we use standard techniques. See, for instance, [2,7,9,18,19].

Lemma 3.2. *Operator T_i satisfies compression type condition (i) if there exist constants $0 < q < Q$ such that*

$$q < \min_{\substack{t \in [\tau/4, 3\tau/4] \\ x_i \in [q/4, q] \\ x_{j \neq i} \in [q/4, Q]^2}} f_i(t,x) \cdot A_i \quad \text{and} \quad \max_{\substack{t \in [0,\tau] \\ x \in [0,Q]^3}} f_i(t,x) \cdot B_i + \|g_i\| < Q, \tag{3.2}$$

and T_i satisfies expansion type condition (ii) if there exist constants $0 < q < Q$ such that

$$Q < \min_{\substack{t \in [\tau/4, 3\tau/4] \\ x_i \in [Q/4, Q] \\ x_{j \neq i} \in [q/4, Q]^2}} f_i(t,x) \cdot A_i \quad \text{and} \quad \max_{\substack{t \in [0,\tau] \\ x_i \in [0,q] \\ x_{j \neq i} \in [0,Q]^2}} f_i(t,x) \cdot B_i + \|g_i\| < q. \tag{3.3}$$

Proof. Let $\overline{K}_{q,Q} = \{x \in K : q \leq \|x_i\| \leq Q,\ i = 1,2,3\}$. We show a proof for compression type condition. Proof for expansion type condition is similar.

Let $\|x_i\| = Q$ and $\Omega = [0,\tau] \times [0,Q]^3$. We show that $\|T_i x\| \leq \|x_i\|$. It is known that this implies $T_i x \neq \lambda x_i$ for $\lambda > 1$. Consider

$$\begin{aligned}
\|T_i x\| &\leq \max_{t \in [0,\tau]} \int_0^\tau G_i(t,s) f_i(s, x_1(s), x_2(s), x_3(s)) \mathrm{d}s + \|g_i\| \\
&\leq \max_{(t,x) \in \Omega} f_i(t,x) \cdot B_i + \|g_i\| < Q = \|x_i\|.
\end{aligned}$$

Now, suppose to contrary that there exists x_i with $\|x_i\| = q$ such that $T_i x + \mu \mathrm{h} = x_i$ for $\mu > 0$ and $\mathrm{h} : t \mapsto 1$. Since $x \in \overline{K}_{q,Q}$, we have

$$x_j(t) \geq \frac{1}{4} \|x_j\| \geq \frac{1}{4} q, \quad t \in [\tau/4, 3\tau/4],\ j = 1,2,3.$$

Let $\omega = [\tau/4, 3\tau/4] \times [q/4, q] \times [q/4, Q]^2$. We get

$$\begin{aligned} x_i(t) &= \int_0^\tau G_i(t,s) f_i(s, x_1(s), x_2(s), x_3(s)) \mathrm{d}s + g_i(t) + \mu \\ &\geq \int_{\tau/4}^{3\tau/4} G_i(t,s) f_i(s, x_1(s), x_2(s), x_3(s)) \mathrm{d}s + g_i(t) + \mu \\ &\geq \min_{(t, x_i, x_{j\neq i}) \in \omega} f_i(t,x) \cdot A_i + g_i(t) + \mu \\ &> q + g_i(t) + \mu, \end{aligned}$$

which gives contradiction. □

Let us show examples of f_i, $i = 1,2,3$, that satisfy (3.2) and (3.3) for sufficiently small q and sufficiently large Q, i.e. there exist $q_i < Q_i$ such that f_i satisfies (3.2) or (3.3) for $0 < q \leq q_i$ and $Q_i \leq Q < +\infty$. The ability to choose such q and Q is used to define proper $\overline{K}_{r,R}$ in the proof of the main result.

Let us define

$$u_{ij}^w = \begin{cases} u, & i = j, \\ w, & i \neq j. \end{cases}$$

We use notation u_{ij}^w to denote that i-th element of a triple $(u_{i1}^w, u_{i2}^w, u_{i3}^w)$ is u and j-th element $(j \neq i)$ is w, e.g. $(u_{11}^w, u_{12}^w, u_{13}^w) = (u, w, w)$ and $(u_{21}^0, u_{22}^0, u_{23}^0) = (0, u, 0)$.

Example 3.3. Let f_i be non-decreasing with respect to all x_i, $i = 1,2,3$. Function f_i satisfies (3.2) for sufficiently small q and sufficiently large Q if

$$1 < \lim_{u \to 0+} \inf_{t \in [\tau/4, 3\tau/4]} \frac{f_i(t,u,u,u)}{u} \cdot \frac{A_i}{4}, \qquad \lim_{u \to +\infty} \sup_{t \in [0,\tau]} \frac{f_i(t,u,u,u)}{u} \cdot B_i < 1,$$

and satisfies (3.3) for sufficiently small q and sufficiently large Q if $a_i = b_i = 0$ and

$$\begin{aligned} \forall w \in [0, +\infty) \ \lim_{u \to 0+} \sup_{t \in [0,\tau]} \frac{f_i(t, u_{i1}^w, u_{i2}^w, u_{i3}^w)}{u} = 0, \\ 1 < \lim_{u \to +\infty} \inf_{t \in [\tau/4, 3\tau/4]} \frac{f_i(t, u_{i1}^0, u_{i2}^0, u_{i3}^0)}{u} \cdot \frac{A_i}{4}. \end{aligned} \tag{3.4}$$

For proof see Proposition 3.6.

Example 3.4. Let f_i be bounded with respect to x_i and non-decreasing with respect to every $x_{j\neq i}$, $j = 1,2,3$. Function f_i satisfies (3.2) for sufficiently small q and sufficiently large Q if

$$1 < \lim_{w \to 0+} \inf_{\substack{t \in [\tau/4, 3\tau/4] \\ u \in [0,+\infty)}} \frac{f_i(t, u_{i1}^w, u_{i2}^w, u_{i3}^w)}{w} \cdot \frac{A_i}{4}, \qquad \lim_{w \to +\infty} \sup_{\substack{t \in [0,\tau] \\ u \in [0,+\infty)}} \frac{f_i(t, u_{i1}^w, u_{i2}^w, u_{i3}^w)}{w} \cdot B_i < 1,$$

and satisfies (3.3) for sufficiently small q and sufficiently large Q if $a_i = b_i = 0$ and (3.4).

Example 3.5. Let f_i be bounded with respect to every $x_{j\neq i}$, $j = 1,2,3$. Function f_i satisfies (3.2) for sufficiently small q and sufficiently large Q if

$$1 < \lim_{x_i \to 0+} \inf_{\substack{t \in [\tau/4, 3\tau/4] \\ x_{j\neq i} \in [0,+\infty)^2}} \frac{f_i(t, x_1, x_2, x_3)}{x_i} \cdot \frac{A_i}{4}, \quad \lim_{x_i \to +\infty} \sup_{\substack{t \in [0,\tau] \\ x_{j\neq i} \in [0,+\infty)^2}} \frac{f_i(t, x_1, x_2, x_3)}{x_i} \cdot B_i < 1,$$

and satisfies (3.3) for sufficiently small q and sufficiently large Q if $a_i = b_i = 0$ and

$$\lim_{x_i \to 0+} \sup_{\substack{t\in[0,\tau] \\ x_{j\neq i}\in[0,+\infty)^2}} \frac{f_i(t,x_1,x_2,x_3)}{x_i} \cdot B_i < 1, \quad 1 < \lim_{x_i\to+\infty} \inf_{\substack{t\in[\tau/4,3\tau/4] \\ x_{j\neq i}\in[0,+\infty)^2}} \frac{f_i(t,x_1,x_2,x_3)}{x_i} \cdot \frac{A_i}{4}.$$

Proposition 3.6. *Function f_i from Example 3.3 satisfies inequalities* (3.2) *and* (3.3) *for sufficiently small q and sufficiently large Q.*

Proof. First, we show that f_i satisfies (3.2). Let us denote

$$\underline{f_0} = \lim_{u\to 0+} \inf_{t\in[\tau/4,3\tau/4]} \frac{f_i(t,u,u,u)}{u}, \quad \overline{f_\infty} = \lim_{u\to+\infty} \sup_{t\in[0,\tau]} \frac{f_i(t,u,u,u)}{u}.$$

Choose $\varepsilon > 0$ and $\delta > 0$ such that

$$1 < (\underline{f_0} - \varepsilon)A_i/4 \quad \text{and} \quad (\overline{f_\infty} + \delta)B_i < 1.$$

Then there exist positive constants r and p such that

$$\begin{aligned} f_i(t,u,u,u) &\geq (\underline{f_0} - \varepsilon)u, \quad (t,u) \in [\tau/4, 3\tau/4] \times (0,r], \\ f_i(t,u,u,u) &\leq (\overline{f_\infty} + \delta)u, \quad (t,u) \in [0,\tau] \times [p,+\infty). \end{aligned}$$

We denote $M = \max\{f_i(t,u,u,u) : t \in [0,\tau],\ u \in [0,p]\}$. Then

$$f_i(t,u,u,u) \leq M + (\overline{f_\infty} + \delta)u, \quad (t,u) \in [0,\tau] \times [0,+\infty).$$

We choose $q \in (0,r]$, define

$$Q = \frac{B_iM + \|g_i\|}{1 - B_i(\overline{f_\infty} + \delta)} + q$$

and let $\overline{K}_{q,Q} = \{x \in K : q \leq \|x_i\| \leq Q,\ i = 1,2,3\}$. Observe that

$$\begin{aligned} \max_{\substack{t\in[0,\tau] \\ x\in[0,Q]^3}} f_i(t,x) \cdot B_i + \|g_i\| &\leq \max_{t\in[0,\tau]} f_i(t,Q,Q,Q) \cdot B_i + \|g_i\| \leq B_iM + B_i(\overline{f_\infty} + \delta)Q + \|g_i\| \\ &= (B_iM + \|g_i\|) + \frac{(B_iM + \|g_i\|)B_i(\overline{f_i^\infty} + \delta)}{1 - B_i(\overline{f_\infty} + \delta)} + q\,B_i(\overline{f_\infty} + \delta) \\ &= \frac{B_iM + \|g_i\|}{1 - B_i(\overline{f_\infty} + \delta)} + q\,B_i(\overline{f_\infty} + \delta) < Q \end{aligned}$$

and

$$\min_{\substack{t\in[\tau/4,3\tau/4] \\ x_i\in[q/4,q] \\ x_{j\neq i}\in[q/4,Q]^2}} f_i(t,x) \cdot A_i \geq \min_{t\in[\tau/4,3\tau/4]} f_i(t,q/4,q/4,q/4) \cdot A_i \geq A_i(\underline{f_0} - \varepsilon)q/4 > q.$$

Now, we show that f_i satisfies (3.3). Recall that $a_i = b_i = 0$ implies $\|g_i\| = 0$. Let us denote

$$\underline{f_\infty} = \lim_{u\to+\infty} \inf_{t\in[\tau/4,3\tau/4]} \frac{f_i(t,u_{i1}^0,u_{i2}^0,u_{i3}^0)}{u}.$$

Choose $\varepsilon > 0$ such that $(\overline{f_\infty} - \varepsilon)A_i/4 > 1$. Then there exist positive constants r and p such that

$$f_i(t, u_{i1}^w, u_{i2}^w, u_{i3}^w) \leq B_i^{-1}u, \quad (t,u,w) \in [0,\tau] \times (0,r] \times [0,+\infty)^2,$$
$$f_i(t, u_{i1}^0, u_{i2}^0, u_{i3}^0) \geq (\underline{f_\infty} - \varepsilon)u, \quad (t,u) \in [\tau/4, 3\tau/4] \times [p,+\infty),$$

We choose $q \in (0,r]$, $Q \in [4p+q, +\infty)$ and let $\overline{K}_{q,Q} = \{x \in K : q \leq \|x_i\| \leq Q,\ i = 1,2,3\}$. Observe that

$$\max_{\substack{t\in[0,\tau] \\ x_i \in [0,q] \\ x_{j\neq i}\in[0,Q]^2}} f_i(t,x) \cdot B_i \leq \max_{t\in[0,\tau]} f_i(t, q_{i1}^Q, q_{i2}^Q, q_{i3}^Q) \cdot B_i < q$$

and

$$\min_{\substack{t\in[\tau/4,3\tau/4] \\ x_i\in[Q/4,Q] \\ x_{j\neq i}\in[q/4,Q]^2}} f_i(t,x) \cdot A_i \geq \min_{t\in[\tau/4,3\tau/4]} f_i\Big(t, (Q/4)_{i1}^0, (Q/4)_{i2}^0, (Q/4)_{i3}^0\Big) \cdot A_i \geq (\underline{f_\infty} - \varepsilon)A_i Q/4 > Q.$$

Finally, note that constants q and Q could be chosen as small and as large as desired, respectively. □

The main result of this paper is following.

Theorem 3.7. *If for every f_i, $i = 1,2,3$, exist $q_i < Q_i$ such that f_i satisfies (3.2) or (3.3) for $0 < q \leq q_i$ and $Q_i \leq Q < +\infty$, then boundary value problem (1.1), (1.2) has at least one positive solution.*

Proof. We denote $r = \min\{q_i : i = 1,2,3\}$, $R = \max\{Q_i : i = 1,2,3\}$ and let

$$\overline{K}_{r,R} = \{x \in K : r \leq \|x_i\| \leq R,\ i = 1,2,3\}.$$

By Lemma 3.2, each T_i satisfies compression type condition (i) or expansion type condition (ii) in $\overline{K}_{r,R}$. Therefore, by Krasnosel'skiĭ–Precup fixed point theorem, operator T has a fixed point in $\overline{K}_{r,R}$, which implies that boundary value problem (1.1), (1.2) has at least one positive solution. □

Let us show applicability of Theorem 3.7 in following example. Here and in Section 4, we round numbers to three decimal places unless we can calculate the numbers exactly.

Example 3.8. Consider system of differential equations

$$\begin{aligned} x_1'' + x_1^2(t + x_2x_3)^3 &= 0, \quad t \in (0,\tau), \\ x_2'' + \Big(x_1 t + x_3^{1/3}\Big)\frac{\exp(-x_2) + 1}{2} &= 0, \quad t \in (0,\tau), \\ x_3'' + \frac{80x_3 t}{x_3^3 + 1} + 7\sin(x_1 - x_2) + 7 &= 0, \quad t \in (0,\tau), \end{aligned} \tag{3.5}$$

with boundary conditions

$$\begin{aligned} x_1(0) &= 3x_1(1/5) - x_1(1/2), & x_1(\tau) &= \frac{1}{2}\int_0^\tau t^2 x_1(t)\mathrm{d}t, \\ x_2(0) &= a_2, & x_2(\tau) &= \int_0^\tau (\tau - t)x_2(t)\mathrm{d}t + b_2, \\ x_3(0) &= x_3(1/2) + a_3, & x_3(\tau) &= b_3, \end{aligned} \tag{3.6}$$

where a_2, b_2, a_3, $b_3 \geq 0$. Observe that $1/5$ and $1/2$ appear in the multi-point boundary conditions in first and third line of (3.6). Hence τ is greater than $1/2$.

In this example, the Green's functions (and intervals where assumptions (A1)–(A3) are valid) are as follows (recall that G_0 is given by (2.1)):

$$G_1(t,s) = \frac{1}{\tau/10 - \tau^4/60} \begin{vmatrix} \tau - t & 1/10 & -\tau^4/24 \\ t & -1/10 & \tau - \tau^4/8 \\ G_0(t,s) & G_0(1/2,s) - 3G_0(1/5,s) & -(s\tau^3 - s^4)/24 \end{vmatrix}, \tau \in (1/2, 6^{1/3}),$$

$$G_2(t,s) = \frac{1}{\tau^2(1-\tau^2/6)} \begin{vmatrix} \tau - t & \tau & -\tau^3/3 \\ t & 0 & \tau - \tau^3/6 \\ G_0(t,s) & 0 & -(s^3 - 3s^2\tau + 2s\tau^2)/6 \end{vmatrix}, \quad \tau \in (1/2, \sqrt{6}),$$

$$G_3(t,s) = \frac{2}{\tau} \begin{vmatrix} \tau - t & 1/2 & 0 \\ t & -1/2 & \tau \\ G_0(t,s) & -G_0(1/2,s) & 0 \end{vmatrix}, \quad \tau \in (1/2, +\infty).$$

Observe that $f_1(t,x) = x_1^2(t + x_2x_3)^3$ is non-decreasing with respect to all x_i, $i = 1,2,3$, and $a_1 = b_1 = 0$, and

$$\forall w \in [0,+\infty) \lim_{u\to 0+} \sup_{t\in[0,\tau]} \frac{u^2(t+w^2)^3}{u} = 0, \qquad \lim_{u\to+\infty} \inf_{t\in[\tau/4,3\tau/4]} \frac{u^2(t+0)^3}{u} = +\infty.$$

We do not need to calculate B_1 and A_1. But we need $A_1 > 0$, which is true for $\tau \in (1/2, 6^{1/3})$. Therefore (see Example 3.3), f_1 satisfies (3.3) for $\tau \in (1/2, 6^{1/3})$.

Next, $f_2(t,x) = (x_1 t + x_3^{1/3})(\exp(-x_2) + 1)/2$ is bounded with respect to x_2, non-decreasing with respect to x_1, x_3 and

$$\lim_{w\to 0+} \inf_{\substack{t\in[\tau/4,3\tau/4] \\ u\in[0,+\infty)}} \left(wt + w^{1/3}\right) \frac{\exp(-u)+1}{2w} = +\infty,$$

$$\lim_{w\to+\infty} \sup_{\substack{t\in[0,\tau] \\ u\in[0,+\infty)}} \left(wt + w^{1/3}\right) \frac{\exp(-u)+1}{2w} = \tau.$$

We expand $G_2(t,s)$ along the second column and consider

$$\begin{aligned} B_2 &= \sup_{t\in[0,\tau]} \int_0^\tau G_2(t,s)\mathrm{d}s \\ &= \sup_{t\in[0,\tau]} \frac{1}{\tau^2(1-\tau^2/6)} \int_0^\tau \left(\frac{\tau\, t(s^3 - 3s^2\tau + 2s\tau^2)}{6} + \tau\left(\tau - \frac{\tau^3}{6}\right) G_0(t,s) \right) \mathrm{d}s \\ &= \sup_{t\in[0,\tau]} \frac{\tau\, t(\tau^2 - 12) - 2t^2(\tau^2 - 6)}{4(\tau^2 - 6)} = \begin{cases} (144\tau^2 - 24\tau^4 + \tau^6)/(32(\tau^2-6)^2), & 1/2 < \tau < 2, \\ \tau^4/(4(6-\tau^2)), & 2 \leq \tau < \sqrt{6}. \end{cases} \end{aligned}$$

Calculations show that $\tau B_2 < 1$ for $\tau \in (1/2, 1.612)$. Therefore (see Example 3.4), f_2 satisfies (3.2) for $\tau \in (1/2, 1.612)$.

Next, $f_3(t,x) = 80x_3t/(x_3^3 + 1) + 7\sin(x_1 - x_2) + 7$ is bounded with respect to x_1, x_2 and

$$\lim_{x_3\to 0+} \inf_{\substack{t\in[\tau/4,3\tau/4] \\ x_1,x_2\in[0,+\infty)}} \frac{80x_3t}{(x_3^3+1)x_3} + \frac{7\sin(x_1-x_2)+7}{x_3} = 20\tau,$$

$$\lim_{x_3\to+\infty} \sup_{\substack{t\in[0,\tau]\\ x_1,x_2\in[0,+\infty)}} \frac{80x_3t}{(x_3^3+1)x_3} + \frac{7\sin(x_1-x_2)+7}{x_3} = 0.$$

We expand $G_3(t,s)$ along the third column and consider

$$\begin{aligned} A_3 &= \inf_{t\in[\tau/4,3\tau/4]} \int_{\tau/4}^{3\tau/4} G_3(t,s)\mathrm{d}s = \inf_{t\in[\tau/4,3\tau/4]} \frac{2}{\tau}\int_{\tau/4}^{3\tau/4} \Big(\tau(\tau-t)G_0(1/2,s) + \frac{\tau}{2}G_0(t,s)\Big)\,\mathrm{d}s \\ &= \inf_{t\in[\tau/4,3\tau/4]} \frac{-16t^2-\tau(8-15\tau+2\tau^2)+2t(4+\tau^2)}{32} \\ &= \begin{cases} (-12\tau+28\tau^2-3\tau^3)/64, & 1/2<\tau\le 2(2-\sqrt{3}) \text{ or } 2(2+\sqrt{3})<\tau, \\ (-4\tau+12\tau^2-\tau^3)/64, & 2(2-\sqrt{3})<\tau\le 2(2+\sqrt{3}). \end{cases} \end{aligned}$$

Calculations show that $1 < 5\tau A_3$ for $\tau \in (1.197, 8.877)$. Therefore (see Example 3.5), f_3 satisfies (3.2) for $\tau \in (1.197, 8.877)$.

Finally, we consider interval

$$(1/2, 6^{1/3}) \cap (1/2, 1.612) \cap (1.197, 8.877) = (1.197, 1.612).$$

Each f_i satisfies either (3.2) or (3.3) for sufficiently small q, sufficiently large Q and $\tau \in (1.197, 1.612)$. Therefore, by Theorem 3.7, boundary value problem (4.3), (4.4) has at least one positive solution for $\tau \in (1.197, 1.612)$.

4 Numerical result for Dirichlet boundary conditions

In this section, we consider problem (1.1) with boundary conditions

$$x_i(0) = a_i,\ x_i(\tau) = b_i, \tag{4.1}$$

and show examples where is compared theoretical estimation of τ with result obtained numerically. Note that here $G_i = G_0$, $A_i = \tau^2/16$ and $B_i = \tau^2/8$ for every $i = 1,2,3$.

For numerical result let us consider the initial conditions

$$x_i(0) = a_i, \quad x_i'(0) = c_i \in \mathbb{R}, \quad i = 1,2,3. \tag{4.2}$$

Let $c = (c_1, c_2, c_3)$ and $x^c = (x_1^c, x_2^c, x_3^c)$ be a solution of initial value problem (1.1), (4.2). We denote by $t_1(c) > 0$ the positive argument for which $x^c(t_1(c)) = (b_1, b_2, b_3)$ holds for the first time. Such $t_1(c)$ exists if and only if boundary value problem (1.1), (4.1) has a solution for $t_1(c) = \tau$. Thus set of values of the map $c \mapsto t_1(c)$ determines values of the τ. We assume that if there is no c such that $t_1(c) = \tau > 0$, then $t_1(c) = 0$.

In case of one equation this method is known as the shooting method. We do the following in our case. We fix c_1 and consider $t_1(c_1,\cdot,\cdot)$. If problem (1.1), (4.1) has a solution for $t_1(c) = \tau$, then $t_1(c_1,\cdot,\cdot)$ is everywhere zero except one point.

To obtain the result we are using "brute force", i.e. go through all possible choices. To make count of choices less, we consider meshes with step sizes θ_i for c_i, $i = 1,2,3$. To make sure to "shoot somewhere", we consider weakened conditions

$$\Big(x_2(t_1(c)) - b_2\Big)^2 + \Big(x_3(t_1(c)) - b_3\Big)^2 < \varepsilon^2.$$

Thus for every c_1 there exists a set $\Omega_\varepsilon \subset \mathbb{R}^2$ such that $t_1(c) > 0$ for $(c_2, c_3) \in \Omega_\varepsilon$. We denote $t_M(c_1) = \max\{t_1(c) : (c_2, c_3) \in \mathbb{R}^2\}$. Numerical result is a discrete plot $c_1 \mapsto t_M(c_1)$.

In the following examples we compare the results. Examples emphasize that Theorem 3.7 gives sufficient conditions.

Example 4.1. Consider system of differential equations

$$\begin{aligned} x_1'' + (x_1 t^3 + x_3)^{1/2}\, x_2^{1/3} &= 0, \quad t \in (0,\tau), \\ x_2'' + (x_1 t^3 + x_3^{1/2})\frac{\exp(-x_2) + 1}{10} &= 0, \quad t \in (0,\tau), \\ x_3'' + \frac{16^2 x_3^4 + x_3}{1 + x_3^3}\Big(2 + \sin(x_1 t + x_2)\Big) &= 0, \quad t \in (0,\tau), \end{aligned} \tag{4.3}$$

with boundary conditions

$$\begin{aligned} x_1(0) &= 0.2, & x_1(\tau) &= 0, \\ x_2(0) &= 0, & x_2(\tau) &= 0.2, \\ x_3(0) &= 0, & x_3(\tau) &= 0. \end{aligned} \tag{4.4}$$

Here $f_1(t,x) = (x_1 t^3 + x_3)^{1/2}\, x_2^{1/3}$ is non-decreasing with respect to all x_i, $i = 1,2,3$, and

$$\lim_{u \to 0+} \inf_{t \in [\tau/4, 3\tau/4]} \frac{(ut^3 + u)^{1/2}\, u^{1/3}}{u} = +\infty, \qquad \lim_{u \to +\infty} \sup_{t \in [0,\tau]} \frac{(ut^3 + u)^{1/2} u^{1/3}}{u} = 0.$$

Therefore, f_1 satisfies (3.2) for $\tau \in (0, +\infty)$.

Next, $f_2(t,x) = (x_1 t^3 + x_3^{1/2})(\exp(-x_2) + 1)/10$ is bounded with respect to x_2, non-decreasing with respect to x_1, x_3 and

$$\lim_{w \to 0+} \inf_{\substack{t \in [\tau/4, 3\tau/4] \\ u \in [0,+\infty)}} \left(wt^3 + w^{1/2}\right) \frac{\exp(-u) + 1}{10w} = +\infty,$$

$$\lim_{w \to +\infty} \sup_{\substack{t \in [0,\tau] \\ u \in [0,+\infty)}} \left(wt^3 + w^{1/2}\right) \frac{\exp(-u) + 1}{10w} = \frac{\tau^3}{5}.$$

Calculations show that $B_2 \tau^3/5 < 1$ for $\tau \in (0, 40^{1/5})$. Therefore, f_2 satisfies (3.2) for $\tau \in (0, 40^{1/5})$.

Next, $f_3(t,x) = (16^2 x_3^4 + x_3)(2 + \sin(x_1 t + x_2))/(1 + x_3^3)$ is bounded with respect to x_1, x_2, and

$$\lim_{x_3 \to 0+} \sup_{\substack{t \in [0,\tau] \\ x_1, x_2 \in [0,+\infty)}} \frac{16^2 x_3^4 + x_3}{(1 + x_3^3) x_3}\Big(2 + \sin(x_1 t + x_2)\Big) = 3,$$

$$\lim_{x_3 \to +\infty} \inf_{\substack{t \in [\tau/4, 3\tau/4] \\ x_1, x_2 \in [0,+\infty)}} \frac{16^2 x_3^4 + x_3}{(1 + x_3^3) x_3}\Big(2 + \sin(x_1 t + x_2)\Big) = 16^2.$$

Calculations show that $3B_3 < 1 < 16^2 A_3/4$ for $\tau \in (1/2,\, 2\sqrt{2/3})$. Therefore, f_3 satisfies (3.3) for $\tau \in (1/2,\, 2\sqrt{2/3})$.

Each f_i satisfies either (3.2) or (3.3) for sufficiently small q, sufficiently large Q and $\tau \in (1/2, 2\sqrt{2/3})$. Therefore, by Theorem 3.7, the theoretical result is $1/2 < \tau < 2\sqrt{2/3}$, or approximately $1/2 < \tau < 1.633$.

Since $a_2 \leq b_2$ and $a_3 \leq b_3$, we consider non-negative c_2 and c_3. For numerical result we make meshes in interval $[-1,1]$ for c_1, $[0,2]$ for c_2 and $[0.001, 2.001]$ for c_3 with step sizes $\theta_1 = \theta_2 = \theta_3 = 0.1$ and $\varepsilon = 0.1$. The result is illustrated in Figure 4.1. Numerical result shows that $0.200 \leq \tau \leq 2.579$.

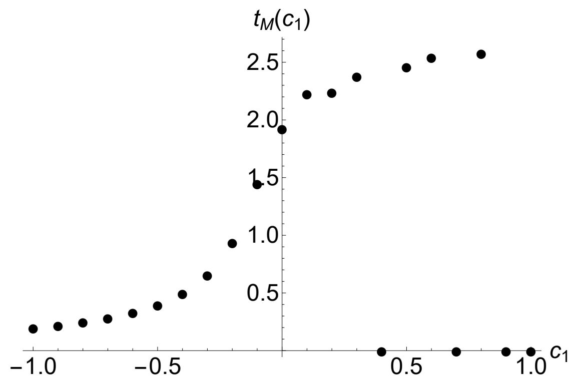

Figure 4.1: Graph of the $c_1 \mapsto t_M(c_1)$ for problem (4.3), (4.4).

Example 4.2. Consider system of differential equations

$$\begin{aligned} x_1'' + (x_1x_2x_3)^{1/4} &= 0, \quad t \in (0,\tau), \\ x_2'' + \frac{1}{1+tx_2} + \frac{1}{1+x_1x_3} &= 0, \quad t \in (0,\tau), \\ x_3'' + + \frac{(15+4t)x_3}{1+x_3^2} + \cos x_1 \sin x_2 + 1 &= 0, \quad t \in (0,\tau), \end{aligned} \tag{4.5}$$

with boundary conditions

$$\begin{aligned} x_1(0) &= 1, \quad x_1(\tau) = 0, \\ x_2(0) &= 0, \quad x_2(\tau) = 1, \\ x_3(0) &= 1, \quad x_3(\tau) = 1. \end{aligned} \tag{4.6}$$

Here $f_1(t,x) = (x_1x_2x_3)^{1/4}$ is non-decreasing with respect to all x_i, $i = 1,2,3$, and

$$\lim_{u\to 0+} \inf_{t\in[\tau/4,3\tau/4]} \frac{u^{3/4}}{u} = +\infty, \qquad \lim_{u\to+\infty} \sup_{t\in[0,\tau]} \frac{u^{3/4}}{u} = 0.$$

Therefore, f_1 satisfies (3.2) for $\tau \in (0,+\infty)$.

Next, $f_2(t,x) = (1+tx_2)^{-1} + (1+x_1x_3)^{-1}$ is bounded with respect to x_1, x_3 and

$$\lim_{x_2\to 0+} \inf_{\substack{t\in[\tau/4,3\tau/4] \\ x_1,x_3\in[0,+\infty)}} \frac{1}{(1+tx_2)x_2} + \frac{1}{(1+x_1x_3)x_2} = +\infty,$$

$$\lim_{x_2\to +\infty} \sup_{\substack{t\in[0,\tau] \\ x_1,x_3\in[0,+\infty)}} \frac{1}{(1+tx_2)x_2} + \frac{1}{(1+x_1x_3)x_2} = 0.$$

Therefore, f_2 satisfies (3.2) for $\tau \in (0,+\infty)$.

Next, $f_3(t,x) = (15+4t)x_3/(1+x_3^2) + \cos x_1 \sin x_2 + 1$ is bounded with respect to x_1, x_2, and

$$\lim_{x_3\to 0+} \inf_{\substack{t\in[\tau/4,3\tau/4] \\ x_1,x_2\in[0,+\infty)}} \frac{(15+4t)x_3}{(1+x_3^2)x_3} + \frac{\cos x_1 \sin x_2 + 1}{x_3} = 15 + \tau,$$

$$\lim_{x_3\to+\infty} \sup_{\substack{t\in[0,\tau]\\ x_1,x_2\in[0,+\infty)}} \frac{(15+4t)x_3}{(1+x_3^2)x_3} + \frac{\cos x_1 \sin x_2 + 1}{x_3} = 0.$$

Calculations show that $1 < (15+\tau)A_3/4$ for $\tau \in (1.944, +\infty)$. Therefore, f_3 satisfies (3.2) for $\tau \in (1.944, +\infty)$.

Each f_i satisfies (3.2) for sufficiently small q, sufficiently large Q and $\tau \in (1.944, +\infty)$. Therefore, by Theorem 3.7, the theoretical result is $\tau > 1.944$.

For numerical result we make meshes in interval $[-7, 0]$ for c_1, $[0, 7]$ for c_2 and c_3 with step sizes $\theta_1 = 1$, $\theta_2 = \theta_3 = 0.1$ and $\varepsilon = 0.1$. The result is illustrated in Figure 4.2. Numerical result shows that τ could be less than 1.944.

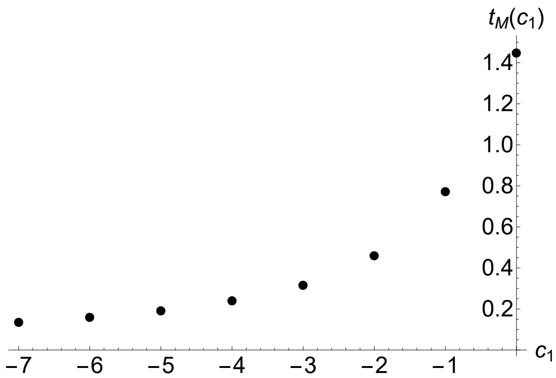

Figure 4.2: Graph of the $c_1 \mapsto t_M(c_1)$ for problem (4.5), (4.6).

Remark 4.3. There is no ground to say that this method is not suitable for nonlocal conditions, for instance, $x(0) = \varphi_i[x_i] + a_i$, $x(\tau) = b_i$. But, since we are using “brute force” (which is long itself), in case of nonlocal conditions program needs much smaller step size to get nonzero $t_M(c_1)$, and hence much more time to run, which makes the program inefficient.

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Ground state sign-changing solution for a logarithmic Kirchhoff-type equation in $\mathbb{R}^3$

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Abstract. We investigate the following logarithmic Kirchhoff-type equation:

$$\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 dx\right)[-\Delta u + V(x)u] = |u|^{p-2}u\ln|u|, \qquad x \in \mathbb{R}^3,$$

where $a, b > 0$ are constants, $4 < p < 2^* = 6$. Under some appropriate hypotheses on the potential function V, we prove the existence of a positive ground state solution, a ground state sign-changing solution and a sequence of solutions by using the constraint variational methods, topological degree theory, quantitative deformation lemma and symmetric mountain pass theorem. Our results complete those of Gao et al. [*Appl. Math. Lett.* **139**(2023), 108539] with the case of $4 < p < 6$.

Keywords: Kirchhoff-type equation, logarithmic nonlinearity, ground state sign-changing solution, variational methods, topological degree theory.

2020 Mathematics Subject Classification: 35J50, 35J61.

1 Introduction and main result

In this work, we are concerned with the existence of ground state sign-changing solutions for the following logarithmic Kirchhoff-type equation

$$\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 dx\right)[-\Delta u + V(x)u] = |u|^{p-2}u\ln|u|, \qquad x \in \mathbb{R}^3, \tag{1.1}$$

where $a, b > 0$ are constants, $4 < p < 6$. Besides, we shall impose the following conditions on potential function V:

(V_1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\lim_{|x|\to\infty} V(x) = +\infty$;

(V_2) There exists a constant V_0 such that $\inf_{x\in\mathbb{R}^3} V(x) \geq V_0 > 0$.

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As is known to all, Kirchhoff [12] first proposed the following Kirchhoff model given by the stationary analogue of equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0,$$

where ρ is the mass density, P_0 is the initial tension, h represents the area of the cross-section, E is the Young modulus of the material and L is the length of the string. The above model is an extension of the classical D'Alembert wave equation by taking into account the changes in the length of the string during the transverse vibrations. After that, Lions [13] derived the following Kirchhoff equation by using the functional analysis method

$$u_{tt}-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u=f(x,u). \tag{1.2}$$

This model is used to describe the chord length variation of elastic strings caused by lateral vibration, where u is displacement, f is external force, b is initial tension force and a is related to inherent properties of strings (see [1, 2, 5, 7] and the references therein). The corresponding problem associated with equation (1.2) is called as the Kirchhoff-type problem.

In the past years, logarithmic nonlinearity appears frequently in partial differential equations, which has numerous applications to quantum optics, quantum mechanics, nuclear physics, transport and diffusion phenomenon etc (see [22] and the references therein). Therefore, many scholars studied the following Kirchhoff-type problem with logarithmic nonlinearity

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=|u|^{p-2}u\log u^2,\qquad x\in\mathbb{R}^3, \tag{1.3}$$

where $4<p<6$ and $V\in C(\mathbb{R}^3,\mathbb{R})$. By using the constrained variational method, deformation lemma and topological degree theory, Hu and Gao [10] proved that equation (1.3) owns both positive solution and sign-changing solution under different types of potential (coercive potential and periodic potential). Wen, Tang and Chen [20] verified that equation (1.3) in smooth bounded domain $\Omega\subset\mathbb{R}^3$ has a ground state solution and a ground state sign-changing solution, besides, the energy of sign-changing solution is larger than twice of the ground state energy.

In particular, letting $a=1, b=0$ and $p=2$ in equation (1.3), it leads to the following logarithmic Schrödinger equation

$$-\Delta u+V(x)u=u\log u^2,\qquad x\in\mathbb{R}^N. \tag{1.4}$$

Equation (1.4) has received much attention in mathematical analysis and applications. Ji and Szulkin [11] got infinitely many solutions by adapting some arguments of the fountain theorem when the potential is coercive (i.e. $\lim_{|x|\to\infty}V(x)=+\infty$), and in the case of bounded potential (i.e. $\lim_{|x|\to\infty}V(x)=V_\infty\in(-1,+\infty)$), they obtained a ground state solution. By using the direction derivative and constrained minimization method, Shuai [16] proved the existence of positive and sign-changing solutions of equation (1.4) under different types of potential (coercive potential and periodic potential). When the potential is radially symmetric, the author constructed infinitely many radial nodal solutions. Zhang and Zhang [24] proved the existence, uniqueness, non-degeneracy and some qualitative properties of positive solutions of equation (1.4) when the potential $V\in C^2(\mathbb{R}^N,\mathbb{R})$ is radially symmetric and allowed to be singular at $x=0$ and repulsive at infinity(i.e. $\lim_{|x|\to\infty}V(x)=-\infty$). When potential

$V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\lim_{|x|\to\infty} V(x) = V_\infty$ and $V(x) < V_\infty + \log 2$, Feng, Tang and Zhang [8] proved that equation (1.4) has a positive bound state solution.

After that, inspired by [10], Gao, Jiang and Liu et al. [9] studied the existence of solutions to equation (1.1) for the first time, and proved that equation (1.1) has only trivial solution for large $b > 0$ and two positive solutions for small $b > 0$ and $2 < p < 4$. To the best of our knowledge, there is no result for the existence of positive ground state, ground state sign-changing solutions and sequence of solutions of equation (1.1) with $4 < p < 6$. Inspired by the above literature, we are interested in the existence of positive ground state solutions, ground state sign-changing solutions and sequence of solutions for equation (1.1).

Equation (1.1) is formally associated with the energy functional $I : H \to \mathbb{R}$ defined by

$$\begin{aligned} I(u) = \frac{a}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx\right)^2 \\ - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p \ln|u|dx + \frac{1}{p^2}\int_{\mathbb{R}^3}|u|^p dx, \end{aligned} \tag{1.5}$$

with $I(0) = 0$, where Sobolev space H is defined as follows:

$$H := \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < +\infty\right\}.$$

endowed with the inner product

$$\langle u, v\rangle := \int_{\mathbb{R}^3}(\nabla u \cdot \nabla v + V(x)uv)dx, \qquad \forall u, v \in H$$

and endowed with the norm

$$\|u\|^2 := \langle u, u\rangle = \int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx.$$

Denote $|u|_k = \left(\int_{\mathbb{R}^3}|u|^k dx\right)^{1/k}$ the norm of $u \in L^k(\mathbb{R}^3)$ for $k \geq 1$, the $C, C_1, C_2, \ldots$ represent several different positive constants. A elementary computation, we have

$$\lim_{t\to 0}\frac{t^{p-1}\ln|t|}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{t^{p-1}\ln|t|}{t^{q-1}} = 0,$$

where $4 < p < q < 6$. Therefore, for arbitrarily $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|t^{p-1}\ln|t|| \leq \varepsilon|t| + C_\varepsilon|t|^{q-1}, \qquad \forall\, t \in \mathbb{R}\backslash\{0\}. \tag{1.6}$$

By (1.6) and [19, Lemma 3.10], we get that $I \in C^1(H, \mathbb{R})$ and the Fréchet derivative of I is given by

$$\langle I'(u), v\rangle = (a + b\|u\|^2)\int_{\mathbb{R}^3}(\nabla u\nabla v + V(x)uv)dx - \int_{\mathbb{R}^3}|u|^{p-2}uv\ln|u|dx, \tag{1.7}$$

for all $u, v \in H$. $u \in H$ is a weak solution of equation (1.1) if and only if u is a critical point of I. Additional, if $u \in H$ is a weak solution of equation (1.1) with $u^\pm \neq 0$, then u is called a sign-changing solution of equation (1.1), where

$$u^+ := \max\{u(x), 0\}, \qquad u^- := \min\{u(x), 0\}.$$

From (1.7), we know

$$\langle I'(u),u\rangle = a\|u\|^2 + b\|u\|^4 - \int_{\mathbb{R}^3} |u|^p \ln|u| dx \tag{1.8}$$

and

$$\langle I'(u),u^\pm\rangle = (a+b\|u\|^2)\|u^\pm\|^2 - \int_{\mathbb{R}^3} |u^\pm|^p \ln|u^\pm| dx. \tag{1.9}$$

By virtue of (1.8) and (1.9), it is noticed that if $u \not\equiv 0$, then

$$\begin{aligned} I(u) &= I(u^+) + I(u^-) + \frac{b}{2}\|u^+\|^2\|u^-\|^2, \\ \langle I'(u),u^+\rangle &= \langle I'(u^+),u^+\rangle + b\|u^+\|^2\|u^-\|^2, \\ \langle I'(u),u^-\rangle &= \langle I'(u^-),u^-\rangle + b\|u^+\|^2\|u^-\|^2. \end{aligned}$$

In this paper, our main purpose is to seek the ground state sign-changing solution for equation (1.1). As we all known, there are some very interesting results for the existence and multiplicity of sign-changing solutions of the following Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \qquad x \in \mathbb{R}^N. \tag{1.10}$$

However, these methods of seeking sign-changing solutions dependent on the following decomposition

$$J(u) = J(u^+) + J(u^-), \tag{1.11}$$

and

$$\langle J'(u),u^+\rangle = \langle J'(u^+),u^+\rangle, \qquad \langle J'(u),u^-\rangle = \langle J'(u^-),u^-\rangle, \tag{1.12}$$

where J is the energy functional of equation (1.10) given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx.$$

However, it follows from (1.5) that the energy functional I does not possess the same decompositions as (1.11) and (1.12). Indeed, a direct calculation yields that

$$I(u) > I(u^+) + I(u^-),$$

and

$$\langle I'(u),u^+\rangle > \langle I'(u^+),u^+\rangle, \qquad \langle I'(u),u^-\rangle > \langle I'(u^-),u^-\rangle$$

for $u^\pm \neq 0$. Therefore, the method of getting sign-changing solutions for the local problem (1.10) does not seem applicable to equation (1.1). In order to overcome this difficulty, we follow in [4] by the following Nehari manifold and the nodal Nehari sets respectively

$$\mathcal{N} := \left\{u \in H\backslash\{0\} : \langle I'(u),u\rangle = 0\right\},$$

and

$$\mathcal{M} := \left\{u \in H, u^\pm \neq 0 : \langle I'(u),u^\pm\rangle = 0\right\}.$$

It is well known that the existence of positive ground state and sign-changing solutions to equation (1.1) can be transformed into studying the following minimization problems respectively

$$c := \inf_{u\in\mathcal{N}} I(u) \quad \text{and} \quad m := \inf_{u\in\mathcal{M}} I(u).$$

Now, we state the main results.

Theorem 1.1. *Assume that (V_1)–(V_2) hold and $4 < p < 6$, then equation (1.1) possesses a positive ground state solution $\bar{u} \in \mathcal{N}$ such that $I(\bar{u}) = c$.*

Theorem 1.2. *Assume that (V_1)–(V_2) hold and $4 < p < 6$, then equation (1.1) has a ground state sign-changing solution $u_* \in \mathcal{M}$ with precisely two nodal domains such that $I(u_*) = m$. Moreover, $m > 2c$.*

Theorem 1.3. *Assume that (V_1)–(V_2) hold and $4 < p < 6$, then equation (1.1) owns a sequence of solutions of $\{u_n\}$ with $I(u_n) \to +\infty$ as $n \to \infty$.*

Remark 1.4. To our best knowledge, our results are up to date. Compared with [9], we study the case of $4 < p < 6$. Moreover, we consider the ground state sign-changing solution and a sequence of high energy solutions for equation (1.1).

The remaining of paper is organized as follows. In Section 2, we show some necessary logarithmic inequalities and important lemmas. In Section 3, we prove Theorems 1.1–1.3 by the maximum principle, quantitative deformation lemma, topological degree theory and symmetric mountain pass theorem.

2 Some preliminary results

Firstly, because of the existence of logarithmic nonlinearity, the following lemmas will be used to obtain vital estimates for our problem.

Lemma 2.1. *The following inequalities hold*

$$(1 - x^s) + sx^s \ln x > 0, \qquad \forall x \in (0,1) \cup (1, +\infty), s > 0; \tag{2.1}$$

$$\ln x \le \frac{1}{e\sigma} x^\sigma, \qquad \forall x \in (0, +\infty), \sigma > 0. \tag{2.2}$$

Proof. Define $f(x) := (1 - x^s) + sx^s \ln x$, then $f'(x) := s^2 x^{s-1} \ln x$, it's easy to see that the function $f(x)$ is decreasing on $(0,1)$ and increasing on $(1, +\infty)$. So $f(x) > f(1) = 0$, i.e.

$$(1 - x^s) + sx^s \ln x > 0.$$

Thus, (2.1) is true. The proof of (2.2) is similar to that of (2.1), here we omit it. □

Next, we give the following lemma by the conclusions of [3].

Lemma 2.2. *Under the assumptions (V_1)–(V_2), then the embedding $H \hookrightarrow L^q(\mathbb{R}^3)$ is compact for $q \in [2, 6)$.*

By virtue of Lemma 2.2, we define the following Sobolev embedding constants

$$S_q = \inf_{u \in H \setminus \{0\}} \frac{\|u\|^q}{|u|_q^q}, \qquad q \in [2, 6]. \tag{2.3}$$

As is known to all, the logarithmic nonlinearity $|u|^{p-2} u \ln |u|$ satisfies neither the well-known Nehari type monotonicity condition in [23] nor (AR) condition in [17]. Therefore, we will establish an energy inequality related to $I(u)$, $I(su^+ + tu^-)$, $\langle I'(u), u^+ \rangle$ and $\langle I'(u), u^- \rangle$ in order to overcome the this difficulty.

Lemma 2.3. *For all $u \in H$ and $s, t \geq 0$, there holds*

$$I(u) \geq I(su^+ + tu^-) + \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle. \tag{2.4}$$

Proof. It follows from (1.9) that (2.4) holds for $u = 0$, then we only consider the case when $u \in H\backslash\{0\}$. Set

$$\Omega^+ = \{u \in \mathbb{R}^3 : u(x) \geq 0\}, \qquad \Omega^- = \{u \in \mathbb{R}^3 : u(x) < 0\}.$$

For all $u \in H\backslash\{0\}$ and $s, t \geq 0$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^3} |su^+ + tu^-|^p \ln|su^+ + tu^-| dx \\
&= \int_{\Omega^+} |su^+ + tu^-|^p \ln|su^+ + tu^-| dx + \int_{\Omega^-} |su^+ + tu^-|^p \ln|su^+ + tu^-| dx \\
&= \int_{\Omega^+} |su^+|^p \ln|su^+| dx + \int_{\Omega^-} |tu^-|^p \ln|tu^-| dx \\
&= \int_{\mathbb{R}^3} (|su^+|^p \ln|su^+| + |tu^-|^p \ln|tu^-|) dx.
\end{aligned} \tag{2.5}$$

It follows from (1.5), (1.8), (2.1) and (2.5) that

$$\begin{aligned}
&I(u) - I(su^+ + tu^-) \\
&= \frac{a}{2}\left(\|u\|^2 - \|su^+ + tu^-\|^2\right) + \frac{b}{4}\left(\|u\|^4 - \|su^+ + tu^-\|^4\right) \\
&\quad + \frac{1}{p^2}\int_{\mathbb{R}^3} \left(|u|^p - |su^+ + tu^-|^p\right) dx \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^3} \left(|u|^p \ln|u| - |su^+ + tu^-|^p \ln|su^+ + tu^-|\right) dx \\
&= \frac{a(1-s^2)}{2}\|u^+\|^2 + \frac{a(1-t^2)}{2}\|u^-\|^2 + \frac{b(1-s^4)}{4}\|u^+\|^4 + \frac{b(1-t^4)}{4}\|u^-\|^4 \\
&\quad + \frac{b(1-s^2t^2)}{2}\|u^+\|^2\|u^-\|^2 + \frac{(1-s^p)}{p^2}\int_{\mathbb{R}^3} |u^+|^p dx + \frac{(1-t^p)}{p^2}\int_{\mathbb{R}^3} |u^-|^p dx \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3} \left(|u^+|^p \ln|u^+| - |su^+|^p \ln|u^+| - |su^+|^p \ln s\right) dx \\
&\quad - \frac{1}{p}\int_{\mathbb{R}^3} \left(|u^-|^p \ln|u^-| - |tu^-|^p \ln|u^-| - |tu^-|^p \ln t\right) dx \\
&= \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle \\
&\quad + a\left[\left(\frac{1-s^2}{2} - \frac{1-s^p}{p}\right)\|u^+\|^2 + \left(\frac{1-t^2}{2} - \frac{1-t^p}{p}\right)\|u^-\|^2\right] \\
&\quad + b\left[\left(\frac{1-s^4}{4} - \frac{1-s^p}{p}\right)\|u^+\|^4 + \left(\frac{1-t^4}{4} - \frac{1-t^p}{p}\right)\|u^-\|^4\right] \\
&\quad + b\left(\frac{1-s^2t^2}{2} - \frac{1-s^p}{p} - \frac{1-t^p}{p}\right)\|u^+\|^2\|u^-\|^2 \\
&\quad + \frac{(1-s^p) + ps^p \ln s}{p^2}\int_{\mathbb{R}^3} |u^+|^p dx + \frac{(1-t^p) + pt^p \ln t}{p^2}\int_{\mathbb{R}^3} |u^-|^p dx.
\end{aligned}$$

Since the function $f(x) = \frac{1-a^x}{x}$ is monotonically decreasing on $(0, +\infty)$ for $a \in (0,1) \cup (1, +\infty)$. It follows from the above equation that

$$\begin{aligned}
&I(u) - I(su^+ + tu^-) \\
&\geq \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle + b\left(\frac{1-s^2t^2}{2} - \frac{1-s^p}{p} - \frac{1-t^p}{p}\right)\|u^+\|^2\|u^-\|^2 \\
&= \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle \\
&\quad + b\left[\frac{(s^2-t^2)^2}{4} + \left(\frac{1-s^4}{4} - \frac{1-s^p}{p}\right) + \left(\frac{1-t^4}{4} - \frac{1-t^p}{p}\right)\right]\|u^+\|^2\|u^-\|^2 \\
&\geq \frac{1-s^p}{p}\langle I'(u), u^+\rangle + \frac{1-t^p}{p}\langle I'(u), u^-\rangle.
\end{aligned}$$

Hence, (2.4) holds for all $u \in H$ and $s, t \geq 0$. □

Let $s = t$ in (2.4), we can obtain the following corollary.

Corollary 2.4. *For all $u \in H$ and $t \geq 0$, there holds*

$$I(u) \geq I(tu) + \frac{1-t^p}{p}\langle I'(u), u\rangle.$$

According to Lemma 2.3 and Corollary 2.4, we have the following lemmas.

Lemma 2.5. *For all $u \in \mathcal{M}$, there holds $I(u) = \max_{s,t\geq 0} I(su^+ + tu^-)$.*

Lemma 2.6. *For all $u \in \mathcal{N}$, there holds $I(u) = \max_{t\geq 0} I(tu)$.*

By Lemmas 2.3 and 2.5, we have the following lemma.

Lemma 2.7. *For all $u \in \mathcal{M}$ and $s, t \geq 0$, there holds $I(u) \geq I(su^+ + tu^-)$, and the equality sign holds if and only if $s = t = 1$.*

Lemma 2.8. *For any $u \in H$ with $u^\pm \neq 0$, there exists a unique positive numbers pair (s_0, t_0) such that $s_0u^+ + t_0u^- \in \mathcal{M}$.*

Proof. We firstly prove that there exists positive numbers pair (s_0, t_0) such that $s_0u^+ + t_0u^- \in \mathcal{M}$. For any $u \in H$ with $u^\pm \neq 0$, let

$$g(s,t) = \langle I'(su^+ + tu^-), su^+\rangle, \qquad h(s,t) = \langle I'(su^+ + tu^-), tu^-\rangle.$$

From (1.9), one gets

$$g(s,t) = (a + b\|su^+ + tu^-\|^2)\|su^+\|^2 - \int_{\mathbb{R}^3} |su^+|^p \ln|su^+| dx; \tag{2.6}$$

$$h(s,t) = (a + b\|su^+ + tu^-\|^2)\|tu^-\|^2 - \int_{\mathbb{R}^3} |tu^-|^p \ln|tu^-| dx. \tag{2.7}$$

Let $t = s$ in (2.6), then

$$\begin{aligned}
g(s,s) &= (a + b\|su\|^2)\|su^+\|^2 - \int_{\mathbb{R}^3} |su^+|^p \ln|su^+| dx \\
&= as^2\|u^+\|^2 + bs^4\|u^+\|^4 + bs^4\|u^+\|^2\|u^-\|^2 \\
&\quad - s^p \int_{\mathbb{R}^3} |u^+|^p \ln|u^+| dx - s^p \ln s \int_{\mathbb{R}^3} |u^+|^p dx.
\end{aligned}$$

Obviously $g(s,s)$ is continuous, it is easy to verify that $g(s,s) > 0$ when $0 < s < 1$ small enough and $g(s,s) < 0$ when $s > 1$ large enough. Similarly, $h(t,t) > 0$ when $0 < t < 1$ small enough and $h(t,t) < 0$ when $t > 1$ large enough. Therefore, there exists $0 < r < R$ such that

$$g(r,r) > 0, h(r,r) > 0; \qquad g(R,R) < 0, h(R,R) < 0. \tag{2.8}$$

It follows from (2.6)–(2.8) that we have

$$\begin{aligned} &g(r,t) > 0, \quad &g(R,t) < 0, \quad &\forall t \in [r,R], \\ &h(s,r) > 0, \quad &h(s,R) < 0, \quad &\forall s \in [r,R]. \end{aligned}$$

Based on Miranda's Theorem [14], there exist $r < s_0, t_0 < R$ such that $g(s_0,t_0) = h(s_0,t_0) = 0$, which implies that $s_0u^+ + t_0u^- \in \mathcal{M}$.

Next, we prove the uniqueness of (s_0,t_0). By contradiction, we suppose that there are two pairs positive numbers $(s_1,t_1),(s_2,t_2)$ with $s_1 \neq s_2, t_1 \neq t_2$ such that $g(s_1,t_1) = g(s_2,t_2) = 0, h(s_1,t_1) = h(s_2,t_2) = 0$. Let $s = \frac{s_1}{s_2}$ and $t = \frac{t_1}{t_2}$, then $s \neq 1$ and $t \neq 1$. From Lemma 2.7, we know

$$I(s_1u^+ + t_1u^-) = I(s(s_2u^+) + t(t_2u^-)) < I(s_2u^+ + t_2u^-). \tag{2.9}$$

Similarly, one has

$$I(s_2u^+ + t_2u^-) < I(s_1u^+ + t_1u^-),$$

which contradicts (2.9). Therefore, (s_0,t_0) is unique. □

Lemma 2.9. *For any $u \in H$ with $u \neq 0$, there exists a unique positive number $t_0 > 0$ such that $t_0u \in \mathcal{N}$.*

Proof. Define a function $g(t) = \langle I'(tu), tu\rangle$ on $(0,+\infty)$, then

$$\begin{aligned} g(t) &= a\|tu\|^2 + b\|tu\|^4 - \int_{\mathbb{R}^3} |tu|^p \ln|tu|dx \\ &= at^2\|u\|^2 + bt^4\|u\|^4 - t^p \int_{\mathbb{R}^3} |u|^p \ln|u|dx - t^p \ln t \int_{\mathbb{R}^3} |u|^p dx. \end{aligned} \tag{2.10}$$

Combined with (1.8), we know

$$g(t) = t^p\langle I'(u), u\rangle + a(t^2 - t^p)\|u\|^2 + b(t^4 - t^p)\|u\|^4 - t^p \ln t \int_{\mathbb{R}^3} |u|^p dx.$$

If $u \in \mathcal{N}$, then $t_0 = 1$. Therefore, we only consider the existence of t_0 when $u \notin \mathcal{N}$. Since $4 < p < 6$ and in view of (2.3), we have $\int_{\mathbb{R}^3} |u|^p dx \leq S_p^{-1}\|u\|^p < +\infty$. It follows from (2.10) that $g(t) > 0$ for $0 < t < 1$ small enough and $g(t) < 0$ for $t > 1$ large enough. Since $g(t)$ is continuous, there exists $t_0 > 0$ such that $g(t_0) = \langle I'(t_0u), t_0u\rangle = 0$, i.e. $t_0u \in \mathcal{N}$. As a similar argument of Lemma 2.8, we can obtain the uniqueness of t_0. □

Lemma 2.10. *Assume there exists $u \in H$ with $u^\pm \neq 0$ such that $\langle I'(u), u^\pm\rangle \leq 0$, then the unique positive numbers pair (s_0,t_0) obtained in Lemma 2.8 satisfies* $0 < s_0, t_0 \leq 1$.

Proof. From Lemma 2.8, there exists a unique positive numbers pair (s_0,t_0) such that $s_0u^+ + t_0u^- \in \mathcal{M}$. Without loss of generally, we may suppose that $s_0 \geq t_0 > 0$. Since $s_0u^+ + t_0u^- \in \mathcal{M}$, we have

$$\begin{aligned} I'(s_0u^+ + t_0u^-), s_0u^+\rangle &= as_0^2\|u^+\|^2 + bs_0^4\|u^+\|^4 + bs_0^2t_0^2\|u^+\|^2\|u^-\|^2 \\ &\quad - \int_{\mathbb{R}^3} |s_0u^+|^p \ln|s_0u^+|dx. \end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^3} |s_0 u^+|^p \ln |s_0 u^+| dx &= a s_0^2 \|u^+\|^2 + b s_0^4 \|u^+\|^4 + b s_0^2 t_0^2 \|u^+\|^2 \|u^-\|^2 \\ &\leq a s_0^2 \|u^+\|^2 + b s_0^4 \|u^+\|^4 + b s_0^4 \|u^+\|^2 \|u^-\|^2.\end{aligned} \tag{2.11}$$

Since $\langle I'(u), u^+ \rangle \leq 0$, one has

$$a\|u^+\|^2 + b\|u^+\|^4 + b\|u^+\|^2\|u^-\|^2 \leq \int_{\mathbb{R}^3} |u^+|^p \ln |u^+| dx.$$

Multiplying the both sides of the above equation with $-s_0^p$, then

$$- s_0^p \int_{\mathbb{R}^3} |u^+|^p \ln |u^+| dx \leq -a s_0^p \|u^+\|^2 - b s_0^p \|u^+\|^4 - b s_0^p \|u^+\|^2 \|u^-\|^2. \tag{2.12}$$

It follows from (2.11) and (2.12) that

$$s_0^p \ln s_0 \int_{\mathbb{R}^3} |u^+|^p dx \leq a(s_0^2 - s_0^p)\|u^+\|^2 + b(s_0^4 - s_0^p)\|u^+\|^4 + b(s_0^4 - s_0^p)\|u^+\|^2\|u^-\|^2.$$

Clearly, if $s_0 > 1$, the left-hand side of the above inequality is positive, while the right-hand side of the above inequality is always negative. This is a contradiction. Therefore, $s_0 \leq 1$. Similarly, we can also obtain $t_0 \leq 1$. □

Lemma 2.11. *The following minimax characterization hold*

$$\inf_{u \in \mathcal{N}} I(u) = c = \inf_{u \in H \setminus \{0\}} \max_{t \geq 0} I(tu),$$

and

$$\inf_{u \in \mathcal{M}} I(u) = m = \inf_{u \in H, u^{\pm} \neq 0} \max_{s,t \geq 0} I(su^+ + tu^-).$$

Moreover,

$$c > 0 \quad \text{and} \quad m > 0 \quad \text{are achieved.}$$

Proof. Firstly, we prove the second equality since the first equality is similar. On one hand, it follows from Lemma 2.5 that

$$\inf_{u \in H, u^{\pm} \neq 0} \max_{s,t \geq 0} I(su^+ + tu^-) \leq \inf_{u \in \mathcal{M}} \max_{s,t \geq 0} I(su^+ + tu^-) = \inf_{u \in \mathcal{M}} I(u) = m. \tag{2.13}$$

On the other hand, for all $u \in H$ with $u^{\pm} \neq 0$, Lemma 2.8 implies that there exists a unique positive numbers pair (s_0, t_0) such that $s_0 u^+ + t_0 u^- \in \mathcal{M}$. Let $v := s_0 u^+ + t_0 u^- \in \mathcal{M}$, we have

$$m = \inf_{v \in \mathcal{M}} I(v) \leq I(s_0 u^+ + t_0 u^-) \leq \max_{s,t \geq 0} I(su^+ + tu^-),$$

which implies that

$$m = \inf_{v \in \mathcal{M}} I(v) \leq \inf_{u \in H, u^{\pm} \neq 0} \max_{s,t \geq 0} I(su^+ + tu^-). \tag{2.14}$$

Thus, the conclusion directly follows from (2.13) and (2.14).

Next, we prove that $m > 0$ is achieved. Let $\{u_n\} \subset \mathcal{M}$ be a minimizing sequence, i.e. $I(u_n) \to m$ as $n \to \infty$. In light of (1.5) and (1.8), one has

$$\begin{aligned} m + o(1) &= I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \\ &= a\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{p}\right)\|u_n\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3} |u_n|^p dx \\ &\geq a\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2. \end{aligned}$$

This implies that $\{u_n\}$ is bounded in H. Thus, up to a subsequence, there exists $u_* \in H$ such that

$$\begin{cases} u_n^{\pm} \rightharpoonup u_*^{\pm}, & \text{in } H, \\ u_n^{\pm} \to u_*^{\pm}, & \text{in } L^q(\mathbb{R}^3),\ 2 \leq q < 6, \\ u_n^{\pm}(x) \to u_*^{\pm}(x), & \text{a.e. in } \mathbb{R}^3. \end{cases}$$

Since $\{u_n\} \subset \mathcal{M}$, we have $\langle I'(u_n), u_n^{\pm}\rangle = 0$. In light of (1.9), (2.2) and (2.3), for all $q \in (p, 6)$ and taking $\sigma = q - p$ in (2.2), we have

$$\begin{aligned} aS_q^{2/q}\left(\int_{\mathbb{R}^3} |u_n^{\pm}|^q dx\right)^{2/q} &\leq a\|u_n^{\pm}\|^2 \leq a\|u_n^{\pm}\|^2 + b\|u_n\|^2\|u_n^{\pm}\|^2 \\ &\leq \int_{\mathbb{R}^3} (|u_n^{\pm}|^p \ln|u_n^{\pm}|)^+ dx \\ &\leq \frac{1}{e(q-p)}\int_{\mathbb{R}^3} |u_n^{\pm}|^q dx. \end{aligned} \tag{2.15}$$

Thus,

$$\int_{\mathbb{R}^3} |u_n^{\pm}|^q dx \geq C > 0.$$

By Lemma 2.2, we get

$$\int_{\mathbb{R}^3} |u_*^{\pm}|^q dx \geq C > 0, \tag{2.16}$$

which implies that $u_*^{\pm} \neq 0$.

Since $\langle I'(u_n), u_n\rangle = \langle I'(u_n), u_n^+\rangle + \langle I'(u_n), u_n^-\rangle = 0$, in view of (2.3) and (2.15), we have

$$a\|u_n\|^2 \leq a\|u_n\|^2 + b\|u_n\|^4 \leq \int_{\mathbb{R}^3} (|u_n|^p \ln|u_n|)^+ dx \leq C\int_{\mathbb{R}^3} |u_n|^q dx \leq CS_q^{-1}\|u_n\|^q, \tag{2.17}$$

which implies that

$$\|u_n\| \geq C > 0.$$

If $\|u_n\| \to 0$ as $n \to \infty$, from (2.17) we know $\int_{\mathbb{R}^3} |u_n|^q dx \to 0$. Using Lemma 2.2 we get $\int_{\mathbb{R}^3} |u_*|^q dx = 0$, which is in contradiction with (2.16). Therefore

$$m = \lim_{n\to\infty}\left[a(\frac{1}{2} - \frac{1}{p})\|u_n\|^2 + b(\frac{1}{4} - \frac{1}{p})\|u_n\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3} |u_n|^p dx\right] \geq C > 0.$$

By the Lebesgue dominated convergence theorem and the weak semi-continuity of norm, we have

$$\begin{aligned} a\|u_*^{\pm}\|^2 + b\|u_*\|^2\|u_*^{\pm}\|^2 &\leq \liminf_{n\to\infty}\left(a\|u_n^{\pm}\|^2 + b\|u_n\|^2\|u_n^{\pm}\|^2\right) \\ &= \liminf_{n\to\infty}\int_{\mathbb{R}^3} |u_n^{\pm}|^p \ln|u_n^{\pm}| dx \\ &= \int_{\mathbb{R}^3} |u_*^{\pm}|^p \ln|u_*^{\pm}| dx. \end{aligned}$$

Together with (1.9), it shows that

$$\langle I'(u_*), u_*^{\pm}\rangle \leq 0.$$

According to Lemma 2.10, there are two positive constants $0 < s_0, t_0 \leq 1$ such that $s_0u_*^+ + t_0u_*^- \in \mathcal{M}$. Define $\tilde{u} := s_0u_*^+ + t_0u_*^-$, it follows from (1.5), (1.8) and weak semi-continuity of norm that

$$\begin{aligned}
m &\leq I(\tilde{u}) - \frac{1}{p}\langle I'(\tilde{u}), \tilde{u}\rangle \\
&= a\left(\frac{1}{2} - \frac{1}{p}\right)\|s_0u_*^+ + t_0u_*^-\|^2 + b\left(\frac{1}{4} - \frac{1}{p}\right)\|s_0u_*^+ + t_0u_*^-\|^4 \\
&\quad + \frac{1}{p^2}\int_{\mathbb{R}^3}|s_0u_*^+ + t_0u_*^-|^p dx \\
&= a\left(\frac{1}{2} - \frac{1}{p}\right)(s_0^2\|u_*^+\|^2 + t_0^2\|u_*^-\|^2) \\
&\quad + b\left(\frac{1}{4} - \frac{1}{p}\right)(s_0^4\|u_*^+\|^4 + t_0^4\|u_*^-\|^4 + 2s_0^2t_0^2\|u_*^+\|^2\|u_*^-\|^2) \\
&\quad + \frac{1}{p^2}\left(s_0^p\int_{\mathbb{R}^3}|u_*^+|^p dx + t_0^p\int_{\mathbb{R}^3}|u_*^-|^p dx\right) \\
&\leq a\left(\frac{1}{2} - \frac{1}{p}\right)\|u_*\|^2 + b\left(\frac{1}{4} - \frac{1}{p}\right)\|u_*\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3}|u_*|^p dx \\
&\leq \liminf_{n\to\infty}\left[a\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + b\left(\frac{1}{4} - \frac{1}{p}\right)\|u_n\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3}|u_n|^p dx\right] \\
&\leq \liminf_{n\to\infty}\left[I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle\right] \\
&= m.
\end{aligned}$$

This means that $s_0 = t_0 = 1$, i.e. $\tilde{u} = u_* \in \mathcal{M}$ and $I(u_*) = m > 0$. By a similar argument as above, we have that $c > 0$ is achieved. □

Lemma 2.12. *The minimizers of* $\inf_{u\in\mathcal{N}} I(u)$ *and* $\inf_{u\in\mathcal{M}} I(u)$ *are critical points of* I.

Proof. According to Lemma 2.11, we have $u_* = u_*^+ + u_*^- \in \mathcal{M}$ and $I(u_*) = m$, Therefore it is only necessary to prove that $I'(u_*) = 0$. Arguing by contradiction, assume that $I'(u_*) \neq 0$. Then, there exist $\delta > 0$ and $\gamma > 0$ such that

$$\|I'(u)\| \geq \gamma, \qquad \forall \|u - u_*\| \leq 3\delta \text{ and } u \in H.$$

Let $D := \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{1}{2}, \frac{3}{2}\right)$, by Lemma 2.7, one has

$$\tilde{m} := \max_{(s,t)\in\partial D} I(su_*^+ + tu_*^-) < m. \tag{2.18}$$

Set $\varepsilon := \min\{(m - \tilde{m})/3, \delta\gamma/8\}$ and $S_\delta := B(u_*, \delta)$. By applying [19, Lemma 2.3], there exists a deformation $\eta \in ([0,1] \times H, H)$ such that

(i) $\eta(1, u) = u$, if $u \notin I^{-1}([m - 2\varepsilon, m + 2\varepsilon]) \cap S_{2\delta}$;

(ii) $\eta(1, I^{m+\varepsilon} \cap S_\delta) \subset I^{m-\varepsilon}$;

(iii) $I(\eta(1, u)) \leq I(u)$, $\forall u \in H$.

From (iii) and Lemma 2.7, for each $s, t > 0$ with $|s-1|^2 + |t-1|^2 \geq \delta^2/\|u_*\|^2$, one has

$$I(\eta(1, su_*^+ + tu_*^-)) \leq I(su_*^+ + tu_*^-) < I(u_*) = m. \tag{2.19}$$

By Lemma (2.3), we have $I(su_*^+ + tu_*^-) \leq I(u_*) = m$ for $s, t > 0$. According to (ii), one has

$$I(\eta(1, su_*^+ + tu_*^-)) \leq m - \varepsilon, \qquad \forall s, t > 0, |s-1|^2 + |t-1|^2 < \delta^2/\|u_*\|^2. \tag{2.20}$$

Thus, from (2.19) and (2.20), we get

$$\max_{(s,t)\in \bar{D}} I(\eta(1, su_*^+ + tu_*^-)) < m. \tag{2.21}$$

Let $h(s,t) = su_*^+ + tu_*^-$, we prove that $\eta(1, h(D)) \cap \mathcal{M} \neq \emptyset$, which contradicts the definition of m. Define

$$k(s,t) := \eta(1, h(s,t)),$$

$$\Phi(s,t) := \left(\langle I'(h(s,t)), u_*^+\rangle, \langle I'(h(s,t)), u_*^-\rangle\right) := \left(\Phi_1(s,t), \Phi_2(s,t)\right),$$

and

$$\Psi(s,t) := \left(\frac{1}{s}\langle I'(k(s,t)), (k(s,t))^+\rangle, \frac{1}{t}\langle I'(k(s,t)), (k(s,t))^-\rangle\right),$$

where

$$\begin{aligned}\Phi_1(s,t) &= \frac{1}{s}\langle I'(su_*^+ + tu_*^-), su_*^+\rangle \\ &= a(s - s^{p-1})\|u_*^+\|^2 + b(s^3 - s^{p-1})\|u_*^+\|^4 + b(st^2 - s^{p-1})\|u_*^+\|^2\|u_*^-\|^2 \\ &\quad - s^{p-1}\ln s \int_{\mathbb{R}^3} |u_*^+|^p dx,\end{aligned}$$

and

$$\begin{aligned}\Phi_2(s,t) &= \frac{1}{t}\langle I'(su_*^+ + tu_*^-), tu_*^-\rangle \\ &= a(t - t^{p-1})\|u_*^-\|^2 + b(t^3 - t^{p-1})\|u_*^-\|^4 + b(ts^2 - t^{p-1})\|u_*^+\|^2\|u_*^-\|^2 \\ &\quad - t^{p-1}\ln t \int_{\mathbb{R}^3} |u_*^-|^p dx.\end{aligned}$$

Obviously, Φ is C^1 functions. Moreover, by a direct calculation we have

$$\left.\frac{\partial \Phi_1(s,t)}{\partial s}\right|_{(1,1)} = a(2-p)\|u_*^+\|^2 + b(4-p)\|u_*^+\|^4 + b(2-p)\|u_*^+\|^2\|u_*^-\|^2 - \int_{\mathbb{R}^3} |u_*^+|^p dx,$$

and

$$\left.\frac{\partial \Phi_1(s,t)}{\partial t}\right|_{(1,1)} = 2b\|u_*^+\|^2\|u_*^-\|^2.$$

Similarly, we obtain

$$\left.\frac{\partial \Phi_2(s,t)}{\partial s}\right|_{(1,1)} = 2b\|u_*^+\|^2\|u_*^-\|^2,$$

and

$$\left.\frac{\partial \Phi_2(s,t)}{\partial t}\right|_{(1,1)} = a(2-p)\|u_*^-\|^2 + b(4-p)\|u_*^-\|^4 + b(2-p)\|u_*^+\|^2\|u_*^-\|^2 - \int_{\mathbb{R}^3} |u_*^-|^p dx.$$

Let

$$M = \begin{bmatrix} \frac{\partial \Phi_1(s,t)}{\partial s}\Big|_{(1,1)} & \frac{\partial \Phi_2(s,t)}{\partial s}\Big|_{(1,1)} \\ \frac{\partial \Phi_1(s,t)}{\partial t}\Big|_{(1,1)} & \frac{\partial \Phi_2(s,t)}{\partial t}\Big|_{(1,1)} \end{bmatrix},$$

then we have that

$$\begin{aligned} \det M &= \frac{\partial \Phi_1(s,t)}{\partial s}\Big|_{(1,1)} \times \frac{\partial \Phi_2(s,t)}{\partial t}\Big|_{(1,1)} - \frac{\partial \Phi_1(s,t)}{\partial t}\Big|_{(1,1)} \times \frac{\partial \Phi_2(s,t)}{\partial s}\Big|_{(1,1)} \\ &= \left[a(2-p)\|u_*^+\|^2 + b(4-p)\|u_*^+\|^4 + b(2-p)\|u_*^+\|^2\|u_*^-\|^2 - \int_{\mathbb{R}^3} |u_*^+|^p dx\right] \\ &\quad \times \left[a(2-p)\|u_*^-\|^2 + b(4-p)\|u_*^-\|^4 + b(2-p)\|u_*^+\|^2\|u_*^-\|^2 - \int_{\mathbb{R}^3} |u_*^-|^p dx\right] \\ &\quad - 4b^2\|u_*^+\|^4\|u_*^-\|^4 \\ &> 0. \end{aligned}$$

Hence, the solution of equation (1.1) is the unique isolated zero point of $\Phi(s,t)$. Then, the topological degree theory [6, 21] implies that $\deg(\Phi, D, 0) = 1$. Combining (2.18) with (i), we have that $h = k$ on ∂D, then we obtain

$$\deg(\Phi, D, 0) = \deg(\Psi, D, 0) = 1.$$

So, $\Psi(s_0, t_0) = 0$ for some $(s_0, t_0) \in D$, and

$$\eta(1, h(s_0, t_0)) = k(s_0, t_0) \in \mathcal{M},$$

which is a contradiction with (2.21). So we get that $I'(u_*) = 0$. Similarly, we can prove that any minimizer of $\inf_{u\in\mathcal{N}} I(u)$ are a critical point of $I(u)$. □

3 Proof of theorems

Firstly, we prove the existence of positive ground state solutions for equation (1.1).

Proof of Theorem 1.1. According to Lemma 2.11 and Lemma 2.12, there exists $\bar{u} \in \mathcal{N}$ such that

$$I(\bar{u}) = c, \qquad I'(\bar{u}) = 0.$$

Now, we only need to prove that $\bar{u}$ is a positive solution of equation (1.1). Indeed, replacing $I(u)$ with the functional

$$\begin{aligned} I(\bar{u}) = \frac{a}{2}\int_{\mathbb{R}^3}(|\nabla \bar{u}|^2 + V(x)\bar{u}^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}(|\nabla \bar{u}|^2 + V(x)\bar{u}^2)dx\right) \\ - \frac{1}{p}\int_{\mathbb{R}^3} |\bar{u}^+|^p \ln|\bar{u}^+| dx + \frac{1}{p^2}\int_{\mathbb{R}^3} |\bar{u}^+|^p dx. \end{aligned}$$

In this way we can get a solution $\bar{u}$ such that

$$\left(a + b\int_{\mathbb{R}^3} |\nabla \bar{u}|^2 + V(x)\bar{u}^2 dx\right)[-\Delta \bar{u} + V(x)\bar{u}] = |\bar{u}^+|^{p-2}\bar{u}^+ \ln|\bar{u}^+|, \qquad x \in \mathbb{R}^3. \tag{3.1}$$

Multiplying the both sides of (3.1) with u^-, we deduce that

$$a\|\bar{u}^-\|^2 + b\|\bar{u}^-\|^4 + b\|\bar{u}^+\|^2\|\bar{u}^-\|^2 = 0.$$

It follows that $\bar{u}^-(x) = 0$, and then $\bar{u}(x) \geq 0$ for a.e. $x \in \mathbb{R}^3$. The regularity theory of elliptic equation implies that $\bar{u} \in C^2(\mathbb{R}^3)$ is nonnegative classical solution of equation (1.1). Since $p \in (4,6)$, we know that $\lim_{\bar{u}\to 0^+} \bar{u}^{p-1}\ln \bar{u} = 0$. It makes sense to denote $\bar{u}^{p-1}\ln \bar{u} = 0$ for $\bar{u} = 0$. Let $\Omega^+ = \{\bar{u} \in \mathbb{R}^3 : \bar{u}(x) \geq 0\}$. Then $\bar{u}(x)$ is positive solution in $\mathbb{R}^3$ if $\partial\Omega^+ = \emptyset$. In the following, we prove that $\partial\Omega^+ = \emptyset$. Otherwise, let $x_0 \in \partial\Omega^+$ and $B_\rho(x_0) = \{x \in \mathbb{R}^3 : |x - x_0| < \rho\}$ with small $\rho > 0$. Define

$$\alpha = \left(a + b\int_{\mathbb{R}^3} |\nabla\bar{u}|^2 + V(x)\bar{u}^2 dx\right) > 0,$$

and

$$c(x) = \left(a + b\int_{\mathbb{R}^3} |\nabla\bar{u}|^2 + V(x)\bar{u}^2 dx\right) V(x) - \bar{u}^{p-1}\ln|\bar{u}|.$$

Then, $\bar{u}|_{B_\rho(x_0)}$ is nontrivial solution of the following boundary value problem

$$-\alpha\triangle v + c(x)v = 0, \qquad x \in B_\rho(x_0) \quad \text{and} \quad v(x) = \bar{u}(x) \quad \text{for } x \in \partial B_\rho(x_0).$$

Under the assumptions, we see that $c(x) > 0$ in $B_\rho(x_0)$ for $\rho > 0$ small enough. By the maximum principle [18], we see that $\bar{u}(x) > 0$ for all $x \in B_\rho(x_0)$, which contradicts to that $x_0 \in \partial\Omega^+$. In conclusion, $\bar{u}$ is a positive ground state solution of equation (1.1). Thus the proof of Theorem 1.1 is completed. □

Secondly, we verify that equation (1.1) has a ground state sign-changing solution with precisely two nodal domains.

Proof of Theorem 1.2. In light of Lemma 2.11 and Lemma 2.12, there exists $u_* \in \mathcal{M}$ such that

$$I(u_*) = m, \qquad I'(u_*) = 0. \tag{3.2}$$

Now, we show that u_* has exactly two nodal domains. Set $u_* = u_1 + u_2 + u_3$, where

$$u_1 \geq 0, \qquad u_2 \leq 0, \qquad \Omega_1 \cap \Omega_2 = \emptyset, \qquad u_1|_{\mathbb{R}^3\setminus\Omega_1} = u_2|_{\mathbb{R}^3\setminus\Omega_2} = u_3|_{\Omega_1\cup\Omega_2} = 0, \tag{3.3}$$

$$\Omega_1 := \{x \in \mathbb{R}^3 : u_1 > 0\}, \qquad \Omega_2 := \{x \in \mathbb{R}^3 : u_2 < 0\},$$

and Ω_1, Ω_2 are connected open subsets of $\mathbb{R}^3$. Letting $v = u_1 + u_2$, then $v^+ = u_1$ and $v^- = u_2$, i.e. $v^\pm \neq 0$. Note that $I'(u_*) = 0$, by a straightforward calculation, we can obtain

$$\langle I'(v), v^+\rangle = -b\|v^+\|^2\|u_3\|^2, \tag{3.4}$$

and

$$\langle I'(v), v^-\rangle = -b\|v^-\|^2\|u_3\|^2. \tag{3.5}$$

It follows from (1.5), (1.8), (2.4), (3.2)–(3.5) that

$$\begin{aligned} m = I(u_*) &= I(u_*) - \frac{1}{p}\langle I'(u_*), u_*\rangle \\ &= I(v) + I(u_3) + \frac{b}{2}\|v\|^2\|u_3\|^2 - \frac{1}{p}\left[\langle I'(v), v\rangle + \langle I'(u_3), u_3\rangle + 2b\|v\|^2\|u_3\|^2\right] \end{aligned}$$

$$
\begin{aligned}
&\geq \sup_{s,t\geq 0}\left[I(sv^+ + tv^-) + \frac{1-s^p}{p}\langle I'(v), v^+\rangle + \frac{1-t^p}{p}\langle I'(v), v^-\rangle\right] \\
&\quad + I(u_3) - \frac{1}{p}\langle I'(v), v\rangle - \frac{1}{p}\langle I'(u_3), u_3\rangle \\
&\geq \sup_{s,t\geq 0}\left[I(sv^+ + tv^-) + \frac{bs^p}{p}\|v^+\|^2\|u_3\|^2 + \frac{bt^p}{p}\|v^-\|^2\|u_3\|^2\right] \\
&\quad + a\left(\frac{1}{2}-\frac{1}{p}\right)\|u_3\|^2 + b\left(\frac{1}{4}-\frac{1}{p}\right)\|u_3\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3}|u_3|^p dx \\
&\geq \max_{s,t\geq 0} I(sv^+ + tv^-) + a\left(\frac{1}{2}-\frac{1}{p}\right)\|u_3\|^2 \\
&\geq m + a\left(\frac{1}{2}-\frac{1}{p}\right)\|u_3\|^2,
\end{aligned}
$$

which implies that $u_3 = 0$. Therefore, u_* has exactly two nodal domains.

Next, we show that the energy of sign-changing solution of equation (1.1) is strictly larger than twice of the energy of positive ground state solution. By Lemma 2.9, there exist $s_*, t_* > 0$ such that $s_*u_*^+, t_*u_*^- \in \mathcal{N}$. Then it follows from (3.2) and Lemma 2.7 that

$$
\begin{aligned}
m = I(u_*) &\geq I(s_*u_*^+ + t_*u_*^-) \\
&= I(s_*u_*^+) + I(t_*u_*^-) + 2bs_*^2t_*^2\|u_*^+\|^2\|u_*^-\|^2 \\
&> I(s_*u_*^+) + I(t_*u_*^-) \geq 2c > 0.
\end{aligned}
$$

To sum up, u_* is a ground state sign-changing solution of equation (1.1) with precisely two nodal domains. Besides, $m > 2c$. Then, the proof of Theorem 1.2 is completed. □

Finally, we prove that equation (1.1) has a sequence of solutions to infinity by the following symmetric mountain pass theorem:

Theorem 3.1 ([15, Theorem 9.12])**.** *Let E be an infinite dimensional Banach space, and let $I \in C^1(E, \mathbb{R})$ be even, satisfying (PS) condition and $I(0) = 0$. If $E = V \bigoplus X$, where V is finite dimensional and I satisfies*

- *(i) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$,*
- *(ii) for each finite dimensional subspace $\tilde{E} \subset E$, there exists an $R = R(\tilde{E})$ such that $I \leq 0$ on $\tilde{E}\backslash B_{R(\tilde{E})}$,*

then I possesses an unbounded sequence critical values.

Lemma 3.2. *Assume that (V_1)–(V_2) hold, then I satisfies (PS) condition.*

Proof. Let $\{u_n\} \subset H$ be a sequence with $\{I(u_n)\}$ bounded and $I'(u_n) \to 0$. We first claim that $\{u_n\}$ is bounded in H. Indeed,

$$
\begin{aligned}
C + o(1)\|u_n\| &\geq I(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \\
&= a\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2 + b\left(\frac{1}{4}-\frac{1}{p}\right)\|u_n\|^4 + \frac{1}{p^2}\int_{\mathbb{R}^3}|u_n|^p dx \\
&\geq a\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2.
\end{aligned}
$$

This implies that $\{u_n\}$ is bounded in H. Going if necessary up to subsequence, we may assume that there exists $u \in H$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H, \\ u_n \to u, & \text{in } L^q(\mathbb{R}^3),\ 2 \le q < 6, \\ u_n(x) \to u(x), & \text{a.e. in } \mathbb{R}^3. \end{cases} \tag{3.6}$$

By $\lim_{n\to\infty} \|\langle I'(u_n), u_n - u\rangle\| \le \lim_{n\to\infty} \|I'(u_n)\| \|u_n - u\| = 0$ and $\|\langle I'(u), u_n - u\rangle\| \le \lim_{n\to\infty} \|I'(u_n)\| \|u_n - u\| = 0$, we deduce that

$$\langle I'(u_n) - I'(u), u_n - u\rangle = \langle I'(u_n), u_n - u\rangle - \langle I'(u), u_n - u\rangle \to 0, \quad \text{as } n \to \infty.$$

On the other hand, it follows from (3.6) and Hölder's inequality that

$$\begin{aligned} \lim_{n\to\infty} \langle u, u_n - u\rangle &= \lim_{n\to\infty} \int_{\mathbb{R}^3} \nabla u[\nabla u_n - \nabla u] + V(x)u(u_n - u)dx \\ &\le \int_{\mathbb{R}^3} \nabla u[\nabla u - \nabla u]dx + \left(\int_{\mathbb{R}^3} (V(x)u)^2 dx\right)^{1/2} |u_n - u|_2 \\ &= 0. \end{aligned}$$

Therefore, by some preliminary calculations, one has

$$\begin{aligned} &(a + b\|u_n\|^2)\|u_n - u\|^2 \\ &\quad= (a + b\|u_n\|^2) \int_{\mathbb{R}^3} [\nabla u_n \nabla(u_n - u) + V(x)u_n(u_n - u)]dx \\ &\qquad - (a + b\|u\|^2 + b\|u_n\|^2 - b\|u\|^2) \int_{\mathbb{R}^3} [\nabla u \nabla(u_n - u) + V(x)u(u_n - u)]dx \\ &\quad= \langle I'(u_n), u_n - u\rangle + b(\|u\|^2 - \|u_n\|^2) \int_{\mathbb{R}^3} [\nabla u \nabla(u_n - u) + V(x)u(u_n - u)]dx \\ &\qquad + \int_{\mathbb{R}^3} |u_n|^{p-2}u_n(u_n - u)\ln|u_n|dx - \langle I'(u), u_n - u\rangle - \int_{\mathbb{R}^3} |u|^{p-2}u(u_n - u)\ln|u|dx \\ &\quad= \langle I'(u_n) - I'(u), u_n - u\rangle + b(\|u\|^2 - \|u_n\|^2)\langle u, u_n - u\rangle \\ &\qquad + \int_{\mathbb{R}^3} (|u_n|^{p-2}u_n \ln|u_n| - |u|^{p-2}u\ln|u|)(u_n - u)dx. \end{aligned}$$

We obtain the conclusion if the last term of the above formula tend to zero as $n \to +\infty$. Indeed, in view of (1.7), (3.6) and Hölder's inequality, for any $\varepsilon > 0$ small enough we deduce that

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} (|u_n|^{p-2}u_n \ln|u_n| - |u|^{p-2}u\ln|u|)(u_n - u)dx \right| \\ &\qquad \le \int_{\mathbb{R}^3} \left(||u_n|^{p-1}\ln|u_n|| + ||u|^{p-1}\ln|u||\right) |u_n - u|dx \\ &\qquad \le \int_{\mathbb{R}^3} \left[\varepsilon(|u_n| + |u|) + C_\varepsilon(|u_n|^{q-1} + |u|^{q-1})\right] |u_n - u|dx \\ &\qquad \le 4\varepsilon\left(|u_n|_2^2 + |u|_2^2\right) + C_\varepsilon\left(|u_n|_q^{q-1} + |u|_q^{q-1}\right)|u_n - u|_q \\ &\qquad \le \varepsilon C + C_\varepsilon|u_n - u|_q. \end{aligned}$$

These estimates show that $u_n \to u$ in H, so I satisfies (PS) condition. □

Proof of Theorem 1.3. In Theorem 3.1, let $E = H$ and the functional I given by (1.5). By Lemma 3.2, the functional I satisfies (PS) condition, so we just need to verify that I satisfies conditions (i) and (ii) of Theorem 3.1. Note that

$$\begin{aligned} I(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \ln|u| dx \\ &\geq \frac{a}{2}\|u\|^2 - \frac{1}{p}\int_{\{x:|u|\geq 1\}} |u|^p \ln|u| dx \\ &\geq \frac{a}{2}\|u\|^2 - C_1\|u\|^{p+\sigma}, \end{aligned}$$

where $0 < \sigma < 6 - p$. Thus, we can choose $\rho > 0$ and $\alpha > 0$ small enough such that $I|_{\partial B_\rho} \geq \alpha > 0$.

We suppose that $\tilde{E}$ is a finite dimensional subspace of H, and for $u \in \tilde{E}\backslash\{0\}$, define $v = u/\|u\|$, then $\|v\| = 1$. one has

$$\begin{aligned} I(u) &= \|u\|^p\left(\frac{a}{2}\|u\|^{2-p} + \frac{b}{4}\|u\|^{4-p} - \frac{1}{p}\int_{\mathbb{R}^3} |v|^p \ln|v| dx + \frac{1}{p^2}\int_{\mathbb{R}^3} |v|^p dx - \frac{1}{p}(\ln\|u\|)\int_{\mathbb{R}^3} |v|^p dx\right) \\ &\leq \|u\|^p\left(\frac{a}{2}\|u\|^{2-p} + \frac{b}{4}\|u\|^{4-p} + C_1\|v\|^2 + C_2\|v\|^q + \frac{S_p^{-1}}{p^2}\|v\|^p - \frac{1}{p}(\ln\|u\|)\int_{\mathbb{R}^3} |v|^p dx\right) \\ &= \|u\|^p\left(\frac{a}{2}\|u\|^{2-p} + \frac{b}{4}\|u\|^{4-p} + C - \frac{1}{p}(\ln\|u\|)\int_{\mathbb{R}^3} |v|^p dx\right). \end{aligned}$$

Thus, there exists an $R = R(\tilde{E})$ large enough such that $I \leq 0$ on $\tilde{E}\backslash B_{R(\tilde{E})}$.

To sum up, all conditions of Theorem 3.1 are satisfied. Therefore, equation (1.1) owns a sequence of solutions $\{u_n\}$ with $I(u_n) \to +\infty$ as $n \to \infty$. This completes the proof of Theorem 1.3. □

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3-dimensional piecewise linear and quadratic vector fields with invariant spheres

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Abstract. We consider the class $\mathcal{X}$ of 3-dimensional piecewise smooth vector fields that admit a first integral which leaves invariant any sphere centered at the origin. In this class, we prove that a linear vector field does not admit isolated invariant cones. Moreover, we provide the existence of at least ten 1-parameter families of crossing closed trajectories for quadratic vector fields in $\mathcal{X}$.

Keywords: piecewise smooth vector fields with invariant spheres, invariant cones, 1-parameter families of closed trajectories.

2020 Mathematics Subject Classification: 34C07, 34C23, 37C27.

1 Introduction

Differential equations and dynamical systems can be used to model natural phenomena and we can obtain information about it from their solutions. An interesting tool used to understand the behavior of the solutions of a dynamical system is the existence of first integrals because, when they exist, the trajectories of the corresponding vector field remain restricted to the level surfaces of these functions. We say that a n-dimensional differential system is completely integrable when it has $n-1$ independent first integrals and the orbits of it are obtained just intersecting the level sets of the first integrals. Moreover, if it has less than $n-1$ first integrals, it is said to be partially integrable. The $2n$-dimensional Hamiltonian systems are particular cases of partially integrable systems, for which we commonly study their behavior restricted to their invariant level sets. The study of Hamiltonian systems has many applications and it is very important in mechanics, for example, as we can see in [28].

Observe that, if the system restricted to an invariant level set of the first integral has a hyperbolic closed trajectory, then the original system has a 1-parameter family of hyperbolic

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periodic orbits. As we will work with 3-dimensional piecewise smooth vector fields having a first integral that keeps invariant all the spheres centered at origin, in fact we deal with 1-parameter radial families. For more details about how to consider 3-dimensional smooth vector fields (resp. 3-dimensional piecewise smooth vector fields) with invariant spheres as 1-parameter radial family see for instance Section 5 of [4] (resp. [5]). In [6] it was proved that the behavior of a homogeneous vector field restricted to an invariant sphere of radius $\rho = 1$ is topologically equivalent to the behavior of the same system restricted to any other level. So, when a homogeneous vector field restricted to an invariant sphere has a limit cycle (resp. a center), the 3-dimensional vector field has an isolated (resp. non-isolated) invariant cone fulfilled of closed trajectories. On the other hand, the behavior of non-homogeneous vector fields could be totally different in distinct levels of invariant spheres (see again [6]). In this case, each hyperbolic closed trajectory restricted to an invariant sphere of radius ρ generates a 1-parameter radial family of closed trajectories of the 3-dimensional vector field near the sphere of radius ρ. So, it has locally a topological invariant cylinder near the sphere of radius ρ fulfilled of closed trajectories. In general, it is very difficult to classify the invariant surfaces generated by these 1-parameter families, as they can have very different behavior depending on the vector field. Understanding it certainly depends on the knowledge on the behavior of the vector fields restricted to each invariant sphere.

As these invariant surfaces are generated by closed trajectories of the restricted vector field, this problem is strictly related to the Hilbert's 16th problem, presented by D. Hilbert in 1900, at the International Congress of Mathematicians, in Paris. The second part of the Hilbert's 16th problem asks for an estimation of the maximal number of limit cycles that a planar polynomial vector field can have, being one of the most important open problems in Qualitative Theory of Ordinary Differential Equations and Dynamical Systems. For more details we refer the reader to [20].

In the last years, many classes of piecewise smooth dynamical systems have also been studied and a rigorous formulation of their qualitative properties was given by Filippov, in [15]. This theory is very important in many areas of science, see for instance [12]. Note that, the Hilbert's 16th problem has been extended to piecewise polynomial vector fields in a natural way (see for example [16, 24]). In part of this paper we will analyze the existence of (crossing) invariant cones for piecewise linear and quadratic vector fields. This dynamics also appears in 3-dimensional piecewise linear systems as in [7–9].

In this work, we consider 3-dimensional piecewise differential vector fields with a separation set given by $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$, that is

$$Y(x, y, z) = \begin{cases} X^+(x, y, z), & z \geq 0, \\ X^-(x, y, z), & z \leq 0. \end{cases} \tag{1.1}$$

As Y can be multi-valued in Σ, we will follow the Filippov's convention on the escaping and sliding regions, see again [15].

In the piecewise smooth case, as in the smooth one, the integrability of the vector fields $X^{\pm}$ is an important tool used to understand the behavior of the trajectories of $Y = (X^+, X^-)$, in the classification of phase portraits, and also to answer questions related to the existence of crossing limit cycles (i.e. isolated crossing periodic orbits). See Section 2.2 and also [27] for more details. Furthermore, when both $X^{\pm}$ have the same first integral the dimension of the phase space where the trajectories of the piecewise smooth vector field $Y = (X^+, X^-)$ are defined is reduced by one. This property has motivated us to study 3-dimensional piecewise smooth vector fields partially integrable (that is, having both $X^{\pm}$ the same first integral $H : \mathbb{R}^3 \to \mathbb{R}$) restricted to invariant level sets of H, as we explain on the following.

Let $\mathfrak{X}$ be the class of smooth vector fields $X : \mathbb{R}^3 \to \mathbb{R}^3$ that admits $H(x, y, z) = x^2 + y^2 + z^2$ as a first integral. This class was previously studied in [6]. Note that all the spheres centered at the origin with radius ρ, $\mathbb{S}^2_\rho = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = \rho^2\}$, are invariant by the flow of $X \in \mathfrak{X}$. We denote by $\mathfrak{X}_n$ (resp. $\mathfrak{X}^H_n$) the class of polynomials (resp. homogeneous polynomials) vector fields of degree n in $\mathfrak{X}$. In this work, we consider the class of piecewise differential vector fields given by (1.1) such that $X^{\pm} \in \mathfrak{X}$. We denote this class by $\mathcal{X}$ and by $\mathcal{X}_n$ (resp. $\mathcal{X}^H_n$) when $X^{\pm} \in \mathfrak{X}_n$ (resp. $X^{\pm} \in \mathfrak{X}^H_n$). Hence, if $Y \in \mathcal{X}$, then any sphere centered at the origin is invariant by the flow of the piecewise differential system Y. Observe that we can consider invariant ellipsoids instead of invariant spheres. Although all the results can be easily generalized to this case, we have preferred not to do it here, to avoid repetitions. In the sequel, we describe the results that we have obtained for piecewise linear and quadratic (homogeneous and nonhomogeneous) differential systems in $\mathcal{X}$. We remark that, in general, the 3-dimensional homogeneous vector fields will not be when we consider them projected to a 2-dimensional space.

Before introducing our main results, we recall some properties about homogeneous vector fields $X \in \mathfrak{X}^H$ proved in [6]. $X \in \mathfrak{X}_1$ is homogeneous and it writes in the form

$$X(\mathrm{x}; a_1, a_2, a_3) = (-a_1 y - a_2 z, a_1 x - a_3 z, a_2 x + a_3 y), \tag{1.2}$$

where, $\mathrm{x} = (x, y, z)$. Moreover, (1.2) has (generically) only a line of equilibrium points passing through the origin. Further, when we consider the restriction of (1.2) to the invariant spheres $\mathbb{S}^2_\rho$, we conclude that (1.2) has only two equilibrium points on each sphere which are centers and antipodals of each other (see Lemma 3.1, for more details). It means that the 3-dimensional smooth vector field (1.2) has a continuous of invariant cones fulfilled of non-isolated closed trajectories. In Proposition 4.3 we show that a quadratic homogeneous vector field $X \in \mathfrak{X}^H_2$ can present an isolated invariant cone, fulfilled of closed trajectories, showing an important difference between linear and quadratic homogeneous vector fields in the class $\mathfrak{X}$.

Using (1.2) we can see that each 3-dimensional piecewise linear system $Y = (X^+, X^-) \in \mathcal{X}_1$ is of the form

$$Y(x, y, z) = \begin{cases} X^+(\mathrm{x}; a_1^+, a_2^+, a_3^+), & z \geq 0, \\ X^-(\mathrm{x}; a_1^-, a_2^-, a_3^-), & z \leq 0, \end{cases} \tag{1.3}$$

with

$$X^{\pm}(\mathrm{x}; a_1^{\pm}, a_2^{\pm}, a_3^{\pm}) = (-a_1^{\pm} y - a_2^{\pm} z, a_1^{\pm} x - a_3^{\pm} z, a_2^{\pm} x + a_3^{\pm} y). \tag{1.4}$$

As explained in Section 2.3, we use the stereographic projection to study the local behavior of $Y \in \mathcal{X}$ restricted to the invariant spheres. Moreover, the projection of a linear (resp. quadratic) vector field defined on an invariant sphere is a quadratic (resp. cubic) planar vector field. We observe that they lose the property of homogeneity once projected. Usually, the behavior of piecewise smooth vector fields is richer than the behavior of the smooth ones. This property made us to look for isolated invariant cones in $\mathcal{X}_1$. However, the next result proves that they do not exist.

Theorem 1.1. *No piecewise differential system $Y \in \mathcal{X}_1$, given by (1.3), admits an isolated invariant cone.*

We prove it in Section 3, where we also show the possible phase portraits of (1.3), restricted to the invariant sphere $\mathbb{S}^2_\rho$, with respect to the admissibility of its equilibria (see Figures 3.3

and 3.4). We point out that the existence of crossing invariant cones for piecewise linear vector fields which are continuous in the separation set Σ was studied in [8,9]. But the results cannot be applied to our study because the continuity condition is not satisfied.

Inspired by the homogeneity property of the linear vector fields in $\mathcal{X}$, we study some families in $\mathcal{X}_2^H$ in Section 4, where we prove that they can present isolated and non-isolated crossing invariant cones, showing an important difference between piecewise linear and quadratic homogeneous vector fields in $\mathcal{X}$. We prove it considering the restriction of a piecewise smooth vector field $Y \in \mathcal{X}_2^H$ to the sphere of radius $\rho = 1$ and showing that they can present centers for some specific values of the coefficients and crossing limit cycles for others. Another difference can be observed when we compare the piecewise quadratic homogeneous vector fields defined on $\mathbb{S}_1^2$ and on $\mathbb{R}^2$. To see it, we recall briefly the concept of reversible vector field defined in open regions of $\mathbb{R}^m$. Let $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ be a $\mathcal{C}^r$-involution. It means that $\varphi \circ \varphi(\mathrm{x}) = \mathrm{x}$, where $\mathrm{x} \in \mathbb{R}^m$. Let $\mathrm{Fix}(\varphi) = \{\mathrm{x}; \varphi(\mathrm{x}) = \mathrm{x}\}$. We say that a differential vector field X, defined in $\mathbb{R}^m$, is φ-reversible if $D\varphi \circ X = -X \circ \varphi$, where $D\varphi$ denotes the Jacobian matrix of φ. We say that X is reversible with respect to a line (resp. a point) when $\mathrm{Fix}(\varphi)$ is a line (resp. a point). We refer [23], for an interesting survey about reversible differential systems. We recall that any quadratic homogeneous vector field defined in $\mathbb{R}^2$ is reversible with respect to the origin and, because of that, it does not have an equilibrium point of center type (see [2]). Thus, we cannot consider the center-focus problem for piecewise quadratic homogeneous vector fields on the plane. Note that the concept of reversibility can also be considered for piecewise smooth vector fields. For more details see Section 2.2.

Finally, we have also analyzed the local behavior of a piecewise quadratic vector field $\mathcal{X}_2$, proving the following result.

Theorem 1.2. *There exist at least ten* 1*-parameter radial families of invariant crossing closed trajectories in the quadratic family* $\mathcal{X}_2$*, near the radius* $\rho = 1$.

For proving Theorem 1.2, see Section 5, we consider the restriction of a piecewise smooth vector field $Y \in \mathcal{X}_2$ to the invariant sphere of radius $\rho = 1$ and we show that it has 10 hyperbolic crossing limit cycles on the sphere $\mathbb{S}_1^2$. Since these crossing limit cycles are hyperbolic on $\mathbb{S}_1^2$, they are normally hyperbolic with respect to the radial direction. This implies that $Y \in \mathcal{X}_2$ has at least ten 1-parameter radial families of crossing periodic orbits which cross the sphere of radius ρ in isolated closed trajectories, with $1 - \varepsilon < \rho < 1 + \varepsilon$ for ε sufficiently small. So, the 3-dimensional vector field $Y \in \mathcal{X}_2$ has invariant surfaces, foliated by crossing closed trajectories, which are locally topologically equivalent to cylinders. The global structure of each invariant surface is due to the birth or death of limit cycles. For example, this surface is topologically equivalent to a sphere when we have exactly two Hopf points in $\mathbb{S}_{\rho_*}^2$ and $\mathbb{S}_{\rho^*}^2$, being $\rho_* < 1 < \rho^*$.

This paper is structured as follows. Section 2 is devoted to recalling the tools used to prove our main results. In Section 3 we study piecewise linear vector fields with invariant spheres and we also prove Theorem 1.1. In Section 4 we give some families of centers for piecewise continuous quadratic homogeneous vector fields, in the sphere $\mathbb{S}_1^2$. Finally, in Section 5 we prove Theorem 1.2.

2 Preliminary results

This section is dedicated to recall some concepts and bifurcation techniques for piecewise smooth vector fields, that we use in the proofs of the results of this paper. Firstly, we recall the integrability concept and the Filippov's convention for piecewise smooth vector fields. After that, we consider a smooth vector field $X : \mathbb{R}^3 \to \mathbb{R}^3$ having $H(x, y, z) = x^2 + y^2 + z^2$ as a first integral and define a piecewise smooth vector field with the same property. Considering that the center-focus problem and local cyclicity will be studied projecting each 3-dimensional piecewise smooth vector field defined on the invariant sphere into a planar one, we also recall some definitions and the computation algorithm of the center conditions (or Lyapunov constants) for planar piecewise smooth vector fields.

2.1 Integrability

Let $P : \mathbb{R}^m \to \mathbb{R}^m$ such that $P(\mathrm{x}) = (P_1(\mathrm{x}), \ldots, P_m(\mathrm{x}))$, where $\mathrm{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and P_i, $i = 1, \ldots, m$, are polynomials in the variables x_i with real coefficients. Let n be the maximum between the degrees of P_i, $i = 1, \ldots, m$ and consider an m-dimensional differential system

$$\dot{\mathrm{x}} = P(\mathrm{x}). \tag{2.1}$$

Let $U \subset \mathbb{R}^m$ be an open subset. If there exists a non-constant analytic function $H : U \to \mathbb{R}$ such that

$$\langle P(\mathrm{x}), \nabla H(\mathrm{x}) \rangle = \sum_{i=1}^{m} P_i(\mathrm{x}) \frac{\partial H}{\partial x_i}(\mathrm{x}) = 0, \quad \text{for } \mathrm{x} \in U,$$

then (2.1) is *partially integrable* on U and H is a *first integral* of (2.1) on U. Moreover, if P has $m - 1$ independent first integrals then P is called a *completely integrable* system. In [14], it was proved that any m-dimensional linear system has $m - 1$ independent first integrals and then this is an example of a class of completely integrable systems.

It is worth to say that when a system P is completely integrable its trajectories are determined by the intersection of the level sets of its first integrals, see [13] for more details about it. Moreover, each $X \in \mathfrak{X}$ has at least one first integral and, in Section 3 we will see that the key point for the proof of Theorem 1.1 is the existence of a second first integral for $X^{\pm} \in \mathfrak{X}_1$ and to have a good knowledge of how the levels of these first integrals interact with the separation curve of piecewise system (1.3).

2.2 Filippov vector fields

In this subsection we recall the definition of a piecewise smooth vector field under the Filippov's convention (see [15] for more details). We restrict our attention to piecewise smooth vector fields defined in $\mathbb{R}^m$, the same definitions can be extended easily to m-dimensional manifolds.

Let $\mathrm{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and consider $f : \mathbb{R}^m \to \mathbb{R}$ a $\mathcal{C}^r$-class function such that $0 \in \mathbb{R}$ is a regular value of f. Therefore, $\Sigma = f^{-1}(0) = \{\mathrm{x} \in \mathbb{R}^m : f(\mathrm{x}) = 0\}$ is an embedded codimension one submanifold of $\mathbb{R}^m$. Consider $\Sigma^+ = f^{-1}([0, +\infty)) = \{\mathrm{x} \in \mathbb{R}^m : f(\mathrm{x}) \geq 0\}$, $\Sigma^- = f^{-1}((-\infty, 0]) = \{\mathrm{x} \in \mathbb{R}^m : f(\mathrm{x}) \leq 0\}$ and the piecewise smooth vector field with separation set Σ defined by

$$Y(\mathrm{x}) = \begin{cases} X^+(\mathrm{x}), & \mathrm{x} \in \Sigma^+, \\ X^-(\mathrm{x}), & \mathrm{x} \in \Sigma^-, \end{cases} \tag{2.2}$$

where $X^{\pm}$ are smooth vector fields defined on $\Sigma^{\pm}$. The equilibrium points of X^+ and X^- located in Σ^+ and Σ^-, respectively, are called admissible (or visible) equilibrium points or simply equilibrium points of (2.2). On the other hand, the equilibrium points of X^+ and X^-

located in Σ^- and Σ^+, respectively, are called non-admissible (or invisible) equilibrium points of (2.2).

The Lie derivative of f with respect to the vector field $X^\pm$ at the point $p \in \Sigma$ is defined by $X^\pm f(p) = X^\pm(p) \cdot \nabla f(p)$, where the dot stands for the scalar or usual product on $\mathbb{R}^m$. The successive Lie derivatives are given by $(X^\pm)^n f(p) = X^\pm(p) \cdot \nabla (X^\pm)^{n-1} f(p)$, $n \geq 2$. When $X^+ f(p) = X^- f(p)$, for all $p \in \Sigma$, we say that (2.2) is a refractive system (on Σ). For more details about refractive systems we refer the reader to [3,5].

On the following, we recall the definitions of tangency points and tangency sets of (2.2). We say that $p \in \Sigma$ is a fold point of Y if $X^+ f(p) = 0, (X^+)^2 f(p) \neq 0$ and $X^- f(p) \neq 0$ (or $X^- f(p) = 0, (X^-)^2 f(p) \neq 0$ and $X^+ f(p) \neq 0$). Hence, p is a fold-fold point when $X^+ f(p) = 0$, $X^- f(p) = 0$, $(X^+)^2 f(p) \neq 0$, and $(X^-)^2 f(p) \neq 0$. We also define the tangency set of $X^\pm$ with Σ by $S_{X^\pm} = \{p \in \Sigma : \ X^\pm f(p) = 0\}$ and the tangency set of Y by $S_Y = S_{X^+} \cup S_{X^-}$.

As usual, we consider the crossing region $\Sigma^c = \{p \in \Sigma : (X^+ f(p))(X^- f(p)) > 0\}$, the sliding region $\Sigma^s = \{p \in \Sigma : X^+ f(p) < 0, \ X^- f(p) > 0\}$ and the escaping region $\Sigma^e = \{p \in \Sigma : X^+ f(p) > 0, \ X^- f(p) < 0\}$. So Σ is the disjoint union $\Sigma^c \cup \Sigma^s \cup \Sigma^e \cup S_Y$ and following the Filippov's convention we define the Filippov vector field $F_Y(p)$ on $\Sigma^s \cup \Sigma^e$ by

$$F_Y(p) = \frac{1}{X^- f(p) - X^+ f(p)} \left(X^- f(p) X^+(p) - X^+ f(p) X^-(p) \right).$$

We also recall that a crossing trajectory is an orbit that have isolated crossing points of intersection with the separation set Σ. Moreover, a crossing limit cycle is an isolated crossing periodic orbit.

Finally, we say that (2.2) is time φ-reversible if $\mathrm{Fix}(\varphi) \subset \Sigma^c$ and $D\varphi \circ Y = -Y \circ \varphi$, where φ is an $\mathcal{C}^r$-involution defined in $\mathbb{R}^m$. As in the smooth case, each piecewise reversible vector field presents a certain symmetry. For more details, see [21].

2.3 Orthogonal change of coordinates and stereographic projection

We say that a change of coordinates is orthogonal when the matrix of it is orthogonal, in other words, if M is this matrix it must satisfy $M^t = M^{-1}$. This kind of change of coordinates keeps all the spheres invariant and using it we can assume that the equilibrium point of a smooth vector field, that always exists on each invariant sphere $\mathbb{S}^2_\rho$, can be located at any (x_0, y_0, z_0) that we choose. Note that, when we consider piecewise smooth vector fields on invariant spheres, this kind of change of coordinates (on the whole sphere) allows us to assume that some equilibrium point of the Filippov vector field or some fold point can be located at any $(x_0, y_0, 0) \in \Sigma$.

To study local behaviors, we use the stereographic projection with respect to the point $(0, -\rho, 0)$. It allows us to consider planar vector fields instead of 3-dimensional ones restricted to spheres. In the following, we define the piecewise projected vector field. Consider the stereographic projection, $\mathfrak{p} : \mathbb{S}^2_\rho \setminus \{(0, -\rho, 0)\} \to \mathbb{R}^2$, on the plane $\{(x, y, z) \in \mathbb{R}^3 : y = \rho\}$ given by $\mathfrak{p}(x, y, z) = 2\rho(x, z)/(y + \rho)$. We define the projected vector field associated to $X \in \mathfrak{X}$ by

$$\mathcal{P}_X(\mathrm{u}) = d\mathfrak{p}_{\mathfrak{p}^{-1}(\mathrm{u})} \circ X \circ \mathfrak{p}^{-1}(\mathrm{u}),$$

where $X = X_{|_{\mathbb{S}^2_\rho}}$, $\mathrm{u} = (u, v)$ and $\mathfrak{p}(\mathrm{x}) = \mathrm{u}$. Note that, this stereographic projection sends the separation set $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ of a piecewise smooth vector field $Y \in \mathcal{X}$ to

$\{(u,v)\in\mathbb{R}^2 : v=0\}$. Thus, the projection $\mathcal{P}_Y:\mathbb{R}^2\to\mathbb{R}^2$ of (1.1) is written as

$$\mathcal{P}_Y(\mathrm{u})=\begin{cases}\mathcal{P}_{X^+}(\mathrm{u}),\ v\geq 0,\\ \mathcal{P}_{X^-}(\mathrm{u}),\ v\leq 0,\end{cases}\tag{2.3}$$

where $X^{\pm}=X^{\pm}_{|_{\mathbb{S}^2_\rho}}$, $\mathrm{u}=(u,v)$. Besides, $\mathfrak{p}$ preserves closed curves and contact between curves contained on its domain of definition, so $p\in\mathbb{S}^2_\rho$ is said to be a monodromic equilibrium point of (1.1) if $q=\mathfrak{p}(p)$ is a monodromic equilibrium point of (2.3).

2.4 Lyapunov constants and local cyclicity for planar piecewise systems

In this section, we will recall the stability algorithm (see [16, 17] and references therein) for planar piecewise smooth vector fields of the form

$$Y(x,y)=\begin{cases}X^+(x,y),\ y\geq 0,\\ X^-(x,y),\ y\leq 0,\end{cases}\tag{2.4}$$

having both $X^{\pm}$ an equilibrium point of nondegenerate center-focus type at the origin. That is,

$$X^{\pm}(x,y)=\left(\alpha^{\pm}x-\beta^{\pm}y+\sum_{k=2}^{n}P_k^{\pm}(x,y),\beta^{\pm}x+\alpha^{\pm}y+\sum_{k=2}^{n}Q_k^{\pm}(x,y)\right),$$

with $P_k^{\pm}$ and $Q_k^{\pm}$ homogeneous polynomials of degree k in the variables x and y. We have assumed that both linear parts are in Jordan's normal form. Furthermore, we follow the Filippov's convention to define the trajectories of Y on the separation set $\Sigma=\{(x,y)\in\mathbb{R}^2 : y=0\}$ and we assume $\beta^{\pm}\neq 0$ as the non degeneracy condition for each $X^{\pm}$. Using polar coordinates, $(x,y)=(r\cos\theta,r\sin\theta)$, we write system (2.4) as

$$\begin{cases}\dot r=R^+(r,\theta), & \theta\in[0,\pi],\\ \dot r=R^-(r,\theta), & \theta\in[\pi,2\pi],\end{cases}$$

where the dot represents the derivative with respect to θ.

Consider $r^{\pm}(\theta,r_0)$ the solution of $\dot r=R^{\pm}(r,\theta)$ with initial condition $r^{\pm}(0,r_0)=r_0$ and $r_0>0$ sufficiently small. The expansion in Taylor's series of the solution $r^{\pm}(\theta,r_0)$ can be written as

$$r^{\pm}(\theta,r_0)=r_0+\sum_{k=1}^{\infty}r_k^{\pm}(\theta)r_0^k,$$

with $r_k^{\pm}(0)=0$, for all $k\geq 1$, and with r^+ defined for $\theta\in[0,\pi]$ and r^- defined for $\theta\in[\pi,2\pi]$. The *Poincaré half-return maps* are defined by

$$\Pi^+(r_0)=r^+(\pi,r_0),$$
$$\widetilde{\Pi}^-(r_0)=r^-(-\pi,r_0),$$

where $\widetilde{\Pi}^-$ denotes the inverse of Π^- since both $r^{\pm}$ are defined with initial condition $\theta=0$ and $r_0>0$ sufficiently small. The *displacement function*, which is analytic, is given by

$$\Delta(r_0) = \widetilde{\Pi}^-(r_0) - \Pi^+(r_0) = \sum_{k=1}^{\infty} L_k r_0^k,$$

for r_0 small enough. When $\alpha^+\alpha^- \neq 0$ the origin is a hyperbolic equilibrium point. Otherwise $L_1 = 0$ and, for $k \geq 2$, we can define the k-th *Lyapunov constant* by $L_k \neq 0$, when $L_1 = \cdots = L_{k-1} = 0$. In this case, if there exists $k \geq 2$ so that $L_k \neq 0$, then the origin of system (2.4) is a weak focus of order k. Otherwise the origin is a center. For more details see for instance [16,17]. Usually, to simplify computations we take $\alpha^+ = \alpha^- = 0$. Note that on the smooth case the first non-vanishing Lyapunov constant has always odd subscript while in the piecewise class this property does not hold. Recall that, for analytical vector fields, the classical *Hopf bifurcation* occurs when one limit cycle of small amplitude bifurcates from a weak focus of first order (with the above notation it occurs when $L_1 = L_2 = 0$ and $L_3 \neq 0$), while the limit cycles arise from a higher-order weak focus in the degenerate Hopf bifurcation (see [1] for more details). Moreover, for piecewise smooth vector fields, in [11] it is shown that one more limit cycle appears moving the equilibrium points on Σ. Because a sliding or escaping segment is created adding adequately some perturbative parameters which implies that the displacement function of a piecewise-smooth vector field could be of the form

$$\Delta(r_0) = \widetilde{\Pi}^-(r_0) - \Pi^+(r_0) = L_0 + \sum_{k=1}^{\infty} L_k r_0^k,$$

with r_0 sufficiently small. So, by the derivative division algorithm, it is possible to obtain one limit cycle more when $L_0 \neq 0$. This is known as a pseudo-Hopf type bifurcation. Because in [22], this limit cycle bifurcation was called *pseudo-Hopf* near a fold-fold point and proved previously in [15]. For more details see [10,19]. We notice that a weak focus of order k, generically, unfolds exactly k limit cycles. Note that when we deal with continuous or refractive perturbations we do have not pseudo-Hopf type bifurcations because, in these cases, we never have sliding or escaping segments on Σ.

As we deal with polynomial perturbations of a piecewise center, we can use the Implicit Function Theorem to obtain hyperbolic crossing limit cycles of small amplitude in a neighborhood of the origin of (2.4). In this case, like in the analytical one, when we perturb a center under the condition $\alpha^{\pm} = 0$, the expressions of L_k are polynomials that vanish when the perturbative parameters do. Therefore, we can compute the Taylor series of $L_2, \ldots, L_l$ with respect to the perturbative parameters. We denote by $L_i^{[1]}$, $i = 2, \ldots, l$ their linear parts. Consequently, if the matrix $[L_2^{[1]}, \ldots, L_l^{[1]}]$, with respect to the perturbative parameters, has rank $l-1$, as we have previously explained, adding the traces and the sliding or escaping segments we can get l small amplitude hyperbolic crossing limit cycles in a neighborhood of the origin. For more details see [17] and references therein.

We can also study bifurcations of small amplitude limit cycles for piecewise smooth vector fields using the Melnikov's method. It is also used to study global bifurcations that occur near one-parameter families of periodic orbits. In particular, the first Melnikov Function and the first-order of the Lyapunov constants are related and we know that if, after perturbing a center, the rank of the matrix defined by the coefficients of $[L_2^{[1]}, \ldots, L_m^{[1]}]$, with respect to the parameters, is $l-1$, where $m > l$, then there exist l hyperbolic crossing limit cycles bifurcating from this center, when we also use the trace and the sliding parameters. For more details, see [18].

3 Piecewise linear vector fields on invariant spheres

In this section, we study piecewise linear vector fields defined on invariant spheres. We prove Theorem 1.1 and we provide all phase portraits of piecewise smooth vector fields $Y \in \mathcal{X}$ when we restrict on a invariant sphere. From the complete analysis of the phase portraits we can have a more complete result, Proposition 3.3, that also shows the nonexistence of other type of limit cycles on the invariant spheres, different from the crossing ones. In fact, Theorem 1.1 can be also thought as a corollary of it. Of course, the nonexistence of limit cycles on spheres, by the homogeneity, proves immediately the nonexistence of any kind of isolated invariant cones.

At first we summarize some results about smooth vector fields presented in [6] that we use in what follows.

Lemma 3.1. *Let $X \in \mathfrak{X}_1$. The following statements are true.*

(a) If $p \in \mathbb{S}^2_\rho = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2=\rho^2\}$ is an equilibrium point of system (1.2), then p is a center.

(b) Any $X \in \mathfrak{X}_1$ is completely integrable with the second first integral

$$\widetilde{H}(x,y,z) = a_3x - a_2y + a_1z. \tag{3.1}$$

(c) $X \in \mathfrak{X}_1$ has only two equilibrium points of center type on each sphere $\mathbb{S}^2_\rho$ which are antipodal of each other.

(d) The equilibrium points of (1.2) are $(0,0,\rho)$ if, and only if, $a_2 = a_3 = 0$. In this case, the second first integral of (1.2), given by (3.1), is of the form $\widetilde{H}(x,y,z) = a_1z$.

(e) Suppose that $a_3 \neq 0$. Then the equilibrium points of (1.2) are of the form

$$\{(x,y,z) \in \mathbb{R}^3 : y = -(a_2/a_3)x, z = (a_1/a_3)x\}.$$

(f) Suppose that $a_2 \neq 0$. Then the equilibrium points of (1.2) are of the form

$$\{(x,y,z) \in \mathbb{R}^3 : x = -(a_3/a_2)y, z = -(a_1/a_2)y\}.$$

(g) System (1.2) is invariant by the change of coordinates $(x,y,z,t) \mapsto (-x,-y,-z,t)$.

(h) The phase portrait of any $X \in \mathfrak{X}_1$ on $\mathbb{S}^2_\rho$, with $\rho > 0$, is topologically equivalent to the one on $\mathbb{S}^2_1$.

So, (1.2) has (generically) only a line of equilibrium points passing through the origin. As we observed in the introduction, by Lemma 3.1, we conclude that the 3-dimensional smooth vector field (1.2) has a continuous of invariant cones fulfilled of non-isolated closed trajectories. One of these cones is illustrated in Figure 3.1.

Now we consider the 3-dimensional piecewise smooth vector fields $Y = (X^+, X^-) \in \mathcal{X}_1$ given by (1.3), with separation set $\Sigma = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$. Observe that, when we restrict our study to an invariant sphere $\mathbb{S}^2_\rho$ we deal with a piecewise smooth vector field defined on $\mathbb{S}^2_\rho$ with separation set $\{(x,y,z) \in \mathbb{S}^2_\rho : z = 0\}$. Sure that there will be no doubt, to simplify the notation we will continue calling the separation set and the vector fields Y and $X^\pm$ restricted to the sphere $\mathbb{S}^2_\rho$ by Σ, Y, and $X^\pm$.

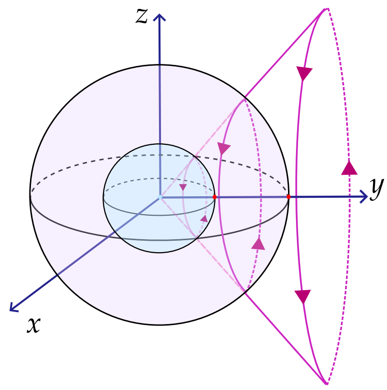

Figure 3.1: Invariant cone of the linear vector field $X(x,y,z) = (z, 0, -x) \in \mathfrak{X}_1$.

Firstly, we use Lemma 3.1 to analyze the possible positions of the equilibrium points of (1.4) with respect to the separation set Σ. By Lemma 3.1(d), the equilibria of the linear systems $X^{\pm}$, defined by (1.4), are $(0,0,\rho)$ if, and only if, $a_2^{\pm} = a_3^{\pm} = 0$. Note that the item (d) of Lemma 3.1 also implies that Σ is invariant by the flow of $X^{\pm}$ and that $(0,0,\pm\rho)$ are the unique equilibria of $X^{\pm}$ on each sphere when $a_2^{\pm} = a_3^{\pm} = 0$. So, on the following we assume that $(a_2^{\pm})^2 + (a_3^{\pm})^2 \neq 0$. We do all the calculations assuming that $a_3^{\pm} \neq 0$, the case $a_2^{\pm} \neq 0$ is analogous. Under this condition, Lemma 3.1(e) implies that the equilibria of $X^{\pm}$ are of the form $\{(x,y,z) \in \mathbb{R}^3 : y = -(a_2^{\pm}/a_3^{\pm})x, z = (a_1^{\pm}/a_3^{\pm})x\}$. Hence, the equilibria of $X^{\pm}$ are on the separation set Σ if, and only if, $a_1^{\pm} = 0$. Moreover, by Lemma 3.1(c), both $X^{\pm}$ have two equilibrium points of center type on each sphere. So, if $a_1^{\pm} \neq 0$ we conclude that the vector field $X^{\pm}$ has one admissible and one non-admissible equilibrium point.

Following [27], we use the first integrals $H(x,y,z) = x^2 + y^2 + z^2$ and $\widetilde{H}^{\pm}(x,y,z)$, given by (3.1), of the linear vector fields (1.4) to calculate a difference map, on Σ, defined below. With this map we can analyze and describe the behavior of the levels curves of (3.1) on $\mathbb{S}^2_\rho$ and how these levels interact with the separation set Σ. It allows us to know the behavior of the trajectories of (1.4) on each sphere $\mathbb{S}^2_\rho$, and, in particular, see if any system (1.3) admits crossing limit cycles on $\mathbb{S}^2_\rho$.

Lemma 3.2. *No piecewise differential system $Y \in \mathcal{X}_1$, given by (1.3), admits crossing limit cycles restricting the dynamics on each fixed sphere $\mathbb{S}^2_\rho$, with $\rho > 0$.*

Proof. As we saw before, $a_2^{\pm} = a_3^{\pm} = 0$ implies that Σ is invariant by the flow of (1.4). Therefore, in this case we cannot define a difference map using (3.1). Then, on the following, we assume that $(a_2^{\pm})^2 + (a_3^{\pm})^2 \neq 0$. We do all the calculations assuming that $a_3^{\pm} \neq 0$. The case $a_2^{\pm} \neq 0$ is analogous.

Let $p = (x_0, y_0, 0) \in \Sigma \cap \mathbb{S}^2_\rho$. Then, there exist $k^{\pm}$ such that $\widetilde{H}^{\pm}(p) = k^{\pm}$. The half-return maps $\pi^{\pm}(p) = q^{\pm} = (x_1^{\pm}, y_1^{\pm}, 0)$ satisfy

$$
\begin{aligned}
H(q^{\pm}) &= \rho^2, \\
\widetilde{H}^{+}(q^{+}) &= a_3^{+} x_1^{+} - a_2^{+} y_1^{+} = k^{+}, \\
\widetilde{H}^{-}(q^{-}) &= a_3^{-} x_1^{-} - a_2^{-} y_1^{-} = k^{-}.
\end{aligned}
$$

Solving the systems of equations

$$
\{H(q^{+}) = \rho^2, \widetilde{H}^{+}(q^{+}) = k^{+}\}, \qquad \{H(q^{-}) = \rho^2,\ \widetilde{H}^{-}(q^{-}) = k^{-}\}
$$

we obtain the solutions

$$q^{\pm}=\left(-\frac{((a_2^{\pm})^2-(a_3^{\pm})^2)x_0+2a_2^{\pm}a_3^{\pm}y_0}{(a_2^{\pm})^2+(a_3^{\pm})^2},\frac{((a_2^{\pm})^2-(a_3^{\pm})^2)y_0-2a_2^{\pm}a_3^{\pm}x_0}{(a_2^{\pm})^2+(a_3^{\pm})^2},0\right).$$

So, the difference map, $d(p)=\pi^{+}(p)-\pi^{-}(p):\Sigma\to\mathbb{R}$, is such that

$$\begin{aligned}d(p)=\big(&2(a_2^-a_3^+-a_3^-a_2^+)((a_2^-a_3^++a_2^+a_3^-)x_0-(a_2^-a_2^+-a_3^-a_3^+)y_0),\\&-2(a_2^-a_3^+-a_3^-a_2^+)((a_2^-a_2^+-a_3^-a_3^+)x_0+(a_2^-a_3^++a_2^+a_3^-)y_0),0\big).\end{aligned}$$

Consequently, it is identically zero if, and only if, $a_2^+a_3^-=a_2^-a_3^+$. Hence, either all the crossing trajectories of (1.3), on $\mathbb{S}^2_\rho$, are closed or none of them are, which concludes the proof. □

Proof of Theorem 1.1. It is a direct consequence of the fact that system (1.3) does not admit isolated crossing periodic orbits, by Lemmas 3.1(a) and 3.2. □

The remainder of this section is devoted to describing the behavior of any piecewise smooth vector field (1.3) restricted to the sphere of radius ρ, that is $\mathbb{S}^2_\rho$. We show also that no piecewise differential system $Y\in\mathcal{X}_1$, given by (1.3), admits sliding limit cycles on each fixed sphere $\mathbb{S}^2_\rho$, with $\rho>0$. Moreover, we provide the possible phase portraits of $Y\in\mathcal{X}_1$ and, consequently, we will prove the following result.

Proposition 3.3. *A piecewise differential system $Y\in\mathcal{X}_1$, given by* (1.3)*, does admit neither a crossing nor any other type of limit cycle on each fixed invariant sphere $\mathbb{S}^2_\rho$, with $\rho>0$.*

We start studying the behavior of the tangency lines of (1.4) assuming that $a_3\neq 0$, under the condition $(a_2^{\pm})^2+(a_3^{\pm})^2\neq 0$.

Lemma 3.4. *The tangency lines of* (1.4) *are given by*

$$S_{X^{\pm}}=\left\{(x,y,z)\in\mathbb{R}^3: y=-(a_2^{\pm}/a_3^{\pm})x, z=0\right\}.$$

Moreover, these tangency lines intersect the sphere $\mathbb{S}^2_\rho$ at the points

$$\left\{x=-\rho a_3^{\pm}/\sqrt{(a_2^{\pm})^2+(a_3^{\pm})^2},\ y=\ \rho a_2^{\pm}/\sqrt{(a_2^{\pm})^2+(a_3^{\pm})^2}\right\}$$

and their antipodals. Then, (1.3) *has two fold points on $\mathbb{S}^2_\rho$, for all $\rho\in\mathbb{R}$, $\rho\neq 0$. Besides, one of these tangency points is visible and the other one is invisible unless that $a_1^{\pm}=0$. Finally, when $S_{X^+}=S_{X^-}$ we have two fold-fold points of $Y=(X^+,X^-)$ on each sphere and it occurs if, and only if, $a_2^+a_3^-=a_2^-a_3^+$.*

Proof. Let $f:\mathbb{R}^3\to\mathbb{R}$ given by $f(x,y,z)=z$. So, $\Sigma=\{(x,y,z)\in\mathbb{R}^3; z=0\}=f^{-1}(0)$. Thus, $X^{\pm}f=X^{\pm}\cdot\nabla f=a_2^{\pm}x+a_3^{\pm}y$ and the first part of the result follows. With straightforward computations we prove the other statements. □

It is important to note that the symmetry of the problem guarantees that if one equilibrium point of X^+ or X^- remains on Σ, then so does the other. Moreover, if an equilibrium point of X^+ coincides with a tangency or an equilibrium point of X^- the other one also coincides. Besides, the change of coordinates $(x,y,z)\mapsto(x,y,-z)$ allows us to change the behavior of the southern and northern hemispheres and then we can fix the behavior in one of them in the next analysis.

As we saw in Section 2, we can define the projected vector field associated to (1.3), on the sphere $\mathbb{S}^2_\rho$, using (2.3). It is of the form

$$\mathcal{P}_Y(u,v) = \begin{cases} \mathcal{P}_{X^+}(u,v), \; v \geq 0, \\ \mathcal{P}_{X^-}(u,v), \; v \leq 0, \end{cases}$$

where,

$$\mathcal{P}_{X^\pm}(u,v) = \big(-4\rho^2 a_1^\pm - 4\rho a_2^\pm v - a_1^\pm u^2 + 2a_3^\pm uv + a_1^\pm v^2, \\ 4\rho^2 a_3^\pm + 4\rho a_2^\pm u - a_3^\pm u^2 - 2a_1^\pm uv + a_3^\pm v^2 \big).$$

The projected Filippov vector field is 1-dimensional and it is well defined at the points $(u,0)$ for which $(\mathcal{P}_{X^+} f)(\mathcal{P}_{X^-} f)(u) = (4\rho^2 a_3^+ + 4\rho a_2^+ u - a_3^+ u^2)(4\rho^2 a_3^- + 4\rho a_2^- u - a_3^- u^2) < 0$. In this case, we have

$$F_Y(u) = \frac{(4\rho^2 + u^2)\left((a_1^- a_3^+ - a_1^+ a_3^-)u^2 + 4(a_1^+ a_2^- - a_1^- a_2^+)u\rho + 4(a_1^+ a_3^- - a_1^- a_3^+)\rho^2\right)}{(a_3^- - a_3^+)u^2 + 4\rho(a_2^+ - a_2^-)u + 4\rho^2(a_3^+ - a_3^-)}. \tag{3.2}$$

Now, we summarize the key points of the proof of Proposition 3.3, providing after the necessary technical lemmas.

Using the two first integrals of $X^\pm$ we prove, in Lemma 3.5, that there exist only 10 possible behaviors for the levels curves of (1.3) on each sphere $\mathbb{S}^2_\rho$, concerning the admissibility of equilibrium points of $Y \in \mathcal{X}_1$, which are the ones in Figure 3.2. After that, we study the behavior of the Filippov vector field, F_Y, given by (3.2). Note that, (3.2) is not defined when Σ is a trajectory of X^+ or X^- on $\mathbb{S}^2_\rho$. In Lemma 3.7, we conclude that if the equilibrium points of both $X^\pm$ stay on Σ ($a_1^+ = a_1^- = 0$) or if there exists $\lambda \in \mathbb{R}$ such that $X^+ = \lambda X^-$, then the Filippov vector field (3.2) is identically zero. In Lemma 3.9, we prove that (3.2) has two symmetric equilibrium points r_1 and r_2 if, and only if, $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$, $a_1^+ a_1^- < 0$ and $a_2^+ a_3^- - a_2^- a_3^+ \neq 0$. In this case, the equilibrium points r_1 and r_2 have the same (1-dimensional) stability and they are stable (resp. unstable) if $(a_2^+ a_3^- - a_2^- a_3^+)(a_1^+ - a_1^-) > 0$ (resp. < 0). In addition, (3.2) can have isolated equilibrium points only when the sliding and escaping segments are delimited by two tangency points of the same type otherwise both vector fields X^+ and X^- point towards the same direction on Σ. Moreover, (3.2) does not have isolated equilibrium points when $Y \in \mathcal{X}_1$ has fold-fold points or the equilibrium points of X^+ or X^- stay on Σ.

Now, changing the time orientation of the piecewise smooth vector field (1.3), if it is necessary, we can fix an orientation for the vector field X^- in $\Sigma^- = \{(x,y,z) \in \mathbb{S}^2_\rho : \; z \leq 0\}$ and choose between two different ones for X^+ in $\Sigma^+ = \{(x,y,z) \in \mathbb{S}^2_\rho : \; z \geq 0\}$. Doing this, in Figure 3.2 we draw the possible phase portraits for system (1.3) on the sphere $\mathbb{S}^2_\rho$, with respect to the admissibility of equilibrium points of $Y \in \mathcal{X}_1$, which are the ones in Figures 3.3 and 3.4. Note that, in Figures 3.3 and 3.4 we do not distinguish the cases in which it is possible to have connections (see Remark 3.6), because it will not be necessary to conclude the proof of Proposition 3.3.

Joining the information about the positions of the equilibrium points on each sphere $\mathbb{S}^2_\rho$ with the property of the difference map detailed on Lemma 3.2, we can classify the possible behavior of the invariant curves of piecewise smooth vector fields (1.3) on $\mathbb{S}^2_\rho$, with respect to the admissibility of its equilibrium points.

Lemma 3.5. *With respect to the admissibility of equilibrium points, the behavior of the level curves of (1.3) on each fixed sphere $\mathbb{S}^2_\rho$, $\rho > 0$, are shown in Figure 3.2.*

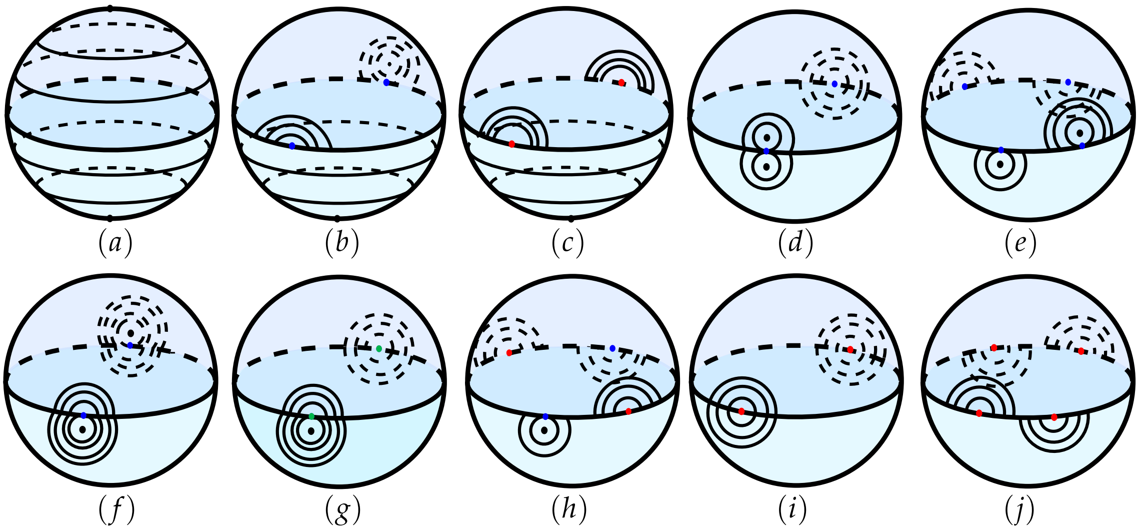

Figure 3.2: Invariant curves of $Y \in \mathcal{X}_1$. The blue, red, green and black dots indicate, respectively, tangency points; equilibrium points of $X^{\pm}$ on Σ; the coincidence of equilibrium and tangency points on Σ; the admissible center points of $X^{\pm}$.

Proof. Let $Y = (X^+, X^-) \in \mathcal{X}_1$. Denote by $p_i^{\pm}$ and $q_i^{\pm}$, with $i = 1, 2$ the equilibrium and the tangency points of $X^{\pm}$, respectively. On the following we assume that $p_1^{\pm} = (x_0, y_0, z_0)$ is such that $z_0 \geq 0$ and $p_2^{\pm} = (x_0, y_0, z_0)$ is such that $z_0 \leq 0$. We show the possible behaviors of the level curves of $Y \in \mathcal{X}_1$ on $\mathbb{S}^2_\rho$ in Figure 3.2. We divide the analysis into three cases depending on the position of the equilibrium points of X^- on $\mathbb{S}^2_\rho$.

Firstly, if $p_1^- = (0, 0, \rho)$ and $p_2^- = -p_1^-$ the trajectories of X^- on $\mathbb{S}^2_\rho$ are parallel to Σ. The same property holds when $p_1^+ = (0, 0, \rho)$ and $p_2^+ = -p_1^+$. If $p_i^+ = (x_0, y_0, z_0)$, $i = 1, 2$, are such that $z_0 \neq \pm\rho$ and $z_0 \neq 0$, then we have one admissible and one non-admissible center for X^+ on $\mathbb{S}^2_\rho$ and therefore, two tangency points, $q_i^+ \in \Sigma$, $i = 1, 2$, one visible and one invisible. Finally, if $p_i^+ = (x_0, y_0, 0)$, $i = 1, 2$, both equilibrium points of X^+ are on Σ. We draw the invariant curves of these cases in Figure 3.2 $(a) - (c)$.

Now we consider the case where $p_i^- = (x_0, y_0, z_0)$, $i = 1, 2$, with $z_0 \neq 0$ and $z_0 \neq \rho$. Then, we have one admissible and one non-admissible center for X^- on $\mathbb{S}^2_\rho$ and therefore, two tangency points, $q_i^- \in \Sigma$, $i = 1, 2$, one visible and one invisible, respectively. Here, as we have already considered the case where the trajectories of X^- are parallel to Σ, using the change of coordinates $(x, y, z) \mapsto (x, y, -z)$ explained before, we only need to consider the following two behaviors of X^+ on $\mathbb{S}^2_\rho$. If $p_i^+ = (x_0, y_0, z_0)$, $i = 1, 2$, are such that $z_0 \neq \rho$ and $z_0 \neq 0$, we have one admissible and one non-admissible center for X^+ and therefore, two tangency points $q_i^+ \in \Sigma$, $i = 1, 2$, one visible and one invisible, respectively. Hence, we have three new global behaviors depending on the relative position of $q_i^{\pm}$, $i = 1, 2$ that occur when $q_1^+ = q_1^-$ and $q_2^+ = q_2^-$, when they do not coincide and when $q_1^+ = q_2^-$ and $q_2^+ = q_1^-$. Finally, if $p_i^+ = (x_0, y_0, 0)$, $i = 1, 2$, the two equilibrium points of X^+ are on Σ and we have two new global behaviors depending on the positions of these equilibrium points, that is $p_i^+ = q_i^-$ or $p_i^+ \neq q_i^-$, $i = 1, 2$. We show the invariant curves of these cases in Figure 3.2 $(d) - (h)$.

We finish the analysis considering the case where the two centers of X^- on $\mathbb{S}^2_\rho$ are on Σ, it means that $p_i^- = (x_0, y_0, 0)$, $i = 1, 2$. Using the change of coordinates $(x, y, z) \mapsto (x, y, -z)$, we can restrict to the case in which the two equilibrium points of X^+, p_i^+ for $i = 1, 2$, are also on

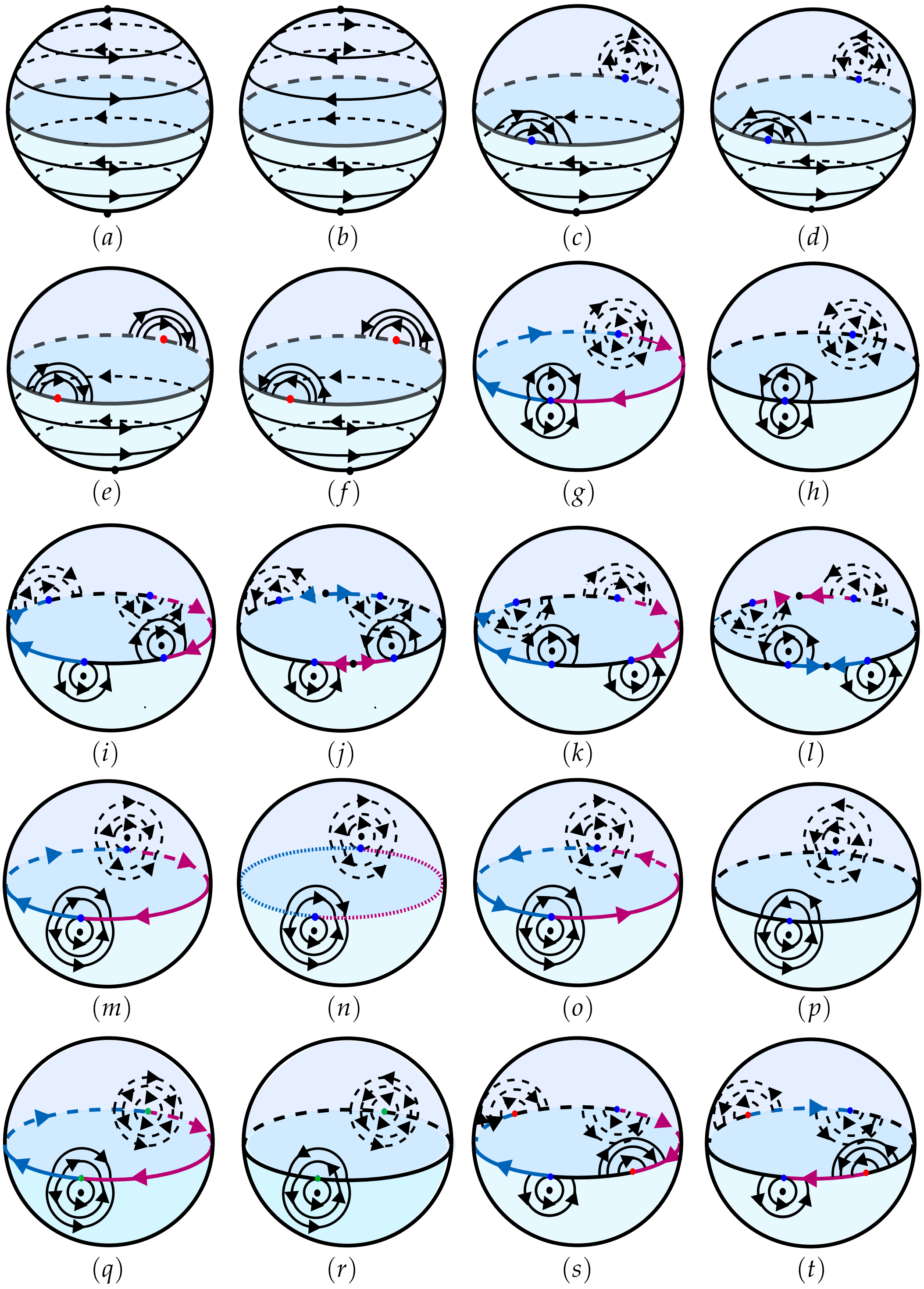

Figure 3.3: Possible phase portraits of $Y \in \mathcal{X}_1$. The gray, blue, pink and black segments indicate, respectively, that the Filippov's convention does not apply; the escaping, sliding, and the crossing regions. Moreover, the blue, red, green and black dots indicate, respectively, tangency points; equilibrium points of $X^{\pm}$ on Σ; the coincidence of equilibrium and tangency points on Σ; the admissible center points of $X^{\pm}$ or the critical points of the Fillipov vector field F_Y.

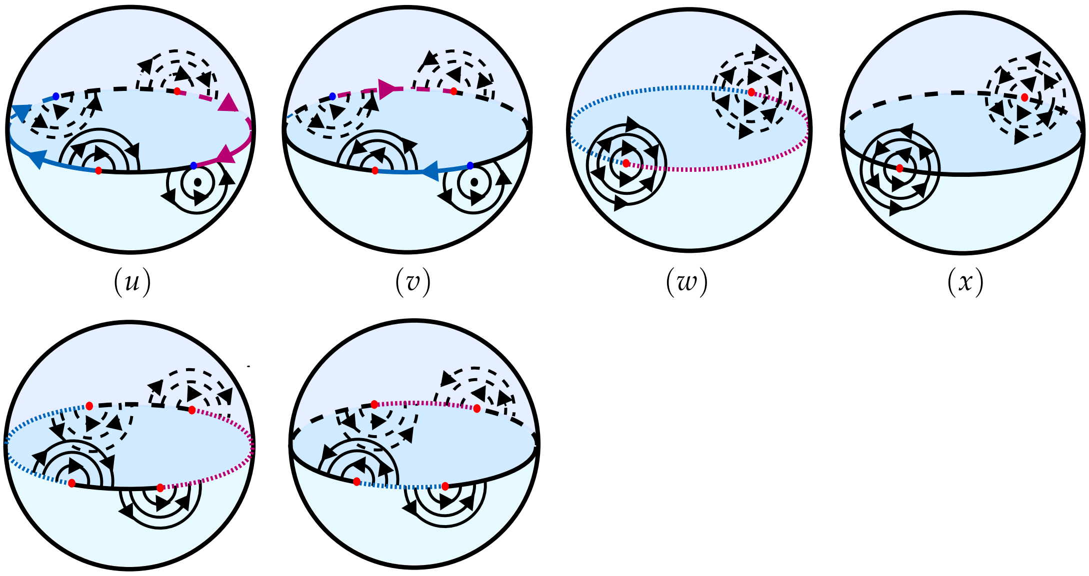

Figure 3.4: Possible phase portraits of $Y \in \mathcal{X}_1$. The blue, pink and black segments indicate, respectively, the escaping, sliding, and the crossing regions. Moreover, the blue, red and black dots indicate, respectively, tangency points; equilibrium points of $X^{\pm}$ on Σ; the admissible center points of $X^{\pm}$.

Σ. Here we have two new global behaviors depending on the positions of these equilibrium points: $p_i^+ = p_i^-$ or $p_i^+ \neq p_i^-$. We draw the invariant curves of these cases in Figure 3.2 (i)–(j). □

Remark 3.6. Note that the tangency points of $X^{\pm}$ are antipodal of each other. Therefore, the tangency lines $S_{X^{\pm}}$ of $X^{\pm}$ are contained in the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and pass through the origin. Observe that when these tangency lines are perpendicular $Y \in \mathcal{X}_1$ admits a tangential connection. It occurs because the trajectories of $X^{\pm}$ are restricted to the level curves of (3.1), on $\mathbb{S}^2_\rho$. Thus, depending on the relative position of the tangency lines, the behavior illustrated in the cases (e), (h), and (j) of Figure 3.2 are not unique. But for our purpose we do not need to distinguish the cases in which there are or not separatrix connections.

As in the above analysis, we only have considered the level curves of $X^{\pm}$ we have not taken into account the behavior of the Filippov vector field. On the following lemmas we describe the behavior of it using the projected Filippov vector field (3.2) associated to (1.3) restricted to the sphere $\mathbb{S}^2_\rho$, because it is 1-dimensional.

Lemma 3.7. *The Filippov vector field* (3.2) *is well defined when* $(\mathcal{P}_{X^+}f)(\mathcal{P}_{X^-}f) < 0$. *In this case, it is identically zero if, and only if,* $a_1^- a_3^+ - a_1^+ a_3^- = 0$ *and* $a_1^- a_2^+ - a_1^+ a_2^- = 0$.

Proof. It follows since (3.2) is a rational function and its numerator is identically zero if, and only if, $(a_1^- a_3^+ - a_1^+ a_3^-)u^2 + 4\rho(a_1^+ a_2^- - a_1^- a_2^+)u + 4\rho^2(a_1^+ a_3^- - a_1^- a_3^+) \equiv 0$. □

Remark 3.8. The geometric implication of Lemma 3.7 is the following. Firstly, we note that $a_2^+ = a_2^- = 0$ and $a_3^+ = a_3^- = 0$ imply that $a_1^- a_3^+ - a_1^+ a_3^- = 0$ and $a_1^- a_2^+ - a_1^+ a_2^- = 0$. But, in this case, $(\mathcal{P}_{X^+}f)(\mathcal{P}_{X^-}f)$ is identically zero and then (3.2) is not defined for these values of the coefficients. In addition, $a_1^+ = a_1^- = 0$ implies that $a_1^- a_3^+ - a_1^+ a_3^- = 0$ and $a_1^- a_2^+ - a_1^+ a_2^- = 0$

and then (3.2) vanishes identically when the equilibrium points of both X^+ and X^- are on Σ. Finally, when $(a_1^+)^2 + (a_1^-)^2 \neq 0$, the conditions $a_1^- a_3^+ - a_1^+ a_3^- = 0$ and $a_1^- a_2^+ - a_1^+ a_2^- = 0$ imply that $a_2^- a_3^+ - a_3^- a_2^+ = 0$ and then X^+ and X^- are multiple of each other, which means that the equilibrium points and tangency lines of X^+ and X^- coincide. So, we have that (3.2) is identically zero if, and only if, the equilibrium points of both $X^\pm$ are on Σ or if X^+ and X^- are multiple of each other.

Note that when $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$, the projected Filippov vector field (3.2) can have at most two real roots, for $i = 1, 2$, given by

$$r_i = \frac{2\rho\left((a_1^+ a_2^- - a_1^- a_2^+) - (-1)^i \sqrt{(a_1^+ a_2^- - a_1^- a_2^+)^2 + (a_1^+ a_3^- - a_1^- a_3^+)^2}\right)}{a_1^+ a_3^- - a_1^- a_3^+} \tag{3.3}$$

if $(\mathcal{P}_{X^+} f)(\mathcal{P}_{X^-} f)(r_i) < 0$. In addition, $(\mathcal{P}_{X^+} f)(\mathcal{P}_{X^-} f)(r_i) > 0$ means that X^+ and X^- are parallel at a crossing point and then (3.2) is not defined at this point. On the other hand, when $a_1^+ a_3^- - a_1^- a_3^+ = 0$ the projected Filippov vector field has a unique possible real root at the origin. The symmetry of the problem ensures that the other root is situated at infinity. We can avoid this and suppose that $(0, 0)$ is not an equilibrium point of (3.2) making the same orthogonal change of coordinates in X^+ and X^- that put $(0, \rho, 0)$ in (x_0, y_0, z_0) with $y_0 \neq \rho$, as it was done previously. So, without loss of generality, we only analyze the case $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$ and study the stability of the equilibrium points of the Filippov vector field, when it is well defined.

Lemma 3.9. *The Filippov vector field (3.2) is well defined when $(\mathcal{P}_{X^+} f)(\mathcal{P}_{X^-} f) < 0$. In this case, it has two symmetric equilibrium points r_1 and r_2 defined in (3.3) if, and only if, $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$, $a_1^+ a_1^- < 0$ and $a_2^+ a_3^- - a_2^- a_3^+ \neq 0$. The equilibrium points r_1 and r_2 have the same (1-dimensional) stability. Moreover, they are stable (resp. unstable) if $(a_2^+ a_3^- - a_2^- a_3^+)(a_1^+ - a_1^-) > 0$ (resp. < 0).*

Proof. As we saw before, when $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$, the Filippov vector field can have at most two real roots r_1 and r_2 given in (3.3). Note that, for $i = 1, 2$, we have

$$(\mathcal{P}_{X^+} f)(\mathcal{P}_{X^-} f)(r_i) = \frac{64\rho^4 a_1^+ a_1^- (a_2^+ a_3^- - a_2^- a_3^+)^2}{(a_1^+ a_3^- - a_1^- a_3^+)^4}$$
$$\left(\sqrt{(a_1^+ a_2^- - a_1^- a_2^+)^2 + (a_1^+ a_3^- - a_1^- a_3^+)^2} - (-1)^i (a_1^+ a_2^- - a_1^- a_2^+)\right)^2.$$

As we are assuming $a_1^+ a_3^- - a_1^- a_3^+ \neq 0$, then

$$\sqrt{(a_1^+ a_2^- - a_1^- a_2^+)^2 + (a_1^+ a_3^- - a_1^- a_3^+)^2} \pm (a_1^+ a_2^- - a_1^- a_2^+)$$

are always different from zero. Consequently, the projected vector field (3.2) is defined at r_1 and r_2 if, and only if, $a_1^+ a_1^- < 0$ and $a_2^+ a_3^- - a_2^- a_3^+ \neq 0$. It means that, when the equilibrium points of both X^+ and X^- are on Σ and when the tangential points of X^+ and X^- coincide, the projected Filippov vector field does not have isolated equilibrium points.

When $a_2^+ a_3^- - a_2^- a_3^+ \neq 0$ and $a_1^+ a_1^- < 0$ we study the stability of these equilibrium points. As $4\rho^2 + u^2$ is a positive factor of (3.2), we can study the stability of the equilibrium points of $F_Y(u)/(4\rho^2 + u^2)$. In this case, the derivative with respect to u is nonvanishing for all u, because it is

$$-\frac{4\rho(4\rho^2 + u^2)(a_2^+ a_3^- - a_2^- a_3^+)(a_1^+ - a_1^-)}{((a_3^- - a_3^+)u^2 + 4\rho(a_2^+ - a_2^-)u + 4\rho^2(a_3^+ - a_3^-))^2}.$$

Thus, r_1 and r_2 have the same stability which depends on the sign of

$$(a_2^+ a_3^- - a_2^- a_3^+)(a_1^+ - a_1^-). \qquad \square$$

Remark 3.10. As we saw above, (3.2) is not defined when Σ is a trajectory of X^+ or X^-. Moreover (3.2) does not have isolated equilibrium points neither when it has fold-fold points nor when the equilibrium points of X^+ or X^- stay on Σ. Besides this, the Filippov vector field (3.2) can have equilibrium points only when the sliding and escaping segments are delimited by two tangency points of the same type, otherwise both vector fields X^+ and X^- point on the same direction.

Now, changing the time orientation of the piecewise smooth vector field (1.3), if it is necessary, we can fix a time orientation for the vector field X^- and choose two different ones for X^+ on $\mathbb{S}^2_\rho$. Hence, when we add a time orientation in Figure 3.2 we obtain the possible behaviors for system (1.3), that are depicted in Figures 3.3 and 3.4. Note that Figures 3.3 and 3.4 do not take into account connections of (1.3). These elements do not influence in the existence of limit cycles. Moreover, the nonexistence of limit cycles in $\mathbb{S}^2_\rho$ is not related to the existence of connections. This is due to the arrangement of tangency points, admissible and non-admissible equilibrium points, and, as (1.3) is completely integrable, the difference map does not have isolated zeros.

With this analysis we conclude that system (1.3) has neither limit cycles nor crossing limit cycles on the spheres $\mathbb{S}^2_\rho$, with $\rho > 0$. So, the proof of Proposition 3.3 follows.

4 Centers and limit cycles for piecewise continuous quadratic homogeneous vector fields

In this section, inspired by the homogeneity property of linear vector fields with invariant spheres, we study the center-focus problem for piecewise quadratic homogeneous vector fields in $\mathcal{X}_2^H$. Because of the difficulty of the problem, we restrict our attention to the class of continuous homogeneous vector fields and give some families of centers in Proposition 4.3. Even with this restriction, in Proposition 4.5 we exhibit a system in $\mathcal{X}_2^H$ with a weak focus of third-order at the point $(0,1,0)$ from which 2 small amplitude crossing limit cycles bifurcate on $\mathbb{S}^2_1$ with a continuous perturbation in $\mathcal{X}_2^H$. Note that with a continuous perturbation, we cannot produce a sliding segment and then it is natural that we do not reach the maximum upper bound for the number of small amplitude limit cycles that can bifurcate from a generic weak focus of third-order. Moreover, in this section we only consider the perturbation in $\mathcal{X}_2^H$ and, in the next section we deal with a general quadratic perturbation in $\mathcal{X}_2$.

On the following, we recall some assumptions given in [6], for a quadratic homogeneous vector field $X \in \mathfrak{X}_2^H$. Firstly, doing an orthogonal change of coordinates we can assume, without loss of generality, that $(0,\rho,0) \in \mathbb{S}^2_\rho$ is an equilibrium point of $X \in \mathfrak{X}_2^H$. With this assumption, we can write X in the form

$$\begin{aligned} \dot{x} &= -a_4xy - a_5xz - (a_6+a_7)yz - a_8z^2, \\ \dot{y} &= a_4x^2 + a_6xz - a_9z^2, \\ \dot{z} &= a_5x^2 + a_7xy + a_8xz + a_9yz. \end{aligned} \tag{4.1}$$

Observe that the equilibrium point $(0,\rho,0)$ of (4.1) is located at the origin after projection and let J be the Jacobian matrix associated to the projected vector field $\mathcal{P}_X$ at the origin. Therefore,

$(0,\rho,0)$ is of nondegenerate center-focus type if, and only if, the trace of J is zero and its determinant is positive. A straightforward computation shows that it occurs if, and only if, $a_4 - a_9 = 0$ and $a_6a_7 + a_7^2 - a_9^2 > 0$. We also assume $a_7 \neq 0$, otherwise $a_6a_7 + a_7^2 - a_9^2 = -a_9^2 \leq 0$. Hence, with these assumptions $(0,\rho,0)$ is a weak focus of (4.1). Doing $w^2 = a_6a_7 + a_7^2 - a_9^2$, and $\varrho = (w^2 + a_4a_9)/a_7$ the projected system $\mathcal{P}_X$ is of the form:

$$\begin{aligned} \dot{u} &= -4a_4u - 4\varrho v - 4a_5uv - 4a_8v^2 - a_4u^3 - (\varrho - 2a_7)u^2v + (a_4 + 2a_9)uv^2 + \varrho v^3, \\ \dot{v} &= 4a_7u + 4a_9v + 4a_5u^2 + 4a_8uv - a_7u^3 - (2a_4 + a_9)u^2v - (2\varrho - a_7)uv^2 + a_9v^3. \end{aligned} \tag{4.2}$$

Now, the trace and the determinant of J are $-4(a_4 - a_9)$ and $16w^2$, respectively. The next theorem was proved in [6] and gives the conditions to have a center of (4.1) at the point $(0,1,0)$, on the sphere $\mathbb{S}_1^2$.

Theorem 4.1 ([6])**.** *The equilibrium point $(0,1,0)$ of system (4.1) is a nondegenerate center if, and only if, $a_7 \neq 0$, $a_4 = a_9$, and $a_4a_5a_8a_9 + a_5a_6a_7a_8 + a_5^2a_7a_9 + a_5a_8a_9^2 - a_7a_8^2a_9 = 0$.*

Next we will show an important difference between polynomial homogeneous vector fields defined on the sphere $\mathbb{S}_1^2$ and on the plane. Also in our special case that the dynamics is restricted on a invariant sphere. Firstly, we recall that a planar quadratic homogeneous vector field does not have limit cycles. The following example shows a quadratic homogeneous vector field $X \in \mathfrak{X}_2^H$ which has at least one limit cycle on the sphere $\mathbb{S}_1^2$. It occurs because the projected vector field (4.2) is a planar cubic non-homogeneous vector field. Fore more results about quadratic homogeneous vector fields defined on invariant spheres we refer the reader to [25,26]. As in the previous section, this limit cycle forces the existence of an invariant cone fulfilled of periodic orbits for (4.1).

Proposition 4.2. *The quadratic homogeneous vector field (4.1) has at least one limit cycle bifurcating from $(0,1,0)$ on the sphere $\mathbb{S}_1^2$.*

Proof. Consider the quadratic homogeneous vector field (4.1) and its projection (4.2) with the parameters values $(a_4, a_5, a_7, a_8, a_9, w) = (1+\varepsilon, 1, 1, 0, 1, 1)$. Note that with these values, (4.2) writes as the following cubic vector field

$$\begin{aligned} \dot{u} &= (-4+\varepsilon)u - 4(2+\varepsilon)v - 4uv - (1+\varepsilon)u^3 + \varepsilon u^2v + (3+\varepsilon)uv^2 + (2+\varepsilon)v^3, \\ \dot{v} &= 4u + 4v + 4u^2 - u^3 - (3+\varepsilon)u^2v - (3+\varepsilon)uv^2 + v^3. \end{aligned} \tag{4.3}$$

As we observed before, the origin is an equilibrium point of (4.3). Let J be the Jacobian matrix associated to (4.3) at the origin. As the trace of J is ε and its determinant is $16 + 12\varepsilon$, then the origin is a weak focus for $\varepsilon = 0$. Note that we can use the algorithm explained in Section 2.4 to calculate the Lyapunov constants of analytical vector fields assuming that $\mathcal{P}_{X^+}$ and $\mathcal{P}_{X^-}$ are both defined by (4.3), because it is a generalization of the algorithm presented in Chapter IX of [1]. So, when $\varepsilon = 0$, we calculate the first Lyapunov constant of (4.3) being $L_3 = 4 \neq 0$. Thus, by the classical Hopf bifurcation, there exist values of ε for which (4.3) has one limit cycle bifurcating from the origin. □

On the following we will focus our attention on the center-focus problem that appears naturally for the piecewise smooth system

$$Y(x,y,z) = \begin{cases} X^+(x,y,z), \ z \geq 0, \\ X^-(x,y,z), \ z \leq 0, \end{cases} \tag{4.4}$$

where we obtain $X^{\pm}$ doing $a_i = a_i^{\pm}$ in (4.1) and assuming that $p = (0,1,0) \in \Sigma = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$ is of center type for both X^+ and X^- on $\mathbb{S}^2_{\rho}$. Here, as we commented above, because of the number of free parameters, we also assume that the system (4.4) is continuous but not differentiable on the separation set Σ. Note that, system (4.4), and the projected associated systems $\mathcal{P}_Y = (\mathcal{P}_{X^+}, \mathcal{P}_{X^-})$, where $\mathcal{P}_{X^{\pm}}$ are obtained doing $a_i = a_i^{\pm}$ in (4.2), are continuous on its separation set if, and only if, $a_4^- = a_4^+$, $a_5^- = a_5^+$, and $a_7^- = a_7^+$. Consequently, on the following, we are assuming these conditions.

As we are interested in exhibiting some families of centers for this family of piecewise smooth vector fields we use the method explained in Section 2.4, to calculate the Lyapunov constants for the projected system $\mathcal{P}_Y$. To do that, we need to consider $\mathcal{P}_{X^{\pm}}$ in its Jordan canonical form.

Note that the change of coordinates $\{\mathrm{u} = v, \mathrm{v} = (cu + dv)/w\}$, where $c = 4a_7$ and $d = 4a_9$ puts the linear part of (4.2) in its Jordan canonical form

$$\begin{aligned}
\dot{\mathrm{u}} &= \mathrm{v} + \frac{a_9\,(a_5a_9 - a_7a_8)}{wa_7{}^2}\mathrm{u}^2 - \frac{(2a_5a_9 - a_7a_8)}{a_7{}^2}\mathrm{uv} + \frac{a_5w}{a_7{}^2}\mathrm{v}^2 + \frac{wa_9}{2a_7{}^2}\mathrm{u}^3 \\
&\quad + \frac{(a_7{}^2 + a_9{}^2 - 2w^2)}{4a_7{}^2}\mathrm{u}^2\mathrm{v} - \frac{w^2}{4a_7{}^2}\mathrm{v}^3, \\
\dot{\mathrm{v}} &= -\mathrm{u} + \frac{(a_5a_9 - a_7a_8)(a_7{}^2 + a_9{}^2)}{a_7{}^2w^2}\mathrm{u}^2 - \frac{(a_5a_7{}^2 + 2a_5a_9{}^2 - a_7a_8a_9)}{wa_7{}^2}\mathrm{uv} \\
&\quad + \frac{a_5a_9}{a_7{}^2}\mathrm{v}^2 + \frac{(a_7{}^2 + a_9{}^2)}{4a_7{}^2}\mathrm{u}^3 + \frac{(2a_7{}^2 + 2a_9{}^2 - w^2)}{4a_7{}^2}\mathrm{uv}^2 - \frac{wa_9}{2a_7{}^2}\mathrm{v}^3.
\end{aligned} \tag{4.5}$$

Moreover, the change of coordinates $\{\mathrm{u} = v, \mathrm{v} = (cu + dv)/w^{\pm}\}$ puts $\mathcal{P}_{X^{\pm}}$ in the canonical form and the separation set $\Sigma = \{(u,v) \in \mathbb{R}^2 : v = 0\}$ becomes $\widetilde{\Sigma} = \{(\mathrm{u},\mathrm{v}); \mathrm{u} = 0, \mathrm{v} = cu/w^{\pm}\}$. Consequently, after this change of coordinates, we deal with the piecewise smooth system

$$\mathcal{P}_Y(\mathrm{u},\mathrm{v}) = \begin{cases} \mathcal{P}_{X^+}(\mathrm{u},\mathrm{v}),\ \mathrm{u} \geq 0, \\ \mathcal{P}_{X^-}(\mathrm{u},\mathrm{v}),\ \mathrm{u} \leq 0, \end{cases} \tag{4.6}$$

where $\mathcal{P}_{X^{\pm}}$ are obtained doing $a_i = a_i^{\pm}$ in (4.5) and then, in polar coordinates, it is written as

$$\begin{cases} \dot{r} = R^+(r,\theta),\ \theta \in [-\pi/2, \pi/2], \\ \dot{r} = R^-(r,\theta),\ \theta \in [-\pi/2, -3\pi/2]. \end{cases}$$

Therefore, we use the technique shown in Section 2.4 after a rotation of angle $\pi/2$, to calculate the Lyapunov constants of (4.6). Note that after the change of coordinates $\{\mathrm{u} = v, \mathrm{v} = (cu + dv)/w^{\pm}\}$ the separation set of (4.6), $\widetilde{\Sigma} = \{(\mathrm{u},\mathrm{v}); \mathrm{u} = 0, \mathrm{v} = cu/w^{\pm}\}$, is parameterized in two different ways when $w^+ \neq w^-$ and then the continuity condition must be considered before doing it and we also take it into account when we compute the Lyapunov constants.

On the following, we give some families of centers for the piecewise smooth vector field (4.4). Some of these centers appear in a family of reversible vector fields with respect to a line (see the definition in Section 2.2).

Proposition 4.3. *The piecewise continuous vector field (4.4) has a center at the equilibrium point $(0,1,0)$, on $\mathbb{S}^2_1$, if $a_7^{\pm} \neq 0$, $a_4^{\pm} = a_9^{\pm}$ and one of the following conditions is satisfied:*

(a) $a_8^- = -a_8^+$, $a_9^- = 0$, *and* $w^+ = w^-$;

(b) $a_7^- = \pm w$, $a_9^- = 0$, *and* $w^+ = w^-$;

(c) $a_8^+ = a_8^-$, $(a_5^-)^2 a_7^- a_9^- - a_5^-(a_7^-)^2 a_8^- + a_5^- a_8^-(a_9^-)^2 + a_5^- a_8^- w^2 - a_7^-(a_8^-)^2 a_9^- = 0$, *and* $w^+ = w^-$;

(d) $a_5^- = 0$ *and* $a_9^- = 0$.

Proof. In case (a), the piecewise projected continuous vector field $\mathcal{P}_Y = (\mathcal{P}_{X^+}, \mathcal{P}_{X^-})$ is reversible with respect to the separation set. The case (b) follows because in polar coordinates we have $dr/dt = 0$ for both $\mathcal{P}_{X^\pm}$, which implies that the difference map defined in Σ, in a neighborhood of the origin, is zero. In case (c), the vector field $\mathcal{P}_Y$ is smooth and it satisfies the condition given on Theorem 4.1. Finally, case (d) follows because both vector fields $\mathcal{P}_{X^\pm}$ are reversible with respect to the u-axis and then the difference map defined in Σ, in a neighborhood of the origin, is zero which concludes the proof. □

If $w^+ = w^-$ the change of coordinates that puts the system (4.2) on form (4.5) is the same for X^+ and X^- and then the parametrization of $\widetilde{\Sigma}$ coincides before this change of coordinates. In this case, the only possible center families for (4.4) are that given on items (a)-(d) of Proposition 4.3.

Proposition 4.4. *The piecewise continuous vector field* (4.4) *with* $w^+ = w^-$ *has a center at the equilibrium point* $(0,1,0)$, *on* $\mathbb{S}_1^2$, *if, and only if,* $a_7^\pm \neq 0$, $a_4^\pm = a_9^\pm$ *and one of the conditions (a), (b), (c), or (d) of Proposition 4.3 is satisfied.*

Proof. To simplify the notation of this proof, we eliminate the superscript $\pm$ when the corresponding coefficients of X^+ and X^- are equal. Hence, we consider $w^+ = w^- = w$, $a_4^+ = a_4^- = a_4$, $a_5^+ = a_5^- = a_5$, $a_7^+ = a_7^- = a_7$, and $a_9^+ = a_9^- = a_9$. According to the proof of Proposition 4.3, all the families detailed in the statement have a center at the origin. Consequently, we only need to check that these are the only ones when $w^+ = w^- = w$. To do that, we compute four Lyapunov constants using the method explained in Section 2.4 for system (4.6), with the statement assumptions, and we obtain

$$
\begin{aligned}
L_2 &= \frac{2}{3wa_7} a_9(a_8^+ - a_8^-),\\
L_3 &= \frac{\pi}{8w^3a_7^3}\big(2a_5^2a_7a_9(a_7^2 + a_9^2 + w^2) - a_5(a_7^4a_8^+ - 2a_8^+a_9^4 - 4a_8^+a_9^2w^2\\
&\qquad - (a_8^+ + a_8^-)w^4) - 2a_7a_9(a_7^2(a_8^+)^2 + (a_8^-)^2a_9^2 + (a_8^+)^2w^2)\big),\\
L_4 &= \frac{4}{45a_7^4w^5} a_5((a_8^+)^2 - (a_8^-)^2)(a_7^2 - w^2)(4a_7^4 + 3a_7^2w^2 + 2w^4),\\
L_5 &= \frac{\pi}{12w^3a_7^3}(a_5^2 + (a_8^+)^2 - w^2)\big(2a_5^2a_7a_9 - a_5a_7^2a_8^+ - a_5a_7^2a_8^-\\
&\qquad + 2a_5a_8^+a_9^2 + a_5a_8^+w^2 + a_5a_8^-w^2 - 2a_7(a_8^+)^2a_9\big).
\end{aligned}
$$

When we solve the system of equations $\mathcal{S}_L = \{L_2 = \cdots = L_5 = 0\}$ we obtain the real solutions given on statements (a)-(d) of Proposition 4.3 and four more complex solutions given by $\left\{a_5 = \pm\sqrt{-(a_8^-)^2 + w^2}, a_7 = \pm\,\mathrm{i}\sqrt{a_9^2 + w^2}, a_8^+ = a_8^-\right\}$ and $\{a_7 = \pm\,\mathrm{i}\,a_9, a_8^+ = a_8^-, w = 0\}$. As we are interested in real families with $w \neq 0$ we conclude the proof. □

Proposition 4.5. *Consider system* (4.4) *with* $a_5^- = 1, a_7^- = 1, a_8^+ = 3, a_8^- = 1, a_9^- = 0$, *and* $w^+ = w^- = 2$. *Then, the equilibrium point* $p = (0,1,0)$ *is a weak focus of third-order and there exist* 2 *small amplitude limit cycles, on* $\mathbb{S}_1^2$, *bifurcating from* p *with a continuous perturbation in* $\mathcal{X}_2^H$.

Proof. Using the expressions of L_i, $i = 2, \dots, 5$ given in the above proposition we conclude that, for these values of parameters, we have $L_2 = 0$ and $L_3 = 15\pi/16 \neq 0$. Hence, adding the trace parameter and using the derivation-division algorithm (see more details in [29]) we obtain 2 small amplitude crossing limit cycles bifurcating from the equilibrium point $(0, 1, 0)$ on $\mathbb{S}_1^2$. □

We emphasize that, when we deal with a continuous perturbation, we do not have the sliding parameter to get the maximum upper bound for the number of small amplitude crossing limit cycles bifurcating from a center or a weak focus, as we have explained in Section 2.4. Because of that, the maximum number of limit cycles that we can obtain bifurcating from the weak focus of third-order, in the last result, is 2.

5 Local cyclicity for quadratic vector fields in $\mathfrak{X}_2$ with piecewise smooth perturbation in $\mathcal{X}$

In this section, we study the local cyclicity of centers and weak focus families of quadratic smooth vector fields with piecewise quadratic perturbations in $\mathcal{X}_2$. The continuous or refractive perturbation cases are also analyzed. We show the results that we have obtained in Propositions 5.1 and 5.2. The proof of Theorem 1.2 follows directly from Proposition 5.2(c).

On the following, we summarize some assumptions and results given in [6] for a quadratic vector field $X \in \mathfrak{X}_2$ which will be useful in the sequence. Firstly, the behavior of system X can be totally different in two different levels of invariant spheres. Hence, we restrict our analysis to the unit sphere $\mathbb{S}_1^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. In this case any $X \in \mathfrak{X}_2$ can be written in its canonical form

$$\begin{aligned} \dot{x} &= -a_1 y - a_2 z - a_4 xy - a_5 xz - a_{10} y^2 - (a_6 + a_7) yz - a_8 z^2, \\ \dot{y} &= a_1 x - a_3 z + a_4 x^2 + a_{10} xy + a_6 xz - a_{11} yz - a_9 z^2, \\ \dot{z} &= a_2 x + a_3 y + a_5 x^2 + a_7 xy + a_8 xz + a_{11} y^2 + a_9 yz. \end{aligned} \tag{5.1}$$

Note that $(0, 1, 0)$ is an equilibrium point of (5.1) if, and only if, $a_1 + a_{10} = 0$, $a_3 + a_{11} = 0$. Consequently, to have the origin as an equilibrium point of the projected vector field $\mathcal{P}_X$ associated to (5.1), we assume these conditions on the following. Next we will impose the conditions that ensure that $(0, 1, 0)$ is an equilibrium point of nondegenerate center-focus type on the sphere $\mathbb{S}_1^2$. We will do that analyzing the trace and the determinant of the Jacobian matrix J associated to the projected vector field $\mathcal{P}_X$ at the equilibrium point $(0, 0)$. Recall that $\mathcal{P}_X$ has an equilibrium point of nondegenerate center-focus type at origin if, and only if, the trace of J is zero and its determinant is positive. It occurs when $a_4 = a_9$ and $a_2 a_6 + a_6 a_7 + 2a_2 a_7 + a_2^2 + a_7^2 - a_9^2 > 0$. As explained in [6], due to the high number of free parameters, we will restrict our analysis adding two extra conditions: $a_9 = 0$ and $a_2 + a_7 = 1$. With these assumptions, the projected vector field $\mathcal{P}_X$ has a weak focus at the origin if, and only if, $a_4 = 0$ and $a_6 + 1 > 0$. Moreover, with these restrictions the projected vector field has a center-focus point at the origin with a Jacobian matrix in Jordan normal form. Doing $a_4 = 0$ and $w^2 = a_6 + 1$, with $w \neq 0$, the following system is obtained from (5.1)

$$\begin{aligned} \dot{x} &= -a_1 y - (1 - a_7) z - a_5 xz + a_1 y^2 + (1 - a_7 - w^2) yz - a_8 z^2, \\ \dot{y} &= a_1 x + a_{11} z - a_1 xy + (w^2 - 1) xz - a_{11} yz, \\ \dot{z} &= (1 - a_7) x - a_{11} y + a_5 x^2 + a_7 xy + a_8 xz + a_{11} y^2. \end{aligned} \tag{5.2}$$

After a time reparameterization and the change of coordinates $u \to wu$, the corresponding projected system obtained from (5.2) is

$$\begin{aligned}\dot{u} &= -v - \frac{a_1}{2}u^2 - \frac{a_5}{w}uv - \frac{a_1+2a_8}{2w^2}v^2 + \frac{2a_7-w^2}{4}u^2v \\ &\quad + \frac{w^2+2a_7-2}{4w^2}v^3 - \frac{a_1w^2}{8}u^4 - \frac{a_{11}w}{4}u^3v - \frac{a_{11}}{4w}uv^3 + \frac{a_1}{8w^2}v^4, \\ \dot{v} &= u + \frac{(2a_5-a_{11})w}{2}u^2 + a_8uv - \frac{a_{11}}{2w}v^2 - \frac{(2a_7-1)w^2}{4}u^3 \\ &\quad - \frac{2w^2+2a_7-3}{4}uv^2 + \frac{w^3a_{11}}{8}u^4 - \frac{w^2a_1}{4}u^3v - \frac{a_1}{4}uv^3 - \frac{a_{11}}{8w}v^4.\end{aligned} \tag{5.3}$$

In items (a) and (b) of Proposition 5.2, we show that with a continuous (resp. refractive) perturbation in $\mathcal{X}_2$ we obtain 5 (resp. 6) crossing limit cycles bifurcating from a center family of (5.2). Moreover, when we consider a piecewise quadratic general perturbation in $\mathcal{X}_2$ we obtain 10 limit cycles, as we will see in item (c) of Proposition 5.2, which proves Theorem 1.2. We also exhibit a piecewise quadratic perturbation of a weak focus in Proposition 5.1 for a fixed value of w.

In the following, we will describe the type of piecewise smooth perturbation of $X \in \mathfrak{X}_2$ that we will consider and which are the conditions that will make this perturbation continuous or refractive.

Let $X = X(\mathrm{x}, \mathrm{a}) \in \mathfrak{X}_2$ given by (5.2) where $\mathrm{x} = (x, y, z)$ and $\mathrm{a} = (a_1, a_5, a_7, a_8, a_{11}, w)$. Denoting $\mathrm{a} + \varepsilon^{\pm} = (a_1 + \varepsilon_1^{\pm}, \ldots, w + \varepsilon_6^{\pm})$ we consider the piecewise smooth perturbation of X defined by

$$Y(\mathrm{x}, \varepsilon) = \begin{cases} X(\mathrm{x}; \mathrm{a} + \varepsilon^{+}), \, z \geq 0, \\ X(\mathrm{x}; \mathrm{a} + \varepsilon^{-}), \, z \leq 0, \end{cases} \tag{5.4}$$

and the projected vector field associated, defined by (2.3), which is of the form

$$\mathcal{P}_Y(\mathrm{u}, \varepsilon) = \begin{cases} \mathcal{P}_X(\mathrm{u}; \mathrm{a} + \varepsilon^{+}), \, v \geq 0, \\ \mathcal{P}_X(\mathrm{u}; \mathrm{a} + \varepsilon^{-}), \, v \leq 0, \end{cases} \tag{5.5}$$

where $\mathrm{u} = (u, v)$ and $\mathcal{P}_X(\mathrm{u}, 0)$ is given by (5.3).

So, when $\varepsilon = 0$ we have the unperturbed analytical systems (5.2) and (5.3). Following the same idea of the last section, we will say that the perturbation of the vector field Y is continuous (resp. refractive) if (5.4) is continuous (resp. refractive) in the separation set. With a straightforward computation we see that (5.4) is continuous (resp. refractive) if, and only if, $\varepsilon_i^{+} = \varepsilon_i^{-}$, for $i = 1, 2, 3, 4$ (resp. $\varepsilon_i^{+} = \varepsilon_i^{-}$, for $i = 2, 3, 4$).

Note that the origin is on the boundary of two crossing segments of the perturbative system (5.5) as well as of the unperturbed one. This is because we assumed that the origin is an equilibrium point of the center type for the unperturbed system and the perturbative parameters do not change the linear part of it. If we assume $a_3 = -a_{11} + \varepsilon_7$ instead of $a_3 = -a_{11}$ we can create a sliding segment in a neighborhood of the origin, because in this case, the projected system is of the form

$$\left(-\frac{a_4}{w}u - v + \mathcal{O}_2(u, v), \frac{\varepsilon_7}{w} + u + \mathcal{O}_2(u, v)\right).$$

Thus, we also use the perturbative parameter ε_7 when we deal with a piecewise perturbation instead of piecewise continuous or piecewise refractive ones, to obtain one more crossing limit cycle of small amplitude creating from a sliding or escaping segment, as it was explained in Section 2.

Now we are able to prove the last results. It is important to note that we confine the dynamics to an invariant sphere of fixed radius, which remains unchanged with the considered perturbations because the notion of 2-dimensional limit cycle make no sense if the perturbations do not keep the 2-dimensional spheres invariant.

Proposition 5.1. *Consider the system*

$$\begin{aligned}\dot{x} &= 2\alpha y + \frac{9}{20}z - xz - 2\alpha y^2 - \frac{89}{20}yz - \alpha z^2,\\ \dot{y} &= -2\alpha x + 2z + 2\alpha xy + 3xz - 2yz,\\ \dot{z} &= -\frac{9}{20}x - 2y + x^2 + \frac{29}{20}xy + \alpha xz + 2y^2.\end{aligned} \tag{5.6}$$

Then, for $\alpha = \pm\sqrt{857/488}$ there exists a piecewise quadratic perturbation in $\mathcal{X}$ such that at least 9 hyperbolic crossing limit cycles of small amplitude bifurcate from the equilibrium point $(0,1,0)$ on $\mathbb{S}^2_1$.

Proof. Let $\alpha = \pm\sqrt{857/488}$. Note that system (5.6) is obtained doing $a_1 = -2\alpha, a_4 = 0, a_5 = 1, a_7 = 29/20, a_8 = \alpha, a_{11} = 2$, and $w = 2$ in (5.2). It was proved in [6] that the equilibrium point $p = (0,1,0)$ of (5.6) is a weak focus of fourth-order and that there exist 4 small amplitude limit cycles, on $\mathbb{S}^2_1$, bifurcating from p considering an analytical perturbation of (5.6) inside family (5.2). Now we consider a piecewise smooth perturbation $(a_1^\pm, a_5^\pm, a_7^\pm, a_8^\pm, a_{11}^\pm, w^\pm) = (-2\alpha + \varepsilon_1^\pm, 1 + \varepsilon_2^\pm, 29/20 + \varepsilon_3^\pm, \alpha + \varepsilon_4^\pm, 2 + \varepsilon_5^\pm, 2 + \varepsilon_6^\pm)$ in the piecewise projected system (5.3). As we saw before, we consider the separation set $\{(u,v) \in \mathbb{R}^2 : v = 0\}$ of the projected system and then we consider the perturbative parameter $\varepsilon^+ = (\varepsilon_1^+, \ldots, \varepsilon_6^+)$ for $v > 0$, $\varepsilon^- = (\varepsilon_1^-, \ldots, \varepsilon_6^-)$ for $v < 0$, and joining all $\varepsilon = (\varepsilon_1^+, \ldots, \varepsilon_6^+, \varepsilon_1^-, \ldots, \varepsilon_6^-)$. Let $L_i(\varepsilon)$ be the corresponding Lyapunov constants. Using the method explained in Section 2.4 we compute the Taylor series of these Lyapunov constants up to first-order with respect to ε, $L_i^{[1]}(\varepsilon)$, and we write $L_i(\varepsilon) = L_i^{[1]}(\varepsilon) + \mathcal{O}_2(\varepsilon)$, where

$$\begin{aligned}L_2^{[1]}(\varepsilon) =& -2\varepsilon_2^- + 2\varepsilon_5^- - \varepsilon_6^- + 2\varepsilon_2^+ - 2\varepsilon_5^+ + \varepsilon_6^+,\\ L_3^{[1]}(\varepsilon) =&\ 15616\varepsilon_2^- - 15616\varepsilon_5^- + 7808\varepsilon_6^- - 15616\varepsilon_2^+ + 15616\varepsilon_5^+ - 7808\varepsilon_6^+\\ &+ 13\pi\sqrt{104554}\left(-2\varepsilon_2^- + \varepsilon_5^- - 2\varepsilon_2^+ + \varepsilon_5^+\right) + \pi\left(427\varepsilon_1^- + 854\varepsilon_4^- + 427\varepsilon_1^+ + 854\varepsilon_4^+\right),\\ L_4^{[1]}(\varepsilon) =& -5916624\varepsilon_2^- + 1112640\varepsilon_3^- + 5433936\varepsilon_5^- - 2314584\varepsilon_6^- + 5916624\varepsilon_2^+ - 1112640\varepsilon_3^+\\ &- 5433936\varepsilon_5^+ + 2314584\varepsilon_6^+ + \sqrt{104554}\left(-80\varepsilon_1^- - 160\varepsilon_4^- + 80\varepsilon_1^+ + 160\varepsilon_4^+\right.\\ &\left.+\pi\left(9750\varepsilon_2^- - 4875\varepsilon_5^- + 9750\varepsilon_2^+ - 4875\varepsilon_5^+\right)\right) + \pi\left(-160125\varepsilon_1^- - 320250\varepsilon_4^-\right.\\ &\left.-160125\varepsilon_1^+ - 320250\varepsilon_4^+\right),\end{aligned} \tag{5.7}$$

we omit the expressions of $L_i^{[1]}(\varepsilon)$, $5 \le i \le 9$, because of their size.

We get $L_2^{[1]}(0) = \cdots = L_8^{[1]}(0) = 0$ and $L_9^{[1]}(0) \neq 0$. Hence, as the matrix formed with the coefficients of $(L_2^{[1]}, \ldots, L_8^{[1]})$ with respect to ε has rank 7, we obtain eight hyperbolic crossing limit cycles of small amplitude bifurcating from the origin adding the trace parameter and using the Implicit Function Theorem and then the derivation-division algorithm (see again [29]). Finally, adding the sliding parameter we obtain the ninth hyperbolic crossing limit cycle of small amplitude. □

Proposition 5.2. *Consider the system*

$$\begin{aligned}\dot{x} &= -\frac{4}{5}y - \frac{13}{8}z - \frac{5}{2}xz + \frac{4}{5}y^2 - \frac{59}{8}yz - z^2,\\ \dot{y} &= \frac{4}{5}x + 2z - \frac{4}{5}xy + 8xz - 2yz,\\ \dot{z} &= \frac{13}{8}x - 2y + \frac{5}{2}x^2 - \frac{5}{8}xy + xz + 2y^2.\end{aligned} \tag{5.8}$$

(a) There exists a continuous quadratic perturbation of (5.8) in $\mathcal{X}$ such that at least 5 hyperbolic crossing limit cycles of small amplitude bifurcate from the equilibrium point $(0,1,0)$ on $\mathbb{S}^2_1$.

(b) There exists a refractive quadratic perturbation of (5.8) in $\mathcal{X}$ such that at least 6 hyperbolic crossing limit cycles of small amplitude bifurcate from the equilibrium point $(0,1,0)$ on $\mathbb{S}^2_1$.

(c) There exists a piecewise quadratic perturbation of (5.8) in $\mathcal{X}$ such that at least 10 hyperbolic crossing limit cycles of small amplitude bifurcate from the equilibrium point $(0,1,0)$ on $\mathbb{S}^2_1$.

Proof. It was proved in [6] that system (5.2), with $w \neq 1$, has a center at the origin when its coefficients satisfy the conditions,

$$a_1 = \frac{w^2-1}{w^2+1}a_8, \quad a_4 = 0, \quad a_5 = \frac{w^2+1}{w^2-1}a_{11}, \quad \text{and} \quad a_7 = \frac{1}{w^2+1} - \frac{1}{(w^2+1)}a_8^2 - \frac{w^2+1}{(w^2-1)^2}a_{11}^2,$$

exhibiting an inverse integral factor for the system. Moreover, there exists an analytical perturbation inside family (5.2) such that at least 3 small amplitude limit cycles bifurcate from the equilibrium point $(0,1,0)$ on $\mathbb{S}^2_1$. Thus, as system (5.8) is obtained doing $a_1 = 4/5, a_4 = 0, a_5 = 5/2, a_7 = -5/8, a_8 = 1, a_{11} = 2$, and $w = 3$ in (5.2) it has a center at $(0,1,0)$. So, we take the parameter values $(a_1, a_5, a_7, a_8, a_{11}, w)$ satisfying it and we consider the piecewise smooth perturbation $(a_1, a_5, a_7, a_8, a_{11}, w) = (4/5 + \varepsilon_1^{\pm}, 5/2 + \varepsilon_2^{\pm}, -5/8 + \varepsilon_3^{\pm}, 1 + \varepsilon_4^{\pm}, 2 + \varepsilon_5^{\pm}, 3 + \varepsilon_6^{\pm})$ in the projected system (5.3). As we saw before, we consider the separation set $\{(u,v) \in \mathbb{R}^2 : v = 0\}$ of the projected system and then we consider the perturbative parameter $\varepsilon^+ = (\varepsilon_1^+, \dots, \varepsilon_6^+)$ for $v > 0$, $\varepsilon^- = (\varepsilon_1^-, \dots, \varepsilon_6^-)$ for $v < 0$ and $\varepsilon = (\varepsilon_1^+, \dots, \varepsilon_6^+, \varepsilon_1^-, \dots, \varepsilon_6^-)$. We denote by $L_i(\varepsilon)$, the corresponding Lyapunov constants. When $\varepsilon = 0$ the origin is a center and then $L_i(0) = 0$ for all i. Using the method explained in Section 2.4 we compute the Taylor series of these Lyapunov constants up to first-order with respect to ε, $L_i^{[1]}(\varepsilon)$, and we write $L_i(\varepsilon) = L_i^{[1]}(\varepsilon) + \mathcal{O}_2(\varepsilon)$ where

$$\begin{aligned}L_2^{[1]}(\varepsilon) =& -48\varepsilon_2^- + 33\varepsilon_5^- - 36\varepsilon_6^- + 48\varepsilon_2^+ - 33\varepsilon_5^+ + 36\varepsilon_6^+,\\ L_3^{[1]}(\varepsilon) =& -7680\varepsilon_2^- + 5280\varepsilon_5^- - 5760\varepsilon_6^- + 7680\varepsilon_2^+ - 5280\varepsilon_5^+ + 5760\varepsilon_6^+ + \pi\,(675\varepsilon_1^- - 56\varepsilon_2^-\\ & -540\varepsilon_4^- + 70\varepsilon_5^- - 102\varepsilon_6^- + 675\varepsilon_1^+ - 56\varepsilon_2^+ - 540\varepsilon_4^+ + 70\varepsilon_5^+ - 102\varepsilon_6^+),\\ L_4^{[1]}(\varepsilon) =& \,6380640\varepsilon_1^- - 263540576\varepsilon_2^- - 50641200\varepsilon_3^- - 6099840\varepsilon_4^- + 152862163\varepsilon_5^-\\ & - 159148902\varepsilon_6^- - 6380640\varepsilon_1^+ + 263540576\varepsilon_2^+ + 50641200\varepsilon_3^+ + 6099840\varepsilon_4^+\\ & - 152862163\varepsilon_5^+ + 159148902\varepsilon_6^+ + \pi\,(13668750\varepsilon_1^- - 1134000\varepsilon_2^- - 10935000\varepsilon_4^-\\ & +1417500\varepsilon_5^- - 2065500\varepsilon_6^- + 13668750\varepsilon_1^+ - 1134000\varepsilon_2^+ - 10935000\varepsilon_4^+\\ & +1417500\varepsilon_5^+ - 2065500\varepsilon_6^+).\end{aligned} \tag{5.9}$$

We omit the expressions of $L_i^{[1]}(\varepsilon)$, $5 \le i \le 12$ because of their size.

In the case (a) (resp. (b)) we consider a continuous (resp. refractive) perturbation of this center family. As we saw above, it implies that $\varepsilon_i^+ = \varepsilon_i^-$, for $i = 1,2,3,4$ (resp. $\varepsilon_i^+ = \varepsilon_i^-$, for $i = 2,3,4$). With this assumption, the matrix formed by the coefficients of $(L_2^{[1]}, \dots, L_7^{[1]})$ with respect to ε has rank 5 (resp. 6). Adding the trace parameter and using the Melnikov theory, as we have explained in the Section 2.4, we obtain 5 (resp. 6) hyperbolic crossing limit cycles of small amplitude bifurcating from the origin.

Finally, in the case (c), the proof follows because the matrix formed by the coefficients of $(L_2^{[1]}, \ldots, L_{12}^{[1]})$ with respect to ε has rank 9, so adding the trace parameter and using the Melnikov theory, as we explained in the Section 2.4, we get 9 hyperbolic crossing limit cycles of small amplitude bifurcating from the origin. Adding the sliding or escaping segments we obtain one more crossing limit cycle and the proof follows. □

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Existence of positive solutions for generalized quasilinear elliptic equations with Sobolev critical growth

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Abstract. In this paper, we are devoted to establishing that the existence of positive solutions for a class of generalized quasilinear elliptic equations in $\mathbb{R}^N$ with Sobolev critical growth, which have appeared from plasma physics, as well as high-power ultrashort laser in matter. To begin, by changing the variable, quasilinear equations are transformed into semilinear equations. The positive solutions to semilinear equations are then presented using the Mountain Pass Theorem for locally Lipschitz functionals and the Concentration-Compactness Principle. Finally, an inverse translation reveals the presence of positive solutions to the original quasilinear equations.

Keywords: variational methods, Sobolev critical growth, locally Lipschitz functional.

2020 Mathematics Subject Classification: 35J20, 35J62.

1 Introduction

In this paper, we aim at studying the existence of positive solutions for the following generalized quasilinear elliptic equations

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $g \in C^1(\mathbb{R}, \mathbb{R}^+)$ is an even function and $g'(t) \geq 0$ for all $t \geq 0$ and $g(0) = 1$, the potential $V \in C(\mathbb{R}^N, \mathbb{R})$, h is a measurable function defined on $\mathbb{R}^N \times \mathbb{R}$. These equations are closely related to the existence of standing wave solutions for the following quasilinear Schrödinger equations

$$i\partial_t z = -\Delta z + E(x)z - \sigma(x, |z|^2)z - \Delta[l(|z|^2)]l'(|z|^2)z, \tag{1.2}$$

where $z : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $E : \mathbb{R}^N \to \mathbb{R}$ is a potential function and $\sigma : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $l : \mathbb{R} \to \mathbb{R}$ are suitable functions. Notice that equation (1.2) can be reduced to elliptic equations with the

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following formal structure (see [15]) by setting $z(x,t) = \exp(-iFt)u(x)$, where $F \in \mathbb{R}$ and u is a real function,

$$-\Delta u + V(x)u - \Delta[l(u^2)]l'(u^2)u = h(x,u), \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Furthermore, we take $g^2(u) = 1 + \frac{(l'(u^2))^2}{2}$, then (1.3) turns into (1.1) (see [16,28]).

To the best of our knowledge, the quasilinear equation (1.1) have been utilized to simulate a range of physical phenomena corresponding to various types of $g(u)$ in several fields of mathematical physics. For instance, the case $g^2(u) = 1 + 2u^2$, that is, $l(u) = u$ in (1.3), we get

$$-\Delta u + V(x)u - u\Delta(u^2) = h(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

which simulates the time evolution of the condensate wave function in a superfluid film (see [20]). For equation (1.1), if we take $g^2(u) = 1 + \frac{u^2}{2(1+u^2)}$, that is, $l(u) = \sqrt{1+u}$, we get

$$-\Delta u + V(x)u - [\Delta\sqrt{1+u^2}]\frac{u}{2\sqrt{1+u^2}}u = h(x,u), \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

Equation (1.5) is often used as a model of the self-channeling of a high-power ultrashort laser in matter (see [8, 12, 25]). For more physical backgrounds about equation (1.1), readers can refer to [7, 19, 23, 24] and the references within. So, the study for general quasilinear elliptic equation (1.1) is meaningful and important.

In [21], the quasilinear equation (1.4) was firstly transformed to a semilinear one by using a change of variable. Then, they chose an Orlicz space as the working space and obtained the existence of positive solutions to equation (1.4) by using the Mountain Pass theorem. Afterwards, the same change of variable was applied in [14,30,31], but the usual Sobolev space framework was used as the working space. For example, Silva and Vieira in [30] obtained the existence of positive solutions of equation (1.4) in an asymptotically periodic condition with critical growth. In [28], Shen and Wang used a new change of variable developed by (1.4) to show the existence of positive solutions of equation (1.1) when $h(x,u)$ was superlinear and subcritical. Following that, by applying the same modification in variable as in [28] and the classical Mountain Pass Theorem, Deng et al. in [16] proved the existence of a positive solution for equation (1.1) where nonlinearity was critical growth, and Shi and Chen in [29] proved the existence of positive solutions of equation (1.1) when nonlinearity was periodic or asymptotically periodic cases with critical growth. On the other hand, Candela et al. in [9] considered the more general quasilinear ellptic equation:

$$-\operatorname{div}(A(x,u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p}A_u(x,u)|\nabla u|^p + V(x)|u|^{p-2}u = h(x,u), \quad \text{in } \mathbb{R}^N,$$

with $p > 1$ and $A : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that $A_u(x,u) = \frac{\partial A}{\partial u}(x,u)$. When $A(x,u) = g^2(u)$ and $p = 2$, the above equation turns to (1.1) with $N = 3$. They employed an entirely new approach to deal with (1.1) because the arguments of change of variable frequently need $g(u)$ to meet certain particular assumptions, and the features of $g(u)$ directly affect the hypotheses on the nonlinear term $h(x,u)$. By using the Mountain Pass Theorem with the weak Cerami Palais Smale condition, they established the existence of week-bounded solutions under certain appropriate hypotheses on $V(x)$ and $h(x,u)$, which are independent of $g(u)$.

It is worth pointing out that the continuity of nonlinearity is always required in these aforementioned papers. It seems that there are no results concerning equation (1.1) with discontinuous nonlinearities. Actually, many free boundary problems and obstacle problems arising from physics can be described with nonlinear partial differential equations with discontinuous nonlinearities. For more problems with discontinuous nonlinearities, readers can refer to [1,3,6,10,22,26,35] and their references. Hence, our goal is to discuss the existence of positive solutions for problem (1.1) with discontinuous nonlinearities. In this paper, we consider equation (1.1) with $h(x,t) = \kappa f(x,t) + g(t)|G(t)|^{2^*-2}G(t)$, where $\kappa > 0$, $G(t) = \int_0^t g(s)ds$, $2^* = \frac{2N}{N-2}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a discontinuous function. We rewrite equation (1.1) as follows:

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = \kappa f(x,u) + g(u)|G(u)|^{2^*-2}G(u), \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

The hypotheses on the function V and f are the following:

(V_1) There exist a function $V_p(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\mathbb{Z}^N$-periodic with respect to variable x, satisfying

$$V_p(x) \geq V_0, \qquad \forall x \in \mathbb{R}^N,$$

and a function $W(x) \in L^{\frac{N}{2}}(\mathbb{R}^N) \cap C(\mathbb{R}^N, \mathbb{R})$ with $W(x) \geq 0$ such that

$$V(x) = V_p(x) - W(x) \geq W_0, \qquad \forall x \in \mathbb{R}^N,$$

where V_0, W_0 are positive constants and the inequality $W(x) \geq 0$ is strict on a subset of positive measure in $\mathbb{R}^N$.

(f_1) $f(x,t)$ is a measurable function defined on $\mathbb{R}^N \times \mathbb{R}$ and the functions

$$\underline{f}(x,t) := \lim_{\delta \downarrow 0} \text{ess inf}\{f(x,s); |t-s| < \delta\}$$

and

$$\overline{f}(x,t) := \lim_{\delta \downarrow 0} \text{ess sup}\{f(x,s); |t-s| < \delta\},$$

are N-mensurable (see [11]).

(f_2) $f(x,t) \equiv 0$ if $t \leq 0$ and $\limsup_{t \to 0^+} \frac{f(x,t)}{t} = 0$, uniformly in $x \in \mathbb{R}^N$.

(f_3) There are $C > 0$ and $q \in (2, 2^*)$ such that

$$|f(x,t)| \leq C(1 + g(t)|G(t)|^{q-1}), \qquad \forall (x,t) \in \mathbb{R}^N \times [0, \infty).$$

(f_4) There exists $\theta \in (2, 2^*)$ such that

$$0 \leq \theta g(t)F(x,t) \leq G(t)\min\{f(x,t), \varrho\}, \qquad \forall \varrho \in \partial_t F(x,t) \quad \text{and} \quad \forall (x,t) \in \mathbb{R}^N \times [0, \infty),$$

where $F(x,t) = \int_0^t f(x,s)ds$ and $\partial_t F(x,t) := [\underline{f}(x,t), \overline{f}(x,t)]$ denotes the generalized gradient of $F(x,t)$ with respect to variable t (see [4]).

Here, we provide a nonlinearity f that satisfies the assumptions above as following: fixed $T > 0$, let us consider the function

$$f(x,t) = \begin{cases} 0, & t \in (-\infty, 0], \\ g(t)(G(t))^{q-2}\left[G(t) - \arctan(G(t))\right], & t \in (0, T], \\ g(t)(G(t))^{q-2}\left[G(t) - \mu \arctan(G(t))\right], & t \in (T, +\infty), \end{cases}$$

where $0 < \mu < 1$.

The asymptotic periodicity of f at infinity is given by the following condition:

(f_5) There exists a function $f_p(x,t) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $\mathbb{Z}^N$-periodic with respect to variable x, such that $f_p(x,t) \equiv 0$ if $t \le 0$ and $\frac{f_p(x,t)}{g(t)G(t)}$ is nondecreasing for all $t > 0$.

(f_6) There exists $\nu \in (2, 2^*)$ such that

$$0 \le \nu g(t) F_p(x,t) \le G(t) f_p(x,t), \qquad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $F_p(x,t) = \int_0^t f_p(x,s) ds$.

(f_7) $F(x,t) \ge F_p(x,t) = \int_0^t f_p(x,s) ds$, $\forall (x,t) \in \mathbb{R}^N \times [0,\infty)$ and

$$|f(x,t) - f_p(x,t)| \le \pi(x) g(t) |G(t)|^{q-1}, \qquad \forall (x,t) \in \mathbb{R}^N \times [0,\infty),$$

$$|\varrho - f_p(x,t)| \le \pi(x) g(t) |G(t)|^{q-1}, \qquad \forall (x,t) \in \mathbb{R}^N \times [0,\infty), \text{ and } \varrho \in \partial_t F(x,t),$$

where $\pi(x) > 0$ for all $x \in \mathbb{R}^N$, $\pi(x) \in L^\infty(\mathbb{R}^N)$, and $\pi(x) \to 0$ as $|x| \to \infty$.

Next, we provide a suitable example of function $f(x,t)$ that satisfies the assumptions (f_1)–(f_7). Fixed $T > 0$, let

$$f(x,t) = \begin{cases} 0, & t \in (-\infty, 0], \\ f_p(x,t), & t \in (0, T], \\ f_p(x,t) + \mu \exp(-|x|) g(t) (G(t))^{q-2} \arctan(G(t)), & t \in (T, +\infty), \end{cases}$$

where $0 < \mu < 1$ and

$$f_p(x,t) = \begin{cases} 0, & t \in (-\infty, 0], \\ g(t)(G(t))^{q-2} \left[G(t) - \arctan(G(t)) \right], & t \in (0, +\infty). \end{cases}$$

Since $f(x,t)$ is discontinuous, inspired by [11] and [27], we give the definition of solutions for (1.6). We say a function u is a solution for the multivalued problem (1.6) if it satisfies

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u - g(u)|G(u)|^{2^*-2}G(u) \in \kappa \hat{f}(x,u), \quad \text{a.e. in } \mathbb{R}^N, \tag{1.7}$$

where $\hat{f}$ is the multivalued function

$$\hat{f}(x,t) = [\underline{f}(x,t), \overline{f}(x,t)].$$

Below, we describe the main results of this paper.

Theorem 1.1 (The non periodic case)**.** *Assume that (V_1) and (f_1)–(f_7) are satisfied. Then, there exists $\kappa^* > 0$ such that the problem (1.6) possesses a positive solution for all $\kappa \ge \kappa^*$.*

When f is $\mathbb{Z}^N$ periodic and $V = V_p$ given by (V_1), problem (1.6) can turns to the following periodic problem:

$$-\operatorname{di} v(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V_p(x)u = \kappa f(x,u) + g(u)|G(u)|^{2^*-2}G(u), \quad \text{in } \mathbb{R}^N. \tag{1.8}$$

For periodic problem (1.8), we may state:

Theorem 1.2 (The periodic case)**.** *Assume that (f_1)–(f_4) are satisfied and f is $\mathbb{Z}^N$ periodic. Then, there exists $\kappa^* > 0$ such that the problem (1.8) possesses a positive solution for all $\kappa \ge \kappa^*$.*

Remark 1.3. As is known to all, the discontinuity of nonlinearity causes a lack of functional differentiability. In this paper, as f is discontinuous, the modified energy functional associated with (1.6) is only locally Lipschitz continuous. The classical variational methods cannot be directly utilized for nonsmooth functionals. For smooth functionals, it is essential that the energy functional can be studied on the Nehari manifold and that the mountain pass level is equal to the minimum of the energy functional on the Nehari manifold. However, these results are not valid for nonsmooth cases. Hence, the proof of the existence of solutions for equation (1.6) is more difficult.

Remark 1.4. Similar equations have been considered in [9]. However, our assumptions on nonliearities are critical growth and discontinuous.

Below we give a sketch of the proofs of our main results:

1) Firstly, we make a change of variable to reduce the quasilinear problem (1.6) to a semilinear problem (2.1) which can be well defined in $H^1(\mathbb{R}^N)$ and satisfies the geometric hypotheses of the Mountain Pass Theorem. Hence, we get a $(PS)_c$ sequence associated with the minimax level. And by using standard arguments, we show that the weak convergence limit of $(PS)_c$ sequence is a solution of the problem (2.1).

2) Furthermore, for adopting the similar technical scheme due to [30], we assume this solution is trivial. Thereby, we get a nontrivial critical point of the functional associated with the periodic case, and use the nontrivial critical point to construct a special path to prove that the maximum of the functional associated with (2.1) is strictly less than the one of the functional associated with the periodic case, which is a contradiction.

3) Hence, we obtain the existence of nontrivial solutions of the problem (2.1). Finally, by Lemma 2.2, Theorem 1.1 is proved.

The outline of the article is as follows: in Section 2, we introduce the variational setting associated with problem (1.6) and some basic knowledge of the critical point theory of locally Lipschitz continuous functionals. In Section 3, we prove the geometric structure of the Mountain Pass Theorem and some preliminary lemmas. In Section 4 and Section 5, we prove Theorem 1.1 and Theorem 1.2, respectively.

Throughout this paper, we make use of the following notations:

- M, C, C_ε denote positive constants, which may vary from line to line.
- $L^p(\mathbb{R}^N)$ denotes the Lebesgue space with the norm $\|\cdot\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}}$ for $1 \leq p \leq \infty$.
- The dual space of a Banach space X will be denoted by X^*.
- The strong (respectively, weak) convergence is denoted by $\to$ (respectively, $\rightharpoonup$).
- $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$.
- Denote the function space $D^{1,2}(\mathbb{R}^N) := \{v \in L^{2^*}(\mathbb{R}^N) : |\nabla v| \in L^2(\mathbb{R}^N)\}$. Here, S is the best constant that verifies
$$S\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{\frac{2}{2^*}} \leq \int_{\mathbb{R}^N} |\nabla u|^2, \quad \text{for all } u \in D^{1,2}(\mathbb{R}^N).$$
- Denote the function space $H^1(\mathbb{R}^N) = \{v \in L^2(\mathbb{R}^N) : |\nabla v| \in L^2(\mathbb{R}^N)\}$ with the usual norm
$$\|v\|_{H^1}^2 = \int_{\mathbb{R}^N} \left(|\nabla v|^2 + |v|^2\right).$$

2 Variational setting and preliminary knowledge

From the variational point of view, we note that we may not directly apply the variational method to deal with the problem (1.6), since the functional associated with (1.6) may not be well defined in general $H^1(\mathbb{R}^N)$. The first difficulty associated with (1.6) is to find an appropriate function space where the functional responding to (1.6) is well defined. In order to overcome this difficulty, we we change the variables $u = G^{-1}(v)$, where G is defined as

$$v = G(u) = \int_0^u g(t)dt$$

by Shen and Wang in [28].

Now, we present some important properties about the functions g, G and G^{-1}, which proofs can be found in [16].

Lemma 2.1. *The functions $g(s)$ and $G(s) = \int_0^s g(t)dt$ enjoy the following properties.*

(i) *$G(s)$ and $G^{-1}(t)$ are odd and strictly increasing.*

(ii) *For all $s \geq 0$, $t \geq 0$,*

$$G(s) \leq g(s)s, \qquad G^{-1}(t) \leq \frac{t}{g(0)} = t.$$

(iii) *For all $t \geq 0$, $\frac{G^{-1}(t)}{t}$ is nonincreasing and*

$$\lim_{t\to 0} \frac{G^{-1}(t)}{t} = \frac{1}{g(0)} = 1, \qquad \lim_{t\to\infty} \frac{G^{-1}(t)}{t} = \begin{cases} \frac{1}{g(\infty)}, & \text{if } g \text{ is bounded,} \\ o(1), & \text{if } g \text{ is unbounded.} \end{cases}$$

(iv) *Denote $T(t) = \frac{G^{-1}(t)}{g(G^{-1}(t))}$, then $t^2T'(t) \leq T(t)t$, $\forall t \in \mathbb{R}$.*

After the change of variable $u = G^{-1}(v)$, the problem(1.6) can be rewritten as follows:

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} + |v|^{2^*-2}v, \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

As a consequence of Lemma 2.1, the functional associated with (2.1) is well defined in $H^1(\mathbb{R}^N)$.

Lemma 2.2. *From Lemma 2.1, direct calculations demonstrate that $u = G^{-1}(v)$ shall be a solution of the equation (1.6) when v is a solution of the problem (2.1). That is to say, $v \in H^1(\mathbb{R}^N)$ satisfies*

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} - |v|^{2^*-2}v \in \frac{\hat{f}(x, G^{-1}(v))}{g(G^{-1}(v))} \quad \text{a.e. in } \mathbb{R}^N, \tag{2.2}$$

where

$$\frac{\hat{f}(x, G^{-1}(v))}{g(G^{-1}(v))} = \left[\frac{\underline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}, \frac{\overline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}\right].$$

From the above commentaries, in order to find solutions to equation (1.6), it suffices to study the existence of solutions to equation (2.1). The second difficulty associated with (2.1) is that the classical critical theory for smooth functionals cannot be directly applied to (2.1) since the function $f(x, G^{-1}(t))$ is discontinuous. To study nonsmooth problems like (2.1), we will apply the critical point theory of locally Lipschitz continuous functionals developed by Clarke [13]. For the convenience of the readers, here we provide some relevant knowledge of the critical point theory of locally Lipschitz continuous functionals.

Let X be a real Banach space and $I : X \to \mathbb{R}$.

Definition 2.3 ([27])**.** If given $v \in X$ there exists an open neighborhood $U := U_v \subset X$ and some constant $C_U > 0$ such that

$$|I(v_1) - I(v_2)| \leq C_U \|v_1 - v_2\|_X, \qquad v_i \in U, \ \ i = 1, 2.$$

We call that I is locally Lipschitz continuous ($I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ for short).

Definition 2.4 ([27])**.** The generalized directional derivative of $I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ at v in the direction of $\tilde{v} \in X$ is defined by

$$I^\circ(v; \tilde{v}) = \limsup_{u \to 0 \ t \downarrow 0} \frac{I(v + u + t\tilde{v}) - I(v + u)}{t}.$$

The definition 2.4 implies that $I^\circ(v; .)$ is continuous, convex and its subdifferential at $w \in X$ is given by

$$\partial I^\circ(v; w) = \{\mu \in X^*; I^\circ(v; u) \geq I^\circ(v; w) + \langle \mu, u - w \rangle, \quad \forall u \in X\},$$

where X^* is the dual of X and $\langle ., . \rangle$ is the duality paring between X^* and X.

Definition 2.5. ([27]) The general gradient of $I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ at v is the set

$$\partial I(v) = \{\mu \in X^*; I^\circ(v; u) \geq \langle \mu, u \rangle, \quad \forall u \in X\}.$$

Since $I^\circ(v; 0) = 0$, $\partial I(v)$ is the subdifferential of $I^\circ(v; 0)$. Moreover, $\partial I(v) \subset X^*$ is convex, nonempty and weak* compact. If I is C^1 functional, $\partial I(v) = \{I'(v)\}$. We denote by $\lambda_I(u)$ the following real number

$$\lambda_I(v) := \min\{\|\mu\|_{X^*};\ \mu \in \partial I(v)\}.$$

Definition 2.6 ([27])**.** An element $v \in X$ is a critical point of I if $0 \in \partial I(v)$ or equivalently, when $\lambda_I(v) = 0$.

Lemma 2.7. *If $I_1 \in C^1(X, \mathbb{R})$ and $I_2 \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$, then*

$$\partial(I_1 + I_2)(v) = \{I_1'(v)\} + \partial I_2(v), \qquad \forall v \in X.$$

Lemma 2.8 ([13] and [34])**.** *Let Y be a Banach space and $j : Y \to X$ be a continuously differentiable function. Then $I \circ j$ is locally Lipschitz and*

$$\partial(I \circ j)(v) \subset \partial I(j(v)) \circ j'(v), \qquad \forall v \in Y.$$

Lemma 2.9 ([13])**.** *Let $I \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbb{R})$ with $I(0) = 0$ and X be a Banach space. Suppose there are constants $\alpha,\ \rho > 0$ and function $e \in X$, such that* *(i)* $I(v) \geq \alpha$, *for all* $v \in X$ *with* $\|v\|_X = \rho$,

(ii) $I(e) < 0$ *and* $\|e\|_X > r$.

Let

$$c = \inf_{\gamma \in \Gamma_I} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma_I = \{\gamma \in C([0,1], X) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Then, $c \geq \alpha$ *and there exists a sequence* $\{v_n\} \subset X$ *satisfying* $I(v_n) \to c$ *and* $\lambda_I(v_n) \to 0$. *The sequence* $\{v_n\}$ *is called a* $(PS)_c$ *sequence for* I.

3 Some preliminary lemmas

Hypotheses (f_1)–(f_3) imply that, for any $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\begin{aligned} |f(x,t)| &\leq \varepsilon|t| + C_\varepsilon g(t)|G(t)|^{q-1}, \qquad \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^N, \\ |F(x,t)| &\leq \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{q}|G(t)|^q, \qquad \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^N. \end{aligned} \tag{3.1}$$

From the second inequality of (3.1) and Lemma 2.1-(ii), we can prove

$$\Psi(v) = \int_{\mathbb{R}^N} F(x, G^{-1}(v)) \leq \int_{\mathbb{R}^N} \left(\frac{\varepsilon}{2}|v|^2 + \frac{C_\varepsilon}{q}|v|^q\right) \leq C(\|v\|_2 + \|v\|_q), \tag{3.2}$$

so functional Ψ is well defined in $H^1(\mathbb{R}^N)$. However, in order to apply variational methods for locally Lipschitz functionals, it is preferable to deal with the functional Ψ in a more appropriate space, that is $\Psi : L^\Phi(\mathbb{R}^N) \to \mathbb{R}$, for $\Phi(t) = |t|^2 + |t|^q$, where $L^\Phi(\mathbb{R}^N)$ denotes the Orlicz space associated with the N-function Φ. In this paper, we are working in $\mathbb{R}^N$ and the conditions on f yield

$$|F(x, G^{-1}(t))| \leq C(|t|^2 + |t|^q), \qquad \forall t \in \mathbb{R}, \tag{3.3}$$

then Ψ is not well defined in $L^p(\mathbb{R}^N)$. The above estimate involving the function F suggests that the best space to work is the Orlicz space $L^\Phi(\mathbb{R}^N)$. In bounded domains, the Orlicz space $L^\Phi(\mathbb{R}^N)$ is not necessary. In this case, (3.2) implies that the functional Ψ is well defined in $L^p(\Omega)$. Since $2 < q < 2^*$, we obtain that the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^\Phi(\mathbb{R}^N)$ is continuous and Φ satisfies Δ_2 condition which ensures that $L^\Phi(\mathbb{R}^N)$ and $L^{\tilde{\Phi}}(\mathbb{R}^N)$ are reflexive spaces ($\tilde{\Phi}$ is the conjugate function of Φ (see [17])). Hence, given $\varsigma \in (L^\Phi(\mathbb{R}^N))^*$, we get

$$\varsigma(v) = \int_{\mathbb{R}^N} u_\varsigma v, \qquad \forall v \in L^\Phi(\mathbb{R}^N),$$

for some $u_\varsigma \in L^{\tilde{\Phi}}(\mathbb{R}^N)$. Essentially, by the definition of Φ and (f_1)–(f_3) the conditions below occur:

$$\begin{aligned} |\zeta| &\leq \varepsilon|t| + C_\varepsilon|t|^{q-1} \leq C\Phi'(|t|), \qquad \forall \zeta \in \partial_t F(x, G^{-1}(t)), \\ |F(x, G^{-1}(t))| &\leq \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{q}|t|^{q-1} \leq C\Phi(t), \end{aligned} \tag{3.4}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$. Here, $\partial_t F(x, G^{-1}(t))$ denotes the generalized gradient of $F(x, G^{-1}(t))$ with respect to variable t. The above information involving Ψ and Φ is crucial in the below.

The next two lemmas establish important properties of the functional Ψ given in (3.2).

Lemma 3.1 ([2, Theorem 4.1] and [5, Theorem 4.2]). *Assume (3.4). Then, the functional* $\Psi : L^{\Phi}(\mathbb{R}^N) \to \mathbb{R}$ *given by*

$$\Psi(v) = \int_{\mathbb{R}^N} F(x, G^{-1}(v)), \qquad v \in L^{\Phi}(\mathbb{R}^N),$$

is well defined and $\Psi \in \mathrm{Lip}_{\mathrm{loc}}(L^{\Phi}(\mathbb{R}^N), \mathbb{R})$*. Furthermore,*

$$\partial\Psi(v) \subset \partial_t F(x, G^{-1}(v)), \qquad \forall\, v \in L^{\Phi}(\mathbb{R}^N),$$

in the sense that for every $\varrho^* \in \partial\Psi(v) \subset (L^{\Phi}(\mathbb{R}^N))^* \cong L^{\tilde{\Phi}}(\mathbb{R}^N)$ *there exists* $\varrho \in L^{\tilde{\Phi}}(\mathbb{R}^N)$ *such that*

$$\varrho(x) \in \partial_t F(x, G^{-1}(v(x))) = \left[\frac{\underline{f}(x, G^{-1}(v(x)))}{g(G^{-1}(v(x)))}, \frac{\overline{f}(x, G^{-1}(v(x)))}{g(G^{-1}(v(x)))} \right] \quad \text{a.e. in } \mathbb{R}^N$$

and

$$\langle \varrho^*, v \rangle = \int_{\mathbb{R}^N} \varrho v, \qquad \forall\, v \in L^{\Phi}(\mathbb{R}^N).$$

As a similar consequence of Proposition 2.3 in [5], we obtain the following lemma and the proof will be omitted. More details can be found in [5].

Lemma 3.2. *Assume (3.4). If* $\{v_n\} \subset H^1(\mathbb{R}^N)$ *satisfies* $v_n \rightharpoonup v$ *in* $H^1(\mathbb{R}^N)$ *and* $\varrho_n \in \partial\Psi(v_n)$ *satisfies* $\varrho_n \overset{*}{\rightharpoonup} \varrho$ *in* $(H^1(\mathbb{R}^N))^*$*, then* $\varrho \in \partial\Psi(v)$*.*

We consider $H^1(\mathbb{R}^N)$ endowed with the following norm

$$\|v\|^2 = \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)|v|^2 \right).$$

Under the assumption (V_1), the norm $\|\cdot\|$ is equivalent to the standard norm $\|\cdot\|_{H^1}$.

In order to get the positive solutions, we consider the functional corresponding to (2.1) given by $J(v) = Q(v) - \kappa\Psi(v)$, $v \in H^1(\mathbb{R}^N)$, where

$$Q(v) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)|G^{-1}(v)|^2 \right) - \frac{1}{2^*} \int_{\mathbb{R}^N} (v^+)^{2^*}$$

and

$$\Psi(v) = \int_{\mathbb{R}^N} F(x, G^{-1}(v)).$$

By standard arguments, we get the functional $Q \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and

$$\langle Q'(v), \varphi \rangle = \int_{\mathbb{R}^N} \left(\nabla u \nabla \varphi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \varphi \right) - \int_{\mathbb{R}^N} (v^+)^{2^*-1} \varphi,$$

for all $v, \varphi \in H^1(\mathbb{R}^N)$. Then, by Lemma 3.1, $J \in \mathrm{Lip}_{\mathrm{loc}}(H^1(\mathbb{R}^N), \mathbb{R})$ and

$$\partial J(v) = \{Q'(v)\} - \kappa \partial\Psi(v). \tag{3.5}$$

Lemma 3.3. *Assume that* (V_1) *and* (f_1)–(f_3) *are satisfied. Then there exist* $\rho, \alpha > 0$*, such that*

$$J(v) \geq \alpha, \qquad \forall v \in H^1(\mathbb{R}^N) \text{ with } \|v\| = \rho.$$

Proof. Since $\lim_{|t|\to 0} \frac{G^{-1}(t)}{t} = 1$, by (3.1), for any $\varepsilon > 0$, there is C_ε such that

$$|F(x, G^{-1}(t))| \le \varepsilon |t|^2 + C_\varepsilon |t|^q, \ \forall t \in \mathbb{R}.$$

From Lemma 2.1, we have

$$\lim_{s\to+\infty} \frac{G^{-1}(t)}{t} = \begin{cases} \dfrac{1}{g(\infty)}, & \text{if } g \text{ is bounded,} \\ o(1), & \text{if } g \text{ is unbounded.} \end{cases}$$

If g is bounded, in view of that $\frac{G^{-1}(t)}{t}$ is nonincreasing, we get

$$\begin{aligned} J(v) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)|G^{-1}(v)|^2\right) - \kappa \int_{\mathbb{R}^N} F(x, G^{-1}(v)) - \frac{1}{2^*}\int_{\mathbb{R}^N} (v^+)^{2^*} \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + W_0|G^{-1}(v)|^2\right) - \kappa \int_{\mathbb{R}^N} (\varepsilon|v|^2 + C_\varepsilon |v|^q) - \frac{1}{2^*}\int_{\mathbb{R}^N} (v^+)^{2^*} \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \left(\frac{W_0}{2g(\infty)} - \kappa\varepsilon\right)\int_{\mathbb{R}^N} |v|^2 - \kappa C_\varepsilon \int_{\mathbb{R}^N} |v|^q - \frac{1}{2^*}\int_{\mathbb{R}^N} |v|^{2^*}. \end{aligned} \tag{3.6}$$

If g is unbounded, we set $\Upsilon(t) := -\frac{1}{2}W_0|G^{-1}(t)|^2 + \kappa F(x, G^{-1}(t))$, then

$$\lim_{t\to 0} \frac{\Upsilon(t)}{t^2} = -\frac{W_0}{2} < 0, \qquad \lim_{t\to+\infty} \frac{\Upsilon(t)}{t^{2^*}} = 0.$$

Therefore,

$$\begin{aligned} J(v) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)|G^{-1}(v)|^2\right) - \kappa \int_{\mathbb{R}^N} F(x, G^{-1}(v)) - \frac{1}{2^*}\int_{\mathbb{R}^N} (v^+)^{2^*} \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 - \int_{\mathbb{R}^N} \Upsilon(v) - \frac{1}{2^*}\int_{\mathbb{R}^N} (v^+)^{2^*} \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \left(\frac{W_0}{2} - \varepsilon\right)\int_{\mathbb{R}^N} |v|^2 - C_\varepsilon \int_{\mathbb{R}^N} |v|^{2^*} - \frac{1}{2^*}\int_{\mathbb{R}^N} |v|^{2^*}. \end{aligned} \tag{3.7}$$

By (3.6), (3.7) and Sobolev's inequality, we get

$$J(v) \ge C\|v\|^2 - C\|v\|^q - C\|v\|^{2^*}.$$

Since $2 < q < 2^*$, taking $\rho > 0$ sufficiently small, we conclude that there exists $\alpha > 0$ such that

$$J(v) \ge \alpha, \qquad \forall v \in H^1(\mathbb{R}^N) \text{ with } \|v\| = \rho.$$

This proof is completed. □

Lemma 3.4. *Suppose that (V_1) and (f_4) are satisfied. Then, for all $\kappa > 0$, there exists function $e \in H^1(\mathbb{R}^N)$ such that $J(e) \le 0$ and $\|e\| > \rho$.*

Proof. Fixing $\phi \in H^1(\mathbb{R}^N)$ with $\phi \ge 0$ and $\phi \not\equiv 0$, by Lemma 2.1-(i), (ii), we get

$$\begin{aligned} J(t\phi) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|t\nabla\phi|^2 + V(x)|G^{-1}(t\phi)|^2\right) - \kappa \int_{\mathbb{R}^N} F(x, G^{-1}(t\phi)) - \frac{1}{2^*}\int_{\mathbb{R}^N} (t\phi)^{2^*} \\ &\le \frac{t^2}{2}\|\phi\|^2 - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} \phi^{2^*}. \end{aligned}$$

Then, we can choose some t_0 large enough such that $\|t_0\phi\| > \rho$ and $J(t_0\phi) < 0$. The lemma is completed when $e = t_0\phi$. □

Note that $J(0) = 0$ and by Lemma 3.3 and Lemma 3.4, (i) and (ii) of Lemma 2.9 are satisfied. Thereby, we may define

$$c_\kappa = \inf_{\gamma\in\Gamma_J} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma_J = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

By Lemma 2.9, there exists a sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ satisfying $J(v_n) \to c_\kappa$ and $\lambda_J(v_n) \to 0$. Namely, the sequence $\{v_n\}$ is a $(PS)_{c_\kappa}$ sequence for functional J.

Lemma 3.5. *Assume that (V_1), (f_1) and (f_4) hold. Then any $(PS)_{c_\kappa}$ sequence for J is bounded in $H^1(\mathbb{R}^N)$.*

Proof. Let $\{v_n\} \subset H^1(\mathbb{R}^N)$ be a $(PS)_{c_\kappa}$ sequence for J, that is,

$$J(v_n) \to c_\kappa \quad \text{and} \quad \lambda_J(v_n) \to 0, \quad \text{as } n \to \infty.$$

Then, there exists $w_n \in \partial J(v_n) \subset (H^1(\mathbb{R}^N))^*$ such that

$$\|w_n\|_* = \lambda_J(v_n) = o_n(1)$$

and

$$w_n = Q'(v_n) - \varrho_n,$$

where $\|w_n\|_* := \|w_n\|_{(H^1(\mathbb{R}^N))^*}$ and $\varrho_n \in \partial\Psi(v_n) \subset L^{\tilde{\Phi}}(\mathbb{R}^N)$.

Therefore, we obtain that

$$\begin{aligned} c + 1 + \|v_n\| &\geq J(v_n) - \frac{1}{\theta}\langle w_n, v_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |\nabla v_n|^2 + \int_{\mathbb{R}^N} V(x)G^{-1}(v_n)\left(\frac{1}{2}G^{-1}(v_n) - \frac{1}{\theta}\frac{v_n}{g(G^{-1}(v_n))}\right) \\ &\quad - \kappa\int_{\mathbb{R}^N}\left(F(x, G^{-1}(v_n)) - \frac{1}{\theta}\varrho_n v_n\right) - \left(\frac{1}{2^*} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N} |v_n^+|^{2^*}. \end{aligned}$$

From (f_4) and $\varrho_n(x) \in \left[\frac{\underline{f}(x,G^{-1}(v_n(x)))}{g(G^{-1}(v_n(x)))}, \frac{\overline{f}(x,G^{-1}(v_n(x)))}{g(G^{-1}(v_n(x)))}\right]$ a.e. in $\mathbb{R}^N$, we have

$$\frac{1}{\theta}\varrho_n v_n \geq F(x, G^{-1}(v_n)) \geq 0 \quad \text{a.e. in } \mathbb{R}^N.$$

Hence, by Lemma 2.1-(ii), we get

$$c + 1 + \|v_n\| \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right). \tag{3.8}$$

For all $t \geq 1$, by (f_4), we can verify that there exists some $C > 0$ such that $CF(x,t) \geq (G(t))^\theta \geq (G(t))^2$. Then

$$\begin{aligned} \int_{\{|G^{-1}(v_n)|>1\}} V(x)v_n^2 &\leq \kappa C\int_{\{|G^{-1}(v_n)|>1\}} F(x, G^{-1}(v_n)) \\ &\leq \kappa C\int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) + \frac{C}{2^*}\int_{\mathbb{R}^N} |v_n^+|^{2^*} \\ &= C\left[\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right) - J(v_n)\right] \\ &= C\left[\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right) - c_\kappa + o_n(1)\right]. \end{aligned} \tag{3.9}$$

For $\{|G^{-1}(v_n)| \le 1\}$, by Lemma 2.1-(ii) and $g'(t) \ge 0$ for all $t \ge 0$, we have

$$\begin{aligned}\frac{1}{g^2(1)} \int_{\{|G^{-1}(v_n)|\le 1\}} V(x)v_n^2 &\le \int_{\{|G^{-1}(v_n)|\le 1\}} V(x)|G^{-1}(v_n)|^2 \\ &\le \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2.\end{aligned} \tag{3.10}$$

By (3.8)–(3.10), we deduce that

$$\begin{aligned}\|v_n\|^2 &= \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|v_n|^2\right) \\ &\le C \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2\right) + C \\ &\le C\|v_n\| + C,\end{aligned}$$

which implies that the sequence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. □

Next, the following lemma shows the behavior of c_κ associated with the parameter κ.

Lemma 3.6. *Suppose that (V_1) and (f_4) are satisfied, then $\lim_{\kappa\to+\infty} c_\kappa = 0$.*

Proof. Since $J(v)$ is nonsmooth functional, unlike the method used to prove Lemma 3.1 in [29], we will not use the Nehari manifold. For ϕ given by Lemma 3.4, it follows that there is $t_\kappa > 0$ satisfying

$$J(t_\kappa\phi) = \max_{t\ge 0} J(t\phi) \ge \alpha > 0.$$

Then, we have

$$\frac{t_\kappa^2}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(t_\kappa\phi)|^2 \ge \kappa\int_{\mathbb{R}^N} F(x, G^{-1}(t_\kappa\phi)) + \frac{t_\kappa^{2^*}}{2^*}\int_{\mathbb{R}^N}\phi^{2^*}.$$

By (f_4), we get

$$\frac{t_\kappa^2}{2}\left(\int_{\mathbb{R}^N} |\nabla\phi|^2 + \int_{\mathbb{R}^N} V(x)|\phi|^2\right) \ge \frac{t_\kappa^{2^*}}{2^*}\int_{\mathbb{R}^N}\phi^{2^*},$$

which implies that t_κ is bounded.

Next, we will prove that $t_\kappa \to 0$ as $\kappa \to +\infty$. Suppose, by contradiction, that there exists a sequence $\kappa_n \to +\infty$ and a constant $\bar{t} > 0$ such that $t_{\kappa_n} \to \bar{t}$ as $n \to \infty$. The boundedness of t_{κ_n} implies that there is $M > 0$ such that

$$\frac{t_{\kappa_n}^2}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 + \int_{\mathbb{R}^N} V(x)|G^{-1}(t_{\kappa_n}\phi)|^2 \le M.$$

Hence,

$$\kappa_n\int_{\mathbb{R}^N} F(x, G^{-1}(t_{\kappa_n}\phi)) + \frac{t_{\kappa_n}^{2^*}}{2^*}\int_{\mathbb{R}^N}\phi^{2^*} \le M.$$

If $\bar{t} > 0$, we have that

$$\lim_{n\to\infty}\left[\kappa_n\int_{\mathbb{R}^N} F(x, G^{-1}(t_{\kappa_n}\phi)) + \frac{t_{\kappa_n}^{2^*}}{2^*}\int_{\mathbb{R}^N}\phi^{2^*}\right] = +\infty$$

which is absurd. Thus, we have $t_\kappa \to 0$ as $\kappa \to +\infty$.

Observe that

$$J(t_\kappa\phi) \le \frac{t_\kappa^2}{2}\int_{\mathbb{R}^N} |\nabla\phi|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(t_\kappa\phi)|^2 \le \frac{t_\kappa^2}{2}\|\phi\|^2.$$

Due to $t_\kappa \to 0$ as $\kappa \to +\infty$, we get $c_\kappa \le J(t_\kappa\phi) \to 0$ as $\kappa \to +\infty$, which finishes the proof. □

Lemma 3.7. *Suppose that* (V_1) *and* (f_1)*–*(f_3) *are satisfied. Let* $\{v_n\} \subset H^1(\mathbb{R}^N)$ *be a* $(PS)_{c_\kappa}$ *sequence for* J *with* $v_n \rightharpoonup 0$ *in* $H^1(\mathbb{R}^N)$*. Then there is* $\kappa^* > 0$*. When* $\kappa > \kappa^*$*, there exists a sequence* $\{y_n\} \subset \mathbb{R}^N$ *and* $\delta > 0$ *such that*

$$\limsup_{n\to\infty} \int_{B_1(y_n)} |v_n|^2 \geq \delta > 0.$$

Proof. By Lemma 3.5, there exists a constant $\kappa^* > 0$ satisfying

$$c_\kappa < \frac{1}{N} S^{\frac{N}{2}},$$

for all $\kappa > \kappa^*$. Suppose, by contradiction, that $\{v_n\}$ is vanishing. Then, from Lions compactness lemma [33], we deduce that $v_n \to 0$ in $L^r(\mathbb{R}^N)$ for all $2 < r < 2^*$. From $|G^{-1}(v_n)| \leq |v_n|$, we get $G^{-1}(v_n) \to 0$ in $L^r(\mathbb{R}^N)$ for all $2 < r < 2^*$. Since $\{v_n\}$ is a $(PS)_{c_\kappa}$ sequence for J, there exists $w_n \in \partial J(v_n)$ with $\|w_n\|_* = \lambda_J(v_n) = o_n(1)$ and $w_n = Q'(v_n) - \varrho_n$, where $\varrho_n \in \partial\Psi(v_n)$. By (3.1) and Lemma 3.1, we have

$$\int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} \varrho_n v_n \to 0, \quad \text{as } n \to \infty. \tag{3.11}$$

Therefore, by (3.11), we have

$$\begin{aligned} c_\kappa + o_n(1) &= J(v_n) - \frac{1}{2}\langle w_n, v_n\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left[|G^{-1}(v_n)|^2 - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n\right] + \frac{1}{N}\int_{\mathbb{R}^N} |v_n^+|^{2^*}. \end{aligned} \tag{3.12}$$

We claim that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\left[|G^{-1}(v_n)|^2 - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n\right] = 0. \tag{3.13}$$

For proving (3.13), we only verify that

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\left[|G^{-1}(v_n)|^2 - |v_n|^2\right] &= 0, \\ \lim_{n\to\infty} \int_{\mathbb{R}^N} V(x)\left[|v_n|^2 - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n\right] &= 0. \end{aligned} \tag{3.14}$$

For $\delta > 0$ to be chosen later, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} V(x)\left[|G^{-1}(v_n)|^2 - |v_n|^2\right] \\ &\quad = \int_{\{|v_n|>\delta\}} V(x)\left[|G^{-1}(v_n)|^2 - |v_n|^2\right] + \int_{\{|v_n|\leq\delta\}} V(x)\left[|G^{-1}(v_n)|^2 - |v_n|^2\right]. \end{aligned}$$

By Lemma 2.1-(ii) and (V_1), we get

$$\begin{aligned} \int_{\{|v_n|>\delta\}} V(x)\left[|G^{-1}(v_n)|^2 - |v_n|^2\right] &\leq 2\|V\|_\infty \int_{\{|v_n|>\delta\}} |v_n|^2 \\ &\leq \frac{2\|V\|_\infty}{\delta^{r-2}} \int_{\mathbb{R}^N} |v_n|^r = o_n(1), \end{aligned} \tag{3.15}$$

where $2 < r < 2^*$.

On the other hand, given $\varepsilon > 0$, by Lemma 2.1-(iii), we choose $\delta > 0$ so that

$$\left|\left(\frac{G^{-1}(s)}{s}\right)^2 - 1\right| < \varepsilon, \quad \text{if } |s| \leq \delta.$$

Then, we have

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\{0<|v_n|\le\delta\}} V(x)|v_n|^2\left|\left(\frac{G^{-1}(v_n)}{v_n}\right)^2-1\right|\\&\le\|V\|_\infty\limsup_{n\to\infty}\int_{\{0<|v_n|\le\delta\}}|v_n|^2\left|\left(\frac{G^{-1}(v_n)}{v_n}\right)^2-1\right|\\&\le\varepsilon\limsup_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^2.\end{aligned}$$

Hence

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\mathbb{R}^N}V(x)\left[|G^{-1}(v_n)|^2-|v_n|^2\right]\\&\le\limsup_{n\to\infty}\int_{\{|v_n|>\delta\}}V(x)\left[|G^{-1}(v_n)|^2-|v_n|^2\right]+\varepsilon\limsup_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^2.\end{aligned}$$

As $\varepsilon>0$ is arbitrary and $\{v_n\}\subset H^1(\mathbb{R}^N)$ is bounded, using (3.15), we have the first limit in (3.14).

By Lemma 2.1-(ii), (iii) and the fact that

$$(G^{-1})'(s)=\frac{1}{g(G^{-1}(s))}\to 1\quad\text{as } s\to 0,$$

the verification of the second limit in (3.14) is similar to the first one. Therefore, our claim (3.13) is true.

Then, by (3.12) and (3.13), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n^+|^{2^*}=Nc_\kappa. \tag{3.16}$$

From the fact $\langle w_n,v_n\rangle=o_n(1)$ and the second limit in (3.11), we get

$$\int_{\mathbb{R}^N}|\nabla v_n|^2+\int_{\mathbb{R}^N}V(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n-\int_{\mathbb{R}^N}|v_n^+|^{2^*}=o_n(1). \tag{3.17}$$

From the definition of $G^{-1}(s)$ and (V_1), the second integral in (3.17) is nonnegative. Then, we have

$$\int_{\mathbb{R}^N}|\nabla v_n|^2\le\int_{\mathbb{R}^N}|v_n^+|^{2^*}+o_n(1). \tag{3.18}$$

By the definition of S, (3.16) and (3.18), it follows that

$$\int_{\mathbb{R}^N}|v_n^+|^{2^*}\le\int_{\mathbb{R}^N}|v_n|^{2^*}\le S^{-\frac{2^*}{2}}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2\right)^{\frac{2^*}{2}}\le S^{-\frac{2^*}{2}}\left(\int_{\mathbb{R}^N}|v_n^+|^{2^*}+o_n(1)\right)^{\frac{2^*}{2}}.$$

Taking $n\to\infty$ in the above inequality, in view of (3.16), we get

$$Nc_\kappa\le S^{-\frac{2^*}{2}}(Nc_\kappa)^{\frac{2^*}{2}},$$

that is,

$$c_\kappa\ge\frac{1}{N}S^{\frac{N}{2}},$$

which is a contradiction. Hence $\{v_n\}$ is non-vanishing. This concludes the proof. □

4 Proof of Theorem 1.1

In the following, we will prove that there exists $v \in H^1(\mathbb{R}^N)$ is a positive solution of problem (1.6). With this aim in mind, we need to show that there is $v \in H^1(\mathbb{R}^N)$ and $v > 0$ such that

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} - v^{2^*-1} \in \left[\frac{\underline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}, \frac{\overline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}\right] \quad \text{a.e. in } \mathbb{R}^N.$$

By Lemma 3.3 and Lemma 3.4, the functional J satisfies all hypotheses of Lemma 2.9. Then, by Lemma 2.9 and Lemma 3.5 , there exists a bounded sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ satisfying

$$J(v_n) \to c_\kappa \geq \alpha > 0 \quad \text{and} \quad \lambda_J(v_n) \to 0, \quad \text{as } n \to \infty,$$

where

$$c_\kappa = \inf_{\gamma \in \Gamma_J} \max_{t \in [0,1]} J(\gamma(t)),$$

and

$$\Gamma_J = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

Therefore, there exists $w_n \in \partial J(v_n)$ such that $\|w_n\|_* = \lambda_J(v_n)$, $w_n = Q'(v_n) - \varrho_n$ where $\varrho_n \in \partial\Psi(v_n)$. For all $\psi \in H^1(\mathbb{R}^N)$,

$$\langle w_n, \psi\rangle = \langle Q'(v_n), \psi\rangle - \langle \varrho_n, \psi\rangle, \qquad \forall n \in \mathbb{N}.$$

Since $H^1(\mathbb{R}^N)$ is reflexive, taking a subsequence if necessary, there exists $v \in H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. Thus, we obtain

$$\langle \varrho_n, \psi\rangle = \langle Q'(v_n), \psi\rangle - \langle w_n, \psi\rangle \to \langle Q'(v), \psi\rangle, \quad \text{as } n \to \infty,$$

that is, $\varrho_n \overset{*}{\rightharpoonup} Q'(v)$ in $(H^1(\mathbb{R}^N))^*$. By Lemma 3.2, we get $Q'(v) \in \partial\Psi(v)$. Then, there exits $\varrho \in \partial\Psi(v)$ such that $Q'(v) = \varrho$ and

$$\int_{\mathbb{R}^N}\left(\nabla v \nabla \psi + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\psi\right) - \int_{\mathbb{R}^N}(v^+)^{2^*-1}\psi = \int_{\mathbb{R}^N}\varrho\psi, \qquad \forall \psi \in H^1(\mathbb{R}^N),$$

where

$$\varrho(x) \in \left[\frac{\underline{f}(x, G^{-1}(v(x)))}{g(G^{-1}(v(x)))}, \frac{\overline{f}(x, G^{-1}(v(x)))}{g(G^{-1}(v(x)))}\right] \quad \text{a.e. in } \mathbb{R}^N.$$

Taking $\psi = v^- := \min\{v, 0\}$, we obtain

$$\int_{\mathbb{R}^N}\left(|\nabla v^-|^2 + V(x)\frac{G^{-1}(v^-)}{g(G^{-1}(v^-))}v^-\right) \leq 0,$$

which implies $v^- \equiv 0$. Thus, we get $v = v^+ \geq 0$ satisfying

$$\begin{cases} -\Delta v + V(x)\dfrac{G^{-1}(v)}{g(G^{-1}(v))} = \varrho + v^{2^*-1} & \text{in } \mathbb{R}^N, \\ v \in H^1(\mathbb{R}^N). \end{cases}$$

Furthermore, since $\varrho \in L^{\tilde{\Psi}}(\mathbb{R}^N) \subset L^{\frac{q}{q-1}}_{\text{loc}}(\mathbb{R}^N)$, the elliptic regularity theory gives that $v \in W^{2,\frac{2^*}{2^*-1}}_{\text{loc}}(\mathbb{R}^N)$ and v satisfies

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \varrho + v^{2^*-1} \quad \text{a.e. in } \mathbb{R}^N,$$

that is,

$$-\Delta v+V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}-v^{2^*-1}\in\left[\frac{\underline{f}(x,G^{-1}(v))}{g(G^{-1}(v))},\frac{\overline{f}(x,G^{-1}(v))}{g(G^{-1}(v))}\right]\quad\text{a.e. in } \mathbb{R}^N.$$

Finally, in order to prove Theorem 1.1, it suffices to verify that v is nontrivial. Suppose, by contradiction, that v is trivial. Then, we claim that in this case $\{v_n\}$ is also a $(PS)_{c_\kappa}$ sequence for J_p defined by

$$J_p(v)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2+V_p(x)|G^{-1}(v)|^2\right)-\frac{1}{2^*}\int_{\mathbb{R}^N}(v^+)^{2^*}-\int_{\mathbb{R}^N}F_p(x,G^{-1}(v)),$$

for $v\in H^1(\mathbb{R}^N)$ and J_p possesses a nontrivial critical point. It is well known that $J_p\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$, with

$$\langle J_p'(v),\varphi\rangle=\int_{\mathbb{R}^N}\left(\nabla u\nabla\varphi+V_p(x)\frac{G^{-1}(v)}{g(G^{-1}(v))}\varphi\right)-\int_{\mathbb{R}^N}(v^+)^{2^*-1}\varphi-\int_{\mathbb{R}^N}\frac{f_p(x,G^{-1}(v))}{g(G^{-1}(v))}\varphi,$$

for all $\varphi\in H^1(\mathbb{R}^N)$.

Lemma 4.1. *If $\{v_n\}$ is given by the above, then*

$$\varrho_n-\Psi_p'(v_n)\to 0\quad\textit{and}\quad\Psi(v_n)-\Psi_p(v_n)\to 0,$$

where

$$\Psi(v)=\int_{\mathbb{R}^N}F(x,G^{-1}(v))\quad\textit{and}\quad\Psi_p(v)=\int_{\mathbb{R}^N}F_p(x,G^{-1}(v)).$$

Proof. For any $\varphi\in H^1(\mathbb{R}^N)$ with $\|\varphi\|\le 1$, by (f_7), we obtain

$$\begin{aligned}\left|\langle\varrho_n-\Psi_p'(v_n),\varphi\rangle\right|&\le\int_{\mathbb{R}^N}\left|\varrho_n-\frac{f_p(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\right||\varphi|\\&\le\int_{\mathbb{R}^N}\pi(x)|v_n|^{q-1}|\varphi|\\&\le\left(\int_{\mathbb{R}^N}|\pi(x)|^{\frac{q}{q-1}}|v_n|^q\right)^{\frac{q-1}{q}}\|\varphi\|_q\\&\le C\left(\int_{\mathbb{R}^N}|\pi(x)|^{\frac{q}{q-1}}|v_n|^q\right)^{\frac{q-1}{q}}.\end{aligned}$$

Since $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, there is $M>0$ with $\|v_n\|_q\le M$ for all $n\in\mathbb{N}$. Using the fact that $\pi(x)\to 0$ as $|x|\to+\infty$, given $\varepsilon>0$ there is $R_\varepsilon>0$ such that $|\pi(x)|\le\varepsilon$ for $|x|>R_\varepsilon$. Since $H^1(B_{R_\varepsilon}(0))\hookrightarrow L^q(B_{R_\varepsilon}(0))$ is compact, we have $v_n\to 0$ in $L^q(B_{R_\varepsilon}(0))$. Thus, there is $n_0\in\mathbb{N}$ satisfying $\|v_n\|_{L^q(B_{R_\varepsilon}(0))}\le\varepsilon$, for all $n\ge n_0$, and so,

$$\begin{aligned}\int_{\mathbb{R}^N}|\pi(x)|^{\frac{q}{q-1}}|v_n|^q&=\int_{B_{R_\varepsilon}(0)}|\pi(x)|^{\frac{q}{q-1}}|v_n|^q+\int_{B^c_{R_\varepsilon}(0)}|\pi(x)|^{\frac{q}{q-1}}|v_n|^q\\&\le\|\pi(x)\|_\infty^{\frac{q}{q-1}}\int_{B_{R_\varepsilon}(0)}|v_n|^q+\varepsilon^{\frac{q}{q-1}}\int_{\mathbb{R}^N}|v_n|^q\\&\le\varepsilon^q\|\pi(x)\|_\infty^{\frac{q}{q-1}}+\varepsilon^{\frac{q}{q-1}}M^q.\end{aligned}$$

As ε is arbitrary,

$$\varrho_n - \Psi_p'(v_n) \to 0 \quad \text{in } (H^1(\mathbb{R}^N))^*.$$

A similar argument guarantees that

$$\Psi(v_n) - \Psi_p(v_n) \to 0 \quad \text{in } \mathbb{R}. \qquad \square$$

Since $W(x) \in L^{\frac{N}{2}}(\mathbb{R}^N)$ and $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, we can conclude

$$\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^2 \le \int_{\mathbb{R}^N} W(x)|v_n|^2 \to 0. \tag{4.1}$$

From Lemma 4.1 and (4.1), we deduce

$$\begin{aligned}
\left|J(v_n) - J_p(v_n)\right| &= \left|\frac{1}{2}\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^2 + \kappa \int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) - F_p(x, G^{-1}(v_n))\right| \\
&\le \frac{1}{2}\int_{\mathbb{R}^N} W(x)|G^{-1}(v_n)|^2 + \kappa \left|\int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) - F_p(x, G^{-1}(v_n)\right| \\
&\le \frac{1}{2}\int_{\mathbb{R}^N} W(x)|v_n|^2 + \kappa \left|\Psi(v_n) - \Psi_p(v_n)\right| \\
&= o_n(1),
\end{aligned} \tag{4.2}$$

which shows that $J_p(v_n) \to c_\kappa$ as $n \to \infty$.

On the other hand, note that $w_n = Q'(v_n) - \varrho_n$ and $\|w_n\|_* = \lambda_J(v_n) = o_n(1)$, where $\varrho_n \in \partial\Psi(v_n)$. From Lemma 4.1 and (4.1), for $\varphi \in H^1(\mathbb{R}^N)$ with $\|\varphi\| \le 1$, we obtain

$$\begin{aligned}
&\left|\langle w_n, \varphi\rangle - \langle J_p'(v_n), \varphi\rangle\right| \\
&\quad= \left|\int_{\mathbb{R}^N} W(x)\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\varphi + \kappa \int_{\mathbb{R}^N}\left(\varrho_n\varphi - \frac{f_p(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi\right)\right| \\
&\quad\le \int_{\mathbb{R}^N} W(x)\left|\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\right||\varphi| + \kappa\left|\int_{\mathbb{R}^N}\left(\varrho_n\varphi - \frac{f_p(x, G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi\right)\right| \\
&\quad\le \left(\int_{\mathbb{R}^N} W(x)|v_n|^2\right)^{\frac{1}{2}} \|w\|_{\frac{N}{2}}\|\varphi\|_{2^*}^{\frac{1}{2}} + \kappa\left|\langle \varrho_n - \Psi_p'(v_n), \varphi\rangle\right| \\
&\quad= o_n(1),
\end{aligned} \tag{4.3}$$

which shows that $J_p'(v_n) \to 0$, as $n \to \infty$. Thus, by (4.2) and (4.3), $\{v_n\}$ is a $(PS)_{c_\kappa}$ sequence for J_p.

As we suppose that v is trivial, by Lemma 3.7, there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and $\delta > 0$ such that

$$\limsup_{n\to\infty} \int_{B_1(y_n)} |v_n|^2 \ge \delta > 0.$$

So, we can find a sequence $\{z_n\} \subset \mathbb{Z}^N$ such that $|z_n - y_n| < \sqrt{N}$ for all $n \in \mathbb{N}$ and

$$\limsup_{n\to\infty} \int_{B_{1+\sqrt{N}}(z_n)} |v_n|^2 \ge \limsup_{n\to\infty} \int_{B_1(y_n)} |v_n|^2 \ge \delta > 0. \tag{4.4}$$

Since $v_n \to v$ in $L^s_{\text{loc}}(\mathbb{R}^N)$ for all $s \in [1, 2^*)$ and $v = 0$, we may suppose that $|z_n| \to \infty$ up to a subsequence. Denote $\hat{v}_n(x) = v_n(x + z_n)$. Since $\{v_n\}$ is a $(PS)_{c_\kappa}$ sequence for J_p, in view of the periodicities of V_p and f_p, $\{\hat{v}_n\}$ is also a $(PS)_{c_\kappa}$ sequence for J_p. As $\{v_n\}$ is bounded in

$H^1(\mathbb{R}^N)$, it follows that $\{\hat{v}_n\}$ is also bounded in $H^1(\mathbb{R}^N)$. Without loss of generality, we may suppose that

$$\begin{cases} \hat{v}_n \rightharpoonup \hat{v} & \text{in } H^1(\mathbb{R}^N), \\ \hat{v}_n \to \hat{v} & \text{in } L^r_{\text{loc}}(\mathbb{R}^N),\ \forall\, r \in [1, 2^*), \\ \hat{v}_n \to \hat{v} & \text{a.e. in } \mathbb{R}^N, \end{cases}$$

then $J_p'(\hat{v}) = 0$. By (4.4), going to a subsequence if necessary, there exists $n_1 \in \mathbb{N}$ such that

$$\int_{B_{1+\sqrt{N}}(z_n)} |v_n|^2 \geq \frac{\delta}{2} > 0, \qquad \forall\, n \geq n_1.$$

Since $\hat{v}_n(x) = v_n(x + z_n)$ and $\hat{v}_n \to \hat{v}$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, we get

$$\int_{B_{1+\sqrt{N}}(0)} |\hat{v}|^2 = \lim_{n\to\infty} \int_{B_{1+\sqrt{N}}(0)} |\hat{v}_n|^2 = \lim_{n\to\infty} \int_{B_{1+\sqrt{N}}(z_n)} |v_n|^2 \geq \frac{\delta}{2} > 0,$$

which shows $\hat{v} \not\equiv 0$. Besides,

$$0 = \langle J_p'(\hat{v}), \hat{v}^- \rangle = \int_{\mathbb{R}^N} \left(|\nabla \hat{v}^-|^2 + V(x) \frac{G^{-1}(\hat{v}^-)}{g(G^{-1}(\hat{v}^-))} \hat{v}^- \right),$$

which implies $\hat{v} = \hat{v}^+ \geq 0$. Thus, by Fatou's Lemma and (f_6), we have

$$\begin{aligned} c_\kappa &= \limsup_{n\to\infty} \left[J_p(\hat{v}_n) - \frac{1}{2} \langle J_p'(\hat{v}_n), \hat{v}_n \rangle \right] \\ &= \frac{1}{2} \limsup_{n\to\infty} \int_{\mathbb{R}^N} V_p(x) \left[|G^{-1}(\hat{v}_n)|^2 - \frac{G^{-1}(\hat{v}_n)}{g(G^{-1}(\hat{v}_n))} \hat{v}_n \right] + \frac{1}{N} \limsup_{n\to\infty} \int_{\mathbb{R}^N} |\hat{v}_n^+|^{2^*} \\ &\quad - \kappa \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left[F_p(x, G^{-1}(\hat{v}_n)) - \frac{1}{2} \frac{f_p(x, G^{-1}(\hat{v}_n))}{g(G^{-1}(\hat{v}_n))} \hat{v}_n \right] \\ &\geq \frac{1}{2} \int_{\mathbb{R}^N} V_p(x) \left[|G^{-1}(\hat{v})|^2 - \frac{G^{-1}(\hat{v})}{g(G^{-1}(\hat{v}))} \hat{v} \right] + \frac{1}{N} \int_{\mathbb{R}^N} |\hat{v}^+|^{2^*} \\ &\quad - \kappa \int_{\mathbb{R}^N} \left[F_p(x, G^{-1}(\hat{v})) - \frac{1}{2} \frac{f_p(x, G^{-1}(\hat{v}))}{g(G^{-1}(\hat{v}))} \hat{v} \right] \\ &= J_p(\hat{v}) - \frac{1}{2} \langle J_p'(\hat{v}), \hat{v} \rangle = J_p(\hat{v}). \end{aligned}$$

Thus, $\hat{v}$ is a nontrivial critical point of J_p and $J_p(\hat{v}) \leq c_\kappa$.

Claim 4.2. $\hat{v} > 0$ in $\mathbb{R}^N$.

For proving the result, we adapt the same ideas used in [30]. Since $\hat{v}$ is a critical point of J_p (namely $J_p'(\hat{v}) = 0$), $\hat{v}$ is a weak solution of the following equation

$$-\Delta \hat{v} = \zeta, \quad \text{a.e. in } \mathbb{R}^N, \tag{4.5}$$

where

$$\zeta(x, \hat{v}) = \hat{v}^{2^*-1} + \kappa \frac{f_p(x, G^{-1}(\hat{v}))}{g(G^{-1}(\hat{v}))} - V_p(x) \frac{G^{-1}(\hat{v})}{g(G^{-1}(\hat{v}))}.$$

From the conditions (V), (f_7), (3.1) and the Lemma 2.1-(ii), we get

$$\begin{aligned}&\left|\hat{v}^{2^*-1}+\kappa\frac{f_p(x,G^{-1}(\hat{v}))}{g(G^{-1}(\hat{v}))}-V_p(x)\frac{G^{-1}(\hat{v})}{g(G^{-1}(\hat{v}))}\right|\\&\quad\le|\hat{v}|^{2^*-1}+\kappa\left(\varepsilon|\hat{v}|+C_\varepsilon|\hat{v}|^{q-1}+\pi(x)|\hat{v}|^{q-1}\right)+V_p(x)|\hat{v}|\\&\quad\le C\left(|\hat{v}|^{2^*-1}+|\hat{v}|^{q-1}+|\hat{v}|\right)\\&\quad\le C\left(|\hat{v}|^{2^*-1}+1\right).\end{aligned}\tag{4.6}$$

Using a result concluded by Brézis–Kato (see [32]), it yields that $\zeta(x,\hat{v})\in L^r(B_R(0))$ for every $r\in[1,+\infty)$, with $R>0$ arbitrary. By standard elliptic regularity theory, we get that $\hat{v}\in W^{2,r}(B_R(0))$. So, there exits some $\sigma\in(0,1)$ such that $\hat{v}\in C^{1,\sigma}_{\mathrm{loc}}(\mathbb{R}^N)$.

Arguing by contradiction, we assume that there exits $x_0\in\mathbb{R}^N$ such that $\hat{v}(x_0)=0$. Meanwhile, we have

$$\begin{aligned}-\Delta\hat{v}(x)+b(x)\hat{v}(x)=V_p(x)&\left(\frac{\hat{v}(x)}{g(G^{-1}(\hat{v}(x)))}-\frac{G^{-1}(\hat{v}(x))}{g(G^{-1}(\hat{v}(x)))}\right)\\&+\hat{v}^{2^*-1}(x)+\kappa\frac{f_p(x,G^{-1}(\hat{v}(x)))}{g(G^{-1}(\hat{v}(x)))},\end{aligned}\tag{4.7}$$

where $b(x):=\frac{V_p(x)}{g(G^{-1}(\hat{v}(x)))}\ge0$, for $x\in\mathbb{R}^N$. Combining Lemma 2.1-(i) and (ii), we get $\frac{\hat{v}(x)}{g(G^{-1}(\hat{v}(x)))}-\frac{G^{-1}(\hat{v}(x))}{g(G^{-1}(\hat{v}(x)))}\ge0$. By the hypotheses of $V_p(x)$, we know $-\Delta\hat{v}(x)+b(x)\hat{v}(x)\ge0$. In view of (V_1), $b(x)$ is continuous in $\mathbb{R}^N$. Thus, applying the Maximum Principle for the weak solution (see [18]) on an arbitrary ball centered in x_0, we get that $\hat{v}\equiv0$. This is a contradiction.

Claim 4.3. There exists a curve $\gamma(t):[0,1]\to H^1(\mathbb{R}^N)$ such that

$$\begin{cases}\gamma(0)=0,\ J_p(\gamma(1))<0,\ \hat{v}\in\gamma([0,1]),\\\gamma(t)(x)>0,\ \forall x\in\mathbb{R}^N,\ t\in(0,1],\\\max_{t\in[0,1]}J_p(\gamma(t))=J_p(\hat{v}).\end{cases}\tag{4.8}$$

Defining the function $\tilde{\gamma}(t)(x)=t\hat{v}(x)$ for $t\ge0$, we have

$$\begin{aligned}J_p(\tilde{\gamma}(t))=J_p(t\hat{v})&=\frac12\int_{\mathbb{R}^N}\left(|t\nabla\hat{v}|^2+V_p(x)|G^{-1}(t\hat{v})|^2\right)-\kappa\int_{\mathbb{R}^N}F_p(x,G^{-1}(t\hat{v}))-\frac{1}{2^*}\int_{\mathbb{R}^N}|t\hat{v}|^{2^*}\\&\le\frac{t^2}{2}\int_{\mathbb{R}^N}\left(|\nabla\hat{v}|^2+V_p(x)|\hat{v}|^2\right)-\frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}|\hat{v}|^{2^*}.\end{aligned}$$

Therefore, we may choose a sufficiently large constant $L>1$ such that $J_p(\tilde{\gamma}(L))<0$ with $\tilde{\gamma}(t)(x)>0$, for all $(x,t)\in\mathbb{R}^N\times(0,L]$. Furthermore, since $\hat{v}$ is a critical point of J_p, set $\varsigma(t)=J_p(t\hat{v})$ and we may write

$$\begin{aligned}\varsigma'(t)&=t\int_{\mathbb{R}^N}|\nabla\hat{v}|^2+\int_{\mathbb{R}^N}V_p(x)\frac{G^{-1}(t\hat{v})}{g(G^{-1}(t\hat{v}))}\hat{v}-\kappa\int_{\mathbb{R}^N}\frac{f_p(x,G^{-1}(t\hat{v}))}{g(G^{-1}(t\hat{v}))}\hat{v}-t^{2^*-1}\int_{\mathbb{R}^N}|\hat{v}|^{2^*}\\&=t\left\{\int_{\mathbb{R}^N}|\nabla\hat{v}|^2+\int_{\mathbb{R}^N}\left[V_p(x)\frac{G^{-1}(t\hat{v})}{g(G^{-1}(t\hat{v}))t\hat{v}}-\kappa\frac{f_p(x,G^{-1}(t\hat{v}))}{g(G^{-1}(t\hat{v}))t\hat{v}}-(t\hat{v})^{2^*-2}\right]\hat{v}^2\right\}.\end{aligned}$$

As a direct consequence of Lemma 2.1-(iv) and (f_5), fixed $x \in \mathbb{R}^N$, the function $\eta : (0, +\infty) \to \mathbb{R}$ defined by

$$\eta(t) = V_p(x)\frac{G^{-1}(t)}{g(G^{-1}(t))t} - \kappa\frac{f_p(x, G^{-1}(t))}{g(G^{-1}(t))t} - t^{2^*-2}$$

is decreasing.

Since $\hat{v}$ is a critical point of J_p, we have $\varsigma'(1) = 0$. Moreover, $\varsigma(t) > 0$ for $0 < t < 1$ and $\varsigma(t) < 0$ for $t > 1$. Hence,

$$J_p(\hat{v}) = \varsigma(1) = \max_{t\geq 0}\varsigma(t) = \max_{t\geq 0} J_p(t\hat{v}) = \max_{t\in[0,L]} J_p(t\hat{v}) = \max_{t\in[0,L]} J_p(\tilde{\gamma}(t)).$$

Let $\gamma(t) = \tilde{\gamma}(tL)$. We can check the curve $\gamma(t)$ satisfies (4.8). From $J(\phi) \leq J_p(\phi)$ for all $\phi \in H^1(\mathbb{R}^N)$, we get $\gamma \in \Gamma_J$.

Due to the fact that $\gamma \in \Gamma_J$ satisfies (4.8) and the inequality $W(x) \geq 0$ is strict on a subset of positive measure in $\mathbb{R}^N$, we deduce that

$$c_\lambda \leq \max_{t\in[0,1]} J(\gamma(t)) := J(\gamma(\bar{t})) < J_p(\gamma(\bar{t})) \leq \max_{t\in[0,1]} J_p(\gamma(t)) = J_p(\hat{v}) \leq c_\lambda,$$

which is absurd.

Thus, we conclude that v is a nontrivial solution to problem (2.1). An argument similar to Claim 4.2 shows $v > 0$ in $\mathbb{R}^N$. By Lemma 2.2, problem (1.6) possesses a positive solution $u = G^{-1}(v)$.

5 Proof of Theorem 1.2

The following section gives the proof of Theorem 1.2. First, note that the lemmas in Section 3 are not dependent on the periodicity of function f, but only on its growth, meaning all of them are also valid here. As f satisfies (f_1)–(f_4), by Lemmas 3.3, 3.4 and 3.5, there is a bounded $(PS)_{c_\kappa}$ sequence for J, denoted by $\{v_n\} \subset H^1(\mathbb{R}^N)$, that is,

$$J(v_n) \to c_\kappa \geq \alpha > 0 \ \text{ and } \ \lambda_J(v_n) \to 0, \quad \text{as } n \to \infty.$$

Consider $w_n \in \partial J(v_n)$ such that $\|w_n\|_* = \lambda_J(v_n) = o_n(1)$ and $w_n = Q'(v_n) - \varrho_n$, where $\varrho_n \in \partial\Psi(v_n)$. Without loss of generality, we may suppose that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. If v is nontrivial, then Theorem 1.2 is proved. Indeed, repeating the analogous arguments as in the initial steps of the proof of Theorem 1.1, we can instantly obtain that $v = v^+ \geq 0$ and satisfies

$$-\Delta v + V_p(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \varrho + v^{2^*-1} \quad \text{a.e. } \mathbb{R}^N,$$

that is,

$$-\Delta v + V_p(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} - v^{2^*-1} \in \left[\frac{\underline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}, \frac{\overline{f}(x, G^{-1}(v))}{g(G^{-1}(v))}\right] \quad \text{a.e. in } \mathbb{R}^N.$$

By the argument similar to the one used in Claim 4.2, we can show $v > 0$. Then, $u = G^{-1}(v)$ will be a positive solution of problem (1.8).

Hence, in order to prove Theorem 1.2, it suffices to assume that $v = 0$.

In view of Lemma 3.6, it follows that there exists κ^* such that $c_\kappa < \frac{1}{N}S^{\frac{N}{2}}$ for all $\kappa > \kappa^*$. Furthermore, by Lemma (3.8), there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and $\delta > 0$ such that

$$\limsup_{n\to\infty} \int_{B_1(y_n)} |v_n|^2 \geq \delta > 0, \qquad \text{for all } n \in \mathbb{N}. \tag{5.1}$$

Since $v_n \to v$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $v = 0$, we may suppose that $|y_n| \to \infty$ up to a subsequence. As in the proof of Theorem 1.1, without loss of generality, we can suppose that $\{y_n\} \subset \mathbb{Z}^N$. Defining $\tilde{v}_n(x) = v_n(x + y_n)$, we get $\|\tilde{v}_n\| = \|v_n\|$. Then, taking a subsequence if necessary, there exists $\tilde{v} \in H^1(\mathbb{R}^N)$ such that

$$\begin{cases} \tilde{v}_n \rightharpoonup \tilde{v} & \text{in } H^1(\mathbb{R}^N), \\ \tilde{v}_n \to \tilde{v} & \text{in } L^r_{\text{loc}}(\mathbb{R}^N),\ \forall\, r \in [1, 2^*), \\ \tilde{v}_n \to \tilde{v} & \text{a.e. in } \mathbb{R}^N. \end{cases}$$

The fact that

$$\int_{B_1(0)} |\tilde{v}|^2 = \lim_{n\to\infty} \int_{B_1(0)} |\tilde{v}_n|^2 = \lim_{n\to\infty} \int_{B_1(y_n)} |v_n|^2,$$

and (5.1) imply that $\tilde{v} \neq 0$.

Now, we claim $\tilde{v}$ is a nontrivial solution of periodic problem.

First, we note that $\varrho_n \in \partial\Psi(v_n)$. By the definition of $\partial\Psi(v_n)$,

$$\Psi^\circ(v_n, \psi) \geq \langle \varrho_n, \psi \rangle, \qquad \forall \psi \in L^\Phi(\mathbb{R}^N).$$

Since $H^1(\mathbb{R}^N) \hookrightarrow L^\Phi(\mathbb{R}^N)$ is continuous, a simple change variable implies

$$\begin{aligned} \Psi^\circ(v_n; \psi(\cdot - y_n)) &\geq \langle \varrho_n, \psi(\cdot - y_n) \rangle \\ &= \int_{\mathbb{R}^N} \varrho_n \psi(\cdot - y_n) \\ &= \int_{\mathbb{R}^N} \varrho_n(\cdot + y_n)\psi \\ &= \langle \tilde{\varrho}_n, \psi \rangle, \end{aligned} \tag{5.2}$$

where $\tilde{\varrho}_n = \varrho_n(\cdot + y_n)$. Meanwhile, we can easily verify

$$\Psi(v_n + h + t\psi(\cdot - y_n)) = \Psi(\tilde{v}_n + h(\cdot + y_n) + t\psi) \quad \text{and} \quad \Psi(v_n + h) = \Psi(\tilde{v}_n + h(\cdot + y_n)),$$

where $h \in H^1(\mathbb{R}^N)$ and $t \in \mathbb{R}$. Thus, directly calculations demonstrate

$$\Psi^\circ(v_n + \psi(\cdot - y_n)) = \Psi^\circ(\tilde{v}_n + \psi). \tag{5.3}$$

By (5.2) and (5.3), we get

$$\Psi^\circ(\tilde{v}_n + \psi) \geq \langle \tilde{\varrho}_n, \psi \rangle, \qquad \forall \psi \in H^1(\mathbb{R}^N),$$

which shows $\tilde{\varrho}_n \in \partial\Psi(\tilde{v}_n)$. Furthermore, for all $\psi \in H^1(\mathbb{R}^N)$, we have

$$\begin{aligned} \langle w_n, \psi(\cdot - y_n) \rangle &= \langle Q'(v_n), \psi(\cdot - y_n) \rangle - \langle \varrho_n, \psi(\cdot - y_n) \rangle \\ &= \langle Q'(\tilde{v}_n), \psi \rangle - \langle \tilde{\varrho}_n, \psi \rangle. \end{aligned}$$

Setting $\langle w_n, \psi(\cdot - y_n) \rangle = \langle \tilde{w}_n, \psi \rangle$, we assert

$$\tilde{w}_n = Q'(\tilde{v}_n) - \tilde{\varrho}_n. \tag{5.4}$$

Claim 5.1. $\tilde{w}_n \in \partial J(\tilde{v}_n)$.

Similarly, by change of variables, we get

$$J^\circ(v_n; \psi(\cdot - y_n)) = J^\circ(\tilde{v}_n; \psi). \tag{5.5}$$

And as $w_n \in \partial J(v_n)$, then

$$J^\circ(v_n; \psi(\cdot - y_n)) \geq \langle w_n, \psi(\cdot - y_n)\rangle = \langle \tilde{w}_n, \psi\rangle, \qquad \forall \psi \in H^1(\mathbb{R}^N). \tag{5.6}$$

Combining (5.5) and (5.6), we have

$$J^\circ(\tilde{v}_n; \psi) \geq \langle \tilde{w}_n, \psi\rangle, \qquad \forall \psi \in H^1(\mathbb{R}^N),$$

which shows $\tilde{w}_n \in \partial J(\tilde{v}_n)$.

Moreover, by definition of $\tilde{w}_n$, we get

$$\|\tilde{w}_n\|_* = \sup_{\psi \in H^1(\mathbb{R}^N)} \frac{|\langle \tilde{w}_n, \psi\rangle|}{\|\psi\|} \leq \|w_n\|_*, \qquad \forall n \in \mathbb{N}.$$

Therefore,

$$0 \leq \|\tilde{w}_n\|_* \leq \|w_n\|_* = \lambda_J(v_n) = o_n(1). \tag{5.7}$$

By (5.4), we get

$$\langle \tilde{w}_n, \psi\rangle = \langle Q'(\tilde{v}_n), \psi\rangle - \langle \tilde{\varrho}_n, \psi\rangle, \qquad \forall \psi \in H^1(\mathbb{R}^N). \tag{5.8}$$

From (5.7) and (5.8), we obtain

$$\langle \tilde{\varrho}_n, \psi\rangle = \langle Q'(\tilde{v}_n), \psi\rangle - \langle \tilde{w}_n, \psi\rangle \to \langle Q'(\tilde{v}), \psi\rangle, \quad \text{as } n \to \infty,$$

that is, $\tilde{\varrho}_n \overset{*}{\rightharpoonup} Q'(\tilde{v})$ in $(H^1(\mathbb{R}^N))^*$.

This limit together with Lemma 3.2 shows that $Q'(\tilde{v}) \in \partial\Psi(v)$. Thereby, $Q'(\tilde{v}) = \tilde{\varrho} \in \partial\Psi(\tilde{v})$, and so,

$$\int_{\mathbb{R}^N} \left(\nabla\tilde{v}\nabla\psi + V_p(x)\frac{G^{-1}(\tilde{v})}{g(G^{-1}(\tilde{v}))}\psi\right) - \int_{\mathbb{R}^N} (\tilde{v}^+)^{2^*-1}\psi = \int_{\mathbb{R}^N} \tilde{\varrho}\psi, \qquad \forall\ \psi \in H^1(\mathbb{R}^N),$$

where

$$\tilde{\varrho}(x) \in \left[\frac{\underline{f}(x, G^{-1}(\tilde{v}(x)))}{g(G^{-1}(\tilde{v}(x)))}, \frac{\overline{f}(x, G^{-1}(\tilde{v}(x)))}{g(G^{-1}(\tilde{v}(x)))}\right] \quad \text{a.e. in } \mathbb{R}^N.$$

Repeating the analogous steps of the proof of Theorem 1.1, $\tilde{v} = \tilde{v}^+ \geq 0$ and satisfies

$$-\Delta\tilde{v} + V_p(x)\frac{G^{-1}(\tilde{v})}{g(G^{-1}(\tilde{v}))} = \tilde{\varrho} + \tilde{v}^{2^*-1} \quad \text{a.e. } \mathbb{R}^N,$$

that is,

$$-\Delta\tilde{v} + V_p(x)\frac{G^{-1}(\tilde{v})}{g(G^{-1}(\tilde{v}))} - \tilde{v}^{2^*-1} \in \left[\frac{\underline{f}(x, G^{-1}(\tilde{v}))}{g(G^{-1}(\tilde{v}))}, \frac{\overline{f}(x, G^{-1}(\tilde{v}))}{g(G^{-1}(\tilde{v}))}\right] \quad \text{a.e. in } \mathbb{R}^N.$$

Similarly, we also have $\tilde{v} > 0$ in $\mathbb{R}^N$ by the analogous argument used in Claim 4.2. Since (1.8) is only the periodic case for (1.6), Lemma 2.2 is also valid. Hence, we can see that $u = G^{-1}(\tilde{v})$ will be a positive solution of problem (1.8).

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Global bifurcation of positive solutions for a superlinear p-Laplacian system

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Abstract. We are concerned with the principal eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda\theta_1\varphi_p(v), & x \in \Omega, \\ -\Delta_p v = \lambda\theta_2\varphi_p(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega \end{cases} \tag{P}$$

and the global structure of positive solutions for the system

$$\begin{cases} -\Delta_p u = \lambda f(v), & x \in \Omega, \\ -\Delta_p v = \lambda g(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{Q}$$

where $\varphi_p(s) = |s|^{p-2}s$, $\Delta_p s = \text{div}(|\nabla s|^{p-2}\nabla s)$, $\lambda > 0$ is a parameter, $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded domain with smooth boundary $\partial\Omega$, $f, g : \mathbb{R} \to (0, \infty)$ are continuous functions with p-superlinear growth at infinity. We obtain the principal eigenvalue of (P) by using a nonlinear Krein–Rutman theorem and the unbounded branch of positive solutions for (Q) via bifurcation technology.

Keywords: p-Laplacian, principal eigenvalue, positive solutions, bifurcation.

2020 Mathematics Subject Classification: 35B32, 35B40, 35J92.

1 Introduction

In this paper, we are concerned with the global structure of positive solutions for the system

$$\begin{cases} -\Delta_p u = \lambda f(v), & x \in \Omega, \\ -\Delta_p v = \lambda g(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Delta_p s = \operatorname{div}(|\nabla s|^{p-2}\nabla s)$, $\lambda > 0$ is a parameter, $\Omega \subset \mathbb{R}^N$, $N > 2$, is a bounded domain with smooth boundary $\partial\Omega$, $f, g : \mathbb{R} \to (0, \infty)$ are continuous functions with p-superlinear growth at infinity.

The bifurcation behavior for p-Laplacian scalar equation has been investigated by many authors, see [11, 12, 22, 23, 30] for finite interval ($N = 1$) and [10, 18] for bounded domain ($N > 1$). For example, in [18], Fleckinger and Reichel established the global solution branches for the problem

$$\begin{cases} -\Delta_p u = \lambda(1 + u^q), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda \geqslant 0$ and $q > p - 1$. Let $p^* := Np/(N - p)$ if $N > p$ and $p^* := \infty$ if $n \leqslant p$. They obtained, in supercritical case, that is, $q > p^* - 1$, then there exists an unbounded continuum $\mathscr{C}^+ \subset [0, +\infty) \times C_0^1(\bar{\Omega})$ of solutions of (1.2). Moreover, in subcritical case, that is, $q \in (p - 1, p^* - 1)$, then $\mathscr{C}^+$ is bounded in the λ-direction and becomes unbounded near $\lambda = 0$ under some additional conditions.

In [8], Chhetri and Girg studied global structure of the semilinear system

$$\begin{cases} -\Delta u = \lambda f(v), & x \in \Omega, \\ -\Delta v = \lambda g(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

they make the following assumptions:

(H1) $f, g \in C(\mathbb{R}, (0, \infty))$ are continuous and non-decreasing functions;

(C2) f and g satisfy

$$\lim_{s \to +\infty} \frac{f(s)}{s} = +\infty, \qquad \lim_{s \to +\infty} \frac{g(s)}{s} = +\infty.$$

Under (H1) and (C2), they obtained the global behavior of positive solutions set of (1.3) by using bifurcation technology. To be more precise, in supercritical case, they obtained a component of positive solutions for (1.3), emanating from the origin, which is bounded in positive λ-direction. If in addition, Ω is convex, and $f, g \in C^1$ satisfy certain subcriticality conditions, they showed that the component must bifurcate from infinity at $\lambda = 0$.

As for p-Laplacian system (1.1), the first result of which we are aware concerning is the one by Hai and Shivaji in [20], by means of sub- and supersolutions, it was proved that (1.1) has a large positive solution (u, v) for $\lambda > \lambda_0$ with some p-sublinear conditions for f and g. Quite recently, there are some authors concerned with the positive solutions for p-Laplacian systems, refer to [9, 15, 17, 26, 27, 29] and references therein. But all of them only obtain the positive solution and do not provide any information about the global structure of positive solutions set.

The global structure is very useful for computing the numerical solutions of differential equations as it can be used to guide numerical work. For example, it can be used to estimate the u-interval in advance in applying the finite difference method and when applying the shooting method, it can be used to restrict the range of initial values that need to be considered.

As we all know, if we want to get global structure of positive solutions for (1.1) by using bifurcation technology, it is necessary to investigate the eigenvalue of corresponding eigenvalue problem. However, to the best of our knowledge, the spectral theory of the corresponding p-Laplacian system has not yet been established. In fact, for $p = 2$, the principal eigenvalue ν_1 of linear system corresponding to (1.3) can be directly expressed as

$$\nu_1 = \hat{\eta}_1 / \sqrt{\theta_1 \theta_2},$$

see [7, Prop. B.1], where $\hat{\eta}_1$ is the principal eigenvalue of scalar equation. However, this result do not hold for $p \neq 2$ because p-Laplacian operator is neither self-adjoint linear nor symmetric. In order to overcome this, we introduce a nonlinear version of Krein–Rutman theorem established by Arapostathis [2]. Let $\varphi_p(s) = |s|^{p-2}s$. By using the nonlinear Krein–Rutman theorem, we obtain that

$$\begin{cases} -\Delta_p u = \lambda \theta_1 \varphi_p(v), & x \in \Omega, \\ -\Delta_p v = \lambda \theta_2 \varphi_p(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

has a positive simple eigenvalue, expressed as μ_1, having the smallest absolute value, that is to say, μ_1 is the principal eigenvalue of (1.4).

The additional difficulty is the bifurcation results in dealing with semilinear boundary value problems [8] cannot be applied directly to quasilinear problems. Hence, we use the jumps of the index of the trivial solution to obtain a branch of nontrivial solutions. In addition to that, since Δ_p is asymmetric, the proof used in Theorem 1.1 of [8] do not applicable to (1.1). To this end, we adopt a new approach, with the help of sub- and supersolutions, to prove the nonexistence of solution for (1.1).

Let $X = C(\bar{\Omega}) \times C(\bar{\Omega})$, it is easy to know X is a Banach space endowed with the norm $\|(u_1, u_2)\| = \|u_1\|_C + \|u_2\|_C$, where $\|u\|_C$ is equipped with the supremum norm. By a solution of (1.1), we mean a $(\lambda, (u, v))$ that solves (1.1) in the weak sense, that is, $(u, v) \in E$, where $E := W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, and satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \omega \mathrm{d}x = \lambda \int_\Omega f(v) \omega \mathrm{d}x,$$
$$\int_\Omega |v|^{p-2} \nabla v \nabla \varpi \mathrm{d}x = \lambda \int_\Omega g(u) \varpi \mathrm{d}x$$

for all $(\omega, \varpi) \in E$. We denote Π of the form

$$\Pi = \overline{\{(\lambda, (u, v)) \in \mathbb{R} \times E \mid (\lambda, (u, v)) \text{ solution of } (1.1)\}}.$$

If $(\lambda, (u, v)) \in \Pi$ and $u > 0$, $v > 0$, then we say that $(\lambda, (u, v))$ is a *positive solution* of (1.1). By a *continuum* of solutions of (1.1) we mean a subset $\mathscr{K} \subset \Pi$ which is closed and connected. By a *component* of solutions set Π we mean a continuum which is maximal with respect to inclusion ordering. We say that λ_∞ is a *bifurcation point* from infinity if the solution set Π contains a sequence $(\lambda_n, (u_n, v_n))$ such that $\lambda_n \to \lambda_\infty$ and $\|(u_n, v_n)\| \to +\infty$ as $n \to +\infty$. We say that a continuum $\mathscr{C}$ bifurcates from infinity at $\lambda \in \mathbb{R}$ if there exists a sequence of solutions $(\lambda_n, (u_n, v_n)) \in \mathscr{C}$ such that $\lambda_n \to \lambda_\infty$ and $\|(u_n, v_n)\| \to +\infty$ as $n \to +\infty$.

Let $\eta_1 > 0$ be the principal eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda \varphi_p(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

by [1], η_1 is simple, isolated, and the unique positive eigenvalue having a nonnegative eigenfunction χ_1.

Further we assume that:

(H2) f and g satisfy

$$\lim_{s\to+\infty}\frac{f(s)}{\varphi_p(s)}=+\infty,\qquad \lim_{s\to+\infty}\frac{g(s)}{\varphi_p(s)}=+\infty.$$

We first state a nonexistence result in Theorem 1.1, which holds under weaker assumptions than (H1)–(H2). Theorem 1.2 gives a existence result when f and g have supercritical growth at infinite. Specifically, there is an unbounded branch $\mathscr{C}$ of positive solutions for (1.1), bifurcating from infinity and going through trivial solution, which is bounded in positive λ-direction. Moreover, when Ω is convex with C^2 boundary and $p\in(1,2)$, f and g satisfy certain subcritical growth restrictions, Theorem 1.3 shows that $\lambda=0$ is the unique bifurcation point from infinity for the continuum $\mathscr{C}$ obtained in Theorem 1.2.

Theorem 1.1. *Let $m>0$, and suppose $f(s),g(s)\geqslant m\varphi_p(s)$ for all $s>0$, then (1.1) has no solution for $\lambda\geqslant\bar\lambda:=\eta_1/m$.*

Theorem 1.2 (Supercritical case)**.** *Let (H1)–(H2) hold. Then there exists an unbounded component $\mathscr{C}\subset\Pi$ satisfying the following:*

(a) *$(\lambda,(u,v))\in\mathscr{C}$ is positive whenever $\lambda\in(0,\bar\lambda)$;*

(b) *(0,(0,0)) is the only element belonging to $\mathscr{C}$ with $\lambda=0$;*

(c) $\mathrm{Proj}_{\lambda\in[0,+\infty)}\mathscr{C}\overset{def}{=}\{\lambda\in[0,+\infty)\mid\exists(u,v)\in E \text{ with } (\lambda,(u,v))\in\mathscr{C}\}\subset[0,\bar\lambda)$;

(d) *any sequence $(\lambda_n,(u_n,v_n))\in\mathscr{C}$ such that $\|(u_n,v_n)\|\to+\infty$ as $n\to+\infty$ and $\lambda_n>0$ must satisfy $\lambda_n\to0^+$ as $n\to+\infty$.*

Theorem 1.3 (Subcritical case)**.** *Assume that (H1)–(H2) hold. Let $p\in(1,2)$, $N>2$ and Ω be convex with C^2 boundary, f,g satisfy*

$$\lim_{s\to+\infty}\frac{f(s)}{s^{q_1}}=C,\qquad \lim_{s\to+\infty}\frac{g(s)}{s^{q_2}}=D, \tag{1.6}$$

for some positive constants C,D, and $q_1q_2>(p-1)^2$ satisfy

$$\max\left\{\frac{p(q_1+p-1)}{q_1q_2-(p-1)^2}-\frac{N-p}{p-1},\ \frac{p(q_2+p-1)}{q_1q_2-(p-1)^2}-\frac{N-p}{p-1}\right\}\geqslant 0.$$

Then $\mu_1=0$ is the unique bifurcation point from infinity, for the continuum $\mathscr{C}\subset\Pi$ from Theorem 1.2.

Corollary 1.4. *We may obtain the number of positive solutions of (1.1) from Theorem 1.3:*

(i) *(1.1) has no positive solution for $\lambda\geqslant\bar\lambda$;*

(ii) *there exists $\underline{\lambda}<\bar\lambda$ such that (1.1) has at least two positive solutions for each $\lambda\in(0,\underline{\lambda})$.*

Remark 1.5. It is worth remarking that Theorem 1.1 and Theorem 1.2 are direct generalizations of the results in [8]. However, Theorem 1.3 only holds for $p\in(1,2)$ because *a priori estimates* established in [4] is not available to $p=2$.

2 Preliminaries

Next we state some notations from [2].

An *ordered Banach space* is a real Banach space W with a cone K. When the interior of K, denoted as int$K$, is nonempty, we call W a *strongly ordered* Banach space. As usual, we write $x \preceq y$ if $y - x \in K$. A continuous map $T : W \to W$ is

- order-preserving or increasing if $x \preceq y \Rightarrow T(x) \preceq T(y)$;
- homogeneous of degree one, or 1-homogeneous, if $T(tx) = tT(x)$ for all $t \geqslant 0$.

Lemma 2.1 ([4, Lemma 1.1]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,\beta}$ for some $\beta \in (0,1)$ and $g \in L^\infty(\Omega)$. Then the problem*

$$\begin{cases} \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \omega dx = \lambda \int_\Omega f(u)\omega dx, & \forall \omega \in C_c^\infty(\Omega), \\ u \in W_0^{1,p}(\Omega), \quad p > 1 \end{cases} \tag{2.1}$$

has a unique solution $u \in C_0^1(\bar{\Omega})$. Moreover, if we define the operator $K : L^\infty(\Omega) \to C_0^1(\bar{\Omega}) : g \mapsto u$ where u is the unique solution of (2.1), then K is continuous, compact and order-preserving.

Now we give a definition of weak sub- and supersolutions of (1.1), which is defined by [20].

Definition 2.2. We say that (α_u, α_v) is a weak subsolution of problem (1.1) if (α_u, α_v) satisfies

$$\int_\Omega |\nabla \alpha_u|^{p-2}\nabla \alpha_u \nabla \omega \mathrm{d}x \leqslant \lambda \int_\Omega f(\alpha_v)\omega \mathrm{d}x,$$
$$\int_\Omega |\nabla \alpha_v|^{p-2}\nabla \alpha_v \nabla \omega \mathrm{d}x \leqslant \lambda \int_\Omega g(\alpha_u)\omega \mathrm{d}x$$

for all $\omega \in W_0^{1,p}(\Omega)$ with $\omega \geqslant 0$. Similarly, we say that (β_u, β_v) is a weak supersolution of problem (1.1) if (β_u, β_v) satisfies

$$\int_\Omega |\nabla \beta_u|^{p-2}\nabla \beta_u \nabla \omega \mathrm{d}x \geqslant \lambda \int_\Omega f(\beta_v)\omega \mathrm{d}x,$$
$$\int_\Omega |\nabla \beta_v|^{p-2}\nabla \beta_v \nabla \omega \mathrm{d}x \geqslant \lambda \int_\Omega f(\beta_u)\omega \mathrm{d}x$$

for all $\omega \in W_0^{1,p}(\Omega)$ with $\omega \geqslant 0$.

Proposition 2.3 ([34, Theorem 14.D]). *Let Y be a Banach space with $Y \neq \{0\}$ and let $F : Y \to Y$ be compact. Then the solution component $\mathfrak{C} \subset \mathbb{R} \times Y$ of the equation*

$$x = \lambda F(x)$$

which contains $(0,0) \in \mathbb{R} \times Y$ is unbounded as are both subsets

$$\mathfrak{C}_\pm \overset{def}{=} \mathfrak{C} \cap (\mathbb{R}_\pm \times Y),$$

where $\mathbb{R}_+ \overset{def}{=} [0,\infty)$ and $\mathbb{R}_- \overset{def}{=} (-\infty, 0]$.

Definition 2.4 ([33]). Let Z be a Banach space and $\{C_n \mid n = 1,2,\cdots\}$ be a certain infinite collection of subset of Z. Then the superior limit of $\mathcal{D}$ of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \{x \in Z \mid \exists \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \to x\}.$$

Lemma 2.5 ([33]). *Let Z be a Banach space with the norm $\|\cdot\|_Z$, let $\{C_n\}$ be a family of closed subsets of Z. Assume that:*

(i) *there exist $z_n \in C_n$, $n = 1, 2, \dots,$ and $z^* \in Z$, such that $z_n \to z^*$;*

(ii) $d_n = \sup\{\|x\|_Z \mid x \in C_n\} = \infty$;

(iii) *for every $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relatively compact set of Z, where*

$$B_R = \{x \in Z \mid \|x\|_Z \leqslant R\},$$

then there exists an unbounded component $\mathcal{C}$ in $\mathcal{D}$ and $z^* \in \mathcal{C}$.

The following nonlinear version of the Krein–Rutman theorem is firstly established by Mahadevan [25] and corrected by Arapostathis [2].

Let

$$\sigma_+(T) := \{\lambda > 0 : T(x) = \lambda x,\ x \in K \setminus \{0\}\}.$$

Consider the following hypotheses:

(B1) If $x \in \partial K \setminus \{0\}$, then $x - \beta T(x) \notin K$ for all $\beta > 0$.

(B2) If $x - y \in \partial K \setminus \{0\}$, then $x - y - \beta(T(x) - T(y)) \notin K$ for all $\beta > 0$.

Lemma 2.6 ([2, Theorem 3]). *Let W be strongly ordered, and $T : K \to K$ be an order-preserving, 1-homogeneous map with $\sigma_+(T) \neq \emptyset$.*

(i) If (B1) holds, then $T(\operatorname{int} K) \subset \operatorname{int} K$, $\sigma_+(T)$ is a singleton, and all eigenvectors lie in $\operatorname{int} K$.

(ii) If (B2) holds, then the unique eigenvalue in $\sigma_+(T)$ is simple.

3 Eigenvalue problems

Consider the eigenvalue of problem

$$\begin{cases} -\Delta_p u = \lambda\theta_1\varphi_p(v), & x \in \Omega, \\ -\Delta_p v = \lambda\theta_2\varphi_p(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{3.1}$$

where $\theta_1, \theta_2 > 0$, then (3.1) is equivalent to

$$-\Delta_p \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix} \begin{pmatrix} \varphi_p(u) \\ \varphi_p(v) \end{pmatrix},$$

that is,

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\Delta_p^{-1} \begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix} \begin{pmatrix} \varphi_p(u) \\ \varphi_p(v) \end{pmatrix}.$$

It is equivalent to

$$U = -\Delta_p^{-1} A\varphi_p(U) =: HU,$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix}.$$

Obviously, H is 1-homogeneous by the property of φ_p and φ_p^{-1}. Lemma 2.1 shows H is a continuous, positively compact operator, and by the strictly increasing property of φ_p and $(-\Delta_p)^{-1}$ we know that H is strictly increasing.

Taking the cone to be

$$P = \left\{ w \in C_0^1(\overline{\Omega}) : w \geqslant 0 \text{ in } \Omega,\ \frac{\partial w}{\partial n} \leqslant 0 \text{ on } \partial\Omega \right\}.$$

Lemma 3.1 ([32, Lemma 2.2.1])**.** *Let $u, v \in C^1(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$, $v|_{\partial\Omega} \geqslant 0$, $u > 0$ in Ω, and $\partial_{\mathbf{n}} u|_{\partial\Omega} < 0$. Then there exists a positive constant $\varepsilon > 0$ such that $u + \varepsilon v > 0$ in Ω.*

By Lemma 3.1, the interior of P can be expressed as

$$\operatorname{int} P = \left\{ w \in C_0^1(\overline{\Omega}) : w > 0 \text{ in } \Omega,\ \frac{\partial w}{\partial n} < 0 \text{ on } \partial\Omega \right\}.$$

Let

$$K = P \times P.$$

Obviously, (B1) can be obtained by letting $y = 0$ in (B2), therefore, we only show that H satisfies (B2). In fact, since K is a closed set, if we choose $V_1, V_2 \in K$ such that $V_1 - V_2 \in \partial K \setminus \{0\}$, then $V_1 - V_2 \in K$, that is to say, $V_1 \succeq V_2$. The strong maximum principle shows $H(V_1), H(V_2) \in K$. In addition, by the property that H is strictly increasing we have $H(V_1) \succeq H(V_2)$, this means $H(V_1) - H(V_2) \in K$ with $H(V_1) - H(V_2) \neq \{0\}$. On the other hand, $V_1 - V_2 \in \partial K \setminus \{0\}$, then any neighborhood of $V_1 - V_2$ contains some points that do not belong to K, this means

$$V_1 - V_2 - \beta(H(V_1) - H(V_2)) \notin K$$

for all $\beta > 0$.

Lemma 3.2. *System (3.1) has a positive and simple eigenvalue $\mu_1 > 0$ with a eigenfunction $U_1 = (\phi_1, \psi_1) > 0$.*

Proof. By Lemma 2.1, H is compact, then $\sigma_+(T) \neq \emptyset$. By (i) of Lemma 2.6, $\sigma_+(T)$ is a singleton, denoted by λ_1, and all eigenvectors lie in $\operatorname{int} K$, that is, λ_1 is unique. Moreover, (B2) shows λ_1 is simple. Therefore, H has a unique positive eigenvector (eigenfunction) $U_1 = (\phi_1, \psi_1)$, which satisfies

$$HU_1 = \lambda_1 U_1,$$

Subsequently, $\mu_1 = 1/\lambda_1$ is a simple, isolated and positive eigenvalue of (1.4). □

4 Auxiliary results

Proof of Theorem 1.1. Suppose that there is a sequence of $\{\lambda_n\}$ such that $\lambda_n \to \infty$ as $n \to \infty$, and (1.1) has a positive solution (u_n, v_n) when $\lambda = \lambda_n$. Then there is a $\lambda_{n_0} > 0$ such that $\lambda_n > \frac{\eta_1}{m}$ for all $n > n_0$. We can choose a suitable $\epsilon_0 > 0$ such that $\lambda^* = \eta_1 + \epsilon_0$ and

$$\lambda_n m > \lambda^*, \qquad n > n_0.$$

Then

$$\begin{aligned}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla \omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(v_n)\omega \mathrm{d}x \\ = \lambda_n \int_\Omega f(v_n)\omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(v_n)\omega \mathrm{d}x \\ \geqslant \lambda_n \int_\Omega m\varphi_p(v_n)\omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(v_n)\omega \mathrm{d}x \\ > 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla \omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(u_n)\omega \mathrm{d}x \\ = \lambda_n \int_\Omega g(u_n)\omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(u_n)\omega \mathrm{d}x \\ \geqslant \lambda_n \int_\Omega m\varphi_p(u_n)\omega \mathrm{d}x - \lambda^* \int_\Omega \varphi_p(u_n)\omega \mathrm{d}x \\ > 0.\end{aligned}$$

That is, (u_n, v_n) is a weak supersolution of the problem

$$\begin{cases} -\Delta_p u = (\eta_1 + \epsilon_0)\varphi_p(v), & x \in \Omega, \\ -\Delta_p v = (\eta_1 + \epsilon_0)\varphi_p(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

On the other hand $t(\chi_1, \chi_1)$, $t > 0$, is a subsolution of (4.1). Letting $t > 0$ be such that $t(\chi_1, \chi_1) \leqslant (u_n, v_n)$, then by the method of sub- and super-solutions, for every $n > n_0$, (4.1) has a positive solution (x_n, y_n). (The proof of the existence of (x_n, y_n) have been showed, see [16, 20] for details. We omit it here.) On the other hand, since $\epsilon_0 > 0$ is arbitrary, this contradicts with the fact that η_1 is isolated. □

Now, consider an asymptotically positively homogeneous system of the form

$$\begin{cases} -\Delta_p u = \lambda\theta_1\varphi_p(v^+) + \lambda\widetilde{f}(v), & x \in \Omega, \\ -\Delta_p v = \lambda\theta_2\varphi_p(u^+) + \lambda\widetilde{g}(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{4.2}$$

where $x^+ \overset{\text{def}}{=} \max\{0, x\}$ is the positive part of x, and θ_1, θ_2 are defined above. The nonlinear perturbations $\widetilde{f}, \widetilde{g} : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

(A1) $\widetilde{f}$ and $\widetilde{g}$ are continuous, non-negative, and bounded functions;

(A2) $\theta_1\varphi_p(y^+) + \widetilde{f}(y) > 0$, $\theta_2\varphi_p(x^+) + \widetilde{g}(x) > 0$ for all $x, y \in \mathbb{R}$.

Let

$$\mathscr{T} = \{(\lambda, (u, v)) \in \mathbb{R} \times E \mid (\lambda, (u, v)) \text{ solution of } (4.2)\},$$

then we prove the following bifurcation result.

Proposition 4.1. *If μ_∞ is a bifurcation point from infinity for (4.2), then $\mu_\infty = \mu_1$. Moreover, for any sequence $(\lambda_j, (u_j, v_j)) \in \mathbb{R} \times E$ with $\lambda_j \to \mu_1$ and $\|(u_j, v_j)\| \to +\infty$ as $j \to +\infty$, there exists a subsequence $(\lambda_{j_k}, (u_{j_k}, v_{j_k}))$ such that*

$$\lim_{j_k \to +\infty} \frac{(u_{j_k}, v_{j_k})}{\|(u_{j_k}, v_{j_k})\|} = \frac{(\phi_1, \psi_1)}{\|(\phi_1, \psi_1)\|},$$

where the convergence is in E.

Proof. The operator equation corresponding to the system (4.2) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \lambda(-\Delta_p^{-1}) \begin{pmatrix} \theta_1 \varphi_p(v^+) + \widetilde{f}(v) \\ \theta_2 \varphi_p(u^+) + \widetilde{g}(u) \end{pmatrix}. \tag{4.3}$$

Let $(\lambda_j, (u_j, v_j)) \in \mathbb{R} \times E$ be a solution of (4.2) such that $\|(u_j, v_j)\| \to +\infty$ and $\lambda_j \to \mu_\infty$. Then $(\hat{u}_j, \hat{v}_j) = \frac{(u_j, v_j)}{\|(u_j, v_j)\|}$ satisfies

$$\hat{u}_j = \lambda_j(-\Delta_p^{-1}) \left(\theta_1 \varphi_p(\hat{v}_j^+) + \frac{\widetilde{f}(v_j)}{\|(u_j, v_j)\|} \right), \tag{4.4}$$

$$\hat{v}_j = \lambda_j(-\Delta_p^{-1}) \left(\theta_2 \varphi_p(\hat{u}_j^+) + \frac{\widetilde{g}(u_j)}{\|(u_j, v_j)\|} \right). \tag{4.5}$$

It then follows from (A1) that the right hand side of (4.4) and (4.5) are bounded in X (independent of j). Hence $\|\hat{u}_j\|_{C^1}$ and $\|\hat{v}_j\|_{C^1}$ are bounded (independent of j), and there exists subsequence of $\hat{u}_j$ and $\hat{v}_j$ converging to $\hat{u}$ and $\hat{v}$ and satisfying

$$\begin{cases} -\Delta_p \hat{u} = \mu_\infty \theta_1 \varphi_p(\hat{v}^+), & x \in \Omega, \\ -\Delta_p \hat{v} = \mu_\infty \theta_2 \varphi_p(\hat{u}^+), & x \in \Omega, \\ \hat{u} = 0 = \hat{v}, & x \in \partial\Omega. \end{cases} \tag{4.6}$$

Suppose $\mu_\infty \leqslant 0$. Since $\hat{u}^+, \hat{v}^+ \geqslant 0$, it follows by applying the maximum principle to (4.6) that $\hat{u} \equiv 0$ and repeating the same argument we get $\hat{v} \equiv 0$ as well. This leads to a contradiction since $\|(\hat{u}, \hat{v})\| = 1$.

For $\mu_\infty > 0$, we distinguish two cases: the first case is $\hat{v}^+ \equiv 0$ and $\hat{u}^+ \equiv 0$, and the second is one of $\hat{v}^+ \not\equiv 0$ or $\hat{u}^+ \not\equiv 0$ holding. In the first case, we get $\hat{u} \equiv 0$, a contradiction as before. In the other case, we get $\hat{u} > 0$ from maximum principle and $\hat{v} > 0$ by repeating the same argument. Thus μ_∞ and $\hat{u}, \hat{v} > 0$ satisfies the eigenvalue problem (3.1).

However, we already discussed that (3.1) has precisely one eigenvalue μ_1 with componentwise positive eigenfunction (ϕ_1, ψ_1). Therefore, it must be that $\mu_\infty = \mu_1$ and

$$(\hat{u}, \hat{v}) = \frac{(\phi_1, \psi_1)}{\|(\phi_1, \psi_1)\|}.$$

This concludes the proof of Proposition 4.1. □

Lemma 4.2. *Let (A1)–(A2) hold and Λ be a compact interval with $\mu_1 \notin \Lambda$. Then there is a $M_\Lambda > 0$ such that all solutions $(\lambda, (u, v))$ of (4.2) with $\lambda \in \Lambda$ must satisfy $\|(u, v)\| < M_\Lambda$.*

Proof. Assume to the contrary that there exist sequences $\lambda_j \in \Lambda$ and $(u_j, v_j) \in E$ satisfying (4.2) with $\|(u_j, v_j)\| \to \infty$ as $j \to \infty$. Then there is a subsequence $(\lambda_{j_k}, (u_{j_k}, v_{j_k}))$ of $(\lambda_j, (u_j, v_j))$ satisfying $\lambda_{j_k} \to \tilde{\lambda}$ and $\|(u_{j_k}, v_{j_k})\| \to \infty$. Dividing (4.2) by $\|(u_{j_k}, v_{j_k})\|$, then the same argument as in the proof of Proposition 4.1, we obtain that $\tilde{\lambda} = \mu_1 \in \Lambda$, which contradicts $\mu_1 \notin \Lambda$. □

Lemma 4.3. *Let (A1)–(A2) hold, then for $\tau \in [0, 1]$, the system*

$$\begin{cases} -\Delta_p u = \lambda\theta_1\varphi_p(v^+) + \lambda\widetilde{f}(v) + \tau\|(u,v)\|^p, & x \in \Omega, \\ -\Delta_p v = \lambda\theta_2\varphi_p(u^+) + \lambda\widetilde{g}(u) + \tau\|(u,v)\|^p, & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega \end{cases} \tag{4.7}$$

has no solution for $\lambda > \nu := \mu_1 / \min\{\theta_1, \theta_2\}$.

Proof. Suppose that there is a sequence of $\{\lambda_n\}$ such that $\lambda_n \to \infty$ as $n \to \infty$, and (4.7) has a positive solution (u_n, v_n) when $\lambda = \lambda_n$. Then there is a $\lambda_{n_0} > 0$ such that $\lambda_n > \frac{\mu_1}{\min\{\theta_1, \theta_2\}}$ for all $n > n_0$. We can choose a suitable $\epsilon_0 > 0$ such that $\mu^* = \mu_1 + \epsilon_0$ and

$$\lambda_n \min\{\theta_1, \theta_2\} > \mu^*, n > n_0.$$

Since (u_n, v_n) is the positive solution of (4.7), then $u_n^+ = u_n,\ v_n^+ = v_n$, and

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\omega \mathrm{d}x - \mu^* \int_\Omega \theta_1\varphi_p(v_n)\omega \mathrm{d}x \\ &\quad = \lambda_n \int_\Omega \theta_1\varphi_p(v_n^+)\omega \mathrm{d}x + \lambda_n \int_\Omega \widetilde{f}(v_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x - \mu^* \int_\Omega \varphi_p(v_n)\omega \mathrm{d}x \\ &\quad = \lambda_n \int_\Omega \theta_1\varphi_p(v_n)\omega \mathrm{d}x - \mu^* \int_\Omega \varphi_p(v_n)\omega \mathrm{d}x + \lambda_n \int_\Omega \widetilde{f}(v_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x \\ &\quad > \lambda_n \int_\Omega \widetilde{f}(v_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x \\ &\quad > 0. \end{aligned}$$

Similarly,

$$\begin{aligned} &\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla\omega \mathrm{d}x - \mu^* \int_\Omega \theta_2\varphi_p(u_n)\omega \mathrm{d}x \\ &\quad = \lambda_n \int_\Omega \theta_2\varphi_p(u_n^+)\omega \mathrm{d}x + \lambda_n \int_\Omega \widetilde{g}(u_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x - \mu^* \int_\Omega \theta_2\varphi_p(u_n)\omega \mathrm{d}x \\ &\quad = \lambda_n \int_\Omega \theta_2\varphi_p(u_n)\omega \mathrm{d}x - \mu^* \int_\Omega \theta_2\varphi_p(u_n)\omega \mathrm{d}x + \lambda_n \int_\Omega \widetilde{g}(u_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x \\ &\quad > \lambda_n \int_\Omega \widetilde{g}(v_n)\omega \mathrm{d}x + \int_\Omega \tau\|(u_n, v_n)\|^p\omega \mathrm{d}x \\ &\quad > 0. \end{aligned}$$

That is, (u_n, v_n) is a weak supersolution of the problem

$$\begin{cases} -\Delta_p u = (\mu_1 + \epsilon_0)\theta_1\varphi_p(v), & x \in \Omega, \\ -\Delta_p v = (\mu_1 + \epsilon_0)\theta_2\varphi_p(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega. \end{cases} \tag{4.8}$$

On the other hand $t(\phi_1, \psi_1)$, $t > 0$, is a subsolution of (4.8). Letting $t > 0$ be such that $t(\phi_1, \psi_1) \leqslant (u_n, v_n)$, then the same argument with the proof of Theorem 1.1 we obtain a contradiction with that μ_1 is isolated. □

Define the operator $L : X \to X$ by

$$Q_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \theta_1 \varphi_p(v^+) \\ \theta_2 \varphi_p(u^+) \end{pmatrix}$$

and

$$T_\lambda(u,v) := (-\Delta_p^{-1}) \circ \lambda Q_1(u,v).$$

It is well known that T_λ is completely continuous in X. By Lemma 3.2, μ_1 is a isolated eigenvalue of (3.1), therefore, there is no nontrivial solution of (3.1) in $\lambda \in (\mu_1 - \delta, \mu_1) \cup (\mu_1, \mu_1 + \delta)$ for some $\delta > 0$, that is, T_λ has no fixed point in $\partial B_r(0)$ for arbitrary r-ball $B_r(0)$ when $\lambda \in (\mu_1 - \delta, \mu_1) \cup (\mu_1, \mu_1 + \delta)$. Subsequently, the Leray–Schauder degree $\deg(I - T_\lambda, B_r(0), (0,0))$ is well defined arbitrary r-ball $B_r(0)$ with $\lambda \in (\mu_1 - \delta) \cup (\mu_1, \mu_1 + \delta)$.

Define the operator $Q : X \to X$ by

$$Q \begin{pmatrix} u \\ v \end{pmatrix} = Q_1 \begin{pmatrix} u \\ v \end{pmatrix} + Q_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \theta_1 \varphi_p(v^+) \\ \theta_2 \varphi_p(u^+) \end{pmatrix} + \begin{pmatrix} \widetilde{f}(v) \\ \widetilde{g}(u) \end{pmatrix}.$$

Then it is clear that Q is continuous operator and problem (4.2) can be equivalently written as

$$(u,v) = (-\Delta_p^{-1}) \circ \lambda Q(u,v) := \mathcal{F}_\lambda(u,v).$$

Since $-\Delta_p^{-1} : X \to X$ is compact, then $\mathcal{F}_\lambda : X \to X$ is completely continuous. Let

$$\Upsilon_\lambda^p(u,v) = (u,v) - (-\Delta_p^{-1}) \circ \mathcal{F}_\lambda(\tau, u, v) = (u,v) - (-\Delta_p^{-1}) \circ \lambda(Q_1(u,v) + Q_2(u,v)).$$

By (A1), $\widetilde{f}, \widetilde{g}$ are bounded, then we have $Q_2(u,v) \to (0,0)$ for (u,v) large enough, that is, $\Upsilon_\lambda^p(u,v)$ close to $T_\lambda(u,v)$ when $(u,v) \to (+\infty, +\infty)$, and subsequently, $\Upsilon_\lambda^p(u,v)$ has no nontrivial solution when $\lambda \in (\mu_1 - \delta) \cup (\mu_1, \mu_1 + \delta)$. Let $(y,z) = (u,v)/\|(u,v)\|^p$, then $(u,v) \to (+\infty, +\infty)$ is equivalent to $(y,z) \to (0,0)$.

$$\begin{aligned} \Psi_\lambda^p(y,z) = \frac{\Upsilon_\lambda^p(u,v)}{\|(u,v)\|^p} &= (y,z) - \|(y,z)\|^p \mathcal{F}_\lambda\left(\frac{(y,z)}{\|(y,z)\|^p}\right) \\ &= (y,z) - \|(y,z)\|^p (-\Delta_p^{-1}) \circ \lambda \left[Q_1\left(\frac{(y,z)}{\|(y,z)\|^p}\right) + Q_2\left(\frac{(y,z)}{\|(y,z)\|^p}\right)\right]. \end{aligned} \tag{4.9}$$

Obviously, $\Psi_\lambda^p(y,z) = (0,0)$ has no nontrivial solution when (y,z) small enough, a similar argument with T_λ we have Leray–Schauder degree $\deg(\Psi_\lambda^p, B_r(0), (0,0))$ is well defined with r small enough for $\lambda \in (\mu_1 - \delta, \mu_1) \cup (\mu_1, \mu_1 + \delta)$.

Next we will show μ_1 is a bifurcation point, defined in Section 1, from infinity of (4.2). By Rabinowitz [28], it is equivalent to show that

$$\deg(\Psi_{r_1}^p, B_r(0), (0,0)) \neq \deg(\Psi_{r_2}^p, B_r(0), (0,0)),$$

where $r_1 \in [\mu_1 - \delta, \mu_1), r_2 \in (\mu_1, \mu_1 + \delta]$.

Lemma 4.4. *μ_1 is a bifurcation point from infinity of problem* (4.2).

Proof. By (4.9), one has that μ_1 is a bifurcation from infinity for (4.2) if and only if it is a bifurcation from the trivial solution for $\Psi^p_\lambda(u,v)=0$. Next we show

$$\deg(\Psi^p_\lambda, B_r(0),(0,0))=1, \qquad \lambda\in[\mu_1-\delta,\mu_1)$$

and

$$\deg(\Psi^p_\lambda, B_r(0),(0,0))=0, \qquad \lambda\in(\mu_1,\mu_1+\delta].$$

Suppose $(\mu_1,(0,0))$ is not a bifurcation point of (4.2), then there are $\delta,\rho_0>0$ such that for $|\lambda-\mu_1|\leqslant\delta$ and $0<\rho<\rho_0$, that is, equation

$$(y,z)-\mathscr{F}(\lambda,y,z)\neq 0$$

for $\|(y,z)\|_X=\rho$, where

$$\mathscr{F}(\lambda,y,z):=\|(y,z)\|^p\mathcal{F}_\lambda\left(\frac{(y,z)}{\|(y,z)\|^p}\right).$$

Consider the homotopy problem

$$(y,z)-\tau\mathscr{F}(\lambda,y,z)\neq 0,$$

where $\tau\in[0,1]$, then by the invariance of the degree of the compact, the same argument with [5, Corollary 3.2], we have

$$\deg(I-\mathscr{F}(\lambda,y,z),B_\rho(0),(0,0))=\deg(I,B_\rho(0),(0,0))=1 \tag{4.10}$$

for $\lambda\in[\mu_1-\delta,\mu_1)$.

Next we show, for $\lambda\in(\mu_1,\mu_1+\delta]$,

$$\deg(\Psi^p_\lambda(y,z),B_\rho(0),(0,0))=0.$$

Let $\check{\lambda}=\max\{\mu_1+\delta,\nu\}$. Define

$$\Phi(\lambda,(y,z))=\Psi^p_\lambda(y,z)$$

and consider the following one parameter family of operators

$$\Phi((1-\sigma)(\mu_1+\delta)+\sigma\check{\lambda},(y,z)), \qquad \sigma\in[0,1].$$

Obviously,

$$(1-\sigma)(\mu_1+\delta)+\sigma\check{\lambda}\in[\mu_1+\delta,\check{\lambda}].$$

By Lemma 4.2, the solutions of (4.2) satisfy

$$\|(u,v)\|<M(\varepsilon).$$

Since $(y,z)=(u,v)/\|(u,v)\|^p$, then all solutions of (4.2) satisfy

$$\|(y,z)\|=\frac{1}{\|(u,v)\|^{p-1}}>\frac{1}{[M(\varepsilon)]^{p-1}}.$$

Hence the problem (4.2) with $\lambda = (1-\sigma)(\mu_1+\delta)+\sigma\check{\lambda}$ does not have any solution on ∂B_ρ with $0<\rho<1/[M(\varepsilon)]^{p-1}$. Then by the homotopy invariance of degree with respect to $\sigma \in [0,1]$, we have

$$\deg(\Phi(\mu_1+\delta,(y,z)),B_\rho,(0,0)) = \deg(\Phi(\check{\lambda},(y,z)),B_\rho,(0,0)). \tag{4.11}$$

Obviously, we wish to show that

$$\deg(\Phi(\check{\lambda},(y,z)),B_\rho,(0,0)) = 0.$$

This would be trivial if $\Phi(\check{\lambda},(y,z)) = (0,0)$ has no solution on B_ρ. Therefore, we construct an admissible homotopy connecting $\Phi(\check{\lambda},(y,z))$ to an operator which does not have any solution on B_ρ for $0<\rho<1/[M(\varepsilon)]^{p-1}$. To this end, for $\tau \in [0,1]$, consider the operator

$$\Phi(\check{\lambda},(y,z)) - \tau\xi$$

with $(0,0) \neq \xi := (-\Delta_p)^{-1}(\chi_\Omega,\chi_\Omega)$, where χ_Ω stands for the characteristic function of Ω, that is,

$$\chi_\Omega(x) = \begin{cases} 1, & x\in\Omega, \\ 0, & x \notin \Omega. \end{cases}$$

First we show that, for all $\tau \in [0,1]$ and $0<\rho<1/[M(\varepsilon)]^{p-1}$,

$$\Phi(\check{\lambda},(y,z)) - \tau\xi = (0,0) \tag{4.12}$$

does not have any solution on ∂B_ρ. Indeed, assume to the contrary that there exists a solution (y,z) of (4.12) with $\|(y,z)\| = \rho > 0$. Then $(u,v) = (y,z)/\|(y,z)\|^p$ must satisfy (4.7), which is absurd due to Lemma 4.3. Therefore, (4.2) does not have any nontrivial solution for all $\tau \in [0,1]$. Moreover, since $(0,0)$ is not a solution of (4.2), then

$$\deg(\Phi(\check{\lambda},(y,z)) - \tau\xi, B_\rho,(0,0)) = 0 \quad \text{for all } \tau\in[0,1].$$

Then homotopy invariance of degree with respect to $\tau \in [0,1]$ yields

$$\deg(\Phi(\check{\lambda},(y,z)),B_\rho,(0,0)) = \deg(\Phi(\check{\lambda},(y,z)) - \xi, B_\rho,(0,0)) = 0.$$

Since this holds for any $0<\rho<1/[M(\varepsilon)]^{p-1}$, it follows from (4.11) that

$$\deg(\Phi(\mu_1+\delta,(y,z)),B_\rho,(0,0)) = \deg(\Phi(\check{\lambda},(y,z)),B_\rho,(0,0)) = 0.$$

This combine (4.10) we have μ_1 is a bifurcation point of (4.2) from infinity. □

Lemma 4.5. *Let (A1)–(A2) hold. Then μ_1 is the unique bifurcation point from infinity for* (4.2)*. Moreover, there exists a continuum $\mathscr{D} \subset \mathscr{T}$ bifurcating from infinity at μ_1 and satisfies the following:*

(i) *if $(\lambda,(u,v)) \in \mathscr{D}$ and $\lambda > 0$ then $u>0$ and $v>0$;*

(ii) *for $\lambda = 0$, $(u,v) = (0,0)$ is the only solution of* (4.2) *and (0,(0,0))$\in \mathscr{D}$;*

(iii) $\mathrm{Proj}_\lambda \mathscr{C} \overset{def}{=} \{\lambda \in \mathbb{R} \mid \exists\, (u,v) \in E \text{ with } (\lambda,(u,v)) \in \mathscr{D}\}$ *is bounded from above and unbounded from below.*

Proof. By Lemma 4.4 and Proposition 4.1, μ_1 is the unique bifurcation point from infinity for (4.2). It is easy to see that operator $\mathcal{F}_\lambda : X \to X$ satisfies the hypotheses of Proposition 2.3. Then there exist unbounded continua

$$\mathscr{D}_\pm \subset \widehat{\mathscr{S}} \stackrel{\text{def}}{=} \{(\lambda,(u,v)) \in \mathbb{R} \times E \mid (\lambda,(u,v)) \text{ solution of } (4.2)\}$$

containing $(0,(0,0))$. By the nonexistence result of Theorem 1.1

$$\mathscr{D}_+ \subset ([0,\lambda^*) \times E),$$

and thus $\mathscr{D}_+$ must be unbounded in the Banach space E-direction. Then $\mathscr{D} \stackrel{\text{def}}{=} \mathscr{D}_+ + \mathscr{D}_-$ is a continuum containing (0,(0,0)). By Proposition 4.1, μ_1 is the only bifurcation point from infinity for (4.2) and $\mathscr{D}_+$ is unbounded in the E-direction, hence $\mathscr{D}_+$ must bifurcate from infinity at μ_1. To conclude the proof of Lemma 4.5, it remains to verify that $\mathscr{D}$ satisfies the properties (i)–(iii).

It follows from assumption (A2) and maximum principle that $u,v > 0$ whenever $(\lambda,(u,v)) \in \mathscr{D}$ and $\lambda > 0$, this implies part (i). For $\lambda = 0, (u,v) = (0,0)$ is the only solution of (4.2) and $(0,(0,0)) \in \mathscr{D}$, hence part (ii) holds. Applying Proposition 2.3, we see that $\mathscr{D}_-$ must be unbounded in $\mathbb{R} \times E$. However, by part (ii) and the fact that ν_1 is the unique bifurcation point from infinity for (4.3), we see that $\mathscr{D}_-$ must be unbounded in the negative λ-direction, hence $(-\infty,\nu_1) \subset \mathrm{Proj}_\lambda \mathscr{D}$. This completes the proof of Lemma 4.5. □

5 Proof of Theorem 1.2

Step 1. Approximation problems

Fix $n \in \mathbb{N}$ and define $f_n(s), g_n(s) : \mathbb{R} \to (0,\infty)$ by

$$f_n(s) \stackrel{\text{def}}{=} \begin{cases} f(s); & s \leqslant n, \\ \frac{f(n)}{\varphi_p(n)}\varphi_p(s); & s > n, \end{cases}$$

$$g_n(s) \stackrel{\text{def}}{=} \begin{cases} g(s); & s \leqslant n, \\ \frac{g(n)}{\varphi_p(n)}\varphi_p(s); & s > n, \end{cases}$$

Then f_n and g_n are continuous functions. Note that, $f_n(s) = f(s)$ for $s \leqslant n$, $\lim_{s\to n^-} f_n(s) = f(n)$, hence f_n is continuous. On the other hand, by assumption (H2),

$$\lim_{s\to\infty} \frac{f_n(s)}{\varphi_p(s)} = \lim_{s\to\infty} \frac{\frac{f(n)}{\varphi_p(n)}\varphi_p(s)}{\varphi_p(s)} = \frac{f(n)}{\varphi_p(n)} \to \infty$$

as $n \to \infty$, then f_n approaches f. Similarly, g_n approaches g as $n \to \infty$.

For each n, we consider the following problem

$$\begin{cases} -\Delta_p u = \lambda f_n(v), & x \in \Omega, \\ -\Delta_p v = \lambda g_n(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{5.1}$$

by above argument, which approaches (1.1) as $n \to +\infty$. We will use Lemma 4.5 to treat (5.1) and thus we rewrite (5.1) in the form of system (4.2) as

$$\begin{cases} -\Delta_p u = \frac{f(n)}{\varphi_p(n)}\varphi_p(v^+) + \lambda \widetilde{f}_n(v), & x \in \Omega, \\ -\Delta_p v = \frac{g(n)}{\varphi_p(n)}\varphi_p(u^+) + \lambda \widetilde{g}_n(u), & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{5.2}$$

where

$$\widetilde{f}_n(y) \stackrel{\text{def}}{=} f_n(y) - \frac{f(n)}{\varphi_p(n)}\varphi_p(y^+),$$
$$\widetilde{g}_n(y) \stackrel{\text{def}}{=} g_n(y) - \frac{g(n)}{\varphi_p(n)}\varphi_p(y^+).$$

We note that $\widetilde{f}_n(y)$ and $\widetilde{g}_n(y)$ are bounded in $\mathbb{R}$. Indeed, since $f_n(y)$ is nondecreasing and $f_n(x) = f(x) > 0$ for $s \leqslant n$, we get

$$|\widetilde{f}_n(x)| \leqslant \sup_{x\in\mathbb{R}} \left| f_n(x) - \frac{f(n)}{\varphi_p(n)}\varphi_p(x^+) \right| \leqslant \max_{x\in[0,n]} \left| f_n(x) - \frac{f(n)}{\varphi_p(n)}\varphi_p(x^+) \right| + f(0) = +\infty,$$

where the constant only depends on n. And a same argument we can get $\widetilde{g}_n$ is bounded.

Since $f_n, g_n > 0$, it is easy to see that (5.2) satisfies the hypotheses of Lemma 4.5 by taking $\theta_1 = \frac{f(n)}{\varphi_p(n)}$, $\theta_2 = \frac{g(n)}{\varphi_p(n)}$ Moreover, by Lemma 4.5, there is a $\mu_{1,n}$, such that $\nu_{1,n}$ is the unique bifurcation point from infinity for (5.2) and there exists a continuum $\mathscr{C}_n$ of positive solutions of (5.2), which bifurcates from infinity at $\mu_{1,n}$ and satisfies the properties (i)–(iii) of Lemma 4.5. In particular, $(0,(0,0)) \in \mathscr{C}_n$, $\mathscr{C}_n$ is bounded above by the hyperplane $\lambda = \bar{\lambda}$.

Step 2. Passing to the limit

Now we verify $\{\mathscr{C}_n\}$ satisfying the conditions of Lemma 2.5. By the definition of continuum, $\mathscr{C}_n$ is closed.

Since all of $\mathscr{C}_n$ contain $(0,(0,0))$, we can choose $z_n \in \mathscr{C}_n$ such that $z_n = (0,(0,0))$ for each $n = 1, 2, \dots$. Naturally, $z_n \to z^* = (0,(0,0))$, the condition (i) of Lemma 2.5 is satisfied.

Obviously, because of the unboundedness of $\{\mathscr{C}_n\}$, then

$$d_n = \sup\{|\mu| + \|(u,v)\| \mid (\mu,(u,v)) \in \mathscr{C}_n\} = +\infty,$$

(ii) of Lemma 2.5 holds.

(iii) in Lemma 2.5 can be deduced directly from the Arzelà–Ascoli theorem and the definition of f_n, g_n.

Therefore, the superior limit of $\{\mathscr{C}_n\}$ contains a component $\mathscr{C} \subset \Pi$ joining $(0,(0,0))$ with infinity, and it follows from $u, v > 0$ for $\lambda > 0$ whenever $(\lambda,(u,v)) \in \mathscr{C}$, which establishes (a). Part (b) follows from $(0,(0,0)) \in \mathscr{C}$ and $f(0), g(0) > 0$. (c) in Theorem 1.2 can be deduced directly from the Theorem 1.1.

6 Proof of Theorem 1.3

Now we only show if the conditions of Theorem 1.3 are satisfied, then the unique point from infinity must be $\lambda = 0$.

In order to do this, we use a *rescaling* technology below, which is used by Ambrosetti et al. [3] to prove the scalar case as $p = 2$ and by Drábek et al. [6] to deal with the semipositone p-Laplacian system.

Let

$$F(s) := f(s) - C|s|^{q_1}, \qquad G(s) := g(s) - D|s|^{q_2},$$

where C, D was defined in (1.6). Let

$$\lambda = \gamma^{\sigma}, \qquad w_1 = \gamma^{\kappa_1} u, \qquad w_2 = \gamma^{\kappa_2} v,$$

where $\sigma, \kappa_1, \kappa_2$ are parameters and provided by [6, Proof of Theorem 4.3]. Then (1.1) can be translated to the form

$$\begin{cases} -\Delta_p w_1 = \tilde{F}(\gamma, w_2), & x \in \Omega, \\ -\Delta_p w_2 = \tilde{G}(\gamma, w_1), & x \in \Omega, \\ w_1 = 0 = w_2, & x \in \partial\Omega \end{cases} \tag{6.1}$$

with

$$\tilde{F}(\gamma, s_2) := \gamma^{\kappa_2 q_1} F(s_2/\gamma^{\kappa_2}) + C|s_2|^{q_1},$$
$$\tilde{G}(\gamma, s_1) := \gamma^{q_2} G(s_1/\gamma) + D|s_1|^{q_2}.$$

Then, by a directly result of [4, Theorem 1.1], there is a $M > 0$ such that for all positive solutions $(w_1, w_2) \in C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega})$ of (6.1) we have

$$\|w_1\|_{C^1} + \|w_2\|_{C^1} \leqslant M. \tag{6.2}$$

Now let $\lambda_n \in \mathbb{R}$ be a decreasing sequence with $\lambda_1 < \bar{\lambda}$ such that $\lambda_n \to 0^+$ as $n \to \infty$. Then for each n, we can get γ_n and the sequence $w_{1,n}, w_{2,n}$ of the solution for (6.1), such that

$$\|w_{1,n}\|_{C^1} + \|w_{2,n}\|_{C^1} \leqslant M_n. \tag{6.3}$$

Now let $n \to \infty$, then $\lambda_n \to 0^+$ and $\gamma_n = \lambda_n^{\sigma} \to 0^+$, then $(u_n, v_n) = (w_{1,n}/\gamma_n^{\kappa_1}, w_{2,n}/\gamma_n^{\kappa_2})$ gives the solution of (1.1) and $\|(u_n, v_n)\| \to \infty$ as $n \to \infty$.

Finally, we prove $\mathscr{C}$ must bifurcate from infinity at $\lambda \to 0^+$. Now let $(\lambda_n, (u_n, v_n)) \in \mathscr{C}$ with $\|(\mu_n, (u_n, v_n))\| \to +\infty$ as $n \to +\infty$ and $\lambda_n > 0$ for all $n \in \mathbb{N}$. Suppose to the contrary that $\lambda_n \to \lambda' > 0$ as $n \to +\infty$, then there exists a closed and bounded interval I such that $\lambda' \in I$. By above proof,

$$\|(u_n, v_n)\| \leqslant M < +\infty$$

for all λ', a contradiction to $\|(u_n, v_n)\| \to +\infty$ as $n \to +\infty$, which completes the proof of Theorem 1.3.

Declarations

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Global phase portraits of quintic reversible uniform isochronous centers

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Abstract. This paper studies the global phase portraits of uniform isochronous quintic centers at the origin with time reversibility such that their nonlinear part is not homogeneous. By using Poincaré compactification, we obtain all possible phase portraits of this quintic polynomial differential system.

Keywords: quintic polynomial system, uniform isochronous centers, global phase portraits, time reversibility.

2020 Mathematics Subject Classification: 35C15, 35Q51.

1 Introduction

In the qualitative theory of planar differential systems, one of the classical problems is to study the global phase portraits of polynomial differential systems. The global phase portraits of polynomial differential systems have been extensively investigated, see for example [7, 9, 11, 20–27].

The isochronous center's interest started the works of Huygens [15]. The isochronicity phenomena occurred in many physical problems [10]. In the past few decades the study of isochronicity, specially in the case of polynomial differential systems, has been driven by the diffusion of more powerful methods of computerized analysis [1, 8, 13, 16, 28].

We assume that p is a center, then p is a *uniform isochronous center* if the system, in polar coordinates $x = r\cos\theta, y = r\sin\theta$, is of the form $\dot{r} = G(\theta, r), \dot{\theta} = k, k \in \mathbb{R}\backslash\{0\}$. That is, the angular velocity of the orbits of an uniform isochronous center does not depend on the radius [13].

Proposition 1.1. *Assume that a planar differential polynomial system $\dot{x} = P(x,y), \dot{y} = Q(x,y)$ of degree n has a center at the origin of coordinates. Then, this center is uniform isochronous if and only if by doing a linear change of variables and a rescaling of time it can be written into the form*

$$\dot{x} = -y + xf(x,y), \qquad \dot{y} = x + yf(x,y). \tag{1.1}$$

Where $f(x,y)$ is a polynomial in x and y of degree $n-1$, and $f(0,0) = 0$.

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See for instance [18] for a proof of Proposition 1.1.

Recently, the global phase portraits of differential systems with uniform isochronous centers has attracted scholars' attention, for example [2, 8, 12, 17–19]. In 1999, Chavarriga et al. study the phase portraits of the quadratic polynomial differential system S_2 at P_{33} of [8]. Collins [12] found that differential systems with uniform isochronous cubic centers may have three global different portraits. In 2016, Itikawa and Llibre [19] study the phase portraits of uniform isochronous quartic centers. The first studies on some of these phase portraits are due to Algaba et al. [3]. The phase portraits of uniform isochronous quartic centers whose nonlinear part is homogeneous and not homogeneous were studied in [19] and [18], respectively. However, there are some mistakes in [18], which are corrected in [5]. Until now there are few results about the global phase portraits of differential system with uniform isochronous of degree 5 [2]. In this paper, we will study the global phase portraits of uniform isochronous quintic centers at the origin with time reversibility such that their nonlinear part is not homogeneous. We say that systems (1.1) reversible with respect to the y-axis if it is invariant under the transformation $(x, y, t) \mapsto (-x, y, -t)$. For this case, the differential system (1.1) of quintic reversible uniform isochronous centers can be written as

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = -y + x(a_1x + a_2xy + a_3x^3 + a_4xy^2 + a_5xy^3 + a_6x^3y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = x + y(a_1x + a_2xy + a_3x^3 + a_4xy^2 + a_5xy^3 + a_6x^3y), \end{cases} \tag{1.2}$$

where $a_i \in \mathbb{R}$, $i = 1, 2, 3, 4, 5, 6$, with $a_1^2 + a_2^2 + a_3^2 + a_4^2 \neq 0$ and $a_5^2 + a_6^2 \neq 0$.

If $a_3 \neq 0$, by a scaling $(x, y) \to \left(a_3^{-1/3}x, a_3^{-1/3}y\right)$, we can assume $a_3 = 1$, then system (1.2) becomes

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = -y + x(a_1x + a_2xy + x^3 + a_4xy^2 + a_5xy^3 + a_6x^3y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = x + y(a_1x + a_2xy + x^3 + a_4xy^2 + a_5xy^3 + a_6x^3y). \end{cases} \tag{1.3}$$

And if $a_3 = 0$, then system (1.2) becomes

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = -y + x(a_1x + a_2xy + a_4xy^2 + a_5xy^3 + a_6x^3y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} = x + y(a_1x + a_2xy + a_4xy^2 + a_5xy^3 + a_6x^3y). \end{cases} \tag{1.4}$$

In what follows we state our main results.

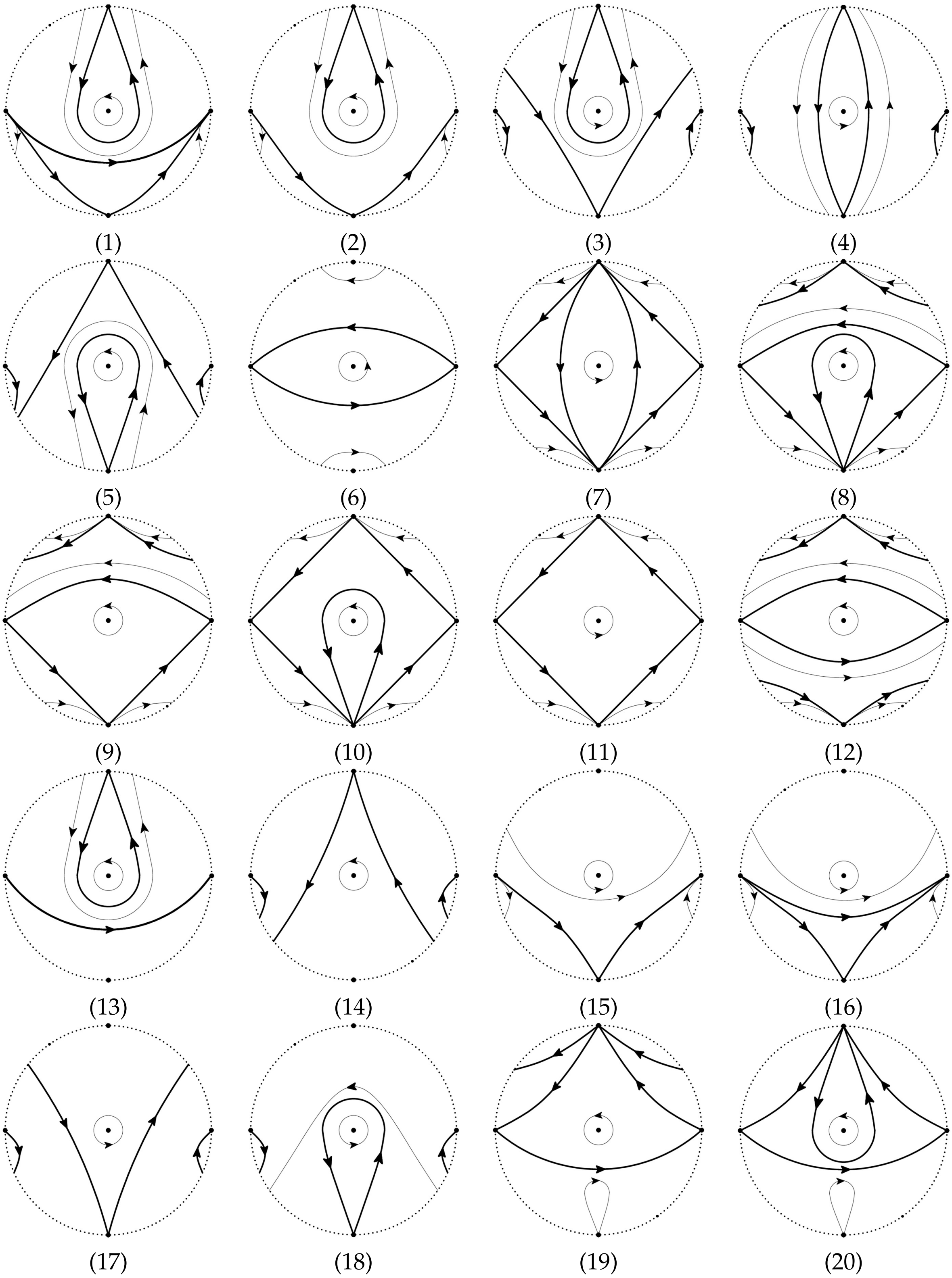
Theorem 1.2. *The phase portrait in the Poincaré disk of uniform isochronous quintic centers with time reversibility is topologically equivalent to one of the following 67 possibilities global phase portraits of Figure 1.1.*

The rest of this paper is organized as follows. In Section 2, we characterize the global phase portraits of system (1.2) in the Poincaré disc, that is we prove Theorem 1.2.

2 Proof of the results

In this section, we will prove Theorem 1.2. In order to obtain all possible phase portraits in the Poincaré disc for the uniform isochronous system of degree 5, we shall study the finite and infinite singular points of system (1.2).

By discussing the coefficient a_3 of system (1.2), we divide into the following two cases: if $a_3 \neq 0$, by parameter and time scale transformation, the system (1.2) can be given by system (1.3); if $a_3 = 0$, the system (1.2) can be given by system (1.4). Next, we will study the phase portraits of system (1.3) and system (1.4).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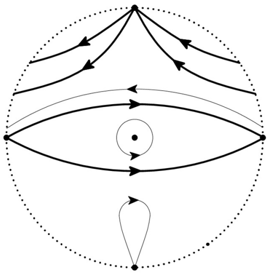
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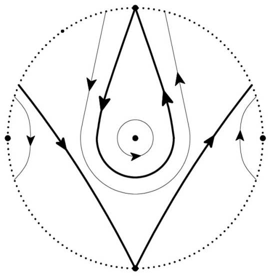
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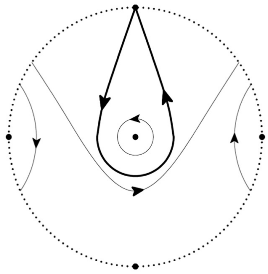
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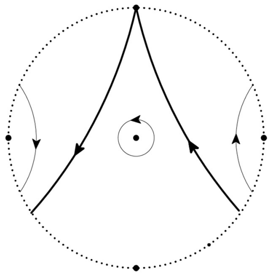
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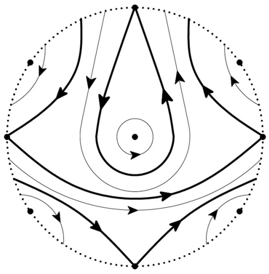
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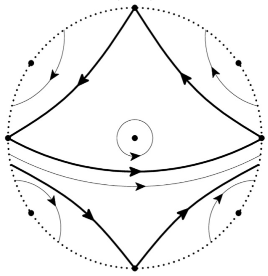
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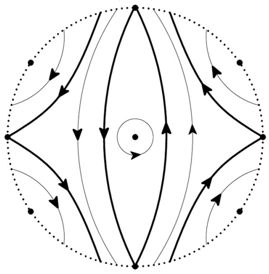
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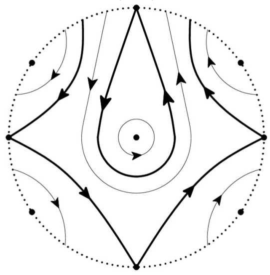
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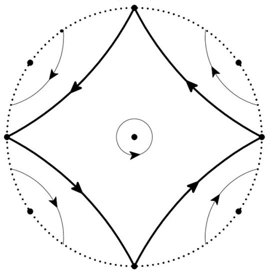
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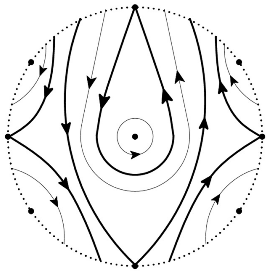
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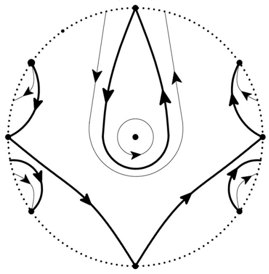
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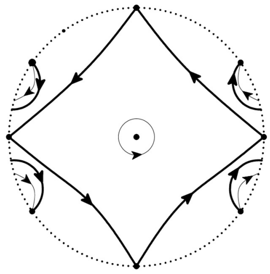
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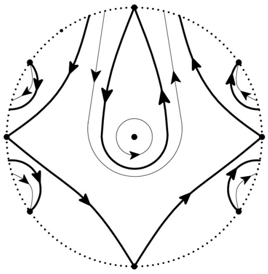
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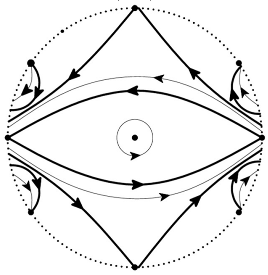
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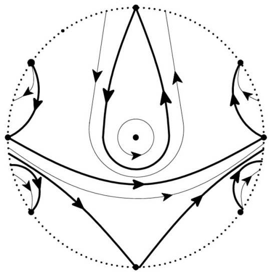
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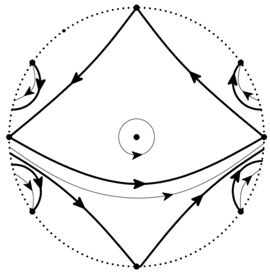
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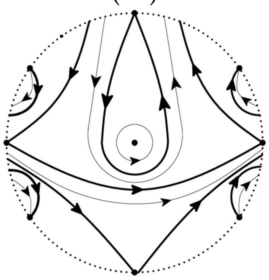
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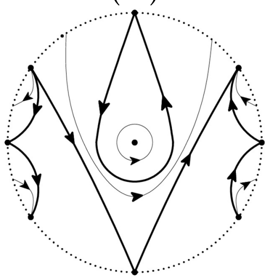
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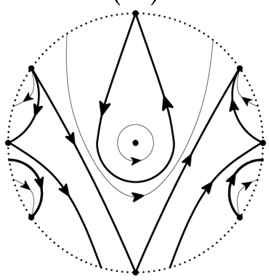
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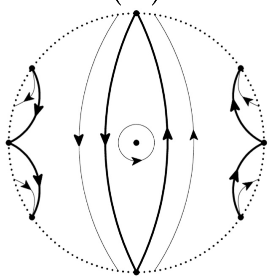
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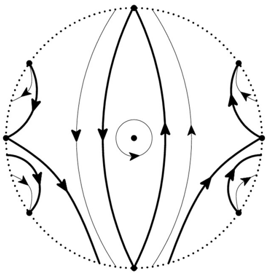
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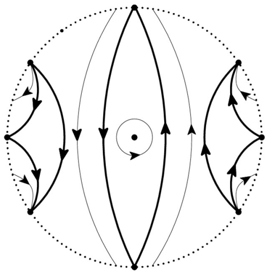
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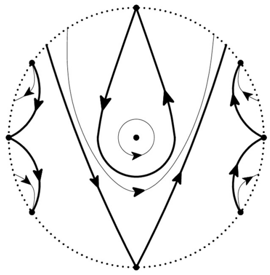
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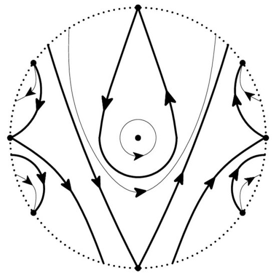
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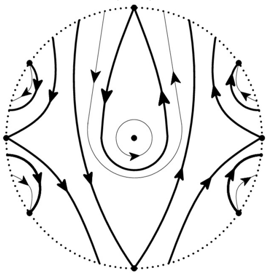
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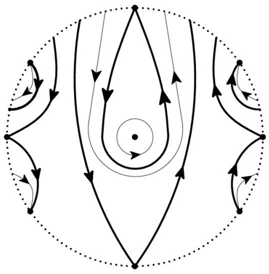
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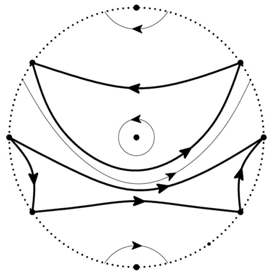
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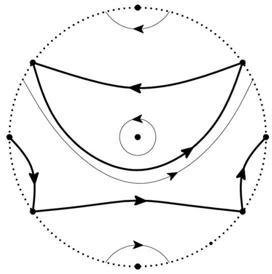
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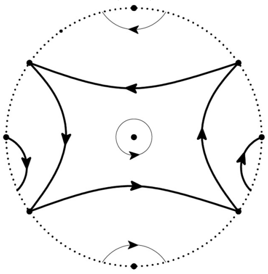
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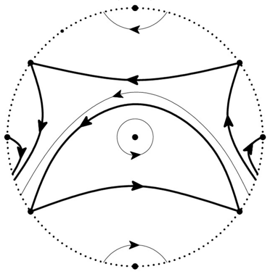
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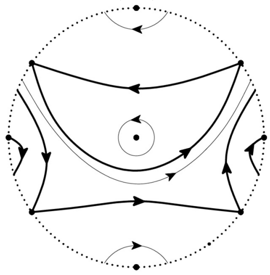
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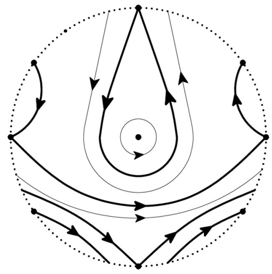
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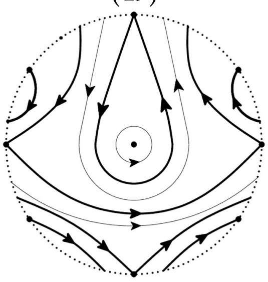
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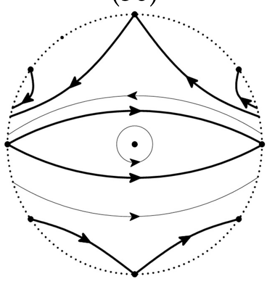
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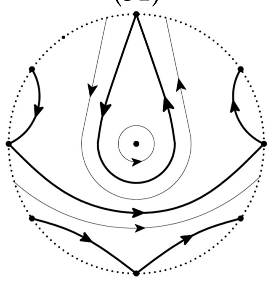
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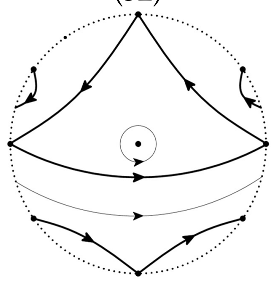
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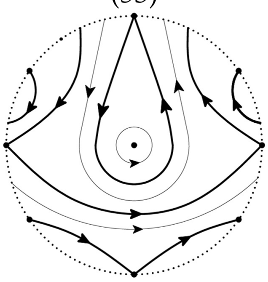
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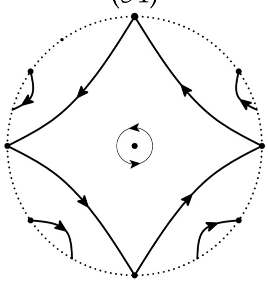
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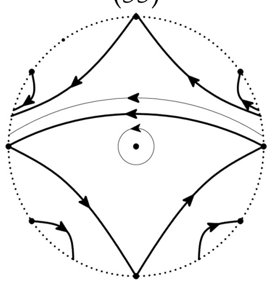
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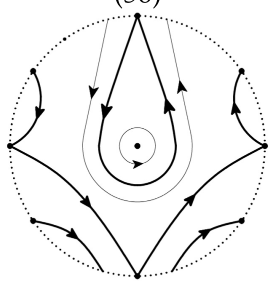
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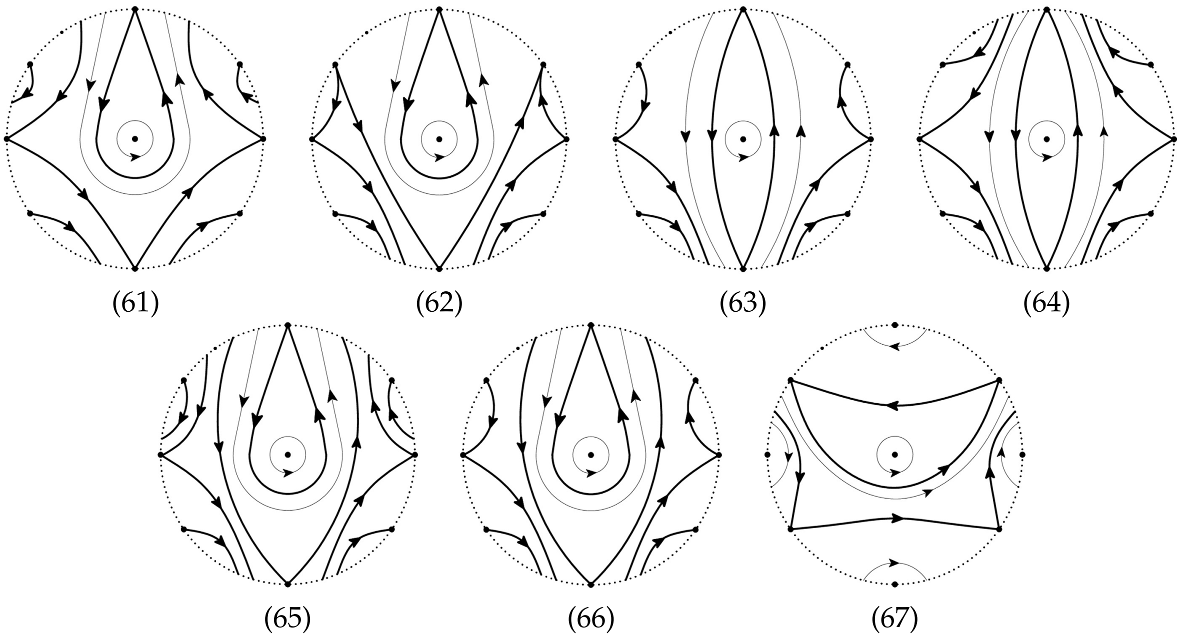


Figure 1.1: Global phase portraits of system (1.2)

Table 1.1: The corresponding relationship between the global phase portraits of system (1.3) and Figure 1.1

Figure 1.1	System (1.3)
(1)–(5)	$a_6 = 0, \ \ a_5 > 0;$
	$a_5 = 0, \ \ a_6 > 0, \ \ a_4 = 0;$
	$a_5 a_6 > 0, \ \ a_5 > 0.$
(6)	$a_6 = 0, \ \ a_5 < 0;$
	$a_5 = 0, \ \ a_6 < 0, \ \ a_4 = 0, \ \ a_6 < -{a_2}^2/4;$
	$a_5 a_6 > 0, \ \ a_6 < 0.$
(7)–(12)	$a_5 = 0, \ \ -{a_2}^2/4 \le a_6 < 0, \ \ a_4 = 0.$
(13)–(14)	$a_5 = 0, \ \ a_6 > 0, \ \ a_4 > 0.$
(15)–(18)	$a_5 = 0, \ \ a_6 > 0, \ \ a_4 < 0.$
(19)–(21)	$a_5 = 0, \ \ a_6 < 0, \ \ a_4 \neq 0.$
(25)–(30)	$a_4 a_6 - a_5 = 0, \ \ a_5 a_6 < 0, \ \ a_6 < 0, \ \ 4(a_5 - a_6) \ge {a_2}^2.$
(31)–(46)	$a_4 a_6 - a_5 = 0, \ \ a_5 a_6 < 0, \ \ a_6 < 0, \ \ a_2 \neq 0, \ \ 4(a_5 - a_6) < {a_2}^2.$
(47)–(51)	$a_4 a_6 - a_5 = 0, \ \ a_5 a_6 < 0, \ \ a_6 > 0;$
	$a_4 a_6 - a_5 \neq 0, \ \ a_5 a_6 < 0, \ \ a_6 > 0.$
(52)–(66)	$a_4 a_6 - a_5 \neq 0, \ \ a_5 a_6 < 0, \ \ a_6 < 0.$

In polar coordinates defined by $(x, y) = (r\cos\theta, r\sin\theta)$, a planar differential system (1.2) with an uniform isochronous center at the origin always can be written as $\dot{r} = p(r, \theta), \dot{\theta} = 1$. Hence such systems have no finite points except the origin.

By using Poincaré compactification in [14], in the local chart U_1, we obtain

$$\begin{cases} \dot{u} = (1+u^2)v^4, \\ \dot{v} = (uv^4 - a_1 v^3 - a_2 u v^3 - a_3 v - a_4 u^2 v - a_5 u^3 - a_6 u)v, \end{cases} \tag{2.1}$$

Table 1.2: The corresponding relationship between the global phase portraits of system (1.4) and Figure 1.1

Figure 1.1	System (1.4)
(1)–(5)	$a_5 = 0,\ \ a_6 > 0.$
(6)	$a_5 = 0,\ \ a_6 < -{a_2}^2/4;$
	$a_5a_6 > 0.$
(7)–(12)	$a_5 = 0,\ \ -{a_2}^2/4 \le a_6 < 0,\ \ a_4 = 0.$
(19)–(21)	$a_5 = 0,\ \ a_6 < 0,\ \ a_4 \neq 0.$
(22)	$a_5a_6 > 0,\ \ a_6 > 0.$
(23)–(24)	$a_5 = 0,\ \ a_6 > 0,\ \ a_4 \neq 0.$
(25)–(30)	$a_4 = 0,\ \ a_5a_6 < 0,\ \ a_6 < 0,\ \ 4(a_5 - a_6) \ge {a_2}^2.$
(31)–(46)	$a_4 = 0,\ \ a_5a_6 < 0,\ \ a_6 < 0,\ \ a_2 \neq 0,\ \ 4(a_5 - a_6) < {a_2}^2.$
(52)–(66)	$a_5a_6 < 0,\ \ a_6 < 0,\ \ a_4 \neq 0.$
(29), (67)	$a_5a_6 < 0,\ \ a_6 > 0,\ \ a_4 \neq 0;$
	$a_5a_6 < 0,\ \ a_6 > 0,\ \ a_4 = 0.$

and therefore all the points $(u, 0)$ for all $u \in \mathbb{R}$ are infinite singular points of the system (2.1) in U_1. In order to obtain the local phase portraits near the infinity, we make a transformation $\mathrm{d}s = v\mathrm{d}t$ and obtain the following system

$$\begin{cases} u' = (1+u^2)v^3, \\ v' = uv^4 - a_1v^3 - a_2uv^3 - a_3v - a_4u^2v - a_5u^3 - a_6u. \end{cases} \tag{2.2}$$

Where the prime denotes derivative with respect to s and the system (2.2) has infinite singular points in the u-axis.

In the local chart U_2, we obtain

$$\begin{cases} \dot{u} = -(1+u^2)v^4, \\ \dot{v} = (-uv^4 - a_1uv^3 - a_2uv^2 - a_3u^3v - a_4u^2v - a_5u - a_6u^3)v. \end{cases} \tag{2.3}$$

After the rescaling of time $\mathrm{d}s = v\mathrm{d}t$, we obtain

$$\begin{cases} u' = -(1+u^2)v^4, \\ v' = -uv^4 - a_1uv^3 - a_2uv^2 - a_3u^3v - a_4u^2v - a_5u - a_6u^3. \end{cases} \tag{2.4}$$

It is obvious that the system (2.4) has the only singular point $O_{u_2}(0,0)$.

2.1 Global phase portraits of system (1.3)

In this section, we discuss the global phase portraits of system (1.3). According to the number of the singular points in the u-axis, the system can be divided into the following five cases.

Case I: $a_6 = 0$, $a_5 \neq 0$.

In the chart U_1, the system (2.2) has one singular point in the u-axis, that is $O_{U_1}\ (0,0)$, the corresponding linear part of system (2.2) is

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

By the Theorem 2.19 of [14], we have the statements: if $a_5 > 0$, then O_{U_1} $(0,0)$ is a stable node; if $a_5 < 0$, then O_{U_1} $(0,0)$ is a saddle.

For chart U_2, the system (2.4) has only one singular point O_{U_2} $(0,0)$, the corresponding linear part of system (2.4) is

$$\begin{pmatrix} 0 & 0 \\ -a_5 & 0 \end{pmatrix}.$$

It is easy to find that O_{U_2} $(0,0)$ is a nilpotent singularity. By the Theorem 3.5 of [14], we have the statements: if $a_5 > 0$, then O_{U_2} $(0,0)$ is a nilpotent saddle; if $a_5 < 0$, and the system (2.4) is symmetry about v-axis, then O_{U_2} $(0,0)$ is a center. If the same situation appears again, we will not explain in detail.

By the above analysis, if $a_5 > 0$, the local phase portrait of system (1.3) is shown in Figure 2.1. Since $v' \mid_{v=0} = -a_5 u^3$: when $u > 0$, $v' < 0$; when $u < 0$, $v' > 0$, the direction of the local phase portrait through the disc is shown in the Figure 2.1.

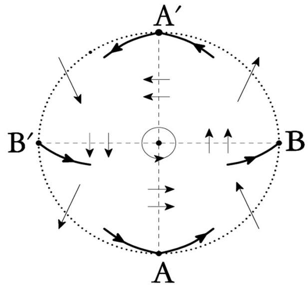


Figure 2.1: Local phase portrait of system (1.3) on the Poincaré disk of Case I for $a_5 > 0$.

In Figure 2.1, there are four singularities on the equator, i.e. A, B, A', B', and the direction between any two points is shown. Firstly, we consider the point A_1. Since the system (1.3) is symmetry about y axis, there are only four possibilities for the ω-limit set of unstable manifold: a singularity on the arc $\overset{\frown}{BA'}$, a point A, a point B, and itself (return to point A_1 after bypassing the periodic orbit around the origin, forming a homoclinic orbit). They are shown in Figure 2.2 (1)–(4).

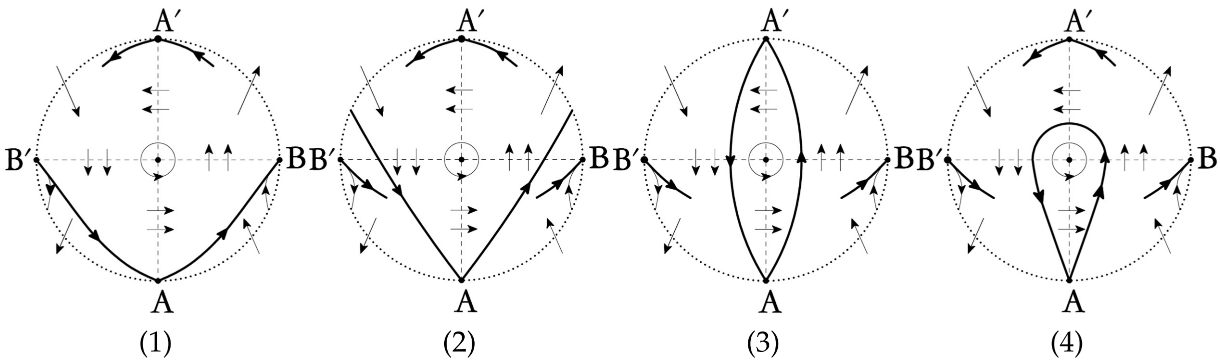


Figure 2.2: Consider the point A in Figure 2.1, four possibilities for the ω limit set of an unstable manifold.

The ω-limit set of the unstable manifold at point A' in Figure 2.2 (1) can only be itself (bypass the origin and returns to A' again, forming a homoclinic orbit). Therefore, there are two possibility global phase portraits on the Poincaré disk, as shown in Figure 1.1 (1) and (2).

The ω-limit set of the unstable manifold at point A' in Figure 2.2 (2) can only be itself (bypass the origin and returns to A' again, forming a homoclinic orbit). Next, we consider a singular point on the arc $\overset{\frown}{AB'}$, and its α-limit set can only be a point B'. Based on the symmetry of the original system, the ω-limit set of a singular point on the arc $\overset{\frown}{AB}$ can only be a point B, and its phase portrait is equivalent to the Figure 1.1 (3).

The α-limit set of a singular point on the arc $\overset{\frown}{AB'}$ in Figure 2.2 (3) can only be the point B'. Based on the symmetry of the original system, the ω-limit set of a singular point on the arc $\overset{\frown}{AB}$ can only be the point B, its phase portrait is equivalent to the Figure 1.1 (4).

The α-limit set of the stable manifold at point A' in Figure 2.2 (4) can only be a singular point on the arc $\overset{\frown}{AB}$. Based on the symmetry of the original system, the ω-limit set of a singular point on the arc $\overset{\frown}{AB'}$ can only be the point B, and its phase portrait is equivalent to the Figure 1.1 (5).

If $a_5 < 0$, the local phase portrait of the system (1.3) on the Poincaré disk is shown in Figure 2.3, which has four singularities on its equator.

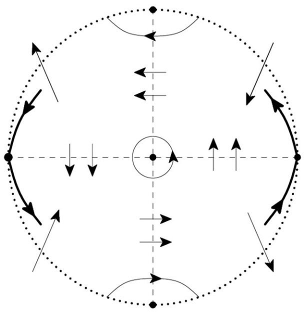


Figure 2.3: The local phase portraits of system (1.3) on the Poincaré disk of Case I for $a_5 < 0$.

By the same argument as Figure 2.1, Figure 2.3 has only one global phase portrait, as shown in the Figure 1.1 (6). This is the only one global phase portrait that can be determined. Therefore, there are 6 possible global phase portraits of system (1.3) in Case I, as shown in Figure 1.1 (1)–(6).

Case II: $a_5 = 0$**,** $a_6 \neq 0$**.**

In chart U_1, O_{U_1} $(0,0)$ is a singular point of system (2.2), and its linear part is

$$\begin{pmatrix} 0 & 0 \\ -a_6 & -1 \end{pmatrix}.$$

Applying the Theorem 2.19 of [14], we have the statements: if $a_6 > 0$, then O_{U_1} $(0,0)$ is a stable node; if $a_6 < 0$, then O_{U_1} $(0,0)$ is an unstable node.

For the chart U_2, the system (2.4) can be written as

$$\begin{cases} u' = -(1+u^2)v^3, \\ v' = -uv^4 - a_1uv^3 - a_2uv^2 - u^3v - a_4uv - a_6u^3. \end{cases} \tag{2.5}$$

The origin O_{U_2} $(0,0)$ of system (2.5) is a singular point and its linear part is identically zero, so the O_{U_2} $(0,0)$ is degenerate.

To investigate the local phase portraits of the degenerate singular points, we use the quasi-homogeneous directional blow up(or (α,β)-blow up) technique [4,6] . Since the choice of the exponents α and β depends on the coefficients a_4 of system (2.5), we need to consider whether a_4 is zero. Thus we apply a (3,2)-blow up and (1,1)-blow up to system (2.5) if $a_4 \neq 0$ and $a_4 = 0$, respectively.

Case II.1: $a_4 = 0$**.** When $a_4 = 0$, applying a $(1,1)$-blow up to system (2.5). Firstly, we apply blow-up $(u,v) \mapsto (\bar{u},\bar{u}\bar{v})$ in the positive u-direction. After division by $\bar{u}^2$, we get,

$$\begin{cases} \bar{u}' = -(1+\bar{u}^2)\bar{u}\bar{v}^3, \\ \bar{v}' = -(-\bar{v}^4 + a_1\bar{u}\bar{v}^3 + a_2\bar{v}^2 + \bar{u}\bar{v} + a_6). \end{cases} \tag{2.6}$$

Since in the line $\bar{u} = 0$, we have

$$\bar{v}^4 - a_2\bar{v}^2 - a_6 = 0.$$

Next, we only need to discuss the existence of the roots of the above equation.

(a) If $\sqrt{{a_2}^2 + 4a_6} > a_2$, i.e. $a_6 > 0$, the equilibrium of (2.6) are $P_1(0, \sqrt{(a_2 + \sqrt{{a_2}^2+4a_6})/2})$ and $P_2(0, -\sqrt{(a_2 + \sqrt{{a_2}^2+4a_6})/2})$. The corresponding linear part of system (3.11) at P_1 is

$$\begin{pmatrix} -\left(\sqrt{\left(a_2 + \sqrt{{a_2}^2+4a_6}\right)/2}\right)^3 & 0 \\ * & 2\sqrt{\left(a_2 + \sqrt{{a_2}^2+4a_6}\right)/2}\sqrt{{a_2}^2+4a_6} \end{pmatrix},$$

where "$*$" stands for the formula about parameters a_2 and a_6. Applying the Theorem 2.15 of [14], we have P_1 is a saddle. By the same argument, P_2 is a saddle.

(b) If $\sqrt{{a_2}^2 + 4a_6} < a_2$, i.e. $-{a_2}^2/4 < a_6 < 0, a_2 > 0$, the singular points of system (2.6) in the line $\bar{u} = 0$ are $P_1(0, \sqrt{(a_2 + \sqrt{{a_2}^2+4a_6})/2})$, $P_2(0, -\sqrt{(a_2 + \sqrt{{a_2}^2+4a_6})/2})$, $P_3(0, \sqrt{(a_2 - \sqrt{{a_2}^2+4a_6})/2})$ and $P_4(0,\ \sqrt{(a_2 - \sqrt{{a_2}^2+4a_6})/2})$. By the Theorem 2.15 of [14], P_1 is a saddle, P_2 is a saddle, P_3 is a stable node, and P_4 is an unstable node.

(c) If ${a_2}^2 + 4a_6 < 0$, i.e. $-{a_2}^2/4 > a_6$, then $\bar{v}^4 - a_2\bar{v}^2 - a_6 = 0$ has no roots, that is, the system (2.6) has no singular point in $\bar{v}$-axis.

(d) If ${a_2}^2 + 4a_6 = 0$, i.e. $-{a_2}^2/4 = a_6$, the singular points of system (2.6) in the line $\bar{u} = 0$ are $P_1\left(0, \sqrt{a_2/2}\right)$, $P_2\left(0, \sqrt{a_2/2}\right)$. By Theorem 3.5 of [14], P_1 is a saddle-node, and P_2 is a saddle-node.

Consider the blow-up $(u,v) \mapsto (-\bar{u},\bar{u}\bar{v})$ in the negative u-direction. After cancelling a common factor $\bar{u}^2$, we obtain

$$\begin{cases} \bar{u}' = (1+\bar{u}^2)\bar{u}\bar{v}^3, \\ \bar{v}' = -\bar{v}^4 + a_1\bar{u}\bar{v}^3 + a_2\bar{v}^2 + \bar{u}\bar{v} + a_6. \end{cases} \tag{2.7}$$

It can be verified that system (2.7) and system (2.6) have the same number and type of singularities.

In addition, we apply blow-up $(u,v) \mapsto (\bar{u}\bar{v}, \bar{v})$ in the positive v-direction as well as $(u,v) \mapsto (\bar{u}\bar{v}, -\bar{v})$ in the negative v-direction. After division by $\bar{v}^2$, we get, respectively,

$$\begin{cases} \bar{u}' = -1 + \bar{u}^2(a_1\bar{v} + a_2 + a_3\bar{u}^2\bar{v} + a_6\bar{u}^2), \\ \bar{v}' = -\bar{u}\bar{v}(\bar{v}^2 + a_1\bar{v} + a_2 + a_3\bar{u}^2\bar{v} + a_6\bar{u}^2), \end{cases} \tag{2.8}$$

and

$$\begin{cases} \bar{u}' = 1 + \bar{u}^2(-a_1\bar{v} + a_2 - a_3\bar{u}^2\bar{v} + a_6\bar{u}^2), \\ \bar{v}' = \bar{u}\bar{v}(\bar{v}^2 - a_1\bar{v} + a_2 - a_3\bar{u}^2\bar{v} + a_6\bar{u}^2). \end{cases} \tag{2.9}$$

It is obvious that the origin of system (2.8) and system (2.9) are not a singular point.

The blow up procedure and local phase portrait of the system (2.5) at the origin are shown in Figure 2.4 in Case (a). The trajectories of the circle is shown in Figure 2.4 (1). Retracting the circle to the origin and obtaining the trajectories in the uov plane near the origin, see Figure 2.4 (2). Considering the transformation d$s = v$dt, the u axis is filled with singular points, and the negative v-axis direction is reversed, as shown in Figure 2.4 (3).

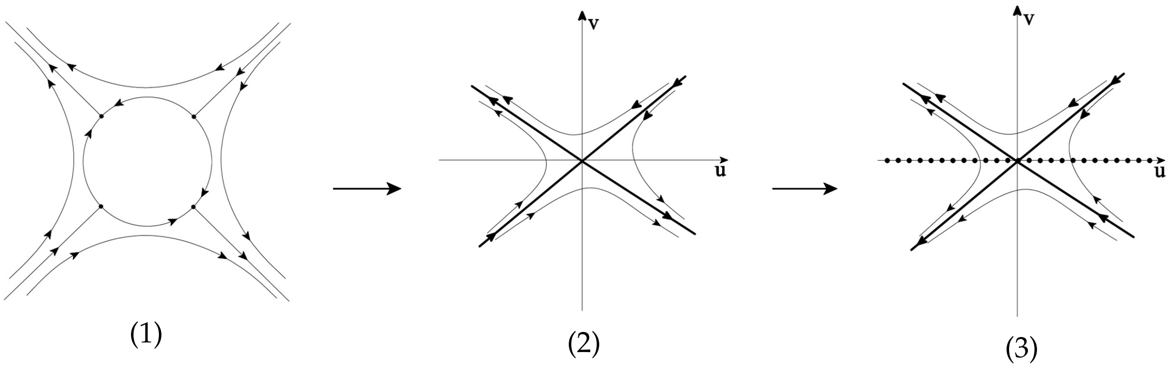
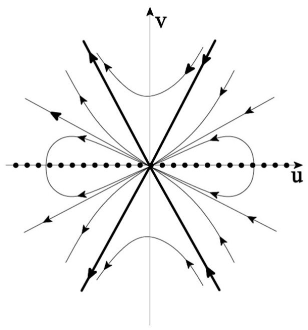
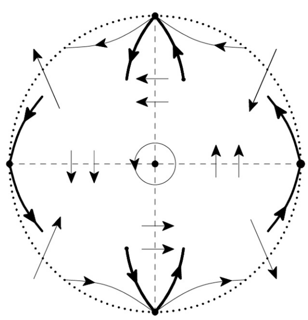


Figure 2.4: The local phase portrait of system (2.3) at the origin. The horizontal axis (3) is filled with singular points.

By the same argument as Case (a), we can obtain the local phase portrait of the system (2.5) of Case (b) and Case (d) at the origin and is shown in Figure 2.5(a). Then the local phase portrait of (1.3) of Case (a), Case (b) (d) and Case (c) are shown in Figure 2.1, Figure 2.5(b) and Figure 2.3, respectively.



(a) The local phase portrait of the system (2.5) of Case (b) at the origin.

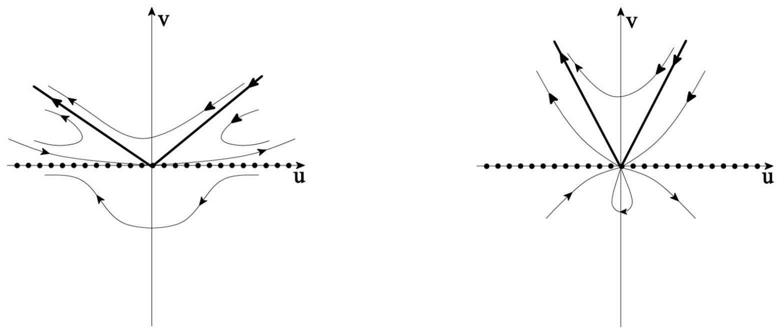


(b) Local phase portrait of system (1.3) on the Poincaré disk of Case (b).

Figure 2.5: Local phase portraits of systems (2.5) and (1.3).

For Case II.1, according to the symmetry and the direction through the Poincaré disk of system (1.3), we can obtain that, for Case (a), Figure 2.1 corresponds to the possible global phase portraits are Figure 1.1 (1)–(5); for Case (b) and Case (d), Figure 2.5(b) corresponds to the possible global phase portraits are Figure 1.1 (7)–(12); for Case (c), Figure 2.3 corresponds to the possible global phase portraits is Figure 1.1 (6). Therefore, there are 12 possible global phase portraits of system (1.2) of Case II.1, see Figure 1.1 (1)–(12).

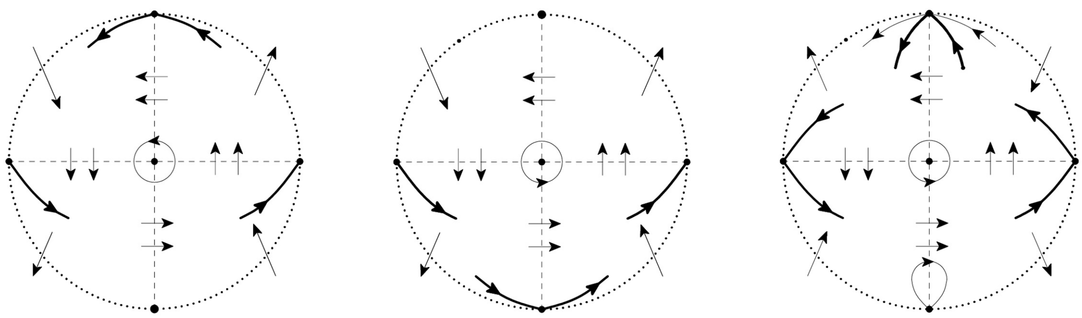
Case II.2: $a_4 \neq 0$. Since $a_4 \neq 0$, applying a $(3,2)$-blow up to system (2.5). By the same argument as Case II.1, we get the local phase portrait of system (2.5) at the origin, see Figure 2.6. Then the local phase portraits of $a_6 > 0, a_4 < 0$ and $a_6 < 0, a_4 < 0$ of the system (2.5) are topologically equivalent to (1) and (2) in Figure 2.6, respectively.



(1) $a_6 > 0, a_4 > 0$ (2) $a_6 < 0, a_4 > 0$

Figure 2.6: The local phase portrait of system (2.5) at the origin of Subcase II.2.

From the above analysis, we characterize the local phase portrait of system (1.3) on the Poincaré disk, see Figure 2.7 (1)–(3), and they are topologically equivalent to the possible global phase portraits Figure 1.1 (13)–(14), (15)–(18), (19)–(21) of Theorem 1.2, respectively. Therefore we can obtain the global phase portraits for Subcase II.2 shown in Figure 1.1 (13)–(21) of Theorem 1.2.



(1) $a_6 > 0$, $a_4 > 0$. (2) $a_6 > 0$, $a_4 < 0$. (3) $a_6 < 0$.

Figure 2.7: Local phase portrait of system (1.3) on the Poincaré disk of Subcase II.2.

Case III: $a_5 a_6 > 0$.

In the chart U_1, we consider the system (2.2). Since $a_5 a_6 > 0$, the system (2.2) has only one singular point $O_{U_1}(0,0)$, and its linear part of system (2.2) is

$$\begin{pmatrix} 0 & 0 \\ -a_6 & -1 \end{pmatrix}.$$

Its type of singular point is the same as the Case II.

In the chart U_2, the origin is the only one singular point of system (2.4). Its linear part is

$$\begin{pmatrix} 0 & 0 \\ -a_5 & 0 \end{pmatrix}.$$

Applying the Theorem 3.5 of [14], we have if $a_5 > 0$, then the origin $O_{U_2}(0,0)$ is a saddle; and if $a_5 < 0$, then $O_{U_2}(0,0)$ is a center.

According to the above analysis, we characterize the local phase portrait of system (1.3) on the Poincaré disk, see Figure 2.1 and Figure 2.3. Consequently the possible global phase portraits can referenced by Case I, that is, they are shown in Figure 1.1(1)–(6) of Theorem 1.2.

Case IV: $a_4a_6 - a_5 = 0$, $a_5a_6 < 0$.

In the chart U_1, we consider the system (2.2). It is easy to find that there are three singular points, $O_{U_1}(0,0)$, $P_1(\sqrt{-a_6/a_5}, 0)$ and $P_2(\sqrt{-a_6/a_5}, 0)$.

For the point $O_{U_1}(0,0)$, we have the same results as the Case II. For the points P_1 and P_2, their linear part is

$$\begin{pmatrix} 0 & 0 \\ 2a_6 & 0 \end{pmatrix}.$$

By the Theorem 3.5 of [14], if $a_6 > 0$, then P_1 and P_2 are saddles. If $a_6 < 0$,

- **(a)** For $a_2 \neq 0$, we have the following statements hold.
  - **(a.1)** If $4(a_5 - a_6) \geq a_2^2$, then P_1 and P_2 are centers;
  - **(a.2)** If $4(a_5 - a_6) < a_2^2$, then the phase portrait of P_1 and P_2 of the system (2.2) consists of one hyperbolic and one elliptic sector.
- **(b)** If $a_1 \neq 0$, $a_2 = 0$, and $9a_1^2a_5 < 16a_6(a_6 - a_5)$, then P_1 and P_2 are centers.
- **(c)** If $a_1 = 0$, $a_2 = 0$, and $4(a_5 - a_6) > 9$, then P_1 and P_2 are centers.

In the chart U_2, the origin $O_{U_2}(0,0)$ is the only one singular point of system (2.4). By the same argument as the Case III, we have if $a_5 > 0$, then the origin $O_{U_2}(0,0)$ is a saddle, and if $a_5 < 0$, then the origin $O_{U_2}(0,0)$ is a center.

Except Case (a.2), the local phase portraits of system (1.3) on the Poincaré disk is shown in Figure 2.8.

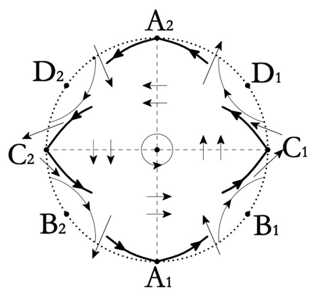


Figure 2.8: The local phase portrait of system (1.3) on the Poincaré disk except Case (a.2).

In Figure 2.8, there are eight singularities on the equator, i.e. $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2$, and the direction between any two points is shown. Firstly, we consider the point A_1. Since the system (1.3) is symmetry about y axis, there are only five possibilities for the ω-limit set of unstable manifold: a singularity on the arc $\overset{\frown}{B_1C_1}$, a point C_1, a singular on the arc $\overset{\frown}{D_1A_2}$, a point A_2, and itself (bypass the origin and return to A_1 again, forming a homoclinic orbit). As shown in Figure 2.9 (1)–(5).

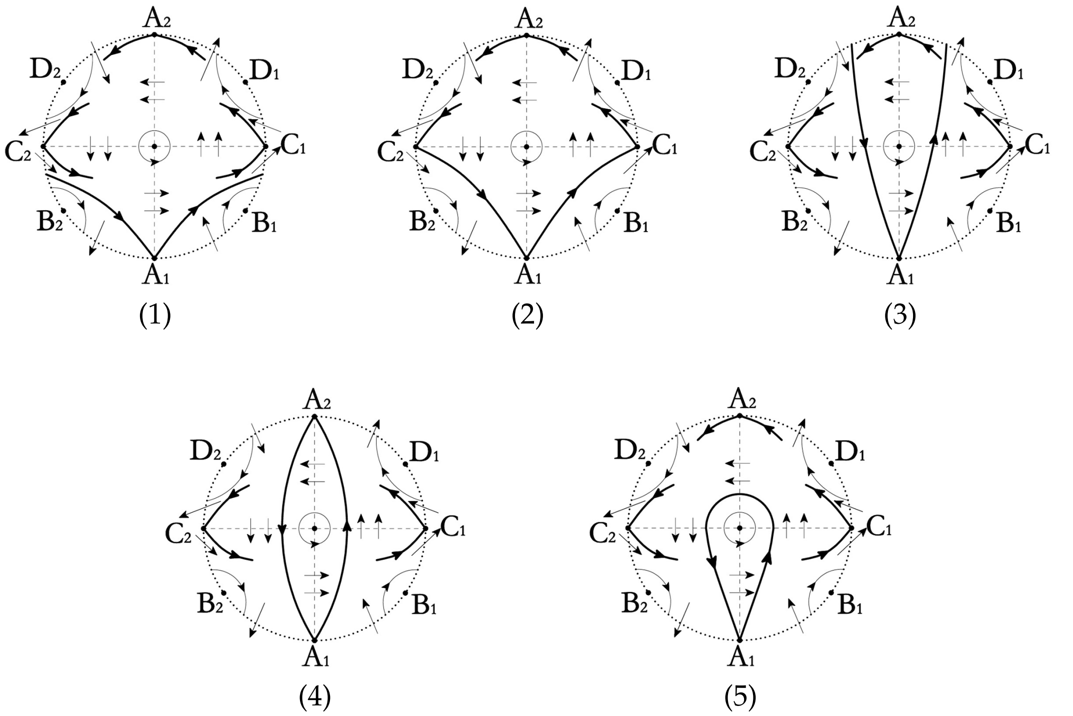


Figure 2.9: Consider the point A_1 of Figure 2.8, five possibilities for the ω-limit set of an unstable manifold at this point.

Figure 2.9 (1): Consider the point C_2, the ω-limit set of the unstable manifold at this point can only be the point C_1. Next, we consider the unstable manifold at point C_1. According to the symmetry of the original system, there are three possibilities for the ω-limit set of an unstable manifold at the point C_1, which are a point on $\overset{\frown}{D_1A_2}$, A_2 and C_2.

When the ω-limit set of an unstable manifold at point C_1 is a singular point on arc $\overset{\frown}{D_1A_2}$, ω-limit set of the unstable manifold at this point can only be a itself (bypass the origin and return to A_2 again, forming a homoclinic orbit), C_2 and a singular point on arc $\overset{\frown}{D_2C_2}$, respectively. According to the symmetry of the original system, the global phase portraits are equivalent to Figure 1.1 (25), (26) and (27), respectively.

Figure 2.9 (2): Consider the unstable manifold at point C_1, there are three possibilities for the ω-limit set on point C_1, which are a singular on the arc $\overset{\frown}{D_1A_2}$, A_2 and C_2. When the ω-limit set of an unstable manifold at C_1 is a singular point on arc $\overset{\frown}{D_1A_2}$, ω-limit set of the unstable manifold at this point can only be a itself (bypass the origin and return to A_2 again, forming a homoclinic orbits), C_2 and a singular point on arc $\overset{\frown}{D_2C_2}$, respectively. According to the symmetry of the original system, the global phase portraits are equivalent to the Figure 1.1 (29), (28) and (26), respectively.

Figure 2.9 (3): Consider the point A_2, the ω-limit set of the unstable manifold at this point can only be a itself (bypass the origin and return to A_2 again, forming a homoclinic orbit). The α-limit set of the stable manifold at point C_1 can only be a singular on the arc $\widehat{A_1B_1}$. The ω-limit set of the unstable manifold at point C_1 can only be a singular on the arc $\widehat{D_1A_2}$. According to the symmetry of the original system, the global phase portrait is equivalent to the Figure 1.1 (30).

Figure 2.9 (4): Consider the point C_1, the ω-limit set of the unstable manifold at this point can only be a singular on the arc $\widehat{D_1A_2}$. The α-limit set of the stable manifold at point C_1 can only be a singular on the arc $\widehat{A_1B_1}$. According to the symmetry of the original system, the global phase portrait is equivalent to the Figure 1.1 (27).

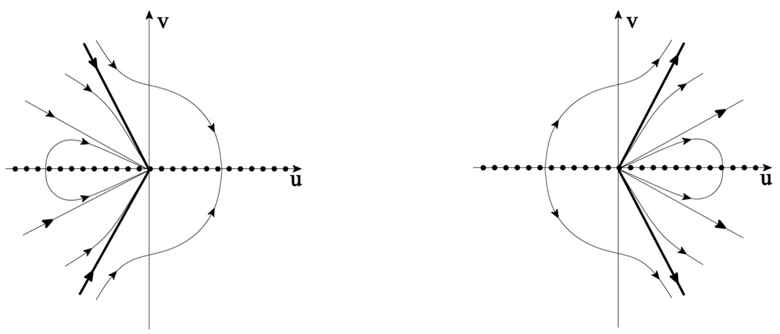
Figure 2.9 (5): Consider the point C_1, α-limit set of the unstable manifold at this point can only be a singular on the arc $\widehat{A_1B_1}$. Based on the the symmetry, ω-limit set of the unstable manifold at the point C_2 can only be a a singular on the arc $\widehat{A_1B_2}$. Fixed the unstable manifold of C_1, there are three possibilities for the ω-limit set on the C_1 , which are a singular on the arc $\widehat{D_1A_2}$, A_2 and C_2. Then the global phase portraits are equivalent to the Figure 1.1 (30) (29) (27).

Therefore, the local phase portraits Figure 2.8 have 6 possible global phase portraits as shown in Figure 1.1 (25)–(30).

For case (a.2): $a_2 \neq 0$, $4(a_5 - a_6) < a_2^2$, the local phase portrait of P_1 and P_2 of the system (2.2) consists of one hyperbolic and one elliptic sector by using blowing up. Moving P_1 to the origin through the change of coordinates $(u, v) \mapsto \left(u + \sqrt{-a_6/a_5}, v\right)$, the system (2.2) becomes

$$\begin{cases} u' = [1 + (u + 2^2)]v^3, \\ v' = uv^4 + \sqrt{-\dfrac{a_6}{a_5}}v^4 - a_1v^3 - a_2uv^2 - a_2\sqrt{-\dfrac{a_6}{a_5}}v^2 - 2a_4\sqrt{-\dfrac{a_6}{a_5}}uv \\ \qquad - a_4u^2v - a_5u^3 - 3a_5\sqrt{-\dfrac{a_6}{a_5}}u^2 + 2a_6u. \end{cases} \tag{2.10}$$

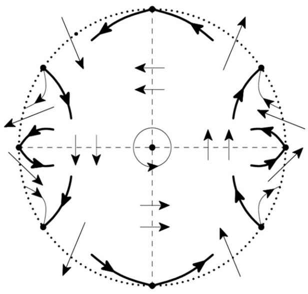
We apply a (2,1)-blow up to the system (2.1), and the local phase portrait of point P_1 in the system (2.1) is shown in Figure 2.10 (1). By the same argument, the local phase portrait of point P_2 in the system (2.1) is shown in Figure 2.10 (2). In the chart U_2, we have the same results as the Case IV.



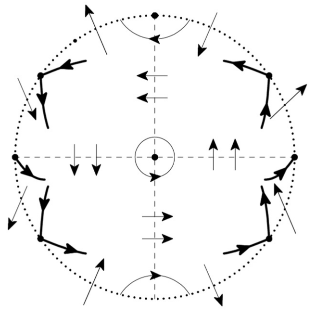
(1) P_1 (2) P_2

Figure 2.10: Local phase portrait of system (2.1) at point P_1 and point P_2 . The horizontal axis is filled with singular points.

After the analysis of case (a.2): $a_2 \neq 0$, $4(a_5 - a_6) < a_2^2$, we characterize the local phase portrait of system (1.3) on the Poincaré disk, see Figure 2.11 (1). By the symmetry of the system and the directions of the Poincaré disc, the possible global phase portraits of system (1.3) are shown in Figure 1.1 (31)–(46) of Theorem 1.2.



(1) case (a.2).



(2) $a_6 > 0$.

Figure 2.11: Local phase portrait of system (1.3) on the Poincaré disk of Case IV.

Using the similar argument as previous cases, if $a_6 > 0$, the local phase portrait of system (1.3) is Figure 2.11 (2), then the global phase portraits in the Poincaré disk are Figure 1.1 (47)–(51).

For Case IV, the possible global phase portrait of system (1.3) is shown in Figure 1.1 (25)–(51) of Theorem 1.2.

Case V: $a_4a_6 - a_5 \neq 0$, $a_5a_6 < 0$.

In the chart U_1, we consider the system (2.2). It is easy to find that there are three singular points in the u-axis: $O_{U_1}(0,0)$, $P_1(\sqrt{-a_6/a_5}, 0)$ and $P_2(\sqrt{-a_6/a_5}, 0)$. For the point $O_{U_1}(0,0)$, its singular point is the same to the system (2.2), please refer to Case II(i) and (ii) for details.

For the points P_1 and P_2, their linear part is

$$\begin{pmatrix} 0 & 0 \\ 2a_6 & \frac{a_4a_6 - a_5}{a_5} \end{pmatrix}.$$

If $a_4a_6 - a_5 \neq 0$, we have the following statements hold.

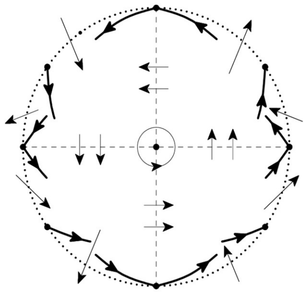
(A.1) If $a_6 < 0$ and $a_4a_6 - a_5 > 0$, then P_1 and P_2 are unstable nodes.

(A.2) If $a_6 < 0$ and $a_4a_6 - a_5 < 0$, then P_1 and P_2 are stable nodes.

(A.3) If $a_6 > 0$, then P_1 and P_2 are saddles.

In the chart U_2 the origin is the only one singular point of system (2.4). By the same argument as the Case III, if $a_6 < 0$, then the origin is a saddle, and if $a_6 > 0$, then the origin is a center.

Based on the above analysis, we characterize the local phase portrait of system (1.2) on the poincaré disk. If $a_6 > 0$, it is shown in Figure 2.11 (2); if $a_6 < 0$, Case (A.1) and Case (A.2) are equivalent to Figure 2.12.



$a_6 < 0, a_4 a_6 < a_5$

Figure 2.12: Local phase portrait of system (1.3) on the Poincaré disk of Case (A.1) and (A.2).

By the same argument as Figure 2.8, the possible global phase portraits of Figure 2.12 are Figure 1.1 (52)–(66). For Case IV, the possible global phase portraits of Figure 2.11 (2) are Figure 1.1 (47)–(51).

Therefore, the possible global phase portraits of Case V of system (1.3) are Figure 1.1(47)-(66).

2.2 Global phase portraits of system (1.4)

In this section, we discuss the global phase portraits of system (1.4). For $a_6 = 0$, $a_5 \neq 0$, the global phase portraits of system (1.4) has been studied in [2], thus we only investigate the following situations.

Case i: $a_5 = 0$**,** $a_6 \neq 0$**.**

In the chart U_1, the origin $O_{U_1}(0,0)$ of system (2.2) is a singular point and its linear part is

$$\begin{pmatrix} 0 & 0 \\ -a_6 & 0 \end{pmatrix}.$$

By the Theorem 3.5 of [14], if $a_6 < 0$, then $O_{U_1}(0,0)$ is a saddle, and if $a_6 > 0$, then $O_{U_1}(0,0)$ is a center.

In the chart U_2, the origin $O_{U_2}(0,0)$ is the only one singular point of system (2.4). Due to in the chart U_2, the coefficient a_3 does not work, the conclusion of system (1.4) is same to system (1.3), as shown Case II.1 and Case II.2 in Section 2.1.

Based on the above analysis, the corresponding local and global phase portraits of system (1.4) in Case I can be summarized as follows:

i.1 $a_4 = 0$. If $a_6 > 0$, the local phase portrait of system (1.4) on the Poincaré disk is Figure 2.3, but it needs to be rotated $\pi/2$ clockwise rotation, and all possible global phase portraits are topologically equivalent to Figure 1.1 (6) and (22). If $a_6 < 0$, the local phase portrait of system (1.4) is Figure 2.5(a) and Figure 2.3, and all possible global phase portrait is Figure 1.1 (7)–(12) and (6).

i.2 $a_4 \neq 0$. If $a_6 > 0$, the local phase portrait of system (1.4) on the Poincaré disk is Figure 2.13, all possible global phase portraits are topologically equivalent to Figure 1.1 (23) and (24). If $a_6 < 0$, the local phase portrait of system (1.4) is Figure 2.7 (3), and all possible global phase portraits are Figure 1.1 (19)–(21).

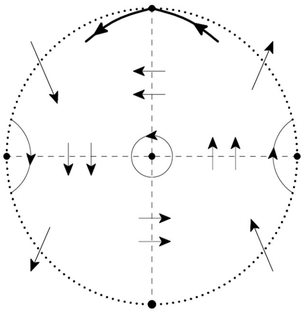


Figure 2.13: Local phase portrait of system (1.4) on the Poincaré disk of Case i.2 for $a_6 > 0$.

Therefore, the system (1.4) has 13 possible global phase portraits in Case i, as shown in Figure 1.1 (6)–(12) and (19)–(24).

Case ii: $a_5 a_6 > 0$**.**

If $a_6 > 0$, the local phase portrait of system (1.4) on the Poincaré disk is Figure 2.3, but it needs to be rotated $\pi/2$ clockwise rotation, all possible global phase portraits are topologically equivalent to Figure 1.1 (6) and (22). If $a_6 < 0$, the local phase portrait of system (1.4) is Figure 2.3, and all possible global phase portrait is Figure 1.1(6).

Therefore, the system (1.4) has 2 possible global phase portraits in Case ii, as shown in Figure 1.1 (6) and (22).

Case iii: $a_5 a_6 < 0$**,** $a_4 \neq 0$**.**

If $a_6 > 0$, the local phase portrait of system (1.4) on the Poincaré disk is Figure 2.14, all possible global phase portraits are topologically equivalent to Figure 1.1 (29) and (67). If $a_6 < 0$, the local phase portrait of system (1.4) is Figure 2.12, and all possible global phase portraits are Figure 1.1 (52)–(66).

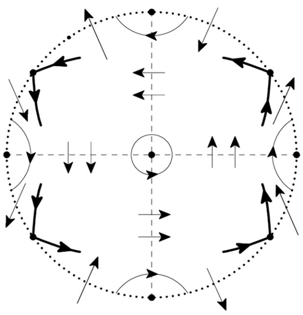


Figure 2.14: Local phase portrait of system (1.4) on the Poincaré disk of Case iii for $a_6 > 0$.

Therefore, the system (1.4) has 17 possible global phase portraits in Case iii, as shown in Figure 1.1 (29) and (52)–(67).

Case iv: $a_5 a_6 < 0$**,** $a_4 = 0$**.**

If $a_6 > 0$, the local phase portrait of system (1.4) is same to Case iii: $a_6 > 0$, as shown in Figure 2.14, its corresponding all possible global phase portraits are Figure 1.1 (29) and (67). If $a_6 < 0$, the local phase portrait of system (1.4) is same to Case IV: $a_6 < 0$, as shown in Figure

2.8 and Figure 2.11 (1), and its corresponding all possible global phase portraits are Figure 1.1 (25)–(30) and (31)–(46).

Consequently, the system (1.4) has 22 possible global phase portraits in Case iv, as shown in Figure 1.1 (25)–(46) and (67).

To sum up, the system (1.2) has 67 possible global phase portraits.

This completes the proof of Theorem 1.2.

Acknowledgements

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Existence and exponential stability for the wave equation with nonlinear interior source and localized viscoelastic boundary feedback

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Abstract. In this work, we aim to investigate an integro-differential model that involves localized viscoelastic effects at the boundary of the domain under the history framework. We have established that the equation is well-posed and exhibits exponential stability when a localized admissible kernel is applied, along with the δ-condition.

Keywords: wave equation, localized boundary feedback, exponential stability.

2020 Mathematics Subject Classification: 35L05, 35L20, 35B40.

1 Introduction

1.1 The model and literature overview

We consider the following problem

$$\begin{cases} u_{tt} - \Delta u + f(u) = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} + \displaystyle\int_0^\infty g(s)a(x)u_t(x,t-s)\,ds = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,-t) = u^0(x,-t), \quad u_t(x,0) = u_t^0(x) & \text{in } \Omega \times (0,\infty) \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is an open bounded domain with a sufficiently smooth boundary $\Gamma = \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, $u^0 : \Omega \times [0,\infty) \to \mathbb{R}$ is the prescribed past history of u. We denote by ω, ω_0, ω_1 the intersection of Ω with a neighborhood of Γ, Γ_0, Γ_1 in $\mathbb{R}^d$, respectively. In addition, $a = a(x)$, is real valued non-negative function, responsible for the localized dissipative effect, $f : \mathbb{R}^+ \to \mathbb{R}^+$ represents a source term and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative function having the form

$$g(s) = \int_s^\infty \mu(\tau)\,d\tau, \tag{1.2}$$

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where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is an integrable function. Other assumptions on the functions f, g, μ and a will be precisely stated ahead.

It is worth mentioning that the study of stabilization of evolution equations subjected to boundary dissipation has been gaining more attention in the academic world over the past few years. In the absence of the viscolelastic term

$$\int_0^\infty g(s)a(x)u_t(x,t-s)ds,$$

problem (1.1) has been handled by many authors when a frictional damping term (linear or not) at the boundary is included; see for instance [7, 9, 26, 28, 40] among others. Related to viscoelastic boundary conditions, Aassila and Cavalcanti [2] studied the following problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} + \displaystyle\int_0^t k(t-s,x)u'(s)\,ds + a(x)g(u') = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \quad u'(x) = u_1(x) & \text{in } \Omega \times (0,\infty) \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, $a : \Gamma_1 \to \mathbb{R}^+$ is such that $a(x) \geq a_0 > 0$. Under the following assumptions on functions k and g

$$k \geq 0, \qquad k' \leq 0, \qquad k'' \geq \alpha k' \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \tag{1.4}$$

$$C_1|x|^p \leq |g(x)| \leq C_2|x|^{1/p},\ |x| \leq 1; \qquad C_3|x| \leq |g(x)| \leq C_4|x|,\ |x| \geq 1, \tag{1.5}$$

for some positive constants α, $C_i (1 \leq i \leq 4)$, the authors obtained the energy decays exponentially if $p = 1$ and decays polynomially if $p > 1$ when $u_0 = 0$ in Γ_1, extending the work of [23] to the case of nonlinear frictional dampings at boundary. Park and collaborators, in [35], considered a similar extension to a nonlinear boundary condition of memory type with the same assumption on k but without the above assumption on u_0. They also included a nonlinear source term $|u|^\rho u$ acting on the domain Ω, which turns the problem more subtle than those previously cited. For other problems in connection with viscoelastic and dynamic boundary conditions, the reader is referred to [10], [1, 11, 19, 20, 25] and references therein.

Nowadays a question that has been extensively investigated is the role of the kernel k in a viscoelastic term of type

$$\int_0^t k(t-s,x)w(s)ds \qquad (w = u \text{ or } w = u') \tag{1.6}$$

acting on the domain and/or the boundary to provide existence, as well stability of solutions. A reasonably large class of them has been carried out for many authors. Indeed, since the highly cited article of Dafermos [15], a flurry of work has been done with increasing kernels $k \in L^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^+)$ satisfying $k(s) > 0$ and conditions like $1 - \int_0^\infty k(s)\,ds > 0$, together with the classical conditions (1.4) and improvements of them to provided existence and stability of solutions, we quote for instance [1, 10, 21, 29, 33, 36, 38] among others. A generalization of condition (1.4) was considered by Alabau-Boussouira and Cannarsa in [3] (see also [30–32]), where the main assumption is that the kernel k solves a suitable differential inequality. Other refinements of such condition are also discussed in [24, 27, 34].

Efforts are being made to achieve a less restricted assumption on the memory kernel k. Indeed, in [12], the authors introduced a general class of kernels called admissible kernels. These kernels are allowed not being strictly decreasing and can be locally flat while still fulfilling the so-called δ-condition: for some $\delta > 0$, there exists $C \geq 1$ such that

$$k(t+s) \leq Ce^{-\delta t}k(s)$$

for every $t \geq 0$ and $s > 0$. On these conditions, some authors have explored questions related to existence and stability of solutions, see for example [6,13,14].

1.2 Contribution and article structure

As mentioned earlier, previous research on viscoelastic dissipation at boundaries has mainly focused on the standard assumptions for the kernel, k. However, we have not found any studies that explore the effects of a more general memory kernel at the system's boundary, nor in a localized framework. Therefore, this paper's main contribution is its novel approach to this topic. We consider the past history framework together with a localized admissible kernel under the δ-condition to show exponential stability without the inclusion of frictional damping, unlike some of the articles mentioned earlier. However, this approach presents certain technical difficulties that must be addressed to obtain an observability inequality, which is crucial to proving the exponential stability of the problem.

Indeed, to demonstrate the exponential stability, we draw inspiration from the works of Dehman, Gérard, Lebeau [16] and Dehman, Lebeau, and Zuazua [17]. The key step in this approach involves establishing the observability inequality:

$$E(0) \leq C\left(\int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta^t(s)|^2\,ds\,d\Gamma\,dt\right)$$

for all $t \geq T_0$.

To prove this statement we employ a contradiction argument and seek a sequence (u_m, η_m^t) of weak solutions to the equivalent problem (2.2) such that $E_m(0) = 1$. By utilizing a boundary observability theorem by Duyckaerts, Zhang, and Zuazua [18], we aim to derive the desired contradiction by showing that $E_m(0) \to 0$ as $m \to \infty$. However, challenges arise due to the nature of μ satisfying the δ-condition, making it difficult to establish that the convergence

$$\lim_{m\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta_m^t|^2\,ds\,d\Gamma\,dt = 0$$

implies

$$\lim_{m\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty \mu(s)a(x)|\eta_m^t|^2\,ds\,d\Gamma\,dt = 0,$$

which is usual in this kind of problem, and is a crucial step for completing our contradiction argument.

Based on the above statements, this article is structured as follows: Section 2 discusses the well-posedness of problem (1.1) by introducing the well-known relative displacement history variable introduced by [15] to obtain an equivalent problem, as is typical in this kind of approach. In Section 3, the exponential stability of the solution is established by demonstrating an appropriate observability inequality.

2 Existence and uniqueness of solution

Through this article, we will use basic notations and results from books by [5,8,39].

In this section, we will prove the first result of this paper regarding the existence and uniqueness of solution for the system (2.2). To achieve this, we will introduce an equivalent problem that will enable us to utilize the Semigroups theory, as well the main assumptions and notations to be used throughout this paper.

As in the pioneer work of [15], and by following [22], we introduce the following new variable corresponding to the relative displacement history

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \qquad x \in \Omega,\ s > 0,\ t \geq 0, \tag{2.1}$$

in order to translate (1.1) into the autonomous problem

$$\begin{cases} u_{tt} - \Delta u + f(u) = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} + \displaystyle\int_0^\infty \mu(s) a(x) \eta^t(x,s)\, ds = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \eta^t_s + \eta^t_t = u_t & \text{in } \Omega \times (0,\infty) \times (0,\infty), \\ u(x,-t) = u^0(x,-t) & \text{in } \Omega \times (0,\infty) \\ u(x,0) = u_0(x) = u^0(x,0), \quad u_t(x,0) = u_1(x) = u^0_t(x) & \text{in } \Omega, \\ \eta^t(x,s) = 0 & \text{on } \Gamma_0 \times (0,\infty) \times (0,\infty), \\ \eta^0(x,s) = \eta_0(x,s) = u^0(x,0) - u^0(x,-s) & \text{on } \Omega \times (0,\infty) \end{cases} \tag{2.2}$$

in the two variables $u = u(t)$ and $\eta = \eta^t(s)$.

In the sequel, to state our main results on the well-posedness and asymptotic behavior of problem (1.1), let us consider the following assumptions and notations:

A.1. $a : \Gamma_1 \to \mathbb{R}^+ \in L^\infty(\Gamma_1) \cap C(\overline{\omega}_1)$ is such that

- *i.* $a(x) \geq 0$ on Γ_1;
- *ii.* $a(x) \geq a_0 > 0$ in $\omega_1 \subset\subset \Gamma_1$.

A.2. $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative function having the form

$$g(s) = \int_s^\infty \mu(\tau)\, d\tau, \tag{2.3}$$

where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a pointwise absolutely continuous function, nonincreasing, integrable and such that

- *i.* $l = \int_0^\infty \mu(s)\, ds \in (0,1)$;
- *ii.* there exists a strictly increasing sequence $\{s_n\}$, with $s_0 = 0$, either finite or converging to $s_\infty \in (0,\infty]$, such that μ has jumps at $s = s_n,\ n > 0$.

A.3. $f \in C^2(\mathbb{R})$ satisfies

- *i.* $f(0) = 0$;

ii. the primitive $F(s) = \int_0^s f(\tau)\, d\tau$ is such that

$$-\frac{\gamma|s|^2}{2} \le F(s) \le f(s)s + \frac{\gamma|s|^2}{2}, \tag{2.4}$$

$\gamma \in [0, \lambda_1[$, where $\lambda_1 > 0$ is the first eigenvalue corresponding to the Laplacian operator with Dirichlet boundary condition;

iii. there exists $c > 0$ such that

$$|f^{(j)}(s)| \le c(1+|s|)^{p-j}, \quad \forall s \in \mathbb{R},\ j = 1, 2, \tag{2.5}$$

where

$$p \ge 1 \text{ if } n = 2 \quad \text{and} \quad 1 \le p < \frac{n}{n-2} \text{ if } n \ge 3. \tag{2.6}$$

Remark 2.1.

1. Notice that the function μ defined in Assumption A.2 can be unbounded in a neighborhood of zero. Moreover, μ is differentiable almost everywhere, and $\mu'(s) \le 0$ for almost every s.

2. Observe that the growth condition on f implies that

$$|f(s)| \le c(p)|s| + c(p)|s|^p. \tag{2.7}$$

We still note that (2.4) implies $f'(0) + \gamma \ge 0$ as well.

Now, consider $A : D(A) \subset H^1_{\Gamma_0}(\Omega) \to H^{-1}(\Omega)$ the operator $Au = -\Delta u$, with $D(A) = \{u \in H^1_{\Gamma_0}(\Omega),\ \partial_\nu u_{|\Gamma_1} = 0\}$, $h : \mathcal{M} \to L^2(\Gamma_1)$, $h(w(s)) = \int_0^\infty \mu(s)a(x)w(s)\, ds$ and $N : L^2(\Omega) \to L^2(\Omega)$ be the Neumann map

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ Ng = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial(Ng)}{\partial\nu} = g & \text{on } \Gamma_1. \end{cases}$$

Therefore, we have that

$$N^* A^* v = -v_{|\Gamma_1}, \qquad \forall v \in D(A^{\frac{1}{2}}) \tag{2.8}$$

as well as the system (2.2) is equivalent to

$$\begin{cases} u_{tt} + A(u - N[h(\eta)]) + f(u) = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \eta^t_s + \eta^t_t = u_t & \text{in } \Omega \times (0,\infty) \times (0,\infty), \\ u(x,-t) = u^0(x,-t) & \text{in } \Omega \times (0,\infty) \\ u(x,0) = u_0(x) = u^0(x,0), \quad u_t(x,0) = u_1(x) = u^0_t(x) & \text{in } \Omega, \\ \eta^t(x,s) = 0 & \text{on } \Gamma_0 \times (0,\infty) \times (0,\infty), \\ \eta^0(x,s) = \eta_0(x,s) = u^0(x,0) - u^0(x,-s) & \text{on } \Omega \times (0,\infty). \end{cases} \tag{2.9}$$

Next, let a be a function satisfying Assumption A.1, and define the μ-weighted space with values in $L^2(\Gamma_1)$ as

$$\mathcal{M} = \left\{ \eta : \mathbb{R}^+ \to L^2(\Gamma_1);\ \int_0^\infty \mu(s) \|\sqrt{a}\eta(s)\|^2 < \infty \right\}, \tag{2.10}$$

which is a Hilbert space endowed with the inner-product

$$(\eta, \zeta)_{\mathcal{M}} = \int_0^\infty \mu(s) \int_{\Gamma_1} \sqrt{a}\eta(s)\sqrt{a}\zeta(s)\, d\Gamma\, ds.$$

Throughout this article, $\mathcal{H}$ represents the energy space

$$\mathcal{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times \mathcal{M},$$

where $H^1_{\Gamma_0}(\Omega) := \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$, and $\mathcal{H}$ is endowed with the inner product

$$\langle (u_1, v_1, \eta_1), (u_2, v_2, \eta_2) \rangle_{\mathcal{H}} = \int_\Omega (\nabla u_1 \nabla u_2 + v_1 v_2)\, dx + \int_0^\infty \mu(s) \int_{\Gamma_1} \sqrt{a}\eta_1 \sqrt{a}\eta_2\, d\Gamma\, ds.$$

Therefore, denoting $U = (u, u_t, \eta)^T$ we can write, equivalently, the system (2.9) in the form

$$\begin{cases} \frac{d}{dt}U(t) + \mathcal{S}U(t) + \mathcal{F}(U(t)) = 0, \\ U(0) = (u_0, u_1 \eta_0), \end{cases} \tag{2.11}$$

where $\mathcal{S} : D(\mathcal{S}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{S} \begin{bmatrix} u \\ v \\ \eta \end{bmatrix} = \begin{bmatrix} -v \\ A(u - N[h(\eta)]) \\ v - \eta^t_s \end{bmatrix},$$

$$D(\mathcal{S}) = \left\{ (u, v, \eta) \in \mathcal{H} :\ v \in H^1_{\Gamma_0},\ u - N[h(\eta)] \in D(A),\ \eta^t_s \in \mathcal{M},\ \eta(0) = 0 \right\},$$

which is well-defined in view of the previous explanation, and $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is set by

$$\mathcal{F}(U) = (0, f(u), 0)^T,$$

being also well-defined by virtue of the growth condition on f and standard Sobolev embeddings. The Hadamard well-posedness of problem (2.11) and, consequently, of the original system (1.1), reads as follows.

Theorem 2.2. *Under the Assumptions A.1–A.3 we have:*

(i) *If $U_0 = (u_0, u_1, \eta_0) \in D(\mathcal{S})$, then there exists a unique regular solution $U = (u, u_t, \eta)$ of (2.11) such that*

$$u \in W^{2,\infty}(0, T; L^2(\Omega)) \cap W^{1,\infty}(0, T; H^1_{\Gamma_0}(\Omega)), \quad \eta \in W^{1,\infty}(0, T; \mathcal{M}),$$

with $U(t) = (u(t), u_t(t), \eta^t) \in D(\mathcal{S})$, for all $t \in [0, T]$, for a given $T > 0$.

(ii) *If $U_0 = (u_0, u_1, \eta_0) \in \mathcal{H}$, then there exists a unique mild solution $U = (u, u_t, \eta)$ of (2.11) such that*

$$u \in C^1([0, T]; L^2(\Omega)) \cap C([0, T]; H^1_{\Gamma_0}(\Omega)), \quad \eta \in C([0, T], \mathcal{M}),$$

for all $T > 0$ given.

(iii) *Moreover, these solutions are continuously dependent of the initial data, in the norm of* $C([0,T],\mathcal{H})$*, for all* $T>0$.

Proof. To establish this result, firstly we shall prove that $\mathcal{S}$ is monotone and $I-\mathcal{S}$ is surjective on the space $\mathcal{H}$. Indeed, for $\eta \in D(\mathcal{S})$, define

$$\mathbb{J}[\eta] = \sum_{n\geq 1} (\mu(s_n^-) - \mu(s_n^+))\|\eta(s_n)\|^2_{\mathcal{M}},$$

which is a nonpositive quantity in view of Assumption A.2. Following [37, Lemma 3.4], one notices that $\eta \in D(\mathcal{S})$ satisfies

$$2(\eta_s,\eta)_{\mathcal{M}} = \int_0^\infty \mu'(s)\|\eta(s)\|^2_{\mathcal{M}}\,ds + \mathbb{J}[\eta].$$

Let

$$\begin{bmatrix} u_1 \\ v_1 \\ \eta_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \\ \eta_2 \end{bmatrix} \in D(\mathcal{S}).$$

Then

$$\begin{aligned} &\left\langle \mathcal{S}\begin{bmatrix} u_1 \\ v_1 \\ \eta_1 \end{bmatrix} - \mathcal{S}\begin{bmatrix} u_2 \\ v_2 \\ \eta_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_1 \\ \eta_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \\ \eta_2 \end{bmatrix}\right\rangle \\ &= \langle -v_1+v_2, u_1-u_2\rangle_{H^1_{\Gamma_0}} + \langle Av_1 - Av_2, v_1-v_2\rangle_{L^2(\Omega)} \\ &\quad - \langle A(N[h(\eta_1)-h(\eta_2)]), v_1-v_2\rangle_{L^2(\Omega)} + (v_1-v_2,\eta_1-\eta_2)_{L^2(\Omega)} \\ &\quad - ((\eta_1)_s - (\eta_2)_s, \eta_1-\eta_2)_{\mathcal{M}} \\ &= -\frac{1}{2}\mathbb{J}[\eta] \geq 0, \end{aligned}$$

which shows that $\mathcal{S}$ is monotone.

Next, we will prove that $I-\mathcal{S}$ is surjective. To this end, we show the equation

$$(I-\mathcal{S})\left(\begin{bmatrix} u \\ v \\ \eta \end{bmatrix}\right) = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

has a solution

$$\begin{bmatrix} u \\ v \\ \eta \end{bmatrix} \in \mathcal{H}, \quad \text{for any} \quad h = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \in \mathcal{H}.$$

The above equation is equivalent to write

$$\begin{cases} u+v = h_1, \\ v - A(u - N[h(\eta)]) = h_2 \\ \eta + \eta_s - v = h_3. \end{cases} \tag{2.12}$$

Combining the above identities we deduce in the weak space

$$\begin{cases} -A(-v - N[h(\eta)]) + v = h_2 + A(h_1) \\ \eta + \eta_s - v = h_3. \end{cases} \tag{2.13}$$

Denote

$$R(v,\eta) = (T_0 + T_1 + T_2)(v,\eta),$$
$$T_0(v,\eta) = (0, \eta + s), \quad T_1(v,\eta) = (Av + v, \eta), \quad T_2(v,\eta) = (A(N(h(\eta))), -v).$$

It is well-known that T_0 is maximal monotone in $H^1_{\Gamma_0} \times \mathcal{M}$. Also, T_1 is monotone and from the Lax–Milgram Theorem follows that it is surjective, therefore maximal monotone in $H^1_{\Gamma_0} \times \mathcal{M}$. Furthermore, T_2 is monotone and Lipschitz continuous in $H^1_{\Gamma_0} \times \mathcal{M}$. Then, using standard perturbation results in [4], we conclude that $R = (T_0 + T_1 + T_2)$ is maximal monotone and coercive, therefore the left hand term in (2.13) is surjective. Then, (2.13) possesses a unique solution $(v,\eta) \in H^1_{\Gamma_0}(\Omega) \times \mathcal{M}$. Since $u = v + h_1$ we obtain $u \in H^1_{\Gamma_0}(\Omega)$, which implies that $I - \mathcal{S}$ is surjective.

Next, to finish the proof we observe that from Assumption A.3, for a given $T > 0$, f generates a locally Lipschitz perturbation on the phase space $\mathcal{H}$ which after some standard calculations guarantees, by using the Kato's Theorem, the existence of a unique strong solution $U \in W^{1,\infty}([0,T], \mathcal{H})$ such that $U(t) \in D(S)$ for all $t \in [0,T]$. Moreover, this solution continuously depends on the initial data for any $T > 0$. □

3 Asymptotic stability result

In this section the goal is to establish the exponential stability result concerning problem (2.2).

Denoting $U = (u, u_t, \eta)$ the unique global solution to the problem (2.11) as stated in Theorem 2.2, then the couple (u,η) is the corresponding solution to the equivalent system (2.2). The associated energy functional is given by

$$E(t) = \frac{1}{2}\left[|u_t|^2 + |\nabla u|^2 + \|\eta\|^2_{\mathcal{M}} + 2\int_\Omega \int_0^u f(\tau)\, d\tau dx \right]. \tag{3.1}$$

A straightforward computation provides the identity

$$\frac{d}{dt}E(t) = \frac{1}{2}\int_{\Gamma_1}\int_0^\infty \mu'(s)a(x)|\eta^t(s)|^2 ds\, d\Gamma,$$

which, in view on Assumption A.2, implies that $E(t)$ is a non-increasing function for all $t > 0$ and satisfies the identity

$$E(T) - E(0) = \frac{1}{2}\int_0^T \int_{\Gamma_1}\int_0^\infty \mu'(s)a(x)|\eta^t(s)|^2 ds\, d\Gamma\, dt$$

for all $T > 0$.

In order to obtain the desired stability, we need to make some complementary assumptions on the given functions, as well to make some remarks and comments which will be necessary to prove the exponential stability.

Concerning the memory kernel $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ defined in Assumption A.2 it is also assumed that

A.4. (i) there exist $\delta > 0$ and $C \geq 1$ such that

$$\mu(t+s) \leq Ce^{-\delta t}\mu(s) \tag{3.2}$$

for every $t \geq 0$ and almost every $s > 0$;

(ii) μ is not completely flat, that is, the set

$$D = \{s > 0,\ \mu'(s) < 0\}$$

has positive Lebesgue measure.

Remark 3.1.

a. A kernel μ satisfying Assumption A.4(i) is said to fulfill the δ-condition;

b. Particularly, the δ-condition implies that, for each $t \geq 0$

$$|N_t = \{s \in \mathbb{R}^+,\ Ce^{-\delta t}\mu(s) - \mu(t+s) < 0\}| = 0, \tag{3.3}$$

where $|\cdot|$ stands for the Lebesgue measure of the set.

c. If $S_\infty = \sup\{s,\ \mu(s) > 0\} < \infty$, then μ fulfills the δ-condition for every $\delta > 0$;

d. When $C = 1$, (3.2) is equivalent to the well-known condition in the literature $\mu'(s) + \delta\mu(s) \leq 0$, for almost every $s > 0$;

e. Regarding Assumption A.4(ii), it is fairly easy to show that there exists $\alpha > 0$ large enough such that the set

$$N = \{s \in \mathbb{R}^+,\ \alpha\mu'(s) + \mu(s) < 0\} \tag{3.4}$$

has positive Lebesgue measure.

In view of the aforementioned considerations, the stability result reads as follows.

Theorem 3.2. *Assume that Assumptions A.1–A.4 are in force and let $R > 0$ be a given constant. If $E(0) \leq R$, there exist $T_0 > 0$ and constants $C_0, \lambda > 0$, depending on R, verifying*

$$E(t) \leq C_0 E(0) e^{-\lambda t}, \quad \forall t > T_0. \tag{3.5}$$

As mentioned earlier, an important step to prove estimate (3.5) relies on obtaining an observability inequality through a contradiction argument. To accomplish this it is needed, among other tools, to obtain the following convergence

$$\lim_{n\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty \mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt = 0,$$

for a sequence $\{(u_n, \eta_n^t)\}$ of solutions to the problem (2.2), which is not an easy task since μ satisfies the δ-condition $A.4(i)$. The proof of this convergence is stated in the following result:

Lemma 3.3. *Let $\{(u_n, \eta_n^t)\}$ be a sequence of solutions to the problem (2.2). By assuming Assumption A.4, if*

$$\lim_{n\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt = 0,$$

then

$$\lim_{n\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty \mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt = 0.$$

Proof. First one notices that, according to Remark 4.1(e), as

$$\lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_0^\infty \mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt = \lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_N \mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt + \lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_{\mathbb{R}^+\setminus N} \mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt,$$

we have that

$$\lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_N \mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt \le \lim_{n\to\infty}\alpha\int_0^T\int_{\Gamma_1}\int_0^\infty -\mu'(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt = 0. \tag{3.6}$$

Next, suppose that $\lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_{\mathbb{R}^+\setminus N}\mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt \neq 0$. Thus, there exists n_1 large enough such that

$$\int_{\mathbb{R}^+\setminus N}\mu(s)a(x)\|\eta_{n_1}^t\|^2_{L^2(\Gamma_1)}ds > 0,$$

for all $t \ge 0$. To not overload the notation, the index n_1 shall be omitted in the next calculations.

As in [12] consider, for $\eta_0 \in \mathcal{M}$, $U(t) = \mathcal{R}(t)(0,0,\eta_0)$. Therefore, as μ satisfies (3.2), if $\tilde{C}_1 > \max\{1, \tilde{C}\}$, one gets

$$\begin{aligned}
0 < \int_{\mathbb{R}^+\setminus N}\mu(s)a(x)\|\eta_0\|^2_{L^2(\Gamma_1)}ds &\le \int_0^\infty \mu(s)a(x)\|\eta_0\|^2_{L^2(\Gamma_1)}ds \\
&\le \int_0^\infty \mu(s-t)a(x)\|\eta_0\|^2_{L^2(\Gamma_1)}ds \\
&\le 2\|a\|\left[\int_t^\infty \mu(s)(\tilde{C}\|\eta^t(s)\|^2_{L^2(\Gamma_1)} + \|u(t)\|^2)ds\right] \\
&\le 2\|a\|\left[\tilde{C}\int_0^\infty \mu(s+t)\|\eta_0(s)\|^2_{L^2(\Gamma_1)} + \int_0^\infty \mu(s)\|u(t)\|^2 ds\right] \\
&< \tilde{C}_1(C+M)e^{-\delta t}\|\eta_0\|^2_{\mathcal{M}},
\end{aligned}$$

where $M = \|\mathcal{R}(t)\|$.

Particularly, for $t > 0$ fixed and $\eta_0(s) = \chi_{N_t}\phi(s)$, where $\|\phi\|_{L^2(\Gamma_1)} = 1$, we obtain

$$\int_0^\infty [1 - \tilde{C}_1(C+M)]e^{-\delta t}\chi_{N_t}(s)ds < 0. \tag{3.7}$$

On the other hand, for any fixed $t > 0$, define

$$\mathcal{N}_t = \{s \in \mathbb{R}^+,\ \mu(t+s) - \tilde{C}_1(C+M)e^{-\delta t}\mu(s) > 0\}.$$

Thus, from Remark 4.1(b) follows that

$$0 = \int_{\mathcal{N}_t}\mu(t+s) - \tilde{C}_1(C+M)e^{-\delta t}\mu(s)\, ds \le \tilde{C}_1(C+M)e^{-\delta t}|\mathcal{N}_t|,$$

which contradicts (3.7) and shows that

$$\lim_{n\to\infty}\int_0^T\int_{\Gamma_1}\int_{\mathbb{R}^+\setminus N}\mu(s)a(x)|\eta_n^t|^2 ds\, d\Gamma\, dt = 0.$$

□

Next, to the aim of obtaining the desired observability inequality which lead us to the proof of Theorem 3.2, and in order to use an appropriate boundary observability inequality in our arguments it is considered, for each $k \in \mathbb{N}$, the following approximation of problem (2.2):

$$\begin{cases} \partial_{tt}u^k - \Delta u^k + f^k(u^k) = 0 & \text{in } \Omega \times (0,\infty), \\ u^k = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u^k}{\partial \nu} + \displaystyle\int_0^\infty \mu(s) a(x) \eta^{t,k}(x,s) ds = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \partial_s \eta^{t,k} + \partial_t \eta^{t,k} = \partial_t u^k & \text{in } \Omega \times (0,\infty) \times (0,\infty), \\ u^k(x,-t) = u^0(x,-t) & \text{in } \Omega \times (0,\infty) \\ u^k(x,0) = u_0^k(x) = u^0(x,0), \quad \partial_t u^k(x,0) = u_1^k(x) = \partial_t u^0(x) & \text{in } \Omega, \\ \eta^{t,k}(x,s) = 0 & \text{on } \Gamma_0 \times (0,\infty) \times (0,\infty), \\ \eta^{0,k}(x,s) = \eta_0^k(x,s) = u_0^k(x,0) - u_0^k(x,-s) & \text{on } \Omega \times (0,\infty), \end{cases} \tag{3.8}$$

where the function f^k is defined by

$$f^k(s) = \begin{cases} f(s), & |s| \le k \\ f(k), & s \ge k \\ f(-k), & s \le -k. \end{cases}$$

Notice that, for each k, f^k is Lipschitz continuous on $\mathbb{R}$ and the associated energy functional is given by

$$E^k(t) = \frac{1}{2}\left[|\partial_t u^k|^2 + |\nabla u^k|^2 + \|\eta^{t,k}\|^2_{\mathcal{M}} + 2\int_\Omega \int_0^{u^k} f^k(\tau)\, d\tau dx \right]. \tag{3.9}$$

An observability inequality to the truncated problem (3.8) shall be provided by the next result.

Proposition 3.4. *Let us take Assumptions A.1-A.4 and let $R > 0$ be a given constant. The solution (u^k, η^k) of (3.8) satisfies the following inequality*

$$E^k(0) \le C \left(\int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s) a(x) |\eta^{t,k}(s)|^2 \, ds d\Gamma dt \right), \tag{3.10}$$

for all $T \ge T_0$ and some constant C depending only on $U_0 = (u_0, u_1, \eta_0)$, provided that $E^k(0) \le R$.

Proof. To prove (3.10) we argue by contradiction. Indeed, if such inequality does not hold, there exist $T > T_0 > 0$, $R > 0$ and a sequence $\{(u_n^k, \eta_n^{t,k})\}$ of solutions to

$$\begin{cases} \partial_{tt}u_n^k - \Delta u_n^k + f^k(u_n^k) = 0 & \text{in } \Omega \times (0,\infty), \\ u_n^k = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u_n^k}{\partial \nu} + \displaystyle\int_0^\infty \mu(s) a(x) \eta_n^{t,k}(x,s) ds = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \partial_s \eta_n^{t,k} + \partial_t \eta_n^{t,k} = \partial_t u_n^k & \text{in } \Omega \times (0,\infty) \times (0,\infty), \\ u_n^k(x,-t) = u^0(x,-t) & \text{in } \Omega \times (0,\infty) \\ u_n^k(x,0) = u_{0n}^k(x) = u_n^0(x,0), \; \partial_t u_n^k(x,0) = u_{1n}^k(x) = \partial_t u_n^0(x) & \text{in } \Omega, \\ \eta_n^{t,k}(x,s) = 0 & \text{on } \Gamma_0 \times (0,\infty) \times (0,\infty), \\ \eta_n^{0,k}(x,s) = \eta_{0n}^k(x,s) = u_{0n}^k(x,0) - u_{0n}^k(x,-s) & \text{on } \Omega \times (0,\infty), \end{cases} \tag{3.11}$$

such that $E_n^k(0) \le R$, which satisfies

$$\lim_{n\to\infty} \frac{E_n^k(0)}{\int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta_n^{t,k}(s)|^2 ds d\Gamma dt} = \infty. \tag{3.12}$$

Since $E_n^k(t) \le E_n^k(0) \le R$ for all $t \ge 0$, from (3.12) one gets

$$\lim_{n\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta_n^{t,k}(s)|^2 ds d\Gamma dt = 0, \tag{3.13}$$

and also guarantees the existence of a subsequence of $\{(u_n^k, \eta_n^{t,k})\}$, still denoted by $\{(u_n^k, \eta_n^{t,k})\}$, such that

$$\begin{aligned} u_n^k &\overset{*}{\rightharpoonup} u^k \quad \text{in } L^\infty(0,T;H_0^1(\Omega)), \\ \partial_t u_n^k &\overset{*}{\rightharpoonup} \partial_t u^k \quad \text{in } L^\infty(0,T;L^2(\Omega)), \end{aligned} \tag{3.14}$$

when $n \to \infty$. By using compactness arguments we also obtain

$$u_n^k \to u^k \text{ in } L^2(0,T;L^2(\Omega)). \tag{3.15}$$

In the sequel, with respect to the limit function u^k, the proof is twofold: $u^k \neq 0$ and $u^k = 0$.

Case I: $u^k \neq 0$. Taking in mind (3.13), (3.14) and Lemma 3.3, from (3.11) one obtains, when $n \to \infty$

$$\begin{cases} \partial_{tt}u^k + \Delta u^k + f^k(u^k) = 0 & \text{in } \Omega \times (0,\infty), \\ u^k = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u^k}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ u^k(x,-t) = u^{0,k}(x,-t) & \text{in } \Omega \times (0,\infty) \\ u^k(x,0) = u_0^k(x) = u^{0,k}(x,0), \quad \partial_t u^k(x,0) = u_1^k(x) = \partial_t u^{0,k}(x) & \text{in } \Omega, \end{cases} \tag{3.16}$$

Since f^k is globally Lipschitz, for each $k \in \mathbb{N}$ we find by the boundary observability theorem due to the Theorem 2.2 in [18] that $u^k = 0$, which presents the desired contradiction.

Case II: $u^k = 0$. Denote

$$\alpha_n = \left(E_n^k(0)\right)^{\frac{1}{2}}, \quad v_n^k = \frac{1}{\alpha_n} u_n^k, \quad \zeta_n^k = \frac{1}{\alpha_n}\eta_n^k. \tag{3.17}$$

Whereupon, $\{(v_n^k, \zeta_n^k)\}$ is solution of the normalized problem

$$\begin{cases} \partial_{tt}v_n^k - \Delta v_n^k + \dfrac{1}{\alpha_n} f^k(\alpha_n v_n^k) = 0 & \text{in } \Omega \times (0,\infty), \\ v_n^k = 0 & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial v_n^k}{\partial \nu} + \displaystyle\int_0^\infty \mu(s)a(x)\zeta_n^{t,k}(x,s)ds = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ \partial_s \zeta_n^{t,k} + \partial_t \zeta_n^{t,k} = \partial_t v_n^k & \text{in } \Omega \times (0,\infty) \times (0,\infty), \\ v_n^k(x,-t) = v^0(x,-t) & \text{in } \Omega \times (0,\infty) \\ v_n^k(x,0) = v_{0n}^k(x) = v_n^0(x,0), \ \partial_t v_n^k(x,0) = v_{1n}^k(x) = \partial_t v_n^0(x) & \text{in } \Omega, \\ \zeta_n^{t,k}(x,s) = 0 & \text{on } \Gamma_0 \times (0,\infty) \times (0,\infty), \\ \zeta_n^{0,k}(x,s) = \zeta_{0n}^k(x,s) = v_{0n}^k(x,0) - v_{0n}^k(x,-s) & \text{on } \Omega \times (0,\infty), \end{cases} \tag{3.18}$$

whose energy functional is defined by

$$E^{v_n^k}(t) = \frac{1}{2}\left[|\partial_t v_n^k|^2 + |\nabla v_n^k|^2 + \|\zeta_n^{t,k}\|_{\mathcal{M}}^2 + 2\int_\Omega \int_0^{v_n^k} f^k(\tau)\, d\tau dx\right]. \tag{3.19}$$

Further, as $E^{v_n^k}(t) = \frac{1}{\alpha_n^2} E_n^k(t)$ for all $t \geq 0$ we deduce

$$E^{v_n^k}(0) = \frac{1}{\alpha_n^2} E_n^k(0) = 1 \tag{3.20}$$

for all $n > 0$, and also the existence of a subsequence $\{(v_n^k, \zeta_n^k)\}$ such that such that

$$\begin{aligned} v_n^k &\overset{*}{\rightharpoonup} v^k && \text{in } L^\infty(0,T; H_0^1(\Omega)),\\ \partial_t v_n^k &\overset{*}{\rightharpoonup} \partial_t v^k && \text{in } L^\infty(0,T; L^2(\Omega)),\\ v_n^k &\to v^k && \text{in } L^2(0,T; L^2(\Omega)), \end{aligned} \tag{3.21}$$

since $E^{v_n^k}(t) \leq E^{v_n^k}(0)$ for all $t \geq 0$. Moreover, by combining (3.13) and Lemma 3.3 we get

$$\lim_{n\to\infty} \int_0^T \int_{\Gamma_1} \int_0^\infty \mu(s) a(x) |\zeta_n^{t_k}|^2 ds\, d\Gamma\, dt = 0. \tag{3.22}$$

If we show that $E^{v_n^k}(T)$ goes to zero uniformly for each k fixed the desired contradiction is proved, since

$$E^{v_n^k}(T) = E^{v_n^k}(0) + \int_0^T \int_{\Gamma_1} \int_0^\infty \mu'(s) a(x) \zeta_n^{t,k}\, ds d\Gamma dt.$$

Indeed, for this purpose observe that, for an eventual subsequence, $\alpha_n \to \alpha$, where $\alpha \geq 0$. Therefore we separate the proof in two subcases: $\alpha > 0$ and $\alpha = 0$.

If $\alpha > 0$, since we have $\alpha_n v^n = u_n^k \to 0$ strongly in $L^2(0, T, L^2(\Omega))$, passing to the limit in (3.18) when $n \to \infty$, and taking (3.21) and (3.22) into account, we arrive at

$$\begin{cases} \partial_{tt} v^k - \Delta v^k + \frac{1}{\alpha} f^k(0) = 0 & \text{in } \Omega \times (0,\infty),\\ v^k = 0 & \text{on } \Gamma_0 \times (0,\infty),\\ \dfrac{\partial v^k}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0,\infty),\\ v^k(x,-t) = v^0(x,-t) & \text{in } \Omega \times (0,\infty)\\ v^k(x,0) = v_0^k(x) = v^0(x,0),\ \partial_t v^k(x,0) = v_1^k(x) = \partial_t v^0(x) & \text{in } \Omega \end{cases} \tag{3.23}$$

which implies, as in the Case I, that $v^k = 0$.

Now, consider $\alpha = 0$. Firstly notice that, by Taylor's formula, we have

$$\begin{aligned} \frac{1}{\alpha_n} f(\alpha_n v_n^k) &= \frac{f'(0)\alpha_n v_n^k}{\alpha_n} + \frac{R(\alpha_n v_n^k)}{\alpha_n},\\ \frac{|R(\alpha_n v_n^k)|}{\alpha_n} &\leq \frac{\alpha_n^2 |v_n^k|^2}{\alpha_n} + \frac{\alpha_n^p |v_n^k|^p}{\alpha_n}. \end{aligned} \tag{3.24}$$

Next, by defining the set $\Omega_n^t = \{x \in \Omega \text{ s.t. } |u_n^k(x,t)| > k\}$, we have, thanks to assumption $A.3$

and Sobolev's embbeding,

$$
\begin{aligned}
&\left\|\frac{1}{\alpha_n} f^k(v_n^k) - \frac{1}{\alpha_n} f(v_n^k)\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad = \left\|\frac{1}{\alpha_n} f^k(u_n^k) - \frac{1}{\alpha_n} f(u_n^k)\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad = \frac{1}{\alpha_n^2}\int_0^T \int_{\Omega_n^t} \left|f^k(u_n^k) - f(u_n^k)\right|^2 \, dxdt \\
&\quad \leq c\frac{1}{\alpha_n^2}\int_0^T \int_{\Omega_n^t} \left|f^k(u_n^k)\right|^2 \, dxdt + \frac{1}{\alpha_n^2}\int_0^T \int_{\Omega_n^t} \left|f(u_n^k)\right|^2 \, dxdt \\
&\quad \leq c\frac{1}{\alpha_n^2}\int_0^T \int_{\Omega_m^t} (|k|^2 + |k|^{2p}) \, dxdt + \frac{1}{\alpha_n^2}\int_0^T \int_{\Omega_n^t} (|u_n^k|^2 + |u_n^k|^{2p}) \, dxdt \\
&\quad \leq c\alpha_n^{2(p-1)} \left\|v_m^k\right\|^{2p}_{L^{2p}(0,T;L^{2p}(\Omega))} \longrightarrow 0.
\end{aligned}
\tag{3.25}
$$

Also, it is not difficult to see that, up to a subsequence,

$$
\frac{R(\alpha_n v_n^k)}{\alpha_n} \rightharpoonup 0 \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{3.26}
$$

Therefore, from (3.24) – (3.26), and since

$$
\frac{1}{\alpha_n} f^k(\alpha_n v_n^k) - f'(0)v_n^k = \frac{1}{\alpha_n} f^k(u_n^k) - f'(0)v_k = \frac{1}{\alpha_n} f^k(u_n^k) - \frac{1}{\alpha_n} f(u_n^k) + \frac{1}{\alpha_n} f(u_n^k) - f'(0)v_k,
$$

one obtain

$$
\frac{1}{\alpha_n} f^k(\alpha_n v_n^k) - (f^k)'(0)v_k \to 0 \quad \text{in } L^2(0,T;L^2(\Omega)), \tag{3.27}
$$

By passing to the limit in (3.18) when $n \to \infty$, and taking (2.9), (3.22), (3.25) and (3.27) into account, we arrive at

$$
\begin{cases}
\partial_{tt} v^k - \Delta v^k + \frac{1}{\alpha}(f^k)'(0)v^k = 0 & \text{in } \Omega \times (0,\infty), \\
v^k = 0 & \text{on } \Gamma_0 \times (0,\infty), \\
\dfrac{\partial v^k}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0,\infty), \\
v^k(x,-t) = v^0(x,-t) & \text{in } \Omega \times (0,\infty) \\
v^k(x,0) = v_0^k(x) = v^0(x,0), \quad \partial_t v^k(x,0) = v_1^k(x) = \partial_t v^0(x) & \text{in } \Omega
\end{cases}
\tag{3.28}
$$

allowing us to conclude, as before, that $v^k = 0$. Thus, convergences (3.14) and (3.15) read as

$$
\begin{aligned}
v_n^k &\overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T : H_0^1(\Omega)), \\
\partial_t v_n^k &\overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T;L^2(\Omega)), \\
v_n^k &\to 0 \quad \text{in } L^2(0,T;L^2(\Omega)).
\end{aligned}
\tag{3.29}
$$

Besides that,

$$
\frac{1}{\alpha_n} f^k(\alpha_n v_n^k) \to 0 \quad \text{in } L^2(0,T,L^2(\Omega)). \tag{3.30}
$$

In light of these calculations, consider now $\phi_n^k(x,t) = \int_0^\infty \mu(s)\zeta_n^{t,k}(x,s)\, ds$ and $\theta \in C^\infty(0,T)$; $0 \le \theta < 1$; $\theta(t) = 1$ in $(\varepsilon, T-\varepsilon)$. Multiplying the first equation of (3.18) by $\psi_n = \theta\phi_n^k$ and integrating by parts, we infer

$$\begin{aligned}\mu_0 \int_0^T \int_\Omega |\partial_t v_n^k|^2 \theta dxdt = & -\int_0^T \int_\Omega \partial_t v_n^k \left(\int_0^\infty \mu(s)\partial_s v_n^k ds\right)\theta dxdt + \int_0^T \int_\Omega \nabla v_n^k \nabla \phi_n^k \theta dxdt \\ & -\int_0^T \int_\Omega \partial_t v_n^k \phi_n^k \theta_t dxdt + \int_0^T \int_{\Gamma_1} a(x)\left(\int_0^\infty \mu(s)\zeta_n^{t,k} ds\right)^2 \theta\, d\Gamma dt \\ & + \int_0^T \int_\Omega \frac{1}{\alpha_n} f^k(\alpha_n v_n^k)\phi_n^k \theta\, dxdt = I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{3.31}$$

From convergences (3.22), (3.29) and (3.30) it is not hard to conclude that

$$\lim_{n\to\infty} I_1 = \cdots = \lim_{n\to\infty} I_5 = 0.$$

Thus, $\lim_{n\to\infty} \int_\varepsilon^{T-\varepsilon} |\partial_t v_n^k|^2 dxdt = 0$, that is,

$$\lim_{n\to\infty} \int_0^T |\partial_t v_n^k|^2 dxdt = 0. \tag{3.32}$$

The next step is to show that the potential energy converges to zero. To this aim, we multiply the first equation of (3.18) by θv_n^k and integrate by parts to get

$$\begin{aligned}\int_0^T \int_\Omega |\nabla v_n^k|^2 \theta dxdt = & \int_0^T \int_\Omega |\partial_t v_n^k|^2 \theta dxdt + \int_0^T \int_\Omega \partial_t v_n^k v_n^k \theta_t dxdt \\ & - \int_0^T \int_{\Gamma_1} a(x) \int_0^\infty \mu(s)\zeta_n^{t,k} v_n^k \theta ds\, d\Gamma dt \\ & - \int_0^T \int_\Omega \frac{1}{\alpha_n} f^k(\alpha_n v_n^k) v_n^k \theta\, dxdt \\ = & J_1 + J_2 + J_3 + J_4,\end{aligned} \tag{3.33}$$

which, through an analysis similar to the performed previously, produces

$$\lim_{n\to\infty} \int_0^T \int_\Omega |\nabla v_n^k|^2 dxdt = 0. \tag{3.34}$$

Therefore, since $E^{v_n^k}(t)$ is non-increasing, from (3.30)–(3.34) we conclude that

$$\lim_{n\to\infty} E^{v_n^k}(T) = 0,$$

which concludes this proof. □

Proof of Theorem 3.2. Notice that since $C > 0$ in (3.10) does not depend on k, by arguing similarly to [7, Lemma 2.1 and Proposition 2.1] one can pass (3.10) to limit to obtain the observability inequality

$$E(0) \le C\left(\int_0^T \int_{\Gamma_1} \int_0^\infty -\mu'(s)a(x)|\eta^t(s)|^2\, dsd\Gamma dt\right) \tag{3.35}$$

and, consequently, the desired exponential stability. □

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Multiple normalized solutions for $(2, q)$-Laplacian equation problems in whole $\mathbb{R}^N$

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Abstract. This paper considers the existence of multiple normalized solutions of the following $(2, q)$-Laplacian equation:

$$\begin{cases} -\Delta u - \Delta_q u = \lambda u + h(\epsilon x) f(u), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2, \end{cases}$$

where $2 < q < N$, $\epsilon > 0, a > 0$ and $\lambda \in \mathbb{R}$ is a Lagrange multiplier which is unknown, h is a continuous positive function and f is also continuous satisfying L^2-subcritical growth. When ϵ is small enough, we show that the number of normalized solutions is at least the number of global maximum points of h by Ekeland's variational principle.

Keywords: normalized solution, multiplicity, $(2, q)$-Laplacian, variational methods.

2020 Mathematics Subject Classification: 35A15, 35B38, 35J60, 35J20.

1 Introduction

This paper is devoted to the existence of multiple normalized solutions, with $X := H^1(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$, of the following $(2, q)$-Laplacian equation:

$$-\Delta u - \Delta_q u = \lambda u + h(\epsilon x) f(u), \text{ in } \mathbb{R}^N \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2, \tag{1.2}$$

where $\epsilon, a > 0$, $\Delta_q u = \mathrm{div}(|\nabla u|^{q-2} \nabla u)$ is the q-Laplacian of u, $2 < q < N$ and $\lambda \in \mathbb{R}$ is a Lagrange multiplier which is unknown. The continuous function f satisfies the following conditions:

(f_1) f is odd and $\lim_{t \to 0} \frac{|f(t)|}{|t|^{p-1}} = \alpha > 0$ for some $p \in \left(2, 2 + \frac{4}{N}\right)$;

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(f_2) There exist some constants $c_1, c_2 > 0$ and $p_1 \in (q, q + \frac{2q}{N})$ such that $|f(t)| \le c_1 + c_2|t|^{p_1-1}$, $\forall t \in \mathbb{R}$;

(f_3) the mapping $t \mapsto \frac{f(t)}{t^{q-1}}$ is a non-decreasing function when $t > 0$.

Hereafter, the continuous function h satisfies the following assumptions:

(h_1) $0 < h_0 = \inf_{x\in\mathbb{R}^N} h(x) \le \max_{x\in\mathbb{R}^N} h(x) = h_{\max}$;

(h_2) $h_\infty = \lim_{|x|\to+\infty} h(x) < h_{\max}$;

(h_3) $h^{-1}(\{h_{\max}\}) = \{e_1, e_2, \dots, e_l\}$ with $e_1 = 0$ and $e_j \ne e_k$ when $j \ne k$.

In particular, since restriction of (1.2), we are seeking normalized solutions to (1.1), which corresponds to seek critical points of the following functional

$$I_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q dx - \int_{\mathbb{R}^N} h(\epsilon x)F(u) dx$$

on the sphere

$$S(a) := \left\{ u \in X := H^1(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N) : |u|_2^2 = \int_{\mathbb{R}^N} |u|^2\, dx = a^2 \right\}, \tag{1.3}$$

where $|\cdot|_\tau$ denotes the usual norm on $L^\tau(\mathbb{R}^N)$ for $\tau \in [1, +\infty)$ and $D^{1,q}(\mathbb{R}^N) := \{u \in L^{q^*}(\mathbb{R}^N) : \nabla u \in L^q(\mathbb{R}^N)\}$ with semi-norm $\|u\|_{D^{1,q}(\mathbb{R}^N)} = \|\nabla u\|_q$. Moreover, $\|u\|_X = \|u\|_{H^1(\mathbb{R}^N)} + \|u\|_{D^{1,q}(\mathbb{R}^N)}$. It is well known that $I_\epsilon \in C^1(X, \mathbb{R})$ and

$$\langle I'_\epsilon(u), \varphi\rangle = \int_{\mathbb{R}^N} \nabla u \nabla\varphi dx + \int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u\nabla\varphi dx - \int_{\mathbb{R}^N} h(\epsilon x) f(u)\varphi dx$$

for all $u, \varphi \in X$.

The equation (1.1) is related to the general reaction-diffusion system

$$\partial_t u - \Delta_p u - \Delta_q u = f(x, u). \tag{1.4}$$

The system has wide range of applications in physics and related sciences, such as biophysics, chemical reaction and plasma physics. In such applications, the function u describes a concentration, the (p,q)-Laplacian term in (1.4) corresponds to the diffusion as $\operatorname{div}\left[\left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right)\nabla u\right] = \Delta_p u + \Delta_q u$, whereas the term $f(x,u)$ is the reaction and relates to sources and loss processes. Another model related to the (p,q)-Laplacian operator concerns the Lavrentiev gap phenomenon, which involved variational functions with non-standard (p,q) growth conditions, e.g., in [9,30].

The stationary version of equation (1.4)

$$-\Delta_p u - \Delta_q u = f(x,u), \quad x \in \mathbb{R}^N$$

has been extensively studied. Where $N \ge 3, 1 < p < q < N$, C. J. He et al. in [11] proved the existence of solution by mountain pass theorem and the concentration–compactness principle when f does not satisfy the Ambrosetti–Rabinowitz condition and they derived the regularity of weak solutions in [12]. Furthermore, when nonlinear function f is discontinuous and satisfies the Ambrosetti–Rabinowitz condition, the authors in [31] showed the existence of solution by mountain pass theorem and the concentration-compactness principle. Moreover, some researchers had studied the existence results for the nonlinear function f involving the critical Sobolev exponent in a bounded domain. G. B. Li et al. [21] studied $f = |u|^{p^*-2}u + \mu|u|^{r-2}u$ and obtained infinitely many weak solutions by genus theorem when $1 < r < q < p < N, \mu > 0$. Later on, in [28], the authors proved multiplicity of positive solutions by using the Lusternik–Schnirelman category theorem where $p < r < p^*$. [13] proved some nonexistence results where $N \geq 2, 1 < q < p < N$ and $1 < r < p^*$. Finally, we refer the interested readers works [8,29] for a development of the existence theory for various problems of the (p,q)-Laplacian.

In literature, the following equation

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2 \end{cases} \tag{1.5}$$

has been widely studied by many researchers. In the L^2-subcritical problem, namely $2 < p < 2 + \frac{4}{N}$, it is well konwn that the functional

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p \mathrm{d}x, u \in H^1(\mathbb{R}^N)$$

is bounded from below on the set $\left\{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = \int_{\mathbb{R}^N} |u|^2 \, \mathrm{d}x = a^2\right\}$, so we can found a solution as a global minimizer on the sphere, see [24]. While in the L^2-supercritical problem, namely $2 + \frac{4}{N} < p < \frac{2N}{N-2}$, $E|_{S(a)}$ is unbounded from below. One of the main difficulties in dealing with normalized solutions is proving the Palais–Smale condition, as a compactness property. Jeanjean in [14] got one normalized solution by a mountain pass structure for an auxiliary functional. Furthermore, in [5], the authors obtained infinitely many normalized solutions by using linking geometry for a stretched functional. More results about L^2-supercritical problem can be found in [6,15]. Regarding the critical case, we cite the articles [7,23]. Furthermore, in a recent paper, Yang and Baldelli [27] considered the following equation

$$\begin{cases} -\Delta u - \Delta_q u + \lambda u = |u|^{p-2}u, \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2 \end{cases}$$

in all the possible cases, where $2 < p < \min\{2^*, q^*\}$ and $1 < q < N$. They showed a ground state solution by using Ekeland's variational principle in L^2-subcritical case, while in L^2-critical case, they proved existence and nonexistence results, at last, they get a solution by using a natural constraint approach in L^2-supercritical case.

In addition, the multiplicity of normalized solutions has been wildly researched. For example, Jeanjean and Lu [18] studied the following problem

$$\begin{cases} -\Delta u = \lambda u + h(u), \text{ in } \mathbb{R}^N, \\ u > 0, \quad \displaystyle\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2, \end{cases}$$

they obtained multiple normalized solutions by the variational methods and genus theory. More information about multiplicity of normalized solutions by using genus theory and deformation arguments, see [2,16,17]. Particularly, without use of the genus theory, the authors [19] studied the following problem

$$\begin{cases} -\Delta u + \lambda u = (I_\alpha * [h(\epsilon x)|u|^{\frac{N+\alpha}{N}}])h(\epsilon x)|u|^{\frac{N+\alpha}{N}-2}u + \mu|u|^{q-2}u, \quad x \in \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2. \end{cases}$$

They showed multiple normalized solutions by Ekeland's variational principle when ϵ small enough, $\mu, a > 0$, $2 < q < 2 + \frac{4}{N}$, $\lambda \in \mathbb{R}$ and h is a continuous positive function satisfying (h_1)–(h_3).

This paper is devoted to study the problem (1.1)–(1.2), which has not been studied in our knowledge. In order to get the existence of multiple normalized solutions for (1.1), we will follow the variational methods in [19]. Moreover, since the workspace is $X = H^1(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$, it will be more complicated to obtain the strong $L^2(\mathbb{R}^N)$ convergence of the selected Palais-Smale sequence in X.

The main result of this paper is the following:

Theorem 1.1. *Assume that f satisfies (f_1)–(f_3) and h satisfies (h_1)–(h_3). Then, there exists ϵ_0 such that (1.1) has at least l couples weak solutions $(u_j, \lambda_j) \in X \times \mathbb{R}$ for $0 < \epsilon < \epsilon_0$. Moreover, $\lambda_j < 0$ and $I_\epsilon(u_j) < 0$ for $j = 1, 2, \ldots, l$.*

Now, we will give the outline about this paper. In Section 2, we prove a compactness theorem in the autonomous case. In Section 3, we use the compactness theorem to study the non-autonomous case. Finally, we give the proof of Theorem 1.1 in Section 4.

2 The autonomous case

Firstly, we consider the existence of normalized solution $(u, \lambda) \in X \times \mathbb{R}$, where $X = H^1(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N)$, for the problem below

$$\begin{cases} -\Delta u - \Delta_q u = \lambda u + \mu f(u), \\ \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = a^2, \end{cases} \tag{2.1}$$

where $a, \mu > 0$, $\lambda \in \mathbb{R}$ and f satisfies (f_1)–(f_3). It is well known that the critical point of the functional

$$J_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x - \int_{\mathbb{R}^N} \mu F(u) \mathrm{d}x$$

is a solution to the problem (2.1), which is restricted to the sphere $S(a)$, where $F(t) = \int_0^t f(s)\mathrm{d}s$. Next, we will show that problem (2.1) has a normalized solution.

Lemma 2.1 ([20, Lemma 2.7]). *Assume that $k > 1$, Ω is an open set in $\mathbb{R}^N$, $\alpha, \beta > 0$ and $\Theta \in C(\Omega \times \mathbb{R}^N, \mathbb{R}^N)$ satisfying*

(1) $\alpha|\xi|^k \le \Theta(x, \xi)\xi$, $\forall (x, \xi) \in \Omega \times \mathbb{R}^N$,

(2) $|\Theta(x, \xi)| \le \beta|\xi|^{k-1}, \forall (x, \xi) \in \Omega \times \mathbb{R}^N$,

(3) $(\Theta(x, \xi) - \Theta(x, \eta))(\xi - \eta) > 0$, $\forall (x, \xi) \in \Omega \times \mathbb{R}^N$ with $\xi \neq \eta$,

(4) $\Theta(x, \gamma\xi) = \gamma|\gamma|^{k-2}\Theta(x, \xi)$, $\forall (x, \xi) \in \Omega \times \mathbb{R}^N$ and $\gamma \in \mathbb{R} \setminus \{0\}$.

Consider $(u_n), u \in W^{1,k}(\Omega)$, then $\nabla u_n \to \nabla u$ in $L^k(\Omega)$ if and only if

$$\lim_{n \to \infty} \int_\Omega (\Theta(x, \nabla u_n(x)) - \Theta(x, \nabla u(x)))(\nabla u_n(x) - \nabla u(x))\, \mathrm{d}x = 0.$$

Lemma 2.2. *The functional J_μ restricts to $S(a)$ is bounded from below.*

Proof. From the conditions (f_1)–(f_2), we can infer that there exist some constants $C_1, C_2 > 0$ such that

$$|F(t)| \leq C_1|t|^p + C_2|t|^{p_1}, \quad \forall t \in \mathbb{R}.$$

By the L^q-Gagliardo–Nirenberg inequality [1, Theorem 2.1], we get that

$$|u|_l \leq C|\nabla u|_q^{\nu_{l,q}}|u|_2^{(1-\nu_{l,q})}, \quad \forall u \in D^{1,q}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \tag{2.2}$$

for some positive constant $C > 0$, where $\nu_{l,q} = \frac{Nq(l-2)}{l[Nq-2(N-q)]}$, $l \in (2, q^* = \frac{Nq}{N-q})$. Hence,

$$\begin{aligned} J_\mu(u) \geq \frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\mathrm{d}x - CC_1a^{(1-\nu_{p,q})p}\left(\int_{\mathbb{R}^N}|\nabla u|^q\mathrm{d}x\right)^{\frac{\nu_{p,q}p}{q}} \\ - CC_2a^{(1-\nu_{p_1,q})p_1}\left(\int_{\mathbb{R}^N}|\nabla u|^q\mathrm{d}x\right)^{\frac{\nu_{p_1,q}p_1}{q}}. \end{aligned} \tag{2.3}$$

As $p \in (2, 2+\frac{4}{N})$, $p_1 \in (q, q+\frac{2q}{N})$, clearly $\nu_{p,q}p, \nu_{p_1,q}p_1 < q$, which ensures the boundedness of J_μ from below. If J_μ is not bound from below, then there is u such that

$$\frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\mathrm{d}x - C\left(\int_{\mathbb{R}^N}|\nabla u|^q\mathrm{d}x\right)^{\frac{\nu_{p,q}p}{q}} - C\left(\int_{\mathbb{R}^N}|\nabla u|^q\mathrm{d}x\right)^{\frac{\nu_{p_1,q}p_1}{q}} \to -\infty,$$

which is a contradiction since $\nu_{p,q}p, \nu_{p_1,q}p_1 < q$. □

This lemma ensures that $m_\mu(a) := \inf_{u\in S(a)} J_\mu(u)$ is well defined.

Lemma 2.3. *Let $\mu, a > 0$, then $m_\mu(a) < 0$.*

Proof. By (f_1), we can deduce $\lim_{t\to 0}\frac{pF(t)}{t^p} = \alpha > 0$, which implies that, for some $\delta > 0$,

$$\frac{pF(t)}{t^p} \geq \frac{\alpha}{2} \tag{2.4}$$

for all $t \in [0, \delta]$. Let $0 < u_0 \in S(a) \cap L^\infty(\mathbb{R}^N)$, we set

$$H(u_0, r)(x) = e^{\frac{Nr}{2}}u_0(e^r x), \quad \forall x \in \mathbb{R}^N, \ \forall r \in \mathbb{R}.$$

It is well known that

$$\int_{\mathbb{R}^N}|H(u_0,r)(x)|^2\mathrm{d}x = a^2.$$

Furthermore, by a direct calculation, we have

$$\int_{\mathbb{R}^N}F(H(u_0,r)(x))\mathrm{d}x = e^{-Nr}\int_{\mathbb{R}^N}F(e^{\frac{Nr}{2}}u_0(x))\mathrm{d}x.$$

Then, for $r < 0$ and $|r|$ big enough, we have

$$0 \leq e^{\frac{Nr}{2}}u_0(x) \leq \delta, \quad \forall x \in \mathbb{R}^N.$$

Furthermore, by (2.4), we derive

$$\int_{\mathbb{R}^N}F(H(u_0,r)(x))\mathrm{d}x \geq \frac{\alpha}{2p}e^{\frac{(p-2)Nr}{2}}\int_{\mathbb{R}^N}|u_0(x)|^p\mathrm{d}x,$$

so,

$$J_\mu(H(u_0,r)) \leq \frac{e^{2r}}{2}\int_{\mathbb{R}^N}|\nabla u_0|^2\mathrm{d}x + \frac{e^{\frac{Nqr}{2}+rq-rN}}{2}\int_{\mathbb{R}^N}|\nabla u_0|^q\mathrm{d}x - \frac{\mu\alpha e^{\frac{(p-2)Nr}{2}}}{2p}\int_{\mathbb{R}^N}|u_0(x)|^p\mathrm{d}x.$$

Since $q>2, p\in\left(2,2+\frac{4}{N}\right)$, increasing $|r|$ if necessary, we get that

$$\frac{e^{2r}}{2}\int_{\mathbb{R}^N}|\nabla u_0|^2\mathrm{d}x + \frac{e^{\frac{Nqr}{2}+rq-rN}}{2}\int_{\mathbb{R}^N}|\nabla u_0|^q\mathrm{d}x - \frac{\mu\alpha e^{\frac{(p-2)Nr}{2}}}{2p}\int_{\mathbb{R}^N}|u_0(x)|^p\mathrm{d}x = A_r < 0,$$

then

$$J_\mu(H(u_0,r)) \leq A_r < 0,$$

showing that $m_\mu(a)<0$. □

Lemma 2.4. *If $\mu>0$, $a>0$, then*

(i) $a\mapsto m_\mu(a)$ is a continuous mapping;

(ii) if $a_1\in(0,a)$ and $a_2=\sqrt{a^2-a_1^2}$, we have $m_\mu(a)<m_\mu(a_1)+m_\mu(a_2)$.

Proof. (i) Let $a>0$ and $(a_n)\subset(0,+\infty)$ such that $a_n\to a$, we need to prove that $m_\mu(a_n)\to m_\mu(a)$. There exists $u_n\in S(a_n)$ such that $m_\mu(a_n)\leq J_\mu(u_n)<m_\mu(a_n)+\frac{1}{n}$ for every $n\in\mathbb{N}^+$. Firstly, we deduce from Lemma 2.3 that $m_\mu(a_n)<0$. Then by Lemma 2.2, we can get that (u_n) is bounded in X. Now considering $v_n:=\frac{a}{a_n}u_n\in S(a)$, since the boundedness of (u_n) and $a_n\to a$, we have

$$\begin{aligned}
m_\mu(a) &\leq J_\mu(v_n)\\
&= J_\mu(u_n)+\frac{1}{2}\left(\frac{a^2}{a_n^2}-1\right)\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x+\frac{1}{q}\left(\frac{a^q}{a_n^q}-1\right)\int_{\mathbb{R}^N}|\nabla u_n|^q\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}\Big(\mu F(u_n)\mathrm{d}x-\mu F(\frac{a}{a_n}u_n)\Big)\mathrm{d}x\\
&= J_\mu(u_n)+o_n(1).
\end{aligned}$$

Let $n\to+\infty$, we can get $m_\mu(a)\leq\lim_{n\to+\infty}\inf m_\mu(a_n)$. In the same manner, let (w_n) be a bounded minimizing sequence of $m_\mu(a)$ and $z_n:=\frac{a_n}{a}w_n\in S(a_n)$, then we have

$$m_\mu(a_n)\leq J_\mu(z_n)=J_\mu(w_n)+o_n(1)\Longrightarrow \lim_{n\to+\infty}\sup m_\mu(a_n)\leq m_\mu(a),$$

so we get $m_\mu(a_n)\to m_\mu(a)$.

(ii) For any fix $a_1\in(0,a)$, we first claim that

$$m_\mu(\theta a_1)<\theta^2 m_\mu(a_1),\ \forall\theta>1. \tag{2.5}$$

Let $(u_n)\subset S(a_1)$ be a minimizing sequence for $m_\mu(a_1)$, then $u_n(\theta^{-\frac{2}{N}}x)\in S(\theta a_1)$. Since $\theta>1$ and $\frac{2(N-q)}{N}<\frac{2(N-2)}{N}<2$, we have

$$\begin{aligned}
m_\mu(\theta a_1)-\theta^2 J_\mu(u_n) &\leq J_\mu(u_n(\theta^{-\frac{2}{N}}x))-\theta^2 J_\mu(u_n)\\
&=\frac{\theta^{\frac{2(N-2)}{N}}-\theta^2}{2}|\nabla u_n|_2^2+\frac{\theta^{\frac{2(N-q)}{N}}-\theta^2}{q}|\nabla u_n|_q^q\leq 0.
\end{aligned}$$

As a consequence $m_\mu(\theta a_1) \le \theta^2 m_\mu(a_1)$. If $m_\mu(\theta a_1) = \theta^2 m_\mu(a_1)$, we will have $|\nabla u_n|_2^2 \to 0$ and $|\nabla u_n|_q^q \to 0$ as $n \to +\infty$, which can indicates that $\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \to 0$ by inequality (2.2). Then,

$$\begin{aligned}
0 &> m_\mu(a_1) \\
&= \lim_{n\to+\infty} J_\mu(u_n) = \frac{1}{2}\lim_{n\to+\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x + \frac{1}{q}\lim_{n\to+\infty}\int_{\mathbb{R}^N}|\nabla u_n|^q\mathrm{d}x - \lim_{n\to+\infty}\int_{\mathbb{R}^N}\mu F(u_n)\mathrm{d}x \\
&= 0,
\end{aligned}$$

which is a contradiction. So we get $m_\mu(\theta a_1) < \theta^2 m_\mu(a_1)$. In the same manner, we can get

$$m_\mu(\theta a_2) < \theta^2 m_\mu(a_2), \quad \forall \theta > 1. \tag{2.6}$$

Finally, apply (2.5) with $\theta = \frac{a}{a_1} > 1$ and (2.6) with $\theta = \frac{a}{a_2} > 1$ respectively, we get

$$m_\mu(a) = \frac{a_1^2}{a^2} m_\mu\left(\frac{a}{a_1}a_1\right) + \frac{a_2^2}{a^2} m_\mu\left(\frac{a}{a_2}a_2\right) < m_\mu(a_1) + m_\mu(a_2). \qquad \square$$

Next, we will show the compactness theorem on $S(a)$ which is useful for studying the autonomous and the nonautonomous case.

Proposition 2.5. *Assume that $(u_n) \subset S(a)$ is a minimizing sequence of $m_\mu(a)$. Then, for some subsequence, either*

(i) (u_n) is strongly convergent,

or

(ii) there exists a sequence $v_n(\cdot) = u(\cdot + y_n)$ with $|y_n| \to +\infty$ and $(y_n) \subset \mathbb{R}^N$, which is strongly convergent to a function $v \in S(a)$ with $J_\mu(v) = m_\mu(a)$.

Proof. It is easy to obtain the boundedness of sequence (u_n) by Lemma 2.2, then there is a subsequence $u_n \rightharpoonup u$ in X, which is still denoted as itself. For the case of $u \neq 0$ and $|u|_2 = b$, by the Brézis–Lieb lemma in [26], we can deduce that $b \in (0, a)$ and

$$\begin{aligned}
|u_n|_2^2 &= |u|_2^2 + |u_n - u|_2^2 + o_n(1), \\
|\nabla u_n|_2^2 &= |\nabla u|_2^2 + |\nabla(u_n - u)|_2^2 + o_n(1).
\end{aligned}$$

Moreover, according to the assumption of f, we can deduce

$$\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^N} F(u)\mathrm{d}x + \int_{\mathbb{R}^N} F(u_n - u)\mathrm{d}x + o_n(1).$$

Now, we will prove $\nabla u_n \to \nabla u$ a.e. on $\mathbb{R}^N$, up to subsequences. Choose $\psi \in C_0^\infty(\mathbb{R}^N)$ satisfying $0 \le \psi \le 1$ in $\mathbb{R}^N$, $\psi(x) = 1$ for every $x \in B_1(0)$ and $\psi(x) = 0$ for every $x \in \mathbb{R}^N \setminus B_2(0)$. Take $R > 1$ and define $\psi_R(x) = \psi(x/R)$. Using the $\langle J'_\mu(u), \phi\rangle$ with $u = u_n$ and $\phi = (u_n - u)\psi_R$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left[\nabla u_n - \nabla u + |\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u\right](\nabla u_n - \nabla u)\psi_R\mathrm{d}x \\
&\quad= \langle J'_\mu(u_n), (u_n - u)\psi_R\rangle - \int_{\mathbb{R}^N}\nabla u_n u_n \nabla\psi_R\mathrm{d}x - \int_{\mathbb{R}^N}|\nabla u_n|^{q-2}\nabla u_n u_n\nabla\psi_R dx \\
&\qquad+ \int_{\mathbb{R}^N}\mu f(u_n)u_n\psi_R\mathrm{d}x + \int_{\mathbb{R}^N}\nabla u_n u\nabla\psi_R + \int_{\mathbb{R}^N}|\nabla u_n|^{q-2}\nabla u_n u\nabla\psi_R\mathrm{d}x \\
&\qquad- \int_{\mathbb{R}^N}\mu f(u_n)u\psi_R\mathrm{d}x - \int_{\mathbb{R}^N}\nabla u_n\nabla u\psi_R\mathrm{d}x - \int_{\mathbb{R}^N}|\nabla u|^{q-2}\nabla u\nabla u_n\psi_R\mathrm{d}x. \\
&\qquad+ \int_{\mathbb{R}^N}|\nabla u|^q\psi_R\mathrm{d}x + \int_{\mathbb{R}^N}|\nabla u|^2\psi_R\mathrm{d}x.
\end{aligned}$$

Since, $(u_n) \subset S(a)$ and $(J_\mu|_{S(a)})'(u_n) \to 0$, we have $\langle J'_\mu(u_n), (u_n - u)\psi_R\rangle \to 0$ as $n \to \infty$. Moreover, combining with the definition of ψ_R and $u_n \rightharpoonup u$ in X, we can get, as $n \to \infty$,

$$\int_{\mathbb{R}^N} \nabla u_n u_n \nabla \psi_R \mathrm{d}x - \int_{\mathbb{R}^N} \nabla u_n u \nabla \psi_R \mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n u_n \nabla \psi_R \mathrm{d}x - \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n u \nabla \psi_R \mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^N} \mu f(u_n) u_n \psi_R \mathrm{d}x - \int_{\mathbb{R}^N} \mu f(u_n) u \psi_R \mathrm{d}x \to 0,$$

$$\int_{\mathbb{R}^N} |\nabla u|^{q-2}\nabla u \nabla u_n \psi_R \mathrm{d}x \to \int_{\mathbb{R}^N} |\nabla u|^q \psi_R \mathrm{d}x,$$

$$\int_{\mathbb{R}^N} \nabla u_n \nabla u \psi_R \mathrm{d}x \to \int_{\mathbb{R}^N} |\nabla u|^2 \psi_R \mathrm{d}x.$$

So,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[\nabla u_n - \nabla u + |\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u\right] (\nabla u_n - \nabla u)\psi_R \mathrm{d}x = 0,$$

which is equivalent to

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (\nabla u_n - \nabla u)^2 \psi_R dx = 0, \ \lim_{n\to\infty} \int_{\mathbb{R}^N} (\nabla u_n - \nabla u)^q \psi_R \mathrm{d}x = 0.$$

Then, by Lemma 2.1 for $\Theta(x, \xi) = |\xi|^{k-2}\xi$ with $k = 2, k = q$, we have $\nabla u_n \to \nabla u$ in $L^2(B_2(0))$ and $L^q(B_2(0))$, which ensures that $\nabla u_n \to \nabla u$ a.e. on $\mathbb{R}^N$, up to subsequence. Now, applying Brézis–Lieb lemma in [26] again, we obtain

$$|\nabla u_n|_q^q = |\nabla u|_q^q + |\nabla(u_n - u)|_q^q + o_n(1).$$

Let $v_n = u_n - u$ and $|v_n|_2 = d_n \to d$, we can get that $a^2 = b^2 + d^2$ and $d_n \in (0, a)$ for n big enough. So,

$$m_\mu(a) + o_n(1) = J_\mu(u_n) = J_\mu(u) + J_\mu(v_n) + o_n(1) \geq m_\mu(d_n) + m_\mu(b) + o_n(1).$$

By the continuity of $a \mapsto m_\mu(a)$ (see Lemma 2.4(i)), we have

$$m_\mu(a) \geq m_\mu(d) + m_\mu(b),$$

which is contradicted to the conclusion of Lemma 2.4(ii), where $a^2 = b^2 + d^2$. This asserts that $|u|_2 = a$.

Combining with $|u_n|_2 = |u|_2 = a$, $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ is reflexive, we can get

$$u_n \to u \text{ in } L^2(\mathbb{R}^N). \tag{2.7}$$

Combining with the inequality (2.2) and $(f_1) - (f_2)$, we get

$$\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \to \int_{\mathbb{R}^N} F(u)\mathrm{d}x. \tag{2.8}$$

So

$$m_\mu(a) = J_\mu(u_n) + o_n(1) = J_\mu(u) + J_\mu(v_n) + o_n(1) \geq \frac{1}{2}|\nabla v_n|_2^2 + \frac{1}{q}|\nabla v_n|_q^q + m_\mu(a) + o_n(1),$$

which indicates $|\nabla v_n|_2^2, |\nabla v_n|_q^q \leq o_n(1)$. So we have $v_n \to 0$ in X, which means $u_n \to u$ in X.

Let us assume $u=0$, i.e., $u_n \rightharpoonup 0$ in X. Then, for some $\varsigma, r>0$ and $\{y_n\}\subset \mathbb{R}^N$, we have

$$\int_{B_r(y_n)} |u_n|^2 \mathrm{d}x \geq \varsigma, \ \forall y_n \in \mathbb{R}^N. \tag{2.9}$$

Otherwise we must have $u_n \to 0$ in $L^k(\mathbb{R}^N), \forall k\in(2,2^*)$, which implies $F(u_n)\to 0$ in $L^1(\mathbb{R}^N)$. But it contradicts to the fact that

$$0 > m_\mu(a) + o_n(1) = J_\mu(u_n) \geq -\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x.$$

Then (2.9) holds. Since $u=0$, combining with the inequality (2.9) and the Sobolev embedding, we can infer that (y_n) is unbounded. Then we consider $v_n(x)=u(x+y_n)$, which is easy to check that (v_n) is also a minimizing sequence of $m_\mu(a)$ and $(v_n)\subset S(a)$. So, there holds $v_n \rightharpoonup v$ in X, where $v\in X\setminus\{0\}$. According to the proof of the first part, we deduce that $v_n \to v$ in X. □

Lemma 2.6. *Assume* (f_1)*–*(f_3) *hold,* $\mu>0$*. Then, problem* (2.1) *has a positive radial solution* u *and* $\lambda<0$*.*

Proof. We can assume that there is a bounded minimizing sequence $(u_n)\subset S(a)$ of $m_\mu(a)$ by Lemma 2.2. Then, applying Proposition 2.5, we can deduce $m_\mu(a)=J_\mu(u)$, where $u\in S(a)$. Thus, we can get that there exists a constant $\lambda_a\in\mathbb{R}$ such that

$$J'_\mu(u) = \lambda_a \Psi'(u) \text{ in } X', \tag{2.10}$$

where $\Psi(u) := \int_{\mathbb{R}^N} |u|^2\mathrm{d}x$. Then, according to (2.10),

$$-\Delta u - \Delta_q u = \lambda_a u + \mu f(u), \quad x\in\mathbb{R}^N,$$

and

$$\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^q\mathrm{d}x - \int_{\mathbb{R}^N} \lambda_a u^2\mathrm{d}x - \int_{\mathbb{R}^N} \mu f(u)u\mathrm{d}x = 0.$$

By (f_3), it is easy to obtain $qF(t)\leq f(t)t$ when $t\geq 0$, furthermore, since $m_\mu(a)=J_\mu(u)<0$, we get

$$\begin{aligned}
0 &> J_\mu(u) - \frac{1}{q}\Big(\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u|^q\mathrm{d}x - \int_{\mathbb{R}^N} \lambda_a u^2\mathrm{d}x - \int_{\mathbb{R}^N} \mu f(u)u\mathrm{d}x\Big) \\
&= (\frac{1}{2}-\frac{1}{q})\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} \lambda_a u^2\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} \mu f(u)u\mathrm{d}x - \int_{\mathbb{R}^N} \mu F(u)\mathrm{d}x \\
&\geq \frac{1}{q}\int_{\mathbb{R}^N} \lambda_a u^2\mathrm{d}x,
\end{aligned}$$

which implies that $\lambda_a<0$.

Next, we will show that u is positive. From the definition of $J_\mu(u)$, we have $J_\mu(|u|)=J_\mu(u)$. Moreover we can get $|u|\in S(a)$. Then, we deduce

$$m_\mu(a) = J_\mu(u) = J_\mu(|u|) \geq m_\mu(a).$$

Then we have $J_\mu(|u|)=m_\mu(a)$. Therefore, we replace u by $|u|$. If u^* is the Schwarz's Symmetrization of u [22, Section 3.3], we have

$$\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x \geq \int_{\mathbb{R}^N} |\nabla u^*|^2\mathrm{d}x, \quad \int_{\mathbb{R}^N} |\nabla u|^q\mathrm{d}x \geq \int_{\mathbb{R}^N} |\nabla u^*|^q\mathrm{d}x$$

and
$$\int_{\mathbb{R}^N} F(u)\mathrm{d}x = \int_{\mathbb{R}^N} F(u^*)\mathrm{d}x.$$
It is easy to check that $u^* \in S(a)$ and $J_\mu(u^*) = m_\mu(a)$. Thus, we replace u by u^*.

Next, we prove $u(x)$ is positive for all $x \in \mathbb{R}^N$. Firstly, we assume that the conclusion is false, then there is $x_0 \in \mathbb{R}^N$ satisfying $u(x_0) = 0$. Furthermore, we can assume that there is $x_1 \in \mathbb{R}^N$ satisfying $u(x_1) > 0$ by $u \neq 0$. Thus, we can find a ball with a sufficiently large radius $R > 0$ such that $x_0, x_1 \in B_R(0)$. Then, combining with the Harnack Inequality ([10, Theorem 8.20]), we can infer there is a constant $C > 0$ such that
$$\sup_{y\in B_R(0)} u(y) \leq C \inf_{y\in B_R(0)} u(y),$$
which contradicts to the fact that
$$\sup_{y\in B_R(0)} u(y) \geq u(x_1) > 0 \quad \text{and} \quad \inf_{y\in B_R(0)} u(y) = u(x_0) = 0. \qquad \square$$

The next corollary is obtained by Lemma 2.6.

Corollary 2.7. *Fix $a > 0$ and let $0 \leq \mu_1 < \mu_2$. Then, $m_{\mu_2}(a) < m_{\mu_1}(a) < 0$.*

Proof. Let $u_{\mu_1} \in S(a)$ satisfy $J_{\mu_1}(u_{\mu_1}) = m_{\mu_1}(a)$, then
$$m_{\mu_2}(a) \leq J_{\mu_2}(u_{\mu_1}) < J_{\mu_1}(u_{\mu_1}) = m_{\mu_1}(a). \qquad \square$$

3 The nonautonomous case

Next, we will show some properties of $I_\epsilon : X \to \mathbb{R}$,
$$I_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x)F(u)\mathrm{d}x,$$
which is restricted to $S(a)$.

Firstly, we define $I_{\max}, I_\infty : X \to \mathbb{R}$ as
$$I_{\max}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\mathrm{d}x - \int_{\mathbb{R}^N} h_{\max}F(u)\mathrm{d}x$$
and
$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\mathrm{d}x - \int_{\mathbb{R}^N} h_\infty F(u)\mathrm{d}x.$$
Moreover, Lemma 2.2 guarantees that
$$m_\infty(a) = \inf_{u\in S(a)} I_\infty(u),\ m_\epsilon(a) = \inf_{u\in S(a)} I_\epsilon(u),\ m_{\max}(a) = \inf_{u\in S(a)} I_{\max}(u).$$
Then, according to Corollary 2.7 and $h_\infty < h_{\max}$, we can immediately get
$$m_{\max}(a) < m_\infty(a) < 0. \tag{3.1}$$

Now, we fix $0 < \rho_1 = \frac{1}{2}(m_\infty(a) - m_{\max}(a))$.

Lemma 3.1. $\lim_{\epsilon\to 0^+} m_\epsilon(a) \leq m_{\max}(a)$. *Hence, there exists $\epsilon_0 > 0$ such that $m_\epsilon(a) < m_\infty(a)$ for all $0 < \epsilon < \epsilon_0$.*

Proof. Let $u_0 \in S(a)$ satisfying $I_{\max}(u_0) = m_{\max}(a)$. A simple calculus gives that

$$m_\epsilon(a) \leq I_\epsilon(u_0) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_0|^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u_0|^q dx - \int_{\mathbb{R}^N} h(\epsilon x)F(u_0)dx.$$

Letting $\epsilon \to 0^+$ and applying (h_3) we can get

$$\limsup_{\epsilon\to 0^+} m_\epsilon(a) \leq \lim_{\epsilon\to 0^+} I_\epsilon(u_0) = I_{\max}(u_0) = m_{\max}(a).$$

According to (3.1), we obtain $m_\epsilon(a) < m_\infty(a)$ for ϵ small enough. □

The following two lemmas will be used to prove $(PS)_c$ condition for I_ϵ at some levels.

Lemma 3.2. *Assume that $(u_n) \subset S(a)$ is a minimizing sequence with $I_\epsilon(u_n) \to c$ and $c < m_{\max}(a) + \rho_1 < 0$. If $u_n \rightharpoonup u$ in X, then $u \neq 0$.*

Proof. Firstly, we assume the conclusion is false, i.e., $u \equiv 0$. Then, we have

$$c = m_\epsilon(a) = I_\epsilon(u_n) + o_n(1) = I_\infty(u_n) + \int_{\mathbb{R}^N} (h_\infty - h(\epsilon x))\, F(u_n)dx + o_n(1).$$

According to (h_2), there exist some constants $\xi, R > 0$ such that

$$h_\infty \geq h(x) - \xi, \quad |x| > R.$$

Thus, we have the following estimate

$$c = I_\epsilon(u_n) + o_n(1) \geq I_\infty(u_n) + \int_{B_{R/\epsilon}(0)} (h_\infty - h(\epsilon x))\, F(u_n)dx - \xi \int_{B^c_{R/\epsilon}(0)} F(u_n)dx + o_n(1).$$

Recalling that (u_n) is bounded in X, then for some constant $C > 0$, there holds

$$\int_{\mathbb{R}^N} F(u_n)dx \leq C_1 \left(\int_{\mathbb{R}^N} |\nabla u|^q dx\right)^{\frac{\nu_{p,q}p}{q}} + C_2 \left(\int_{\mathbb{R}^N} |\nabla u|^q dx\right)^{\frac{\nu_{p_1,q}p_1}{q}} \leq C.$$

By the fact of $u_n \to 0$ in $L^l(B_{R/\epsilon}(0))$ when $l \in [1, 2^*)$, one has

$$c = I_\epsilon(u_n) + o_n(1) \geq I_\infty(u_n) - \xi C > m_\infty(a) - \xi C + o_n(1),$$

which combines with the arbitrariness of $\xi > 0$, we can get

$$c \geq m_\infty(a),$$

which contradicts to the fact that $c < m_{\max}(a) + \rho_1 < m_\infty(a)$. So, we can get that $u \neq 0$. □

Lemma 3.3. *Assume that $(u_n) \subset S(a)$ is a $(PS)_c$ sequence of I_ϵ satisfying $u_n \rightharpoonup u_\epsilon$ in X when $c < m_{\max}(a) + \rho_1 < 0$, that is, as $n \to +\infty$,*

$$I_\epsilon(u_n) \to c \quad \text{and} \quad \|I_\epsilon|'_{S(a)}(u_n)\| \to 0.$$

Then there holds

$$\liminf_{n\to+\infty} |u_n - u_\epsilon|_2^2 \geq \beta,$$

where $u_n \nrightarrow u_\epsilon$ in X and $\beta > 0$ independent of $\epsilon \in (0, \epsilon_0)$.

Proof. Firstly, defining the functional $\Psi : X \to \mathbb{R}$ with $\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx$, we can see $S(a) = \Psi^{-1}(\{a^2/2\})$. According to [26, Proposition 5.12], there exist $(\lambda_n) \subset \mathbb{R}$ such that

$$\|I_\epsilon'(u_n) - \lambda_n \Psi'(u_n)\|_{X'} \to 0 \quad \text{as } n \to +\infty.$$

(u_n) is bounded in X since I_ϵ is bounded from below and coercive as J_μ, which ensures that (λ_n) is bounded, then there exists λ_ϵ such that $\lambda_n \to \lambda_\epsilon$ as $n \to +\infty$. Thus, we have

$$I_\epsilon'(u_\epsilon) - \lambda_\epsilon \Psi'(u_\epsilon) = 0 \quad \text{in } X',$$

and

$$\|I_\epsilon'(v_n) - \lambda_\epsilon \Psi'(v_n)\|_{X'} \to 0 \quad \text{as } n \to +\infty,$$

where $v_n := u_n - u_\epsilon$. According to (f_3), we can get $qF(t) \le f(t)t$ when $t \ge 0$. Then we have

$$0 > \rho_1 + m_{\max}(a) > c = \liminf_{n\to+\infty} I_\epsilon(u_n) = \liminf_{n\to+\infty}\left(I_\epsilon(u_n) - \frac{1}{q}\langle I_\epsilon'(u_n), u_n\rangle + \frac{1}{q}\lambda_n a^2\right) \ge \frac{1}{q}\lambda_\epsilon a^2,$$

which implies that

$$\limsup_{\epsilon\to 0} \lambda_\epsilon \le \frac{q(\rho_1 + m_{\max}(a))}{a^2} < 0.$$

Then, there is a constant λ^* satisfying $\lambda_\epsilon < \lambda^* < 0$, which is independent of ϵ. Therefore,

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} |\nabla v_n|^q dx - \lambda_\epsilon \int_{\mathbb{R}^N} |v_n|^2 dx = \int_{\mathbb{R}^N} h(\epsilon x) f(v_n) v_n dx + o_n(1),$$

and

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} |\nabla v_n|^q dx - \lambda^* \int_{\mathbb{R}^N} |v_n|^2 dx \le \int_{\mathbb{R}^N} h(\epsilon x) f(v_n) v_n dx + o_n(1).$$

According to (f_1), we get $f(t) < \varepsilon t, \forall \varepsilon > 0$ if t small enough, which combines with (f_2) to give

$$\int_{\mathbb{R}^N} f(v_n) v_n dx \le C_2 \int_{\mathbb{R}^N} |v_n|^{p_1} dx + \varepsilon \int_{\mathbb{R}^N} |v_n|^2 dx \le C_2 \int_{\mathbb{R}^N} |v_n|^{p_1} dx.$$

So, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{\mathbb{R}^N} |\nabla v_n|^q dx + C_0 \int_{\mathbb{R}^N} |v_n|^2 dx \\ \le h_{\max} \int_{\mathbb{R}^N} f(v_n) v_n dx \le C_2 h_{\max} \int_{\mathbb{R}^N} |v_n|^{p_1} dx + o_n(1)\end{aligned}$$

for some constant $C_0 > 0$ independent of $\epsilon \in (0, \epsilon_0)$. Since $v_n \nrightarrow 0$ in X, we can assume that $\liminf_{n\to+\infty} \|v_n\|_X > C > 0$. Thus, there holds

$$\liminf_{n\to+\infty} |v_n|_{p_1}^{p_1} \ge C_3 \tag{3.2}$$

for some constant $C_3 > 0$. By (2.2), we can deduce

$$C_3 \le \liminf_{n\to+\infty} |v_n|_{p_1}^{p_1} \le C(\liminf_{n\to+\infty} |v_n|_2)^{(1-\nu_{p_1,q})p_1} K^{\nu_{p_1,q} p_1}, \tag{3.3}$$

where $K > 0$ is independent of $\epsilon \in (0, \epsilon_0)$ with $\|v_n\| \le K$ for all $n \in \mathbb{N}$. Then, combining with (3.2), and (3.3), we achieve the proof. □

Next, we consider $0 < \rho < \min\{\frac{1}{2}, \frac{\beta}{a^2}\}(m_\infty(a) - m_{\max}(a))$.

Lemma 3.4. *Assume that $0 < \epsilon < \epsilon_0$ and $c < m_{\max}(a) + \rho$. Then, I_ϵ restricted to $S(a)$ satisfies the $(PS)_c$ condition.*

Proof. Firstly, we can get that (u_n) is bounded by Lemma 2.2, then let $(u_n) \subset S(a)$ be $(PS)_c$ sequence of I_ϵ with $u_n \rightharpoonup u_\epsilon$, where $u_\epsilon \neq 0$ by Lemma 3.2 and $c < m_{\max}(a) + \rho$. Set $v_n = u_n - u_\epsilon$. If $v_n \to 0$ in X, the proof is complete. If $v_n \nrightarrow 0$ in X and $|u_\epsilon|_2 = b$, by Lemma 3.3, we have

$$\liminf_{n\to+\infty} |v_n|_2^2 \geq \beta \tag{3.4}$$

for some $\beta > 0$ which is independent of $\epsilon \in (0, \epsilon_0)$.

Let $|v_n|_2 = d_n \to d \geq \beta^{\frac{1}{2}}$, we have $a^2 = b^2 + d^2$. From $d_n \in (0, a)$ for n large enough, we can deduce

$$c + o_n(1) = I_\epsilon(u_n) = I_\epsilon(v_n) + I_\epsilon(u_\epsilon) + o_n(1) \geq m_\infty(d_n) + m_{\max}(b) + o_n(1).$$

Applying Lemma 2.4(i) and inequality (2.5), letting $n \to +\infty$, we get

$$m_{\max}(a) + \rho > c \geq m_\infty(d) + m_{\max}(b) \geq \frac{d^2}{a^2} m_\infty(a) + \frac{b^2}{a^2} m_{\max}(a).$$

Then

$$\rho \geq \frac{d^2}{a^2}(m_\infty(a) - m_{\max}(a)) \geq \frac{\beta}{a^2}(m_\infty(a) - m_{\max}(a)),$$

which is contradicted to the fact of $\rho < \frac{\beta}{a^2}(m_\infty(a) - m_{\max}(a))$. Then, it holds $v_n \to 0$ in X, that is, $u_n \to u_\epsilon$ in X, which implies that $u_\epsilon \in S(a)$ and

$$-\Delta u_\epsilon - \Delta_q u_\epsilon = \lambda_\epsilon u_\epsilon + h(\epsilon x) f(u_\epsilon), \quad x \in \mathbb{R}^N.$$

□

4 Multiplicity result

In the following, we do some technical stuff. Let $\rho_0, r_0 > 0$, e_j be defined in (h_3), satisfying:

- $\overline{B_{\rho_0}(e_i)} \cap \overline{B_{\rho_0}(e_j)} = \emptyset$ for $i \neq j$ and $i, j \in \{1, \ldots, l\}$.
- $\bigcup_{i=1}^{l} B_{\rho_0}(e_i) \subset B_{r_0}(0)$.
- $K_{\frac{\rho_0}{2}} = \bigcup_{i=1}^{l} \overline{B_{\frac{\rho_0}{2}}(e_i)}$.

Set $\kappa : \mathbb{R}^N \to \mathbb{R}^N$ with

$$\kappa(x) := \begin{cases} x, & \text{if } |x| \leq r_0, \\ r_0 \frac{x}{|x|}, & \text{if } |x| > r_0. \end{cases}$$

Now we consider the function $G_\epsilon : X \backslash \{0\} \to \mathbb{R}^N$ with

$$G_\epsilon(u) := \frac{\int_{\mathbb{R}^N} \kappa(\epsilon x)|u|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x},$$

Then, we will get the existence of (PS) sequences of I_ϵ, which is restricted to $S(a)$ by the next two lemmas.

Lemma 4.1. *Decreasing ϵ_0 if necessary, there exists a positive constant $\delta_0 < \rho$ such that*

$$G_\epsilon(u) \in K_{\frac{\rho_0}{2}}, \quad \forall \epsilon \in (0, \epsilon_0),$$

where $u \in S(a)$ and $I_\epsilon(u) \leq m_{\max}(a) + \delta_0$.

Proof. We assume that the conclusion is false, so there exist $\delta_n \to 0$, $u_n \in S(a)$ and $\epsilon_n \to 0$ such that

$$I_{\epsilon_n}(u_n) \leq m_{\max}(a) + \delta_n$$

and

$$G_{\epsilon_n}(u_n) \notin K_{\frac{\rho_0}{2}}.$$

Firstly, we know

$$m_{\max}(a) \leq I_{\max}(u_n) \leq I_{\epsilon_n}(u_n) \leq m_{\max}(a) + \delta_n,$$

then,

$$I_{\max}(u_n) \to m_{\max}(a), \quad \text{as } n \to \infty.$$

We will analyze the following two cases by Proposition 2.5.

(i) $u_n \to u$ in X, where $u \in S(a)$. According to the Lebesgue dominated convergence theorem, we can deduce that

$$G_{\epsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \kappa(\epsilon_n x)|u_n|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x} \to \frac{\int_{\mathbb{R}^N} \kappa(0)|u|^2 \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x} = 0 \in K_{\frac{\rho_0}{2}},$$

which contradicts to $G_{\epsilon_n}(u_n) \notin K_{\frac{\rho_0}{2}}$ for n large.

(ii) There exists a sequence $v_n(\cdot) = u(\cdot + y_n)$ with $|y_n| \to +\infty$ and $(y_n) \subset \mathbb{R}^N$, which is convergent in X for some $v \in S(a)$. Then, we can also study the following two cases:

When $|\epsilon_n y_n| \to +\infty$, we can deduce that

$$I_{\epsilon_n}(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla v_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon_n x + \epsilon_n y_n) F(v_n) \mathrm{d}x \to I_\infty(v).$$

Since $I_{\epsilon_n}(u_n) \leq m_{\max}(a) + \delta_n$, there holds

$$m_{\max}(a) \geq I_\infty(v) \geq m_\infty(a),$$

which contradicts to (3.1).

When $\epsilon_n y_n \to y$ for some $y \in \mathbb{R}^N$, we get

$$I_{\epsilon_n}(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla v_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon_n x + \epsilon_n y_n) F(v_n) \mathrm{d}x \to I_{h(y)}(v),$$

then we obtain

$$m_{h(y)}(a) \leq m_{\max}(a). \tag{4.1}$$

If $h(y) < h_{\max}$, Corollary 2.7 implies that $m_{h(y)}(a) > m_{\max}(a)$, which contradicts to (4.1). Thus, it holds $h(y) = h_{\max}$, which means $y = e_i$ for some $i = 1, \ldots, l$. Then we have

$$G_{\epsilon_n}(u_n) = \frac{\displaystyle\int_{\mathbb{R}^N} \kappa(\epsilon_n x)|u_n|^2 \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x} = \frac{\displaystyle\int_{\mathbb{R}^N} \kappa(\epsilon_n x + \epsilon_n y_n)|v_n|^2 \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |v_n|^2 \mathrm{d}x} \to \frac{\displaystyle\int_{\mathbb{R}^N} \kappa(y)|v|^2 \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |v|^2 \mathrm{d}x} = e_i \in K_{\frac{\rho_0}{2}},$$

which contradicts to $G_{\epsilon_n}(u_n) \notin K_{\frac{\rho_0}{2}}$ for n large. □

Next, we introduce some notations:

- $\theta_\epsilon^i := \{u \in S(a); |G_\epsilon(u) - e_i| \le \rho_0\}$,
- $\partial\theta_\epsilon^i := \{u \in S(a); |G_\epsilon(u) - e_i| = \rho_0\}$,
- $\eta_\epsilon^i := \inf_{u\in\theta_\epsilon^i} I_\epsilon(u)$,
- $\tilde{\eta}_\epsilon^i := \inf_{u\in\partial\theta_\epsilon^i} I_\epsilon(u)$.

Lemma 4.2. *Let* $0 < \delta_0 < \rho < \min\{\frac{1}{2}, \frac{\beta}{a^2}\}(m_\infty(a) - m_{\max}(a))$. *Then, there holds*

$$\eta_\epsilon^i < m_{\max}(a) + \rho \quad and \quad \eta_\epsilon^i < \tilde{\eta}_\epsilon^i, \quad \forall \epsilon \in (0, \epsilon_0).$$

Proof. By Proposition (2.5), we set that

$$m_{\max}(a) = I_{\max}(u), \ I'_{\max}(u) = 0,$$

where $u \in S(a)$. Let $u_\epsilon^i : \mathbb{R}^N \to \mathbb{R}$ be $u_\epsilon^i(x) = u(x - e_i/\epsilon)$ for $1 \le i \le l$. By direct calculation, we get

$$I_\epsilon(u_\epsilon^i(x)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x + e_i)F(u)\mathrm{d}x,$$

which implies that

$$\limsup_{\epsilon\to 0} I_\epsilon(u_\epsilon^i(x)) \le I_{\max}(u) = m_{\max}(a). \tag{4.2}$$

If $\epsilon \to 0^+$, there holds

$$G_\epsilon(u_\epsilon^i) = \frac{\displaystyle\int_{\mathbb{R}^N} \kappa(\epsilon x)|u_\epsilon^i|^2 \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |u_\epsilon^i|^2 \mathrm{d}x} = \frac{\displaystyle\int_{\mathbb{R}^N} \kappa(\epsilon x + e_i)|u|^2 \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x} \to e_i.$$

Then we can infer that $u_\epsilon^i \in \theta_\epsilon^i$ when ϵ is small enough. Moreover, by (4.2),

$$I_\epsilon(u_\epsilon^i(x)) \le m_{\max}(a) + \frac{\delta_0}{4}, \quad \forall \epsilon \in (0, \epsilon_0).$$

From this, decreasing ϵ_0 if necessary,

$$\eta_\epsilon^i \le m_{\max}(a) + \frac{\delta_0}{4}, \quad \forall \epsilon \in (0, \epsilon_0).$$

Then,

$$\eta_\epsilon^i \le m_{\max}(a) + \rho, \quad \forall \epsilon \in (0, \epsilon_0),$$

showing the first inequality.

If there holds $u \in \partial\theta_\epsilon^i$, i.e.,

$$u \in S(a) \quad \text{and} \quad |G_\epsilon(u) - e_i| = \rho_0 > \frac{\rho_0}{2},$$

which implies $G_\epsilon(u) \notin K_{\frac{\rho_0}{2}}$. Then, combining with Lemma 4.1, we have

$$I_\epsilon(u) > m_{\max}(a) + \frac{\delta_0}{2}, \quad \forall u \in \partial\theta_\epsilon^i, \quad \forall \epsilon \in (0, \epsilon_0),$$

and so,

$$\tilde{\eta}_\epsilon^i \ge m_{\max}(a) + \frac{\delta_0}{2}, \quad \forall \epsilon \in (0, \epsilon_0),$$

from which it follows that

$$\eta_\epsilon^i < \tilde{\eta}_\epsilon^i, \quad \forall \epsilon \in (0, \epsilon_0).$$

□

4.1 Proof of Theorem 1.1

By Ekeland's variational principle, we can get that there exists a sequence $(u_n^i) \subset S(a)$ such that

$$I_\epsilon(u_n^i) \to \eta_\epsilon^i$$

and

$$I_\epsilon(v) - I_\epsilon(u_n^i) \geq -\frac{1}{n}\|v - u_n^i\|, \quad \forall v \in \theta_\epsilon^i \quad \text{with} \quad v \neq u_n^i$$

for each $i \in \{1, \ldots, l\}$. Then, we get $u_n^i \in \theta_\epsilon^i \setminus \partial\theta_\epsilon^i$ for n large enough by Lemma 4.2.

Given $v \in T_{u_n^i} S(a) = \{w \in X : \int_{\mathbb{R}^N} u_n^i w \mathrm{d}x = 0\}$, we can define the path $\sigma : (-\xi, \xi) \to S(a)$ with

$$\sigma(t) = a \frac{(u_n^i + tv)}{|u_n^i + tv|_2},$$

where $\xi > 0$. It is obvious to know that $\sigma \in C^1((-\xi, \xi), S(a))$ and we have

$$\sigma(t) \in \theta_\epsilon^i \backslash \partial\theta_\epsilon^i, \ \forall t \in (-\xi, \xi), \ \sigma(0) = u_n^i \text{ and } \sigma'(0) = v.$$

Then we get

$$I_\epsilon(\sigma(t)) - I_\epsilon(u_n^i) \geq -\frac{1}{n}\|\sigma(t) - u_n^i\|$$

for $t \in (-\xi, \xi)$, which implies that

$$\begin{aligned}
\frac{I_\epsilon(\sigma(t)) - I_\epsilon(\sigma(0)))}{t} &= \frac{I_\epsilon(\sigma(t)) - I_\epsilon(u_n^i)}{t} \\
&\geq -\frac{1}{n}\left\|\frac{\sigma(t) - u_n^i}{t}\right\| \\
&= -\frac{1}{n}\left\|\frac{\sigma(t) - \sigma(0)}{t}\right\|, \quad \forall t \in (0, \xi).
\end{aligned}$$

Taking the limit of $t \to 0^+$, we have

$$\langle I_\epsilon'(u_n^i), v\rangle \geq -\frac{1}{n}\|v\|.$$

Then, we can replace v by $-v$ to deduce

$$\sup\{|\langle I_\epsilon'(u_n^i), v\rangle| : \|v\| \leq 1\} \leq \frac{1}{n},$$

which implies that

$$I_\epsilon(u_n^i) \to \eta_\epsilon^i \quad \text{and} \quad \|I_\epsilon|'_{S(a)}(u_n^i)\| \to 0 \quad \text{as } n \to +\infty,$$

which means $(u_n^i) \subset S(a)$ is a $(PS)_{\eta_\epsilon^i}$ sequence of I_ϵ. Combining with Lemma 3.4 and $\eta_\epsilon^i < m_{\max}(a) + \rho$, we can infer that there is u^i such that $u_n^i \to u^i$ in X. So, we have

$$u^i \in \theta_\epsilon^i, \ I_\epsilon(u^i) = \eta_\epsilon^i \quad \text{and} \quad I_\epsilon|'_{S(a)}(u^i) = 0.$$

According to our assumptions, we have

$$G_\epsilon(u^i) \in \overline{B_{\rho_0}(e_i)}, \quad G_\epsilon(u^j) \in \overline{B_{\rho_0}(e_j)}$$

and

$$\overline{B_{\rho_0}(e_i)} \cap \overline{B_{\rho_0}(e_j)} = \emptyset \text{ for } i \neq j,$$

which means $u^i \neq u^j$ for $i \neq j$ while $1 \leq i, j \leq l$. Thus, for any $\epsilon \in (0, \epsilon_0)$, I_ϵ has at least l nontrivial critical points, i.e.,

$$-\Delta u^i - \Delta_q u^i = \lambda_i u^i + h(\epsilon x) f(u^i), \quad \forall i \in \{1, 2, \dots, l\},$$

which ensures

$$\int_{\mathbb{R}^N} |\nabla u^i|^2 \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u^i|^q \mathrm{d}x - \int_{\mathbb{R}^N} \lambda_i |u^i|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x) f(u^i) u^i \mathrm{d}x = 0.$$

Combining with $I_\epsilon(u^i) < 0$, we have

$$\begin{aligned} 0 &> I_\epsilon(u^i) - \frac{1}{q}\Big(\int_{\mathbb{R}^N} |\nabla u^i|^2 \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla u^i|^q \mathrm{d}x - \int_{\mathbb{R}^N} \lambda_i |u^i|^2 \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x) f(u^i) u^i \mathrm{d}x \Big) \\ &= \left(\frac{1}{2} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} \lambda_i |u^i|^2 \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} h(\epsilon x) f(u^i) u^i \mathrm{d}x - \int_{\mathbb{R}^N} h(\epsilon x) F(u^i) \mathrm{d}x \\ &\geq \frac{1}{q} \int_{\mathbb{R}^N} \lambda_i |u^i|^2 \mathrm{d}x, \end{aligned}$$

which implies $\lambda_i < 0$. This proves the desired result.

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Symmetric nonlinear solvable system of difference equations

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Abstract. We show the theoretical solvability of the system of difference equations

$$x_{n+k} = \frac{y_{n+l}y_n - cd}{y_{n+l} + y_n - c - d}, \quad y_{n+k} = \frac{x_{n+l}x_n - cd}{x_{n+l} + x_n - c - d}, \quad n \in \mathbb{N}_0,$$

where $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, $l < k$, $c, d \in \mathbb{C}$ and $x_j, y_j \in \mathbb{C}$, $j = \overline{0, k-1}$. For several special cases of the system, we give some detailed explanations on how some formulas for their general solutions can be found in closed form, that is, we show their practical solvability. To do this, among other things, we use the theory of homogeneous linear difference equations with constant coefficients and the product-type difference equations with integer exponents, which are theoretically solvable.

Keywords: symmetric system of difference equations, solvable system, solution in closed form.

2020 Mathematics Subject Classification: 39A20, 39A45.

1 Introduction

Finding general solutions to difference equations and systems of difference equations is a classical problem which can be traced back to the beginning of the 18th century, [5,8,9]. During the century many important results on the problem have been obtained [10, 16, 18–20]. A majority of the results were on linear difference equations and systems of difference equations, but some of them were also on the nonlinear ones (see also [6, 12, 17, 21, 22]). For some later presentations and applications of the equations, see [11, 13, 23, 26, 37]. Since the solvability

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theory for linear difference equations was essentially completed during that time, and since it is practically impossible to find some general methods for solving nonlinear equations, interest in the topic diminished during the 19th century. Although from time to time, some solvable difference equations occurred, for example, in computational mathematics [7], problem books [2, 14, 24, 25] and in some popular journals for a wide audience. Solvable difference equations are also useful in some comparison results [4, 15, 40]. One can study the invariants of the equations and systems, as it was the case, e.g., in [28–30, 32, 38, 39], but only some of their very special classes were considered therein.

During the last two decades there has been a renewed interest in the area. It seems mostly because of the use of some computer calculations. We have analysed some of the recent papers and given many comments and theoretical explanations related to them (see, for example, [47] where some of the analyses, comments and explanations are given). An interesting fact is that the solvability of almost all of the recently presented classes of solvable difference equations and systems rely on the solvability of some linear ones (see, for example, [3, 35, 47, 49] and the related references therein). However, it is of some interest to enlarge the list of solvable nonlinear difference equations and systems which are not obtained from linear ones in an obvious way.

There has been some interest in systems of difference equations which are close to symmetric ones since the mid of the nineties [27, 31, 33, 34, 38, 39], which attracted our attention. We have devoted a part of our investigation also in this direction (see, e.g., [41–47]).

Motivated by the equation

$$x_n = \frac{x_{n-s}x_{n-t} + a}{x_{n-s} + x_{n-t}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $s, t \in \mathbb{N}$, $a \in \mathbb{C}$ and $x_{-j} \in \mathbb{C}$, $j = \overline{0, \max\{s, t\}}$ ([36, 48]), in [49] we studied the equation

$$x_{n+s} = \frac{x_{n+t}x_n - ab}{x_{n+t} + x_n - a - b}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $s \in \mathbb{N}$, $t \in \mathbb{N}_0$, $t < s$, $a, b \in \mathbb{C}$ and $x_j \in \mathbb{C}$, $j = \overline{0, s-1}$, and showed its theoretical solvability. Equation (1.1) is a natural generalization of its special case with $s = 1$ and $t = 2$, which can be obtained by using the secant method [7]

$$x_n = \frac{x_{n-2}f(x_{n-1}) - x_{n-1}f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}, \quad n \in \mathbb{N}_0,$$

for $f(x) = x^2 - a$ (see, e.g., [15]).

On the other hand, motivated by our studies of the systems which stem from equation (1.1) (see the nonlinear systems of difference equations in [41–45]), in [46] we investigated a nonlinear system of difference equations which is related to equation (1.2), showed its solvability and discussed some special cases of the system in detail.

Here, we continue above mentioned investigations on solvability by studying the system

$$x_{n+k} = \frac{y_{n+l}y_n - cd}{y_{n+l} + y_n - c - d}, \quad y_{n+k} = \frac{x_{n+l}x_n - cd}{x_{n+l} + x_n - c - d}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, $l < k$, $c, d \in \mathbb{C}$ and $x_j, y_j \in \mathbb{C}$, $j = \overline{0, k-1}$, which is a symmetric relative to equation (1.2) and has not been considered in the literature yet.

Definition 1.1. We say that a nonlinear difference equation or system is *theoretically solvable* if by some changes of variables it can be transformed to a linear difference equation or system with constant coefficients. If the general solution to the linear difference equation or system can be found in closed form, we say that the nonlinear difference equation or system is *practically solvable*.

Remark 1.2. Not all linear difference equations with constant coefficients are practically solvable. For example, the difference equation

$$x_{n+5} - 6x_{n+1} + 3x_n = 0, \quad n \in \mathbb{N}_0,$$

is one of them, since we cannot find the roots of the associated characteristic polynomial

$$q_5(\lambda) = \lambda^5 - 6\lambda + 3$$

by radicals (see, e.g., [50]), due to the famous result by Abel and Ruffini [1].

Here we show the theoretical solvability of system (1.3), and give a detailed explanation on how in some cases can be found the general solution, that is, how to show their practical solvability.

2 Main results

Here we state and prove our main results.

Theorem 2.1. *Suppose $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, $l < k$, and $c, d \in \mathbb{C}$. Then, system (1.3) is theoretically solvable.*

Proof. Suppose $c \neq d$. Note that

$$x_{n+k} - d = \frac{(y_{n+l} - d)(y_n - d)}{y_{n+l} + y_n - c - d},$$
$$x_{n+k} - c = \frac{(y_{n+l} - c)(y_n - c)}{y_{n+l} + y_n - c - d},$$
$$y_{n+k} - d = \frac{(x_{n+l} - d)(x_n - d)}{x_{n+l} + x_n - c - d},$$
$$y_{n+k} - c = \frac{(x_{n+l} - c)(x_n - c)}{x_{n+l} + x_n - c - d},$$

for $n \in \mathbb{N}_0$.

Dividing the first two relations we get

$$\frac{x_{n+k} - d}{x_{n+k} - c} = \frac{(y_{n+l} - d)(y_n - d)}{(y_{n+l} - c)(y_n - c)},$$

for $n \in \mathbb{N}_0$, whereas dividing the last two relations we get

$$\frac{y_{n+k} - d}{y_{n+k} - c} = \frac{(x_{n+l} - d)(x_n - d)}{(x_{n+l} - c)(x_n - c)},$$

for $n \in \mathbb{N}_0$.

Now we define two auxiliary sequences as follows

$$\zeta_n = \frac{x_n - d}{x_n - c}, \quad \eta_n = \frac{y_n - d}{y_n - c}, \tag{2.1}$$

for $n \in \mathbb{N}_0$.

They obviously satisfy the relations

$$\zeta_{n+k} = \eta_{n+l}\eta_n, \quad \eta_{n+k} = \zeta_{n+l}\zeta_n, \tag{2.2}$$

for $n \in \mathbb{N}_0$, which yields that ζ_n and η_n are two solutions to the equation

$$\omega_{n+2k} = \omega_{n+2l}\omega_{n+l}^2\omega_n, \quad n \in \mathbb{N}_0, \tag{2.3}$$

a product-type difference equation with integer exponents, which is theoretically solvable. Hence, such one is the system (1.3).

Note also that from (2.1) we have

$$x_n = \frac{c\zeta_n - d}{\zeta_n - 1}, \quad y_n = \frac{c\eta_n - d}{\eta_n - 1}, \tag{2.4}$$

for $n \in \mathbb{N}_0$.

Now suppose $c = d$. Note that

$$x_{n+k} - c = \frac{(y_{n+l} - c)(y_n - c)}{y_{n+l} + y_n - 2c}, \tag{2.5}$$

$$y_{n+k} - c = \frac{(x_{n+l} - c)(x_n - c)}{x_{n+l} + x_n - 2c}, \tag{2.6}$$

for $n \in \mathbb{N}_0$.

Now we define the two auxiliary sequences

$$\zeta_n = \frac{1}{x_n - c}, \quad \eta_n = \frac{1}{y_n - c}, \tag{2.7}$$

for $n \in \mathbb{N}_0$.

Combining (2.5)–(2.7), we get

$$\zeta_{n+k} = \eta_{n+l} + \eta_n, \quad \eta_{n+k} = \zeta_{n+l} + \zeta_n, \tag{2.8}$$

for $n \in \mathbb{N}_0$, which implies that ζ_n and η_n satisfy the equation

$$\omega_{n+2k} - \omega_{n+2l} - 2\omega_{n+l} - \omega_n = 0, \tag{2.9}$$

for $n \in \mathbb{N}_0$, which according to Definition 1.1 means that system (1.3) is theoretically solvable in this case. □

A natural problem is to find special cases of system (1.3) for which it is possible to find some closed-form formulas for their general solutions.

The polynomial

$$q_{2k}(\lambda) = \lambda^{2k} - \lambda^{2l} - 2\lambda^l - 1,$$

is the characteristic one associated to equation (2.9) [11,13,23,25]. Note that

$$q_{2k}(\lambda) = \lambda^{2k} - (\lambda^l + 1)^2 = (\lambda^k - \lambda^l - 1)(\lambda^k + \lambda^l + 1). \tag{2.10}$$

In some cases it is possible to find its roots (but, of course, not always [1]), for instance, if $0 \le l < k \le 4$ (there are ten cases). In all these cases, among other ones, equation (2.9) is practically solvable. Now we present the general solution to system (1.3) in some of these cases.

Theorem 2.2. *Consider the system* (1.3) *with* $k = 1$, $l = 0$ *and* $c, d \in \mathbb{C}$.

(a) If $c = d$, $x_0 \ne c \ne y_0$, *then*

$$x_{2m} = c + \frac{x_0 - c}{4^m}, \tag{2.11}$$

$$x_{2m+1} = c + \frac{y_0 - c}{2 \cdot 4^m}, \tag{2.12}$$

$$y_{2m} = c + \frac{y_0 - c}{4^m}, \tag{2.13}$$

$$y_{2m+1} = c + \frac{x_0 - c}{2 \cdot 4^m}, \tag{2.14}$$

for $m \in \mathbb{N}_0$.

(b) If $c \ne d$, *then well-defined solutions to the system are given by*

$$x_{2m} = \frac{c\left(\frac{x_0-d}{x_0-c}\right)^{4^m} - d}{\left(\frac{x_0-d}{x_0-c}\right)^{4^m} - 1}, \tag{2.15}$$

$$x_{2m+1} = \frac{c\left(\frac{y_0-d}{y_0-c}\right)^{2\cdot 4^m} - d}{\left(\frac{y_0-d}{y_0-c}\right)^{2\cdot 4^m} - 1}, \tag{2.16}$$

$$y_{2m} = \frac{c\left(\frac{y_0-d}{y_0-c}\right)^{4^m} - d}{\left(\frac{y_0-d}{y_0-c}\right)^{4^m} - 1}, \tag{2.17}$$

$$y_{2m+1} = \frac{c\left(\frac{x_0-d}{x_0-c}\right)^{2\cdot 4^m} - d}{\left(\frac{x_0-d}{x_0-c}\right)^{2\cdot 4^m} - 1}, \tag{2.18}$$

for $m \in \mathbb{N}_0$.

Proof. (a) First, note that (2.8) is

$$\zeta_{n+1} = 2\eta_n, \quad \eta_{n+1} = 2\zeta_n,$$

for $n \in \mathbb{N}_0$. Thus

$$\zeta_{n+2} = 4\zeta_n, \quad \eta_{n+2} = 4\eta_n,$$

for $n \in \mathbb{N}_0$, which yields

$$\zeta_{2m} = 4^m\zeta_0, \quad \zeta_{2m+1} = 4^m\zeta_1, \quad \eta_{2m} = 4^m\eta_0, \quad \eta_{2m+1} = 4^m\eta_1,$$

for $m \in \mathbb{N}_0$. This and (2.7) imply (2.11)–(2.14), under the assumption $c = d$.

(b) If we assume that $c \ne d$, then from (2.3) we have

$$\zeta_{n+2} = \zeta_n^4, \quad \eta_{n+2} = \eta_n^4, \quad n \in \mathbb{N}_0.$$

Therefore

$$\zeta_{2m} = \zeta_0^{4^m}, \quad \zeta_{2m+1} = \zeta_1^{4^m}, \quad \eta_{2m} = \eta_0^{4^m}, \quad \eta_{2m+1} = \eta_1^{4^m},$$

for $m \in \mathbb{N}_0$.

These four relations, the transformation in (2.1), and (2.4), imply (2.15)–(2.18), completing the proof. □

Remark 2.3. Assume that in Theorem 2.2, $c = d$, and that $x_0 = c$ or $y_0 = c$. Note that from (1.3) we have

$$x_1 = \frac{y_0^2 - c^2}{2(y_0 - c)} \tag{2.19}$$

and

$$y_1 = \frac{x_0^2 - c^2}{2(x_0 - c)}. \tag{2.20}$$

Hence, if $x_0 = c$, then from (2.20) we see that y_1 is not defined, whereas if $y_0 = c$, then from (2.19) we see that x_1 is not defined.

Corollary 2.4. *The system* (1.3) *with* $c, d \in \mathbb{C}$, $l = 0$ *and* $k \in \mathbb{N} \setminus \{1\}$ *is practically solvable.*

Proof. Under these conditions, we have

$$x_{n+k} = \frac{y_n^2 - cd}{2y_n - c - d}, \quad y_{n+k} = \frac{x_n^2 - cd}{2x_n - c - d}, \quad n \in \mathbb{N}_0,$$

which is a system with interlacing indices ([47]).

Let

$$x_m^{(j)} = x_{mk+j}, \quad y_m^{(j)} = y_{mk+j},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, k-1}$.

Then, $(x_m^{(j)}, y_m^{(j)})_{m \in \mathbb{N}_0}$, $j = \overline{0, k-1}$, are k solutions to the system

$$x_{m+1} = \frac{y_m^2 - cd}{2y_m - c - d}, \quad y_{m+1} = \frac{x_m^2 - cd}{2x_m - c - d}, \quad m \in \mathbb{N}_0.$$

Note that it is the system (1.3) with $k = 1$ and $l = 0$.

Thus, if $c = d$, $x_0^{(j)} \neq c \neq y_0^{(j)}$, $j = \overline{0, k-1}$, by Theorem 2.2 we get

$$x_{2m}^{(j)} = c + \frac{x_0^{(j)} - c}{4^m},$$

$$x_{2m+1}^{(j)} = c + \frac{y_0^{(j)} - c}{2 \cdot 4^m},$$

$$y_{2m}^{(j)} = c + \frac{y_0^{(j)} - c}{4^m},$$

$$y_{2m+1}^{(j)} = c + \frac{x_0^{(j)} - c}{2 \cdot 4^m},$$

for $m \in \mathbb{N}_0$, $j = \overline{0, k-1}$, whereas if $c \neq d$, then well-defined solutions to the system are

$$x_{2m}^{(j)} = \frac{c\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{4^m} - d}{\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{4^m} - 1},$$

$$x_{2m+1}^{(j)} = \frac{c\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{2\cdot 4^m} - d}{\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{2\cdot 4^m} - 1},$$

$$y_{2m}^{(j)} = \frac{c\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{4^m} - d}{\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{4^m} - 1},$$

$$y_{2m+1}^{(j)} = \frac{c\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{2\cdot 4^m} - d}{\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{2\cdot 4^m} - 1}$$

for $m \in \mathbb{N}$, $j = \overline{0, k-1}$.

Hence, if $c = d$ we have

$$x_{2mk+j} = c + \frac{x_j - c}{4^m},$$

$$x_{(2m+1)k+j} = c + \frac{y_j - c}{2 \cdot 4^m},$$

$$y_{2mk+j} = c + \frac{y_j - c}{4^m},$$

$$y_{(2m+1)k+j} = c + \frac{x_j - c}{2 \cdot 4^m},$$

for $m \in \mathbb{N}_0$, $j = \overline{0, k-1}$, whereas if $c \neq d$ we have

$$x_{2mk+j} = \frac{c\left(\frac{x_j-d}{x_j-c}\right)^{4^m} - d}{\left(\frac{x_j-d}{x_j-c}\right)^{4^m} - 1},$$

$$x_{(2m+1)k+j} = \frac{c\left(\frac{y_j-d}{y_j-c}\right)^{2\cdot 4^m} - d}{\left(\frac{y_j-d}{y_j-c}\right)^{2\cdot 4^m} - 1},$$

$$y_{2mk+j} = \frac{c\left(\frac{y_j-d}{y_j-c}\right)^{4^m} - d}{\left(\frac{y_j-d}{y_j-c}\right)^{4^m} - 1},$$

$$y_{(2m+1)k+j} = \frac{c\left(\frac{x_j-d}{x_j-c}\right)^{2\cdot 4^m} - d}{\left(\frac{x_j-d}{x_j-c}\right)^{2\cdot 4^m} - 1}$$

for $m \in \mathbb{N}$, $j = \overline{0, k-1}$. □

Theorem 2.5. *The system (1.3) with $c, d \in \mathbb{C}$, $l = 1$ and $k = 2$ is practically solvable.*

Proof. Suppose $c \neq d$. From (2.3) we get that $(\zeta_n)_{n\in\mathbb{N}_0}$ and $(\eta_n)_{n\in\mathbb{N}_0}$ are the solutions to

$$\omega_{n+4} = \omega_{n+2}\omega_{n+1}^2\omega_n, \quad n \in \mathbb{N}_0, \tag{2.21}$$

with the initial values

$$\zeta_0, \quad \zeta_1, \quad \zeta_2 = \eta_1\eta_0, \quad \zeta_3 = \zeta_1\zeta_0\eta_1, \tag{2.22}$$

$$\eta_0, \quad \eta_1, \quad \eta_2 = \zeta_1\zeta_0, \quad \eta_3 = \eta_1\eta_0\zeta_1, \tag{2.23}$$

respectively.

Rewrite (2.21) as follows

$$\omega_n = \omega_{n-2}^{a_1}\omega_{n-3}^{b_1}\omega_{n-4}^{c_1}\omega_{n-5}^{d_1}, \quad n \geq 5, \tag{2.24}$$

where

$$a_1 := 1, \quad b_1 := 2, \quad c_1 := 1, \quad d_1 := 0. \tag{2.25}$$

Further, we have

$$\begin{aligned}\omega_n &= (\omega_{n-4}\omega_{n-5}^2\omega_{n-6})^{a_1}\omega_{n-3}^{b_1}\omega_{n-4}^{c_1}\omega_{n-5}^{d_1} \\ &= \omega_{n-3}^{b_1}\omega_{n-4}^{a_1+c_1}\omega_{n-5}^{2a_1+d_1}\omega_{n-6}^{a_1} \\ &= \omega_{n-3}^{a_2}\omega_{n-4}^{b_2}\omega_{n-5}^{c_2}\omega_{n-6}^{d_2},\end{aligned} \tag{2.26}$$

for $n \geq 6$, where $a_2 := b_1$, $b_2 := a_1 + c_1$, $c_2 := 2a_1 + d_1$ and $d_2 := a_1$.

A simple inductive argument shows that

$$\omega_n = \omega_{n-k-1}^{a_k}\omega_{n-k-2}^{b_k}\omega_{n-k-3}^{c_k}\omega_{n-k-4}^{d_k}, \tag{2.27}$$

for $n \geq k + 4$, and

$$a_k = b_{k-1}, \quad b_k = a_{k-1} + c_{k-1}, \quad c_k = 2a_{k-1} + d_{k-1}, \quad d_k = a_{k-1} \tag{2.28}$$

for $k \geq 2$.

For $k = n - 4$ from (2.27) and (2.28), we get

$$\begin{aligned}\omega_n &= \omega_3^{a_{n-4}}\omega_2^{b_{n-4}}\omega_1^{c_{n-4}}\omega_0^{d_{n-4}} \\ &= \omega_3^{a_{n-4}}\omega_2^{a_{n-3}}\omega_1^{a_{n-2}-a_{n-4}}\omega_0^{a_{n-5}},\end{aligned} \tag{2.29}$$

for $n \geq 6$, whereas from (2.28), we get

$$a_k = a_{k-2} + 2a_{k-3} + a_{k-4}, \tag{2.30}$$

for $k \geq 5$, and

$$a_1 = 1, \quad a_2 = 2, \quad a_3 = 2, \quad a_4 = 4. \tag{2.31}$$

The polynomial

$$q_4(\lambda) = \lambda^4 - \lambda^2 - 2\lambda - 1 = (\lambda^2 - \lambda - 1)(\lambda^2 + \lambda + 1), \tag{2.32}$$

is the characteristic one associated to (2.30), with the zeros

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2} \quad \text{and} \quad \lambda_{3,4} = \frac{-1 \pm i\sqrt{3}}{2}. \tag{2.33}$$

Since

$$a_{k-4} = a_k - a_{k-2} - 2a_{k-3}, \tag{2.34}$$

by using (2.31) we can find a_k for $k \le 0$, and get

$$a_{-4} = a_{-3} = a_{-2} = 0, \quad a_{-1} = 1 \quad \text{and} \quad a_0 = 0. \tag{2.35}$$

Using [41, Lemma 1] we obtain

$$a_n = \frac{\lambda_1^{n+4}}{q_4'(\lambda_1)} + \frac{\lambda_2^{n+4}}{q_4'(\lambda_2)} + \frac{\lambda_3^{n+4}}{q_4'(\lambda_3)} + \frac{\lambda_4^{n+4}}{q_4'(\lambda_4)}, \tag{2.36}$$

for $n \in \mathbb{Z}$.

Since

$$q_4'(\lambda) = 4\lambda^3 - 2\lambda - 2 = 2(2\lambda^3 - \lambda - 1) = 2(\lambda - 1)(2\lambda^2 + 2\lambda + 1),$$

we have

$$q_4'(\lambda_1) = 5 + 3\sqrt{5}, \quad q_4'(\lambda_2) = 5 - 3\sqrt{5}, \tag{2.37}$$

$$q_4'(\lambda_3) = 3 - i\sqrt{3}, \quad q_4'(\lambda_4) = 3 + i\sqrt{3}. \tag{2.38}$$

Using (2.37) and (2.38) in (2.36) imply

$$\begin{aligned} a_n &= \frac{\lambda_1^{n+4}}{5 + 3\sqrt{5}} + \frac{\lambda_2^{n+4}}{5 - 3\sqrt{5}} + \frac{\lambda_3^{n+4}}{3 - i\sqrt{3}} + \frac{\lambda_4^{n+4}}{3 + i\sqrt{3}} \\ &= \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{2\sqrt{5}} + \frac{\lambda_3^{n+2} - \lambda_4^{n+2}}{2i\sqrt{3}}, \end{aligned} \tag{2.39}$$

for $n \in \mathbb{Z}$. Employing this formula it is not difficult to check that (2.29) holds for all $n \in \mathbb{N}_0$.

Relations (2.22) and (2.29) imply

$$\begin{aligned} \zeta_n &= \zeta_3^{a_{n-4}} \zeta_2^{a_{n-3}} \zeta_1^{a_{n-2} - a_{n-4}} \zeta_0^{a_{n-5}} \\ &= (\eta_1 \zeta_0 \zeta_1)^{a_{n-4}} (\eta_1 \eta_0)^{a_{n-3}} \zeta_1^{a_{n-2} - a_{n-4}} \zeta_0^{a_{n-5}} \\ &= \zeta_0^{a_{n-4} + a_{n-5}} \zeta_1^{a_{n-2}} \eta_0^{a_{n-3}} \eta_1^{a_{n-3} + a_{n-4}}, \end{aligned} \tag{2.40}$$

for $n \in \mathbb{N}_0$, and due to the symmetry

$$\eta_n = \eta_0^{a_{n-4} + a_{n-5}} \eta_1^{a_{n-2}} \zeta_0^{a_{n-3}} \zeta_1^{a_{n-3} + a_{n-4}}, \tag{2.41}$$

for $n \in \mathbb{N}_0$.

By some simple calculation and use of the Viète formulas we get

$$\begin{aligned} a_n + a_{n-1} &= \frac{(\lambda_1 + 1)\lambda_1^{n+1} - (\lambda_2 + 1)\lambda_2^{n+1}}{2\sqrt{5}} + \frac{(\lambda_3 + 1)\lambda_3^{n+1} - (\lambda_4 + 1)\lambda_4^{n+1}}{2i\sqrt{3}} \\ &= \frac{\lambda_1^{n+3} - \lambda_2^{n+3}}{2\sqrt{5}} - \frac{\lambda_3^{n+3} - \lambda_4^{n+3}}{2i\sqrt{3}}, \end{aligned} \tag{2.42}$$

for $n \in \mathbb{Z}$.

From (2.39)–(2.42) we get

$$\zeta_n = \zeta_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \zeta_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \eta_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \eta_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}},$$
$$\eta_n = \eta_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \eta_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \zeta_0^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \zeta_1^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}},$$

for $n \in \mathbb{N}_0$, from which together with (2.1) with $n = 0, 1$, we get

$$\begin{aligned} \zeta_n = {} & \left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \\ & \times \left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}, \end{aligned} \tag{2.43}$$

$$\begin{aligned} \eta_n = {} & \left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \\ & \times \left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}, \end{aligned} \tag{2.44}$$

for $n \in \mathbb{N}_0$.

Combining (2.4), (2.43) and (2.44), we have

$$x_n = \frac{c\left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} - d}{\left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} - 1}$$
$$y_n = \frac{c\left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} - d}{\left(\frac{y_0-d}{y_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{y_1-d}{y_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} \left(\frac{x_0-d}{x_0-c}\right)^{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}} \left(\frac{x_1-d}{x_1-c}\right)^{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}} - 1},$$

for $n \in \mathbb{N}_0$.

Now assume that $c = d$. In this case, we have

$$\zeta_{n+2} = \eta_{n+1} + \eta_n, \quad \eta_{n+2} = \zeta_{n+1} + \zeta_n, \tag{2.45}$$

for $n \in \mathbb{N}_0$, implying that $(\zeta_n)_{n\in\mathbb{N}_0}$ and $(\eta_n)_{n\in\mathbb{N}_0}$ are the two solutions to the equation

$$\omega_{n+4} - \omega_{n+2} - 2\omega_{n+1} - \omega_n = 0, \tag{2.46}$$

for $n \in \mathbb{N}_0$, with the initial values

$$\zeta_0, \quad \zeta_1, \quad \zeta_2 = \eta_1 + \eta_0, \quad \zeta_3 = \eta_1 + \zeta_1 + \zeta_0, \tag{2.47}$$

$$\eta_0, \quad \eta_1, \quad \eta_2 = \zeta_1 + \zeta_0, \quad \eta_3 = \zeta_1 + \eta_1 + \eta_0, \tag{2.48}$$

respectively (see (2.45)).

If we write equation (2.46) in the form

$$\begin{aligned} \omega_n &= \omega_{n-2} + 2\omega_{n-3} + \omega_{n-4} + 0 \cdot \omega_{n-5} \\ &= a_1\omega_{n-2} + b_1\omega_{n-3} + c_1\omega_{n-4} + d_1\omega_{n-5}, \end{aligned}$$

where a_1, b_1, c_1, d_1 are given in (2.25), then by a simple inductive argument we can prove that

$$\omega_n = a_k\omega_{n-k-1} + b_k\omega_{n-k-2} + c_k\omega_{n-k-3} + d_k\omega_{n-k-4}, \tag{2.49}$$

for $n \geq k+4$, where $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$, satisfy (2.28). Thus (2.39) holds.

For $k = n-4$ we have

$$\omega_n = a_{n-4}\omega_3 + a_{n-3}\omega_2 + (a_{n-2} - a_{n-4})\omega_1 + a_{n-5}\omega_0, \tag{2.50}$$

for $n \geq 6$, from which along with (2.47), we get

$$\begin{aligned}\zeta_n &= a_{n-4}(\eta_1 + \zeta_1 + \zeta_0) + a_{n-3}(\eta_1 + \eta_0) + (a_{n-2} - a_{n-4})\zeta_1 + a_{n-5}\zeta_0 \\ &= (a_{n-4} + a_{n-5})\zeta_0 + a_{n-2}\zeta_1 + a_{n-3}\eta_0 + (a_{n-3} + a_{n-4})\eta_1,\end{aligned} \tag{2.51}$$

for $n \in \mathbb{N}_0$. Therefore

$$\eta_n = (a_{n-4} + a_{n-5})\eta_0 + a_{n-2}\eta_1 + a_{n-3}\zeta_0 + (a_{n-3} + a_{n-4})\zeta_1, \tag{2.52}$$

for $n \in \mathbb{N}_0$.

Combining (2.39), (2.42), (2.51) and (2.52) it follows that

$$\zeta_n = \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c},$$

$$\eta_n = \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c},$$

for $n \in \mathbb{N}_0$.

Thus

$$x_n = \frac{c\left(\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c}\right) + 1}{\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c}},$$

$$y_n = \frac{c\left(\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c}\right) + 1}{\frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} - \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{y_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} + \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{y_1 - c} + \frac{\frac{\lambda_1^{n-1}-\lambda_2^{n-1}}{2\sqrt{5}} + \frac{\lambda_3^{n-1}-\lambda_4^{n-1}}{2i\sqrt{3}}}{x_0 - c} + \frac{\frac{\lambda_1^{n}-\lambda_2^{n}}{2\sqrt{5}} - \frac{\lambda_3^{n}-\lambda_4^{n}}{2i\sqrt{3}}}{x_1 - c}},$$

for $n \in \mathbb{N}_0$, where we have used the change of variables (2.7). □

Corollary 2.6. *The system* (1.3) *with $c, d \in \mathbb{C}$, $k = 2s$, $l = s$, for some $s \in \mathbb{N}$, is practically solvable.*

Proof. Under these conditions we have

$$x_{n+2s} = \frac{y_{n+s}y_n - cd}{y_{n+s} + y_n - c - d}, \quad y_{n+2s} = \frac{x_{n+s}x_n - cd}{x_{n+s} + x_n - c - d}, \quad n \in \mathbb{N}_0,$$

which is a system with interlacing indices ([47]).

Let

$$x_m^{(j)} = x_{ms+j}, \quad y_m^{(j)} = y_{ms+j},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$. Then, $(x_m^{(j)}, y_m^{(j)})_{m\in\mathbb{N}_0}$, $j = \overline{0, s-1}$, are s solutions to the system

$$x_{m+2} = \frac{y_{m+1}y_m - cd}{y_{m+1} + y_m - c - d},$$

$$y_{m+2} = \frac{x_{m+1}x_m - cd}{x_{m+1} + x_m - c - d},$$

for $m \in \mathbb{N}_0$. Note that it is the system (1.3) with $k = 2$ and $l = 1$.

Thus, Theorem 2.5 can be applied, and if $c \neq d$, we get

$$x_m^{(j)} = \frac{c\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{x_1^{(j)}-d}{x_1^{(j)}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{y_1^{(j)}-d}{y_1^{(j)}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-d}{\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{x_1^{(j)}-d}{x_1^{(j)}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{y_1^{(j)}-d}{y_1^{(j)}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-1},$$

$$y_m^{(j)} = \frac{c\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{y_1^{(j)}-d}{y_1^{(j)}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{x_1^{(j)}-d}{x_1^{(j)}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-d}{\left(\frac{y_0^{(j)}-d}{y_0^{(j)}-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{y_1^{(j)}-d}{y_1^{(j)}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\left(\frac{x_0^{(j)}-d}{x_0^{(j)}-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{x_1^{(j)}-d}{x_1^{(j)}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-1},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$, whereas if $c = d$, we get

$$x_m^{(j)} = \frac{c\left(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-c}+\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-c}\right)+1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-c}+\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-c}},$$

$$y_m^{(j)} = \frac{c\left(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-c}+\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-c}\right)+1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_1^{(j)}-c}+\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_0^{(j)}-c}+\frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_1^{(j)}-c}},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$, that is, if $c \neq d$, we have

$$x_{ms+j} = \frac{c\left(\frac{x_j-d}{x_j-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{x_{s+j}-d}{x_{s+j}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\left(\frac{y_j-d}{y_j-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{y_{s+j}-d}{y_{s+j}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-d}{\left(\frac{x_j-d}{x_j-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{x_{s+j}-d}{x_{s+j}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\left(\frac{y_j-d}{y_j-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{y_{s+j}-d}{y_{s+j}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-1},$$

$$y_{ms+j} = \frac{c\left(\frac{y_j-d}{y_j-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{y_{s+j}-d}{y_{s+j}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\left(\frac{x_j-d}{x_j-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{x_{s+j}-d}{x_{s+j}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-d}{\left(\frac{y_j-d}{y_j-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}-\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{y_{s+j}-d}{y_{s+j}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}+\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}\left(\frac{x_j-d}{x_j-c}\right)^{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}}+\frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}\left(\frac{x_{s+j}-d}{x_{s+j}-c}\right)^{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}}-\frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}-1},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$, whereas if $c = d$, we get

$$x_{ms+j} = \frac{c\left(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j} - c} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j} - c}\right) + 1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j} - c} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j} - c}},$$

$$y_{ms+j} = \frac{c\left(\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j} - c} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j} - c}\right) + 1}{\frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} - \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{y_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} + \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{y_{s+j} - c} + \frac{\frac{\lambda_1^{m-1}-\lambda_2^{m-1}}{2\sqrt{5}} + \frac{\lambda_3^{m-1}-\lambda_4^{m-1}}{2i\sqrt{3}}}{x_j - c} + \frac{\frac{\lambda_1^{m}-\lambda_2^{m}}{2\sqrt{5}} - \frac{\lambda_3^{m}-\lambda_4^{m}}{2i\sqrt{3}}}{x_{s+j} - c}},$$

for $m \in \mathbb{N}_0$ and $j = \overline{0, s-1}$. □

Remark 2.7. Theorem 2.2, Corollary 2.4, Theorem 2.5 and Corollary 2.6, show the practical solvability of system (1.3) in the following six cases: $k = 1$, $l = 0$; $k = 2$, $l = 0$; $k = 2$, $l = 1$; $k = 3$, $l = 0$; $k = 4$, $l = 0$ and $k = 4$, $l = 2$. Practical solvability of the system (1.3) in the cases: $k = 3$, $l = 1$; $k = 3$, $l = 2$; $k = 4$, $l = 1$ and $k = 4$, $l = 3$ is shown similarly, but with more technical details.

Remark 2.8. Employing the formulas for the general solutions to system (1.3), one can describe their well-defined solutions. The standard problem is left to the reader as an exercise.

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Existence and uniqueness of Carathéodory and Filippov solutions for discontinuous systems of differential equations

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Abstract. We use essential limits inferior and superior of the nonlinear part of a discontinuous ODE to introduce some novel transversality conditions which imply that Filippov solutions are Carathéodory solutions. We also prove some uniqueness criteria based on different Lipschitz conditions on different parts of the domain separated from one another by boundaries which satisfy certain transversality conditions.

Keywords: discontinuous differential equations, Carathéodory solutions, Filippov solutions, differential inclusions.

2020 Mathematics Subject Classification: 34A36.

1 Introduction

Consider the initial value problem

$$x' = f(t,x), \qquad t \in I = [t_0, t_0 + L], \quad x(t_0) = x_0, \tag{1.1}$$

where $t_0, L \in \mathbb{R}$, $L > 0$, $x_0 \in \mathbb{R}^n$ ($n \in \mathbb{N}$) and $f = (f_1, f_2, \dots, f_n) : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ need not be continuous. In this paper, we prove new existence and uniqueness results on Carathéodory and Filippov solutions to (1.1).

Assume that for a.a. $t \in I$ the mapping $f(t, \cdot)$ is locally essentially bounded. A Filippov solution of (1.1) is defined as an absolutely continuous function $x : I \longrightarrow \mathbb{R}^n$ such that $x(t_0) = x_0$ and

$$x'(t) \in \bigcap_{\varepsilon>0} \bigcap_{m(N)=0} \overline{\text{co}}\, f\left(t, B_\varepsilon(x(t)) \setminus N\right) \quad \text{for a.a. } t \in I, \tag{1.2}$$

where m is the Lebesgue measure, $\overline{\text{co}}$ means closed convex hull and $B_\varepsilon(x) = \{y \in \mathbb{R}^n : \|y - x\| < \varepsilon\}$. Here and henceforth, we denote by $\|x\|$ the usual norm of a vector $x \in \mathbb{R}^n$. Observe that, in the scalar case ($n = 1$), we have $B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$.

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Filippov solutions satisfy

$$x'(t) \in \prod_{j=1}^{n} \left[\operatorname*{ess\,lim\,inf}_{y \to x(t)} f_j(t,y), \operatorname*{ess\,lim\,sup}_{y \to x(t)} f_j(t,y) \right] \quad \text{for a.a. } t \in I, \tag{1.3}$$

where ess lim inf and ess lim sup stand for the essential limit inferior and superior, respectively. Namely, for each $j \in \{1, 2, \dots, n\}$,

$$\operatorname*{ess\,lim\,inf}_{y \to x} f_j(t,y) = \lim_{\varepsilon \to 0^+} \operatorname*{ess\,inf}_{0<\|x-y\|<\varepsilon} f_j(t,y) = \sup_{\varepsilon>0} \operatorname*{ess\,inf}_{0<\|x-y\|<\varepsilon} f_j(t,y).$$

The essential limit superior is defined analogously.

With this information on Filippov solutions we shall deduce some sufficient conditions for the existence of Carathédory solutions in terms of essential limits.

This paper is organized as follows. In Section 2 we prove (1.3). In Section 3 we introduce novel transversality conditions on $f(t,x)$ in terms of essential limits which ensure that, first, Filippov solutions of (1.1) exist and, second, every Filippov solution is a Carathéodory solution. In Section 4, we deduce new uniqueness results for both Carathéodory and Filippov solutions of (1.1).

2 Preliminaries

This section is mainly devoted to proving that Filippov solutions of (1.1) satisfy (1.3). We thank the anonymous reviewer of a previous version of this paper for having brought to our attention reference [9], where Filippov himself uses (1.3) omiting its proof.

Proposition 2.1. *Let $f = (f_1, f_2, \dots, f_n) : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an arbitrary function.*

For all $t \in I$ and every $x \in \mathbb{R}^n$ such that $f(t, \cdot)$ is essentially bounded on a neighborhood of x, we have

$$\bigcap_{\varepsilon>0} \bigcap_{m(N)=0} \overline{\text{co}}\, f\,(t, B_\varepsilon(x) \setminus N) \subset \prod_{j=1}^{n} \left[\operatorname*{ess\,lim\,inf}_{y \to x} f_j(t,y), \operatorname*{ess\,lim\,sup}_{y \to x} f_j(t,y) \right]. \tag{2.1}$$

Moreover, in the scalar case ($n = 1$) we have

$$\bigcap_{\varepsilon>0} \bigcap_{m(N)=0} \overline{\text{co}}\, f\,(t, B_\varepsilon(x) \setminus N) = \left[\operatorname*{ess\,lim\,inf}_{y \to x} f(t,y), \operatorname*{ess\,lim\,sup}_{y \to x} f(t,y) \right]. \tag{2.2}$$

Proof. For each $j \in \{1, 2, \dots, n\}$ and each sufficiently small $\varepsilon > 0$ there exist $c_*(j)$, $c^*(j) \in \mathbb{R}$, essential lower and upper bounds of the set

$$A_\varepsilon(j) = \{f_j(t,y) \,:\, 0 < \|x-y\| < \varepsilon\},$$

i.e., there exists a null measure set N_ε such that

$$c_*(j) \le f_j(t,y) \le c^*(j) \quad \text{provided that } 0 < \|x-y\| < \varepsilon,\, y \notin N_\varepsilon.$$

We may (and we do) assume that $x \in N_\varepsilon$ and that N_ε does not depend on j. Hence

$$\overline{\text{co}}\, f\,(t, B_\varepsilon(x) \setminus N_\varepsilon) \subset \prod_{j=1}^{n} [c_*(j), c^*(j)],$$

which implies that

$$\bigcap_{m(N)=0} \overline{\text{co}}\, f\,(t, B_\varepsilon(x) \setminus N) \subset \prod_{j=1}^{n} [c_*(j), c^*(j)].$$

Since $c_*(j)$ and $c^*(j)$ were arbitrary essential lower and upper bounds of $A_\varepsilon(j)$, we deduce that

$$\bigcap_{m(N)=0} \overline{\text{co}}\, f\,(t, B_\varepsilon(x) \setminus N) \subset \prod_{j=1}^{n} \left[\operatorname*{ess\,inf}_{0<\|x-y\|<\varepsilon} f_j(t,y),\ \operatorname*{ess\,sup}_{0<\|x-y\|<\varepsilon} f_j(t,y) \right].$$

Finally, since ε was fixed arbitrarily, we conclude that

$$\bigcap_{\varepsilon>0} \bigcap_{m(N)=0} \overline{\text{co}}\, f\,(t, B_\varepsilon(x) \setminus N) \subset \prod_{j=1}^{n} \left[\text{ess} \liminf_{y \to x} f_j(t,y), \text{ess} \limsup_{y \to x} f_j(t,y) \right]. \tag{2.3}$$

Next we prove (2.2) in the scalar case. Notice that if

$$\xi \in \left[\text{ess} \liminf_{y \to x} f(t,y), \text{ess} \limsup_{y \to x} f(t,y) \right]$$

then for each $\varepsilon > 0$ and each $N \subset \mathbb{R}$, $m(N) = 0$, we have

$$\xi \geq \text{ess} \liminf_{y \to x} f(t,y) \geq \operatorname*{ess\,inf}_{B_\varepsilon(x)} f(t,y) \geq \inf_{B_\varepsilon(x) \setminus N} f(t,y),$$

and, analogously,

$$\xi \leq \sup_{B_\varepsilon(x) \setminus N} f(t,y).$$

Hence, for each $\varepsilon > 0$ and each null measure set $N \subset \mathbb{R}$, we have

$$\xi \in \left[\inf_{B_\varepsilon(x) \setminus N} f(t,y),\ \sup_{B_\varepsilon(x) \setminus N} f(t,y) \right] = \overline{\text{co}} f\,(t, B_\varepsilon(x) \setminus N)\,,$$

as desired. □

In applications we shall often have some more assumptions on $f(t,x)$, which yield a clearer version of (2.2). An interesting particular case is considered in our next lemma.

Lemma 2.2. *Assume that $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfies that for a.a. $t \in I$ there is a null measure set $N(t)$ such that the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous.*

Then, for a.a. $t \in I$ and every $x \in \mathbb{R}^n$ such that $f(t,\cdot)$ is essentially bounded on a neighborhood of x, we have

$$\text{ess} \liminf_{y \to x} f(t,y) = \liminf_{y \to x,\, y \notin N(t)} f(t,y), \tag{2.4}$$

and

$$\text{ess} \limsup_{y \to x} f(t,y) = \limsup_{y \to x,\, y \notin N(t)} f(t,y). \tag{2.5}$$

Proof. Let us prove (2.4). The proof of (2.5) is analogous and we omit it.

Les us fix $t \in I$ such that the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous and $N(t)$ is null. Observe that (2.4) is obviously true in case $x \in \mathbb{R}^n \setminus N(t)$ because the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous at x, so we assume that $x \in N(t)$.

By definition, we have

$$\operatorname{ess}\liminf_{y\to x} f(t,y) = \lim_{\varepsilon\to 0^+} \operatorname*{ess\,inf}_{0<\|x-y\|<\varepsilon} f(t,y),$$

so it suffices to check that for all sufficiently small $\varepsilon > 0$ we have

$$\eta := \operatorname*{ess\,inf}_{0<\|x-y\|<\varepsilon} f(t,y) = \inf_{0<\|x-y\|<\varepsilon,\, y\notin N(t)} f(t,y) =: \iota. \tag{2.6}$$

Take any $\varepsilon > 0$ such that $f(t,\cdot)$ is essentially bounded on $B_\varepsilon(x)$. Clearly, ι is an essential lower bound for the set $\{f(t,y) : 0 < \|x-y\| < \varepsilon\}$, hence $\iota \le \eta$.

Now assume, reasoning by contradiction, that $\iota < \eta$. The definition of ι guarantees that we can find $y_0 \in \mathbb{R}^n \setminus N(t)$, $0 < \|x-y_0\| < \varepsilon$, such that

$$\iota \le f(t,y_0) < \eta.$$

Since the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous at y_0, there is a neighborhood of y_0 relative to $\mathbb{R}^n \setminus N(t)$ of points y satisfying $0 < \|x-y\| < \varepsilon$ for which we have $f(t,y) < \eta$, so η cannot be an essential lower bound for the set $\{f(t,y) : 0 < \|x-y\| < \varepsilon\}$, a contradiction. The proof of (2.6) is complete. □

3 Existence of Carathéodory solutions

In this section we use a deep existence result due to Filippov [10, Theorem 8, page 85] along with Proposition 2.1 and Lemma 2.2, to establish a new existence result of Carathéodory solutions for (1.1). We recall that a Carathéodory solution of (1.1) is an absolutely continuous function $x : I \longrightarrow \mathbb{R}^n$ such that $x(t_0) = x_0$ and $x'(t) = f(t,x(t))$ for a.a. $t \in I$.

For the convenience of the reader, we gather the main ingredients we need from [10, Theorem 8, page 85] in the following proposition.

Proposition 3.1. *Assume that $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies the following conditions.*

(i) The function $f(t,x)$ is measurable;

(ii) There exists $\psi \in L^1(I)$ such that for a.a. $t \in I$ and all $x \in \mathbb{R}^n$ we have

$$|f(t,x)| \le \psi(t).$$

Then, problem (1.1) has at least one Filippov solution.

We shall also employ the following result, which follows from [3, Lemma 5.8.13].

Lemma 3.2. *Let a, $b \in \mathbb{R}$, $a < b$. If $\varphi : [a,b] \longrightarrow \mathbb{R}$ is almost everywhere differentiable on $[a,b]$, then for each null measure set $A \subset \mathbb{R}$ there exists a null measure set $B \subset \varphi^{-1}(A)$ such that*

$$\varphi'(t) = 0 \quad \textit{for all } t \in \varphi^{-1}(A) \setminus B.$$

We are now in a position to prove a result on the existence of Carathéodory solutions for (1.1).

Theorem 3.3. *In the conditions of Proposition 3.1, assume also that there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, and differentiable mappings $\tau_k : [a_k,b_k] \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $[a_k,b_k] \subset I$, such that for a.a. $t \in I$ the following conditions hold:*

(a) *There exists a null measure set $N(t) \subset \mathbb{R}^n$ such that the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous;*

(b) *For each $x \in N(t)$ there exists $k \in \mathcal{C}$ such that $t \in [a_k, b_k]$, $\tau_k(t,x) \in A_k$, and*

$$\nabla \tau_k(t,x) \cdot (1,z) \neq 0 \quad \textit{for all } z \in \prod_{j=1}^{n} \left[\liminf_{y\to x,\, y\notin N(t)} f_j(t,y),\ \limsup_{y\to x,\, y\notin N(t)} f_j(t,y) \right]. \tag{3.1}$$

Then, problem (1.1) has at least one Carathéodory solution, which is also a Filippov solution.

Proof. By virtue of Proposition 2.1, problem (1.1) has at least one Filippov solution $x : I \longrightarrow \mathbb{R}^n$, which, according to (1.2), (2.1) and (2.4)–(2.5), satisfies

$$x'(t) \in \prod_{j=1}^{n} \left[\liminf_{y\to x(t),\, y\notin N(t)} f_j(t,y),\ \limsup_{y\to x(t),\, y\notin N(t)} f_j(t,y) \right] \quad \text{for a.a. } t \in I.$$

We shall prove that x is a Carathéodory solution of (1.1).

Let $E \subset I$ be a null measure set such that, first, conditions (a) and (b) hold for all $t \in I \setminus E$, and, second,

$$x'(t) \in \prod_{j=1}^{n} \left[\liminf_{y\to x(t),\, y\notin N(t)} f_j(t,y),\ \limsup_{y\to x(t),\, y\notin N(t)} f_j(t,y) \right] \quad \text{for all } t \in I \setminus E.$$

Observe that for each $t \in I \setminus E$ such that $x(t) \notin N(t)$ condition (a) ensures that the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is continuous at $x(t)$ and therefore

$$x'(t) \in \prod_{j=1}^{n} \left[\liminf_{y\to x(t),\, y\notin N(t)} f_j(t,y),\ \limsup_{y\to x(t),\, y\notin N(t)} f_j(t,y) \right] = \{f(t,x(t))\}.$$

Hence, it suffices to prove that the set $J = \{t \in I \setminus E \,:\, x(t) \in N(t)\}$ is null.

We deduce from condition (b) that

$$J \subset \bigcup_{k\in\mathcal{C}} \{t \in [a_k,b_k] \setminus E \,:\, \tau_k(t,x(t)) \in A_k\},$$

so the proof is reduced to showing that each $J_k = \{t \in [a_k,b_k] \setminus E \,:\, \tau_k(t,x(t)) \in A_k\}$ is a null measure set. For an arbitrarily fixed $k \in \mathcal{C}$, we define $\varphi(t) = \tau_k(t,x(t))$ for all $t \in [a_k,b_k]$, so that $J_k \subset \varphi^{-1}(A_k)$ and it suffices to prove that $\varphi^{-1}(A_k)$ is null. Since $m(A_k) = 0$, Lemma 3.2 guarantees the existence of a set $B \subset \varphi^{-1}(A_k)$, with $m(B) = 0$, such that for every $t \in \varphi^{-1}(A_k) \setminus B$ we have $\varphi'(t) = 0$, i.e.

$$\frac{d}{dt}\tau_k(t,x(t)) = 0. \tag{3.2}$$

Let us prove that $\varphi^{-1}(A_k) \subset B \cup E$, thus showing that $\varphi^{-1}(A_k)$ is null. Reasoning by contradiction, we assume that there is some $t \in \varphi^{-1}(A_k)$ such that $t \notin B \cup E$, and then we can use the chain rule in (3.2) to deduce that

$$\nabla \tau_k(t,x(t)) \cdot (1,x'(t)) = 0,$$

a contradiction with condition (3.1). □

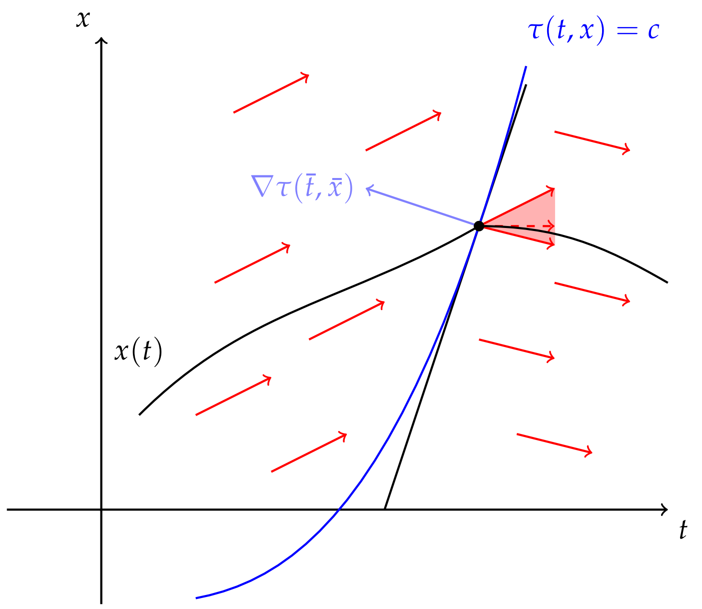

Figure 3.1: Visualization of the transversality condition (3.1) at $(\bar{t},\bar{x}) = (\bar{t}, x(\bar{t}))$.

Remark 3.4. Observe that, under the assumptions of Theorem 3.3, every Filippov solution of (1.1) is in fact a Carathéodory solution.

Note that, in general, Carathéodory solutions need not be Filippov solutions. Indeed, the constant function $x(t) \equiv 0$ is a Carathéodory solution of the initial value problem

$$x' = f(x) = \begin{cases} 1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \qquad t \in [0,1], \quad x(0) = 0, \tag{3.3}$$

but it is not a Filippov solution. Readers can find in [15] a good account on the relations between Carathéodory and Filippov solutions.

The transversality condition (3.1) is based on an original idea by Bressan and Shen [6], later improved and applied by the authors in [12, 13]. All those papers assumed that

$$f(t,x) = F(t, g_1(\tau_1(t,x),x), g_2(\tau_2(t,x),x), \dots, g_N(\tau_N(t,x),x)) \quad \text{for some } N \in \mathbb{N},$$

for functions F and g_k under suitable conditions, a technical drawback which we avoid in this paper.

Figure 3.1 can help readers to have a clearer intuition of what (3.1) means, at least in the very specific setting of one dimension and just one discontinuity curve $\tau(t,x) = c$. Note that vectors $(1,z)$ in condition (3.1) are represented as the red triangle in the figure, and condition (3.1) means that the red triangle cannot contain tangent vectors to $\tau(t,x) = c$ at $(\bar{t},\bar{x})$.

In addition, in [12, 13], we first look for Krasovskij solutions and then we use a transversality condition to prove that they are Carathéodory solutions. In this paper, we use Filippov solutions instead of Krasovskij's, thus getting a milder transversality condition in terms of essential limits. Both transversality conditions are compared in our next example.

Example 3.5. Consider the initial value problem (3.3). Theorem 3.3 ensures that (3.3) has at least one Carathéodory solution. Indeed, the function f is continuous on $\mathbb{R} \setminus \{0\}$ and so conditions (a) and (b) in Theorem 3.3 hold with $\mathcal{C} = \{1\}$, $N = A_1 = \{0\}$ and $\tau_1(t,x) = x$, since

$$\left[\liminf_{y\to 0} f(y), \limsup_{y\to 0} f(y)\right] = \{1\}.$$

The main results in [12,13] do not apply because they are based on the larger Krasovskij envelope

$$\mathcal{K}f(x) := \left[\min\left\{f(x), \liminf_{y\to x} f(y)\right\}, \max\left\{f(x), \limsup_{y\to x} f(y)\right\}\right],$$

so the transversality condition in [12,13], namely,

$$\nabla\tau_1(t,x)\cdot(1,z) = z \neq 0 \quad \text{for all } z \in \mathcal{K}f(0) = [0,1],$$

fails (at $z = 0$).

We also stress that the information provided by Proposition 2.1 and Lemma 2.2 concerning the Filippov envelope is useful in order to reduce the regularity required to the function f. Note that in the previous mentioned papers [12,13], it was basically assumed that for a.a. $t \in I$ there exists a null measure set $N(t) \subset \mathbb{R}^n$ such that $f(t,\cdot)$ is continuous on $\mathbb{R}^n \setminus N(t)$, instead of the weaker assumption (a) in Theorem 3.3. We highlight that Theorem 3.3 can be even applied to functions f which are discontinuous at every point of its domain, as shown by the following example.

Example 3.6. Any planar system of the form

$$\begin{cases} x' = f_1(t,x,y), & x(0) = 0, \\ y' = f_2(t,x,y), & y(0) = 0, \end{cases}$$

where f_1 is continuous and bounded and f_2 is measurable, bounded and its restriction to $[0,L] \times (\mathbb{R} \setminus A) \times \mathbb{R}$ is continuous with A a null measure set, has at least one absolutely continuous solution defined in the interval $[0,L]$ provided that $f_1(t,x,y) \neq 0$ for all (t,x,y) such that $x \in A$.

Indeed, it suffices to apply Theorem 3.3 with $\mathcal{C} = \{1\}$, $\tau_1(t,x,y) = x$, $A_1 = A$ and $N(t) = A \times \mathbb{R}$. Note that the transversality condition (3.1) can be written in this case as

$$z_1 \neq 0 \quad \text{for all } (z_1,z_2) \in \prod_{j=1}^{2} \left[\liminf_{(u,v)\to(x,y),\, u\notin A} f_j(t,u,v), \limsup_{(u,v)\to(x,y),\, u\notin A} f_j(t,u,v)\right],$$

for each $(x,y) \in \mathbb{R}^2$ such that $x \in A$. Since f_1 is continuous,

$$\left[\liminf_{(u,v)\to(x,y),\, u\notin A} f_1(t,u,v), \limsup_{(u,v)\to(x,y),\, u\notin A} f_1(t,u,v)\right] = \{f_1(t,x,y)\},$$

and thus the conclusion follows from the fact that f_1 does not vanish at the points (t,x,y) with $x \in A$.

For instance, we can choose $f_1(t,x,y) = \cos^2(x\,y) + e^{t-x^2-y^2}$ and $f_2(t,x,y) = \varphi(x)e^{\sin(t+y)}$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is given by

$$\varphi(x) = \chi_{\mathbb{Q}}(x) - \chi_{\mathbb{R}\setminus\mathbb{Q}}(x),$$

where χ_B denotes the characteristic function of the set $B \subset \mathbb{R}$. It is worth mentioning that φ is discontinuous at every point which, to the best of the authors' knowledge, falls outside the scope of earlier existence results. Observe, however that its restriction to the set $\mathbb{R} \setminus \mathbb{Q}$ is continuous and therefore Theorem 3.3 applies.

Our existence result applies for discontinuous ODE-systems associated with two-phase flows, that is, initial value problem (1.1) with a nonlinearity f which is discontinuous over a single hypersurface $\Sigma(t)$ defined as

$$\Sigma(t) := \{x \in \mathbb{R}^n \, : \, \tau(t,x) = 0\},$$

where $\tau : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a differentiable mapping. More precisely, let us consider an initial value problem of type

$$x' = f(t,x) = \begin{cases} f^+(t,x) & \text{if } x \in \Sigma^+(t), \\ f^-(t,x) & \text{if } x \in \Sigma^-(t), \end{cases} \qquad t \in I, \quad x(t_0) = x_0, \tag{3.4}$$

where

$$\Sigma^+(t) = \{x \in \mathbb{R}^n \, : \, \tau(t,x) > 0\} \text{ and } \Sigma^-(t) = \{x \in \mathbb{R}^n \, : \, \tau(t,x) < 0\},$$

and $f^\pm : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are L^1–bounded Carathéodory mappings. Note that the definition of f on $\Sigma(t)$ is not relevant in order to apply Theorem 3.3, so we may assume that either $f(t,x) = f^+(t,x)$ or $f(t,x) = f^-(t,x)$ on $\Sigma(t)$.

As a straightforward consequence of Theorem 3.3, we obtain the following existence result for (3.4).

Corollary 3.7. *Assume that for a.a. $t \in I$ and for each $x \in \Sigma(t)$ we have*

$$\nabla\tau(t,x) \cdot (1,z) \neq 0 \quad \textit{for all } z \in \overline{\text{co}} \left\{ f^-(t,x), f^+(t,x) \right\}. \tag{3.5}$$

Then problem (3.4) has at least one Carathéodory solution.

Note that we need less regularity on $f^\pm$ than some related results (cf. Step 1 of [4, Theorem 1], where they are required to be locally Lipschitz continuous in x instead of merely continuous in x).

Let us now focus on the scalar case of (1.1), i.e., $n = 1$. By the implicit function theorem, if τ is regular enough, the discontinuity curve $\tau(t,x) = c$ can be seen, at least locally, as the graph of a time-dependent curve $x = \gamma(t)$ provided that $\frac{\partial \tau}{\partial x}(t,x) \neq 0$. Note that the transversality condition (3.1) implies that $\nabla\tau(t,x) \neq (0,0)$ over the discontinuity points of f which satisfy $\tau(t,x) = c$, where c belongs to a suitable null measure set.

With this in mind, we have the following alternative version of Theorem 3.3.

Corollary 3.8. *In the conditions of Proposition 3.1 and in the case $n = 1$, assume also that there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, and differentiable mappings $\gamma_k : [a_k, b_k] \subset I \longrightarrow \mathbb{R}$ such that for a.a. $t \in I$ the following conditions hold:*

(a) the restriction of $f(t,\cdot)$ to the set $\mathbb{R} \setminus N(t)$ is continuous, where

$$N(t) = \bigcup_{\{k \in \mathcal{C} \, : \, t \in [a_k,b_k]\}} \bigcup_{c \in A_k} \{\gamma_k(t) + c\} \, ;$$

(b) for each $k \in \mathcal{C}$ such that $t \in [a_k, b_k]$, and each $c \in A_k$, we have either

$$\gamma_k'(t) < \liminf_{y \to \gamma_k(t), \, y \notin N(t)} f(t, y + c) \tag{3.6}$$

or

$$\gamma_k'(t) > \limsup_{y \to \gamma_k(t), \, y \notin N(t)} f(t, y + c). \tag{3.7}$$

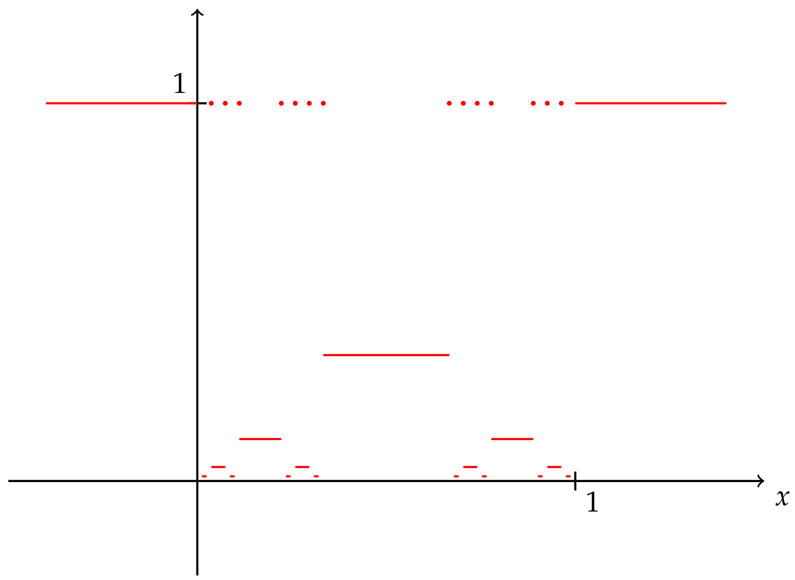

Figure 3.2: Approximate plot of $x \mapsto f(0,x)$, discontinuous at every point of Cantor's ternary set.

Then, problem (1.1) *has at least one Carathéodory solution.*

Proof. It suffices to apply Theorem 3.3 with $\tau_k(t,x) = x - \gamma_k(t)$. □

Example 3.9. Let C denote Cantor's ternary set. We have

$$[0,1] \setminus C = \bigcup_{n=1}^{\infty} (a_n, b_n),$$

where $(a_n, b_n) \cap (a_m, b_m) = \emptyset$ if $n \neq m$.

Define $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ as $f(t,x) = b_n - a_n$ provided that $x + t \in (a_n, b_n)$ for some $n \in \mathbb{N}$, and $f(t,x) = 1$ otherwise. See Figure 3.2 for a plot of $x \mapsto f(0,x)$.

Corollary 3.8 guarantees that the corresponding initial value problem (1.1) (with $t_0 = x_0 = 0$ and $L = 1$) has at least one Carathéodory solution. To prove it, just define $\mathcal{C} = \{1\}$, $A_1 = C$, and $\gamma_1(t) = -t$ for $t \in [0,1]$. In this case, $N(t) = -t + C$ for all $t \in [0,1]$ and the restriction of $f(t,\cdot)$ to $\mathbb{R} \setminus (-t + C)$ is continuous; moreover, condition (3.6) holds for all $t \in [0,1]$.

Observe that the set $-t + C$ is not countable for any t, so discontinuities of f cannot be covered by countable unions of curves, as required, for instance, in Corollary 3.7 or Corollary 3.8 in [7]. Remarkably, solutions are increasing on $[0,1]$, so they cross every line $x + t = c$, $c \in C$.

The previous result does not cover some situations in which the set of Filippov solutions is a subset of that of Carathéodory solutions for (1.1). Indeed, one may easily verify that every Filippov solution of the initial value problem

$$x' = f(t,x) = \begin{cases} 0, & \text{if } x > t, \\ 1, & \text{if } x = t, \\ 2, & \text{if } x < t, \end{cases} \qquad t \in [0,1], \quad x(0) = 0, \tag{3.8}$$

is in fact a Carathéodory solution. Nevertheless, f is discontinuous over the line $x = \gamma(t) := t$, $t \in [0,1]$, which does not satisfy neither (3.6) nor (3.7), since

$$\gamma'(t) = 1 \in [0,2] = \left[\liminf_{y\to t} f(t,y), \limsup_{y\to t} f(t,y)\right].$$

Note that γ is a solution of the initial value problem (3.8).

In the following result, we admit that f be discontinuous over the graphs of a countable family of solutions of the differential equation $x' = f(t,x)$.

Proposition 3.10. *In the conditions of Proposition 3.1 and in the case $n = 1$, assume also that there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, $j \in \mathcal{D} \subset \mathbb{N}$, and differentiable mappings $\gamma_k : [a_k, b_k] \subset I \longrightarrow \mathbb{R}$ and $\psi_j : [\tilde{a}_j, \tilde{b}_j] \subset I \longrightarrow \mathbb{R}$ such that for a.a. $t \in I$ the following conditions hold:*

(a) the restriction of $f(t,\cdot)$ to the set $\mathbb{R} \setminus N(t)$ is continuous, where

$$N(t) = N_1(t) \cup N_2(t), \qquad N_1(t) = \bigcup_{k\in\mathcal{C}} \bigcup_{c_k\in A_k} \{\gamma_k(t) + c_k\} \quad \text{and} \quad N_2(t) = \bigcup_{j\in\mathcal{D}} \{\psi_j(t)\};$$

(b) for each $k \in \mathcal{C}$ and each $c_k \in A_k$, the function γ_k satisfies that for a.a. $t \in [a_k, b_k]$ either (3.6) or (3.7) holds;

(c) for each $j \in \mathcal{D}$, $\psi_j'(t) = f(t, \psi_j(t))$ for a.a. $t \in [\tilde{a}_j, \tilde{b}_j]$.

Then, problem (1.1) has at least one Carathéodory solution.

Proof. It follows from Proposition 3.1 that problem (1.1) has at least one Filippov solution x. Let us prove that x is a Carathéodory solution.

It can be shown (just as in the proof of Theorem 3.3) that the set

$$J^{\gamma} = \bigcup_{k\in\mathcal{C}} \{t \in [a_k, b_k] : x(t) - \gamma_k(t) \in A_k\}$$

has Lebesgue measure zero. Suppose that there exists $j \in \mathcal{D}$ such that $m(J_j^{\psi}) > 0$, with

$$J_j^{\psi} = \{t \in [\tilde{a}_j, \tilde{b}_j] : x(t) = \psi_j(t)\} \quad \text{and} \quad J^{\psi} = \bigcup_{j\in\mathcal{D}} J_j^{\psi}.$$

Then $x'(t) = \psi_j'(t)$ for a.a. $t \in J_j^{\psi}$. By the definition of ψ_j, we have that $\psi_j'(t) = f(t, \psi_j(t))$ for a.a. $t \in J_j^{\psi}$ and thus $x'(t) = f(t, x(t))$ for a.a. $t \in J_j^{\psi}$. Hence, $x'(t) = f(t, x(t))$ for a.a. $t \in J^{\psi}$.

Finally, since the restriction of $f(t,\cdot)$ to $\mathbb{R} \setminus N(t)$ is continuous at $x(t)$ for a.a. $t \in I \setminus (J^{\gamma} \cup J^{\psi})$, we conclude that x is a Carathéodory solution of (1.1). □

We shall say that both the functions γ_k satisfying (3.6) or (3.7) and the functions of type ψ_j are *admissible discontinuity curves*(cf. [8]).

Note that the previous result is sharp in the following sense: if there exists a differentiable function $\gamma : [t_0, t_1] \subset I \to \mathbb{R}$ which is not an admissible discontinuity curve and such that $\gamma(t_0) = x_0$ and for each $t \in [t_0, t_1]$, $f(t,\cdot)$ is discontinuous at $\gamma(t)$, then γ can be extended to a Filippov solution of (1.1) which is not a solution in the sense of Carathéodory.

4 Uniqueness of Carathéodory and Filippov solutions

In this section we show that a similar transversality condition to that employed in Theorem 3.3 can be used to deduce uniqueness of Filippov or Carathéodory solutions of (1.1). Note that recently Fjordholm [11] gave necessary and sufficient conditions for the uniqueness of Filippov solutions in the scalar autonomous case of (1.1), complementing the results for Carathéodory solutions due to Binding [2]. Our results concern non-autonomous discontinuous systems.

First, we use some transversality conditions to prove uniqueness of Filippov solutions of (1.1). Surprisingly enough, it does not guarantee uniqueness of Carathéodory solutions.

Basically, we assume that $f(t,x)$ satisfies a Lipschitz condition with respect to x in every gap delimited in $I \times \mathbb{R}^n$ by a set of hypersurfaces (not necessarily a finite set) where f may be discontinuous and where some suitable transversality conditions hold.

We need some notation for our first main result on uniqueness. Let $A \subset \mathbb{R}$ be a non-empty set. We will say that $x_0 \in A$ is a *left-isolated point*(*right-isolated point*) of A if there is $\delta > 0$ such that $(x_0 - \delta, x_0) \cap A = \emptyset$ ($(x_0, x_0 + \delta) \cap A = \emptyset$). In the sequel, we denote by $\mathcal{I}^-(A)$ the set of left-isolated points of A and by $\mathcal{I}^+(A)$ that of right-isolated points. Obviously, if $x_0 \in \mathcal{I}^-(A) \cap \mathcal{I}^+(A)$, it is an isolated point of the subset A.

Theorem 4.1. *In the conditions of Proposition 3.1, assume also that there exist continuously differentiable mappings $\tau_k : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ ($k = 1, 2, \ldots, m$, $m \in \mathbb{N}$) and countable closed sets $A_k \subset \mathbb{R}$ satisfying that $A_k = \mathcal{I}^-(A_k) \cup \mathcal{I}^+(A_k)$ such that the following conditions hold:*

(i) For each $k \in \{1, 2, \ldots, m\}$ and each $(t,x) \in \tau_k^{-1}(A_k)$, there exists $\varepsilon > 0$ such that for a.a. $s \in (t, t+\varepsilon)$ and all $\zeta \in B_\varepsilon(x)$ we have

$$\nabla \tau_k(s,\zeta) \cdot (1,z) > 0 \quad \text{for all } z \in \prod_{j=1}^{n} \left[\operatorname{ess} \liminf_{y \to \zeta} f_j(s,y), \operatorname{ess} \limsup_{y \to \zeta} f_j(s,y) \right] \tag{4.1}$$

if $\tau_k(t,x) \in \mathcal{I}^+(A_k) \setminus \mathcal{I}^-(A_k)$;

$$\nabla \tau_k(s,\zeta) \cdot (1,z) < 0 \quad \text{for all } z \in \prod_{j=1}^{n} \left[\operatorname{ess} \liminf_{y \to \zeta} f_j(s,y), \operatorname{ess} \limsup_{y \to \zeta} f_j(s,y) \right] \tag{4.2}$$

if $\tau_k(t,x) \in \mathcal{I}^-(A_k) \setminus \mathcal{I}^+(A_k)$; and either (4.1) or (4.2) if $\tau_k(t,x) \in \mathcal{I}^-(A_k) \cap \mathcal{I}^+(A_k)$.

(ii) For each connected component, $\mathcal{O}$, of the set $I \times \mathbb{R}^n \setminus \bigcup_{k=1}^m \tau_k^{-1}(A_k) = \bigcap_{k=1}^m \tau_k^{-1}(\mathbb{R} \setminus A_k)$, there exists $l \in L^1(I)$ such that for a.a. $t \in I$ and all x, y such that (t,x), $(t,y) \in \mathcal{O}$ we have

$$\|f(t,x) - f(t,y)\| \le l(t) \|x - y\|. \tag{4.3}$$

Then, problem (1.1) has exactly one Filippov solution.

Proof. We can assume, without loss of generality, that $A_k = \mathcal{I}^+(A_k)$ for every $k = 1, 2, \ldots, m$. Indeed, if for some k we have $\mathcal{I}^-(A_k) \neq \emptyset$, then we replace the set A_k by two sets, namely, $A_{k,1} = \mathcal{I}^+(A_k)$ (which satisfies (4.1) with τ_k) and $A_{k,2} = -\mathcal{I}^-(A_k)$ (which satisfies $A_{k,2} = \mathcal{I}^+(A_{k,2})$) and we define a new function $\tilde{\tau}_k = -\tau_k$. Now, condition (4.2) for $\mathcal{I}^-(A_k)$ and τ_k implies condition (4.1) for $A_{k,2}$ and $\tilde{\tau}_k$.

By virtue of Proposition 3.1, problem (1.1) has at least one Filippov solution. Let us prove uniqueness. Let $x(t)$ and $y(t)$ be Filippov solutions of (1.1); we shall prove that $x(t) = y(t)$ for all $t \in I$. Reasoning by contradiction, we assume that there exists some $t_1 \in [t_0, t_0 + L)$ and $\rho > 0$ such that $x(t_1) = y(t_1)$ and $\|x(t) - y(t)\| > 0$ for all $t \in (t_1, t_1 + \rho)$.

Let $z(t)$ be an arbitrary solution of (1.1) such that $z(t_1) = x(t_1)$. Observe that for all $t \in I$, $t \geq t_1$, we have

$$\|z(t) - x(t_1)\| \leq \int_{t_1}^{t} \|z'(r)\| \, dr \leq \int_{t_1}^{t} \psi(r) \, dr,$$

so for any $\varepsilon > 0$ there exists $\mu > 0$ (independent of the solution $z(t)$) such that $z(t) \in B_\varepsilon(x(t_1))$ for all $t \in [t_1, t_1 + \mu]$.

Let $k \in \{1, 2, \ldots, m\}$ be fixed. If $\tau_k(t_1, x(t_1)) \notin A_k$, then there exists an open interval $I_k \subset \mathbb{R} \setminus A_k$ such that $\tau_k(t_1, z(t_1)) = \tau_k(t_1, x(t_1)) \in I_k$ (observe that I_k does not depend on the specific solution $z(t)$). Since τ_k is continuous at $(t_1, x(t_1))$, there exists $r_k > 0$ (independent of the solution $z(t)$) such that $\tau_k(t, z(t)) \in I_k$ for all $t \in [t_1, t_1 + r_k)$, or, equivalently, $(t, z(t)) \in \tau_k^{-1}(I_k)$ for all $t \in [t_1, t_1 + r_k)$.

If $\tau_k(t_1, x(t_1)) =: a \in A_k = \mathcal{I}^+(A_k)$, then there is $\delta > 0$ such that $(a, a + \delta) \cap A_k = \emptyset$ and we define $I_k = (a, a + \delta)$ (which does not depend on the solution $z(t)$). Then, assumption (i) implies that there exists $\varepsilon > 0$ such that for a.a. $s \in (t_1, t_1 + \varepsilon)$ and $\zeta \in B_\varepsilon(x(t_1))$ we have

$$\nabla \tau_k(s, \zeta) \cdot (1, z) > 0 \quad \text{for all } z \in \prod_{j=1}^{n} \left[\operatorname*{ess\,lim\,inf}_{y \to \zeta} f_j(s, y), \operatorname*{ess\,lim\,sup}_{y \to \zeta} f_j(s, y) \right]. \tag{4.4}$$

Take a sufficiently small $r_k \in (0, \varepsilon)$ such that $z(t) \in B_\varepsilon(x(t_1))$ for all $t \in [t_1, t_1 + r_k)$. For a.a. $t \in [t_1, t_1 + r_k)$ we use the chain rule and (4.4) to deduce that

$$\frac{d}{dt} \tau_k(t, z(t)) = \nabla \tau_k(t, z(t)) \cdot (1, z'(t)) > 0,$$

because, as a Filippov solution, $z(t)$ satisfies (1.3). Note that the composition $\tau_k(\cdot, z(\cdot))$ is absolutely continuous, so the previous inequality implies that $\tau_k(t, z(t)) > a$ for all $t \in (t_1, t_1 + r_k)$.

Let $r = \min\{r_1, r_2, \ldots, r_m\}$; we have proven that for every $t \in (t_1, t_1 + r)$ we have

$$(t, z(t)) \in \bigcap_{k=1}^{m} \tau_k^{-1}(I_k),$$

and r does not depend on the specific solution $z(t)$ such that $z(t_1) = x(t_1)$.

We know from [10, Theorem 9] that the set of all Filippov solutions of (1.1) with initial condition $(t_1, x(t_1))$ is a connected subset of $\mathcal{C}([t_1, t_1 + r])$. Hence, for a fixed $t^* \in (t_1, t_1 + r)$ the set

$$S(t^*) = \{(t^*, z(t^*)) \, : \, z(t) \text{ solution}, \, z(t_1) = x(t_1)\}$$

is a connected subset of $I \times \mathbb{R}^n$. This implies that the set

$$S = \{(t, z(t)) \, : \, z(t) \text{ solution}, \, z(t_1) = x(t_1) \text{ and } t \in (t_1, t_1 + r)\}$$

is a connected subset of $I \times \mathbb{R}^n$ because it is the union of all the graphs $\{(t, z(t)) : t \in (t_1, t_1 + r)\}$ which are connected and each contains a point in $S(t^*)$. Therefore, S must be inside one of the connected components of $\bigcap_{k=1}^{m} \tau_k^{-1}(I_k)$, which is contained in a connected component of $I \times \mathbb{R}^n \setminus \bigcup_{k=1}^{m} \tau_k^{-1}(A_k)$. Now condition (ii) ensures the existence of some $l \in L^1(I)$ such that

$$\|x(t) - y(t)\| \leq \int_{t_1}^{t} \|f(s, x(s)) - f(s, y(s))\| \, ds \leq \int_{t_1}^{t} l(s) \|x(s) - y(s)\| \, ds, \quad t \in [t_1, t_1 + r],$$

and we deduce from Gronwall's inequality that $\|x - y\| = 0$ on $[t_1, t_1 + r]$, a contradiction. $\square$

Remark 4.2. The mappings τ_k may be defined in $[a_k, b_k] \times \mathbb{R}^n$, with $[a_k, b_k] \subset I$, instead of the whole $I \times \mathbb{R}^n$. Indeed, the set $\{a_1, b_1, a_2, b_2, \dots, a_m, b_m\}$ defines a partition of the interval $[t_0, t_0 + L]$ and it suffices to apply Theorem 4.1 in each subinterval defined by the partition in order to obtain uniqueness of Filippov solutions to (1.1).

Remark 4.3. The Lipschitz type condition (4.3) can be replaced, for instance, by one-sided Lipschitz, Osgood's or Montel–Tonelli's conditions (see [1]) in such a way that uniqueness of Filippov solutions is proven in a similar manner.

Observe that, in the hypotheses of Theorems 3.3 and 4.1, although existence of Carathéodory solutions for the initial value problem (1.1) is guaranteed, uniqueness cannot be ensured, as shown by the Cauchy problem (3.3). Hence, to obtain uniqueness of Carathéodory solutions, it is necessary to reinforce the assumptions on Theorem 4.1. Obviously, it would be sufficient to ensure that the set of Carathéodory solutions and the set of Filippov solutions coincide (as pointed out in Remark 3.4, Filippov solutions are Carathéodory solutions, so it only remains to ensure the reverse inclusion). In case of autonomous systems, some comparison between them can be found in [15]. In general, this can be directly obtained if one assumes that f is a selection of the Filippov envelope or, even less, if for a.e. t and all x,

$$\operatorname{ess}\liminf_{y \to x} f_j(t,y) \le f_j(t,x) \le \operatorname{ess}\limsup_{y \to x} f_j(t,y), \qquad j = 1, \dots, n, \tag{4.5}$$

which provides uniqueness of Carathéodory solutions as a straightforward consequence of Theorem 4.1.

Corollary 4.4. *In the conditions of Theorem 4.1, if in addition f satisfies (4.5), then problem (1.1) has exactly one Carathéodory solution.*

This simple uniqueness criterion can be useful in practice. In particular, it enables us to establish uniqueness of Carathéodory solutions for discontinuous ODE-systems associated with two-phase flows, namely, initial value problems of type (3.4).

Corollary 4.5. *Let $f^{\pm} : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be L^1-bounded Carathéodory mappings. Assume that $f^{\pm}$ are Lipschitz continuous in x and that for a.a. $t \in I$ and for each $x \in \Sigma(t)$, there exists $\varepsilon > 0$ such that for all $s \in (t, t+\varepsilon)$ and all $\zeta \in B_\varepsilon(x)$ we have either*

$$\nabla\tau(s,\zeta) \cdot (1,z) > 0 \quad \textit{for all } z \in \overline{\text{co}}\left\{f^-(s,\zeta), f^+(s,\zeta)\right\}$$

or

$$\nabla\tau(s,\zeta) \cdot (1,z) < 0 \quad \textit{for all } z \in \overline{\text{co}}\left\{f^-(s,\zeta), f^+(s,\zeta)\right\}.$$

Then problem (3.4) has a unique Carathéodory solution.

Once existence is guaranteed, uniqueness of both Carathéodory and Filippov solutions can be obtained simultaneously, without assuming (4.5), provided that there exists a unique Krasovskij solution of (1.1). To do so, we shall strengthen the transversality condition (i) in Theorem 4.1.

Theorem 4.6. *In the conditions of Theorems 3.3 and 4.1, assume that hypothesis (i) is replaced by the following one:*

(i) For each $k \in \{1,2,\dots,m\}$ and each $(t,x) \in \tau_k^{-1}(A_k)$, there exists $\varepsilon > 0$ such that for a.a. $s \in (t,t+\varepsilon)$ and all $\zeta \in B_\varepsilon(x)$ we have*

$$\nabla \tau_k(s,\zeta)\cdot(1,z) > 0 \quad \text{for all } z \in \mathcal{K}f(s,\zeta) \tag{4.6}$$

if $\tau_k(t,x) \in \mathcal{I}^+(A_k) \setminus \mathcal{I}^-(A_k)$;

$$\nabla \tau_k(s,\zeta)\cdot(1,z) < 0 \quad \text{for all } z \in \mathcal{K}f(s,\zeta), \tag{4.7}$$

if $\tau_k(t,x) \in \mathcal{I}^-(A_k) \setminus \mathcal{I}^+(A_k)$; and either (4.6) or (4.7) if $\tau_k(t,x) \in \mathcal{I}^-(A_k) \cap \mathcal{I}^+(A_k)$ (where $\mathcal{K}f$ denotes the Krasovskij envelope of f).

Then problem (1.1) has exactly one Krasovskij solution, which is also the unique Carathéodory and Filippov solution of (1.1).

Proof. It follows, in an analogous way to the proof of Theorem 4.1, that the differential inclusion

$$x' \in \mathcal{K}f(t,x), \quad t \in I, \; x(t_0) = x_0, \tag{4.8}$$

has a unique solution.

To conclude, it suffices to observe that any Carathéodory, Filippov or Krasovskij solution of (1.1) is, in particular, an absolutely continuous solution of (4.8). □

Remark 4.7. Note that the Krasovskij envelope of the function f satisfies that

$$\begin{aligned}\mathcal{K}f(s,\zeta) &:= \bigcap_{\varepsilon>0} \overline{\text{co}}\, f\left(s, B_\varepsilon(\zeta)\right) \\ &\subset \prod_{j=1}^{n} \left[\min\left\{f_j(s,\zeta), \liminf_{y\to\zeta} f_j(s,y)\right\}, \max\left\{f_j(s,\zeta), \limsup_{y\to\zeta} f_j(s,y)\right\}\right].\end{aligned}$$

Example 4.8. Let $\lfloor\cdot\rfloor$ be the floor function. The system

$$\begin{cases} x' = f_1(x,y), & x(0)=0, \\ y' = f_2(x,y), & y(0)=0, \end{cases}$$

with $f = (f_1,f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(x,y) = \begin{cases} \left(5x + \dfrac{1}{\lfloor 1/(1-x^2-y^2)\rfloor}, 5y + \dfrac{1}{\lfloor 1/(1-x^2-y^2)\rfloor}\right), & \text{if } x^2+y^2<1, \\ \left(5x\, e^{1-x^2-y^2}, 5y\, e^{1-x^2-y^2}\right), & \text{otherwise,} \end{cases}$$

has exactly one Carathéodory solution in any interval $[0,L]$ ($L>0$).

Observe that Theorem 4.6 can be applied with $m=1$, $\tau_1(x,y) = x^2+y^2$ and the closed countable set $A_1 = \{1-1/(k+1) : k \in \mathbb{N}\} \cup \{1\}$ (clearly, $A_1 = \mathcal{I}^+(A_1)$ and for any open interval I the set $\tau_1^{-1}(I)$ is empty or connected). Indeed, in order to check condition (i*), notice that

$$\frac{1}{\lfloor 1/(1-x^2-y^2)\rfloor} = \frac{1}{j} \quad (j \in \mathbb{N}) \quad \text{if and only if} \quad 1-\frac{1}{j} \le x^2+y^2 < 1-\frac{1}{j+1}.$$

Hence, for $j \in \mathbb{N}$, $j \geq 2$, and $x^2 + y^2 = 1 - 1/j$ fixed, we have that

$$\begin{aligned}\mathcal{K}f(x,y) &\subset \prod_{i=1}^{2}\left[\liminf_{(z_1,z_2)\to(x,y)} f_i(z_1,z_2),\ \limsup_{(z_1,z_2)\to(x,y)} f_i(z_1,z_2)\right]\\ &\subset \left[5x+\frac{1}{j}, 5x+\frac{1}{j-1}\right]\times\left[5y+\frac{1}{j}, 5y+\frac{1}{j-1}\right].\end{aligned}$$

It follows that for $(z_1,z_2) \in \mathcal{K}f(x,y)$,

$$\nabla\tau_1(x,y)\cdot(z_1,z_2) = (2x,2y)\cdot(5x+a_j, 5y+b_j) = 10(x^2+y^2) + 2(xa_j + yb_j)$$

where $a_j, b_j \in [1/j, 1/(j-1)]$. Then

$$\nabla\tau_1(x,y)\cdot(z_1,z_2) \geq 10\left(1-\frac{1}{j}\right) - 2\left(|x|+|y|\right)\frac{1}{j-1} > 10\left(1-\frac{1}{j}\right) - 4 \geq 1$$

for all $(z_1,z_2) \in \mathcal{K}f(x,y)$.

Note that f is continuous at any (x,y) such that $x^2+y^2 = 1$, so $\mathcal{K}f(x,y) = \{f(x,y)\}$ and

$$\nabla\tau_1(x,y)\cdot(f_1(x,y), f_2(x,y)) = (2x,2y)\cdot(5x,5y) = 10.$$

Finally, by continuity, we deduce that condition (i*) holds. Moreover, f is Lipschitz continuous in the sets $\mathcal{O}_- = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$, $\mathcal{O}_+ = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 1\}$ and

$$\mathcal{O}_j = \left\{(x,y) \in \mathbb{R}^2 : 1-\frac{1}{j} < x^2+y^2 < 1-\frac{1}{j+1}\right\} \quad (j \in \mathbb{N},\ j \geq 2).$$

Therefore, Theorem 4.6 is applicable and the conclusion follows.

As a consequence of Theorem 4.6, we can also deduce a discontinuous version of a classical uniqueness criterion due to Norris and Driver [14]. Observe that condition (c) is slightly stronger than condition (i^*) in Theorem 4.6 because we need to use Theorem 3.3.

Corollary 4.9. *Let $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a measurable function satisfying the following hypotheses:*

(a) There exist a constant $K > 0$ and functions $g_k : \mathbb{R} \longrightarrow \mathbb{R}$ and $\tau_k : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$, for $k = 1, 2, \ldots, m$, such that

$$\|f(t,x) - f(t,y)\| \leq K\|x-y\| + K\sum_{k=1}^{m}|g_k(\tau_k(t,x)) - g_k(\tau_k(t,y))|$$

for all $(t,x), (t,y) \in I \times \mathbb{R}^n$.

(b) Each function $g_k : \mathbb{R} \longrightarrow \mathbb{R}$ is bounded in $\mathbb{R}$ and Lipschitz continuous in each bounded interval contained in $\mathbb{R} \setminus A_k$, where A_k is a countable closed subset of $\mathbb{R}$ such that $A_k = \mathcal{I}^-(A_k) \cup \mathcal{I}^+(A_k)$.

(c) Each function $\tau_k : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuously differentiable and for each $(t,x) \in \tau_k^{-1}(A_k)$, there exists $\varepsilon > 0$ such that for a.a. $s \in [t, t+\varepsilon)$ and all $\zeta \in B_\varepsilon(x)$ we have

$$\nabla\tau_k(s,\zeta)\cdot(1,z) > 0 \quad \textit{for all } z \in \mathcal{K}f(s,\zeta) \tag{4.9}$$

if $\tau_k(t,x) \in \mathcal{I}^+(A_k) \setminus \mathcal{I}^-(A_k)$;

$$\nabla\tau_k(s,\zeta)\cdot(1,z) < 0 \quad \textit{for all } z \in \mathcal{K}f(s,\zeta), \tag{4.10}$$

if $\tau_k(t,x) \in \mathcal{I}^-(A_k) \setminus \mathcal{I}^+(A_k)$; and either (4.9) or (4.10) if $\tau_k(t,x) \in \mathcal{I}^-(A_k) \cap \mathcal{I}^+(A_k)$ (where $\mathcal{K}f$ denotes the Krasovskij envelope of f).

Then the initial value problem

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{4.11}$$

has a unique local Carathéodory solution.

Proof. Existence follows from Theorem 3.3. Let us prove uniqueness as a consequence of Theorem 4.6.

Consider an arbitrary collection of open bounded intervals $I_k \subset \mathbb{R} \setminus A_k$, $k = 1, 2, \dots, m$, such that

$$\mathcal{O} = \bigcap_{k=1}^{m} \tau_k^{-1}(I_k) \neq \emptyset.$$

For each $k \in \{1, 2, \dots, m\}$, the function g_k is Lipschitz continuous in the interval I_k, so there is $L_k > 0$ such that

$$|g_k(z_1) - g_k(z_2)| \le L_k |z_1 - z_2| \quad \text{for all } z_1, z_2 \in I_k.$$

Hence, for all $(t,x), (t,y) \in \mathcal{O}$, we have that $\tau_k(t,x), \tau_k(t,y) \in I_k$ and thus

$$\sum_{k=1}^{m} |g_k(\tau_k(t,x)) - g_k(\tau_k(t,y))| \le \sum_{k=1}^{m} L_k |\tau_k(t,x) - \tau_k(t,y)| \le C \|x - y\|,$$

for some positive constant C, since the functions τ_k are Lipschitz continuous w.r.t. x on any compact set which contains the graphs of the solutions. Therefore, for all $(t,x), (t,y) \in \mathcal{O}$ we have

$$\|f(t,x) - f(t,y)\| \le (K + KC) \|x - y\|.$$

The conclusion follows now from Theorem 4.6 guaranteeing uniqueness of solutions for (4.11) in the interval $I = [t_0, t_0 + \delta]$. □

In view of the general assumptions on Theorem 3.3 concerning existence of Carathéodory solutions, one may wonder whether a Lipschitz continuous condition outside the set of discontinuities of f implies uniqueness of Carathéodory solutions of (1.1).

Theorem 4.10. *In the conditions of Proposition 3.1, assume also that there exist null measure sets $A_k \subset \mathbb{R}$, $k \in \mathcal{C} \subset \mathbb{N}$, and differentiable mappings $\tau_k : [a_k, b_k] \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $[a_k, b_k] \subset I$, such that for a.a. $t \in I$ the following conditions hold:*

(a) There exists a null measure set $N(t) \subset \mathbb{R}^n$ such that the restriction of $f(t,\cdot)$ to $\mathbb{R}^n \setminus N(t)$ is locally Lipschitz continuous, i.e., for each compact set $K \subset \mathbb{R}^n$ there exists $l_K \in L^1(I)$ such that for a.a. $t \in I$ and all $x, y \in K \cap (\mathbb{R}^n \setminus N(t))$ we have

$$\|f(t,x) - f(t,y)\| \le l_K(t) \|x - y\|.$$

(b) For each $x \in N(t)$ there exists $k \in \mathcal{C}$ such that $t \in [a_k, b_k]$, $\tau_k(t,x) \in A_k$ and

$$\nabla \tau_k(t,x) \cdot (1, z) \neq 0 \quad \text{for all } z \in \mathcal{K}f(t,x). \tag{4.12}$$

Then, problem (1.1) has exactly one Carathéodory solution, which is also the unique Filippov solution.

Proof. Existence follows from Theorem 3.3. For uniqueness, first note that if x is a Carathéodory solution, then it is a solution in the sense of Krasovskij and so the transversality condition (4.12) implies that

$$m\left(\{t \in I : x(t) \in N(t)\}\right) = 0.$$

Let $x(t)$ and $y(t)$ be Carathéodory solutions of (1.1); we shall prove that $x(t) = y(t)$ for all $t \in I$. Reasoning by contradiction, we assume that there exists some $t_1 \in [t_0, t_0 + L)$ such that $x(t_1) = y(t_1)$ and $\|x - y\| > 0$ on $(t_1, t_1 + \rho)$ for some $\rho > 0$. Note that $x(t), y(t) \in \mathbb{R}^n \setminus N(t)$ for a.a. $t \in (t_1, t_1 + \rho)$. Hence, there exists $l \in L^1(I)$ such that for a.a. $t \in (t_1, t_1 + \rho)$ we have

$$\|f(t, x(t)) - f(t, y(t))\| \le l(t)\|x(t) - y(t)\|.$$

Then

$$\|x(t) - y(t)\| \le \int_{t_1}^{t} \|f(s, x(s)) - f(s, y(s))\|\, ds \le \int_{t_1}^{t} l(s)\|x(s) - y(s)\|\, ds, \quad t \in [t_1, t_1 + \rho),$$

and we deduce from Gronwall's inequality that $\|x - y\| = 0$ on $[t_1, t_1 + \rho)$, a contradiction. □

Remark 4.11. Note that the transversality condition (4.12) cannot be replaced by (3.1) in Theorem 4.10 in order to ensure uniqueness of Carathéodory solutions for (1.1), as shown once again by Example 3.3. Nevertheless, it is enough to ensure uniqueness of Filippov solutions for (1.1).

Example 4.12. The planar system

$$\begin{cases} x' = f_1(t, x, y), & x(0) = 0, \\ y' = f_2(t, x, y), & y(0) = 0, \end{cases}$$

where $f_1(t, x, y) = \cos^2(x\,y) + e^{t - x^2 - y^2}$ and $f_2(t, x, y) = \left(\chi_{\mathbb{Q}}(x) - \chi_{\mathbb{R}\setminus\mathbb{Q}}(x)\right) e^{\sin(t+y)}$, already considered in Example 3.6, has a unique Carathéodory solution. Indeed, the restriction of the function $f = (f_1, f_2)$ to $I \times (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$ is clearly locally Lipschitz continuous with respect to (x, y) and, moreover, condition (b) in Theorem 4.10 can be verified as in Example 3.6 since f_1 is continuous.

Unlike assumption (a) in Theorem 3.3, which is a reasonable condition to obtain existence for discontinuous ODEs, condition (a) in Theorem 4.10 imposes strong restrictions on the discontinuities of f (for instance, f cannot have jump discontinuities).

Finally, let us observe that the following simple result, which leans on local directional Lipschitz conditions in the line of [5], can be useful in many situations with discontinuous nonlinearities.

Theorem 4.13. *In the conditions of Proposition 3.1, assume also that for each $t \in [t_0, t_0 + L)$ and each $\xi \in \mathbb{R}^n$ there exist $\varepsilon = \varepsilon(t, \xi) > 0$ and $l = l_{(t,\xi)} \in L^1(I)$ such that for a.a. $s \in (t, t + \varepsilon)$ we have*

$$\|f(s, x) - f(s, y)\| \le l(s)\|x - y\| \quad \text{for all } x, y \in \prod_{j=1}^{n} \left[\xi_j - \int_t^s \psi(r)\, dr, \xi_j + \int_t^s \psi(r)\, dr\right]. \tag{4.13}$$

Then problem (1.1) has at most one Carathéodory solution.

Proof. Let $x(t)$ and $y(t)$ be Carathéodory solutions of (1.1); we shall prove that $x(t) = y(t)$ for all $t \in I$. Reasoning by contradiction, we assume that there exists some $t_1 \in [t_0, t_0 + L)$ such that $x(t_1) = y(t_1)$ and $\|x - y\| > 0$ on $(t_1, t_1 + \rho)$ for some $\rho > 0$.

Take $\varepsilon \in (0, \rho)$ and $l \in L^1(I)$ in the conditions of (4.13) for the point $(t, \xi) = (t_1, x(t_1))$. We have

$$\|x(t) - y(t)\| \leq \int_{t_1}^{t} \|f(s, x(s)) - f(s, y(s))\|\, ds \leq \int_{t_1}^{t} l(s)\|x(s) - y(s)\|\, ds, \quad t \in [t_1, t_1 + \varepsilon),$$

and then Gronwall's inequality yields $\|x - y\| = 0$ on $[t_1, t_1 + \varepsilon)$, a contradiction. □

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Ground-state solutions of a Hartree–Fock type system involving critical Sobolev exponent

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Abstract. In this paper, ground-state solutions to a Hartree–Fock type system with a critical growth are studied. Firstly, instead of establishing the local Palais–Smale (P.–S.) condition and estimating the mountain-pass critical level, a perturbation method is used to recover compactness and obtain the existence of ground-state solutions. To achieve this, an important step is to get the right continuity of the mountain-pass level on the coefficient in front of perturbing terms. Subsequently, depending on the internal parameters of coupled nonlinearities, whether the ground state is semi-trivial or vectorial is proved.

Keywords: Hartree–Fock systems, ground-state solutions, critical growth.

2020 Mathematics Subject Classification: 35J60, 35A23, 35J50.

1 Introduction

In this paper, we will study the following class of Hartree–Fock (HF) system

$$\begin{cases} -\Delta u + u + \phi_{u,v} u = |u|^{2q-2}u + \beta |v|^q |u|^{q-2} u + \mu(u^5 + \alpha |v|^3 |u| u), & x \in \mathbb{R}^3, \\ -\Delta v + v + \phi_{u,v} v = |v|^{2q-2}v + \beta |u|^q |v|^{q-2} v + \mu(v^5 + \alpha |u|^3 |v| v), & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where the Coulomb term $\phi_{u,v}$ has the following form

$$\phi_{u,v}(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y) + v^2(y)}{|x-y|} \mathrm{d}y, \qquad x \in \mathbb{R}^3, \tag{1.2}$$

$\alpha, \beta, \mu \in \mathbb{R}_+ := [0, \infty)$ are parameters and $q \in (2, 3)$.

It is well known that the (HF) equation is one of the most important equations in quantum physics, condensed matter physics and quantum chemistry. For example, in the study of a molecular system composed of M atomic nucleus interacting with N electrons through Coulomb potential, the (HF) equation is used as an approximation to describe the stationary state, and one can refer to [5] for the specific process of derivation. According to [5], in the system (1.1), $-\Delta u, -\Delta v$ represent the kinetic part of the electronic system, Vu, Vv denote potentials of the action on electronic system by nucleus, $\phi_{u,v} u, \phi_{u,v} v$ represent the electron-electron

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Coulomb interactions, and the power-type nonlinearity describes the effects of exchange and correlation among electrons. For more details on the physical aspects of the Hartree–Fock system, we refer readers to [1,2,8–11] and the references therein.

In mathematics, a particular case of system (1.1), when $\mu = 0$, leads to the following class of Hartree–Fock type system with a cooperative pure power and subcritical nonlinearity

$$\begin{cases} -\Delta u + u + \phi_{u,v} u = |u|^{2q-2}u + \beta |v|^q |u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\Delta v + v + \phi_{u,v} v = |v|^{2q-2}v + \beta |u|^q |v|^{q-2}v, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

which has been studied by d'Avenia, Maia and Siciliano in [5]. In the case that $q \in (3/2, 3)$, they showed the existence of semitrivial and vectorial ground state depending on parameters involved. Furthermore, they also derived the asymptotic behavior of ground states with respect to the parameter β.

In view of conclusions obtained in [5], we considered the Sobolev critical case $q = 3$. However, combining the Pohozaev identity and Nehari manifold, it could be proved that the system (1.3) has no nontrivial solution when $q = 3$. Motivated by the above facts, we would like to consider the system (1.1), which is obtained through a Sobolev perturbation basing on the above system (1.3). It is well known that since Brezis and Nirenberg published their famous paper [3] in 1983, elliptic equations or systems with Sobolev critical growth have been researched extensively. The usual strategy to achieve the ground-state solution to these critical problems is establishing the local (P.-S.) condition and verifying that the ground-state energy belongs to the interval where the (P.-S.) condition holds. Differently, in this paper, we will achieve the existence of ground-state solutions to the system (1.1) with a perturbation method.

Before stating our main results, we introduce the variational setting used in this paper. Firstly, let $H^1_r(\mathbb{R}^3) = \{w \in H^1(\mathbb{R}^3) : w(x) = w(|x|)\}$ and $\|w\|_1^2 = \int_{\mathbb{R}^3} [|\nabla w|^2 + w^2]$ for $w \in H^1_r(\mathbb{R}^3)$. Then our working space is $H := H^1_r(\mathbb{R}^3) \times H^1_r(\mathbb{R}^3)$ endowed with the norm

$$\|(u,v)\| = \left(\|u\|_1^2 + \|v\|_1^2\right)^{1/2}, \qquad (u,v) \in H.$$

It is well known that the embedding $H^1_r(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $s \in [2,6]$ and compact for $s \in (2,6)$. Hence the same conclusions hold for the embedding $H \hookrightarrow L^s(\mathbb{R}^3) \times L^s(\mathbb{R}^3)$ for $s \in [2,6]$. Throughout this paper, denote the norm endowed in $L^s(\mathbb{R}^3)$ by $|\cdot|_s$: $|w|_s = \left[\int_{\mathbb{R}^3} |w|^s\right]^{1/s}$ for $w \in L^s(\mathbb{R}^3)$. While the norm of $L^s(\mathbb{R}^3) \times L^s(\mathbb{R}^3)$ is $|(u,v)|_s = \left(|u|_s^s + |v|_s^s\right)^{1/s}$ for $(u,v) \in L^s(\mathbb{R}^3) \times L^s(\mathbb{R}^3)$. Subsequently, we will give the energy functional corresponding to the system (1.1). According to the Hardy–Littlewood–Sobolev inequality, the nonlocal term $\int_{\mathbb{R}^3} \phi_{u,v}(u^2+v^2)$ is well defined in H. Therefore, we could define the energy functional related to the system (1.1) as

$$\begin{aligned} J_\mu(u,v) &= \frac{1}{2}\|(u,v)\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u,v}(u^2+v^2) - \frac{1}{2q}\left[|u|_{2q}^{2q} + |v|_{2q}^{2q} + 2\beta \int_{\mathbb{R}^3} |u|^q |v|^q\right] \\ &\quad - \frac{\mu}{6}\left[|u|_6^6 + |v|_6^6 + 2\alpha \int_{\mathbb{R}^3} |u|^3 |v|^3\right] \\ &=: \frac{1}{2}A(u,v) + \frac{1}{4}B(u,v) - \frac{1}{2q}C(u,v) - \frac{\mu}{6}D(u,v), \quad (u,v) \in H. \end{aligned} \tag{1.4}$$

Via a standard proof, there also holds that $J_\mu \in C^1(H, \mathbb{R})$ with

$$\begin{aligned}\left\langle J'_\mu(u,v),(\varphi,\psi)\right\rangle = &\int_{\mathbb{R}^3}(\nabla u\cdot\nabla\varphi+\nabla v\cdot\nabla\psi+u\varphi+v\psi)+\int_{\mathbb{R}^3}\phi_{u,v}(u\varphi+v\psi)\\ &-\int_{\mathbb{R}^3}\left[|u|^{2q-2}u\varphi+|v|^{2q-2}v\psi+\beta(|v|^q|u|^{q-2}u\varphi+|u|^q|v|^{q-2}v\psi)\right]\\ &-\mu\int_{\mathbb{R}^3}\left[u^5\varphi+v^5\psi+\alpha(|v|^3|u|u\varphi+|u|^3|v|v\psi)\right], \qquad (u,v),(\varphi,\psi)\in H.\end{aligned}$$

Hence, finding solutions of the system (1.1) is equivalent to seeking critical points of the functional J_μ in H. Furthermore, to achieve the ground-state solution to the system (1.1), we may consider the ground state of the energy functional J_μ, and the Nehari manifold is used in this paper. Now, let I_μ be the related Nehari functional, that is, $I_\mu(u,v) := \langle J'_\mu(u,v),(u,v)\rangle$, $(u,v)\in H$. Then adopting notations given in (1.4) it could be rewritten as

$$I_\mu(u,v) = A(u,v)+B(u,v)-C(u,v)-\mu D(u,v), \qquad (u,v)\in H. \tag{1.5}$$

Let us denote by $\mathcal{N}_\mu$ the Nehari manifold associated to the functional J_μ, namely

$$\mathcal{N}_\mu = \{(u,v)\in H\setminus\{(0,0)\} : I_\mu(u,v)=0\},$$

and define the ground-state energy as

$$d(\mu) = \inf_{\mathcal{N}_\mu} J_\mu.$$

In this context, the ground-state solution to be found in this paper is a radial ground state whose energy is minimal among all other radial ones.

Now, we formulate our first result for the system (1.1).

Theorem 1.1. *Assume that $q\in(2,3)$. Then for any given $\alpha,\beta\in\mathbb{R}_+$, there exists $\mu_0>0$ such that the system (1.1) has a ground-state solution $(u_\mu,v_\mu)\neq(0,0)$ for all $\mu\in[0,\mu_0)$.*

An important step to prove Theorem 1.1 via perturbation methods is estimating the distance between the (P.–S.)$_{m(\mu)}$ sequence of the functional J_μ and the ground-state critical points set of the functional J_0 for μ small enough. Here $m(\mu)$ is the mountain-pass level of the functional J_μ. To achieve this, we first verify the fact that $m(\mu)=d(\mu)$ and get the right continuity of $m(\cdot)$ at $\mu=0$ by showing that $\lim_{\mu\to0^+} d(\mu)=d(0)$ subsequently, where the implicit function theorem is used.

Basing on the existence of ground-state radial solutions, motivated by [4] and [5], we also consider whether the ground state obtained above is semitrivial or vectorial and get the following conclusion. Here we say that $(u,v)\neq(0,0)$ is semitrivial if $u=0$ or $v=0$, and (u,v) is vectorial if $u\neq0$ and $v\neq0$.

Theorem 1.2. *Assume that $q\in(2,3)$ and $\mu\in[0,\mu_0)$, where μ_0 is given by Theorem 1.1. Let (u_μ,v_μ) be the ground state achieved in Theorem 1.1.*

(i) If $0\leqslant\alpha<3, 0\leqslant\beta<2^{q-1}-1$, then (u_μ,v_μ) is semitrivial.

(ii) If $\alpha>3,\beta>2^{q-1}-1$, then (u_μ,v_μ) is vectorial.

In view of Theorem 1.2, there is an open question that whether the ground state obtained in Theorem 1.1 is semitrivial or vectorial in the cases that $(\alpha,\beta) \in (0,3] \times [2^{q-1}-1,\infty)$ or $(\alpha,\beta) \in [3,\infty) \times (0,2^{q-1}-1]$. This is caused by the non-homogeneity of the nonlinearity in the system (1.1).

This paper is organized as follows. In Section 2, we give some preliminaries to get the existence of ground state via the perturbation method, subsequently, Theorems 1.1 and 1.2 are proved in Sections 3 and 4 respectively. Throughout this paper, $C_i (i = 0,1,2,\dots)$ represent some positive constants which may be different from line to line.

2 Preliminaries

In this section, we first give some inequalities about the four functionals A, B, C, and D by the following lemma.

Lemma 2.1. *There exist some constants C_0, C_1, C_2 independent of μ such that for any $(u,v) \in H$, the following inequalities hold*

$$B(u,v) \leqslant C_0[A(u,v)]^2, \tag{2.1}$$

$$C(u,v) \leqslant C_1|(u,v)|_{2q}^{2q} \leqslant C_2[A(u,v)]^q, \tag{2.2}$$

$$D(u,v) \leqslant C_1|(u,v)|_6^6 \leqslant C_2[A(u,v)]^3. \tag{2.3}$$

Proof. For (2.1), it follows from (1.2) that $\phi_{u,v} \in D^{1,2}(\mathbb{R}^3)$ is a weak solution to the equation $-\Delta\phi_{u,v} = u^2 + v^2$ for all $(u,v) \in H$. Consequently,

$$B(u,v) = \int_{\mathbb{R}^3} \phi_{u,v}(u^2+v^2) = \int_{\mathbb{R}^3} |\nabla\phi_{u,v}|^2.$$

By the Hölder inequality and the Sobolev embedding, there exists a constant $C_0 > 0$ independent of (u,v) such that

$$\int_{\mathbb{R}^3} \phi_{u,v}u^2 \leqslant |\phi_{u,v}|_6 |u|_{12/5}^2 \leqslant C_0|\nabla\phi_{u,v}|_2\|u\|_1^2.$$

Similarly, we get

$$\int_{\mathbb{R}^3} \phi_{u,v}v^2 \leqslant C_0|\nabla\phi_{u,v}|_2\|v\|_1^2.$$

Thus

$$|\nabla\phi_{u,v}|_2^2 = \int_{\mathbb{R}^3} \phi_{u,v}(u^2+v^2) \leqslant C_0|\nabla\phi_{u,v}|_2\|(u,v)\|^2 = C_0|\nabla\phi_{u,v}|_2 A(u,v),$$

which implies that (2.1) holds.

By the Hölder inequality and the embedding that $H \hookrightarrow L^s(\mathbb{R}^3) \times L^s(\mathbb{R}^3)$ for $s \in [2,6]$,

$$C(u,v) \leqslant |u|_{2q}^{2q} + |v|_{2q}^{2q} + 2\beta|u|_{2q}^q|v|_{2q}^q \leqslant \max\{\beta,1\}\left(|u|_{2q}^q + |v|_{2q}^q\right)^2 \leqslant C_1|(u,v)|_{2q}^{2q} \leqslant C_2[A(u,v)]^q.$$

Hence (2.2) holds. Similarly, (2.3) holds. □

Next, we prove that the functional J_μ has a mountain pass geometry structure for all $\mu \in \mathbb{R}_+$. Let

$$\Gamma_\mu = \{\gamma \in C([0,1],H) : \gamma(0) = 0, J_\mu(\gamma(1)) < 0\},$$

and

$$m(\mu) = \inf_{\gamma\in\Gamma_\mu} \max_{t\in[0,1]} J_\mu(\gamma(t)).$$

Then we could prove that both Γ_μ and $m(\mu)$ are well defined.

Lemma 2.2. *Assume $\mu \in \mathbb{R}_+$. Then $\Gamma_\mu \neq \emptyset$ and $m(\mu) > 0$.*

Proof. First, for any $(u,v) \in H\backslash\{(0,0)\}$, we define a fiber mapping corresponding to the functional J_μ as follows:

$$\begin{aligned} g_{u,v}(t) &= J_\mu(t(u,v)) \\ &= \frac{t^2}{2}A(u,v) + \frac{t^4}{4}B(u,v) - \frac{t^{2q}}{2q}C(u,v) - \frac{\mu}{6}t^6 D(u,v), \quad t \in \mathbb{R}. \end{aligned} \tag{2.4}$$

Since $q \in (2,3)$, there exists a sufficiently small positive number δ depending on μ such that $g_{u,v}(t) > 0, t \in (0,\delta)$. Moreover, note that $g_{u,v}(t) \to -\infty$ as $t \to \infty$. Then there exists $t_0 > 0$ such that $J_\mu(t_0(u,v)) = g_{u,v}(t_0) < 0$. Let $\gamma_0(t) = tt_0(u,v), t \in [0,1]$. Then $\gamma_0 \in \Gamma_\mu$.

For $\mu \in \mathbb{R}_+$, it follows from inequalities (2.2) and (2.3) that

$$J_\mu(u,v) \geqslant \frac{1}{2}A(u,v) - \frac{1}{2q}C_2[A(u,v)]^q - \frac{1}{6}\mu C_2[A(u,v)]^3, \quad (u,v) \in H.$$

Therefore, there exists $\rho > 0$ depending on μ such that if $0 < \|(u,v)\|^2 = A(u,v) < \rho^2$, then $J_\mu(u,v) > 0$. Moreover,

$$\alpha_\mu := \inf_{\|(u,v)\|=\rho} J_\mu(u,v) > 0.$$

Furthermore, by the standard process one can deduce that $m(\mu) \geqslant \alpha_\mu > 0$. □

Lemma 2.3. *Suppose that $(u,v) \in H\backslash\{(0,0)\}$. Then the following conclusions hold:*

(i) for any $\mu \in \mathbb{R}_+$, there exists a unique $t(\mu) > 0$ such that $t(\mu)(u,v) \in \mathcal{N}_\mu$, $I_\mu(t(u,v)) > 0, t \in (0,t(\mu))$ and $I_\mu(t(u,v)) < 0, t \in (t(\mu),\infty)$. Furthermore,

$$J_\mu(t(\mu)(u,v)) = \max_{t\in\mathbb{R}_+} J_\mu(t(u,v));$$

(ii) the function $t(\cdot)\colon \mathbb{R}_+ \to (0,\infty)$ is differentiable and

$$t'(\mu) = -\frac{t^5(\mu)D(u,v)}{2A(u,v) + 4t^2(\mu)B(u,v) + 2qt^{2q-2}(\mu)C(u,v) + 6\mu t^4(\mu)D(u,v)}. \tag{2.5}$$

Moreover, $t(\cdot)$ is decreasing in μ.

Proof. (i) Assume $\mu \in \mathbb{R}_+$. For each $(u,v) \in H\backslash\{(0,0)\}$, recall the definition of $g_{u,v}$ given in (2.4). Then

$$g'_{u,v}(t) = tA(u,v) + t^3B(u,v) - t^{2q-1}C(u,v) - \mu t^5 D(u,v), \qquad t \in \mathbb{R}_+, \tag{2.6}$$

which yields that

$$g'_{u,v}(t)/t \to A(u,v) > 0, t \to 0^+, \qquad g'_{u,v}(t) \to -\infty,\ t \to \infty. \tag{2.7}$$

Therefore, there exists $t(\mu) > 0$ satisfying $g'_{u,v}(t(\mu)) = 0$, and so $t(\mu)u \in \mathcal{N}_\mu$. Furthermore, it follows from (2.6) that

$$t^{-2}(\mu)A(u,v) - t^{2q-4}(\mu)C(u,v) - \mu t^2(\mu)D(u,v) = -B(u,v).$$

Because the function $t \mapsto t^{-2}A(u,v) - t^{2q-4}C(u,v) - \mu t^2 D(u,v)$ is decreasing in t, then $g'_{(u,v)}(t) = 0$ has a unique positive root. Hence, $t(\mu)$ is the unique positive critical point of $g_{u,v}$. Combining this with (2.7) and (1.5), we know that (i) holds.

(ii) Let $H(t,\mu) = I_\mu(t(u,v)), (t,\mu) \in (-\delta,\infty) \times (-\delta,\infty)$ for some $\delta > 0$. Then it follows from (1.5) that

$$H(t,\mu) = t^2 A(u,v) + t^4 B(u,v) - t^{2q} C(u,v) - \mu t^6 D(u,v), \qquad (t,\mu) \in (-\delta,\infty) \times (-\delta,\infty).$$

For any $(t,\mu) \in (-\delta,\infty) \times (-\delta,\infty)$ for some $\delta > 0$, we have

$$\frac{\partial H}{\partial t}(t,\mu) = 2tA(u,v) + 4t^3 B(u,v) - 2qt^{2q-1} C(u,v) - 6\mu t^5 D(u,v) \tag{2.8}$$

and

$$\frac{\partial H}{\partial \mu}(t,\mu) = -t^6 D(u,v).$$

Note that $H(t(\mu),\mu) = 0$ i.e. $I_\mu(t(\mu)(u,v)) = 0$ for $\mu \in [0,1]$. Then it could be derived from (2.8) and (1.5) that

$$\frac{\partial H}{\partial t}(t(\mu),\mu) = -2t(\mu)A(u,v) - (2q-4)t^{2q-1}(\mu)C(u,v) - 2\mu t^5(\mu)D(u,v) < 0.$$

Therefore, by the implicit function theorem, we can obtain that $t(\cdot) : \mathbb{R}_+ \to (0,\infty)$ is continuous and differentiable, and (2.5) holds. It then follows from (2.5) directly that for given $(u,v) \in H \backslash \{(0,0)\}$, $t(\cdot)$ is decreasing in μ. □

Lemma 2.4. $\inf_{\mu \in [0,1]} \operatorname{dist}(0, \mathcal{N}_\mu) > 0$.

Proof. Assume $\mu \in [0,1]$. Given $(u,v) \in \mathcal{N}_\mu$, it follows from (1.5), (2.2) and (2.3) that

$$\begin{aligned} A(u,v) &\leqslant A(u,v) + B(u,v) \\ &= C(u,v) + \mu D(u,v) \\ &\leqslant C_2 \left[[A(u,v)]^q + [A(u,v)]^3 \right]. \end{aligned}$$

Therefore there exists a $\sigma > 0$ independent of μ such that $\|(u,v)\|^2 = A(u,v) \geqslant \sigma$ for all $(u,v) \in \mathcal{N}_\mu$. The proof is complete. □

In view of Lemma 2.4, since $2q > 4$, then for any $(u,v) \in \mathcal{N}_\mu$, it holds that

$$J_\mu(u,v) = J_\mu(u,v) - \frac{1}{4} I_\mu(u,v) \geqslant \frac{1}{4}\|(u,v)\|^2, \qquad (u,v) \in \mathcal{N}_\mu, \tag{2.9}$$

from which one can also derive that $d(\mu)$ is well defined for all $\mu \in \mathbb{R}_+$. Moreover, recall the definition of $m(\mu)$. Then by Lemma 2.2 and (i) of Lemma 2.3, we can prove that following lemma via a standard process similarly to the proof of [12, Theorem 4.2, p. 73].

Lemma 2.5. *For any $\mu \in \mathbb{R}_+$, it holds that $m(\mu) = d(\mu)$.*

Note that due to definitions of J_μ, Γ_μ and $m(\mu)$ for $\mu \in \mathbb{R}_+$, it could be concluded that $m(\cdot)$ is decreasing on $\mathbb{R}_+$. Then by Lemma 2.5, $d(\cdot)$ is also decreasing on $\mathbb{R}_+$. Now, we will prove the continuity of $m(\cdot)$ at $\mu = 0$. Actually, by the above lemma, it is sufficient to illustrate the right continuity of $d(\cdot)$ at $\mu = 0$. Hence, we have the following lemma.

Lemma 2.6. *$d(\cdot)$ is right continuous at $\mu = 0$.*

Proof. Assume $\{\mu_n\} \subset [0,1]$ satisfies that $\mu_n \to 0^+$ as $n \to \infty$. Then, according to the definition of $d(\mu_n)$, for each $\varepsilon \in (0, d(0))$, there exists $(u_n, v_n) \in \mathcal{N}_{\mu_n}$ such that

$$J_{\mu_n}(u_n, v_n) \leqslant d(\mu_n) + \varepsilon/2, \qquad n \in \mathbb{N}. \tag{2.10}$$

Combining this with (2.9), one gets that

$$A(u_n, v_n) = \|(u_n, v_n)\|^2 \leqslant 4d(\mu_n) + 2\varepsilon < 6d(0). \tag{2.11}$$

Thus, there exist $(u, v) \in H$ and a subsequence of $\{(u_n, v_n)\}$ (still denoted by $\{(u_n, v_n)\}$) such that $(u_n, v_n) \rightharpoonup (u, v)$. Moreover, $(u, v) \neq (0, 0)$. Otherwise, it follows from $(u_n, v_n) \in \mathcal{N}_{\mu_n}$, (2.2), the compact embedding $H \hookrightarrow L^{2q}(\mathbb{R}^3) \times L^{2q}(\mathbb{R}^3)$, (2.3) and (2.11) that

$$A(u_n, v_n) + B(u_n, v_n) = C(u_n, v_n) + \mu_n D(u_n, v_n) \leqslant C_1 |(u_n, v_n)|_{2q}^{2q} + \mu_n C_2 [A(u_n, v_n)]^3 \to 0,$$

which contradicts to Lemma 2.4. Hence $(u, v) \neq (0, 0)$. Consequently, noting $(u_n, v_n) \to (u, v)$ in $L^{2q}(\mathbb{R}^3) \times L^{2q}(\mathbb{R}^3)$, there exists some $N_0 > 0$ such that for $n > N_0$,

$$C(u_n, v_n) \geqslant |(u_n, v_n)|_{2q}^{2q} \geqslant |(u, v)|_{2q}^{2q}/2 > 0,$$

which implies that for some positive number C_3 independent of n such that

$$C(u_n, v_n) \geqslant C_3 > 0, \qquad n \in \mathbb{N} \tag{2.12}$$

Now, for given $n > N_0$, according to (i) of Lemma 2.3, let $t_n(\mu)$ satisfy that $t_n(\mu)(u_n, v_n) \in \mathcal{N}_\mu$ for all $\mu \in [0, \mu_n]$, and define

$$h_n(\mu) = J_\mu(t_n(\mu)(u_n, v_n)), \qquad \mu \in [0, \mu_n].$$

Since $t_n(\mu)(u_n, v_n) \in \mathcal{N}_\mu$, one could derive that

$$\begin{aligned} h_n'(\mu) &= \left\langle J_\mu'(t_n(\mu)(u_n, v_n)), (u_n, v_n) \right\rangle t_n'(\mu) - \frac{1}{6} t_n^6(\mu) \left[|u_n|_6^6 + |v_n|_6^6 + 2\alpha \int_{\mathbb{R}^3} |u_n|^3 |v_n|^3 \right] \\ &= -\frac{1}{6} t_n^6(\mu) D(u_n, v_n), \quad \mu \in [0, \mu_n]. \end{aligned}$$

Hence, from $t_n(\mu_n) = 1$, (ii) of Lemma 2.3, (2.3) and (2.11), we arrive at

$$\begin{aligned} & J_0(t_n(0)(u_n, v_n)) - J_{\mu_n}(u_n, v_n) \\ &\quad = h_n(0) - h_n(\mu_n) \\ &\quad = -\int_0^{\mu_n} h_n'(s) \mathrm{d}s \\ &\quad = \frac{1}{6} \int_0^{\mu_n} t_n^6(s) D(u_n, v_n) \mathrm{d}s \\ &\quad \leqslant \frac{1}{6} t_n^6(0) \mu_n C_2 [A(u_n, v_n)]^3 \leqslant 36 t_n^6(0) \mu_n C_2 d^3(0). \end{aligned} \tag{2.13}$$

Next, we shall illustrate that $\{t_n(0)\}$ is bounded. Indeed, due to $I_0(u_n, v_n) > I_\mu(u_n, v_n) = 0$ and (i) of Lemma 2.3, it holds that $t_n(0) > 1$. Moreover, by (2.12), (2.1) and (2.11) one can deduce that

$$C_3 t_n^{2q}(0) \leqslant t_n^{2q}(0) C(u_n, v_n) = t_n^2(0) A(u_n, v_n) + t_n^4(0) B(u_n, v_n) \leqslant 6d(0) t_n^2(0) + 36 C_2 d^2(0) t_n^4(0).$$

Since $q > 2$, this implies that there exists some C_4 independent of n such that $1 < t_n(0) \leqslant C_4$ for $n \in \mathbb{N}$.

Subsequently, combining this with (2.13), it holds that

$$J_0(t_n(0)(u_n, v_n)) - J_{\mu_n}(u_n, v_n) \leqslant 36C_4^6\mu_n C_2 d^3(0).$$

Furthermore, as a consequence of Lemma 2.5, the fact that $m(0) \geqslant m(\mu)$ for $\mu \geqslant 0$ and (2.10), we also get that

$$0 \leqslant d(0) - d(\mu_n) \leqslant J_0(t_n(0)(u_n, v_n)) - J_{\mu_n}(u_n, v_n) + \varepsilon/2 \leqslant 36C_4^6\mu_n C_2 d^3(0) + \varepsilon/2.$$

Thus,

$$0 \leqslant \limsup_{n\to\infty}[d(0) - d(\mu_n)] \leqslant \varepsilon/2,$$

which yields that $d(\mu_n) \to d(0)$ as a consequence of the arbitrariness of ε. The proof is complete. □

3 Proof of Theorem 1.1

Lemma 3.1. *Assume $\mu \in (0,1]$ and $\{(u_n^\mu, v_n^\mu)\}$ is a (P.–S.)$_{m(\mu)}$ sequence for the functional J_μ. Then*

$$\lim_{\mu\to 0}\lim_{n\to\infty} \operatorname{dist}\big((u_n^\mu, v_n^\mu), K\big) = 0,$$

where

$$K = \{(u,v) \in H : J_0'(u,v) = 0, J_0(u,v) = m(0)\}.$$

Proof. This proof is motivated by [13] and [6]. Firstly, by the mountain pass theorem and the fact that J_0 satisfies the (P.–S.) condition on H, it holds that $K \neq \emptyset$.

Secondly, for any $\mu \in (0,1]$, since $\{(u_n^\mu, v_n^\mu)\}$ is a (P.–S.)$_{m(\mu)}$ sequence for the functional J_μ, we have

$$m(\mu) + 1 + \|(u_n^\mu, v_n^\mu)\| \geqslant J_\mu(u_n^\mu, v_n^\mu) - \frac{1}{4}I_\mu(u_n^\mu, v_n^\mu).$$

Thus, similarly to (2.9) we can derive that

$$m(0) + 1 + \|(u_n^\mu, v_n^\mu)\| \geqslant \frac{1}{4}\|(u_n^\mu, v_n^\mu)\|^2. \tag{3.1}$$

Therefore, there is a constant $C_5 > 0$ independent of μ and n such that $\|(u_n^\mu, v_n^\mu)\| \leqslant C_5$ for all $n \in \mathbb{N}$ and $\mu \in (0,1]$.

Now, assume that $\{\mu_i\}$ satisfies $\mu_i \to 0$ as $i \to \infty$. Denote the (P.–S.)$_{m(\mu_i)}$ sequence of the functional J_{μ_i} by $\{(u_n^{\mu_i}, v_n^{\mu_i})\}$. Furthermore, for any given i, we could find $n_i > i$ such that

$$|J_{\mu_i}(u_{n_i}^{\mu_i}, v_{n_i}^{\mu_i}) - m(\mu_i)| \leqslant \frac{1}{i}, \qquad \left\|J_{\mu_i}'(u_{n_i}^{\mu_i}, v_{n_i}^{\mu_i})\right\| \leqslant \frac{1}{i}.$$

Denote $\{(u_{n_i}^{\mu_i}, v_{n_i}^{\mu_i})\}$ by $\{(u_i, v_i)\}$. Then by (2.3), the uniform boundedness of the sequence $\{(u_n^\mu, v_n^\mu)\}$, Lemma 2.6 and $\mu_i \to 0$, we can derive that

$$\begin{aligned}|J_0(u_i, v_i) - m(0)| &\leqslant |J_{\mu_i}(u_i, v_i) - m(\mu_i)| + \frac{\mu_i}{6}\left[|u_i|_6^6 + |v_i|_6^6 + 2\alpha\int_{\mathbb{R}^3}|u_i|^3|v_i|^3\right] + m(0) - m(\mu_i)\\ &\leqslant \frac{1}{i} + \frac{\mu_i}{6}C_2C_5^6 + m(0) - m(\mu_i) \to 0, \qquad i \to \infty.\end{aligned}$$

Similarly,

$$\|J_0'(u_i, v_i)\| \leqslant \|J_0'(u_i, v_i)\| + \mu_i C_6 \|(u_i, v_i)\|^5 \to 0, \qquad i \to \infty,$$

where C_6 is some positive constant independent of i. These yield that $\{(u_i, v_i)\}$ is a (P.-S.)$_{m(0)}$ sequence of J_0. Due to the fact that J_0 satisfies the (P.–S.) condition on H, then there exists $(u_0, v_0) \in K$ and up to a subsequence still denoted by $\{(u_i, v_i)\}$ such that $(u_i, v_i) \to (u_0, v_0)$ as $i \to \infty$. Thus, one can derive that

$$\lim_{i\to\infty} \lim_{n\to\infty} \operatorname{dist}\big((u_n^{\mu_i}, v_n^{\mu_i}), K\big) \leqslant \lim_{i\to\infty} \operatorname{dist}\big((u_i, v_i), K\big) = 0.$$

In the end, by the arbitrariness for $\{\mu_i\}$, the proof is complete. □

Proof of Theorem 1.1. Assume $\mu \in [0,1]$ and $\{(u_n^\mu, v_n^\mu)\}$ is a (P.–S.)$_{m(\mu)}$ sequence of the functional J_μ. Similarly to (3.1), we can derive that $\{(u_n^\mu, v_n^\mu)\}$ is uniformly bounded for $\mu \in [0,1]$, then there exists $(u_\mu, v_\mu) \in H$ and a subsequence for $\{(u_n^\mu, v_n^\mu)\}$ still denoted by $\{(u_{n_i}^\mu, v_{n_i}^\mu)\}$ such that $(u_{n_i}^\mu, v_{n_i}^\mu) \rightharpoonup (u_\mu, v_\mu)$ as $i \to \infty$ and $J_\mu'(u_\mu, v_\mu) = 0$.

In what follows, we will prove that there exists $\mu_0 > 0$, such that $(u_\mu, v_\mu) \neq (0,0)$ for $\mu \in [0, \mu_0]$. Indeed, since $m(0) > 0$ and K is nonempty and compact, $\delta_0 := \operatorname{dist}((0,0), K) = \min_{(u,v)\in K} \|(u,v)\| > 0$. According to Lemma 3.1,

$$\lim_{\mu\to 0} \lim_{i\to\infty} \operatorname{dist}\big((u_{n_i}^\mu, v_{n_i}^\mu), K\big) = 0.$$

Hence, for any given $\delta < \delta_0/2$, there exists some $\mu_0 = \mu_0(\delta)$ satisfying: for any $\mu \in (0, \mu_0)$ there is $i_0 = i_0(\mu)$ such that

$$\operatorname{dist}\big((u_{n_i}^\mu, v_{n_i}^\mu), K\big) < \delta, \qquad i > i_0.$$

Thus, for fixed $\mu \in (0, \mu_0)$, by the compactness of K, one can obtain a sequence $\{(u_i^\mu, v_i^\mu)\} \subset K$ such that $\|(u_{n_i}^\mu, v_{n_i}^\mu) - (u_i^\mu, v_i^\mu)\| \leqslant \delta$ for $i > i_0$. Moreover, noting that there is $(u_0^\mu, v_0^\mu) \in K$ such that $(u_i^\mu, v_i^\mu) \to (u_0^\mu, v_0^\mu)$ as $i \to \infty$, it also holds that $(u_{n_i}^\mu, v_{n_i}^\mu) \in B_{2\delta}(u_0^\mu, v_0^\mu)$ for i large. Therefore, the facts that $\overline{B_{2\delta}(u_0^\mu, v_0^\mu)}$ is closed weakly and $(u_{n_i}^\mu, v_{n_i}^\mu) \rightharpoonup (u_\mu, v_\mu)$ lead to

$$(u_\mu, v_\mu) \in \overline{B_{2\delta}(u_0^\mu, v_0^\mu)}.$$

Thereby, owing to the choosing of δ,

$$\|(u_\mu, v_\mu)\| \geqslant \|(u_0^\mu, v_0^\mu)\| - 2\delta > 0, \qquad \mu \in (0, \mu_0).$$

In the end, we will prove that (u_μ, v_μ) is a ground-state solution to the system (1.1). Actually, it is sufficient to prove that $J_\mu(u_\mu, v_\mu) = d(\mu)$ since $(u_\mu, v_\mu) \neq (0,0)$ and $J_\mu'(u_\mu, v_\mu) = 0$. To achieve this, we calculate the following inequalities:

$$\begin{aligned}
d(\mu) &\leqslant J_\mu(u_\mu, v_\mu) \\
&= J_\mu(u_\mu, v_\mu) - I_\mu(u_\mu, v_\mu)/4 \\
&= A(u_\mu, v_\mu)/4 + (q-2)C(u_\mu, v_\mu)/(4q) + \mu D(u_\mu, v_\mu)/12 \\
&\leqslant \liminf_{i\to\infty} \big[A(u_{n_i}^\mu, v_{n_i}^\mu)/4 + (q-2)C(u_{n_i}^\mu, v_{n_i}^\mu)/(4q) + \mu D(u_{n_i}^\mu, v_{n_i}^\mu)/12\big] \\
&= \liminf_{i\to\infty} [J_\mu(u_{n_i}^\mu, v_{n_i}^\mu) - I_\mu(u_{n_i}^\mu, v_{n_i}^\mu)/4] = m(\mu).
\end{aligned}$$

Hence, it follows from Lemma 2.5 that $J_\mu(u_\mu, v_\mu) = d(\mu)$. Therefore, (u_μ, v_μ) is a ground-state solution to the system (1.1). □

4 Proof of Theorem 1.2

Lemma 4.1. *Let $p \in [2,\infty)$, and $\sigma \geqslant 0$. Define*

$$h_\sigma(s) = s^p + (1-s)^p + 2\sigma s^{p/2}(1-s)^{p/2}, \qquad s \in [0,1].$$

Then

(i) *if $\sigma < 2^{p-1}-1$, then $h_\sigma(s) < 1$ for all $s \in (0,1)$;*

(ii) *if $\sigma > 2^{p-1}-1$, then $h_\sigma(1/2) > 1$.*

Proof. For (i) one can refer to [4, Lemma 2.7] or [5, Lemma 2.4], while (ii) could be derived through a direct calculation. □

Lemma 4.2. *Assume that $q \in (2,3)$, $\mu > 0$, $0 \leqslant \alpha < 3, 0 \leqslant \beta < 2^{q-1}-1$. If $(u,v) \in H$ is a ground-state radial solution to the system (1.1) with $J_\mu(u,v) = d(\mu)$, then $u = 0$ or $v = 0$.*

Proof. Suppose by contradiction, $u \neq 0$ and $v \neq 0$. Then replacing (u,v) with $(|u|,|v|)$, by a regularity process and the maximum principle, one could also assume that $u > 0$ and $v > 0$. Now, let (ρ,θ) be the polar form of (u,v), that is, that is,

$$(u,v) = (\rho\cos\theta, \rho\sin\theta), \qquad \rho = \rho(x) > 0, \qquad \theta = \theta(x) \in (0,\pi/2).$$

Then on one aspect, by the convexity inequality for gradients in [7], there also holds that $\rho = \sqrt{u^2+v^2} \in H_r^1(\mathbb{R}^3)$. On the other aspect, through calculations, we could get that

$$\nabla u = (\cos\theta)\nabla\rho - \rho(\sin\theta)\nabla\theta, \qquad \nabla v = (\sin\theta)\nabla\rho + \rho(\cos\theta)\nabla\theta.$$

Hence, it follows from definitions of functionals A, B, C and D given in (1.4),

$$\begin{aligned}
A(u,v) &= \int_{\mathbb{R}^3} \left[|\nabla\rho|^2 + \rho^2|\nabla\theta|^2 + \rho^2\right] = A(\rho,0) + \int_{\mathbb{R}^3} \rho^2|\nabla\theta|^2,\\
B(u,v) &= \lambda\int_{\mathbb{R}^3} \phi_{u,v}(u^2+v^2) = \lambda\int_{\mathbb{R}^3} \phi_{\rho,0}\rho^2 = B(\rho,0),\\
C(u,v) &= \int_{\mathbb{R}^3} \rho^{2q}\left[\cos^{2q}\theta + \sin^{2q}\theta + 2\beta\cos^q\theta\sin^q\theta\right] = \int_{\mathbb{R}^3} \rho^{2q} h_\beta(\cos^2\theta),
\end{aligned}$$

and similarly there holds

$$D(u,v) = \int_{\mathbb{R}^3} \rho^6 h_\alpha(\cos^2\theta).$$

Furthermore, since $\theta \in (0,1)$, then by Lemma 4.1 it holds that

$$C(u,v) < |\rho|_{2q}^{2q} = C(\rho,0), \quad D(u,v) < D(\rho,0).$$

Note that by (i) of Lemma 2.3 there exists some $t(\mu) > 0$ such that $t(\mu)(\rho,0) \in \mathcal{N}_\mu$. Then

$$\begin{aligned}
d(\mu) &\leqslant J_\mu(t(\mu)(\rho,0))\\
&= \frac{1}{2}t^2(\mu)A(\rho,0) + \frac{1}{4}t^4(\mu)B(\rho,0) - \frac{1}{2q}t^{2q}(\mu)C(\rho,0) - \frac{1}{6}\mu t^6(\mu)D(\rho,0)\\
&< \frac{1}{2}t^2(\mu)A(u,v) + \frac{1}{4}t^4(\mu)B(u,v) - \frac{1}{2q}t^{2q}(\mu)C(u,v) - \frac{1}{6}\mu t^6(\mu)D(u,v)\\
&= J_\mu(t(\mu)(u,v)) < J_\mu(u,v) = d(\mu).
\end{aligned}$$

This is absurd. Thus, it could only hold that $u = 0$ or $v = 0$. □

Lemma 4.3. *Assume that $q \in (2,3)$, $\mu > 0$, $\alpha > 3, \beta > 2^{q-1} - 1$. If $(u,v) \in H$ is a ground-state radial solution to the system (1.1) with $J_\mu(u,v) = d(\mu)$, then $u \neq 0$ and $v \neq 0$.*

Proof. Arguing by contradiction, we suppose that $v = 0$. Define $(u_0, v_0) = (u/\sqrt{2}, v/\sqrt{2})$. Then $(u_0, v_0) \in H \backslash \{(0,0)\}$. Similarly to the proof of Lemma 4.2, one could derive that

$$A(u_0, v_0) = A(u,0), \qquad B(u_0, v_0) = B(u,0)$$

and

$$C(u_0, v_0) = h_\beta(1/2)C(u,0), \qquad D(u_0, v_0) = h_\alpha(1/2)D(u,0).$$

Moreover, by (i) of Lemma 2.3, there exists a unique $t(\mu) > 0$ such that $t(\mu)(u_0, v_0) \in \mathcal{N}_\mu$. Now, we make the following calculation

$$\begin{aligned} &J_\mu(t(\mu)(u_0, v_0)) \\ &\quad = \frac{1}{2}A(u_0, v_0)t^2(\mu) + \frac{1}{4}B(u_0, v_0)t^4(\mu) - \frac{1}{2q}C(u_0, v_0)t^{2q}(\mu) - \frac{1}{6}\mu D(u_0, v_0)t^6(\mu) \\ &\quad = \frac{1}{2}A(u,0)t^2(\mu) + \frac{1}{4}B(u,0)t^4(\mu) - \frac{1}{2q}h_\beta(1/2)C(u,0)t^{2q}(\mu) - \frac{1}{6}\mu h_\alpha(1/2)D(u,0)t^6(\mu) \\ &\quad < \frac{1}{2}A(u,0)t^2(\mu) + \frac{1}{4}B(u,0)t^4(\mu) - \frac{1}{2q}C(u,0)t^{2q}(\mu) - \frac{1}{6}\mu D(u,0)t^6(\mu) = J_\mu(t(\mu)(u,0)). \end{aligned}$$

Consequently,

$$d(\mu) \leqslant J_\mu(t(\mu)(u_0, v_0)) < J_\mu(t(\mu)(u,0)) \leqslant J_\mu(u,0) = d(\mu),$$

which is a contraction. Thus, it holds that $u \neq 0$ and $v \neq 0$. □

Proof of Theorem 1.2. According to Lemmas 4.2 and 4.3, one can get Theorem 1.2 directly. □

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Two regularity criteria of the 3D magneto-micropolar equations in Vishik spaces

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Abstract. In this paper, we utilize the Littlewood–Paley decomposition theory to establish two regularity criteria for the 3D magneto-micropolar equations in Vishik spaces, specifically focusing on the gradient of the velocity field.

Keywords: magneto-micropolar equations, regularity criteria, Vishik spaces.

2020 Mathematics Subject Classification: 35Q35, 35B40, 76A15.

1 Introduction

In this paper, we study the magneto-micropolar system in the whole space $\mathbb{R}^3$:

$$\begin{cases} \partial_t u - (\mu+\chi)\Delta u + (u\cdot\nabla)u - (b\cdot\nabla)b - \chi\nabla\times\omega + \nabla p = 0, \\ \partial_t \omega - \gamma\Delta\omega - \kappa\nabla\nabla\cdot\omega + 2\chi\omega + (u\cdot\nabla)\omega - \chi\nabla\times u = 0, \\ \partial_t b - \nu\Delta b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x),\ \omega(x,0) = \omega_0(x), b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $(x,t)\in\mathbb{R}^3\times\mathbb{R}^+$, $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$, $\omega(x,t) = (\omega_1(x,t), \omega_2(x,t), \omega_3(x,t))$, $b(x,t) = (b_1(x,t), b_2(x,t), b_3(x,t))$ and p denote the fluid velocity, micro-rotational velocity, magnetic field and scalar pressure, respectively. Here, μ is the kinematic viscosity, χ is the vortex viscosity, $\frac{1}{\nu}$ is the magnetic Reynolds number, while κ and γ denote angular viscosities. This model has been used to study microelectrode fluid motion in the presence of a magnetic field. It was first proposed by Galdi and Rionero [8] to address microscopic physical phenomena, such as the motion of animal blood, liquid crystals, and dilute aqueous polymer solutions, which cannot be accurately described by the classical Navier–Stokes equations for incompressible viscous fluids. These fluids are characterized by asymmetric stress tensors, which is why they are referred to as asymmetric fluids. Due to the complex physical background and the richness of the phenomena involved, incompressible micropolar fluids have been extensively studied (see [1, 2, 4, 5, 12, 17, 19, 23, 24] and references therein).

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Function spaces are essential tools for studying and solving systems of fluid mechanics equations. By means of Sobolev spaces [13], researchers can effectively characterize solutions to fluid dynamics problems and investigate their existence, uniqueness, and regularity. To further refine the analysis of local behavior and regularity, especially when dealing with nonlinear partial differential equations, Morrey-type spaces are employed (see [6,16,21,27]). These spaces better capture the local integrability and smoothness of solutions, aiding in the study of conditions for the emergence of local singularities and vortex structures, while also providing more precise integral estimates for nonlinear terms. The concept of weak solutions relies on the weak formulation within function spaces. By examining the well-posedness of fluid mechanics equations in various function spaces, researchers can not only ensure the reasonableness and solvability of these systems but also gain a deeper understanding of solution regularity, nonlinear characteristics, stability, and the feasibility of numerical computations. For more details, please refer to [9,10,14].

Rojas-Medar and Boldrini [18] used the Galerkin method to prove, for the first time, the existence of weak solutions to the 2D and 3D magnetic differential equations (1.1). Yuan [25] showed that if $\nabla u \in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$, then the weak solution is smooth on $\mathbb{R}\times[0,T]$. Subsequently, Gala [7], Zhang et al. [28] and Xu [22] extended the regularity criterion to Morrey–Campanato spaces, Triebel–Lizorkin spaces and Besov spaces, respectively. Yuan and Li [26] further refined the results of Xu [22]. Recently, Wu [20] established the regularity of the weak solution to this system by imposing specific conditions on the partial derivatives of the velocity and magnetic field components. Additionally, Qin and Zhang [15] obtained optimal decay estimates for the higher-order derivatives of the strong solution to the system (1.1).

The aim of this paper is to study the regularity criterion for the solution of the magneto-micropolar equations (1.1). Understanding this criterion is crucial for comprehending the physical laws governing magneto-micropolar motion. Notably, $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)\subset \dot{V}^0_{\infty,\infty,\theta}(\mathbb{R}^3)$, where the Vishik spaces $\dot{V}^s_{p,r,\theta}(\mathbb{R}^3)$ are introduced as a class of Banach spaces (see Definition 2.3). Consequently, we anticipate that weak solution exhibit corresponding smoothness in such Banach spaces. In this paper, we demonstrate that to ensure the regularity of weak solution to (1.1), it is sufficient to impose certain conditions on the fluid's velocity field. This finding also indirectly suggests that, in the study of weak solution regularity, the fluid velocity u plays a more significant role than both the microscopic rotational velocity ω of the particles and the magnetic field b.

Our main result of the paper is stated as follows:

Theorem 1.1. *Let $(u_0,\omega_0,b_0)\in H^1(\mathbb{R}^3)$ and $\nabla\cdot u_0=\nabla\cdot b_0=0$. Assume that (u,ω,b) is a weak solution to the system (1.1) on the interval $[0,T]$. If the velocity gradient ∇u satisfies one of the following conditions:*

$$\nabla u \in L^1\Big((0,T;\dot{V}^{\frac{3}{p}}_{p,r,1}(\mathbb{R}^3)\Big),\qquad p\geq 1, \tag{1.2}$$

$$\nabla u \in L^{\frac{2p}{2p-3}}(0,T;\dot{V}^0_{p,r,1}(\mathbb{R}^3)),\qquad p\geq \frac{3}{2}, \tag{1.3}$$

then the weak solution (u,ω,b) is smooth on $[0,T]$.

Remark 1.2. Notice that for $\theta\in[1,\infty]$, we have $\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)\subset\dot{V}^0_{\infty,\infty,\theta}(\mathbb{R}^3)$. Therefore, Theorem 1.1 can be viewed as a further improvement of [24].

The rest of this paper is organized as follows. Section 2 reviews some preliminaries. Section 3 is devoted to the proof of Theorem 1.1.

2 Preliminaries

Let us begin with a brief review of the definition of Littlewood–Paley decomposition, as detailed in [3]. Let χ be a smooth, radially non-increasing function that takes values in $[0,1]$ and is supported within the ball $|\xi| \leq \frac{4}{3}$. Define φ in terms of χ by setting $\varphi(\xi) := \chi(\frac{\xi}{2}) - \chi(\xi)$, so that φ is supported in the annulus $\{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. These functions satisfy the following partition of unity:

$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^3; \qquad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \neq 0.$$

Let $h = \check{\varphi}, \tilde{h} = \check{\chi}$, where $\check{\varphi}$ and $\check{\chi}$ denote the inverse Fourier transforms of φ and χ, respectively. The dyadic blocks $\dot{\Delta}_j u$ and low-frequency cut-off $\dot{S}_j u$ can then be defined as:

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{3j}\int_{\mathbb{R}^3} h(2^j y)u(x-y)\mathrm{d}y,$$

$$\dot{S}_j u = \sum_{k\leq j-1} \Delta_j u = 2^{3j}\int_{\mathbb{R}^3} \tilde{h}(2^j y)u(x-y)\mathrm{d}y, \qquad j \in \mathbb{Z}.$$

According to the Bony decomposition, any distribution $u \in \mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{P}(\mathbb{R}^3)$ can be expressed as:

$$u = \sum_{j=-\infty}^{+\infty} \dot{\Delta}_j u, \qquad u \in \mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{P}(\mathbb{R}^3),$$

where $\mathcal{P}(\mathbb{R}^3)$ denotes the set of polynomials.

Recall the definition of the homogeneous Besov spaces [3], which are based on the Littlewood–Paley decomposition.

Definition 2.1. Let $p, r \in [1,\infty]$ and $s \in \mathbb{R}$. The homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^3)$ are defined as

$$\dot{B}^s_{p,r}(\mathbb{R}^3) := \left\{ f \in \mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{P}(\mathbb{R}^3) : \|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} < \infty \right\},$$

where

$$\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} := \begin{cases} \left(\sum_{j=1}^{\infty} 2^{jrs}\|\dot{\Delta}_j f\|^r_{L^p}\right)^{\frac{1}{r}}, & r \neq \infty, \\ \sup_{j\in\mathbb{Z}} \|\dot{\Delta}_j f\|_{L^p}, & r = \infty. \end{cases}$$

We also recall the Bernstein inequality, which plays a key role in the proof of the main result, see [3].

Lemma 2.2. *Let $k \geq 0$ and $1 \leq a, b \leq \infty$. Then the following inequality holds*

$$\sum_{|\alpha|=k} \|\partial^\alpha \dot{\Delta}_j u\|_{L^b} \leq C 2^{kj+3j(\frac{1}{a}-\frac{1}{b})}\|\dot{\Delta}_j u\|_{L^a},$$

where $C > 0$ is a constant depending only on k, a, b.

Next, we introduce a class of Banach spaces, known as Vishik spaces [11], which generalize the homogeneous Besov spaces.

Definition 2.3. Let $p, r \in [1, \infty]$, $s \in \mathbb{R}$ and $\theta \in [1, r]$. The Vishik spaces $\dot{V}^s_{p,r,\theta}(\mathbb{R}^3)$ are defined as

$$\dot{V}^s_{p,r,\theta}(\mathbb{R}^3) := \left\{ f \in \mathcal{S}'(\mathbb{R}^3) \setminus \mathcal{P}(\mathbb{R}^3) : \|f\|_{\dot{V}^s_{p,r,\theta}(\mathbb{R}^3)} < \infty \right\},$$

where

$$\|f\|_{\dot{V}^s_{p,r,\theta}(\mathbb{R}^3)} := \begin{cases} \sup_{N \in \mathbb{N}^*} \frac{(\sum_{j=-N}^{N} 2^{j\theta s} \|\dot{\Delta}_j f\|^{\theta}_{L^p})^{\frac{1}{\theta}}}{N^{\frac{1}{\theta} - \frac{1}{r}}}, & \theta \neq \infty, \\ \|f\|_{B^0_{p,\infty}(\mathbb{R}^3)}, & \theta = \infty. \end{cases}$$

3 The proof of Theorem 1.1

Proof. By taking the L^2 inner product of the first equation, the second equation and the third equation of (1.1) with u, ω and b, respectively, summing the results, and then integrating with respect to t, we obtain

$$\|u\|^2_{L^2} + \|\omega\|^2_{L^2} + \|b\|^2_{L^2} + 2\int_0^T (\mu\|\nabla u\|^2_{L^2} + \gamma\|\nabla\omega\|^2_{L^2} + \nu\|\nabla b\|^2_{L^2})\mathrm{d}t \le C(u_0, \omega_0, b_0).$$

The first equation, as well as the second and third equations in (1.1), are multiplied by $-\Delta u$, $-\Delta\omega$ and $-\Delta b$, respectively, and then integrated over $\mathbb{R}^3$ with respect to x, which yields

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|^2_{L^2} + \|\nabla\omega\|^2_{L^2} + \|\nabla b\|^2_{L^2}) + (\mu+\chi)\|\Delta u\|^2_{L^2} + \gamma\|\Delta\omega\|^2_{L^2} + \nu\|\Delta b\|^2_{L^2} \\ &\qquad + 2\chi\|\nabla\omega\|^2_{L^2} + \kappa\|\nabla\nabla\cdot\omega\|^2_{L^2} \\ &\quad = \int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x - \int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot\Delta u\mathrm{d}x - \chi\int_{\mathbb{R}^3}\nabla\times\omega\cdot\Delta u\mathrm{d}x - \chi\int_{\mathbb{R}^3}\nabla\times u\cdot\Delta\omega\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^3}(u\cdot\nabla)\omega\cdot\Delta\omega\mathrm{d}x + \int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot\Delta b\mathrm{d}x - \int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot\Delta b\mathrm{d}x \\ &\quad =: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + I_7(t). \end{aligned} \tag{3.1}$$

According to the Littlewood–Paley decomposition theory, it can be obtained that

$$\nabla u = \sum_{j<-N}\dot{\Delta}_j\nabla u + \sum_{j=-N}^{N}\dot{\Delta}_j\nabla u + \sum_{j>N}\dot{\Delta}_j\nabla u, \tag{3.2}$$

where N is to be determined. Without loss of generality, we first estimate $I_5(t)$. Using integration by parts and (3.2), we have that

$$\begin{aligned} I_5(t) &= \int_{\mathbb{R}^3}(u\cdot\nabla)\omega\cdot\Delta\omega\mathrm{d}x \\ &= -\int_{\mathbb{R}^3}\partial_k u_i\partial_i\omega_j\partial_k\omega_j\mathrm{d}x - \int_{\mathbb{R}^3}u_i\partial_k\partial_i\omega_j\partial_k\omega_j\mathrm{d}x \\ &\le \int_{\mathbb{R}^3}|\nabla\omega|^2|\nabla u|\mathrm{d}x \\ &\le \sum_{j<-N}\int_{\mathbb{R}^3}|\nabla\omega|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x + \sum_{j=-N}^{N}\int_{\mathbb{R}^3}|\nabla\omega|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x + \sum_{j>N}\int_{\mathbb{R}^3}|\nabla\omega|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x \\ &=: I_{51} + I_{52} + I_{53}. \end{aligned} \tag{3.3}$$

Below we estimate I_{51}–I_{53} separately. For I_{51}, by means of the Hölder inequality and the Bernstein inequality, it follows that

$$\begin{aligned}
I_{51}(t) &= \sum_{j<-N} \int_{\mathbb{R}^3} |\nabla\omega|^2 |\dot{\Delta}_j \nabla u| \mathrm{d}x \\
&\le \sum_{j<-N} \|\dot{\Delta}_j \nabla u\|_{L^\infty} \|\nabla\omega\|_{L^2}^2 \\
&\le C \sum_{j<-N} 2^{\frac{3j}{2}} \|\dot{\Delta}_j \nabla u\|_{L^2} \|\nabla\omega\|_{L^2}^2 \\
&\le C 2^{-\frac{3N}{2}} \|\nabla u\|_{L^2} \|\nabla\omega\|_{L^2}^2. \qquad (3.4)
\end{aligned}$$

For I_{52}. Let $p \ge 1$. By the Hölder inequality, the Bernstein inequality and the definition of Vishik spaces, we have

$$\begin{aligned}
I_{52}(t) &= \sum_{j=-N}^{N} \int_{\mathbb{R}^3} |\nabla\omega|^2 |\dot{\Delta}_j \nabla u| \mathrm{d}x \\
&\le \sum_{j=-N}^{N} \|\dot{\Delta}_j \nabla u\|_{L^\infty} \|\nabla\omega\|_{L^2}^2 \\
&\le C \sum_{j=-N}^{N} 2^{\frac{3j}{p}} \|\dot{\Delta}_j \nabla u\|_{L^p} \|\nabla\omega\|_{L^2}^2 \\
&\le C N^{1-\frac{1}{r}} \sup_{N\in\mathbb{N}^*} \frac{\sum_{j=-N}^{N} 2^{\frac{3j}{p}} \|\dot{\Delta}_j \nabla u\|_{L^p}}{N^{1-\frac{1}{r}}} \|\nabla\omega\|_{L^2}^2 \\
&\le C N^{1-\frac{1}{r}} \|\nabla u\|_{V_{p,r,1}^{\frac{3}{p}}} \|\nabla\omega\|_{L^2}^2. \qquad (3.5)
\end{aligned}$$

For I_{53}. From the Hölder inequality, the Bernstein inequality and space embedding relation, it follows that

$$\begin{aligned}
I_{53}(t) &= \sum_{j>N} \int_{\mathbb{R}^3} |\nabla\omega|^2 |\dot{\Delta}_j \nabla u| \mathrm{d}x \\
&\le C \sum_{j>N} \|\dot{\Delta}_j \nabla u\|_{L^3} \|\nabla\omega\|_{L^2} \|\nabla\omega\|_{L^6} \\
&\le C \sum_{j>N} 2^{\frac{j}{2}} \|\dot{\Delta}_j \nabla u\|_{L^2} \|\nabla\omega\|_{L^2} \|\Delta\omega\|_{L^2} \\
&\le C \Big(\sum_{j>N} 2^{-j} \Big)^{\frac{1}{2}} \Big(\sum_{j>N} 2^{2j} \|\dot{\Delta}_j \nabla u\|_{L^2}^2 \Big)^{\frac{1}{2}} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} \\
&\le C 2^{-\frac{N}{2}} \|\nabla u\|_{B^1_{2,2}} \|\nabla\omega\|_{L^2} \|\Delta\omega\|_{L^2} \\
&\le C 2^{-\frac{N}{2}} \|\Delta u\|_{L^2} \|\nabla\omega\|_{L^2} \|\Delta\omega\|_{L^2}. \qquad (3.6)
\end{aligned}$$

Combining (3.4)–(3.6), there are

$$\begin{aligned}
I_5(t) &\le C 2^{-\frac{3N}{2}} \|\nabla u\|_{L^2} \|\nabla\omega\|_{L^2}^2 + C N^{1-\frac{1}{\sigma}} \|\nabla u\|_{V_{p,r,1}^{\frac{3}{p}}} \|\nabla\omega\|_{L^2}^2 + C 2^{-\frac{N}{2}} \|\nabla\omega\|_{L^2} \|\Delta u\|_{L^2} \|\Delta\omega\|_{L^2} \\
&\le C 2^{-\frac{3N}{2}} (\|\nabla u\|_{L^2}^3 + \|\nabla\omega\|_{L^2}^3) + C N^{1-\frac{1}{r}} \|\nabla u\|_{V_{p,r,1}^{\frac{3}{p}}} \|\nabla\omega\|_{L^2}^2 \\
&\quad + C 2^{-\frac{N}{2}} \|\nabla\omega\|_{L^2} (\|\Delta u\|_{L^2}^2 + \|\Delta\omega\|_{L^2}^2). \qquad (3.7)
\end{aligned}$$

Similarly, we have

$$I_1(t) \le C2^{-\frac{3N}{2}}\|\nabla u\|_{L^2}^3 + CN^{1-\frac{1}{\sigma}}\|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}}\|\nabla u\|_{L^2}^2 + C2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^2. \tag{3.8}$$

Using the Hölder inequality and the Young inequality, one obtains that

$$\begin{aligned} I_3(t) + I_4(t) = & -\chi\int_{\mathbb{R}^3}\nabla\times\omega\cdot\Delta u \mathrm{d}x - \chi\int_{\mathbb{R}^3}\nabla\times u\cdot\Delta\omega \mathrm{d}x \\ & \le \chi(\|\nabla\omega\|_{L^2}\|\Delta u\|_{L^2} + \|\nabla u\|_{L^2}\|\Delta\omega\|_{L^2}) \\ & \le \frac{\mu+\chi}{4}\|\Delta u\|_{L^2}^2 + \frac{\gamma}{4}\|\Delta\omega\|_{L^2}^2 + C\|\nabla u\|_{L^2} + C\|\nabla\omega\|_{L^2}. \end{aligned} \tag{3.9}$$

Similar to I_5, we have

$$\begin{aligned} I_2(t) + I_6(t) + I_7(t) \le & \int_{\mathbb{R}^3}|\nabla b|^2|\nabla u|\mathrm{d}x \\ & \le C2^{-\frac{3N}{2}}(\|\nabla u\|_{L^2}^3 + \|\nabla b\|_{L^2}^3) + CN^{1-\frac{1}{\sigma}}\|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}}\|\nabla b\|_{L^2}^2 \\ & \quad + C2^{-\frac{N}{2}}\|\nabla b\|_{L^2}(\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2). \end{aligned} \tag{3.10}$$

Combining (3.2) and (3.7)–(3.10), one has

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}t}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{\mu+\chi}{2}\|\Delta u\|_{L^2}^2 + \frac{\gamma}{2}\|\Delta\omega\|_{L^2}^2 + \nu\|\Delta b\|_{L^2}^2 \\ & \qquad + 2\chi\|\nabla\omega\|_{L^2}^2 + \kappa\|\nabla\nabla\cdot\omega\|_{L^2}^2 \\ & \quad \le C2^{-\frac{3N}{2}}(\|\nabla u\|_{L^2}^3 + \|\nabla\omega\|_{L^2}^3 + \|\nabla b\|_{L^2}^3) + CN^{1-\frac{1}{r}}\|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}}(\|\nabla u\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \\ & \qquad + C2^{-\frac{N}{2}}(\|\nabla u\|_{L^2} + \|\nabla\omega\|_{L^2} + \|\nabla b\|_{L^2})(\|\Delta u\|_{L^2}^2 + \|\Delta\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) \\ & \qquad + C(\|\nabla u\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2) \\ & \quad \le C2^{-\frac{3N}{2}}(\|\nabla u\|_{L^2} + \|\nabla\omega\|_{L^2} + \|\nabla b\|_{L^2})(\|\nabla u\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \\ & \qquad + CN^{1-\frac{1}{r}}\|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}}(\|\nabla u\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \\ & \qquad + C2^{-\frac{N}{2}}(\|\nabla u\|_{L^2} + \|\nabla\omega\|_{L^2} + \|\nabla b\|_{L^2})(\|\Delta u\|_{L^2}^2 + \|\Delta\omega\|_{L^2}^2 + \|\Delta b\|_{L^2}^2). \end{aligned} \tag{3.11}$$

We fix a large enough N, which obeys

$$C2^{-\frac{N}{2}}(\|\nabla u\|_{L^2} + \|\nabla\omega\|_{L^2} + \|\nabla b\|_{L^2}) \le \frac{1}{4}\min\{\mu+\chi, \gamma, 2\nu\},$$

i.e.

$$N \ge 4 + \frac{2lnC + 2ln(\|\nabla u\|_{L^2} + \|\nabla\omega\|_{L^2} + \|\nabla b\|_{L^2}) - 2\min\{\mu+\chi,\gamma,2\nu\}}{ln2}.$$

Taking

$$N = \left[4 + \frac{2lnC + 2ln(\|\nabla u\|_{L^2} + \|\nabla\omega\|_{L^2} + \|\nabla b\|_{L^2}) - 2\min\{\mu+\chi,\gamma,2\nu\}}{ln2}\right] + 1,$$

it follows from (3.11) that

$$\frac{1}{2}\frac{\mathrm{d}t}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\left(\frac{\mu+\chi}{2}-\frac{1}{4}\min\{\mu+\chi,\gamma,2\nu\}\right)\|\Delta u\|_{L^2}^2+2\chi\|\nabla\omega\|_{L^2}^2$$
$$+\left(\frac{\gamma}{2}-\frac{1}{4}\min\{\mu+\chi,\gamma,2\nu\}\right)\|\Delta\omega\|_{L^2}^2$$
$$+\left(\nu-\frac{1}{4}\min\{\mu+\chi,\gamma,2\nu\}\right)\|\Delta b\|_{L^2}^2+\kappa\|\nabla\nabla\cdot\omega\|_{L^2}^2$$
$$\leq C\left(1+\|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}}\right)(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2). \tag{3.12}$$

Using the Gronwall inequality yields

$$\sup_{0\leq t\leq T}(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+C\int_0^T(\|\Delta u\|_{L^2}^2+\|\Delta\omega\|_{L^2}^2+\|\Delta b\|_{L^2}^2)(t)\,\mathrm{d}t$$
$$\leq \exp\left(CT+C\int_0^T\|\nabla u\|_{V^{\frac{3}{p}}_{p,r,1}}\mathrm{d}t\right)(\|\nabla u_0\|_{L^2}^2+\|\nabla\omega_0\|_{L^2}^2+\|\nabla b_0\|_{L^2}^2).$$

Using the hypothetical condition (1.2), we get

$$u,\omega,b\in L^\infty(0,T;H^1(\mathbb{R}^3))\cap L^2(0,T;H^2(\mathbb{R}^3)).$$

For the regularity criterion (1.3), we focus our analysis on I_5. Similarly, by applying the theory of Littlewood–Paley decompositions, it follows that

$$I_5(t)\leq\sum_{j<-N}\int_{\mathbb{R}^3}|\nabla u|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x+\sum_{j=-N}^{N}\int_{\mathbb{R}^3}|\nabla u|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x+\sum_{j>N}\int_{\mathbb{R}^3}|\nabla u|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x$$
$$=:J_{51}+J_{52}+J_{53}. \tag{3.13}$$

Using $I_{51}(t)$ and $I_{53}(t)$, one obtains that

$$J_{51}(t)+J_{53}(t)\leq C2^{-\frac{3N}{2}}\|\nabla u\|_{L^2}^3+C2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^2. \tag{3.14}$$

Let $p\geq\frac{3}{2}$. From the Hölder inequality, the definition of Vishik spaces, the Gagliardo–Nirenberg inequality and the Young inequality, we have

$$J_{52}(t)=\sum_{j=-N}^{N}\int_{\mathbb{R}^3}|\nabla u|^2|\dot{\Delta}_j\nabla u|\mathrm{d}x$$
$$\leq\sum_{j=-N}^{N}\|\dot{\Delta}_j\nabla u\|_{L^p}\|\nabla u\|^2_{L^{\frac{2p}{p-1}}}$$
$$\leq CN^{1-\frac{1}{r}}\sup_{N\in\mathbb{N}^*}\frac{\sum_{j=-N}^{N}\|\dot{\Delta}_j\nabla u\|_{L^p}}{N^{1-\frac{1}{r}}}\|\nabla u\|_{L^2}^{2-\frac{3}{p}}\|\Delta u\|_{L^2}^{\frac{3}{p}}$$
$$\leq CN^{1-\frac{1}{r}}\|\nabla u\|_{V^0_{p,r,1}}\|\nabla u\|_{L^2}^{2-\frac{3}{p}}\|\Delta u\|_{L^2}^{\frac{3}{p}}$$
$$\leq\frac{\mu+\chi}{4}\|\Delta u\|_{L^2}^2+CN^{\frac{(r-1)2p}{(2p-3)r}}\|\nabla u\|_{V^0_{p,r,1}}^{\frac{2p}{2p-3}}\|\Delta u\|_{L^2}^2. \tag{3.15}$$

Combining (3.13)–(3.15), yields

$$I_5(t) \le C2^{-\frac{3N}{2}}\|\nabla u\|_{L^2}^3 + \frac{\mu+\chi}{4}\|\Delta u\|_{L^2}^2 + CN^{\frac{(r-1)2p}{(2p-3)r}}\|\nabla u\|_{V^0_{p,r,1}}^{\frac{2p}{2p-3}}\|\Delta u\|_{L^2}^2 + C2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^2.$$

The analysis that follows is similar to that of the regularity criterion (1.3), except for a slight difference in the choice of N, which is omitted here. We leave the details to the interested reader.

Based on the above analysis, we complete the proof of Theorem 1.1. □

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Normalized solutions for Schrödinger equations with potential and general nonlinearities involving critical case on large convex domains

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Abstract. In this paper, we study the following Schrödinger equations with potentials and general nonlinearities

$$\begin{cases} -\Delta u + V(x)u + \lambda u = |u|^{q-2}u + \beta f(u), \\ \int |u|^2 dx = \Theta, \end{cases}$$

both on $\mathbb{R}^N$ as well as on domains Ω_r where $\Omega_r \subset \mathbb{R}^N$ is an open bounded convex domain and $r > 0$ is large. The exponent satisfies $2 + \frac{4}{N} \leq q \leq 2^* = \frac{2N}{N-2}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfies L^2-subcritical or L^2-critical growth. This paper generalizes the conclusion of Bartsch et al. in [4]. Moreover, we consider the Sobolev critical case and L^2-critical case of the above problem.

Keywords: Schrödinger equations, normalized solutions, variational methods, mixed nonlinearity.

2020 Mathematics Subject Classification: 35A15, 35B09, 35B38, 35J50.

1 Introduction and main results

This paper studies the existence of normalized solutions for the following Schrödinger equations with potentials and general nonlinearities

$$\begin{cases} -\Delta u + V(x)u + \lambda u = |u|^{q-2}u + \beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r, \end{cases} \tag{1.1}$$

where $\Omega_r \subset \mathbb{R}^N$ is either all of $\mathbb{R}^N$ or a bounded smooth convex domain, $N \geq 3$, $2 + \frac{4}{N} \leq q \leq 2^* = \frac{2N}{N-2}$, the mass $\Theta > 0$ and the parameter $\beta \in \mathbb{R}$ are prescribed. The frequency λ is unknown and to be determined.

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Such problems are motivated in particular by searching for solitary waves (stationary states) in nonlinear equations of the Schrödinger type. Specifically, consider the following nonlinear Schrödinger equation

$$\begin{cases} -i\dfrac{\partial}{\partial t}\Psi = \Delta\Psi - V(x)\Psi + f\left(|\Psi|^2\right)\Psi = 0, & (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ \Psi = \Psi(x,t), & (x,t) \in \mathbb{C}, \end{cases}$$

where $N \geq 1$. Researchers are interested in finding the existence of standing wave solutions to the above equations, that is, $\Psi(x,t) = e^{i\lambda t}u(x), \lambda \in \mathbb{R}$, and $u : \mathbb{R}^N \to \mathbb{R}$, so we get the equation

$$-\Delta u + (V(x) + \lambda)u = Q(u), \qquad x \in \mathbb{R}^N,$$

where $Q(u) = f(|u|^2)u$. For physical reasons, we focus on the existence of normalized solutions for the following problem

$$\begin{cases} -\Delta u + (V(x) + \lambda)u = Q(u), & x \in \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = \Theta, & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

For more physical background about the above equation, please refer to [9,16].

If potential $V(x)$ in (1.2) is constant, we call (1.2) is autonomous. In this case, recalling paper [21], Jeanjean developed an approach based on the Pohozaev identity which has been used successfully in recent years. The key to this method is to find a bounded Palais–Smale sequences by using the transformation $s * u(x) = e^{\frac{sN}{2}}u(e^s x)$. After that, by weakening the conditions in [21], Jeanjean [22] and Bieganowski [8] improved these results. Of course, these articles only consider the problem of a single nonlinear term. Recently, there have been many studies on mixed nonlinear terms. For example, Soave [26,27] studied normalized solution of (1.2) with mixed nonlinearity $f(|u|)u = \mu|u|^{q-2}u + |u|^{p-2}u$, $2 < p < 2 + \frac{4}{N} < q \leq 2^* = \frac{2N}{N-2}$. Specifically, Soave in [26] obtained many results of existence and non-existence. More precisely, if $2 < q < p = 2 + \frac{4}{N}$, that is, the leading nonlinearity is L^2-critical and a L^2-subcritical lower order term. (1.2) had a real-valued positive and radially symmetric solution for some $\lambda < 0$ in $\mathbb{R}^N$ provided $\mu > 0$ and $\Theta > 0$ small enough. Moreover, if $\mu < 0$, (1.2) had no solution. If $2 + \frac{4}{N} = q < p < 2^*$, that is, the leading term is L^2-critical and L^2-supercritical, (1.2) had a real-valued positive, radially symmetric solution for some $\lambda < 0$ in $\mathbb{R}^N$ provided $\mu > 0$ and μ, Θ satisfy the appropriate conditions. If $2 < q < 2 + \frac{4}{N} < p < 2^*$, that is, the leading term L^2-subcritical and L^2-supercritical, (1.2) also had a real-valued positive and radially symmetric solution for some $\lambda < 0$ in $\mathbb{R}^N$ provided $\Theta > 0, \mu < 0$ and μ, Θ satisfy the appropriate conditions. Soave in [27] considered the Sobolev critical case and obtained some similar results. In particular, the Sobolev critical case also has been considered in [1,2,24,25](see also the references therein). It is worth mentioning that many researchers are also interested in the existence of normalized multiple solutions. In [23], Jeanjean et al. obtained the existence of normalized multiple solutions for Sobolev critical case in (1.2). For more results on this aspect, please refer to [5–7,10,29] and its references.

If (1.2) is non-autonomous, Ikoma and Miyamoto in [19] considered question (1.2) with $V(x) \in C(\mathbb{R}^N), 0 \not\equiv V(x) \leq 0, V(x) \to 0(|x| \to \infty)$, they obtained some existence and non-existence results. After that, Ding and Zhong in [14] proved the existence of normalized solutions to the following Schrödinger equation

$$\begin{cases} -\Delta u(x) + V(x)u(x) + \lambda u(x) = g(u(x)), & x \in \mathbb{R}^N, \\ 0 \leq u(x) \in H^1(\mathbb{R}^N),\ N \geq 3, \end{cases}$$

where g satisfies:

$(G1)$ $g : \mathbb{R} \to \mathbb{R}$ is C^1 and odd.

$(G2)$ There exists some $(\alpha, \beta) \in \mathbb{R}^2_+$ satisfying $2 + \frac{4}{N} < \alpha \le \beta < \frac{2N}{N-2}$ such that

$$\alpha G(s) \le g(s)s \le \beta G(s) \text{ with } G(s) = \int_0^s g(t)dt.$$

$(G3)$ The functional defined by $\widetilde{G}(s) := \frac{1}{2} g(s)s - G(s)$ is of class C^1 and

$$\widetilde{G}'(s)s \ge \alpha \widetilde{G}(s), \forall s \in \mathbb{R},$$

where α is given by $(G2)$.

Note that, $(G3)$ plays a crucial role in the uniqueness of t_u(see [14] or [21, Lemma 2.9]). However, we do not need this condition, since we directly perform scaling and complex calculations on energy functionals. Recently, Bartsch et al. in [4] considered following Schrödinger equations with potentials and inhomogeneous nonlinearities on large convex domains

$$\begin{cases} -\Delta u + V(x)u + \lambda u = |u|^{q-2}u + \beta |u|^{p-2}u, \\ \int |u|^2 dx = \Theta, \end{cases}$$

they developed a robust method to study the existence of normalized solutions of nonlinear Schrödinger equations with potential. Under the stimulation of [4], our goal is to generalize its conclusion to general nonlinear terms and the Sobolev critical case.

In order to state our main results, we introduce some notations. Set $s_+ = \max\{s, 0\}$, $s_- = \min\{s, 0\}$ for $s \in \mathbb{R}$. The Aubin–Talenti constant [3] is denoted by S, that is, S is the best constant in the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, where $\mathcal{D}^{1,2}(\mathbb{R}^N)$ denotes the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{\mathcal{D}^{1,2}} := \|\nabla u\|_2$. It is well known [28] that the optimal constant is achieved by (any multiple of)

$$U_{\varepsilon,y}(x) = [N(N-2)]^{\frac{N-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2} \right)^{\frac{N-2}{2}}, \qquad \varepsilon > 0,\ y \in \mathbb{R}^N, \tag{1.3}$$

which are the only positive classical solutions to the critical Lane–Emden equation

$$-\Delta w = w^{2^*-1}, \qquad w > 0 \text{ in } \mathbb{R}^N.$$

Let $C_{N,s}$ be the best constant in the Gagliardo–Nirenberg inequality

$$\|u\|_s^s \le C_{N,s} \|u\|_2^{\frac{2s-N(s-2)}{2}} \|\nabla u\|_2^{\frac{N(s-2)}{2}}, \qquad 2 < s < 2^*.$$

For some results, we expect that V is C^1 and consider the function

$$\widetilde{V} : \mathbb{R}^N \to \mathbb{R}, \qquad \widetilde{V}(x) = \nabla V(x) \cdot x.$$

For $\Omega \subset \mathbb{R}^N$ and $r > 0$, let

$$\Omega_r = \left\{ rx \in \mathbb{R}^N : x \in \Omega \right\}$$

and

$$S_{r,\Theta} := S_\Theta \cap H_0^1(\Omega_r) = \left\{ u \in H_0^1(\Omega_r) : \|u\|^2_{L^2(\Omega_r)} = \Theta \right\}.$$

From now on we assume that $\Omega \subset \mathbb{R}^N$ is a bounded smooth convex domain with $0 \in \Omega$.

Our assumptions on V are:

(V_0) $V \in C^1(\mathbb{R}^N) \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ is bounded and $\|V_-\|_{\frac{N}{2}} < S$.

$(\widetilde{V_0})$ $V \in C^1(\mathbb{R}^N) \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ is bounded and $\|V_-\|_{\frac{N}{2}} < \frac{N(q-p_2)-2[N(p_2-2)-4]}{N(q-p_2)} S$.

$(\widehat{V_0})$ $V \in C^1(\mathbb{R}^N) \cap L^{\frac{N}{2}}(\mathbb{R}^N)$ is bounded and $\|V_-\|_{\frac{N}{2}} < \left(1 - \frac{NC_N\Theta^{\frac{2}{N}}}{N+2}\right) S$.

(V_1) V is of class $C^1, \lim_{|x|\to\infty} V(x) = 0$, and there exists $\rho \in (0,1)$ such that

$$\liminf_{|x|\to\infty} \inf_{y\in B(x,\rho|x|)} (x \cdot \nabla V(y))e^{\tau|x|} > 0 \quad \text{for any } \tau > 0.$$

Remark 1.1. In order to obtain the existence of normalized solutions in $\mathbb{R}^N$ by taking $\Omega = B_1$, the unit ball centered at the origin in $\mathbb{R}^N$, and analyzing the compactness of the solutions $u_{r,\Theta}$ established in Theorems 1.3, 1.4 and 1.5 as r tends to infinity, we require the condition (V_1).

Now, we make the following assumptions on the nonlinearity f:

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and f is odd.

(f_2) There exists some $(p_1, p_2) \in \mathbb{R}^2_+$ satisfying $2 < p_2 \le p_1 < 2 + \frac{4}{N}$ such that

$$p_2F(\tau) \le f(\tau)\tau \le p_1F(\tau) \text{ with } F(\tau) = \int_0^\tau f(t)dt.$$

$(\widetilde{f_2})$ There exists some $(p_1, p_2) \in \mathbb{R}^2_+$ satisfying $2 < p_2 < p_1 = 2 + \frac{4}{N}$ such that

$$p_2F(\tau) \le f(\tau)\tau \le p_1F(\tau).$$

Remark 1.2. If $f(u) = \sum_{i=1}^m a_i|u|^{\sigma_i-2}u$, where $a_i > 0$ and $2 < \sigma_i < 2 + \frac{4}{N}$, then the assumption (f_1) can be weakened to $f \in C(\mathbb{R}, \mathbb{R})$ and f is odd. In order to ensure the boundedness of Palais–Smale sequence under constraint conditions in Lemma 3.3, we need to slightly strengthen the conditions for the nonlinear term f, that is, $f \in C^1(\mathbb{R}, \mathbb{R})$.

The main results of this paper are as follows. Firstly, we consider the Sobolev subcritical case, that is, $2 + \frac{4}{N} < q < 2^*$.

Theorem 1.3 (case $\beta \le 0$)**.** *Assume V satisfies (V_0), is of class C^1 and $\widetilde{V}$ is bounded, f satisfies (f_1)–(f_2). There hold:*

(i) For every $\Theta > 0$, there exists $r_\Theta > 0$ such that (1.1) on Ω_r with $r > r_\Theta$ has a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} > 0$ in Ω_r and positive energy $I_r(u_{r,\Theta}) > 0$. Moreover, there exists $C_\Theta > 0$ such that

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

(ii) If in addition $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$, then there exists $\widetilde{\Theta} > 0$ such that

$$\liminf_{r\to\infty} \lambda_{r,\Theta} > 0 \quad \textit{for any } 0 < \Theta < \widetilde{\Theta}.$$

Theorem 1.4 (case $\beta > 0$)**.** *Assume V satisfies (V_0), f satisfies (f_1)–(f_2) and set*

$$\Theta_V = \left[\frac{1 - \|V_-\|_{\frac{N}{2}} S^{-1}}{2N(q-p_1)}\right]^{\frac{N}{2}} \left[\frac{q(4-N(p_1-2))}{C_{N,q}}\right]^{\frac{4-N(p_1-2)}{2(q-p_1)}} \left[\frac{N(q-2)-4}{\alpha\beta C_{N,p_1}}\right]^{\frac{N(q-2)-4}{2(q-p_1)}}.$$

Then the following hold for $0 < \Theta < \Theta_V$:

(i) There exists $r_\Theta > 0$ such that (1.1) on Ω_r with $r > r_\Theta$ has a local minimum type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} > 0$ in Ω_r and negative energy $I_r(u_{r,\Theta}) < 0$.

(ii) There exists $C_\Theta > 0$ such that

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta, \qquad \liminf_{r\to\infty} \lambda_{r,\Theta} > 0.$$

Theorem 1.5 (case $\beta > 0$)**.** *Assume V satisfies (V_0), is of class C^1 and $\widetilde{V}$ is bounded, f satisfies (f_1)–(f_2). Set*

$$\widetilde{\Theta}_V = \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} \left(\frac{C_{N,q}}{q} A_{p_1,q} + \frac{C_{N,q}}{q}\right)^{-\frac{N}{2}} \left(\frac{\alpha\beta q C_{N,p_1}}{C_{N,q} A_{p_1,q}}\right)^{\frac{N(q-2)-4}{2N(q-p_1)}},$$

where

$$A_{p_1,q} = \frac{(q-2)(N(q-2)-4)}{(p_1-2)(4-N(p_1-2))}.$$

Then the following hold for $0 < \Theta < \widetilde{\Theta}_V$:

(i) There exists $\widetilde{r}_\Theta > 0$ such that (1.1) in Ω_r admits for $r > r_\Theta$ a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} > 0$ in Ω_r and positive energy $I_r(u_{r,\Theta}) > 0$. Moreover, there exists $C_\Theta > 0$ such that

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

(ii) There exists $0 < \bar{\Theta} \le \widetilde{\Theta}_V$ such that

$$\liminf_{r\to\infty} \lambda_{r,\Theta} > 0 \quad \text{for any } 0 < \Theta \le \bar{\Theta}.$$

If $\Omega = \mathbb{R}^N$, (V_1) is significant for obtaining the following results.

Theorem 1.6 (case $\beta > 0$)**.** *Assume V satisfies (V_0)–(V_1). Then problem (1.1) with $\Omega = \mathbb{R}^N$ admits for any $0 < \Theta < \Theta_V$, where Θ_V is as in Theorem 1.4, a solution $(\lambda_\Theta, u_\Theta)$ with $u_\Theta > 0, \lambda_\Theta > 0$, and $I(u_\Theta) < 0$.*

Theorem 1.7 (case $\beta > 0$)**.** *Assume V satisfies (V_0)–(V_1). Then (1.1) with $\Omega = \mathbb{R}^N$ admits for $0 < \Theta < \bar{\Theta}, \bar{\Theta} > 0$ as in Theorem 1.5 (ii), a solution $(\lambda_\Theta, u_\Theta)$ with $u_\Theta > 0, \lambda_\Theta > 0$, and $I(u_\Theta) > 0$. Moreover, $\lim_{\Theta\to 0} I(u_\Theta) = \infty$.*

Theorem 1.8 (case $\beta \le 0$)**.** *Assume V satisfies (V_0)–(V_1), and $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$. Then problem (1.1) with $\Omega = \mathbb{R}^N$ admits for $0 < \Theta < \widetilde{\Theta}, \widetilde{\Theta} > 0$ as in Theorem 1.3, a solution $(\lambda_\Theta, u_\Theta)$ with $u_\Theta > 0, \lambda_\Theta > 0$, and $I(u_\Theta) > 0$. Moreover, $\lim_{\Theta\to 0} I(u_\Theta) = \infty$.*

For the Sobolev critical case, that is $q = 2^*$, we have the following results.

Theorem 1.9 (case $\beta > 0$). *Assume V satisfies (V_0), f satisfies (f_1)–(f_2). Set*

$$\Theta_V = \left(\frac{1}{N\alpha\beta C_{N,p_1}}\right)^{\frac{4}{2p_1-N(p_1-2)}} \left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}\cdot\frac{4-N(p_1-2)}{2p_1-N(p_1-2)}}.$$

Then the following hold for $0 < \Theta < \Theta_V$:

(i) There exists $r_\Theta > 0$ such that (6.2) on Ω_r with $r > r_\Theta$ has a local minimum type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} > 0$ in Ω_r and negative energy $\mathcal{I}_r(u_{r,\Theta}) < 0$.

(ii) There exists $C_\Theta > 0$ such that

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta, \qquad \liminf_{r\to\infty} \lambda_{r,\Theta} > 0.$$

Theorem 1.10 (case $\beta \le 0$). *Assume V satisfies (V_0), is of class C^1 and $\widetilde{V}$ is bounded, f satisfies (f_1)–(f_2). There hold:*

(i) There exists $r_\Theta > 0$ such that (7.1) on Ω_r with $r > r_\Theta$ has a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} \ge 0$ in Ω_r and positive energy $\mathcal{I}_r(u_{r,\Theta}) > 0$.

(ii) There exists $C_\Theta > 0$ such that

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

Theorem 1.11 (case $\beta > 0$). *Assume V satisfies (V_0), is of class C^1 and $\widetilde{V}$ is bounded, f satisfies (f_1)–(f_2). Set*

$$\widetilde{\Theta}_V = \left(\frac{\alpha\beta C_{N,p_1} S^{\frac{2^*}{2}}}{A_{p_1}}\right)^{-\frac{4}{2p_1-N(p_1-2)}} \left[\frac{S^{\frac{2^*}{2}}}{2\cdot 2^*}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right)(2^* A_{p_1}+1)\right]^{\frac{2[2\cdot 2^*-N(p_1-2)]}{(2^*-2)[2p_1-N(p_1-2)]}}$$

where

$$A_{p_1} = \frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))}.$$

Then the following hold for $0 < \Theta < \widetilde{\Theta}_V$:

(i) There exists $\widetilde{r}_\Theta > 0$ such that (8.1) in Ω_r admits for $r > r_\Theta$ a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ with $u_{r,\Theta} \ge 0$ in Ω_r and positive energy $\mathcal{I}_r(u_{r,\Theta}) > 0$.

(ii) There exists $C_\Theta > 0$ such that

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

For the L^2-critical case, that is $p_1 = 2 + \frac{4}{N}$ or $q = 2 + \frac{4}{N}$, we have the following results.

Theorem 1.12 (case $\beta > 0$ and $p_1 = 2 + \frac{4}{N}$). *Assume V satisfies $(\widetilde{V_0})$, f satisfies (f_1) and $(\widetilde{f_2})$. Set*

$$\widetilde{\Theta}_V = \left[\frac{N(q-p_2)-4}{N\alpha\beta(q-p_2)C_N}\right]^{\frac{N}{2}}.$$

Then the following hold for $0 < \Theta < \widetilde{\Theta}_V$:

(i) *There exists* $\tilde{r}_\Theta > 0$ *such that* (1.1) *in* Ω_r *admits for* $r > r_\Theta$ *a mountain pass type solution* $(\lambda_{r,\Theta}, u_{r,\Theta})$ *with* $u_{r,\Theta} > 0$ *in* Ω_r *and positive energy* $I_r(u_{r,\Theta}) > 0$. *Moreover, there exists* $C_\Theta > 0$ *such that*

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

(ii) *There exists* $0 < \bar{\Theta} \le \widetilde{\Theta}_V$ *such that*

$$\liminf_{r\to\infty} \lambda_{r,\Theta} > 0 \quad \textit{for any } 0 < \Theta \le \bar{\Theta}.$$

Theorem 1.13 (case $\beta \le 0$ and $p_1 = 2 + \frac{4}{N}$)**.** *Assume* V *satisfies* (V_0), *is of class* C^1 *and* $\widetilde{V}$ *is bounded,* f *satisfies* (f_1) *and* $(\widetilde{f_2})$. *Set*

$$\widehat{\Theta}_V = \left[\frac{(N-2)q - 2N}{2N\alpha\beta(q-p_2)C_N}\right]^{\frac{N}{2}}.$$

Then the following hold for $0 < \Theta < \widehat{\Theta}_V$:

(i) *There exists* $r_\Theta > 0$ *such that* (1.1) *on* Ω_r *with* $r > r_\Theta$ *has a mountain pass type solution* $(\lambda_{r,\Theta}, u_{r,\Theta})$ *with* $u_{r,\Theta} > 0$ *in* Ω_r *and positive energy* $I_r(u_{r,\Theta}) > 0$. *Moreover, there exists* $C_\Theta > 0$ *such that*

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta.$$

(ii) *If in addition* $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$, *then there exists* $\widetilde{\Theta} > 0$ *such that*

$$\liminf_{r\to\infty} \lambda_{r,\Theta} > 0 \quad \textit{for any } 0 < \Theta < \widetilde{\Theta}.$$

Theorem 1.14. *(**case** $\beta > 0$ **and** $q = 2 + \frac{4}{N}$) Assume* V *satisfies* $(\widehat{V_0})$, f *satisfies* (f_1)–(f_2) *and set*

$$\Theta_V = \left(\frac{N+2}{NC_N}\right)^{\frac{N}{2}}.$$

Then the following hold for $0 < \Theta < \Theta_V$:

(i) *There exists* $r_\Theta > 0$ *such that* (1.1) *on* Ω_r *with* $r > r_\Theta$ *has a global minimum type solution* $(\lambda_{r,\Theta}, u_{r,\Theta})$ *with* $u_{r,\Theta} > 0$ *in* Ω_r *and negative energy* $I_r(u_{r,\Theta}) < 0$.

(ii) *There exists* $C_\Theta > 0$ *such that*

$$\limsup_{r\to\infty} \max_{x\in\Omega_r} u_{r,\Theta}(x) < C_\Theta, \qquad \liminf_{r\to\infty} \lambda_{r,\Theta} > 0.$$

Remark 1.15.

(i) Theorems 1.3–1.11 are valid if $2 = p_2 < p_1 < 2 + \frac{4}{N}$ in (f_2). Moreover, the proof of Theorems 1.6–1.8 is very similar to [4], so we omit it in this paper.

(ii) Our conclusion also applies to $p_1 = p_2 = 2 + \frac{4}{N}$ if $2 + \frac{4}{N} < q < 2^*$, such as $f(u) = |u|^{\frac{4}{N}} u$. Therefore, our results cover certain conclusions in [26].

Remark 1.16. Theorems 1.4 and 1.9 (resp. Theorems 1.5 and 1.11) both require some limitations on Θ_V (resp. $\widetilde{\Theta}_V$), although their values are different, they all stem from changes in the geometric structure of the energy functional. In addition, there are still some unknown results for the Sobolev critical case, that is, $\liminf_{r\to\infty}\lambda_{r,\Theta} > 0$ may not necessarily hold when $\beta > 0$ or $\beta \leq 0$. In fact, the methods and techniques in Theorem 1.4 (or Theorem 1.5) cannot be applied to the Sobolev critical case since $\frac{(N-2)q-2N}{2Nq} = 0$, thus $\lambda_\Theta > 0$ cannot be obtained $0 < \Theta \leq \bar{\Theta}$.

Remark 1.17. In this paper, whether in subcritical or critical situations, the monotonicity trick in [20] is one of the keys to get the conclusion. Proposition 3.2 does not ensure the existence of a mountain pass solution for the original problem obtained when $s = 1$. However, it gives the existence of a sequence $s_n \to 1^-$, with a corresponding sequence of mountain pass critical points u_{r,s_n} of I_{r,s_n}, constrained on $S_{r,\Theta}$. We aim to show that u_{r,s_n} strongly converges to a constrained critical point of I_r. For this purpose, it is sufficient to prove that u_{r,s_n} is bounded in $H_0^1(\Omega_r)$, thanks to Proposition 3.1 in [15].

The structure of this paper is arranged as follows. In section 2, we provide some ideas in the proof of main theorems. In section 3, we obtain the mountain pass type positive solution in the case $\beta \leq 0$ and have completed the proof of Theorem 1.3. If $\beta > 0$, there are two situations, that is, Theorems 1.4 and 1.5. We get the two results in sections 3 and 4 by using different geometric analysis. After that, we consider the Sobolev critical case. Finally, we consider the L^2-critical case and give some comments.

2 Preliminary

Consider the problem

$$\begin{cases} -\Delta u + V(x)u + \lambda u = |u|^{q-2}u + \beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r, \end{cases} \tag{2.1}$$

where $N \geq 3$, $2 + \frac{4}{N} \leq q \leq 2^* = \frac{2N}{N-2}$, the mass $\Theta > 0$ and the parameter $\beta \in \mathbb{R}$ are prescribed. The frequency λ is unknown and to be determined. The energy functional $I_r : H_0^1(\Omega_r) \to \mathbb{R}$ is defined by

$$I_r(u) = \frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} V(x)u^2 dx - \frac{1}{q}\int_{\Omega_r} |u|^q dx - \beta \int_{\Omega_r} F(u)dx \tag{2.2}$$

and the mass constraint manifold is defined by

$$S_{r,\Theta} = \left\{ u \in H_0^1(\Omega_r) : \|u\|_2^2 = \Theta \right\}. \tag{2.3}$$

If $\Omega = \mathbb{R}^N$, the energy functional $I : H_0^1(\Omega_r) \to \mathbb{R}$ is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx - \frac{1}{q}\int_{\mathbb{R}^N} |u|^q dx - \beta \int_{\mathbb{R}^N} F(u)dx \tag{2.4}$$

and the mass constraint manifold is defined by

$$S_\Theta = \left\{ u \in H_0^1(\mathbb{R}^N) : \|u\|_2^2 = \Theta \right\}. \tag{2.5}$$

The proof idea of Theorem 1.3 is as follows. In order to find a mountain pass type solution $(\lambda_{r,\Theta}, u_{r,\Theta})$, we first need to analyze the geometric structure of the energy functional corresponding to the equation (3.1). In Lemma 3.1, we perform precise geometric analysis on the energy functional corresponding to (3.1) and know that the energy functional $I_{r,s}$ has a global maximum. Next, we obtain the bounded Palais–Smale sequence by using [11, Theorem 1] and get a solution $(\lambda_{r,s}, u_{r,s})$ for (3.1). Finally, we consider Lagrange multiplier and establish an a priori estimate for the solutions of (1.1). Theorem 1.4 is relatively simple because the energy functional has a local minimum, which can be proved using the method of constrained minimization. The proof of Theorem 1.5 is similar to Theorem 1.3, but the geometric structures of the two cases are significantly different and require refined estimate of energy.

Note that, there are some differences between the proof of Lemma 5.3 and Lemma 3.4, and we cannot directly use the method of Lemma 3.4, even if q can be reduced to p_2 according to condition (f_2) and $p_2 < 2 + \frac{4}{N} < q < 2^*$. More precisely, it then follows from $\beta > 0$ and (f_2) that

$$\begin{aligned}
&\frac{1}{N}\int_{\Omega_r}|\nabla u|^2dx - \frac{1}{2N}\int_{\partial\Omega_r}|\nabla u|^2(x\cdot\mathbf{n})d\sigma - \frac{1}{2N}\int_{\Omega_r}(\nabla V\cdot x)u^2dx \\
&\quad= \frac{(q-2)s}{2q}\int_{\Omega_r}|u|^q dx + s\int_{\Omega_r}(\frac{\beta}{2}f(u)u - \beta F(u))dx \\
&\quad\geq \frac{p_2-2}{2}\left(\frac{1}{2}\int_{\Omega_r}|\nabla u|^2dx + \frac{1}{2}\int_{\Omega_r}Vu^2dx - m_{r,s}(\Theta)\right).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\frac{p_2-2}{2}m_{r,s}(\Theta) &\geq \frac{p_2-2}{2}\left(\frac{1}{2}\int_{\Omega_r}|\nabla u|^2dx + \frac{1}{2}\int_{\Omega_r}Vu^2dx\right) - \frac{1}{N}\int_{\Omega_r}|\nabla u|^2dx \\
&\quad + \frac{1}{2N}\int_{\partial\Omega_r}|\nabla u|^2(x\cdot\mathbf{n})d\sigma + \frac{1}{2N}\int_{\Omega_r}(\nabla V\cdot x)u^2dx \\
&\geq \frac{N(p_2-2)-4}{4N}\int_{\Omega_r}|\nabla u|^2dx - \Theta\left(\frac{1}{2N}\|\nabla V\cdot x\|_\infty + \frac{p_2-2}{4}\|V\|_\infty\right).
\end{aligned}$$

However, this method is useless because $\frac{N(p_2-2)-4}{4N} < 0$, we cannot obtain that $\int_{\Omega_r}|\nabla u|^2dx$ is uniformly bounded in s and r.

3 Proof of Theorem 1.3

In this section, we assume $\beta \leq 0$ and the assumptions of Theorem 1.3 hold. In order to obtain a bounded Palais–Smale sequence, we will use the monotonicity trick inspired by [20]. For $\frac{1}{2} \leq s \leq 1$, we define the functional $I_{r,s}: S_{r,\Theta} \to \mathbb{R}$ by

$$I_{r,s}(u) = \frac{1}{2}\int_{\Omega_r}|\nabla u|^2dx + \frac{1}{2}\int_{\Omega_r}Vu^2dx - \frac{s}{q}\int_{\Omega_r}|u|^q dx - \beta\int_{\Omega_r}F(u)dx. \tag{3.1}$$

Note that if $u \in S_{r,\Theta}$ is a critical point of $I_{r,s}$, then there exists $\lambda \in \mathbb{R}$ such that (λ, u) is a solution of the equation

$$\begin{cases} -\Delta u + Vu + \lambda u = s|u|^{q-2}u + \beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r}|u|^2dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r. \end{cases} \tag{3.2}$$

Lemma 3.1. *For any $\Theta > 0$, there exist $r_\Theta > 0$ and $u^0, u^1 \in S_{r_\Theta,\Theta}$ such that*

(i) $I_{r,s}(u^1) \le 0$ *for any* $r > r_\Theta$ *and* $s \in \left[\frac{1}{2}, 1\right]$,

$$\left\|\nabla u^0\right\|_2^2 < \left[\frac{2q}{N(q-2)C_{N,q}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}} < \left\|\nabla u^1\right\|_2^2$$

and

$$I_{r,s}\left(u^0\right) < \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2N(q-2)}\left[\frac{2q\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

(ii) If $u \in S_{r,\Theta}$ *satisfies*

$$\|\nabla u\|_2^2 = \left[\frac{2q}{N(q-2)C_{N,q}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}},$$

then there holds

$$I_{r,s}(u) \ge \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2N(q-2)}\left[\frac{2q\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

(iii) Set

$$m_{r,s}(\Theta) = \inf_{\gamma\in\Gamma_{r,\Theta}} \sup_{t\in[0,1]} I_{r,s}(\gamma(t))$$

with

$$\Gamma_{r,\Theta} = \left\{\gamma \in C\left([0,1], S_{r,\Theta}\right) : \gamma(0) = u^0, \gamma(1) = u^1\right\}.$$

Then

$$\frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2N(q-2)}\left[\frac{2q\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}} \le m_{r,s}(\Theta) \le h(T_\Theta),$$

where $h(T_\Theta) = \max_{t\in\mathbb{R}^+} h(t)$, *the function* $h : \mathbb{R}^+ \to \mathbb{R}$ *being defined by*

$$h(t) = \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)t^2\theta\Theta - \alpha\beta C_{N,p_1}\Theta^{\frac{p_1}{2}}\theta^{\frac{N(p_1-2)}{4}}t^{\frac{N(p_1-2)}{2}} - \frac{1}{2q}\Theta^{\frac{q}{2}}|\Omega|^{\frac{2-q}{2}}t^{\frac{N(q-2)}{2}}.$$

Here θ is the principal eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω, and $|\Omega|$ is the volume of Ω.

Proof. (i) Clearly, the set $S_{r,\Theta}$ is path connected. Since $v_1 \in S_{1,\Theta}$ be the positive eigenfunction associated to θ and note that θ is the principal eigenvalue of $-\Delta$, then

$$\int_\Omega |\nabla v_1|^2\, dx = \theta\Theta. \tag{3.3}$$

By the Hölder inequality, we know that

$$\Theta = \int_\Omega |v_1(x)|^2 dx \le \left(\int_\Omega |v_1(x)|^q dx\right)^{\frac{2}{q}} \cdot |\Omega|^{\frac{q-2}{q}},$$

which implies

$$\int_{\Omega} |v_1(x)|^q dx \geq \Theta^{\frac{q}{2}} \cdot |\Omega|^{\frac{2-q}{2}}. \tag{3.4}$$

According to (f_2), there exists a constant $\alpha > 0$ such that

$$F(\tau) \leq \alpha \tau^{p_1}. \tag{3.5}$$

For $x \in \Omega_{\frac{1}{t}}$ and $t > 0$, define $v_t(x) := t^{\frac{N}{2}} v_1(tx)$. Using (3.3), (3.4), (3.5) and $\frac{1}{2} \leq s \leq 1$, it holds

$$\begin{aligned}
I_{\frac{1}{t},s}(v_t) &\leq \frac{1}{2}\int_{\Omega_r} |\nabla v_t|^2 dx + \frac{1}{2}\int_{\Omega_r} V v_t^2 dx - \frac{1}{2q}\int_{\Omega_r} |v_t|^q dx - \alpha\beta \int_{\Omega_r} |v_t|^{p_1} dx \\
&\leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) \int_{\Omega_r} |\nabla v_t|^2 dx - \alpha\beta C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1-2)}{4}} \left(\int_{\Omega_r} |\nabla v_t|^2 dx\right)^{\frac{N(p_1-2)}{4}} \\
&\quad - \frac{1}{2q}\int_{\Omega_r} |v_t|^q dx \\
&\leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \int_{\Omega} |\nabla v_1|^2 dx - \alpha\beta C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1-2)}{4}} \left(t^2 \int_{\Omega} |\nabla v_1|^2 dx\right)^{\frac{N(p_1-2)}{4}} \\
&\quad - \frac{1}{2q} t^{\frac{N(q-2)}{2}} \int_{\Omega} |v_1|^q dx \\
&\leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta \Theta - \alpha\beta C_{N,p_1} \Theta^{\frac{p_1}{2}} \theta^{\frac{N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{2}} - \frac{1}{2q} t^{\frac{N(q-2)}{2}} \Theta^{\frac{q}{2}} \cdot |\Omega|^{\frac{2-q}{2}} \\
&=: h(t).
\end{aligned} \tag{3.6}$$

Note that since $2 < p_1 < 2 + \frac{4}{N} < q < 2^*$ and $\beta \leq 0$ there exist $0 < T_\Theta < t_0$ such that $h(t_0) = 0, h(t) < 0$ for any $t > t_0, h(t) > 0$ for any $0 < t < t_0$ and $h(T_\Theta) = \max_{t \in \mathbb{R}^+} h(t)$. As a consequence, there holds

$$I_{r,s}(v_{t_0}) = I_{\frac{1}{t_0},s}(v_{t_0}) \leq h(t_0) = 0 \tag{3.7}$$

for any $r \geq \frac{1}{t_0}$ and $s \in \left[\frac{1}{2}, 1\right]$. Moreover, there exists $0 < t_1 < T_\Theta$ such that

$$h(t) < \frac{(N(q-2)-4)\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2N(q-2)} \left[\frac{2q\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2) C_{N,q} \Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}} \tag{3.8}$$

for $t \in [0, t_1]$. On the other hand, it follows from the Gagliardo–Nirenberg inequality and the Hölder inequality that

$$\begin{aligned}
I_{r,s}(u) &\geq \frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} V u^2 dx - \frac{1}{q}\int_{\Omega_r} |u|^q dx \\
&\geq \frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2} \int_{\Omega_r} |\nabla u|^2 dx - \frac{C_{N,q} \Theta^{\frac{2q-N(q-2)}{4}}}{q} \left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(q-2)}{4}}.
\end{aligned} \tag{3.9}$$

Define

$$g(t) := \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t - \frac{C_{N,q} \Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-2)}{4}}$$

and

$$\widetilde{t} = \left[\frac{2q}{N(q-2) C_{N,q}} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) \Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}},$$

it is easy to see that g is increasing on $(0,\widetilde{t})$ and decreasing on $(\widetilde{t},\infty)$, and

$$g(\widetilde{t}) = \frac{(N(q-2)-4)\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)}{2N(q-2)}\left[\frac{2q\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)}{N(q-2)C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

For $r \geq \widetilde{r}_\Theta := \max\left\{\frac{1}{t_1}, \sqrt{\frac{2\theta\Theta}{\widetilde{t}}}\right\}$, we have $v_{\frac{1}{\widetilde{r}_\Theta}} \in S_{r,\Theta}$ and

$$\begin{aligned}\|\nabla v_{\frac{1}{\widetilde{r}_\Theta}}\|_2^2 &= \left(\frac{1}{\widetilde{r}_\Theta}\right)^2 \|\nabla v_1\|_2^2 \\ &< \left[\frac{2q}{N(q-2)C_{N,q}}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}}. \end{aligned} \tag{3.10}$$

Moreover, there holds

$$I_{\widetilde{r}_\Theta,s}\left(v_{\frac{1}{\widetilde{r}_\Theta}}\right) \leq h\left(\frac{1}{\widetilde{r}_\Theta}\right) \leq h\left(t_1\right). \tag{3.11}$$

Setting $u^0 = v_{\frac{1}{\widetilde{r}_\Theta}}, u^1 = v_{t_0}$ and

$$r_\Theta = \max\left\{\frac{1}{t_0}, \widetilde{r}_\Theta\right\}. \tag{3.12}$$

Combining (3.7), (3.8), (3.10) and (3.11), (i) holds.

(ii) By (3.9) and a direct calculation, (ii) holds.

(iii) Since $I_{r,s}\left(u^1\right) \leq 0$ for any $\gamma \in \Gamma_{r,\Theta}$, we have

$$\|\nabla\gamma(0)\|_2^2 < \tilde{t} < \|\nabla\gamma(1)\|_2^2.$$

It then follows from (3.9) that

$$\begin{aligned}&\max_{t\in[0,1]} I_{r,s}(\gamma(t)) \geq g(\widetilde{t}) \\ &= \frac{(N(q-2)-4)\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)}{2N(q-2)}\left[\frac{2q\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)}{N(q-2)C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}\end{aligned}$$

for any $\gamma \in \Gamma_{r,\Theta}$, hence the first inequality in (iii) holds. Now we define a path $\gamma \in \Gamma_{r,\Theta}$ by

$$\gamma(\tau)(x) = \left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta}\right)^{\frac{N}{2}} v_1\left(\left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta}\right)x\right)$$

for $\tau \in [0,1]$ and $x \in \Omega_r$. Then by (3.6) we have $m_{r,s}(\Theta) \leq h(T_\Theta)$, where $h(T_\Theta) = \max_{t\in\mathbb{R}^+} h(t)$. Note that T_Θ is independent of r and s. □

By using Lemma 3.1, the energy functional $I_{r,s}$ possesses the mountain pass geometry. To obtain bounded Palais–Smale sequence, we recall a proposition from [11,13].

Proposition 3.2 (see [11, Theorem 1]). *Let $(E,\langle\cdot,\cdot\rangle)$ and $(H,(\cdot,\cdot))$ be two infinite-dimensional Hilbert spaces and assume there are continuous injections*

$$E \hookrightarrow H \hookrightarrow E'.$$

Let

$$\|u\|^2 = \langle u, u\rangle, \quad |u|^2 = (u, u) \quad \text{for } u \in E,$$

and

$$S_\mu = \left\{u \in E : |u|^2 = \mu\right\}, \qquad T_u S_\mu = \{v \in E : (u, v) = 0\} \quad \text{for } \mu \in (0, +\infty).$$

Let $I \subset (0, +\infty)$ *be an interval and consider a family of* C^2 *functionals* $\Phi_\rho : E \to \mathbb{R}$ *of the form*

$$\Phi_\rho(u) = A(u) - \rho B(u), \quad \text{for } \rho \in I,$$

with $B(u) \geq 0$ *for every* $u \in E$*, and*

$$A(u) \to +\infty \quad \text{or} \quad B(u) \to +\infty \quad \text{as } u \in E \text{ and } \|u\| \to +\infty. \tag{3.13}$$

Suppose moreover that Φ'_ρ *and* Φ''_ρ *are* τ*-Hölder continuous,* $\tau \in (0, 1]$*, on bounded sets in the following sense: for every* $R > 0$ *there exists* $M = M(R) > 0$ *such that*

$$\left\|\Phi'_\rho(u) - \Phi'_\rho(v)\right\| \leq M\|u - v\|^\tau \quad \text{and} \quad \left\|\Phi''_\rho(u) - \Phi''_\rho(v)\right\| \leq M\|u - v\|^\tau \tag{3.14}$$

for every $u, v \in B(0, R)$*. Finally, suppose that there exist* $w_1, w_2 \in S_\mu$ *independent of* ρ *such that*

$$c_\rho := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\rho(\gamma(t)) > \max\left\{\Phi_\rho(w_1), \Phi_\rho(w_2)\right\} \quad \text{for all } \rho \in I,$$

where

$$\Gamma = \left\{\gamma \in C\left([0, 1], S_\mu\right) : \gamma(0) = w_1, \gamma(1) = w_2\right\}.$$

Then for almost every $\rho \in I$*, there exists a sequence* $\{u_n\} \subset S_\mu$ *such that*

(i) $\Phi_\rho(u_n) \to c_\rho$*,*

(ii) $\Phi'_\rho\big|_{S_\mu}(u_n) \to 0$*,*

(iii) $\{u_n\}$ *is bounded in* E*.*

Lemma 3.3. *For any* $\Theta > 0$*, let* $r > r_\Theta$*, where* r_Θ *is defined in Lemma 3.1. Then problem (3.1) has a solution* $(\lambda_{r,s}, u_{r,s})$ *for almost every* $s \in \left[\frac{1}{2}, 1\right]$*. Moreover,* $u_{r,s} \geq 0$ *and* $I_{r,s}(u_{r,s}) = m_{r,s}(\Theta)$*.*

Proof. By Proposition 3.2, it follows that

$$A(u) = \frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} V(x)u^2 dx - \beta\int_{\Omega_r} F(u)dx \quad \text{and} \quad B(u) = \frac{1}{q}\int_{\Omega_r} |u|^q dx.$$

Note that the assumptions in Proposition 3.2 hold due to $\beta \leq 0$ and Lemma 3.1. Hence, for almost every $s \in \left[\frac{1}{2}, 1\right]$, there exists a bounded Palais–Smale sequence $\{u_n\}$ satisfying

$$I_{r,s}(u_n) \to m_{r,s}(\Theta) \quad \text{and} \quad I'_{r,s}(u_n)\big|_{T_{u_n} S_{r,\Theta}} \to 0,$$

where $T_{u_n} S_{r,\Theta}$ denotes the tangent space of $S_{r,\Theta}$ at u_n. Then

$$\lambda_n = -\frac{1}{\Theta}\left(\int_{\Omega_r} |\nabla u_n|^2 dx + \int_{\Omega_r} V(x)u_n^2 dx - \beta\int_{\Omega_r} f(u_n)u_n dx - s\int_{\Omega_r} |u_n|^q dx\right)$$

is bounded and

$$I'_{r,s}(u_n) + \lambda_n u_n \to 0 \quad \text{in } H^{-1}(\Omega_r). \tag{3.15}$$

Moreover, since $\{u_n\}$ is a bounded Palais–Smale sequence, there exist $u_0 \in H_0^1(\Omega_r)$ and $\lambda \in \mathbb{R}$ such that, up to a subsequence,

$$\begin{aligned} \lambda_n &\to \lambda && \text{in } \mathbb{R}, \\ u_n &\rightharpoonup u_0 && \text{in } H_0^1(\Omega_r), \\ u_n &\to u_0 && \text{in } L^t(\Omega_r) \text{ for all } 2 \le t < 2^*, \end{aligned}$$

where u_0 satisfies

$$\begin{cases} -\Delta u_0 + V u_0 + \lambda u_0 = s\,|u_0|^{q-2}\,u_0 + \beta f(u_0) & \text{in } \Omega_r \\ u_0 \in H_0^1(\Omega_r), \quad \int_{\Omega_r} |u_0|^2\,dx = \Theta. \end{cases}$$

Using (3.15), we have

$$I'_{r,s}(u_n)\,u_0 + \lambda_n \int_{\Omega_r} u_n u_0 dx \to 0 \quad \text{as } n \to \infty$$

and

$$I'_{r,s}(u_n)\,u_n + \lambda_n \Theta \to 0 \quad \text{as } \; n \to \infty.$$

Note that

$$\begin{aligned} \lim_{n\to\infty} \int_{\Omega_r} V(x) u_n^2 dx &= \int_{\Omega_r} V(x) u_0^2 dx, \\ \lim_{n\to\infty} \int_{\Omega_r} f(u_n) u_n dx &= \int_{\Omega_r} f(u_0) u_0 dx, \\ \lim_{n\to\infty} \int_{\Omega_r} f(u_n) u_0 dx &= \int_{\Omega_r} f(u_0) u_0 dx, \end{aligned}$$

so we get $u_n \to u_0$ in $H_0^1(\Omega_r)$, hence $I_{r,s}(u_0) = m_{r,s}(\Theta)$.

Now, we show that $u_{r,s} \ge 0$. In order to obtain it, we only need to modify the proof of Proposition 3.2. In fact, for almost every $s \in \left[\frac{1}{2}, 1\right]$, the derivative $m'_{r,s}$ with respect to s is well defined since the function $s \mapsto m_{r,s}$ is nonincreasing, where $m_{r,s}$ denotes $m_{r,s}(\Theta)$ for fixed Θ. Let s be such that $m'_{r,s}$ exists and $\{s_n\} \subset \left[\frac{1}{2}, 1\right]$ be a monotone increasing sequence converging to s. Similar to the proof of Proposition 3.2, there exist $\{\gamma_n\} \subset \Gamma_{r,\Theta}$ and $K = K\left(m'_{r,s}\right)$ such that:

(i) if $I_{r,s}(\gamma_n(t)) \ge m_{r,s} - \left(2 - m'_{r,s}\right)(s - s_n)$, then $\int_{\Omega_r} |\nabla \gamma_n(t)|^2\,dx \le K$.

(ii) $\max_{t\in[0,1]} I_{r,s}(\gamma_n(t)) \le m_{r,s} - \left(2 - m'_{r,s}\right)(s - s_n)$.

Letting $\widetilde{\gamma}_n(t) = |\gamma_n(t)|$ for any $t \in [0,1]$, it follows that $\{\widetilde{\gamma}_n\} \subset \Gamma_{r,\Theta}$. Observe that $\|\nabla |u|\|_2^2 \le \|\nabla u\|_2^2$ for any $u \in H^1(\mathbb{R}^N)$. Now we have:

(I) if $I_{r,s}(\widetilde{\gamma}_n(t)) \ge m_{r,s} - \left(2 - m'_{r,s}\right)(s - s_n)$, then $I_{r,s}(\gamma_n(t)) \ge m_{r,s} - \left(2 - m'_{r,s}\right)(s - s_n)$. By (i), there holds $\int_{\Omega_r} |\nabla \gamma_n(t)|^2\,dx \le K$, and hence $\int_{\Omega_r} |\nabla \widetilde{\gamma}_n(t)|^2\,dx \le K$. Thus (i) also holds for $\widetilde{\gamma}_n$.

(II) $\max_{t\in[0,1]} I_{r,s}(\widetilde{\gamma}_n(t)) \le \max_{t\in[0,1]} I_{r,s}(\gamma_n(t)) \le m_{r,s} - \left(2 - m'_{r,s}\right)(s - s_n)$.

By replacing γ_n with $\widetilde{\gamma}_n$ in the proof of Proposition 3.2, we obtain a nonnegative bounded Palais–Smale sequence $\{u_n\}$. Consequently, there exists a nonnegative normalized solution to (3.1) for almost every $s \in \left[\frac{1}{2}, 1\right]$ as above. □

In order to obtain a solution of (1.1), we need to prove a uniform estimate for the solutions of (3.1) established in Lemma 3.3.

Lemma 3.4. *If* $(\lambda_{r,s}, u_{r,s}) \in \mathbb{R} \times S_{r,\Theta}$ *is a solution of* (3.1) *established in Lemma* 3.3 *for some r and s, then*

$$\int_{\Omega_r} |\nabla u|^2 dx \leq \frac{4N}{N(q-2)-4}\left[\frac{q-2}{2}h(T_\Theta) + \Theta\left(\frac{1}{2N}\|\widetilde{V}\|_\infty + \frac{q-2}{4}\|V\|_\infty\right)\right],$$

where the constant $h(T_\Theta)$ *is defined in (iii) of Lemma* 3.1 *and is independent of r and s.*

Proof. For simplicity, we denote $(\lambda_{r,s}, u_{r,s})$ as (λ, u) in this lemma. Since u is a solution of (3.1), we have

$$\int_{\Omega_r} |\nabla u|^2 dx + \int_{\Omega_r} V(x)u^2 dx = s\int_{\Omega_r} |u|^q dx + \beta\int_{\Omega_r} f(u)u dx - \lambda\int_{\Omega_r} |u|^2 dx. \tag{3.16}$$

The Pohozaev identity implies

$$\begin{aligned}&\frac{N-2}{2N}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2N}\int_{\partial\Omega_r} |\nabla u|^2 (x\cdot\mathbf{n}) d\sigma + \frac{1}{2N}\int_{\Omega_r} \widetilde{V}(x)u^2 dx + \frac{1}{2}\int_{\Omega_r} Vu^2 dx\\ &\quad = -\frac{\lambda}{2}\int_{\Omega_r} |u|^2 dx + \frac{s}{q}\int_{\Omega_r} |u|^q dx + \beta\int_{\Omega_r} F(u)dx,\end{aligned}$$

where $\mathbf{n}$ denotes the outward unit normal vector on $\partial\Omega_r$. It then follows from $\beta \leq 0$ and (f_2) that

$$\begin{aligned}&\frac{1}{N}\int_{\Omega_r} |\nabla u|^2 dx - \frac{1}{2N}\int_{\partial\Omega_r} |\nabla u|^2 (x\cdot\mathbf{n}) d\sigma - \frac{1}{2N}\int_{\Omega_r} (\nabla V\cdot x)u^2 dx\\ &\quad = \frac{(q-2)s}{2q}\int_{\Omega_r} |u|^q dx + \int_{\Omega_r} (\frac{\beta}{2}f(u)u - \beta F(u))dx\\ &\quad \geq \frac{(q-2)s}{2q}\int_{\Omega_r} |u|^q dx + \frac{\beta(q-2)}{2}\int_{\Omega_r} F(u)dx\\ &\quad = \frac{q-2}{2}\left(\frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} Vu^2 dx - m_{r,s}(\Theta)\right).\end{aligned}$$

Consequently, we have

$$\begin{aligned}\frac{q-2}{2}m_{r,s}(\Theta) &\geq \frac{q-2}{2}\left(\frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} Vu^2 dx\right) - \frac{1}{N}\int_{\Omega_r} |\nabla u|^2 dx\\ &\quad + \frac{1}{2N}\int_{\partial\Omega_r} |\nabla u|^2 (x\cdot\mathbf{n}) d\sigma + \frac{1}{2N}\int_{\Omega_r} (\nabla V\cdot x)u^2 dx\\ &\geq \frac{N(q-2)-4}{4N}\int_{\Omega_r} |\nabla u|^2 dx - \Theta\left(\frac{1}{2N}\|\nabla V\cdot x\|_\infty + \frac{q-2}{4}\|V\|_\infty\right),\end{aligned}$$

where the last inequality holds since $x\cdot\mathbf{n}(x) \geq 0$ for any $x \in \partial\Omega_r$ due to the convexity of Ω_r. Using Lemma 3.1, we have

$$\frac{N(q-2)-4}{4N}\int_{\Omega_r} |\nabla u|^2 dx - \Theta\left(\frac{1}{2N}\|\nabla V\cdot x\|_\infty + \frac{q-2}{4}\|V\|_\infty\right) \leq \frac{q-2}{2}h(T_\Theta),$$

which implies

$$\int_{\Omega_r} |\nabla u|^2 dx \leq \frac{4N}{N(q-2)-4}\left[\frac{q-2}{2}h(T_\Theta) + \Theta\left(\frac{1}{2N}\|\widetilde{V}\|_\infty + \frac{q-2}{4}\|V\|_\infty\right)\right].$$

This completes the proof of lemma. □

Now, we obtain a solution of (1.1) by letting $s \to 1$.

Lemma 3.5. *For every $\Theta > 0$, problem (1.1) has a solution (λ_r, u_r) provided $r > r_\Theta$ where r_Θ is as in Lemma 3.1. Moreover, $u_r \geq 0$ in Ω_r.*

Proof. By using Lemma 3.3, there is a nonnegative solution $(\lambda_{r,s}, u_{r,s})$ to (3.1) for almost every $s \in \left[\frac{1}{2}, 1\right]$. In view of Lemma 3.4, $\{u_{r,s}\}$ is bounded. By an argument similar to that in Lemma 3.3, there exist $u_r \in S_{r,\Theta}$ and λ_r such that, going if necessary to a subsequence,

$$\lambda_{r,s} \to \lambda_r \quad \text{and} \quad u_{r,s} \to u_r \quad \text{in } H_0^1(\Omega_r) \quad \text{as } s \to 1.$$

Hence u_r is a nonnegative solution of problem (1.1). □

Next, we will consider the Lagrange multiplier. we first establish an a priori estimate for the solutions of (1.1).

Lemma 3.6. *If $\{(\lambda_r, u_r)\}$ is a family of nonnegative solutions of (1.1) such that $\|u_r\|_{H^1} \leq C$ with $C > 0$ independent of r, then $\limsup_{r\to\infty} \|u_r\|_\infty < \infty$.*

Proof. Using the regularity theory of elliptic partial differential equations, we know that $u_r \in C(\Omega_r)$. Assume to the contrary that there exist a sequence, for simplicity denoted by $\{u_r\}$, and $x_r \in \Omega_r$ such that

$$M_r := \max_{x\in\Omega_r} u_r(x) = u_r(x_r) \to \infty \quad \text{as } r \to \infty.$$

Suppose without loss of generality that, up to a subsequence, $\lim_{r\to\infty} \frac{x_r}{|x_r|} = (1, 0, \ldots, 0)$. Set

$$v_r(x) = \frac{u_r(x_r + \tau_r x)}{M_r} \quad \text{for } x \in \Sigma^r := \left\{x \in \mathbb{R}^N : x_r + \tau_r x \in \Omega_r\right\},$$

where $\tau_r = M_r^{\frac{2-q}{2}}$. Then $\tau_r \to 0$ as $r \to \infty$, $\|v_r\|_{L^\infty(\Sigma^r)} \leq 1$, and v_r satisfies

$$-\Delta v_r + \tau_r^2 V(x_r + \tau_r x) v_r + \tau_r^2 \lambda_r v_r = |v_r|^{q-2} v_r + \beta M_r^{1-q} f(M_r v_r) \quad \text{in } \Sigma^r. \tag{3.17}$$

In fact, since u_r is a nonnegative solution of (1.1), we obtain

$$\begin{aligned} &-\Delta u_r(x_r + \tau_r x) + V(x_r + \tau_r x) u_r(x_r + \tau_r x) + \lambda_r u_r(x_r + \tau_r x) \\ &\quad = |u_r(x_r + \tau_r x)|^{q-2} u_r(x_r + \tau_r x) + \beta f(u_r(x_r + \tau_r x)) \quad \text{in } \Omega_r, \end{aligned}$$

then by a direct calculation and the definition of $v_r(x)$, τ_r, we know that (3.17) holds. In view of (1.1), the Gagliardo–Nirenberg inequality and $\|u_r\|_{H^1} \leq C$ with C independent of r, we infer that the sequence $\{\lambda_r\}$ is bounded. It then follows from the regularity theory of elliptic partial differential equations and the Arzelà–Ascoli theorem that there exists v such that, up to a subsequence

$$v_r \to v \quad \text{in } H_0^1(\Sigma) \quad \text{and} \quad v_r \to v \quad \text{in } C_{loc}^\beta(\Sigma) \text{ for some } \beta \in (0,1),$$

where $\Sigma := \lim\limits_{r\to\infty} \Sigma^r$.

Similar to the proof of [4, Lemma 2.7], we have

$$\liminf_{r\to\infty} \frac{\operatorname{dist}(x_r, \partial\Omega_r)}{\tau_r} = \liminf_{r\to\infty} \frac{|y_r - x_r|}{\tau_r} \geq d > 0,$$

where $y_r \in \partial\Omega_r$ is such that $\operatorname{dist}(x_r, \partial\Omega_r) = |y_r - x_r|$ for any large r. As a result, by letting $r \to \infty$ in (3.17), we obtain that $v \in H_0^1(\Sigma)$ is a nonnegative solution of

$$-\Delta v = |v|^{q-2}v \quad \text{in } \Sigma,$$

where

$$\Sigma = \begin{cases} \mathbb{R}^N & \text{if } \liminf_{r\to\infty} \frac{\operatorname{dist}(x_r,\partial\Omega_r)}{\tau_r} = \infty, \\ \{x \in \mathbb{R}^N : x_1 > -d\} & \text{if } \liminf_{r\to\infty} \frac{\operatorname{dist}(x_r,\partial\Omega_r)}{\tau_r} > 0. \end{cases}$$

It then follows from the Liouville theorems (see [17]) that $v = 0$ in $H_0^1(\Sigma)$, which contradicts $v(0) = \lim_{r\to\infty} v_r(0) = 1$. □

Clearly, the proof of Lemma 3.6 does not depend on β.

Lemma 3.7. *Let $(\lambda_{r,\Theta}, u_{r,\Theta})$ be the solution of (1.1) from Lemma 3.5. If $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$, then there exists $\bar{\Theta} > 0$ such that*

$$\liminf_{r\to\infty} \lambda_{r,\Theta} > 0 \quad \textit{for } 0 < \Theta < \bar{\Theta}.$$

Proof. Let $(\lambda_{r,\Theta}, u_{r,\Theta})$ be the solution of (1.1) established in Theorem 3.5. By the regularity theory of elliptic partial differential equations, we have $u_{r,\Theta} \in C(\Omega_r)$. Using Lemma 3.6, it holds

$$\limsup_{r\to\infty} \max_{\Omega_r} u_{r,\Theta} < \infty.$$

Setting

$$Q(\Theta) = \liminf_{r\to\infty} \max_{\Omega_r} u_{r,\Theta},$$

we claim that there is $\Theta_1 > 0$ such that $Q(\Theta) > 0$ for any $0 < \Theta < \Theta_1$. Assume to the contrary that there exists a sequence $\{\Theta_k\}$ tending to 0 as $k \to \infty$ such that $Q(\Theta_k) = 0$ for any k, that is,

$$\liminf_{r\to\infty} \max_{\Omega_r} u_{r,\Theta_k} = 0 \quad \text{for any } k. \tag{3.18}$$

As a consequence of (iii) in Lemma 3.1, for any $r > r_{\Theta_k}$, we have

$$I_r(u_{r,\Theta_k}) = m_{r,1}(\Theta_k) \to \infty \quad \text{as } k \to \infty. \tag{3.19}$$

For any given k, it follows from (3.18) and $u_{r,\Theta_k} \in S_{r,\Theta_k}$ that, up to a subsequence,

$$\int_{\Omega_r} |u_{r,\Theta_k}|^s\,dx = \int_{\Omega_r} |u_{r,\Theta_k}|^{s-2}\,|u_{r,\Theta_k}|^2\,dx \le \left|\max_{\Omega_r} u_{r,\Theta_k}\right|^{s-2} \Theta_k \to 0 \quad \text{as } r \to \infty \tag{3.20}$$

for any $s > 2$. Hence, for any given large k, there exists $\bar{r}_k > r_{\Theta_k}$ such that

$$\left|\frac{1}{q}\int_{\Omega_r} |u_{r,\Theta_k}|^q dx + \beta \int_{\Omega_r} f(u_{r,\Theta_k})dx\right| < \frac{m_{r,1}(\Theta_k)}{2} \text{ for any } r \ge \bar{r}_k.$$

In view of (3.19) and $I_r(u_{r,\Theta_k}) = m_{r,1}(\Theta_k)$, we further have

$$\int_{\Omega_r} |\nabla u_{r,\Theta_k}|^2\,dx + \int_{\Omega_r} V(x)u_{r,\Theta_k}^2 dx \ge \frac{m_{r,1}(\Theta_k)}{2} \text{ for any large } k \text{ and } r \ge \bar{r}_k. \tag{3.21}$$

It follows from (3.18), (3.20) and (3.21) that there exists $r_k \ge \bar{r}_k$ with $r_k \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \max_{\Omega_{r_k}} u_{r_k,\Theta_k} = 0, \tag{3.22}$$

$$\int_{\Omega_{r_k}} |u_{r_k,\Theta_k}|^s \, dx \le \left| \max_{\Omega_{r_k}} u_{r_k,\Theta_k} \right|^{s-2} \Theta_k \to 0 \quad \text{as } k \to \infty \text{ for any } s > 2 \tag{3.23}$$

and

$$\int_{\Omega_{r_k}} |\nabla u_{r_k,\Theta_k}|^2 \, dx + \int_{\Omega_r} V u_{r_k,\Theta_k}^2 dx \to \infty \quad \text{as } k \to \infty. \tag{3.24}$$

By (1.1), (3.23) and (3.24), we have

$$\lambda_{r_k,\Theta_k} \to -\infty \quad \text{as } k \to \infty. \tag{3.25}$$

Now (1.1) implies

$$-\Delta u_{r_k,\Theta_k} + V(x) u_{r_k,\Theta_k} + \lambda_{r_k,\Theta_k} u_{r_k,\Theta_k} = |u_{r_k,\Theta_k}|^{q-2} u_{r_k,\Theta_k} + \beta f(u_{r_k,\Theta_k}),$$

so

$$-\Delta u_{r_k,\Theta_k} + \left(\|V\|_\infty + \frac{\lambda_{r_k,\Theta_k}}{2} \right) u_{r_k,\Theta_k} \ge -\frac{\lambda_{r_k,\Theta_k}}{2} u_{r_k,\Theta_k} + |u_{r_k,\Theta_k}|^{q-2} u_{r_k,\Theta_k} + \beta f(u_{r_k,\Theta_k}).$$

Using (3.25) and (3.22), it follows that

$$-\Delta u_{r_k,\Theta_k} + \left(\|V\|_\infty + \frac{\lambda_{r_k,\Theta_k}}{2} \right) u_{r_k,\Theta_k} \ge 0$$

for large k. Let θ_{r_k} be the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω_{r_k}, and $v_{r_k} > 0$ be the corresponding normalized eigenfunction. It follows that

$$\left(\theta_{r_k} + \|V\|_\infty + \frac{\lambda_{r_k,\Theta_k}}{2} \right) \int_{\Omega_{r_k}} u_{r_k,\Theta_k} v_{r_k} dx \ge 0.$$

Since $\int_{\Omega_{r_k}} u_{r_k,\Theta_k} v_{r_k} dx > 0$, we have

$$\theta_{r_k} + \|V\|_\infty + \frac{\lambda_{r_k,\Theta_k}}{2} \ge 0,$$

which contradicts (3.25) for large k. Hence the claim holds, that is, there exists $\Theta_1 > 0$ such that

$$Q(\Theta) = \liminf_{r \to \infty} \max_{\Omega_r} u_{r,\Theta} > 0 \tag{3.26}$$

for any $0 < \Theta < \Theta_1$.

We consider $H^1(\Omega_r)$ as a subspace of $H^1(\mathbb{R}^N)$ for any $r > 0$. It follows from Lemma 3.4 that the set of solutions $\{u_{r,\Theta} : r > r_\Theta\}$ established in Lemma 3.5 is bounded in $H^1(\mathbb{R}^N)$, so there exist $u_\Theta \in H^1(\mathbb{R}^N)$ and $\lambda_\Theta \in \mathbb{R}$ such that up to a subsequence:

$$\begin{aligned} &\lambda_{r,\Theta} \to \lambda_\Theta, \\ &u_{r,\Theta} \rightharpoonup u_\Theta \quad \text{in } H^1(\mathbb{R}^N), \\ &u_{r,\Theta} \to u_\Theta \quad \text{in } L^k_{loc}(\mathbb{R}^N) \text{ for all } 2 \le k < 2^*, \\ &u_{r,\Theta} \to u_\Theta \quad \text{a.e. in } \mathbb{R}^N \end{aligned}$$

and u_Θ is a solution of the equation

$$-\Delta u + V(x) u + \lambda_\Theta u = |u|^{q-2} u + \beta f(u) \quad \text{in } \mathbb{R}^N.$$

Hence,

$$\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx + \int_{\mathbb{R}^N} V(x)u_\Theta^2 dx + \lambda_\Theta \int_{\mathbb{R}^N} u_\Theta^2 dx = \int_{\mathbb{R}^N} |u_\Theta|^q\,dx + \beta \int_{\mathbb{R}^N} f(u_\Theta)u_\Theta dx \tag{3.27}$$

and the Pohozaev identity gives

$$\begin{aligned}&\frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx + \frac{1}{2N}\int_{\mathbb{R}^N} \widetilde{V} u_\Theta^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u_\Theta^2 dx + \frac{\lambda_\Theta}{2}\int_{\mathbb{R}^N} u_\Theta^2 dx\\ &= \frac{1}{q}\int_{\mathbb{R}^N} |u_\Theta|^q\,dx + \beta\int_{\mathbb{R}^N} F(u_\Theta)dx.\end{aligned} \tag{3.28}$$

It follows from (3.27), (3.28), (f_2), the Gagliardo–Nirenberg inequality and the fact $\beta \leq 0$ that

$$\begin{aligned}&\frac{1}{N}\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx + \frac{1}{2N}\int_{\mathbb{R}^N} \widetilde{V}(x)u_\Theta^2 dx\\ &= \left(\frac{1}{2}-\frac{1}{q}\right)\int_{\mathbb{R}^N} |u_\Theta|^q\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N}(f(u_\Theta)u_\Theta - 2F(u_\Theta))dx\\ &\leq \frac{C_{N,q}(q-2)}{2q}\left(\int_{\mathbb{R}^N} u_\Theta^2 dx\right)^{\frac{2q-N(q-2)}{4}}\left(\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx\right)^{\frac{N(q-2)}{4}}.\end{aligned}$$

By using the Hölder inequality, we have

$$\left(\frac{1}{N} - \frac{\|\widetilde{V}_+\|_{\frac{N}{2}} S^{-1}}{2N}\right)\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx \leq \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx + \frac{1}{2N}\int_{\mathbb{R}^N} \widetilde{V}(x)u_\Theta^2 dx.$$

Therefore,

$$\begin{aligned}&\left(\frac{1}{N} - \frac{\|\widetilde{V}_+\|_{\frac{N}{2}} S^{-1}}{2N}\right)\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx\\ &\leq \frac{C_{N,q}(q-2)}{2q}\left(\int_{\mathbb{R}^N} u_\Theta^2 dx\right)^{\frac{2q-N(q-2)}{4}}\left(\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx\right)^{\frac{N(q-2)}{4}}.\end{aligned}$$

If $u_\Theta \neq 0$, Using $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$, we obtain that

$$\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx \geq \left[\frac{q\left(2-\|\widetilde{V}_+\|_{\frac{N}{2}} S^{-1}\right)}{NC_{N,q}(q-2)}\right]^{\frac{4}{N(q-2)-4}} \Theta^{\frac{q(N-2)-2N}{N(q-2)-4}}. \tag{3.29}$$

Next, it follows from (3.5), (3.27), (3.28), (3.29), (f_2) and $2+\frac{4}{N} < q < 2^*$ that

$$\begin{aligned}\left(\frac{1}{q}-\frac{1}{2}\right)\lambda_\Theta\int_{\mathbb{R}^N} u_\Theta^2 dx &= \left(\frac{N-2}{2N}-\frac{1}{q}\right)\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx + \frac{1}{2N}\int_{\mathbb{R}^N} \widetilde{V}(x)u_\Theta^2 dx\\ &\quad + \left(\frac{1}{2}-\frac{1}{q}\right)\int_{\mathbb{R}^N} V(x)u_\Theta^2 dx - \frac{\beta}{q}\int_{\mathbb{R}^N}(qF(u_\Theta) - f(u_\Theta)u_\Theta))\,dx\\ &\leq \frac{(N-2)q-2N}{2Nq}\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx + \frac{\|\widetilde{V}\|_\infty}{2N}\Theta + \frac{(q-2)\|V\|_\infty}{2q}\Theta\\ &\quad - \frac{\beta(q-p_2)\alpha}{q} C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\mathbb{R}^N} |\nabla u_\Theta|^2\,dx\right)^{\frac{N(p_1-2)}{4}}\\ &\to -\infty \quad \text{as } \Theta \to 0,\end{aligned}$$

since $\frac{(N-2)q-2N}{2Nq} < 0$. Therefore, if $u_\Theta \neq 0$ for $\Theta > 0$ small there exists $\Theta_0 > 0$ such that $\lambda_\Theta > 0$ for $0 < \Theta < \Theta_0$.

In order to complete the proof, we consider the case that there is a sequence $\Theta_k \to 0$ such that $u_{\Theta_k} = 0$ for any k. Assume without loss of generality that $u_\Theta = 0$ for any $\Theta \in (0, \Theta_1)$. Let $x_{r,\Theta} \in \Omega_r$ be such that $u_{r,\Theta}(x_{r,\Theta}) = \max_{\Omega_r} u_{r,\Theta}$. In view of (3.26), there holds $|x_{r,\Theta}| \to \infty$ as $r \to \infty$. Otherwise, there exists $x_0 \in \mathbb{R}$ such that, up to a subsequence, $x_{r,\Theta} \to x_0$, and hence $u_\Theta(x_0) \geq d_\Theta > 0$. This contradicts $u_\Theta = 0$. We claim that $\operatorname{dist}(x_{r,\Theta}, \partial\Omega_r) \to \infty$ as $r \to \infty$. Arguing by contradiction we assume that $\liminf_{r\to\infty} \operatorname{dist}(x_{r,\Theta}, \partial\Omega_r) = l < \infty$. It follows from (3.26) that $l > 0$. Let $w_r(x) = u_{r,\Theta}(x + x_{r,\Theta})$ for any $x \in \Sigma^r := \{x \in \mathbb{R}^N : x + x_{r,\Theta} \in \Omega_r\}$. Then w_r is bounded in $H^1(\mathbb{R}^N)$, and there is $w \in H^1(\mathbb{R}^N)$ such that $w_r \rightharpoonup w$ as $r \to \infty$. By the regularity theory of elliptic partial equations and $\liminf_{r\to\infty} u_{r,\Theta}(x_{r,\Theta}) > d_\Theta > 0$, we infer that $w(0) \geq d_\Theta > 0$. Assume without loss of the generality that, up to a subsequence,

$$\lim_{r\to\infty} \frac{x_{r,\Theta}}{|x_{r,\Theta}|} = e_1.$$

Setting

$$\Sigma = \left\{x \in \mathbb{R}^N : x \cdot e_1 < l\right\} = \left\{x \in \mathbb{R}^N : x_1 < l\right\},$$

we have $\phi(\cdot - x_{r,\Theta}) \in C_c^\infty(\Omega_r)$ for any $\phi \in C_c^\infty(\Sigma)$ and r large enough. It then follows that

$$\begin{aligned}\int_{\Omega_r} \nabla u_{r,\Theta} \nabla \phi(\cdot - x_{r,\Theta})\, dx + \int_{\Omega_r} V u_{r,\Theta} \phi(\cdot - x_{r,\Theta})\, dx + \lambda_{r,\Theta} \int_{\Omega_r} u_{r,\Theta} \phi(\cdot - x_{r,\Theta})\, dx \\ = \int_{\Omega_r} |u_{r,\Theta}|^{q-2} u_{r,\Theta} \phi(\cdot - x_{r,\Theta})\, dx + \beta \int_{\Omega_r} f(u_{r,\Theta}) \phi(\cdot - x_{r,\Theta})\, dx. \end{aligned} \tag{3.30}$$

Since $|x_{r,\Theta}| \to \infty$ as $r \to \infty$, it holds

$$\begin{aligned}\left| \int_{\Omega_r} V u_{r,\Theta} \phi(\cdot - x_{r,\Theta})\, dx \right| &\leq \int_{\text{Supp}\,\phi} |V(\cdot + x_{r,\Theta}) w_r \phi|\, dx \\ &\leq \|w_r\|_{2^*} \|\phi\|_{2^*} \left(\int_{\text{Supp}\,\phi} |V(\cdot + x_{r,\Theta})|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \\ &\leq \|w_r\|_{2^*} \|\phi\|_{2^*} \left(\int_{\mathbb{R}^N \setminus B_{\frac{|x_{r,\Theta}|}{2}}} |V|^{\frac{N}{2}} dx \right)^{\frac{2}{N}} \\ &\to 0 \quad \text{as } r \to \infty. \end{aligned} \tag{3.31}$$

Letting $r \to \infty$ in (3.30), we obtain for $\phi \in C_c^\infty(\Sigma)$:

$$\int_\Sigma \nabla w \cdot \nabla \phi dx + \lambda_\Theta \int_\Sigma w \phi dx = \int_\Sigma |w|^{q-2} w \phi dx + \beta \int_\Sigma f(w) \phi dx.$$

Thus $w \in H_0^1(\Sigma)$ is a weak solution of the equation

$$-\Delta w + \lambda_\Theta w = |w|^{q-2} w + \beta f(w) \quad \text{in } \Sigma. \tag{3.32}$$

Hence we obtain a nontrivial nonnegative solution of (3.32) on a half space which is impossible by the Liouville theorem (see [17]). This proves that $\operatorname{dist}(x_{r,\Theta}, \partial\Omega_r) \to \infty$ as $r \to \infty$. A similar argument as above shows that (3.32) holds for $\Sigma = \mathbb{R}^N$. Now we argue as in the case $u_\Theta \neq 0$ above that there exists Θ_2 such that $\lambda_\Theta > 0$ for any $0 < \Theta < \Theta_2$.

Setting $\bar{\Theta} = \min\{\Theta_0, \Theta_1, \Theta_2\}$, the proof is complete. □

Proof of Theorem 1.3. The proof is an immediate consequence of Lemmas 3.5, 3.6 and 3.7. □

4 Proof of Theorem 1.4

In this section, we assume that the assumptions of Theorem 1.4 hold. Since $\beta > 0$,

$$\begin{aligned} I_r(u) &\geq \frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right)\int_{\Omega_r}|\nabla u|^2dx - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q}\left(\int_{\Omega_r}|\nabla u|^2dx\right)^{\frac{N(q-2)}{4}} \\ &\quad - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\Omega_r}|\nabla u|^2dx\right)^{\frac{N(p_1-2)}{4}} \\ &= h_1(t), \end{aligned}$$

where

$$\begin{aligned} h_1(t) &:= \frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right)t^2 - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q}t^{\frac{N(q-2)}{2}} - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}t^{\frac{N(p_1-2)}{2}} \\ &= t^{\frac{N(p_1-2)}{2}}\left[\frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right)t^{\frac{4-N(p_1-2)}{2}} - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q}t^{\frac{N(q-p_1)}{2}}\right] \\ &\quad - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}t^{\frac{N(p_1-2)}{2}}. \end{aligned}$$

Consider

$$\psi(t) := \frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right)t^{\frac{4-N(p_1-2)}{2}} - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q}t^{\frac{N(q-p_1)}{2}}.$$

Note that ψ admits a unique maximum at

$$\bar{t} = \left[\frac{q(4-N(p_1-2))\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)}{2N(q-p_1)C_{N,q}}\right]^{\frac{2}{N(q-2)-4}}\Theta^{\frac{N(q-2)-2q}{2(N(q-2)-4)}}.$$

By a direct calculation, we obtain

$$\psi(\bar{t}) = \left[\frac{1-\|V_-\|_{\frac{N}{2}} S^{-1}}{2N(q-p_1)}\right]^{\frac{N(q-p_1)}{N(q-2)-4}}\left[\frac{q(4-N(p_1-2))}{C_{N,q}}\right]^{\frac{4-N(p_1-2)}{N(q-2)-4}}[N(q-2)-4].$$

Hence,

$$\psi(\bar{t}) > \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}$$

as long as

$$\Theta_V = \left[\frac{1-\|V_-\|_{\frac{N}{2}} S^{-1}}{2N(q-p_1)}\right]^{\frac{N}{2}}\left[\frac{q(4-N(p_1-2))}{C_{N,q}}\right]^{\frac{4-N(p_1-2)}{2(q-p_1)}}\left[\frac{N(q-2)-4}{\alpha\beta C_{N,p_1}}\right]^{\frac{N(q-2)-4}{2(q-p_1)}}.$$

Now, let $0 < \Theta < \Theta_V$ be fixed, we obtain

$$\psi(\bar{t}) > \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} \tag{4.1}$$

and $h_1(\bar{t}) > 0$. In view of $2 < p_1 < 2+\frac{4}{N} < q < 2^*$ and (4.1), there exist $0 < R_1 < T_\Theta < R_2$ such that $h_1(t) < 0$ for $0 < t < R_1$ and for $t > R_2, h_1(t) > 0$ for $R_1 < t < R_2$, and $h_1(T_\Theta) = \max_{t\in\mathbb{R}^+} h_1(t) > 0$.

Define

$$\mathcal{V}_{r,\Theta} = \left\{u \in S_{r,\Theta} : \|\nabla u\|_2^2 \le T_\Theta^2\right\}.$$

Let θ be the principal eigenvalue of operator $-\Delta$ with Dirichlet boundary condition in Ω, and let $|\Omega|$ be the volume of Ω.

Lemma 4.1.

(i) If $r < \frac{\sqrt{C\Theta}}{T_\Theta}$, then $\mathcal{V}_{r,\Theta} = \emptyset$.

(ii) If

$$r > \max\left\{\frac{\sqrt{C\Theta}}{T_\Theta}, \left(\frac{\theta\left(1+\|V\|_{\frac{N}{2}}S^{-1}\right)}{2\alpha_1\beta}\Theta^{\frac{2-p_2}{2}}|\Omega|^{\frac{p_2-2}{2}}\right)^{\frac{2}{N(p_2-2)+4}}\right\},$$

then $\mathcal{V}_{r,\Theta} \neq \emptyset$ and

$$e_{r,\Theta} := \inf_{u\in\mathcal{V}_{r,\Theta}} I_r(u) < 0$$

is attained at some interior point $u_r > 0$ of $\mathcal{V}_{r,\Theta}$. As a consequence, there exists a Lagrange multiplier $\lambda_r \in \mathbb{R}$ such that (λ_r, u_r) is a solution of (2.1). Moreover $\liminf_{r\to\infty}\lambda_r > 0$ holds true.

Proof. (i) The Poincaré inequality implies there exists a positive constant C (only depending on Ω) such that

$$\int_{\Omega_r}|\nabla u|^2dx = \frac{1}{r^2}\int_\Omega|\nabla u|^2dx \ge \frac{C}{r^2}\int_\Omega|u|^2dx = \frac{C\Theta}{r^2}$$

for any $u \in S_{r,\Theta}$. Since T_Θ is independent of r, there holds $\mathcal{V}_{r,\Theta} = \emptyset$ if and only if $r < \frac{\sqrt{C\Theta}}{T_\Theta}$.

(ii) Let $v_1 \in S_{1,\Theta}$ be the positive normalized eigenfunction corresponding to θ. Setting

$$r_\Theta = \max\left\{\frac{\sqrt{C\Theta}}{T_\Theta}, \left[\frac{\theta\left(1+\|V\|_{\frac{N}{2}}S^{-1}\right)}{2\alpha_1\beta}\Theta^{\frac{2-p_2}{2}}|\Omega|^{\frac{p_2-2}{2}}\right]^{\frac{2}{N(p_2-2)+4}}\right\}. \tag{4.2}$$

Now, we construct for $r > r_\Theta$ a function $u_r \in S_{r,\Theta}$ such that $u_r \in \mathcal{V}_{r,\Theta}$ and $I_r(u_r) < 0$. Clearly,

$$\int_\Omega|\nabla v_1|^2dx = \theta\Theta,\ \Theta = \int_\Omega|v_1|^2dx \le \left(\int_\Omega|v_1|^{p_2}dx\right)^{\frac{2}{p_2}}|\Omega|^{\frac{p_2-2}{p_2}}.$$

Define $u_r \in S_{r,\Theta}$ by $u_r(x) = r^{-\frac{N}{2}}v_1\left(r^{-1}x\right)$ for $x \in \Omega_r$. Then

$$\int_{\Omega_r}|\nabla u_r|^2dx = r^{-2}\theta\Theta \quad\text{and}\quad \int_{\Omega_r}|u_r|^{p_2}dx \ge r^{\frac{N(2-p_2)}{2}}\Theta^{\frac{p_2}{2}}|\Omega|^{\frac{2-p_2}{2}}. \tag{4.3}$$

According to (f_2), there exists a constant $\alpha_1 > 0$ such that

$$F(\tau) \ge \alpha_1\tau^{p_2}. \tag{4.4}$$

By (4.2), (4.3), (4.4), $2 < p_2 < 2+\frac{4}{N}$ and a direct calculation we have $u_r \in \mathcal{V}_{r,\Theta}$ and

$$\begin{aligned} I_r(u_r) &\le \frac{1}{2}\left(1+\|V\|_{\frac{N}{2}}S^{-1}\right)r^{-2}\theta\Theta - \alpha_1\beta r^{\frac{N(2-p_2)}{2}}\Theta^{\frac{p_2}{2}}|\Omega|^{\frac{2-p_2}{2}} \\ &< 0. \end{aligned}$$

It then follows from the Gagliardo–Nirenberg inequality that

$$I_r(u_r) \geq \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) \int_{\Omega_r} |\nabla u|^2 dx - C_{N,p_1}\beta\Theta^{\frac{2p_1-N(p_1-2)}{4}} \left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(p_1-2)}{4}}$$
$$- \frac{C_{N,q}}{q}\Theta^{\frac{2q-N(q-2)}{4}} \left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(q-2)}{4}}. \tag{4.5}$$

As a consequence I_r is bounded from below in $\mathcal{V}_{r,\Theta}$. By the Ekeland principle there exists a sequence $\{u_{n,r}\} \subset \mathcal{V}_{r,\Theta}$ such that

$$I_r(u_{n,r}) \to \inf_{u \in \mathcal{V}_{r,\Theta}} I_r(u), \; I_r'(u_{n,r})|_{T_{u_{n,r}} S_{r,\Theta}} \to 0 \quad \text{as } n \to \infty.$$

Consequently there exists $u_r \in H_0^1(\Omega_r)$ such that $u_{n,r} \rightharpoonup u_r$ in $H_0^1(\Omega_r)$ and

$$u_{n,r} \to u_r \quad \text{in } L^k(\Omega_r) \text{ for all } 2 \leq k < 2^*.$$

Moreover, $\|\nabla u_r\|_2^2 \leq \liminf_{n\to\infty} \|\nabla u_{n,r}\|_2^2 \leq T_\Theta^2$, that is, $u_r \in \mathcal{V}_{r,\Theta}$. Note that

$$\int_{\Omega_r} V u_{n,r}^2 dx \to \int_{\Omega_r} V u_r^2 dx \text{ as } n \to \infty,$$

hence

$$e_{r,\Theta} \leq I_r(u_r) \leq \liminf_{n\to\infty} I_r(u_{n,r}) = e_{r,\Theta}.$$

It follows that $u_{n,r} \to u_r$ in $H_0^1(\Omega_r)$, so $I_r(u_r) < 0$. Therefore u is an interior point of $\mathcal{V}_{r,\Theta}$ because $I_r(u) \geq h_1(T_\Theta) > 0$ for any $u \in \partial\mathcal{V}_{r,\Theta}$ by (4.5). The Lagrange multiplier theorem implies that there exists $\lambda_r \in \mathbb{R}$ such that (λ_r, u_r) is a solution of (2.1). Moreover,

$$\begin{aligned}\lambda_r\Theta &= \int_{\Omega_r} |u_r|^q\, dx + \beta\int_{\Omega_r} f(u_r)u_r dx - \int_{\Omega_r} |\nabla u_r|^2\, dx - \int_{\Omega_r} V u_r^2 dx \\ &= \int_{\Omega_r} |u_r|^q\, dx + \beta\int_{\Omega_r} f(u_r)u_r dx - \frac{2}{q}\int_{\Omega_r} |u_r|^q dx - 2\beta\int_{\Omega_r} F(u_r) dx - 2I_r(u_r) \\ &> -2I_r(u_r) = -2e_{r,\Theta}.\end{aligned} \tag{4.6}$$

It follows from the definition of $e_{r,\Theta}$ that $e_{r,\Theta}$ is nonincreasing with respect to r. Hence, $e_{r,\Theta} \leq e_{r_\Theta,\Theta} < 0$ for any $r > r_\Theta$ and $0 < \Theta < \Theta_V$. In view of (4.6), we have $\liminf_{r\to\infty} \lambda_r > 0$. Finally, the strong maximum principle implies $u_r > 0$. □

Proof of Theorem 1.4. The proof is a direct consequence of Lemma 4.1 and Lemma 3.6. □

5 Proof of Theorem 1.5

In this subsection we assume that the assumptions of Theorem 1.5 hold. For $s \in \left[\frac{1}{2}, 1\right]$, $\beta > 0$, we define the functional $J_{r,s} : S_{r,\Theta} \to \mathbb{R}$ by

$$J_{r,s}(u) = \frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} V u^2 dx - s\left(\frac{1}{q}\int_{\Omega_r} |u|^q dx + \beta\int_{\Omega_r} F(u) dx\right).$$

Note that if $u \in S_{r,\Theta}$ is a critical point of $J_{r,s}$ then there exists $\lambda \in \mathbb{R}$ such that (λ, u) is a solution of the problem

$$\begin{cases} -\Delta u + Vu + \lambda u = s|u|^{q-2}u + s\beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r, \end{cases} \tag{5.1}$$

Lemma 5.1. *For $0<\Theta<\widetilde{\Theta}_V$ where $\widetilde{\Theta}_V$ is defined in Theorem 1.5, there exist $\widetilde{r}_\Theta>0$ and $u^0,u^1\in S_{r_\Theta,\Theta}$ such that*

(i) For $r>\widetilde{r}_\Theta$ and $s\in\left[\frac{1}{2},1\right]$ we have $J_{r,s}\left(u^1\right)\le 0$ and

$$J_{r,s}\left(u^0\right)<\frac{N(q-2)-4}{4}\left(\frac{2\left(1-\|V_-\|_{\frac{N}{2}}S\right)}{N(q-2)}\right)^{\frac{N(q-2)}{N(q-2)-4}}A^{\frac{4}{4-N(q-2)}}\Theta^{\frac{N(q-2)-2q}{N(q-2)-4}},$$

where

$$A=\left(\frac{C_{N,q}(q-2)(N(q-2)-4)}{q(p_1-2)(4-N(p_1-2))}+\frac{C_{N,q}}{q}\right).$$

Moreover,

$$\|\nabla u^0\|_2^2<\left[\frac{2\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)}{N(q-2)A}\right]^{\frac{4}{N(q-2)-4}}\Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}$$

and

$$\left\|\nabla u^1\right\|_2^2>\left[\frac{2\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)}{N(q-2)A}\right]^{\frac{4}{N(q-2)-4}}\Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}.$$

(ii) If $u\in S_{r,\Theta}$ satisfies

$$\|\nabla u\|_2^2=\left[\frac{2\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)}{N(q-2)A}\right]^{\frac{4}{N(q-2)-4}}\Theta^{\frac{N(q-2)-2q}{N(q-2)-4}},$$

then there holds

$$J_{r,s}(u)\ge\frac{N(q-2)-4}{4}\left[\frac{2\left(1-\|V_-\|_{\frac{N}{2}}S\right)}{N(q-2)}\right]^{\frac{N(q-2)}{N(q-2)-4}}A^{\frac{4}{4-N(q-2)}}\Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}.$$

(iii) Let

$$m_{r,s}(\Theta)=\inf_{\gamma\in\Gamma_{r,\Theta}}\sup_{t\in[0,1]}J_{r,s}(\gamma(t)),$$

where

$$\Gamma_{r,\Theta}=\left\{\gamma\in C\left([0,1],S_{r,\Theta}\right):\gamma(0)=u^0,\gamma(1)=u^1\right\}.$$

Then

$$m_{r,s}(\Theta)\ge\frac{N(q-2)-4}{4}\left[\frac{2\left(1-\|V_-\|_{\frac{N}{2}}S\right)}{N(q-2)}\right]^{\frac{N(q-2)}{N(q-2)-4}}A^{\frac{4}{4-N(q-2)}}\Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}$$

and

$$m_{r,s}(\Theta)\le\frac{N(q-2)-4}{2}\left(\frac{\theta\left(1+\|V\|_{\frac{N}{2}}S^{-1}\right)}{N(q-2)}\right)^{\frac{N(q-2)}{N(q-2)-4}}(4q)^{\frac{4}{N(q-2)-4}}|\Omega|^{\frac{2(q-2)}{N(q-2)-4}}\Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}.$$

where θ is the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω.

Proof. Let $v_1 \in S_{1,\Theta}$ be the positive normalized eigenfunction of $-\Delta$ with Dirichlet boundary condition in Ω associated to θ, then we have

$$\int_{\Omega} |\nabla v_1|^2 \, dx = \theta\Theta. \tag{5.2}$$

By the Hölder inequality, we know

$$\int_{\Omega} |v_1(x)|^{p_2} dx \geq \Theta^{\frac{p_2}{2}} \cdot |\Omega|^{\frac{2-p_2}{2}}. \tag{5.3}$$

Setting $v_t(x) = t^{\frac{N}{2}} v_1(tx)$ for $x \in B_{\frac{1}{t}}$ and $t > 0$. Using (4.4), (5.2), (5.3) and $\frac{1}{2} \leq s \leq 1$, we get

$$\begin{aligned} J_{\frac{1}{t},s}(v_t) &\leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta\Theta - \frac{\beta}{2}\alpha_1 t^{\frac{N(p_2-2)}{2}} \int_{\Omega} |v_1|^{p_2} dx - \frac{1}{2q} t^{\frac{N(q-2)}{2}} \Theta^{\frac{q}{2}} \cdot |\Omega|^{\frac{2-q}{2}} \\ &\leq h_2(t), \end{aligned} \tag{5.4}$$

where

$$h_2(t) = \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta\Theta - \frac{1}{2q} t^{\frac{N(q-2)}{2}} \Theta^{\frac{q}{2}} \cdot |\Omega|^{\frac{2-q}{2}}.$$

A simple computation shows that $h_2(t_0) = 0$ for

$$t_0 := \left[\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) q\theta\Theta^{\frac{2-q}{2}} |\Omega|^{\frac{q-2}{2}}\right]^{\frac{2}{N(q-2)-4}}$$

and $h_2(t) < 0$ for any $t > t_0, h_2(t) > 0$ for any $0 < t < t_0$. Moreover, $h_2(t)$ achieves its maximum at

$$t_\Theta = \left[\frac{4q\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)\theta}{N(q-2)} \Theta^{\frac{2-q}{2}} |\Omega|^{\frac{q-2}{2}}\right]^{\frac{2}{N(q-2)-4}}.$$

This implies

$$J_{r,s}(v_{t_0}) = J_{\frac{1}{t_0},s}(v_{t_0}) \leq h_2(t_0) = 0 \tag{5.5}$$

for any $r \geq \frac{1}{t_0}$ and $s \in \left[\frac{1}{2}, 1\right]$. There exists $0 < t_1 < t_\Theta$ such that for any $t \in [0, t_1]$,

$$h_2(t) < \frac{N(q-2)-4}{4} \left(\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S\right)}{N(q-2)}\right)^{\frac{N(q-2)}{N(q-2)-4}} A^{\frac{4}{4-N(q-2)}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}. \tag{5.6}$$

On the other hand, it follows from (3.5), the Gagliardo–Nirenberg inequality and the Hölder inequality that

$$\begin{aligned} J_{r,s}(u) \geq{}& \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) \int_{\Omega_r} |\nabla u|^2 dx - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} \left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(q-2)}{4}} \\ &- \alpha\beta C_{N,p_1} \Theta^{\frac{2p_1-N(p_1-2)}{4}} \left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(p_1-2)}{4}}. \end{aligned} \tag{5.7}$$

Define

$$
\begin{aligned}
g_1(t) &:= \frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right) t - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-2)}{4}} - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{4}} \\
&= t^{\frac{N(p_1-2)}{4}}\left[\frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right) t^{\frac{4-N(p_1-2)}{4}} - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-p_1)}{4}}\right] \\
&\quad - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{4}}.
\end{aligned}
$$

In view of $2<p<2+\frac{2}{N}<q<2^*$ and the definition of $\widetilde{\Theta}_V$, there exist $0<l_1<l_M<l_2$ such that $g_1(t)<0$ for any $0<t<l_1$ and $t>l_2$, $g_1(t)>0$ for $l_1<t<l_2$ and $g_1(l_M)=\max\limits_{t\in\mathbb{R}^+} g_1(t)>0$. Let

$$
t_2=\left(\frac{\alpha\beta q C_{N,p_1}(p_1-2)(4-N(p_1-2))}{C_{N,q}(q-2)(N(q-2)-4)}\right)^{\frac{4}{N(q-p_1)}}\Theta^{\frac{N-2}{N}}.
$$

Then by a direct calculation, we have $g_1''(t)\le 0$ if and only if $t\ge t_2$. Hence

$$
\max_{t\in\mathbb{R}^+} g_1(t)=\max_{t\in[t_2,\infty)} g_1(t).
$$

Note that for any $t\ge t_2$,

$$
\begin{aligned}
g_1(t) &= \frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right) t - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-2)}{4}} - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{4}} \\
&= \frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right) t - \alpha\beta C_{N,p_1}\Theta^{\frac{(q-p_1)(N-2)}{4}}\cdot\Theta^{\frac{2q-N(q-2)}{4}} t^{\frac{N(p_1-2)}{4}} \\
&\quad - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-2)}{4}} \\
&\ge \frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right) t - \left(\frac{C_{N,q}(q-2)(N(q-2)-4)}{q(p_1-2)(4-N(p_1-2))}+\frac{C_{N,q}}{q}\right)\Theta^{\frac{2q-N(q-2)}{4}}\cdot t^{\frac{N(q-2)}{4}} \\
&=: g_2(t). \qquad (5.8)
\end{aligned}
$$

Now, we will determine the value of $\widetilde{\Theta}_V$. In fact, $g_1(l_M)=\max_{t\in\mathbb{R}^+} g_1(t)>0$ as long as $g_2(t_2)>0$, that is,

$$
\begin{aligned}
g_2(t_2) &= \frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right)\left(\frac{\alpha\beta q C_{N,p_1}}{C_{N,q}A_{p_1,q}}\right)^{\frac{4}{N(q-p_1)}}\Theta^{\frac{N-2}{N}} - \left(\frac{C_{N,q}}{q}A_{p_1,q}+\frac{C_{N,q}}{q}\right)\cdot\Theta \\
&\quad \cdot\left(\frac{\alpha\beta q C_{N,p_1}}{C_{N,q}A_{p_1,q}}\right)^{\frac{q-2}{q-p_1}} \\
&> 0,
\end{aligned}
$$

where

$$
A_{p_1,q}=\frac{(q-2)(N(q-2)-4)}{(p_1-2)(4-N(p_1-2))}.
$$

Hence, we take

$$
\widetilde{\Theta}_V=\frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}}\left(\frac{C_{N,q}}{q}A_{p_1,q}+\frac{C_{N,q}}{q}\right)^{-\frac{N}{2}}\left(\frac{\alpha\beta q C_{N,p_1}}{C_{N,q}A_{p_1,q}}\right)^{\frac{N(q-2)-4}{2N(q-p_1)}}.
$$

Let

$$A = \left(\frac{C_{N,q}(q-2)(N(q-2)-4)}{q(p_1-2)(4-N(p_1-2))} + \frac{C_{N,q}}{q} \right)$$

and

$$t_g = \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)A} \right]^{\frac{4}{N(q-2)-4}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}},$$

so that $t_g > t_2$ by the definition of $\widetilde{\Theta}_V$, $\max_{t \in [t_2, \infty)} g_2(t) = g_2(t_g)$ and

$$\begin{aligned} \max_{t \in \mathbb{R}^+} g_1(t) &\geq \max_{t \in [t_2, \infty)} g_2(t) \\ &= \frac{(N(q-2)-4)}{4} \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S\right)}{N(q-2)} \right]^{\frac{N(q-2)}{N(q-2)-4}} A^{\frac{4}{4-N(q-2)}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}. \end{aligned}$$

Set $\bar{r}_\Theta = \max\left\{\frac{1}{t_1}, \sqrt{\frac{2\theta\Theta}{t_g}}\right\}$, then $v_{\frac{1}{\bar{r}_\Theta}} \in S_{r,\Theta}$ for any $r > \bar{r}_\Theta$, and

$$\left\| \nabla v_{\frac{1}{\bar{r}_\Theta}} \right\|_2^2 = \left(\frac{1}{\bar{r}_\Theta} \right)^2 \|\nabla v_1\|_2^2 < t_g = \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)A} \right]^{\frac{4}{N(q-2)-4}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}. \tag{5.9}$$

Moreover,

$$J_{\bar{r}_\Theta,s}\left(v_{\frac{1}{\bar{r}_\Theta}}\right) \leq h_2\left(\frac{1}{\bar{r}_\Theta}\right) \leq h_2(t_1). \tag{5.10}$$

Let $u^0 = v_{\frac{1}{\bar{r}_\Theta}}, u^1 = v_{t_0}$ and

$$\widetilde{r}_\Theta = \max\left\{ \frac{1}{t_0}, \bar{r}_\Theta \right\}.$$

Then the statement (i) holds by (5.5), (5.6), (5.9), (5.10).

(ii) holds by (5.8) and a direct calculation.

(iii) In view of $J_{r,s}(u^1) \leq 0$ for any $\gamma \in \Gamma_{r,\Theta}$ and the definition of t_0, we have

$$\|\nabla\gamma(0)\|_2^2 < t_g < \|\nabla\gamma(1)\|_2^2.$$

It then follows from (5.8) that

$$\begin{aligned} \max_{t \in [0,1]} J_{r,s}(\gamma(t)) &\geq g_2(t_g) \\ &= \frac{N(q-2)-4}{4} \left[\frac{2\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)} \right]^{\frac{N(q-2)}{N(q-2)-4}} A^{\frac{4}{4-N(q-2)}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}} \end{aligned}$$

for any $\gamma \in \Gamma_{r,\Theta}$, hence the first inequality in (iii) holds. We define a path $\gamma : [0,1] \to S_{r,\Theta}$ by

$$\gamma(t) : \Omega_r \to \mathbb{R}, \qquad x \mapsto \left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta} \right)^{\frac{N}{2}} v_1\left(\left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta} \right) x \right).$$

Then $\gamma \in \Gamma_{r,\Theta}$, and the second inequality in (iii) follows from (5.4). □

Lemma 5.2. *Assume* $0 < \Theta < \widetilde{\Theta}_V$ *where* $\widetilde{\Theta}_V$ *is given in Theorem 1.5. Let* $r > \widetilde{r}_\Theta$*, where* $\widetilde{r}_\Theta$ *is defined in Lemma 5.1. Then problem (5.1) admits a solution* $(\lambda_{r,s}, u_{r,s})$ *for almost every* $s \in \left[\frac{1}{2}, 1\right]$*. Moreover, there hold* $u_{r,s} > 0$ *and* $J_{r,s}(u_{r,s}) = m_{r,s}(\Theta)$*.*

Proof. The proof is similar to the Lemma 3.3. We omit it here. □

Lemma 5.3. *For fixed* $\Theta > 0$ *the set of solutions* $u \in S_{r,\Theta}$ *of (5.1) is bounded uniformly in* s *and* r*.*

Proof. Since u is a solution of (5.1), we have

$$\int_{\Omega_r} |\nabla u|^2 dx + \int_{\Omega_r} V u^2 dx = s \int_{\Omega_r} |u|^q dx + s\beta \int_{\Omega_r} f(u) u dx - \lambda \int_{\Omega_r} |u|^2 dx.$$

The Pohozaev identity implies

$$\begin{aligned}\frac{N-2}{2N} \int_{\Omega_r} |\nabla u|^2 dx &+ \frac{1}{2N} \int_{\partial\Omega_r} |\nabla u|^2 (x \cdot \mathbf{n}) d\sigma + \frac{1}{2N} \int_{\Omega_r} \widetilde{V}(x) u^2 + \frac{1}{2} \int_{\Omega_r} V u^2 dx \\ &= -\frac{\lambda}{2} \int_{\Omega_r} |u|^2 dx + \frac{s}{q} \int_{\Omega_r} |u|^q dx + s\beta \int_{\Omega_r} F(u) dx.\end{aligned}$$

It then follows from $\beta > 0$ and (f_2) that

$$\begin{aligned}\frac{1}{N} \int_{\Omega_r} |\nabla u|^2 dx &- \frac{1}{2N} \int_{\partial\Omega_r} |\nabla u|^2 (x \cdot \mathbf{n}) d\sigma - \frac{1}{2N} \int_{\Omega_r} (\nabla V \cdot x) u^2 dx \\ &= \frac{(q-2)s}{2q} \int_{\Omega_r} |u|^q dx + s \int_{\Omega_r} \left(\frac{\beta}{2} f(u) u - \beta F(u)\right) dx \\ &\geq \frac{q-2}{2} \left(\frac{1}{2} \int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega_r} V u^2 dx - m_{r,s}(\Theta) \right) + s \frac{\beta(p_2 - q)}{2} \int_{\Omega_r} F(u) dx.\end{aligned}$$

Using Gagliardo–Nirenberg inequality, (3.5) and (iii) in Lemma 5.1, we have

$$\begin{aligned}\frac{q-2}{2} m_{r,s}(\Theta) \geq{}& \frac{N(p_2-2)-4}{4N} \int_{\Omega_r} |\nabla u|^2 dx - \Theta \left(\frac{1}{2N} \|\nabla V \cdot x\|_\infty + \frac{p_2 - 2}{4} \|V\|_\infty \right) \\ &+ \frac{s\alpha\beta(p_2-q)}{2} C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1-2)}{4}} \left(\int_{\Omega_r} |\nabla u|^2 dx \right)^{\frac{N(p_1-2)}{4}}.\end{aligned}$$

Since $2 < p_1 < 2 + \frac{4}{N}$, we can bound $\int_{\Omega_r} |\nabla u|^2 dx$ uniformly in s and r. □

Lemma 5.4. *Assume* $0 < \Theta < \widetilde{\Theta}_V$*, where* $\widetilde{\Theta}_V$ *is given in Theorem 1.5, and let* $r > \widetilde{r}_\Theta$*, where* $\widetilde{r}_\Theta$ *is defined in Lemma 5.1. Then the following hold:*

(i) Equation (2.1) admits a solution $(\lambda_{r,\Theta}, u_{r,\Theta})$ *for every* $r > \widetilde{r}_\Theta$ *such that* $u_{r,\Theta} > 0$ *in* Ω_r*.*

(ii) There is $0 < \bar{\Theta} \leq \widetilde{\Theta}_V$ *such that*

$$\liminf_{r \to \infty} \lambda_{r,\Theta} > 0 \quad \textit{for any } 0 < \Theta < \bar{\Theta}.$$

Proof. The proof of (i) is similar to that of Lemma 3.5, we omit it. As be consider $H_0^1(\Omega_r)$ as a subspace of $H^1(\mathbb{R}^N)$ for every $r > 0$. In view of Lemma 5.3, there are λ_Θ and $u_\Theta \in H^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_{r,\Theta} \rightharpoonup u_\Theta \quad \text{in } H^1(\mathbb{R}^N) \quad \text{and} \quad \lim_{r \to \infty} \lambda_{r,\Theta} \to \lambda_\Theta.$$

Arguing by contradiction, we assume that $\lambda_{\Theta_n} \le 0$ for some sequence $\Theta_n \to 0$. Let θ_r be the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω_r and let $v_r > 0$ be the corresponding normalized eigenfunction. Testing (2.1) with v_r, it holds

$$(\theta_r + \lambda_{r,\Theta_n}) \int_{\Omega_r} u_{r,\Theta_n} v_r dx + \int_{\Omega_r} V u_{r,\Theta_n} v_r dx \ge 0.$$

In view of $\int_{\Omega_r} u_{r,\Theta_n} v_r dx > 0$ and $\theta_r = r^{-2}\theta_1$, there holds

$$\max_{x\in\mathbb{R}^N} V + \lambda_{r,\Theta_n} + r^{-2}\theta_1 \ge 0.$$

Hence there exists $C > 0$ independent of n such that $|\lambda_{\Theta_n}| \le C$ for any n.

Case 1. There is subsequence denoted still by $\{\Theta_n\}$ such that $u_{\Theta_n} = 0$. We first claim that there exists $d_n > 0$ for any n such that

$$\liminf_{r\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} u^2_{r,\Theta_n} dx \ge d_n. \tag{5.11}$$

Otherwise, the concentration compactness principle implies for every n that

$$u_{r,\Theta_n} \to 0 \text{ in } L^t(\mathbb{R}^N) \quad \text{as } r \to \infty, \quad \text{for all } 2 < t < 2^*.$$

By the diagonal principle, (2.1) and $|\lambda_{r,\Theta_n}| \le 2C$ for large r, there exists $r_n \to \infty$ such that

$$\int_{\Omega_r} |\nabla u_{r_n,\Theta_n}|^2 dx \le C$$

for some C independent of n, contradicting (iii) in Lemma 5.1 for large n. As a consequence (5.11) holds, and there is $z_{r,\Theta_n} \in \Omega_r$ with $|z_{r,\Theta_n}| \to \infty$ such that

$$\int_{B(z_{r,\Theta_n},1)} u^2_{r,\Theta_n} dx \ge \frac{d_n}{2}.$$

Moreover, dist $(z_{r,\Theta_n}, \partial\Omega_r) \to \infty$ as $r \to \infty$ by an argument similar to that in Lemma 3.7. Now, for n fixed let $v_r(x) = u_{r,\Theta_n}(x + z_{r,\Theta_n})$ for $x \in \Sigma^r := \{x \in \mathbb{R}^N : x + z_{r,\Theta_n} \in \Omega_r\}$. It follows from Lemma 5.3 that there is $v \in H^1(\mathbb{R}^N)$ with $v \ne 0$ such that $v_r \rightharpoonup v$. Observe that for every $\phi \in C_c^\infty(\mathbb{R}^N)$ there is r large such that $\phi(\cdot - z_{r,\Theta_n}) \in C_c^\infty(\Omega_r)$ due to dist $(z_{r,\Theta_n}, \partial\Omega_r) \to \infty$ as $r \to \infty$. It follows that

$$\begin{aligned}\int_{\Omega_r} \nabla u_{r,\Theta_n} \nabla\phi(\cdot - z_{r,\Theta_n}) dx + \int_{\Omega_r} V u_{r,\Theta_n}\phi(\cdot - z_{r,\Theta_n}) dx + \lambda_{r,\Theta_n}\int_{\Omega_r} u_{r,\Theta_n}\phi(\cdot - z_{r,\Theta_n}) dx \\ = \int_{\Omega_r} |u_{r,\Theta_n}|^{q-2} u_{r,\Theta_n}\phi(\cdot - z_{r,\Theta_n}) dx + \beta\int_{\Omega_r} f(u_{r,\Theta_n})\phi(\cdot - z_{r,\Theta_n}) dx.\end{aligned} \tag{5.12}$$

Using $|z_{r,\Theta_n}| \to \infty$ as $r \to \infty$, it follows that

$$\begin{aligned}\left|\int_{\Omega_r} V u_{r,\Theta_n}\phi(\cdot - z_{r,\Theta_n}) dx\right| &\le \int_{\operatorname{Supp}\phi} |V(\cdot + z_{r,\Theta_n}) v_r\phi| dx \\ &\le \|v_r\|_{2^*}\|\phi\|_{2^*}\left(\int_{\mathbb{R}^N\setminus B_{\frac{|z_{r,\Theta_n}|}{2}}} |V|^{\frac{N}{2}} dx\right)^{\frac{2}{N}} \\ &\to 0 \quad \text{as } r \to \infty.\end{aligned}$$

Letting $r \to \infty$ in (5.12), we get for every $\phi \in C_c^\infty(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} \nabla v \cdot \nabla \phi dx + \lambda_{\Theta_n} \int_{\mathbb{R}^N} v\phi dx = \int_{\mathbb{R}^N} |v|^{q-2} v\phi dx + \beta \int_{\mathbb{R}^N} f(v)\phi dx.$$

Therefore $v \in H^1(\mathbb{R}^N)$ is a weak solution of the equation

$$-\Delta v + \lambda_{\Theta_n} v = \beta f(v) + |v|^{q-2} v \quad \text{in } \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} |\nabla v|^2 dx + \lambda_{\Theta_n} \int_{\mathbb{R}^N} |v|^2 dx = \beta \int_{\mathbb{R}^N} f(v) v dx + \int_{\mathbb{R}^N} |v|^q dx.$$

The Pohozaev identity implies

$$\frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{\lambda_{\Theta_n}}{2} \int_{\mathbb{R}^N} |v|^2 dx = \beta \int_{\mathbb{R}^N} F(v) dx + \frac{1}{q} \int_{\mathbb{R}^N} |v|^q dx,$$

hence

$$\begin{aligned} &\frac{\lambda_{\Theta_n}}{N} \int_{\mathbb{R}^N} |v|^2 dx \\ &\quad = \frac{\beta(N-2)}{2N} \int_{\mathbb{R}^N} \left[\frac{2N}{N-2} F(v) - f(v) v \right] dx + \frac{2N - q(N-2)}{2Nq} \int_{\mathbb{R}^N} |v|^q dx \\ &\quad \geq \frac{\beta(N-2)}{2N} \left(\frac{2N}{N-2} - p_1 \right) \int_{\mathbb{R}^N} F(v) dx + \frac{2N - q(N-2)}{2Nq} \int_{\mathbb{R}^N} |v|^q dx. \end{aligned} \tag{5.13}$$

We have $\lambda_{\Theta_n} > 0$ because of $2 < p_1 < 2 + \frac{4}{N} < q < 2^*$, which is a contradiction.

Case 2. $u_{\Theta_n} \neq 0$ for n large. Note that u_{Θ_n} satisfies

$$-\Delta u_{\Theta_n} + V u_{\Theta_n} + \lambda_{\Theta_n} u_{\Theta_n} = \beta f(u_{\Theta_n}) + |u_{\Theta_n}|^{q-2} u_{\Theta_n}. \tag{5.14}$$

If $v_{r,\Theta_n} := u_{r,\Theta_n} - u_{\Theta_n}$ satisfies

$$\limsup_{r \to \infty} \max_{z \in \mathbb{R}^N} \int_{B(z,1)} v_{r,\Theta_n}^2 dx = 0, \tag{5.15}$$

then the concentration compactness principle implies $u_{r,\Theta_n} \to u_{\Theta_n}$ in $L^t(\mathbb{R}^N)$ for any $2 < t < 2^*$. It then follows from (2.1) and (5.14) that

$$\begin{aligned} \int_{\Omega_r} |\nabla u_{r,\Theta_n}|^2 dx + \Theta_n \lambda_{r,\Theta_n} &= \beta \int_{\Omega_r} f(u_{r,\Theta_n}) u_{r,\Theta_n} dx + \int_{\Omega_r} |u_{r,\Theta_n}|^q dx - \int_{\Omega_r} V u_{r,\Theta_n}^2 dx \\ &\to \beta \int_{\mathbb{R}^N} f(u_{\Theta_n}) u_{r,\Theta_n} dx + \int_{\mathbb{R}^N} |u_{\Theta_n}|^q dx - \int_{\mathbb{R}^N} V u_{\Theta_n}^2 dx \\ &= \int_{\mathbb{R}^N} |\nabla u_{\Theta_n}|^2 dx + \lambda_{\Theta_n} \int_{\mathbb{R}^N} u_{\Theta_n}^2 dx. \end{aligned}$$

Using $\lambda_{r,\Theta_n} \to \lambda_{\Theta_n}$ as $r \to \infty$, we further have

$$\int_{\Omega_r} |\nabla u_{r,\Theta_n}|^2 dx + \Theta_n \lambda_{\Theta_n} \to \int_{\mathbb{R}^N} |\nabla u_{\Theta_n}|^2 dx + \lambda_{\Theta_n} \int_{\mathbb{R}^N} u_{\Theta_n}^2 dx \quad \text{as } r \to \infty. \tag{5.16}$$

Using (5.16), (iii) in Lemma 5.1 and $|\lambda_{\Theta_n}| \leq C$ for large n, there holds

$$\int_{\mathbb{R}^N} |\nabla u_{\Theta_n}|^2 dx \to \infty \quad \text{as } n \to \infty.$$

By (5.14) and the Pohozaev identity

$$\frac{N-2}{2N}\int_{\mathbb{R}^N}|\nabla u_{\Theta_n}|^2\,dx+\frac{1}{2N}\int_{\mathbb{R}^N}\widetilde{V}u_{\Theta_n}^2dx+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u_{\Theta_n}^2dx+\frac{\lambda_\Theta}{2}\int_{\mathbb{R}^N}u_{\Theta_n}^2dx$$
$$=\frac{1}{q}\int_{\mathbb{R}^N}|u_{\Theta_n}|^q\,dx+\beta\int_{\mathbb{R}^N}F(u_{\Theta_n})dx.$$

It holds that

$$\begin{aligned}0&\le\frac{(2-q)\lambda_{\Theta_n}}{2q}\int_{\mathbb{R}^N}u_{\Theta_n}^2dx\\&\le\frac{(N-2)q-2N}{2Nq}\int_{\mathbb{R}^N}|\nabla u_{\Theta_n}|^2\,dx+\frac{\|\widetilde{V}\|_\infty}{2N}\Theta_n+\frac{(q-2)\|V\|_\infty}{2q}\Theta_n\\&\to-\infty\quad\text{as } n\to\infty.\end{aligned}$$

Therefore (5.15) cannot occur. Consequently there exist $d_n>0$ and $z_{r,\Theta_n}\in\Omega_r$ with $|z_{r,\Theta_n}|\to\infty$ as $r\to\infty$ such that

$$\int_{B(z_{r,\Theta_n},1)}v_{r,\Theta_n}^2dx>d_n.$$

Then $\widetilde{v}_{r,\Theta_n}:=v_{r,\Theta_n}(\cdot+z_{r,\Theta_n})\rightharpoonup\widetilde{v}_{\Theta_n}\neq 0$, and $\widetilde{v}_{\Theta_n}$ is a nonnegative solution of

$$-\Delta v+\lambda_{\Theta_n}v=\beta f(v)v+|v|^{q-2}v\quad\text{in }\mathbb{R}^N.$$

In fact, we have $\liminf_{r\to\infty}\operatorname{dist}(z_{r,\Theta_n},\partial\Omega_r)=\infty$ by the Liouville theorem on the half space. It follows from an argument similar to that of (5.13) that $\lambda_{\Theta_n}>0$ for large n, which is a contradiction. □

Proof of Theorem 1.5. The proof is a direct consequence of Lemma 5.4 and Lemma 3.6. □

6 Proof of Theorem 1.9

In this section we assume that the assumptions of Theorem 1.9 hold. Define the functional $\mathcal{I}_r:S_{r,\Theta}\to\mathbb{R}$ by

$$\mathcal{I}_r(u)=\frac{1}{2}\int_{\Omega_r}|\nabla u|^2dx+\frac{1}{2}\int_{\Omega_r}Vu^2dx-\frac{1}{2^*}\int_{\Omega_r}|u|^{2^*}dx-\beta\int_{\Omega_r}F(u)dx.\tag{6.1}$$

Note that if $u\in S_{r,\Theta}$ is a critical point of $\mathcal{I}_{r,s}$, then there exists $\lambda\in\mathbb{R}$ such that (λ,u) is a solution of the equation

$$\begin{cases}-\Delta u+Vu+\lambda u=|u|^{2^*-2}u+\beta f(u), & x\in\Omega_r,\\ \int_{\Omega_r}|u|^2dx=\Theta, u\in H_0^1(\Omega_r), & x\in\Omega_r.\end{cases}\tag{6.2}$$

Since $\beta>0$,

$$\begin{aligned}\mathcal{I}_r(u)\ge&\frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)\int_{\Omega_r}|\nabla u|^2dx-\frac{1}{2^*\cdot S^{\frac{2^*}{2}}}\left(\int_{\Omega_r}|\nabla u|^2dx\right)^{\frac{2^*}{2}}\\&-\alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\Omega_r}|\nabla u|^2dx\right)^{\frac{N(p_1-2)}{4}}\\=&\widetilde{h}_1(t),\end{aligned}$$

where

$$\widetilde{h_1}(t) := \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t^2 - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}} t^{2^*} - \alpha\beta C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{2}}.$$

Consider

$$\widehat{\psi}(t) := \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t^2 - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}} t^{2^*}.$$

Note that $\widehat{\psi}$ admits a unique maximum at

$$\widehat{t} = \left[\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) S^{\frac{2^*}{2}}\right]^{\frac{1}{2^*-2}}.$$

By a direct calculation, we obtain

$$\widehat{\psi}(\widehat{t}) = \frac{1}{N}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}}.$$

Hence,

$$\widehat{\psi}(\widehat{t}) > \alpha\beta C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1-2)}{4}} \widehat{t}^{\frac{N(p_1-2)}{2}}$$

as long as

$$\Theta_V = \left(\frac{1}{N\alpha\beta C_{N,p_1}}\right)^{\frac{4}{2p_1 - N(p_1-2)}} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2} \cdot \frac{4 - N(p_1-2)}{2p_1 - N(p_1-2)}}.$$

Now, let $0 < \Theta < \Theta_V$ be fixed, we obtain

$$\widehat{\psi}(\widehat{t}) > \alpha\beta C_{N,p_1} \Theta^{\frac{2p_1 - N(p_1-2)}{4}} \widehat{t}^{\frac{N(p_1-2)}{2}}$$

and $\widetilde{h_1}(\widehat{t}) > 0$. In view of $2 < p_1 < 2 + \frac{4}{N} < 2^*$, there exist $0 < \widetilde{R_1} < \widetilde{T_\Theta} < \widetilde{R_2}$ such that $\widetilde{h_1}(t) < 0$ for $0 < t < \widetilde{R_1}$ and for $t > \widetilde{R_2}$, $\widetilde{h_1}(t) > 0$ for $\widetilde{R_1} < t < \widetilde{R_2}$, and $\widetilde{h_1}(\widetilde{T_\Theta}) = \max_{t \in \mathbb{R}^+} \widetilde{h_1}(t) > 0$.

Define

$$\widetilde{\mathcal{V}}_{r,\Theta} = \left\{u \in S_{r,\Theta} : \|\nabla u\|_2^2 \le \widetilde{T_\Theta}^2\right\}.$$

Let θ be the principal eigenvalue of operator $-\Delta$ with Dirichlet boundary condition in Ω, and let $|\Omega|$ be the volume of Ω.

Lemma 6.1.

(i) If $r < \frac{\sqrt{C\Theta}}{\widetilde{T_\Theta}}$, then $\widetilde{\mathcal{V}}_{r,\Theta} = \emptyset$.

(ii) If

$$r > \max\left\{\frac{\sqrt{C\Theta}}{\widetilde{T_\Theta}}, \left(\frac{\theta\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)}{2\alpha_1\beta} \Theta^{\frac{2-p_2}{2}} |\Omega|^{\frac{p_2-2}{2}}\right)^{\frac{2}{N(p_2-2)+4}}\right\}$$

then $\widetilde{\mathcal{V}}_{r,\Theta} \neq \emptyset$ and

$$\widetilde{e}_{r,\Theta} := \inf_{u \in \widetilde{\mathcal{V}}_{r,\Theta}} \mathcal{I}_r(u) < 0$$

is attained at some interior point $u_r > 0$ of $\widetilde{\mathcal{V}}_{r,\Theta}$. As a consequence, there exists a Lagrange multiplier $\lambda_r \in \mathbb{R}$ such that (λ_r, u_r) is a solution of (6.1). Moreover $\liminf_{r\to\infty} \lambda_r > 0$ holds true.

Proof. (i) The proof is similar to the Lemma 4.1. (ii) Let $v_1 \in S_{1,\Theta}$ be the positive normalized eigenfunction corresponding to θ. Setting

$$r_\Theta = \max\left\{\frac{\sqrt{C\Theta}}{\widetilde{T_\Theta}}, \left(\frac{\theta\left(1+\|V\|_{\frac{N}{2}}S^{-1}\right)}{2\alpha_1\beta}\Theta^{\frac{2-p_2}{2}}|\Omega|^{\frac{p_2-2}{2}}\right)^{\frac{2}{N(p_2-2)+4}}\right\}. \tag{6.3}$$

Now, we construct for $r > r_\Theta$ a function $u_r \in S_{r,\Theta}$ such that $u_r \in \widetilde{\mathcal{V}}_{r,\Theta}$ and $\mathcal{I}_r(u_r) < 0$. By (6.3), (4.3), (4.4), $2 < p_2 < 2 + \frac{4}{N}$ and a direct calculation, we have $u_r \in \widetilde{\mathcal{V}}_{r,\Theta}$ and

$$\begin{aligned}\mathcal{I}_r(u_r) &\leq \frac{1}{2}\int_{\Omega_r}|\nabla u_r|^2 dx + \frac{1}{2}\int_{\Omega_r} V u_r^2 dx - \frac{1}{2^*}\int_{\Omega_r}|u_r|^{2^*}dx - \alpha_1\beta\int_{\Omega_r}|u_r|^{p_2}dx\\ &\leq \frac{1}{2}\left(1+\|V\|_{\frac{N}{2}}S^{-1}\right) r^{-2}\theta\Theta - \alpha_1\beta r^{\frac{N(2-p_2)}{2}}\Theta^{\frac{p_2}{2}}|\Omega|^{\frac{2-p_2}{2}}\\ &< 0.\end{aligned}$$

It then follows from the Gagliardo–Nirenberg inequality that

$$\begin{aligned}\mathcal{I}_r(u_r) \geq{}& \frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)\int_{\Omega_r}|\nabla u|^2dx - C_{N,p_1}\beta\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\Omega_r}|\nabla u|^2dx\right)^{\frac{N(p_1-2)}{4}}\\ &- \frac{1}{2^*\cdot S^{\frac{2^*}{2}}}\left(\int_{\Omega_r}|\nabla u|^2dx\right)^{\frac{2^*}{2}}.\end{aligned} \tag{6.4}$$

As a consequence $\mathcal{I}_r$ is bounded from below in $\widetilde{\mathcal{V}}_{r,\Theta}$. By the Ekeland principle there exists a sequence $\{u_{n,r}\} \subset \widetilde{\mathcal{V}}_{r,\Theta}$ such that

$$\mathcal{I}_r(u_{n,r}) \to \inf_{u\in\widetilde{\mathcal{V}}_{r,\Theta}} \mathcal{I}_r(u),\ \mathcal{I}_r'(u_{n,r})|_{T_{u_{n,r}}S_{r,\Theta}} \to 0 \quad \text{as } n\to\infty \tag{6.5}$$

Consequently there exists $u_r \in H_0^1(\Omega_r)$ such that

$$u_{n,r} \rightharpoonup u_r \quad \text{in } H_0^1(\Omega_r)$$

and

$$u_{n,r} \to u_r \quad \text{in } L^k(\Omega_r) \text{ for all } 2 \leq k < 2^*. \tag{6.6}$$

We claim now that the weak limit u_r does not vanish identically. Suppose by contradiction that $u_r \equiv 0$. Since $\{u_{n,r}\}$ is bounded in $H^1(\Omega_r)$, up to a subsequence we have that $\|\nabla u_{n,r}\|_2^2 \to \ell \in \mathbb{R}$. Using (f_2), (6.5), (6.6), we have

$$\begin{aligned}\langle \mathcal{I}_r'(u_{n,r}), u_{n,r}\rangle &= \int_{\Omega_r}|\nabla u_{n,r}|^2dx + \int_{\Omega_r}Vu_{n,r}^2dx - \int_{\Omega_r}|u_{n,r}|^{2^*}dx - \beta\int_{\Omega_r}f(u_{n,r})u_{n,r}dx\\ &\to 0,\end{aligned}$$

hence

$$\|u_{n,r}\|_{2^*}^{2^*} = \|\nabla u_{n,r}\|_2^2 \to \ell$$

as well. Therefore, by the Sobolev inequality $\ell \geq S\ell^{\frac{2}{2^*}}$, and we deduce that either $\ell = 0$, or $\ell \geq S^{\frac{N}{2}}$. Let us suppose at first that $\ell \geq S^{N/2}$. Since $\mathcal{I}_r(u_{n,r}) \to \tilde{e}_{r,\Theta} < 0$, we have that

$$\begin{aligned}0 > \tilde{e}_{r,\Theta} + o(1) &= I_r(u_{n,r})\\ &= \frac{1}{2}\int_{\Omega_r}|\nabla u_{n,r}|^2dx + \frac{1}{2}\int_{\Omega_r}Vu_{n,r}^2dx - \frac{1}{2^*}\int_{\Omega_r}|u_{n,r}|^{2^*}dx - \beta\int_{\Omega_r}F(u_{n,r})dx\\ &= \frac{1}{N}\|\nabla u_{n,r}\|_2^2 + o(1) = \frac{\ell}{N} + o(1),\end{aligned}$$

which is not possible. If instead $\ell = 0$, we have $\|u_{n,r}\|_{2^*} \to 0$, $\|\nabla u_{n,r}\|_2 \to 0$ and $F(u_{n,r}) \to 0$. But then $\mathcal{I}_r(u_{n,r}) \to 0 \neq \widetilde{e}_{r,\Theta}$, which gives again a contradiction. Thus, u_r does not vanish identically.

Since $\{u_{n,r}\}$ is a bounded minimization sequence for $\mathcal{I}_r(u_{n,r})|_{S_{r,\Theta}}$, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that for every $\varphi \in H^1(\Omega_r)$,

$$\int_{\Omega_r} \nabla u_{n,r} \cdot \nabla \varphi + \int_{\Omega_r} V u_{n,r}\varphi + \lambda_n u_{n,r}\varphi - \beta f(u_{n,r})\varphi - |u_{n,r}|^{2^*-2} u_{n,r}\varphi = o(1)\|\varphi\| \tag{6.7}$$

as $n \to \infty$, by the Lagrange multipliers rule. Choosing $\varphi = u_{n,r}$, we deduce that $\{\lambda_n\}$ is bounded as well, and hence up to a subsequence $\lambda_n \to \lambda_r \in \mathbb{R}$. Moreover, passing to the limit in (6.7) by weak convergence, we obtain

$$-\Delta u_r + V u_r + \lambda_r u_r = |u_r|^{2^*-2} u_r + \beta f(u_r), \qquad x \in \Omega_r.$$

Recalling that $v_{n,r} = u_{n,r} - u_r \rightharpoonup 0$ in $H_0^1(\Omega_r)$, we know

$$\|\nabla u_{n,r}\|_2^2 = \|\nabla u_r\|_2^2 + \|\nabla v_{n,r}\|_2^2 + o(1).$$

By the Brézis–Lieb lemma [12], we have

$$\|u_{n,r}\|_{2^*}^{2^*} = \|u_r\|_{2^*}^{2^*} + \|v_{n,r}\|_{2^*}^{2^*} + o(1).$$

Moreover,

$$\|\nabla u_r\|_2^2 \le \liminf_{n\to\infty} \|\nabla u_{n,r}\|_2^2 \le \widetilde{T_\Theta}^2,$$

that is, $u_r \in \widetilde{\mathcal{V}}_{r,\Theta}$. Note that

$$\int_{\Omega_r} V u_{n,r}^2 dx \to \int_{\Omega_r} V u_r^2 dx \quad \text{as } n \to \infty,$$

hence

$$\|v_{n,r}\|_{2^*}^{2^*} = \|\nabla v_{n,r}\|_2^2 \to \ell$$

as well. Therefore, $\ell \ge S\ell^{\frac{2}{2^*}}$, and we deduce that either $\ell = 0$, or $\ell \ge S^{\frac{N}{2}}$. Let us suppose at first that $\ell \ge S^{N/2}$. Since $\mathcal{I}_r(v_{n,r}) \to 0$, we have that

$$\begin{aligned} o(1) &= \mathcal{I}_r(v_{n,r}) \\ &= \frac{1}{2}\int_{\Omega_r} |\nabla v_{n,r}|^2 dx + \frac{1}{2}\int_{\Omega_r} V v_{n,r}^2 dx - \frac{1}{2^*}\int_{\Omega_r} |v_{n,r}|^{2^*} dx - \beta \int_{\Omega_r} F(v_{n,r}) dx \\ &= \frac{1}{2}\|\nabla v_{n,r}\|_2^2 + o(1) = \frac{\ell}{N} + o(1), \end{aligned}$$

which is not possible. If instead $\ell = 0$, we have that $u_{n,r} \to u_r$ in $H_0^1(\Omega_r)$, so $\mathcal{I}_r(u_r) < 0$. Therefore u is an interior point of $\widetilde{\mathcal{V}}_{r,\Theta}$ because $\mathcal{I}_r(u) \ge \widetilde{h}_1(\widetilde{T_\Theta}) > 0$ for any $u \in \partial\widetilde{\mathcal{V}}_{r,\Theta}$ by (6.4). The Lagrange multiplier theorem implies that there exists $\lambda_r \in \mathbb{R}$ such that (λ_r, u_r) is a solution of (6.1). Moreover,

$$\begin{aligned} \lambda_r\Theta &= \int_{\Omega_r} |u_r|^{2^*} dx + \beta\int_{\Omega_r} f(u_r)u_r dx - \frac{2}{2^*}\int_{\Omega_r} |u_r|^{2^*} dx - 2\beta\int_{\Omega_r} F(u_r)dx - 2I_r(u_r) \\ &= \frac{2^*-2}{2^*}\int_{\Omega_r} |u_r|^{2^*} dx + \beta\int_{\Omega_r} [f(u_r)u_r - 2F(u_r)]dx - 2I_r(u_r) \\ &> -2\mathcal{I}_r(u_r) = -2\widetilde{e}_{r,\Theta}. \end{aligned} \tag{6.8}$$

It follows from the definition of $\widetilde{e}_{r,\Theta}$ that $\widetilde{e}_{r,\Theta}$ is nonincreasing with respect to r. Hence, $\widetilde{e}_{r,\Theta} \le \widetilde{e}_{r_\Theta,\Theta} < 0$ for any $r > r_\Theta$ and $0 < \Theta < \Theta_V$. In view of (6.8), we have

$$\liminf_{r\to\infty} \lambda_r > 0.$$

Finally, the strong maximum principle implies $u_r > 0$. □

Proof of Theorem 1.9. The proof is a direct consequence of Lemma 6.1 and Lemma 3.6. □

7 Proof of Theorem 1.10

In this subsection, we assume $\beta \le 0$ and the assumptions of Theorem 1.10 hold. Consider the following equation

$$\begin{cases} -\Delta u + Vu + \lambda u = |u|^{2^*-2}u + \beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r. \end{cases} \tag{7.1}$$

For $\frac{1}{2} \le s \le 1$, we define the functional $\mathcal{I}_{r,s} : S_{r,\Theta} \to \mathbb{R}$ by

$$\mathcal{I}_{r,s}(u) = \frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} Vu^2 dx - \frac{s}{2^*}\int_{\Omega_r} |u|^{2^*} dx - \beta\int_{\Omega_r} F(u)dx. \tag{7.2}$$

Note that if $u \in S_{r,\Theta}$ is a critical point of $\mathcal{I}_{r,s}$, then there exists $\lambda \in \mathbb{R}$ such that (λ, u) is a solution of the equation

$$\begin{cases} -\Delta u + Vu + \lambda u = s|u|^{2^*-2}u + \beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r. \end{cases} \tag{7.3}$$

Lemma 7.1. *For any $\Theta > 0$, there exist $r_\Theta > 0$ and $u^0, u^1 \in S_{r_\Theta,\Theta}$ such that*

(i) $\mathcal{I}_{r,s}(u^1) \le 0$ *for any $r > r_\Theta$ and $s \in \left[\frac{1}{2}, 1\right]$,*

$$\|\nabla u^0\|_2^2 < \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N-2}{2}} S^{\frac{N}{2}} < \left\|\nabla u^1\right\|_2^2$$

and

$$\mathcal{I}_{r,s}\left(u^0\right) < \frac{1}{N}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}}.$$

(ii) If $u \in S_{r,\Theta}$ satisfies

$$\|\nabla u\|_2^2 = \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N-2}{2}} S^{\frac{N}{2}},$$

then there holds

$$\mathcal{I}_{r,s}(u) \ge \frac{1}{N}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}}.$$

(iii) Set

$$\widetilde{m}_{r,s}(\Theta) = \inf_{\gamma\in\widetilde{\Gamma}_{r,\Theta}} \sup_{t\in[0,1]} \mathcal{I}_{r,s}(\gamma(t))$$

with

$$\widetilde{\Gamma}_{r,\Theta} = \left\{\gamma \in C\left([0,1], S_{r,\Theta}\right) : \gamma(0) = u^0, \gamma(1) = u^1\right\}.$$

Then

$$\frac{1}{N}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)^{\frac{N}{2}}S^{\frac{N}{2}}\leq \widetilde{m}_{r,s}(\Theta)\leq \mathbf{h}(\mathbf{h}_\Theta),$$

where $\mathbf{h}(\mathbf{h}_\Theta)=\max_{t\in\mathbb{R}^+}h(t)$, *the function* $\mathbf{h}:\mathbb{R}^+\to\mathbb{R}$ *being defined by*

$$\mathbf{h}(t)=\frac{1}{2}\left(1+\|V\|_{\frac{N}{2}}S^{-1}\right)t^2\theta\Theta-\alpha\beta C_{N,p_1}\Theta^{\frac{p_1}{2}}\theta^{\frac{N(p_1-2)}{4}}t^{\frac{N(p_1-2)}{2}}-\frac{1}{2\cdot 2^*}t^{2^*}\Theta^{\frac{2^*}{2}}\cdot|\Omega|^{\frac{2-2^*}{2}}.$$

Here θ *is the principal eigenvalue of* $-\Delta$ *with Dirichlet boundary conditions in* Ω, *and* $|\Omega|$ *is the volume of* Ω.

Proof. (i) By the Hölder inequality,

$$\int_\Omega |v_1(x)|^{2^*}dx\geq \Theta^{\frac{2^*}{2}}\cdot|\Omega|^{\frac{2-2^*}{2}}. \tag{7.4}$$

For $x\in\Omega_{\frac{1}{t}}$ and $t>0$, define $v_t(x):=t^{\frac{N}{2}}v_1(tx)$. Using (3.3), (7.4), (3.5) and $\frac{1}{2}\leq s\leq 1$, it holds

$$\begin{aligned}\mathcal{I}_{\frac{1}{t},s}(v_t)&\leq \frac{1}{2}\left(1+\|V\|_{\frac{N}{2}}S^{-1}\right)t^2\theta\Theta-\alpha\beta C_{N,p_1}\Theta^{\frac{p_1}{2}}\theta^{\frac{N(p_1-2)}{4}}t^{\frac{N(p_1-2)}{2}}\\&\quad-\frac{1}{2\cdot 2^*}t^{\frac{N(2^*-2)}{2}}\Theta^{\frac{2^*}{2}}\cdot|\Omega|^{\frac{2-2^*}{2}}\\&=:\mathbf{h}(t).\end{aligned}\tag{7.5}$$

Note that since $2<p_1<2+\frac{4}{N}<q=2^*$ and $\beta\leq 0$ there exist $0<\mathbf{h}_\Theta<t_0$ such that $\mathbf{h}(t_0)=0, \mathbf{h}(t)<0$ for any $t>t_0, \mathbf{h}(t)>0$ for any $0<t<t_0$ and $\mathbf{h}(\mathbf{h}_\Theta)=\max_{t\in\mathbb{R}^+}\mathbf{h}(t)$. As a consequence, there holds

$$\mathcal{I}_{r,s}(v_{t_0})=\mathcal{I}_{\frac{1}{t_0},s}(v_{t_0})\leq \mathbf{h}(t_0)=0 \tag{7.6}$$

for any $r\geq\frac{1}{t_0}$ and $s\in\left[\frac{1}{2},1\right]$. Moreover, there exists $0<t_1<\mathbf{h}_\Theta$ such that

$$\mathbf{h}(t)<\frac{1}{N}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)^{\frac{N}{2}}S^{\frac{N}{2}} \tag{7.7}$$

for $t\in[0,t_1]$. On the other hand, it follows from the Sobolev inequality and the Hölder inequality that

$$\mathcal{I}_{r,s}(u)\ \geq\ \frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)\int_{\Omega_r}|\nabla u|^2dx-\frac{1}{2^*\cdot S^{\frac{2^*}{2}}}\left(\int_{\Omega_r}|\nabla u|^2dx\right)^{\frac{2^*}{2}}. \tag{7.8}$$

Define

$$\mathbf{g}(t):=\frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)t-\frac{1}{2^*\cdot S^{\frac{2^*}{2}}}t^{\frac{2^*}{2}}$$

and

$$\widetilde{t}=\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)^{\frac{N-2}{2}}S^{\frac{N}{2}},$$

it is easy to see that $\mathbf{g}$ is increasing on $(0,\widetilde{t})$ and decreasing on $(\widetilde{t},\infty)$, and

$$\mathbf{g}(\widetilde{t})=\frac{1}{N}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}\right)^{\frac{N}{2}}S^{\frac{N}{2}}.$$

For $r \geq \widetilde{r}_\Theta := \max\left\{\frac{1}{t_1}, \sqrt{\frac{2\theta\Theta}{\widetilde{t}}}\right\}$ we have $v_{\frac{1}{\widetilde{r}_\Theta}} \in S_{r,\Theta}$ and

$$\|\nabla v_{\frac{1}{\widetilde{r}_\Theta}}\|_2^2 = \left(\frac{1}{\widetilde{r}_\Theta}\right)^2 \|\nabla v_1\|_2^2 < \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N-2}{2}} S^{\frac{N}{2}}. \tag{7.9}$$

Moreover, there holds

$$\mathcal{I}_{\widetilde{r}_\Theta,s}\left(v_{\frac{1}{\widetilde{r}_\Theta}}\right) \leq \mathbf{h}\left(\frac{1}{\widetilde{r}_\Theta}\right) \leq \mathbf{h}\left(t_1\right). \tag{7.10}$$

Setting $u^0 = v_{\frac{1}{\widetilde{r}_\Theta}}, u^1 = v_{t_0}$ and

$$r_\Theta = \max\left\{\frac{1}{t_0}, \widetilde{r}_\Theta\right\}. \tag{7.11}$$

Since (7.6), (7.7), (7.8) and (7.9), then (i) holds.

(ii) By (7.8) and a direct calculation, (ii) holds.

(iii) Since $\mathcal{I}_{r,s}\left(u^1\right) \leq 0$ for any $\gamma \in \Gamma_{r,\Theta}$, we have

$$\|\nabla\gamma(0)\|_2^2 < \widetilde{t} < \|\nabla\gamma(1)\|_2^2.$$

It then follows from (7.8) that

$$\max_{t\in[0,1]} \mathcal{I}_{r,s}(\gamma(t)) \geq \mathbf{g}(\widetilde{t}) = \frac{1}{N}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{N}{2}} S^{\frac{N}{2}}$$

for any $\gamma \in \widetilde{\Gamma}_{r,\Theta}$, hence the first inequality in (iii) holds. Now we define a path $\gamma \in \widetilde{\Gamma}_{r,\Theta}$ by

$$\gamma(\tau)(x) = \left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta}\right)^{\frac{N}{2}} v_1\left(\left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta}\right)x\right)$$

for $\tau \in [0,1]$ and $x \in \Omega_r$. Then by (7.5) we have $\widetilde{m}_{r,s}(\Theta) \leq \mathbf{h}(\mathbf{h}_\Theta)$, where $\mathbf{h}(\mathbf{h}_\Theta) = \max_{t\in\mathbb{R}^+} \mathbf{h}(t)$. Note that $\mathbf{h}_\Theta$ is independent of r and s. □

Using Proposition 3.2 to $\mathcal{I}_{r,s}$, it follows that

$$A(u) = \frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} V(x)u^2 dx - \beta\int_{\Omega_r} F(u)dx \quad \text{and} \quad B(u) = \frac{1}{2^*}\int_{\Omega_r} |u|^{2^*} dx.$$

Hence, for almost every $s \in \left[\frac{1}{2}, 1\right]$, there exists a bounded Palais–Smale sequence $\{u_n\}$ satisfying

$$\mathcal{I}_{r,s}\left(u_n\right) \to \widetilde{m}_{r,s}(\Theta) \quad \text{and} \quad \mathcal{I}'_{r,s}\left(u_n\right)|_{T_{u_n}S_{r,\Theta}} \to 0.$$

Next, we are devoted to proving compactness.

Lemma 7.2. *If $\beta \leq 0$ and the assumptions of Theorem 1.10 hold, then $\widetilde{m}_{r,s}(\Theta) < \frac{\zeta}{N} S^{\frac{N}{2}}$, where $\zeta = s^{-\frac{2}{2^*-2}}$.*

Proof. Let U_ε be defined by $U_\varepsilon(x) := \left(\frac{\varepsilon}{\varepsilon^2+|x|^2}\right)^{\frac{N-2}{2}}$ (up to a scalar factor, U_ε is the bubble centered in the origin, with concentration parameter $\varepsilon > 0$, defined in (1.3)). Let also $\varphi \in C_c^\infty(\Omega_r)$ be a radial cut-off function with $\varphi \equiv 1$ in $B_1, \varphi \equiv 0$ in B_2^c, and φ radially decreasing. We define

$$u_\varepsilon(x) := \varphi(x)U_\varepsilon(x), \quad \text{and} \quad v_\varepsilon(x) := \sqrt{\Theta}\frac{u_\varepsilon(x)}{\|u_\varepsilon\|_2}.$$

Notice that $u_\varepsilon \in C_c^\infty(\Omega_r)$, and $v_\varepsilon \in S_{r,\Theta}$. Let us recall the following useful estimates (see [27, Lemma A.1]):

$$\|\nabla u_\varepsilon\|_2^2 = K_1 + O\left(\varepsilon^{N-2}\right), \tag{7.12}$$

$$\|u_\varepsilon\|_{2^*}^2 = \begin{cases} K_2 + O\left(\varepsilon^N\right), & \text{if } N \geq 4, \\ K_2 + O\left(\varepsilon^2\right), & \text{if } N = 3, \end{cases} \tag{7.13}$$

$$\|u_\varepsilon\|_2^2 = \begin{cases} \varepsilon^2 K_3 + O\left(\varepsilon^{N-2}\right), & \text{if } N \geq 5, \\ \omega\varepsilon^2|\log\varepsilon| + O\left(\varepsilon^2\right), & \text{if } N = 4, \\ \omega\left(\int_0^2 \varphi(r)dr\right)\varepsilon + O\left(\varepsilon^2\right), & \text{if } N = 3, \end{cases} \tag{7.14}$$

$$\|u_\varepsilon\|_q^q = \varepsilon^{N-\frac{N-2}{2}q}\left(K_4 + O\left(\varepsilon^{(N-2)q-N}\right)\right)$$
$$\text{if } N \geq 4 \text{ and } q \in (2, 2^*), \text{ and if } N = 3 \text{ and } q \in (3, 6). \tag{7.15}$$

as $\varepsilon \to 0$. Since U_ε is extremal for the Sobolev inequality, we have that $\frac{K_1}{K_2} = S$. Therefore, using $\frac{1}{2} \leq s \leq 1$, we have

$$\begin{aligned}\mathcal{I}_{r,s}(tv_\varepsilon) &\leq \frac{t^2}{2}\int_{\Omega_r}|\nabla v_\varepsilon|^2dx + \frac{t^2}{2}\int_{\Omega_r}Vv_\varepsilon^2dx - \frac{st^{2^*}}{2^*}\int_{\Omega_r}|v_\varepsilon|^{2^*}dx - \alpha\beta t^{p_1}\int_{\Omega_r}|v_\varepsilon|^{p_1}dx \\ &=: h_3(t).\end{aligned}$$

Clearly, $h_3(t) > 0$ for $t > 0$ small and $h_3(t) \to -\infty$ as $t \to \infty$, so $h_3(t)$ attains its maximum at some $t_\varepsilon > 0$ with $h_3'(t_\varepsilon) = 0$. Then, observing that the function

$$t \mapsto \frac{t^2}{2}\int_{\Omega_r}|\nabla v_\varepsilon|^2dx + \frac{t^2}{2}\int_{\Omega_r}Vv_\varepsilon^2dx - \frac{st^{2^*}}{2^*}\int_{\Omega_r}|v_\varepsilon|^{2^*}dx - \alpha\beta t^{p_1}\int_{\Omega_r}|v_\varepsilon|^{p_1}dx$$

is increasing on the interval of

$$\left[0, \left[\frac{-2^*\alpha\beta(p_1-2)\|v_\varepsilon\|_{p_1}^{p_1}}{s(2^*-2)\|v_\varepsilon\|_{2^*}^{2^*}}\right]^{\frac{1}{2^*-p_1}}\right].$$

This fact combined with (7.12)-(7.15) implies that there exist $\delta_1, \delta_2 > 0$, independent of $\varepsilon > 0$, such that

$$\delta_1 \leq t_\varepsilon \leq \delta_2.$$

Moreover, observing that the function

$$t \mapsto \frac{t^2}{2}\int_{\Omega_r}|\nabla v_\varepsilon|^2dx - \frac{st^{2^*}}{2^*}\int_{\Omega_r}|v_\varepsilon|^{2^*}dx$$

is increasing on the interval of

$$\left[0, \left(\frac{\|\nabla v_\varepsilon\|_2^2}{s\|v_\varepsilon\|_{2^*}^{2^*}}\right)^{\frac{1}{2^*-2}}\right].$$

Using (7.12)–(7.15) and the fact that $\frac{K_1}{K_2} = S$, if $N = 3$, the same estimate holds eventually, using $\|u_\varepsilon\|_{2^*}^2 = K_2 + O(\varepsilon^2)$ instead of $\|u_\varepsilon\|_{2^*}^2 = K_2 + O(\varepsilon^N)$. Therefore, the maximum level is

$$\begin{aligned}
h_3(t_\varepsilon) &\le \frac{t_\varepsilon^2}{2}\|\nabla v_\varepsilon\|_2^2 + \frac{t_\varepsilon^2}{2}\int_{\Omega_r} V v_\varepsilon^2 dx - \frac{s t_\varepsilon^{2^*}}{2^*}\int_{\Omega_r} |v_\varepsilon|^{2^*} dx - \alpha\beta t_\varepsilon^{p_1}\int_{\Omega_r} |v_\varepsilon|^{p_1} dx \\
&\le \frac{1}{N s^{\frac{2}{2^*-2}}}\left(\frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_{2^*}^2}\right)^{\frac{2^*}{2^*-2}} + \frac{\int_{\Omega_r} V v_\varepsilon^2 dx}{2}\left(\frac{\|\nabla v_\varepsilon\|_2^2}{s\|v_\varepsilon\|_{2^*}^{2^*}}\right)^{\frac{2}{2^*-2}} - \alpha\beta\|v_\varepsilon\|_{p_1}^{p_1}\left(\frac{\|\nabla v_\varepsilon\|_2^2}{s\|v_\varepsilon\|_{2^*}^{2^*}}\right)^{\frac{p_1}{2^*-2}} \\
&= \frac{1}{N s^{\frac{2}{2^*-2}}}\left(\frac{\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^2}\right)^{\frac{N}{2}} + \frac{\int_{\Omega_r} V v_\varepsilon^2 dx}{2 s^{\frac{2}{2^*-2}}}\left(\Theta^{\frac{2-2^*}{2}} \cdot \frac{\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^{2^*}} \cdot \frac{\|u_\varepsilon\|_2^{2^*}}{\|u_\varepsilon\|_2^2}\right)^{\frac{2}{2^*-2}} \\
&\quad - \frac{\alpha\beta}{s^{\frac{p_1}{2^*-2}}}\left(\Theta^{\frac{2-2^*}{2}} \cdot \frac{\|\nabla u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2^*}^{2^*}} \cdot \frac{\|u_\varepsilon\|_2^{2^*}}{\|u_\varepsilon\|_2^2}\right)^{\frac{p_1}{2^*-2}} \cdot \Theta^{\frac{p_1}{2}}\frac{\|u_\varepsilon\|_{p_1}^{p_1}}{\|u_\varepsilon\|_2^{p_1}} \\
&\le \frac{1}{N s^{\frac{2}{2^*-2}}}\left[\frac{K_1 + O\left(\varepsilon^{N-2}\right)}{K_2 + O\left(\varepsilon^N\right)}\right]^{\frac{N}{2}} + \frac{\max_{x\in\Omega_r} V(x)}{2 s^{\frac{2}{2^*-2}}} \cdot \|u_\varepsilon\|_2^2 \cdot \frac{\|\nabla u_\varepsilon\|_2^{\frac{4}{2^*-2}}}{\|u_\varepsilon\|_{2^*}^{\frac{2\cdot 2^*}{2^*-2}}} \\
&\quad - \frac{\alpha\beta}{s^{\frac{p_1}{2^*-2}}} \cdot \|u_\varepsilon\|_{p_1}^{p_1} \cdot \frac{\|\nabla u_\varepsilon\|_2^{\frac{2p_1}{2^*-2}}}{\|u_\varepsilon\|_{2^*}^{\frac{p_1\cdot 2^*}{2^*-2}}} \\
&= \frac{1}{N s^{\frac{2}{2^*-2}}} S^{\frac{N}{2}} + O\left(\varepsilon^{N-2}\right) + C_1\|u_\varepsilon\|_2^2 + C_2\|u_\varepsilon\|_{p_1}^{p_1} \\
&= \frac{1}{N s^{\frac{2}{2^*-2}}} S^{\frac{N}{2}}
\end{aligned}$$

as $\varepsilon \to 0$, where $\zeta = s^{-\frac{2}{2^*-2}}$ and $C_1 \ge 0$, $C_2 \ge 0$ because of $\beta \le 0$. In the penultimate equal sign, we used

$$\frac{1}{N}\left[\frac{K_1 + O\left(\varepsilon^{N-2}\right)}{K_2 + O\left(\varepsilon^N\right)}\right]^{\frac{N}{2}} = \frac{1}{N}\left[\frac{K_1}{K_2} + O\left(\varepsilon^{N-2}\right)\right]^{\frac{N}{2}} = \frac{S^{\frac{N}{2}}}{N} + O\left(\varepsilon^{N-2}\right).$$

This completes the proof. □

Lemma 7.3. *For any $\Theta > 0$, let $r > r_\Theta$, where r_Θ is defined in Lemma 7.1. Then problem (7.3) has a solution $(\lambda_{r,s}, u_{r,s})$ for almost every $s \in \left[\frac{1}{2}, 1\right]$. Moreover, $u_{r,s} \ge 0$ and $\mathcal{I}_{r,s}(u_{r,s}) = \widetilde{m}_{r,s}(\Theta)$.*

Proof. Based on the previous analysis, we know that, for almost every $s \in \left[\frac{1}{2}, 1\right]$, there exists a bounded Palais–Smale sequence $\{u_n\}$ satisfying

$$\mathcal{I}_{r,s}(u_n) \to \widetilde{m}_{r,s}(\Theta) \quad \text{and} \quad \mathcal{I}'_{r,s}(u_n)\big|_{T_{u_n} S_{r,\Theta}} \to 0. \tag{7.16}$$

Then

$$\lambda_n = -\frac{1}{\Theta}\left(\int_{\Omega_r} |\nabla u_n|^2 dx + \int_{\Omega_r} V(x) u_n^2 dx - \beta\int_{\Omega_r} f(u_n) u_n dx - s\int_{\Omega_r} |u_n|^{2^*} dx\right)$$

is bounded and

$$\mathcal{I}'_{r,s}(u_n) + \lambda_n u_n \to 0 \quad \text{in } H^{-1}(\Omega_r). \tag{7.17}$$

Moreover, since $\{u_n\}$ is a bounded Palais–Smale sequence, there exist $u_0 \in H_0^1(\Omega_r)$ and $\lambda \in \mathbb{R}$ such that, up to a subsequence,

$$\begin{aligned} &\lambda_n \to \lambda \quad \text{in } \mathbb{R}, \\ &u_n \rightharpoonup u_0 \quad \text{in } H_0^1(\Omega_r), \\ &u_n \to u_0 \quad \text{in } L^t(\Omega_r) \text{ for all } 2 \le t < 2^*, \end{aligned} \tag{7.18}$$

where u_0 satisfies

$$\begin{cases} -\Delta u_0 + V u_0 + \lambda u_0 = s\,|u_0|^{2^*-2} u_0 + \beta f(u_0) & \text{in } \Omega_r, \\ u_0 \in H_0^1(\Omega_r), \quad \int_{\Omega_r} |u_0|^2\,dx = \Theta. \end{cases}$$

Using (7.17), we have

$$\mathcal{I}'_{r,s}(u_n)\,u_0 + \lambda_n \int_{\Omega_r} u_n u_0 dx \to 0 \quad \text{as } n \to \infty$$

and

$$I'_{r,s}(u_n)\,u_n + \lambda_n \Theta \to 0 \quad \text{as } \ n \to \infty.$$

Note that

$$\begin{aligned} \lim_{n\to\infty} \int_{\Omega_r} V(x) u_n^2 dx &= \int_{\Omega_r} V(x) u_0^2 dx, \\ \lim_{n\to\infty} \int_{\Omega_r} f(u_n) u_n dx &= \int_{\Omega_r} f(u_0) u_0 dx, \\ \lim_{n\to\infty} \int_{\Omega_r} f(u_n) u_0 dx &= \int_{\Omega_r} f(u_0) u_0 dx. \end{aligned}$$

Now, we show that $u_n \to u_0$ in $H_0^1(\Omega_r)$. Firstly, note that the weak limit u_0 does not vanish identically. Suppose by contradiction that $u_0 \equiv 0$. Since $\{u_n\}$ is bounded in $H^1(\Omega_r)$, up to a subsequence we have that $\|\nabla u_n\|_2^2 \to \ell \in \mathbb{R}$. Using (f_2), (7.17), (7.18), we have

$$\begin{aligned} \langle \mathcal{I}'_{r,s}(u_n), u_n \rangle &= \int_{\Omega_r} |\nabla u_n|^2 dx + \int_{\Omega_r} V u_n^2 dx - s \int_{\Omega_r} |u_n|^{2^*} dx - \beta \int_{\Omega_r} f(u_n) u_n dx \\ &\to 0, \end{aligned}$$

hence

$$s\|u_n\|_{2^*}^{2^*} = \|\nabla u_n\|_2^2 \to \ell$$

as well. Therefore, $\ell \ge s^{-\frac{2}{2^*}} S \ell^{\frac{2}{2^*}}$, and we deduce that either $\ell = 0$, or $\ell \ge s^{-\frac{2}{2^*-2}} S^{\frac{N}{2}}$. Let us suppose at first that $\ell \ge s^{-\frac{2}{2^*-2}} S^{\frac{N}{2}}$. Since $\mathcal{I}_{r,s}(u_n) \to \widetilde{m}_{r,s}(\Theta) < \frac{\zeta}{N} S^{\frac{N}{2}}$, we have that

$$\begin{aligned} \frac{\zeta}{N} S^{\frac{N}{2}} > \widetilde{m}_{r,s}(\Theta) &\leftarrow \mathcal{I}_{r,s}(u_n) + o(1) \\ &= \frac{1}{2} \int_{\Omega_r} |\nabla u_n|^2 dx + \frac{1}{2} \int_{\Omega_r} V u_n^2 dx - \frac{s}{2^*} \int_{\Omega_r} |u_n|^{2^*} dx - \beta \int_{\Omega_r} F(u_n) dx \\ &= \frac{\ell}{N} \ge s^{-\frac{2}{2^*-2}} \frac{S^{\frac{N}{2}}}{N}, \end{aligned}$$

which is not possible. If instead $\ell = 0$, we have $\|u_n\|_{2^*} \to 0$, $\|\nabla u_n\|_2 \to 0$ and $F(u_n) \to 0$. But then $I_{r,s}(u_n) \to 0 \neq \widetilde{m}_{r,s}(\Theta)$, which gives again a contradiction. Thus, u_r does not vanish identically.

Since $\{u_{n,r}\}$ is a bounded minimization sequence for $\mathcal{I}_r(u_{n,r})|_{S_{r,\Theta}}$, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that for every $\varphi \in H^1(\Omega_r)$,

$$\int_{\Omega_r} \nabla u_{n,r} \cdot \nabla \varphi + \int_{\Omega_r} V u_{n,r}\varphi + \lambda_n u_{n,r}\varphi - \beta f(u_{n,r})\varphi - s\,|u_{n,r}|^{2^*-2} u_{n,r}\varphi = o(1)\|\varphi\| \qquad (7.19)$$

as $n \to \infty$, by the Lagrange multipliers rule. Choosing $\varphi = u_{n,r}$, we deduce that $\{\lambda_n\}$ is bounded as well, and hence up to a subsequence $\lambda_n \to \lambda_r \in \mathbb{R}$. Moreover, passing to the limit in (7.19) by weak convergence, we obtain

$$-\Delta u_r + V u_r + \lambda_r u_r = s|u_r|^{2^*-2}u_r + \beta f(u_r), \qquad x \in \Omega_r.$$

Recalling that $v_{n,r} = u_{n,r} - u_r \rightharpoonup 0$ in $H_0^1(\Omega_r)$, we know

$$\|\nabla u_{n,r}\|_2^2 = \|\nabla u_r\|_2^2 + \|\nabla v_{n,r}\|_2^2 + o(1).$$

By the Brézis–Lieb lemma [12], we have

$$\|u_{n,r}\|_{2^*}^{2^*} = \|u_r\|_{2^*}^{2^*} + \|v_{n,r}\|_{2^*}^{2^*} + o(1).$$

Note that

$$\int_{\Omega_r} V v_{n,r}^2 dx \to 0 \quad \text{as } n \to \infty,$$

hence

$$s\|v_{n,r}\|_{2^*}^{2^*} = \|\nabla v_{n,r}\|_2^2 \to \ell$$

as well. Therefore, by the Sobolev inequality $\ell \geq s^{-\frac{2}{2^*}} S \ell^{\frac{2}{2^*}}$, and we deduce that either $\ell = 0$, or $\ell \geq s^{-\frac{2}{2^*-2}} S^{\frac{N}{2}}$. Let us suppose at first that $\ell \geq s^{-\frac{2}{2^*-2}} S^{\frac{N}{2}}$. Since $\mathcal{I}_{r,s}(u_n) \to \widetilde{m}_{r,s}(\Theta) < \frac{\zeta}{N} S^{\frac{N}{2}}$, we have that

$$\begin{aligned}\frac{\zeta}{N} S^{\frac{N}{2}} > \widetilde{m}_{r,s}(\Theta) &\leftarrow \mathcal{I}_{r,s}(v_n) + o(1)\\ &= \frac{1}{2}\int_{\Omega_r} |\nabla v_n|^2 dx + \frac{1}{2}\int_{\Omega_r} V v_n^2 dx - \frac{s}{2^*}\int_{\Omega_r} |v_n|^{2^*} dx - \beta \int_{\Omega_r} F(v_n) dx\\ &= \frac{\ell}{N} \geq s^{-\frac{2}{2^*-2}} \frac{S^{\frac{N}{2}}}{N},\end{aligned}$$

which is not possible. If instead $\ell = 0$, we have that $u_{n,r} \to u_r$ in $H_0^1(\Omega_r)$, so $\mathcal{I}_r(u_r) > 0$.

Similar to the proof of Lemma 3.3, we also obtain that $u_{r,s} \geq 0$. □

In order to obtain a solution of (7.1), we also need to prove a uniform estimate for the solutions of (7.3) established in Lemma 7.3. Similar to the proof of Lemma 3.4 and Lemma 3.5, we obtain the following lemmas.

Lemma 7.4. *If $(\lambda, u) \in \mathbb{R} \times S_{r,\Theta}$ is a solution of (7.3) established in Lemma 7.3 for some r and s, then*

$$\int_{\Omega_r} |\nabla u|^2 dx \leq \frac{4N}{N(2^*-2)-4}\left(\frac{2^*-2}{2}\mathbf{h}(\mathbf{h}_\Theta) + \Theta\left(\frac{1}{2N}\|\widetilde{V}\|_\infty + \frac{2^*-2}{4}\|V\|_\infty\right)\right),$$

where the constant $\mathbf{h}(\mathbf{h}_\Theta)$ is defined in (iii) of Lemma 7.1 and is independent of r and s.

Lemma 7.5. *For every $\Theta > 0$, problem (7.3) has a solution (λ_r, u_r) provided $r > r_\Theta$ where r_Θ is as in Lemma 7.1. Moreover, $u_r \geq 0$ in Ω_r.*

Proof of Theorem 1.10. The proof is an immediate consequence of Lemmas 7.5 and 3.6. □

8 Proof of Theorem 1.11

In this subsection, we assume $\beta > 0$ and the assumptions of Theorem 1.10 hold. Consider the following equation

$$\begin{cases} -\Delta u + Vu + \lambda u = |u|^{2^*-2}u + \beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r. \end{cases} \tag{8.1}$$

For $\frac{1}{2} \le s \le 1$, we define the functional $\mathcal{J}_{r,s} : S_{r,\Theta} \to \mathbb{R}$ by

$$\mathcal{J}_{r,s}(u) = \frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} Vu^2 dx - \frac{s}{2^*}\int_{\Omega_r} |u|^{2^*} dx - s\beta \int_{\Omega_r} F(u) dx. \tag{8.2}$$

Note that if $u \in S_{r,\Theta}$ is a critical point of $\mathcal{J}_{r,s}$, then there exists $\lambda \in \mathbb{R}$ such that (λ, u) is a solution of the equation

$$\begin{cases} -\Delta u + Vu + \lambda u = s|u|^{2^*-2}u + s\beta f(u), & x \in \Omega_r, \\ \int_{\Omega_r} |u|^2 dx = \Theta, u \in H_0^1(\Omega_r), & x \in \Omega_r. \end{cases} \tag{8.3}$$

Lemma 8.1. *For any $\Theta > 0$, there exist $\widehat{r}_\Theta > 0$ and $u^0, u^1 \in S_{r_\Theta,\Theta}$ such that*

(i) For $r > \widehat{r}_\Theta$ and $s \in \left[\frac{1}{2}, 1\right]$ we have $\mathcal{J}_{r,s}\left(u^1\right) \le 0$ and

$$\mathcal{J}_{r,s}\left(u^0\right) < \widehat{A}^{-\frac{2}{2^*-2}} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}} \left[\frac{2^*-2}{2}\left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}}\right],$$

where

$$\widehat{A} = S^{-\frac{2^*}{2}} \left[\frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))} + \frac{1}{2^*}\right].$$

Moreover,

$$\|\nabla u^0\|_2^2 < \left[\frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2^*\widehat{A}}\right]^{\frac{2}{2^*-2}}, \qquad \left\|\nabla u^1\right\|_2^2 > \left[\frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2^*\widehat{A}}\right]^{\frac{2}{2^*-2}}.$$

(ii) If $u \in S_{r,\Theta}$ satisfies

$$\|\nabla u\|_2^2 = \left[\frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2^*\widehat{A}}\right]^{\frac{2}{2^*-2}},$$

then there holds

$$\mathcal{J}_{r,s}(u) \ge \widehat{A}^{-\frac{2}{2^*-2}} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}} \left[\frac{2^*-2}{2}\left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}}\right].$$

(iii) Let

$$\widehat{m}_{r,s}(\Theta) = \inf_{\gamma \in \widehat{\Gamma}_{r,\Theta}} \sup_{t \in [0,1]} \mathcal{J}_{r,s}(\gamma(t)),$$

where

$$\widehat{\Gamma}_{r,\Theta} = \left\{ \gamma \in C\left([0,1], S_{r,\Theta}\right) : \gamma(0) = u^0, \gamma(1) = u^1 \right\}.$$

Then

$$\widehat{m}_{r,s}(\Theta) \geq \widehat{A}^{-\frac{2}{2^*-2}} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}} \left[\frac{2^*-2}{2} \left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}} \right]$$

and

$$\widehat{m}_{r,s}(\Theta) \leq \frac{N(2^*-2)-4}{2} \left(\frac{\theta\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)}{N(2^*-2)} \right)^{\frac{N(2^*-2)}{N(2^*-2)-4}} (4 \cdot 2^*)^{\frac{4}{N(2^*-2)-4}} |\Omega|^{\frac{2(2^*-2)}{N(2^*-2)-4}} \cdot \Theta^{\frac{N(2^*-2)-2\cdot 2^*}{N(2^*-2)-4}},$$

where θ is the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω.

Proof. (i) By the Hölder inequality, we know

$$\int_\Omega |v_1(x)|^{2^*} dx \geq \Theta^{\frac{2^*}{2}} \cdot |\Omega|^{\frac{2-2^*}{2}}. \tag{8.4}$$

For $x \in \Omega_{\frac{1}{t}}$ and $t > 0$, define $v_t(x) := t^{\frac{N}{2}} v_1(tx)$. Using (5.2), (8.4), (4.4) and $\frac{1}{2} \leq s \leq 1$, it holds

$$\begin{aligned} \mathcal{J}_{\frac{1}{t},s}(v_t) &\leq \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta \Theta - \frac{\beta}{2} \alpha_1 t^{\frac{N(p_2-2)}{2}} \int_\Omega |v_1|^{p_2} dx - \frac{1}{2\cdot 2^*} t^{\frac{N(2^*-2)}{2}} \Theta^{\frac{2^*}{2}} \cdot |\Omega|^{\frac{2-2^*}{2}} \\ &\leq \widehat{h}_2(t), \end{aligned} \tag{8.5}$$

where

$$\widehat{h}_2(t) = \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) t^2 \theta \Theta - \frac{1}{2\cdot 2^*} t^{\frac{N(2^*-2)}{2}} \Theta^{\frac{2^*}{2}} \cdot |\Omega|^{\frac{2-2^*}{2}}.$$

A simple computation shows that $\widehat{h}_2(t_0) = 0$ for

$$t_0 := \left[\left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right) 2^* \theta \Theta^{\frac{2-2^*}{2}} |\Omega|^{\frac{2^*-2}{2}} \right]^{\frac{2}{N(2^*-2)-4}}$$

and $\widehat{h}_2(t) < 0$ for any $t > t_0, \widehat{h}_2(t) > 0$ for any $0 < t < t_0$. Moreover, $\widehat{h}_2(t)$ achieves its maximum at

$$t_\Theta = \left[\frac{4\cdot 2^* \left(1 + \|V\|_{\frac{N}{2}} S^{-1}\right)\theta}{N(2^*-2)} \Theta^{\frac{2-2^*}{2}} |\Omega|^{\frac{2^*-2}{2}} \right]^{\frac{2}{N(2^*-2)-4}}.$$

This implies

$$\mathcal{J}_{r,s}(v_{t_0}) = J_{\frac{1}{t_0},s}(v_{t_0}) \leq \widehat{h}_2(t_0) = 0 \tag{8.6}$$

for any $r \geq \frac{1}{t_0}$ and $s \in \left[\frac{1}{2}, 1\right]$. There exists $0 < t_1 < t_\Theta$ such that for any $t \in [0, t_1]$,

$$\widehat{h}_2(t) < A^{-\frac{2}{2^*-2}} \left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}} \left[\frac{2^*-2}{2} \left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}} \right]. \tag{8.7}$$

On the other hand, it follows from (3.5) that

$$\mathcal{J}_{r,s}(u) \geq \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)\int_{\Omega_r} |\nabla u|^2 dx - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}}\left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{2^*}{2}}$$
$$- \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\Omega_r} |\nabla u|^2 dx\right)^{\frac{N(p_1-2)}{4}}.$$

Define

$$\begin{aligned}\widehat{g}_1(t) &:= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}} t^{\frac{2^*}{2}} - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{4}} \\ &= t^{\frac{N(p_1-2)}{4}}\left[\frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t^{\frac{4-N(p_1-2)}{4}} - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}} t^{\frac{2\cdot 2^*-N(p_1-2)}{4}}\right] \\ &\quad - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{4}}.\end{aligned}$$

In view of $2 < p < 2 + \frac{2}{N} < q < 2^*$ and the definition of $\widetilde{\Theta}_V$, there exist $0 < l_1 < l_M < l_2$ such that $\widehat{g}_1(t) < 0$ for any $0 < t < l_1$ and $t > l_2, \widehat{g}_1(t) > 0$ for $l_1 < t < l_2$ and $\widehat{g}_1(l_M) = \max_{t\in\mathbb{R}^+}\widehat{g}_1(t) > 0$. Let

$$t_2 = \left(\frac{\alpha\beta C_{N,p_1} S^{\frac{2^*}{2}} N(p_1-2)(4-N(p_1-2))}{4(2^*-2)}\right)^{\frac{4}{2\cdot 2^*-N(p_1-2)}} \Theta^{\frac{2p_1-N(p_1-2)}{2\cdot 2^*-N(p_1-2)}}.$$

Then by a direct calculation, we have $g_1''(t) \leq 0$ if and only if $t \geq t_2$. Hence

$$\max_{t\in\mathbb{R}^+}\widehat{g}_1(t) = \max_{t\in[t_2,\infty)}\widehat{g}_1(t).$$

Note that for any $t \geq t_2$,

$$\begin{aligned} g_1(t) &= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}} t^{\frac{2^*}{2}} - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} t^{\frac{N(p_1-2)}{4}} \\ &= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t - \alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}} \cdot t^{\frac{N(p_1-2)}{4}} - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}} t^{\frac{2^*}{2}} \\ &= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t - \frac{4(2^*-2)}{S^{\frac{2^*}{2}} N(p_1-2)(4-N(p_1-2))} \cdot t_2^{\frac{2\cdot 2^*-N(p_1-2)}{4}} \cdot t^{\frac{N(p_1-2)}{4}} \\ &\quad - \frac{1}{2^* \cdot S^{\frac{2^*}{2}}} t^{\frac{2^*}{2}} \\ &\geq \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)t - S^{-\frac{2^*}{2}}\left[\frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))} + \frac{1}{2^*}\right] t^{\frac{2^*}{2}} \\ &=: \widehat{g}_2(t). \end{aligned} \tag{8.8}$$

Now, we will determine the value of $\widetilde{\Theta}_V$. In fact, $\widehat{g}_1(l_M) = \max_{t\in\mathbb{R}^+}\widehat{g}_1(t) > 0$ as long as

$\widehat{g}_2(t_2) > 0$, that is,

$$\begin{aligned}\widehat{g}_2(t_2) &= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right) t_2 - S^{-\frac{2^*}{2}}\left[\frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))} + \frac{1}{2^*}\right] t_2^{\frac{2^*}{2}} \\ &= \frac{1}{2}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)\left(\frac{\alpha\beta C_{N,p_1} S^{\frac{2^*}{2}}}{A_{p_1}}\right)^{\frac{4}{2\cdot 2^* - N(p_1-2)}} \Theta^{\frac{2p_1 - N(p_1-2)}{2\cdot 2^* - N(p_1-2)}} \\ &\quad - S^{-\frac{2^*}{2}}\left(A_{p_1} + \frac{1}{2^*}\right)\left(\frac{\alpha\beta C_{N,p_1} S^{\frac{2^*}{2}}}{A_{p_1}}\right)^{\frac{2\cdot 2^*}{2\cdot 2^* - N(p_1-2)}} \Theta^{\frac{2^*[2p_1 - N(p_1-2)]}{2[2\cdot 2^* - N(p_1-2)]}} \\ &> 0,\end{aligned}$$

where

$$A_{p_1} = \frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))}.$$

Hence, we take

$$\widetilde{\Theta}_V = \left(\frac{\alpha\beta C_{N,p_1} S^{\frac{2^*}{2}}}{A_{p_1}}\right)^{-\frac{4}{2p_1 - N(p_1-2)}}\left[\frac{S^{\frac{2^*}{2}}}{2\cdot 2^*}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)(2^* A_{p_1} + 1)\right]^{\frac{2[2\cdot 2^* - N(p_1-2)]}{(2^*-2)[2p_1 - N(p_1-2)]}}.$$

Let

$$\widehat{A} = S^{-\frac{2^*}{2}}\left[\frac{4(2^*-2)}{N(p_1-2)(4-N(p_1-2))} + \frac{1}{2^*}\right],\ t_g = \left[\frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2^*\widehat{A}}\right]^{\frac{2}{2^*-2}},$$

so that $t_g > t_2$ by the definition of $\widetilde{\Theta}_V, \max_{t\in[t_2,\infty)} \widehat{g}_2(t) = \widehat{g}_2(t_g)$ and

$$\max_{t\in\mathbb{R}^+} \widehat{g}_1(t) \geq \max_{t\in[t_2,\infty)} \widehat{g}_2(t) = \widehat{A}^{-\frac{2}{2^*-2}}\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}}\left[\frac{2^*-2}{2}\left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}}\right].$$

Set $\bar{r}_\Theta = \max\left\{\frac{1}{t_1}, \sqrt{\frac{2\theta\Theta}{t_g}}\right\}$, then $v_{\frac{1}{\bar{r}_\Theta}} \in S_{r,\Theta}$ for any $r > \bar{r}_\Theta$, and

$$\left\|\nabla v_{\frac{1}{\bar{r}_\Theta}}\right\|_2^2 = \left(\frac{1}{\bar{r}_\Theta}\right)^2 \|\nabla v_1\|_2^2 < t_g = \left[\frac{\left(1 - \|V_-\|_{\frac{N}{2}} S^{-1}\right)}{2^*\widehat{A}}\right]^{\frac{2}{2^*-2}}. \tag{8.9}$$

Moreover,

$$\mathcal{J}_{\bar{r}_\Theta,s}\left(v_{\frac{1}{\bar{r}_\Theta}}\right) \leq \widehat{h}\left(\frac{1}{\bar{r}_\Theta}\right) \leq \widehat{h}(t_1). \tag{8.10}$$

Let $u^0 = v_{\frac{1}{\bar{r}_\Theta}}, u^1 = v_{t_0}$ and

$$\widetilde{r}_\Theta = \max\left\{\frac{1}{t_0}, \bar{r}_\Theta\right\}.$$

Then the statement (i) holds by (8.6), (8.7), (8.9), (8.10).

(ii) holds by (8.8) and a direct calculation.

(iii) In view of $J_{r,s}(u^1) \leq 0$ for any $\gamma \in \Gamma_{r,\Theta}$ and the definition of t_0, we have

$$\|\nabla\gamma(0)\|_2^2 < t_g < \|\nabla\gamma(1)\|_2^2.$$

It then follows from (8.8) that

$$\max_{t\in[0,1]} J_{r,s}(\gamma(t)) \geq g_2(t_g) = \widehat{A}^{-\frac{2}{2^*-2}}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1}\right)^{\frac{2^*}{2^*-2}}\left[\frac{2^*-2}{2}\left(\frac{1}{2^*}\right)^{\frac{2^*}{2^*-2}}\right]$$

for any $\gamma \in \Gamma_{r,\Theta}$, hence the first inequality in (iii) holds. We define a path $\gamma : [0,1] \to S_{r,\Theta}$ by

$$\gamma(t) : \Omega_r \to \mathbb{R}, \quad x \mapsto \left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta}\right)^{\frac{N}{2}} v_1\left(\left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta}\right)x\right).$$

Then $\gamma \in \Gamma_{r,\Theta}$, and the second inequality in (iii) follows from (8.5). □

Using Proposition 3.2 to $\widetilde{J}_{r,s}$, it follows that

$$A(u) = \frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} V(x)u^2 dx \quad \text{and} \quad B(u) = \frac{1}{2^*}\int_{\Omega_r} |u|^{2^*} dx + \beta\int_{\Omega_r} F(u)dx.$$

Hence, for almost every $s \in \left[\frac{1}{2},1\right]$, there exists a bounded Palais–Smale sequence $\{u_n\}$ satisfying

$$\mathcal{J}_{r,s}(u_n) \to \widetilde{m}_{r,s}(\Theta) \quad \text{and} \quad \mathcal{J}'_{r,s}(u_n)\big|_{T_{u_n}S_{r,\Theta}} \to 0.$$

Similar to the proof of Lemmas 7.2 and 7.3, we have the following lemmas.

Lemma 8.2. *If $\beta > 0$ and the assumptions of Theorem 1.11 hold, then $\widetilde{m}_{r,s}(\Theta) < \frac{\zeta}{N}S^{\frac{N}{2}}$, where $\zeta = s^{-\frac{2}{2^*-2}}$.*

Lemma 8.3. *Assume $0 < \Theta < \widetilde{\Theta}_V$ where $\widetilde{\Theta}_V$ is given in Theorem 1.11, let $r > r_\Theta$, where r_Θ is defined in Lemma 8.1. Then problem (8.3) has a solution $(\lambda_{r,s}, u_{r,s})$ for almost every $s \in \left[\frac{1}{2},1\right]$. Moreover, $u_{r,s} \geq 0$ and $\mathcal{J}_{r,s}(u_{r,s}) = \widetilde{m}_{r,s}(\Theta)$.*

In order to obtain a solution of (8.1), we also need to prove a uniform estimate for the solutions of (8.3) established in Lemma 8.3.

Lemma 8.4. *For fixed $\Theta > 0$ the set of solutions $u \in S_{r,\Theta}$ of (8.3) is bounded uniformly in s and r.*

Proof. Since u is a solution of (8.3), we have

$$\int_{\Omega_r} |\nabla u|^2 dx + \int_{\Omega_r} Vu^2 dx = s\int_{\Omega_r} |u|^{2^*} dx + s\beta\int_{\Omega_r} f(u)u dx - \lambda\int_{\Omega_r} |u|^2 dx.$$

The Pohozaev identity implies

$$\frac{N-2}{2N}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2N}\int_{\partial\Omega_r} |\nabla u|^2 (x\cdot\mathbf{n}) d\sigma + \frac{1}{2N}\int_{\Omega_r} \widetilde{V}(x)u^2 + \frac{1}{2}\int_{\Omega_r} Vu^2 dx$$
$$= -\frac{\lambda}{2}\int_{\Omega_r} |u|^2 dx + \frac{s}{2^*}\int_{\Omega_r} |u|^{2^*} dx + s\beta\int_{\Omega_r} F(u)dx$$

where $\mathbf{n}$ denotes the outward unit normal vector on $\partial\Omega_r$. It then follows from $\beta > 0$ and (f_2) that

$$\frac{1}{N}\int_{\Omega_r} |\nabla u|^2 dx - \frac{1}{2N}\int_{\partial\Omega_r} |\nabla u|^2 (x\cdot\mathbf{n}) d\sigma - \frac{1}{2N}\int_{\Omega_r} (\nabla V\cdot x)u^2 dx$$
$$\leq \frac{2^*-2}{2}\left(\frac{1}{2}\int_{\Omega_r} |\nabla u|^2 dx + \frac{1}{2}\int_{\Omega_r} Vu^2 dx - \widehat{m}_{r,s}(\Theta)\right) + s\frac{\beta(p_2-2^*)}{2}\int_{\Omega_r} F(u)dx.$$

Using Gagliardo–Nirenberg inequality, (3.5) and (iii) in Lemma 8.1, we have

$$\frac{2^*-2}{2}\widehat{m}_{r,s}(\Theta) \geq \frac{N(p_2-2)-4}{4N}\int_{\Omega_r}|\nabla u|^2dx - \Theta\left(\frac{1}{2N}\|\nabla V\cdot x\|_\infty + \frac{p_2-2}{4}\|V\|_\infty\right) + \frac{s\alpha\beta(p_2-2^*)}{2}C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\Omega_r}|\nabla u|^2dx\right)^{\frac{N(p_1-2)}{4}}.$$

Since $2<p_1<2+\frac{4}{N}$, we can bound $\int_{\Omega_r}|\nabla u|^2dx$ uniformly in s and r. □

Lemma 8.5. *Assume $0<\Theta<\widetilde{\Theta}_V$, where $\widetilde{\Theta}_V$ is given in Theorem 1.11, and let $r>\widetilde{r}_\Theta$, where $\widetilde{r}_\Theta$ is defined in Lemma 8.1. Then equation (8.3) admits a solution $(\lambda_{r,\Theta},u_{r,\Theta})$ for every $r>\widetilde{r}_\Theta$ such that $u_{r,\Theta}>0$ in Ω_r.*

Proof. The proof of lemma is similar to the Lemma 7.5. □

Proof of Theorem 1.11. The proof is an immediate consequence of Lemmas 8.5 and 3.6. □

9 Mass critical case

9.1 Proof of Theorem 1.12

This subsection considers the case of $p_1=2+\frac{4}{N}$, so we need to modify the proof of Theorem 1.5.

Lemma 9.1. *For $0<\Theta<\widetilde{\Theta}_V$ where $\widetilde{\Theta}_V$ is defined in Theorem 1.12, there exist $\widetilde{r}_\Theta>0$ and $u^0,u^1\in S_{r_\Theta,\Theta}$ such that*

(i) For $r>\widetilde{r}_\Theta$ and $s\in\left[\frac{1}{2},1\right]$ we have $J_{r,s}\left(u^1\right)\leq 0$ and

$$J_{r,s}\left(u^0\right) < \frac{(N(q-2)-4)\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}-2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}}\left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

Moreover,

$$\|\nabla u^0\|_2^2 < \left[\frac{2q}{N(q-2)C_{N,q}}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}-2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}}$$

and

$$\left\|\nabla u^1\right\|_2^2 > \left[\frac{2q}{N(q-2)C_{N,q}}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}-2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}}.$$

(ii) If $u\in S_{r,\Theta}$ satisfies

$$\|\nabla u\|_2^2 = \left[\frac{2q}{N(q-2)C_{N,q}}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}-2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}},$$

then there holds

$$J_{r,s}(u) \geq \frac{(N(q-2)-4)\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}-2\alpha\beta C_N\Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}}\left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

(iii) Let

$$m_{r,s}(\Theta) = \inf_{\gamma\in\Gamma_{r,\Theta}} \sup_{t\in[0,1]} J_{r,s}(\gamma(t)),$$

where

$$\Gamma_{r,\Theta} = \left\{\gamma \in C\left([0,1], S_{r,\Theta}\right) : \gamma(0) = u^0, \gamma(1) = u^1\right\}.$$

Then

$$m_{r,s}(\Theta) \geq \frac{(N(q-2)-4)\left(1-\|V_-\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}} \left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}$$

and

$$m_{r,s}(\Theta) \leq \frac{N(q-2)-4}{2}\left(\frac{\theta\left(1+\|V\|_{\frac{N}{2}} S^{-1}\right)}{N(q-2)}\right)^{\frac{N(q-2)}{N(q-2)-4}} (4q)^{\frac{4}{N(q-2)-4}} |\Omega|^{\frac{2(q-2)}{N(q-2)-4}} \Theta^{\frac{N(q-2)-2q}{N(q-2)-4}}.$$

where θ is the principal eigenvalue of $-\Delta$ with Dirichlet boundary condition in Ω.

Proof. We only need to modify the proof of Lemma 5.1. There exists $0 < t_1 < t_\Theta$ such that for any $t \in [0, t_1]$,

$$h_2(t) < \frac{(N(q-2)-4)\left(1-\|V_-\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}} \left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}. \tag{9.1}$$

On the other hand, it follows from (3.5), the Gagliardo–Nirenberg inequality and the Hölder inequality that

$$J_{r,s}(u) = \left[\frac{1-\|V_-\|_{\frac{N}{2}} S^{-1}}{2} - \alpha\beta C_N \Theta^{\frac{2}{N}}\right] \|\nabla u\|_2^2 - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} \|\nabla u\|_2^{\frac{N(q-2)}{2}}. \tag{9.2}$$

Define

$$g_1(t) := \left[\frac{1-\|V_-\|_{\frac{N}{2}} S^{-1}}{2} - \alpha\beta C_N \Theta^{\frac{2}{N}}\right] t - \frac{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}{q} t^{\frac{N(q-2)}{4}}$$

and

$$t_g = \left[\frac{2q}{N(q-2)C_{N,q}}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}},$$

it is easy to see that g_1 is increasing on $(0, t_g)$ and decreasing on (t_g, ∞), and

$$g_1(t_g) = \frac{(N(q-2)-4)\left(1-\|V_-\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}} \left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}.$$

Set $\bar{r}_\Theta = \max\left\{\frac{1}{t_1}, \sqrt{\frac{2\theta\Theta}{t_g}}\right\}$, then $v_{\frac{1}{\bar{r}_\Theta}} \in S_{r,\Theta}$ for any $r > \bar{r}_\Theta$, and

$$\begin{aligned}\left\|\nabla v_{\frac{1}{\bar{r}_\Theta}}\right\|_2^2 &= \left(\frac{1}{\bar{r}_\Theta}\right)^2 \|\nabla v_1\|_2^2 \\ &< t_g = \left[\frac{2q}{N(q-2)C_{N,q}}\left(1-\|V_-\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right)\Theta^{\frac{q(N-2)-2N}{4}}\right]^{\frac{4}{N(q-2)-4}}.\end{aligned} \tag{9.3}$$

Then the statement (i) holds by (5.5), (9.1), (5.9), (9.3).

(ii) holds by (9.2) and a direct calculation.

(iii) In view of $J_{r,s}\left(u^1\right) \le 0$ for any $\gamma \in \Gamma_{r,\Theta}$ and the definition of t_0, we have

$$\|\nabla\gamma(0)\|_2^2 < t_g < \|\nabla\gamma(1)\|_2^2.$$

It then follows from (9.2) that

$$\begin{aligned}\max_{t\in[0,1]} J_{r,s}(\gamma(t)) &\ge g_1\left(t_g\right)\\ &= \frac{(N(q-2)-4)\left(1-\|V_-\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right)^{\frac{N(q-2)}{N(q-2)-4}}}{2[N(q-2)]^{\frac{N(q-2)}{N(q-2)-4}}} \left[\frac{2q}{C_{N,q}\Theta^{\frac{2q-N(q-2)}{4}}}\right]^{\frac{4}{N(q-2)-4}}\end{aligned}$$

for any $\gamma \in \Gamma_{r,\Theta}$, hence the first inequality in (iii) holds. We define a path $\gamma : [0,1] \to S_{r,\Theta}$ by

$$\gamma(t) : \Omega_r \to \mathbb{R}, \qquad x \mapsto \left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta}\right)^{\frac{N}{2}} v_1\left(\left(\tau t_0 + (1-\tau)\frac{1}{\widetilde{r}_\Theta}\right)x\right).$$

Then $\gamma \in \Gamma_{r,\Theta}$, and the second inequality in (iii) follows from (5.4). □

Lemma 9.2. *For fixed $\Theta > 0$ the set of solutions $u \in S_{r,\Theta}$ of (5.1) is bounded uniformly in s and r.*

Proof. We only need to modify the proof of Lemma 5.3. Using the Gagliardo–Nirenberg inequality, (3.5) and (iii) in Lemma 5.1, we have

$$\begin{aligned}\frac{q-2}{2} m_{r,s}(\Theta) \ge & \left[\frac{N(p_2-2)-4}{4N} - \frac{s\alpha\beta(q-p_2)}{2} C_N \Theta^{\frac{2}{N}}\right] \int_{\Omega_r} |\nabla u|^2 dx\\ & - \Theta\left(\frac{1}{2N}\|\nabla V \cdot x\|_\infty + \frac{p_2-2}{4}\|V\|_\infty\right).\end{aligned}$$

Since $0 < \Theta < \widetilde{\Theta}_V$, we can bound $\int_{\Omega_r} |\nabla u|^2 dx$ uniformly in s and r. □

Proof of Theorem 1.12. The proof is an immediate consequence of Lemmas 5.4 and 3.6. □

9.2 Proof of Theorem 1.13

Firstly, we modify the proof of Lemma 3.1. Using (3.3), (3.4), (3.5) and $\frac{1}{2} \le s \le 1$, it holds

$$\begin{aligned} I_{\frac{1}{t},s}(v_t) &\le \frac{1}{2}\left(1 + \|V\|_{\frac{N}{2}} S^{-1} - 2\alpha\beta C_N \Theta^{\frac{2}{N}}\right) t^2 \theta\Theta - \frac{1}{2q} t^{\frac{N(q-2)}{2}} \Theta^{\frac{q}{2}} \cdot |\Omega|^{\frac{2-q}{2}}\\ &=: h(t).\end{aligned}$$

Note that since $2 + \frac{4}{N} < q < 2^*$ and $\beta \le 0$, there exist $0 < T_\Theta < t_0$ such that $h(t_0) = 0, h(t) < 0$ for any $t > t_0, h(t) > 0$ for any $0 < t < t_0$ and $h(T_\Theta) = \max_{t\in\mathbb{R}^+} h(t)$.

Lemma 9.3. *Let $(\lambda_{r,\Theta}, u_{r,\Theta})$ be the solution of (1.1) from Lemma 3.5. If $\|\widetilde{V}_+\|_{\frac{N}{2}} < 2S$, then there exists $\bar{\Theta} > 0$ such that*

$$\liminf_{r\to\infty} \lambda_{r,\Theta} > 0 \quad \textit{for } 0 < \Theta < \bar{\Theta}.$$

Proof. We only need to modify the proof of Lemma 3.7. It follows from (3.5), (3.27), (3.28), (3.29), (f_2) and $2+\frac{4}{N}<q<2^*$ that

$$\begin{aligned}\left(\frac{1}{q}-\frac{1}{2}\right)\lambda_\Theta\int_{\mathbb{R}^N}u_\Theta^2dx &\le \left[\frac{(N-2)q-2N}{2Nq}-\frac{\beta(q-p_2)\alpha C_N}{q}\Theta^{\frac{2}{N}}\right]\int_{\mathbb{R}^N}|\nabla u_\Theta|^2\,dx\\ &\quad+\frac{\|\widetilde{V}\|_\infty}{2N}\Theta+\frac{(q-2)\|V\|_\infty}{2q}\Theta\\ &\to -\infty \quad \text{as } \Theta\to 0,\end{aligned}$$

since $0<\Theta<\widehat{\Theta}_V$. □

Proof of Theorem 1.13. The proof is an immediate consequence of Lemmas 3.5, 3.6 and 9.3. □

9.3 Proof of Theorem 1.14

Similarly, we only need to modify the proof of Theorem 1.4. Since $\beta>0$, it follows from the Gagliardo–Nirenberg inequality and the Hölder inequality that

$$\begin{aligned}I_r(u)&=\frac{1}{2}\int_{\Omega_r}|\nabla u|^2dx+\frac{1}{2}\int_{\Omega_r}V(x)u^2dx-\frac{N}{2N+4}\int_{\Omega_r}|u|^{2+\frac{4}{N}}dx-\beta\int_{\Omega_r}F(u)dx\\ &\ge\frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}-\frac{NC_N\Theta^{\frac{2}{N}}}{N+2}\right)\int_{\Omega_r}|\nabla u|^2dx\\ &\quad-\alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}\left(\int_{\Omega_r}|\nabla u|^2dx\right)^{\frac{N(p_1-2)}{4}}\\ &=h_1(t),\end{aligned}$$

where

$$h_1(t):=\frac{1}{2}\left(1-\|V_-\|_{\frac{N}{2}}S^{-1}-\frac{NC_N\Theta^{\frac{2}{N}}}{N+2}\right)t^2-\alpha\beta C_{N,p_1}\Theta^{\frac{2p_1-N(p_1-2)}{4}}t^{\frac{N(p_1-2)}{2}}.$$

In view of $2<p_1<2+\frac{4}{N}$, there exists $T_\Theta>0$ such that $h_1(t)<0$ for $0<t<T_\Theta$ and $h_1(t)>0$ for $t>T_\Theta$.

Proof of Theorem 1.14. The proof is a direct consequence of Lemma 4.1 and Lemma 3.6. □

10 Final comments

Some similar result (Theorems 1.3, 1.4, 1.5, but there are subtle changes in the assumptions) can be proved for the following class of problem

$$\begin{cases}-\Delta u+V(x)u+\lambda u=w(u)+\beta|u|^{p-2}u, & x\in\Omega,\\ \int_\Omega|u|^2dx=\Theta, u\in H_0^1(\Omega), & x\in\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is either all of $\mathbb{R}^N$ or a bounded smooth convex domain, $N\ge 3$, $2<p<2+\frac{4}{N}$, the mass $\Theta>0$ and the parameter $\beta\in\mathbb{R}$ are prescribed. Nonlinearity w satisfies:

(W_1) $w\in C^1(\mathbb{R},\mathbb{R})$ and w is odd.

(W_2) There exists some $(p_1,p_2)\in\mathbb{R}^2_+$ satisfying $2+\frac{4}{N}<p_2\le p_1<2^*$ such that

$$p_2W(\tau)\le w(\tau)\tau\le p_1W(\tau)\quad\text{with } W(\tau)=\int_0^\tau w(t)dt.$$

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Positive solutions for concave-convex type problems for the one-dimensional ϕ-Laplacian

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Abstract. Let $\Omega = (a,b) \subset \mathbb{R}$, $0 \leq m,n \in L^1(\Omega)$, $\lambda, \mu > 0$ be real parameters, and $\phi : \mathbb{R} \to \mathbb{R}$ be an odd increasing homeomorphism. In this paper we consider the existence of positive solutions for problems of the form

$$\begin{cases} -\phi\,(u')' = \lambda m(x) f(u) + \mu n(x) g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f, g : [0,\infty) \to [0,\infty)$ are continuous functions which are, roughly speaking, sublinear and superlinear with respect to ϕ, respectively. Our assumptions on ϕ, m and n are substantially weaker than the ones imposed in previous works. The approach used here combines the Guo–Krasnoselskiĭ fixed-point theorem and the sub-supersolutions method with some estimates on related nonlinear problems.

Keywords: elliptic one-dimensional problems, ϕ-Laplacian, positive solutions.

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1 Introduction

Let $\Omega = (a,b) \subset \mathbb{R}$, $m,n \in L^1(\Omega)$ and $\lambda, \mu > 0$ be a real parameters. In this article we consider problems of the form

$$\begin{cases} -\phi\,(u')' = \lambda m\,(x)\, f\,(u) + \mu n(x) g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism and $f, g : [0,\infty) \to [0,\infty)$ are continuous functions which are, roughly speaking, sublinear and superlinear with respect to ϕ, respectively. When the nonlinearities f and g are concave and convex, the problem (1.1) with

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$\phi(x) = x$ was first studied by Ambrosetti, Brezis and Cerami in their celebrated paper [1]. More precisely, in that article the authors studied the N-dimensional problem

$$\begin{cases} -\Delta u = \lambda u^q + u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

with $0 < q < 1 < p$ and Ω a bounded domain in $\mathbb{R}^N$. They proved the following facts: there exists $\Lambda > 0$ such that: if $\lambda \in (0, \Lambda)$ then (1.2) has at least two positive solutions, if $\lambda = \Lambda$ there is at least one positive solution, and if $\lambda > \Lambda$ then there are no positive solutions.

Several authors have studied generalizations of (1.2), see for instance [2, 11, 14] and their references, where the corresponding problem for the p-Laplacian is considered. Also, in [12] the authors have treated the N-dimensional problem for the ϕ-Laplacian operator.

Regarding the one-dimensional ϕ-Laplacian problem that we will deal with in this article, Wang in [15, Theorem 1.2] and [16, Theorem 1.2] studied the case $m = n \geq 0$, $m \not\equiv 0$ on any subinterval in Ω, $m \in C(\overline{\Omega})$ and $\lambda = \mu$. In these papers it is proved that there exist $\lambda_0, \lambda_1 > 0$ such that if $\lambda \in (0, \lambda_0)$, then (1.1) has at least two positive solutions; and if $\lambda > \lambda_1$, then there are no positive solutions. Let us note that the hypothesis on ϕ imposed in [15, 16] are much stronger than the ones that we shall require here. More precisely, Wang assumes

(Φ) There exist increasing homeomorphisms $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ such that $\psi_1(t)\,\phi(x) \leq \phi(tx) \leq \psi_2(t)\,\phi(x)$ for all $t, x > 0$.

On other hand, (1.1) is also considered in [8] with $m = n \geq 0$, $m \not\equiv 0$ on any subinterval in Ω and $\lambda = \mu$ like in [15, 16]. However, the regularity assumptions for m allow some $m \in L^1_{loc}(\Omega)$. Regarding the hypothesis on ϕ they require that

(Φ') There exist an increasing homeomorphism $\psi_1 : [0, \infty) \to [0, \infty)$ and a function $\psi_2 : [0, \infty) \to [0, \infty)$ such that $\psi_1(t)\,\phi(x) \leq \phi(tx) \leq \psi_2(t)\,\phi(x)$ for all $t, x > 0$.

The authors prove that there exist $\lambda_1 \geq \lambda_0 > 0$ such that (1.1) has at least two positive solutions for $\lambda \in (0, \lambda_0)$, one positive solution for $\lambda \in [\lambda_0, \lambda_1]$, and no positive solution for $\lambda > \lambda_1$.

In this article, employing the method of sub and supersolutions and the Guo–Krasnoselskiĭ fixed-point theorem along with some estimates for related problems, we shall prove that there are at least two positive solutions for $\lambda \approx 0$, under much weaker assumptions on ϕ, m and n. Moreover, as a consequence of Theorem 4.4 we shall see that (Φ) and (Φ') are in fact equivalent.

To be more precise, let us introduce the following hypothesis.

(F) There exist $c_0, t_0, q > 0$ such that

$$f(t) \geq c_0 t^q \text{ for all } t \in [0, t_0] \quad \text{and} \quad \lim_{t \to 0^+} \frac{t^q}{\phi(t)} = \infty. \tag{1.3}$$

(G1) There exist $c_1, t_1, r_1 > 0$ such that

$$g(t) \leq c_1 t^{r_1} \text{ for all } t \in [0, t_1] \quad \text{and} \quad \lim_{t \to 0^+} \frac{t^{r_1}}{\phi(t)} = 0. \tag{1.4}$$

(G2) There exist $c_2, t_2, r_2 > 0$ such that

$$g(t) \geq c_2 t^{r_2} \text{ for all } t \geq t_2 \quad \text{and} \quad \lim_{t\to\infty} \frac{t^{r_2}}{\phi(t)} = \infty. \tag{1.5}$$

Note that when $\phi(t) = |t|^{p-2}\,t$, $f(u) = u^q$ and $g(u) = u^r$, the limits in (F) and (G1) are satisfied if and only if $0 < q < p-1 < r$. Let us set $\mathcal{C}_0^1(\overline{\Omega}) := \{u \in \mathcal{C}^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ and

$$\mathcal{P}^\circ := \left\{ u \in \mathcal{C}_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega \text{ and } u'(b) < 0 < u'(a) \right\}.$$

Our main result is the following theorem:

Theorem 1.1. *Let $0 \leq m, n \in L^1(\Omega)$.*

(I) Assume that $m \not\equiv 0$ and (F) and (G1) hold. Then for all $\mu > 0$ there exists $\lambda_0(\mu) > 0$ such that (1.1) has a solution $u_\lambda \in \mathcal{P}^\circ$ for all $0 < \lambda < \lambda_0(\mu)$. Moreover, the solutions u_λ can be chosen such that

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0. \tag{1.6}$$

(II) Assume that $n \not\equiv 0$ and (G1) and (G2) hold. Then for all $\mu > 0$ there exists $\lambda_1(\mu) > 0$ such that (1.1) has a solution $v_\lambda \in \mathcal{P}^\circ$ for all $0 < \lambda < \lambda_1(\mu)$. Furthermore, there exists $\rho > 0$ such that $\|v_\lambda\|_\infty > \rho$ for all $0 < \lambda < \lambda_1(\mu)$.

(III) Assume that $\{\lambda > 0 : (1.1)$ has a solution in $\mathcal{P}^\circ\} \neq \emptyset$ and (F) holds for all $t_0 > 0$. Let

$$\Lambda := \sup\{\lambda > 0 : (1.1) \text{ has a solution in } \mathcal{P}^\circ\}.$$

Then, for $0 < \lambda < \Lambda$ (1.1) has at least one solution in $\mathcal{P}^\circ$.

As an immediate consequence of the above theorem we have the following

Corollary 1.2. *Let $\mu > 0$ and $0 \leq m, n \in L^1(\Omega)$ with $m, n \not\equiv 0$. Assume that (F), (G1) and (G2) hold. Then (1.1) has at least two solutions in $\mathcal{P}^\circ$ for $\lambda \approx 0$.*

The rest of the paper is organized as follows. In the next section we state some necessary facts about nonlinear problems involving the ϕ-Laplacian, and in Section 3 we prove our main results. Finally, in Section 4 we introduce some concepts about Orlicz spaces indices which we use to prove Theorem 4.4 (and, in particular, the equivalence of (Φ) and (Φ')), and at the end of the section we give several examples of functions ϕ illustrating our conditions and their relations with the ones used in the previous works. Let us mention that all the ϕ's constructed in Example (e) satisfy conditions (F), (G1) and (G2) but do not fulfill condition (Φ).

2 Preliminaries

Let $\phi : \mathbb{R} \to \mathbb{R}$ be an odd increasing homeomorphism. We start considering problems of the form

$$\begin{cases} -\phi(v')' = h(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is well known that for all $h \in L^1(\Omega)$, (2.1) possesses a unique solution $v \in \mathcal{C}_0^1(\overline{\Omega})$ such that $\phi(v')$ is absolutely continuous and that the equation holds pointwise *a.e.* $x \in \Omega$. Furthermore, the solution operator $\mathcal{S}_\phi \colon L^1(\Omega) \to \mathcal{C}^1(\overline{\Omega})$ is completely continuous and nondecreasing, see [3, Lemma 2.1] and [6, Lemma 2.2].

We need now to introduce some notation. For $0 \le h \in L^1(\Omega)$ with $h \not\equiv 0$, set

$$\mathcal{A}_h := \{x \in \Omega : h(y) = 0 \text{ a.e. } y \in (a,x)\},$$
$$\mathcal{B}_h := \{x \in \Omega : h(y) = 0 \text{ a.e. } y \in (x,b)\},$$

and

$$\begin{aligned} \alpha_h &:= \begin{cases} \sup \mathcal{A}_h & \text{if } \mathcal{A}_h \neq \emptyset, \\ a & \text{if } \mathcal{A}_h = \emptyset, \end{cases} \qquad \beta_h := \begin{cases} \inf \mathcal{B}_h & \text{if } \mathcal{B}_h \neq \emptyset, \\ b & \text{if } \mathcal{B}_h = \emptyset, \end{cases} \\ \underline{\theta}_h &:= \min\left\{\frac{1}{\beta_h - a}, \frac{1}{b - \alpha_h}\right\}, \qquad \overline{\theta}_h := \frac{\alpha_h + \beta_h}{2}. \end{aligned} \tag{2.2}$$

We observe that $\underline{\theta}_h$ is well defined because $h \not\equiv 0$, and $\alpha_h < \beta_h$ (and so, $\overline{\theta}_h \in (\alpha_h, \beta_h)$). We also write

$$\delta_\Omega(x) := \operatorname{dist}(x, \partial\Omega) = \min(x - a, b - x).$$

We shall utilize the following estimates on several occasions in the sequel. For the proof, see [6, Lemma 2.3 and (2.6)] and [7, Corollary 2.2].

Lemma 2.1. *Let $0 \le h \in L^1(\Omega)$ with $h \not\equiv 0$.*

(i) In $\overline{\Omega}$ it holds that

$$\underline{\theta}_h \min\left\{\int_a^{\overline{\theta}_h} \phi^{-1}\left(\int_y^{\overline{\theta}_h} h\right) dy, \int_{\overline{\theta}_h}^b \phi^{-1}\left(\int_{\overline{\theta}_h}^y h\right) dy\right\} \delta_\Omega \le \mathcal{S}_\phi(h) \le \phi^{-1}\left(\int_a^b h\right) \delta_\Omega. \tag{2.3}$$

(ii) In $\overline{\Omega}$ it holds that

$$\mathcal{S}_\phi(h) \ge \underline{\theta}_h \left\|\mathcal{S}_\phi(h)\right\|_\infty \delta_\Omega. \tag{2.4}$$

(iii) For $M > 0$ there exists $c > 0$ not depending on M such that it holds that

$$\min\left\{\int_a^{\overline{\theta}_h} \phi^{-1}\left(\int_y^{\overline{\theta}_h} Mh\right) dy, \int_{\overline{\theta}_h}^b \phi^{-1}\left(\int_{\overline{\theta}_h}^y Mh\right) dy\right\} \ge c\phi^{-1}(cM). \tag{2.5}$$

Observe that, since $\overline{\theta}_h \in (\alpha_h, \beta_h)$, the constant that appears in the first term of the inequalities in (2.3) is strictly positive. Note also that, since $\underline{\theta}_h \|\delta_\Omega\|_\infty \ge 1/2$, using the lower bound of (2.3) and taking into account the monotonicity of the infinite norm we get

$$\frac{1}{2}\min\left\{\int_a^{\overline{\theta}_h} \phi^{-1}\left(\int_y^{\overline{\theta}_h} h\right) dy, \int_{\overline{\theta}_h}^b \phi^{-1}\left(\int_{\overline{\theta}_h}^y h\right) dy\right\} \le \left\|\mathcal{S}_\phi(h)\right\|_\infty. \tag{2.6}$$

Observe also that for h as in Lemma 2.1 $\mathcal{S}_\phi(h) \in \mathcal{P}^\circ$.

Let $h : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function (that is, $h(x,\cdot)$ is continuous for *a.e.* $x \in \Omega$ and $h(\cdot,\xi)$ is measurable for all $\xi \in \mathbb{R}$). We now consider problems of the form

$$\begin{cases} -\phi(u')' = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

We shall say that $v \in C(\overline{\Omega})$ is a *subsolution* of (2.7) if there exists a finite set $\Sigma \subset \Omega$ such that $\phi(v') \in AC_{loc}(\overline{\Omega}\setminus\Sigma)$, $v'(\tau^+) := \lim_{x\to\tau^+} v'(x) \in \mathbb{R}$, $v'(\tau^-) := \lim_{x\to\tau^-} v'(x) \in \mathbb{R}$ for each $\tau \in \Sigma$, and

$$\begin{cases} -\phi\left(v'\right)' \leq h\left(x, v\left(x\right)\right) & a.e.\ x \in \Omega, \\ v \leq 0 \text{ on } \partial\Omega, & v'(\tau^-) < v'(\tau^+) \text{ for each } \tau \in \Sigma. \end{cases} \tag{2.8}$$

If the inequalities in (2.8) are inverted, we shall say that v is a *supersolution* of (2.7).

For the sake of completeness, we state an existence result in the presence of well-ordered sub and supersolutions, and a particular case of the well-known Guo–Krasnoselskiĭ fixed-point theorem (for a proof, see e.g. [13, Theorem 7.16] and [4, Theorem 2.3.4], respectively).

Lemma 2.2. *Let v and w be sub and supersolutions respectively of (2.7) such that $v \leq w$ in Ω. Suppose there exists $g \in L^1(\Omega)$ such that*

$$|h\left(x, \xi\right)| \leq g\left(x\right) \quad \textit{for a.e. } x \in \Omega \textit{ and all } \xi \in \left[v\left(x\right), w\left(x\right)\right].$$

Then there exists $u \in C_0^1(\overline{\Omega})$ solution of (2.7) with $v \leq u \leq w$ in Ω.

Lemma 2.3. *Let X be a Banach space and let K be a cone in X. Let $\Omega_1, \Omega_2 \subset X$ be two open sets with $0 \in \Omega_1$ and $\Omega_1 \subset \Omega_2$. Suppose that $T : K \cap (\Omega_2 \setminus \Omega_1) \to K$ is a completely continuous operator and*

$$\begin{aligned} \|Tv\| &\geq \|v\|, \quad \textit{for } v \in K \cap \partial\Omega_2, \\ \|Tv\| &\leq \|v\|, \quad \textit{for } v \in K \cap \partial\Omega_1. \end{aligned}$$

Then, T has a fixed point in $K \cap (\Omega_2 \setminus \Omega_1)$.

3 Proof of the main results

3.1 Proof of item (I)

We start this section with two lemmas concerning sub and supersolutions that shall be used to prove item (I) of Theorem 1.1.

Lemma 3.1. *Let $m, n \in L^1(\Omega)$ such that $0 \not\equiv m + n \geq 0$. Assume that (G1) holds. Then for all $\mu > 0$ there exists $\lambda_0(\mu) > 0$ such that for each $0 < \lambda < \lambda_0(\mu)$ there exists $w_\lambda \in \mathcal{P}^\circ$ supersolution of (1.1). Moreover,*

$$\lim_{\lambda\to 0^+} \|w_\lambda\|_\infty = 0. \tag{3.1}$$

Proof. Let c_1, t_1, r_1 be given by (G1). Let us define $c_\Omega := \max_{\overline{\Omega}} \delta_\Omega$. By the continuity of ϕ^{-1} and the fact that $\phi^{-1}(0) = 0$, there exists $K_0 > 0$ such that

$$\phi^{-1}\left(\kappa \int_a^b m(s) + n(s) ds\right) \leq \frac{t_1}{c_\Omega} \quad \text{for all } \kappa \leq K_0. \tag{3.2}$$

We observe that by the second condition on (1.4), for $\rho > 0$ fixed we have

$$\lim_{t\to 0^+} \frac{[\phi^{-1}(\rho t)]^{r_1}}{t} = 0. \tag{3.3}$$

We now define

$$\epsilon := \frac{1}{c_1 \mu c_\Omega^{r_1}}, \qquad \rho := \int_a^b m(s) + n(s) ds.$$

We can deduce from (3.3) that there exists $K_1 = K_1(\epsilon, \rho) > 0$ such that

$$[\phi^{-1}(\kappa\rho)]^{r_1} \leq \kappa\epsilon \quad \text{for all } \kappa \leq K_1. \tag{3.4}$$

Let $C = \max_{[0,t_1]} f(t)$ and choose $\lambda_0 > 0$ such that

$$\lambda_0 C \leq \min\{K_0, K_1\}. \tag{3.5}$$

Also, for each $0 < \lambda < \lambda_0$, pick κ_λ such that

$$\lambda C \leq \kappa_\lambda \leq \min\{K_0, K_1\}, \tag{3.6}$$

and for such κ_λ define $w_\lambda := \mathcal{S}_\phi(\kappa_\lambda(m+n))$. Since $\kappa_\lambda \leq K_0$, the upper bound in (2.3) and (3.2) tell us that $\|w_\lambda\|_\infty \leq t_1$. Taking into account (3.4), (3.5) and (3.6), employing (G1) and the upper bound in (2.3) we deduce that

$$\begin{aligned}\lambda m(x)f(w_\lambda) + \mu n(x)g(w_\lambda) &\leq \lambda m(x)C + c_1\mu n(x)w_\lambda^{r_1} \\ &\leq \kappa_\lambda m(x) + c_1\mu n(x)\left[\phi^{-1}(\kappa_\lambda \int_a^b m(s)+n(s)ds)\delta_\Omega\right]^{r_1} \\ &\leq \kappa_\lambda(m(x)+n(x)) = -\phi(w_\lambda')' \quad \text{in } \Omega,\end{aligned}$$

and hence w_λ is a supersolution of (1.1).

In order to prove (3.1), we choose κ_λ satisfying (3.6) and such that $\kappa_\lambda \to 0$ when $\lambda \to 0^+$. Hence, using the second inequality (2.3) we get that

$$0 \leq w_\lambda(x) = \mathcal{S}_\phi(\kappa_\lambda(m+n)) \leq \phi^{-1}\left(\int_a^b \kappa_\lambda(m+n)\right)\delta_\Omega(x) \to 0$$

uniformly in $\overline{\Omega}$ when $\lambda \to 0^+$. Thus, $\lim_{\lambda\to 0^+} \|w_\lambda\|_\infty = 0$. □

Lemma 3.2. *Let $0 \leq m, n \in L^1(\Omega)$ with $m \not\equiv 0$. Assume that (F) holds. Then for all $\lambda, \mu > 0$ (1.1) has a subsolution $v \in \mathcal{P}^\circ$.*

Proof. Let $\lambda, \mu > 0$ and let c_0, t_0, q be given by (F). Recall that $c_\Omega := \max_{\overline{\Omega}} \delta_\Omega$. Since ϕ^{-1} is continuous and $\phi^{-1}(0) = 0$, there exists $\varepsilon_0 > 0$ such that

$$\phi^{-1}\left(\varepsilon \int_a^b m(s)\delta_\Omega^q(s)ds\right) \leq \frac{t_0}{c_\Omega} \quad \text{for all } \varepsilon \leq \varepsilon_0. \tag{3.7}$$

By the second condition in (1.3), for $\rho > 0$ fixed

$$\lim_{t\to 0^+} \frac{[\phi^{-1}(\rho t)]^q}{t} = \infty. \tag{3.8}$$

Let us define

$$M := \frac{1}{\lambda c_0 c^q},$$

where c is the constant in (2.5) with $h = m\delta_\Omega^q$. It follows from (3.8) that there exists $\varepsilon_1 = \varepsilon_1(M, \rho)$ such that

$$[\phi^{-1}(\varepsilon\rho)]^q \geq M\varepsilon \quad \text{for all } \varepsilon \leq \varepsilon_1. \tag{3.9}$$

Let us choose

$$0 < \varepsilon < \min\{\varepsilon_0, \varepsilon_1\} \tag{3.10}$$

and for such ε define $v := \mathcal{S}_\phi(\varepsilon m\delta_\Omega^q)$. Since $\varepsilon \le \varepsilon_0$, the upper bound of Lemma 2.1 and (3.7) tell us that $\|v\|_\infty \le t_0$. Consequently, taking into account (3.9) and (3.10), employing (F) and (2.5) we deduce that

$$\lambda m(x)f(v) + \mu n(x)g(v) \ge \lambda c_0 m(x)v^q \ge \lambda c_0 m(x)[c\phi^{-1}(c\varepsilon)\delta_\Omega]^q \ge \varepsilon m(x)\delta_\Omega^q \quad \text{in } \Omega.$$

In other words, v is a subsolution of (1.1). □

Proof of Theorem 1.1 (I). Given $\mu > 0$, let $\lambda_0(\mu)$ be as in Lemma 3.1. For $0 < \lambda < \lambda_0(\mu)$, let $w_\lambda \in \mathcal{P}^\circ$ be a supersolution provided by the aforementioned lemma, and let $v_\lambda \in \mathcal{P}^\circ$ be a subsolution given by Lemma 3.2 with ε_λ chosen such that $\varepsilon_\lambda m(x)\delta_\Omega^q(x) \le \kappa_\lambda(m(x)+n(x))$ for $a.e. x \in \Omega$. It follows that v_λ, w_λ are a pair of well-ordered sub and supersolutions of (1.1). Hence, Lemma 2.2 gives a solution of (1.1) $u_\lambda \in \mathcal{P}^\circ$. Moreover, (1.6) follows from (3.1). □

3.2 Proof of item (II)

Proof of Theorem 1.1 (II). We shall use Lemma 2.3 with the operator

$$Tv := \mathcal{S}_\phi(\lambda m(x)f(v) + \mu n(x)g(v)),$$

the cone

$$\mathcal{K} := \{v \in C(\overline{\Omega}) : v \ge \underline{\theta}_n \|v\|_\infty \delta_\Omega\}$$

($\underline{\theta}_n$ as in (2.2)) and the open balls $B_R(0), B_\rho(0) \subset C(\overline{\Omega})$ with $0 < \rho < R$. Observe that $C_0^1(\overline{\Omega}) \cap (\mathcal{K} \setminus \{0\}) \subset \mathcal{P}^\circ$ and that any fixed point of T belongs to $C_0^1(\overline{\Omega})$.

Let c_2, t_2 and r_2 be given by (G2). We consider the function $h := c_2\mu\,(\underline{\theta}_n)^{r_2}\, n\delta_\Omega^{r_2}$. Taking into account (2.5), we can find $c = c(\mu) > 0$ such that for all $M > 0$

$$\min\left\{\int_a^{\overline{\theta}_n} \phi^{-1}(M\int_y^{\overline{\theta}_n} h)dy, \int_{\overline{\theta}_n}^b \phi^{-1}(M\int_{\overline{\theta}_n}^y h)dy\right\} \ge c\phi^{-1}(cM). \tag{3.11}$$

On other hand, the second condition in (G2) is equivalent to

$$\lim_{t\to\infty} \frac{\phi^{-1}(\rho t^{r_2})}{t} = \infty$$

for all fixed $\rho > 0$, and then there exists $\overline{t} > 0$ such that

$$\phi^{-1}(ct^{q_2}) \ge \frac{2t}{c} \quad \text{for all } t \ge \overline{t}. \tag{3.12}$$

Let us fix $R > \max\{t_2, \overline{t}\}$. Taking into account that $\mathcal{S}_\phi$ and ϕ^{-1} are nondecreasing, the inequality (2.6), (G2), (3.11) and (3.12) we obtain that for $v \in \mathcal{K} \cap \partial B_R(0)$,

$$\begin{aligned}
\|Tv\|_\infty &= \|\mathcal{S}_\phi(\lambda m(x)f(v)) + \mu n(x)g(v))\| \ge \|S_\phi(\mu n(x)g(v))\|_\infty \\
&\ge \frac{1}{2}\min\left\{\int_a^{\overline{\theta}_n} \phi^{-1}\left(\int_y^{\overline{\theta}_n} \mu n g(v)\right)dy, \int_{\overline{\theta}_n}^b \phi^{-1}\left(\int_{\overline{\theta}_n}^y \mu n g(v)\right)dy\right\} \\
&\ge \frac{1}{2}\min\left\{\int_a^{\overline{\theta}_n} \phi^{-1}\left(c_2\mu\int_y^{\overline{\theta}_n} n v^{r_2}\right)dy, \int_{\overline{\theta}_n}^b \phi^{-1}\left(c_2\mu\int_{\overline{\theta}_n}^y n v^{r_2}\right)dy\right\} \\
&\ge \frac{1}{2}\min\left\{\int_a^{\overline{\theta}_n} \phi^{-1}\left(c_2\mu\,(\underline{\theta}_n\|v\|_\infty)^{r_2}\int_y^{\overline{\theta}_n} n\delta_\Omega^{r_2}\right)dy, \int_{\overline{\theta}_n}^b \phi^{-1}\left(c_2\mu\,(\underline{\theta}_n\|v\|_\infty)^{r_2}\int_{\overline{\theta}_n}^y n\delta_\Omega^{r_2}\right)dy\right\} \\
&\ge \frac{1}{2}c\phi^{-1}(c\|v\|_\infty^{r_2}) \\
&\ge \|v\|_\infty.
\end{aligned}$$

That is, $\|Tv\|_\infty \geq \|v\|_\infty$ for such v.

On other side, let $N := c_1 \int_a^b n$. The second condition in (G1) implies that there exists $\underline{t} > 0$ such that $\phi(t/c_\Omega) > \mu N t^{r_1}$ for all $t \in (0, \underline{t})$. Set $C := \max_{[0,R]} f(t)$ and $M := \int_a^b m$. Let $0 < \rho < \min\{\underline{t}, R/2, t_1\}$ be fixed and define

$$\lambda_1 := \frac{\phi(\rho/c_\Omega) - \mu N \rho^{r_1}}{MC}. \tag{3.13}$$

Note that $\lambda_1 > 0$ by our election of $\underline{t}$.

Now, taking into account (2.3), (G1), (3.13) and the monotonicity of ϕ^{-1} we see for $0 < \lambda \leq \lambda_1$ and all $v \in \mathcal{K} \cap \partial B_\rho(0)$,

$$\begin{aligned} Tv &\leq \phi^{-1}\left(\int_a^b \lambda m(x) f(v) + \mu n(x) g(v) dx\right) \delta_\Omega \\ &\leq \phi^{-1}\left(\lambda C \int_a^b m(x)dx + c_1 \mu \int_a^b n(x) v^{r_1} dx\right) \delta_\Omega \\ &\leq \phi^{-1}\left(\lambda_1 MC + \mu N \rho^{r_1}\right) \delta_\Omega \\ &\leq \rho \text{ in } \Omega. \end{aligned}$$

This tells us that $\|Tv\|_\infty \leq \rho = \|v\|_\infty$ for all $v \in \mathcal{K} \cap \partial B_\rho(0)$.

Thus, Lemma 2.3 says that T has a fixed point in $\mathcal{K} \cap (\overline{B_R(0)} \setminus B_\rho(0))$. □

3.3 Proof of item (III)

Proof of Theorem 1.1 (III). In order to prove (III) we combine Lemma 3.2 and the inequality (2.4). Let $0 < \lambda < \Lambda$. By the definition of Λ there exists $\overline{\lambda} \in (\lambda, \Lambda]$ and $u_{\overline{\lambda}} \in \mathcal{P}^\circ$ solution of (1.1) associated to $\overline{\lambda}$. Since $\lambda < \overline{\lambda}$ it follows that $u_{\overline{\lambda}}$ is a supersolution (1.1) associated to λ. Now, thanks to Lemma 3.2 there exists $\varepsilon > 0$ such that $v = \mathcal{S}_\phi(\varepsilon m \delta_\Omega^q)$ is a subsolution of (1.1) associated to λ. Moreover, taking ε smaller if necessary, we get that $v \leq u_{\overline{\lambda}}$. Now, (III) follows from Lemma 2.2. □

4 Comments about the hypothesis

Let us introduce some concepts about Orlicz spaces indices. Given a nonbounded, increasing, continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$, we define

$$M(t, \phi) := \sup_{x>0} \frac{\phi(tx)}{\phi(x)}.$$

This function is nondecreasing and submultiplicative with $M(1, \phi) = 1$. Then, thanks to e.g. [9, Chapter 11], the following limits exist:

$$\alpha_\phi := \lim_{t \to 0^+} \frac{\ln M(t, \phi)}{\ln t}, \qquad \beta_\phi := \lim_{t \to \infty} \frac{\ln M(t, \phi)}{\ln t},$$

and moreover, $0 \leq \alpha_\phi \leq \beta_\phi \leq \infty$. These numbers are called **Orlicz space indices** or **Matuszewska–Orlicz's indices**, who introduced them in [10].

As usual, we say that ϕ satisfies the Δ_2 condition if there exists $k > 0$ such that

$$\phi(2x) \leq k\phi(x) \quad \text{for all } x \geq 0.$$

Remark 4.1.

(i) For $\varepsilon > 0$, there exists $t_1 > 0$ such that $\phi(tx) \leq t^{\alpha_\phi - \varepsilon}\phi(x)$ for all $x > 0$ and $t \in [0, t_1]$.

(ii) Suppose that $\beta_\phi < \infty$. Then, for $\varepsilon > 0$, there exists $t_2 > 0$ such that $\phi(tx) \leq t^{\beta_\phi + \varepsilon}\phi(x)$ for all $x > 0$ and $t \in [t_2, \infty)$. So, if $\beta_\phi < \infty$ then ϕ satisfies the Δ_2 condition.

(iii) If $x^{-p}\phi(x)$ is nondecreasing for all $x > 0$, then $\alpha_\phi \geq p$.

(iv) If $x^{-p}\phi(x)$ is nonincreasing for all $x > 0$, then $\beta_\phi \leq p$.

(v) The following relationships between the Orlicz space indices of ϕ and ϕ^{-1} hold:

$$\beta_\phi = \frac{1}{\alpha_{\phi^{-1}}} \quad \text{and} \quad \alpha_\phi = \frac{1}{\beta_{\phi^{-1}}}.$$

As usual, we set $1/0 = \infty$ and $1/\infty = 0$.

We shall need the next two useful lemmas to prove Theorem 4.4 below.

Lemma 4.2 ([5, page 34])**.** *If $0 < \alpha_\phi \leq \beta_\phi < \infty$ then there exist $C, p, q > 0$ such that*

$$C^{-1} \min\{t^p, t^q\}\phi(x) \leq \phi(tx) \leq C \max\{t^p, t^q\}\phi(x) \quad \textit{for all } t, x \geq 0.$$

Lemma 4.3 ([9, Theorem 11.7])**.** *The function ϕ satisfies the Δ_2 condition if and only if the constant β_ϕ is finite.*

Theorem 4.4. *The following hypothesis for ϕ are equivalent:*

(i) $0 < \alpha_\phi \leq \beta_\phi < \infty$.

(ii) (Φ).

(iii) (Φ').

Proof. It is obvious that (ii) implies (iii), and Lemma 4.2 shows that (i) implies (ii). Let us prove that (iii) implies (i).

Since $\alpha_\phi = 1/\beta_{\phi^{-1}}$, Lemma 4.3 and Remark 4.1 (v) tell us that $\alpha_\phi > 0$ if and only if ϕ^{-1} satisfies Δ_2. Let us check that the first inequality in (Φ') implies that ϕ^{-1} satisfies Δ_2. Indeed, taking into account that

$$\psi_1(t)\phi(x) \leq \phi(xt) \quad \text{for all } t, x > 0,$$

setting $y = \phi(x)$ and $s = \psi(t)$ we get that

$$sy \leq \phi(\psi_1^{-1}(s)\phi^{-1}(y)) \quad \text{for all } s, y > 0.$$

Since ϕ^{-1} is increasing its follows that

$$\phi^{-1}(sy) \leq \psi_1^{-1}(s)\phi^{-1}(y) \quad \text{for all } s, y > 0.$$

This implies that ϕ^{-1} satisfies Δ_2. Thus, $\alpha_\phi > 0$. Moreover, the second inequality in (Φ') implies that ϕ satisfies Δ_2. Then, $\beta_\phi < \infty$. □

The following two lemmas will be useful to compare the indices α_ϕ and β_ϕ with our hypotheses (F), (G1) and (G2) stated in Section 1.

Lemma 4.5. *Let $q > 0$.*

(i) If $\lim_{t\to 0^+} \dfrac{t^q}{\phi(t)} = 0$ *then* $\alpha_\phi \le q$.

(ii) If $\lim_{t\to \infty} \dfrac{t^q}{\phi(t)} = 0$ *then* $\beta_\phi \ge q$.

(iii) If $\lim_{t\to 0^+} \dfrac{t^q}{\phi(t)} = \infty$ *then* $\beta_\phi \ge q$.

(iv) If $\lim_{t\to \infty} \dfrac{t^q}{\phi(t)} = \infty$ *then* $\alpha_\phi \le q$.

Proof. We start proving (i). If $\alpha_\phi > q$, by Remark 4.1 (i) there exists $t_1 > 0$ such that

$$\phi(tx) \le t^q \phi(x) \quad \text{for all } x > 0 \text{ and } t \in (0, t_1).$$

Let us set $C = \phi(1)^{-1}$ and fix $x = 1$. Using the above inequality we have that $C \le \frac{t^q}{\phi(t)}$ for all $t \in (0, t_1)$, which contradicts that $\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = 0$. Therefore, we must have $\alpha_\phi \le q$. Item (ii) follows similarly. Indeed, if $\beta_\phi < q$, by Remark 4.1 (ii) we have that there exists $t_1 > 0$ such that

$$\phi(tx) \le t^q \phi(x) \quad \text{for all } x > 0 \text{ and } t > t_1.$$

We now again define $C = \phi(1)^{-1}$ and fix $x = 1$. Employing the above inequality we have that $C \le \frac{t^q}{\phi(t)}$ for all $t > t_1$, contradicting that $\lim_{t\to\infty} \frac{t^q}{\phi(t)} = 0$. Thus, $\beta_\phi \ge q$.

We prove (iii). We notice first that

$$\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = \infty \quad \text{if and only if} \quad \lim_{t\to 0^+} \frac{t^{1/q}}{\phi^{-1}(t)} = 0. \tag{4.1}$$

Indeed, the first limit is true if for every sequence $\{t_k\}$ with $0 < t_k \to 0$, it holds that $\frac{t_k^q}{\phi(t_k)} \to \infty$. Thus, taking $s_k = \phi(t_k)$ we have that $0 < s_k \to 0$ and $\frac{[\phi^{-1}(s_k)]^q}{s_k} \to \infty$. Since $h(t) = t^{1/q}$ is continuous and converges to ∞ as $t \to \infty$, it follows that $\frac{\phi^{-1}(s_k)}{s_k^{1/q}} \to \infty$, which is equivalent to $\frac{s_k^{1/q}}{\phi^{-1}(s_k)} \to 0$. Since $0 \le \frac{t^{1/q}}{\phi^{-1}(t)}$ for all $t > 0$ it follows that $\lim_{t\to 0^+} \frac{t^{1/q}}{\phi^{-1}(t)} = 0$. Now, from (4.1) and item (i) we deduce that $\alpha_{\phi^{-1}} \le 1/q$, and recalling Remark 4.1 (v) we get that $\beta_\phi \ge q$, and (iii) holds. Analogously, (iv) follows from (ii), taking into account that

$$\lim_{t\to \infty} \frac{t^q}{\phi(t)} = \infty \quad \text{if and only if} \quad \lim_{t\to \infty} \frac{t^{1/q}}{\phi^{-1}(t)} = 0,$$

and using again Remark 4.1 (v). □

Lemma 4.6. *Let $\phi : [0, \infty) \to [0, \infty)$ be a nonbounded, increasing, continuous function with $\phi(0) = 0$.*

(i) If $q < \alpha_\phi$ *then* $\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = \infty$.

(ii) If $q > \beta_\phi$ *then* $\lim_{t\to \infty} \frac{t^q}{\phi(t)} = \infty$.

(iii) If $q < \alpha_\phi$ *then* $\lim_{t\to \infty} \frac{t^q}{\phi(t)} = 0$.

(iv) If $q > \beta_\phi$ then $\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = 0$.

Let us note that the reciprocals of items (i) and (ii) of the above lemma *are not true*, see Example (e.1) below.

Proof. Let us begin by proving (i). Let $\varepsilon > 0$ such that $\alpha_\phi - \varepsilon > q$. By Remark 4.1 (i) there exists $t_1 > 0$ such that $\phi(tx) \le t^{\alpha_\phi - \varepsilon}\phi(x)$ for all $x > 0$ and $t < t_1$. Taking $x = 1$ we get that $\frac{1}{t^{\alpha_\phi-\varepsilon}} \le \frac{\phi(1)}{\phi(t)}$ for $t < t_1$. Multiplying by t^q on both sides and taking limit as $t \to 0^+$ it follows that

$$\lim_{t\to 0^+} \frac{t^q}{t^{\alpha_\phi-\varepsilon}} \le \lim_{t\to 0^+} \frac{\phi(1)t^q}{\phi(t)}.$$

Since $q < \alpha_\phi - \varepsilon$, the first limit is infinite, and so also the second one. Thus, (i) is proved.

Analogously, let $\varepsilon > 0$ such that $\beta_\phi + \varepsilon < q$. By Remark 4.1 (ii) there exists $t_1 > 0$ such that $\phi(tx) \le t^{\beta_\phi-\varepsilon}\phi(x)$ for all $x > 0$ and $t > t_1$. Taking $x = 1$ we have $\frac{1}{t^{\beta_\phi-\varepsilon}} \le \frac{\phi(1)}{\phi(t)}$ for $t < t_1$. Multiplying by t^q on both sides and taking limit as $t \to \infty$ we get

$$\lim_{t\to\infty} \frac{t^q}{t^{\beta_\phi+\varepsilon}} \le \lim_{t\to\infty} \frac{\phi(1)t^q}{\phi(t)}.$$

Since $q > \beta_\phi + \varepsilon$, the first limit is infinite, and thus also the second one.

On other hand, (iii) follows from (ii) noting that

$$\lim_{t\to\infty} \frac{t^q}{\phi(t)} = 0 \quad \text{if and only if} \quad \lim_{t\to\infty} \frac{t^{1/q}}{\phi^{-1}(t)} = \infty,$$

and taking into account that $\alpha_\phi > q$ if and only if $\beta_{\phi^{-1}} < 1/q$. Similarly, (iv) follows from (i) noting that

$$\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = 0 \quad \text{if and only if} \quad \lim_{t\to 0^+} \frac{t^{1/q}}{\phi^{-1}(t)} = \infty,$$

and recalling that $\beta_\phi < q$ if and only if $\alpha_{\phi^{-1}} > 1/q$. □

Corollary 4.7. *Let q, r_1 and r_2 be given by (F), (G1) and (G2) respectively.*

1. *Suppose that α_ϕ is positive.*

   (a) If $q < \alpha_\phi$ then the limit in (F) holds.

2. *Suppose that β_ϕ is finite.*

   (a) If $r_1 > \beta_\phi$ then the limit in (G1) holds.
   (b) If $r_2 > \beta_\phi$ then the limit in (G2) holds.

4.1 Examples

Let us conclude the article with some examples of functions ϕ. We suppose $x \ge 0$ and we extend the function oddly.

a. Let

$$\phi(x) = x^{p_1} + x^{p_2}, \quad \text{with } p_1 \ge p_2 > 0.$$

Since $\phi(x)/x^{p_1}$ is nonincreasing and $\phi(x)/x^{p_2}$ is nondecreasing, we see that $\beta_\phi < \infty$ and $\alpha_\phi > 0$.

b. Let
$$\phi(x) = \frac{x^{p_1}}{1+x^{p_2}}, \quad \text{with } p_1 > p_2 > 0.$$
Since $\phi(x)/x^{p_1}$ is nonincreasing and $\phi(x)/x^{p_1-p_2}$ is nondecreasing, we get that $\beta_\phi < \infty$ and $\alpha_\phi > 0$.

c. Let
$$\phi(x) = x\left(|\ln x| + 1\right).$$
We have that $\phi(x)/x^2$ is nonincreasing. Then, $\beta_\phi < \infty$. Furthermore, given $p \in (0,1)$ there exists $T > 0$ such that
$$\phi(tx) \le t^p \phi(x) \quad \text{for } t \in [0,T] \text{ and all } x \ge 0.$$
This inequality implies that $\alpha_\phi \ge 1$.

d. Let
$$\phi(x) := x - \ln(x+1).$$
As in the above example, $\phi(x)/x^2$ is nonincreasing and then $\beta_\phi < \infty$. Also, there exist $C, T > 0$ such that
$$\phi(tx) \le \; Ct\phi(x) \quad \text{for } t \in [0,T] \text{ and all } x \ge 0.$$
The above inequality implies that $\alpha_\phi \ge 1$. Moreover, since
$$\lim_{t\to\infty} \frac{t^q}{\phi(t)} = \infty \quad \text{for all } q > 1,$$
thanks to Lemma 4.5 (iv) we deduce that $\alpha_\phi = 1$.

e. Let $h : (0,\infty) \to (1,\infty)$ be an increasing differentiable function such that $\lim_{t\to 0^+} h(t) = 1$,
$$\lim_{t\to\infty} \frac{qt^{q-1}h(t)}{h'(t)} = \infty \quad \text{for all } q > 0, \tag{4.2}$$
and there exists $p_1 > 0$ such that
$$\lim_{t\to 0^+} \frac{qt^{q-1}h(t)}{h'(t)} = \begin{cases} 0 & \text{if } q > p_1, \\ \infty & \text{if } q < p_1. \end{cases} \tag{4.3}$$
Define
$$\phi(x) := (\ln(h(x)))^p, \quad \text{with } p > 0.$$
By (4.2), ϕ satisfies the limit in (G2). Moreover, from Lemma 4.5 (iv) we can deduce that $\alpha_\phi = 0$. Then ϕ *does not satisfy* the hypothesis (Φ) (and (Φ')) at the introduction. And since (4.3) holds it follows that
$$\lim_{t\to 0^+} \frac{t^q}{\phi(t)} = \begin{cases} 0 & \text{if } q > pp_1. \\ \infty & \text{if } q < pp_1. \end{cases}$$
Therefore, ϕ satisfies the limits in (F) and (G1). Let us exhibit next a few particular cases.

e.1 Let
$$\phi(x) := (\ln(x+1))^p, \quad \text{with } p > 0.$$
A few computations show that $h(x) = x+1$ satisfies (4.2) and (4.3). Moreover, we can see that $\phi(x)/x^p$ is nonincreasing and thus $\beta_\phi \leq p$, and since
$$\lim_{t\to 0^+} \frac{t^q}{\ln(t+1)} = \infty \quad \text{for all } q < 1,$$
by Lemma 4.5 it follows that $\beta_\phi = p$. This shows that the reciprocals of the items (i) and (ii) in Lemma 4.6 *are not true*.

e.2 Let
$$\phi(x) := \text{arcsinh}(x) = \ln\left(\sqrt{x^2+1} + x\right).$$
One can see that $h(x) = \sqrt{x^2+1} + x$ satisfies (4.2) and (4.3).

e.3 Let
$$\phi(x) := \ln(\ln(x+1) + 1).$$
One can verify that $h(x) = \ln(x+1) + 1$ satisfies (4.2) and (4.3).

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Analysis of stochastic SEIR(S) models with random total populations and variable diffusion rates

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Abstract. A stochastic SEIR(S) model with random total population, overall saturation constant $K > 0$ and general, local Lipschitz continuous diffusion rates is presented. We prove the existence of unique, Markovian, continuous time solutions w.r.t. filtered, complete probability spaces on certain, bounded 4D prisms. The total population $N(t)$ is governed by kind of stochastic logistic equations, which allows to have an asymptotically stable maximum population constant $K > 0$. Under natural conditions on our SEIR(S) model, we establish asymptotic stochastic and moment stability of the disease-free and endemic equilibria. Those conditions naturally depend on the basic reproduction number $\mathcal{R}_0$, the growth parameter $\mu > 0$ and environmental noise intensity σ_5^2 coupled with the maximum threshold K^2 of total population $N(t)$. For the mathematical proofs, the technique of appropriate Lyapunov functionals $V(S(t), E(t), I(t), R(t))$ is exploited. Some numerical simulations of the expected Lyapunov functionals $\mathbb{E}[V(S, E, I, R)]$ depending on several parameters and time t support our findings.

Keywords: stochastic SEIR(S) model, stochastic differential equations, random transition functions, variable diffusion rates, Lyapunov functionals, asymptotic stability.

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1 Introduction to stochastic SEIR(S) model based on SDEs

Research on epidemic modeling has gone quite far since the seminal contributions of Kermack and McKendrick [15]. The random, erratic nature of evolution of populations forces us to incorporate stochastic terms in modeling and analysis. For modeling of diseases with

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sensitive, randomly fluctuating transmissions such as COVID, here we suggest to make use of and analyze the stochastic SEIR(S) models based on Itô-interpreted SDEs

$$\begin{aligned} dS &= \Big(-\beta SI + \mu(K-S) + \alpha I + \zeta R \Big) dt - \sigma_1 SI \cdot F_1\big(S,E,I,R\big) dW_1 \\ &\quad + \sigma_4 R \cdot F_4(S,E,I,R) dW_4 + \sigma_5 S(K-N) dW_5 \\ dE &= \Big(\beta SI - (\mu+\eta)E \Big) dt + \sigma_1 SI \cdot F_1\big(S,E,I,R\big) dW_1 - \sigma_2 E \cdot F_2\big(S,E,I,R\big) dW_2 \\ &\quad + \sigma_5 E(K-N) dW_5 \\ dI &= \Big(\eta E - (\alpha+\gamma+\mu) I \Big) dt + \sigma_2 E \cdot F_2\big(S,E,I,R\big) dW_2 - \sigma_3 I \cdot F_3\big(S,E,I,R\big) dW_3 \\ &\quad + \sigma_5 I(K-N) dW_5 \\ dR &= \Big(\gamma I - (\mu+\zeta) R \Big) dt + \sigma_3 I \cdot F_3\big(S,E,I,R\big) dW_3 - \sigma_4 R \cdot F_4\big(S,E,I,R\big) dW_4 \\ &\quad + \sigma_5 R(K-N) dW_5, \end{aligned} \tag{1.1}$$

driven by independent, standard Wiener processes $W_k = (W_k(t))_{t \geq 0}$ on a complete, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and the total initial population (note that they can be supposed to be nonrandom since start values are known from real-time data)

$$0 < N(0) := S(0) + E(0) + I(0) + R(0) < K$$

with nonrandom constant $K > 0$ of maximum possible threshold for total population. These models (1.1) are stochastic generalizations of deterministic counterparts in mathematical epidemiology (cf. [21, 22]). For an introduction to mathematical models in population biology and epidemiology, see the textbooks [2, 5, 6]. In biological modeling Itô calculus has to be used since the dynamics of offsprings can only depend on its past, parental generations. For an overview on the theory of Itô-interpreted stochastic differential equations (SDEs), see [1,3,10,11,14,23,24,33] for stochastic calculus with Wiener processes. Deterministic model variants of SEIR(S), SI, SIR, SIS, etc. are well-understood nowadays. The construction and analysis of dynamics along Lyapunov functions plays a key role in understanding those models, cf. [7, 9, 13, 17–19]. This is also the case with stochastic settings, cf. [12, 30–32, 34, 36]. Extinction, ergodicity, stability and recurrence of some random SEIR(S) models with constant or absent F_k are studied in [35, 37–39], restricted to unbounded cones $\mathbb{R}^d_+$. Our models (1.1) allow all solutions to live exclusively a.s. on bounded prisms of $\mathbb{R}^4$ or $\mathbb{R}^5$, resp., which represents a real requirement for biologically relevant application (due to finite resources in real life of organisms).

To the best of our knowledge, the class of SEIR(S) models (1.1) is fairly new to the literature. Our model focuses on the possible sensitivity of diseases to random transitions between compartments S of susceptible, E of exposed, I of infected, and R of recovering sub-populations, which are controlled by noise intensity functions $\sigma_k F_k$ in a fairly general manner. Those random transitions can be interpreted as random perturbations of the incidence terms βSI (i.e. **direct contact terms**), motivated by the CLT (=Central Limit Theorem, cf. Shiryaev [33]). Moreover, we allow a possible return of a share of the recovered sub-populations R to the susceptible ones as an expression for the possible loss of immunity w.r.t. the modelled disease-type, represented by the parameter ζ, and a possible switch of the infected sub-populations I to the susceptible ones, represented by the parameter α. The parameter γ stands for the rate of transitions from the infected to the recovering sub-populations. Our main focus in this paper is to verify several qualitative properties such as the boundedness and stability of all dynamics of (1.1) on certain positive prisms - properties which relate to biologically relevant models in order to to be able to replicate real scenarios.

We shall show that the new SEIR(S) model (1.1) is well-defined ($\mathbb{P}$-a.s.) on 4D prism

$$\mathbb{D} = \Big\{(S,E,I,R) \in \mathbb{R}^4_+ : \ 0 < S,E,I,R < K, \ S+E+I+R < K\Big\}.$$

For this purpose, first we shall analyze the total population $N(t)$ of the SEIR(S) model (1.1) at time t, which is defined by

$$N(t) = S(t) + E(t) + I(t) + R(t).$$

The understanding of their dynamics plays a crucial role in establishing qualitative properties of the solutions of SDEs (1.1). By summing up all equations of SEIR(S) model (1.1), the SDE for the total population $N = S + E + I + R$ is found to be of the form

$$dN(t) = \mu(K-N)\,dt + \sigma_5 N(K-N)\,dW_5 \tag{1.2}$$

on natural domain $\mathbb{D}_0 = \big(0,K\big)$, which can be treated in a separated fashion from the original system (1.1).

The paper is organized as follows. Section 2 studies the boundedness of the total population N and proves the existence of strong solutions of SDE (1.2) on open domain $\mathbb{D}_0 = (0,K)$ for all times $t \geq 0$. Section 3 is devoted to establish the existence of unique, Markovian, continuous time, strong solutions of the SEIR(S) model (1.1) on certain, positive 4D prisms. There we present the two types of equilibrium solutions, namely the disease-free and the endemic ones. Section 4 investigates stochastic stability of the disease-free equilibrium and the endemic equilibrium of SEIR(S) model (1.1). As usual, the associated basic reproduction number decides in which stable state the system is in (in the long-term sense). Moreover, we also discuss the moment and stochastic stability of its saturation equilibrium $n^* = K$ for the SDE (1.2) of the total populations. Finally, Section 5 is reporting on some graphical illustrations of simulation results related to the associated mean Lyapunov functionals depending on diverse parameters. Section 6 concludes the paper with a brief summary and outlook. An appendix recalls a general standard result on the existence of bounded, unique solutions of systems of Itô SDEs and the structure of associated infinitesimal generator, which plays a key role in our studies.

2 Existence of bounded, unique solution of (1.2) on $\mathbb{D}_0 = (0,K)$

The proof of existence of global, unique solutions of nonlinear SDE (1.2) is far from trivial, due to the quadratic nonlinearity in its diffusion term. For the sake of abbreviation, take $\sigma = \sigma_5$. Let $N(t_0) = N_0 \in \mathbb{D}_0$ with $\mathbb{D}_0 = (0,K)$ and $\mathbb{D}_r = \left(\frac{1}{r}, K - \frac{1}{r}\right), r > 1/K$. Now, consider the events $[N(t) = n]$. Define

$$n \in \mathbb{D}_0 \quad \mapsto \quad V(n) := c - \ln\Big(n(K-n)\Big) = c - \ln(n) - \ln(K-n).$$

Choose c sufficiently large such that $V \geq 0$ on $\mathbb{D}_0$. e.g. $c = \ln\left(\frac{K^2}{4}\right)$. The infinitesimal generator $\mathcal{L}$ of SDE (1.2) applied to the function V (see the general formula (A.2) in appendix) takes the form

$$\forall n \in \mathbb{D}_0: \quad \mathcal{L}V(n) = \mu(K-n)\left(\frac{-1}{n} + \frac{1}{K-n}\right) + \frac{1}{2}\sigma^2 n^2 (K-n)^2 \left[\frac{1}{n^2} + \frac{1}{(K-n)^2}\right]$$
$$= -\mu\left(\frac{K-n}{n}\right) + \mu + \frac{1}{2}\sigma^2(K-n)^2 + \frac{1}{2}\sigma^2 n^2.$$
$$\Longrightarrow \quad \mathcal{L}V(n) \overset{\mu>0}{\leq} \mu + \frac{1}{2}\sigma^2(K-n)^2 + \frac{1}{2}\sigma^2 n^2 = \mu + \frac{1}{2}\sigma^2\Big[(K-n)^2 + n^2\Big]$$
$$< \mu + \frac{1}{2}\sigma^2 K^2 =: c_0 \quad \text{since} \quad g(n) := (K-n)^2 + n^2 < K^2 \quad \text{on } \mathbb{D}_0.$$

Hence, by Dynkin's formula (1965) (cf. Dynkin [8]), we arrive at

$$\forall t \geq 0: \quad \mathbb{E}\Big[V\Big(N(t)\Big)\Big] = \mathbb{E}\Big[V\Big(N(0)\Big)\Big] + \mathbb{E}\Big[\int_0^t \mathcal{L}V(s)\,ds\Big] \leq \mathbb{E}\Big[V\Big(N(0)\Big)\Big] + c_0 \cdot t \; < \; +\infty$$

with constant $c_0 = \mu + \sigma^2 K^2/2$. Obviously, we have

$$\lim_{r\to+\infty} \inf_{t\geq 0, n\in\partial\mathbb{D}_r} V(n) = \lim_{r\to+\infty} \min\Big(V\Big(\frac{1}{r}\Big), V\Big(K - \frac{1}{r}\Big)\Big) = c - \lim_{r\to+\infty} \ln\Big(\frac{1}{r}\Big(K - \frac{1}{r}\Big)\Big) = +\infty.$$

By the remark below Theorem A.1, there exists exactly one strong, global, continuous time, unique Markovian solution $N = (N(t))_{t\geq 0}$ of SDE (1.2) with $N(t) \in \mathbb{D}_0 = (0, K)$ (a.s.) for all $t \geq 0$. This gives the positivity of N (a.s.) and boundedness $N(t) < K$ (a.s.). Of course, the equilibrium $n^* = K$ represents a solution itself (i.e. the trivial solution). Consequently, we verified the following theorem.

Theorem 2.1 (Solvability and boundedness of total population SDE (1.2))**.** *Assume that either* $N(0) = K$ *or* $N(0) \in (0, K)$ *(a.s.) is independent of sigma-algebra* $\sigma(W) = \sigma(W(t) : t \geq 0)$ *with*

$$\mathbb{E}\Big[\ln\Big(N(0)(K - N(0))\Big)\Big] \; < \; +\infty.$$

Then, *there is a unique, strong solution process* $N = (N(t))_{t\geq 0}$ *satisfying SDE* (1.2) *and* $\forall$ *nonrandom* $0 < T < +\infty$ $\forall 0 < N(0) < K$

$$\sup_{0\leq t\leq T} \mathbb{E}\Big[\ln\Big(N(t)(K - N(t))\Big)\Big] \leq \mathbb{E}\Big[\ln\Big(N(0)(K - N(0))\Big)\Big] + \Big(\mu + \frac{\sigma^2}{2}K^2\Big) \cdot T < +\infty.$$

3 Existence of bounded, unique solution of (1.1) on 4D prism $\mathbb{D}$

The following theorem establishes the existence of strong, unique solutions of SEIR(S) models (1.1) bounded to stay on certain positive prisms (a.s.).

Theorem 3.1 (Existence theorem of unique solutions of SEIR(S) model on prisms)**.** *Let* $\big(S(t_0), E(t_0), I(t_0), R(t_0)\big) = (S_0, E_0, I_0, R_0) \in \mathbb{D}$ *with*

$$\mathbb{D} = \Big\{(S, E, I, R) \in \mathbb{R}_+^4 : 0 < S, E, I, R < K, \; S + E + I + R < K\Big\}.$$

Consider the stochastic SEIR(S) model with random, nonconstant total populations

$$\begin{aligned} dS &= \big(-\beta SI + \mu(K-S) + \alpha I + \zeta R\big)dt - \sigma_1 SI \cdot F_1(S,E,I,R)dW_1 \\ &\quad + \sigma_4 R \cdot F_4(S,E,I,R)dW_4 + \sigma_5 S(K-N)dW_5 \\ dE &= \big(\beta SI - (\mu+\eta)E\big)dt + \sigma_1 SI \cdot F_1(S,E,I,R)dW_1 - \sigma_2 E \cdot F_2(S,E,I,R)dW_2 \\ &\quad + \sigma_5 E(K-N)dW_5 \\ dI &= \big(\eta E - (\alpha+\gamma+\mu)I\big)dt + \sigma_2 E \cdot F_2(S,E,I,R)dW_2 - \sigma_3 I \cdot F_3(S,E,I,R)dW_3 \\ &\quad + \sigma_5 I(K-N)dW_5 \\ dR &= \big(\gamma I - (\mu+\zeta)R\big)dt + \sigma_3 I \cdot F_3(S,E,I,R)dW_3 - \sigma_4 R \cdot F_4(S,E,I,R)dW_4 \\ &\quad + \sigma_5 R(K-N)dW_5 \\ dN &= \mu(K-N)\,dt + \sigma_5 N(K-N)\,dW_5. \end{aligned} \tag{3.1}$$

Assume that all constants $\alpha, \beta, \eta, \gamma, \zeta, \mu \geq 0$ *and*

(i) $(S_0,\ E_0,\ I_0,\ R_0) \in \mathbb{D}$ *is independent of* $\sigma\big(W_k : 1 \leq k \leq 5\big)$,

(ii) $\forall k = 1,\ 2,\ 3,\ 4,\ 5$: $F_k \in C^0_{locLip}(\mathbb{D})$ *(i.e. local Lipschitz continuous on interior* $\mathbb{D}$*)* $\bigcap C^0(\overline{\mathbb{D}})$,

(iii) $\mathbb{E}[V(S_0,E_0,I_0,R_0)] < +\infty$ *with*

$$V(S,E,I,R) = \begin{cases} R - \ln(R) + I - \ln(I) + E - \ln(E) + S - \ln(S), \\ +K - S - E - I - R - \ln(K-S-E-I-R), \end{cases}$$

(iv) $\sup_{(S,E,I,R)\in\mathbb{D}} \frac{S^2 I^2 F_1^2(S,E,I,R)}{E^2} + \sup_{(S,E,I,R)\in\mathbb{D}} \frac{E^2 F_2^2(S,E,I,R)}{I^2} + \sup_{(S,E,I,R)\in\mathbb{D}} \frac{I^2 F_3^2(S,E,I,R)}{R^2} +$ $+ \sup_{(S,E,I,R)\in\mathbb{D}} \frac{R^2 F_4^2(S,E,I,R)}{S^2} < +\infty$.

Then, *the stochastic SEIR model* (1.1) *with random total population size* $N(t)$ *admits*

(1) a unique, continuous time, Markovian, global strong solution $\big(S(t),\ E(t),\ I(t),\ R(t)\big)$ *on* $t \geq t_0$,

(2) an a.s. $\mathbb{D}$*-invariant solution (i.e. a.s. uniform boundedness of solutions on positive cone of* $\mathbb{R}^4$*)*,

(3) a uniform estimate of moments ($\forall T < +\infty$ *nonrandom)*

$$\sup_{0\leq t\leq T} \mathbb{E}[V(S(t),E(t),S(t),R(t))] \leq \mathbb{E}[V(S_0,E_0,S_0,R_0)] + [\beta K + 4\mu + \eta + \alpha + \gamma + \zeta + c_1] \cdot T,$$

where c_1 *is an appropriate constant (one may extract that from proof below).*

Proof. Define

$$\mathbb{D}_n := \Big\{ (S,E,I,R,N) \in \mathbb{R}^5_+ : e^{-n} < S,E,I,R < K - e^{-n},\ N = S+E+I+R < K(1-e^{-n}) \Big\}$$

for $n \in \mathbb{N}$. Then, due to its local Lipschitz continuous drift and diffusion coefficients, system (3.1) has a unique solution up to stopping time $\tau(\mathbb{D}_n)$ hitting the boundary of open sets $\mathbb{D}_n$ (see [3,11,16]). Furthermore, define

$$\begin{aligned} V(S,E,I,R,N) &= \begin{cases} R - \ln(R) + I - \ln(I) + E - \ln(E) + S - \ln(S) \\ +K - N - \ln(K-N) \end{cases} \\ &= V_1(S,E,I,R) + V_2(N) \end{aligned}$$

where $V_1(S,E,I,R) = R - \ln(R) + I - \ln(I) + E - \ln(E) + S - \ln(S)$,

$$V_2(N) = \widetilde{V}_2(S,E,I,R) = K - S - E - I - R - \ln(K-S-E-I-R) \ = \ K - N - \ln(K-N)$$

on $\widetilde{\mathbb{D}} = \big\{ (S,E,I,R,N) \in \mathbb{R}^5_+ : 0 < S,E,I,R < K, N = S+E+I+R < K \big\}$.

Suppose that $\mathbb{E}\big[V(S_0, E_0, I_0, R_0, N_0)\big] < +\infty$. Note that $V(S,E,I,R,N) \geq 5$ for $(S,E,I,R,N) \in \widetilde{\mathbb{D}}$ (this fact will be used below to estimate $\mathcal{L}V$). Now, calculate the infinitesimal generator $\mathcal{L}V$ applied to our SEIR(S) model 3.1 using the general formula (A.2) as stated in appendix). One encounters with the 2nd differential operator

$$\begin{aligned}
\mathcal{L}V(S,E,I,R,N) &= (-\beta SI + \mu(K-S) + \alpha I + \zeta R)\,\frac{\partial V}{\partial S} + (\beta SI - (\mu+\eta)E)\,\frac{\partial V}{\partial E} \\
&+ (\eta E - (\alpha+\gamma+\mu)I)\,\frac{\partial V}{\partial I} + (\gamma I - (\mu+\zeta)R)\,\frac{\partial V}{\partial R} + \mu(K-N)\frac{\partial V}{\partial N} \\
&+ \frac{\sigma_1^2}{2} S^2 I^2 [F_1(S,E,I,R)]^2 \left(\frac{\partial^2 V}{\partial S^2} - 2\frac{\partial^2 V}{\partial S \partial E} + \frac{\partial^2 V}{\partial E^2}\right) \\
&+ \frac{\sigma_2^2}{2} E^2 [F_2(S,E,I,R)]^2 \left(\frac{\partial^2 V}{\partial E^2} - 2\frac{\partial^2 V}{\partial E \partial I} + \frac{\partial^2 V}{\partial I^2}\right) \\
&+ \frac{\sigma_3^2}{2} I^2 [F_3(S,E,I,R)]^2 \left(\frac{\partial^2 V}{\partial I^2} - 2\frac{\partial^2 V}{\partial I \partial R} + \frac{\partial^2 V}{\partial R^2}\right) \\
&+ \frac{\sigma_4^2}{2} R^2 [F_4(S,E,I,R)]^2 \left(\frac{\partial^2 V}{\partial R^2} - 2\frac{\partial^2 V}{\partial R \partial S} + \frac{\partial^2 V}{\partial S^2}\right) \\
&+ \frac{\sigma_5^2}{2} [K-N]^2 \left(S^2\frac{\partial^2 V}{\partial S^2} + 2ES\frac{\partial^2 V}{\partial S \partial E} + E^2\frac{\partial^2 V}{\partial E^2}\right) \\
&+ \sigma_5^2 [K-N]^2 \left(SI\frac{\partial^2 V}{\partial S \partial I} + SR\frac{\partial^2 V}{\partial S \partial R} + SN\frac{\partial^2 V}{\partial S \partial N}\right) \\
&+ \sigma_5^2 [K-N]^2 \left(EI\frac{\partial^2 V}{\partial E \partial I} + ER\frac{\partial^2 V}{\partial E \partial R} + EN\frac{\partial^2 V}{\partial E \partial N} + IR\frac{\partial^2 V}{\partial I \partial R} + IN\frac{\partial^2 V}{\partial I \partial N} + RN\frac{\partial^2 V}{\partial R \partial N}\right) \\
&+ \frac{\sigma_5^2}{2} [K-N]^2 \left(I^2\frac{\partial^2 V}{\partial I^2} + R^2\frac{\partial^2 V}{\partial R^2} + N^2\frac{\partial^2 V}{\partial N^2}\right)
\end{aligned} \tag{3.2}$$

for any twice continuously differentiable function $V \in C^2(\widetilde{\mathbb{D}})$. Next, an application $\mathcal{L}V$ to our specific functional V yields that

$$\begin{aligned}
\mathcal{L}V(S,E,I,R,N) &= \mathcal{L}V_1(S,E,I,R) + \mathcal{L}V_2(N), \\
\mathcal{L}V_1(S,E,I,R) &= \mu(K-S-E-I-R) + \beta I - \frac{1}{S}\Big(\mu(K-S) + \alpha I + \zeta R\Big) - \frac{1}{E}\,\beta SI \\
&\quad + \mu + \eta - \frac{1}{I}\,\eta E + \alpha + \gamma + \mu - \frac{1}{R}\,\gamma I + \mu + \zeta + \frac{\sigma_1^2}{2}\, I^2\, F_1^2(S,E,I,R) \\
&\quad + \frac{\sigma_4^2}{2}\,\frac{R^2\, F_4^2(S,E,I,R)}{S^2} + \frac{\sigma_1^2}{2}\,\frac{S^2\, I^2\, F_1^2(S,E,I,R)}{E^2} + \frac{\sigma_2^2}{2}\, F_2^2(S,E,I,R) \\
&\quad + \frac{\sigma_2^2}{2}\,\frac{E^2\, F_2^2(S,E,I,R)}{I^2} + \frac{\sigma_3^2}{2}\, F_3^2(S,E,I,R) \\
&\quad + \frac{\sigma_3^2}{2}\,\frac{I^2\, F_3^2(S,E,I,R)}{R^2} + \frac{\sigma_4^2}{2}\, F_4^2(S,E,I,R) + 2\sigma_5^2(K-N)^2 \\
&\leq \mu(K-S-E-I-R) + \beta I + 3\mu + \alpha + \gamma + \eta + \zeta + 2\sigma_5^2(K-N)^2 \\
&\quad + \frac{\sigma_1^2}{2} \sup_{(S,E,I,R)\in\mathbb{D}} \frac{S^2 I^2 F_1^2(S,E,I,R)}{E^2} + \frac{\sigma_1^2}{2} \max_{(S,E,I,R)\in\mathbb{D}} I^2 F_1^2(S,E,I,R) \\
&\quad + \frac{\sigma_4^2}{2} \sup_{(S,E,I,R)\in\mathbb{D}} \frac{R^2 F_4^2(S,E,I,R)}{S^2} + \frac{\sigma_4^2}{2} \max_{(S,E,I,R)\in\mathbb{D}} F_4^2(S,E,I,R)
\end{aligned}$$

$$+\frac{\sigma_3^2}{2}\sup_{(S,E,I,R)\in\mathbb{D}}\frac{I^2F_3^2(S,E,I,R)}{R^2}+\frac{\sigma_3^2}{2}\max_{(S,E,I,R)\in\mathbb{D}}F_3^2(S,E,I,R)$$
$$+\frac{\sigma_2^2}{2}\sup_{(S,E,I,R)\in\mathbb{D}}\frac{E^2F_2^2(S,E,I,R)}{I^2}+\frac{\sigma_2^2}{2}\max_{(S,E,I,R)\in\mathbb{D}}F_2^2(S,E,I,R)<+\infty$$

since Theorem 3.1 (iv). Similarly, we find

$$\mathcal{L}V_2(N)=\mathcal{L}\widetilde{V}_2(S,\,E,\,I,\,R)=-\mu(K-N)+\mu+\frac{\sigma_5^2}{2}N^2.$$

Note that we may estimate $\sigma_5^2[4(K-N)^2+N^2]\leq 4\sigma_5^2K^2/5$ on $0<N<K$. Thus, we have

$$\mathcal{L}V(S,E,I,R,N)\;\leq\;\beta I+4\mu+\eta+\alpha+\gamma+\zeta+c_1\;\leq\;\beta K+4\mu+\eta+\alpha+\gamma+\zeta+c_1$$

on $\widetilde{\mathbb{D}}$, where

$$c_1=\begin{cases}\frac{2\sigma_5^2}{5}K^2+\frac{\sigma_1^2}{2}\left[\sup_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}\frac{S^2I^2F_1^2(S,E,I,R)}{E^2}+\max_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}I^2F_1^2(S,E,I,R)\right]\\ +\frac{\sigma_4^2}{2}\left[\sup_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}\frac{R^2F_4^2(S,E,I,R)}{S^2}+\max_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}F_4^2(S,E,I,R)\right]\\ +\frac{\sigma_3^2}{2}\left[\sup_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}\frac{I^2F_3^2(S,E,I,R)}{R^2}+\max_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}F_3^2(S,E,I,R)\right]\\ +\frac{\sigma_2^2}{2}\left[\sup_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}\frac{E^2F_2^2(S,E,I,R)}{I^2}+\max_{(S,E,I,R,N)\in\widetilde{\mathbb{D}}}F_2^2(S,E,I,R)\right].\end{cases} \tag{3.3}$$

Note that c_1 is finite due to hypotheses (ii) and (iv). Next, let $\tau_n(t):=\min(\tau(\mathbb{D}_n),t)$ where $\tau(\mathbb{D}_n)$ is the stopping time of the first exit from the domain $\mathbb{D}_n$. An application of Dynkin's formula [8] (1965) provides us the estimate

$$\begin{aligned}&\mathbb{E}[V(S(t),E(t),I(t),R(t),N(t))]\\ &\quad=\mathbb{E}[V(S_0,E_0,I_0,R_0,N_0)]+\mathbb{E}\left[\int_0^{\tau_n(t)}\mathcal{L}V(S(s),E(s),I(s),R(s),N(s))\,ds\right]\\ &\quad\leq\mathbb{E}[V(S_0,E_0,I_0,R_0,N_0)]+[\beta K+4\mu+\eta+\alpha+\gamma+\zeta+c_1]\cdot\mathbb{E}[\tau_n(t)]\\ &\quad\leq\mathbb{E}[V(S_0,E_0,I_0,R_0,N_0)]+[\beta K+4\mu+\eta+\alpha+\gamma+\zeta+c_1]\cdot t\quad\text{since } \tau_n(t)\leq t,\end{aligned}$$

for all nonrandom times $t>t_0$, as long as the solution $\big(S(s),E(s),I(s),R(s),N(s)\big)$ on $\widetilde{\mathbb{D}}$. Note that $\forall n\in\mathbb{N}:n>0$ and $n>\ln(K)/5$

$$\inf_{(S,E,I,R,N)\,\in\,\partial\,\mathbb{D}_n}V(S,E,I,R,N)>5n-\ln(K). \tag{3.4}$$

Recall that we have defined the stopping time $\tau_n(t):=\min\{t,\tau(\mathbb{D}_n)\}$ based on the stopping time $\tau(\mathbb{D}_n)$ arriving the first time at the boundary of $\mathbb{D}_n$. Now, apply the above estimate to get to

$$
\begin{aligned}
0 \;&\leq\; \mathbb{P}\Big(\Big[\tau(\widetilde{\mathbb{D}}) < t\Big]\Big) \overset{\mathbb{D}_n \subseteq \widetilde{\mathbb{D}}}{\leq} \mathbb{P}\Big(\Big[\tau(\mathbb{D}_n) < t\Big]\Big) = \mathbb{P}\Big(\Big[\tau_n(t) < t\Big]\Big) \\
&= \mathbb{E}\big[\mathbf{1}_{\tau_n(t)<t}\big] \quad \text{where } \mathbf{1} \text{ is the indicator function} \\
&\leq \mathbb{E}\left[\frac{V\Big(S(\tau(\mathbb{D}_n)), E(\tau(\mathbb{D}_n)), I(\tau(\mathbb{D}_n)), R(\tau(\mathbb{D}_n)), N(\tau(\mathbb{D}_n))\Big)}{\inf_{(S,E,I,R,N)\in\partial\mathbb{D}_n} V(S,E,I,R,N)} \cdot \mathbf{1}_{\tau_n<t}\right] \\
&\overset{(3)}{\leq} \frac{\mathbb{E}[V(S_0,E_0,I_0,R_0.N_0)] + \mathbb{E}[\tau_n(t)]\cdot[\beta K + 4\mu + \eta + \alpha + \gamma + \zeta + c_1]}{\inf_{(x,y,z,v,w)\in\partial\mathbb{D}_n} V(x,y,z,v,w)} \\
&\overset{(3.4)}{\leq} \frac{\mathbb{E}[V(S_0,E_0,I_0,R_0,N_0)] + t[\beta K + 4\mu + \eta + \alpha + \gamma + \zeta + c_1]}{5n - \ln(K)} \longrightarrow 0 \quad \text{as } n \to \infty
\end{aligned}
$$

for all $(S_0, E_0, I_0, R_0, N_0) \in \mathbb{D}_n$, and for all fixed, nonrandom $t \in [s, \infty)$. Thus

$$
\Longrightarrow \quad \mathbb{P}\Big(\Big[\tau(\widetilde{\mathbb{D}}) < t\Big]\Big) = \lim_{n\to+\infty} \mathbb{P}\Big(\Big[\tau(\mathbb{D}_n) < t\Big]\Big) = 0
$$

for all adapted $(S_0, E_0, I_0, R_0, N_0) \in \widetilde{\mathbb{D}}$ and all $t \geq t_0$. That means that

$$
\mathbb{P}\Big(\Big[\tau(\widetilde{\mathbb{D}}) = +\infty\Big]\Big) = 1.
$$

This proves the invariance property and global existence of solutions $\big(S(t), I(t), R(t), N(t)\big)$ on $\widetilde{\mathbb{D}}$ for any finite time t. Thus, the proof of Theorem (3.1) is complete. □

4 Asymptotic moment and stochastic stability, stability exponents

Let $p > 0$ be a real constant. Consider the d-dimensional, autonomous, Itô-interpreted SDEs

$$
dX(t) = a\big(X(t)\big)dt + b\big(X(t)\big)\,dW(t). \tag{4.1}
$$

Definition 4.1. SDE (4.1) has a **globally asymptotically p-th moment stable** equilibrium (solution) $X = x^*$ if and only if $a(x^*) = b(x^*) = 0$ and $\forall\, X(s) \in L^p(\Omega, \mathcal{F}_s, \mathbb{P})$, $s \geq 0$, $X(s) \neq x^*$ we have

$$
\lim_{t\to+\infty} \mathbb{E}\Big[\|X_{s,X(s)}(t) - x^*\|_d^p\Big] = 0
$$

(where d is the state-space dimension of the stochastic process X).

Definition 4.2. The equilibrium solution x^* of SDE (4.1) is **stochastically stable** (stable in probability) iff, for every $\varepsilon > 0$ and $s \geq t_0$, we have

$$
\lim_{x_0\to x^*} \mathbb{P}\left(\left[\sup_{t_0\leq s<\infty} \|X_{s,x_0}(t) - x^*\| \geq \varepsilon\right]\right) = 0 \tag{4.2}
$$

where $X_{s,x_0}(t)$ denotes the solution of SDE (4.1) satisfying $X(s) = x_0$ at time $t \geq s$.

Definition 4.3. The equilibrium solution x^* of SDE (4.1) is said to be **(locally) asymptotically stochastically stable** iff it is stochastically stable and

$$
\forall x_0 \in N(x^*) : \mathbb{P}\left(\left[\lim_{t\to\infty} X_{s,x_0}(t) = x^*\right]\right) = 1. \tag{4.3}
$$

Definition 4.4. The equilibrium solution x^* of SDE (4.1) is said to be **globally asymptotically stochastically stable** iff it is stochastically stable and, for every x_0 and every s, we have

$$\mathbb{P}\left(\left[\lim_{t\to\infty} X_{s,x_0}(t) = x^*\right]\right) = 1. \tag{4.4}$$

Theorem 4.5 (Stability theorem of Arnold [3])**.** *Assume that the SDE (4.1) has a unique solution started at every nonrandom x_0 in the nonrandom, a.s. invariant, open neighborhood $N(x^*) \subseteq \mathbb{R}^d$.* **Then**, *the equilibrium solution $x^* \in \overline{N(x^*)} \subseteq \mathbb{R}^d$ for SDE (4.1) is stochastically stable if $\exists$ positive definite*

$$V = V(t,x) \in C^{1,2}\Big([t_0,\infty) \times N(x^*), \mathbb{R}^1_+\Big)$$

on $N(x^*)$ such that $\forall\,(t,x) \in [t_0,\infty) \times N(x^*)$:

$$\mathcal{L}V(t,x) \le 0.$$

If additionally V is decrescent on $N(x^*)$ and

$$\forall\,(t,x) \in [t_0,\infty) \times N(x^*) \setminus \{x^*\} : \quad \mathcal{L}V(t,x) < 0,$$

then *x^* is (locally) asymptotically stochastically stable for SDE (4.1).*

We also call the equilibrium x^* of SDE (4.1) to be **globally asymptotically stochastically stable** iff it is asymptotically stochastically stable and $\mathcal{L}V < 0$ on the entire domain $\mathbb{D}$ where the dynamics of X live on (a.s.) (i.e., in this case, we may extend $N(x^*) = \mathbb{D}$ as the relevant neighborhood of x^* in above definition of stochastic stability). Note that the equilibria x^* do not have to be in the neighborhood $N(x^*)$, but $x^* \in \overline{N(x^*)}$. In fact, the Theorem 4.5 remains valid for the cases like neighborhoods of the form $N(K) = (0,K)$ or $N(K) = [\varepsilon, K)$ with equilibrium $x^* = K$ (or multidimensional variants of those examples) in order to cover the important cases of semi-stability too. For the SDE of the total population N of our SEIR(S) model, we can establish both stochastic and moment stability of the saturation constant $x^* = K$.

Theorem 4.6 (Stability of equilibrium $n^* = K$ for total populations)**.** *Consider the SDE for random total population*

$$dN = \mu(K-N)\,dt + \sigma_5 N(K-N)\,dW_5. \tag{4.5}$$

Then, *the equilibrium point $n^* = K$ of SDE (4.5) is*

(1) global asymptotically stochastically stable if $2\mu > K^2\sigma_5^2$,

(2) p-th moment exponentially stable if $2\mu > (2p-1)K^2\sigma_5^2$ and $p \ge \frac{1}{2}$, and

(3) almost surely asymptotically stable if $2\mu > (2p-1)K^2\sigma_5^2$ and $p \ge \frac{1}{2}$.

Proof. The proof for the equilibrium $n^* = K$ is naturally divided into the items (1)–(3).

(1) Let $n \in (0,K)$ for $n^* = K$. Define the Lyapunov function V by

$$n \in (0,K) \mapsto V(n) = (K-n)^2.$$

Then, we find that

$$\begin{aligned}\mathcal{L}V(n) &= \mu(K-n)\cdot 2(K-n)\cdot(-1)+\frac{1}{2}\sigma_5^2 n^2 (K-n)^2\cdot 2\\ &= (-2\mu+\sigma_5^2 n^2)\,V(n)\\ &\overset{\text{on }\mathbb{D}_0}{\leq} (-2\mu+\sigma_5^2 K^2)\,V(n).\end{aligned}$$

Now, note that $\mathcal{L}V(n) < 0$ for all $n \in (0,K)$ if $2\mu > K^2\sigma_5^2$. Therefore, an application of Arnold's Stability Theorem 4.5 confirms the claim of stochastic stability.

(2) Next, consider the Lyapunov function

$$n \in (0,K) \mapsto V(n) = (K-n)^{2p} = \left[(K-n)^2\right]^p.$$

Then, for $p \geq \frac{1}{2}$, we have

$$\begin{aligned}\mathcal{L}V(n) &= \mu(K-n)\cdot 2p(K-n)^{2p-1}\cdot(-1)+\frac{1}{2}\sigma_5^2 n^2(K-n)^2\cdot 2p(2p-1)(K-n)^{2p-2}\\ &= 2p\left[-\mu+\frac{1}{2}(2p-1)\sigma_5^2 n^2\right]\cdot V(n)\\ &\leq 2p\left[-\mu+\frac{1}{2}(2p-1)\sigma_5^2 K^2\right]\cdot V(n).\end{aligned}$$

An application of Dynkin's formula (cf. [8]) will give the conclusion that

$$\begin{aligned}\mathbb{E}\Big[V\Big(N(t)\Big)\Big] &= \mathbb{E}\Big[|K-N(t)|^{2p}\Big]\\ &\leq \mathbb{E}\Big[V\Big(N(s)\Big)\Big]\cdot e^{2p\left[-\mu+\frac{1}{2}(2p-1)\sigma_5^2K^2\right](t-s)} \qquad (4.6)\\ &\overset{t\to+\infty}{\longrightarrow} 0\end{aligned}$$

if $2\mu > (2p-1)K^2\sigma_5^2$ and $p \geq \frac{1}{2}$.

(3) The property of a.s. asymptotical stability of $n^* = K$ follows directly from the item (2) due to the fact that all exponentially moment stable equilibria also possess a.s. asymptotically stable pathwise solutions (for a proof of this fact, see [26]). This completes the proof of Theorem 4.6. □

As a by-product of the previous proof, we gain the following result on the asymptotic behavior of Lyapunov functionals $V(n) = |K-n|^{2p} = \|K-n\|_{\mathbb{R}^1}^{2p}$.

Theorem 4.7 (Uniform estimation of moment V-exponents)**.** *Consider Itô SDEs* (4.5) *for random total population* $N = (N(t))_{t\geq 0}$*.* **Then**

$$\forall p>0\ \forall N(0)=n_0\in(0,K):\ \lambda_{2p}(n_0) := \lim_{t\to+\infty}\frac{\ln\left(\mathbb{E}\left[|K-N(t)|^{2p}\right]\right)^{1/2p}}{t} \leq -\mu+\frac{1}{2}\max(2p-1,0)\sigma_5^2K^2,$$

which represents a uniform estimation of moment V-exponents $\lambda_{2p}(n_0)$ on $(0,K)$.

Proof. Return to the identity (4.6) with $V(n) = |K - n|^{2p}$. Taking $2p$-th root and the natural logarithm yield that

$$\ln\left(\mathbb{E}\left[|K - N(t)|^{2p}\right]\right)^{1/2p} \leq \ln\left(\mathbb{E}\left[|K - N(0)|^{2p}\right]\right)^{1/2p} + \left(-\mu + \frac{1}{2}\max(2p-1,0)\sigma_5^2 K^2\right) \cdot t.$$

Thus, dividing by $t > 0$ and taking the limit as $t \to +\infty$ confirms the conclusion on the asymptotic behavior of $\ln\left(\mathbb{E}[V(N(t))]\right)/t$ in the moment sense. □

Remark 4.8 (Moment V-exponents)**.** The moment V-exponents λ_{2p} measure the speed of exponential convergence of total populations $N(t)$ to the saturation constant K as time $t \to +\infty$ in the $2p$-th moment sense. The definition of moment V-exponents is made in a consistent manner (to incorporate the deterministic case). In passing, note that nonlinear V-exponents may depend on initial quantity $N(0) = n_0$, whereas V-exponents for linear systems do not depend on $N(0) = n_0$. Remarkably, for sufficiently small powers p or noise intensities σ_5 or very small constants $K > 0$, we find exponentially stable $2p$-moments (i.e. exponential moment convergence of $N(t)$ to equilibrium $n^* = K$) due to the birth parameter $\mu > 0$ in our model.

The following lemma states the form of all existing equilibria (trivial solutions) of SEIR(S) models (1.1). Its proof is an elementary exercise of algebra, hence it is omitted here.

Lemma 4.9 (Disease-free and endemic equilibria)**.** *For the drift coefficients of our SEIR(S) model* (1.1)*, we have two equilibrium points. One is disease-free and the other is the endemic equilibrium. The disease-free equilibrium of* (1.1) *is given by*

$$(S_1, E_1, I_1, R_1) = (K, 0, 0, 0) \in \overline{\mathbb{D}}$$

with its total sum $N_1 := S_1 + E_1 + I_1 + R_1 = K$ *and the endemic equilibrium by*

$$(S_2, E_2, I_2, R_2) \in \overline{\mathbb{D}},$$

$$\begin{aligned}
\text{where} \quad S_2 &= \frac{(\mu+\eta)(\alpha+\gamma+\mu)}{\beta\eta} \\
E_2 &= \frac{(\mu+\zeta)(\alpha+\gamma+\mu)}{\beta\eta}\left[\frac{\beta\eta K - (\mu+\eta)(\alpha+\gamma+\mu)}{(\mu+\zeta)(\alpha+\gamma+\mu)+\eta(\gamma+\mu+\zeta)}\right] \\
I_2 &= \left(\frac{\mu+\zeta}{\beta}\right)\left[\frac{\beta\eta K - (\mu+\eta)(\alpha+\gamma+\mu)}{(\mu+\zeta)(\alpha+\gamma+\mu)+\eta(\gamma+\mu+\zeta)}\right] \\
R_2 &= \frac{\gamma}{\beta}\left[\frac{\beta\eta K - (\mu+\eta)(\alpha+\gamma+\mu)}{(\mu+\zeta)(\alpha+\gamma+\mu)+\eta(\gamma+\mu+\zeta)}\right].
\end{aligned} \tag{4.7}$$

The disease-free equilibrium is also an equilibrium of the diffusion coefficients. For biologically meaningful occurrence of endemic equilibrium (i.e. $(S_2, E_2, I_2, R_2) \in \overline{\mathbb{D}}$*), we need to require that*

$$\beta\eta K > (\mu+\eta)(\alpha+\gamma+\mu) \tag{$*$}$$

– a condition, which is equivalent to $\mathcal{R}_0 > 1$*. For the classic concept of endemic equilibrium of both drift and diffusion terms at the same location, vanishing* $F_k(S_2, E_2, I_2, R_2) = 0$ *are imposed. Moreover, at the endemic equilibrium, we have total sum*

$$N_2 := S_2 + E_2 + I_2 + R_2 = K.$$

Proof. Elementary calculus exercise. □

Biologists and Ecologists usually express the qualitative state of systems in terms of **basic reproduction numbers** $\mathcal{R}_0$. For our SEIR(S) model (1.1), this quantity takes the form

$$\mathcal{R}_0 = \frac{\beta\eta K}{(\mu+\eta)(\alpha+\gamma+\mu)}.$$

Indeed, as we shall see below, this quantity decides about the long-term **stability mode** in which the stochastic SEIR(S) models is and the value $\mathcal{R}_0 = 1$ serves as a bifurcation parameter (cf. stability analysis in what follows). Moreover, the endemic equilibrium (S_2, E_2, I_2, R_2) can be expressed in terms of $\mathcal{R}_0$ by

$$(S_2, E_2, I_2, R_2) = \left(\frac{K}{\mathcal{R}_0}, (\mu+\zeta)\frac{K}{\mathcal{R}_0}\rho, \frac{(\mu+\zeta)}{\beta}\rho, \frac{\gamma}{\beta}\rho\right)$$

with

$$\rho := \frac{\mathcal{R}_0 - 1}{\frac{(\mu+\zeta)}{(\mu+\eta)} + \frac{(\gamma+\mu+\zeta)\mathcal{R}_0}{\beta K}}.$$

Clearly, the 2nd, 3rd and 4th components of (S_2, E_2, I_2, R_2) are positive iff $\rho > 0$ iff $\mathcal{R}_0 > 1$.

First, for stability investigation of SEIR(S) models (1.1), note that all equilibria $(S_k^*, E_k^*, I_k^*, R_k^*)$ of SDEs (1.1) possess the total sum

$$N_k^* = S_k^* + E_k^* + I_k^* + R_k^* = K$$

and are at the boundary of domain $\mathbb{D}$, i.e.

$$N_k^* \in \overline{\mathbb{D}}.$$

This is also the unique equilibrium of dynamics $N = (N(t))_{t\geq 0}$ of total populations governed by SDE (1.2). Consequently, it remains to prove that the asymptotic stability of the disease-free equilibrium when reproduction number $\mathcal{R}_0 < 1$ and the endemic equilibrium when reproduction number $\mathcal{R}_0 > 1$.

Asymptotic stability of general epidemic or environmental systems has already been investigated in [4, 13, 20, 31, 32, 34], but not our SEIR(S) model (1.1) to the best of our knowledge. These investigations are associated to appropriate Lyapunov functions or functionals (cf. [4,7,9,10,17–19] among others).

Theorem 4.10 (Asymptotic stochastic stability of disease-free equilibrium)**.** *The disease-free equilibrium solution $(S_1, E_1, I_1, R_1) = (K, 0, 0, 0)$ of (1.1) is (globally) asymptotically stochastically stable if*

$$\sigma^2 K^2 < 2\mu, \qquad \zeta \geq 0, \qquad \beta K \leq \alpha. \tag{4.8}$$

Proof. We shall apply Theorem 4.5. For this purpose, define the Lyapunov function

$$\begin{aligned} V_4(S, E, I, R) &= \frac{1}{2}(S - K + E + I + R)^2 + KE + KI + KR \\ &= \frac{1}{2}(K - N)^2 + KE + KI + KR = \hat{V}_4(E, I, R, N) \end{aligned}$$

on $\mathbb{D}$. The infinitesimal generator $\mathcal{L}$ (cf. (3.2) and generally presented one in Appendix A) acting on the Lyapunov function V_4 can be written as:

$$
\begin{aligned}
\mathcal{L}V_4(S,E,I,R) &= \big(-\beta SI + \mu(K-S) + \alpha I + \zeta R\big)(S-K+E+I+R) \\
&\quad + \big(\beta SI - (\mu+\eta)E\big)(S-K+E+I+R+K) \\
&\quad + \big(\eta E - (\alpha+\gamma+\mu)I\big)(S-K+E+I+R+K) \\
&\quad + \big(\gamma I - (\mu+\zeta)R\big)(S+E+I+R+K-K) + \frac{\sigma_5^2}{2}N^2(K-N)^2 \\
&= -\mu(S+E+I+R-K)^2 + K\big[\beta SI - \mu(E+I+R) - \alpha I - \zeta R\big] \\
&\quad + \frac{\sigma_5^2}{2}N^2(K-N)^2 \\
&= -\mu(N-K)^2 - \mu(KE+KI+KR) + K\big[(\beta S-\alpha)I - \zeta R\big] + \frac{\sigma_5^2}{2}N^2(K-N)^2 \\
&\overset{0<\delta<1}{=} -\mu(1-\delta)(N-K)^2 - \mu(KE+KI+KR) + K\big[(\beta K-\alpha)I - \zeta R\big] \\
&\quad - \Big[\delta\mu - \frac{\sigma_5^2}{2}N^2\Big](K-N)^2 \\
&\leq -\mu(1-\delta)V_4 + K\big[(\beta K-\alpha)I - \zeta R\big] - \big[\delta\mu - \frac{\sigma_5^2}{2}K^2\big](K-N)^2.
\end{aligned}
$$

Now, let $\delta \to 1-$. Then, from some $\delta > 0$ onwards as $\delta \uparrow 1$, we find a $\delta_0 < 1$ such that, for all $\delta \in (\delta_0, 1]$, we have $-\delta\mu + \sigma_5^2 K^2/2 \leq 0$ by hypothesis $2\mu > \sigma_5^2 K^2$.

Thus, $\mathcal{L}V_4(S,E,I,R) \leq 0$ is indeed negative-definite on $\mathbb{D}$ under the presumptions that $\beta K - \alpha \leq 0$, $\zeta \geq 0$ and $2\mu > \sigma_5^2 K^2$. It remains to apply stochastic stability Theorem 4.5 to confirm Theorem 4.10. □

Remark 4.11 (Role of basic reproduction number)**.** One of the most important quantities in epidemiology is the basic reproduction number $\mathcal{R}_0$, expected number of secondary infections produced when one infected individual entered a fully susceptible population [15]. It usually determines whether there is an epidemic or not. If $\mathcal{R}_0 < 1$ then the outbreak will disappear. On the contrary, if $\mathcal{R}_0 > 1$ then the epidemic will spread a population. Recall that the basic reproduction number of our SEIR(S) model is $\mathcal{R}_0 = \frac{\eta\beta K}{(\mu+\eta)(\alpha+\gamma+\mu)}$. Later we will see that this number $\mathcal{R}_0$, the magnitude of $\mu > 0$ and the parameter $\sigma_5^2 K^2$ involving environmental noise intensity σ_5 decide about whether the disease-free or the endemic equilibrium is (asymptotically) stochastically stable (cf. Theorems 4.10 and 4.15).

Remark 4.12 (Possible extinction of disease)**.** Theorem 4.10 concludes that, if $\alpha - \beta K \geq 0$ and the environmental noise level σ_5^2 is so small such that $2\mu \geq \sigma_5^2 K^2$, then the disease will die out. This statement does not contradict to the fact $\mathcal{R}_0 < 1$. Because the stability condition $\alpha - \beta K \geq 0$ can be written in terms of the basic reproduction number as follows

$$\beta K \leq \alpha < (\alpha+\gamma+\mu)\frac{(\mu+\eta)}{\eta} \quad \Rightarrow \quad \frac{\eta\beta K}{(\mu+\eta)(\alpha+\gamma+\mu)} = \mathcal{R}_0 < 1.$$

Corollary 4.13 (Exponential moment stability of disease-free equilibrium)**.** *Since*

$$\mathcal{L}V_4(S,E,I,R) \leq -\mu V_4(S,E,I,R)$$

under condition that $\beta K \le \alpha$, $\zeta \ge 0$ *and* $\mu \ge \sigma_5^2 K^2$, *by Dynkin's formula, the disease-free equilibrium* $(K,0,0,0)$ *is exponentially moment* V*-stable [26,27] with rate* $-\mu$, *i.e.* $\forall t \ge 0$:

$$\mathbb{E}\Big[V_4(S(t),E(t),I(t),R(t))\Big] \le \mathbb{E}\Big[V_4(S(0),E(0),I(0),R(0))\Big]\cdot \exp\Big(-\mu\cdot t\Big),$$

$$\text{hence} \quad \lim_{t\to\infty}\frac{\ln\left[\mathbb{E}[V_4(S(t),E(t),I(t),R(t))]\right]}{t} = -\mu < 0. \tag{4.9}$$

Proof. Recall the structure of associated infinitesimal generator $\mathcal{L}$ and the computations of $\mathcal{L}V_4$ in the proof of Theorem 4.10. There we have found that

$$\mathcal{L}V_4(S,E,I,R) = -\mu(N-K)^2 - \mu(KE+KI+KR) + K\big[(\beta S-\alpha)I - \zeta R\big] + \frac{\sigma_5^2}{2}N^2(K-N)^2.$$

Under $\mu \ge \sigma_5^2 K^2$, we further estimate

$$\begin{aligned}\mathcal{L}V_4(S,E,I,R) &= -\frac{\mu}{2}(N-K)^2 - \mu(KE+KI+KR) + K\big[(\beta S-\alpha)I-\zeta R\big] - \frac{\mu-\sigma_5^2N^2}{2}(K-N)^2\\ &\le -\mu V_4 + \underbrace{K\big[(\beta S-\alpha)I-\zeta R\big] - \frac{\mu-\sigma_5^2K^2}{2}(K-N)^2}_{\le\ 0 \text{ on } \mathbb{D} \text{ since } \beta K\le\alpha,\ \zeta\ge 0,\ \mu\ge\sigma_5^2K^2}\\ &\le -\mu V_4.\end{aligned}$$

Then, Dynkin's formula [8] gives

$$\begin{aligned}\mathbb{E}\big[V_4\big(S(t),E(t),I(t),R(t)\big)\big] &= \mathbb{E}\big[V_4\big(S_0,E_0,I_0,R_0\big)\big] + \mathbb{E}\Big[\int_0^t \mathcal{L}V_4\big(S(s),E(s),I(s),R(s)\big)ds\Big]\\ &\le \mathbb{E}\big[V_4\big(S_0,E_0,I_0,R_0\big)\big] - \mu\mathbb{E}\Big[\int_0^t V_4\big(S(s),E(s),I(s),R(s)\big)ds\Big].\end{aligned}$$

Now, apply the well-known Bellman–Gronwall lemma to the dynamics of

$$v(t) := \mathbb{E}\big[V_4\big(S(t),E(t),I(t),R(t)\big)\big]$$

to conclude exponentially moment V-stability with $V = V_4$ (for the general concept of moment V-stability, see [26,27]). This finishes the proof of Corollary 4.13. □

Remark 4.14 (Extension of exponential stability at reduced rates)**.** There is a verification of a small extension of the range of exponential stability of disease-free equilibrium possible for the case $\sigma_5^2K^2/2 < \mu < \sigma_5^2K^2$. However, this is verified only at reduced rate $-\mu + \sigma_5^2K^2/2$ of exponential convergence, compared to rate $-\mu < 0$ of Corollary 4.13. For this, one may establish the estimates $\mathcal{L}V_4 \le [-\mu + \sigma_5^2K^2/2]V_4$ from the above proof.

Now, let us turn to the study of asymptotic stability of the endemic equilibrium.

Theorem 4.15 (Asymptotic stochastic stability of endemic equilibrium)**.** *Assume that*

$$\beta\eta K > (\mu+\eta)(\alpha+\gamma+\mu)$$

(i.e. $\mathcal{R}_0 > 1$*) and* $2\mu \ge \sigma_5^2K^2$. **Then**, *the endemic equilibrium solution* (S_2,E_2,I_2,R_2) *of the system (1.1) is (globally) stochastically stable on*

$$\mathbb{D} = \big\{(S,E,I,R) : S>0, E>0, I>0, R>0, S+E+I+R<K\big\}.$$

If even $2\mu > \sigma_5^2K^2$, **then** *the endemic equilibrium* (S_2,E_2,I_2,R_2) *of* (1.1) *is (globally) asymptotically stochastically stable on* $\mathbb{D}$.

Proof. Introduce the function

$$V_5(S,E,I,R) = \begin{cases} S - S_2 + E - E_2 + I - I_2 + R - R_2 \\ -(S_2 + E_2 + I_2 + R_2)\ln\left(\dfrac{S+E+I+R}{S_2+E_2+I_2+R_2}\right) \end{cases} \tag{4.10}$$

on $\mathbb{D}$. Note that $V_5 \geq 0$ on $\mathbb{D}$ and $V_5 = 0 \iff S+E+I+R = K$ on $\mathbb{D}$ (by elementary calculus applied to $V_5(S,E,I,R) = \widetilde{V}_5(N)$ with $N = S+E+I+R \in (0,K]$. Actually $\widetilde{V}_5$ is strictly decreasing in $N \in [0,K)$). Thus, it is fairly easy to recognize that V_5 possesses all properties of a Lyapunov function on $\mathbb{D}$. Then, we have $\forall (S,E,I,R) \in \mathbb{D}$

$$\begin{aligned} \mathcal{L}V_5(S,E,I,R) &= \big(-\beta SI + \mu(K-S) + \alpha I + \zeta R\big)\left(1 - \frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right) \\ &\quad + \big(\beta SI - (\mu+\eta)E\big)\left(1 - \frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right) \\ &\quad + \big(\eta E - (\alpha+\gamma+\mu)I\big)\left(1 - \frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right) \\ &\quad + \big(\gamma I - (\mu+\zeta)R\big)\left(1 - \frac{S_2+E_2+I_2+R_2}{(S+E+I+R)}\right) + \frac{\sigma_5^2}{2}N^2(K-N)^2 \cdot \frac{K}{N^2} \\ &= \left(1 - \frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right)\mu(K-S-E-I-R) + \frac{\sigma_5^2}{2}K(K-N)^2. \end{aligned} \tag{4.11}$$

Note that, with $N = S+E+I+R$, we find that

$$\mu(K-N) = \mu(K-S-E-I-R) \overset{\text{LEM 4.9}}{=} -\mu(S+E+I+R-S_2-E_2-I_2-R_2).$$

Hence, we arrive at

$$\begin{aligned} \mathcal{L}V_5(S,E,I,R) &= -\mu(S+E+I+R-S_2-E_2-I_2-R_2)\left(1 - \frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right) \\ &\quad + \frac{\sigma_5^2}{2}K(K-N)^2 \\ &= -\mu\frac{(S-S_2+E-E_2+I-I_2+R-R_2)^2}{S+E+I+R} + \frac{\sigma_5^2}{2}K(K-N)^2 \\ &= -\mu\frac{(K-N)^2}{N} + \frac{\sigma_5^2}{2}KN\frac{(K-N)^2}{N} \\ &\overset{\sigma_5^2 \geq 0}{\leq} -\left(\mu - \frac{\sigma_5^2}{2}K^2\right)\frac{(K-N)^2}{N} \leq 0 \end{aligned} \tag{4.12}$$

since $0 < N < K$ on $\mathbb{D}$ and by hypothesis $2\mu \geq \sigma_5^2 K^2$.

Therefore, by Theorem 4.5, the endemic equilibrium (S_2, E_2, I_2, R_2) is stochastically stable (globally on $\mathbb{D}$) if $\mathcal{R}_0 > 1$ and $2\mu \geq \sigma_5^2 K^2$. Moreover, when additionally $2\mu > \sigma_5^2 K^2$, a careful look again at estimation (4.12) yields that

$$\mathcal{L}V_5(S,E,I,R) \leq -\left(\mu - \frac{\sigma_5^2}{2}K^2\right)\frac{(K-N)^2}{N} < 0$$

on $\mathbb{D}$. Consequently, by Theorem 4.5, the endemic equilibrium (S_2, E_2, I_2, R_2) of SDEs (1.1) indeed is asymptotically stochastically stable (globally on $\mathbb{D}$) if $\mathcal{R}_0 > 1$ and $2\mu > \sigma_5^2 K^2$. This conclusion completes the proof of Theorem 4.15. □

Remark 4.16 (Nonlinear distance measure to endemic equilibrium)**.** The function V_5 of the form (4.10) measures the distance of solutions (S, E, I, R) to the endemic equilibrium (S_2, E_2, I_2, R_2) in a nonlinear fashion.

Corollary 4.17 (Exponential moment stability of endemic equilibrium)**.** *Assume that*

$$\beta\eta K > (\mu+\eta)(\alpha+\gamma+\mu)$$

(i.e. $\mathcal{R}_0 > 1$*),* $2\mu > \sigma_5^2 K^2$ *and the initial total population* $0 < N(0) := S(0)+E(0)+I(0)+R(0) < K$ *is nonrandom.*
Then, *the endemic equilibrium solution* (S_2, E_2, I_2, R_2) *of system* (1.1) *is exponentially moment* V_5*-stable with rate* $-(\mu-\sigma_5^2K^2/2)N(0)/K$*, i.e.* $\forall t \geq 0$:

$$\mathbb{E}[V_5(S(t),E(t),I(t),R(t))] \leq \mathbb{E}[V_5(S(0),E(0),I(0),R(0))] \cdot \exp\left(-\left(\mu-\frac{\sigma_5^2K^2}{2}\right)\frac{N(0)}{K}\cdot t\right),$$

hence

$$\lim_{t\to+\infty} \frac{\ln\left[\mathbb{E}[V_5(S(t),E(t),I(t),R(t))]\right]}{t} \leq -\left(\mu-\frac{\sigma_5^2K^2}{2}\right)\frac{N(0)}{K} < 0.$$

Proof. Define the total population $N(t) = S(t)+E(t)+I(t)+R(t)$ for $t \geq 0$. Suppose that $N(0)$ is nonrandom. Recall that $S_2+E_2+I_2+R_2 = K$ by Lemma 4.9. Now, return to the proof of Theorem 4.15 where we have computed

$$\begin{aligned}\mathcal{L}V_5(S,E,I,R) &= \left(1-\frac{S_2+E_2+I_2+R_2}{S+E+I+R}\right)\mu(K-S-E-I-R)\\ &= \left(1-\frac{K}{N}\right)\left(\mu-\frac{\sigma_5^2}{2}K^2\right)(K-N) = -\left(\mu-\frac{\sigma_5^2}{2}K^2\right)\cdot\frac{(N-K)^2}{N}\\ &\leq -\left(\mu-\frac{\sigma_5^2}{2}K^2\right)\frac{N(0)}{K}\cdot V_5(S,E,I,R) \quad \text{for } N \geq N(0)\end{aligned}$$

since the total population $N(t)$ is monotonically increasing for our SEIR model and Lyapunov functional $V_5(S,E,I,R) = N-K-K\cdot\ln\left[\frac{N}{K}\right] =: \widetilde{V}_5(N)$ on $\mathbb{D}$ with monotonically decreasing $\widetilde{V}_5(N)$ in N (calculate $\widetilde{V}_5'(N) = (N-K)/N < 0$ on $N \in (0,K)$ and the simple calculus fact that

$$-\frac{(N-K)^2}{N} < -\frac{N}{K}\widetilde{V}_5(N) < -\frac{N(0)}{K}\widetilde{V}_5(N)$$

for all $N \geq N(0)$. Finally, with nonrandom initial $N(0) = S(0)+E(0)+I(0)+R(0) < K$, apply Dynkin's formula to arrive at

$$\begin{aligned}\mathbb{E}\left[\widetilde{V}_5(N(t))\right] &= \mathbb{E}\left[\widetilde{V}_5(N(0))\right] + \mathbb{E}\left[\int_0^t \mathcal{L}\widetilde{V}_5(N(s))\,ds\right]\\ &\leq \mathbb{E}\left[\widetilde{V}_5(N(0))\right] - \left(\mu-\frac{\sigma_5^2}{2}K^2\right)\frac{N(0)}{K}\int_0^t \mathbb{E}\left[\widetilde{V}_5(N(s))\right]ds.\end{aligned}$$

It remains to use the well-known Bellman–Gronwall lemma to conclude that

$$v(t) := \mathbb{E}\left[\widetilde{V}_5(N(t))\right]$$

(recall that $V_5(S(t),E(t),I(t),R(t)) = \widetilde{V}_5(N(t))$) in order to verify exponential moment stability along functional V_5 with a "least" rate estimated by $-(\mu-\frac{\sigma_5^2}{2}K^2)\frac{N(0)}{K}$. □

Remark 4.18 (A.s. stability and rates of exponential stability)**.** Since exponential moment stability also implies a.s. asymptotic stability, from Corollary 4.17, we also gain the conclusion on a.s. asymptotic V_5-stability of the endemic equilibrium (S_2, E_2, I_2, R_2) on $\overline{\mathbb{D}}$ under the hypothesis that $\mathcal{R}_0 > 1$ and $2\mu > \sigma_5^2 K^2$ (by dissipative techniques from [27]). Besides, the continuous time and discrete time moment attractivity exponents of other appropriate functionals $V \geq 0$ can also be estimated by some results from [26]. But, this would sprinkle the frame of this paper. Note, it is common that the rates of stability or attractivity of nonlinear dynamical systems depend on the initial values like $N(0)$ above (in contrast to linear systems).

5 Illustrations of moment functionals and reproduction number

Here we illustrate the behavior of moment Lyapunov functionals along the solutions of SEIR(S) model (1.1) and the structure of reproduction number. First, we plot the 2D surface of reproduction number $\mathcal{R}_0$ depending on growth parameter μ and transition parameters $\alpha + \gamma$. The conceivable hyperplane $\mathcal{R}_0 = 1$ decides whether the system (1.1) has an asymptotically stable disease-free or endemic equilibrium. For examples, above the hyperplane $\mathcal{R}_0 = 1$ we locate the region where the endemic equilibrium is asymptotically stochastically stable (similar below that plane for stability of disease-free equilibrium). Figure 5.1 shows that increasing μ stabilizes the dynamics of SEIR model (1.1) toward the disease-free equilibrium. This also happens with increasing the transition parameter sum $\alpha + \gamma$, but at a much slower scale. For sufficiently small μ and small $\alpha + \gamma$, the endemic equilibrium is asymptotically stochastically stable since the reproduction number is well above the hyperplane $\mathcal{R}_0 = 1$, as clearly seen in left corner of Figure 5.1.

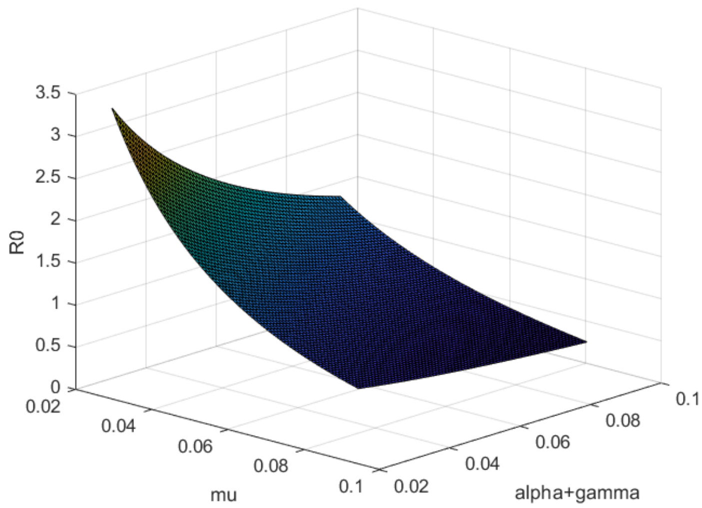

Figure 5.1: Reproduction number $\mathcal{R}_0(\mu, \alpha+\gamma)$ with $\beta = 25 \cdot 10^{-5}$, $\eta = 0.005$, $\zeta = 0.002$, $K = 1000$ depending on $\mu = mu$ and $\alpha + \gamma = r$.

Next, we illustrate the dynamics of total population process $N = (N(t))_{t \geq 0}$ in pathwise (a.s.) and mean sense. Figure 5.2 shows several paths of total population $N(t)$ generated by

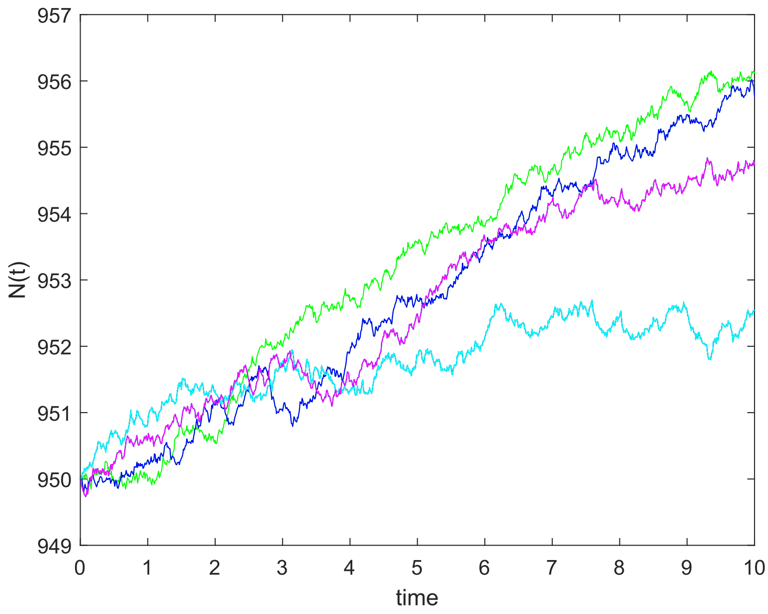

Figure 5.2: Several trajectories of total population with $\mu = 10^{-2}$, $K = 1000$, $\sigma = 10^{-5}$, $T = 10$, step size $h = 10^{-2}$, started at $N(0) = 950$.

the Euler–Maruyama method in MATLAB. This demonstrates the variety and erratic effect of noise on the solution-paths.

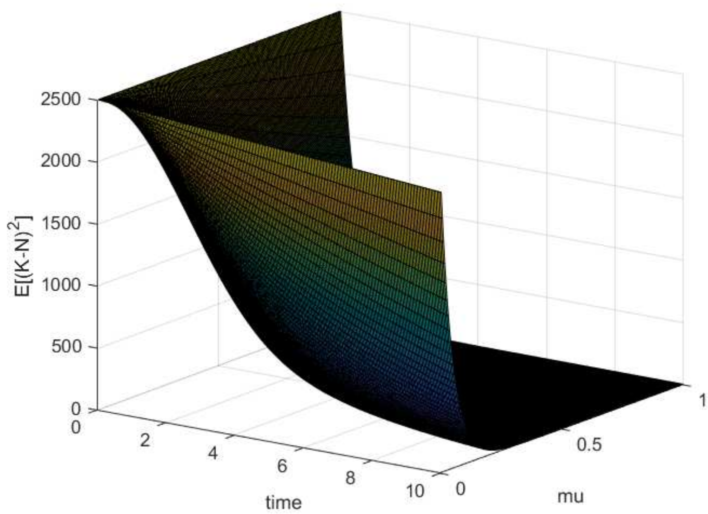

Figure 5.3: Expected Lyapunov functional $\mathbb{E}[K - N(t)]^2$ versus t and μ with $K = 1000$, $\sigma = 10^{-5}$, $T = 10$, $M = 10^6$, step size $h = 10^{-2}$, started at $N(0) = 950$.

Figure 5.3 displays the expected Lyapunov functional $\mathbb{E}[K - N(t)]^2$ versus time t and parameter μ, generated by $M = 10^6$ samples started with same total population size $N(0) = 950 < K = 10^3$ and discretized by standard Euler–Maruyama method with uniform step size $h = 10^{-2}$ in MATLAB. As seen there, the dynamics stabilize with increasing parameter μ and with advancing time t. The decline of 2D surface of $\mathbb{E}[K - N(t)]^2$ in time t and μ also confirms Theorem 4.6 that, for sufficiently large μ, we find asymptotically stable equilibrium $n^* = K$ for total population.

In Figure 5.4 the expected Lyapunov functional $\mathbb{E}[K - N(10)]^2$ versus parameters μ and σ is depicted, generated by $M = 10^6$ samples started with same total population size $N(0) = 950 < K = 10^3$ and discretized by standard Euler–Maruyama method with uniform step size $h = 10^{-2}$ in MATLAB. Clearly, we reckon that the dynamics of that functional is "destabilized" with increasing noise intensity σ and "stabilized" with increasing parameter μ. This gives us some statistical evidence for our Theorems 4.7 (i.e. decline of moments with growing $\mu > 0$) and 4.6 (i.e. the destabilizing effect of growing σ^2 on moments and stability). Of course, care is needed since growing variance with increasing σ^2 reduces our confidence in the estimation process and perhaps larger sample sizes are needed to confirm simulation results. All in all, larger noise intensities reveal a fairly nontrivial, nonlinear dependence of functionals $\mathbb{E}[K - N(10)]^2$ on model parameters (μ, σ).

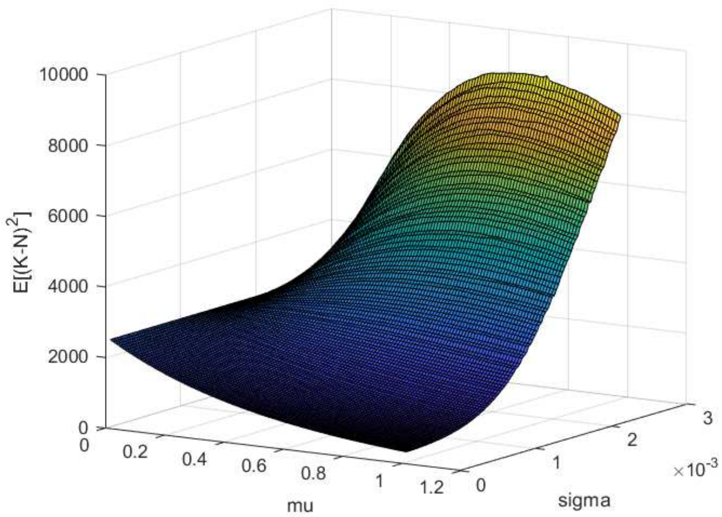

Figure 5.4: Expected Lyapunov functional $\mathbb{E}[K - N(10)]^2$ versus μ and σ with $K = 1000$, $T = 10$, $M = 10^6$, step size $h = 10^{-2}$, started at $N(0) = 950$.

Figure 5.5 plots the 2D surface of expected Lyapunov functional $\mathbb{E}[K-N(t)]^2$ versus time t and parameter σ with $\mu = 0.09$, generated by $M = 10^6$ samples started with same total population size $N(0) = 950 < K = 10^3$ and discretized by standard Euler–Maruyama method with step size $h = 10^{-2}$ in MATLAB. For small $\sigma > 0$, the surface of this functional declines at lower right corner of Figure 5.5. That is an empirical indicator that the SEIR(S) model is in the stable regime. However, for larger, increasing values of $\sigma > 0$, the 2D surface gets "destablized" as time t advances, as we especially reckon at upper right corner of Figure 5.5.

We could continue with showing more and more simulation results. Clearly, we have demonstrated the applicability of our analysis and have suggested to plot 2D surface of multi-dimensional expected Lyapunov functionals in order to get empirical evidence about which stable or unstable mode the SEIR(S) model is in. Eventually, by Figure 5.6, we display 2D surfaces of expected Lyapunov functional $m(t, p) = (\mathbb{E}[|K - N(t)|^{2p}])^{1/2p}$ depending on powers $p \geq 0.5$ and time t, while $\mu = 1.0$, $K = 1000$ and $\sigma = 10^{-5}$ are fixed. This shows the depen-

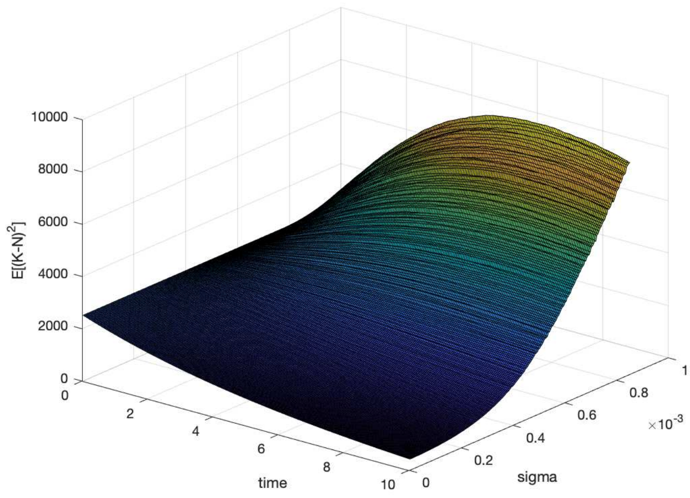

Figure 5.5: Expected Lyapunov functional $\mathbb{E}[K-N(t)]^2$ versus time t and σ with $\mu = 0.09$, $K = 1000$, $T = 10$, $M = 10^6$, step size $h = 10^{-2}$, started at $N(0) = 950$.

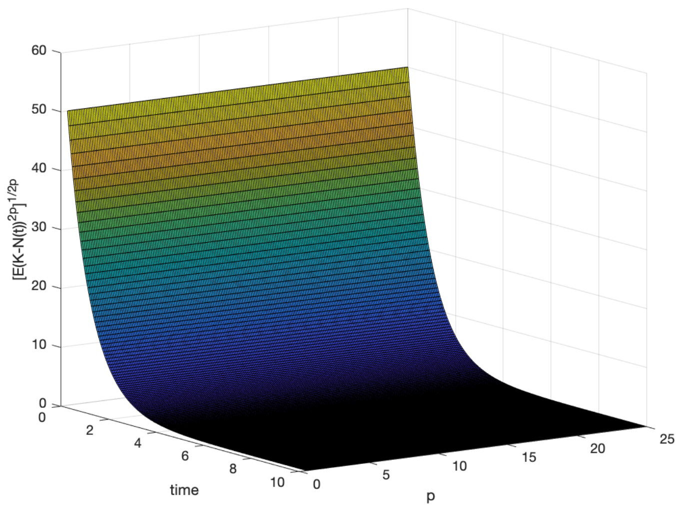

Figure 5.6: Expected Lyapunov functional $\left(\mathbb{E}[K-N(t)]^{2p}\right)^{1/2p}$ versus time t and power p with $\mu = 1.0$, $K = 1000$, $T = 10$, $M = 10^6$, step size $h = 0.05$, started at $N(0) = 950$.

dence of expected Lyapunov functionals $m(t,p)$ on powers $p \geq 0.5$ and time $t \geq 0$. As time t advances,the depicted hyperplane declines toward zero, giving some evidence of the stable mode of our SEIR(S) model since $2\mu > \sigma^2 K^2$ in our simulation. There are only small changes in p in that range. The decline of 2D-surface $m(t,p)$ in Figure 5.6 with increasing time t confirms the findings of Theorems 4.6 and 4.7 on asymptotic stability of equilibrium $n^* = N$ using Lyapunov functionals. The simulation has been conducted with the Euler–Maruyama method using step size $h = 0.05$ in MATLAB. The crude step size $h = 0.05$ is applied since the limited computational capacity of our computers, and we put our main emphasis on large sample sizes M and fine discretization of parameter space $p \in (0, 25]$ in order to get more statistical evidence (instead of higher numerical accuracy). To get some assurance of our graphical plots and stable computations, we repeated the experiments to get some confirmation by much smaller step sizes h (but at the expense of reduced sample sizes).

Similar experiments can be conducted for other functionals of biological interest like the convergence to all their equilibria. Such an endeavor is left to interested reader.

6 Summary, conclusions and outlook

This paper introduced a stochastic SEIR(S) model (1.1) based on Itô stochastic differential equations (SDEs) with a deterministic maximum saturation constant $K > 0$. The main emphasis is on the incorporation of possible random transitions from one compartment to another (sub-populations). As one of the major differences to previously introduced SEIR(S) models, our model (1.1) possesses a random total population $N = (N(t))_{t\geq 0}$, which itself is governed by a logistic Itô SDE with the equilibrium $n^* = K$. It was shown that the total population $N(t)$ is a.s. positive and bounded by the saturation constant $K > 0$ - a requirement for the practical relevance of any SEIR(S) models. Moreover, conditions have been worked out for the asymptotic stochastic and moment stability of the equilibrium K of the total population process $N = (N(t))_{t\geq 0}$. The analysis of dynamics of the total population N is essential for the understanding and qualitative control of the solutions of SEIR models (1.1).

The paper proves the existence of unique, strong solutions (S, E, I, R) of original SEIR(S) models (1.1) on bounded, positive prisms $\mathbb{D} \subset \mathbb{R}^4_+$ for all adapted, initial data residing inside $\mathbb{D}$ (with finite initial "energy"). We have also verified reasonable criteria for the asymptotic stochastic and moment stability of the disease-free and the endemic equilibria of (1.1). As commonly expected, the basic reproduction number $\mathcal{R}_0$ decides about the stable character of the equilibria ($\mathcal{R}_0 < 1$ for stability of the disease-free equilibrium and $\mathcal{R}_0 > 1$ for stability of the endemic equilibrium). Finally, we illustrated our major findings w.r.t. declining moment Lyapunov functionals, depending on several parameters. Very recently during submission of this paper, it came to our attention that there is already a generalization of SEIR(S) models with stochastic transmission by [36]. However, his model only allows back-and-forth transitions from S to E to S and there is no back coupling from R or I back to S and E, and he does not incorporate general functions F_k controlling the rates of nonlinearities (i.e. just the case of constant rates in the incidence terms). Moreover, a verification of a.s. exponential stability of equilibria is only conducted there. Our model also admits random transitions from the remaining population $K - N$ to the sub-populations S, E, I, R with $N = S + E + I + R$.

There are plenty of possible generalizations. One could try out Levy-type- or jump-processes for the random noise sources or Markovian switching or non-Markovian regimes. However, all generalizations should be done through semi-martingale theory due to the continuity requirement of the underlying integration operator in biologically relevant applications.

Itô calculus interpretations are the commonly adopted models for the sake of the fact that the offspring populations should only depend on the past, i.e. the closest parental generations. At the end, statistical matching to real data would decide on the relevance of each SEIR(S) model. We leave the practical execution of all of those ideas to the interested reader. We are convinced that our model class already offers enough flexibility and interesting phenomena, however restricted to Markovian modeling by this contribution.

A Appendix: A general existence result of solutions of SDEs

Consider d-dimensional, Itô-interpreted stochastic differential equations (SDEs) of the form

$$dX(t) = f\big(X(t),t\big)dt + g\big(X(t),t\big)dW(t) \tag{A.1}$$

with initial value $X(t_0) = X_0, t_0 \le t \le T < +\infty$, where $f : \mathbb{R}^d \times [t_0,T] \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times [t_0,T] \to \mathbb{R}^{d\times m}$ are Borel measurable functions, $W = \{W(t)\}_{t\ge t_0}$ is a $\mathbb{R}^m-$valued Wiener process and X_0 is a $\mathbb{R}^d-$valued random variable. Recall that its infinitesimal generator $\mathcal{L}$ associated with the above SDE (A.1) is given by

$$\mathcal{L} \;=\; \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{j=1}^{m}\sum_{i=1}^{d}\sum_{k=1}^{d} g_i^j(x,t)g_k^j(x,t)\frac{\partial^2}{\partial x_i \partial x_k}. \tag{A.2}$$

Theorem A.1 (Improved version of a theorem from Khas'minskii (1980))**.** *Assume that*

(i) $f,g \in C^0_{locLip(L)}\big(\mathbb{D}\times[0,T]\big)$,

(ii) $(\mathbb{D}_r)_{r>0}$ *nondecreasing, bounded, connected, all* $\mathbb{D}_r \subseteq \mathbb{R}^d$ *and* $\mathbb{D} = \cup_{r>0}\mathbb{D}_r$,

(iii) $\sigma\big(X(0)\big)$ *is independent of* $\sigma\big(W(s) : s \le T\big)$ *and* $X(0) \in \mathbb{D}$,

(iv) $\exists\, V \in C^{2,1}\big(\mathbb{D}\times[0,T]\big)$ *with* $V : \mathbb{D}\times[0,T] \to \mathbb{R}^1_+$, $\exists\, a \in L^1\big([0,T]\big)$

$$\forall\, x \in \mathbb{D}\;\forall\, t \in [0,T]: \quad \mathcal{L}V(x,t) \le a \cdot V(x,t),$$

(v) $\mathbb{E}\big[V\big(X(0),0\big)\big] < +\infty$,

(vi) $\inf_{t>0,\, x\in\partial\mathbb{D}_r} V(x,t) \overset{r\to+\infty}{\longrightarrow} +\infty$.

Then, $\exists$ *strong, unique, continuous time, Markovian solution* X *of SDE* (A.1) *with* $X(0) = X_0$ *and* $X(t) \in \mathbb{D}$ *for all* $t > 0$.

Remark A.2 (Linear versus exponential moment bounds)**.** The conclusion of Theorem A.1 remains valid if one replaces the assumption (iv) by the hypothesis

(iv)' $\exists\, V \in C^{2,1}\big(\mathbb{D}\times[0,T]\big)$ with $V : \mathbb{D}\times[0,T] \to \mathbb{R}^1_+$, $\exists\, a \in L^1\big([0,T]\big)$

$$\forall\, x \in \mathbb{D}\;\forall t \in [0,T]: \quad \mathcal{L}V(x,t) \le c_0,$$

where c_0 is an appropriate constant. In this case, one is able to prove the uniform boundedness

$$\sup_{0\le t\le T} \mathbb{E}\Big[V\big(X(t),t\big)\Big] \le \mathbb{E}\Big[V\big(X(0),0\big)\Big] + [c_0]_+ \cdot T$$

of the moments along the functionals V of solutions X. In contrast to that fact, the original assumption (iv) of Theorem A.1 with any constant $a(t) = c_1$ guarantees the uniform exponential bounds

$$\sup_{0 \le t \le T} \mathbb{E}\Big[V(X(t),t)\Big] \le \mathbb{E}\Big[V(X(0),0)\Big] \cdot \exp\left([c_1]_+ \cdot T\right)$$

of the moments $\mathbb{E}[V(X(t),t)]$ as worst-case estimate. Here, $[\cdot]_+$ denotes the nonnegative part of the inscribed mathematical expression.

Remark A.3 (Comment on uniqueness of solutions)**.** Uniqueness of strong solutions X of SDEs (A.1) with local Lipschitz continuous coefficients f, g on open, connected sets $\mathbb{D} \subseteq \mathbb{R}^d$ can only be lost when the solutions explode on the boundary of $\mathbb{D}$. Common (nonrandom) equilibria x^* of both f and g are considered unique solutions $X = x^*$ of SDEs (A.1) itself, sometimes called **trivial solutions** or **equilibrium solutions** (i.e., in this case, applied to SDEs with extended drift and diffusion coefficients vanishing on entire $\mathbb{D}$). In our paper the existence of local solutions is established for SDEs with Lipschitz coefficients inside the open prism $\mathbb{D}$. The uniqueness of such local solutions inside $\mathbb{D}$ is clear from standard texts on SDEs (such as [3], [14] and [23]) since the closed prism $\overline{\mathbb{D}}$ is a compact set and we do not hit the boundary of $\mathbb{D}$ at any finite time, provided that we start inside the prism (that latter is what we presumed anyway). Recall that the equilibria of our SEIR(S) model are located on the boundary of the open prism $\mathbb{D}$. Hence, they can not be reached in any finite time from the interior of $\mathbb{D}$. Moreover, we have proved the boundedness of moments along certain Lyapunov functionals V, which implies that the solutions can not hit the boundary of the prism $\mathbb{D}$. This is obvious from the application of Khasminskij's Theorem A.1 in this appendix. We just had to construct and verify a related Lyapunov functional V and the appropriate set $\mathbb{D}$ for our SEIR(S) model.

Contribution statement

The first author suggested the work on this SEIR(S) model (1.1) and its extension (3.1). The 2nd and 3rd authors simulated all pictures (Figures 5.1–5.6) of this paper by MATLAB and checked the calculations for the related Lyapunov functionals.

Interest statement

There is not any conflict of interest and any other competitive interest of whatever nature.

Data statement

We have not used any real-world data. All our plotted data have arisen from the simulations.

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Topological dimensions of random attractors for a stochastic reaction-diffusion equation with delay

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Abstract. The aim of this paper is to obtain an estimation of Hausdorff as well as fractal dimensions of random attractors for a stochastic reaction-diffusion equation with delay. The stochastic equation is firstly transformed into a delayed random partial differential equation by means of a random conjugation, which is then recast into an auxiliary Hilbert space. For the obtained equation, it is firstly proved that it generates a random dynamical system (RDS) in the auxiliary Hilbert space. Then it is shown that the equation possesses random attractors by a uniform estimate of the solution and the asymptotic compactness of the generated RDS. After establishing the variational equation in the auxiliary Hilbert space and the almost surely differentiable properties of the RDS, upper estimates of both Hausdorff and fractal dimensions of the random attractors are obtained.

Keywords: Hausdorff dimension, fractal dimension, random dynamical system, random attractors, delay, stochastic reaction-diffusion equations.

2020 Mathematics Subject Classification: 37L25, 37L55, 37B55, 60H15, 35R60.

1 Introduction

Existence and estimation of topological dimensions of attractors play important roles in the study of the long time behavior of deterministic or random dynamical systems. For many infinite dimensional systems generated by deterministic or stochastic partial differential equations and delay differential equations, the existence of attractors can reduce the essential part of the flow to a compact set. The finite dimensionality of the attractors, which represents the

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number of degrees of freedom presented in the long term dynamics of the system can further simplify global dynamics of complex nonlinear systems and hence it is of great significance.

The theory of attractors for deterministic infinite dimensional dynamical systems has been well established (see the monograph [26]). On the other hand, the study of random attractors for RDSs dates back to the pioneer works [15,16,24], where H. Crauel, F. Flandoli, B. Schmalfuß, amongst others, generalized the concept of global attractors of infinite dimensional dissipative systems and established the basic framework of random attractors for infinite dimensional RDSs. Since then, the existence, dimension estimation and qualitative properties of random attractors for various stochastic nonlinear evolution equations or stochastic functional differential equations have been investigated by many researchers. For example, for the stochastic reaction-diffusion equation without time delay, Caraballo et al. [7], Gao et al. [25] and Li and Guo [33] explored the existence of global attractors on bounded domains. In [2], [42] and [45], the authors obtained the existence of global attractors on unbounded domains. For the stochastic reaction diffusion equation with delay, the existence of random attractors and their structure have been studied in [5,8,12,32,41] and the references therein.

Criteria for the finite Hausdorff dimensionality of attractors for deterministic fluid dynamics models have been derived by Douady and Oesterle [20], which were later generalized by Constantin, Foiaș and Temam [13] (see also Temam [40]). Then, it was further extended to the stochastic case in [17] and [37], where the RDS is first linearized and the global Lyapunov exponents of the linearized mapping is later examined. The main difficulty of this method lies in controlling the difference between the original nonlinear RDS and its linearization, since in the stochastic case, the attractor is a random set which is not uniformly bounded. Debussche showed that the random attractors of many random dynamical systems generated by dissipative evolution equations have finite Hausdorff dimension by an ergodicity argument in [18] and further derived a precise bound on the dimension by combining the method of linearization and Lyapunov exponents in [19]. With respect to the fractal dimensionality of random sets, Langa proved the finite fractal dimensionality of the random attractor associated to a model from fluid dynamics in [30]. Langa and Robinson generalized the method in [19] to the fractal dimension by requiring differentiability of RDS in [31]. Recently, the above established framework was generalized and adopted to various stochastic and random evolution equations. For instance, Fan proved the existence of random attractor and obtained an upper bound of the Hausdorff and fractal dimension of the random attractor for a stochastic wave equations in [23] by using the method in [19]. In the recent work [46], Zhou and Zhao proved the finiteness of fractal dimension of random attractor for stochastic damped wave equation with linear multiplicative white noise.

Despite the fact that the finite Hausdorff and fractal dimensionality of attractors for abstract RDSs and applications to stochastic partial differential equations (SPDEs) have been extensively and intensively studied, to the best of our knowledge, the estimation of dimensions of SPDEs with delay, i.e., the stochastic partial functional differential equations (SPFDEs) have not been extensively studied. There are only some early results on the existence and local stability of solutions [6,27,39] and recent results on the existence and qualitative properties of random attractors [28,29,32,41,45]. Indeed, the dimension estimation of attractors for delayed partial differential equations is scarce even for the deterministic case. To this respect, the only works about dimensions of attractors for partial functional differential equations (PFDEs) we could find are [38] and the very recent work [36]. In this paper, we make an attempt to estimate topological dimensions of random attractors for a stochastic delayed reaction-diffusion equation. Specifically, we consider the following SPFDE with additive noise

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) - \mu u(x,t) + f\left(u(x,t-\tau)\right) + \sum_{j=1}^{m} g_j(x)\frac{\mathrm{d}w_j(t)}{\mathrm{d}t}, t > 0, x \in \mathcal{O}, \\ u_0(x,s) = \phi(x,s), -\tau \le s \le 0, x \in \mathcal{O} \\ u_0(x,t) = 0, -\tau \le t, x \in \partial\mathcal{O}, \end{cases} \tag{1.1}$$

where $\mathcal{O} \subseteq \mathbb{R}^N$ is a bounded open domain with smooth boundary $\partial\mathcal{O}$, $\{w_j\}_{j=1}^m$ are mutually independent two-sided real-valued Wiener process on an appropriate probability space to be specified below. Equation (1.2) can model many processes from chemistry or mathematical biology. For instance, it can be used to describe the evolution of mature populations for age-structured species, where Δu and μu represent the spatial diffusion and the death rate of mature individuals, τ is a positive number, representing the maturation time. The maturation time $f\left(u(x,t-\tau)\right)$ represents birth rate, $\sum_{j=1}^m g_j\frac{\mathrm{d}w_j(t)}{\mathrm{d}t}$ stands for the random perturbations or environmental effects.

Let $\mathbb{X} = L^2(\mathcal{O})$ be the space of square Lesbegue integrable functions on $\mathcal{O}$ with its usual norm $\|\cdot\|_{\mathbb{X}}$ and inner product $(\cdot,\cdot)_{\mathbb{X}}$, $\mathcal{C} = C([-\tau,0],\mathbb{X})$ be the space of continuous function from $[-\tau,0]$ to $\mathbb{X}$ with the usual supremum norm $\|\cdot\|_{\mathcal{C}}$ and $A = \Delta$. Let $u \in C([-\tau,T],\mathbb{X})$ and for each $t \in [0,T]$ define the function $u_t : [-\tau,0] \to \mathbb{X}$ by $u_t(\xi) = u(t+\xi)$ for $\xi \in [0,T]$. Then, we can rewrite the term $\tilde{f}(u(t-\tau)) = f(u_t)$ for any $u \in C([-\tau,T]$, by simply defining $f(\phi) = \tilde{f}(\phi(-\tau))$, for $\phi \in \mathcal{C}$ (notice that we are identifying the function $\tilde{f}$ in problem (1.1) with its associated Nemitskii operator: $\tilde{f}(u(x,t-\tau) \equiv \tilde{f}(u(t-\tau))(x)$ for all $x \in \mathcal{O}$ and $t \in [0,T]$). However, in order to deal with weak solutions of problem (1.1), we need to have the functional $\tilde{f}$ well defined in a bigger space than $\mathcal{C}$, namely we will extend the definition of $\tilde{f}$ to the Hilbert space $\mathcal{L} \triangleq L^2([-\tau,0],\mathbb{X})$ of all square Lebesgue integral functions from $[-\tau,0]$ to $\mathbb{X}$ equipped with the inner product $(\varphi,\psi)_{\mathcal{L}} = [\int_{-\tau}^0 (\varphi(s),\psi(s))_{\mathbb{X}}^2 ds]^{1/2}$ and norm $\|\varphi\|_{\mathcal{L}} = [\int_{-\tau}^0 \|\varphi(s)\|_{\mathbb{X}}^2 ds]^{1/2}$ for all $\varphi \in \mathcal{L}$. This can be done by imposing appropriate assumptions on the function f (or equivalently on $\tilde{f}$). This is explained in details in the next section (see also [10, 11]). From now on we will identify the notation of f and its extension to the space $\mathcal{L}$.

Then (1.1) can be written as the following abstract SPFDE in $\mathbb{X} = L^2(\mathcal{O})$

$$\frac{du(t)}{dt} = Au(t) - \mu u(t) + f\left(u_t\right) + \sum_{j=1}^{m} g_j \frac{\mathrm{d}w_j(t)}{\mathrm{d}t}. \tag{1.2}$$

The main difficulty for studying the topological dimensions of (1.2) lies in the fact that the natural phase spaces for deterministic or stochastic PFDEs are Banach spaces while all the above mentioned theories are established for dynamical systems in Hilbert spaces. Hence, in [38], the authors associated the deterministic PFDE with a nonlinear semigroup on a product space, i.e. a Hilbert space. In this paper, we extend the method established in [38] and [36] to the stochastic case. Nevertheless, the extension is not trivial since the RDSs are nonautonomous in nature and the random attractor is not uniformly bounded. In [38], the authors assumed that the deterministic PFDEs are dissipative which directly implies the existence of attractors in the auxiliary Hilbert space. In this paper, we will firstly prove the existence of a random attractor for (1.2) in the auxiliary Hilbert space and then provide explicit upper bounds of the Hausdorff and fractal dimensionality for the obtained attractor.

The rest of this paper is organized as follows. In Section 2, we introduce some notation, hypotheses and recast (1.2) into a Hilbert space. In Section 3, we prove the obtained auxiliary equation admits a global mild solution which generates a RDS and possesses a random attractor under certain conditions. In Section 4, we obtain an upper bound of the Hausdorff and fractal dimensions for the random attractor of the auxiliary equation, which directly implies the finite dimensionality of the original equation (1.2). Finally, we conclude the paper by pointing out some potential research directions.

2 Auxiliary equation

In this paper, we consider the canonical probability space $(\Omega, \mathcal{F}, P)$ with

$$\Omega = \{\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in C(\mathbb{R}; \mathbb{R}^m) : \omega(0) = 0\}$$

and $\mathcal{F}$ being the Borel σ-algebra induced by the compact open topology of Ω, while P being the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then, we identify $W(t)$ with $\omega(t)$, i.e.,

$$W(t) \equiv (\omega_1(t), \omega_2(t), \ldots, \omega_m(t)) \quad \text{for } t \in \mathbb{R},$$

and the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad t \in \mathbb{R}.$$

We now follow the idea of [21] to transform (1.2) into a pathwise deterministic equation. The same idea has been adopted by many authors when dealing with random attractors or invariant manifolds for various stochastic evolution equations, such as [22,28,32,34]. Consider the stochastic stationary solution of the one dimensional Ornstein–Uhlenbeck equation

$$\mathrm{d}z_j + \mu z_j \mathrm{d}t = \mathrm{d}w_j(t), \qquad j = 1, \ldots, m, \tag{2.1}$$

which is given by

$$z_j(t) \triangleq z_j(\theta_t\omega_j) = -\mu \int_{-\infty}^{0} e^{\mu s} (\theta_t\omega_j)(s)\mathrm{d}s, \qquad t \in \mathbb{R}. \tag{2.2}$$

By Definition 3.4 (in Section 3), one can see that the random variable $|z_j(\omega_j)|$ is tempered and $z_j(\theta_t\omega_j)$ is P-a.e. ω continuous. Therefore, Proposition 4.3.3 in [1] implies that there exists a tempered function $0 < r(\omega) < \infty$ such that

$$\sum_{j=1}^{m} |z_j(\omega_j)|^2 \le r(\omega), \tag{2.3}$$

where $r(\omega)$ satisfies, for P-a.e. $\omega \in \Omega$,

$$r(\theta_t\omega) \le e^{\frac{\mu}{2}|t|} r(\omega), \qquad t \in \mathbb{R}. \tag{2.4}$$

Combining (3.11) with (2.4), we obtain that for P-a.e. $\omega \in \Omega$,

$$\sum_{j=1}^{m} |z_j(\theta_t\omega_j)|^2 \le e^{\frac{\mu}{2}|t|} r(\omega), \qquad t \in \mathbb{R}. \tag{2.5}$$

Moreover, we have

$$\sum_{j=1}^{m} |z_j(\theta_\xi\omega_j)|^2 \le e^{\frac{\mu\tau}{2}} r(\omega), \tag{2.6}$$

for any $\xi \in [-\tau, 0]$ and P-a.e. $\omega \in \Omega$. Putting $z(\theta_t\omega) = \sum_{j=1}^m g_j z_j(\theta_t\omega_j)$, we have

$$\mathrm{d}z + \mu z \mathrm{d}t = \sum_{j=1}^m g_j \mathrm{d}w_j.$$

Take the transformation $v(t) = u(t) - z(\theta_t\omega)$. Then, simple computation gives

$$\frac{dv(t)}{dt} = Av(t) - \mu v(t) + f(v_t + z(\theta_{t+\cdot}\omega)) + Az(\theta_t\omega), \tag{2.7}$$

where $\theta_{t+\cdot}\omega$ is defined as $\theta_{t+\xi}\omega$ for $\xi \in [-\tau, 0]$.

Throughout the remaining part of this paper, we always impose the following assumptions on A and the nonlinear term f:

Hypothesis A1 $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a densely defined linear operator that generates a strongly continuous compact semigroup $S(t)$ on $\mathbb{X}$. Moreover, $\varrho \triangleq s(\tilde{A}) - \mu < 0$, where $s(\tilde{A})$ is defined by $s(\tilde{A}) := \sup\{\Re\lambda : \lambda \in \sigma(\tilde{A})\}$ representing the spectral bound of the linear operator $\tilde{A}$.

Hypothesis A2 $f : \mathcal{C} \to \mathbb{X}$ is Lipschitz continuous with $\mathbf{0}$ being a fixed point, that is, $f(\mathbf{0}) = \mathbf{0}$ and there exists $L_f > 0$ such that

$$\|f(\phi) - f(\varphi)\|_{\mathbb{X}} \leq L_f \|\phi - \varphi\|_{\mathcal{C}},$$

for any $\phi, \varphi \in \mathcal{C}$. Moreover, there exists $m_0 \geqslant 0$ and $C_f > 0$ such that for all $l \in [0, m_0], 0 \leqslant t, u$ and $v \in C([-\tau, t]; \mathbb{X})$, the following inequality holds

$$\int_0^t e^{ls} \|f(u_s) - f(v_s)\|_{\mathbb{X}}^2 \,\mathrm{d}s \leqslant C_f^2 \int_{-\tau}^t e^{ls} \|u(s) - v(s)\|_{\mathbb{X}}^2 \,\mathrm{d}s. \tag{2.8}$$

Remark 2.1. Notice that, thanks to Hypothesis A2, given $u \in C^0([-\tau, T]; \mathbb{X})$, the function $f_u : t \in [0, T] \to \mathbb{X}$ defined by $f_u(t) = f(u_t)\ \forall\, t \in [0, T]$, is measurable (see Bensoussan et al. [4]) and, in fact, belongs to $L^\infty(0, T; \mathbb{X})$. Then, thanks to (2.8), the mapping

$$\mathcal{F} : u \in C^0([-\tau, T]; X) \to f_u \in L^2(0, T; \mathbb{X})$$

has a unique extension to a mapping $\widetilde{\mathcal{F}}$ which is uniformly continuous from $L^2(-\tau, T; \mathbb{X})$ into $L^2(0, T; \mathbb{X})$. From now on, we will denote $f(u_t) = \widetilde{\mathcal{F}}(u)(t)$ for each $u \in L^2(-h, T; \mathbb{X})$, and thus, $\forall\, t \in [0, T]$, $\forall\, u, v \in L^2(-\tau, T; \mathbb{X})$, we will have

$$\int_0^t e^{ls} \|f(u_s) - f(v_s)\|_{\mathbb{X}}^2 \,\mathrm{d}s \leqslant C_f^2 \int_{-\tau}^t e^{ls} \|u(s) - v(s)\|_{\mathbb{X}}^2 \,\mathrm{d}s.$$

Remark 2.2. Observe that considering the abstract formulation of our original problem with a functional f satisfying Assumption A2, we not only are considering the case of constant delay $(f(u_t) = f(u(t - \tau))$ but also the distributed delay one as well, that is, when $f(u_t) = \int_\tau^0 g(s, u(t+s))ds$, for an appropriate Lipschitz function g (see Caraballo and Real [9] for more information).

Since for P-a.e. $\omega \in \Omega$, (2.7) is a path-wise deterministic equation, by similar techniques as [10, Theorem 2.3] and [43, Theorem 8], we have the following results on the existence of solutions to (2.7).

Lemma 2.3. *Assume that* ***Hypotheses A1–A2*** *hold. Then, for any initial condition* $(\phi, \phi(0)) \in \mathcal{L} \times \mathbb{X}$*, there exists a solution* $v(\cdot, \omega, \phi)$ *to problem* (2.7) *with* $v(\cdot, \omega, \phi) \in L^2(-r, T; \mathbb{X}) \cap L^\infty(0, T; \mathbb{X}) \cap C([-r, T]; \mathbb{X}), \forall T > 0$ *and P-a.e.* $\omega \in \Omega$.

In order to estimate topological dimensions of the random attractors of (1.2), unlike previous works [5, 8, 12, 32, 41], where v_t is taken as the state and $\mathcal{L}$ as state space for the above obtained pathwise deterministic delayed equation (2.7), we take $V(t) = (v_t, v(t))$ as state space and recast the equation into an auxiliary product space $H = \mathcal{L} \times \mathbb{X}$ equipped with the inner product

$$\langle (\phi, l), (\psi, k) \rangle = \int_{-\tau}^{0} (\phi(s), \psi(s))_{\mathbb{X}} ds + (h, k)_{\mathbb{X}} \quad \text{for } (\phi, l), (\psi, k) \in H$$

and norm

$$\|(\phi, l)\| = \langle (\phi, l), (\phi, l) \rangle^{1/2} \quad \text{for } (\phi, l) \in H,$$

making H a Hilbert space and hence we can overcome the lack of Hilbert space geometry in applying the abstract theory established in [18, 19, 30, 31]. Furthermore, recasting (1.2) into the Hilbert space H also facilitates us to construct an appropriate variational equation. Take $V(t) = (v_t, v(t))$,

$$\hat{f}(t, \theta_t\omega, v_t) \triangleq Az(\theta_t\omega) + f\left(v_t + z(\theta_{t+\cdot}\omega)\right) \tag{2.9}$$

and

$$F(t, \theta_t\omega, V(t)) = (0, \hat{f}(t, \theta_t\omega, v_t)). \tag{2.10}$$

We consider the following auxiliary random partial differential equation on H.

$$\begin{cases} \dfrac{dV(t)}{dt} = \tilde{A}V(t) - \tilde{L}V(t) + F(t, \theta_t\omega, V(t)), \\ V(0) = (\phi, l), \quad (\phi, l) \in H, \end{cases} \tag{2.11}$$

where operator $\tilde{A}$ is defined as

$$\tilde{A} := \begin{pmatrix} \frac{d}{dt} & 0 \\ 0 & A \end{pmatrix}, \tag{2.12}$$

with domain

$$D(\tilde{A}) = \left\{ (\phi, l) \in H : \phi \text{ is differentiable on } [-\tau, 0], \dot{\phi} \in \mathcal{L} \text{ and } h = \phi(0) \in D(A) \right\}.$$

The linear operator $\tilde{L}$ is defined by

$$\tilde{L} := \begin{pmatrix} 0 & 0 \\ 0 & \mu I \end{pmatrix}.$$

It follows from the definition of $\tilde{L}$ that

$$\|\tilde{L}\| \triangleq \sup_{\varphi \in H, \|\varphi\| = 1} \|\tilde{L}\varphi\| \leq \mu. \tag{2.13}$$

It follows from **Hypothesis A1**, Lemma 3.6, Theorem 3.25 in [3] that the operator $(\tilde{A}, D(\tilde{A}))$ is closed and densely defined on H, and generates a strongly continuous semigroup $\tilde{S}(t)$ given by

$$\tilde{S}(t) := \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix},$$

where $(T_0(t))_{t\geq 0}$ is the nilpotent left shift semigroup on $\mathcal{L}$, and $S_t : \mathbb{X} \to \mathcal{L}$ is defined by

$$(S_t x)(\xi) := \begin{cases} S(t+\xi)x & \text{if } -t < \xi \leq 0, \\ 0 & \text{if } -\tau \leq \xi \leq -t. \end{cases}$$

Moreover, by Theorem 4.11 in [3], we have

$$\|\tilde{S}(t)\| \leqslant e^{s(\tilde{A})t}, \qquad t \geqslant 0.$$

Furthermore, it follows from Pazy [35, Theorem 6.1.5] that (2.11) admits a global classical solution which can be represented by a integral equation based on the variation of constants formula.

Theorem 2.4. *Assume that* **Hypothesis A1** *holds and f is continuously differentiable. Then, for each $(\phi, l) \in H$, there exists a continuous function $V(\cdot, \omega, (\phi, l)) : [0, \infty) \to H$ such that*

$$V(t, \omega, (\phi, l)) = e^{-\tilde{L}t}\tilde{S}(t)(\phi, l) + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)F(s, \theta_s\omega, V(s, \omega, (\phi, l)))ds, \qquad t \geqslant 0 \quad (2.14)$$

for P-a.e. $\omega \in \Omega$. Moreover, if $(\phi, l) \in D(\tilde{A})$, then $V(t, \omega, (\phi, l))$ is a strong solution of (2.11).

3 Random attractors

This section is devoted to showing the existence of random attractors for the auxiliary equation (2.11). In the sequel, we first introduce the concept of random attractor and random dynamical systems following [1] and [15, 16, 24]. Subsequently, we prove the existence of tempered pullback attractors for the auxiliary equation (2.11) by first establishing a uniform estimation for the solution and then proving that the RDS generated by (1.2) is pullback asymptotically compact. Unlike the previous works [5, 8, 12, 41], we prove the uniform a priori estimates of the solution by using the semigroup approach instead of taking inner product.

Definition 3.1. Let $\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}$ be a family of measure preserving transformations such that $(t, \omega) \mapsto \theta_t\omega$ is measurable and $\theta_0 = \text{id}$, $\theta_{t+s} = \theta_t\theta_s$, for all $s, t \in \mathbb{R}$. The flow θ_t together with the probability space $\left(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}}\right)$ is called a metric dynamical system.

It follows from Definition 3.1 that $\left(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}}\right)$ is a metric dynamical system, where $(\Omega, \mathcal{F}, P)$ is defined in Section 2. Moreover, θ is ergodic. For a given separable Hilbert space $(H, \|\cdot\|_H)$, denote by $\mathcal{B}(H)$ the Borel σ-algebra generated by open subsets in H.

Definition 3.2. A mapping $\Phi : \mathbb{R}^+ \times \Omega \times H \to H$ is said to be a random dynamical system (RDS) on a complete separable metric space (H, d) with Borel σ-algebra $\mathcal{B}(H)$ over the metric dynamical system $\left(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}^+}\right)$ if

(i) $\Phi(\cdot, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times H \to H$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(H), \mathcal{B}(H))$-measurable;

(ii) $\Phi(0, \omega, \cdot)$ is the identity on H for P-a.e. $\omega \in \Omega$;

(iii) $\Phi(t+s, \omega, \cdot) = \Phi(t, \theta_s\omega, \cdot) \circ \Phi(s, \omega, \cdot)$, for all $t, s \in \mathbb{R}^+$ for P-a.e. $\omega \in \Omega$.

A RDS Φ is continuous or differentiable if $\Phi(t, \omega, \cdot) : H \to H$ is continuous or differentiable for all $t \in \mathbb{R}^+$ and P-a.e. $\omega \in \Omega$.

Definition 3.3. A set-valued map $\Omega \ni \omega \mapsto D(\omega) \in 2^H$, such that $D(\omega)$ is closed, is said to be a random set in H if the mapping $\omega \mapsto d(x, D(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for any $x \in H$, where $d(x, D(\omega)) \triangleq \inf_{y \in D(\omega)} \mathrm{d}(x, y)$ is the distance in H between the element x and the set $D(\omega) \subset H$.

Definition 3.4. A random set $\{D(\omega)\}_{\omega \in \Omega}$ of H is called tempered with respect to $(\theta_t)_{t \in \mathbb{R}}$ if for P-a.e. $\omega \in \Omega$,

$$\lim_{t \to \infty} e^{-\beta t} d\left(D\left(\theta_{-t}\omega\right)\right) = 0, \quad \text{for all } \beta > 0,$$

where $d(D) = \sup_{x \in D} \|x\|_H$.

Definition 3.5. Let $\mathcal{D} = \{D(\omega) \subset H, \omega \in \Omega\}$ be a family of random sets. A random set $K(\omega) \in \mathcal{D}$ is said to be a $\mathcal{D}$-pullback absorbing set for Φ if for P-a.e. $\omega \in \Omega$ and for every $B \in \mathcal{D}$, there exists $T = T(B, \omega) > 0$ such that

$$\Phi\left(t, \theta_{-t}\omega, B\left(\theta_{-t}\omega\right)\right) \subseteq K(\omega) \quad \text{for all } t \geq T.$$

If, in addition, for all $\omega \in \Omega, K(\omega)$ is measurable in Ω with respect to $\mathcal{F}$, then we say K is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

Definition 3.6. A RDS Φ is said to be $\mathcal{D}$-pullback asymptotically compact in H if for P-a.e. $\omega \in \Omega$, $\{\Phi\left(t_n, \theta_{-t_n}\omega, x_n\right)\}_{n \geq 1}$ has a convergent subsequence in H whenever $t_n \to \infty$ and $x_n \in D\left(\theta_{-t_n}\omega\right)$ for any given $D \in \mathcal{D}$.

Definition 3.7. A compact random set $\mathcal{A}(\omega)$ is said to be a $\mathcal{D}$-pullback random attractor associated to the RDS Φ if it satisfies the invariance property

$$\Phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}\left(\theta_t \omega\right), \quad \text{for all } t \geq 0$$

and the pullback attracting property

$$\lim_{t \to \infty} \operatorname{dist}\left(\Phi\left(t, \theta_{-t}\omega, D\left(\theta_{-t}\omega\right)\right), \mathcal{A}(\omega)\right) = 0, \quad \text{for all } t \geq 0, D \in \mathcal{D}, P - a.e.\ \omega \in \Omega.$$

where $\operatorname{dist}(\cdot, \cdot)$ denotes the Hausdorff semidistance

$$\operatorname{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} \mathrm{d}(x, y), \qquad A, B \subset H.$$

Lemma 3.8 ([15, Theorem 3.11])**.** *Let (θ, Φ) be a continuous random dynamical system. Suppose that Φ is $\mathcal{D}$-pullback asymptotically compact and has a closed pullback $\mathcal{D}$-absorbing random set $K = \{K(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$. Then it possesses a random attractor $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$, where*

$$\mathcal{A}(\omega) = \cap_{\tau \geq 0} \overline{\cup_{t \geq \tau} \Phi\left(t, \theta_{-t}\omega, K\left(\theta_{-t}\omega\right)\right)}.$$

For convenience, we introduce the following Gronwall inequality in [5] that will be frequently used in our subsequent proofs.

Lemma 3.9. *Let $T > 0$ and u, α, f and g be non-negative continuous functions defined on $[0, T]$ such that*

$$u(t) \leq \alpha(t) + f(t) \int_0^t g(r) u(r) dr, \quad \textit{for } t \in [0, T].$$

Then,

$$u(t) \leq \alpha(t) + f(t) \int_0^t g(r) \alpha(r) e^{\int_r^t f(\tau) g(\tau) d\tau} dr, \quad \textit{for } t \in [0, T].$$

Apparently, under the conjugation transformation induced by (2.2), no exceptional sets appear in the equation (2.11). By the uniqueness of solution to (2.11) for each $\omega \in \Omega$, we can see the mapping $\Phi(\cdot,\cdot,\cdot) : \mathbb{R}^+ \times \Omega \times H \to H$ defined by

$$\Phi(t,\omega,(\phi,\phi(0))) = V(t,\omega,(\phi,\phi(0))) \tag{3.1}$$

generates a RDS, which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(H), \mathcal{B}(H))$-measurable. Let P_1 and P_2 be the projections of H onto $\mathcal{L}$ and $\mathbb{X}$ respectively. Then, by Theorem 3.1 and Proposition 3.2 in [38], we have

$$v_t(\cdot,\omega,\phi) = P_1 V(t,\omega,(\phi,\phi(0))) \tag{3.2}$$

and

$$v(t,\omega,\phi) = P_2 V(t,\omega,(\phi,\phi(0))) \tag{3.3}$$

for $t \geqslant 0$ and P-a.e. $\omega \in \Omega$, where $v(t,\omega,\phi)$ is the solution to (2.7). Therefore, the solution of (1.2) can be represented by

$$\begin{aligned} u_t(\cdot,\omega,\phi) &= v_t(\cdot,\omega,\phi) + z(\theta_{t+\cdot}\omega) = P_1[V(t,\omega,(\phi,\phi(0))) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega))] \\ &\triangleq P_1\Psi(t,\omega,(\psi,\psi(0))) \end{aligned} \tag{3.4}$$

where the mapping $\Psi : \mathbb{R}^+ \times \Omega \times H \to H$ is defined by

$$\begin{aligned} \Psi(t,\omega,(\psi,\psi(0))) &\triangleq \Phi(t,\omega,(\phi,\phi(0))) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega)) \\ &= V(t,\omega,(\phi,\phi(0))) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega)) \end{aligned} \tag{3.5}$$

and $(\psi,\psi(0)) = (\phi,\phi(0))) + (z(\theta_\cdot\omega), z(\omega))$. By the cocycle property of z and Φ, we can see that Ψ is a RDS on H. In the following, we show the existence of random attractor for Ψ.

Lemma 3.10. *Assume that **Hypotheses A1–A2** hold and $\varrho \triangleq s(\tilde{A}) - \mu < \frac{-\mu}{2}$, $\varrho + L_f < 0$, then there exists $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ satisfying that, for any $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there is $T_B(\omega) > 0$ such that*

$$\Psi\left(t,\theta_{-t}\omega, B\left(\theta_{-t}\omega\right)\right) \subseteq K(\omega) \quad \textit{for all } t \geqslant T_B(\omega),$$

that is, $\{K(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for Ψ in $\mathcal{D}$.

Proof. We first derive uniform estimates of V by (2.14) and then obtain the existence of an absorbing set for Ψ given by $\Psi(t,\omega,(\phi,\phi(0)) = V(t,\omega,(\phi,\phi(0))) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega))$. It follows from (2.14) that, for any $t > 0$,

$$\begin{aligned} &\|V(t,\omega,(\phi,\phi(0)))\| \\ &\quad = \left\| e^{-\tilde{L}t}\tilde{S}(t)(\phi,\phi(0)) + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)F(s,\theta_s\omega, V(s,\omega,(\phi,\phi(0))))ds \right\| \\ &\quad \le e^{\varrho t}\|(\phi,\phi(0))\| + \int_0^t e^{\varrho(t-s)}\|\tilde{f}(s,\theta_s\omega, v_s(\cdot,\omega,\phi)))\|_{\mathbb{X}} ds \\ &\quad \le e^{\varrho t}\|(\phi,\phi(0))\| + \int_0^t e^{\varrho(t-s)}(\|Az(\theta_s\omega)\|_{\mathbb{X}} + L_f\|z(\theta_{s+\cdot}\omega)\|_{\mathcal{L}})ds \\ &\qquad + L_f\int_0^t e^{\varrho(t-s)}\|v_s(\cdot,\omega,\phi)\|_{\mathcal{L}} ds \\ &\quad \le e^{\varrho t}\|(\phi,\phi(0))\| + \int_0^t e^{\varrho(t-s)}(\|Az(\theta_s\omega)\|_{\mathbb{X}} + L_f\|z(\theta_{s+\cdot}\omega)\|_{\mathcal{L}})ds \\ &\qquad + L_f\int_0^t e^{\varrho(t-s)}\|V(s,\omega,(\phi,\phi(0)))\| ds \end{aligned} \tag{3.6}$$

for P-a.e. $\omega \in \Omega$. For the sake of simplicity, we denote $\varpi(\omega) = (\phi(\cdot,\omega), \phi(0,\omega))$. By replacing ω by $\theta_{-t}\omega$, we derive from (3.6) that, for all $t \geq 0$,

$$\begin{aligned} \|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\| \leq e^{\varrho t}\|\varpi(\theta_{-t}\omega)\| + L_f \int_0^t e^{\varrho(t-s)}\|V(s,\theta_{-t}\omega,\varpi(\theta_{-t}\omega)\|ds \\ + \int_0^t e^{\varrho(t-s)}(\|Az(\theta_{-t}\theta_s\omega)\|_{\mathbb{X}} + L_f\|z(\theta_{-t}\theta_{s+\cdot}\omega)\|_{\mathcal{L}})ds. \end{aligned} \tag{3.7}$$

Since $g_j \in \mathbb{X}$, $Ag_j \in \mathbb{X}$ and $z(\omega) = \sum_{j=1}^m g_j z_j(\omega_j)$, it follows from (2.5) and (2.6) that there exists a constant c such that $p_1(\omega) \triangleq \|Az(\omega)\|_{\mathbb{X}} + L_f\|z(\theta_{\cdot}\omega)\|_{\mathcal{L}} \leq c\sum_{j=1}^m |z_j(\omega_j)|^2$. Therefore, it follows from (2.4) and (2.5) that

$$\int_0^t e^{\varrho(t-s)} p_1(\theta_{s-t}\omega)\,\mathrm{ds} \leq c\int_0^t e^{(\varrho+\frac{\mu}{2})(t-s)} r(\omega)\mathrm{ds} \leq cr(\omega), \tag{3.8}$$

where the second inequality follows from the assumption that $\rho + \frac{\mu}{2} < 0$. Incorporating (3.8) into (3.7) gives rise to

$$\|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega)\| \leq e^{\varrho t}\|\varpi(\theta_{-t}\omega)\| + L_f\int_0^t e^{\varrho(t-s)}\|V(s,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\|ds + cr(\omega). \tag{3.9}$$

Multiplying both sides of (3.9) by $e^{-\varrho t}$,

$$\begin{aligned} &e^{-\varrho t}\|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\| \\ &\quad \leq \|\varpi(\theta_{-t}\omega)\| + L_f\int_0^t e^{-\varrho s}\|V(s,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\|ds + ce^{-\varrho t}r(\omega). \end{aligned} \tag{3.10}$$

Hence, by the Gronwall inequality (Lemma 3.9), we have

$$\begin{aligned} e^{-\varrho t}\|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\| &\leq \|\varpi(\theta_{-t}\omega)\| + ce^{-\varrho t}r(\omega) \\ &\quad + L_f\int_0^t e^{L_f(t-s)}(\|\varpi(\theta_{-s}\omega)\| + ce^{-\varrho s}r(\omega))ds \\ &\leq \|\varpi(\theta_{-t}\omega)\| + ce^{-\varrho t}r(\omega) + L_f\|\varpi(\theta_{-t}\omega)\|\int_0^t e^{L_f(t-s)}\mathrm{ds} \\ &\quad + cL_f r(\omega)\int_0^t e^{L_f(t-s)}e^{-\varrho s}\mathrm{ds}. \end{aligned} \tag{3.11}$$

Therefore, we have

$$\begin{aligned} \|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\| \leq e^{\varrho t}\|\varpi(\theta_{-t}\omega)\| + cr(\omega) + e^{\varrho t}(e^{L_f t}-1)\|\varpi(\theta_{-t}\omega)\| \\ + \frac{cL_f}{\varrho + L_f}[e^{(L_f+\varrho)t} - 1]r(\omega). \end{aligned} \tag{3.12}$$

Note that $\Psi(t,\omega,\chi(\omega)) = V(t,\omega,\varpi(\omega)) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega))$ and $\chi(\omega) = \varpi(\omega) + (z(\theta_{\cdot}\omega), z(\omega))$. The above estimate (3.12) implies that, for all $t \geq 0$

$$\begin{aligned} \|\Psi(t,\theta_{-t}\omega,\chi(\theta_{-t}\omega))\| &\leq \|V(t,\theta_{-t}\omega,\varpi(\theta_{-t}\omega))\| + \|(z(\theta_{-t}\theta_{t+\cdot}\omega), z(\theta_{-t}\theta_t\omega))\| \\ &\leq e^{\varrho t}\|\varpi(\theta_{-t}\omega)\| + 2cr(\omega) + e^{\varrho t}(e^{L_f t}-1)\|\varpi(\theta_{-t}\omega)\| \\ &\quad + \frac{cL_f}{\varrho + L_f}[e^{(L_f+\varrho)t} - 1]r(\omega). \end{aligned} \tag{3.13}$$

Therefore, if $\chi \in \mathcal{D}(\theta_{-t}\omega)$ and $L_f + \varrho < 0$, then there exists a $T_{\mathcal{D}} > 0$ such that, for all $t \geq T_D(\omega)$,

$$e^{\varrho t}\|\varpi(\theta_{-t}\omega)\| + e^{\varrho t}(e^{L_f t} - 1)\|\varpi(\theta_{-t}\omega)\| + \frac{cL_f}{\varrho + L_f}e^{(L_f+\varrho)t}r(\omega) \leq c_1(\omega), \tag{3.14}$$

which, along with (3.13) shows that, for all $t \geq T_{\mathcal{D}}(\omega)$

$$\|\Psi(t, \theta_{-t}\omega, \chi(\theta_{-t}\omega))\| \leq 2cr(\omega) + \frac{-cL_f}{\varrho + L_f}r(\omega) + c_1(\omega). \tag{3.15}$$

Given $\omega \in \Omega$, define

$$K(\omega) = \{\varphi \in H : \|\varphi\| \leq 2cr(\omega) + \frac{-cL_f}{\varrho + L_f}r(\omega) + c_1(\omega)\}. \tag{3.16}$$

Then, $K = \{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Furthermore, (3.15) implies that $K(\omega)$ is a random absorbing set for the RDS Ψ in $\mathcal{D}$. □

Lemma 3.11. *Assume that* ***Hypotheses A1–A2*** *are satisfied and $\varrho \triangleq s(\tilde{A}) - \mu < \frac{-\mu}{2}$, $\varrho + L_f < 0$. Then, the RDS Ψ is $\mathcal{D}$-pullback asymptotically compact for $t > \tau$ (the time delay), i.e., for P-a.e. $\omega \in \Omega$, the sequence $\{\Psi(t_n, \theta_{-t_n}\omega, \phi_n(\theta_{-t_n}\omega))\}$ has a convergent subsequence provided $t_n \to \infty$, $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\phi_n(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$.*

Proof. Take an arbitrary random set $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, a sequence $t_n \to +\infty$ and $\phi_n \in B(\theta_{-t_n}\omega)$. We have to prove that $\{\Psi(t_n, \theta_{-t_n}\omega, \phi_n)\}$ is precompact. Since $\{K(\omega)\}$ is a random absorbing set for Ψ, there exists $T > 0$ such that, for all $\omega \in \Omega$,

$$\Psi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega) \tag{3.17}$$

for all $t \geq T$. Because $t_n \to +\infty$, we can choose $n_1 \geq 1$ such that $t_{n_1} - 1 \geq T$. Applying (3.17) for $t = t_{n_1} - 1$ and $\omega = \theta_{-1}\omega$, we find that

$$\eta_1 \triangleq \Psi\left(t_{n_1} - 1, \theta_{-t_{n_1}}\omega, \phi_{n_1}\right) \in K(\theta_{-1}\omega) \tag{3.18}$$

Similarly, we can choose a subsequence $\{n_k\}$ of $\{n\}$ such that $n_1 < n_2 < \cdots < n_k \to +\infty$ with $t_{n_k} \geq k$ and

$$\eta_k \triangleq \Psi\left(t_{n_k} - k, \theta_{-t_{n_k}}\omega, \phi_{n_k}\right) \in K(\theta_{-k}\omega) \tag{3.19}$$

Hence, by the assumptions we conclude that the sequence

$$\{\Psi(k, \theta_{-k}\omega, \eta_k)\} \text{ is precompact.} \tag{3.20}$$

On the other hand, by (3.19) we have

$$\Psi(k, \theta_{-k}\omega, \eta_k) = \Psi(k, \theta_{-k}\omega, \Psi(t_{n_k} - k, \theta_{-t_{n_k}}\omega, \phi_{n_k})) = \Psi\left(t_{n_k}, \theta_{-t_{n_k}}\omega, \phi_{n_k}\right), \tag{3.21}$$

for all $k \geq 1$. Combining (3.20) and (3.21), we obtain that the sequence $\left\{\Psi\left(t_{n_k}, \theta_{-t_{n_k}}\omega, \phi_{n_k}\right)\right\}$ is precompact. Therefore, $\{\Psi(t_n, \theta_{t_n}\omega, \phi_n)\}$ is precompact, which completes the proof. □

Lemma 3.10 says that the continuous RDS Ψ has a random absorbing set while Lemma 3.11 tells us that (θ, Ψ) is pullback asymptotically compact in H. Thus, it follows from Lemma 3.8 that the continuous RDS (θ, Ψ) possesses a random attractor. Namely, we obtain the following result.

Theorem 3.12. *Assume that **Hypotheses A1–A2** are satisfied and* $\varrho \triangleq s(\tilde{A}) - \mu < \frac{-\mu}{2}$, $\varrho + L_f < 0$, *then the continuous RDS* Ψ *admits a* $\mathcal{D}$*-pullback attractor* $\mathcal{A}_\Psi(\omega)$ *in* H *belonging to the class* $\mathcal{D}$.

Moreover, by Theorem 3.12, the relationship between the RSDs Φ and Ψ defined by (3.5) as well as Proposition 3.2 in [38], we have the following result about the existence of random attractors for equation (1.2).

Corollary 3.13. *Assume that **Hypotheses A1–A2** are satisfied and* $\varrho \triangleq s(\tilde{A}) - \mu < \frac{-\mu}{2}$, $\varrho + L_f < 0$. *Then, the continuous RDS* $P_1\Psi$ *generated by* (1.2) *admits a pullback attractor* $P_1\mathcal{A}_\Psi(\omega)$ *in* P_1H. *Moreover,* $\mathcal{A}_\Phi(\omega) \triangleq \{\zeta | \zeta = \chi - (z(\theta_{t+\cdot}\omega), z(\theta_t\omega)), \chi \in \mathcal{A}_\Psi(\omega)\}$ *is a random attractor of* Φ.

4 Topological dimensions of random attractors

The aim of this section is to estimate the Hausdorff and fractal dimensions for the attractor of (1.2). Denote by $d_H(\mathcal{A}_\Psi(\omega))$ and $d_F(\mathcal{A}_\Psi(\omega))$ the Hausdorff and fractal dimensions of a random set $\mathcal{A}_\Psi(\omega)$ respectively. We only need to prove that there exist constants d_H and d_F such that $d_H(\mathcal{A}_\Psi(\omega)) \leq d_H$ and $d_F(\mathcal{A}_\Psi(\omega)) \leq d_F$, since by Theorem 3.1 and Proposition 3.2 in [38], the topological dimensions of attractor $P_1\mathcal{A}_\Psi(\omega)$ for (1.2) satisfy $d_H(P_1\mathcal{A}_\Psi(\omega)) \leq d_H$ and $d_F(P_1\mathcal{A}_\Psi(\omega)) \leq d_F$, i.e., the random attractors of (1.2) have finite Hausdorff and fractal dimensions less than those of (2.11). In the sequel, we investigate the Hausdorff and fractal dimensions for the random attractor $\mathcal{A}_\Psi(\omega)$ of (2.11).

We first recall the concepts of Hausdorff and fractal dimensions of the attractor $\mathcal{A}_\Psi(\omega) \subset H$. More details can be found in [19] and [31]. The Hausdorff dimension of the compact set $\mathcal{A}_\Psi(\omega) \subset H$ is

$$d_H(\mathcal{A}_\Psi(\omega)) = \inf\{d : \mu_H(\mathcal{A}_\Psi(\omega), d) = 0\}$$

where, for $d \geq 0$,

$$\mu_H(\mathcal{A}_\Psi(\omega), d) = \lim_{\varepsilon \to 0} \mu_H(\mathcal{A}_\Psi(\omega), d, \varepsilon)$$

denotes the d-dimensional Hausdorff measure of the set $\mathcal{A}_\Psi(\omega) \subset H$, where

$$\mu_H(\mathcal{A}_\Psi(\omega), d, \varepsilon) = \inf \sum_i r_i^d$$

and the infimum is taken over all coverings $\mathcal{K} = \{B_i\}_{i \in I}$ of $\mathcal{A}_\Psi(\omega)$ by balls B_i of radius $r_i \leq \varepsilon$ and the sum is over all balls of $\mathcal{K}$. It can be shown that there exists $d_H(\mathcal{A}_\Psi(\omega)) \in [0, +\infty]$ such that $\mu_H(\mathcal{A}_\Psi(\omega), d) = 0$ for $d > d_H(\mathcal{A}_\Psi(\omega))$ and $\mu_H(\mathcal{A}_\Psi(\omega), d) = \infty$ for $d < d_H(\mathcal{A}_\Psi(\omega))$. $d_H(\mathcal{A}_\Psi(\omega))$ is called the Hausdorff dimension of $\mathcal{A}_\Psi(\omega)$.

The fractal dimension (or capacity) of $\mathcal{A}_\Psi(\omega)$ is defined as

$$d_F(\mathcal{A}_\Psi(\omega)) = \inf\{d > 0 : \mu_F(\mathcal{A}_\Psi(\omega), d) = 0\},$$

where

$$\mu_F(\mathcal{A}_\Psi(\omega), d) = \limsup_{\varepsilon \to 0} \varepsilon^d n_F(\mathcal{A}_\Psi(\omega), \varepsilon)$$

and $n_F(\mathcal{A}_\Psi(\omega), \varepsilon)$ is the minimum number of balls of radius $\leqslant \varepsilon$ which is necessary to cover $\mathcal{A}_\Psi(\omega)$.

Take a covering of $\mathcal{A}_\Psi(\omega)$ by balls of radii less than ε:

$$\mathcal{A}_\Psi(\omega) \subset \bigcup_{i=1} B(u_i, r_i), \qquad r_i \leq \varepsilon, \ u_i \in H$$

where $B(u_i, r_i)$ denotes the ball in H of center u_i and radius r_i. Let $\theta = \theta_1$ and define

$$\Psi(\omega)\phi = \Psi(1, \omega, \phi) \tag{4.1}$$

for any $\phi \in H$ and P-a.e. $\omega \in \Omega$. Then, it follows from the invariance of $\mathcal{A}_\Psi(\omega)$ that

$$\mathcal{A}_\Psi(\theta\omega) \subset \bigcup_{i=1} \Psi(\omega) B(u_i, r_i).$$

In order to approximate $\Psi(\omega)$ by a linear map, we impose the following almost surely uniformly differentiable assumption of $\Psi(\omega)$ on the attractor $\mathcal{A}_\Psi(\omega)$.

Hypothesis A3 The mapping $\Psi(\omega)$ is $\mathbb{P}$ almost surely differentiable on $\mathcal{A}_\Psi(\omega)$, that is, $\mathbb{P}$ almost surely, for every u in $\mathcal{A}_\Psi(\omega)$, there exists a continuous linear operator $D\Psi(\omega, u) : H \to H$, such that if $u, u+h \in \mathcal{A}_\Psi(\omega)$, then

$$\|\Psi(\omega)(u+h) - \Psi(\omega)u - D\Psi(\omega, u) \cdot h\| \leq K(\omega)\|h\|^{1+\alpha},$$

where $K(\omega)$ is a random variable such that

$$K(\omega) \geq 1, \quad \text{for all } \omega \in \Omega,$$

$\mathbb{E}(\ln K) < \infty$ and $\alpha > 0$.

We follow [40, Chapter 5] to give following definitions. For the bounded linear operator $D\Psi(\omega, u)$ on H and $n \in \mathbb{N}$, we set

$$\alpha_n(D\Psi(\omega, u)) = \sup_{\substack{G \subset H \\ \dim \leqslant n}} \inf_{\substack{\phi \in G \\ \|\phi\|=1}} \|D\Psi(\omega, u)\phi\|$$

and

$$\omega_n(D\Psi(\omega, u)) = \alpha_1(D\Psi(\omega, u)) \ldots \alpha_n(D\Psi(\omega, u)),$$

where $\alpha_n(D\Psi(\omega, u))$ are the square roots of the eigenvalues of $D\Psi(\omega, u)^* D\Psi(\omega, u)$ corresponding to orthogonal eigenvectors e_n, which are in decreasing order. We set

$$\alpha_\infty(D\Psi(\omega, u)) = \inf_n \alpha_n(D\Psi(\omega, u))$$

and further make the following assumptions.

Hypothesis A4 For every $d \in \mathbb{N}$, there exists an integrable random variable $\bar{\omega}_d$, such that $\mathbb{P}$ almost surely,

$$\omega_d(D\Psi(\omega, u)) \leq \bar{\omega}_d(\omega)$$

for any $u \in \mathcal{A}_\Psi(\omega)$ and

$$\mathbb{E}(\ln(\bar{\omega}_d)) < 0.$$

Under the above assumptions, we have the following results concerning the dimension estimation of random attractors $\mathcal{A}_\Psi(\omega)$ for Ψ, of which the proof is given in [19, 31].

Lemma 4.1. *Assume that* ***Hypotheses A3–A4*** *are satisfied. Then,* $\mathbb{P}$*-a.s.*

$$d_{\mathrm{H}}(\mathcal{A}_\Psi(\omega)) \leqslant d$$

and

$$d_{\mathrm{F}}(\mathcal{A}_\Psi(\omega)) \leqslant \gamma$$

for any γ *such that*

$$\gamma > \frac{\mathbb{E}\left[\max_{1 \leqslant j \leqslant d}(dq_j - jq_d)\right]}{-\mathbb{E}q_d},$$

where $q_j = \log \bar{\omega}_j$.

In the following, we verify **Hypothesis A3–A4**. We first establish the following result, which is a key ingredient to prove the $\mathbb{P}$ almost surely uniform differentiability results of $\Psi(\omega)$.

Proposition 4.2. *If $f : \mathcal{L} \to H$ is twice continuously differentiable, then for each $\varpi \in \mathcal{A}_\Psi(\omega)_\Phi$ and $h \in H$, there exists a continuous function $U^{\varpi,h}(t,\omega) : [0,\infty) \times \Omega \to H$ such that*

$$U^{\varpi,h}(t,\omega) = e^{-\tilde{L}t}\tilde{S}(t)h + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)(0, D\tilde{f}(P_1\Phi(s,\omega,\varpi))\, P_1U(s,\omega))ds, \quad t \geqslant 0. \quad (4.2)$$

Moreover, if $h \in D(\tilde{A})$, then $U(t,\omega)$ is a strong solution of the following variational equation on H.

$$\begin{cases} \dfrac{dU(t,\omega)}{dt} = \tilde{A}U(t,\omega) - \tilde{L}U(t,\omega) + (0, D\tilde{f}(P_1\Phi(s,\omega,\varpi))\, P_1U(t,\omega)), \\ U(0,\omega) = h \in H, \end{cases} \quad (4.3)$$

where operators $\tilde{A}$ and $\tilde{f}$ are defined by (2.12) and (2.9), $\Phi(t,\omega,\varpi)$ is the RDS defined by (3.5) with initial condition h.

Proof. Let

$$L_1(\omega) = \sup_{\varsigma \in P_1\mathcal{A}_\Phi(\omega)} |D\tilde{f}(\varsigma)|, \quad (4.4)$$

where

$$|D\tilde{f}(\varsigma)| = \sup_{\|\eta\|_{\mathcal{L}} \leqslant 1} \|D\tilde{f}(\varsigma)\eta\|_{\mathbb{X}}. \quad (4.5)$$

Since $\tilde{f}$ is C^1 and $P_1\mathcal{A}_\Psi(\omega)$ is compact, then $L_1(\omega) < \infty$. Given any $h \in D(\tilde{A})$, define $F_\chi : H \to H$ by

$$F_\varpi(h) = (0, D\tilde{f}(P_1\Phi(t,\omega,\varpi))\, P_1h), \qquad t \geqslant 0,\ h \in H.$$

It follows from the invariance of $\mathcal{A}_\Phi(\omega)$ under Φ and $\varpi \in \mathcal{A}_\Phi(\omega)$ that $\Phi(s,\omega,\varpi) \in \mathcal{A}_\Phi(\omega)$ and hence $P_1\Phi(t,\omega,\varpi) \in P_1\mathcal{A}_\Phi(\omega)$ and $|Df(P_1\Phi(t,\omega,\varpi))| \leqslant L_1(\omega) < \infty$, for all $t \geqslant 0$. This implies that $F_\varpi(\cdot)$ is Lipschitz continuous on H. Therefore the conclusion follows from Pazy [35, Theorem 6.1.5]. □

Now, we establish the following almost surely uniform differentiability results of $\Psi(\omega)$ on the random attractor $\mathcal{A}_\Psi(\omega)$.

Theorem 4.3. *The mapping $\Psi(\omega)$ is $\mathbb{P}$ almost surely differentiable on $\mathcal{A}_\Psi(\omega)$, that is, $\mathbb{P}$ almost surely, for every u in $\mathcal{A}_\Psi(\omega)$, there exist a continuous linear operator $D\Psi(\omega,u) : H \to H$, such that if $u, u+h \in \mathcal{A}_\Psi(\omega)$, then*

$$\|\Psi(\omega)(u+h) - \Psi(\omega)u - D\Psi(\omega,u)\cdot h\| \leq K(\omega)\|h\|^{1+\alpha}$$

where $K(\omega)$ is a random variable such that

$$K(\omega) \geq 1, \qquad w \in \Omega$$

and $\alpha > 0$ is a number.

Proof. We first claim that for any constant $t > 0$ and $\chi, \chi + h \in \mathcal{A}_\Psi(\omega)$, there exists a constant $L(t) > 0$ such that

$$\|\Psi(t,\omega,\chi) - \Psi(t,\omega,\chi+h)\|_X \leqslant L(t)\|h\|.$$

By Theorem 2.4 and the relationship $\Psi(t,\omega,\chi(\omega)) = \Phi(t,\omega,\varpi(\omega)) + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega))$ with $\varpi(\omega) = \chi(\omega) - (z(\theta_\cdot\omega), z(\omega))$, we have

$$\Psi(t,\omega,\chi) = e^{-\tilde{L}t}\tilde{S}(t)\chi + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)F(s,\theta_s\omega,\Phi(s,\omega,\varpi))ds + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega)), \quad (4.6)$$

$$\begin{aligned}\Psi(t,\omega,\chi+h) = e^{-\tilde{L}t}\tilde{S}(t)(\chi+h) + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)F(s,\theta_s\omega,\Phi(s,\omega,\varpi+h)))ds \\ + (z(\theta_{t+\cdot}\omega), z(\theta_t\omega)),\end{aligned} \quad (4.7)$$

from which it follows that

$$\begin{aligned}\Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi) = e^{-\tilde{L}t}\tilde{S}(t)h + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)[F(s,\theta_s\omega,\Phi(s,\omega,\varpi+h)) \\ - F(s,\theta_s\omega,\Phi(s,\omega,\varpi))]ds.\end{aligned} \quad (4.8)$$

Since $\|\tilde{S}(t)\| \leqslant e^{s(\tilde{A})t}, t \geqslant 0$, we have

$$\begin{aligned}&\|\Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi)\| \\ &\leq e^{\varrho t}\|h\| + L_f \int_0^t e^{\varrho(t-s)}\|P_1[\Phi(s,\omega,\varpi+h) - \Phi(s,\omega,\varpi)]\|ds \\ &= e^{\varrho t}\|h\| + L_f \int_0^t e^{\varrho(t-s)}\|P_1[\Psi(s,\omega,\chi+h) - \Psi(s,\omega,\chi)]\|ds \\ &\leq e^{\varrho t}\|h\| + L_f \int_0^t e^{\varrho(t-s)}\|\Psi(s,\omega,\chi+h) - \Psi(s,\omega,\chi)\|ds.\end{aligned} \quad (4.9)$$

Multiplying both sides of (4.9) by $e^{-\varrho t}$ and taking into account the Gronwall inequality, we obtain

$$e^{-\varrho t}\|\Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi)\| \leq e^{L_f t}\|h\|, \quad (4.10)$$

and hence

$$\|\Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi)\| \leq e^{(L_f+\varrho)t}\|h\|. \quad (4.11)$$

Therefore, the claim holds by taking $L(t) = e^{(L_f+\varrho)t}$.

Next we prove that, for any $t > 0$, there exist $K(\omega) \geq 1$ and $\alpha > 0$ such that, if $\chi, \chi + h \in \mathcal{A}_\Psi(\omega)$, then

$$\|\Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi) - U^{\chi+h,\chi}(t,\omega)\| \leq K(\omega)\|h\|^{1+\alpha}. \quad (4.12)$$

Let

$$L_2(\omega) := \sup_{\xi \in \overline{co}\mathcal{A}_\Psi(\omega)(w)} \left|D^2 f(P_1\xi)\right|, \quad (4.13)$$

where $\overline{co}\mathcal{A}_\Psi(\omega)$ represents the closed convex hull of $\mathcal{A}_\Psi(\omega)$. Since f is C^2 and $\mathcal{A}_\Psi(\omega)$ is compact, $L_2 < \infty$. By Proposition 4.2, we have

$$U^{\chi,h}(t,\omega) = e^{-\tilde{L}t}\tilde{S}(t)h + \int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)(0, D\tilde{f}\,(P_1\Phi(s,\omega,\varpi))\,P_1U(s))ds, \qquad t \geqslant 0.$$

For notation simplicity, we denote $y(t,\omega) \triangleq \Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi) = \Phi(s,\omega,\varpi+h) - \Phi(s,\omega,\varpi)$ and $w(t,\omega) \triangleq \Psi(t,\omega,\chi+h) - \Psi(t,\omega,\chi) - U^{\varpi,h}(t,\omega)$. Then, it follows from (4.6) and (4.7) that

$$
\begin{aligned}
&\|w(t,\omega)\| \\
&\quad = \Big\| \int_0^t e^{-\tilde{L}(t-s)} \tilde{S}(t-s)\{0, \tilde{f}\left(P_1\Phi(s,\omega,\varpi+h)\right) \\
&\qquad - \tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) - D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) P_1U(s,\omega)\}ds \| \\
&\quad \le \int_0^t e^{\varrho(t-s)} \|\tilde{f}\left(P_1\Phi(s,\omega,\varpi+h)\right) - \tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) - D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) P_1U(s,\omega)\|_{\mathbb{X}} ds \\
&\quad \le \int_0^t e^{\varrho(t-s)} \int_0^1 |D\tilde{f}\left(P_1(\Phi(s,\omega,\varpi)+\vartheta y(s,\omega))\right) - D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)| d\vartheta \|P_1 y(s,\omega)\|_{\mathcal{L}} ds \\
&\qquad + \int_0^t e^{\varrho(t-s)} \|D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) P_1 w(s,\omega)\|_{\mathbb{X}} ds \\
&\quad \le \int_0^t e^{\varrho(t-s)} \int_0^1 \int_0^1 |D^2\tilde{f}\left(P_1(\Phi(s,\omega,\varpi)+\lambda\vartheta y(s,\omega))\right)| \lambda\vartheta d\lambda d\vartheta \|P_1 y(s,\omega)\|_{\mathcal{L}}^2 ds \\
&\qquad + \int_0^t e^{\varrho(t-s)} \|D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right) P_1 w(s,\omega)\|_{\mathbb{X}} ds \\
&\quad \le \int_0^t e^{\varrho(t-s)} \int_0^1 \int_0^1 |D^2\tilde{f}\left(P_1(\Phi(s,\omega,\varpi)+\lambda\vartheta y(s,\omega))\right)| d\lambda d\vartheta \|y(s,\omega)\|^2 ds \\
&\qquad + \int_0^t e^{\varrho(t-s)} |D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)| \|w(s,\omega)\| ds.
\end{aligned} \tag{4.14}
$$

Since $\chi, \chi+h \in \mathcal{A}_\Psi(\omega)$, it follows from the invariance Corollary 3.13 that $\varpi, \varpi+h \in \mathcal{A}_\Phi(\omega)$. Therefore, the invariance of $\mathcal{A}_\Phi(\omega)$ under Φ implies that $\Phi(t,\omega,\varpi), \Phi(t,\omega,\varpi+h) \in \mathcal{A}_\Psi(\omega)$ for all $t \geqslant 0$. Therefore, $P_1\Phi(t,\omega,\varpi) + \lambda\vartheta y(s,\omega)) \in \overline{co}\left(\mathcal{A}_\Phi(\omega)\right)$, for all $\vartheta, \lambda \in [0,1]$, where $\overline{co}\mathcal{A}_\Phi(\omega)$ represents the closed convex hull of $\mathcal{A}_\Phi(\omega)$. Thus, it follows from (4.13) and (4.10) and the fact f is C^2 that

$$
\|w(t,\omega)\| \leqslant L_2(\omega) \int_0^t e^{[2(L_f+\varrho)+\varrho](t-s)} \|h\|^2 ds + L_1(\omega) \int_0^t e^{\varrho(t-s)} \|w(s,\omega)\| ds. \tag{4.15}
$$

Multiplying both sides of (4.15) by $e^{-\varrho t}$ yields that

$$
e^{-\varrho t}\|w(t,\omega)\| \leqslant \frac{-L_2(\omega)e^{-\varrho t}}{2(L_f+\varrho)+\varrho}\|h\|^2 + L_1(\omega) \int_0^t e^{-\varrho s} \|w(s,\omega)\| ds, \tag{4.16}
$$

which implies, by the Gronwall inequality, that

$$
e^{-\varrho t}\|w(t,\omega)\| \leqslant \frac{-L_2(\omega)e^{-\varrho t}}{2(L_f+\varrho)+\varrho}\|h\|^2 + \frac{L_1(\omega)e^{L_1(\omega)t}}{-(L_f+\varrho)}(e^{-(\varrho+L_1)t} - 1)\|h\|^2. \tag{4.17}
$$

Therefore, we have

$$
\|w(t,\omega)\| \leqslant \frac{-L_2(\omega)}{2(L_f+\varrho)+\varrho}(1 + \frac{L_1(\omega)(1-e^{(L_1(\omega)+\varrho)t}}{-(L_f+\varrho)})\|h\|^2. \tag{4.18}
$$

Take $D\Psi(\omega)h \triangleq U^{\varpi,h}(1,\omega)$, then it follows from (4.2) that $D\Psi(\omega)$ is linear and continuous.

Moreover, we have

$$\begin{aligned}&\|\Psi(\omega)(\chi+h)-\Psi(\omega)\chi-D\Psi(\omega,\chi)\cdot h\|\\&=\|\Psi(1,\omega,\chi+h)-\Psi(1,\omega\chi)-U^{\chi,h}(1,\omega)\|\\&\leq\frac{-L_2(\omega)}{2(L_f+\varrho)+\varrho}\left(1+\frac{L_1(\omega)(1-e^{(L_1(\omega)+\varrho)}}{-(L_f+\varrho)}\right)\|h\|^2,\end{aligned}\tag{4.19}$$

what implies that the statement of Theorem 4.3 holds by taking $\alpha = 1$ and $K(\omega) = \frac{-L_2(\omega)}{2(L_f+\varrho)+\varrho}(1+\frac{L_1(\omega)(1-e^{(L_1(\omega)+\varrho)}}{-(L_f+\varrho)})$. □

We can now prove the main results of this paper.

Theorem 4.4. *Assume that* ***Hypotheses A1–A4*** *as well as conditions of Theorem 3.12 are satisfied, $f:\mathcal{L}\to H$ is twice continuously differentiable and there exists $L_3(\omega)$ such that for all $\phi\in\mathcal{L}$ and a.e. $\omega\in\Omega$ such that*

$$(D\tilde{f}\,(P_1\Phi(t,\omega,\phi))\,\varphi_j,\varphi_j(0))<L_3(\omega),\tag{4.20}$$

where $\varphi_j\in\mathcal{C}, j=1,2,\ldots,m$ is a sequence of unit orthogonal vectors. Then, the Hausdorff and fractal dimensions of the random attractor $\mathcal{A}_\Psi(\omega)$ of (2.11) are bounded by

$$d<[\frac{L_1(\omega)}{c}]^{\frac{1}{1+2/N}},\tag{4.21}$$

where c is a positive constant, N is dimension of the spatial domain, $L_1(\omega)$ is defined by (4.4).

Proof. The existence of random attractor $\mathcal{A}_\Psi(\omega)$ has been proved in Theorem 2.4 under **Hypotheses A1–A2**. We show in the sequel the finite dimensionality of $\mathcal{A}_\Psi(\omega)$. Let $\varpi\in\mathcal{A}_\Psi(\omega)$, $h_i=(\phi_i,\phi_i(0))\in D(\tilde{A})$ and $U_i^{\varpi,h_i}(t,\omega)$ be defined by

$$U_i^{\varpi,h_i}(t,\omega)=e^{-\tilde{L}t}\tilde{S}(t)h_i+\int_0^t e^{-\tilde{L}(t-s)}\tilde{S}(t-s)(0,D\tilde{f}\,(P_1\Phi(s,\omega,\varpi))P_1U_i^{\varpi,h_i}(s,\omega))ds,\quad t\geqslant 0.\tag{4.22}$$

It follows from Proposition 4.2 that $U_i^{\varpi,h_i}(t,\omega)$ satisfies the following variational equation on H.

$$\begin{cases}\dfrac{dU_i^{\varpi,h_i}(t,\omega)}{dt}=\tilde{A}U_i^{\varpi,h_i}(t,\omega)-\tilde{L}U_i^{\varpi,h_i}(t,\omega)+(0,D\tilde{f}\,(P_1\Phi(s,\omega,\varpi))\,P_1U_i^{\varpi,h_i}(t,\omega)),\\U_i^{\varpi,h_i}(0,\omega)=h_i\in H.\end{cases}\tag{4.23}$$

Define a family of random maps $U_i(t,\omega):H\to H, i=1,\ldots,m$ by $U_i(t,\omega)h_i=U^{\varpi,h_i}(t,\omega)$. By a similar argument to that for (2.40) in [40] Chapter V, we obtain

$$\frac{1}{2}\frac{d}{dt}\,|U_1(t,\omega)\wedge\cdots\wedge U_m(t,\omega)|^2_{\wedge^m H}=|U_1(t,\omega)\wedge\cdots\wedge U_m(t,\omega)|^2_{\wedge^m H}\,\mathrm{Tr}\,(G(t)\circ Q_m(t))\,,\tag{4.24}$$

where $|\cdot|_{\wedge^m H}$ represents the exterior product and

$$Q_m(t)=Q_m\,(t,\varpi;h_1,\ldots,h_m)\tag{4.25}$$

is the orthogonal projection of H onto the space spanned by $\{U_j(t,\omega)\}_{j=1,2,\ldots,m}$ and $G(t)=G(t,\omega):H\to H$ is defined by

$$G(t,\omega)h_i=\tilde{A}h_i-\tilde{L}h_i+(0,D\tilde{f}\,(P_1\Phi(s,\omega,\varpi))\,P_1h_i).\tag{4.26}$$

Therefore

$$\begin{aligned} &|U_1(t,\omega)\wedge\cdots\wedge U_m(t,\omega)|_{\wedge^m H} \\ &\quad = |U_1(0,\omega)\wedge\cdots\wedge U_m(0,\omega)|_{\wedge^m H}\exp\left(\int_0^t \operatorname{Tr}\left(G(s,\omega)\circ Q_m(s)\right) ds\right) \\ &\quad = |h_1\wedge\cdots\wedge h_m|_{\wedge^m H}\exp\left(\int_0^t \operatorname{Tr}\left(G(s,\omega)\circ Q_m(s)\right) ds\right). \end{aligned} \tag{4.27}$$

Let us fix an orthonormal basis $\{e_j = (\varphi_j, \varphi_j(0))\}_{j=1,2,\dots,m}$ of the span $\{U_j(t,\omega)\}_{j=1,2,\dots,m}$. Then, we have

$$\begin{aligned} \operatorname{Tr}\left(G(s,\omega)\circ Q_m(s)\right) &= \sum_{j=1}^m \left\langle G(s,\omega)e_j, e_j\right\rangle \\ &= \sum_{j=1}^m \left\langle \tilde{A}e_j - \tilde{L}e_j + (0, D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)P_1 e_j), e_j\right\rangle \\ &= \sum_{j=1}^m \left\langle (\dot{\varphi}_j, A\varphi_j(0) - \mu\varphi_j(0) + D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)\varphi_j), (\varphi_j, \varphi_j(0))\right\rangle. \end{aligned} \tag{4.28}$$

By the definition of the inner product,

$$\begin{aligned} &\left\langle (\dot{\varphi}_j - L\varphi_j, A\varphi_j(0) - \mu\varphi_j(0) + D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)\varphi_j), (\varphi_j, \varphi_j(0))\right\rangle \\ &\quad = \int_{-\tau}^0 \left(\frac{d}{dr}\varphi_j(s+r), \varphi_j(s+r)\right) dr + \left(A\varphi_j(0), \varphi_j(0)\right) - \mu\|\varphi_j(0)\|_{\mathbb{X}} \\ &\qquad + \left(D\tilde{f}\left(P_1\Phi(s,\omega,\varpi)\right)\varphi_j, \varphi_j(0)\right) \\ &\quad \leqslant \|\varphi_j(s)\|_{\mathbb{X}}^2 - \|\varphi_j(s-\tau)\|_{\mathbb{X}}^2 - \left(-A\varphi_j(0), \varphi_j(0)\right) + L_3(\omega). \end{aligned} \tag{4.29}$$

Incorporating (4.29) into (4.28) yields

$$\operatorname{Tr}\left(G(s,\omega)\circ Q_m(s)\right) \leqslant \sum_{j=1}^m (\|\varphi_j(s)\|_{\mathbb{X}}^2 - \|\varphi_j(s-\tau)\|_{\mathbb{X}}^2) - \sum_{j=1}^m \left(-A\varphi_j(0), \varphi_j(0)\right) + L_3(\omega). \tag{4.30}$$

In order to estimate the evolution of the volume under the random maps $U_i(t,\omega) : H \to H, i = 1,\dots,m$, i.e., the norm of $|U_1(t,\omega)\wedge\cdots\wedge U_m(t,\omega)|_{\wedge^m H}$ defined by (4.27), we introduce two quantities $q_m(t,\omega)$ and $q_m(\omega)$, which are defined by

$$q_m(t,\omega) \triangleq \sup_{\chi\in\mathcal{A}_\Psi(\omega), h_i\in D(\tilde{A}), \|h_i\|_H\leq 1} \frac{1}{t}\int_0^t \operatorname{Tr}\left(G(s,\omega)\circ Q_m(s)\right) ds$$

and

$$q_m(\omega) \triangleq \lim_{t\to\infty} q_m(t,\omega)$$

respectively. Now we keep in mind that

$$q_m(t,\omega) \leqslant \frac{1}{t}\int_0^t \left[\sum_{j=1}^m \left(\|\varphi_j(s)\|_{\mathbb{X}}^2 - \|\varphi_j(s-\tau)\|_{\mathbb{X}}^2\right) - \sum_{j=1}^m \left(-A\varphi_j(0), \varphi_j(0)\right) + L_3(\omega)\right] ds, \tag{4.31}$$

and

$$\frac{1}{t}\int_0^t \sum_{j=1}^m \left(\|\varphi_j(s)\|_{\mathbb{X}}^2 - \|\varphi_j(s-\tau)\|_{\mathbb{X}}^2\right) ds = 0. \tag{4.32}$$

Moreover, by [40] Chapter VI Section 2.1, there exists a $c > 0$ such that

$$\sum_{i=1}^{m} \left(-A\varphi_j(0), \varphi_j(0)\right) \geq cm^{1+\frac{2}{N}}.$$

Then, we have $q_m \leq -cm^{1+\frac{2}{N}} + L_3(\omega)$. Therefore, the Hausdorff and fractal dimensions of the random attractor $\mathcal{A}_\Psi(\omega)$ of (2.11) obtained in Theorem 3.12 are bounded by the first integer such m that $q_m \leq 0$, that is,

$$d < \left[\frac{L_3(\omega)}{c}\right]^{\frac{1}{1+2/N}}, \tag{4.33}$$

completing the proof. □

5 Conclusions

In this paper, we have estimated the topological dimensions of random attractor for the stochastic delayed semilinear partial differential equation (1.2). In order to overcome the difficulty caused by the lack of Hilbert geometry, we recast the equation into a Hilbert space. One naturally wonders, whether we can estimate the dimension of attractors for SPFDEs in their natural phase space, i.e. Banach spaces. This requires to establish the general framework to estimate the dimension of attractors of RDS in Banach spaces, which will be studied in the near future. Moreover, there are also SPFDEs on infinite domains which can model the spatial-temporal patterns for the mature population of age-structured species under random perturbations. The existence of random attractors for a stochastic nonlocal delayed reaction-diffusion equation on a semi-infinite interval have been studied in [29]. However, little attention has been paid to the estimation of topological dimensions of random attractor for the equation therein, which also deserves much effort in the future.

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On the solvability of a higher-order semilinear ODE

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Abstract. The existence of at least one or two nontrivial solutions to a general higher-order boundary value problem is established by using variational tools. Two of the results are obtained without any asymptotic behaviour at infinity of potential F of the nonlinear term f, which is a key condition in the available literature, when applying critical point theorems. Moreover, F may change sign. The last result is stated when the nonlinearity has asymptotic behaviour at both infinity and zero.

Keywords: ODE, higher-order, semilinear, Brézis–Nirenberg's linking theorem, Mountain Pass theorem, variational method.

2020 Mathematics Subject Classification: 34A12, 34B15.

1 Introduction

In this paper, we investigate the existence of at least one or two solutions for the boundary value problem

$$\begin{cases} u^{(2n)} + A_{n-1}u^{(2n-2)} + \cdots + A_1 u'' + A_0 u + f(x,u) = 0 \text{ in } \Omega = (0, L) \\ u = u'' = \cdots = u^{(2n-2)} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $A_0, A_1, \ldots, A_{n-1}$ are some given real constants, f is a continuous function on $\overline{\Omega} \times \mathbb{R}$ and $n \geq 2$.

The existence of solutions for fourth-order problems ($n = 2$), which describe the deflection of an elastic beam with supported ends, has been extensively studied in the literature (see for example [2–4, 7, 8, 11, 15, 16] and the literature cited therein).

We mention the paper [12], where (1.1) (case $n = 2$) was treated under the assumption $A_1^2 > 4A_0$ by variational tools. The authors obtained existence and multiplicity results if the potential $F(x,s) = \int_0^s f(x,t)dt$ satisfies an asymptotic behaviour at zero and for some $C > 0$ and $p > 2$

$$F(x,s) \geq C|s|^p, \qquad \forall\, x \in \Omega,\ s \in \mathbb{R}. \tag{1.2}$$

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The case $A_1^2 = 4A_0$ was treated in [15].

The existence of solutions to sixth-order equations ($n = 3$) was investigated in [8] by using Clark's theorem provided the coefficients A_0, A_1, A_2 satisfy some relations in the particular case when $f(x,s) = a(x)s^3$. Here $a(x)$ is a continuous positive and even function.

In [13], using two Brézis–Nirenberg's linking theorems, the existence of at least two or three solutions was obtained, where $F \geq 0$ has an asymptotic behaviour at zero and satisfies

$$\frac{F(x,s)}{s^2} \to +\infty, \quad \text{uniformly with respect to } x \text{ as } |s| \to \infty. \tag{1.3}$$

Note that condition (1.2) implies the weaker super-quadratic condition (1.3). Also in the new paper [1] infinitely many solutions to equation (1.1) (case $n = 3, \Omega = (0,1)$) are obtained in the case when the nonlinear term f has an oscillating behaviour and the following restriction holds

$$\max\{A_2k, A_2k - A_1k^2, A_2k - A_1k^2 + A_0k^3\} < 1, \tag{1.4}$$

where $k = 1/\pi^2$.

For further results on sixth-order equations we refer the reader to [5,9,14,16–18].

The existence results of this paper are obtained for a general $2n-$ order equation by variational methods and hold under different assumptions on the coefficients.

We impose here suitable conditions on the coefficients $A_0, \dots, A_{n-1}$, allowing to define several norms equivalent to the usual norm of the working space. One of the condition we impose (relation (2.5)) represents a generalization to the higher-order case of condition $A_1^2 > 4A_0$ which plays a role in the works [16] and [12].

We see that even we restrict ourselves to the case $n = 3$ our conditions imposed to the coefficients are different from the condition (1.4) or from the results obtained in the above mentioned papers.

Moreover, we note that our first two main results are stated without any asymptotic behaviour at infinity. More precisely, we prove by using the Brézis–Nirenberg's linking theorem that an existence result holds without any behaviour at infinity if $F \geq 0$ (Theorem 3.1). By using Ekeland's variational principle we show (Theorem 3.4) that a result holds if F may change sign and if no asymptotic behaviour at infinity is required. The last existence result uses the Mountain Pass theorem and is stated when F may change sign and f satisfies an asymptotic behaviour at both zero and infinity (f behavies at $\pm\infty$ as $|s|^p, p > 1$).

2 Auxiliary results and variational settings

We consider the Hilbert space

$$H(\Omega) = \{u \in H^n(\Omega) \mid u = u'' = \cdots = u^{(2n-4)} = 0 \text{ on } \partial\Omega\}$$

endowed with the standard inner product

$$(u,v)_{H^n(\Omega)} = \int_\Omega \left(uv + u'v' + u''v'' + \cdots + u^{(n)}v^{(n)}\right)dx$$

and standard norm

$$\|u\|_{H^n(\Omega)} = (u,u)^{\frac{1}{2}}_{H^n(\Omega)}.$$

For the sake of simplicity we consider $n = 4k, k = 1,2,3,\dots,$ unless otherwise stated.

We recall the meaning of a weak solution to (1.1).

Definition 2.1. A weak solution of (1.1) is a function $u \in H(\Omega)$ such that

$$\int_\Omega \Big(u^{(n)}v^{(n)} - A_{n-1}u^{(n-1)}v^{(n-1)} + \cdots - A_1 u'v' + A_0 uv + f(x,u)v \Big) dx = 0, \qquad \forall\, v \in H(\Omega).$$

A classical solution of (1.1) is a function $u \in C^{2n}(\overline{\Omega})$ that satisfies (1.1).

We note that since f is a continuous function on $\overline{\Omega} \times \mathbb{R}$, it follows that a weak solution of (1.1) belongs to $C^{2n}(\overline{\Omega})$ (to get the result imitate the proof in [17]).

We also recall that the set of functions

$$\left\{ \sin \frac{m\pi x}{L},\, m \in \mathbb{N}, m \geq 1 \right\}$$

is a complete orthogonal basis in $H(\Omega)$.

The symbol $P(\xi) = \xi^{2n} - A_{n-1}\xi^{2n-2} + \cdots + A_2\xi^4 - A_1\xi^2 + A_0$ of the differential operator $L(u) = u^{(2n)} + A_{n-1}u^{(2n-2)} + \cdots + A_2 u^{(4)} + A_1 u'' + A_0 u$ plays an important role in the sequel.

Problem (1.1) has a variational structure and weak solutions in the space $H(\Omega)$ can be found as critical points of the functional

$$J : H(\Omega) \to \mathbb{R}$$

$$J(u) = \frac{1}{2}\int_\Omega \Big((u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0 u^2 \Big) dx + \int_\Omega F(x,u)dx,$$

which is Fréchet differentiable and its Fréchet derivative is given by

$$\langle J'(u), v \rangle = \int_\Omega \Big(u^{(n)}v^{(n)} - A_{n-1}u^{(n-1)}v^{(n-1)} + \cdots - A_1 u'v' + A_0 uv + f(x,u)v \Big) dx,$$

for all $v \in H(\Omega)$.

Throughout the paper C denotes a universal positive constant depending on the indicated quantities, unless otherwise specified.

The next lemmas are fundamental tools in proving our existence result.

First we point out some Poincaré-type inequalities.

Lemma 2.2 ([10]). *The following relations hold true for any $u \in H(\Omega)$.*

$$\int_\Omega (u^{(k)})^2 dx \leq \left(\frac{L}{\pi}\right)^2 \int_\Omega (u^{(k+1)})^2 dx, \qquad k = 0,1,2,\ldots,n-1. \tag{2.1}$$

$$\int_\Omega u^2 dx \leq \left(\frac{L}{\pi}\right)^{2k} \int_\Omega (u^{(k)})^2 dx, \qquad k = 1,2,\ldots,n. \tag{2.2}$$

In particular,

$$\int_\Omega u^2 dx \leq \left(\frac{L}{\pi}\right)^{2n} \int_\Omega (u^{(n)})^2 dx. \tag{2.3}$$

An immediate consequence of Lemma 2.2 is the inequality

$$C(L,n)\|u\|_{H^n(\Omega)} \leq \int_\Omega (u^{(n)})^2 dx \leq \|u\|_{H^n(\Omega)}, \tag{2.4}$$

which shows that the scalar product

$$(u,v)_{H(\Omega)} = \int_\Omega u^{(n)}v^{(n)} dx$$

induces a norm equivalent (denoted $\|\cdot\|_{H(\Omega)}$) to the norm $\|\cdot\|_{H^n(\Omega)}$ in the space $H(\Omega)$.

The next lemma is an extension of Lemma 8, [16] and is proved by different means.

Lemma 2.3. *Let $u \in H(\Omega)$.*

a). Suppose that $A_0, A_2, \dots, A_{n-4} \geq 0$, $A_1, A_3, \dots, A_{n-3} \leq 0$, $A_{n-2}, A_{n-1} > 0$ and

$$A_{n-1}^2 < 4A_{n-2}. \tag{2.5}$$

Then there exists a constant k such that

$$\int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0 u^2 \right] dx \geq k\|u\|^2_{H^n(\Omega)}. \tag{2.6}$$

A similar estimate holds for $A_0 < 0$ but under the restriction

$$A_{n-1}^2 < 4A_{n-2}A^*, \tag{2.7}$$

where $A^* = 1 + A_0\left(\frac{L}{\pi}\right)^{2n} > 0$.

b). The same estimate (2.6) holds if for some index $j = 2, 4, \dots, n-2$

$$\frac{A_{n-j-1}^2}{A_{n-j}} < 4A_{n-j-2}, \tag{2.8}$$

where

$$A_1, A_3, \dots, A_{n-j-3}, A_{n-j+1}, \dots, A_{n-1} < 0,$$
$$A_0, A_2, \dots, A_{n-j-2}, A_{n-j+2} \dots, A_{n-2} \geq 0, \quad A_{n-j-1}, A_{n-j} > 0.$$

c). Similarly, (2.6) holds if for some index $j = 1, 3, \dots, n-1$ (2.8) is fulfilled, where

$$A_1, A_3, \dots, A_{n-j-3}, A_{n-j+1}, \dots, A_{n-1} \leq 0,$$
$$A_0, A_2, \dots, A_{n-j-2}, A_{n-j+2} \dots, A_{n-2} \geq 0, \quad A_{n-j-1} < 0, A_{n-j} > 0.$$

Remark 2.4.

1. Of course if $A_{n-1} \leq 0, A_{n-2} \geq 0, \dots, A_1 \leq 0, A_0 \geq 0$, then Lemma 2.3 is always true, i.e., there is nothing to prove.

2. We easily see that if $n = 2$ (Case a).) then we obtain exactly Lemma 8, [16] for bounded domains, i.e., our result is a direct extension to the higher-order case.

3. Note that Lemma 2.5 and Lemma 2.6 can also be seen as extensions of Lemma 8, [16] and hold for bounded domains Ω as well when $\Omega = \mathbb{R}$.

Proof. a). We see that for any real α

$$\int_\Omega \left(u^{(n)} + \alpha u^{(n-1)}\right)^2 dx = \int_\Omega \left((u^{(n)})^2 - 2\alpha(u^{(n-1)})^2 + \alpha^2(u^{(n-2)})^2\right) dx.$$

It follows that for any α the quantity

$$Q_\alpha = \int_\Omega \left((u^{(n)})^2 - 2\alpha(u^{(n-1)})^2 + \alpha^2(u^{(n-2)})^2\right) dx$$

is positive.

For arbitrary $\varepsilon > 0$ and by the assumptions

$$
\begin{aligned}
&\int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0u^2\right] dx \\
&\quad \geq \int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + A_{n-2}(u^{(n-2)})^2\right] dx \\
&\quad = \left\{\varepsilon \int_\Omega \left[(u^{(n)})^2 + (u^{(n-1)})^2 + (u^{(n-2)})^2\right] dx \right.\\
&\qquad + (1-\varepsilon)\int_\Omega \left[(u^{(n)})^2 - \frac{A_{n-1}+\varepsilon}{1-\varepsilon}(u^{(n-1)})^2 + \frac{1}{4}\left(\frac{A_{n-1}+\varepsilon}{1-\varepsilon}\right)^2 (u^{(n-1)})^2\right] dx \\
&\qquad \left. + \left[A_{n-2} - \varepsilon - \frac{1}{4}\frac{(A_{n-1}+\varepsilon)^2}{1-\varepsilon}\right] \int_\Omega (u^{(n-2)})^2 dx \right\} \\
&\quad \geq \varepsilon \int_\Omega (u^{(n)})^2 dx + (1-\varepsilon) Q_{\frac{A_{n-1}+\varepsilon}{1-\varepsilon}} + \left[A_{n-2} - \varepsilon - \frac{1}{4}\frac{(A_{n-1}+\varepsilon)^2}{1-\varepsilon}\right] \int_\Omega (u^{(n-2)})^2 dx.
\end{aligned}
$$

Choosing ε sufficiently small, using that $Q_{\frac{A_{n-1}+\varepsilon}{1-\varepsilon}} \geq 0$, (2.5) and the equivalence of norms $\|\cdot\|_{H^n(\Omega)}$ and $\|\cdot\|_{H(\Omega)}$ we get the result.

b). and c). Follows from case a). □

Lemma 2.5. *Let $u \in H(\Omega)$ and $A_0 > 1$.*

Suppose that for an index i and j,

$$
A_i^2 < -4A_j, \qquad \frac{A_i^2}{-4A_j} \leq A_0 - 1, \tag{2.9}
$$

where $i = 2, 3, \ldots, \frac{n}{2}$, $A_i \neq A_j$, $1 \leq j \leq n-1$, $A_j < 0$, $A_i < 0$ if i is even and $A_i > 0$ if i is odd.

Then there exist the constants $k_{i,j} > 0$ such that

$$
\int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0u^2\right] dx \geq k_{i,j}\|u\|^2_{H^n(\Omega)}. \tag{2.10}
$$

Proof. a). For the sake of simplicity we consider $j = 1$ and $i = 2$, i.e.,

$$
A_1, A_2 < 0, \quad A_4, \ldots, A_{n-2} \geq 0, \quad A_3, \ldots, A_{n-1} \leq 0
$$

and

$$
\frac{A_2^2}{-4A_1} \leq A_0 - 1, \qquad A_2^2 < -4A_1.
$$

We are going to prove the required inequality for $u \in H^n(\mathbb{R})$ by using the Fourier transform.

Taking in particular $u \in H(\Omega) \cap H^n(\mathbb{R})$ we get the inequalities for bounded domains Ω.

Let $\hat{u}(\xi)$ be the Fourier transform of $u(x) \in H^n(\mathbb{R})$.

First observe that by Parseval's identity we get

$$
\begin{aligned}
&\int_\mathbb{R} \left((u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0u^2\right) dx \\
&\quad = \int_\mathbb{R} \left(\xi^{2n} - A_{n-1}\xi^{2n-2} + \cdots - A_1\xi^2 + A_0\right) \|\hat{u}(\xi)\|^2 d\xi.
\end{aligned} \tag{2.11}
$$

By using elementary inequalities we get for all $\xi \in \mathbb{R}$

$$\begin{aligned} A_2\xi^4 &\le \frac{A_2^2}{-4A_1}\xi^6 + (-A_1)\xi^2 \le \frac{A_2^2}{-4A_1}\xi^{2n} + (-A_1)\xi^2 + \frac{A_2^2}{-4A_1} \\ &\le \frac{A_2^2}{-4A_1}\xi^{2n} + (-A_1)\xi^2 + A_0 - 1 \\ &\le \frac{A_2^2}{-4A_1}\xi^{2n} - A_{n-1}\xi^{2n-2} + \cdots - A_3\xi^6 + (-A_1)\xi^2 + A_0 - 1. \end{aligned}$$

Hence

$$\begin{aligned} &\xi^{2n} - A_{n-1}\xi^{2n-2} + \cdots + A_2\xi^4 - A_1\xi^2 + A_0 \\ &\quad \ge \left(1 - \frac{A_2^2}{-4A_1}\right)\xi^{2n} + 1 \ge \left(1 - \frac{A_2^2}{-4A_1}\right)\left(\xi^{2n} + 1\right). \end{aligned} \tag{2.12}$$

It can be easily checked that $\forall\, \xi \in \mathbb{R}$

$$\xi^{2n} + 1 \ge \frac{1}{n}\left(1 + \xi^2 + \cdots + \xi^{2n}\right). \tag{2.13}$$

From (2.12) and (2.13) we get

$$\xi^{2n} - A_{n-1}\xi^{2n-2} + \cdots + A_2\xi^4 - A_1\xi^2 + A_0 \ge \frac{1}{n}\left(1 - \frac{A_2^2}{-4A_1}\right)\left(1 + \xi^2 + \cdots + \xi^{2n}\right). \tag{2.14}$$

Now from (2.11) and (2.14) we obtain

$$\begin{aligned} &\int_{\mathbb{R}} \left((u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0u^2\right)dx \\ &\quad \ge \frac{1}{n}\left(1 - \frac{A_2^2}{-4A_1}\right)\int_{\mathbb{R}}\left(1 + \xi^2 + \cdots + \xi^{2n}\right)\|\hat{u}(\xi)\|^2 d\xi \\ &\quad = \frac{1}{n}\left(1 - \frac{A_2^2}{-4A_1}\right)\int_{\mathbb{R}}\left(u^2 + (u')^2 + \cdots + \left(u^{(2n)}\right)^2\right)dx \\ &\quad = k_{2,1}\|u\|^2_{H^n(\mathbb{R})}, \end{aligned}$$

which is the desired result. □

Lemma 2.6. *Let $u \in H(\Omega)$ and $A_0 > 1$.*

Suppose that for an index $i = 1, 3, \ldots, (n/2) - 1$, $A_i > 0$ and for an index $j = 2, 4, \ldots, n-2$, $A_j > 0$ the following inequality be fulfilled

$$A_i^2 < 4A_j, \qquad \frac{A_i^2}{4A_j} + A_j \le A_0 - 1, \tag{2.15}$$

where the rest of coefficients

$$A_1, A_3, \ldots, A_{i-2}, A_{i+2}, \ldots, A_{n-1} \le 0$$

and

$$A_2, A_4, \ldots, A_{j-2}, A_{j+2}, \ldots, A_{n-2} \ge 0.$$

Then there exist the constants $k_{i,j} > 0$ such that

$$\int_{\Omega}\left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0u^2\right]dx \ge k_{i,j}\|u\|^2_{H^n(\Omega)}. \tag{2.16}$$

The proof is similar to the proof of Lemma 2.5 and hence is omitted.

Lemma 2.7. *Let $u \in H(\Omega)$.*

Suppose that $A_0, A_2, \dots, A_{n-2} \geq 0$, $A_1, A_3, \dots, A_{n-1} \geq 0$, and

$$1 - A_{n-1}\left(\frac{L}{\pi}\right)^2 - A_{n-3}\left(\frac{L}{\pi}\right)^6 - \dots - A_1\left(\frac{L}{\pi}\right)^{2n-2} > 0. \tag{2.17}$$

Then there exists a constant $k_1 > 0$ such that

$$\int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \dots - A_1(u')^2 + A_0u^2\right]dx \geq k_1\|u\|^2_{H^n(\Omega)}. \tag{2.18}$$

A similar result holds if $A_0, A_2, \dots, A_{n-2} < 0$ and $A_1, A_3, \dots, A_{n-1} \geq 0$ under the assumption

$$1 - A_{n-1}\left(\frac{L}{\pi}\right)^2 + A_{n-2}\left(\frac{L}{\pi}\right)^4 - \dots - A_1\left(\frac{L}{\pi}\right)^{2n-2} + A_0\left(\frac{L}{\pi}\right)^{2n} > 0. \tag{2.19}$$

The next four lemmas gives conditions on parameters $A_i, i = 0, 1, \dots, n-1$ when the functional J is bounded below and satisfies the Palais–Smale condition. We recall here what means that J satisfies the Palais–Smale condition.

Definition 2.8. Let X be a Banach space and $J \in C^1(X, \mathbb{R})$. We say that J satisfies a Palais–Smale condition if any sequence $\{u_m\}$ in X for which $J(u_m)$ is bounded and $J'(u_m) \to 0$ as $m \to \infty$, has a convergent subsequence.

Lemma 2.9. *Let $u \in H(\Omega)$ and let $\alpha > 0$ be a constant. Suppose that $F \geq 0$, $A_0, A_2, \dots, A_{n-4} \geq 0$, $A_1, A_3, \dots, A_{n-3} \leq 0$, $A_{n-2}, A_{n-1} > 0$ and*

$$\frac{\alpha+1}{\alpha}A^2_{n-1} < 4A_{n-2}. \tag{2.20}$$

Then J is bounded below and satisfies the Palais–Smale condition.

A similar statement holds for $A_0 < 0$ but under the restriction

$$\frac{\alpha+1}{\alpha}A^2_{n-1} < 4A_{n-2}A^*, \tag{2.21}$$

where $A^* = 1 + \frac{\alpha+1}{\alpha}A_0\left(\frac{L}{\pi}\right)^{2n} > 0$.
The same conclusion holds if we are under the hypotheses of the case b). or case c). of Lemma 2.3.

Proof. We observe that for any $\alpha > 0$ we can write $J(u)$ as a sum of

$$J(u) = \frac{1}{2}\frac{1}{\alpha+1}\int_\Omega \left(u^{(n)}\right)^2 dx + \frac{\alpha}{\alpha+1}J_1(u),$$

where

$$J_1(u) = \frac{1}{2}\int_\Omega \left[(u^{(n)})^2 - \frac{\alpha+1}{\alpha}A_{n-1}(u^{(n-1)})^2 + \dots + \frac{\alpha+1}{\alpha}A_0u^2 + 2\frac{\alpha+1}{\alpha}F\right]dx.$$

Since (2.20) holds we can use Lemma 2.3 and the positivity of F to get that $J_1(u)$ is bounded below which implies that $J(u)$ is bounded below.

We now show that $J(u)$ satisfies the Palais–Smale condition.

Suppose that $\{u_m\}$ is a Palais–Smale sequence, i.e., there exists a constant $C > 0$ such that

$$|J(u_m)| \leq C \quad \text{and} \quad J'(u_m) \to 0 \quad \text{as } m \to \infty.$$

Since $J_1(u)$ is bounded below we get that there exists a constant $C_1 > 0$ such that

$$C > \frac{1}{2}\frac{1}{\alpha+1}\int_\Omega (u_m^{(n)})^2 dx - C_1,$$

which implies that $\{u_m\}$ is a bounded sequence in $H(\Omega)$.

Since

$$J(u) = \frac{1}{2}(u,u)_{H(\Omega)} - \frac{1}{2}\int_\Omega \Big[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0u^2\Big]dx + \int_\Omega F dx,$$

we see that

$$J'(u) = u + K(u),$$

where

$$K : H(\Omega) \to H(\Omega)$$

is defined by

$$\langle K(u), v\rangle = -\int_\Omega \Big[A_{n-1}u^{(n-1)}v^{(n-1)} + \cdots + A_1u'v' - A_0uv - f(x,u)v\Big]dx.$$

Using the fact that the Sobolev imbedding $H(\Omega) \hookrightarrow C^{n-1}(\overline{\Omega})$ is compact we get that K is a complete continuous operator. Since $J'(u_m) \to 0$ as $m \to \infty$ it follows that

$$u_m = J'(u_m) - K(u_m)$$

is a convergent sequence and hence $J(u)$ satisfies the Palais–Smale condition. □

Using the same techniques we can prove

Lemma 2.10. *Let $u \in H(\Omega)$, $A_0 > 1$ and let $\alpha > 0$ be a constant.*

Suppose that for an index i and j,

$$\frac{\alpha+1}{\alpha}A_i^2 < -4A_j, \qquad \frac{A_i^2}{-4A_j} \leq A_0 - \frac{\alpha}{\alpha+1}, \tag{2.22}$$

where $i = 2,3,\ldots,\frac{n}{2}$, $A_i \neq A_j$, $1 \leq j \leq n-1$, $A_j < 0$, $A_i < 0$ if i is even and $A_i > 0$ if i is odd. Then J is bounded below and satisfies the Palais–Smale condition.

Lemma 2.11. *Let $u \in H(\Omega)$, $A_0 > 1$ and let $\alpha > 0$ be a constant.*

Suppose that for an index $i = 1,3,\ldots,(n/2)-1$, $A_i > 0$ and for an index $j = 2,4,\ldots,n-2$, $A_j > 0$ the following inequality be fulfilled

$$\frac{\alpha+1}{\alpha}A_i^2 < 4A_j, \qquad \frac{A_i^2}{4A_j} + A_j \leq A_0 - \frac{\alpha}{\alpha+1}, \tag{2.23}$$

where the rest of coefficients

$$A_1, A_3, \ldots, A_{i-2}, A_{i+2}, \ldots, A_{n-1} \leq 0,$$

and

$$A_2, A_4, \ldots, A_{j-2}, A_{j+2}, \ldots, A_{n-2} \geq 0.$$

Then J is bounded below and satisfies the Palais–Smale condition.

Lemma 2.12. *Let $u \in H(\Omega)$ and let $\alpha > 0$ be a constant.*

Suppose that $A_0, A_2, \dots, A_{n-2} \geq 0$, $A_1, A_3, \dots, A_{n-1} \geq 0$, and

$$1 - \frac{\alpha+1}{\alpha}\left[A_{n-1}\left(\frac{L}{\pi}\right)^2 + A_{n-3}\left(\frac{L}{\pi}\right)^6 + \cdots + A_1\left(\frac{L}{\pi}\right)^{2n-2}\right] > 0. \tag{2.24}$$

Then J is bounded below and satisfies the Palais–Smale condition. A similar result holds if $A_0, A_2, \dots, A_{n-2} < 0$ and $A_1, A_3, \dots, A_{n-1} \geq 0$ under the assumption

$$1 - \frac{\alpha+1}{\alpha}\left[A_{n-1}\left(\frac{L}{\pi}\right)^2 - A_{n-2}\left(\frac{L}{\pi}\right)^4 - \cdots - A_1\left(\frac{L}{\pi}\right)^{2n-2} - A_0\left(\frac{L}{\pi}\right)^{2n}\right] > 0. \tag{2.25}$$

The main tool in our approach is the Brézis–Nirenberg's linking theorem [6].

Theorem 2.13. *Suppose that $J \in C^1(H, \mathbb{R})$ satisfies the Palais–Smale condition and has a local linking at 0. Assume that J is bounded below and $\inf_H J < 0$. Then J has at least two nontrivial critical points.*

For the sake of completeness we recall the definition of local linking.

Let the Banach space H has a direct sum decomposition $H = X \oplus Y$, where X is finite dimensional.

Definition 2.14. The functional J is said to have a local linking at 0 if for some $\rho > 0$,

$$J(x) \leq 0, \qquad \forall\, x \in X,\ \|x\| \leq \rho,$$

and

$$J(y) \geq 0, \qquad \forall\, y \in Y,\ \|y\| \leq \rho.$$

3 Main results

Our existence results read.

Theorem 3.1. *Let the function $F \geq 0$, $\forall\, x \in \Omega$, $s \in \mathbb{R}$ satisfy*

$$F(x,s) \leq K|s|^p, \qquad p > 2,\ \forall\, x \in \Omega,\ s \in \mathbb{R}, s \text{ small}, \tag{3.1}$$

where $K > 0$ is a constant. Suppose that we are under hypotheses of either Lemma 2.9, Lemma 2.10, Lemma 2.11 or Lemma 2.12. If in addition there exists a natural number $m \neq 0$ such that

$$P\left(\frac{m\pi}{L}\right) < 0, \tag{3.2}$$

then the boundary value problem (1.1) has at least two nontrivial solutions.

Proof. The proof uses the Brézis–Nirenberg's linking theorem (Theorem 2.13). Hence we have to show that J satisfies the condition imposed in Theorem 2.13.

Since we are under the hypotheses of either Lemma 2.9, Lemma 2.10, Lemma 2.11 or Lemma 2.12 it follows that J is bounded below and satisfies the Palais–Smale condition.

We now follow the proof of Lemma 8, [13] and show that $\inf_{H(\Omega)} J < 0$.

We see that $P\left(\frac{m\pi}{L}\right) \to \infty$ and since (3.2) holds we get that there exists a finite set of natural numbers $\{m_1, m_2, \dots, m_k\}$ such that $P\left(\frac{m_i\pi}{L}\right) < 0$, $i = 1, 2, \dots, k$.

Introducing the finite dimensional space

$$X = \operatorname{span}\left\{ \sin \frac{m_1 \pi x}{L}, \dots, \sin \frac{m_k \pi x}{L} \right\}$$

we see that any $\varphi \in X$ can be written

$$\varphi(x) = c_1 \sin \frac{m_1 \pi x}{L} + \cdots + c_k \sin \frac{m_k \pi x}{L}$$

and its norm in $L^2(\Omega)$ is given by

$$\|\varphi\|_X^2 = c_1^2 + \cdots + c_k^2 = \rho^2,$$

where $c_1, \dots, c_k$ are real constants.

By (3.1) and Hölder's inequality we get for sufficiently small $\rho > 0$

$$\begin{aligned} \int_\Omega F(x, \varphi(x)) dx &\le K \int_\Omega |\varphi(x)|^p dx \\ &\le K \int_\Omega \left[\left(c_1^2 + \cdots + c_k^2 \right)^{\frac{1}{2}} \left(\sin \frac{m_1 \pi x}{L} + \cdots + \sin \frac{m_k \pi x}{L} \right)^{\frac{1}{2}} \right]^p \\ &\le C(K, k, p, L) \left(c_1^2 + \cdots + c_k^2 \right)^{\frac{p}{2}} = C(K, k, p, L) \rho^p. \end{aligned}$$

Hence

$$\begin{aligned} J(\varphi) &\le \frac{L}{4} \sum_{i=1}^{k} P\left(\frac{m_i \pi}{L} \right) c_i^2 + C(K, k, p, L) \rho^p \\ &\le \frac{L}{4} \alpha \rho^2 + C(K, k, p, L) \rho^p = \rho^2 \left(\frac{L}{4} \alpha + C(K, k, p, L) \rho^{p-2} \right) < 0, \end{aligned}$$

where $\alpha = \max \left\{ P\left(\frac{m_i \pi}{L}\right), i = 1, 2, \dots, k \right\} < 0$ by hypothesis.

We now show that J has a local linking at 0.

By the above estimation, we see that for sufficiently small ρ

$$J(u) \le 0, \qquad \forall\, u \in X,\ \|u\| \le \rho.$$

Also since for any $u \in Y = X^\perp$ (bear in mind that $P\left(\frac{m_{k+1}\pi}{L}\right) \ge 0$)

$$J(u) \ge \frac{1}{2} P\left(\frac{m_{k+1} \pi}{L} \right) \|u\|_{L^2(\Omega)}^2 + \int_\Omega F(x, u) dx \ge 0,$$

we get that J has a local linking at 0 and the proof follows. □

Immediate consequences of Theorem 3.1 are the following.

Corollary 3.2. *Suppose that $P(0) > 0$ and that P takes negative values. The problem (1.1) has at least two nontrivial solutions in $\Omega = (0, L)$ provided the following relation holds true*

$$\frac{m\pi}{\xi_2} < L < \frac{m\pi}{\xi_1} \quad \textit{for some natural number } m \neq 0. \tag{3.3}$$

Here $0 < \xi_1 < \xi_2$ are the first (the smallest) two positive roots of P. Note that P may have other roots.

Corollary 3.3. *Suppose that $P(0) < 0$ and let $\xi_1 > 0$ be the smallest root of P (P may have other roots). The problem (1.1) has at least two nontrivial solutions in $\Omega = (0, L)$ provided the following relation holds true*

$$L > \frac{m\pi}{\xi_1} \quad \text{for some natural number } m \neq 0. \tag{3.4}$$

We note that the uniqueness results presented in [12, 13] as well as our Theorem 3.1 are stated under the restriction $F \geq 0$ and

$$\lim_{s\to 0} \frac{F(s)}{s^2} = 0. \tag{3.5}$$

The next result is stated when F may change sign and (3.5) is weakened.

Theorem 3.4. *Let the function F satisfy*

$$F(x,s) \geq -K_1|s|^p - K_2, \qquad \forall\, x \in \Omega,\ s \in \mathbb{R}, \tag{3.6}$$

where $0 < p < 2$, and $K_1, K_2 > 0$.

Suppose that $A_{n-1} \leq 0, A_{n-2} \geq 0, \dots, A_1 \leq 0, A_0 \geq 0$ holds or we are under hypotheses of either Lemma 2.3, Lemma 2.5, Lemma 2.6 or Lemma 2.7. If in addition one of the following relation holds

$$\lim_{s\to 0^+} \frac{f(x,s)}{s^\alpha} = q(x) \quad \text{uniformly in } \overline{\Omega}, \tag{3.7}$$

where $q(x) \leq 0$, $\|q\|_{L^\infty(\Omega)} > 0$, $0 < \alpha < 1$

$$\lim_{s\to 0} \frac{F(x,s)}{s^2} = \beta(x) \in L^1(\Omega), \quad \text{uniformly in } \overline{\Omega}, \tag{3.8}$$

where

$$\int_\Omega \beta(x) \sin^2 \frac{\pi x}{L} dx + \frac{L}{4} P\left(\frac{\pi}{L}\right) < 0, \tag{3.9}$$

then the boundary value problem (1.1) has at least one nontrivial solution.

Proof. We choose $\rho > 0$ arbitrary but fixed and denote by

$$B_\rho = \{u \in H(\Omega) \mid \|u\|_{H(\Omega)} < \rho\}.$$

We first note that one of the relations (3.7) or (3.8) assures that

$$\mu = \inf_{\overline{B}_\rho} J(u) < 0.$$

Indeed, suppose that (3.7) holds.

We can choose the positive function $\varphi(x) = \sin \frac{\pi x}{L} \in H(\Omega)$ such that

$$\int_\Omega q(x) \varphi^{\alpha+1}(x) dx < 0.$$

Hence

$$\begin{aligned}
\lim_{s\to 0^+} \frac{J(s\varphi)}{s^{\alpha+1}} &= \frac{1}{2} \lim_{s\to 0^+} s^{1-\alpha} \int_\Omega \Big((\varphi^{(n)})^2 - A_{n-1}(\varphi^{(n-1)})^2 + \cdots + A_0\varphi^2 \Big) dx \\
&\quad + \lim_{s\to 0^+} \int_\Omega \frac{F(x, s\varphi)}{s^{\alpha+1}} dx \\
&= \int_\Omega \lim_{s\to 0^+} \frac{F(x,s\varphi)}{s^{\alpha+1}} dx = \int_\Omega \lim_{s\to 0^+} \frac{f(x,s\varphi)\varphi}{(\alpha+1)s^\alpha} dx \\
&= \frac{1}{\alpha+1} \int_\Omega q(x)\varphi^{\alpha+1}(x) dx < 0.
\end{aligned}$$

Similarly if (3.8) holds we see that

$$\lim_{s\to 0}\frac{J(s\varphi)}{s^2}=\frac{L}{4}P\left(\frac{\pi}{L}\right)+\int_\Omega \beta(x)\sin^2\frac{\pi x}{L}dx<0.$$

By relation (3.6), Cauchy's inequality with ε and (2.3)

$$\begin{aligned}\int_\Omega F(x,u)dx &\geq -\varepsilon\int_\Omega u^2dx-\int_\Omega\left(C(p,\varepsilon)K_1^{\frac{2}{2-p}}+K_2\right)dx \\ &\geq -\varepsilon\left(\frac{L}{\pi}\right)^{2n}\|u\|^2_{H(\Omega)}-C(p,\varepsilon,K_1,K_2,L).\end{aligned} \tag{3.10}$$

Hence if we are under hypotheses of either Lemma 2.3, Lemma 2.5, Lemma 2.6 or Lemma 2.7 we can combine (3.10) with one of relations (2.6), (2.10), (2.16) or (2.18) to get (by choosing ε sufficiently small) that $J(u)$ is bounded below on $\overline{B}_\rho$ by a negative constant.

According to the Remark, inequalities of type (2.6) are always true if $A_{n-1}\leq 0, A_{n-2}\geq 0,\dots,A_1\leq 0, A_0\geq 0$ and hence again we obtain that $J(u)$ is bounded below.

From Ekeland's variational principle it follows that there exists a minimizing sequence $\{u_m\}\subset\overline{B}_\rho$ such that

$$J(u_m)\to\mu \quad\text{and}\quad J'(u_m)\to 0,\quad \text{as } m\to\infty.$$

Since $\{u_m\}$ is bounded we can extract (by using the Sobolev imbedding) a subsequence still denoted $\{u_m\}$ such that

$$\begin{aligned} u_m &\rightharpoonup u_0 \quad \text{weakly in } H(\Omega),\\ u_m &\to u_0 \quad \text{strongly in } C^{n-1}(\overline{\Omega}).\end{aligned}$$

Arguing as in the proof Lemma 2.9 we get that $\{u_m\}$ converges strongly to u_0 in $H(\Omega)$.

As a consequence there exists $u_0\in H(\Omega)$ such that $J'(u_0)=0$, $J(u_0)<0$ i.e., problem (1.1) has at least a nontrivial solution. □

The last existence result shows that if we impose some asymptotic assumptions to f we can allow $p>2$ in (3.6). The proof uses the Mountain Pass theorem and the following two lemmas.

The first lemma shows when $J(u)$ has a mountain pass structure

Lemma 3.5. *Suppose that we are under one of the assumptions of Lemma 2.3, Lemma 2.5 or Lemma 2.6. Let F satisfy*

$$F(x,s)\leq C|s|^t,\qquad \forall\,(x,s)\in\Omega\times\mathbb{R}, \tag{3.11}$$

where $C>0, t>2$ and relation (3.7) holds.

Then

1. *there exist two positive constants ρ and η such that*

$$J(u)|_{\|u\|=\rho}\geq\eta, \tag{3.12}$$

2. *there exists $e\in H(\Omega)$ satisfying $\|u\|>\rho$ and $J(e)<0$.*

Here

$$\|u\|^2 = \int_\Omega \left[(u^{(n)})^2 - A_{n-1}(u^{(n-1)})^2 + \cdots - A_1(u')^2 + A_0 u^2 \right] dx$$

is a norm since we work under the assumptions of Lemma 2.3, Lemma 2.5 or Lemma 2.6.

We also note that $J(u)$ becomes

$$J(u) = \frac{1}{2}\|u\|^2 + \int_\Omega F(x,u)dx.$$

Proof. For a proof see [10]. □

We can now apply the Mountain Pass theorem in $H(\Omega)$ to find a Cerami type sequence, i.e.,

$$\text{there exists } \{u_m\} \subset H(\Omega) \text{ such that } J(u_m) \to \lambda \text{ and } \|J'(u_m)\|_{H^*(\Omega)} \to 0. \tag{3.13}$$

The next lemma gives the boundedness of the sequence $\{u_m\}$.

Lemma 3.6. *Suppose that we are under the hypotheses of Lemma 3.5. If in addition there exist the constants $\theta \in (0,2), K_1 \in \mathbb{R}, K_2 > 0$ such that*

$$f(x,s)s \geq K_1|s|^\theta - K_2, \qquad \forall\, x \in \Omega,\ |s| > M, \tag{3.14}$$

for some $M > 0$, then the sequence $\{u_m\}$ defined by (3.13) is bounded in $H(\Omega)$.

Proof. We argue by contradiction and suppose that $\|u_m\| \to \infty$. Let $w_m = \frac{u_m}{\|u_m\|}$. Obviously $\{w_m\}$ is a bounded sequence and we can extract a subsequence, still denoted $\{w_m\}$, such that

$$w_m \to w \quad \text{strongly in } \mathrm{C}^{\mathrm{n}-1}(\overline{\Omega}).$$

For each fixed m we define

$$\Omega_m^1 = \{x \in \Omega \mid u_m(x) \leq M\} \quad \text{and} \quad \Omega_m^2 = \{x \in \Omega \mid u_m(x) > M\}.$$

By the continuity of f there exists a constant $C_1 > 0$ such that

$$\int_{\Omega_m^1} f(x,u_m)u_m dx \geq -C_1. \tag{3.15}$$

Since

$$\langle J'(u_m), u_m \rangle = \|u_m\|^2 + \int_\Omega f(x,u_m)u_m dx,$$

we get by combining (3.14) and (3.15) that

$$\begin{aligned} \langle J'(u_m), u_m \rangle &\geq \|u_m q|^2 - C_1 - \int_{\Omega_m^2} (K_1|u_m|^\theta - K_2)dx \\ &\geq \|u_m\|^2 - C_1 - |K_1| \int_{\Omega_m^2} |u_m|^\theta dx - K_2 \operatorname{meas}(\Omega). \end{aligned} \tag{3.16}$$

Using (3.16) and the fact that $\langle J'(u_m), u_m \rangle \to 0$, as $m \to \infty$ it follows that

$$\begin{aligned} \infty = \lim_{m\to\infty} \frac{\|u_m\|^2}{\|u_m\|^\theta} &\leq \lim_{m\to\infty} \left(\frac{\langle J'(u_m), u_m \rangle}{\|u_m\|^\theta} + |K_1| \int_\Omega |w_m|^\theta dx + \frac{C_1 + K_2 \operatorname{meas}(\Omega)}{\|u_m\|^\theta} \right) \\ &= |K_1| \int_\Omega |w|^\theta dx < \infty, \end{aligned}$$

which is a contradiction.

Hence we conclude that the sequence $\{u_m\}$ is bounded. □

The last existence result reads

Theorem 3.7. *Suppose that we are under one of the assumptions of Lemma 2.3, Lemma 2.5 or Lemma 2.6 and that relation (3.7) holds. Let $p, q, r > 1$ be such that $p \geq \sigma = \max\{q, r\}$ and $L_1, L_2, L_3 \in L^\infty(\Omega)$. If in addition*

$$\lim_{|s| \to 0} \frac{f(x,s)}{|s|^p} = L_1(x) \tag{3.17}$$

and

$$\lim_{s \to \infty} \frac{f(x,s)}{s^q} = L_2(x) > 0, \qquad \lim_{s \to -\infty} \frac{f(x,s)}{|s|^r} = L_3(x) < 0, \tag{3.18}$$

uniformly in $\overline{\Omega}$, then the boundary value problem (1.1) has at least a nontrivial solution.

Proof. Combining relations (3.17) and (3.18) we get that there exists a constant $C > 0$ such that for sufficiently large M

$$-sf(x,s) \leq C|s|^{\sigma+1}, \qquad \forall\, x \in \Omega,\ |s| > M. \tag{3.19}$$

Integrating (3.19) one has

$$-F(x,s) = -\int_0^1 f(x, us) s du \leq \frac{C}{\sigma+1}|s|^{\sigma+1}, \qquad \forall\, x \in \Omega,\ |s| > M.$$

We can now apply Lemma 3.5 to get a sequence $\{u_m\}$ that satisfies (3.13).

On the other hand, in view of (3.18) we see that (3.14) is satisfied and hence $\{u_m\}$ is bounded. As a consequence $u_m \to u_0$ in $C^{n-1}(\overline{\Omega})$ and the proof follows. □

Finally, we give some examples as an application of our results.

Example 1. Let F satisfy (3.1) and suppose that (3.3) holds with $m = 1$. Then the boundary value problem

$$\begin{cases} u^{(2n)} + Au^{(4)} + Bu'' + Cu + f(x,u) = 0 \text{ in } \Omega = (0, L) \\ u = u'' = \cdots = u^{(2n-2)} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.20}$$

has at least two nontrivial solutions in $H(\Omega)$. Here $A < 0, B = 0, C > 0$, (2.25) holds and

$$\left(\frac{-2A}{n}\right)^{\frac{n(n-2)}{4}} + A\left(\frac{-2A}{n}\right)^{\frac{n-2}{2}} + C < 0. \tag{3.21}$$

In particular, the result holds if $n = 4, A = -2, 0 < C < 1, L = 2$.

The proof follows from Corollary 3.2. Since $P(\xi) = \xi^{2n} + A\xi^4 + C$ we study the function $\varphi(t) = t^{\frac{n}{2}} + At + C$. We can check that φ attains its minimum at $t_0 = (-2A/n)^{\frac{n-2}{2}}$. Imposing $\varphi(t_0) < 0$, i.e., (3.21) we see that P has (at least) two positive roots.

Consider $n = 3$. Then P becomes $P(\xi) = \xi^6 - A\xi^4 + B\xi^2 - C$. If

$$A > 0, \quad B < 0, \quad 0 > C > \gamma = \frac{1}{27}\left[9AB - 2A^3 - 2\left(A^2 - 3B\right)^{\frac{3}{2}}\right],$$

then P has precisely two positive roots $0 < \xi_1 < \xi_2$. As a consequence (3.20) has at least two nontrivial solutions in $H(\Omega)$ if (3.3) holds with $m = 1$.

The reader is referred to Appendix A, [13] where the authors give detailed conditions on parameters A, B, C and L which guarantee the existence of at least one or two positive solutions of $P(\xi) = \xi^6 - A\xi^4 + B\xi^2 - C$.

Example 2. Let F satisfy (3.1) and suppose that (3.4) holds (here ξ_1 is the unique solution of $P(\xi) = 0$). Consider the boundary value problem (3.20), where $C < 0$. Suppose that one of the following relations holds true

$$A, B \geq 0 \quad \text{and} \quad (2.25) \tag{3.22}$$

$$A > 0, \quad B < 0 \quad \text{and} \quad (2.25) \tag{3.23}$$

$$A < 0, \quad B > 0, \quad (3.4) \quad \text{and} \quad n\left(\frac{-2A}{n(n-1)}\right)^{\frac{n-1}{n-2}} + 2A\left(\frac{-2A}{n(n-1)}\right)^{\frac{1}{n-2}} + B > 0. \tag{3.24}$$

Then the boundary value problem (3.20) has at least two nontrivial solutions in Ω.

The proof follows from Corollary 3.3 by using the same techniques as in Example 1.

Example 3. In as similar way we can conclude that if F satisfies (3.1) and that (3.4) holds (here ξ_1 is the unique solution of $P(\xi) = 0$), then the problem

$$\begin{cases} u^{(2n)} + Au^{(2n-2)} + Bu^{(2n-4)} + Cu + f(x,u) = 0 \text{ in } \Omega = (0, \mathrm{L}) \\ u = u'' = \cdots = u^{(2n-2)} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.25}$$

has at least two nontrivial solutions in Ω. Here $A, B > 0, C < 0$ and we are under the assumptions of Lemma 2.9.

Example 4. Arguing as before, if $A, B > 0$, $A^2 > 4B$, F satisfies (3.1) and if (3.3) holds, it follows that the problem

$$\begin{cases} u^{(6)} + Au^{(4)} + Bu'' + f(x,u) = 0 \text{ in } \Omega = (0, \mathrm{L}) \\ u = u'' = u^{(4)} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.26}$$

has at least two nontrivial solutions in Ω.

Example 5. The functions $F_1(s) = \ln(1 + \ln(1 + \cdots + \ln(1 + |s|^p)))$, $p > 2$ and $F_2(s) = |s|(\arctan|s|^p + \ln(1 + |s|^p))$, $p > 1$ satisfy (3.1). Hence, under the requirements of Theorem 3.1 problem (1.1) (with f replaced by $f_1 = F_1'$ or $f_2 = F_2'$) has at least two nontrivial solutions in Ω.

It is easy to check that F_1, F_2 don't satisfy (1.3) and hence this existence result cannot be deduced from the corresponding results presented in [12] or [13] even if we restrict ourselves to the particular cases $n = 2$ or $n = 3$.

We can see that $F_3(s) = s^p - Cs^2$, where $p > 2$ is even and $C > 0$ changes sign and does not fulfill the restriction (3.5) imposed in [12, 13], but fulfills the requirements of Theorem 3.4 with $\beta = -C < 0$. Again we conclude that problem (1.1) (with f replaced by $f_3 = F_3'$) has at least a nontrivial solution if (3.9) is satisfied.

Example 6. Let $C > 0, q > 2, \alpha \in (0, 1)$. Then the function f_4

$$f_4(s) = \begin{cases} -s^q - C\ln(1 + s^\alpha), & s > 0 \\ |s|^q, & s \leq 0 \end{cases}$$

satisfies the requirements of Theorem 3.7. Hence the boundary value problem (1.1) with f replaced by f_4 has at least one solution.

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Global multiplicity of positive solutions for anisotropic (p,q)-Robin boundary value problems with an indefinite potential term

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Abstract. We consider a nonlinear Robin problem driven by the anisotropic (p,q)-Laplacian plus an indefinite potential term. In the reaction, we have the competing effects of a parametric concave (sublinear) term perturbed by a superlinear one (concave-convex problem). We prove the existence and multiplicity result for positive solutions which is global with respect to the parameter. We also show the existence of a minimal positive solution and determine its dependence on the parameter.

Keywords: nonlinear nonhomogeneous differential operator, indefinite potential, positive solutions, truncations and comparisons, nonlinear regularity, (p,q)-Laplacian, Robin boundary.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following parametric anisotropic Robin boundary value problem:

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) + \xi(z)(u(z))^{p(z)-1} = \lambda(u(z))^{\tau(z)-1} + f(z,u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{pq}} + \beta(z)(u(z))^{p(z)-1} = 0 \quad \text{on } \partial\Omega, \ \lambda > 0, \ u > 0, \ 1 < \tau < p. \end{cases} \tag{p_λ}$$

In this problem the variable exponents $p(\cdot)$ and $q(\cdot)$ of the two differential operators are Lipschitz continuous on $\overline{\Omega}$, that is, $p,q \in C^{0,1}(\overline{\Omega})$. Then the two operators are defined by

$$\Delta_{p(z)}u = \operatorname{div}(|Du|^{p(z)-2}Du), \quad \forall u \in W^{1,p(z)}(\Omega),$$

$$\Delta_{q(z)}u = \operatorname{div}(|Du|^{q(z)-2}Du), \quad \forall u \in W^{1,q(z)}(\Omega).$$

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There is also a potential term $\xi(z)(u(z))^{p(z)-1}$ which is in general indefinite since $\xi \in L^\infty(\Omega)$ can be sign-changing (nodal). Therefore the left hand side of (p_λ) is not coercive. In the reaction (right hand side of (p_λ)), we have the combined effects of a parametric "concave" (sublinear) term $\lambda u^{\tau(z)-1}$ with $\tau \in C(\overline{\Omega})$, $\tau_+ = \max_{\overline{\Omega}} \tau < q_- = \min_{\overline{\Omega}} q$ and of a Carathéodory perturbation $f(z,x)$ (that is, for all $x \geq 0$ $z \to f(z,x)$ is measurable and for almost a.a. $z \in \Omega$ $x \to f(z,x)$ is continuous) which is "convex" ($(p_+ - 1)$-superlinear with $p_+ = \max_{\bar{\Omega}} p$) but without satisfying the usual in such cases Ambrosetti–Rabinowitz condition.

In the boundary condition, $\frac{\partial u}{\partial n_{pq}}$ denotes the conormal derivative of u, corresponding to the anisotropic (p,q)-Laplacian. If $u \in C^1(\bar{\Omega})$ then

$$\frac{\partial u}{\partial n_{pq}} = \left(|Du|^{p(z)-2} + |Du|^{q(z)-2}\right)\frac{\partial u}{\partial n}$$

with $n(.)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta \in C^{0,1}(\partial\Omega)$ with $\beta(z) \geq 0$ and either $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

Therefore (p_λ) is an anisotropic version of the classical "concave-convex problem", with an indefinite potential term and Robin boundary condition. Concave-convex problems, were first studied by Ambrosetti–Brezis–Cerami [1], for semilinear Dirichlet problems driven by the Laplacian with no potential term (that is, $\xi \equiv 0$) and a reaction of the form $u \to \lambda u^{\tau-1} + u^{r-1}$ with $1 < \tau < 2 < r$. Their work was extended to p-Laplacian equations by García Azorero–Peral Alonso–Manfredi [10] and Guo–Zhang [14]. Further extensions involved more general nonlinear nonhomogeneous differential operators and more general reactions (see Marano–Marino–Papageorgiou [19], Papageorgiou–Rădulescu [29] and the references therein). For anisotropic problems, there are significantly fewer papers. We mention the works of Papageorgiou–Qin–Rădulescu [24] (Dirichlet problems driven by the anisotropic p-Laplacian) and by Deng [6], Liu–Papageorgiou [18] (Robin problems, in [6] the differential operator is the $p(z)$-Laplacian and $\xi \equiv 0$, while in [18] the equation is driven by the anisotropic (p,q)-Laplacian, with $\xi(z) > 0$ for a.a. $z \in \Omega$, the conditions on $f(z,.)$ are stronger near 0^+ and the authors employ a different superlinearity condition). Using variational tools from the critical point theory, together with truncation and comparison techniques, we prove an existence and multiplicity result for positive solutions which is global in the parameter $\lambda > 0$ (a bifurcation-type theorem). Our result here extends all the aforementioned anisotropic works. We also mention the works of [3], [12], [20], [27] and [28] on isotropic Neumann and Robin problems with indefinite potential term.

Anisotropic problems are interesting from a purely mathematical viewpoint since they exhibit challenging nonlinearities that we do not encounter in isotropic problems. The $p(z)$-Laplace differential operator is not homogeneous in contrast to the p-Laplacian. This excludes from consideration techniques which proved to be very effective in the context of isotropic problems. This makes anisotropic problems in principle more difficult to deal with. Anisotropic equations, proved to be the right mathematical tool to describe various phenomena from physics and engineering. Materials with inhomogeneities, such as electrorheological fluids (also known as "smart fluids"), can not be modelled adequately using the formalism of the classical Lebesgue and Sobolev spaces. They require the use of variable such spaces (a particular case of the so-called "Musielak–Orlicz spaces"). The book of Růžička [33] contains mathematical models of such fluids and the phenomena characterizing them (Winslow effect). Another important application of anisotropic problem is in image restoration, where we try to eliminate the effect of noise. Initially, this problem was approached by smoothing the input, which corresponds to minimizing the energy functional:

$$\varphi_1(u) = \int_\Omega (|Du|^2 + |u-i|^2)dz$$

with $i(.)$ being the input which corresponds to shades of grey in $\Omega \subset \mathbb{R}^n$. We assume that noise is additive, that is, $i = t_0 + n$ with t_0 representing the true image and n the noise which is a random variable with zero mean. It turns out that this approach destroys the small details of the image. To remedy this, it was proposed to use the "total variation smoothing", which corresponds to minimizing the energy functional:

$$\varphi_2(u) = \int_\Omega (|Du| + |u-i|^2)dz.$$

This approach does a good job of preserving the edges of the image (an edge gives rise to a large gradient of $u(\cdot)$). But unfortunately, this approach also introduces edges, where they did not exist before. For this reason Chen–Levine–Rao [4], suggested to consider the energy functional:

$$\varphi_3(u) = \int_\Omega (|Du|^{p(z)} + |u-i|^2)dz$$

with $1 \leq p(z) \leq 2$. This function is close to 1 where there are no edges and close to 2 where there are. Therefore, we have an energy functional which incorporates the positive aspects of both $\varphi_1(.)$ and $\varphi_2(.)$.

More details on the mathematical and physical applications of variable spaces can be found in the books of Cruz Uribe–Fiorenza [5], Diening–Harjulehto–Hästö–Růžička [7], Rădulescu–Repovš [31], Růžička [33].

The Robin boundary condition is a weighted combination of Dirichlet and Neumann boundary conditions and so it is more difficult to handle and for this reason it is less common in the literature. However, it is important from a physical viewpoint since it appears in electromagnetic problems (impedance boundary condition) and in heat transfer problems (convective boundary condition).

2 Mathematical background and hypotheses

In this section, we briefly review some basic facts about variable exponent spaces. A comprehensive presentation of variable exponent Lebesgue and Sobolev spaces can be found in the books of Cruz Uribe–Fiorenza [5], Diening–Harjulehto–Hästö–Růžička [7].

Let $L_1^\infty(\Omega) = \{p \in L^\infty(\Omega) : \operatorname{ess\,inf}_\Omega p \geq 1\}$. For $p \in L_1^\infty(\Omega)$, we set

$$p_- = \operatorname*{ess\,inf}_\Omega p \quad \text{and} \quad p_+ = \operatorname*{ess\,sup}_\Omega p.$$

Also, let $M(\Omega) = \{u : \Omega \to \mathbb{R} : u(\cdot) \text{ is measurable}\}$. As usual, we identify two functions which differ on a set of zero measure.

Given $p \in L_1^\infty(\Omega)$, we define the following variable exponent Lebesgue space

$$L^{p(z)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u|^{p(z)}\, dz < +\infty \right\}.$$

We equip $L^{p(z)}(\Omega)$ with the following norm (known as the Luxemburg norm)

$$\|u\|_{p(z)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|u|}{\lambda} \right)^{p(z)} dz \leq 1 \right\}.$$

Also, we introduce the variable exponent Sobolev spaces as follows:

$$W^{1,p(z)}(\Omega) = \left\{ u \in L^{p(z)}(\Omega) : |Du| \in L^{p(z)}(\Omega) \right\}.$$

We equip this space with the following norm:

$$\|u\|_{1,p(z)} = \|u\|_{p(z)} + \|Du\|_{p(z)}.$$

An equivalent norm of $W^{1,p(z)}(\Omega)$ is given by:

$$\|u\|_{1,p(z)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\left(\frac{|Du|}{\lambda} \right)^{p(z)} + \left(\frac{|u|}{\lambda} \right)^{p(z)} \right) dz \le 1 \right\}.$$

We define $W_0^{1,p(z)}(\Omega)$ as the closure in the $\|\cdot\|_{1,p(z)}$ norm of all compactly supported $W^{1,p(z)}(\Omega)$-functions.

When $p \in L_1^\infty(\Omega)$ and $p_- > 1$, then the spaces $L^{p(z)}(\Omega)$, $W^{1,p(z)}(\Omega)$, and $W_0^{1,p(z)}(\Omega)$ are all separable, reflexive, and uniformly convex.

If $p, p' \in L_1^\infty(\Omega)$ and $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$, then $L^{p(z)}(\Omega)^* = L^{p'(z)}(\Omega)$, and we have the following Hölder-type inequality:

$$\int_\Omega |uv|\, dz \le \left(\frac{1}{p_-} + \frac{1}{p'_-} \right) \|u\|_{p(z)} \|v\|_{p'(z)}$$

for all $u \in L^{p(z)}(\Omega)$, $v \in L^{p'(z)}(\Omega)$.

We set

$$p^*(z) = \begin{cases} \frac{Np(z)}{N-p(z)}, & \text{if } p(z) < N, \\ +\infty, & \text{if } p(z) \ge N. \end{cases}$$

Theorem 2.1. *If $p, q \in C(\bar{\Omega})$, $p_+ < N$ and $1 \le q(z) \le p^*(z)$ (resp. $1 \le q(z) < p^*(z)$) for all $z \in \bar{\Omega}$, then $W^{1,p(z)}(\Omega)$ and $W_0^{1,p(z)}(\Omega)$ are embedded continuously (resp. compactly) into $L^{q(z)}(\Omega)$.*

We set

$$p^\partial(z) = \begin{cases} \frac{(N-1)p(z)}{N-p(z)}, & \text{if } p(z) < N, \\ +\infty, & \text{if } p(z) \ge N. \end{cases}$$

Theorem 2.2. *If $p \in C(\bar{\Omega})$, $p_- > 1$ and $q \in C(\partial\Omega)$ satisfies the condition*

$$1 \le q(z) < p^\partial(z) \quad \text{for all } z \in \partial\Omega$$

then $W^{1,p(z)}(\Omega)$ embedded compactly into $L^{q(z)}(\partial\Omega)$. In particular, $W^{1,p(z)}(\Omega)$ embedded compactly into $L^{p(z)}(\partial\Omega)$.

We introduce the following modular functions:

$$\rho(u) = \int_\Omega |u|^{p(z)}\, dz \quad \text{for all } u \in L^{p(z)}(\Omega),$$

$$\hat{\rho}(u) = \int_\Omega \left(|Du|^{p(z)} + |u|^{p(z)} \right) dz \quad \text{for all } u \in W^{1,p(z)}(\Omega).$$

We have the following properties.

Proposition 2.3.

(a) *For every* $u \in L^{p(z)}(\Omega)$, $u \neq 0$, *we have* $\|u\|_{p(z)} = \lambda \iff \rho\left(\frac{u}{\lambda}\right) = 1$;

(b) $\|u\|_{p(z)} < 1$ *(resp.* $= 1, > 1$) $\iff \rho(u) < 1$ *(resp.* $= 1, > 1$);

(c) $\|u\|_{p(z)} < 1 \Rightarrow \|u\|_{p(z)}^{p_+} \leq \rho(u) \leq \|u\|_{p(z)}^{p_-}$ *and* $\|u\|_{p(z)} > 1 \Rightarrow \|u\|_{p(z)}^{p_-} \leq \rho(u) \leq \|u\|_{p(z)}^{p_+}$;

(d) $\|u_n\|_{p(z)} \to 0 \iff \rho(u_n) \to 0$;

(e) $\|u_n\|_{p(z)} \to +\infty \iff \rho(u_n) \to +\infty$.

Similarly, we have the following implications when $p \in C^{0,1}(\overline{\Omega})$.

Proposition 2.4.

(a) *For every* $u \in W^{1,p(z)}(\Omega)$, $u \neq 0$, *we have* $\|u\|_{1,p(z)} = \lambda \iff \hat{\rho}\left(\frac{u}{\lambda}\right) = 1$;

(b) $\|u\|_{1,p(z)} < 1$ *(resp.* $= 1, > 1$) $\iff \hat{\rho}(u) < 1$ *(resp.* $= 1, > 1$);

(c) $\|u\|_{1,p(z)} < 1 \Rightarrow \|u\|_{1,p(z)}^{p_+} \leq \hat{\rho}(u) \leq \|u\|_{1,p(z)}^{p_-}$ *and* $\|u\|_{1,p(z)} > 1 \Rightarrow \|u\|_{1,p(z)}^{p_-} \leq \hat{\rho}(u) \leq \|u\|_{1,p(z)}^{p_+}$;

(d) $\|u_n\|_{1,p(z)} \to 0 \iff \hat{\rho}(u_n) \to 0$;

(e) $\|u_n\|_{1,p(z)} \to +\infty \iff \hat{\rho}(u_n) \to +\infty$.

Let $\beta \in L^\infty(\partial\Omega)$ with $\beta_- := \inf_{z\in\partial\Omega} \beta(z) > 0$, and for any $u \in W^{1,p(z)}(\Omega)$, define

$$\|u\|_\beta := \inf\left\{\tau > 0 : \int_\Omega \left(\frac{|\nabla u|}{\tau}\right)^{p(z)} dz + \int_{\partial\Omega} \beta(z) \left(\frac{|u|}{\tau}\right)^{p(z)} d\sigma \leq \tau\right\}.$$

Proposition 2.5. *Let* $\rho_\beta(u) = \int_\Omega |\nabla u|^{p(z)}\, dz + \int_{\partial\Omega} \beta(z)|u|^{p(z)}\, d\sigma$ *with* $\beta_- > 0$, *where* $d\sigma$ *is the measure on the boundary of* Ω. *For any* $u, u_k \in W^{1,p(z)}(\Omega)$ $(k = 1, 2, \dots)$, *we have that*

(a) $\|u\|_\beta \leq 1 \Rightarrow \|u\|_\beta^{p_-} \leq \rho_\beta(u) \leq \|u\|_\beta^{p_+}$;

(b) $\|u\|_\beta \geq 1 \Rightarrow \|u\|_\beta^{p_+} \leq \rho_\beta(u) \leq \|u\|_\beta^{p_-}$;

(c) $\|u_k\|_\beta \to 0 \iff \rho_\beta(u_k) \to 0$ *(as* $k \to \infty$);

(d) $\|u_k\|_\beta \to \infty \iff \rho_\beta(u_k) \to \infty$ *(as* $k \to \infty$).

Proposition 2.6. *(see [32]) If there is a vector* $l \in \mathbb{R}^n \setminus \{0\}$ *such that for any* $z \in \Omega$ *the function* $f(t) = q(z + tl)$ *is monotone for* $t \in I_z = \{t : z + tl \in \Omega\}$, *then*

$$0 < \mu^* = \inf_{u\neq 0} \frac{\int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz}{\int_\Omega \frac{1}{q(z)} |u|^{q(z)} dz}.$$

Theorem 2.7. *For any* $u \in W^{1,p(z)}(\Omega)$, *let*

$$\|u\|_\partial := \|\nabla u\|_{p(z)} + \|u\|_\beta.$$

Then $\|u\|_\partial$ *is a norm on* $W^{1,p(z)}(\Omega)$ *which is equivalent to*

$$\|u\|_{1,p(z)} = \|\nabla u\|_{p(z)} + \|u\|_{p(z)}.$$

The Banach space $C^1(\overline{\Omega})$ is an ordered with a positive (order) cone C_+ which is defined by

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Given $u : \Omega \to \mathbb{R}$ is measurable, then we define

$$u^+(z) = \max\{u(z), 0\}, \quad u^-(z) = \max\{-u(z), 0\} \quad \text{for all } z \in \Omega.$$

These are measurable functions and $u = u^+ - u^-$, $|u| = u^+ + u^-$. Moreover, if $u \in W^{1,p(.)}(\Omega)$, then $u^\pm \in W^{1,p(.)}(\Omega)$. Suppose $u, v : \Omega \to \mathbb{R}$ are measurable functions with $u(z) \leq v(z)$ for a.a. $z \in \Omega$. We define

$$[u, v] = \{h \in W^{1,p(.)}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\},$$

$$\operatorname{int}_{C^1(\overline{\Omega})}[u, v] = \text{the interior in } C^1(\overline{\Omega}) \text{ of } [u, v] \cap C^1(\overline{\Omega}),$$

$$[u) = \{h \in W^{1,p(.)}(\Omega) \mid u(z) \leq h(z) \text{ for a.a. } z \in \Omega\}.$$

If $u(.)$ is a measurable function, then we write $0 \prec u$ if for all $K \subseteq \Omega$ compact we have $0 < c_K \leq u(z)$ for a.a. $z \in K$.

Let X be a Banach space and $\varphi \in C^1(X)$. We say that $\varphi(.)$ satisfies the "C-condition", if it has the following property:

"Every sequence $\{u_n\}_{n \in \mathbb{N}} \subset X$ such that

- $\{\varphi(u_n)\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ is bounded,
- $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^*,

admits a strongly convergent subsequence."

This is a compactness-type condition on $\varphi(\cdot)$ which compensates for the fact that the ambient space X need not be locally compact (being in general infinite-dimensional). By K_φ we denote the critical set of $\varphi(.)$, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Now we are ready to state our hypotheses on the data of problem (p_λ):

***H*₀**: $p, q \in C^{0,1}(\bar{\Omega})$, $\tau \in C(\bar{\Omega})$ and $1 < \tau(z) \leq \tau_+ < q_- \leq q_+ < p(z) < N$ for all $z \in \bar{\Omega}$. $p_+ < \frac{Np_-}{N - p_-}$, there exists $d \in \mathbb{R}^N$ such that $t \to q(z + td)$ is monotone on $I_z = \{t : \ z + td \in \Omega\}$, $\xi \in L^\infty(\Omega)$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$ and $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

***H*₁**: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies the following conditions:

- (i) $0 \leq f(z, x) \leq a(z)(1 + x^{r(z)-1})$ for almost every $z \in \Omega$ and all $x \geq 0$, where $a \in L^\infty(\Omega)$, $r \in C(\bar{\Omega})$ and $p_+ < r(z) < p(z)^*$ for all $z \in \bar{\Omega}$;
- (ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x \to +\infty} \frac{F(z,x)}{x^{p_+}} = +\infty$ uniformly for almost every $z \in \Omega$;

(iii) if for every $\lambda > 0$, we define $e(z,x) = f(z,x)x - p_+F(z,x)$ and

$$\beta_\lambda(z,x) = \lambda\left(1 - \frac{p_+}{\tau(z)}\right)x^{\tau(z)} + \xi(z)\left(\frac{p_+}{p(z)} - 1\right)x^{p(z)} + e(z,x),\ x \geq 0,$$

then there exists $\mu \in L^1(\Omega)$ such that

$$\beta_\lambda(z,x) \leq \beta_\lambda(z,y) + \mu(z)$$

for almost all $z \in \Omega$ and all $0 \leq x \leq y$;

(iv) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{q(z)-1}} = 0$ uniformly for almost every $z \in \Omega$.

Remark 2.8. Since we look for positive solutions and all the above hypotheses concern the positive semiaxis, we may assume that $f(z,x) = 0$ for almost every $z \in \Omega$, all $x \leq 0$. Hypotheses $H_1(iii)$ is satisfied if there exists $M > 0$ such that for a.a. $z \in \Omega$

$$x \to \frac{\lambda x^{\tau(z)-1} + f(z,x) - \xi(z)x^{p(z)-1}}{x^{p_+-1}}$$

is nondecreasing on $x \geq M$ (see [16]).

The following function satisfies hypotheses H_1 above

$$f(z,x) = \begin{cases} (x^+)^{s(z)-1}, & \text{if } x \leq 1, \\ x^{p_+-1}\ln x + x^{\mu(z)-1}, & \text{if } 1 < x \end{cases}$$

with $s \in C(\bar{\Omega})$, $q(z) < s(z)$ for all $z \in \bar{\Omega}$, $\mu \in C(\bar{\Omega})$, $\mu(z) \leq p_+$ for all $z \in \Omega$. This function fails to satisfy the Ambrosetti–Rabinowitz condition (see [2]).

Let $p \in C^{0,1}(\overline{\Omega})$ and consider the operator $V : W^{1,p(z)}(\Omega) \to (W^{1,p(z)}(\Omega))^*$ defined by

$$\langle V(u), h\rangle = \int_\Omega \left(|\nabla u|^{p(z)-2}(Du, Dh)_{\mathbb{R}} + |\nabla u|^{q(z)-2}(Du, Dh)_{\mathbb{R}}\right)dz, \qquad \forall u, h \in W^{1,p(z)}(\Omega).$$

This operator has the following properties (see [13]).

Proposition 2.9. *The map $V : W^{1,p(z)}(\Omega) \to (W^{1,p(z)}(\Omega))^*$ defined above is bounded(that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone, too) and for type $(S)_+$, that is*

$$u_n \rightharpoonup u \ \text{(weakly) in } W^{1,p(z)}(\Omega) \text{ and } \limsup_{n\to\infty}\langle V(u_n), (u_n - u)\rangle \leq 0 \Rightarrow u_n \to u \ \text{ in } W^{1,p(z)}(\Omega).$$

3 Positive solutions

We introduce the following two sets:

$$\mathfrak{L} := \{\lambda > 0 : \text{problem } (p_\lambda) \text{ has a positive solution}\},$$
$$S_\lambda := \text{set of positive solutions of } (p_\lambda).$$

Our first goal is to establish some basic properties of $\mathfrak{L}$. From now on $\|\cdot\| := \|\cdot\|_{1,p(z)}$.

Let $\theta > \|\xi\|_\infty$, $\lambda > 0$, and consider the functional $\hat{\varphi}_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \int_\Omega \frac{|\nabla u|^{p(z)}}{p(z)} dz + \int_\Omega \frac{|\nabla u|^{q(z)}}{q(z)} dz + \int_\Omega \frac{\xi(z)}{p(z)} |u|^{p(z)} dz + \int_\Omega \frac{\theta}{p(z)} (u^-)^{p(z)} dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)} |u|^{p(z)} d\sigma - \int_\Omega \frac{\lambda}{\tau(z)} (u^+)^{\tau(z)} dz - \int_\Omega F(z, u^+) dz$$

for all $u \in W^{1,p(z)}(\Omega)$.

Proposition 3.1. *If hypotheses H_0 and H_1 hold, and $\lambda > 0$, then $\hat{\varphi}_\lambda(\cdot)$ satisfies the C-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p(z)}(\Omega)$ such that

$$|\hat{\varphi}_\lambda(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{3.1}$$

$$(1 + \|u_n\|)\hat{\varphi}'_\lambda(u_n) \to 0 \quad \text{in } W^{1,p(z)}(\Omega)^* \text{ as } n \to \infty. \tag{3.2}$$

From (3.2) we have

$$|\langle \hat{\varphi}'_\lambda(u_n), h\rangle| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W^{1,p(z)}(\Omega), \text{ all } n \in \mathbb{N}, \tag{3.3}$$

with $\varepsilon_n \to 0^+$, which implies

$$\Big| \langle V(u_n), h\rangle + \int_\Omega \xi(z)|u_n|^{p(z)-2} u_n h dz - \int_\Omega \theta (u_n^-)^{p(z)-1} h dz - \lambda \int_\Omega (u_n^+)^{\tau(z)-1} h dz + \int_{\partial\Omega} \beta(z)|u_n|^{p(z)-2} u_n h d\sigma - \int_\Omega f(z, u_n^+) h dz \Big| \leq \frac{\varepsilon_n \|h\|}{(1 + \|u_n\|)} \tag{3.4}$$

for all $h \in W^{1,p(z)}(\Omega), n \in \mathbb{N}$.

In (3.4), we choose $h = -u_n^- \in W^{1,p(z)}(\Omega)$. We have

$$\Big| \int_\Omega |Du_n^-|^{p(z)} dz + \int_\Omega |Du_n^-|^{q(z)} dz + \int_\Omega \xi(z)(u_n^-)^{p(z)} dz + \int_\Omega \theta (u_n^-)^{p(z)} dz + \int_{\partial\Omega} \beta(z)(u_n^-)^{p(z)} d\sigma \Big| \leq \varepsilon_n \tag{3.5}$$

for all $n \in \mathbb{N}$.

Then,

$$\Big| \int_\Omega |Du_n^-|^{p(z)} dz + \int_\Omega (\xi(z) + \theta)(u_n^-)^{p(z)} dz + \int_{\partial\Omega} \beta(z)(u_n^-)^{p(z)} d\sigma \Big| \leq \varepsilon_n \tag{3.6}$$

which implies

$$u_n^- \to 0 \quad \text{in } W^{1,p(z)}(\Omega) \quad (\text{recall that } \theta > \|\xi\|_\infty). \tag{3.7}$$

In (3.4), we choose $h = u_n^+ \in W^{1,p(z)}(\Omega)$. We have

$$\Big| \int_\Omega |Du_n^+|^{p(z)} dz + \int_\Omega |Du_n^+|^{q(z)} dz + \int_\Omega \xi(z)(u_n^+)^{p(z)} dz - \lambda \int_\Omega (u_n^+)^{\tau(z)} dz - \int_\Omega f(z, u_n^+) u_n^+ dz + \int_{\partial\Omega} \beta(z)(u_n^+)^{p(z)} d\sigma \Big| \leq \varepsilon_n \tag{3.8}$$

for all $n \in \mathbb{N}$.

On the other hand, from (3.1), (3.7) we have

$$\left| \int_\Omega \frac{p_+}{p(z)} |Du_n^+|^{p(z)} dz + \int_\Omega \frac{p_+}{q(z)} |Du_n^+|^{q(z)} dz + \int_\Omega \frac{p_+}{p(z)} \xi(z)(u_n^+)^{p(z)} dz \right.$$
$$\left. - \lambda \int_\Omega \frac{p_+}{\tau(z)} (u_n^+)^{\tau(z)} dz - \int_\Omega p_+ F(z, u_n^+) u_n^+ dz + \int_{\partial\Omega} \frac{p_+}{p(z)} \beta(z)(u_n^+)^{p(z)} d\sigma \right| \le M_2 \quad (3.9)$$

for some $M_2 > 0$, all $n \in \mathbb{N}$.

From (3.8) and (3.9) it follows that

$$\int_\Omega \left(\frac{p_+}{p(z)} - 1 \right) |Du_n^+|^{p(z)} dz + \int_\Omega \left(\frac{p_+}{q(z)} - 1 \right) |Du_n^+|^{q(z)} dz + \int_\Omega \xi(z) \left(\frac{p_+}{p(z)} - 1 \right) (u_n^+)^{p(z)} dz$$
$$+ \int_{\partial\Omega} \left(\frac{p_+}{p(z)} - 1 \right) \beta(z)(u_n^+)^{p(z)} d\sigma - \lambda \int_\Omega \left(\frac{p^+}{\tau(z)} - 1 \right) (u_n^+)^{\tau(z)} dz + \int_\Omega e(z, u_n^+) dz \le M_3 \quad (3.10)$$

for some $M_3 > 0$, all $n \in \mathbb{N}$.

Recall $\beta_\lambda(z,x) = \lambda(1 - \frac{p_+}{\tau(z)})x^{\tau(z)} + \xi(z)(\frac{p_+}{p(z)} - 1)x^{p(z)} + e(z,x)$ for all $x \ge 0$. Then from (3.10) we have

$$\int_\Omega \beta_\lambda(z, u_n^+) dz \le M_3 \quad \text{for all } n \in \mathbb{N}. \quad (3.11)$$

Claim. The sequence $\{u_n^+\}_{n\ge1} \subseteq W^{1,p(z)}(\Omega)$ is bounded.

Our argument proceeds through contradiction. So, suppose that the claim is not true. Then passing to a subsequence if necessary, we may assume that

$$\|u_n^+\| \to \infty \quad \text{as } n \to \infty. \quad (3.12)$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}, n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \rightharpoonup y \quad \text{(weakly) in } W^{1,p(z)}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^{p(z)}(\Omega),\ y \ge 0. \quad (3.13)$$

Let $\Omega_+ = \{z \in \Omega : y(z) > 0\}$ and $\Omega_0 = \{z \in \Omega : y(z) = 0\}$. Then $\Omega = \Omega_+ \cup \Omega_0$.

First we assume that $|\Omega_+|_N > 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$). We have $u_n^+(z) \to +\infty$ for a.a. $z \in \Omega_+$ and so on account of hypothesis $H_1(ii)$ we have

$$\int_\Omega \frac{F(z, u_n^+)}{\|u_n^+\|^{p_+}} dz \to +\infty \quad \text{as } n \to +\infty \quad \text{(see [23])}. \quad (3.14)$$

On account of (3.12), we may assume that $\|u_n^+\| \ge 1$ for all $n \in \mathbb{N}$. Then from (3.1) and (3.5), we have

$$\lambda \int_\Omega \frac{1}{\tau(z)} \frac{(u_n^+)^{\tau(z)}}{\|u_n^+\|^{p_+}} dz + \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}} dz \le \varepsilon_n' + \frac{1}{p_-} \int_\Omega \frac{|Du_n^+|^{p(z)}}{\|u_n^+\|^{p_+}} dz + \frac{1}{q_-} \int_\Omega \frac{|Du_n^+|^{q(z)}}{\|u_n^+\|^{p_+}} dz$$
$$+ \frac{\|\xi\|_\infty}{p_-} \int_\Omega \frac{(u_n^+)^{p(z)}}{\|u_n^+\|^{p_+}} dz + \frac{\|\beta\|_\infty}{p_-} \int_{\partial\Omega} \frac{(u_n^+)^{p(z)}}{\|u_n^+\|^{p_+}} d\sigma$$
$$\le \varepsilon_n' + \frac{1}{p_-} \int_\Omega |Dy_n|^{p(z)} dz + \frac{1}{q_- \|u_n\|^{p_+ - q_+}} \int_\Omega |Dy_n|^{q(z)} dz + \frac{\|\xi\|_\infty}{p_-} \int_\Omega (y_n)^{p(z)} dz$$
$$+ \frac{\|\beta\|_\infty}{p_-} \int_{\partial\Omega} (y_n)^{p(z)} d\sigma \le M_4 \quad (3.15)$$

for some $M_4 > 0$ with $\varepsilon'_n \to 0$.

Comparing (3.14) and (3.15), we have a contradiction. We can assume that $y \equiv 0$, that is $|\Omega|_N = |\Omega_0|_N$. We define

$$\hat{\varphi}_\lambda(t_n u_n^+) := \max\{\hat{\varphi}_\lambda(t u_n^+) : 0 \le t \le 1\}. \tag{3.16}$$

Let $v_n = \eta^{1/p_-} y_n$ for all $n \in \mathbb{N}$, with $\eta \ge 1$. Then, we have

$$v_n \rightharpoonup 0 \quad \text{in } W^{1,p(z)}(\Omega) \quad \text{and} \quad \int_\Omega F(z, v_n)\, dz \to 0 \quad \text{as } n \to \infty \quad (\text{see [23]}). \tag{3.17}$$

Also, we have

$$\int_\Omega \frac{1}{\tau(z)} v_n^{\tau(z)}\, dz \to 0. \tag{3.18}$$

Moreover (3.12) implies that we can find $n_0 \in \mathbb{N}$ such that

$$\frac{\eta^{\frac{1}{p_-}}}{\|u_n^+\|} \in (0, 1] \quad \text{for all } n \ge n_0. \tag{3.19}$$

Hence from (3.16) and (3.19)

$$\begin{aligned}
\hat{\varphi}_\lambda(t_n u_n^+) \ge \hat{\varphi}_\lambda(v_n) &= \int_\Omega \frac{1}{p(z)} |Dv_n|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |Dv_n|^{q(z)} dz + \int_\Omega \frac{1}{p(z)} \xi(z) v_n^{p(z)}\, dz \\
&\quad - \lambda \int_\Omega \frac{1}{\tau(z)} v_n^{\tau(z)} dz + \int_{\partial\Omega} \frac{1}{p(z)} \beta(z) v_n^{p(z)}\, d\sigma - \int_\Omega F(z, v_n)\, dz \quad \text{for all } n \ge n_0 \\
&\ge \frac{1}{p_+} \left(\int_\Omega |Dv_n|^{p(z)} dz + \int_\Omega \xi(z) v_n^{p(z)}\, dz + \int_{\partial\Omega} \beta(z) v_n^{p(z)}\, d\sigma \right) - \int_\Omega F(z, v_n)\, dz \\
&\ge \frac{\eta}{2p_+} \quad (\text{see hypotheses } H_0 \text{ and (3.17)})
\end{aligned}$$

for all $n \ge n_1 \ge n_0$. Since $\eta \ge 1$ is an arbitrary number, we can infer that

$$\hat{\varphi}_\lambda(t_n u_n^+) \to +\infty \quad \text{as } n \to \infty. \tag{3.20}$$

We know that

$$\hat{\varphi}_\lambda(0) = 0 \quad \text{and} \quad \hat{\varphi}_\lambda(u_n^+) \le M_5, \quad \text{all } n \in \mathbb{N}. \tag{3.21}$$

From (3.20) and (3.21) it follows that we can find $n_2 \in \mathbb{N}$ such that

$$t_n \in (0, 1) \quad \text{for all } n \ge n_2. \tag{3.22}$$

Then from (3.16) and (3.22) we infer that

$$t_n \frac{d}{dt} \hat{\varphi}_\lambda(t u_n^+)\Big|_{t = t_n} = 0, \quad \text{then } \langle \hat{\varphi}'_\lambda(t_n u_n^+), t_n u_n^+ \rangle = 0, \quad \forall n \ge n_2,$$

$$
\begin{aligned}
\hat{\varphi}_\lambda(t_n u_n^+) &= \hat{\varphi}_\lambda(t_n u_n^+) - \frac{1}{p_+}\langle \hat{\varphi}'_\lambda(t_n u_n^+), t_n u_n^+\rangle \\
&= \int_\Omega t_n^{p(z)}\left[\frac{1}{p(z)} - \frac{1}{p_+}\right]|Du_n^+|^{p(z)}dz + \int_\Omega t_n^{q(z)}\left[\frac{1}{q(z)} - \frac{1}{p_+}\right]|Du_n^+|^{q(z)}dz \\
&\quad + \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\xi(z)(t_n u_n^+)^{p(z)}dz - \int_\Omega \lambda\left[\frac{1}{\tau(z)} - \frac{1}{p_+}\right](t_n u_n^+)^{\tau(z)}dz \\
&\quad + \int_{\partial\Omega}\left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\beta(z)(t_n u_n^+)^{p(z)}d\sigma + \frac{1}{p_+}\int_\Omega e(z, t_n u_n^+)dz \\
&\le \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right]|Du_n^+|^{p(z)}dz + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p_+}\right]|Du_n^+|^{q(z)}dz + \frac{1}{p_+}\int_\Omega \beta_\lambda(z, t_n u_n^+)dz \\
&\quad + \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\xi(z)(t_n u_n^+)^{p(z)}dz + \int_{\partial\Omega}\left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\beta(z)(t_n u_n^+)^{p(z)}d\sigma \\
&\le \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right]|Du_n^+|^{p(z)}dz + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p_+}\right]|Du_n^+|^{q(z)}dz \\
&\quad + \frac{1}{p_+}\int_\Omega \beta_\lambda(z, u_n^+)dz + \frac{1}{p_+}\|\mu\|_1 + \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\xi(z)(u_n^+)^{p(z)}dz \\
&\quad + \int_{\partial\Omega}\left[\frac{1}{p(z)} - \frac{1}{p_+}\right]\beta(z)(u_n^+)^{p(z)}d\sigma \\
&= \hat{\varphi}_\lambda(u_n^+) - \frac{1}{p_+}\langle \hat{\varphi}_\lambda(u_n^+), u_n^+\rangle + \frac{1}{p_+}\|\mu\|_1.
\end{aligned}
\tag{3.23}
$$

Hence we have,

$$
\begin{aligned}
\hat{\varphi}_\lambda(t_n u_n^+) &\le \hat{\varphi}_\lambda(u_n^+) - \frac{1}{p_+}\langle \hat{\varphi}_\lambda(u_n^+), u_n^+\rangle + \frac{1}{p_+}\|\mu\|_1 \quad \text{for all } n \ge n_2 \text{ (see (3.8))} \\
&\le \hat{\varphi}_\lambda(u_n^+) + \frac{\varepsilon_n}{p_+} + \frac{1}{p_+}\|\mu\|_1
\end{aligned}
\tag{3.24}
$$

(3.20) and (3.24) give us that $\hat{\varphi}_\lambda(u_n^+) \to +\infty$, and this contradicts with (3.21).

Therefore $\{u_n^+\} \subset W^{1,p(z)}(\Omega)$ is bounded. Then from (3.7) and the claim it follows that

$$\{u_n\} \subset W^{1,p(z)}(\Omega) \quad \text{is bounded.}$$

We may assume that

$$u_n \rightharpoonup u \quad \text{in } W^{1,p(z)}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{r(z)}(\Omega) \quad \text{as } n \to \infty. \tag{3.25}$$

In (3.4), we choose $h = u_n - u \in W^{1,p(z)}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.25). Then

$$\lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0 \tag{3.26}$$

(3.26) and Proposition (2.9) give us $u_n \to u$ in $W^{1,p(z)}(\Omega)$. So $\hat{\varphi}_\lambda(\cdot)$ satisfies the C-condition. □

Proposition 3.2. *If hypotheses H_0 and H_1 hold, then $\mathfrak{L} \neq \emptyset$ and we have then $S_\lambda \subset \operatorname{int} C_+$ for every $\lambda \in \mathfrak{L}$.*

Proof. On account of hypotheses $H_1(iv)$, we see that given $\varepsilon > 0$, we can find $C_\varepsilon = C(\varepsilon) > 0$ such that

$$F(z,x) \le \frac{\varepsilon}{q(z)}x^{q(z)} + C_\varepsilon x^{r_+} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0.$$

For every $u \in W^{1,p(z)}(\Omega)$, we have

$$\begin{aligned}\hat{\varphi}_\lambda(u) \geq &\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}\,dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)}|u|^{p(z)}d\sigma + \int_\Omega \frac{\xi(z)}{p(z)}|u|^{p(z)}dz \\ &+ \int_\Omega \frac{\theta}{p(z)}(u^-)^{p(z)}dz - \int_\Omega \frac{\varepsilon}{q(z)}|u|^{q(z)}\,dz - C_\varepsilon \int_\Omega |u|^{r_+}dz - \lambda \int_\Omega \frac{1}{\tau(z)}(u^+)^{\tau(z)}dz \\ \geq &\, \tilde{C}\min\{\|u\|^{p_+}, \|u\|^{p_-}\} + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz - \int_\Omega \frac{\varepsilon}{q(z)}|u|^{q(z)}\,dz \\ &- C_\varepsilon \int_\Omega |u|^{r_+}dz - \lambda \int_\Omega \frac{1}{\tau(z)}(u^+)^{\tau(z)}dz \end{aligned} \tag{3.27}$$

for some $\tilde{C} > 0$ (recall that $\theta > \|\xi\|_\infty$).

Observe next that,

$$\int_\Omega \frac{1}{q(z)}|Du|^{q(z)}\,dz \geq \mu^* \int_\Omega \frac{1}{q(z)}|u|^{q(z)}\,dz \text{ (see Proposition 2.6).} \tag{3.28}$$

$$\int_\Omega \frac{1}{\tau(z)}(u^+)^{\tau(z)}dz \leq \frac{1}{\tau_-}\max\{\|u\|^{\tau_+}_{\tau(z)}, \|u\|^{\tau_-}_{\tau(z)}\}. \tag{3.29}$$

We return to (3.27) and use (3.28) and (3.29). Then for $u \in W^{1,p(z)}(\Omega)$ with $\|u\| \leq 1$, $\varepsilon < \mu^*$ we have

$$\begin{aligned}\hat{\varphi}_\lambda(u) &\geq \tilde{C}\|u\|^{p_+} - \lambda C_1\|u\|^{\tau_-} - C_\varepsilon\|u\|^{r_+} \\ &= (\tilde{C} - \lambda C_1\|u\|^{\tau_- - p_+} - C_\varepsilon\|u\|^{r_+ - p_+})\|u\|^{p_+},\ u \in W^{1,p(z)}(\Omega)\end{aligned} \tag{3.30}$$

for some $C_1 > 0$.

Let us set, for any $t > 0$,

$$k_\lambda(t) = \lambda C_1 t^{\tau_- - p_+} - C_\varepsilon t^{r_+ - p_+}.$$

Since $\tau_- < p_+ < r_+$ we have $\lim_{t\to\infty} k_\lambda(t) = \lim_{t\to 0^+} k_\lambda(t) = \infty$.

Then there exists $t_0 > 0$ satisfying $k'_\lambda(t_0) = 0$. One has

$$\lambda C_1(\tau_- - p_+)t_0^{\tau_- - p_+ - 1} = -C_\varepsilon(r_+ - p_+)t_0^{r_+ - p_+ - 1}$$

$$\Rightarrow t_0 = t_0(\lambda) = \left(\frac{\lambda C_1}{C_\varepsilon}\frac{p_+ - \tau_-}{r_+ - p_+}\right)^{\frac{1}{r_+ - \tau_-}}.$$

Then

$$k_\lambda(t_0) = \lambda C_1\left(\frac{\lambda C_1}{C_\varepsilon}\frac{p_+ - \tau_-}{r_+ - p_+}\right)^{\frac{\tau_- - p_+}{r_+ - \tau_-}} + C_\varepsilon\left(\frac{\lambda C_1}{C_\varepsilon}\frac{p_+ - \tau_-}{r_+ - p_+}\right)^{\frac{r_+ - p_+}{r_+ - \tau_-}}$$

and since $p_+ < \tau_+$ we have $\lim_{\lambda\to 0^+} k_\lambda(t_0) = 0$. So we can find $\lambda_0 > 0$ such that

$$k_\lambda(t_0) < \tilde{C} \quad \text{for all } \lambda \in (0, \lambda_0).$$

Then from (3.30) it follows that

$$\hat{\varphi}_\lambda(u) \geq \hat{m}_\lambda > 0 \quad \text{for all } \|u\| = t_0. \tag{3.31}$$

For $u \in \operatorname{int} C_+$, on account of the superlinearity hypothesis $H_1(ii)$, we have

$$\hat{\varphi}_\lambda(tu) \to -\infty \quad \text{as } t \to \infty. \tag{3.32}$$

Then, (3.31), (3.32) and Proposition (3.1) permit the use of mountain pass theorem. Therefore for every $\lambda \in (0, \lambda_0)$ we can find $u_\lambda \in W^{1,p(z)}(\Omega)$ such that

$$u_\lambda \in K_{\hat{\varphi}_\lambda} \quad \text{and} \quad 0 < \hat{m}_\lambda \leq \hat{\varphi}_\lambda(u_\lambda). \tag{3.33}$$

From (3.33) we have $u_\lambda \neq 0$ (recall that $\hat{\varphi}_\lambda(0) = 0$) and

$$\langle \hat{\varphi}'_\lambda(u_\lambda), h \rangle = 0 \quad \text{for all } h \in W^{1,p(z)}(\Omega). \tag{3.34}$$

Choosing $h = -u_\lambda^- \in W^{1,p(z)}(\Omega)$, we obtain

$$\int_\Omega |Du_\lambda^-|^{p(z)} dz + \int_\Omega |Du_\lambda^-|^{q(z)} dz + \int_\Omega (\theta + \xi(z))(u_\lambda^-)^{p(z)} dz + \int_{\partial\Omega} \beta(z)(u_\lambda^-)^{p(z)} d\sigma = 0$$

$$\Rightarrow \int_\Omega |Du_\lambda^-|^{p(z)} dz + \int_\Omega (\theta + \xi(z))(u_\lambda^-)^{p(z)} dz + \int_{\partial\Omega} \beta(z)(u_\lambda^-)^{p(z)} d\sigma \leq 0$$

$$\Rightarrow u_\lambda \geq 0, \; u_\lambda \neq 0.$$

Then from (3.34) it follows that u_λ is a positive solution (p_λ). From the anisotropic regularity theory (see [8] and [17] for the corresponding isotropic theory) we have

$$u_\lambda \in C_+ \setminus \{0\}.$$

For every $u \in S_\lambda$, we have $u \in C_+ \setminus \{0\}$ and

$$-\Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) + \xi(z) u(z)^{p(z)-1} \geq 0 \quad \text{for a.a. } z \in \Omega,$$

$$\Rightarrow \Delta_{p(z)} u(z) + \Delta_{q(z)} u(z) \leq \|\xi\|_\infty u(z)^{p(z)-1} \quad \text{for a.a. } z \in \Omega,$$

$$\Rightarrow u \in \operatorname{int} C_+ \text{ (see [35] and [26], Proposition A2).}$$

So, we have proved that $(0, \lambda_0) \subseteq \mathfrak{L}$ and so $\mathfrak{L} \neq \emptyset$. Moreover, we have $S_\lambda \subseteq \operatorname{int} C_+$ for all $\lambda > 0$. □

Next, we show that $\mathfrak{L}$ is an interval.

Proposition 3.3. *If hypotheses H_0 and H_1 hold, $\lambda \in \mathfrak{L}$ and $0 < \mu < \lambda$ then $\mu \in \mathfrak{L}$ and given $u_\lambda \in S_\lambda$, we can find $u_\mu \in S_\mu$ such that $u_\mu \leq u_\lambda$.*

Proof. Let us introduce the Carathéodory function $g_\mu(z,x)$ defined by

$$g_\mu(z,x) = \begin{cases} \mu(x^+)^{\tau(z)-1} + f(z,x^+) + \theta(x^+)^{p(z)-1}, & \text{if } x \leq u_\lambda(z), \\ \mu u_\lambda(z)^{\tau(z)-1} + f(z,u_\lambda(z)) + \theta u_\lambda(z)^{p(z)-1}, & \text{if } u_\lambda(z) < x. \end{cases} \tag{3.35}$$

Here $\theta > \|\xi\|_\infty$.

We set $G_\mu(z,x) = \int_0^x g_\mu(z,s)\, ds$ and consider the C^1-functional $\Psi_\mu : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\Psi_\mu(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)}\, dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)}\, dz$$
$$+ \int_\Omega \frac{\theta + \xi(z)}{p(z)} |u|^{p(z)}\, dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)} |u|^{p(z)}\, d\sigma - \int_\Omega G_\mu(z,u)\, dz \tag{3.36}$$

for all $u \in W^{1,p(z)}(\Omega)$. Since $\theta > \|\xi\|_\infty$, it is clear that $\Psi_\mu(\cdot)$ is coercive. Also, using the fact that $W^{1,p(z)}(\Omega) \hookrightarrow L^{p(z)}(\Omega)$ compactly, we see that $\Psi_\mu(\cdot)$ is sequentially weakly lower semicontinuous.

So, by the Weierstrass–Tonelli theorem, there exists $u_\mu \in W^{1,p(z)}(\Omega)$ such that

$$\Psi_\mu(u_\mu) = \inf\left\{\Psi_\mu(u) : u \in W^{1,p(z)}(\Omega)\right\}. \tag{3.37}$$

Since $\tau_+ < p_-$, we see that

$$\Psi_\mu(u_\mu) < 0 = \Psi_\mu(0) \Rightarrow u_\mu \neq 0.$$

From (3.37) we have

$$\Psi'_\mu(u_\mu) = 0 \Rightarrow$$

$$\langle V(u_\mu), h\rangle + \int_\Omega [\theta + \xi(z)]|u_\mu|^{p(z)-2}u_\mu h\,dz + \int_{\partial\Omega} \beta(z)|u_\mu|^{p(z)-2}u_\mu h\,d\sigma = \int_\Omega g_\mu(z,u_\mu)h\,dz \tag{3.38}$$

for all $h \in W^{1,p(z)}(\Omega)$. In (3.38) first we choose $h = -u_\mu^- \in W^{1,p(z)}(\Omega)$. We obtain

$$\begin{aligned}&\int_\Omega |Du_\mu^-|^{p(z)}dz + \int_\Omega |Du_\mu^-|^{q(z)}dz \\ &\qquad + \int_\Omega (\theta + \xi(z))(u_\mu^-)^{p(z)}dz + \int_{\partial\Omega} \beta(z)(u_\mu^-)^{p(z)}d\sigma = \int_\Omega g_\mu(z,u_\mu)u_\mu^-\,dz \\ &\qquad \Rightarrow u_\mu \geq 0, u_\mu \neq 0 \quad (\text{see } (3.35)).\end{aligned}$$

Next, in (3.38) we choose $h = (u_\mu - u_\lambda)^+ \in W^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}&\langle V(u_\mu), (u_\mu - u_\lambda)^+\rangle + \int_\Omega [\theta + \xi(z)]u_\mu^{p(z)-1}(u_\mu - u_\lambda)^+\,dz + \int_{\partial\Omega} \beta(z)|u_\mu|^{p(z)-1}u_\mu(u_\mu - u_\lambda)^+\,d\sigma \\ &= \int_\Omega [\mu u_\lambda^{\tau(z)-1} + f(z,u_\lambda) + \theta u_\lambda^{p(z)-1}](u_\mu - u_\lambda)^+\,dz \quad (\text{see } (3.35)) \\ &\leq \int_\Omega [\lambda u_\lambda^{\tau(z)-1} + f(z,u_\lambda) + \theta u_\lambda^{p(z)-1}](u_\mu - u_\lambda)^+\,dz \quad (\text{since } \mu < \lambda) \\ &= \langle V(u_\lambda), (u_\mu - u_\lambda)^+\rangle + \int_\Omega [\theta + \xi(z)]u_\lambda^{p(z)-1}(u_\mu - u_\lambda)^+\,dz \quad (\text{since } u_\lambda \in S_\lambda).\end{aligned}$$

The monotonicity of $V(\cdot)$ (see Proposition (2.9)) and the fact that $\theta > \|\xi\|_\infty$ imply that $u_\mu \leq u_\lambda, \Rightarrow u_\mu \in [0, u_\lambda], u_\mu \neq 0, \Rightarrow u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ (see (3.35) and (3.38)). □

So, according to Proposition 3.3, the solution multifunction $\lambda \mapsto S_\lambda$ has a kind of weak monotonicity property. We can improve this monotonicity property by adding one more condition on the perturbation $f(z,\cdot)$.

The new hypotheses on $f(z,x)$ are the following:

***H*₂**: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function which is measurable in $z \in \Omega$, for a.a. $z \in \Omega$ we have $f(z,\cdot) \in C^1(\mathbb{R})$,

(i)–(iv) hypotheses $H_2(i)$–(iv) are the same as the corresponding hypotheses $H_1(i)$–(iv), and

(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \to f(z,x) + \hat{\xi}_\rho x^{p(z)-1}$ is nondecreasing on $[0,\rho]$.

Remark 3.4. Hypotheses $H_2(v)$ is a one-sided local Hölder condition on $f(z,\cdot)$. It is satisfied for all $z \in \Omega$, $f(z,x)$ is differentiable and for every $\rho > 0$, we can find $C_{\hat{\rho}} > 0$ such that $f'_x(z,x) \geq -C_{\hat{\rho}} x^{p(z)-1}$ for a.a. $z \in \Omega$ and all $0 \leq x \leq \rho$.

Proposition 3.5. *If hypotheses H_0, H_2 hold, $\lambda \in \mathfrak{L}$, $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ and $\mu \in (0,\lambda)$, then $\mu \in \mathfrak{L}$ and we can find $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that $u_\lambda - u_\mu \in \operatorname{int} C_+$.*

Proof. From Proposition 3.3 we know that $\mu \in \mathfrak{L}$ and there exists $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that

$$u_\lambda - u_\mu \in C_+ \setminus \{0\} \tag{3.39}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_2(v)$. We can always assume that $\hat{\xi}_\rho > \|\xi\|_\infty$. Then we have

$$\begin{aligned}
&-\Delta_{p(z)} u_\mu - \Delta_{q(z)} u_\mu + [\xi(z) + \hat{\xi}_\rho] u_\mu^{p(z)-1} \\
&\quad = \mu u_\mu^{\tau(z)-1} + f(z,u_\mu) + \hat{\xi}_\rho u_\mu^{p(z)-1} \\
&\quad \leq \mu u_\lambda^{\tau(z)-1} + f(z,u_\lambda) + \hat{\xi}_\rho u_\lambda^{p(z)-1} \quad \text{(see (3.39) and hypothesis } H_2(v)) \\
&\quad \leq \lambda u_\lambda^{\tau(z)-1} + f(z,u_\lambda) + \hat{\xi}_\rho u_\lambda^{p(z)-1} \quad (\text{since } \mu < \lambda) \\
&\quad = -\Delta_{p(z)} u_\lambda - \Delta_{q(z)} u_\lambda + [\xi(z) + \hat{\xi}_\rho] u_\lambda^{p(z)-1}.
\end{aligned} \tag{3.40}$$

Note that since $u_\lambda \in \operatorname{int} C_+$ and $\mu < \lambda$, we have

$$0 \prec (\lambda - \mu) u_\lambda^{q(z)-1}. \tag{3.41}$$

Then from (3.40), (3.41) and Proposition 2.4 in [23], we can conclude that

$$u_\lambda - u_\mu \in \operatorname{int} C_+.$$

The proof is now complete. □

Next, we show that for every $\lambda \in \mathfrak{L}$, the solution set S_λ has the smallest element (minimal positive solution). To this end, first, we consider the following auxiliary problem:

$$\begin{cases} -\Delta_{p(z)} u(z) - \Delta_{q(z)} u(z) + |\xi(z)||u(z)|^{p(z)-2} u(z) = \lambda |u(z)|^{\tau(z)-2} u(z) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{pq}} + \beta(z)|u(z)|^{p(z)-1} = 0 \text{ on } \partial\Omega, \ \lambda > 0, \ u > 0. \end{cases} \tag{3.42}$$

Proposition 3.6. *If hypotheses H_0 hold and $\lambda > 0$, then problem (3.42) admits a unique positive solution $\bar{u}_\lambda \in \operatorname{int} C_+$.*

Proof. We consider the C^1-functional $\gamma_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}
\gamma_\lambda(u) = &\int_\Omega \frac{1}{p(z)} |Du|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz + \int_\Omega |\xi(z)||u|^{p(z)} dz \\
&- \lambda \int_\Omega \frac{1}{\tau(z)} (u^+)^{\tau(z)} dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)} |u|^{p(z)} d\sigma
\end{aligned}$$

for all $u \in W^{1,p(z)}(\Omega)$. Evidently, $\gamma_\lambda(\cdot)$ is coercive (since $\tau_+ < p_-$, $q_+ < p_-$.) and sequentially weakly lower semicontinuous. So, we can find $\bar{u}_\lambda \in W^{1,p(z)}(\Omega)$ such that

$$\gamma_\lambda(\bar{u}_\lambda) = \min\{\gamma_\lambda(u) : u \in W^{1,p(z)}(\Omega)\} < 0 = \gamma_\lambda(0) \quad (\text{since } \tau_+ < p_-),$$

which implies $\bar{u}_\lambda \neq 0$. We have

$$\gamma'_\lambda(\bar{u}_\lambda) = 0,$$

which implies

$$\langle V(\bar{u}_\lambda), h\rangle + \int_\Omega |\xi(z)||\bar{u}_\lambda|^{p(z)-2}\bar{u}_\lambda h\,dz - \lambda \int_\Omega (\bar{u}_\lambda^+)^{\tau(z)-1} h\,dz + \int_{\partial\Omega} \beta(z)|\bar{u}_\lambda|^{p(z)-2}\bar{u}_\lambda d\sigma = 0 \tag{3.43}$$

for all $h \in W^{1,p(z)}(\Omega)$.

In (3.43) we choose $h = -\bar{u}_\lambda^- \in W^{1,p(z)}(\Omega)$. Then

$$\int_\Omega |D\bar{u}_\lambda^-|^{p(z)} dz + \int_\Omega |D\bar{u}_\lambda^-|^{q(z)} dz + \int_\Omega |\xi(z)|(\bar{u}_\lambda^-)^{p(z)} dz + \int_{\partial\Omega} \beta(z)(\bar{u}_\lambda^-)^{p(z)} d\sigma = 0$$

$$\Rightarrow \int_\Omega |D\bar{u}_\lambda^-|^{p(z)} dz + \int_\Omega |\xi(z)|(\bar{u}_\lambda^-)^{p(z)} dz + \int_{\partial\Omega} \beta(z)(\bar{u}_\lambda^-)^{p(z)} d\sigma \leq 0$$

which implies $\bar{u}_\lambda \geq 0$, $\bar{u}_\lambda \neq 0$, hence $\bar{u}_\lambda$ is a positive solution of (3.42) (see (3.43)), therefore $\bar{u}_\lambda \in C_+ \setminus \{0\}$ (anisotropic regularity theory).

Therefore

$$\Delta_{p(z)}\bar{u}_\lambda(z) + \Delta_{q(z)}\bar{u}_\lambda(z) \leq \|\xi\|_\infty (\bar{u}_\lambda(z))^{p(z)-1} \quad \text{for a.a. } z \in \Omega,$$

which implies $\bar{u}_\lambda \in \operatorname{int} C_+$ (see Zhang [35]).

Next, we show that this positive solution of (3.42) is unique. Suppose that $\bar{v}_\lambda$ is another positive solution of (3.42). Again we have $\bar{v}_\lambda \in \operatorname{int} C_+$. On account of Proposition 4.1.22 of Papageorgiou, Rădulescu and Repovs [22], p. 274, we have $\frac{\bar{u}_\lambda}{\bar{v}_\lambda}, \frac{\bar{v}_\lambda}{\bar{u}_\lambda} \in L^\infty(\Omega)$. So, we can apply Theorem 2.5 of Takac and Giacomoni [34] and get

$$\begin{aligned} 0 &\leq \int_\Omega \left[\frac{-\Delta_{p(z)}\bar{u}_\lambda - \Delta_{q(z)}\bar{u}_\lambda}{(\bar{u}_\lambda)^{q_- -1}} + \frac{-\Delta_{p(z)}\bar{v}_\lambda - \Delta_{q(z)}\bar{v}_\lambda}{(\bar{v}_\lambda)^{q_- -1}} \right] ((\bar{u}_\lambda)^{q_-} - (\bar{v}_\lambda)^{q_-})\,dz \\ &= \int_\Omega \Bigg[\lambda \left(\frac{1}{(\bar{u}_\lambda)^{q_- - \tau(z)}} - \frac{1}{(\bar{v}_\lambda)^{q_- - \tau(z)}} \right) \\ &\qquad - |\xi(z)| \left((\bar{u}_\lambda)^{p(z)-q_-} - (\bar{v}_\lambda)^{p(z)-q_-} \right) \Bigg] ((\bar{u}_\lambda)^{q_-} - (\bar{v}_\lambda)^{q_-})\,dz, \end{aligned}$$

which implies $\bar{u}_\lambda = \bar{v}_\lambda$ (since $\tau_+ < p_- \leq p(z)$).

Therefore the positive solution $\bar{u}_\lambda \in \operatorname{int} C_+$ of problem (3.42) is unique. □

This solution $\bar{u}_\lambda \in \operatorname{int} C_+$ provides a lower bound for the solution set S_λ.

Proposition 3.7. *If hypotheses H_0, H_1 hold and $\lambda \in \mathfrak{L}$, then $\bar{u}_\lambda \leq u$ for all $u \in S_\lambda$.*

Proof. Let $u \in S_\lambda \subset \operatorname{int} C_+$ and consider the Carathéodory function $\beta_\lambda(z, x)$ defined by

$$\hat{\beta}_\lambda(z,x) = \begin{cases} \lambda (x^+)^{\tau(z)-1}, & \text{if } x \leq u(z), \\ \lambda u(z)^{\tau(z)-1}, & \text{if } u(z) < x. \end{cases} \tag{3.44}$$

We set $\hat{B}_\lambda(z,x) = \int_0^x \hat{\beta}_\lambda(z,s)ds$ and consider the C^1-functional $\tau_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \tau_\lambda(u) = &\int_\Omega \frac{1}{p(z)}|Du|^{p(z)} dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)} dz + \int_\Omega \frac{|\xi(z)|}{p(z)}|u|^{p(z)} dz \\ &+ \int_{\partial\Omega} \frac{\beta(z)}{p(z)}|u|^{p(z)} d\sigma - \int_\Omega \hat{B}_\lambda(z,u) dz \end{aligned}$$

for all $u \in W^{1,p(z)}(\Omega)$.

From (3.44) we see that $\tau_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W^{1,p(z)}(\Omega)$ such that

$$\tau_\lambda(\tilde{u}_\lambda) = \min\left\{\tau_\lambda(u) : u \in W^{1,p(z)}(\Omega)\right\} < 0 = \tau_\lambda(0) \quad (\text{since } \tau_+ < p_-),$$

which implies $\tilde{u}_\lambda \neq 0$.

We have

$$\tau'_\lambda(\tilde{u}_\lambda) = 0,$$

$$\langle V(\tilde{u}_\lambda), h\rangle + \int_\Omega |\xi(z)||\tilde{u}_\lambda|^{p(z)-2}\tilde{u}_\lambda h dz + \int_{\partial\Omega} \beta(z)|\tilde{u}_\lambda|^{p(z)-2}\tilde{u}_\lambda h d\sigma = \int_\Omega \hat{\beta}_\lambda(z,\tilde{u}_\lambda) h dz \tag{3.45}$$

for all $h \in W^{1,p(z)}(\Omega)$. In (3.45) we first choose $h = -\tilde{u}_\lambda^- \in W^{1,p(z)}(\Omega)$ and infer that

$$\tilde{u}_\lambda \geq 0,\ \tilde{u}_\lambda \neq 0.$$

Next, in (3.45) we choose $h = (\tilde{u}_\lambda - u)^+ \in W^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}
&\langle V(\tilde{u}_\lambda), (\tilde{u}_\lambda - u)^+\rangle + \int_\Omega |\xi(z)|(\tilde{u}_\lambda)^{p(z)-1}(\tilde{u}_\lambda - u)^+ dz + \int_{\partial\Omega} \beta(z)(\tilde{u}_\lambda)^{p(z)-1}(\tilde{u}_\lambda - u)^+ d\sigma \\
&= \int_\Omega \lambda u^{\tau(z)-1}(\tilde{u}_\lambda - u)^+ dz \quad (\text{see } (3.44)) \\
&\leq \int_\Omega [\lambda u^{\tau(z)-1} + f(z,u)](\tilde{u}_\lambda - u)^+ dz \quad (\text{since } f \geq 0) \\
&\leq \langle V(u), (\tilde{u}_\lambda - u)^+\rangle + \int_\Omega |\xi(z)| u^{p(z)-1}(\tilde{u}_\lambda - u)^+ dz + \int_{\partial\Omega} \beta(z) u^{p(z)-1}(\tilde{u}_\lambda - u)^+ d\sigma
\end{aligned}$$

(since $u \in S_\lambda$)

$$\Rightarrow \tilde{u}_\lambda \leq u.$$

So, we have proved that

$$\tilde{u}_\lambda \in [0,u]\setminus\{0\}. \tag{3.46}$$

Then it follows from (3.43), (3.44), (3.46) that

$$\begin{aligned}
&\tilde{u}_\lambda \text{ is a positive solution of } (3.42), \\
&\quad \Rightarrow \tilde{u}_\lambda = \bar{u}_\lambda \in \operatorname{int} C_+ \quad (\text{see Proposition } 3.5), \\
&\quad \Rightarrow \bar{u}_\lambda \leq u \text{ for all } u \in S_\lambda.
\end{aligned}$$

The proof is now complete. □

Remark 3.8. Reasoning as in the above proof, we show that $\lambda \mapsto \bar{u}_\lambda$ is increasing, that is, if $0 < \mu < \lambda$, then $\bar{u}_\lambda - \bar{u}_\mu \in C_+ \setminus 0$.

We know that S_λ is downward directed (see Filippakis and Papageorgiou [9] and Papageorgiou, Rădulescu and Repovš [21], and recall that $V(\cdot)$ is monotone (see Proposition 2.9)).

Proposition 3.9. *If hypotheses H_0, H_1 hold and $\lambda \in \mathfrak{L}$, then there exists $u^*_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ such that*

$$u^*_\lambda \leq u \quad \textit{for all } u \in S_\lambda \textit{ (minimal positive solution of } (p_\lambda)).$$

Proof. By Lemma 3.10 of Hu and Papageorgiou [15] (p. 178), we know that we can find $\{u_n\}_{n\geq 1} \subseteq S_\lambda \subseteq \operatorname{int} C_+$ decreasing (recall that S_λ is downward directed) such that

$$\inf_{n\geq 1} u_n = \inf S_\lambda.$$

Since $u_\lambda \leq u_n \leq u_1$ for all $n \in \mathbb{N}$ (see Proposition 3.6), from hypothesis $H_1(i)$ it follows that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \rightharpoonup u_\lambda^* \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_\lambda^* \quad \text{in } L^{r(z)}(\Omega) \quad \text{as } n \to \infty. \tag{3.47}$$

We have

$$\begin{aligned}
&\langle V(u_n), u_n - u_\lambda^*\rangle + \int_\Omega \xi(z) u_n^{p(z)-1}(u_n - u_\lambda^*)\,dz + \int_{\partial\Omega} \beta(z) u_n^{p(z)-1}(u_n - u_\lambda^*)d\sigma \\
&= \lambda \int_\Omega u_n^{\tau(z)-1}(u_n - u_\lambda^*)\,dz + \int_\Omega f(z,u_n)(u_n - u_\lambda^*)\,dz,
\end{aligned}$$

which implies

$$\lim_{n\to\infty} \langle V(u_n), u_n - u_\lambda^*\rangle = 0,$$

and thus

$$u_n \to u_\lambda^* \quad \text{in } W^{1,p(z)}(\Omega) \quad \text{(see Proposition 2.9)}. \tag{3.48}$$

Note that $\bar{u}_\lambda \leq u_\lambda^*$ and so $u_\lambda^* \neq 0$,

$$\begin{aligned}
&\langle V(u_\lambda^*), h\rangle + \int_\Omega \xi(z)(u_\lambda^*)^{p(z)-1} h\,dz + \int_{\partial\Omega} \beta(z)(u_\lambda^*)^{p(z)-1} h d\sigma \\
&= \lambda \int_\Omega (u_\lambda^*)^{\tau(z)-1} h\,dz + \int_\Omega f(z,u_\lambda^*) h\,dz
\end{aligned}$$

for all $h \in W^{1,p(z)}(\Omega)$ (see (3.48)).

It follows that

$$u_\lambda^* \in S_\lambda \subseteq \operatorname{int} C_+ \quad \text{and} \quad u_\lambda^* = \inf S_\lambda.$$

The proof is now complete. □

We set $\lambda^* = \sup \mathfrak{L}$.

Proposition 3.10. *If hypotheses H_0, H_2 hold, then $\lambda^* < \infty$.*

Proof. On account of hypotheses $H_0, H_2(iv)$ and since $\tau^+ < p^-$, we see that we can find $\hat{\lambda} > \lambda^*$ such that

$$\lambda x^{\tau(z)-1} + f(z,x) - \xi(z) x^{p(z)-1} \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.49}$$

Let $\lambda > \hat{\lambda}$ and suppose that $\lambda \in \mathfrak{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$. Let $\Omega_0 \subset\subset \Omega$ (that is, $\Omega_0 \subseteq \overline{\Omega}_0 \subseteq \Omega$) and assume that $\partial\Omega_0$ is a C^2-manifold. We set $m_0 = \min_{\Omega_0} u_\lambda > 0$ (recall that $u_\lambda \in \operatorname{int} C_+$). Also, let $\hat{\xi}_\rho > \|\xi\|_\infty$. Let $m_0^\delta = m_0 + \delta$ for $\delta > 0$ small enough. We have

$$\begin{aligned}
&-\Delta_{p(z)} m_0^\delta - \Delta_{q(z)} m_0^\delta + [\xi(z) + \hat{\xi}_\rho](m_0^\delta)^{p(z)-1} \\
&\leq [\xi(z) + \hat{\xi}_\rho](m_0^\delta)^{p(z)-1} + \chi(\delta) \quad \text{with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
&\leq \hat{\lambda} m_0^{\tau(z)-1} + f(z,m_0) + \hat{\xi}_\rho m_0^{p(z)-1} + \chi(\delta) \quad \text{(see (3.49))} \\
&\leq \hat{\lambda} u_\lambda^{\tau(z)-1} + f(z,u_\lambda) + \hat{\xi}_\rho u_\lambda^{p(z)-1} + \chi(\delta) \quad \text{(see hypothesis } H_2(v)) \\
&\leq \hat{\lambda} u_\lambda^{\tau(z)-1} + f(z,u_\lambda) + \hat{\xi}_\rho u_\lambda^{p(z)-1} - [\lambda - \hat{\lambda}] m_0^{p(z)-1} + \chi(\delta) \\
&\leq -\Delta_{p(z)} u_\lambda - \Delta_{q(z)} u_\lambda + [\xi(z) + \hat{\xi}_\rho] u_\lambda^{p(z)-1} \quad \text{in } \Omega_0, \text{ for } 0 < \delta < 1 \text{ small}.
\end{aligned} \tag{3.50}$$

Note that $\delta \in (0,1)$ small enough, we will have

$$[\lambda - \hat{\lambda}] m_0^{p(z)-1} - \chi(\delta) \geq \eta > 0.$$

So, from (3.50) and Papageorgiou–Qin–Rădulescu [24], Proposition 5 (see also [25], Proposition 6) we have

$$u_\lambda(z) \geq m_{\delta_0} \quad \text{for all } z \in \Omega, \text{ all } 0 < \delta < 1 \text{ small enough}$$

which is a contradiction. Therefore $0 < \lambda^* \leq \hat{\lambda} < \infty$. □

According to this proposition, we have

$$(0, \lambda^*) \subseteq \mathfrak{L} \subseteq (0, \lambda^*]. \tag{3.51}$$

We will show that for all $\lambda \in (0, \lambda^*)$, we have at least two positive smooth solutions for problem (p_λ). To do this we need to strengthen a little the hypotheses on $f(z,\cdot)$. The new conditions on $f(z,x)$ are the following:

H$_3$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, hypotheses $H_3(i)$–(v) are the same as the corresponding hypotheses $H_2(i)$–(v) = $H_1(i)$–(v) and

(vi) for every $m > 0$, there exists $\eta_m > 0$ such that

$$f(z,x) \geq \eta_m > 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq m.$$

Proposition 3.11. *If hypotheses H_0, H_3 hold and $\lambda \in (0, \lambda^*)$, then problem (p_λ) admits at least two positive solutions $u_0, \hat{u} \in \operatorname{int} C_+$, $u_0 \neq \hat{u}$.*

Proof. Let $\eta \in (\lambda, \lambda^*)$. We have $\eta \in \mathfrak{L}$ (see (3.51)) and so we can find $u_\eta \in S_\eta \subseteq \operatorname{int} C_+$. Then according to Proposition 3.5, we can find $u_0 \in S_\lambda \subseteq \operatorname{int} C_+$ such that

$$u_\eta - u_0 \in \operatorname{int} C_+. \tag{3.52}$$

Recall that $\bar{u}_\lambda \leq u_0$ (see Proposition 3.7).

Let $\rho = \|u_0\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_3(v)(H_2(v))$. We can assume that $\hat{\xi}_\rho > \|\xi\|_\infty$. Then we have

$$\begin{aligned}
&-\Delta_{p(z)}\bar{u}_\lambda - \Delta_{q(z)}\bar{u}_\lambda + [\xi(z) + \hat{\xi}_\rho]\bar{u}_\lambda^{p(z)-1} \\
&\quad \leq -\Delta_{p(z)}\bar{u}_\lambda - \Delta_{q(z)}\bar{u}_\lambda + [|\xi(z)| + \hat{\xi}_\rho]\bar{u}_\lambda^{p(z)-1} \\
&\quad = \lambda \bar{u}_\lambda^{\tau(z)-1} + \hat{\xi}_\rho \bar{u}_\lambda^{p(z)-1} \quad \text{(see Proposition 3.6)} \\
&\quad \leq \lambda u_0^{\tau(z)-1} + f(z, \bar{u}_\lambda) + \hat{\xi}_\rho u_0^{p(z)-1} \quad \text{(recall that } f \geq 0) \\
&\quad \leq \lambda u_0^{\tau(z)-1} + f(z, u_0) + \hat{\xi}_\rho u_0^{p(z)-1} \quad \text{(see Proposition 3.7 and hypothesis } H_3(v) = H_2(v)) \\
&\quad = -\Delta_{p(z)}u_0 - \Delta_{q(z)}u_0 + [\xi(z) + \hat{\xi}_\rho]u_0^{p(z)-1} \quad \text{(since } u_0 \in S_\lambda).
\end{aligned} \tag{3.53}$$

On account of hypothesis H_3(vi) and since $u_\lambda \in \operatorname{int} C_+$, we see that

$$0 \prec f(\cdot, \bar{u}_\lambda(\cdot)).$$

Then from (3.53) and Proposition 2.4 in [23] (see also [25], Proposition 7), we can conclude that

$$u_0 - \bar{u}_\lambda \in \text{int } C_+. \tag{3.54}$$

It follows from (3.52) and (3.54) that

$$u_0 \in \text{int}_{C^1(\bar{\Omega})}[\bar{u}_\lambda, u_\eta]. \tag{3.55}$$

As before, let $\theta > \|\xi\|_\infty$ and consider the Carathéodory function $k_\lambda(z,x)$ defined by

$$k_\lambda(z,x) = \begin{cases} \lambda \bar{u}_\lambda(z)^{\tau(z)-1} + f(z, \bar{u}_\lambda(z)) + \theta \bar{u}_\lambda(z)^{p(z)-1}, & \text{if } x < \bar{u}_\lambda(z) \\ \lambda x^{\tau(z)-1} + f(z,x) + \theta x^{p(z)-1}, & \text{if } \bar{u}_\lambda(z) \le x \le u_\eta(z) \\ \lambda u_\eta(z)^{\tau(z)-1} + f(z, u_\eta(z)) + \vartheta u_\eta(z)^{p(z)-1}, & \text{if } u_\eta(z) < x. \end{cases} \tag{3.56}$$

We set $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)ds$ and consider the C^1-functional $\tau_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \tau_\lambda(u) = & \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz + \int_\Omega \frac{1}{p(z)}(\theta + \xi(z))|u|^{p(z)}dz \\ & - \int_\Omega K_\lambda(z,u)dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)}|u|^{p(z)}d\sigma \end{aligned}$$

for all $u \in W^{1,p(z)}(\Omega)$.

From (3.56) and since $\theta > \|\xi\|_\infty$, we infer that $\tau_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in W^{1,p(z)}(\Omega)$ such that

$$\begin{aligned} \tau_\lambda(\tilde{u}_0) &= \min\{\tau_\lambda(u) : u \in W^{1,p(z)}(\Omega)\}, \\ &\Rightarrow \tau'_\lambda(\tilde{u}_0) = 0, \\ &\Rightarrow \langle \tau'_\lambda(\tilde{u}_0), h \rangle = 0 \quad \text{for all } h \in W^{1,p(z)}(\Omega). \end{aligned}$$

Choosing $h = (\bar{u}_\lambda - \tilde{u}_0)^+$ and $h = (\tilde{u}_0 - u_\eta)^+$ and using (3.56), we show as before that

$$\tilde{u}_0 \in [\bar{u}_\lambda, u_\eta] \cap \text{int } C_+.$$

Therefore, we may assume that $\tilde{u}_0 = u_0$ or otherwise, we already have a second positive smooth solution and so, we are done.

Next, we consider the Carathéodory function

$$\hat{k}_\lambda(z,x) = \begin{cases} \lambda \bar{u}_\lambda(z)^{\tau(z)-1} + f(z, \bar{u}_\lambda(z)) + \theta \bar{u}_\lambda(z)^{p(z)-1}, & \text{if } x \le \bar{u}_\lambda(z) \\ \lambda x^{\tau(z)-1} + f(z,x) + \vartheta x^{p(z)-1}, & \text{if } \bar{u}_\lambda(z) < x. \end{cases} \tag{3.57}$$

We define $\hat{K}_\lambda(z,x) = \int_0^x \hat{k}_\lambda(z,s)ds$ and introduce the C^1-functional $\hat{\tau}_\lambda : W^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \hat{\tau}_\lambda(u) = & \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz + \int_\Omega \frac{1}{p(z)}(\theta + \xi(z))|u|^{p(z)}dz \\ & - \int_\Omega \hat{K}_\lambda(z,u)dz + \int_{\partial\Omega} \frac{\beta(z)}{p(z)}|u|^{p(z)}d\sigma \end{aligned}$$

for all $u \in W^{1,p(z)}(\Omega)$.

From (3.56) and (3.57), it is clear that

$$\tau_\lambda\big|_{[\bar{u}_\lambda, u_\eta]} = \hat{\tau}_\lambda\big|_{[\bar{u}_\lambda, u_\eta]}.$$

On account of (3.55), we have that u_0 is a local $C^1(\Omega)$-minimizer of $\hat{\tau}_\lambda$,

$$\Rightarrow u_0 \text{ is a local } W^{1,p(z)}(\Omega) - \text{minimizer of } \hat{\tau}_\lambda.$$

(see Gasiński and Papageorgiou [[13], Proposition 3.3])

Using (3.57), we can easily see that

$$K_{\hat{\tau}_\lambda} \subset [\bar{u}_\lambda) \cap \operatorname{int} C_+. \tag{3.58}$$

Then from (3.57) and the above, we can infer that we may assume that $K_{\hat{\tau}_\lambda}$ is finite or otherwise, we already have an infinity of positive smooth solutions all distinct from u_0 and so, we are done. According to Theorem 5.7.6 of Papageorgiou, Rădulescu, and Repovš [[22], p. 449], we can find $\rho \in (0, 1)$ small such that

$$\hat{\tau}_\lambda(u_0) < \inf\{\hat{\tau}_\lambda(u) : \|u - u_0\| = \rho\} = \hat{m}_\rho. \tag{3.59}$$

On account of hypothesis H_3(ii) for $u \in \operatorname{int} C_+$, we have

$$\hat{\tau}_\lambda(tu) \to -\infty \quad \text{as } t \to +\infty. \tag{3.60}$$

Finally, from (3.57), it follows that

$$\phi_{\hat{\lambda}}\big|_{[\bar{u}_\lambda)} = \hat{\tau}_\lambda\big|_{[\bar{u}_\lambda)} + \hat{\eta} \quad \text{with } \hat{\eta} \in \mathbb{R},$$

$$\Rightarrow \hat{\tau}_\lambda(\cdot) \text{ satisfies the } C\text{-condition} \quad (\text{see Proposition 3.1}). \tag{3.61}$$

Then (3.59), (3.60), (3.61) permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W^{1,p(z)}(\Omega)$ such that

$$\hat{u} \in K_{\hat{\tau}_\lambda} \subset [\bar{u}_\lambda) \cap \operatorname{int} C_+ \quad \text{and} \quad \hat{m}_\rho \leq \hat{\tau}_\lambda(\hat{u}). \tag{3.62}$$

From (3.62) and (3.57), we see that $\hat{u} \in S_\lambda \subseteq \operatorname{int} C_+$, while from (3.62) and (3.59), we have that $\hat{u} \neq u_0$. □

Finally, we show that the critical parameter value λ^* is admissible, that is, λ^*

Proposition 3.12. *If hypotheses H_0, H_1 hold, then $\lambda^* \in \mathfrak{L}$.*

Proof. Let $\{\lambda_n\}_{n\geq 1} \subseteq \mathfrak{L}$ such that $\lambda_n \to \lambda^*$ as $n \to \infty$. From the proof of Proposition 3.3, we know that we can find $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ such that $\varphi_{\hat{\lambda}_n}(u_n) < 0$ for all $n \in \mathbb{N}$.

Also, we have $\varphi'_{\hat{\lambda}_n}(u_n) = 0$, for all $n \in \mathbb{N}$. Then as in the proof of Proposition 3.1, we show that $\{u_n\}_{n\geq 1} \subseteq W^{1,p(z)}(\Omega)$ is bounded.

We may assume that

$$u_n \rightharpoonup u^* \quad \text{in } W^{1,p(z)}(\Omega) \quad \text{and} \quad u_n \to u^* \quad \text{in } L^{r(z)}(\Omega) \quad \text{as } n \to \infty. \tag{3.63}$$

We have

$$\langle V(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p(z)-1} h\, dz + \int_{\partial\Omega} \beta(z) u_n^{p(z)-1} h d\sigma = \lambda_n \int_\Omega u_n^{\tau(z)-1} h\, dz + \int_\Omega f(z, u_n) h\, dz$$

for all $h \in W^{1,p(z)}(\Omega)$, all $n \in \mathbb{N}$.

Choosing $h = u_n - u^*$, passing to the limit as $n \to \infty$ and using (3.63), we obtain

$$u_n \to u^* \quad \text{in } W^{1,p(z)}(\Omega).$$

So, in the limit as $n \to \infty$, we have

$$\begin{aligned}\langle V(u^*), h\rangle &+ \int_\Omega \xi(z)(u^*)^{p(z)-1} h\, dz + \int_{\partial\Omega} \beta(z)(u^*)^{p(z)-1} h d\sigma \\ &= \lambda^* \int_\Omega (u^*)^{q(z)-1} h\, dz + \int_\Omega f(z, u^*) h\, dz\end{aligned}$$

for all $h \in W^{1,p(z)}(\Omega)$.

We have $\bar{u}_{\lambda_1} \leq u_n$ for all $n \in \mathbb{N}$ (see the Remark 3.8),

$$\begin{aligned}&\Rightarrow \bar{u}_{\lambda_1} \leq u^*, \\ &\Rightarrow u^* \in S_{\lambda^*} \subseteq \operatorname{int} C_+ \quad \text{and so } \lambda^* \in \mathfrak{L}.\end{aligned}$$

The proof is now complete. □

Summarizing, we can state the following existence and multiplicity theorem for the problem (p_λ), which is global in the parameter $\lambda > 0$ (a bifurcation-type theorem).

Theorem 3.13. *If hypotheses H_0, H_3, hold, then there exists $\lambda^* > 0$ such that*

(a) for all $\lambda \in (0, \lambda^*)$, problem ($p_\lambda$) has at least two positive solutions

$$u_0, \hat{u} \in \operatorname{int} C_+;$$

(b) for $\lambda = \lambda^*$, problem (p_λ) has at least one positive solution

$$u^* \in \operatorname{int} C_+;$$

(c) for all $\lambda > \lambda^*$, problem (p_λ) has no positive solutions;

(d) for every $\lambda \in \mathfrak{L} = (0, \lambda^*]$, problem ($p_\lambda$) has a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$ and the map $\lambda \mapsto u_\lambda^*$ from $\mathfrak{L} = (0, \lambda^*]$ into $C_+ \setminus \{0\}$ is increasing, that is,

$$0 < \mu \leq \lambda \in \mathfrak{L} \Rightarrow u_\lambda^* - u_\mu^* \in C_+ \setminus \{0\}.$$

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Normal forms for retarded functional differential equations associated with zero-double-Hopf singularity with 1 : 1 resonance

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Abstract. This manuscript introduces a framework that focuses on the singularity of a zero-double-Hopf system with 1 : 1 resonance in general retarded differential equations (RDDs). Initially, practical algorithms are proposed to identify the zero-double-Hopf singularity and the associated generalized eigenspace that corresponds to zero and two pairs of purely imaginary eigenvalues. Subsequently, by utilizing center manifold reduction and normal form techniques, we derive a reduced form of parameterized retarded differential systems up to third-order terms.

Keywords: retarded differential equations, zero-double-Hopf with 1 : 1 resonance, center manifold, normal form.

2020 Mathematics Subject Classification: 34K17, 34K18, 34C20, 34C23.

1 Introduction

In this paper, our primary objective is to analyze the zero-double-Hopf bifurcation with 1 : 1 resonance in relation to the following equation:

$$\dot{z}(t) = A(\epsilon)z(t) + B(\epsilon)z(t-\tau) + F(z(t), z(t-\tau), \epsilon), \tag{1.1}$$

where $z \in \mathbb{R}^n$, $\epsilon \in \mathbb{R}^m$, $A(\epsilon)$, $B(\epsilon) \in C^2(\mathbb{R}^m, \mathcal{M}_{n\times n}(\mathbb{R}))$ and $F \in C^3(\mathbb{R}^{2n+m}, \mathbb{R}^n)$ satisfies

$$F(0,0,\epsilon) = \frac{\partial F}{\partial x}(0,0,\epsilon) = \frac{\partial F}{\partial y}(0,0,\epsilon) = 0.$$

The characteristic equation of (1.1) at $(z,\epsilon) = (0,0)$ is

$$\Delta(\lambda) \equiv \det(\lambda I_n - A - e^{-\lambda\tau}B) = 0, \tag{1.2}$$

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where I_n is the $n \times n$ identity matrix, $A = A(0)$ and $B = B(0)$.

To understand the dynamic behavior of the given differential system (1.1), it is crucial to examine the root distribution in equation (1.2). Several cases can arise, including:

- All roots of equation (1.2) have negative real parts, except for a double or triple zero root, respectively. In these scenarios, if the transversality condition is satisfied, the system (1.1) undergoes a Bogdanov–Takens bifurcation or a triple-zero bifurcation, respectively. The Bogdanov–Takens bifurcation in neutral differential systems was studied in [4], while both the Bogdanov–Takens and triple-zero bifurcations for neutral functional differential equations with multiple delays were explored in [2].

- All roots of equation (1.2) possess negative real parts, except for a pair of purely imaginary roots. In this case, if the transversality condition is met, the system (1.1) undergoes a Hopf bifurcation. This case has been discussed in [10].

- All roots of equation (1.2) exhibit negative real parts, except for a pair of purely imaginary roots and a simple zero root. If the transversality condition is satisfied, the system (1.1) experiences a zero-Hopf bifurcation. This case has been examined for delayed differential equations in [11] and for neutral differential equations in [1].

- All roots of equation (1.2) have negative real parts, except for two pairs of purely imaginary roots $\pm i w_1$ and $\pm i w_2$. In this situation, a double-Hopf bifurcation may occur. The corresponding normal form for scalar DDE has been computed in [3], while the same has been derived for systems of delay differential equations in [9] for the case where $\frac{w_1}{w_2} \notin \mathbb{Q}$.

- All roots of equation (1.2) possess negative real parts, except for two pairs of purely imaginary roots $\pm i w_1$ and $\pm i w_2$, where $w_1 = w_2$. In [14], a double-Hopf bifurcation with 1:1 resonance in a van der Pol oscillator has been studied. Where, the authors established explicit conditions for the characteristic equation to have a pair of purely imaginary roots with multiplicity 2 and they derived the corresponding normal forms up to order 2. In [12], authors presented a framework for studying the double-Hopf singularity with 1:1 resonance in general delayed differential equations. They also derived the corresponding normal form up to the third-order terms. To illustrate the application of their study, they applied these findings to a van der Pol oscillator with delayed feedback.

Explicit expressions for the eigenspace, its dual, and the coefficients of the normal form related to the zero-double-Hopf singularity in retarded differential equations have not been provided thus far. In [7], the authors presented the second-order normal form associated with the zero-Hopf singularity for one-dimensional delayed differential equations. However, it has been demonstrated in [8] that the second-order normal form is insufficient for determining and analyzing the bifurcation diagrams of the zero-Hopf singularity. Consequently, to obtain comprehensive bifurcation diagrams for this singularity, it becomes necessary to compute the third-order normal form, which poses greater challenges and complexities compared to the second-order normal form.

The remaining sections of this paper are organized as follows: Section 2 establishes specific conditions for the examined system to guarantee the existence of the zero-double-Hopf singularity. Section 3 applies the theory of normal forms to compute the normal form up to third-order terms for this singularity. The concluding remarks are presented in the final section of the manuscript.

2 Existence of the zero-double-Hopf with 1 : 1 resonance singularity

In this section, our examination focuses on the existence of the zero-double-Hopf singularity with 1 : 1 resonance in the analyzed retarded differential equation, considering the case where $\epsilon \in \mathbb{R}^3$. We make use of the concepts and notations introduced in [5,6,13] for our investigation.

The system (1.1) can be expressed in the following form:

$$\dot{z}(t) = L(\epsilon)z_t + F(z_t,\epsilon) = L(0)z_t + \hat{F}(z_t,\epsilon), \qquad z_t(\theta) = z(t+\theta), \qquad -\tau \leq \theta \leq 0, \tag{2.1}$$

with $\hat{F}(z_t,\epsilon) = (L(\epsilon)-L(0))z_t + F(z_t,\epsilon)$, $L(\epsilon)\phi = \int_{-\tau}^{0} d[\eta_\epsilon(\theta)]\phi(\theta)$, for all $\phi \in C = C([-\tau,0],\mathbb{R}^n)$, with supreme norm.

In particular, $Lz_t = L(0)z_t = Az(t) + Bz(t-\tau) = \int_{-\tau}^{0} d[\eta_0(\theta)]z_t(\theta)$.

The function η_ϵ is a bounded variation matrix-valued function defined on the interval $[-\tau,0]$ as follows:

$$\eta_\epsilon(\theta) = \begin{cases} A(\epsilon) + B(\epsilon), & \theta = 0, \\ B(\epsilon), & -\tau < \theta < 0, \\ 0, & \theta = -\tau. \end{cases}$$

Let us consider the linear system given by

$$\dot{z}(t) = L(0)z_t. \tag{2.2}$$

As stated in [13], the infinitesimal generator for the solution semigroup defined by the system (2.2) can be represented as:

$$\mathcal{A}_0\phi = \dot{\phi},$$
$$\mathbf{D}(\mathcal{A}_0) = \left\{ \phi \in C : \frac{d\phi}{d\theta} \in C, \dot{\phi}(0) = \mathrm{L}(0)\phi \right\}.$$

The adjoint innner product on $C \times C^*$ is defined by:

$$\langle \psi, \phi \rangle = \psi(0)\phi(0) - \int_{-\tau}^{0} \int_{0}^{s} \psi(\theta - s) d[\eta_0(s)]\phi(\theta) d\theta.$$

where $C^* = C([0,\tau],\mathbb{R}^{n*})$, with $\mathbb{R}^{n*}$ is the space of all row n-vector.

The adjoint $\mathcal{A}_0^*$ of $\mathcal{A}_0$ is defined as follows:

$$\mathcal{A}_0^*\psi = -\dot{\psi},$$
$$\mathbf{D}(\mathcal{A}_0^*) = \left\{ \psi \in C^*, \frac{d\phi}{d\theta} \in C^*, -\dot{\psi}(0) = \int_{-\tau}^{0} \psi(-\theta) d[\eta_0(\theta)] \right\}.$$

Now, it is necessary to impose the following hypotheses:

(**A1**): The infinitesimal generator $\mathcal{A}_0$ possesses a pair of purely imaginary eigenvalues $\lambda = \pm iw$ ($w > 0$) with an algebraic multiplicity of 2 and a geometric multiplicity of 1.

(**A2**): The infinitesimal generator $\mathcal{A}_0$ has a unique eigenvalue $\lambda = 0$.

(**A3**): All eigenvalues of the infinitesimal generator $\mathcal{A}_0$ exhibit negative real parts, except for the simple zero eigenvalue and the two pairs of purely imaginary eigenvalues.

Let P denote the eigenspace of $\mathcal{A}_0$, and let P^* represent the adjoint space of P.

The space $C = C([-\tau, 0], \mathbb{R}^n)$ can be decomposed as $C = P \oplus Q$, where $Q = \{\phi \in C : \langle \psi, \phi \rangle = 0, \forall\, \psi \in P^*\}$.

Let ϕ_1, ϕ_2, and ϕ_3 denote eigenvectors of P.

Therefore, we have $\mathcal{A}_0\phi_1 = iw\phi_1$, $(\mathcal{A}_0 - iwI_n)\phi_2 = \phi_1$, and $\mathcal{A}_0\phi_3 = 0$.
By employing the definition of $\mathcal{A}_0$, we derive the following expressions:

$$\begin{cases} \dot{\phi}_1(\theta) = iw\phi_1(\theta), & -\tau \le \theta < 0, \\ \dot{\phi}_1(0) = L_0\phi_1(\theta), & \theta = 0, \end{cases} \qquad \begin{cases} \dot{\phi}_2(\theta) = iw\phi_2(\theta) + \phi_1(\theta), & -\tau \le \theta < 0, \\ iw\phi_2(0) + \phi_1(0) = L_0\phi_2(\theta), & \theta = 0 \end{cases}$$

and

$$\begin{cases} \dot{\phi}_3(\theta) = 0, & -\tau \le \theta < 0, \\ \dot{\phi}_3(0) = L_0\phi_3(\theta), & \theta = 0. \end{cases}$$

Hence, the eigenvectors ϕ_1, ϕ_2, and ϕ_3 can be written as follows: $\phi_1(\theta) = e^{iw\theta}\phi_1^0$, $\phi_2(\theta) = e^{iw\theta}(\phi_2^0 + \theta\phi_1^0)$, and $\phi_3(\theta) = \phi_3^0$, where $\phi_1^0, \phi_2^0 \in \mathbb{C}^n \setminus \{0\}$ and $\phi_3^0 \in \mathbb{R}^n \setminus \{0\}$. These vectors satisfy the following equations:

$$iw\phi_1^0 = (A + e^{-iw\tau}B)\phi_1^0, \quad (\tau e^{-iw\tau}B + I_n)\phi_1^0 = (A + e^{-iw\tau}B - iwI_n)\phi_2^0 \quad \text{and} \quad (A + B)\phi_3^0 = 0.$$

Consequently, we have $P = \text{span}\{\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2, \phi_3\}$.

Now, let ψ_1, ψ_2, and ψ_3 denote the eigenvectors of $\mathcal{A}_0^*$.

Hence, we have $\mathcal{A}_0^*\psi_2 = -iw\psi_2$, $(\mathcal{A}_0^* + iwI_n)\psi_1 = \psi_2$, and $\mathcal{A}_0^*\psi_3 = 0$.
Accordingly, the eigenvectors ψ_1, ψ_2, and ψ_3 can be represented as follows:
$\psi_2(s) = e^{-iws}\psi_2^0$, $\psi_1(s) = e^{-iws}(\psi_1^0 - s\psi_2^0)$ and $\psi_3(s) = \psi_3^0$, where $\psi_1^0, \psi_2^0 \in \mathbb{C}^{n*} \setminus \{0\}$ and $\psi_3^0 \in \mathbb{R}^{n*} \setminus \{0\}$ satisfying the following equations:

$$\psi_2^0(A + e^{iw\tau}B) = -iw\psi_2^0, \quad \psi_2^0(\tau e^{iw\tau}B + I_n) = \psi_1^0(A + e^{iw\tau}B + iwI_n) \quad \text{and} \quad \psi_3^0(A + B) = 0.$$

So, $P^* = \text{span}\{\bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2, \psi_3\}$.

It is crucial to emphasize that the eigenvectors of P and P^* must fulfill the following conditions:

$$\langle \bar{\psi}_1, \phi_1 \rangle = \langle \psi_1, \bar{\phi}_1 \rangle = \langle \bar{\psi}_2, \phi_2 \rangle = \langle \psi_2, \bar{\phi}_2 \rangle = \langle \phi_3, \psi_3 \rangle = 1 \tag{2.3}$$

and

$$\langle \psi_1, \phi_1 \rangle = \langle \psi_1, \phi_2 \rangle = \langle \psi_1, \bar{\phi}_2 \rangle = \langle \psi_1, \phi_3 \rangle = 0, \tag{2.4}$$
$$\langle \bar{\psi}_1, \bar{\phi}_1 \rangle = \langle \bar{\psi}_1, \phi_2 \rangle = \langle \bar{\psi}_1, \bar{\phi}_2 \rangle = \langle \bar{\psi}_1, \phi_3 \rangle = 0, \tag{2.5}$$
$$\langle \psi_2, \phi_1 \rangle = \langle \psi_2, \bar{\phi}_1 \rangle = \langle \psi_2, \phi_2 \rangle = \langle \psi_2, \phi_3 \rangle = 0, \tag{2.6}$$
$$\langle \bar{\psi}_2, \phi_1 \rangle = \langle \bar{\psi}_2, \bar{\phi}_1 \rangle = \langle \bar{\psi}_2, \bar{\phi}_2 \rangle = \langle \bar{\psi}_2, \phi_3 \rangle = 0, \tag{2.7}$$
$$\langle \psi_3, \phi_1 \rangle = \langle \psi_3, \bar{\phi}_1 \rangle = \langle \psi_3, \phi_2 \rangle = \langle \psi_3, \bar{\phi}_2 \rangle = 0. \tag{2.8}$$

Therefore, we can appropriately select values for ϕ_1^0, ϕ_2^0, $\bar{\phi}_1^0$, $\bar{\phi}_2^0$, ϕ_3^0, $\bar{\psi}_1^0$, $\bar{\psi}_2^0$, ψ_1^0, ψ_2^0, and ψ_3^0 to ensure the satisfaction of equations (2.3), (2.4), (2.5), (2.6), (2.7) and (2.8).

Let $\Phi = (\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2, \phi_3)$ and $\Psi = (\bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2, \psi_3)^T$, then $\dot{\Phi} = \Phi J$ and $\dot{\Psi} = -J\Psi$, where

$$J = \begin{pmatrix} iw & 1 & 0 & 0 & 0 \\ 0 & iw & 0 & 0 & 0 \\ 0 & 0 & -iw & 1 & 0 \\ 0 & 0 & 0 & -iw & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

3 Calculation of the normal form

In this section, our focus is on calculating the normal form up to the third order associated with the zero-double-Hopf singularity. Our approach is based on the methodology introduced by Faria and Magalhães [5,6].

We assume that hypotheses (**A1**), (**A2**), and (**A3**) are satisfied.

Let BC the enlarged space of C, which is defined as follows:

$$BC = \{\phi : [-\tau, 0] \to \mathbb{R}^n : \ \phi \text{ uniformly continuous on } [-\tau, 0) \\ \text{and with a possible jump discontinuity at } 0\}.$$

An element $\psi \in BC$ can be expressed as: $\psi = \phi + X_0 v$, with $\phi \in C$, $v \in \mathbb{R}^n$ and

$$X_0(\theta) = \begin{cases} 0, & -\tau \leq \theta < 0, \\ I_n, & \theta = 0. \end{cases}$$

Let π be the projection defined as

$$\begin{aligned} \pi : BC &\to P \\ \phi + X_0 v &\mapsto \Phi[< \Psi, \phi > + \Psi(0)v]. \end{aligned}$$

The differential system (2.1) can be reformulated as

$$\begin{cases} \dot{x} = Jx + \sum_{j\geq 2} \frac{1}{j!} f_j^1(x, y, \epsilon) \\ \dot{y} = \mathcal{A}_{Q^1} y + \sum_{j\geq 2} \frac{1}{j!} f_j^2(x, y, \epsilon) \end{cases} \tag{3.1}$$

where $f_j^1(x, y, \epsilon) = \Psi(0)\hat{F}_j(\Phi x + y, \epsilon)$ and $f_j^2(x, y, \epsilon) = (I - \pi)\hat{F}_j(\Phi x + y, \epsilon)$, $\hat{F}(\Phi x + y, \epsilon) = \sum_{j\geq 2} \frac{1}{j!} \hat{F}_j(\Phi x + y, \epsilon)$ and $z_t = \Phi x + y$, with $x \in \mathbb{C}^5$ and $y \in Q^1 = \{\phi \in Q : \dot{\phi} \in C\}$, and $\mathcal{A}_{Q^1} \subset \ker(\pi)$, $\mathcal{A}_{Q^1}\phi = \dot{\phi} + X_0(L(0)\phi - \dot{\phi}(0))$.

The normal form associated with P of the system (3.1) can be represented as follows on its center manifold:

$$\dot{x} = Jx + \frac{1}{2} g_2^1(x, 0, \epsilon) + \frac{1}{6} g_3^1(x, 0, \epsilon) + \text{h.o.t.}, \tag{3.2}$$

where g_2^1 and g_3^1 are the second and third order terms in (x, ϵ), respectively.

Let M_j be the operator defined in $V_j^8(\mathbb{C}^5 \times \ker \pi)$ with the range in the same space by $M_j(f, g) = (M_j^1 f, M_j^2 g)$, with $V_j(Y)$ is the the space of homogeneous polynomials with degree j, for a normed space Y, where

$$M_j^1 f = M_j^1 \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = D_x f(x, \epsilon) Jx - Jf(x, \epsilon),$$

$$M_j^2 g = M_j^2 g(x, \epsilon) = D_x g(x, \epsilon) Jx - \mathcal{A}_{Q^1} g(x, \epsilon),$$

with $f(x,\epsilon) \in V_j^8(\mathbb{C}^5)$ and $(x,\epsilon) \in V_j^8(\ker \pi)$.

By using the above notation, $g_2^1(x,0,\epsilon)$ and $g_3^1(x,0,\epsilon)$ can be written as follows:

$$g_2^1(x,0,\epsilon) = \text{Proj}_{\ker(M_2^1)} f_2^1(x,0,\epsilon) = \text{Proj}_{S^1} f_2^1(x,0,\epsilon) + \mathcal{O}(|\epsilon|^2)$$
$$g_3^1(x,0,\epsilon) = \text{Proj}_{\ker(M_3^1)} \tilde{f}_3^1(x,0,\epsilon) = \text{Proj}_{S^2} \tilde{f}_3^1(x,0,\epsilon) + \mathcal{O}(|\epsilon|^2 x),$$

where

$$\tilde{f}_3^1(x,0,\epsilon) = f_3^1(x,0,\epsilon) + \frac{3}{2}[(D_x f_2^1)(x,0,\epsilon)U_2^1(x,\epsilon) + (D_y f_2^1)(x,0,\epsilon)U_2^2(x,\epsilon)]$$

U_2^1 and U_2^2 are defined by:

$$U_2^1(x,\epsilon) = (M_2^1)^{-1}\,\text{Proj}_{Im(M_2^1)} f_2^1(x,0,\epsilon) \quad \text{and} \quad (M_2^2 U_2^2(x,\epsilon)) = f_2^2(x,0,\epsilon).$$

$\ker(M_2^1)$ is spanned by

$$\epsilon_j x_2 e_1,\ x_2 x_5 e_1,\ \epsilon_j x_4 e_3,\ x_4 x_5 e_3,\ \epsilon_1\epsilon_2 e_5,\ \epsilon_1\epsilon_3 e_5,\ \epsilon_2\epsilon_3 e_5,\ \epsilon_j^2 e_5,\ \epsilon_j x_5 e_5,\ x_2 x_4 e_5, x_5^2 e_5,$$

for $j = 1,2,3$, with $(e_1,e_2,e_3,e_4,e_5)^T$ being the canonical basis of $\mathbb{R}^5$.

$\ker(M_3^1)$ is spanned by:

$$\epsilon_j^2 x_2 e_1,\ x_2^2 x_4 e_1,\ x_5^2 x_2 e_1,\ \epsilon_j x_2 x_5 e_1,\ \epsilon_1\epsilon_2 x_2 e_1,\ \epsilon_1\epsilon_3 x_2 e_1,\ \epsilon_2\epsilon_3 x_2 e_1,$$
$$\epsilon_j^2 x_4 e_3,\ x_4^2 x_2 e_3,\ x_5^2 x_4 e_3,\ \epsilon_j x_4 x_5 e_3,\ \epsilon_1\epsilon_2 x_4 e_3,\ \epsilon_1\epsilon_3 x_4 e_3,\ \epsilon_2\epsilon_3 x_4 e_3,$$
$$\epsilon_j^3 e_5,\ \epsilon_1^2\epsilon_2 e_5,\ \epsilon_1^2\epsilon_3 e_5,\ \epsilon_2^2\epsilon_1 e_5,\ \epsilon_2^2\epsilon_3 e_5,\ \epsilon_3^2\epsilon_1 e_5,\ \epsilon_3^2\epsilon_1 e_5,\ \epsilon_j^2 x_5 e_5,\ x_5^3 e_5,$$
$$\epsilon_j x_5^2 e_5,\ \epsilon_j x_2 x_4 e_5,\ \epsilon_1\epsilon_2 x_5 e_5,\ \epsilon_1\epsilon_3 x_5 e_5,\ \epsilon_2\epsilon_3 x_5 e_5,\ x_2 x_4 x_5 e_5,$$

for $j = 1,2,3$, with $(e_1,e_2,e_3,e_4,e_5)^T$ being the canonical basis of $\mathbb{R}^5$.

Consequently, S^1 and S^2 are spanned respectively by

$$\epsilon_j x_2 e_1, x_2 x_5 e_1, \epsilon_j x_4 e_3, x_4 x_5 e_3, \epsilon_j x_5 e_5, x_2 x_4 e_5, x_5^2 e_5$$

and

$$x_2^2 x_4 e_1,\ x_5^2 x_2 e_1,\ x_4^2 x_2 e_3,\ x_5^2 x_4 e_3,\ x_5^3 e_5 x_2 x_4 x_5 e_5, \quad \text{for } j = 1,2,3.$$

Write

$$\frac{1}{2}\hat{F}_2(z_t,\epsilon) = A_1\epsilon_1 z(t) + A_2\epsilon_2 z(t) + A_3\epsilon_3 z(t) + B_1\epsilon_1 z(t-\tau) + B_2\epsilon_2 z(t-\tau) + B_3\epsilon_3 z(t-\tau)$$
$$+ \sum_{i=1}^{n} E_i z_i(t) z(t-\tau) + \sum_{i=1}^{n} F_i z_i(t) z(t) + \sum_{i=1}^{n} K_i z_i(t-\tau) z(t-\tau),$$

with $A(\epsilon) = A + \epsilon_1 A_1 + \epsilon_2 A_2 + \epsilon_3 A_3 + O(|\epsilon|^2)$ and $B(\epsilon) = B + \epsilon_1 B_1 + \epsilon_2 B_2 + \epsilon_3 B_3 + O(|\epsilon|^2)$. So

$$\frac{1}{2}\hat{F}_2(\Phi x,\epsilon) = H_1\epsilon_1 x_1 + H_2\epsilon_2 x_1 + H_3\epsilon_3 x_1 + H_4\epsilon_1 x_2 + H_5\epsilon_2 x_2 + H_6\epsilon_3 x_2 + H_7\epsilon_1 x_3 + H_8\epsilon_2 x_3$$
$$+ H_9\epsilon_3 x_3 + H_{10}\epsilon_1 x_4 + H_{11}\epsilon_2 x_4 + H_{12}\epsilon_3 x_4 + H_{13}\epsilon_1 x_5 + H_{14}\epsilon_2 x_5$$
$$+ H_{15}\epsilon_3 x_5 + H_{16} x_1^2 + H_{17} x_2^2 + H_{18} x_3^2 + H_{19} x_4^2 + H_{20} x_5^2 + H_{21} x_1 x_2 + H_{22} x_1 x_3$$
$$+ H_{23} x_1 x_4 + H_{24} x_1 x_5 + H_{25} x_2 x_3 + H_{26} x_2 x_4 + H_{27} x_2 x_5 + H_{28} x_3 x_4 + H_{29} x_3 x_5$$
$$+ H_{30} x_4 x_5 + \mathcal{O}(|\epsilon|^2 |x|),$$

and

$$\frac{1}{2}\hat{F}_2(\Phi x+y,\epsilon)=\frac{1}{2}\hat{F}_2(\Phi x,\epsilon)+\sum_{j=1}^{5}\sum_{k=1}^{n}(R_{jk}x_jy_k(0)+S_{jk}x_jy_k(-\tau))$$
$$+\sum_{j=1}^{n}(T_jy_j^2(0)+Q_jy_j^2(-\tau))+\sum_{j,k=1}^{n}(P_{jk}y_j(0)y_k(-\tau))+\mathcal{O}(|\epsilon|^2|x|), \quad (3.3)$$

with A_1, A_2, A_3, B_1, B_2, B_3, E_j, F_j, K_j, R_{jk}, S_{jk}, P_{jk}, T_j and Q_j, for $1\le j,k\le n$ are coefficient matrix. The values of H_i for $1\le i\le 30$ are provided in the appendix.

Write

$$\frac{1}{6}\hat{F}_3(z_t,0)=\sum_{i,j=1}^{n}\Omega^1_{i,j}z_i(t)z_j(t)z(t)+\sum_{i,j=1}^{n}\Omega^2_{i,j}z_i(t)z_j(t-\tau)z(t-\tau)$$
$$+\sum_{i,j=1}^{n}\Omega^3_{i,j}z_i(t-\tau)z_j(t)z(t)+\sum_{i,j=1}^{n}\Omega^4_{i,j}z_i(t-\tau)z_j(t-\tau)z(t-\tau),$$

$\Omega^1_{i,j}$, $\Omega^2_{i,j}$, $\Omega^3_{i,j}$ and $\Omega^4_{i,j}$, for $1\le i,j\le n$, are coefficient matrices. So

$$\frac{1}{6}\hat{F}_3(\Phi x,0)=G_1x_1^3+G_2x_2^3+G_3x_3^3+G_4x_4^3+G_5x_5^3+G_6x_1^2x_2+G_7x_1^2x_3+G_8x_1^2x_4+G_9x_1^2x_5$$
$$+G_{10}x_2^2x_1+G_{11}x_2^2x_3+G_{12}x_2^2x_4+G_{13}x_2^2x_5+G_{14}x_3^2x_1+G_{15}x_3^2x_2+G_{16}x_3^2x_4$$
$$+G_{17}x_3^2x_5+G_{18}x_4^2x_1+G_{19}x_4^2x_2+G_{20}x_4^2x_3+G_{21}x_4^2x_5+G_{22}x_5^2x_1+G_{23}x_5^2x_2$$
$$+G_{24}x_5^2x_3+G_{25}x_5^2x_4+G_{26}x_1x_2x_3+G_{27}x_1x_2x_4+G_{28}x_1x_2x_5+G_{29}x_1x_3x_4$$
$$+G_{30}x_1x_3x_5+G_{31}x_1x_4x_5+G_{32}x_2x_3x_4+G_{33}x_2x_3x_5+G_{34}x_2x_4x_5$$
$$+G_{35}x_3x_4x_5+\mathcal{O}(|\epsilon|^2|x|).$$

The values of G_i for $i=1,2,\ldots,35$ are provided in the appendix.

We have:

$$\frac{1}{2}g_2^1(x,0,\epsilon)=\frac{1}{2}\operatorname{Proj}_{S^1}f_2^1(x,0,\epsilon)=\frac{1}{2}\operatorname{Proj}_{S^1}\Psi(0)\hat{F}_2(\Phi x,\epsilon)=\begin{pmatrix}\alpha_1x_2+\alpha_2x_2x_5\\0\\\alpha_3x_4+\alpha_4x_4x_5\\0\\\alpha_6x_5+\alpha_7x_2x_4+\alpha_8x_5^2\end{pmatrix},$$

where $\alpha_1=\bar{\psi}_1^0(H_4\epsilon_1+H_5\epsilon_2+H_6\epsilon_3)$, $\alpha_2=\bar{\psi}_1^0H_{27}$, $\alpha_3=\psi_1^0(H_{10}\epsilon_1+H_{11}\epsilon_2+H_{12}\epsilon_3)$, $\alpha_4=\psi_1^0H_{30}$, $\alpha_5=\psi_3^0(H_{13}\epsilon_1+H_{14}\epsilon_2+H_{15}\epsilon_3)$, $\alpha_6=\psi_3^0H_{26}$ and $\alpha_7=\psi_3^0H_{20}$.

Remark 3.1. We remark that $H_4=\bar{H}_{10}$, $H_5=\bar{H}_{11}$, $H_6=\bar{H}_{12}$, $H_{27}=\bar{H}_{30}$. So, $\alpha_1=\bar{\alpha}_3$ and $\alpha_2=\bar{\alpha}_4$.

Therefore,

$$\frac{1}{2}g_2^1(x,0,\epsilon)=\begin{pmatrix}\alpha_1x_2+\alpha_2x_2x_5\\0\\\bar{\alpha}_1x_4+\bar{\alpha}_2x_4x_5\\0\\\alpha_6x_5+\alpha_7x_2x_4+\alpha_8x_5^2\end{pmatrix}.$$

Now, we will calculate the term $g_3^1(x,0,\epsilon)$.

$$\begin{aligned}\frac{1}{6}g_3^1(x,0,\epsilon) &= \frac{1}{6}\operatorname{Proj}_{\ker(M_3^1)}\tilde{f}_3^1(x,0,\epsilon)\\ &= \frac{1}{6}\operatorname{Proj}_{S^2} f_3^1(x,0,0) + \frac{1}{4}\operatorname{Proj}_{S^2}[(D_x f_2^1)(x,0,0)U_2^1(x,0)\\ &\quad + (D_y f_2^1)(x,0,0)U_2^2(x,0)] + \text{h.o.t.}\end{aligned}$$

First,

$$\frac{1}{6}\operatorname{Proj}_{S^2} f_3^1(x,0,0) = \frac{1}{6}\operatorname{Proj}_{S^2}\Psi(0)\hat{F}_3(\Phi x,0) = \begin{pmatrix}\beta_1 x_2^2 x_4 + \beta_2 x_2 x_5^2\\ 0\\ \beta_3 x_4^2 x_2 + \beta_4 x_4 x_5^2\\ 0\\ \beta_5 x_5^3 + \beta_6 x_2 x_4 x_5\end{pmatrix},$$

where $\beta_1 = \bar{\psi}_1^0 G_{12}$, $\beta_2 = \bar{\psi}_1^0 G_{23}$, $\beta_3 = \psi_1^0 G_{19}$, $\beta_4 = \psi_1^0 G_{25}$, $\beta_5 = \psi_3^0 G_5$ and $\beta_6 = \psi_3^0 G_{34}$.
Since $f_2(x,0,0) = \Psi(0)\hat{F}_2(\Phi x,0)$, then:

$$U_2^1(x,0) = U_2^1(x,\epsilon)\mid_{\epsilon=0} = (M_2^1)^{-1}\operatorname{Proj}_{Im(M_2^1)} f_2^1(x,0,0).$$

Therefore,

$$\frac{1}{4}\operatorname{Proj}_{S^2}[(D_x f_2^1)(x,0,0)U_2^1(x,0)] = \begin{pmatrix}\gamma_1 x_2^2 x_4 + \gamma_2 x_2 x_5^2\\ 0\\ \gamma_3 x_4^2 x_2 + \gamma_4 x_4 x_5^2\\ 0\\ \gamma_5 x_5^3 + \gamma_6 x_2 x_4 x_5\end{pmatrix},$$

where

$$\begin{aligned}\gamma_1 = \frac{1}{2}\bar{\psi}_1^0 H_{21}&\left[-\bar{\psi}_1^0\left(\frac{2}{(iw)^3}H_{22} + \frac{1}{(iw)^2}H_{23} + \frac{1}{(iw)^2}H_{25} + \frac{1}{iw}H_{26}\right)\right.\\ &\left.\quad +\bar{\psi}_2^0\left(\frac{5}{(iw)^4}H_{22} + \frac{2}{(iw)^3}H_{23} + \frac{1}{(iw)^3}H_{25} + \frac{1}{(iw)^2}H_{26}\right)\right]\\ &+ \frac{1}{2}\bar{\psi}_1^0 H_{23}\left[\bar{\psi}_1^0\left(\frac{2}{(iw)^3}H_{16} + \frac{1}{iw}H_{17} - \frac{1}{(iw)^2}H_{21}\right)\right.\\ &\left.\quad + \bar{\psi}_2^0\left(\frac{4}{(iw)^4}H_{16} + \frac{1}{(iw)^2}H_{17} - \frac{1}{(iw)^3}H_{21}\right)\right]\\ &- \bar{\psi}_1^0 H_{17}\bar{\psi}_2^0\left(\frac{1}{(iw)^3}H_{22} + \frac{1}{(iw)^2}H_{23} + \frac{1}{iw}H_{26}\right) + \frac{1}{2iw}\bar{\psi}_1^0 H_{26}\bar{\psi}_2^0 H_{17}\\ &+ \frac{1}{2}\bar{\psi}_1^0 H_{25}\left[\psi_1^0\left(\frac{2}{(iw)^3}H_{22} - \frac{1}{(iw)^2}H_{23} - \frac{1}{(iw)^2}H_{25} + \frac{1}{iw}H_{26}\right)\right.\\ &\left.\quad + \psi_2^0\left(\frac{4}{(iw)^4}H_{22} - \frac{2}{(iw)^3}H_{23} - \frac{1}{(iw)^3}H_{25} + \frac{1}{(iw)^2}H_{26}\right)\right]\\ &+ \frac{1}{2}\bar{\psi}_1^0 H_{28}\left[\psi_1^0\left(\frac{2}{(iw)^3}H_{16} + \frac{1}{iw}H_{17} - \frac{1}{(iw)^2}H_{21}\right)\right.\\ &\left.\quad + \psi_2^0\left(\frac{6}{(iw)^4}H_{16} + \frac{1}{(3iw)^2}H_{17} - \frac{2}{(3iw)^3}H_{21}\right)\right]\\ &+ \bar{\psi}_1^0 H_{19}\psi_2^0\left(\frac{2}{(3iw)^3}H_{16} + \frac{1}{3iw}H_{17} - \frac{2}{(3iw)^2}H_{21}\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\bar{\psi}_1^0 H_{26}\psi_2^0\Big(\frac{1}{(iw)^3}H_{22}-\frac{1}{(iw)^2}H_{23}-\frac{1}{(iw)^2}H_{25}+\frac{1}{iw}H_{26}\Big)\\
&+\frac{1}{2}\bar{\psi}_1^0 H_{30}\psi_3^0\Big(\frac{2}{(2iw)^3}H_{16}+\frac{1}{2iw}H_{17}-\frac{1}{(2iw)^2}H_{21}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\gamma_2 =& \frac{1}{2}\bar{\psi}_1^0 H_{21}\Big(-\frac{1}{iw}\bar{\psi}_1^0+\frac{1}{(iw)^2}\bar{\psi}_2^0\Big)H_{20}-\bar{\psi}_1^0 H_{17}\bar{\psi}_2^0 H_{20}+\frac{1}{2}\bar{\psi}_1^0 H_{25}\Big(\frac{1}{iw}\psi_1^0+\frac{1}{(iw)^2}\psi_2^0\Big)H_{20}\\
&+\bar{\psi}_1^0 H_{29}\Big[\psi_1^0\Big(\frac{1}{2iw}H_{27}-\frac{1}{(2iw)^2}H_{24}\Big)+\psi_2^0\Big(\frac{1}{(2iw)^2}H_{27}-\frac{2}{(2iw)^3}H_{24}\Big)\Big]+\frac{1}{2iw}\bar{\psi}_1^0 H_{26}\psi_2^0 H_{20}\\
&+\frac{1}{2}\bar{\psi}_1^0 H_{30}\psi_2^0\Big(\frac{1}{2iw}H_{27}-\frac{1}{(2iw)^2}H_{24}\Big)+\bar{\psi}_1^0 H_{20}\psi_3^0\Big(\frac{1}{iw}H_{27}-\frac{1}{(iw)^2}H_{24}\Big)
\end{aligned}
$$

$$
\gamma_3=\bar{\gamma}_1, \qquad \gamma_4=\bar{\gamma}_2
$$

$$
\begin{aligned}
\gamma_5 =& \frac{1}{2}\psi_3^0 H_{24}\Big(-\frac{1}{iw}\bar{\psi}_1^0+\frac{1}{(iw)^2}\bar{\psi}_2^0\Big)H_{20}-\frac{1}{2iw}\psi_3^0 H_{27}\bar{\psi}_2^0 H_{20}\\
&+\frac{1}{2}\psi_3^0 H_{29}\Big(\frac{1}{iw}\psi_1^0+\frac{1}{(iw)^2}\psi_2^0\Big)H_{20}+\frac{1}{2iw}\psi_3^0 H_{30}\psi_2^0 H_{20}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_6 =& \frac{1}{2}\psi_3^0 H_{21}\Big[-\bar{\psi}_1^0\Big(\frac{1}{iw}\frac{1}{2iw}H_{29}+\frac{1}{2iw}H_{30}\Big)+\bar{\psi}_2^0\Big(\frac{1}{iw}\frac{1}{(2iw)^2}H_{29}+\frac{1}{(2iw)^2}H_{30}\Big)\Big]\\
&+\frac{1}{2}\psi_3^0 H_{24}\Big[-\bar{\psi}_1^0\Big(\frac{2}{(iw)^3}H_{22}+\frac{1}{(iw)^2}H_{23}+\frac{1}{(iw)^2}H_{25}+\frac{1}{iw}H_{26}\Big)\\
&+\bar{\psi}_2^0\Big(\frac{5}{(iw)^4}H_{22}+\frac{2}{(iw)^3}H_{23}+\frac{1}{(iw)^3}H_{25}+\frac{1}{(iw)^2}H_{26}\Big)\Big]\\
&-\frac{1}{2iw}\psi_3^0 H_{17}\bar{\psi}_2^0 H_{30}-\frac{1}{2}\psi_3^0 H_{27}\bar{\psi}_2^0\Big(\frac{1}{(iw)^3}H_{22}+\frac{1}{(iw)^2}H_{23}+\frac{1}{iw}H_{26}\Big)\\
&+\frac{1}{2}\psi_3^0 H_{28}\Big[\psi_1^0\Big(\frac{1}{2iw}H_{27}-\frac{1}{(2iw)^2}H_{24}\Big)+\psi_2^0\Big(\frac{1}{(2iw)^2}H_{27}-\frac{2}{(2iw)^3}H_{24}\Big)\Big]\\
&+\frac{1}{2}\psi_3^0 H_{29}\Big[\psi_1^0\Big(\frac{2}{(iw)^3}H_{22}-\frac{1}{(iw)^2}H_{23}-\frac{1}{(iw)^2}H_{25}+\frac{1}{iw}H_{26}\Big)\\
&+\psi_2^0\Big(\frac{4}{(iw)^4}H_{22}-\frac{2}{(iw)^3}H_{23}-\frac{2}{(iw)^3}H_{25}+\frac{1}{(iw)^2}H_{26}\Big)\Big]\\
&+\psi_3^0 H_{19}\psi_2^0\Big(\frac{1}{2iw}H_{27}-\frac{1}{(2iw)^2}H_{24}\Big)\\
&+\frac{1}{2}\psi_3^0 H_{30}\psi_2^0\Big(\frac{1}{(iw)^3}H_{22}-\frac{1}{(iw)^2}H_{23}-\frac{1}{(iw)^2}H_{25}+\frac{1}{iw}H_{26}\Big)
\end{aligned}
$$

To compute $\mathrm{Proj}_{S^2} D_y f_2^1(x,0,\epsilon)U_2^2(x,0)$, we define $h=h(x)(\theta)=U_2^2$ and write

$$
\begin{aligned}
h(\theta)=\begin{pmatrix} h^{(1)}(\theta)\\ h^{(2)}(\theta)\\ \vdots\\ h^{(n)}(\theta)\end{pmatrix} =& h_{20000}x_1^2+h_{02000}x_2^2+h_{00200}x_3^2+h_{00020}x_4^2+h_{00002}x_5^2+h_{11000}x_1x_2\\
&+h_{10100}x_1x_3+h_{10010}x_1x_4+h_{10001}x_1x_5+h_{01100}x_2x_3+h_{01010}x_2x_4\\
&+h_{01001}x_2x_5+h_{00110}x_3x_4+h_{00101}x_3x_5+h_{00011}x_4x_5
\end{aligned}
$$

$$= \begin{pmatrix} h^{(1)}_{20000}x_1^2 + h^{(1)}_{02000}x_2^2 + h^{(1)}_{00200}x_3^2 + \cdots + h^{(1)}_{00011}x_4x_5 \\ h^{(2)}_{20000}x_1^2 + h^{(2)}_{02000}x_2^2 + h^{(2)}_{00200}x_3^2 + \cdots + h^{(2)}_{00011}x_4x_5 \\ \vdots \\ h^{(n)}_{20000}x_1^2 + h^{(n)}_{02000}x_2^2 + h^{(n)}_{00200}x_3^2 + \cdots + h^{(n)}_{00011}x_4x_5 \end{pmatrix}$$

where h_{20000},h_{02000}, h_{00200}, h_{00020}, h_{00002}, h_{11000}, h_{10100}, h_{10010}, h_{10001}, h_{01100}, h_{01010}, h_{01001}, h_{00110}, h_{00101}, $h_{00011} \in Q^1$.

The coefficients of h can be determined by solving the equation $(M_2^2 h)(x) = f_2^2(x, 0, 0)$, which can also be written as:

$$D_x hJx - \mathcal{A}_{Q^1}(h) = (I - \pi)\hat{F}_2(\Phi x, 0).$$

Next, by utilizing the definition of $\mathcal{A}_{Q^1}$ and the projection π, we derive the following system of equations:

$$\begin{cases} \dot{h} - D_x hJx = \Phi(\theta)\Psi(0)\hat{F}_2(\Phi x, 0), \\ \dot{h}(0) - Lh = \hat{F}_2(\Phi x, 0). \end{cases} \tag{3.4}$$

Here, $\dot{h}$ denotes the derivative of h with respect to θ. We have:

$$\begin{aligned} \hat{F}_2(\Phi x, 0) = 2(&H_{16}x_1^2 + H_{17}x_2^2 + H_{18}x_3^2 + H_{19}x_4^2 + H_{20}x_5^2 + H_{21}x_1x_2 \\ &+ H_{22}x_1x_3 + H_{23}x_1x_4 + H_{24}x_1x_5 + H_{25}x_2x_3 + H_{26}x_2x_4 \\ &+ H_{27}x_2x_5 + H_{28}x_3x_4 + H_{29}x_3x_5 + H_{30}x_4x_5). \end{aligned}$$

By comparing the coefficients of each monomial, we establish the following relationships: $h_{20000} = \bar{h}_{00200}$, $h_{02000} = \bar{h}_{00020}$, $h_{00110} = \bar{h}_{11000}$, $h_{00101} = \bar{h}_{10001}$, $h_{01001} = \bar{h}_{00011}$ and $h_{10010} = \bar{h}_{01100}$. By substituting (3.4) into the expressions, we find that the coefficients h_{20000}, h_{00200}, h_{10100}, h_{10010}, h_{10001}, h_{00101}, h_{00002}, h_{11000} and h_{00110} satisfy the following equations:

$$\begin{cases} \dot{h}_{20000} - 2iwh_{20000} = 2\Phi(\theta)\Psi(0)H_{16}, \\ \dot{h}_{20000}(0) - L(h_{20000}) = 2H_{16}, \end{cases} \tag{3.5}$$

$$\begin{cases} \dot{h}_{02000} - (h_{11000} + 2iwh_{02000}) = 2\Phi(\theta)\Psi(0)H_{17}, \\ \dot{h}_{02000}(0) - L(h_{02000}) = 2H_{17}, \end{cases} \tag{3.6}$$

$$\begin{cases} \dot{h}_{00002} = 2\Phi(\theta)\Psi(0)H_{20}, \\ \dot{h}_{00002}(0) - L(h_{00002}) = 2H_{20}, \end{cases} \tag{3.7}$$

$$\begin{cases} \dot{h}_{11000} - 2(h_{20000} + iwh_{11000}) = 2\Phi(\theta)\Psi(0)H_{21}, \\ \dot{h}_{11000}(0) - L(h_{11000}) = 2H_{21}, \end{cases} \tag{3.8}$$

$$\begin{cases} \dot{h}_{10100} = 2\Phi(\theta)\Psi(0)H_{22}, \\ \dot{h}_{10100}(0) - L(h_{10100}) = 2H_{22}, \end{cases} \tag{3.9}$$

$$\begin{cases} \dot{h}_{10010} - h_{10100} = 2\Phi(\theta)\Psi(0)H_{23}, \\ \dot{h}_{10010}(0) - L(h_{10010}) = 2H_{23}, \end{cases} \tag{3.10}$$

$$\begin{cases} \dot{h}_{10001} - iwh_{10001} = 2\Phi(\theta)\Psi(0)H_{24}, \\ \dot{h}_{10001}(0) - L(h_{10001}) = 2H_{24}, \end{cases} \tag{3.11}$$

$$\begin{cases} \dot{h}_{01010} - (h_{10010} + h_{01100}) = 2\Phi(\theta)\Psi(0)H_{26}, \\ \dot{h}_{00101}(0) - L(h_{00101}) = 2H_{26}, \end{cases} \tag{3.12}$$

$$\begin{cases} \dot{h}_{01001} - (h_{10001} + iwh_{01001}) = 2\Phi(\theta)\Psi(0)H_{27}, \\ \dot{h}_{00110}(0) - L(h_{00110}) = 2H_{27}. \end{cases} \tag{3.13}$$

By utilizing equations (3.3), we deduce that

$$\frac{1}{4}\,\mathrm{Proj}_{S^2}\, D_y f_2^1 \,|_{y=0,\epsilon=0}\, U_2^2 = \begin{pmatrix} \sigma_1 x_2^2 x_4 + \sigma_2 x_2 x_5^2, \\ 0 \\ \sigma_3 x_4^2 x_2 + \sigma_4 x_4 x_5^2, \\ 0 \\ \sigma_5 x_5^3 + \sigma_6 x_2 x_4 x_5 \end{pmatrix}$$

where

$$\begin{aligned}
\sigma_1 &= \frac{1}{2}\bar{\psi}_1(0)\sum_{j=1}^{n}(R_{2j}h^{(j)}_{01010}(0) + S_{2j}h^{(j)}_{01010}(-\tau) + R_{4j}h^{(j)}_{02000}(0) + S_{4j}h^{(j)}_{02000}(-\tau)),\\
\sigma_2 &= \frac{1}{2}\bar{\psi}_1(0)\sum_{j=1}^{n}(R_{2j}h^{(j)}_{00002}(0) + S_{2j}h^{(j)}_{00002}(-\tau) + R_{5j}h^{(j)}_{01001}(0) + S_{5j}h^{(j)}_{01001}(-\tau)),\\
\sigma_3 &= \frac{1}{2}\psi_1(0)\sum_{j=1}^{n}(R_{2j}h^{(j)}_{00020}(0) + S_{2j}h^{(j)}_{00020}(-\tau) + R_{4j}h^{(j)}_{01010}(0) + S_{4j}h^{(j)}_{01010}(-\tau)),\\
\sigma_4 &= \frac{1}{2}\psi_1(0)\sum_{j=1}^{n}(R_{4j}h^{(j)}_{00002}(0) + S_{4j}h^{(j)}_{00002}(-\tau) + R_{5j}h^{(j)}_{00011}(0) + S_{5j}h^{(j)}_{00011}(-\tau)),\\
\sigma_5 &= \frac{1}{2}\psi_3(0)\sum_{j=1}^{n}(R_{5j}h^{(j)}_{00002}(0) + S_{5j}h^{(j)}_{00002}(-\tau)),\\
\sigma_6 &= \frac{1}{2}\psi_3(0)\sum_{j=1}^{n}(R_{2j}h^{(j)}_{00011}(0) + S_{2j}h^{(j)}_{00011}(-\tau) + R_{5j}h^{(j)}_{01010}(0) + S_{5j}h^{(j)}_{01010}(-\tau)\\
&\quad + R_{4j}h^{(j)}_{01001}(0) + S_{4j}h^{(j)}_{01001}(-\tau)).
\end{aligned}$$

Remark 3.2. $\sigma_1 = \bar{\sigma}_3$ and $\sigma_2 = \bar{\sigma}_4$.

We can explicitly determine the expressions of h_{20000}, h_{02000}, h_{00002}, h_{11000}, h_{10100}, h_{10010}, h_{10001}, h_{01010} and h_{01001} in the same way detailed in [1,11].

By consolidating all the obtained results, we reach the following conclusion:

$$\frac{1}{6}g_3^1(x,0,\epsilon) = \begin{pmatrix} (\beta_1+\gamma_1+\sigma_1)x_2^2x_4 + (\beta_2+\gamma_2+\sigma_2)x_2x_5^2 \\ 0 \\ (\beta_3+\gamma_3+\sigma_3)x_4^2x_2 + (\beta_4+\gamma_4+\sigma_4)x_4x_5^2 \\ 0 \\ (\beta_5+\gamma_5+\sigma_5)x_5^3 + (\beta_6+\gamma_6+\sigma_6)x_2x_4x_5 \end{pmatrix}.$$

Consequently, the system described in equation (3.2) can be reformulated as:

$$\begin{cases} \dot{x}_1 = iwx_1 + x_2 + \alpha_1 x_2 + \alpha_2 x_2 x_5 + (\beta_1+\gamma_1+\sigma_1)x_2^2x_4 + (\beta_2+\gamma_2+\sigma_2)x_2x_5^2, \\ \dot{x}_2 = iwx_2, \\ \dot{x}_3 = -iwx_3 + x_4 + \alpha_3 x_4 + \alpha_4 x_4 x_5 + (\beta_3+\gamma_3+\sigma_3)x_4^2x_2 + (\beta_4+\gamma_4+\sigma_4)x_4x_5^2, \\ \dot{x}_4 = -iwx_4, \\ \dot{x}_5 = \alpha_6 x_5 + \alpha_7 x_2 x_4 + \alpha_8 x_5^2 + (\beta_5+\gamma_5+\sigma_5)x_5^3 + (\beta_6+\gamma_6+\sigma_6)x_2x_4x_5. \end{cases} \tag{3.14}$$

Given that $x_1 = \bar{x}_3$ and $x_2 = \bar{x}_4$, the system (3.14) can be expressed equivalently as:

$$\begin{cases} \dot{x}_1 = iwx_1 + x_2 + \alpha_1 x_2 + \alpha_2 x_2 x_5 + (\beta_1 + \gamma_1 + \sigma_1) x_2^2 x_4 + (\beta_2 + \gamma_2 + \sigma_2) x_2 x_5^2 \\ \dot{x}_2 = iwx_2, \\ \dot{x}_3 = -iwx_3 + x_4 + \bar{\alpha}_1 x_4 + \bar{\alpha}_2 x_4 x_5 + (\bar{\beta}_1 + \bar{\gamma}_1 + \bar{\sigma}_1) x_4^2 x_2 + (\bar{\beta}_2 + \bar{\gamma}_2 + \bar{\sigma}_2) x_4 x_5^2, \\ \dot{x}_4 = -iwx_4, \\ \dot{x}_5 = \alpha_6 x_5 + \alpha_7 x_2 x_4 + \alpha_8 x_5^2 + (\beta_5 + \gamma_5 + \sigma_5) x_5^3 + (\beta_6 + \gamma_6 + \sigma_6) x_2 x_4 x_5. \end{cases} \tag{3.15}$$

Theorem 3.3. *Assuming the validity of assumptions (**A1**), (**A2**), and (**A3**), the retarded differential system (1.1) can be equivalently represented by the reduced system (3.15).*

4 Conclusion

Despite the extensive literature on various types of singularities in retarded differential equations (RDDs), such as the Bogdanov–Takens singularity, Hopf singularity, zero-Hopf singularity, saddle-node singularity, and double-Hopf singularity, the zero-double-Hopf singularity in RDDs remains relatively unexplored. This paper addresses this research gap by providing explicit conditions for the occurrence of the zero-double-Hopf singularity with 1 : 1 resonance in general RDDs. By employing the normal form theory proposed by Faria and Magalhães, we transform the considered RDDs into a system of three ordinary differential equations. Detailed calculations and formulas are presented, facilitating their implementation in symbolic computation systems.

However, an important question arises: How can we analyze the bifurcation diagram associated with this singularity to examine and understand the dynamics of various systems modeled by delay differential equations? Answering this question will be the focus of our future research endeavors.

A Appendix

In this part, we will define the notations:

$$\begin{aligned}
H_1 &= A_1\phi_1(0) + B_1\phi_1(-\tau), H_2 = A_2\phi_1(0) + B_2\phi_1(-\tau), H_3 = A_3\phi_1(0) + B_3\phi_1(-\tau),\\
H_4 &= A_1\phi_2(0) + B_1\phi_2(-\tau), H_5 = A_2\phi_2(0) + B_2\phi_2(-\tau), H_6 = A_3\phi_2(0) + B_3\phi_2(-\tau),\\
H_7 &= A_1\bar{\phi}_1(0) + B_1\bar{\phi}_1(-\tau), H_8 = A_2\bar{\phi}_1(0) + B_2\bar{\phi}_1(-\tau), H_9 = A_3\bar{\phi}_1(0) + B_3\bar{\phi}_1(-\tau),\\
H_{10} &= A_1\bar{\phi}_2(0) + B_1\bar{\phi}_2(-\tau), H_{11} = A_2\bar{\phi}_2(0) + B_2\bar{\phi}_2(-\tau), H_{12} = A_3\bar{\phi}_2(0) + B_3\bar{\phi}_2(-\tau),\\
H_{13} &= A_1\phi_3(0) + B_1\phi_3(-\tau), H_{14} = A_2\phi_3(0) + B_2\phi_3(-\tau), H_{15} = A_3\phi_3(0) + B_3\phi_3(-\tau),\\
H_{16} &= \sum_{i=1}^{n}(E_i\phi_{1i}(0)\phi_1(-\tau) + F_i\phi_{1i}(0)\phi_1(0) + K_i\phi_{1i}(-\tau)\phi_1(-\tau)),\\
H_{17} &= \sum_{i=1}^{n}(E_i\phi_{2i}(0)\phi_2(-\tau) + F_i\phi_{2i}(0)\phi_2(0) + K_i\phi_{2i}(-\tau)\phi_2(-\tau)),\\
H_{18} &= \sum_{i=1}^{n}(E_i\bar{\phi}_{1i}(0)\bar{\phi}_1(-\tau) + F_i\bar{\phi}_{1i}(0)\bar{\phi}_1(0) + K_i\bar{\phi}_{1i}(-\tau)\bar{\phi}_1(-\tau)),\\
H_{19} &= \sum_{i=1}^{n}(E_i\bar{\phi}_{2i}(0)\bar{\phi}_2(-\tau) + F_i\bar{\phi}_{2i}(0)\bar{\phi}_2(0) + K_i\bar{\phi}_{2i}(-\tau)\bar{\phi}_2(-\tau)),
\end{aligned}$$

$$
\begin{aligned}
H_{20} =& \sum_{i=1}^{n}(E_i\phi_{3i}(0)\phi_3(-\tau) + F_i\phi_{3i}(0)\phi_3(0) + K_i\phi_{3i}(-\tau)\phi_3(-\tau)),\\
H_{21} =& \sum_{i=1}^{n}\big(E_i(\phi_{1i}(0)\phi_2(-\tau) + \phi_{2i}(0)\phi_1(-\tau)) + F_i(\phi_{1i}(0)\phi_2(0) + \phi_{2i}(0)\phi_1(0))\\
&+ K_i(\phi_{1i}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_1(-\tau))\big),\\
H_{22} =& \sum_{i=1}^{n}\big(E_i(\phi_{1i}(0)\bar{\phi}_1(-\tau) + \bar{\phi}_{1i}(0)\phi_1(-\tau)) + F_i(\phi_{1i}(0)\bar{\phi}_1(0)\\
&+ \bar{\phi}_{1i}(0)\phi_1(0)) + K_i(\phi_{1i}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1i}(-\tau)\phi_1(-\tau))\big),\\
H_{23} =& \sum_{i=1}^{n}\big(E_i(\bar{\phi}_{2i}(0)\phi_1(-\tau) + \phi_{1i}(0)\bar{\phi}_2(-\tau)) + F_i(\bar{\phi}_{2i}(0)\phi_1(0) + \phi_{1i}(0)\bar{\phi}_2(0))\\
&+ K_i(\bar{\phi}_{2i}(-\tau)\phi_1(-\tau) + \phi_{1i}(-\tau)\bar{\phi}_2(-\tau))\big),\\
H_{24} =& \sum_{i=1}^{n}\big(E_i(\phi_{1i}(0)\phi_3(-\tau) + \phi_{3i}(0)\phi_1(-\tau)) + F_i(\phi_{1i}(0)\phi_3(0) + \phi_{3i}(0)\phi_1(0))\\
&+ K_i(\phi_{1i}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\phi_1(-\tau))\big),\\
H_{25} =& \sum_{i=1}^{n}\big(E_i(\bar{\phi}_{1i}(0)\phi_2(-\tau) + \phi_{2i}(0)\bar{\phi}_1(-\tau)) + F_i(\bar{\phi}_{1i}(0)\phi_2(0) + \phi_{2i}(0)\bar{\phi}_1(0))\\
&+ K_i(\bar{\phi}_{1i}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\bar{\phi}_1(-\tau))\big),\\
H_{26} =& \sum_{i=1}^{n}\big(E_i(\phi_{2i}(0)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\phi_2(-\tau)) + F_i(\phi_{2i}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\phi_2(0))\\
&+ K_i(\phi_{2i}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\phi_2(-\tau))\big),\\
H_{27} =& \sum_{i=1}^{n}\big(E_i(\phi_{2i}(0)\phi_3(-\tau) + \phi_{3i}(0)\phi_2(-\tau)) + F_i(\phi_{2i}(0)\phi_3(0) + \phi_{3i}(0)\phi_2(0))\\
&+ K_i(\phi_{2i}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\phi_2(-\tau))\big),\\
H_{28} =& \sum_{i=1}^{n}\big(E_i(\bar{\phi}_{1i}(0)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_1(-\tau)) + F_i(\bar{\phi}_{1i}(0)\bar{\phi}_2(0) + \bar{\phi}_{2i}(0)\bar{\phi}_1(0))\\
&+ K_i(\bar{\phi}_{1i}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_1(-\tau))\big),\\
H_{29} =& \sum_{i=1}^{n}\big(E_i(\bar{\phi}_{1i}(0)\phi_3(-\tau) + \phi_{3i}(0)\bar{\phi}_1(-\tau)) + F_i(\bar{\phi}_{1i}(0)\phi_3(0) + \phi_{3i}(0)\bar{\phi}_1(0))\\
&+ K_i(\bar{\phi}_{1i}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\bar{\phi}_1(-\tau))\big),\\
H_{30} =& \sum_{i=1}^{n}\big(E_i(\bar{\phi}_{2i}(0)\phi_3(-\tau) + \phi_{3i}(0)\bar{\phi}_2(-\tau)) + F_i(\bar{\phi}_{2i}(0)\phi_3(0) + \phi_{3i}(0)\bar{\phi}_2(0))\\
&+ K_i(\bar{\phi}_{2i}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\bar{\phi}_2(-\tau))\big).\\
G_1 =& \sum_{i,j=1}^{n}\big[\Omega^1_{i,j}\phi_{1i}(0)\phi_{1j}(0)\phi_1(0) + \Omega^2_{i,j}\phi_{1i}(0)\phi_{1j}(-\tau)\phi_1(-\tau) + \Omega^3_{i,j}\phi_{1i}(-\tau)\phi_{1j}(0)\phi_1(0)\\
&+ \Omega^4_{i,j}\phi_{1i}(-\tau)\phi_{1j}(-\tau)\phi_1(-\tau)\big],\\
G_2 =& \sum_{i,j=1}^{n}\big[\Omega^1_{i,j}\phi_{2i}(0)\phi_{2j}(0)\phi_2(0) + \Omega^2_{i,j}\phi_{2i}(0)\phi_{2j}(-\tau)\phi_2(-\tau) + \Omega^3_{i,j}\phi_{2i}(-\tau)\phi_{2j}(0)\phi_2(0)\\
&+ \Omega^4_{i,j}\phi_{2i}(-\tau)\phi_{2j}(-\tau)\phi_2(-\tau)\big],
\end{aligned}
$$

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\begin{aligned}
G_3 = & \sum_{i,j=1}^{n} \big[\Omega^1_{i,j}\bar{\phi}_{1i}(0)\bar{\phi}_{1j}(0)\bar{\phi}_1(0) + \Omega^2_{i,j}\bar{\phi}_{1i}(0)\bar{\phi}_{1j}(-\tau)\bar{\phi}_1(-\tau) + \Omega^3_{i,j}\bar{\phi}_{1i}(-\tau)\bar{\phi}_{1j}(0)\bar{\phi}_1(0) \\
& + \Omega^4_{i,j}\bar{\phi}_{1i}(-\tau)\bar{\phi}_{1j}(-\tau)\bar{\phi}_1(-\tau)\big], \\
G_4 = & \sum_{i,j=1}^{n} \big[\Omega^1_{i,j}\bar{\phi}_{2i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_2(0) + \Omega^2_{i,j}\bar{\phi}_{2i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau) + \Omega^3_{i,j}\bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_2(0) \\
& + \Omega^4_{i,j}\bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_2(-\tau)\big], \\
G_5 = & \sum_{i,j=1}^{n} \big[\Omega^1_{i,j}\phi_{3i}(0)\phi_{3j}(0)\phi_3(0) + \Omega^2_{i,j}\phi_{3i}(0)\phi_{3j}(-\tau)\phi_3(-\tau) + \Omega^3_{i,j}\phi_{3i}(-\tau)\phi_{3j}(0)\phi_3(0) \\
& + \Omega^4_{i,j}\phi_{3i}(-\tau)\phi_{3j}(-\tau)\phi_3(-\tau)\big], \\
G_6 = & \sum_{i,j=1}^{n} \big[\Omega^1_{i,j}\big(\phi_{1i}(0)\phi_{1j}(0)\phi_2(0) + \phi_{1i}(0)\phi_{2j}(0)\phi_1(0) + \phi_{2i}(0)\phi_{1j}(0)\phi_1(0)\big) \\
& + \Omega^2_{i,j}\big(\phi_{1i}(0)\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{1i}(0)\phi_{2j}(-\tau)\phi_1(-\tau) + \phi_{2i}(0)\phi_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(0)\phi_2(0) + \phi_{1i}(-\tau)\phi_{2j}(0)\phi_1(0) + \phi_{2i}(-\tau)\phi_{1j}(0)\phi_1(0)\big) \\
& + \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{1i}(-\tau)\phi_{2j}(-\tau)\phi_1(-\tau) + \phi_{2i}(-\tau)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\big], \\
G_7 = & \sum_{i,j=1}^{n} \big[\Omega^1_{i,j}\big(\phi_{1i}(0)\phi_{1j}(0)\bar{\phi}_1(0) + \phi_{1i}(0)\bar{\phi}_{1j}(0)\phi_1(0) + \bar{\phi}_{1i}(0)\phi_{1j}(0)\phi_1(0)\big) \\
& + \Omega^2_{i,j}\big(\phi_{1i}(0)\phi_{1j}(-\tau)\bar{\phi}_1(-\tau) + \phi_{1i}(0)\bar{\phi}_{1j}(-\tau)\phi_1(-\tau) + \bar{\phi}_{1i}(0)\phi_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(0)\bar{\phi}_1(0) + \phi_{1i}(-\tau)\bar{\phi}_{1j}(0)\phi_1(0) + \bar{\phi}_{1i}(-\tau)\phi_{1j}(0)\phi_1(0)\big) \\
& + \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(-\tau)\bar{\phi}_1(-\tau) + \phi_{1i}(-\tau)\bar{\phi}_{1j}(-\tau)\phi_1(-\tau) + \bar{\phi}_{1i}(-\tau)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\big], \\
G_8 = & \sum_{i,j=1}^{n} \big[\Omega^1_{i,j}\big(\phi_{1i}(0)\phi_{1j}(0)\bar{\phi}_2(0) + \phi_{1i}(0)\bar{\phi}_{2j}(0)\phi_1(0) + \bar{\phi}_{2i}(0)\phi_{1j}(0)\phi_1(0)\big) \\
& + \Omega^2_{i,j}\big(\phi_{1i}(0)\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \phi_{1i}(0)\bar{\phi}_{2j}(-\tau)\phi_1(-\tau) + \bar{\phi}_{2i}(0)\phi_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(0)\bar{\phi}_2(0) + \phi_{1i}(-\tau)\bar{\phi}_{2j}(0)\phi_1(0) + \bar{\phi}_{2i}(-\tau)\phi_{1j}(0)\phi_1(0)\big) \\
& + \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \phi_{1i}(-\tau)\bar{\phi}_{2j}(-\tau)\phi_1(-\tau) + \bar{\phi}_{2i}(-\tau)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\big], \\
G_9 = & \sum_{i,j=1}^{n} \big[\Omega^1_{i,j}\big(\phi_{1i}(0)\phi_{1j}(0)\phi_3(0) + \phi_{1i}(0)\phi_{3j}(0)\phi_1(0) + \phi_{3i}(0)\phi_{1j}(0)\phi_1(0)\big) \\
& + \Omega^2_{i,j}\big(\phi_{1i}(0)\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{1i}(0)\phi_{3j}(-\tau)\phi_1(-\tau) + \phi_{3i}(0)\phi_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(0)\phi_3(0) + \phi_{1i}(-\tau)\phi_{3j}(0)\phi_1(0) + \phi_{3i}(-\tau)\phi_{1j}(0)\phi_1(0)\big) \\
& + \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{1i}(-\tau)\phi_{3j}(-\tau)\phi_1(-\tau) + \phi_{3i}(-\tau)\phi_{1j}(-\tau)\phi_1(-\tau)\big)\big], \\
G_{10} = & \sum_{i,j=1}^{n} \big[\Omega^1_{i,j}\big(\phi_{1i}(0)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{1j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{2j}(0)\phi_1(0)\big) \\
& + \Omega^2_{i,j}\big(\phi_{1i}(0)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{2j}(-\tau)\phi_1(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{1j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{2j}(0)\phi_1(0)\big) \\
& + \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{2j}(-\tau)\phi_1(-\tau)\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
G_{11} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\bar\phi_{1i}(0)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(0)\bar\phi_{1j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{2j}(0)\bar\phi_1(0)\big)\\
&+ \Omega^2_{i,j}\big(\bar\phi_{1i}(0)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\bar\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{2j}(-\tau)\bar\phi_1(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\bar\phi_{1i}(-\tau)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(-\tau)\bar\phi_{1j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{2j}(0)\bar\phi_1(0)\big)\\
&+ \Omega^4_{i,j}\big(\bar\phi_{1i}(-\tau)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\bar\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{2j}(-\tau)\bar\phi_1(-\tau)\big)\big],\\
G_{12} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\bar\phi_{2i}(0)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(0)\bar\phi_{2j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{2j}(0)\bar\phi_2(0)\big)\\
&+ \Omega^2_{i,j}\big(\bar\phi_{2i}(0)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\bar\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{2j}(-\tau)\bar\phi_2(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\bar\phi_{2i}(-\tau)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(-\tau)\bar\phi_{2j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{2j}(0)\bar\phi_2(0)\big)\\
&+ \Omega^4_{i,j}\big(\bar\phi_{2i}(-\tau)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\bar\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{2j}(-\tau)\bar\phi_2(-\tau)\big)\big],\\
G_{13} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{3i}(0)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{3j}(0)\phi_2(0) + \phi_{2i}(0)\phi_{2j}(0)\phi_3(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{3i}(0)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{3j}(-\tau)\phi_2(-\tau) + \phi_{2i}(0)\phi_{2j}(-\tau)\phi_3(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{3i}(-\tau)\phi_{2j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{3j}(0)\phi_2(0) + \phi_{2i}(-\tau)\phi_{2j}(0)\phi_3(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{3i}(-\tau)\phi_{2j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{3j}(-\tau)\phi_2(-\tau) + \phi_{2i}(-\tau)\phi_{2j}(-\tau)\phi_3(-\tau)\big)\big],\\
G_{14} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{1i}(0)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\bar\phi_{1j}(0)\phi_1(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{1i}(0)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\bar\phi_{1j}(-\tau)\phi_1(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{1i}(-\tau)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(0)\phi_1(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{1i}(-\tau)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(-\tau)\phi_1(-\tau)\big)\big],\\
G_{15} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{2i}(0)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\bar\phi_{1j}(0)\phi_2(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{2i}(0)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\bar\phi_{1j}(-\tau)\phi_2(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{2i}(-\tau)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(0)\phi_2(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{2i}(-\tau)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(-\tau)\phi_2(-\tau)\big)\big],\\
G_{16} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\bar\phi_{2i}(0)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\bar\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\bar\phi_{1j}(0)\bar\phi_2(0)\big)\\
&+ \Omega^2_{i,j}\big(\bar\phi_{2i}(0)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\bar\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\bar\phi_{1j}(-\tau)\bar\phi_2(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\bar\phi_{2i}(-\tau)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\bar\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(0)\bar\phi_2(0)\big)\\
&+ \Omega^4_{i,j}\big(\bar\phi_{2i}(-\tau)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\bar\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(-\tau)\bar\phi_2(-\tau)\big)\big],\\
G_{17} = \sum_{i,j=1}^{n} \big[&\Omega^1_{i,j}\big(\phi_{3i}(0)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\phi_{3j}(0)\bar\phi_1(0) + \bar\phi_{1i}(0)\bar\phi_{1j}(0)\phi_3(0)\big)\\
&+ \Omega^2_{i,j}\big(\phi_{3i}(0)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\phi_{3j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(0)\bar\phi_{1j}(-\tau)\phi_3(-\tau)\big)\\
&+ \Omega^3_{i,j}\big(\phi_{3i}(-\tau)\bar\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\phi_{3j}(0)\bar\phi_1(0) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(0)\phi_3(0)\big)\\
&+ \Omega^4_{i,j}\big(\phi_{3i}(-\tau)\bar\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\phi_{3j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1i}(-\tau)\bar\phi_{1j}(-\tau)\phi_3(-\tau)\big)\big],
\end{aligned}
$$

$$\begin{aligned}
G_{18} = \sum_{i,j=1}^{n} \big[& \Omega_{i,j}^{1} \big(\phi_{1i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(0)\phi_{1j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(0)\phi_{1}(0) \big) \\
& + \Omega_{i,j}^{2} \big(\phi_{1i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(0)\phi_{1j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(-\tau)\phi_{1}(-\tau) \big) \\
& + \Omega_{i,j}^{3} \big(\phi_{1i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(-\tau)\phi_{1j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(0)\phi_{1}(0) \big) \\
& + \Omega_{i,j}^{4} \big(\phi_{1i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(-\tau)\phi_{1j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\phi_{1}(-\tau) \big) \big], \\
G_{19} = \sum_{i,j=1}^{n} \big[& \Omega_{i,j}^{1} \big(\phi_{2i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(0)\phi_{2j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(0)\phi_{2}(0) \big) \\
& + \Omega_{i,j}^{2} \big(\phi_{2i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(0)\phi_{2j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(-\tau)\phi_{2}(-\tau) \big) \\
& + \Omega_{i,j}^{3} \big(\phi_{2i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(-\tau)\phi_{2j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(0)\phi_{2}(0) \big) \\
& + \Omega_{i,j}^{4} \big(\phi_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(-\tau)\phi_{2j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\phi_{2}(-\tau) \big) \big], \\
G_{20} = \sum_{i,j=1}^{n} \big[& \Omega_{i,j}^{1} \big(\bar{\phi}_{1i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(0)\bar{\phi}_{1j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_{1}(0) \big) \\
& + \Omega_{i,j}^{2} \big(\bar{\phi}_{1i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_{1j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_{1}(-\tau) \big) \\
& + \Omega_{i,j}^{3} \big(\bar{\phi}_{1i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{1j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_{1}(0) \big) \\
& + \Omega_{i,j}^{4} \big(\bar{\phi}_{1i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{1j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_{1}(-\tau) \big) \big], \\
G_{21} = \sum_{i,j=1}^{n} \big[& \Omega_{i,j}^{1} \big(\phi_{3i}(0)\bar{\phi}_{2j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(0)\phi_{3j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(0)\phi_{3}(0) \big) \\
& + \Omega_{i,j}^{2} \big(\phi_{3i}(0)\bar{\phi}_{2j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(0)\phi_{3j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(0)\bar{\phi}_{2j}(-\tau)\phi_{3}(-\tau) \big) \\
& + \Omega_{i,j}^{3} \big(\phi_{3i}(-\tau)\bar{\phi}_{2j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(-\tau)\phi_{3j}(0)\bar{\phi}_{2}(0) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(0)\phi_{3}(0) \big) \\
& + \Omega_{i,j}^{4} \big(\phi_{3i}(-\tau)\bar{\phi}_{2j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(-\tau)\phi_{3j}(-\tau)\bar{\phi}_{2}(-\tau) + \bar{\phi}_{2i}(-\tau)\bar{\phi}_{2j}(-\tau)\phi_{3}(-\tau) \big) \big], \\
G_{22} = \sum_{i,j=1}^{n} \big[& \Omega_{i,j}^{1} \big(\phi_{1i}(0)\phi_{3j}(0)\phi_{3}(0) + \phi_{3i}(0)\phi_{1j}(0)\phi_{3}(0) + \phi_{3i}(0)\phi_{3j}(0)\phi_{1}(0) \big) \\
& + \Omega_{i,j}^{2} \big(\phi_{1i}(0)\phi_{3j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(0)\phi_{1j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(0)\phi_{3j}(-\tau)\phi_{1}(-\tau) \big) \\
& + \Omega_{i,j}^{3} \big(\phi_{1i}(-\tau)\phi_{3j}(0)\phi_{3}(0) + \phi_{3i}(-\tau)\phi_{1j}(0)\phi_{3}(0) + \phi_{3i}(-\tau)\phi_{3j}(0)\phi_{1}(0) \big) \\
& + \Omega_{i,j}^{4} \big(\phi_{1i}(-\tau)\phi_{3j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(-\tau)\phi_{1j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(-\tau)\phi_{3j}(-\tau)\phi_{1}(-\tau) \big) \big], \\
G_{23} = \sum_{i,j=1}^{n} \big[& \Omega_{i,j}^{1} \big(\phi_{2i}(0)\phi_{3j}(0)\phi_{3}(0) + \phi_{3i}(0)\phi_{2j}(0)\phi_{3}(0) + \phi_{3i}(0)\phi_{3j}(0)\phi_{2}(0) \big) \\
& + \Omega_{i,j}^{2} \big(\phi_{2i}(0)\phi_{3j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(0)\phi_{2j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(0)\phi_{3j}(-\tau)\phi_{2}(-\tau) \big) \\
& + \Omega_{i,j}^{3} \big(\phi_{2i}(-\tau)\phi_{3j}(0)\phi_{3}(0) + \phi_{3i}(-\tau)\phi_{2j}(0)\phi_{3}(0) + \phi_{3i}(-\tau)\phi_{3j}(0)\phi_{2}(0) \big) \\
& + \Omega_{i,j}^{4} \big(\phi_{2i}(-\tau)\phi_{3j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(-\tau)\phi_{2j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(-\tau)\phi_{3j}(-\tau)\phi_{2}(-\tau) \big) \big], \\
G_{24} = \sum_{i,j=1}^{n} \big[& \Omega_{i,j}^{1} \big(\bar{\phi}_{1i}(0)\phi_{3j}(0)\phi_{3}(0) + \phi_{3i}(0)\bar{\phi}_{1j}(0)\phi_{3}(0) + \phi_{3i}(0)\phi_{3j}(0)\bar{\phi}_{1}(0) \big) \\
& + \Omega_{i,j}^{2} \big(\bar{\phi}_{1i}(0)\phi_{3j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(0)\bar{\phi}_{1j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(0)\phi_{3j}(-\tau)\bar{\phi}_{1}(-\tau) \big) \\
& + \Omega_{i,j}^{3} \big(\bar{\phi}_{1i}(-\tau)\phi_{3j}(0)\phi_{3}(0) + \phi_{3i}(-\tau)\bar{\phi}_{1j}(0)\phi_{3}(0) + \phi_{3i}(-\tau)\phi_{3j}(0)\bar{\phi}_{1}(0) \big) \\
& + \Omega_{i,j}^{4} \big(\bar{\phi}_{1i}(-\tau)\phi_{3j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(-\tau)\bar{\phi}_{1j}(-\tau)\phi_{3}(-\tau) + \phi_{3i}(-\tau)\phi_{3j}(-\tau)\bar{\phi}_{1}(-\tau) \big) \big],
\end{aligned}$$

$$
\begin{aligned}
G_{25} = \sum_{i,j=1}^{n} \big[& \Omega^1_{i,j}\big(\bar{\phi}_{2i}(0)\phi_{3j}(0)\phi_3(0) + \phi_{3i}(0)\bar{\phi}_{2j}(0)\phi_3(0) + \phi_{3i}(0)\phi_{3j}(0)\bar{\phi}_2(0)\big) \\
& + \Omega^2_{i,j}\big(\bar{\phi}_{2i}(0)\phi_{3j}(-\tau)\phi_3(-\tau) + \phi_{3i}(0)\bar{\phi}_{2j}(-\tau)\phi_3(-\tau) + \phi_{3i}(0)\phi_{3j}(-\tau)\bar{\phi}_2(-\tau)\big) \\
& + \Omega^3_{i,j}\big(\bar{\phi}_{2i}(-\tau)\phi_{3j}(0)\phi_3(0) + \phi_{3i}(-\tau)\bar{\phi}_{2j}(0)\phi_3(0) + \phi_{3i}(-\tau)\phi_{3j}(0)\bar{\phi}_2(0)\big) \\
& + \Omega^4_{i,j}\big(\bar{\phi}_{2i}(-\tau)\phi_{3j}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\bar{\phi}_{2j}(-\tau)\phi_3(-\tau) + \phi_{3i}(-\tau)\phi_{3j}(-\tau)\bar{\phi}_2(-\tau)\big)\big], \\
G_{26} = \sum_{i,j=1}^{n} \Bigg[& \Omega^1_{i,j}\Bigg(\phi_{1i}(0)\big(\phi_{2j}(0)\bar{\phi}_1(0) + \bar{\phi}_{1j}(0)\phi_2(0)\big) + \phi_{2i}(0)\big(\phi_{1j}(0)\bar{\phi}_1(0) + \bar{\phi}_{1j}(0)\phi_1(0)\big) \\
& + \bar{\phi}_{1i}(0)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big)\Bigg) + \Omega^2_{i,j}\Bigg(\phi_{1i}(0)\big(\phi_{2j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(0)\big(\phi_{1j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar{\phi}_{1i}(0)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big)\Bigg) \\
& + \Omega^3_{i,j}\Bigg(\phi_{1i}(-\tau)\big(\phi_{2j}(0)\bar{\phi}_1(0) + \bar{\phi}_{1j}(0)\phi_2(0)\big) + \phi_{2i}(-\tau)\big(\phi_{1j}(0)\bar{\phi}_1(0) + \bar{\phi}_{1j}(0)\phi_1(0)\big) \\
& + \bar{\phi}_{1i}(-\tau)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big)\Bigg) \\
& + \Omega^4_{i,j}\Bigg(\phi_{1i}(-\tau)\big(\phi_{2j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(-\tau)\big(\phi_{1j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar{\phi}_{1i}(-\tau)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big)\Bigg)\Bigg], \\
G_{27} = \sum_{i,j=1}^{n} \Bigg[& \Omega^1_{i,j}\Bigg(\phi_{1i}(0)\big(\phi_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_2(0)\big) + \phi_{2i}(0)\big(\phi_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_1(0)\big) \\
& + \bar{\phi}_{2i}(0)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big)\Bigg) + \Omega^2_{i,j}\Bigg(\phi_{1i}(0)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(0)\big(\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar{\phi}_{2i}(0)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big)\Bigg) \\
& + \Omega^3_{i,j}\Bigg(\phi_{1i}(-\tau)\big(\phi_{2j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_2(0)\big) + \phi_{2i}(-\tau)\big(\phi_{1j}(0)\bar{\phi}_2(0) + \bar{\phi}_{2j}(0)\phi_1(0)\big) \\
& + \bar{\phi}_{2i}(-\tau)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big)\Bigg) \\
& + \Omega^4_{i,j}\Bigg(\phi_{1i}(-\tau)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(-\tau)\big(\phi_{1j}(-\tau)\bar{\phi}_2(-\tau) + \bar{\phi}_{2j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar{\phi}_{2i}(-\tau)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big)\Bigg)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{28} = \sum_{i,j=1}^{n} \Bigg[& \Omega^1_{i,j} \Big(\phi_{1i}(0)\big(\phi_{2j}(0)\phi_3(0) + \phi_{3j}(0)\phi_2(0)\big) + \phi_{2i}(0)\big(\phi_{1j}(0)\phi_3(0) + \phi_{3j}(0)\phi_1(0)\big) \\
& + \phi_{3i}(0)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big) \Big) + \Omega^2_{i,j} \Big(\phi_{1i}(0)\big(\phi_{2j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(0)\big(\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_1(-\tau)\big) \\
& + \phi_{3i}(0)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big) \Big) \\
& + \Omega^3_{i,j} \Big(\phi_{1i}(-\tau)\big(\phi_{2j}(0)\phi_3(0) + \phi_{3j}(0)\phi_2(0)\big) + \phi_{2i}(-\tau)\big(\phi_{1j}(0)\phi_3(0) + \phi_{3j}(0)\phi_1(0)\big) \\
& + \phi_{3i}(-\tau)\big(\phi_{1j}(0)\phi_2(0) + \phi_{2j}(0)\phi_1(0)\big) \Big) \\
& + \Omega^4_{i,j} \Big(\phi_{1i}(-\tau)\big(\phi_{2j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_2(-\tau)\big) \\
& + \phi_{2i}(-\tau)\big(\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_1(-\tau)\big) \\
& + \phi_{3i}(-\tau)\big(\phi_{1j}(-\tau)\phi_2(-\tau) + \phi_{2j}(-\tau)\phi_1(-\tau)\big) \Big) \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{29} = \sum_{i,j=1}^{n} \Bigg[& \Omega^1_{i,j} \Big(\phi_{1i}(0)\big(\bar\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\bar\phi_2(0)\big) + \bar\phi_{2i}(0)\big(\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_1(0)\big) \\
& + \bar\phi_{1i}(0)\big(\phi_{1j}(0)\bar\phi_2(0) + \bar\phi_{2j}(0)\phi_1(0)\big) \Big) + \Omega^2_{i,j} \Big(\phi_{1i}(0)\big(\bar\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\bar\phi_2(-\tau)\big) \\
& + \bar\phi_{2i}(0)\big(\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar\phi_{1i}(0)\big(\phi_{1j}(-\tau)\bar\phi_2(-\tau) + \bar\phi_{2j}(-\tau)\phi_1(-\tau)\big) \Big) \\
& + \Omega^3_{i,j} \Big(\phi_{1i}(-\tau)\big(\bar\phi_{2j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\bar\phi_2(0)\big) + \bar\phi_{2i}(-\tau)\big(\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_1(0)\big) \\
& + \bar\phi_{1i}(-\tau)\big(\phi_{1j}(0)\bar\phi_2(0) + \bar\phi_{2j}(0)\phi_1(0)\big) \Big) \\
& + \Omega^4_{i,j} \Big(\phi_{1i}(-\tau)\big(\bar\phi_{2j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\bar\phi_2(-\tau)\big) \\
& + \bar\phi_{2i}(-\tau)\big(\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\phi_1(-\tau)\big) \\
& + \bar\phi_{1i}(-\tau)\big(\phi_{1j}(-\tau)\bar\phi_2(-\tau) + \bar\phi_{2j}(-\tau)\phi_1(-\tau)\big) \Big) \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
G_{30} = \sum_{i,j=1}^{n} \Bigg[& \Omega^1_{i,j} \Big(\phi_{1i}(0)\big(\phi_{3j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_3(0)\big) + \phi_{3i}(0)\big(\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_1(0)\big) \\
& + \bar\phi_{1i}(0)\big(\phi_{1j}(0)\phi_3(0) + \phi_{3j}(0)\phi_1(0)\big) \Big) + \Omega^2_{i,j} \Big(\phi_{1i}(0)\big(\phi_{3j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\phi_3(-\tau)\big) \\
& + \phi_{3i}(0)\big(\phi_{1j}(-\tau)\bar\phi_1(-\tau) + \bar\phi_{1j}(-\tau)\phi_1(-\tau)\big) + \bar\phi_{1i}(0)\big(\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_1(-\tau)\big) \Big) \\
& + \Omega^3_{i,j} \Big(\phi_{1i}(-\tau)\big(\phi_{3j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_3(0)\big) + \phi_{3i}(-\tau)\big(\phi_{1j}(0)\bar\phi_1(0) + \bar\phi_{1j}(0)\phi_1(0)\big) \\
& + \bar\phi_{1i}(-\tau)\big(\phi_{1j}(0)\phi_3(0) + \phi_{3j}(0)\phi_1(0)\big) \Big)
\end{aligned}
$$

$$+ \Omega^4_{i,j}\Big(\phi_{1i}(-\tau)\big(\phi_{3j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_3(-\tau)\big)$$
$$+ \phi_{3i}(-\tau)\big(\phi_{1j}(-\tau)\bar{\phi}_1(-\tau) + \bar{\phi}_{1j}(-\tau)\phi_1(-\tau)\big)$$
$$+ \bar{\phi}_{1i}(-\tau)\big(\phi_{1j}(-\tau)\phi_3(-\tau) + \phi_{3j}(-\tau)\phi_1(-\tau)\big)\Big)\Big],$$

$$\begin{aligned}
G_{31} = \sum_{i,j=1}^{n} \Big[&\Omega^1_{i,j}\Big(\phi_{1i}(0)\big(\phi_{3j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\phi_3(0)\big)+\phi_{3i}(0)\big(\phi_{1j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\phi_1(0)\big)\\
&+\bar{\phi}_{2i}(0)\big(\phi_{1j}(0)\phi_3(0)+\phi_{3j}(0)\phi_1(0)\big)\Big)+\Omega^2_{i,j}\Big(\phi_{1i}(0)\big(\phi_{3j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\phi_3(-\tau)\big)\\
&+\phi_{3i}(0)\big(\phi_{1j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\phi_1(-\tau)\big)+\bar{\phi}_{2i}(0)\big(\phi_{1j}(-\tau)\phi_3(-\tau)+\phi_{3j}(-\tau)\phi_1(-\tau)\big)\Big)\\
&+\Omega^3_{i,j}\Big(\phi_{1i}(-\tau)\big(\phi_{3j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\phi_3(0)\big)+\phi_{3i}(-\tau)\big(\phi_{1j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\phi_1(0)\big)\\
&+\bar{\phi}_{2i}(-\tau)\big(\phi_{1j}(0)\phi_3(0)+\phi_{3j}(0)\phi_1(0)\big)\Big)\\
&+\Omega^4_{i,j}\Big(\phi_{1i}(-\tau)\big(\phi_{3j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\phi_3(-\tau)\big)\\
&+\phi_{3i}(-\tau)\big(\phi_{1j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\phi_1(-\tau)\big)\\
&+\bar{\phi}_{2i}(-\tau)\big(\phi_{1j}(-\tau)\phi_3(-\tau)+\phi_{3j}(-\tau)\phi_1(-\tau)\big)\Big)\Big],\\
G_{32} = \sum_{i,j=1}^{n} \Big[&\Omega^1_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\phi_{2j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\phi_2(0)\big)+\phi_{2i}(0)\big(\bar{\phi}_{1j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\bar{\phi}_1(0)\big)\\
&+\bar{\phi}_{2i}(0)\big(\bar{\phi}_{1j}(0)\phi_2(0)+\phi_{2j}(0)\bar{\phi}_1(0)\big)\Big)+\Omega^2_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big)\\
&+\phi_{2i}(0)\big(\bar{\phi}_{1j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)+\bar{\phi}_{2i}(0)\big(\bar{\phi}_{1j}(-\tau)\phi_2(-\tau)+\phi_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big)\\
&+\Omega^3_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\phi_{2j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\phi_2(0)\big)+\phi_{2i}(-\tau)\big(\bar{\phi}_{1j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\bar{\phi}_1(0)\big)\\
&+\bar{\phi}_{2i}(-\tau)\big(\bar{\phi}_{1j}(0)\phi_2(0)+\phi_{2j}(0)\bar{\phi}_1(0)\big)\Big)\\
&+\Omega^4_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big)\\
&+\phi_{2i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\\
&+\bar{\phi}_{2i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\phi_2(-\tau)+\phi_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big)\Big],\\
G_{33} = \sum_{i,j=1}^{n} \Big[&\Omega^1_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\phi_{2j}(0)\phi_3(0)+\phi_{3j}(0)\phi_2(0)\big)+\phi_{2i}(0)\big(\bar{\phi}_{1j}(0)\phi_3(0)+\phi_{3j}(0)\bar{\phi}_1(0)\big)\\
&+\phi_{3i}(0)\big(\bar{\phi}_{1j}(0)\phi_2(0)+\phi_{2j}(0)\bar{\phi}_1(0)\big)\Big)+\Omega^2_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\phi_{2j}(-\tau)\phi_3(-\tau)+\phi_{3j}(-\tau)\phi_2(-\tau)\big)\\
&+\phi_{2i}(0)\big(\bar{\phi}_{1j}(-\tau)\phi_3(-\tau)+\phi_{3j}(-\tau)\bar{\phi}_1(-\tau)\big)+\phi_{3i}(0)\big(\bar{\phi}_{1j}(-\tau)\phi_2(-\tau)+\phi_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big)\\
&+\Omega^3_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\phi_{2j}(0)\phi_3(0)+\phi_{3j}(0)\phi_2(0)\big)\\
&+\phi_{2i}(-\tau)\big(\bar{\phi}_{1j}(0)\phi_3(0)+\phi_{3j}(0)\bar{\phi}_1(0)\big)\\
&+\phi_{3i}(-\tau)\big(\bar{\phi}_{1j}(0)\phi_2(0)+\phi_{2j}(0)\bar{\phi}_1(0)\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\Omega^4_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\phi_{2j}(-\tau)\phi_3(-\tau)+\phi_{3j}(-\tau)\phi_2(-\tau)\big)\\
&+\phi_{2i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\phi_3(-\tau)+\phi_{3j}(-\tau)\bar{\phi}_1(-\tau)\big)\\
&+\phi_{3i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\phi_2(-\tau)+\phi_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
G_{34}=\sum_{i,j=1}^{n}\Big[&\Omega^1_{i,j}\Big(\phi_{2i}(0)\big(\phi_{3j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\phi_3(0)\big)+\phi_{3i}(0)\big(\phi_{2j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\phi_2(0)\big)\\
&+\bar{\phi}_{2i}(0)\big(\phi_{2j}(0)\phi_3(0)+\phi_{3j}(0)\phi_2(0)\big)\Big)\\
&+\Omega^2_{i,j}\Big(\phi_{2i}(0)\big(\phi_{3j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\phi_3(-\tau)\big)\\
&+\phi_{3i}(0)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big)+\bar{\phi}_{2i}(0)\big(\phi_{2j}(-\tau)\phi_3(-\tau)+\phi_{3j}(-\tau)\phi_2(-\tau)\big)\Big)\\
&+\Omega^3_{i,j}\Big(\phi_{2i}(-\tau)\big(\phi_{3j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\phi_3(0)\big)+\phi_{3i}(-\tau)\big(\phi_{2j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\phi_2(0)\big)\\
&+\bar{\phi}_{2i}(-\tau)\big(\phi_{2j}(0)\phi_3(0)+\phi_{3j}(0)\phi_2(0)\big)\Big)\\
&+\Omega^4_{i,j}\Big(\phi_{2i}(-\tau)\big(\phi_{3j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\phi_3(-\tau)\big)\\
&+\phi_{3i}(-\tau)\big(\phi_{2j}(-\tau)\bar{\phi}_2(-\tau)\\
&+\bar{\phi}_{2j}(-\tau)\phi_2(-\tau)\big)+\bar{\phi}_{2i}(-\tau)\big(\phi_{2j}(-\tau)\phi_3(-\tau)+\phi_{3j}(-\tau)\phi_2(-\tau)\big)\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
G_{35}=\sum_{i,j=1}^{n}\Big[&\Omega^1_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\bar{\phi}_{2j}(0)\phi_3(0)+\phi_{3j}(0)\bar{\phi}_2(0)\big)+\bar{\phi}_{2i}(0)\big(\bar{\phi}_{1j}(0)\phi_3(0)+\phi_{3j}(0)\bar{\phi}_1(0)\big)\\
&+\phi_{3i}(0)\big(\bar{\phi}_{1j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\bar{\phi}_1(0)\big)\Big)\\
&+\Omega^2_{i,j}\Big(\bar{\phi}_{1i}(0)\big(\bar{\phi}_{2j}(-\tau)\phi_3(-\tau)+\phi_{3j}(-\tau)\bar{\phi}_2(-\tau)\big)\\
&+\bar{\phi}_{2i}(0)\big(\bar{\phi}_{1j}(-\tau)\phi_3(-\tau)+\phi_{3j}(-\tau)\bar{\phi}_1(-\tau)\big)+\phi_{3i}(0)\big(\bar{\phi}_{1j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big)\\
&+\Omega^3_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\bar{\phi}_{2j}(0)\phi_3(0)+\phi_{3j}(0)\bar{\phi}_2(0)\big)+\bar{\phi}_{2i}(-\tau)\big(\bar{\phi}_{1j}(0)\phi_3(0)+\phi_{3j}(0)\bar{\phi}_1(0)\big)\\
&+\phi_{3i}(-\tau)\big(\bar{\phi}_{1j}(0)\bar{\phi}_2(0)+\bar{\phi}_{2j}(0)\bar{\phi}_1(0)\big)\Big)\\
&+\Omega^4_{i,j}\Big(\bar{\phi}_{1i}(-\tau)\big(\bar{\phi}_{2j}(-\tau)\phi_3(-\tau)+\phi_{3j}(-\tau)\bar{\phi}_2(-\tau)\big)\\
&+\bar{\phi}_{2i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\phi_3(-\tau)\\
&+\phi_{3j}(-\tau)\bar{\phi}_1(-\tau)\big)+\phi_{3i}(-\tau)\big(\bar{\phi}_{1j}(-\tau)\bar{\phi}_2(-\tau)+\bar{\phi}_{2j}(-\tau)\bar{\phi}_1(-\tau)\big)\Big)\Big].
\end{aligned}
$$

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Critical points approaches for multiple solutions of a quasilinear periodic boundary value problem

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Abstract. Optimization problems are omnipresent in the mathematical modeling of real world systems and cover a very extensive range of applications becoming apparent in all branches of Economics, Finance, Materials Science, Astronomy, Physics, Structural and Molecular Biology, Engineering, Computer Science, and Medicine. In this paper, we aim to delve deeper into the multiplicity findings concerning a specific class of quasilinear periodic boundary value problems. In fact, as an optimization problem, we look for the critical points of the energy functional related to the problem. Utilizing a corollary derived from Bonanno's local minimum theorem, we investigate the existence of a one solution under certain algebraic conditions on the nonlinear term. Additionally, we explore conditions that lead to the existence of two solutions, incorporating the classical Ambrosetti-Rabinowitz (AR) condition alongside algebraic criteria. Moreover, by employing two critical point theorems one by Averna and Bonanno, and another by Bonanno, we establish the existence of two and three solutions in a particular scenario. To illustrate our findings, we provide an example.

Keywords: multiple solutions, quasilinear periodic, critical point, variational methods.

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1 Introduction

The target of global optimization is to find the best solution of decision models, in presence of the multiple local solutions. Optimization plays an ever-increasing role in mathematics,

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economics, engineering, health sciences, management and life sciences. Many optimization problems have existence in the real world including space planning, networking, logistic management, financial planning, and risk management. The objective of this paper is to ascertain the existence of solutions for the following quasilinear periodic boundary value problem

$$\begin{cases} -p(u')u'' + \zeta(x)u = \lambda f(x,u(x)) + \mu g(x,u(x)), & \text{a.e. } x \in [0,1], \\ u(1) - u(0) = u'(1) - u'(0) = 0 \end{cases} \tag{$P^{f,g}_{\lambda,\mu}$}$$

where $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, λ is a positive parameter and $\mu \geq 0$. We need the following assumptions:

(Q_1) $p : \mathbb{R} \to (0,\infty)$ is a continuous and nondecreasing on $[0,\infty)$, there exist two positive numbers $M \geq m$ such that

$$m \leq p(x) \leq M, \qquad \forall x \in \mathbb{R} \tag{1.1}$$

(Q_2) $\zeta \in C([0,1])$ and there exist $\zeta_1 \geq \zeta_0 > 0$ such that

$$\zeta_0 \leq \zeta(x) \leq \zeta_1, \qquad \forall x \in [0,1]. \tag{1.2}$$

Exactly, as an optimization problem, we look for the critical points of the energy functional related to the problem which are the solutions of the problem.

In recent years, various fixed-point theorems, critical points, and variational methods have been effectively employed to explore the existence of solutions for quasilinear periodic boundary value problems. References such as [2,9,14,17,20,21,23,26,27] and others have extensively discussed this topic. For instance, Matzakos and Papageorgiou in [21] combined the variational method with techniques involving upper and lower solutions to establish the existence of periodic solutions for quasilinear differential equations. Similarly, Papageorgiou and Papalini in [23] utilized variational arguments, methods from the theory of nonlinear operators of monotone type, and upper and lower solution techniques to demonstrate the existence of at least two nontrivial solutions, one positive and the other negative for the following quasilinear periodic problem

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' = f(x,u(x)) = 0, & x \in [0,b], \\ u(0) = u(b),\ u'(0) = u'(b),\ 2 \leq p < \infty \end{cases}$$

where $f : T \times \mathbb{R} \to \mathbb{R}$ is a function. In [14], the existence of at least three classical solutions for a Dirichlet quasilinear elliptic system was established through the application of variational methods and critical point theory. Similarly, in [17], the utilization of a recent three critical points theorem by Bonanno and Marano led to the confirmation of at least three solutions for quasilinear second order differential equation on a compact interval $[a,b] \subset \mathbb{R}$

$$\begin{cases} -u'' = (\lambda f(x,u) + g(u))h'(u), & \text{in } (a,b), \\ u(a) - u(b) = 0 \end{cases}$$

where $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, was discussed. Shen and Liu, in [26], utilized the symmetric mountain pass theorem and genus properties in critical point theory to explore the existence of infinitely many solutions for second-order quasilinear periodic boundary value problems with impulsive effects.

Meanwhile, Wang et al., in [27], investigated the existence of at least three periodic solutions for the problem ($P^{f,g}_{\lambda,\mu}$) by employing appropriate hypotheses and a three critical points theorem by Ricceri.

Additionally, in [15], variational methods and critical point theorems for smooth functionals defined on reflexive Banach spaces were used to discuss the existence of at least three solutions to an impulsive effects version of the problem ($P^{f,g}_{\lambda,\mu}$). Furthermore, in [19], variational methods were employed to discuss the existence of at least three weak solutions for the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$. In [12], the investigation focused on the existence of infinitely many classical solutions for an impulsive effects version of the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$, utilizing critical point theory. In [13], by using variational methods, the existence of non-zero solutions and the existence of multiple solutions for positive parameter values for the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$, was discussed. Lastly, in [18], the existence of at least one weak solution and infinitely many weak solutions for the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$ was studied based on variational methods.

Our approach employs variational methods, with the primary tools being four local minimum theorems for differentiable functionals. Specifically, we utilize a corollary of Bonanno's local minimum theorem to establish the existence of one solution under certain algebraic conditions on the nonlinear terms, and two solutions for the problem under algebraic conditions alongside the classical Ambrosetti–Rabinowitz (AR) condition on the nonlinear terms (refer to [3]). Furthermore, by leveraging two critical point theorems, one by Averna and Bonanno, and another by Bonanno, we ensure the existence of two and three solutions for the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$.

In comparison to previous findings, we introduce novel assumptions to establish the existence of solutions for the problem ($P^{f,g}_{\lambda,\mu}$), thus extending recent related works.

Here, we present two specific cases of our main results focusing on scenarios with a single impulse.

Theorem 1.1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exist two positive constants γ and η with the property*

$$\sqrt{\frac{2h(2\eta) + 2h(-2\eta) + 8\zeta_1\eta^2}{\min\{m, \zeta_0\}}} < \gamma$$

and

- *there exist $\nu > 2$ and $R > 0$ such that*

$$0 < \nu \int_0^{\xi} \psi(s)\mathrm{d}s \le \xi\psi(\xi)$$

  for all $|\xi| \ge R$.

Then, for each

$$\lambda \in \left(0, \frac{\min\{m, \zeta_0\}\gamma^2}{8\int_0^{\gamma} \psi(s)\mathrm{d}s}\right)$$

and for every function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying the following condition:

- *there exist $\nu > 2$ and $R > 0$ such that*

$$0 < \nu \int_0^{\xi} g(x, s)\mathrm{d}s \le \xi g(x, \xi)$$

  for all $|\xi| \ge R$ and for all $x \in [0, 1]$, *there exists $\delta_\lambda > 0$, for each $\mu \in [0, \delta_\lambda[$, the problem*

$$\begin{cases} -p(u')u'' + \zeta(x)u = \lambda e^{-t}\psi(x) + \mu g(x, u(x)), & a.e.\ x \in [0,1], \\ u(1) - u(0) = u'(1) - u'(0) = 0 \end{cases} \tag{1.3}$$

admits at least two solutions u_1 and u_2 in

$$\{u : [0,1] \to \mathbb{R} : u \text{ is absolutely continuous},\ u(1) = u(0),\ u' \in L^2([0,1]\}$$

such that

$$\max_{x\in[0,1]} |u_1(x)| < \gamma.$$

Theorem 1.2. *Assume that $\psi : \mathbb{R} \to \mathbb{R}$ is a nonnegative and continuous function. Moreover, assume that*

$$\lim_{\xi\to 0^+} \frac{\psi(\xi)}{\xi} = \lim_{|\xi|\to\infty} \frac{\psi(\xi)}{|\xi|} = 0$$

and there exists a positive constant $\bar{\eta}$ such that $\int_0^{\bar{\eta}} \psi(s)ds > 0$. Then, for each $\lambda > \lambda^*$ where

$$\lambda^* = \frac{1}{4\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x)dx} \inf_{\bar{\eta}>0} \frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{\int_0^{\bar{\eta}} \psi(s)ds},$$

the problem (1.3) in the case $\mu = 0$, admits at least one nonnegative and one non zero solution in

$$\{u : [0,1] \to \mathbb{R} : u \text{ is absolutely continuous},\ u(1) = u(0),\ u' \in L^2([0,1]\}.$$

The structure of the paper is outlined as follows:

Section 2 presents our fundamental theorems and revisits relevant definitions. Section 3 discusses and proves the existence of one solution for the problem $(P^{f,g}_{\lambda,\mu})$. Section 4 addresses the existence of two solutions for the problem $(P^{f,g}_{\lambda,\mu})$. Section 5 introduces a new multiplicity result aimed at obtaining at least two and three solutions for the problem $(P^{f,g}_{\lambda,\mu})$, specifically in the case $\mu = 0$.

2 Preliminaries

The main tools utilized to prove our results in Sections 3, 4, and 5 are the following theorems.

For the following notations and results, we refer the reader to [22, 24]. Let X be a real Banach space. We say that a continuously Gâteaux differentiable functional $J : X \to \mathbb{R}$ satisfies the *Palais–Smale condition* (abbreviated as (PS)-condition) if any sequence $\{u_n\}$ such that $\{J(u_n)\}$ is bounded and $\lim_{n\to\infty} \|J'(u_n)\|_{X^*} = 0$ has a convergent subsequence.

Let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Set

$$J = \Phi - \Psi,$$

and fix $r_1, r_2 \in [-\infty, \infty]$ with $r_1 < r_2$. We say that J satisfies the *Palais–Smale condition cut off lower at r_1 and upper at r_2*(in short $^{[r_1]}(\text{PS})^{[r_2]}$-condition) if any sequence $\{u_n\}$ such that $\{J(u_n)\}$ is bounded, $\lim_{n\to\infty} \|J'(u_n)\|_{X^*} = 0$ and $r_1 < \Phi(u_n) < r_2$ for all $n \in \mathbb{N}$, has a convergent subsequence.

Clearly, if $r_1 = -\infty$ and $r_2 = \infty$ it coincides with the classical (PS)-condition. Moreover, if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(\text{PS})^{[r_2]}$, while if $r_1 \in \mathbb{R}$ and $r_2 = \infty$ it is denoted by

${}^{[r_1]}$(PS). Indeed, if Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais–Smale condition cut off upper at r (in short (PS)$^{[r]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that $\{I(u_n)\}$ is bounded, $\lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$ and $\Phi(u_n) < r$ for all $n \in \mathbb{N}$, has a convergent subsequence. Furthermore, if J satisfies ${}^{[r_1]}$(PS)$^{[r_2]}$-condition, then it satisfies ${}^{[\varrho_1]}$(PS)$^{[\varrho_2]}$-condition for all $\varrho_1, \varrho_2 \in [-\infty, \infty]$ such that $r_1 \le \varrho_1 < \varrho_2 \le r_2$.

In particular, we deduce that if J satisfies the classical (PS)-condition, then it satisfies ${}^{[\varrho_1]}$(PS)$^{[\varrho_2]}$-condition for all $\varrho_1, \varrho_2 \in [-\infty, \infty]$ with $\varrho_1 < \varrho_2$.

In the proof of our main results, we will apply the following four theorems.

Theorem 2.1 ([7, Theorem 2.3]). *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions such that $\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\bar{u} \in X$, with $0 < \Phi(\bar{u}) < r$, such that:*

(a_1) $\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$,

(a_2) *for each $\lambda \in \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)}\right)$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)$^{[r]}$-condition.*

Then, for each

$$\lambda \in \Lambda_r = \left(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{\Phi(u)\le r} \Psi(u)} \right),$$

there exists $u_{0,\lambda} \in \Phi^{-1}(0, r)$ such that $I_\lambda(u_{0,\lambda}) \equiv \vartheta_{X^*}$ and $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(0, r)$.

Theorem 2.2 ([7, Theorem 3.2]). *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $r > 0$ and assume that, for each*

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)} \right),$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below. Then, for each

$$\lambda \in \left(0, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)} \right),$$

the functional I_λ admits two distinct critical points.

Theorem 2.3 ([4, Theorem A]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that:*

(b_1) $\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty$ *for all $\lambda \in [0, \infty)$;*

(b_2) *there is $r \in \mathbb{R}$ such that*

$$\inf_X \Phi < r,$$

and

$$\varphi_1(r) < \varphi_2(r)$$

where

$$\varphi_1(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(-\infty,r)}^w} \Psi}{r - \Phi(u)},$$

$$\varphi_2(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \sup_{v\in\Phi^{-1}[r,\infty)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)},$$

and $\overline{\Phi^{-1}(-\infty,r)}^w$ *is the closure of* $\Phi^{-1}(-\infty,r)$ *in the weak topology.*

Then, for each $\lambda \in \left(\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right)$, *the functional* $\Phi + \lambda\Psi$ *has at least three critical points in* X.

Note that $\varphi_1(r)$ in Theorem 2.3 could be 0. In this and similar cases, here and in the sequel, we agree to read $\frac{1}{0}$ as ∞.

We also use the following two critical points theorem.

Theorem 2.4 ([6, Theorem 1.1]). *Let* X *be a reflexive real Banach space, and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two sequentially weakly lower semicontinuous and Gâteaux differentiable functions. Assume that* Φ *is (strongly) continuous and satisfies*

$$\lim_{\|u\|\to\infty} \Phi(u) = \infty.$$

Assume also that there exist two constants r_1 *and* r_2 *such that*

(c_1) $\inf_X \Phi < r_1 < r_2$;

(c_2) $\varphi_1(r_1) < \varphi_2^*(r_1, r_2)$;

(c_3) $\varphi_1(r_2) < \varphi_2^*(r_1, r_2)$, *where* φ_1 *is defined as in Theorem 2.3 and*

$$\varphi_2^*(r_1, r_2) = \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)}.$$

Then, for each

$$\lambda \in \left(\frac{1}{\varphi_2^*(r_1, r_2)}, \min\left\{ \frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_1(r_2)} \right\} \right),$$

the functional $\Phi + \lambda\Psi$ *admits at least two critical points which lie in* $\Phi^{-1}(-\infty, r_1]$ *and* $\Phi^{-1}[r_1, r_2)$, *respectively.*

We remind the reader that Theorem 2.3 and Theorem 2.4 rely on Ricceri's variational principle [25].

For successful application of Theorems 2.1–2.2, we recommend referring to [8] to ensure the existence of at least one and two solutions for elliptic Dirichlet problems with variable exponent. Additionally, for the utilization of Theorems 2.3–2.4, we suggest consulting [10] to guarantee the existence of at least two and three solutions for a boundary value problem on the half-line. Furthermore, for effective implementations of Theorems 2.1–2.4, we advise referring to [11, 16] to explore the existence of multiple solutions for Kirchhoff-type second-order impulsive differential equations on the half-line and to study an elastic beam equation with local nonlinearities, respectively.

In this section, we will present several fundamental definitions, notations, lemmas, and propositions utilized throughout this paper.

Let us start by defining the finite T-dimensional Banach space

$$X = \{u : [0,1] \to \mathbb{R} : u \text{ is absolutely continuous, } u(1) = u(0),\ u' \in L^2([0,1]\},\tag{2.1}$$

which is equipped with the norm

$$\|u\| = \left(\int_0^1 \left(|u'(x)|^2 + |u(x)|^2\right) \mathrm{d}x \right)^{\frac{1}{2}}.\tag{2.2}$$

Clearly, X is a Hilbert space and X^* is the dual space of X.

Setting

$$h(y) = \int_0^y \left(\int_0^\tau p(\xi)\mathrm{d}\xi \right) \mathrm{d}\tau$$

for every $y \in \mathbb{R}$, we have

$$h'(y) = \int_0^y p(\xi)\mathrm{d}\xi \quad \text{and} \quad h''(y) = p(y)$$

for every $y \in \mathbb{R}$.

We define functionals Φ, Ψ for every $u \in X$, as follows

$$\Phi(u) = \int_0^1 h(u'(x))\mathrm{d}x + \frac{1}{2}\int_0^1 \zeta(x)|u(x)|^2\mathrm{d}x\tag{2.3}$$

and

$$\Psi(u) = \int_0^1 \left(\int_0^{u(x)} f(x,\xi)\mathrm{d}\xi \right) \mathrm{d}x + \frac{\mu}{\lambda}\int_0^1 \left(\int_0^{u(x)} g(x,\xi)\mathrm{d}\xi \right) \mathrm{d}x,\tag{2.4}$$

and we put

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for every $u \in X$. Clearly, I_λ is Gâteaux differentiable.

Definition 2.5. We mean by a (weak) solution of the BVP ($P^{f,g}_{\lambda,\mu}$), any function $u \in X$ such that

$$\int_0^1 h'(u'(x))y'(x)\mathrm{d}x + \int_0^1 \zeta(x)u(x)y(x)\mathrm{d}x - \lambda\int_0^1 f(x,u(x))y(x)\mathrm{d}x \\ -\mu\int_0^1 g(x,u(x))y(x)\mathrm{d}x = 0$$

for every $y \in X$.

Lemma 2.6. *If $u \in X$ is a critical point of I_λ in X, iff $u \in X$ is a solution of* ($P^{f,g}_{\lambda,\mu}$).

Proof. If $u \in X$ is a critical point for I_λ, we have

$$\int_0^1 h'(u'(x))y'(x)\mathrm{d}x + \int_0^1 \zeta(x)u(x)y(x)\mathrm{d}x \\ = \lambda\int_0^1 f(x,u(x))y(x)\mathrm{d}x + \mu\int_0^1 g(x,u(x))y(x)\mathrm{d}x$$

for each $y \in X$. This implies that $h' \circ u'$ has a weak derivative which equals $\zeta(x)u(x) - \lambda f(x,u(x)) - \mu g(x,u(x))$ and is thus continuous, so $h' \circ u'$ is $C^1([0,1])$. Since h' is an invertible C^1-function, it follows that u' is also in $C^1([0,1])$, hence x is in $C^2([0,1])$. Set

$$e(x) = -h'(u'(x)) + \int_0^t \zeta(\tau)u(\tau)\mathrm{d}\tau - \lambda \int_0^t f(\tau,u(\tau))\mathrm{d}\tau - \mu \int_0^t g(\tau,u(\tau))\mathrm{d}\tau - C$$

such that $\int_0^1 e(x)\mathrm{d}x = 0$. Let $y(x) = \int_0^t e(\tau)\mathrm{d}\tau$. Then $y(x) \in X$ and $\int_0^1 |e(x)|^2\mathrm{d}x = 0$, that is, $e(x) = 0$ for a.e. $x \in [0,1]$. This shows that

$$\begin{aligned} -(h' \circ u')'(x) + \zeta(x)u(x) &= -h''(u')u''(x) + \zeta(x)u(x) = -p(u'(x))u''(x) + \zeta(x)u(x) \\ &= \lambda f(x,u(x)) + \mu g(x,u(x)) \end{aligned}$$

for all $x \in [0,1]$. Hence we conclude that x is a solution of problem ($P^{f,g}_{\lambda,\mu}$) belongs to $C^2([0,1])$. □

Proposition 2.7 ([27, Proposition 2.3]). *If $p(\cdot)$ satisfies (Q_1), then h' is strongly monotone.*

Proposition 2.8 ([27, Proposition 2.4]). *If $p(\cdot)$ and $\zeta(\cdot)$ satisfy* (1.1) *and* (1.2)*, respectively, then*

(1) *Φ is well-defined in X,*

(2) *Φ is Gâteaux differentiable in X,*

(3) *Φ' is a Lipschitzian operator,*

(4) *Φ is convex in X.*

Put

$$F(x,t) = \int_0^t f(x,s)\mathrm{d}s \quad \text{and} \quad G(x,t) = \int_0^t g(x,s)\mathrm{d}s \quad \text{for all } (x,t) \in [0,1] \times \mathbb{R}.$$

3 Existence of one solution

In this section, we focus on establishing the existence of one solution for the problem ($P^{f,g}_{\lambda,\mu}$). For clarity and convenience, let us define

$$G^\theta = \int_0^1 \sup_{|\xi| \le \theta} G(x,\xi)\mathrm{d}x \quad \text{for all } \theta > 0$$

and

$$G_\eta = \inf_{[0,1]\times[0,\eta]} G(x,\xi) \quad \text{for all } \eta > 0.$$

If g is sign-changing, then clearly $G^\theta \ge 0$ and $G_\eta \le 0$.

For our goal, we fix two positive constants θ and η, put

$$\underline{\delta}_{\lambda,g} = \min\left\{ \frac{\min\{m,\zeta_0\}\theta^2 - 8\lambda \int_0^1 \sup_{|t|\le\theta} F(x,t)\mathrm{d}x}{8G^\theta}, \right.$$
$$\left. \frac{\frac{1}{4}h(2\eta) + \frac{1}{4}h(-2\eta) + \zeta_1\eta^2 - \lambda \int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\eta)\mathrm{d}x}{G_\eta} \right\}$$

and

$$\overline{\delta}_{\lambda,g} = \min\left\{ \underline{\delta}_{\lambda,g}, \frac{1}{\max\left\{0, \frac{8}{\min\{m,\zeta_0\}} \limsup_{|t|\to+\infty} \frac{\sup_{x\in[0,1]} G(x,t)}{x^2}\right\}} \right\} \tag{3.1}$$

where we read $\epsilon/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{|t|\to+\infty} \frac{\sup_{x\in[0,1]} G(x,t)}{x^2} \leq 0,$$

and $G_\eta = G^\theta = 0$.

Theorem 3.1. *Assume that there exist two positive constants γ and η with the property*

$$\sqrt{\frac{2h(2\eta)+2h(-2\eta)+8\zeta_1\eta^2}{\min\{m,\zeta_0\}}} < \gamma$$

such that

(A_1) $f(x,t) \geq 0$ *for every* $(x,t) \in [0,1] \times \left[0,\frac{1}{4}\right] \cup \left[\frac{3}{4},1\right]$,

(A_2) $\frac{\int_0^1 \sup_{|t|\leq\gamma} F(x,t)\mathrm{d}x}{\gamma^2} < \min\{m,\zeta_0\} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\eta)\mathrm{d}x}{2h(2\eta)+2h(-2\eta)+8\zeta_1\eta^2}$,

(A_3) $\min_{x\in[0,1]} \limsup_{|\xi|\to\infty} \frac{F(x,\xi)}{|\xi|^2} \in (-\infty,0]$.

Then, for each

$$\lambda \in \Lambda = \left(\frac{h(2\eta)+h(-2\eta)+4\zeta_1\eta^2}{4\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\eta)\mathrm{d}x}, \frac{\min\{m,\zeta_0\}\gamma^2}{8\int_0^1 \sup_{|t|\leq\gamma} F(x,t)\mathrm{d}x} \right)$$

and for every function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\min_{x\in[0,1]} \limsup_{|\xi|\to\infty} \frac{G(x,\xi)}{|\xi|^2} \in (-\infty,0), \tag{3.2}$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (3.1) such that for each $\mu \in [0,\overline{\delta}_{\lambda,g})$, the problem $(P^{f,g}_{\lambda,\mu})$ admits at least one solution u_λ in X such that

$$\max_{x\in[0,1]} |u_\lambda(x)| < \gamma.$$

Proof. Our objective is to apply Theorem 2.1 to the problem $(P^{f,g}_{\lambda,\mu})$. Consider the functionals Φ and Ψ defined in (2.3) and (2.3), respectively. Our task is to demonstrate that these functionals satisfy the necessary conditions outlined in Theorem 2.1. Since $f, g : [0;1] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, we know that Ψ' is a well-defined and Gâteaux differentiable functional with

$$\Psi'(u)(y) = \int_0^1 f(x,u(x))y(x)\mathrm{d}x + \frac{\mu}{\lambda}\int_0^1 g(x,u(x))y(x)\mathrm{d}x$$

for every $u, y \in X$. Since the embeddings $X \hookrightarrow L^q (q \geq 1)$ and $X \hookrightarrow L^\infty$ are compact (Adams [1]), we have $\Psi' : X \to X^*$ is a continuous and compact operator, and Ψ is sequentially weakly upper semicontinuous. Moreover, Again using the Lebesgue's theorem, from the continuity of h' and the arbitrariness of $\{a_n\}$, we know that Φ is Gâteaux differentiable in X with

$$\Phi'(u)(y) = \int_0^1 h'(u'(x))y'(x)\mathrm{d}x + \int_0^1 \zeta(x)u(x)y(x)\mathrm{d}x \tag{3.3}$$

for every $u, y \in X$. Furthermore, by the definition of Φ, we observe that it is sequentially weakly lower semicontinuous and strongly continuous. Combining this observation with (1.1) and (1.2), we have

$$\int_0^1 h'(u'(x))\mathrm{d}x = \int_0^1 \left(\int_0^{u'(x)} \left(\int_0^\tau p(\xi)\mathrm{d}\xi\right)\mathrm{d}\tau\right)\mathrm{d}x$$

and

$$\begin{aligned}\frac{1}{2}\min\{m,\zeta_0\}\|u\|^2 &\le \frac{m}{2}\int_0^1 |u'(x)|^2\mathrm{d}x + \frac{\zeta_0}{2}\int_0^1 |u(x)|^2\mathrm{d}x \le \Phi(u)\\ &\le \frac{M}{2}\int_0^1 |u'(x)|^2\mathrm{d}x + \frac{\zeta_1}{2}\int_0^1 |u(x)|^2\mathrm{d}x \le \frac{1}{2}\max\{M,\zeta_1\}\|u\|^2\end{aligned} \tag{3.4}$$

for every $u \in X$, which implies that Φ is well-defined in X. By using the first inequality in (3.4), it follows

$$\lim_{\|u\|\to+\infty} \Phi(u) = +\infty,$$

namely Φ is coercive. Further, we claim that Φ admits a continuous inverse on X^*. In fact, by (1.1), (1.2), Proposition 2.7 and (3.3), we have

$$\begin{aligned}\langle \Phi'(u) - \Phi'(y), u - y\rangle &= \int_0^1 (h'(u'(x)) - h'(y'(x)), u'(x) - y'(x))\mathrm{d}x\\ &\quad + \int_0^1 \zeta(x)|u(x) - y(x)|^2\mathrm{d}x\\ &\ge \int_0^1 m|u'(x) - y'(x)|^2\mathrm{d}x + \int_0^1 \zeta_0|u(x) - y(x)|^2\mathrm{d}x\\ &\ge \min\{m,\zeta_0\}\|u(x) - y(x)\|^2\end{aligned}$$

for all $u, y \in X$, which shows that Φ' is uniformly monotone in X. Put $y = 0$, then we have

$$\min\{m,\zeta_0\}\|u\|^2 \le \langle \Phi'(u), u\rangle \le \|\Phi'(u)\|_{X^*}.\|u\|$$

$$\Rightarrow \min\{m,\zeta_0\}\|u\| \le \|\Phi'(u)\|_{X^*}$$

which shows that Φ' is coercive in X. Since Φ' is a Lipschitzian operator, it is hemicontinuous in X. According to Theorem 26.A of [28], Φ admits a continuous inverse on X^*. Additionally, the functional Ψ belongs to $C^1(X, \mathbb{R})$ and has a compact derivative. Given that the embedding $X \hookrightarrow L^q$ (where $q \ge 1$) is compact, there exists a positive constant C such that

$$|u|_{L^q([0,1])} \le C\|u\|$$

and it follows that

$$|u|_{L^2([0,1])} \le C_1\|u\|$$

where C_1 is positive constant. Moreover, for $\lambda > 0$, the functional I_λ is coercive. Indeed, since $\mu < \delta_\lambda$ we can fix κ such that

$$\min_{x\in[0,1]} \limsup_{|\xi|\to\infty} \frac{G(x,\xi)}{|\xi|^2} \in (-\infty, \kappa)$$

and $\mu\kappa < \frac{\min\{m,\zeta_0\}}{2C_1^2}$. Therefore, there exists a positive constant ϱ such that

$$G(x,\xi) \le \kappa\xi^2 + \varrho$$

for each $\xi \in \mathbb{R}$ and $x \in [0,1]$. Now, we fix $0 < \varepsilon < \frac{1}{\lambda C_1^2}\left(\frac{\min\{m,\zeta_0\}}{2} - \mu C_1^2\kappa\right)$. From the assumption (A_3) there is a positive constant ρ_ε such that

$$F(x,\xi) \le \varepsilon\xi^2 + \rho_\varepsilon$$

for every $(x,\xi) \in [0,1]\times\mathbb{R}$. It follows that, for each $u \in X$, we have

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &\ge \frac{1}{2}\min\{m,\zeta_0\}\|u\|^2 - \lambda\int_0^1 [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]\mathrm{d}x \\
&\ge \frac{1}{2}\min\{m,\zeta_0\}\|u\|^2 - \lambda\left(\varepsilon\int_0^1 |u(x)|^2\mathrm{d}x + \rho_\varepsilon\right) - \mu\left(\kappa\int_0^1 |u(x)|^2\mathrm{d}x + \varrho\right) \\
&\ge \left(\frac{1}{2}\min\{m,\zeta_0\} - \lambda C_1^2\varepsilon - \mu C_1^2\kappa\right)\|u\|^2 - \lambda\rho_\varepsilon - \mu\varrho
\end{aligned}$$

and thus

$$\lim_{\|u\|\to\infty} (\Phi(u) - \lambda\Psi(u)) = \infty,$$

which means the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive. Thus, by [5, Proposition 2.1] the functional $I_\lambda = \Phi - \lambda\Psi$ verifies $(\mathrm{PS})^{[r]}$-condition for each $r > 0$ and so the condition (a_2) of Theorem 2.1 is verified. Fix $\lambda \in (0,\lambda^*)$, thus

$$\frac{\int_{\frac14}^{\frac34} F(x,\eta)\mathrm{d}x - \frac{\mu}{\lambda}G_\eta}{\frac14 h(2\eta) + \frac14 h(-2\eta) + \zeta_1\eta^2} > \frac{1}{\lambda}.$$

Put $r = \frac{\min\{m,\zeta_0\}}{8}\gamma^2$ and

$$w(x) = \begin{cases} \eta, & x \in \left[0,\frac14\right], \\ 2\eta x + \frac{\eta}{2}, & x \in \left[\frac14,\frac12\right], \\ -2\eta x + \frac{5\eta}{2}, & x \in \left[\frac12,\frac34\right], \\ \eta, & x \in \left[\frac34,1\right]. \end{cases} \tag{3.5}$$

Clearly, $w \in X$. Then, we have $\Phi(0) = \Psi(0) = 0$,

$$\begin{aligned}
\Phi(w) &= \frac14 h(2\eta) + \frac14 h(-2\eta) + \frac12\int_0^1 \zeta(x)|w(x)|^2\mathrm{d}x \\
&\le \frac14 h(2\eta) + \frac14 h(-2\eta) + \frac{\zeta_1}{2}\int_0^1 |w(x)|^2\mathrm{d}x \\
&= \frac14 h(2\eta) + \frac14 h(-2\eta) + \frac{31\zeta_1\eta^2}{2\times 24} \\
&< \frac14 h(2\eta) + \frac14 h(-2\eta) + \zeta_1\eta^2
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\Phi(w) &\geq \frac{1}{4}h(2\eta)+\frac{1}{4}h(-2\eta)+\frac{\zeta_0}{2}\int_0^1|w(x)|^2\mathrm{d}x\\ &=\frac{1}{4}h(2\eta)+\frac{1}{4}h(-2\eta)+\frac{31\zeta_0\eta^2}{2\times 24}\\ &>\frac{1}{4}h(2\eta)+\frac{1}{4}h(-2\eta)+\frac{\zeta_0\eta^2}{8}.\end{aligned}\tag{3.7}$$

From $\min_{x\in[\frac{1}{4},\frac{3}{4}]}\{w(x)\}=\eta$, $\max_{x\in[\frac{1}{4},\frac{3}{4}]}\{w(x)\}=\frac{3\eta}{2}$ and the assumption (A_1), we have

$$\begin{aligned}\Psi(w) =&\int_0^{\frac{1}{4}}\int_0^{\eta}f(x,\xi)\mathrm{d}\xi\mathrm{d}x+\int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^{w}f(x,\xi)\mathrm{d}\xi\mathrm{d}x+\int_{\frac{3}{4}}^{1}\int_0^{\eta}f(x,\xi)\mathrm{d}\xi\mathrm{d}x\\ &+\frac{\mu}{\lambda}\int_0^1\left(\int_0^{w(x)}g(x,\xi)\mathrm{d}\xi\right)\mathrm{d}x\geq\int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^{\eta}f(x,\xi)\mathrm{d}\xi\mathrm{d}x+\frac{\mu}{\lambda}G_\eta\\ =&\int_{\frac{1}{4}}^{\frac{3}{4}}F(x,\eta)\mathrm{d}x+\frac{\mu}{\lambda}G_\eta.\end{aligned}$$

Thus, by the assumption

$$\sqrt{\frac{8}{\min\{m,\zeta_0\}}\left(\frac{1}{4}h(2\eta)+\frac{1}{4}h(-2\eta)+\zeta_1\eta^2\right)}<\gamma,$$

we have $0<\Phi(w)<r$. For $u\in X$, taking into account

$$|u(x)|\leq\left|\int_{t_1}^{t}u'(\tau)\mathrm{d}\tau\right|+|u(x_1)|\leq\int_0^1|u'(\tau)|\mathrm{d}\tau+|u(x_1)|$$

and

$$|u(x)|\leq\int_0^1|u'(\tau)|\mathrm{d}\tau+\int_0^1|u(x_1)|\mathrm{d}x_1\leq\left(\int_0^1|u'(\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}}+\left(\int_0^1|u(\tau)|^2\mathrm{d}\tau\right)^{\frac{1}{2}},$$

we have

$$\max_{x\in[0,1]}|u(x)|\leq 2\|u\|.\tag{3.8}$$

From the definition of Φ and in view of (3.4) for every $r>0$, one has

$$\begin{aligned}\Phi^{-1}(-\infty,r]=&\ \{u\in X,\ \Phi(x)\leq r\}\\ &\subseteq\left\{u\in X,\ \max_{x\in[0,1]}|u(x)|\leq\sqrt{\frac{8r}{\min\{m,\zeta_0\}}}\right\}\\ &\subseteq\left\{u\in X,\ \max_{x\in[0,1]}|u(x)|\leq\gamma\right\}.\end{aligned}$$

Hence, we have

$$\sup_{\Phi(u)<r}\Psi(u)=\sup_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)\leq\int_0^1\sup_{|t|\leq\gamma}F(x,t)\mathrm{d}x+\frac{\mu}{\lambda}G^{\gamma}.$$

Therefore, we have

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u)}{r}=\frac{\sup_{u\in\Phi^{-1}(-\infty,r]}\left(\int_0^1 F(x,u(x))\mathrm{d}x+\frac{\mu}{\lambda}\int_0^1 G(x,u(x))\mathrm{d}x\right)}{r}$$
$$\leq\frac{\int_0^1\sup_{|t|\leq\gamma}F(x,t)\mathrm{d}x+\frac{\mu}{\lambda}G^{\gamma}}{\frac{\min\{m,\zeta_0\}}{8}\gamma^2} \tag{3.9}$$

and

$$\frac{\Psi(w)}{\Phi(w)}\geq\frac{\int_{\frac14}^{\frac34}F(x,\eta)\mathrm{d}x+\frac{\mu}{\lambda}\int_0^1 G(x,\eta)\mathrm{d}x}{\frac14 h(2\eta)+\frac14 h(-2\eta)+\zeta_1\eta^2} \tag{3.10}$$
$$\geq\frac{\int_{\frac14}^{\frac34}F(x,\eta)\mathrm{d}x+\frac{\mu}{\lambda}G_{\eta}}{\frac14 h(2\eta)+\frac14 h(-2\eta)+\zeta_1\eta^2}.$$

Since

$$\mu<\frac{\min\{m,\zeta_0\}\gamma^2-8\lambda\int_0^1\sup_{|t|\leq\gamma}F(x,t)\mathrm{d}x}{8G^{\gamma}},$$

we have

$$8\frac{\int_0^1\sup_{|t|\leq\gamma}F(x,t)\mathrm{d}x+\frac{\mu}{\lambda}G^{\gamma}}{\min\{m,\zeta_0\}\gamma^2}<\frac{1}{\lambda}.$$

Furthermore,

$$\mu<\frac{\frac14 h(2\eta)+\frac14 h(-2\eta)+\zeta_1\eta^2-\lambda\int_{\frac14}^{\frac34}F(x,\eta)\mathrm{d}x}{G_{\eta}},$$

this means

$$\frac{\int_{\frac14}^{\frac34}F(x,\eta)\mathrm{d}x+\frac{\mu}{\lambda}G_{\eta}}{\frac14 h(2\eta)+\frac14 h(-2\eta)+\zeta_1\eta^2}>\frac{1}{\lambda}.$$

Then,

$$\frac{8}{\min\{m,\zeta_0\}}\frac{\int_0^1\sup_{|t|\leq\gamma}F(x,t)\mathrm{d}x+\frac{\mu}{\lambda}G^{\gamma}}{\gamma^2}<\frac{1}{\lambda}<\frac{\int_{\frac14}^{\frac34}F(x,\eta)\mathrm{d}x+\frac{\mu}{\lambda}G_{\eta}}{\frac14 h(2\eta)+\frac14 h(-2\eta)+\zeta_1\eta^2}. \tag{3.11}$$

Hence, from (3.9)–(3.11), the condition (a_1) of Applying Theorem 2.1 with $\bar{u}=w$ ensures the existence of a local minimum point u_λ for the functional I_λ such that $0<\Phi(u_\lambda)<r$. Thus, u_λ serves as a nontrivial solution to the problem ($P^{f,g}_{\lambda,\mu}$), satisfying

$$\max_{x\in[0,1]}|u_\lambda(x)|<\gamma.$$

□

Now, we illustrate Theorem 3.1 through the following example.

Example 3.2. We consider the following problem

$$\begin{cases}-p(u')u''+u=\lambda f(x,u(x))+\mu g(x,u(x)), & \text{a.e. } x\in[0,1],\\ u(1)-u(0)=u'(1)-u'(0)=0\end{cases} \tag{3.12}$$

where $p(t) = 3 - 2\cos(t)$ for every $t \in \mathbb{R}$, $\zeta(x) = 1$ for each $x \in [0,1]$ and

$$f(x,t) = \begin{cases} \frac{4}{10}e^{-x}t^3, & \text{for every } (x,t) \in [0,1] \times (-\infty, 1), \\ \frac{4}{10t}e^{-x}, & \text{for every } (x,t) \in [0,1] \times [1, +\infty). \end{cases}$$

By the expression of f, we have

$$F(x,t) = \begin{cases} \frac{1}{10}e^{-x}t^4, & \text{for every } (x,t) \in [0,1] \times (-\infty, 1), \\ e^{-x}\left(\frac{4}{10}\ln(t) + \frac{1}{10}\right), & \text{for every } (x,t) \in [0,1] \times [1, +\infty). \end{cases}$$

Hence, $\lim_{|\xi|\to\infty} \frac{F(x,\xi)}{|\xi|^2} = 0$, thus (A_3) holds. Choose $\gamma = 10^{-2}$, and $\eta = 1$. By simple calculations, we obtain $m = 1$, $M = 5$ and $\zeta_0 = \zeta_1 = 1$. Since

$$\frac{\int_0^1 \sup_{|t|\leq\gamma} F(x,t)\mathrm{d}x}{\gamma^2} = \frac{e-1}{10^9 e} < \frac{e^{0.75} - e^{0.25}}{(160 + 80\cos(2))e} = \frac{\min\{m, \zeta_0\} \int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\eta)\mathrm{d}x}{2h(2\eta) + 2h(-2\eta) + 8\zeta_1\eta^2},$$

thus (A_2) holds true, then all conditions in Theorem 3.1 are satisfied. Therefore, it follows that for each

$$\lambda \in \left(\frac{(80 + 40\cos(2))e}{4(e^{0.75} - e^{0.25})}, \frac{10^9 e}{8(e-1)} \right)$$

and for every function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\min_{x\in[0,1]} \limsup_{|\xi|\to\infty} \frac{G(x,\xi)}{|\xi|^2} \in (-\infty, 0),$$

there exists $\overline{\delta}_{\lambda,g} > 0$ such that for each $\mu \in [0, \overline{\delta}_{\lambda,g})$, the problem (3.12) admits at least one solution u_λ in X such that

$$\max_{x\in[0,1]} |u_\lambda(x)| < 10^{-2}.$$

4 Existence of two solutions

In this section, our objective is to establish the existence of two distinct solutions for the problem $(P^{f,g}_{\lambda,\mu})$. The following result is derived by applying Theorem 2.2, without the need for assumption (A_3).

Theorem 4.1. *Assume that there exist two positive constants γ and η with the property*

$$\sqrt{\frac{2h(2\eta) + 2h(-2\eta) + 8\zeta_1\eta^2}{\min\{m, \zeta_0\}}} < \gamma$$

and

(A_4) *there exist $\nu > 2$ and $R > 0$ such that*

$$0 < \nu F(x,\xi) \leq \xi f(x,\xi) \tag{4.1}$$

for all $|\xi| \geq R$ and for all $x \in [0,1]$.

Then, for each

$$\lambda \in \left(0, \frac{\min\{m, \zeta_0\}\gamma^2}{8\int_0^1 \sup_{|t|\le\gamma} F(x,t)\mathrm{d}x}\right),$$

and for every function $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ *satisfying the condition* (A_4)*, there exists* $\delta_\lambda > 0$*, for each* $\mu \in [0, \delta_\lambda[$*, the problem* $(P^{f,g}_{\lambda,\mu})$ *admits at least two solutions* u_1 *and* u_2 *in* X *such that*

$$\max_{x\in[0,1]} |u_1(x)| < \gamma.$$

Proof. Our aim is to apply Theorem 2.2 to the space X with the norm is defined in (2.2) and to the functionals Φ and Ψ defined in the proof of Theorem 3.1. The functional I_λ satisfies the (PS)-condition. Indeed, assume that $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{I_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. Then, there exists a positive constant c_0 such that

$$|I_\lambda(u_n)| \le c_0,\ |I'_\lambda(u_n)| \le c_0 \qquad \forall n \in \mathbb{N}.$$

Therefore, we deduce from the definition of I'_λ and assumption (A_3) that

$$\begin{aligned} c_0 + c_1\|u_n\| \ge \nu I_\lambda(u_n) - I'_\lambda(u_n)(u_n) &\ge \min\{m,\zeta_0\}\left(\frac{\nu}{2}-1\right)\|u_n\|^2 \\ &\quad - \lambda\int_0^1 (\nu F(x,u_n(x)) - f(x,u_n(x))(u_n(x)))\,\mathrm{d}x \\ &\quad - \mu\int_0^1 (\nu G(x,u_n(x)) - g(x,u_n(x))(u_n(x)))\,\mathrm{d}x \\ &\ge \min\{m,\zeta_0\}\left(\frac{\nu}{2}-1\right)\|u_n\|^2 \end{aligned}$$

for some $c_1 > 0$. Since $\nu > 2$, this implies that (u_n) is bounded. Consequently, since X is a reflexive Banach space we have, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in } X.$$

By $I'_\lambda(u_n) \to 0$ and $u_n \rightharpoonup u$ in X, we obtain

$$\left(I'_\lambda(u_n) - I'_\lambda(u)\right)(u_n - u) \to 0. \tag{4.2}$$

From the continuity of f and g, we have

$$\int_0^1 (f(x,u_n(x)) - f(x,u(x)))\,(u_n(x)-u(x))\mathrm{d}x \to 0, \quad \text{as } n\to\infty$$

and

$$\int_0^1 (g(x,u_n(x)) - g(x,u(x)))\,(u_n(x)-u(x))\mathrm{d}x \to 0, \quad \text{as } n\to\infty.$$

Moreover, an easy computation shows

$$\begin{aligned}\left(I'_\lambda(u_n) - I'_\lambda(u)\right)(u_n - u) = &\int_0^1 h'(u'_n(x) - u'(x))(u'_n(x) - u'(x))\mathrm{d}x \\ &+ \int_0^1 \zeta(x)(u_n(x) - u(x))(u_n(x) - u(x))\mathrm{d}x \\ &- \lambda \int_0^1 \left(f(x, u_n(x)) - f(x, u(x))\right)(u_n(x) - u(x))\mathrm{d}x \\ &- \mu \int_0^1 \left(g(x, u_n(x) - g(x, u(x))\right)(u_n(x) - u(x))\mathrm{d}x \\ \geq\; &\min\{m, \zeta_0\}\|u_n - u\|^2.\end{aligned}$$

Thus, the sequence u_n converges strongly to u in X. Therefore, I_λ satisfies the (PS)-condition. Moreover, by integrating the condition (4.1), there exist constants $a_1, a_2, a_3, a_4 > 0$ such that that

$$F(x,t) \geq a_1|t|^\nu - a_2 \quad \text{and} \quad G(x,t) \geq a_3|t|^\nu - a_4$$

for all $x \in [0,1]$ and $t \in \mathbb{R}$. Moreover, for any $u \in X$, one has

$$\frac{1}{2}\min\{m, \zeta_0\}\|u\|^2 \leq \Phi(u) \leq \frac{1}{2}\max\{M, \zeta_1\}\|u\|^2. \tag{4.3}$$

Now, choosing any $u \in X \setminus \{0\}$, for each $\tau > 0$ taking (4.3) into account one has

$$\begin{aligned}I_\lambda(\tau u) &= (\Phi + \lambda\Psi)(\tau u) \\ &\leq \frac{\max\{M, \zeta_1\}}{2}\|\tau u\|^2 - \lambda\int_0^1 F(x, \tau u(x))\mathrm{d}x - \mu\int_0^1 G(x, \tau u(x))\mathrm{d}x \\ &\leq \frac{\max\{M, \zeta_1\}\tau^2}{2}\|u\|^2 - \lambda\tau^\nu a_1\int_0^1 |u(x)|^\nu \mathrm{d}x + \mu\tau^\nu a_3\int_0^1 |u(x)|^\nu \mathrm{d}x - \lambda a_2 - \mu a_4.\end{aligned}$$

Since $\nu > 2$, this condition guarantees that I_λ is unbounded from below. Thus, all hypotheses of Theorem 2.2 are satisfied. Therefore, for each

$$\lambda \in \left(0, \frac{\min\{m, \zeta_0\}\gamma^2}{8\int_0^1 \sup_{|t| \leq \gamma} F(x,t)\mathrm{d}x}\right),$$

the functional I_λ admits two distinct critical points that are solutions of the problem $(P^{f,g}_{\lambda,\mu})$. □

Remark 4.2. Theorem 1.1 immediately follows from Theorem 4.1.

Remark 4.3. In Theorem 2.1, it is observed that if either $f(x,0) \neq 0$ for some $x \in [0,1]$ or $g(x,0) \neq 0$ for some $x \in [0,1]$, or both conditions hold true, then Theorem 4.1 guarantees the existence of two nontrivial solutions for the problem $(P^{f,g}_{\lambda,\mu})$. However, if the condition $f(x,0) \neq 0$ for some $x \in [0,1]$ and $g(x,0) \neq 0$ for some $x \in [0,1]$ does not hold, the second solution u_2 of the problem $(P^{f,g}_{\lambda,\mu})$ may be trivial, but the problem still has at least one nontrivial solution.

Remark 4.4. Using similar arguments as those provided in the proof of [7, Theorem 3.5], the non-triviality of the second solution guaranteed by Theorem 4.1 can also be achieved in the case where $f(x,0) = 0$ for all $x \in [0,1]$, provided that an extra condition at zero is imposed.

Specifically, this condition entails the existence of a non-empty open set $D \subseteq [0,1]$ and a set $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{\xi \to 0^+} \frac{\operatorname{ess\,inf}_{x\in B} F(x,\xi)}{|\xi|^2} = \infty \quad \text{and} \quad \liminf_{\xi \to 0^+} \frac{\operatorname{ess\,inf}_{x\in D} F(x,\xi)}{|\xi|^2} > -\infty.$$

See [18, Theorem 3.1] for more details.

5 Another multiplicity result for the case $\mu = 0$

In this section, we focus on establishing the existence of at least two and three solutions for the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$. To achieve this, we define

$$F^c = \int_0^1 \sup_{|t|\le c} F(x,t)\mathrm{d}x \quad \text{and} \quad F_c = \inf_{x\in[0,1]} F(x,c)$$

for every $c > 0$.

Theorem 5.1. *Assume that there exist two positive constants $\bar\gamma$ and $\bar\eta$ such that*

$$\sqrt{\frac{2h(2\bar\eta) + 2h(-2\bar\eta) + 8\zeta_1\bar\eta^2}{\min\{m,\zeta_0\}}} < \bar\gamma \tag{5.1}$$

and suppose that the assumptions (A_1) and (A_3) in Theorem 3.1 hold. Moreover, assume that

(A_5) $\frac{F^{\bar\gamma}}{\min\{m,\zeta_0\}\bar\gamma^2} < \frac{\frac{1}{2}F_{\bar\eta} - F^{\bar\gamma}}{2h(2\bar\eta)+2h(-2\bar\eta)+8\zeta_1\bar\eta^2}$.

Then, for each

$$\lambda \in \left(\frac{h(2\bar\eta) + h(-2\bar\eta) + 4\zeta_1\bar\eta^2}{2F_{\bar\eta} - 4F^{\bar\gamma}}, \frac{\min\{m,\zeta_0\}\bar\gamma^2}{8F^{\bar\gamma}} \right),$$

the problem ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$ admits at least three solutions in X.

Proof. Put $I_\lambda = \Phi + \lambda\Psi$, where

$$\Phi(u) = \int_0^1 h(u'(x))\mathrm{d}x + \frac{1}{2}\int_0^1 \zeta(x)|u(x)|^2\mathrm{d}x \tag{5.2}$$

and

$$\Psi(u) = -\int_0^1 F(x,u(x))\mathrm{d}x$$

for all $u \in X$. Standard arguments demonstrate that Φ and Ψ are Gâteaux differentiable functionals, and their Gâteaux derivatives at the point $u \in X$ are given by

$$\Phi'(u)(v) = \int_0^1 h'(u'(x))v'(x)\mathrm{d}x + \int_0^1 \zeta(x)u(x)v(x)\mathrm{d}x$$

and

$$\Psi'(u)(v) = -\int_0^1 f(x,u(x))v(x)\mathrm{d}x$$

for all $u, v \in X$, respectively. Hence, a critical point of the functional $\Phi + \lambda\Psi$, gives us a solution of ($P^{f,g}_{\lambda,\mu}$) in the case $\mu = 0$. Our goal is to apply Theorem 2.3 to Φ and Ψ. By sequentially weakly lower semicontinuity of the norm and continuity of h, the functional Φ is sequentially weakly lower semicontinuous. Moreover, from Section 3, Φ is continuously Gâteaux differentiable while Proposition 2.7 gives that its Gâteaux derivative admits a continuous inverse on X^*. The functional $\Psi : X \to \mathbb{R}$ is well defined and is continuously Gâteaux differentiable whose Gâteaux derivative is compact. Then, it is enough to show that Φ and Ψ satisfy (c_1) and (c_2) in Theorem 2.3. Now, we fix $0 < \epsilon < \frac{\min\{m,\zeta_0\}}{2\lambda C_1^2}$. From the assumption (A_3) there is a function $\rho_\varepsilon : [0,1] \to \mathbb{R}$ with $\rho_\epsilon(x) < \infty$ for all $x \in [0,1]$ such that

$$F(x,t) \le \varepsilon t^2 + \rho(x)$$

for every $(x,t) \in [0,1] \times \mathbb{R}$. It follows that for each $u \in X$,

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &\ge \frac{\min\{m,\zeta_0\}}{2}\|u\|^2 - \lambda\int_0^1 F(x,u(x))\mathrm{d}x \\ &\ge \frac{\min\{m,\zeta_0\}}{2}\|u\|^2 - \lambda\epsilon\int_0^1 u^2(x)\mathrm{d}x - \lambda\int_0^1 \rho(x)\mathrm{d}x \\ &\ge \left(\frac{\min\{m,\zeta_0\}}{2} - \lambda C_1^2\epsilon\right)\|u\|^2\mathrm{d}x - \lambda\int_0^1 \rho(x)\mathrm{d}x\end{aligned}$$

and thus

$$\lim_{\|u\|\to\infty}(\Phi(u) + \lambda\Psi(u)) = \infty,$$

which means the functional $I_\lambda = \Phi + \lambda\Psi$ is coercive. Now it remains to show that (c_2) of Theorem 2.3 is fulfilled. Let $\bar{r} = \frac{\min\{m,\zeta_0\}}{8}\bar{\gamma}^2$ and

$$w(x) = \begin{cases} \bar{\eta}, & x \in \left[0, \frac{1}{4}\right], \\ 2\bar{\eta}x + \frac{\bar{\eta}}{2}, & x \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ -2\bar{\eta}x + \frac{5\bar{\eta}}{2}, & x \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ \bar{\eta}, & x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$

Clearly, $w \in X$. Then, we have $\Phi(0) = \Psi(0) = 0$,

$$\Phi(w) < \frac{1}{4}h(2\bar{\eta}) + \frac{1}{4}h(-2\bar{\eta}) + \zeta_1\bar{\eta}^2$$

and

$$\Phi(w) > \frac{1}{4}h(2\bar{\eta}) + \frac{1}{4}h(-2\bar{\eta}) + \frac{\zeta_0\bar{\eta}^2}{8}.$$

Thus by (5.1), $\Phi(w) > \bar{r}$. Moreover

$$\Psi(w) = -\int_0^1 F(x,w(x))\mathrm{d}x \le -\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\bar{\eta})\mathrm{d}x \le -\frac{1}{2}F_{\bar{\eta}}.$$

Taking (3.8) into account, for every $u \in X$ such that $\Phi(u) < \bar{r}$, we have

$$\sup_{x\in[0,1]} |u(x)| \le \bar{\gamma}.$$

Thus,

$$\sup_{\Phi(u)<\bar{r}} \Psi(u) = \sup_{u\in\Phi^{-1}(-\infty,\bar{r})} \int_0^1 F(x,u(x))\mathrm{d}x \leq \int_0^1 \sup_{|t|\leq\bar{\gamma}} F(x,t)\mathrm{d}x = F^{\bar{\gamma}}. \tag{5.3}$$

By simple calculations and from the definition of $\varphi(\bar{r})$, since $\Phi(0) = \Psi(0) = 0$ and $\overline{\Phi^{-1}(-\infty,\bar{r})}^{w} = \Phi^{-1}(-\infty,\bar{r})$, one has

$$\begin{aligned}\varphi_1(\bar{r}) &= \inf_{u\in\Phi^{-1}(]-\infty,\bar{r}[)} \frac{\Psi(u) - \inf_{\overline{\Phi^{-1}(-\infty,\bar{r})}^{w}} \Psi}{\bar{r}-\Phi(u)} \leq \frac{-\inf_{\overline{\Phi^{-1}(-\infty,\bar{r})}^{w}} \Psi}{\bar{r}} \\ &\leq \frac{8}{\min\{m,\zeta_0\}} \frac{\int_0^1 \sup_{|t|\leq\bar{\gamma}} F(x,t)\mathrm{d}x}{\bar{\gamma}^2} = \frac{8F^{\bar{\gamma}}}{\min\{m,\zeta_0\}\bar{\gamma}^2}.\end{aligned}$$

On the other hand, by (5.3), one has

$$\begin{aligned}\varphi_2(\bar{r}) &= \inf_{u\in\Phi^{-1}(-\infty,\bar{r})} \sup_{v\in\Phi^{-1}[\bar{r},\infty)} \frac{\Psi(u)-\Psi(v)}{\Phi(u)-\Phi(v)} \geq \inf_{u\in\Phi^{-1}(-\infty,\bar{r})} \frac{\Psi(u)-\Psi(w)}{\Phi(w)-\Phi(u)} \\ &\geq \frac{\inf_{u\in\Phi^{-1}(-\infty,\bar{r})} \Psi(u)-\Psi(w)}{\Phi(w)-\Phi(u)} \\ &\geq \frac{-\int_0^1 \sup_{|t|\leq\bar{\gamma}} F(x,t)\mathrm{d}x + \int_{\frac{1}{4}}^{\frac{3}{4}} F(x,\bar{\eta})\mathrm{d}x}{\Phi(w)-\Phi(u)} \\ &\geq \frac{2F_{\bar{\eta}} - 4F^{\bar{\gamma}}}{h(2\bar{\eta})+h(-2\bar{\eta})+4\zeta_1\bar{\eta}^2}.\end{aligned}$$

Hence, from (A_5), one has

$$\varphi_1(\bar{r}) < \varphi_2(\bar{r}).$$

Therefore, from Theorem 2.3, taking also into account that

$$\frac{1}{\varphi_2(\bar{r})} \leq \frac{h(2\bar{\eta})+h(-2\bar{\eta})+4\zeta_1\bar{\eta}^2}{2F_{\bar{\eta}}-4F^{\bar{\gamma}}}$$

and

$$\frac{1}{\varphi_1(\bar{r})} \geq \frac{\min\{m,\zeta_0\}\bar{\gamma}^2}{8F^{\bar{\gamma}}},$$

we obtain the desired conclusion. □

Remark 5.2. When the assumption (A_5) of Theorem 5.1 holds, simple calculations show that the condition

(A_6) $\frac{F^{\bar{\gamma}}}{\min\{m,\zeta_0\}\bar{\gamma}^2} < \frac{F_{\bar{\eta}}}{4h(2\bar{\eta})+4h(-2\bar{\eta})+16\zeta_1\bar{\eta}^2}$

implies (A_5) of Theorem 5.1. Hence, if the assumptions (5.1) and (A_6) hold, then for each

$$\lambda \in \left(\frac{h(2\bar{\eta})+h(-2\bar{\eta})+4\zeta_1\bar{\eta}^2}{2F_{\bar{\eta}}}, \frac{\min\{m,\zeta_0\}\bar{\gamma}^2}{8F^{\bar{\gamma}}} \right),$$

the problem $(P^{f,g}_{\lambda,\mu})$ in the case $\mu = 0$ admits at least three solutions.

Now, we present an application of Theorem 2.4, which will be utilized later to derive multiple solutions for the problem $(P^{f,g}_{\lambda,\mu})$ in the case $\mu = 0$, without the need for assumption (A_3).

Theorem 5.3. *Assume that there exist three positive constants $\bar{\gamma}_1$, $\bar{\eta}$ and $\bar{\gamma}_2$ with*

$$\bar{\gamma}_1 < \sqrt{\frac{8}{\min\{m,\zeta_0\}}\left(\frac{1}{4}g(2\bar{\eta}) + \frac{1}{4}g(-2\bar{\eta}) + \frac{\zeta_0\bar{\eta}^2}{8}\right)} \tag{5.4}$$

and

$$\sqrt{\frac{8}{\min\{m,\zeta_0\}}\left(\frac{1}{4}g(2\bar{\eta}) + \frac{1}{4}g(-2\bar{\eta}) + \zeta_1\bar{\eta}^2\right)} < \bar{\gamma}_2 \tag{5.5}$$

such that the assumption (A_5) in Theorem 5.1 holds and

(A_7) $\frac{1}{\min\{m,\zeta_0\}} \max\left\{\frac{F^{\bar{\gamma}_1}}{\bar{\gamma}_1^2}, \frac{F^{\bar{\gamma}_2}}{\bar{\gamma}_2^2}\right\} < \frac{F_{\bar{\eta}}}{4h(2\bar{\eta})+4h(-2\bar{\eta})+16\zeta_1\bar{\eta}^2}$.

Then, for each

$$\lambda \in \Lambda = \left(\frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{2F_{\bar{\eta}}}, \min\left\{\frac{\min\{m,\zeta_0\}\bar{\gamma}_1^2}{8F^{\bar{\gamma}_1}}, \frac{\min\{m,\zeta_0\}\bar{\gamma}_2^2}{8F^{\bar{\gamma}_2}}\right\}\right),$$

the problem $(P^{f,g}_{\lambda,\mu})$ in the case $\mu = 0$ admits at least two solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\max_{x\in[0,1]} |u_{1,\lambda}(x)| < \bar{\gamma}_1$ and $\max_{x\in[0,1]} |u_{2,\lambda}(x)| < \bar{\gamma}_2$.

Proof. Put

$$\overline{f}(x,t) = \begin{cases} f(x,-\bar{\gamma}_2), & \text{if } (x,t) \in [0,1]\times(-\infty,\bar{\gamma}_2), \\ f(x,t), & \text{if } (x,t) \in [0,1]\times[-\bar{\gamma}_2,\bar{\gamma}_2], \\ f(x,\bar{\gamma}_2), & \text{if } (x,t) \in [0,1]\times(\bar{\gamma}_2,\infty). \end{cases}$$

Clearly, $\overline{f} : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Now put $\overline{F}(x,\xi) = \int_0^\xi \overline{f}(x,t)\mathrm{d}x$ for all $(x,\xi) \in [0,1]\times\mathbb{R}$ and take X and Φ as (2.1) and (5.2), respectively, and

$$\Psi(u) = -\int_0^1 \overline{F}(x,u(x))\mathrm{d}x$$

for all $u \in X$. Our goal is to apply Theorem 2.4 to Φ and Ψ. It is well known that $\lim_{\|u\|\to\infty}\Phi(u) = \infty$ and Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = -\int_0^1 \overline{f}(x,u(x))v(x)\mathrm{d}x$$

for any $v \in X$ as well as it is sequentially weakly lower semicontinuous. Furthermore $\Psi' : X \to X^*$ is a compact operator. Thus, it is enough to show that Φ and Ψ satisfy the conditions (c_1), (c_2) and (c_3) in Theorem 2.4. Let

$$\bar{r}_1 = \frac{\min\{m,\zeta_0\}}{8}\bar{\gamma}_1^2, \qquad \bar{r}_2 = \frac{\min\{m,\zeta_0\}}{8}\bar{\gamma}_2^2$$

and w as in the proof of Theorem 5.1. Due to the assumptions (3.4), (5.4) and (5.5) we have $\bar{r}_1 < \Phi(w) < \bar{r}_2$ and $\inf_X \Phi < \bar{r}_1 < \bar{r}_2$. Moreover, arguing as in the proof of Theorem 5.1 and taking also into account Remark 5.2 we obtain

$$\varphi_1(\bar{r}_1) \le \frac{8}{\min\{m,\zeta_0\}}\frac{\int_0^1 \sup_{|t|\le\bar{\gamma}_1} F(x,t)\mathrm{d}x}{\bar{\gamma}_1^2} = \frac{8F^{\bar{\gamma}_1}}{\min\{m,\zeta_0\}\bar{\gamma}_1^2},$$

$$\varphi_1(\bar{r}_2) \leq \frac{8}{\min\{m, \zeta_0\}} \frac{\int_0^1 \sup_{|t| \leq \bar{\gamma}_2} F(x,t) \mathrm{d}x}{\bar{\gamma}_2^2} = \frac{8F^{\bar{\gamma}_2}}{\min\{m, \zeta_0\}\bar{\gamma}_2^2}$$

and

$$\varphi_2^*(\bar{r}_1, \bar{r}_2) \geq \frac{2F_{\bar{\eta}}}{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1 \bar{\eta}^2}.$$

Hence, from (A_7), the conditions (c_2) and (c_3) of Theorem 2.4 hold. Therefore, from Theorem 2.4 we obtain that, for each $\lambda \in \Lambda$, the problem

$$\begin{cases} -p(u')u'' + \zeta(x)u = \lambda \overline{f}(x, u(x)), & \text{a.e. } x \in [0,1], \\ u(1) - u(0) = u'(1) - u'(0) = 0 \end{cases}$$

admits at least two solutions $u_{1,\lambda}$ and $u_{2,\lambda}$ such that $\max_{x\in[0,1]} |u_{1,\lambda}(x)| < \bar{\gamma}_1$ and $\max_{x\in[0,1]} |u_{2,\lambda}(x)| < \bar{\gamma}_2$. Observing that these solutions are also solutions for the problem $(P^{f,g}_{\lambda,\mu})$ in the case $\mu = 0$, the conclusion follows. □

Now, we provide some remarks on our results in this section.

Remark 5.4. In Theorems 5.1 and 5.3, we investigated the critical points of the functional I_λ naturally associated with the problem $(P^{f,g}_{\lambda,\mu})$ in the case $\mu = 0$. It is worth noting that, in general, I_λ can be unbounded from below in X. For instance, when $f(t) = 1 + |t|^{\vartheta-2}t$ for all $t \in \mathbb{R}$ with $\vartheta > 2$, for any fixed $u \in X \backslash 0$ and $\iota \in \mathbb{R}$, we obtain

$$\begin{aligned} I_\lambda(\iota u) &\leq \frac{\max\{m, \zeta_1\}}{2} \|\iota u\|^2 - \lambda \int_0^1 F(x, \iota u(x)) \mathrm{d}x \\ &\leq \frac{\max\{M, \zeta_1\}\iota^2}{2} \|u\|^2 - \lambda \iota C_4 \|u\| - \lambda C_5 \frac{\iota^\vartheta}{\vartheta} \|u\|^\vartheta \to -\infty \end{aligned}$$

where C_4 and C_5 are positive constants, as $\iota \to \infty$. Hence, we can not use direct minimization to find critical points of the functional I_λ.

Remark 5.5. We observe that if f is non-negative, Theorem 5.3 represents a bifurcation result, indicating that the pair $(0, 0)$ belongs to the closure of the set

$$\left\{(u_\lambda, \lambda) \in X \times (0, \infty) : u_\lambda \text{ is a non-trivial solution of } (P^{f,g}_{\lambda,\mu}),\ \ \mu = 0\right\} \subset X \times \mathbb{R}.$$

Indeed, if λ goes to zero, by Theorem 5.3 we have that $\bar{\gamma}_i \to 0$, $i = 1, 2$ and since $\max_{x\in[0,1]} |u_{i,\lambda}(x)| < \bar{\gamma}_i$, $i = 1, 2$, there exist two sequences $\{u_j\}$ in X and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here $u_j = u_{\lambda_j}$) such that

$$\lambda_j \to 0^+ \quad \text{and} \quad \|u_j\| \to 0,$$

as $j \to \infty$. Moreover, since f is nonnegative, $\Psi(u) < 0$ for all $u \in \mathbb{R}$ and thus

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing. Hence, for every $\lambda_1, \lambda_2 \in (0, \lambda^*)$, with $\lambda_1 \neq \lambda_2$, solutions u_{λ_1} and u_{λ_2} ensured by Theorem 2.4 are different.

Remark 5.6. If f is non-negative, then the solutions guaranteed by Theorems 5.1 and 5.3 are also non-negative.

Now, we highlight some results where the function f has separated variables. Specifically, consider the following problem

$$\begin{cases} -p(u')u'' + \zeta(x)u = \lambda\theta(x)f(u(x)), & \text{a.e. } x \in [0,1], \\ u(1) - u(0) = u'(1) - u'(0) = 0 \end{cases} \qquad (P_\lambda^{f,\theta})$$

where $\theta : [0,1] \to \mathbb{R}$ is a non-negative and non-zero function such that $\theta(x) < \infty$ for all $x \in [0,1]$ and $f : \mathbb{R} \to \mathbb{R}$ is a non-negative and continuous function. Put

$$F(\xi) = \int_0^\xi f(s)ds$$

for all $\xi \in \mathbb{R}$.

The following existence results are consequences of Theorems 5.1 and 5.3, respectively, by setting $f(x,t) = \theta(x)f(x)$ for every $(x,t) \in [0,1] \times \mathbb{R}$.

Theorem 5.7. *Assume that there exist two positive constants $\bar{\gamma}$ and $\bar{\eta}$, with*

$$\sqrt{\frac{2h(2\bar{\eta}) + 2h(-2\bar{\eta}) + 8\zeta_1\bar{\eta}^2}{\min\{m,\zeta_0\}}} < \bar{\gamma}$$

such that

(A_8) $\theta(x) \geq 0$ *for every* $x \in [0,1]$ *and* $f(t) \geq 0$ *for every* $t \in [0,\frac{1}{4}] \bigcup [\frac{3}{4},1]$,

(A_9) $\frac{1}{\min\{m,\zeta_0\}} \frac{\int_0^1 \theta(x)\mathrm{d}xF(\bar{\gamma})}{\bar{\gamma}^2} < \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x)\mathrm{d}xF(\bar{\eta})}{2h(2\bar{\eta})+2h(-2\bar{\eta})+8\zeta_1\bar{\eta}^2}$,

(A_{10}) $\limsup_{|\xi|\to\infty} \frac{F(\xi)}{|\xi|^2} \in (-\infty, 0]$.

Then, for each

$$\lambda \in \left(\frac{1}{4\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x)\mathrm{d}x} \frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1\bar{\eta}^2}{F(\bar{\eta})}, \frac{\min\{m,\zeta_0\}}{8\int_0^1 \theta(x)\mathrm{d}x} \frac{\bar{\gamma}^2}{F(\bar{\gamma})} \right),$$

the problem $(P_\lambda^{f,\theta})$ *admits at least three solutions in X.*

Theorem 5.8. *Assume that there exist three positive constants $\bar{\gamma}_1$, $\bar{\eta}$ and $\bar{\gamma}_2$ with*

$$\bar{\gamma}_1 < \sqrt{\frac{8}{\min\{m,\zeta_0\}} \left(\frac{1}{4}g(2\bar{\eta}) + \frac{1}{4}g(-2\bar{\eta}) + \frac{\zeta_0\bar{\eta}^2}{8} \right)}$$

and

$$\sqrt{\frac{8}{\min\{m,\zeta_0\}} \left(\frac{1}{4}g(2\bar{\eta}) + \frac{1}{4}g(-2\bar{\eta}) + \zeta_1\eta^2 \right)} < \bar{\gamma}_2$$

such that

(A_{11}) $\theta(x) \geq 0$ *for every* $x \in [0,1]$ *and* $f(t) \geq 0$ *for every* $t \in [0,\frac{1}{4}] \cup [\frac{3}{4},1]$,

(A_{12}) $\frac{\int_0^1 \theta(x)\mathrm{d}x}{\min\{m,\zeta_0\}} \max\left\{ \frac{F(\bar{\gamma}_1)}{\bar{\gamma}_1^2}, \frac{F(\bar{\gamma}_2)}{\bar{\gamma}_2^2} \right\} < \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x)\mathrm{d}xF(\bar{\eta})}{2h(2\bar{\eta})+2h(-2\bar{\eta})+8\zeta_1\bar{\eta}^2}$.

Then, for each

$$\lambda \in \Lambda = \left(\frac{1}{4\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x) \mathrm{d}x} \frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1 \bar{\eta}^2}{F(\bar{\eta})}, \frac{\min\{m, \zeta_0\}}{8\int_0^1 \theta(x) \mathrm{d}x} \min\left\{ \frac{\bar{\gamma}_1^2}{F(\bar{\gamma}_1)}, \frac{\bar{\gamma}_2^2}{F(\bar{\gamma}_2)} \right\} \right),$$

the problem $(P_\lambda^{f,\theta})$ *admits at least two solutions* $u_{1,\lambda}$ *and* $u_{2,\lambda}$ *such that* $\max_{x\in[0,1]} |u_{1,\lambda}(x)| < \gamma_1$ *and* $\max_{x\in[0,1]} |u_{2,\lambda}(x)| < \gamma_2$.

Now, we point out a special case of Theorem 5.7.

Theorem 5.9. *Assume that*

$$\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi} = \lim_{|\xi|\to\infty} \frac{f(\xi)}{|\xi|} = 0 \tag{5.6}$$

and there exists a positive constant $\bar{\eta}$ *such that* $F(\bar{\eta}) > 0$. *Then, for each* $\lambda > \lambda^*$, *where*

$$\lambda^* = \frac{1}{4\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x) \mathrm{d}x} \inf_{\bar{\eta}>0} \frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1 \bar{\eta}^2}{F(\bar{\eta})},$$

the problem $(P_\lambda^{f,\theta})$ *admits at least one nonnegative and one non zero solution in* X.

Proof. Let $\lambda > \lambda^*$. Then, there is $\bar{\eta} > 0$ such that

$$\lambda > \frac{1}{4\int_{\frac{1}{4}}^{\frac{3}{4}} \theta(x) \mathrm{d}x} \inf_{\bar{\eta}>0} \frac{h(2\bar{\eta}) + h(-2\bar{\eta}) + 4\zeta_1 \bar{\eta}^2}{F(\bar{\eta})}.$$

From (5.6) we obtain

$$\lim_{u\to 0^+} \frac{\sup_{|\xi|\le u} f(\xi)}{u} = \lim_{u\to\infty} \frac{\sup_{|\xi|\le u} f(\xi)}{u} = 0.$$

So we can pick $\bar{\gamma}_1$ and $\bar{\gamma}_2$ such that

$$\bar{\gamma}_1 < \sqrt{\frac{8}{\min\{m, \zeta_0\}} \left(\frac{1}{4} g(2\bar{\eta}) + \frac{1}{4} g(-2\bar{\eta}) + \frac{\zeta_0 \bar{\eta}^2}{8} \right)}$$

and

$$\sqrt{\frac{8}{\min\{m, \zeta_0\}} \left(\frac{1}{4} g(2\bar{\eta}) + \frac{1}{4} g(-2\bar{\eta}) + \zeta_1 \bar{\eta}^2 \right)} < \bar{\gamma}_2,$$

$\frac{\sup_{|\xi|\le\bar{\gamma}_1} f(\xi)}{\bar{\gamma}_1} < \frac{\min\{m,\zeta_0\}}{8\int_0^1 \theta(x)\mathrm{d}x} \frac{\bar{\gamma}_1^2}{F(\bar{\gamma}_1)}$ and $\frac{\sup_{|\xi|\le\bar{\gamma}_2} f(\xi)}{\bar{\gamma}_2} < \frac{\min\{m,\zeta_0\}}{8\int_0^1 \theta(x)\mathrm{d}x} \frac{\bar{\gamma}_2^2}{F(\bar{\gamma}_2)}$. Hence, from Theorem 5.8 we obtain the conclusion. □

Remark 5.10. Theorem 1.2 immediately follows from Theorem 5.9.

6 Ethical statement

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Declarations

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Multiple normalized solutions of a nonlinear Schrödinger–Poisson system with L^2-subcritical growth

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Abstract. In this paper, we study the existence of multiple normalized solutions to the following Schrödinger–Poisson system with general nonlinearities:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = \varepsilon^3 a^2, \end{cases}$$

where ε, $a > 0$, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier, $V(x) : \mathbb{R}^3 \to [0,\infty)$ is a continuous function, and f is a differentiable function satisfying L^2-subcritical growth. Through using the minimization techniques and the Lusternik–Schnirelmann category, we prove that the numbers of normalized solutions are related to the topology of the set where the potential $V(x)$ attains its minimum value.

Keywords: Schrödinger–Poisson system, normalized solutions, Lusternik–Schnirelmann category, variational methods.

2020 Mathematics Subject Classification: 35B09, 35J61.

1 Introduction

In this paper, we are concerned with the existence of multiple normalized solutions to the following Schrödinger–Poisson system with general nonlinearities:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = \varepsilon^3 a^2, \end{cases} \tag{1.1}$$

where ε, $a > 0$, $\lambda \in \mathbb{R}$ is an unknown parameter that appears as a Lagrange multiplier.

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Problem (1.1) arises in the study of the coupled Schrödinger–Poisson system:

$$\begin{cases} i\psi_t - \Delta\psi + V(x)\psi + \phi\psi = g(|\psi|^2)\psi & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |\psi|^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\psi(x,t) : \mathbb{R}^3 \times [0,T]$ is the wave function. Equation (1.2) arises from approximation of the Hartree–Fock equation which describes a quantum mechanical of many particles, see [11,12,24]. Set $\psi(x,t) = e^{i\lambda t}u(x)$ and $u : \mathbb{R}^3 \to \mathbb{R}$, one is led to the equation

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $f(u) = g(|u|^2)u$, $\lambda \in \mathbb{R}$. The system (1.1) was firstly introduced by Benci and Fortunato in [9]. System (1.1) also arises in various fields of physics, for instance, in semiconductor theory (see [10,25,26]), for more details on the physical aspects, we refer the reader to [9] and references therein.

When $\lambda \in \mathbb{R}$ is a fixed parameter, we call (1.1) the fixed frequency problem. In the last decades, the existence, concentration and multiplicity of solutions for the fixed frequency problem (1.1) has been studied by many scholars, for example [2–4, 13, 15, 27, 29] and the references therein.

Recently, the existence and multiplicity of normalized solution are attracted many people's interests. Such solutions have a prescribed L^2-norm, that is, solutions which satisfy $\|u\|_2 = a$ for a priori given $a > 0$. In this case, the parameter $\lambda \in \mathbb{R}$ cannot be fixed but instead appears as a Lagrange multiplier.

When $\varepsilon = 1$, $V(x) = 0$ and $f(u) = |u|^{p-2}u$, normalized solutions of (1.1) can be obtained by considering the critical points of the following functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dy dx - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx$$

on the constraint

$$S(a) = \{u \in H^1(\mathbb{R}^3) : \|u\|_2 = a\}.$$

As far as we know, the first work for normalized solutions to Schrödinger–Poisson system is due to Sánchez and Soler [28], they proved that all the minimizing sequence for σ_a are compact provided that $a \in (0, a_0)$ for a suitable $a_0 > 0$ small enough and $p = \frac{8}{3}$, where σ_a is defined by

$$\sigma_a = \inf_{u \in S(a)} J(u). \tag{1.3}$$

Bellazzini and Siciliano in [5] and [6] proved that σ_a is achieved when $a > 0$ is small and $p \in (2,3)$ and when $a > 0$ is large and $p \in (3, \frac{10}{3})$, respectively. Subsequently, Jeanjean and Luo in [22] sharpened the conclusion of [6] by showing that (1.3) has a minimizer if and only if

$$a \geq a_1 = \inf\{a > 0 : \sigma_a < 0\}.$$

Moreover, for the case of $p = 3$ or $p = \frac{10}{3}$, they proved σ_a has no minimizer for any $a > 0$.

For the L^2-supercritical case, that is, $p \in (\frac{10}{3}, 6)$, the functional $J(u)$ is no more bounded from below on $S(a)$. Bellazzini, Jeanjean and Luo in [7] found critical points of $J(u)$ on $S(a)$ by looking at the mountain-pass level for $a > 0$ sufficiently small. In 2021, Jeanjean and Le in

[21] obtained the existence of two positive solutions for (1.1) with $f(u) = |u|^{p-2}u$ which can be characterized respectively as a local minima and as a mountain pass critical point when $p \in (\frac{10}{3}, 6]$. For the general nonlinearity, Chen, Tang and Yuan in [16] studied the existence of normalized solutions for system (1.1), where $f \in C(\mathbb{R}, \mathbb{R})$ covers the case $f(u) = |u|^{p-2}u$ with $q \in (2, \frac{10}{3}) \cup (\frac{10}{3}, 6)$. When considering more general L^2-supercritical conditions without imposing the monotonicity property on f, Hu, Tang and Jin [20] obtained the existence of normalized solutions for problem (1.1) under suitable assumptions on f.

For the case combining nonlinearity, Kang, Li and Tang in [23] considered system (1.1) with $f(u) = \mu|u|^{q-2}u + |u|^{p-2}u$, where $\mu \in \mathbb{R}$, $2 < q \le \frac{10}{3} \le p < 6$ with $q \neq p$. Under some suitable assumptions on s and μ, they proved some existence, nonexistence and multiplicity of normalized solutions.

When $\varepsilon = 1$ and $V(x) \neq 0$, it is more complicated to deal with the existence of normalized solutions. Zeng and Zhang in [32] considered system (1.1) with $f(u) = |u|^p u$ $(0 < p < \frac{4}{3})$ and unbounded potential, where the potential function $V(x)$ satisfies the following conditions

$$V \in C(\mathbb{R}^N, \mathbb{R}^+), \quad \inf_{x \in \mathbb{R}^N} V(x) = 0 \quad \text{and} \quad \lim_{|x| \to +\infty} V(x) = \infty,$$

with the help of the compactness of Sobolev embedding in the working space, they obtained the existence of normalized solutions.

To our best of knowledge, there is few results for the existence of multiple normalized solutions to Schrödinger–Poisson system (1.1). Motivated by [1], the main purpose of this paper is to study the existence of multiple normalized solutions to (1.1) by using the Lusternik–Schnirelmann category when $V(x)$ satisfies the global conditions:

$$(V) \qquad V \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N), \qquad V(0) = 0, \qquad 0 = \inf_{x \in \mathbb{R}^N} V(x) < \liminf_{|x| \to +\infty} V(x) = V_\infty.$$

By change of variable $x \to \varepsilon x$, problem (1.1) reduces to the following system

$$\begin{cases} -\Delta u + V(\varepsilon x)u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = a^2. \end{cases} \tag{1.4}$$

We assume that $V(x)$ satisfies (V) and f satisfies the following assumptions:

(f_1) f is odd and there exist $q \in (3, \frac{10}{3})$ and $\alpha \in (0, +\infty)$ such that $\lim_{s \to 0} \frac{|f(s)|}{|s|^{q-1}} = \alpha$.

(f_2) There exist constants c_1, c_2, c_3, $c_4 > 0$ and $p \in (3, \frac{10}{3})$ such that

$$|f(s)| \le c_1 + c_2|s|^{p-1} \quad \forall s \in \mathbb{R} \quad \text{and} \quad |f'(s)| \le c_3 + c_4|s|^{p-2} \quad \forall s \in \mathbb{R}.$$

(f_3) There exists $q_1 \in (3, \frac{10}{3})$ and $q > q_1$ such that $f(s)/s^{q_1-1}$ is an increasing function of s on $(0, +\infty)$.

Remark 1.1. The conditions (f_1) and (f_3) imply that $F(t) \ge 0$ for all $t \in \mathbb{R}$. Indeed,

$$\frac{f(s)}{s^{q_1-1}} \ge \lim_{s \to 0^+} \frac{f(s)}{s^{q-1}} s^{q-q_1} = 0, \qquad s > 0,$$

that is, $f(s) \ge 0$. Hence, $F(t) \ge 0$.

An example of a function f that satisfies the above assumption is

$$f(s) = |s|^{q-2}s + |s|^{r-2}s \ln(1 + |s|) \qquad \forall s \in \mathbb{R},$$

for some $r, q \in (3, \frac{10}{3})$ and $r > q$, here (f_2) and (f_3) hold with $p \in (r, \frac{10}{3})$.

A solution u to the problem (1.4) with $\int_{\mathbb{R}^3}|u|^2 = a^2$ can be obtained by looking for critical points of the following functional

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(\varepsilon x)u^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - \int_{\mathbb{R}^3}F(u)\mathrm{d}x, \qquad u \in H^1(\mathbb{R}^3),$$

restricted to sphere

$$S(a) = \{u \in H^1(\mathbb{R}^3) : \|u\|_2 = a\},$$

where $\|\cdot\|_p$ denotes the usual norm in $L^p(\mathbb{R}^3)$ for $p \in [1, +\infty)$.

Moreover, it is easy to see that $J_\varepsilon \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ and

$$J'_\varepsilon(u)v = \int_{\mathbb{R}^3}(\nabla u\nabla v + V(\varepsilon x)uv)\mathrm{d}x + \int_{\mathbb{R}^3}\phi_u uv\mathrm{d}x - \int_{\mathbb{R}^3}f(u)v\mathrm{d}x, \qquad \forall v \in H^1(\mathbb{R}^3).$$

When we study the multiplicity of solutions in the nonautonomous case, we need to use the following sets:

$$M = \{x \in \mathbb{R}^3 : V(x) = 0\}$$

and

$$M_\delta = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, M) \leq \delta\}, \qquad \delta > 0.$$

Now, we state our main result as follows.

Theorem 1.2. *Suppose that f satisfies the conditions (f_1)–(f_3) and that V satisfies (V). Then for each $\delta > 0$, there exist ε_0, $\mu_* > 0$ and $a_* > 0$ such that (1.1) admits at least $\mathrm{cat}_{M_\delta}(M)$ couples $(u_j, \lambda_j) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ of weak solutions for $0 < \varepsilon < \varepsilon_0$, $|V|_\infty < \mu_*$ and $a > a_*$ with $\int_{\mathbb{R}^3}|u_j|^2\mathrm{d}x = a^2$ and $J_\varepsilon(u_j) < 0$.*

Remark 1.3. For $V(0) =: V_0 \neq 0$, $V_0 < V_\infty$, we can also obtain Theorem 1.2.

We recall that, if Y is a closed subset of a topological space X, the Lusternik–Schnirelmann category $cat_X(Y)$ is the least number of closed and contractible sets in X which cover Y. If $X = Y$, we use the notation $\mathrm{cat}(X)$. For more details about this subject, we cite [30].

The organization of this paper is as follows. In Section 2, we study the autonomous problem. In Section 3, we study the nonautonomous case. In this section, we also study the Palais–Smale condition on the sphere $S(a)$ for the energy functional and provide some crucial tools to establish a multiplicity result. In Section 4, we prove the multiplicity and concentration of solutions to problem (1.1).

2 The autonomous case

The following classical Gagliardo–Nirenberg inequality is so crucial in this paper, which can be found in [31]. Precisely, let $l \in [2, 6)$, then

$$|u|_l^l \leq C|u|_2^{(1-\beta_l)l}|\nabla u|_2^{\beta_l l} \quad \text{in } \mathbb{R}^3, \qquad \beta_l = \frac{3(l-2)}{2l}, \tag{2.1}$$

for some positive constant $C = C(3, l) > 0$.

In this section, we list some preliminary lemmas which used later involving the existence of normalized solution for the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \mu u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3}|u|^2\mathrm{d}x = a^2, \end{cases} \tag{2.2}$$

where $a > 0$, $\mu \geq 0$, and $\lambda \in \mathbb{R}$ is unknown parameter that appears as a Lagrange multiplier and f is a continuous function satisfying (f_1)–(f_3).

A solution u to the problem (2.2) corresponds to a critical point of the C^1 functional

$$I_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + \mu u^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x, \qquad u \in H^1(\mathbb{R}^3),$$

on the constraint $S(a)$ given by

$$S(a) = \{u \in H^1(\mathbb{R}^3) : \|u\|_2 = a\}.$$

Our main result in this section is stated as follows.

Theorem 2.1. *Suppose that f satisfies the conditions (f_1)–(f_3). Then, there exists $\mu_* > 0$ and $a_* > 0$ such that problem (2.2) has a couple (u, λ) solution when $0 \leq \mu < \mu_*$ and $a > a_*$, where u is positive.*

The proof of the theorem above will be divided into several lemmas.

Now, we recall some properties of the functions ϕ_u in the following lemma (for a proof see [27], [18] and [17]).

Lemma 2.2. *The following results hold:*

(1) $\phi_u \geq 0$;

(2) there exist some constants C_1, $C_2 > 0$ such that $\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x \leq C_1|u|^4_{\frac{12}{5}} \leq C_2\|u\|^4$;

(3) if $u_n \to u$ in $L^t(\mathbb{R}^3)$, $\forall t \in [2,6)$, then $\int_{\mathbb{R}^3}\phi_{u_n}{u_n}^2\mathrm{d}x \to \int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x$,

where $\|\cdot\|$ denotes the usual norm in $H^1(\mathbb{R}^3)$.

Define N: $H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$N(u) = \int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x.$$

Lemma 2.3 ([33, Lemma 2.2])**.** *Let $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$. Then as $n \to \infty$,*

(1) $N(u_n - u) = N(u_n) - N(u) + o(1)$;

(2) $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$, *in* $(H^1(\mathbb{R}^3))'$.

Lemma 2.4. *The functional I_μ is coercive and bounded from below in $S(a)$.*

Proof. According to (f_1)–(f_2), there is C_1, $C_2 > 0$ such that

$$|F(t)| \leq C_1|t|^q + C_2|t|^p \qquad \forall t \in \mathbb{R}.$$

Then it follows from (2.1) that

$$I_\mu(u) \geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x - CC_1 a^{(1-\beta_q)q}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^{\frac{\beta_q q}{2}}$$
$$- CC_2 a^{(1-\beta_p)p}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^{\frac{\beta_p p}{2}}.$$

Since q, $p \in (2, \frac{10}{3})$, by simple calculation, we get $\beta_q q$, $\beta_p p < 2$, which ensures the coercivity and boundedness of I_μ from below. □

Lemma 2.4 guarantees that the minimization problem

$$I_{\mu,a} = \inf_{u \in S(a)} I_\mu(u)$$

is well defined. In what follows, we are going to establish some properties of I_μ related to the parameter $\mu \geq 0$.

Lemma 2.5. *There exists $\mu_* > 0$, $a_* > 0$ such that $\mathcal{I}_{\mu,a} < 0$ for $0 \leq \mu < \mu_*$ and $a > a_*$.*

Proof. By assumption (f_3), we have

$$f'(t)t - (q_1 - 1)f(t) \geq 0 \qquad \forall t > 0. \tag{2.3}$$

In order to show $t \mapsto \frac{F(t)}{t^{q_1}}$ is increasing on $(0, +\infty)$, we need to prove

$$\frac{d}{dt}\frac{F(t)}{t^{q_1}} = \frac{f(t)t^{q_1} - q_1 F(t) t^{q_1 - 1}}{t^{2q_1}} = \frac{f(t)t - q_1 F(t)}{t^{q_1+1}} \qquad \forall t > 0.$$

Define $h(t) = f(t)t - q_1 F(t)$, clearly, $h(0) = 0$ and (2.3) yields that

$$h'(t) = f'(t)t - (q_1 - 1)f(t) \geq 0 \qquad \forall t > 0,$$

which implies that

$$h(t) = f(t)t - q_1 F(t) \geq 0. \tag{2.4}$$

This leads to $\frac{\mathrm{d}}{\mathrm{dt}}\frac{F(t)}{t^{q_1}} \geq 0$, that is, the function $t \mapsto \frac{F(t)}{t^{q_1}}$ is increasing on $(0, +\infty)$, thus, we have that

$$\frac{F(ts)}{(ts)^{q_1}} \geq \frac{F(s)}{s^{q_1}} \qquad \forall s > 0 \quad \text{and} \quad t \geq 1,$$

which yields that

$$F(ts) \geq t^{q_1} F(s) \qquad \forall s > 0 \quad \text{and} \quad t \geq 1. \tag{2.5}$$

Given $u_0(x) \in S(a) \cap L^\infty(\mathbb{R}^3)$ a nonnegative function, let

$$u_0^\eta(x) = \eta^2 u_0(\eta x) \quad \text{for all } x \in \mathbb{R}^3 \text{ and all } \eta \in \mathbb{R}. \tag{2.6}$$

By simple computation, we have

$$\int_{\mathbb{R}^3} |u_0^\eta(x)|^2 \mathrm{d}x = \eta a^2,$$

that is, $u_0^\eta(x) \in S(\eta^{\frac{1}{2}} a)$. Therefore,

$$I_\mu(u_0^\eta(x)) \leq \frac{\eta^3}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x + \frac{\mu\eta a^2}{2} + \frac{\eta^3}{4}\int_{\mathbb{R}^3} \phi_{u_0} u_0^2 \mathrm{d}x - \eta^{2q_1 - 3}\int_{\mathbb{R}^3} F(u_0(x))\mathrm{d}x.$$

When $q_1 \in (3, \frac{10}{3})$ and $\eta > 0$, $2q_1 - 3 > 3$, for $|\eta|$ large, we deduce that

$$\frac{\eta^3}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x + \frac{\eta^3}{4}\int_{\mathbb{R}^3} \phi_{u_0} u_0^2 \mathrm{d}x - \eta^{2q_1 - 3}\int_{\mathbb{R}^3} F(u_0(x))\mathrm{d}x = A_\eta < 0,$$

thus, we obtain that

$$I_\mu(u_0^\eta(x)) \leq A_\eta + \frac{\mu a^2}{2}.$$

Hence, we fix $\mu_* > 0$ such that

$$I_\mu(u_0^\eta(x)) < 0, \qquad \forall \mu \in [0, \mu_*),$$

showing that $\mathcal{I}_{\mu, t^{\frac{1}{2}} a} < 0$. Thus, for a large enough, $\mathcal{I}_{\mu,a} < 0$. □

Lemma 2.6. *Fix $\mu \in [0, \mu_*)$ and let $a_* < a_1 < a_2$. There holds $\frac{a_1^6}{a_2^6}\mathcal{I}_{\mu,a_2} < \mathcal{I}_{\mu,a_1} < 0$.*

Proof. Let $\xi > 1$ such that $a_2 = \xi a_1$ and $\{u_n\} \subset S(a_1)$ be a nonnegative minimizing sequence with respect to the $\mathcal{I}_{\mu,a_1}$ (because $I_\mu(u) = I_\mu(|u|)$ for all $u \in H^1(\mathbb{R}^3)$), that is,

$$I_\mu(u_n) \to \mathcal{I}_{\mu,a_1} \quad \text{as } n \to +\infty.$$

Set $v_n = \xi^4 u_n(\xi^2 x)$. Obviously $v_n \in S(a_2)$ and

$$\begin{aligned}
\mathcal{I}_{\mu,a_2} &\leq I_\mu(v_n) \\
&= \frac{\xi^6}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{\xi^2\mu^2}{2}\int_{\mathbb{R}^3} u_n^2 dx + \frac{\xi^6}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \xi^{-6}\int_{\mathbb{R}^3} F(\xi^4 u_n(x)) dx \\
&\leq \xi^6\left[\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{\mu^2}{2}\int_{\mathbb{R}^3} u_n^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx\right] - \xi^{-6}\int_{\mathbb{R}^3} F(\xi^4 u_n(x)) dx.
\end{aligned}$$

By (2.5), we deduce that

$$\mathcal{I}_{\mu,a_2} \leq I_\mu(v_n) = \xi^6 I_\mu(u_n) + (\xi^6 - \xi^{4q_1-6})\int_{\mathbb{R}^3} F(u_n(x)) dx.$$

Claim 2.7. There exists a constant $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^3} F(u_n) dx \geq C \quad \text{for } n \geq n_0.$$

Arguing by contradiction that there exists a subsequence of $\{u_n\}$, still denoted by itself, such that

$$\int_{\mathbb{R}^3} F(u_n) dx \to 0 \text{ as } n \to +\infty.$$

Thus, we have

$$0 > \mathcal{I}_{\mu,a} + o_n(1) = I_\mu(u_n) \geq -\int_{\mathbb{R}^3} F(u_n) dx,$$

which is absurd. Thus, Claim 2.7 holds. It is easy to verify that $\xi^6 - \xi^{4q_1-6} < 0$. Hence, we have

$$\mathcal{I}_{\mu,a_2} \leq \xi^6 I_\mu(u_n) + (\xi^6 - \xi^{4q_1-6})C.$$

As $n \to +\infty$, we get

$$\mathcal{I}_{\mu,a_2} < \xi^6 \mathcal{I}_{\mu,a_1} + (\xi^6 - \xi^{4q_1-6})C < \xi^6 \mathcal{I}_{\mu,a_1},$$

that is,

$$\frac{a_1^6}{a_2^6}\mathcal{I}_{\mu,a_2} < \mathcal{I}_{\mu,a_1}.$$

□

The following theorem is a compactness theorem on $S(a)$, which is crucial to study the autonomous case and the nonautonomous case.

Theorem 2.8. *Let $\mu \in [0, \mu_*)$, $a > a_*$ and $\{u_n\} \subset S(a)$ be a minimizing sequence with respect to I_μ. Then, for some subsequence either*

(i) $\{u_n\}$ is strongly convergent in $H^1(\mathbb{R}^3)$;

or

(ii) there exists $\{y_n\} \subset \mathbb{R}^3$ with $|y_n| \to +\infty$ such that $v_n(x) = u_n(x + y_n) \to v$ in $H^1(\mathbb{R}^3)$, where $v \in S(a)$ and $I_\mu(v) = \mathcal{I}_{\mu,a}$.

Proof. Since $\mathcal{I}_\mu$ is coercive on $S(a)$, the sequence $\{u_n\}$ is bounded. Hence, up to a subsequence, still denoted by u_n, we may assume that there exists some $u \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$.

If $u \neq 0$ and $|u|_2 = b \neq a$, we have that $b \in (0,a)$. It follows from Brézis–Lieb lemma (see [30]) that:

$$|u_n|_2^2 = |u_n - u|_2^2 + |u|_2^2 + o_n(1).$$

Moreover, setting $v_n = u_n - u$, $d_n = |v_n|_2$, $t \in (0,1)$, by using mean value theorem, (f_1), (f_2), and Young's inequality, we get

$$\begin{aligned}
|F(v_n+u) - F(v_n) - F(u)| &\leq |F(v_n+u) - F(v_n)| + |F(u)| \\
&\leq |f(v_n+tu)||u| + |F(u)| \\
&\leq [C_1|v_n+tu|^{q-1} + C_2|v_n+tu|^{p-1}]|u| + C_1|u|^{q-1} + C_2|u|^{p-1} \\
&\leq C(|v_n|^{q-1} + |u|^{q-1} + |v_n|^{p-1} + |u|^{p-1})|u| + C_1|u|^{q-1} + C_2|u|^{p-1} \\
&\leq C\varepsilon(|v_n|^q + |v_n|^p) + (C\varepsilon^{-(q-1)} + C_1)|u|^q + (C\varepsilon^{-(p-1)} + C_2)|u|^p.
\end{aligned}$$

Since $\lim_{n\to+\infty} |F(v_n+u) - F(v_n) - F(u)| = 0$ a.e. in $\mathbb{R}^3$, by Lebesgue dominated convergence Theorem, it is easy to get that

$$\int_{\mathbb{R}^3} F(v_n+u)\mathrm{d}x = \int_{\mathbb{R}^3} F(v_n)\mathrm{d}x + \int_{\mathbb{R}^3} F(u)\mathrm{d}x + o_n(1),$$

that is,

$$\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^3} F(u_n-u)\mathrm{d}x + \int_{\mathbb{R}^3} F(u)\mathrm{d}x + o_n(1). \tag{2.7}$$

Suppose that $|v_n|_2 \to d$, then $a^2 = b^2 + d^2$ and $d_n \in (0,a)$ for n large enough. Thus, by Lemma 2.3 and (2.7), we have that

$$\mathcal{I}_{\mu,a} + o_n(1) = I_\mu(u_n) = I_\mu(v_n) + I_\mu(u) + o_n(1) \geq \mathcal{I}_{\mu,d_n} + \mathcal{I}_{\mu,b} + o_n(1).$$

From Lemma 2.6, it follows that

$$\mathcal{I}_{\mu,a} + o_n(1) \geq \frac{d_n^6}{a^6}\mathcal{I}_{\mu,a} + \mathcal{I}_{\mu,b} + o_n(1).$$

As $n \to +\infty$, we arrive at the inequality

$$\mathcal{I}_{\mu,a} \geq \frac{d^6}{a^6}\mathcal{I}_{\mu,a} + \mathcal{I}_{\mu,b}. \tag{2.8}$$

Since $b \in (0,a)$, by Lemma 2.6 and (2.8), we obtain

$$0 > \mathcal{I}_{\mu,a} > \frac{d^6}{a^6}\mathcal{I}_{\mu,a} + \frac{b^6}{a^6}\mathcal{I}_{\mu,a} = \frac{b^6+d^6}{a^6}\mathcal{I}_{\mu,a},$$

which yields that

$$\frac{b^6+d^6}{a^6} > 1.$$

By using $a^2 = b^2 + d^2$, we deduce that

$$b^6 + d^6 > a^6 = (b^2+d^2)^3 = b^6 + d^6 + 3b^2d^4 + 3b^4d^2,$$

which is absurd. Hence, we infer that $|u|_2 = a$, that is, $u \in S(a)$.

As $|u_n|_2 = |u|_2 = a$, $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$, it is easy to verify that

$$u_n \to u \quad \text{in } L^2(\mathbb{R}^3). \tag{2.9}$$

By (2.9) and interpolation theorem in the Lebesgue spaces, one infers that

$$u_n \to u \quad \text{in } L^t(\mathbb{R}^3), \qquad \forall t \in [2,6),$$

which combines with (f_1)–(f_2), we can deduce that

$$\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x \to \int_{\mathbb{R}^3} F(u)\mathrm{d}x. \tag{2.10}$$

Thus, by Lemma 2.2-(3) and $\mathcal{I}_{\mu,a} = \lim_{n\to\infty} I_\mu(u_n)$, we have that $\mathcal{I}_{\mu,a} \geq I_\mu(u)$. Since $u \in S(a)$, it follows that $\mathcal{I}_{\mu,a} = I_\mu(u)$, and then $\lim_{n\to\infty} I_\mu(u_n) = \mathcal{I}_\mu(u)$, which combines with (2.9), (2.10) and Lemma 2.2-(3), we have that $u_n \to u$ in $D^{1,2}(\mathbb{R}^3)$. From (2.9), it follows that $\|u_n\|^2 \to \|u\|^2$, that is, $u_n \to u$ in $H^1(\mathbb{R}^3)$.

If $u = 0$, then $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Similar to Claim 2.7, we prove that there exists $C > 0$ such that

$$\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x \geq C \quad \text{for } n \in \mathbb{N} \text{ large}. \tag{2.11}$$

Next, we prove that there exist $R, \beta > 0$ and $y_n \in \mathbb{R}^3$ such that

$$\int_{B_R(y_n)} |u_n|^2\mathrm{d}x \geq \beta \qquad \forall n \in \mathbb{N}. \tag{2.12}$$

Suppose on the contrary, by Lions' vanishing lemma, we get that $u_n \to 0$ in $L^t(\mathbb{R}^3)$ for all $t \in (2, 2^*)$. Hence, it is easy to check that $F(u_n) \to 0$ in $L^1(\mathbb{R}^3)$, which is contradict with (2.11).

Since $u = 0$, we claim that $\{y_n\}$ is unbounded. Arguing by contradiction that $\{y_n\}$ is bounded, there exists $R_0 > 0$, such that $|y_n| < R_0$. Hence, $B_R(y_n) \subset B_{R+R_0}(0)$. Thus, we have that

$$\int_{B_R(y_n)} |u_n|^2\mathrm{d}x \leq \int_{B_{R+R_0}(0)} |u_n|^2\mathrm{d}x \to 0 \quad \text{as } n \to +\infty,$$

which is contradiction with (2.12). The claim follows.

Setting $\widetilde{u}_n(x) = u(x + y_n)$, clearly $\{\widetilde{u}_n\} \subset S(a)$ and it is also a minimizing sequence with respect to $\mathcal{I}_{\mu,a}$, up to a subsequence, we may assume that there exists $\widetilde{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that

$$\widetilde{u}_n \rightharpoonup \widetilde{u} \quad \text{in } H^1(\mathbb{R}^3) \quad \text{and} \quad \widetilde{u}_n(x) \to \widetilde{u}(x) \quad \text{a.e. in } \mathbb{R}^3.$$

Similarly arguing as the above proof, we can deduce that $\widetilde{u}_n \to \widetilde{u}$ in $H^1(\mathbb{R}^3)$. This completes the proof. □

2.1 Proof of Theorem 2.1

From Lemma 2.4, there exists a bounded minimizing sequence $\{u_n\} \subset S(a)$ with respect to $\mathcal{I}_{\mu,a}$, that is, $I_\mu(u_n) \to \mathcal{I}_{\mu,a}$. By Theorem 2.8, there exists $u \in S(a)$ with $I_\mu(u) = \mathcal{I}_{\mu,a}$. Hence, by the Lagrange multiplier, there exists $\lambda_a \in \mathbb{R}$ such that

$$I_\mu{}'(u) = \lambda_a \Psi'(u) \quad \text{in } (H^1(\mathbb{R}^3))', \tag{2.13}$$

where $\Psi : H^1(\mathbb{R}^3) \to \mathbb{R}$ is given by

$$\Psi(u) = \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x, \qquad u \in H^1(\mathbb{R}^3).$$

By (2.13), we have

$$-\Delta u + \mu u + \phi u = \lambda_a u + f(u) \quad \text{in } \mathbb{R}^3. \tag{2.14}$$

Next, by simple calculation, it is easy to see that $I_\mu(|u|) = I_\mu(u)$. Besides, since $u \in S(a)$ implies that $|u| \in S(a)$, then the following equality holds:

$$\mathcal{I}_{\mu,a} = I_\mu(u) = I_\mu(|u|) \geq \mathcal{I}_{\mu,a},$$

thus, $I_\mu(|u|) = \mathcal{I}_{\mu,a}$. Then we can replace u by $|u|$, thus we may assume that $u \geq 0$, by standard argument, we can prove that $u(x) > 0$ in $\mathbb{R}^3$.

By Theorem 2.1, it is easy to conclude the following corollary.

Corollary 2.9. *Fix $a > a^*$ and let $0 \leq \mu_1 < \mu_2 \leq \mu_*$. There holds $\mathcal{I}_{\mu_1,a} < \mathcal{I}_{\mu_2,a} < 0$.*

Proof. Let $u_{\mu_2,a} \in S(a)$ satisfying $I_{\mu_2}(u_{\mu_2,a}) = \mathcal{I}_{\mu_2,a}$. It is easy to infer that

$$\mathcal{I}_{\mu_1,a} \leq I_{\mu_1}(u_{\mu_2,a}) < I_{\mu_2}(u_{\mu_2,a}) = \mathcal{I}_{\mu_2,a}.$$ □

3 The nonautonomous case

In this section, we will study the nonautonomous case of the Schrödinger–Poisson system (1.4). Hereafter, we will suppose that $|V|_\infty < \mu_*$ and $a > a_*$, where μ_* and a_* was given in section 2. In order to prove some properties of the functional J_ε, we give several useful definitions. We define J_0, $J_\infty : H^1(\mathbb{R}^3) \to \mathbb{R}$ by the following functionals:

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x$$

and

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V_\infty |u|^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x.$$

Furthermore, we denote $\Upsilon_{\varepsilon,a}$, $\Upsilon_{0,a}$ and $\Upsilon_{\infty,a}$:

$$\Upsilon_{\varepsilon,a} = \inf_{u \in S(a)} J_\varepsilon(u), \qquad \Upsilon_{0,a} = \inf_{u \in S(a)} J_0(u), \qquad \Upsilon_{\infty,a} = \inf_{u \in S(a)} J_\infty(u).$$

Since $0 < V_\infty < +\infty$, we deduce from Corollary 2.9 that

$$\Upsilon_{0,a} < \Upsilon_{\infty,a} < 0. \tag{3.1}$$

In the following, we set $0 < \rho_1 = \frac{1}{2}(\Upsilon_{\infty,a} - \Upsilon_{0,a})$.

The following lemma establishes some essential relations involving the levels $\Upsilon_{\varepsilon,a}$, $\Upsilon_{0,a}$ and $\Upsilon_{\infty,a}$.

Lemma 3.1. $\limsup_{\varepsilon \to 0^+} \Upsilon_{\varepsilon,a} \leq \Upsilon_{0,a}$ *and there exists $\varepsilon_0 > 0$ such that $\Upsilon_{\varepsilon,a} < \Upsilon_{\infty,a}$ for all $\varepsilon \in (0, \varepsilon_0)$.*

Proof. Let $u_0 \in S(a)$ with $J_0(u_0) = \Upsilon_{0,a}$, we have that

$$\Upsilon_{\varepsilon,a} \leq J_\varepsilon(u_0) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_0|^2 + V(\varepsilon x)|u_0|^2)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_0}u_0^2\mathrm{d}x - \int_{\mathbb{R}^3}F(u_0)\mathrm{d}x.$$

As $\varepsilon \to 0^+$, we arrive at the inequality

$$\limsup_{\varepsilon\to 0^+}\Upsilon_{\varepsilon,a} \leq \lim_{\varepsilon\to 0^+} J_\varepsilon(u_0) = J_0(u_0) = \Upsilon_{0,a}.$$

By (3.1) and the above inequality, we can obtain that $\Upsilon_{\varepsilon,a} < \Upsilon_{\infty,a}$ for ε small enough. □

Lemma 3.2. *Fix $\varepsilon \in (0, \varepsilon_0)$ and let $\{u_n\} \subset S(a)$ such that $J_\varepsilon(u_n) \to c$ with $c < \Upsilon_{0,a} + \rho_1 < 0$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $u \neq 0$.*

Proof. We argue by contradiction that $u = 0$. From the definition of $J_\varepsilon(u_n)$ and $J_\infty(u_n)$, it follows that

$$\Upsilon_{0,a} + \rho_1 + o_n(1) > c + o_n(1) = J_\varepsilon(u_n) = J_\infty(u_n) + \frac{1}{2}\int_{\mathbb{R}^3}(V(\varepsilon x) - V_\infty)|u_n|^2\mathrm{d}x.$$

From (V), for any given $\zeta > 0$, there exists $R > 0$ such that

$$V(x) \geq V_\infty - \zeta, \qquad \forall\, |x| \geq R.$$

Thus, there holds

$$\Upsilon_{0,a} + \rho_1 + o_n(1) > J_\varepsilon(u_n) \geq J_\infty(u_n) + \frac{1}{2}\int_{B_{R/\varepsilon}(0)}(V(\varepsilon x) - V_\infty)|u_n|^2\mathrm{d}x - \frac{\zeta}{2}\int_{B^c_{R/\varepsilon}(0)}|u_n|^2\mathrm{d}x.$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $u_n \to 0$ in $L^l(B_{R/\varepsilon}(0))$ for all $l \in [1, 2^*)$, we obtain

$$\Upsilon_{0,a} + \rho_1 + o_n(1) \geq J_\infty(u_n) - \zeta C \geq \Upsilon_{\infty,a} - \zeta C$$

for some $C > 0$. Because $\zeta > 0$ is arbitrary, it follows that

$$\Upsilon_{0,a} + \rho_1 \geq \Upsilon_{\infty,a},$$

which is contradict with the definition of ρ_1. The proof is completed. □

Lemma 3.3. *Let $\{u_n\} \subset S(a)$ be a $(PS)_c$ sequence for J_ε restricted to $S(a)$ with $c < \Upsilon_{0,a} + \rho_1 < 0$ and $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$, that is,*

$$J_\varepsilon(u_n) \to c \quad \text{as } n \to +\infty \quad \text{and} \quad \|J_\varepsilon|'_{S(a)}(u_n)\| \to 0 \quad \text{as } n \to +\infty.$$

If $v_n = u_n - u_\varepsilon \nrightarrow 0$ in $H^1(\mathbb{R}^3)$, then there exists $\beta > 0$, such that

$$\liminf_{n\to+\infty}|u_n - u_\varepsilon|_2^2 \geq \beta.$$

Proof. Let the functional $\Psi : H^1(\mathbb{R}^3) \to \mathbb{R}$ be given by

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^3}|u|^2\mathrm{d}x,$$

we have that $S(a) = \Psi^{-1}(\{\frac{a^2}{2}\})$. Then, by Proposition 5.12 in [30], we see that

$$\|J_\varepsilon|'_{S(a)}(u_n)\| = \min_{\lambda\in\mathbb{R}}\|J_\varepsilon{}'(u_n) - \lambda\Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'},$$

thus, there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\|J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} \to 0 \quad \text{as } n \to +\infty.$$

Since

$$\|J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} = \sup_{v \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\langle J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n), v\rangle}{\|v\|} \to 0 \quad \text{as } n \to +\infty.$$

In view of the boundedness of $\{u_n\}$, we can deduce that

$$\frac{\langle J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n), u_n\rangle}{\|u_n\|} \le \|J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} \to 0 \quad \text{as } n \to +\infty,$$

which leads to

$$\lambda_n a^2 = \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(\varepsilon x) u_n^2) \mathrm{d}x + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x - \int_{\mathbb{R}^3} f(u_n) u_n \mathrm{d}x + o_n(1). \tag{3.2}$$

From the boundedness of $\{u_n\} \in H^1(\mathbb{R}^3)$, it follows that $\{\lambda_n\}$ is also a bounded sequence, up to a subsequence, we may assume that $\lambda_n \to \lambda_\varepsilon$ as $n \to +\infty$. Hence, we have that

$$\|J_\varepsilon{}'(u_n) - \lambda_\varepsilon \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} \le \|J_\varepsilon{}'(u_n) - \lambda_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} + |\lambda_n - \lambda_\varepsilon| \|\Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'},$$

which combing with $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$, we can deduce that

$$J_\varepsilon{}'(u_\varepsilon) - \lambda_\varepsilon \Psi'(u_\varepsilon) = 0 \quad \text{in } (H^1(\mathbb{R}^3))'.$$

By using Lemma 2.3, we can prove that

$$J_\varepsilon{}'(u_n) = J_\varepsilon{}'(u_\varepsilon) + J_\varepsilon{}'(v_n) + o_n(1),$$

and

$$\Psi'(u_n) = \Psi'(u_\varepsilon) + \Psi'(v_n) + o_n(1).$$

Hence, we have

$$J_\varepsilon{}'(u_n) - \lambda_\varepsilon \Psi'(u_n) = J_\varepsilon{}'(v_n) - \lambda_\varepsilon \Psi'(v_n) + o_n(1),$$

and so

$$\|J_\varepsilon{}'(v_n) - \lambda_\varepsilon \Psi'(v_n)\|_{(H^1(\mathbb{R}^3))'} \to 0 \quad \text{as } n \to +\infty,$$

which implies that

$$\int_{\mathbb{R}^3} (|\nabla v_n|^2 + V(\varepsilon x)|v_n|^2) \mathrm{d}x + \int_{\mathbb{R}^3} \phi_{v_n} v_n{}^2 \mathrm{d}x - \lambda_\varepsilon \int_{\mathbb{R}^3} |v_n|^2 \mathrm{d}x = \int_{\mathbb{R}^3} f(v_n) v_n \mathrm{d}x + o_n(1).$$

Suppose on the contrary that $|v_n|_2 \to 0$, by interpolation inequality, one infers that

$$v_n \to 0 \quad \text{in } L^t(\mathbb{R}^3), \qquad \forall t \in [2, 6). \tag{3.3}$$

By (f_1), (f_2) and (3.3), we deduce that

$$\int_{\mathbb{R}^3} f(v_n) v_n \mathrm{d}x \le \int_{\mathbb{R}^3} C_1 |v_n|^p + C_2 |v_n|^q \mathrm{d}x \to 0 \quad \text{as } n \to +\infty,$$

and

$$\int_{\mathbb{R}^3} \phi_{v_n} v_n{}^2 \mathrm{d}x \le |v_n|^4_{\frac{12}{5}} \to 0 \quad \text{as } n \to +\infty,$$

and

$$\int_{\mathbb{R}^3} V(\varepsilon x)|v_n|^2 \mathrm{d}x \leq \int_{\mathbb{R}^3} \mu_* |v_n|^2 \mathrm{d}x \to 0 \quad \text{as } n \to +\infty.$$

Hence, we have that

$$\int_{\mathbb{R}^3} |\nabla v_n|^2 \mathrm{d}x \to 0 \quad \text{as } n \to +\infty,$$

which leads to $\|v_n\|_{H^1(\mathbb{R}^3)} \to 0$, which gives a contradiction by $v_n \nrightarrow 0$ in $H^1(\mathbb{R}^3)$. Therefore, there exists $\beta > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

$$\liminf_{n \to +\infty} |u_n - u_\varepsilon|_2^2 \geq \beta. \qquad \square$$

In what follows, we set

$$0 < \rho < \min\left\{\frac{1}{2}, \frac{\beta^3}{a^6}\right\} (\Upsilon_{\infty,a} - \Upsilon_{0,a}) \leq \rho_1. \tag{3.4}$$

Lemma 3.4. *For each $\varepsilon \in (0, \varepsilon_0)$, the functional J_ε satisfies the $(PS)_c$ condition restricted to $S(a)$ for $c < \Upsilon_{0,a} + \rho$.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence for J_ε restricted to $S(a)$ with $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$ and $c < \Upsilon_{0,a} + \rho$. Then, by Proposition 5.12 in [30], there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$\|J_\varepsilon'(u_n) - \lambda_n \Psi'(u_n)\|_{(H^1(\mathbb{R}^3))'} \to 0 \quad \text{as } n \to +\infty.$$

By Lemma 3.3, if $v_n = u_n - u_\varepsilon \nrightarrow 0$ in $H^1(\mathbb{R}^3)$, there exists $\beta > 0$ independent of ε such that

$$\liminf_{n \to +\infty} |v_n|_2^2 \geq \beta.$$

Let $d_n = |v_n|_2$ satisfying that $|v_n|_2 \to d > 0$ and $|u_\varepsilon|_2 = b$, by Brézis-Lieb lemma, we obtain $a^2 = b^2 + d^2$. By Lemma 3.2, we have $b > 0$ and in its proof it was proved that $J_\varepsilon(v_n) \geq \Upsilon_{\infty,d_n} + o_n(1)$, we must have $d_n \in (0, a)$ for n large enough, and so

$$c + o_n(1) = J_\varepsilon(u_n) = J_\varepsilon(v_n) + J_\varepsilon(u_\varepsilon) + o_n(1) \geq \Upsilon_{\infty,d_n} + \Upsilon_{0,b} + o_n(1).$$

By Lemma 2.6, we infer that

$$\rho + \Upsilon_{0,a} > \frac{d_n^6}{a^6}\Upsilon_{\infty,a} + \frac{b^6}{a^6}\Upsilon_{0,a}.$$

As $n \to +\infty$, using $a^2 = b^2 + d^2$, we arrive at the inequality

$$\rho > \frac{d^6}{a^6}\Upsilon_{\infty,a} + \frac{b^6 - a^6}{a^6}\Upsilon_{0,a} > \frac{d^6}{a^6}(\Upsilon_{\infty,a} - \Upsilon_{0,a}) + \frac{3a^2d^4 - 3a^4d^2}{a^6}\Upsilon_{0,a} > \frac{\beta^3}{a^6}(\Upsilon_{\infty,a} - \Upsilon_{0,a}),$$

which is contradict with (3.4). Thus, $v_n \to 0$ in $H^1(\mathbb{R}^3)$, that is, $u_n \to u_\varepsilon$ in $H^1(\mathbb{R}^3)$, which implies that $|u_\varepsilon|_2 = a$ and

$$-\Delta u_\varepsilon + V(\varepsilon x)u_\varepsilon + \phi u_\varepsilon = \lambda_\varepsilon u_\varepsilon + f(u_\varepsilon) \quad \text{in } \mathbb{R}^3,$$

where λ_ε is the limit of some subsequence of $\{\lambda_n\}$. $\square$

4 Multiplicity result

Let $\delta > 0$ be fixed and w be a positive solution of the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \phi u = f(u) + \lambda u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 \mathrm{d}x = a^2, \end{cases}$$

with $J_0(w) = \Upsilon_{0,a}$. Let η be a smooth nonincreasing cut-off function satisfying

$$\eta(s) = \begin{cases} 1, & 0 \le s \le \frac{\delta}{2}, \\ 0, & s \ge \delta. \end{cases}$$

For any $y \in M$, let us define

$$\Psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|) w\left(\frac{\varepsilon x - y}{\varepsilon}\right), \qquad \widetilde{\Psi}_{\varepsilon,y}(x) = a\frac{\Psi_{\varepsilon,y}(x)}{|\Psi_{\varepsilon,y}|_2},$$

and denote Φ_ε: $M \to S(a)$ by $\Phi_\varepsilon(y) = \widetilde{\Psi}_{\varepsilon,y}$. Clearly, $\Phi_\varepsilon(y)$ has a compact support for any $y \in M$.

Lemma 4.1 (See [14, Chapter II, 3.2])**.** *Let I be a C^1-functional defined on C^1-Finsler manifold $\mathcal{V}$. If I is bounded from below and satisfies the (PS) condition, the I has at least $\mathrm{cat}_{\mathcal{V}}(\mathcal{V})$ distinct critical points.*

Lemma 4.2 (See [8, Lemma 4.3])**.** *Let Γ, Ω^+, Ω^- be closed sets with $\Omega^- \subset \Omega^+$. Let $\Phi : \Omega^- \to \Gamma$, $\beta : \Gamma \to \Omega^+$ be two continuous maps such that $\beta \circ \Phi$ is homotopically equivalent to the embedding $\mathrm{Id} : \Omega^- \to \Omega^+$. Then $\mathrm{cat}(\Gamma) \ge \mathrm{cat}_{\Omega^+}(\Omega^-)$.*

Lemma 4.3. *The function Φ_ε has the following property:*

$$\lim_{\varepsilon \to 0} J_\varepsilon(\Phi_\varepsilon(y)) = \Upsilon_{0,a}, \quad \textit{uniformly in } y \in M.$$

Proof. To prove this lemma, we argue by contradiction that there exist $\delta_0 > 0$, $\{y_n\} \subset M$, $\{y_n\}$ is a bounded sequence and $\varepsilon_n \to 0$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - \Upsilon_{0,a}| \ge \delta_0, \qquad \forall n \in \mathbb{N}.$$

Since

$$|\eta(\varepsilon_n z) w(z)|^2 \to |w(z)|^2 \quad \text{a.e. in } \mathbb{R}^3 \text{ as } n \to +\infty,$$

and

$$|\eta(\varepsilon_n z) w(z)|^2 \le |w(z)|^2,$$

by Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} |\Psi_{\varepsilon_n, y_n}|^2 \mathrm{d}x = \lim_{n\to+\infty} \int_{\mathbb{R}^3} |\eta(\varepsilon_n z) w(z)|^2 \mathrm{d}z = \int_{\mathbb{R}^3} |w|^2 \mathrm{d}z = a^2.$$

Then, there exists $N > 0$ such that

$$|\Psi_{\varepsilon_n, y_n}|_2^2 \ge \frac{a^2}{2}, \qquad \forall n > N.$$

Setting $|\Psi_{\varepsilon_n,y_n}|_2^2 \geq C = \min\{\frac{a^2}{2}, |\Psi_{\varepsilon_1,y_1}|_2^2, |\Psi_{\varepsilon_2,y_2}|_2^2, \dots, |\Psi_{\varepsilon_N,y_N}|_2^2\}$.

Since

$$\lim_{n\to+\infty} F(\Phi_{\varepsilon_n}(y_n)) = \lim_{n\to+\infty} F\left(a\frac{\eta(\varepsilon_n z)w(z)}{|\eta(\varepsilon_n z)w(z)|_2}\right) = F(w) \quad \text{a.e. in } \mathbb{R}^3,$$

and by (f_1) and (f_2), we have that

$$|F(\Phi_{\varepsilon_n}(y_n))| = \left|F\left(a\frac{\eta(\varepsilon_n z)w(z)}{|\eta(\varepsilon_n z)w(z)|_2}\right)\right| \leq C_1|w(z)|^p + C_2|w(z)|^q,$$

thus, by Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} F(\Phi_{\varepsilon_n}(y_n))\mathrm{d}x = \lim_{n\to+\infty}\int_{\mathbb{R}^3} F\left(a\frac{\eta(\varepsilon_n z)w(z)}{|\eta(\varepsilon_n z)w(z)|_2}\right)\mathrm{d}z = \int_{\mathbb{R}^3} F(w)\mathrm{d}z.$$

For almost every $z \in \mathbb{R}^3$, we deduce that

$$\begin{aligned}
&\lim_{n\to+\infty} |\nabla\Phi_{\varepsilon_n}(y_n)|^2 \\
&= \lim_{n\to+\infty} \frac{a^2}{|\Psi_{\varepsilon_n,y_n}|_2^2}|\nabla(\eta(\varepsilon_n z)w(z))|^2 \\
&= \lim_{n\to+\infty} |\nabla(\eta(\varepsilon_n z))w(z) + \eta(\varepsilon_n z)\nabla w(z)|^2 \\
&= \lim_{n\to+\infty} [\varepsilon_n^2|\nabla(\eta(\varepsilon_n z))w(z)|^2 + |\eta(\varepsilon_n z)\nabla w(z)|^2 + 2\varepsilon_n\eta(\varepsilon_n z)\nabla(\eta(\varepsilon_n z))w(z)\nabla w(z)] \\
&= \lim_{n\to+\infty} |\nabla w(z)|^2
\end{aligned}$$

and

$$\begin{aligned}
|\nabla\Phi_{\varepsilon_n}(y_n)|^2 &\leq \frac{a^2}{C}[\varepsilon_n^2|\nabla(\eta(\varepsilon_n z))w(z)|^2 + |\eta(\varepsilon_n z)\nabla w(z)|^2 + 2\varepsilon_n\eta(\varepsilon_n z)\nabla(\eta(\varepsilon_n z))w(z)\nabla w(z)] \\
&\leq \frac{a^2}{C}[C_3\varepsilon_n^2|w(z)|^2 + |\nabla w(z)|^2 + C_4\varepsilon_n^2|w(z)|^2|\nabla w(z)|^2],
\end{aligned}$$

by Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} |\nabla\Phi_{\varepsilon_n}(y_n)|^2\mathrm{d}x = \int_{\mathbb{R}^3} |\nabla w|^2\mathrm{d}z.$$

Since

$$\lim_{n\to+\infty} V(\varepsilon_n x)|\Phi_{\varepsilon_n}(y_n)|^2 = \lim_{n\to+\infty} \frac{a^2V(\varepsilon_n z + y_n)}{|\Psi_{\varepsilon_n,y_n}|_2^2}|\eta(\varepsilon_n z)w(z)|^2 = 0 \quad \text{a.e. in } \mathbb{R}^3,$$

and

$$V(\varepsilon_n x)|\Phi_{\varepsilon_n}(y_n)|^2 = \frac{a^2V(\varepsilon_n z + y_n)}{|\Psi_{\varepsilon_n,y_n}|_2^2}|\eta(\varepsilon_n z)w(z)|^2 \leq \frac{a^2}{C}\mu_* W(z)^2,$$

by Lebesgue's dominated convergence theorem, we deduce that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} V(\varepsilon_n x)|\Phi_{\varepsilon_n}(y_n)|^2\mathrm{d}x = 0.$$

Since

$$\begin{aligned}
&\lim_{n\to+\infty} \phi_{\Phi_{\varepsilon_n}(y_n)}\Phi_{\varepsilon_n}(y_n)^2 \\
&\quad = \lim_{n\to+\infty} \frac{\left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n z)w(z)\right|^2\left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n r)w(r)\right|^2}{|z-r|} = \phi_w w^2 \quad \text{a.e. in } \mathbb{R}^3,
\end{aligned}$$

and by Lemma 2.2-(2), we have that

$$\phi_{\Phi_{\varepsilon_n}(y_n)}\Phi_{\varepsilon_n}(y_n)^2 = \frac{\left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n z)w(z)\right|^2 \left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n r)w(r)\right|^2}{|z-r|}$$
$$\leq \frac{a^4}{C^4}\frac{|w(z)|^2|w(r)|^2}{|z-r|} \leq C_5\phi_w w^2,$$

by Lebesgue's dominated convergence theorem, there holds

$$\begin{aligned}\lim_{n\to+\infty}&\int_{\mathbb{R}^3}\phi_{\Phi_{\varepsilon_n}(y_n)}\Phi_{\varepsilon_n}(y_n)^2\mathrm{d}x\\ &= \lim_{n\to+\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n z)w(z)\right|^2 \left|\frac{a}{|\Psi_{\varepsilon_n,y_n}|_2}\eta(\varepsilon_n r)w(r)\right|^2}{|z-r|}\mathrm{d}z\mathrm{d}r\\ &= \int_{\mathbb{R}^3}\phi_w w^2\mathrm{d}z.\end{aligned}$$

Consequently,

$$\lim_{n\to+\infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = J_{0,a}(w) = \Upsilon_{0,a},$$

which is absurd. Hence, we complete the proof. □

For any $\delta > 0$, let $R = R(\delta) > 0$ be such that $M_\delta \subset B_R(0)$. Let χ: $\mathbb{R}^3 \to \mathbb{R}^3$ denote by $\chi(x) = x$ for $|x| \leq R$ and $\chi(x) = \frac{Rx}{|x|}$ for $|x| \geq R$. Hereafter, we are going to consider β_ε: $S(a) \to \mathbb{R}^3$ given by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^3}\chi(\varepsilon x)|u|^2\mathrm{d}x}{a^2}.$$

Lemma 4.4. *The function Φ_ε has the following property:*

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\Phi_\varepsilon(y)) = y, \quad \textit{uniformly in } y \in M.$$

Proof. Suppose on the contrary that there exist $\delta_0 > 0$, $\{y_n\} \subset M$, and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0, \qquad \forall n \in \mathbb{N}. \tag{4.1}$$

By the definition of $\Phi_{\varepsilon_n}(y_n)$ and β_{ε_n}, we have that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^3}(\chi(\varepsilon_n z + y_n) - y_n)|\eta(\varepsilon_n z)w(z)|^2\mathrm{d}z}{|\Psi_{\varepsilon_n,y_n}(\frac{\varepsilon_n+y_n}{\varepsilon_n})|_2^2}.$$

Since $(y_n) \subset M \subset B_R(0)$,

$$\frac{(\chi(\varepsilon_n z + y_n) - y_n)|\eta(\varepsilon_n z)w(z)|^2}{|\Psi_{\varepsilon_n,y_n}(\frac{\varepsilon_n+y_n}{\varepsilon_n})|_2^2} \to 0 \quad \text{a.e. in } \mathbb{R}^3,$$

and

$$\frac{(\chi(\varepsilon_n z + y_n) - y_n)|\eta(\varepsilon_n z)w(z)|^2}{|\Psi_{\varepsilon_n,y_n}(\frac{\varepsilon_n+y_n}{\varepsilon_n})|_2^2} \leq \frac{2R}{C}|w(z)|^2,$$

by Lebesgue's dominated convergence theorem, we deduce that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \to 0, \quad \text{as } n \to +\infty,$$

which attains a contradiction with (4.1). Hence, we complete the proof. □

Proposition 4.5. *Let $\varepsilon_n \to 0$ and $\{u_n\} \subset S(a)$ with $J_\varepsilon(u_n) \to \Upsilon_{0,a}$. Then, there is $\{\widetilde{y}_n\} \subset \mathbb{R}^3$ such that $v_n(x) = u_n(x + \widetilde{y}_n)$ has a strongly convergent subsequence in $H^1(\mathbb{R}^3)$. Moreover, up to a subsequence, $y_n = \varepsilon_n \widetilde{y}_n \to y$ in $\mathbb{R}^3$ for some $y \in M$.*

Proof. Firstly, we claim that there exist $R_0, \tau > 0$ and $\widetilde{y}_n \in \mathbb{R}^3$ such that

$$\int_{B_{R_0}(\widetilde{y}_n)} |u_n|^2 \mathrm{d}x \geq \tau \qquad \forall n \in \mathbb{N}.$$

Otherwise, owing to Lions' vanishing lemma, we have that $u_n \to 0$ in $L^p(\mathbb{R}^3)$ for all $p \in (2, 2^*)$, which implies that $\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x \to 0$. Thus, $\lim_{n\to+\infty} J_{\varepsilon_n}(u_n) \geq 0$, which contradicts with $\lim_{n\to+\infty} J_{\varepsilon_n}(u_n) = \Upsilon_{0,a} < 0$.

Considering $v_n(x) = u_n(x + \widetilde{y}_n)$, up to a subsequence, we may assume that there exists $v \in H^1(\mathbb{R}^3) \setminus \{0\}$ satisfying $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$. Since $\{v_n\} \subset S(a)$ and $J_{\varepsilon_n}(u_n) \geq J_0(u_n) = J_0(v_n) \geq \Upsilon_{0,a}$, there holds that $J_0(v_n) \to \Upsilon_{0,a}$. By Theorem 2.8, $v_n \to v$ in $H^1(\mathbb{R}^3)$ and $v \in S(a)$.

In what follows, we are to prove that $\{y_n\}$ is bounded. Arguing by contradiction that for some subsequence $|y_n| \to +\infty$, the limit

$$\begin{aligned}
\Upsilon_{0,a} &= \lim_{n\to+\infty} J_{\varepsilon_n}(u_n) \\
&= \lim_{n\to+\infty} \left(\frac{1}{2} \int_{\mathbb{R}^3} (|\nabla v_n|^2 + V(\varepsilon_n x + y_n)|v_n|^2)\mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{v_n} {v_n}^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(v_n)\mathrm{d}x \right) \\
&\geq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla v|^2 + V_\infty |v|^2)\mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} \phi_v v^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(v)\mathrm{d}x \\
&\geq \Upsilon_{\infty,a},
\end{aligned}$$

this gives a contradiction due to (3.1). Therefore, we can suppose that $y_n \to y$ in $\mathbb{R}^3$. Similarly discussed as above, we obtain

$$\Upsilon_{0,a} \geq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla v|^2 + V(y)|v|^2)\mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} \phi_v v^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(v)\mathrm{d}x \geq \Upsilon_{V(y),a}.$$

By Corollary 2.9, we know that $\Upsilon_{V(y),a} > \Upsilon_{0,a}$ as $V(y) > 0$. Since $V(y) \geq 0$ for all $y \in \mathbb{R}^3$, the above inequality implies that $V(y) = 0$, that is, $y \in M$. □

Let h: $[0, +\infty) \to [0, +\infty)$ be a function such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$ and set

$$\widetilde{S}(a) = \{u \in S(a) : J_\varepsilon(u) \leq \Upsilon_{0,a} + h(\varepsilon)\}. \tag{4.2}$$

In view of Lemma 4.3, the function $h(\varepsilon) = \sup_{y\in M} |J_\varepsilon(\Phi_\varepsilon(y)) - \Upsilon_{0,a}|$ satisfies that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, $\Phi_\varepsilon(y) \in \widetilde{S}(a)$ for all $y \in M$.

Lemma 4.6. *Let $\delta > 0$ and $M_\delta = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, M) \leq \delta\}$. There holds*

$$\lim_{\varepsilon\to 0} \sup_{u\in\widetilde{S}(a)} \inf_{z\in M_\delta} |\beta_\varepsilon(u) - z| = 0.$$

Proof. Let $\varepsilon_n \to 0$ and $u_n \in \widetilde{S}(a)$ such that

$$\inf_{z\in M_\delta} |\beta_{\varepsilon_n}(u_n) - z| = \sup_{u_n\in\widetilde{S}(a)} \inf_{z\in M_\delta} |\beta_{\varepsilon_n}(u_n) - z| + o_n(1).$$

According to the above equality, it is sufficient to find a sequence $\{y_n\} \subset M_\delta$ such that

$$\lim_{n\to+\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

Since $u_n \in \widetilde{S}(a)$, we obtain

$$\Upsilon_{0,a} \leq J_0(u_n) \leq J_{\varepsilon_n}(u_n) \leq \Upsilon_{0,a} + h(\varepsilon_n) \qquad \forall n \in \mathbb{N},$$

and so,

$$u_n \in S(a) \quad \text{and} \quad J_{\varepsilon_n}(u_n) \to \Upsilon_{0,a}.$$

From Proposition 4.5, it follows that there exists $\{\widetilde{y}_n\} \subset \mathbb{R}^3$ such that $y_n = \varepsilon_n \widetilde{y}_n \to y$ for some $y \in M$ and $v_n(x) = u_n(x + \widetilde{y}_n)$ is strongly convergent to some $v \in H^1(\mathbb{R}^3)$ with $v \neq 0$. Then, $\{y_n\} \subset M_\delta$ for n large enough and

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^3} (\chi(\varepsilon_n z + y_n) - y_n)|v_n|^2 \mathrm{d}z}{a^2},$$

which implies that

$$\beta_{\varepsilon_n}(u_n) - y_n = \frac{\int_{\mathbb{R}^3} (\chi(\varepsilon_n z + y_n) - y_n)|v_n|^2 \mathrm{d}z}{a^2} \to 0 \quad \text{as } n \to +\infty.$$

The proof is completed. □

4.1 Proof of Theorem 1.2.

In what follows, let $\varepsilon \in (0, \varepsilon_0)$. By Lemma 4.3, for any $y \in M$, we have

$$J_\varepsilon(\Phi_\varepsilon(y)) \leq \Upsilon_{0,a} + h(\varepsilon), \qquad h(\varepsilon) \to 0 \, (\varepsilon \to 0),$$

which implies that $\Phi_\varepsilon(M) \subset \widetilde{S}(a)$. By Lemma 4.6, we obtain

$$\mathrm{dist}(\beta_\varepsilon(u), M_\delta) \leq \delta, \qquad \forall u \in \widetilde{S}(a),$$

which leads to $\beta_\varepsilon(\widetilde{S}(a)) \subset M_\delta$. Hence, we have that $\beta_\varepsilon \circ \Phi_\varepsilon(M) \subset M_\delta$. We define $id : M \to M_\delta$. Hereafter, let us define $W : [0,1] \times M \to M_\delta$

$$W(t,y) = t\beta_\varepsilon \circ \Phi_\varepsilon + (1-t)\,\mathrm{id}(y) \qquad t \in [0,1],$$

satisfying $W(0,y) = \mathrm{id}(y)$, $W(1,y) = \beta_\varepsilon \circ \Phi_\varepsilon$, we can conclude that $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopic to the inclusion map $\mathrm{id} : M \to M_\delta$. By Lemma 4.2, it follows that

$$\mathrm{cat}(\widetilde{S}(a)) \geq \mathrm{cat}_{M_\delta}(M).$$

Arguing as Lemma 2.4, we also have that J_ε is bounded from below on $S(a)$. From Lemma 3.4, we have that the functional J_ε satisfies the $(PS)_c$ condition for the $c \in (\Upsilon_{0,a}, \Upsilon_{0,a} + h(\varepsilon))$. By Lemma 4.1, there exists at least $\mathrm{cat}(S(a))$ critical points of J_ε restricted to $S(a)$. Since $\widetilde{S}(a) \subset S(a)$, $\mathrm{cat}(\widetilde{S}(a)) \leq \mathrm{cat}(S(a))$. Then, by the Lusternik–Schnirelmann category theory (see [19] and Theorem 5.20 of [30]), we have that J_ε has at least $\mathrm{cat}_{M_\delta}(M)$ critical points on $S(a)$.

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Limit cycles bifurcations of a Liénard system with a hyperelliptic Hamiltonian of degree five

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Abstract. We deal with limit cycles bifurcating from the period annulus of Liénard system with a hyperelliptic Hamiltonian of degree five under quartic perturbation, where Liénard system has a normal form $\dot{x} = y$, $\dot{y} = x(x-1)(x^2+ax+b)$, $a^2-4b<0$. It is proved that the perturbation of this system can produce at most six limit cycles for $a = b = 2$.

Keywords: Liénard system, Poincaré bifurcation, limit cycles.

2020 Mathematics Subject Classification: 34C05, 34C08, 34C23.

1 Introduction

In the qualitative theory of real planar differential systems, one of research focus is the number and configuration of limit cycles, which belong to the context of the second part of Hilbert's 16th Problem. Until now the problem still remains to be unsolved even though a lot of works have been done. As well known, Arnold [1] proposed a weaker version of this problem, the so-called infinitesimal Hilbert's 16th problem, that is to study the number of isolated zeros of the Abelian integrals obtained from integrating polynomial 1-forms over ovals of polynomial Hamiltonian.

Consider perturbations of the Hamiltonian system

$$\begin{cases} \dot{x} = H_y(x,y) + \varepsilon P(x,y), \\ \dot{y} = -H_x(x,y) + \varepsilon Q(x,y), \end{cases} \tag{1.1}$$

where $H(x,y)$ is a polynomial of degree $n+1$, $P(x,y)$ and $Q(x,y)$ are polynomials of degree m in x, y, and ε is a small parameter.

We assume that there is a family of ovals $\Gamma_h \subset \{(x,y) \mid H(x,y) = h\}$, continuously depending on a parameter $h \in (h_1,h_2)$, then the Abelian integral of system (1.1) is defined as

$$I(h) = \oint_{\Gamma_h} P(x,y)dy - Q(x,y)dx, \tag{1.2}$$

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where Γ_h is the open punctured neighborhood foliated by periodic orbits of system (1.1) as $\varepsilon = 0$. The displacement function $d(h, \varepsilon)$ of system (1.1) is defined on a segment transversal to the flow, which is parameterized by the Hamiltonian value h, then

$$d(h, \varepsilon) = \oint_{\Gamma_h} dH = \varepsilon(I(h) + O(\varepsilon)). \tag{1.3}$$

Hence, if $I(h)$ is not identically zero, then the number of isolated zeros of the Abelian integral $I(h)$ (or be called the first order Melnikov function) gives an upper bound for the number of limit cycles of system (1.1) in any compact region of period annulus. It is well known that the limit cycles bifurcating from the period annulus is called Poincaré bifurcation.

The generalized Liénard system $\dot{x} = y, \dot{y} = f(x) + \varepsilon y g(x)$ of type (m, n) has rich dynamic behavior, where m and n are degrees of polynomials $f(x)$ and $g(x)$, respectively. If $m = 2, 3$ and $\varepsilon = 0$, then Hamiltonian functions of this system are called elliptic. Many authors studied the bifurcations of limit cycles on this system. Dumortier and Li have made a complete investigate for Liénard system of type $(3, 2)$ in a series of papers (see [3–6]), and they proved that the upper bound of number of isolated zeroes of Abelian integrals is five. In [11], the authors also investigated some Liénard systems of type $(3, 2)$ with symmetry, which exist at most two limit cycles.

If $m \geq 4$ and $\varepsilon = 0$, then Hamiltonian functions of the above Liénard system are called hyperelliptic. In [7], Gavrilov and Iliev given the topological classification of hyperelliptic Hamiltonian system of degree five, its normal form of Hamiltonian function is

$$H(x, y) = \frac{1}{2}y^2 + \frac{\lambda\mu}{2}x^2 - \frac{\lambda + \mu + \lambda\mu}{3}x^3 + \frac{1 + \lambda + \mu}{4}x^4 - \frac{1}{5}x^5, \tag{1.4}$$

where there are eleven cases having period annulus. J. Wang (see [15,16]) studied the number of limit cycles of two classes of Liénard systems of type $(4, 3)$ and $(4, 2)$, in which unperturbed systems has a saddle and degenerated polycycle, respectively. The authors of [20] obtained lower bounds of the number of limit cycles for a Liénard system of type $(4, n)$ having two elementary centers, where $20 \leq n \leq 24$.

In this paper, we choose one of eleven cases in [7], that is, we investigation Poincaré bifurcation for a Liénard system of type $(4, 3)$ with hyperelliptic Hamiltonian $H(x, y) = h$, $h \in (h_1, h_2)$ in (1.4) having a pair of conjugated complex critical points. The perturbation system is as follows

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(x-1)(x^2 + ax + b) + \varepsilon(\alpha + \beta x + \gamma x^2 + x^3)y, \quad a^2 - 4b < 0. \end{cases} \tag{1.5}$$

It is easy to know that the unperturbed system of (1.5) has a bounded period annulus surrounding the elementary center $(0, 0)$, corresponding to endpoint h_1, and a homoclinic loop (bounder of period annulus) passing through hyperbolic saddle $(1, 0)$, corresponding to endpoint h_2. By (1.2), we know that the Abelian integral of system (1.5) is

$$I(h) = \int_{\Gamma_h} (\alpha + \beta x + \gamma x^2 + x^3) y dx = \alpha I_0(h) + \beta I_1(h) + \gamma I_2(h) + I_3(h), \tag{1.6}$$

where $I_i(h) = \int_{\Gamma_h} x^i y dx, i = 0, 1, 2, 3$ and Γ_h is the compact component of $H(x, y) = h$, defined by (1.5).

There are many techniques and arguments to tackle the problem of bounding the number of zeroes of Abelian integrals, lots of them are very long and non-trivial, see [2]. Since

Hamiltonian function $H(x,y)$ has the higher degree, a purely algebraic criterion proposed in [8] and [12] are usual methods. These methods can transfer the estimation the number of zeros of Abelian integrals to that of the number of real roots of linear combinations of a tuple $(I_0(h), I_1(h), I_2(h), I_3(h))$ of associated semi-algebraic systems (SAS for short). This criterion can reduce the difficulties of qualitative analysis in limit cycle bifurcating from a center. Nonetheless, it is challenging and very difficult to obtain the cyclicity of this family of period annuli that depends on the variables a, b indeed, which need to verify the problem whether the collection of Abelian integrals is an ECT-system or a Chebyshev system with accuracy k (see [8]). We attempt several values of the variables a, b by using software *Maple*, which lead to a desktop computer to a dead end due to the huge polynomials with huge coefficients and thousands of terms.

In the present paper, we take $a = b = 2$, and use approaches of *real root isolation* and interval analysis to get the number of roots of huge polynomial, as a result, we can obtain the number of zeros of Abelian integrals of system (1.5) for $a = b = 2$. We rewrite system (1.5) as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(x-1)(x^2+2x+2) + \varepsilon(\alpha + \beta x + \gamma x^2 + x^3)y, \end{cases} \tag{1.7}$$

where α, β, γ are arbitrary real constants and $\varepsilon > 0$ is a small parameter, and the first integral of (1.7) is

$$H(x,y) = \frac{1}{2}y^2 + x^2 - \frac{1}{4}x^4 - \frac{1}{5}x^5 = h, \qquad h \in \left(0, \frac{11}{20}\right) \tag{1.8}$$

as $\varepsilon = 0$. The projection interval of the period annulus Γ_h on the x-axis is $(x_0, 1)$, where $x_0 \approx -0.763592319985$, and it is an intersection of the homoclinic loop with the negative half axis of the x-axis. Phase portrait of the unperturbed system of (1.7) see Figure 1.1.

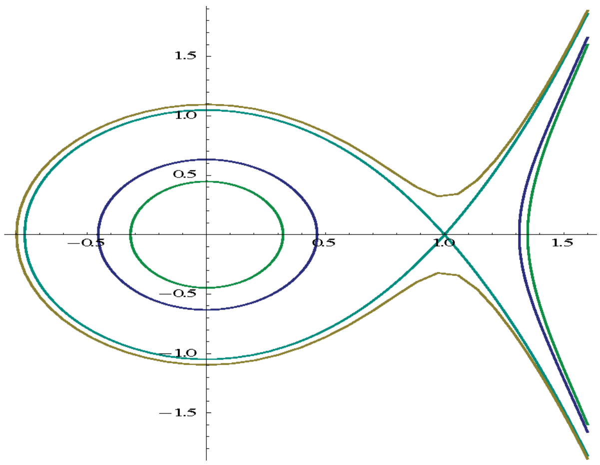

Figure 1.1: Phase portrait of system (1.5) when $\varepsilon = 0$.

The main purpose in this paper is to show that system (1.7) can undergo Poincaré bifurcation from the period annulus surrounding the origin. We can prove that the Abelian integral $I(h)$ has at most six zeros (taken into account muitiplicity), see Proposition 3.1 in Section 3. Proposition 3.1 and the equation (1.3) imply that system (1.7) can produce at most six limit cycles. The main results of this paper as follows.

Theorem 1.1. *The number of limit cycles of system* (1.7) *bifurcating from period annulus surrounding the center is at most six for arbitrary value of parameters* α, β, γ.

Note that unperturbed system of (1.7) has a period annulus surrounding the elementary center, its outer boundary is a saddle loop. According to Roussarie's theorem [14], the upper bound of number of isolated zeros of the Abelian integral $I(h)$ covers the number of limit cycles from center, from period annulus and from the homoclinic loop, therefore we have the following Theorem.

Theorem 1.2. *System (1.7) could give rise to at most six limit cycles in the finite plane surrounding the origin for sufficiently small ε and any parameters α, β, γ.*

The paper is organized as follows. In Section 2, we introduce some definitions and properties of Chebyshev systems. In Section 3, we study the number of zeros of Abelian integral $I(h)$ and obtain the maximal number of limit cycles bifurcating from period annulus by using Chebyshev criterion. Hence Proposition 3.1 is main result of this paper.

2 Preliminary properties

In order to study the number of isolated zeros of Abelian integral $I(h)$ in $h \in (0, \frac{11}{20})$, Grau et al. in [8] give a Chebyshev criterion, which check whether (I_0, I_1, I_2, I_3) in (1.6) is an extended complete Chebyshev system or Chebyshev system with accuracy k. Hence we introduce some preliminary definitions and properties, the reader can refer to [8, 12] or the recent paper [13] for more details.

Definition 2.1. Let $\varphi_0(x), \varphi_1(x), \ldots, \varphi_{n-1}(x)$ be analytic functions on an open interval L of $\mathbb{R}$.

(i) The set of functions $(\varphi_0(x), \varphi_1(x), \ldots, \varphi_{n-1}(x))$ is a Chebyshev system (T-system) with accuracy k on L if any nontrivial linear combination

$$\alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + \cdots + \alpha_{n-1}\varphi_{n-1}(x)$$

has at most $n + k - 1$ isolated zeros for $x \in L$.

(ii) The set of functions $(\varphi_0(x), \varphi_1(x), \ldots, \varphi_{n-1}(x))$ is an extended complete Chebyshev system (ECT-system) on L if for all $m = 1, 2, \ldots, n$, any nontrivial linear combination

$$\alpha_0\varphi_0(x) + \alpha_1\varphi_1(x) + \cdots + \alpha_{m-1}\varphi_{m-1}(x)$$

has at most $m - 1$ isolated zeros on L counted with multiplicities.

(iii) The continuous Wronskian of $(\varphi_0(x), \varphi_1(x), \ldots, \varphi_{m-1}(x))$ at $x \in L$ is

$$W[\varphi_0, \varphi_1, \ldots, \varphi_{m-1}](x) = \mathrm{Det}(\varphi_j^{(i)}(x))_{0\le i,j\le m-1} = \begin{vmatrix} \varphi_0(x) & \cdots & \varphi_{m-1}(x) \\ \varphi_0'(x) & \cdots & \varphi_{m-1}'(x) \\ \vdots & \ddots & \vdots \\ \varphi_0^{(m-1)}(x) & \cdots & \varphi_{m-1}^{(m-1)}(x) \end{vmatrix},$$

where $\varphi_j'(x)$ and $\varphi_j^{(i)}(x) (i \ge 2)$ represent the derivative of one order and the ith order of $\varphi_j(x)$, respectively.

Lemma 2.2 ([10] or [8]). *$(\varphi_0(x), \varphi_1(x), \ldots, \varphi_{n-1}(x))$ is an extended complete Chebyshev system on L if and only if, for each $m = 1, 2, \ldots, n$,*

$$W[\varphi_0, \varphi_1, \ldots, \varphi_{m-1}](x) \neq 0 \quad \textit{for all } x \in L.$$

Now we rewrite the first integral (1.8) as

$$H(x,y) = A(x) + B(x)y^2, \tag{2.1}$$

where

$$A(x) = x^2 - \frac{1}{4}x^4 - \frac{1}{5}x^5, \qquad B(x) = \frac{1}{2}$$

and $H(x,y)$ is an analytic function in open interval. There exists a period annulus filled by the set of ovals $\Gamma_h \in \{(x,y)|H(x,y) = h, h \in (0, \frac{11}{20})\}$ and $H(0,0) = 0$ is a local minimum. It is easy to verify that $xA'(x) = x^2(1-x)(x^2+2x+2) > 0$ for any $x \in (x_0, 1) \setminus 0$ ($x_0 \approx -0.763592319985$). Thus, there exists an analytic involution $z = \sigma(x)$ ($\sigma \circ \sigma = \text{Id}$ and $\sigma \neq \text{Id}$) with $x_0 < z < 0$ such that

$$A(x) = A(\sigma(x)) \quad \text{for all } x \in (0,1)$$

and $\sigma(0) = 0$. Using Theorem A in [12], we get the following lemma.

Lemma 2.3. *Assume that $g_i(x)$ are an analytic function on the interval $(x_0, 1)$, $i = 0, 1, 2, 3$. Denote*

$$\bar{I}_i(h) = \int_{\Gamma_h} g_i(x) y^{2s-1} dx,$$

where Γ_h is the set of periodic orbit surrounding the origin inside the level curve $\{A(x)+B(x)y^{2m} = h\}$ for each $h \in (0, \frac{11}{20})$. Let

$$\varphi_i(x) = \frac{g_i(x)}{A'(x)(B(x))^{\frac{2s-1}{2m}}} - \frac{g_i(\sigma(x))}{A'(\sigma(x))(B(\sigma(x)))^{\frac{2s-1}{2m}}}.$$

If the following statements hold:

- *(i) $W[\varphi_0, \varphi_1, \ldots, \varphi_m](x)$ is not vanish on $(0,1)$ for $m = 1, 2, \ldots, n-2$;*
- *(ii) $W[\varphi_0, \varphi_1, \ldots, \varphi_{n-1}](x)$ has k zeroes on $(0,1)$ counted with multiplicities, and*
- *(iii) $s > m(n+k-2)$,*

then $(\bar{I}_0(h), \bar{I}_1(h), \ldots, \bar{I}_{n-1}(h))$ has at most $n+k-1$ isolated zeros on $(0, \frac{11}{20})$ counted with multiplicities.

To prove Proposition 3.1 in Section 3, note that $I_i(h) = \int_{\Gamma_h} x^i y dx$, $s = 1, m = 1$ and $n = 4$, even if $k = 0$, but the condition $s > 2$ in Lemma 2.3 would not be satisfied. Hence we can not apply Lemma 2.3 directly. We need promote the power y in the integrand of $I_i(h)$ such that the conditions $s > m(n+k-2) = k+2$ hold. By using Lemma 4.1 in [8], we have

Lemma 2.4. *Let Γ_h be an oval inside the level curve $\{A(x) + B(x)y^2 = h\}$. If there exists a function $U(x)$ such that $\frac{U(x)}{A'(x)}$ is analytic at $x = 0$. Then, for any $s \in \mathbb{N}$,*

$$\int_{\Gamma_h} U(x) y^{s-2} dx = \int_{\Gamma_h} V(x) y^s dx,$$

where $V(x) = \frac{2}{s}\left(\frac{B(x)U(x)}{A'(x)}\right)' - \frac{B'(x)U(x)}{A'(x)}$.

In order to get the number of real roots of linear combinations of a tuple $(I_0(h), I_1(h), I_2(h), I_3(h))$ in (1.6), Lemma 2.3 is a main criterion in our paper. By applying this criterion, we can transfer the estimation of the number of real zeroes of Abelian integral to that of the number of real roots of a tuple, which reduce a semi-algebraic systems(SAS) and the reader is referred to see [18,19] for more details. The key to solve of SAS question is to solve polynomial equations.

We suppose that the polynomial equation $W(x,z) = 0$ has a real root (x^*, z^*) at the rectangle domain $D = \{(x,z) \mid (x,z) \in (0,1) \times (x_0, 0)\}$, where the variables x, z also satisfy the equation $q(x,z) = 0$, where $z = \sigma(x)$ is an involution and $\sigma'(x) < 0$. Solving this SAS question is to solve common root of systems of equations of two unknowns $W(x,z) = 0, q(x,z) = 0$. We divide analytic techniques into three steps.

Step 1 (Elimination variable by resultant): We can elimination variable x (or z) by the theory of resultant, that is, variables x, z satisfy the resultant equation

$$R(z) = \text{res}(W(x,z), q(x,y), x) = 0 \quad \text{or} \quad \bar{R}(x) = \text{res}(W(x,z), q(x,y), z) = 0.$$

Step 2 (Interval isolation of real root): Without loss of generality, Assume that the resultant equation $\bar{R}(x) = 0$ has a real root x^* in $[x_1, x_2] \subset [0,1]$ (corresponding z^* in $[z_1, z_2] \subset [x_0, 0]$) by using command realroot in *Maple*. We substitute $x = x_1$ and $x = x_2$ into $p(x,z) = 0$, and let maximal interval of isolation real root of equation $p(x,z) = 0$ be $[z_{11}, z_{12}] \subset [z_1, z_2]$. Thus we minimize the possible existing the rectangle domain of the common roots of $W(x,z) = 0$ and $q(x,y) = 0$.

However, it should be noted that if we solve $\bar{R}(x) = 0$ directly without using interval isolation of real root, then we can only get a approximation of x^*. But the results of these numerical calculation are sometimes unreliable due to the thousands of terms of polynomials with huge coefficients, a famous cautionary example see [17], and example of numerical calculation see Lemma 3.4 [iv] of [9].

Step 3 (Analysis of common real roots): Let the above the rectangle domain be $ABCD$, where $A(x_1, z_{12})$, $B(x_1, z_{11})$, $C(x_2, z_{11})$ and $D(x_2, z_{12})$. We can analysize whether the rectangle domain has a common root or not by positive and negative values of $W(x,z)$ and $q(x,z)$ at vertices A, B, C, D, the involution and the intermediate value theorem of continuous function. The detailed application skills see proof of Lemma 3.2.

3 Poincaré bifurcations of system (1.7)

In this Section, to prove Theorem 1.1, firstly, we will study the number of isolated zeros of Abelian integral $I(h)$ in $h \in (0, \frac{11}{20})$, that is the following Proposition 3.1.

Proposition 3.1. *For arbitrary value of parameters (α, β, γ), the Abelian integral $I(h)$ of system (1.7) has at most six zeros (counting multiplicities) in $h \in (0, \frac{11}{20})$.*

The main tools of proving Proposition 3.1 are Lemmas 2.2–2.4. Hence we need to check the Chebyshev property of the Abelian integral (1.6). Proof of Proposition 3.1 will be given at the end of this section.

Now applying Lemma 2.4, we rewrite $I_i(h)$ in (1.7) as

$$\begin{aligned} I_i(h) &= \frac{1}{h} \int_{\Gamma_h} \left(A(x) + \frac{1}{2} y^2 \right) x^i y dx = \frac{1}{h} \int_{\Gamma_h} \left[x^i A(x) y + \frac{1}{2} x^i y^3 \right] dx \\ &= \frac{1}{h} \int_{\Gamma_h} V_i(x) y^3 dx, \qquad i = 0, 1, 2, 3, \end{aligned}$$

where

$$V_i(x) = \frac{x^i \nu_i(x)}{60(x-1)^2(x^2+2x+2)^2}$$

with

$$\begin{aligned}\nu_i(x) = &(160+40i) - (130+30i)x^2 - (112+28i)x^3 \\ &+ (35+5i)x^4 + (68+9i)x^5 + (34+4i)x^6.\end{aligned}$$

To promote the power y such that the condition $s > 2$ (suppose $k = 0$) is satisfied, by using Lemma 2.4 again, we obtain that

$$\begin{aligned}I_i(h) &= \frac{1}{h^2}\int_{\Gamma_h}\left(A(x)+\frac{1}{2}y^2\right)V_i(x)y^3dx \\ &= \frac{1}{h^2}\int_{\Gamma_h}\left[V_i(x)A(x)y^3+\frac{1}{2}V_i(x)y^5\right]dx = \frac{1}{h^2}\int_{\Gamma_h} g_i(x)y^5dx,\end{aligned}$$

where

$$g_i(x) = \frac{x^i \tau_i(x)}{6000(x-1)^4(x^2+2x+2)^4} \tag{3.1}$$

with

$$\begin{aligned}\tau_i(x) = &(38400+16000i+1600i^2) - (62400+24800i+2400i^2)x^2 \\ &- (51840+21920i+2240i^2)x^3 + (44300+15200i+1300i^2)x^4 \\ &+ (81120+27600i+2400i^2)x^5 + (23992+8324i+804i^2)x^6 \\ &- (41480+11990i+820i^2)x^7 - (37939+10782i+719i^2)x^8 \\ &- (5354+1851i+134i^2)x^9 + (11096+2375i+121i^2)x^{10} \\ &+ (7344+1492i+72i^2)x^{11} + (1836+352i+16i^2)x^{12}.\end{aligned}$$

Denote

$$\bar{I}_i(h) = h^2 I_i(h) = \int_{\Gamma_h} g_i(x)y^5dx, \qquad h \in \left(0, \frac{11}{20}\right).$$

We can see that $g_i(x)$ is analytic on $(x_0, 1)$ and $(I_0(h), I_1(h), I_2(h), I_3(h))$ is an ECT-system or T-system on $(0, \frac{11}{20})$ if and only if so is $(\bar{I}_0(h), \bar{I}_1(h), \bar{I}_2(h), \bar{I}_3(h))$. Therefore, applying Lemma 2.4 to $(\bar{I}_0(h), \bar{I}_1(h), \bar{I}_2(h), \bar{I}_3(h))$ with $s = 3$, we have

$$\bar{\varphi}_i(x,z) = \bar{g}_i(x) - \bar{g}_i(z) = \left(\frac{4\sqrt{2}g_i}{A'}\right)(x) - \left(\frac{4\sqrt{2}g_i}{A'}\right)(z), \qquad i = 0,1,2,3, \tag{3.2}$$

where

$$\bar{g}_i(x) = \frac{\sqrt{2}x^{i-1}\tau_i(x)}{1500(1-x)^5(x^2+2x+2)^5},$$

$$\bar{g}_i(z) = \frac{\sqrt{2}z^{i-1}\tau_i(z)}{1500(1-z)^5(z^2+2z+2)^5}$$

and $z = \sigma(x)$ is an involution function.

On the other hand, due to

$$A(x) - A(z) = \frac{1}{20}(x-z)p(x,z) = 0,$$

where

$$p(x,z) = -20(x+z) + 5(x^3 + x^2z + xz^2 + z^3) + 4(x^4 + x^3z + x^2z^2 + xz^3 + z^4).$$

Since $\sigma(0) = 0$, It turns out that $z = \sigma(x)$ is defined by means of $p(x,z) = 0$. Moreover, we get that

$$\sigma'(x) = \frac{dz}{dx} = -\frac{p'_x(x,z)}{p'_z(x,z)}, \tag{3.3}$$

where

$$p'_x(x,z) = -20 + 15x^2 + 10xz + 5z^2 + 16x^3 + 12x^2z + 8xz^2 + 4z^3,$$
$$p'_z(x,z) = -20 + 5x^2 + 10xz + 15z^2 + 4x^3 + 8x^2z + 12xz^2 + 16z^3.$$

By Lemma 2.2, we find that the Wronskian $W[(\bar{\varphi}_0, \bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)](x)$ has two zeros on $(0,1)$ by interval analysis, which shows that $(\bar{\varphi}_0, \bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)$ is not an ECT-system on $(0,1)$. According to Lemma 2.3(iii), we need to take $s > m(n+k-2) = 4$. Hence we lift the power of y in the integrand of $I_i(h)$ to $2s-1 = 9$. But we find that the associated Wronskian $W[(\varphi_0, \varphi_1, \varphi_2, \varphi_3)](x)$ has three zeros for $x \in (0,1)$. So we further take $s = 6$ and lift the power of y of integrand of $I_i(h)$ to $2s-1 = 11$, fortunately, the associated Wronskian $W[(\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)](x)$ has still three zeros, all of the first order Wronskians $W[(\tilde{\varphi}_i)](x) (i = 0,1,2,3)$, the second order Wronskians $W[(\tilde{\varphi}_2, \tilde{\varphi}_1)](x)$ and the third order Wronskians $W[(\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0)](x)$ have all no zero for $x \in (0,1)$, which give us hope to obtain the number of zeros of linear combinations of the tuple of functions $(\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)$ by Lemma 2.3.

Repeating the above procedures, it follows from Lemma 2.4 that

$$\tilde{I}_i(h) = h^5 I_i(h) = \int_{\Gamma_h} F_i(x) y^{11} dx, \qquad h \in \left(0, \frac{11}{20}\right). \tag{3.4}$$

where

$$F_i(x) = \frac{x^i G_i(x)}{33264000000(x-1)^{10}(x^2+2x+2)^{10}},$$

and $G_i(x)$ is a polynomial of degree 30 in x, we omit it here because the polynomial has a longer expression. Let

$$\tilde{\varphi}_i(x,z) = \left(\frac{32\sqrt{2}F_i}{A'}\right)(x) - \left(\frac{32\sqrt{2}F_i}{A'}\right)(z), \qquad i = 0,1,2,3, \tag{3.5}$$

where $z = \sigma(x)$ is an involution and $x \in (0,1)$.

According to Lemma 2.3, we need to compute the number of zeros of many Wronskians, moreover, each of Wronskians is a the huge polynomial with huge coefficients and hundreds of items. After many attempts to the ordered linear combinations of associated criterion function $(\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)$, finally we find that the tuple of functions $(\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3)$ satisfy the statements in Lemma 2.3. Then we get the following lemma.

Lemma 3.2. $(\tilde{I}_2(h), \tilde{I}_1(h), \tilde{I}_0(h), \tilde{I}_3(h))$ *has at most six isolated zeros on* $(0, \frac{11}{20})$ *counted with multiplicities.*

Proof. It follows from Lemma 2.3 that we need to verify the statements (i) and (ii). For this purpose, we divide the proof into four cases.

Case 1: The fourth Wronskian $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z)$ has three zeros for $(x,z) \in (0,1) \times (x_0, 0)$.

With the help of the computer algebraic system *Maple*, by direct computation, we get that

$$
\begin{aligned}
&W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z) \\
&= -\frac{(x-z)^{10}\omega_4(x,z)}{834198140917968750000000000000x^4z^4q_4(x)q_4(z)(p_z'(x,z))^6},
\end{aligned} \tag{3.6}
$$

where $\omega_4(x,z)$ is a polynomial of degree 228 in (x,z), $p_z'(x,z)$ as in (3.3),

$$
q_4(x) = (x-1)^{38}(x^2+2x+1)^{38}, \qquad q_4(z) = (z-1)^{38}(z^2+2z-1)^{38},
$$

and $z = \sigma(x)$ is an implicit function determined by the polynomial equation $p(x,z) = 0$. Meanwhile, we find that $\omega_4(x,z)$ is symmetric polynomial with respect to x, z.

We assert that $p_z'(x,z) \neq 0$ for any $(x,z) \in (0,1) \times (x_0, 0)$. In fact, by computing the resultant $R(p_z', p, z)$ with respect to z between $p_z'(x,z)$ and $p(x,z)$, we have

$$
\begin{aligned}
R(p_z', p, z) = 8000(4x^3+13x^2+22x+11)(4x^3+5x^2-20) \\
\times (16x^6 - 24x^5 - 7x^4 - 68x^3 + 256x^2 - 296x + 148).
\end{aligned}
$$

It is easy to verify that three polynomial factors in $R(p_z', p, z)$ have not zeros in $x \in (0,1)$. This implies $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z)$ is well defined in $(0,1) \times (x_0, 0)$.

Next we calculate the resultant with respect to z between $\omega_4(x,z)$ and $p(x,z)$ and obtain

$$
R(\omega_4, p, z) = 396154108207169536000000000000(x-1)^{28}(x^2+20+2)^{28}\phi_4(x),
$$

where $\phi_4(x)$ is a polynomial of degree 828 in x.

Note that $\omega_4(x,z)$ and $p(x,z)$ are symmetric polynomials with respect to x, z and $z = \sigma(x)$ is an involution. Therefore the resultant $R(\omega_4, p, z)$(or $R(\omega_4, p, x)$) between $\omega_4(x,z)$ and $p(x,z)$ with respect to x (or z) suffices that the statement $R(\omega_4, q, x) = R(\omega_4, q, z)|_{z=x}$. By command realroot in *Maple*, we get that $\phi_4(x)$ has five isolate zeros $x_i^*(i = 1,2,3,4,5)$ in $(0,1)$ and three isolate zeros $z_i^*(i = 1,2,3)$ in $(x_0, 0)(x_0 \approx -0.763592319985)$. List of the isolation intervals of these zeros are as follows

$$
\begin{aligned}
x_1^* \in [\hat{x}_{11}, \hat{x}_{12}] &= \left[\frac{73134709858141}{140737488355328}, \frac{36567354929071}{70368744177664}\right], \\
x_2^* \in [\hat{x}_{21}, \hat{x}_{22}] &= \left[\frac{42527490909395}{70368744177664}, \frac{85054981818791}{140737488355328}\right], \\
x_3^* \in [\hat{x}_{31}, \hat{x}_{32}] &= \left[\frac{29671532711011}{35184372088832}, \frac{118686130844045}{140737488355328}\right], \\
x_4^* \in [\hat{x}_{41}, \hat{x}_{42}] &= \left[\frac{122355652851087}{140737488355328}, \frac{7647228303193}{8796093022208}\right], \\
x_5^* \in [\hat{x}_{51}, \hat{x}_{52}] &= \left[\frac{137735701579791}{140737488355328}, \frac{8608481348737}{8796093022208}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
z_1^* \in [\hat{z}_{11}, \hat{z}_{12}] &= \left[-\frac{53678146230419}{70368744177664}, -\frac{107356292460837}{140737488355328}\right], \\
z_2^* \in [\hat{z}_{21}, \hat{z}_{22}] &= \left[-\frac{102328736960905}{140737488355328}, -\frac{12791092120113}{17592186044416}\right], \\
z_3^* \in [\hat{z}_{31}, \hat{z}_{32}] &= \left[-\frac{80910761056793}{140737488355328}, -\frac{10113845132099}{17592186044416}\right].
\end{aligned}
$$

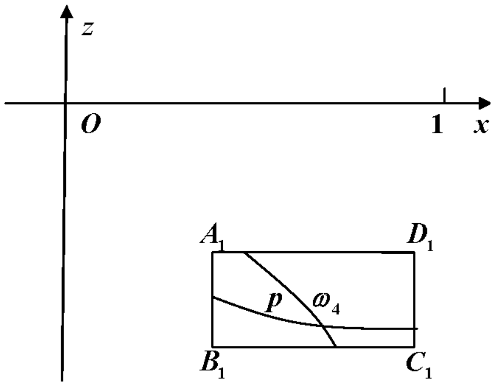

Figure 3.1: The sketch graph of $\omega_4(x,z) = 0$ and $p(x,z) = 0$ intersecting in the rectangle $A_1B_1C_1D_1$.

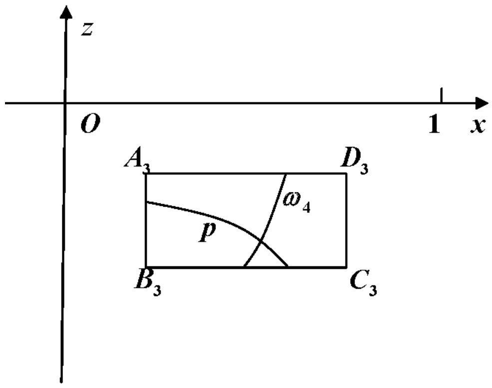

Figure 3.2: The sketch graph of $\omega_4(x,z) = 0$ and $p(x,z) = 0$ intersecting in the rectangle $A_3B_3C_3D_3$.

Since $z = \sigma(x)$ is an involution determined by $p(x,z) = 0$ and satisfy $\sigma'(x) < 0$ for $x \in (0,1)$, which imply that $\omega_4(x,z)$ and $p(x,z) = 0$ have at most three common real roots. Firstly, we substitute $z = \hat{z}_{11}$ and $z = \hat{z}_{12}$ into $p(x,z)$, then equation $p(x,z) = 0$ has one root in interval

$$[x_{11}, x_{12}] = \left[\frac{68867850789901}{70368744177664}, \frac{137735701579803}{140737488355328}\right],$$

and

$$[x_{21}, x_{22}] = \left[\frac{34433925394947}{35184372088832}, \frac{137735701579789}{140737488355328}\right],$$

respectively. It is easy to verify that interval $[x_{11}, x_{12}] \supset [\hat{x}_{51}, \hat{x}_{52}]$ and $[x_{21}, x_{22}] \supset [\hat{x}_{51}, \hat{x}_{52}]$, which shows that $\omega_4(x,z)$ and $p(x,z) = 0$ possibly have common real root.

Let the four vertices of closed rectangle containing the point (x_5^*, z_1^*) be $A_1(\hat{x}_{51}, \hat{z}_{12})$, $B_1(\hat{x}_{51}, \hat{z}_{11})$, $C_1(\hat{x}_{52}, \hat{z}_{11})$ and $D_1(\hat{x}_{52}, \hat{z}_{12})$, see Figure 3.1. Substituting coordinates of four vertices A_1, B_1, C_1, D_1 into $\omega_4(x,z)$, we get that

$$\omega_4(A_1) = -\frac{1103261\cdots6171875}{1000814\cdots7682176}, \qquad \omega_4(B_1) = -\frac{2177946\cdots3395375}{1973759\cdots7196544},$$
$$\omega_4(C_1) = \frac{3939512\cdots4495625}{2780712\cdots3723776}, \qquad \omega_4(D_1) = \frac{3353939\cdots1640625}{2365568\cdots5694464},$$

here we omit digits using dots for brevity because their numerators and denominators are all huge numbers.

Note that $\omega_4(A_1) < 0, \omega_4(B_1) < 0, \omega_4(C_1) > 0$ and $\omega_4(D_1) > 0$, by command fsolve in *Maple*, we find that $\omega_4(x,z)$ has not zero at the sides A_1B_1 and C_1D_1 of rectangle $A_1B_1C_1D_1$ as $x = \hat{x}_{51}$ and $x = \hat{x}_{52}$, respectively, which implies that $\omega_4(x,z) < 0$ at the side A_1B_1 and $\omega_4(x,z) > 0$ at the side C_1D_1. Meanwhile, the zero set of polynomial $\omega_4(x,z)$ intersecting with the sides B_1C_1 and A_1D_1 of rectangle forms a simple curve ended by points $(0.9786710221234176135, \hat{z}_{11})$ and $(0.9786710221234176104, \hat{z}_{12})$, respectively.

Using the same sequence, substituting coordinates of four vertices A_1, B_1, C_1, D_1 into $p(x,z)$, we have

$$p(A_1) = -\frac{1656260\cdots5869779}{9807971\cdots0539264}, \qquad p(B_1) = \frac{9711472\cdots0200531}{9807971\cdots0539264},$$
$$p(C_1) = \frac{5556451\cdots8475385}{6129982\cdots4408704}, \qquad p(D_1) = -\frac{2477410\cdots4753759}{9807971\cdots0539264}.$$

By direct computation, we get that $p(x,z) < 0$ at the side A_1D_1, $p(x,z) > 0$ at the side B_1C_1, a simple curve $p(x,z) = 0$ intersect the side A_1B_1 with point $(\hat{x}_{51}, -0.76281233746184644)$ and the side C_1D_1 with point $(\hat{x}_{52}, -0.76281233746184695)$. According to the Intermediate Value Theorem, curves $\omega_4(x,z) = 0$ intersect transversely with $p(x,z) = 0$ in the rectangle $A_1B_1C_1D_1$, which implies that $\omega_4(x,z) = 0$ and $p(x,z) = 0$ have and only have one common solution in the associated rectangle. Hence $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z)$ has one zero in the rectangle $A_1B_1C_1D_1$.

Secondly, we substitute $z = \hat{z}_{21}$ and $z = \hat{z}_{22}$ into $p(x,z)$, we find that $p(x,z) = 0$ has one root in interval

$$[\bar{x}_{11}, \bar{x}_{12}] = \left[\frac{59343065422023}{70368744177664}, \frac{118686130844047}{140737488355328}\right],$$

and

$$[\bar{x}_{21}, \bar{x}_{22}] = \left[\frac{118686130844043}{140737488355328}, \frac{29671532711011}{35184372088832}\right],$$

respectively. Note that

$$[\min(\bar{x}_{11}, \bar{x}_{21}), \max(\bar{x}_{12}, \bar{x}_{22})] = [\bar{x}_{21}, \bar{x}_{12}] \supset [\hat{x}_{31}, \hat{x}_{32}].$$

By similar discussion and computation to the above procedure and omitting the details for the sake of brevity, we get that $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z)$ has also one zero in the rectangle $A_2B_2C_2D_2$, where the vertices $A_2(\hat{x}_{31}, \hat{z}_{22})$, $B_2(\hat{x}_{31}, \hat{z}_{21}), C_2(\hat{x}_{32}, \hat{z}_{21})$ and $D_2(\hat{x}_{32}, \hat{z}_{22})$.

Finally, we substitute $z = \hat{z}_{31}$ into $p(x,z) = 0$ and get that it has one root just in interval $[\hat{x}_{21}, \hat{x}_{22}]$. Using the same sequence, let $A_3(\hat{x}_{21}, \hat{z}_{32})$, $B_3(\hat{x}_{21}, \hat{z}_{31}), C_3(\hat{x}_{22}, \hat{z}_{31})$ and $D_3(\hat{x}_{22}, \hat{z}_{32})$, see Figure 3.2. In the rectangle $A_3B_3C_3D_3$, we obtain that

$$\omega_4(A_3) < 0, \qquad \omega_4(B_3) < 0, \qquad \omega_4(C_3) > 0, \qquad \omega_4(D_3) > 0,$$

and

$$p(A_3) < 0, \qquad p(B_3) > 0, \qquad p(C_3) < 0, \qquad p(D_3) < 0,$$

here we omit the specific values of $\omega_4(x,z)$ and $p(x,z)$ at A_3, B_3, C_3, D_3 for brevity. By similar analysis, we find that one simple curve $\omega_4(x,z) = 0$ in the rectangle $A_3B_3C_3D_3$ intersect the side A_3D_3 with point $(0.604351994715489, \hat{z}_{32})$ and the side B_3C_3 with point $(0.604351994715488, \hat{z}_{31})$, the other simple curve $p(x,z) = 0$ intersect the side A_3B_3 with point $(\hat{x}_{21}, -0.574905535137251)$ and the side B_3C_3 with point $(0.604351994715491, \hat{z}_{31})$. Since $0.604351994715491 > 0.604351994715488$, it is obvious that curves $\omega_4(x,z) = 0$ and $p(x,z) = 0$ has one intersection in the rectangle $A_3B_3C_3D_3$, which shows that $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z)$ has one zero in the rectangle $A_3B_3C_3D_3$.

To sum up, we prove that $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0, \tilde{\varphi}_3](x,z)$ has three zeros for $(x,z) \in (0,1) \times (x_0, 0)$.

Case 2: The third Wronskian $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0](x,z)$ has no zero for $(x,z) \in (0,1) \times (x_0, 0)$.

Applying the same method as the fourth Wronskian, we obtain the third Wronskian

$$W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0](x,z) = \frac{\sqrt{2}(x-z)^6 \omega_3(x,z)}{57779957812500000000000 x^3 z^3 q_3(x) q_3(z) (p_z'(x,z))^3}, \tag{3.7}$$

where $\omega_3(x,z)$ is a asymmetric polynomial of degree 177 in (x,z), $p_z'(x,z)$ as in (3.3),

$$q_3(x) = (x-1)^{30}(x^2 + 2x + 2)^{30} \quad \text{and} \quad q_3(z) = (z-1)^{30}(z^2 + 2z + 2)^{30}.$$

The resultant with respect to x between $\omega_3(x,z)$ and $p(x,z)$ is

$$R(\omega_3, p, x) = 629407744000000z^2(z-1)^{24}(z^2+2z+2)^{24}\phi_3(z),$$

here $\phi_3(z)$ is a polynomial of degree 636 in z. We find three isolate zeros of $\phi_3(z)$ in $(x_0, 0)$ and one isolate zero in $(0,1)$. These real root isolation intervals are as follows

$$\tilde{z}_1 \in [\tilde{z}_{11}, \tilde{z}_{12}] = \left[-\frac{771}{2024}, -\frac{6167}{8192}\right], \qquad \tilde{z}_2 \in [\tilde{z}_{21}, \tilde{z}_{22}] = \left[-\frac{39}{64}, -\frac{4991}{8192}\right],$$
$$\tilde{z}_3 \in [\tilde{z}_{31}, \tilde{z}_{32}] = \left[-\frac{3753}{16384}, -\frac{7505}{32768}\right], \qquad \tilde{x} \in [\tilde{x}_{11}, \tilde{x}_{12}] = \left[\frac{6279}{8192}, \frac{785}{1024}\right].$$

We take interval end points $\tilde{x}_{11}$ and $\tilde{x}_{12}$ into interval polynomial equation $p(x,z)) = 0$ and get two real roots intervals in $(x_0, 0)$

$$[\bar{z}_{11}, \bar{z}_{12}] = \left[-\frac{5637}{8192}, \frac{1409}{2408}\right], \qquad [\bar{z}_{21}, \bar{z}_{22}] = \left[-\frac{2819}{4096}, \frac{5637}{8192}\right],$$

respectively. We find that each of intervals $[\bar{z}_{i,1}, \bar{z}_{i,2}] (i = 1,2)$ has not intersection with any of the intervals $[\tilde{z}_{i,1}, \tilde{z}_{i,2}]$ for $i = 1,2,3$, which implies that there does not exist value of (x,z) in plane area $D = (0,1) \times (x_0, 0)$ such that both $\omega_3(x,z) = 0$ and $p(x,z) = 0$ hold simultaneously. This shows that $W[\tilde{\varphi}_2, \tilde{\varphi}_1, \tilde{\varphi}_0](x,z) \neq 0$ for $(x,z) \in (0,1) \times (x_0, 0)$.

Case 3: The second Wronskian $W[\tilde{\varphi}_2, \tilde{\varphi}_1](x,z)$ has no zero for $(x,z) \in (0,1) \times (x_0, 0)$.

By similar calculation to Case 2, we get that

$$W[\tilde{\varphi}_2, \tilde{\varphi}_1](x,z) = \frac{(x-z)^3\omega_2(x,z)}{10005187500000000q_2(x)q_2(z)p'_z(x,z)}, \tag{3.8}$$

where $\omega_2(x,z)$ is a asymmetric polynomial of degree 120 in (x,z),

$$q_1(x) = (x-1)^{21}(x^2+2x+2)^{21} \quad \text{and} \quad q_1(z) = (z-1)^{21}(z^2+2z+2)^{21}. \tag{3.9}$$

The resultant with respect to x between $\omega_2(x,z)$ and $p(x,z)$ is

$$R(\omega_1, p, x) = 6400(z-1)^{18}(z^2+2z+2)^{18}\phi_2(z),$$

where $\phi_2(z)$ is a polynomial of degree 426 in z.

Using command realroot in *Maple*, we find that $\phi_2(z)$ has two zero in the interval $(x_0, 0)$ and one zero in $(0,1)$, which are

$$z_1 \in [z_{11}, z_{12}] = \left[-\frac{93303}{131072}, -\frac{46651}{65536}\right], \qquad z_2 \in [z_{21}, z_{22}] = \left[-\frac{85549}{262144}, -\frac{21387}{65536}\right],$$
$$x \in [x_1, x_2] = \left[\frac{39715}{65536}, \frac{79431}{131073}\right].$$

By solving polynomial equations $p([x_i, z]) = 0$ $(i = 1,2)$, we get the following the isolation intervals of real root z

$$[\bar{z}_{11}, \bar{z}_{12}] = \left[-\frac{75527}{131072}, -\frac{37763}{65536}\right] \quad \text{and} \quad [\bar{z}_{21}, \bar{z}_{22}] = \left[-\frac{9441}{16384}, -\frac{75527}{131072}\right].$$

We find that the intervals $[\bar{z}_{11}, \bar{z}_{12}]$ and $[\bar{z}_{21}, \bar{z}_{22}]$ have not intersection with intervals $[z_{11}, z_{12}]$ and $[z_{21}, z_{22}]$. This shows that $\omega_2(x,z) = 0$ and $p(x,z) = 0$ have no common root for $x_0 < z < 0 < x < 1$. Therefore, $W[\bar{\varphi}_2, \bar{\varphi}_1](x,z) \neq 0$ for all $(x,z) \in (0,1) \times (x_0, 0)$.

Case 4: The first Wronskian $W[\tilde{\varphi}_2](x,z)$ has no zero for $(x,z) \in (0,1) \times (x_0, 0)$.

We get easily that the first Wronskian

$$W[\tilde{\varphi}_2](x,z) = \tilde{\varphi}_2(x,z) = \frac{\sqrt{2}(x-z)\omega_1(x,z)}{346500000 q_1(x) q_1(z)}, \tag{3.10}$$

where $\omega_1(x,z)$ is a polynomial of degree 63 in (x,z),

$$q_1(x) = (x-1)^{11}(x^2+2x+2)^{11} \quad \text{and} \quad q_1(z) = (z-1)^{11}(z^2+2z+2)^{11}.$$

The resultant between $\omega_1(x,z)$ and $p(x,z)$ with respect to x is

$$R(\omega_1, p, x) = (z-1)^{10}(z^2+2z+2)^{10}\phi_1(z),$$

where $\phi_1(z)$ is a polynomial of degree 222 in z. It is easy to know that $\phi_1(z) \neq 0$ for $z \in (x_0, 0)$ by command realroot in *Maple* , which implies that $\omega_1(x,z) = 0$ and $p(x,z) = 0$ have no common root. Hence $W[\tilde{\varphi}_2](x,z) \neq 0$ for all $(x,z) \in (0,1) \times (x_0, 0)$.

Summarizing the above cases 1-4, we have verified that the statements (i) and (ii) in Lemma 2.3 are hold. It follows from Lemma 2.3 that Lemma 3.2 holds, thus we finish proof of Lemma 3.2. □

Proof of Proposition 3.1. Since $h^5 I_i(h) = \tilde{I}_i(h)$, $i = 0,1,2,3$, any linear combination of $(\tilde{I}_0, \tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ has the same number of zeros as that occurs with (I_0, I_1, I_2, I_3). It is easy to see that Abelian integral in (1.6) together with Hamiltonian (1.8) is the linear span of generators (I_0, I_1, I_2, I_3). By Lemma 3.2, we know that any linear combination of $(\tilde{I}_0, \tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ has at most six zeros on $(0, \frac{11}{20})$ counted with multiplicities, this implies that any linear combination of (I_0, I_1, I_2, I_3) has also at most six zeros. Therefore, the Abelian integral $I(h)$ has at most six zeros for arbitrary parameters (α, β, γ). The proof of Proposition 3.1 is completed. □

Proof of Theorem 1.1. By Proposition 3.1 and the equation (1.3), if $I(h)$ is not identically zero, then system (1.7) has at most six limit cycles bifurcating from the period annulus of unperturbed system of (1.7) by Poincaré bifurcation. Thus we complete proof of Theorem 1.1. □

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Uniform approximation of a class of impulsive delayed Hopfield neural networks on the half-line

This paper is dedicated to the memory of István Győri

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Abstract. In this work, we investigate a uniform approximation of a nonautonomous delayed CNN-Hopfield-type impulsive system with an associated impulsive differential system where a partial discretization is introduced with the help of piecewise constant arguments. Sufficient conditions are formulated, which imply that the error estimate decays exponentially with time on the half-line $[0, \infty)$. A critical step for the proof of this estimate is to show that, under the assumed conditions, the solutions of the Hopfield impulsive system are exponentially bounded and exponentially stable. A bounded coefficients case is also analyzed under simplified conditions. An example is presented and simulated in order to show the applicability of our conditions.

Keywords: Hopfield neural networks, hybrid equations, impulsive differential equations, numerical approximation of solutions, piecewise constant arguments.

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1 Introduction

Cellular Neural Networks (CNNs) are widely used as mathematical models of the interactions of the neurons in the human brain. For its construction, electrical and chemical properties have been considered. The synapses correspond to the connections of the neurons (excitatory and inhibitory) and are modeled by positive and negative weights. The weighted neural inputs are added up. Then, the so-called activation function defines the amplitude of the response signal of the neuron.

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In [16], John Hopfield proposed a novel type of CNN in order to find how human memory works. They were called *Hopfield cellular neural network*, and it is represented by the following nonlinear system:

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}g_j(x_j(t)) + c_i(t), \quad i = 1, \ldots, m, \tag{1.1}$$

This model corresponds to a mesh of linked neurons, where every neuron is connected to all other neurons without self-connection. The states of the neurons are of binary type, and it depends on whether the neuron's input exceeds a fixed value. This type of CNN has been applied in psychology and combinatorics, among others (see [22]). Since the signals travel at a finite speed between the neurons, time delays are natural to introduce in the models. Without completeness, we refer to [3, 6, 7, 18, 20, 23, 27] for investigations of different classes of delayed CNN models.

In [19], A. D. Myshkis introduced differential equations of the form

$$x'(t) = f(t, x(t), x(\tau(t))),$$

where $\tau(t)$ corresponds to a deviated argument (a discontinuous piecewise constant function). These type of equations are called *Differential Equations with Piecewise Constant Arguments* (*DEPCA*). The research in this new field started in the 80's with the works of S. Busenberg and K. L. Cooke with a model of vertically transmitted diseases (see [8, 26]). There are many fields where this type of equations have been applied (see [5, 10, 17]).

In [2], M. U. Akhmet investigated systems of the form

$$y'(t) = f(t, y(t), y(\gamma(t))), \tag{1.2}$$

where $\gamma(t)$ is a *piecewise constant argument of generalized type*. More precisely, given $(t_n)_{n\in\mathbb{Z}}$ and $(\zeta_n)_{n\in\mathbb{Z}}$ such that $t_n < t_{n+1}, \forall n \in \mathbb{Z}$ with $\lim\limits_{n\to\pm\infty} t_n = \pm\infty$ and $t_n \leq \zeta_n \leq t_{n+1}$, then

$$\gamma(t) = \zeta_n, \quad \text{if } t \in I_n = [t_n, t_{n+1}).$$

When such a function γ is introduced, it generates advanced and delayed arguments in the equation, dividing the interval I_n into two pieces $I_n = I_n^+ \cup I_n^-$, where $I_n^+ = [t_n, \zeta_n]$ corresponds to the advanced, and $I_n^- = [\zeta_n, t_{n+1}]$ to the delayed interval. These equations are known as *Differential Equations with Piecewise Constant Argument of Generalized Type* (*DEPCAG*). In this class of differential equations, the solutions are continuous functions, although γ is a discontinuous function. Integrating (1.2) from t_n to t_{n+1} we obtain a difference equation, giving the character of hybrid to this kind of equations (see also [21]).

The following example will be important for the rest of the work (when $k = 0$). Consider $\gamma(t) = \left[\frac{t+k}{h}\right]h$ with $0 \leq k < h$, where $[\cdot]$ is the greatest integer function. We have

$$\left[\frac{t+k}{h}\right]h = nh, \text{ when } t \in I_n = [nh - k, (n+1)h - k).$$

Hence, $\gamma(t) - t \geq 0 \;\Leftrightarrow\; t \leq nh$ and $\gamma(t) - t \leq 0 \;\Leftrightarrow\; t \geq nh$, that implies

$$I_n^+ = [nh - k, nh], \quad I_n^- = [nh, (n+1)h - k].$$

Now, if additionally a jump condition is applied at the endpoints of the intervals $I_n = [t_n, t_{n+1})$, it defines the class of *Impulsive differential equations with piecewise constant argument of generalized type*, (*IDEPCAG*) (see [1]),

$$\begin{aligned} y'(t) &= f(t, y(t), y(\gamma(t))), \qquad t \neq t_n \\ \Delta y(t_n) := y(t_n) - y(t_n^-) &= J_n(y(t_n^-)), \qquad t = t_n, \quad n \in \mathbb{N}. \end{aligned} \tag{1.3}$$

Definition 1.1 (IDEPCAG solution)**.** A piecewise continuous function $y(t)$ is a solution of (1.3) if:

(i) $y(t)$ is continuous on $I_n = [t_n, t_{n+1})$ with discontinuities of the first kind at t_n with $n \in \mathbb{Z}$, where $y'(t)$ exists at each $t \in \mathbb{R}$ with the possible exception of the points t_n, where the lateral derivatives exist.

(ii) On each interval I_n, the ordinary differential equation

$$y'(t) = f(t, y(t), y(\zeta_n))$$

holds, with $\gamma(t) = \zeta_n$.

(iii) For $t = t_n$, the following impulsive condition holds:

$$\Delta y(t_n) = y(t_n) - y(t_n^-) = J_n(y(t_n^-)),$$

i.e., $y(t_n) = y(t_n^-) + J_n(y(t_n^-))$, where $y(t_n^-)$ denotes the left-hand limit of the function y at t_n.

I. Győri used first DEPCAG to approximate linear delay equations with constant delays in [12]. He defined three variants of approximating DEPCAG and proved the convergence of each method on compact time intervals. See also [14] for further generalization of this approach for other classes of differential equations.

In [9], Cooke and Győri proposed an approximation of a linear delay differential equation

$$x'(t) = \sum_{i=1}^{N} q_i x(t - \tau_i), \qquad t \geq 0, \tag{1.4}$$

$$x(t) = \phi(t), \qquad t \in [-\tau, 0], \tag{1.5}$$

where $q_i \in \mathbb{R}, \tau_i > 0$, and $\phi \in C([-\tau, 0], \mathbb{R})$. Here $C([-\tau, 0], \mathbb{R})$ denotes the space of real-valued continuous functions defined on $[-\tau, 0]$. In order to approximate (1.4)–(1.5), they proposed the following DEPCAG

$$y'(t) = \sum_{i=1}^{N} q_i y([t/h - [\tau_i/h]]h), \qquad t \geq 0, \tag{1.6}$$

$$y(nh) = \phi(nh), \qquad n = k, \ldots, 0. \tag{1.7}$$

In this case, the approximation considered was uniform over the non-compact interval $[0, \infty)$. The main assumption is a condition of asymptotic stability of the trivial solution of (1.4). Note that in [13] I. Győri and F. Hartung extended this result for linear neutral differential equations.

Recently, in [15] F. Hartung investigated the numerical approximation of the following scalar delay differential equation with impulsive self-support condition

$$\begin{aligned} x'(t) &= \alpha x(t) + \beta x(t - \tau), \qquad \text{a.e } \ t \geq 0 \\ x(t) &= c + d, \qquad \text{if } x(t^-) = c \end{aligned} \tag{1.8}$$

with the initial condition

$$x(t) = \varphi(t), \qquad \text{if } t \in [-\tau, 0],$$

where $c, d > 0, \alpha + |\beta| < 0$, $\tau > 0$, $c < \varphi(t)$, for $t \in [-\tau, 0]$, and $\varphi : [-\tau, 0] \to \mathbb{R}$ a Lipschitz continuous function. The approximating equation is an associated DEPCA with a self-support condition

$$\begin{aligned} y'(t) &= \alpha y([t/h]h) + \beta y([t/h]h - [\tau/h]h), \qquad \text{a.e } t \geq 0 \\ y(kh) &= c + d, \qquad \text{if } y(kh^-) \leq c \end{aligned} \tag{1.9}$$

with the initial condition

$$y(t) = \varphi(t), \qquad \text{if } t \in [-\tau, 0].$$

The convergence of (1.9) was proved at every point except the impulsive time moments.

In [24], R. Torres et al. considered the following impulsive Hopfield-type CNN system with impulses

$$\begin{aligned} x_i'(t) &= -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)g_j(x_j(t)) + c_i(t), && t \geq 0, \quad t \neq t_k, \\ \Delta x_i(t_k) &= -p_{i,k}x_i(t_k^-) + e_{i,k} + J_{i,k}(x_i(t_k^-)), && t = t_k, \\ x_i(t_0) &= x_i^0, \end{aligned} \tag{1.10}$$

and the following IDEPCA system

$$\begin{aligned} y_i'(t) &= -a_i(t)y_i(t) + \sum_{j=1}^{m} b_{ij}(t)g_j(y_j(\gamma(t))) + c_i(t), && t \geq 0, \quad t \neq \gamma(t_k) \\ \Delta y_i(\gamma(t_k)) &= -p_{i,k}y_i(\gamma(t_k)^-) + e_{i,k} + J_{i,k}(y_i(\gamma(t_k)^-)), && t = \gamma(t_k), \\ y_i(t_0) &= y_i^0, \end{aligned} \tag{1.11}$$

where $\gamma(t) = [t/h]h$. Assuming an ergodic stability condition over the corresponding linear homogeneous system associated with (1.10), the uniform approximation of (1.10) by the IDEPCA (1.11) was concluded over $[0, \infty)$, where the error of approximation was given by

$$|x_i(t) - y_i(t)| \leq \frac{|x_i^0 - y_i^0|}{1 - \theta_c} + \frac{o_i(h)}{1 - \theta_c},$$

with $o_i(h) \to 0$ as $h \to 0$, and $0 < \theta_c < 1$ were defined in [24].

In [11], M. Elghandouri and K. Ezzinbi, using resolvent operators theory, obtained an approximation of the mild solutions of the delayed semilinear integro-differential equation

$$\begin{aligned} x'(t) &= A(t)x(t) + \int_0^t G(t-s)x(s)ds + f(t, x(t-r)), && t \geq 0, \\ x(t) &= \varphi(t), && t \in [-r, 0], \end{aligned} \tag{1.12}$$

using an integro-differential equation with piecewise constant arguments

$$\begin{aligned} x_h'(t) &= A(t)x_h(t) + \int_0^t G(t-s)x_h(s)ds + f(t, x_h(\gamma_h(t-r))), && t \geq 0, \\ x_h(0) &= \varphi(0), \ x_h(t) = \varphi(kh), && t \in [kh, (k+1)h), \end{aligned} \tag{1.13}$$

with $k = -l, \ldots, -1$, and $\gamma_h(t) = [t/h]h$, on the Banach space $(X, \|\cdot\|)$. The approximation was done over compact and unbounded intervals. They also obtained an exponential error decay by using the stability of the resolvent operator and the Halanay's Inequality.

The interested reader in approximation of solutions of differential equations by using piecewise constant argument can see [25] for an elementary and simple introduction to the subject.

1.1 Aim of the work

In this paper, we use $\gamma(t) = [t/h]\,h$ as the piecewise constant argument function, where $[\cdot]$ is the greatest integer part function, and $h > 0$ is a fixed discretization parameter. We note that γ depends on the selection of h, but for simplicity, this dependence is not indicated explicitly in the notation, but it always should be kept in mind.

We consider a delayed CNN system with impulses

$$\begin{aligned} x_i'(t) &= -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)g_j(x_j(t-\tau_j)) + c_i(t), && t \geq 0, \quad t \neq t_k, \\ \Delta x_i(t_k) &= -p_{i,k}x_i(t_k^-) + e_{i,k} + J_{i,k}(x_i(t_k^-)), && k \in \mathbb{N}, \\ x_i(t) &= \varphi_i(t), && t \in [-\tau, 0]. \end{aligned} \tag{1.14}$$

Similar delayed CNN systems (without impulses) were investigated, e.g., in [7, 18, 20].

For a fixed discretization parameter $h > 0$ we associate to (1.14) the IDEPCA system

$$\begin{aligned} y_i'(t) &= -a_i(t)y_i(t) + \sum_{j=1}^{m} b_{ij}(t)g_j(y_j(\gamma(t) - \gamma(\tau_j))) + c_i(t), && t \geq 0, \quad t \neq \gamma(t_k), \\ \Delta y_i(\gamma(t_k)) &= -p_{i,k}y_i(\gamma(t_k)^-) + e_{i,k} + J_{i,k}(y_i(\gamma(t_k)^-)), && k \in \mathbb{N}, \\ y_i(t) &= \psi_i(t), && t \in [-\tau, 0], \end{aligned} \tag{1.15}$$

where $i = 1, 2, \ldots, m$, $k \in \mathbb{N} = \{1, 2, 3 \ldots\}$, t_k, $p_{i,k}$, $e_{i,k}$ are real sequences, a_i, b_{ij}, c_i are real-valued locally integrable functions on $[0, \infty)$, $J_{i,k} \in C(\mathbb{R}, \mathbb{R})$ and $g_j \in C(\mathbb{R}, \mathbb{R})$ for all $i, j = 1, \ldots, m$ and $k \in \mathbb{N}$; the constant delays satisfy $\tau_i \geq 0$ and $\tau = \max\{\tau_1, \ldots, \tau_m\} > 0$, and the initial functions $\varphi_i, \psi_i : [-\tau, 0] \to \mathbb{R}$ are continuous for $i = 1, \ldots, m$.

We note that the initial time in system (1.14) is fixed to be 0. This does not affect the generality of the problem, but it simplifies the definition of the approximation in (1.15), since, in this way, the initial time is a member of the mesh points of the piecewise constant approximation, and $\gamma(t) \geq 0$, and hence $\gamma(t) - \gamma(\tau_j) \geq -\tau_j \geq -\tau$ for $t \geq 0$ and $j = 1, \ldots, m$. Also, the impulse times t_k form a strictly monotone increasing sequence of positive reals, and they are approximated by $\gamma(t_k)$ for $k \in \mathbb{N}$ for the sake of easier computation of the numerical scheme.

For simplicity of the notation we introduce $t_0 = 0$, so the sequence t_k is defined for $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

The main goal of this manuscript is to show that the solutions of (1.15) approximate that of (1.14) uniformly on $[0, \infty)$, i.e.,

$$\sup_{t \in [0,\infty)} |x_i(t) - y_i(t)| \to 0, \quad \text{as } h \to 0+, \quad i = 1, \ldots, m,$$

assuming also $\varphi_i = \psi_i$ for $i = 1, \ldots, m$, and we show that, under certain conditions, the error estimate goes to 0 as $t \to \infty$ with an exponential speed.

Remark 1.2. This paper extends the work of [24] for the case when $\tau_i > 0$ in (1.14). Moreover, in this work, we assume a different set of conditions and we use M-matrix technique to get our main results. Another improvement corresponds to the exponential error decay of the approximation, see Theorem 3.1 below. A key step to obtain the main result is to show that, under the assumed conditions, the solutions of (1.14) are exponentially bounded (see Lemma 2.2) and are exponentially stable (see Lemma 2.4, below).

1.2 Hypotheses and main assumptions

In this manuscript, we will use the following assumptions on the parameters of problem (1.14):

(H1) Let $g_j \in C(\mathbb{R},\mathbb{R})$ be such that $g_j(0)=0$, and there exist constants $L_j \geq 0$ such that

$$|g_j(u)-g_j(v)| \leq L_j\,|u-v|\,, \qquad u,v\in\mathbb{R}, \quad j=1,2,\ldots,m.$$

(H2) Let $J_{i,k} \in C(\mathbb{R},\mathbb{R})$ be such that $J_{i,k}(0)=0$, and there exist constants $l_{i,k} \geq 0$ such that

$$|J_{i,k}(u)-J_{i,k}(v)| \leq l_{i,k}\,|u-v|\,, \qquad u,v\in\mathbb{R}, \quad i=1,\ldots,m, \quad k\in\mathbb{N}.$$

(H3) There exist positive constants $p_i^*, l_i^*, e_i^*, \underline{\delta}$ and real constants $\underline{p}_i$ for $i=1,\ldots,m$ such that

(i) $\underline{p}_i \leq p_{i,k} \leq p_i^* < 1, \qquad k\in\mathbb{N}, \quad i=1,\ldots,m;$

(ii) $0 \leq l_{ik} \leq l_i^*, \qquad k\in\mathbb{N}, \quad i=1,\ldots,m;$

(iii) $|e_{i,k}| \leq e_i^*, \qquad k\in\mathbb{N}, \quad i=1,\ldots,m;$

(iv) $0 < \underline{\delta} \leq t_{k+1}-t_k, \qquad k\in\mathbb{N}_0.$

(H4) There exist positive constants $\sigma_i, \Lambda_{ij}, c_i^*$ for $i,j=1,\ldots,m$ and ε_0 such that $0<\varepsilon_0<\sigma_i$ for $i=1,\ldots,m$, and

(i) $\sigma_i(t-s) \leq \int_s^t a_i(u)\,du - \sum_{j\in J(s,t)} \ln(1-p_{i,j}), \quad 0\leq s<t, \quad i=1,\ldots,m,$ where $J(s,t)=\{j\in\mathbb{N} : s\leq t_j<t\}$;

(ii) $\int_0^t e^{-(\sigma_i-\varepsilon_0)(t-s)}|b_{ij}(s)|\,ds \leq \Lambda_{ij}, \qquad t\geq 0, \quad i,j=1,\ldots,m;$

(iii) $|c_i(t)| \leq c_i^*, \qquad t\geq 0, \quad i=1,\ldots,m.$

(H5) $\displaystyle\sum_{j=1}^m (1-\underline{p}_i^-)\Lambda_{ij}L_j + \frac{(1-\underline{p}_i^-)l_i^*}{(1-p_i^*)(1-e^{-\sigma_i\underline{\delta}})} < 1, \qquad i=1,\ldots,m,$ where $\underline{p}_i^- = \min\{0,\underline{p}_i\}$.

(H6) There exist positive constants β_1 and β_2 such that

$$|e_{i,k}| \leq e^{-\beta_1 t_k}e_i^*, \qquad k\in\mathbb{N}, \qquad \text{and} \qquad |c_i(t)| \leq e^{-\beta_2 t}c_i^*, \quad t\geq 0, \quad i=1,\ldots,m.$$

(H7) There exist positive constants a_i^* for $i=1,\ldots,m$ such that $a_i(t)\leq a_i^*$, $t\geq 0$, $i=1,\ldots,m$.

(H8) There exist positive constants b_{ij}^* for $i,j=1,\ldots,m$ and L_φ such that

$$|b_{ij}(t)| \leq b_{ij}^*, \qquad t\geq 0, \qquad \text{and} \qquad |\varphi_i(t)-\varphi_i(\bar{t})| \leq L_\varphi|t-\bar{t}|, \qquad t,\bar{t}\in[-\tau,0]$$

for $i,j=1,\ldots,m$.

Remark 1.3. We comment that (H6) and (H8) yield (H3) (iii), (H4) (iii) and (H4) (ii) with $\Lambda_{ij} = \frac{b_{ij}^*}{\sigma_i-\varepsilon_0}$, but they are not assumed in Lemmas 2.2 and 2.4.

2 Auxiliary results

Recall that t_k is a strictly monotone increasing sequence which tends to $+\infty$ as $k \to \infty$. We denote the set of time moments by $\mathcal{T} = \{t_k : k \in \mathbb{N}\}$. Throughout this manuscript, we use the notation $\ell(t)$ for the uniquely defined nonnegative integer with the property that

$$t \in [t_{\ell(t)}, t_{\ell(t)+1}), \qquad t \geq 0. \tag{2.1}$$

Note that if $t \notin \mathcal{T}$, then $t_{\ell(t)} < t$, otherwise $t_{\ell(t)} = t$.

We use the vector notation $\mathbf{x}(t) = (x_1(t), \dots, x_m(t))^T$ throughout the manuscript. For a norm of vector $\mathbf{x} = (x_1, \dots, x_m)^T \in \mathbb{R}^m$ we use the infinity norm $|\mathbf{x}|_\infty = \max\{|x_1|, \dots, |x_m|\}$. The corresponding induced matrix norm is denoted by $\|A\|_\infty$ for $A \in \mathbb{R}^{m\times m}$. For continuous functions $\psi : [-\tau, 0] \to \mathbb{R}$ and $\boldsymbol{\psi} : [-\tau, 0] \to \mathbb{R}^m$ we use the supremum norm $|\psi|_C = \max_{-\tau\leq t\leq 0} |\psi(t)|$ and $|\boldsymbol{\psi}|_C = \max_{-\tau\leq t\leq 0} |\boldsymbol{\psi}(t)|_\infty$, respectively.

The notation $\mathbf{x} \leq \mathbf{y}$ is used for $\mathbf{x} = (x_1, \dots, x_m)^T \in \mathbb{R}^m$ and $\mathbf{y} = (y_1, \dots, y_m)^T \in \mathbb{R}^m$ if the componentwise comparisons $x_i \leq y_i$ hold for all $i = 1, \dots, m$. We note that $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y}$ implies $|\mathbf{x}|_\infty \leq |\mathbf{y}|_\infty$. We say that a matrix $A \in \mathbb{R}^{m\times m}$ is *monotone* if $A\mathbf{x} \leq A\mathbf{y}$ yields $\mathbf{x} \leq \mathbf{y}$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Let $I \in \mathbb{R}^{m\times m}$ denote the identity matrix. We say that the matrix $I - A \in \mathbb{R}^{m\times m}$ is a *nonsingular M-matrix* if $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of A. We refer to [4] for 50 equivalent definitions of a nonsingular M-matrix.

The following variation of constants formula was formulated in [24] for the system (1.14) without delays. It is straightforward to extend it for (1.14).

Lemma 2.1. *The solution* $\mathbf{x}(t) = (x_1(t), \dots, x_m(t))^T$ *of* (1.14) *satisfies*

$$\begin{aligned}
x_i(t) &= e^{-\int_0^t a_i(u)du} \left(\prod_{j=1}^{\ell(t)} (1 - p_{i,j}) \right) \varphi_i(0) \\
&\quad + \sum_{r=1}^{\ell(t)} \left(\prod_{j=r}^{k(t)} (1 - p_{i,j}) \right) \int_{t_{r-1}}^{t_r} e^{-\int_s^t a_i(u)du} G_i(s, \mathbf{x}(s)) ds \\
&\quad + \sum_{r=1}^{\ell(t)} \left(\prod_{j=r+1}^{k(t)} (1 - p_{i,j}) \right) e^{-\int_{t_r}^t a_i(u)du} (J_{i,r}(x_i(t_r^-)) + e_{i,r}) \\
&\quad + \int_{t_{\ell(t)}}^t e^{-\int_s^t a_i(u)du} G_i(s, \mathbf{x}(s)) ds, \qquad i = 1, \dots, m, \quad t \geq 0,
\end{aligned} \tag{2.2}$$

where

$$G_i(t, \mathbf{x}(t)) = \sum_{j=1}^m b_{ij}(t) g_j(x_j(t - \tau_j)) + c_i(t). \tag{2.3}$$

Next we show that under conditions (H1)–(H5) the solutions of (1.14) are bounded on $[0, \infty)$. Moreover, if assumption (H6) holds, the solutions are exponentially bounded.

Lemma 2.2. *Suppose (H1)–(H5) hold. Then all solutions of* (1.14) *are bounded on* $[0, \infty)$*. Moreover, if (H6) holds too, then for every solution* $\mathbf{x}(t)$ *of* (1.14) *there exist positive constants* α_0 *and* K_0 *such that*

$$|\mathbf{x}(t)|_\infty \leq K_0 e^{-\alpha_0 t}, \qquad t \geq -\tau, \tag{2.4}$$

i.e., every solution of (1.14) *is exponentially bounded.*

Proof. 1. First, we prove the boundedness of the solutions. Let $\mathbf{x}(t) = (x_1(t), \dots, x_m(t))^T$ be the solution of (1.14) corresponding to initial condition $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_m)^T$. Then x_i satisfies the variation of constant formula (2.2), where H_i is defined by (2.3).

Now suppose $s \in [t_{r-1}, t_r)$ for some $r \in \mathbb{N}$, $s < t$ and $t \notin \mathcal{T}$. Then we get $s < t_r < t_{\ell(t)} < t < t_{\ell(t)+1}$, $J(s,t) = \{r, r+1, \dots, \ell(t)\}$ if $t > t_r$, and $J(s,t) = \emptyset$ if $t \le t_r$, therefore (H3) (i) and (H4) (i) yield

$$e^{-\int_s^t a_i(u)\,du}\left(\prod_{j=r}^{\ell(t)}(1-p_{i,j})\right) = \exp\left(-\int_s^t a_i(u)\,du + \sum_{j\in J(s,t)} \ln(1-p_{i,j})\right) \le e^{-\sigma_i(t-s)}.$$

If $t = t_{r_0}$ for some $r_0 \in \mathbb{N}$, $s \in [t_{r-1}, t_r)$, $s < t$, then $J(s, t_{r_0}) = \{r, r+1, \dots, r_0 - 1\}$, and

$$e^{-\int_s^{t_{r_0}} a_i(u)\,du}\left(\prod_{j=r}^{\ell(t)}(1-p_{i,j})\right) = (1-p_{i,r_0})\exp\left(-\int_s^{t_{r_0}} a_i(u)\,du + \sum_{j=r}^{r_0-1} \ln(1-p_{i,j})\right) \le (1-\underline{p}_i)e^{-\sigma_i(t_{r_0}-s)}.$$

Hence

$$e^{-\int_s^t a_i(u)\,du}\left(\prod_{j=r}^{\ell(t)}(1-p_{i,j})\right) \le (1-\underline{p}_i^-)e^{-\sigma_i(t-s)}, \quad s \in [t_{r-1}, t_r), \;\; s \le t, \;\; i = 1, \dots, m, \tag{2.5}$$

where $\underline{p}_i^- = \min\{0, \underline{p}_i\}$.

For $s \in (t_{\ell(t)}, t)$ it follows $J(s,t) = \emptyset$, and

$$e^{-\int_s^t a_i(u)\,du} = \exp\left(-\int_s^t a_i(u)\,du + \sum_{j\in J(s,t)} \ln(1-p_{i,j})\right) \le e^{-\sigma_i(t-s)}. \tag{2.6}$$

For $t_r < t$, $t \notin \mathcal{T}$ we get $t_r < t_{r+1} \le t_{\ell(t)} < t$, so $J(t_r, t) = \{r, r+1, \dots, \ell(t)\}$, and therefore (H3) (i) and (H4) (i) imply

$$\begin{aligned} e^{-\int_{t_r}^t a_i(u)\,du}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right) &= \frac{1}{1-p_{i,r}} e^{-\int_{t_r}^t a_i(u)\,du}\left(\prod_{j=r}^{\ell(t)}(1-p_{i,j})\right) \\ &\le \frac{1}{1-p_i^*}\exp\left(-\int_{t_r}^t a_i(u)\,du + \sum_{j\in J(t_r,t)} \ln(1-p_{i,j})\right) \\ &\le \frac{1}{1-p_i^*} e^{-\sigma_i(t-t_r)}. \end{aligned}$$

Finally, if $t = t_{r_0}$ for some $r_0 \in \mathbb{N}$ and $r < r_0$, then it follows $J(t_r, t_{r_0}) = \{r, r+1, \dots, r_0 - 1\}$, $\ell(t) = r_0$, and so

$$\begin{aligned} e^{-\int_{t_r}^{t_{r_0}} a_i(u)\,du}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right) &= \frac{1-p_{i,r_0}}{1-p_{i,r}}\exp\left(-\int_{t_r}^{t_{r_0}} a_i(u)\,du + \sum_{j=r}^{r_0-1} \ln(1-p_{i,j})\right) \\ &\le \frac{1-\underline{p}_i}{1-p_i^*} e^{-\sigma_i(t_{r_0}-t_r)}. \end{aligned}$$

Combining the above two cases, we get

$$e^{-\int_{t_r}^{t} a_i(u)\,du}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\le\frac{1-\underline{p}_i^-}{1-p_i^*}e^{-\sigma_i(t-t_r)},\qquad t\ge t_r,\quad r\in\mathbb{N},\quad i=1,\dots,m. \tag{2.7}$$

Then (2.2), (2.5), (2.6) and (2.7) imply for $t\ge 0$

$$\begin{aligned}|x_i(t)|&\le(1-\underline{p}_i^-)e^{-\sigma_i t}|\varphi_i(0)|+\sum_{r=1}^{\ell(t)}\int_{t_{r-1}}^{t_r}(1-\underline{p}_i^-)e^{-\sigma_i(t-s)}|G_i(s,\mathbf{x}(s))|\,ds\\&\quad+\sum_{r=1}^{\ell(t)}\frac{1-\underline{p}_i^-}{1-p_i^*}e^{-\sigma_i(t-t_r)}(|J_{i,r}(x_i(t_r^-))|+|e_{i,r}|)+\int_{t_{\ell(t)}}^{t}e^{-\sigma_i(t-s)}|G_i(s,\mathbf{x}(s))|\,ds\\&=(1-\underline{p}_i^-)e^{-\sigma_i t}|\varphi_i(0)|+\int_0^t(1-\underline{p}_i^-)e^{-\sigma_i(t-s)}|G_i(s,\mathbf{x}(s))|\,ds\\&\quad+\sum_{r=1}^{\ell(t)}\frac{1-\underline{p}_i^-}{1-p_i^*}e^{-\sigma_i(t-t_r)}(|J_{i,r}(x_i(t_r^-))|+|e_{i,r}|).\end{aligned} \tag{2.8}$$

The assumed relations (H1)–(H4), (2.8), $t\in[t_{\ell(t)},t_{\ell(t)+1})$,

$$|J_{i,r}(x_i(t_r^-))|\le l_{i,r}|x_i(t_r^-)|\le l_i^*|x_i(t_r^-)|,\qquad i=1,\dots,m,\quad r\in\mathbb{N}$$

and

$$|G_i(s,\mathbf{x}(s))|\le\sum_{j=1}^{m}|b_{ij}(s)|L_j|x_j(s-\tau_j)|+|c_i(s)|,\qquad i=1,\dots,m,\quad s\ge 0$$

yield

$$\begin{aligned}|x_i(t)|&\le(1-\underline{p}_i^-)e^{-\sigma_i t}|\varphi_i(0)|\\&\quad+\int_0^t(1-\underline{p}_i^-)e^{-\sigma_i(t-s)}\Big(\sum_{j=1}^{m}|b_{ij}(s)|L_j|x_j(s-\tau_j)|+|c_i(s)|\Big)\,ds\\&\quad+\frac{1-\underline{p}_i^-}{1-p_i^*}\sum_{r=1}^{\ell(t)}e^{-\sigma_i(t-t_r)}(l_i^*|x_i(t_r^-)|+|e_{i,r}|),\qquad i=1,\dots,m,\quad t\ge 0.\end{aligned} \tag{2.9}$$

Using relation $\underline{\delta}\le t_{r+1}-t_r$ for $r\in\mathbb{N}_0$ from (H4) and $t\in[t_{\ell(t)},t_{\ell(t)+1})$, we obtain

$$\begin{aligned}\sum_{r=1}^{\ell(t)}e^{-\sigma_i(t-t_r)}&=\sum_{r=1}^{\ell(t)}e^{-\sigma_i(t-t_{\ell(t)}+(t_{\ell(t)}-t_{\ell(t)-1})+\cdots+(t_{r+1}-t_r))}\\&\le\sum_{r=1}^{\ell(t)}e^{-\sigma_i(t-t_{\ell(t)})}e^{-\sigma_i(\ell(t)-r)\underline{\delta}}\\&\le\sum_{r=1}^{\ell(t)}\left(e^{-\sigma_i\underline{\delta}}\right)^{\ell(t)-r}\\&\le\frac{1}{1-e^{-\sigma_i\underline{\delta}}},\qquad t\ge 0.\end{aligned} \tag{2.10}$$

Combining (2.9) with assumptions (H3), (H4), relation (2.10), and the estimate

$$\int_0^t e^{-\sigma_i(t-s)}\,ds\le\frac{1}{\sigma_i},\qquad t\ge 0, \tag{2.11}$$

we get for $t \geq 0$ and $i = 1, \ldots, m$

$$\begin{aligned}
|x_i(t)| &\leq (1-\underline{p}_i^-)e^{-\sigma_i t}|\varphi_i(0)| \\
&\quad + \int_0^t (1-\underline{p}_i^-)e^{-\sigma_i(t-s)}\Big(\sum_{j=1}^m |b_{ij}(s)|L_j \sup_{-\tau\leq u\leq s}|x_j(u)| + c_i^*\Big)\,ds \\
&\quad + \frac{1-\underline{p}_i^-}{1-p_i^*}\sum_{r=1}^{\ell(t)} e^{-\sigma_i(t-t_r)}(l_i^* \sup_{0\leq u\leq t}|x_i(u)| + e_i^*) \\
&\leq (1-\underline{p}_i^-)|\varphi_i(0)| + \sum_{j=1}^m (1-\underline{p}_i^-)\Lambda_{ij}L_j \sup_{-\tau\leq u\leq t}|x_j(u)| + \frac{(1-\underline{p}_i^-)c_i^*}{\sigma_i} \\
&\quad + \frac{1-\underline{p}_i^-}{(1-p_i^*)(1-e^{-\sigma_i\underline{\delta}})}\Big(l_i^* \sup_{-\tau\leq u\leq t}|x_i(u)| + e_i^*\Big). \qquad (2.12)
\end{aligned}$$

Since the right-hand side of (2.12) is monotone increasing in t, and $|x_i(u)| \leq |\varphi_i|_C \leq (1-\underline{p}_i^-)|\varphi_i|_C$ for $u \in [-\tau, 0]$, (2.12) yields

$$\begin{aligned}
\sup_{-\tau\leq u\leq t}|x_i(u)| &\leq (1-\underline{p}_i^-)|\varphi_i|_C + \sum_{j=1}^m (1-\underline{p}_i^-)\Lambda_{ij}L_j \sup_{-\tau\leq u\leq t}|x_j(u)| + \frac{(1-\underline{p}_i^-)c_i^*}{\sigma_i} \\
&\quad + \frac{1-\underline{p}_i^-}{(1-p_i^*)(1-e^{-\sigma_i\underline{\delta}})}\Big(l_i^* \sup_{-\tau\leq u\leq t}|x_i(u)| + e_i^*\Big), \qquad (2.13)
\end{aligned}$$

for $i = 1, \ldots, m$ and $t \geq 0$. Fix a nonnegative parameter α. Then we introduce the corresponding notations

$$\begin{aligned}
\mathbf{v}^{(\alpha)}(t) &= \Big(\sup_{-\tau\leq u\leq t} e^{\alpha u}|x_1(u)|, \ldots, \sup_{-\tau\leq u\leq t} e^{\alpha u}|x_m(u)|\Big)^T \in \mathbb{R}^m, \qquad t \geq -\tau, \\
\mathbf{a}^{(\alpha)} &= (a_1^{(\alpha)}, \ldots, a_m^{(\alpha)})^T \in \mathbb{R}^m, \quad \text{where} \\
a_i^{(\alpha)} &= (1-\underline{p}_i^-)\left(|\varphi_i|_C + \frac{c_i^*}{\sigma_i-\alpha} + \frac{e_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})}\right), \\
A^{(\alpha)} &= (a_{ij}) \in \mathbb{R}^{m\times m}, \quad a_{ij}^{(\alpha)} = \begin{cases} (1-\underline{p}_i^-)\Lambda_{ij}L_i e^{\alpha\tau_i} + \frac{(1-\underline{p}_i^-)l_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})}, & i = j, \\ (1-\underline{p}_i^-)\Lambda_{ij}L_j e^{\alpha\tau_j}, & i \neq j. \end{cases} \qquad (2.14)
\end{aligned}$$

Hence (2.13) implies the vector inequality

$$\mathbf{v}^{(0)}(t) \leq \mathbf{a}^{(0)} + A^{(0)}\mathbf{v}^{(0)}(t), \qquad t \geq 0.$$

The definition of $\mathbf{a}^{(0)}$ yields $\mathbf{v}^{(0)}(t) \leq \mathbf{a}^{(0)}$ for $t \in [-\tau, 0]$, so

$$\mathbf{v}^{(0)}(t) \leq \mathbf{a}^{(0)} + A^{(0)}\mathbf{v}^{(0)}(t), \qquad t \geq -\tau.$$

Assumption (H5) implies $\|A^{(0)}\|_\infty < 1$, so $I - A^{(0)}$ is a nonsingular M-matrix. Therefore Theorem 6.2.3 in [4] yields that $I - A^{(0)}$ is monotone, and

$$(|x_1(t)|, \ldots, |x_m(t)|)^T \leq \mathbf{v}^{(0)}(t) \leq (I - A^{(0)})^{-1}\mathbf{a}^{(0)}, \qquad t \geq -\tau.$$

It follows

$$|\mathbf{x}(t)|_\infty \leq |(I - A^{(0)})^{-1}\mathbf{a}^{(0)}|_\infty, \qquad t \geq -\tau,$$

i.e., $\mathbf{x}(t)$ is bounded on $[-\tau, \infty)$.

2. Next, we show the exponential boundedness of the solutions under the additional assumption (H6).

We select a positive constant α_0 such that

$$\alpha_0 < \min\{\varepsilon_0, \beta_1, \beta_2\} \quad \text{and} \quad \sum_{j=1}^{m}(1-\underline{p}_i^-)\Lambda_{ij}L_j e^{\alpha_0\tau_j} + \frac{(1-\underline{p}_i^-)l_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha_0)\underline{\delta}})} < 1 \tag{2.15}$$

for $i = 1, \ldots, m$. Note that such α_0 exists since (H5) holds. Multiplying both sides of (2.9) by $e^{\alpha_0 t}$ we get

$$\begin{aligned} e^{\alpha_0 t}|x_i(t)| \le{}& (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha_0)t}|\varphi_i(0)| \\ &+ \int_0^t (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha_0)(t-s)}\Big(\sum_{j=1}^{m}|b_{ij}(s)|L_j e^{\alpha_0\tau_j}e^{\alpha_0(s-\tau_j)}|x_j(s-\tau_j)| + e^{\alpha_0 s}|c_i(s)|\Big)ds \\ &+ \frac{1-\underline{p}_i^-}{1-p_i^*}\sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha_0)(t-t_r)}(l_i^* e^{\alpha_0 t_r}|x_i(t_r^-)| + e^{\alpha_0 t_r}|e_{i,r}|) \end{aligned}$$

for $i = 1, \ldots, m$ and $t \ge 0$. Then (H3), (H4), (H6), $\alpha_0 < \min\{\varepsilon_0, \beta_1, \beta_2\}$, and (2.10) and (2.11) where σ_i is replaced by $\sigma_i - \alpha_0$ imply

$$\begin{aligned} e^{\alpha_0 t}|x_i(t)| \le{}& (1-\underline{p}_i^-)|\varphi_i|_C + \sum_{j=1}^{m}(1-\underline{p}_i^-)\Lambda_{ij}L_j e^{\alpha_0\tau_j}\sup_{-\tau\le u\le t} e^{\alpha_0 u}|x_j(u)| + \frac{(1-\underline{p}_i^-)c_i^*}{\sigma_i-\alpha_0} \\ &+ \frac{(1-\underline{p}_i^-)l_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha_0)\underline{\delta}})}\sup_{-\tau\le u\le t} e^{\alpha_0 u}|x_i(u)| + \frac{(1-\underline{p}_i^-)e_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha_0)\underline{\delta}})} \end{aligned}$$

for $i = 1, \ldots, m$ and $t \ge 0$. Then the monotonicity of the right-hand side and $e^{\alpha_0 t}|x_i(t)| \le |\varphi_i|_C \le (1-\underline{p}_i^-)|\varphi_i|_C$ for $-\tau \le t \le 0$ imply the vector inequality

$$\mathbf{v}^{(\alpha_0)}(t) \le \mathbf{a}^{(\alpha_0)} + A^{(\alpha_0)}\mathbf{v}^{(\alpha_0)}(t), \qquad t \ge -\tau. \tag{2.16}$$

Relation (2.15) yields $\|A^{(\alpha_0)}\|_\infty < 1$, so $I - A^{(\alpha_0)}$ is a nonsingular M-matrix, hence $I - A^{(\alpha_0)}$ is monotone. Therefore

$$(e^{\alpha_0 t}|y_1(t)|, \ldots, e^{\alpha_0 t}|y_m(t)|)^T \le \mathbf{v}^{(\alpha_0)}(t) \le (I - A^{(\alpha_0)})^{-1}\mathbf{a}^{(\alpha_0)}, \qquad t \ge -\tau,$$

so (2.4) holds with

$$K_0 = |(I - A^{(\alpha_0)})^{-1}\mathbf{a}^{(\alpha_0)}|_\infty,$$

i.e., $\mathbf{x}(t)$ is exponentially bounded on $[-\tau, \infty)$. □

Remark 2.3. Let $A^{(0)}$ be defined by (2.14) with $\alpha = 0$. We remark that (H5) can be replaced by the weaker condition $\rho(A^{(0)}) < 1$, and the statement of Lemma 2.2 remains true.

Our next result shows that every solution of (1.14) is exponentially stable.

Lemma 2.4. *Suppose (H1)–(H5) hold. Then there exist positive constants α_0 and K_1 such that*

$$|\mathbf{x}(t) - \bar{\mathbf{x}}(t)|_\infty \le K_1 e^{-\alpha_0 t}|\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}}|_C, \qquad t \ge 0, \tag{2.17}$$

where $\boldsymbol{\varphi}(t) = (\varphi_1, \ldots, \varphi_m)^T$ and $\bar{\boldsymbol{\varphi}} = (\bar{\varphi}_1, \ldots, \bar{\varphi}_m)^T$ are two initial functions in (1.14), and $\mathbf{x}(t) = (x_1(t), \ldots, x_m(t))^T$ and $\bar{\mathbf{x}}(t) = (\bar{x}_1(t), \ldots, \bar{x}_m(t))^T$ are the corresponding solutions of (1.14), respectively, i.e., every solution of (1.14) is exponentially stable.

Proof. Let $(\varphi_1, \dots, \varphi_m)^T$ and $(\bar{\varphi}_1, \dots, \bar{\varphi}_m)^T$ be the vectors of two initial functions in (1.14), and $(x_1(t), \dots, x_m(t))^T$ and $(\bar{x}_1(t), \dots, \bar{x}_m(t))^T$ be the corresponding solutions of (1.14). Then the variation of constant formula (2.2) yields

$$
\begin{aligned}
x_i(t) - \bar{x}_i(t) &= e^{-\int_0^t a_i(u)\,du} \left(\prod_{j=1}^{\ell(t)} (1 - p_{i,j}) \right) (\varphi_i(0) - \bar{\varphi}_i(0)) \\
&\quad + \sum_{r=1}^{\ell(t)} \left(\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) \right) \int_{t_{r-1}}^{t_r} e^{-\int_s^t a_i(u)\,du} \Big(G_i(s, \mathbf{x}(s)) - G_i(s, \bar{\mathbf{x}}(s)) \Big)\, ds \\
&\quad + \sum_{r=1}^{\ell(t)} \left(\prod_{j=r+1}^{\ell(t)} (1 - p_{i,j}) \right) e^{-\int_{t_r}^t a_i(u)\,du} \Big(J_{i,r}(x_i(t_r^-)) - J_{i,r}(\bar{x}_i(t_r^-)) \Big) \\
&\quad + \int_{t_{\ell(t)}}^t e^{-\int_s^t a_i(u)\,du} \Big(G_i(s, \mathbf{x}(s)) - G_i(s, \bar{\mathbf{x}}(s)) \Big)\, ds, \quad i = 1, \dots, m, \quad t \geq 0.
\end{aligned}
$$

We have

$$
|G_i(s, \mathbf{x}(s)) - G_i(s, \bar{\mathbf{x}}(s))| \leq \sum_{j=1}^m |b_{ij}(s)| L_j |x_j(s - \tau_j) - \bar{x}_i(s - \tau_j)|,
$$

hence, similarly to the derivation of (2.9), we get

$$
\begin{aligned}
|x_i(t) - \bar{x}_i(t)| &\leq (1 - \underline{p}_i^-) e^{-\sigma_i t} |\varphi_i(0) - \bar{\varphi}_i(0)| \\
&\quad + \sum_{j=1}^m \int_0^t (1 - \underline{p}_i^-) e^{-\sigma_i (t-s)} |b_{ij}(s)| L_j |x_j(s - \tau_j) - \bar{x}_i(s - \tau_j)|\, ds \\
&\quad + \frac{1 - \underline{p}_i^-}{1 - p_i^*} \sum_{r=1}^{\ell(t)} e^{-\sigma_i (t - t_r)} l_i^* |x_i(t_r^-) - \bar{x}_i(t_r^-)|, \qquad i = 1, \dots, m, \quad t \geq 0. \qquad (2.18)
\end{aligned}
$$

We select a positive constant α_0 such that (2.15) is satisfied. Note that such α_0 exists since (H5) holds. Multiplying both sides of (2.18) by $e^{\alpha_0 t}$ we obtain

$$
\begin{aligned}
& e^{\alpha_0 t} |x_i(t) - \bar{x}_i(t)| \\
&\quad \leq (1 - \underline{p}_i^-) e^{(\alpha_0 - \sigma_i) t} |\varphi_i(0) - \bar{\varphi}_i(0)| \\
&\qquad + \sum_{j=1}^m \int_0^t (1 - \underline{p}_i^-) e^{-(\sigma_i - \alpha_0)(t-s)} |b_{ij}(s)| L_j e^{\alpha_0 \tau_j} e^{\alpha_0 (s - \tau_j)} |x_j(s - \tau_j) - \bar{x}_i(s - \tau_j)|\, ds \\
&\qquad + \frac{1 - \underline{p}_i^-}{1 - p_i^*} \sum_{r=1}^{\ell(t)} e^{-(\sigma_i - \alpha_0)(t - t_r)} l_i^* e^{\alpha_0 t_r} |x_i(t_r^-) - \bar{x}_i(t_r^-)| \qquad (2.19)
\end{aligned}
$$

for $i = 1, \dots, m$ and $t \geq 0$. Introduce the functions

$$
v_i(t) = \sup_{-\tau \leq u \leq t} e^{\alpha_0 u} |x_i(u) - \bar{x}_i(u)|, \qquad i = 1, \dots, m, \quad t \geq -\tau.
$$

Then (2.19) combined with (2.10) where σ_i is replaced by $\sigma_i - \alpha_0$ and

$$
e^{\alpha_0 u} |x_i(u) - \bar{x}_i(u)| \leq e^{\alpha_0 u} |\mathbf{x}(u) - \bar{\mathbf{x}}(u)|_\infty \leq |\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}}|_C \leq (1 - \underline{p}_i^-) |\boldsymbol{\varphi} - \bar{\boldsymbol{\varphi}}|_C, \quad -\tau \leq u \leq 0
$$

imply

$$\begin{aligned} v_i(t) &\le (1-\underline{p}_i^-)|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C + \sum_{j=1}^m \int_0^t (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha_0)(t-s)}|b_{ij}(s)|L_j e^{\alpha_0\tau_j} v_j(s)\,ds \\ &\quad + \frac{1-\underline{p}_i^-}{1-p_i^*}\sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha_0)(t-t_r)} l_i^* v_i(t) \\ &\le (1-\underline{p}_i^-)|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C + \sum_{j=1}^m (1-\underline{p}_i^-)\Lambda_{ij}L_j e^{\alpha_0\tau_j} v_j(t) \\ &\quad + \frac{(1-\underline{p}_i^-)l_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha_0)\underline{\delta}})} v_i(t) \end{aligned}$$

for $i = 1,\ldots,m$ and $t \ge 0$. Therefore, the vector inequality

$$\mathbf{v}(t) \le \mathbf{b} + A^{(\alpha_0)}\mathbf{v}(t), \qquad t \ge -\tau, \tag{2.20}$$

holds, where

$$\begin{aligned} \mathbf{v}(t) &= (v_1(t),\ldots,v_m(t))^T \in \mathbb{R}^m, \qquad t \ge -\tau, \\ \mathbf{b} &= \Big((1-\underline{q}_1^-)|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C,\ldots,(1-\underline{q}_m^-)|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C\Big)^T \in \mathbb{R}^m, \end{aligned} \tag{2.21}$$

and $A^{(\alpha_0)}$ is defined by (2.14). Relation (2.15) yields $\|A^{(\alpha_0)}\|_\infty < 1$, hence $I - A^{(\alpha_0)}$ is a nonsingular M-matrix, so $I - A^{(\alpha_0)}$ is monotone. Therefore (2.20) gives

$$\mathbf{v}(t) \le (I - A^{(\alpha_0)})^{-1}\mathbf{b}, \qquad t \ge -\tau,$$

and hence

$$e^{\alpha_0 t}|\mathbf{x}(t) - \bar{\mathbf{x}}(t)|_\infty \le |\mathbf{v}(t)|_\infty \le \|(I - A^{(\alpha_0)})^{-1}\|_\infty |\mathbf{b}|_\infty = K_1|\boldsymbol{\varphi}-\bar{\boldsymbol{\varphi}}|_C,$$

where $K_1 = \|(I - A^{(\alpha_0)})^{-1}\|_\infty \max\{1-\underline{q}_1^-,\ldots,1-\underline{q}_m^-\}$. This completes the proof of (2.17). □

Remark 2.5. Let $A^{(\alpha)}$ be the matrix defined by (2.14). We note that $\rho(A^{(0)}) < 1$ implies $\rho(A^{(\alpha_0)}) < 1$ for sufficiently small $\alpha_0 > 0$, so assumption (H5) in Lemma 2.4 can be replaced by the weaker condition $\rho(A^{(0)}) < 1$.

Next, we prove an estimate which will be important in the proof of our main result in the next section.

Lemma 2.6. *Suppose (H1)–(H5) hold. Let $\mathbf{x} = (x_1,\ldots,x_m)^T$ be a solution of (1.14), and let $\alpha_0 > 0$ be the corresponding constant from Lemma 2.2. For $0 < \alpha < \alpha_0$ and $u > 0$ define*

$$\begin{aligned} \omega_\alpha(u) = \sup\Big\{ e^{\alpha t}|\mathbf{x}(t) - \mathbf{x}(\bar{t})|_\infty \ :\ \Big(t,\bar{t} \in [t_r,t_{r+1}),\ r \in \mathbb{N} \quad \text{or} \quad t,\bar{t} \in [-\tau,t_1)\Big), \\ \text{and} \quad |\bar{t} - t| \le u \Big\}. \end{aligned} \tag{2.22}$$

(i) Then

$$\lim_{u\to 0+} \omega_\alpha(u) = 0, \qquad 0 < \alpha < \alpha_0. \tag{2.23}$$

(ii) Assume further (H6), (H7) and (H8). Then there exist $M_0 > 0$ and $u_0 > 0$ such that

$$\omega_\alpha(u) \le M_0 u, \qquad 0 < u \le u_0, \quad 0 < \alpha < \alpha_0. \tag{2.24}$$

Proof. (i) It follows from Lemma 2.2 that $\mathbf{x}$ satisfies (2.4). Fix $0 < \alpha < \alpha_0$, $\varepsilon > 0$ and $\bar{u} > 0$. Since $t_k \to \infty$ as $k \to \infty$, there exists k_0 such that

$$K_0 e^{\alpha_0 \bar{u}} e^{(\alpha-\alpha_0)t} < \frac{\varepsilon}{2}, \qquad t \ge t_{k_0}.$$

Then, using (2.4) and the triangle inequality, we get

$$e^{\alpha t}|\mathbf{x}(t) - \mathbf{x}(\bar{t})|_\infty < e^{\alpha t}\left(K_0 e^{-\alpha_0 t} + K_0 e^{-\alpha_0 \bar{t}}\right) \le K_0 e^{(\alpha-\alpha_0)t} + K_0 e^{\alpha_0 u} e^{(\alpha-\alpha_0)t} < \varepsilon,$$

for $t, \bar{t} \ge t_{k_0}$, $|\bar{t} - t| \le u$ and $0 < u < \bar{u}$.

The function $e^{\alpha t} x_i(t)$ is uniformly continuous on the intervals $[t_k, t_{k+1})$ for $k = 1, \dots, k_0 - 1$ and $i = 1, \dots, m$, and on the interval $[-\tau, t_1)$ since it has continuous extension to the closed intervals $[t_k, t_{k+1}]$ and $[-\tau, t_1]$. Therefore, there exists $\delta > 0$ such that

$$|e^{\alpha t}\mathbf{x}(t) - e^{\alpha \bar{t}}\mathbf{x}(\bar{t})|_\infty < \frac{\varepsilon}{2} \qquad \text{and} \qquad \delta < \min\left\{\bar{u}, \frac{\varepsilon}{2(e-1)K_0\alpha}, \frac{1}{\alpha}\right\}$$

if $t, \bar{t} \in [t_r, t_{r+1})$ for some $r \in \{1, \dots, k_0 - 1\}$ or $t, \bar{t} \in [-\tau, t_1)$, and $|\bar{t} - t| \le \delta$. Then (2.4) and the estimate

$$|e^s - 1| \le (e-1)|s|, \qquad |s| \le 1 \tag{2.25}$$

imply

$$\begin{aligned}
e^{\alpha t}|\mathbf{x}(t) - \mathbf{x}(\bar{t})|_\infty &\le |e^{\alpha t}\mathbf{x}(t) - e^{\alpha \bar{t}}\mathbf{x}(\bar{t})|_\infty + |e^{\alpha t} - e^{\alpha \bar{t}}||\mathbf{x}(\bar{t})|_\infty \\
&< \frac{\varepsilon}{2} + |e^{\alpha(t-\bar{t})} - 1| e^{\alpha \bar{t}}|\mathbf{x}(\bar{t})|_\infty \\
&< \frac{\varepsilon}{2} + (e-1)\alpha\delta K_0 \\
&< \varepsilon,
\end{aligned}$$

if $t, \bar{t} \in [t_r, t_{r+1})$ for some $r \in \{1, \dots, k_0 - 1\}$ or $t, \bar{t} \in [-\tau, t_1)$, and $|\bar{t} - t| \le \delta$. Hence $\omega_\alpha(u) \le \varepsilon$ for $0 < \alpha < \alpha_0$ and $0 < u < \delta$, which completes the proof of (2.23).

(ii) Note that it follows from the proof of Lemma 2.2 that $\alpha < \alpha_0 < \beta_2$. Since $x_i(t)$ is continuously differentiable on $[t_r, t_{r+1})$, we get from (1.14) that for $t, \bar{t} \in [t_r, t_{r+1})$ for some $r \in \mathbb{N}_0$

$$e^{\alpha t}(x_i(t) - x_i(\bar{t})) = \int_{\bar{t}}^{t} e^{\alpha t}\left(-a_i(s)x_i(s) + \sum_{j=1}^{m} b_{ij}(s) g_j(x_j(s-\tau_j)) + c_i(s)\right) ds.$$

Define the constant $M = a_i^* K_0 + \sum_{j=1}^{m} b_{ij}^* L_j e^{\alpha_0 \tau_j} K_0 + c_i^*$. Then, using $e^{\alpha s}|x_i(s)| \le e^{\alpha_0 s}|x_i(s)| \le K_0$ from (2.4) and $e^{\alpha s}|c_i(s)| \le e^{\beta_2 s}|c_i(s)| \le c_i^*$ from (H6) and (2.25), we get

$$\begin{aligned}
e^{\alpha t}|x_i(t) - x_i(\bar{t})| &\le \int_{\bar{t}}^{t} e^{\alpha(t-s)}\left(a_i^* e^{\alpha s}|x_i(s)| + \sum_{j=1}^{m} b_{ij}^* L_j e^{\alpha_0 \tau_j} e^{\alpha(s-\tau_j)}|x_j(s-\tau_j)| + e^{\alpha s}|c_i(s)|\right) ds \\
&\le M \int_{\bar{t}}^{t} e^{\alpha(t-s)}\, ds \\
&= M\left(\frac{e^{\alpha(t-\bar{t})} - 1}{\alpha}\right) \\
&\le M(e-1)u, \qquad t, \bar{t} \in [t_r, t_{r+1}), \quad r \in \mathbb{N}_0,
\end{aligned}$$

for $|t - \bar{t}| \le u \le u_0 = \frac{1}{\alpha_0}$.

Assumption (H8) yields

$$e^{\alpha t}|x_i(t) - x_i(\bar{t})| \le |\varphi_i(t) - \varphi_i(\bar{t})| \le L_\varphi |t - \bar{t}|, \qquad t, \bar{t} \in [-\tau, 0].$$

Suppose $-\tau \le \bar{t} \le 0 \le t < t_1$ and $|t - \bar{t}| \le u < u_0$. Then combining the above two cases and $u_0 = \frac{1}{\alpha_0}$ we obtain

$$\begin{aligned} e^{\alpha t}|x_i(t) - x_i(\bar{t})| &\le e^{\alpha t}\Big(|x_i(t) - x_i(0)| + |x_i(0) - x_i(\bar{t})|\Big) \\ &\le M(e-1)t + e^{\alpha u_0} L_\varphi(-\bar{t}) \\ &\le M_0 u, \end{aligned}$$

where $M_0 = \max\{M(e-1), eL_\varphi\}$.

For $-\tau \le t \le 0 \le \bar{t} < t_1$ and $|t - \bar{t}| \le u < u_0$ we get

$$e^{\alpha t}|x_i(t) - x_i(\bar{t})| \le |x_i(t) - x_i(0)| + e^{\alpha \bar{t}}|x_i(0) - x_i(\bar{t})| \le L_\varphi(-t) + M(e-1)\bar{t} \le M_0 u.$$

The proof of (2.24) is completed. □

3 Main results

In this section, we prove that the solutions of (1.15) approximate that of (1.14) uniformly on $[0, \infty)$.

Theorem 3.1. *Suppose (H1)–(H7) hold. Let $\mathbf{x} = (x_1, \dots, x_m)^T$ be the solution of (1.14) corresponding to initial function $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_m)^T$, and $\mathbf{y} = (y_1, \dots, y_m)^T$ be the solution of (1.15) corresponding to $h > 0$ and an initial function $\boldsymbol{\psi} = (\psi_1, \dots, \psi_m)^T$, and let $\alpha_0 > 0$ be the corresponding constant from Lemma 2.2.*

(i) Then for every $0 < \alpha < \alpha_0$ and $\varepsilon > 0$ there exist constants $K_2 > 0$ and $h^* > 0$, and a function $\theta(h)$ such that $\theta(h) \to 0$ as $h \to 0+$, and

$$|\mathbf{x}(t) - \mathbf{y}(t)|_\infty \le e^{-\alpha t} K_2\Big(|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + \theta(h) + \varepsilon\Big), \qquad t \in [-\tau, \infty), \quad 0 < h < h^*. \tag{3.1}$$

(ii) Assume further (H8). Then for every $0 < \alpha < \alpha_0$ there exist constants $K_2 > 0$, $M > 0$ and $\bar{h} > 0$ such that

$$|\mathbf{x}(t) - \mathbf{y}(t)|_\infty \le e^{-\alpha t} K_2\Big(|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + Mh\Big), \qquad t \in [-\tau, \infty), \quad 0 < h < \bar{h}. \tag{3.2}$$

Proof. The variation of constants formula (2.2) applied for problem (1.15) gives

$$\begin{aligned} y_i(t) &= e^{-\int_0^t a_i(u)\,du} \left(\prod_{j=1}^{\ell(t)} (1 - p_{i,j})\right) \psi_i(0) \\ &\quad + \sum_{r=1}^{\ell(t)} \left(\prod_{j=r}^{\ell(t)} (1 - p_{i,j})\right) \int_{t_{r-1}}^{t_r} e^{-\int_s^t a_i(u)\,du} G_i(s, \mathbf{y}(\gamma(s)))\,ds \\ &\quad + \sum_{r=1}^{\ell(t)} \left(\prod_{j=r+1}^{\ell(t)} (1 - p_{i,j})\right) e^{-\int_{\gamma(t_r)}^t a_i(u)\,du} (J_{i,r}(y_i(\gamma(t_r)^-)) + e_{i,r}) \\ &\quad + \int_{t_{\ell(t)}}^t e^{-\int_s^t a_i(u)\,du} G_i(s, \mathbf{y}(\gamma(s)))\,ds, \qquad i = 1, \dots, m, \quad t \ge 0, \end{aligned}$$

where

$$G_i(t,\mathbf{y}(\gamma(t))) = \sum_{j=1}^{m} b_{ij}(t) g_j(y_j(\gamma(t)-\gamma(\tau_j))) + c_i(t), \qquad i=1,\dots,m.$$

Combining it with (2.2) we get

$$\begin{aligned} x_i(t)-y_i(t) &= e^{-\int_0^t a_i(u)\,du}\left(\prod_{j=1}^{\ell(t)}(1-p_{i,j})\right)(\varphi_i(0)-\psi_i(0)) \\ &\quad + \sum_{r=1}^{\ell(t)}\left(\prod_{j=r}^{\ell(t)}(1-p_{i,j})\right)\int_{t_{r-1}}^{t_r} e^{-\int_s^t a_i(u)\,du}\Big(G_i(s,\mathbf{x}(s))-G_i(s,\mathbf{y}(\gamma(s)))\Big)\,ds \\ &\quad + \sum_{r=1}^{\ell(t)}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\Big(e^{-\int_{t_r}^t a_i(u)\,du}(J_{i,r}(x_i(t_r^-))+e_{i,r}) \\ &\qquad - e^{-\int_{\gamma(t_r)}^{\mathbf{t}} a_i(u)\,du}(J_{i,r}(y_i(\gamma(t_r)^-))+e_{i,r})\Big) \\ &\quad + \int_{t_{\ell(t)}}^t e^{-\int_s^t a_i(u)\,du}\Big(G_i(s,\mathbf{x}(s))-G_i(s,\mathbf{y}(\gamma(s)))\Big)\,ds \end{aligned}$$

for $i=1,\dots,m$ and $t\geq 0$. Therefore, (2.5), (2.6) and $|\varphi_i(0)-\psi_i(0)|\leq|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C$ yield

$$\begin{aligned} |x_i(t)-y_i(t)| &\leq (1-\underline{p}_i^-)e^{-\sigma_i t}|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C \\ &\quad + \int_0^t (1-\underline{p}_i^-)e^{-\sigma_i(t-s)}\Big|G_i(s,\mathbf{x}(s))-G_i(s,\mathbf{y}(\gamma(s)))\Big|\,ds + A_i(t) \end{aligned} \tag{3.3}$$

for $i=1,\dots,m$ and $t\geq 0$, where

$$\begin{aligned} A_i(t) = \sum_{r=1}^{\ell(t)}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\Big|&e^{-\int_{t_r}^t a_i(u)\,du}(J_{i,r}(x_i(t_r^-))+e_{i,r}) \\ &-e^{-\int_{\gamma(t_r)}^t a_i(u)\,du}(J_{i,r}(y_i(\gamma(t_r)^-))+e_{i,r})\Big|. \end{aligned}$$

Let α_0 and K_0 be the constants from (2.4). We select a positive constant α such that

$$0<\alpha<\min\{\varepsilon_0,\beta_1,\beta_2,\alpha_0\}$$

and

$$e^{\alpha\underline{\delta}}\sum_{j=1}^{m}(1-\underline{p}_i^-)\Lambda_{ij}L_j e^{\alpha\tau_j} + \frac{(1-\underline{p}_i^-)l_i^* e^{\alpha\underline{\delta}}}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})} < 1, \qquad i=1,\dots,m. \tag{3.4}$$

Note that such α exists since (H5) holds. Multiplying (3.3) with $e^{\alpha t}$ and using (H1) we get

$$\begin{aligned}
&e^{\alpha t}|x_i(t)-y_i(t)| \\
&\quad \le (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha)t}|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C \\
&\qquad + \int_0^t (1-\underline{p}_i^-)e^{\alpha t-\sigma_i(t-s)}\Big|G_i(s,\mathbf{x}(s))-G_i(s,\mathbf{y}(\gamma(s)))\Big|\,ds + e^{\alpha t}A_i(t) \\
&\quad \le (1-\underline{p}_i^-)|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C \\
&\qquad + \sum_{j=1}^m \int_0^t (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha s}\Big|x_j(s-\tau_j)-y_j(\gamma(s)-\gamma(\tau_j))\Big|\,ds \\
&\qquad + e^{\alpha t}A_i(t) \\
&\quad \le (1-\underline{p}_i^-)|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C \\
&\qquad + \sum_{j=1}^m \int_0^t (1-\underline{p}_i^-)e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha s}\Big(\Big|x_j(s-\tau_j)-x_j(\gamma(s)-\gamma(\tau_j))\Big| \\
&\qquad\qquad + \Big|x_j(\gamma(s)-\gamma(\tau_j))-y_j(\gamma(s)-\gamma(\tau_j))\Big|\Big)ds + e^{\alpha t}A_i(t)
\end{aligned} \tag{3.5}$$

for $i=1,\ldots,m$ and $t\ge 0$. We have

$$\begin{aligned}
A_i(t) \le& \sum_{r=1}^{\ell(t)}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\left|e^{-\int_{t_r}^t a_i(u)\,du}-e^{-\int_{\gamma(t_r)}^t a_i(u)\,du}\right||J_{i,r}(x_i(t_r^-))| \\
&+\sum_{r=1}^{\ell(t)}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)e^{-\int_{t_r}^t a_i(u)\,du}e^{-\int_{\gamma(t_r)}^{t_r} a_i(u)\,du} \\
&\quad\times\Big|J_{i,r}(x_i(t_r^-))-J_{i,r}(y_i(\gamma(t_r)^-))\Big| \\
&+\sum_{r=1}^{\ell(t)}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\left|e^{-\int_{t_r}^t a_i(u)\,du}-e^{-\int_{\gamma(t_r)}^t a_i(u)\,du}\right||e_{i,r}|
\end{aligned} \tag{3.6}$$

for $i=1,\ldots,m$ and $t\ge 0$. Assumption (H7), estimates $1-e^{-t}<t$ for $t>0$, the following direct consequence of $|t-\gamma(t)|<h$:

$$t-h<\gamma(t)\le t, \qquad t\in\mathbb{R}, \tag{3.7}$$

and (2.7) imply

$$\begin{aligned}
&\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\left|e^{-\int_{t_r}^t a_i(u)\,du}-e^{-\int_{\gamma(t_r)}^t a_i(u)\,du}\right| \\
&\qquad = e^{-\int_{t_r}^t a_i(u)\,du}\left(\prod_{j=r+1}^{\ell(t)}(1-p_{i,j})\right)\left(1-e^{-\int_{\gamma(t_r)}^{t_r} a_i(u)\,du}\right) \\
&\qquad \le \frac{1-\underline{p}_i^-}{1-p_i^*}e^{-\sigma_i(t-t_r)}(1-e^{-a_i^* h}) \\
&\qquad \le \frac{1-\underline{p}_i^-}{1-p_i^*}e^{-\sigma_i(t-t_r)}a_i^* h, \qquad t\ge t_r.
\end{aligned} \tag{3.8}$$

Then, combining (H2), (H3) (ii), (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned} e^{\alpha t}A_i(t) \le & \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)a_i^* h l_i^*}{1-p_i^*} e^{\alpha t_r}|x_i(t_r^-)| \\ & + \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)l_i^*}{1-p_i^*} e^{\alpha t_r} \\ & \times \Big(|x_i(t_r^-) - x_i(\gamma(t_r)^-)| + |x_i(\gamma(t_r)^-) - y_i(\gamma(t_r)^-)| \Big) \\ & + \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)a_i^* h}{1-p_i^*} e^{\alpha t_r}|e_{i,r}|, \qquad i=1,\dots,m, \quad t\ge 0. \end{aligned} \tag{3.9}$$

For $0<h<\underline{\delta}$ we have $t_{r-1}<\gamma(t_r)\le t_r$, hence (2.22) and (3.7) yield

$$e^{\alpha t_r}|x_i(t_r^-) - x_i(\gamma(t_r)^-)| = \lim_{n\to\infty} e^{\alpha\left(t_r-\frac{1}{n}\right)} \left| x_i\left(t_r - \frac{1}{n}\right) - x_i\left(\gamma(t_r) - \frac{1}{n}\right) \right| \le \omega_\alpha(h)$$

for $h<\underline{\delta}$. Therefore (3.9), (2.10) with σ_i replaced by $\sigma_i-\alpha$, $e^{\alpha t_r}|x_i(t_r^-)| \le e^{\alpha_0 t_r}|x_i(t_r^-)| \le K_0$, $e^{\alpha t_r}|e_{i,r}| \le e_i^*$ and $e^{\alpha t_r} = e^{\alpha(t_r-\gamma(t_r))}e^{\alpha\gamma(t_r)} \le e^{\alpha h}e^{\alpha\gamma(t_r)}$ imply

$$\begin{aligned} e^{\alpha t}A_i(t) \le & \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)a_i^* h l_i^*}{1-p_i^*} K_0 \\ & + \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)l_i^*}{1-p_i^*} \Big(\omega_\alpha(h) + e^{\alpha h} \sup_{0\le u\le t} e^{\alpha u}|x_i(u)-y_i(u)| \Big) \\ & + \sum_{r=1}^{\ell(t)} e^{-(\sigma_i-\alpha)(t-t_r)} \frac{(1-\underline{p}_i^-)a_i^* h e_i^*}{1-p_i^*} \\ \le & \frac{(1-\underline{p}_i^-)(a_i^* h l_i^* K_0)}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})} + \frac{(1-\underline{p}_i^-)l_i^*\omega_\alpha(h)}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})} \\ & + \frac{(1-\underline{p}_i^-)a_i^* h e_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})} \\ & + \frac{(1-\underline{p}_i^-)l_i^* e^{\alpha h}}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})} \sup_{0\le u\le t} e^{\alpha u}|x_i(u)-y_i(u)| \\ \le & d_i^* h + \bar{d}_i\omega_\alpha(h) + \hat{d}_i \sup_{0\le u\le t} e^{\alpha u}|x_i(u)-y_i(u)| \end{aligned} \tag{3.10}$$

for $t\ge 0$, $i=1,\dots,m$ and $0<h<\underline{\delta}$, where

$$d_i^* = \frac{(1-\underline{p}_i^-)(a_i^* l_i^* K_0 + a_i^* e_i^*)}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})}, \qquad \bar{d}_i = \frac{(1-\underline{p}_i^-)l_i^*}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})},$$

and

$$\hat{d}_i = \frac{(1-\underline{p}_i^-)l_i^* e^{\alpha\underline{\delta}}}{(1-p_i^*)(1-e^{-(\sigma_i-\alpha)\underline{\delta}})}.$$

We introduce the functions

$$\eta_i(t,h) = \sum_{j=1}^{m} \int_0^t e^{-(\sigma_i-\alpha)(t-s)} |b_{ij}(s)| L_j e^{\alpha s} \Big| x_j(s-\tau_j) - x_j(\gamma(s)-\gamma(\tau_j)) \Big| ds \tag{3.11}$$

for $t \geq 0,\ h > 0$, and

$$w_i(t) = \max_{-\tau \leq u \leq t} e^{\alpha u}|x_i(u) - y_i(u)|, \qquad t \geq -\tau, \quad h > 0$$

for $i = 1, \ldots, m$. Then estimate (3.5) together with (3.10), (3.11) and

$$e^{\alpha s} = e^{\alpha(s-\gamma(s)+\gamma(\tau_j))} e^{\alpha(\gamma(s)-\gamma(\tau_j))} \leq e^{\alpha(h+\tau_j)} e^{\alpha(\gamma(s)-\gamma(\tau_j))} \tag{3.12}$$

yields

$$\begin{aligned}
e^{\alpha t}|x_i(t) - y_i(t)| &\leq (1 - \underline{p}_i^-)|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + (1 - \underline{p}_i^-)\eta_i(t,h) + e^{\alpha t} A_i(t) \\
&\quad + \sum_{j=1}^m \int_0^t (1 - \underline{p}_i^-) e^{-(\sigma_i - \alpha)(t-s)} |b_{ij}(s)| L_j e^{\alpha s} \\
&\quad \times \Big| x_j(\gamma(s) - \gamma(\tau_j)) - y_j(\gamma(s) - \gamma(\tau_j)) \Big| ds \\
&\leq (1 - \underline{p}_i^-)|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + (1 - \underline{p}_i^-)\eta_i(t,h) + d_i^* h + \bar{d}_i \omega_\alpha(h) + \hat{d}_i w_i(t) \\
&\quad + \sum_{j=1}^m \int_0^t (1 - \underline{p}_i^-) e^{-(\sigma_i - \alpha)(t-s)} |b_{ij}(s)| L_j e^{\alpha(h+\tau_j)} w_j(s)\, ds \\
&\leq (1 - \underline{p}_i^-)|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + (1 - \underline{p}_i^-)\eta_i(t,h) + d_i^* h + \bar{d}_i \omega_\alpha(h) + \hat{d}_i w_i(t) \\
&\quad + \sum_{j=1}^m (1 - \underline{p}_i^-) \Lambda_{ij} L_j e^{\alpha(h_0+\tau_j)} w_j(t)
\end{aligned} \tag{3.13}$$

for $0 < h < \underline{\delta}$, $t \geq 0$ and $i = 1, \ldots, m$.

Define $h_1 = \underline{\delta}/2$, and next we suppose that $0 < h < h_1$, $j \in \{1, \ldots, m\}$ and $r \in \mathbb{N}$. Relation (3.7) implies

$$t_r \leq (s - h) - \tau_j \leq \gamma(s) - \gamma(\tau_j) \leq s - (\tau_j - h) < t_{r+1}, \qquad s \in [t_r + \tau_j + h, t_{r+1} + \tau_j - h).$$

Therefore

$$s - \tau_j \in [t_r, t_{r+1}) \quad \text{and} \quad \gamma(s) - \gamma(\tau_j) \in [t_r, t_{r+1}), \qquad s \in [t_r + \tau_j + h, t_{r+1} + \tau_j - h). \tag{3.14}$$

Moreover, (3.7) yields

$$|s - \tau_j - (\gamma(s) - \gamma(\tau_j))| = |s - \gamma(s) - (\tau_j - \gamma(\tau_j))| \leq h, \qquad s \in [t_r + \tau_j + h, t_{r+1} + \tau_j - h). \tag{3.15}$$

Hence it follows from (2.22), (3.14) and (3.15) that

$$\begin{aligned}
e^{\alpha s}|x_j(s - \tau_j) - x_j(\gamma(s) - \gamma(\tau_j))| &\leq e^{\alpha \tau_j} e^{\alpha(s-\tau_j)} |x_j(s - \tau_j) - x_j(\gamma(s) - \gamma(\tau_j))| \\
&\leq e^{\alpha \tau_j} \omega_\alpha(h), \qquad s \in [t_r + \tau_j + h, t_{r+1} + \tau_j - h).
\end{aligned} \tag{3.16}$$

Similarly, it is easy to check that

$$\begin{aligned}
e^{\alpha s}|x_j(s - \tau_j) - x_j(\gamma(s) - \gamma(\tau_j))| &\leq e^{\alpha \tau_j} e^{\alpha(s-\tau_j)} |x_j(s - \tau_j) - x_j(\gamma(s) - \gamma(\tau_j))| \\
&\leq e^{\alpha \tau_j} \omega_\alpha(h), \qquad s \in [0, t_1 + \tau_j - h).
\end{aligned} \tag{3.17}$$

We define the sets

$$\mathcal{A}_{j,h} = [0, t_1 + \tau_j - h] \cup \bigcup_{r=1}^{\infty} [t_r + \tau_j + h, t_{r+1} + \tau_j - h], \qquad \mathcal{B}_{j,h} = [0, \infty) \setminus \mathcal{A}_{j,h}$$

for $j = 1, \ldots, m$ and $0 < h < h_1$. We use relation (3.16) to estimate the function $\eta_i(t, h)$ defined by (3.11) for $i = 1, \ldots, m$, $t \geq 0$, $0 < h < h_1$, and for $s \in \mathcal{A}_{j,h}$:

$$\begin{aligned}\eta_i(t,h) &= \sum_{j=1}^{m}\Big(\int_{\mathcal{A}_{j,h}\cap[0,t]} e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha s}\Big|x_j(s-\tau_j)-x_j(\gamma(s)-\gamma(\tau_j))\Big|\,ds\\ &\quad+\int_{\mathcal{B}_{j,h}\cap[0,t]} e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha s}\Big|x_j(s-\tau_j)-x_j(\gamma(s)-\gamma(\tau_j))\Big|\,ds\Big)\\ &\le \sum_{j=1}^{m}\Big(\int_0^t e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha\tau_j}\omega_\alpha(h)\,ds\\ &\quad+\int_{\mathcal{B}_{j,h}\cap[0,t]} e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_j e^{\alpha s}\Big|x_j(s-\tau_j)-x_j(\gamma(s)-\gamma(\tau_j))\Big|\,ds\Big).\end{aligned} \tag{3.18}$$

For $s \in \mathcal{B}_{j,h}$ we use a different estimate. Let $0 < h < h_2 = \min\{h_1, 1/\alpha_0\}$. We have from Lemma 2.2

$$\begin{aligned}&e^{\alpha s}|x_j(s-\tau_j)-x_j(\gamma(s)-\gamma(\tau_j))|\\ &\quad\le e^{\alpha\tau_j}e^{\alpha(s-\tau_j)}|x_j(s-\tau_j)| + e^{\alpha(s-\gamma(s)+\gamma(\tau_j))}e^{\alpha(\gamma(s)-\gamma(\tau_j))}|x_j(\gamma(s)-\gamma(\tau_j))|\\ &\quad\le e^{\alpha\tau_j}K_0e^{-(\alpha_0-\alpha)(s-\tau_j)} + e^{\alpha(s-\gamma(s)+\gamma(\tau_j))}K_0e^{-(\alpha_0-\alpha)(\gamma(s)-\gamma(\tau_j))}\\ &\quad\le e^{\alpha\tau_j}K_0e^{-(\alpha_0-\alpha)(s-\tau_j)} + e^{\alpha(h+\tau_j)}K_0e^{-(\alpha_0-\alpha)(s-h-\tau_j)}\\ &\quad\le e^{\alpha\tau_j}(1+e^{\alpha_0 h})K_0e^{-(\alpha_0-\alpha)(s-\tau_j)}\\ &\quad\le e^{\alpha\tau_j}(1+e)K_0e^{-(\alpha_0-\alpha)(s-\tau)}, \qquad j=1,\ldots,m, \quad s\ge 0.\end{aligned} \tag{3.19}$$

Fix $\varepsilon > 0$. Then it follows from (3.19) that there exists $T = T(\varepsilon, \alpha)$ such that

$$e^{\alpha s}|x_j(s-\tau_j)-x_j(\gamma(s)-\gamma(\tau_j))| \le e^{\alpha\tau_j}\frac{\varepsilon}{M_2}, \qquad s\ge T, \quad j=1,\ldots,m, \tag{3.20}$$

where $M_2 = \max_{i=1,\ldots,m}\sum_{j=1}^{m}\Lambda_{ij}L_je^{\alpha\tau_j}$. Let $k_0 \in \mathbb{N}$ be the smallest index such that $t_{k_0} \geq T$. We recall that $h_2 \leq \underline{\delta}/2$. So if $1 \leq r \leq k_0$, then for $s \in [t_r + \tau_j - h, t_r + \tau_j + h]$ and $0 < h < h_2$ we have

$$s-\tau_j \le t_{k_0}+h < t_{k_0+1} \quad\text{and}\quad \gamma(s)-\gamma(\tau_j) \le s-\tau_j+h \le t_{k_0}+2h < t_{k_0+1}. \tag{3.21}$$

Similarly, for $r > k_0$, $s \in [t_r + \tau_j - h, t_r + \tau_j + h]$ and $0 < h < h_2$ we get

$$s-\tau_j \ge t_r-h > t_{r-1} \ge t_{k_0} \ge T, \quad\text{and}\quad \gamma(s)-\gamma(\tau_j) \ge s-h-\tau_j \ge t_r-2h > t_{r-1} \ge t_{k_0}. \tag{3.22}$$

Define the sets

$$\mathcal{C}_{j,h} = \bigcup_{r=1}^{k_0}[t_r+\tau_j-h, t_r+\tau_j+h] \quad\text{and}\quad \mathcal{D}_{j,h} = \bigcup_{r=k_0+1}^{\infty}[t_r+\tau_j-h, t_r+\tau_j+h].$$

Then, clearly, $\mathcal{B}_{j,h} = \mathcal{C}_{j,h} \cup \mathcal{D}_{j,h}$, $j = 1, \ldots, m$. Define the constants $\tilde{b}_{ij} = \tilde{b}_{ij}(\varepsilon)$ by

$$\tilde{b}_{ij} = \max_{0\le u\le t_{k_0+1}}|b_{ij}(u)|, \qquad i,j=1,\ldots,m.$$

Then (H3), (3.18), (3.19), (3.20), (3.21) and (3.22) yield

$$\begin{aligned}
\eta_i(t,h) &\le \sum_{j=1}^{m}\Big(\beta_{ij}^{*}L_je^{\alpha\tau_j}\omega_\alpha(h)\\
&\qquad + \int_{\mathcal{C}_{j,h}\cap[0,t]} e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_je^{\alpha s}\Big|x_j(s-\tau_j)-x_j(\gamma(s)-\gamma(\tau_j))\Big|\,ds\\
&\qquad + \int_{\mathcal{D}_{j,h}\cap[0,t]} e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_je^{\alpha s}\Big|x_j(s-\tau_j)-x_j(\gamma(s)-\gamma(\tau_j))\Big|\,ds\Big)\\
&\le M_2\omega_\alpha(h)+\sum_{j=1}^{m}\sum_{r=1}^{\ell(t)}\int_{t_r+\tau_j-h}^{t_r+\tau_j+h} e^{-(\sigma_i-\alpha)(t-s)}\tilde{b}_{ij}L_je^{\alpha\tau_j}(1+e)K_0\,ds\\
&\qquad +\sum_{j=1}^{m}\int_0^t e^{-(\sigma_i-\alpha)(t-s)}|b_{ij}(s)|L_je^{\alpha\tau_j}\frac{\varepsilon}{M_2}\,ds\\
&\le M_2\omega_\alpha(h)+\sum_{j=1}^{m}\sum_{r=1}^{\ell(t)}\int_{t_r+\tau_j-h}^{t_r+\tau_j+h} e^{-(\sigma_i-\alpha)(t-s)}\tilde{b}_{ij}L_je^{\alpha\tau_j}(1+e)K_0\,ds+\varepsilon
\end{aligned} \tag{3.23}$$

for $t \ge 0$ and $0 < h < h_2$. Relation (2.25) gives

$$e^t - e^{-t} = e^{-t}(e^{2t}-1) \le (e-1)2t, \qquad t\in[0,1],$$

hence, using (H7), (2.10) with σ_i is replaced by $\sigma_i-\alpha$, and (3.23), we get for $i = 1,\dots,m$

$$\begin{aligned}
\eta_i(t,h) &\le M_2\omega_\alpha(h)+\varepsilon\\
&\quad +\sum_{j=1}^{m}\tilde{b}_{ij}L_je^{\alpha\tau}(1+e)K_0\sum_{r=1}^{\ell(t)}\left(\frac{e^{-(\sigma_i-\alpha)(t-t_r-\tau_j-h)}-e^{-(\sigma_i-\alpha)(t-t_r-\tau_j+h)}}{\sigma_i-\alpha}\right)\\
&= M_2\omega_\alpha(h)+\varepsilon\\
&\quad +\sum_{j=1}^{m}\tilde{b}_{ij}L_je^{\alpha\tau}\left(\frac{(1+e)K_0}{\sigma_i-\alpha}\right)\left(e^{(\sigma_i-\alpha)(\tau_j+h)}-e^{(\sigma_i-\alpha)(\tau_j-h)}\right)\sum_{\ell=1}^{\ell(t)}e^{-(\sigma_i-\alpha)(t-t_\ell)}\\
&\le M_2\omega_\alpha(h)+\varepsilon+\sum_{j=1}^{m}\tilde{b}_{ij}L_je^{\alpha\tau}\left(\frac{(1+e)K_0}{\sigma_i-\alpha}\right)\left(\frac{e^{(\sigma_i-\alpha)h}-e^{-(\sigma_i-\alpha)h}}{1-e^{-(\sigma_i-\alpha)\underline{\delta}}}\right)\\
&\le M_2\omega_\alpha(h)+\varepsilon+\tilde{g}_ih, \qquad 0<h<h^*,
\end{aligned} \tag{3.24}$$

where $\tilde{g}_i = \tilde{g}_i(\varepsilon)$ is defined by

$$\tilde{g}_i = \sum_{j=1}^{m}\tilde{b}_{ij}L_je^{\sigma_i\tau}\frac{(1+e)K_0 2(e-1)}{1-e^{-(\sigma_i-\alpha)\underline{\delta}}} \quad \text{and} \quad h^* = \min\left\{h_2, \frac{1}{\sigma_1-\alpha},\dots,\frac{1}{\sigma_m-\alpha}\right\}.$$

Then (3.13) yields for $i = 1,\dots,m$, $t\ge 0$ and $0<h<h^*$

$$\begin{aligned}
w_i(t) &\le (1-\underline{p}_i^-)\Big(|\boldsymbol{\varphi}-\boldsymbol{\psi}|_C + M_2\omega_\alpha(h)+\tilde{g}_ih+\varepsilon\Big)+d_i^*h+\bar{d}_i\omega_\alpha(h)\\
&\quad +\sum_{j=1}^{m}(1-\underline{p}_i^-)\Lambda_{ij}L_je^{\alpha(h_0+\tau_j)}w_j(t)+\hat{d}_iw_i(t).
\end{aligned} \tag{3.25}$$

Relation (3.25) gives the vector inequality

$$\mathbf{w}(t) \le \mathbf{d}(h) + C\mathbf{w}(t), \qquad t\ge 0, \quad 0<h<h^*, \tag{3.26}$$

where

$$\begin{aligned}
\mathbf{w}(t) &= (w_1(t), \dots, w_m(t))^T, \\
\mathbf{d}(h) &= (d_1(h), \dots, d_m(h))^T, \quad \text{where} \\
d_i(h) &= (1 - \underline{p}_i^-)\Big(|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + M_2\omega_\alpha(h) + \tilde{g}_i h + \varepsilon\Big) + d_i^* h + \bar{d}_i \omega_\alpha(h), \\
C = (c_{ij}) \in \mathbb{R}^{m \times m}, &\qquad c_{ij} = \begin{cases} (1 - \underline{p}_i^-)\Lambda_{i,i} L_i e^{\alpha(h_0+\tau_i)} + \hat{d}_i, & i = j, \\ (1 - \underline{p}_i^-)\Lambda_{ij} L_j e^{\alpha(h_0+\tau_j)}, & i \neq j. \end{cases}
\end{aligned} \tag{3.27}$$

Relation (3.4) yields that $\|C\|_\infty < 1$ for $0 < h < h^*$. Then $I - C$ is an M-matrix, hence (3.26) implies

$$\mathbf{w}(t) \leq (I - C)^{-1}\mathbf{d}(h), \qquad t \geq 0, \quad 0 < h < h^*.$$

Therefore, the definitions of $\mathbf{w}(t)$, $\mathbf{d}(h)$ and

$$w_i(t) \leq \max_{-\tau \leq u \leq t} |x_i(u) - y_i(u)| \leq |\boldsymbol{\varphi} - \boldsymbol{\psi}|_C \leq d_i(h), \qquad t \in [-\tau, 0], \quad i = 1, \dots, m$$

gives

$$\mathbf{w}(t) \leq (I - C)^{-1}\mathbf{d}(h), \qquad t \geq -\tau, \quad 0 < h < h^*,$$

which yields (3.1) with

$$K_2 = \|(I - C)^{-1}\|_\infty \max_{i=1,\dots,m} (1 - \underline{p}_i^-) \quad \text{and} \quad \theta(h) = (M_2 + \max_{i=1,\dots,m} \bar{d}_i)\omega_\alpha(h) + h \max_{i=1,\dots,m} (\tilde{g}_i + d_i^*).$$

(ii) To prove (3.2) we now assume (H8) too.

Let M_0 and u_0 be the constants defined by Lemma 2.6 (ii). We consider estimate (3.18) of the proof of part (i). Now, since $b_{ij}(s)$ is bounded by b_{ij}^* for all $s \geq 0$, we estimate the last integral similarly to steps used for the set $C_{j,h}$ in the proof of part (i), but using b_{ij}^* instead of $\tilde{b}_{ij}$:

$$\eta_i(t, h) \leq M_2\omega_\alpha(h) + \sum_{j=1}^{m} \sum_{r=1}^{\ell(t)} \int_{t_r+\tau_j-h}^{t_r+\tau_j+h} e^{-(\sigma_i - \alpha)(t-s)} b_{ij}^* L_j e^{\alpha\tau_j}(1 + e)K_0 \, ds.$$

Then a calculation similar to that used in (3.24) and (2.24) gives

$$\eta_i(t, h) \leq M_2\omega_\alpha(h) + g_i^* h \leq M_2 M_0 h + g_i^* h, \qquad 0 < h < \bar{h}, \tag{3.28}$$

where

$$g_i^* = \sum_{j=1}^{m} b_{ij}^* L_j e^{\sigma_i \tau} \left(\frac{(1 + e)K_0 2(e - 1)}{1 - e^{-(\sigma_i - \alpha)\underline{\delta}}} \right) \quad \text{and} \quad \bar{h} = \min\{u_0, h^*\}.$$

Combining (2.24), (3.13) and (3.28) we get for $i = 1, \dots, m$, $t \geq 0$ and $0 < h < \bar{h}$

$$\begin{aligned}
w_i(t) \leq (1 - \underline{p}_i^-)&\Big(|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + M_2 M_0 h + g_i^* h\Big) + d_i^* h + \bar{d}_i M_0 h \\
&+ \sum_{j=1}^{m} (1 - \underline{p}_i^-)\Lambda_{ij} L_j e^{\alpha(h_0+\tau_j)} w_j(t) + \hat{d}_i w_i(t),
\end{aligned}$$

hence, the vector inequality

$$\mathbf{w}(t) \leq \hat{\mathbf{d}}(h) + C\mathbf{w}(t), \qquad t \geq 0, \quad 0 < h < \bar{h}$$

and therefore

$$\mathbf{w}(t) \le \hat{\mathbf{d}}(h) + C\mathbf{w}(t), \qquad t \ge -\tau, \quad 0 < h < \bar{h} \tag{3.29}$$

holds, where

$$\hat{\mathbf{d}}(h) = (\hat{d}_1(h), \dots, \hat{d}_m(h))^T,$$
$$\hat{d}_i(h) = (1 - \underline{p}_i^-)|\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + \Big((1 - \underline{p}_i^-)(M_2 M_0 + g_i^*) + d_i^* + \bar{d}_i M_0\Big)h.$$

Then (3.29) implies

$$\mathbf{w}(t) \le (I - C)^{-1}\hat{\mathbf{d}}(h), \qquad t \ge -\tau, \quad 0 < h < \bar{h},$$

which proves (3.2) with

$$K_2 = \|(I - C)^{-1}\|_\infty \max_{i=1,\dots,m}(1 - \underline{p}_i^-) \quad \text{and} \quad M = M_2 M_0 + \max_{i=1,\dots,m}(g_i^* + d_i^* + \bar{d}_i M_0). \qquad \square$$

Remark 3.2. We note that relation (3.1) gives for $\boldsymbol{\varphi} = \boldsymbol{\psi}$ that

$$\sup_{t \in [-\tau, \infty)} |\mathbf{x}(t) - \mathbf{y}(t)|_\infty \le K_2(\theta(h) + \varepsilon),$$

which yields that the solutions of (1.15) approximate that of (1.14) uniformly on $[0, \infty)$.

Remark 3.3. Let C be the matrix defined by (3.27). Assumption (H5) in Theorem 3.1 can be replaced by the weaker condition $\rho(C) < 1$, and the statement of the theorem remains true.

Thanks to our main Theorem 3.1 and Lemma 2.2 we can give the following result concerning the *transference of the exponential estimate* of the solutions of (1.14) to the approximate solution.

Proposition 3.4. *Suppose (H1)–(H7) hold. Let* $\mathbf{x} = (x_1, \dots, x_m)^T$ *be the solution of* (1.14) *corresponding to initial function* $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_m)^T$*, and* $\mathbf{y} = (y_1, \dots, y_m)^T$ *be the solution of* (1.15) *corresponding to* $h > 0$ *and an initial function* $\boldsymbol{\psi} = (\psi_1, \dots, \psi_m)^T$*, and let* $\alpha_0 > 0$ *be the corresponding constant from Lemma 2.2.*

(i) Then for every $0 < \alpha < \alpha_0$ *and* $\varepsilon > 0$ *there exist constants* $K_3 > 0$ *and* $h^* > 0$*, and a function* $\theta(h)$ *such that* $\theta(h) \to 0$ *as* $h \to 0+$*, and*

$$|\mathbf{y}(t)|_\infty \le e^{-\alpha t}K_3\Big(1 + |\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + \theta(h) + \varepsilon\Big), \qquad t \in [-\tau, \infty), \quad 0 < h < h^*. \tag{3.30}$$

(ii) Assume further (H8). Then for every $0 < \alpha < \alpha_0$ *there exist constants* $K_3 > 0$*,* $M > 0$ *and* $\bar{h} > 0$ *such that*

$$|\mathbf{y}(t)|_\infty \le e^{-\alpha t}K_3\Big(1 + |\boldsymbol{\varphi} - \boldsymbol{\psi}|_C + Mh\Big), \qquad t \in [-\tau, \infty), \quad 0 < h < \bar{h}. \tag{3.31}$$

Proof. The proof follows immediately from $|\mathbf{z}(t)|_\infty \le |\mathbf{z}(t) - \mathbf{x}(t)|_\infty + |\mathbf{x}(t)|_\infty$, (2.4), (3.1) and (3.2) with $K_3 = \max\{K_0, K_2\}$. $\square$

4 The bounded coefficients case

In the following, we give a practical result concerning the bounded coefficients case for a CNN delayed impulsive system as a simple consequence of Theorem 3.1. In addition to our assumptions (H1)–(H3), we suppose that the coefficient functions $a_i(t)$ are bounded below too, and $b_{ij}(t)$ are bounded. This will allow to simplify conditions (H4)–(H8), as follows:

(H4′) There exist positive constants $\underline{a}_i, a_i^*, c_i^*, \sigma_i$ for $i = 1, \ldots, m$ such that

(i) $\underline{a}_i \leq a_i(t) \leq a_i^*$ for $t \geq 0$ and $i = 1, \ldots, m$;

(ii) $\underline{a}_i - \dfrac{1}{\underline{\delta}} \ln(1 - \underline{p}_i^-) \geq \sigma_i, \quad i = 1, \ldots, m$, where $\underline{p}_i^- = \min\{0, \underline{p}_i\}$;

(iii) $|b_{ij}(t)| \leq b_{ij}^*, \quad t \geq 0, \quad i = 1, \ldots, m.$

(H5′) There exists ε_0 such that $0 < \varepsilon_0 < \sigma_i$ for $i = 1, \ldots, m$, and

$$\sum_{j=1}^{m} \frac{(1 - \underline{p}_i^-) b_{ij}^*}{\sigma_i - \varepsilon_0} L_j + \frac{(1 - \underline{p}_i^-) l_i^*}{(1 - p_i^*)(1 - e^{-\sigma_i \underline{\delta}})} < 1, \qquad i = 1, \ldots, m.$$

(H6′) There exist positive constants β_1, β_2 and c_i^* $(i = 1, \ldots, m)$ such that

$$|e_{i,k}| \leq e^{-\beta_1 t_k} e_i^*, \qquad k \in \mathbb{N}, \qquad \text{and} \qquad |c_i(t)| \leq e^{-\beta_2 t} c_i^*, \qquad t \geq 0, \quad i = 1, \ldots, m.$$

(H7′) There exists a positive constant L_φ such that

$$|\varphi_i(t) - \varphi_i(\bar{t})| \leq L_\varphi |t - \bar{t}|, \qquad t, \bar{t} \in [-\tau, 0], \quad i = 1, \ldots, m.$$

Note that (H4) (i) was used in the proofs of Lemma 2.2, Lemma 2.4 and Theorem 3.1 to prove estimates (2.5), (2.6) and (2.7). Now we show that our boundedness assumptions (H4′) imply the same estimates (2.5), (2.6) and (2.7).

Suppose $\underline{p}_i < 0$, $s \in [t_{r-1}, t_r)$ for some $r \in \mathbb{N}$, and $t \geq t_r$. Then (H3) (iv), (H4′) and the estimates $(\ell(t) - r)\underline{\delta} \leq t_{\ell(t)} - t_r \leq t - s$ yield

$$\begin{aligned} e^{-\int_s^t a_i(u)\,du} \left(\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) \right) &= \exp\left(-\int_s^t a_i(u)\,du + \sum_{j=r}^{\ell(t)} \ln(1 - p_{i,j}) \right) \\ &\leq \exp\left(-\underline{a}_i(t - s) + (\ell(t) - r + 1) \ln(1 - \underline{p}_i) \right) \\ &\leq (1 - \underline{p}_i) \exp\left(-\underline{a}_i(t - s) + \frac{t - s}{\underline{\delta}} \ln(1 - \underline{p}_i) \right) \\ &\leq (1 - \underline{p}_i) e^{-\sigma_i(t-s)}. \end{aligned}$$

If $s < t < t_r$, then $\ell(t) < r$, so $\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) = 1$. In the case $\underline{p}_i \geq 0$ we have $\sigma_i \leq \underline{a}_i$, hence

$$e^{-\int_s^t a_i(u)\,du} \left(\prod_{j=r}^{\ell(t)} (1 - p_{i,j}) \right) \leq e^{-\int_s^t a_i(u)\,du} \leq e^{-\sigma_i(t-s)}.$$

Therefore (2.5) holds. (2.6) and (2.7) can be proved similarly under assumption (H4′).

Using Remark 1.3, we get immediately that (H5′) implies (H5). Hence Theorem 3.1 has the following corollary.

Corollary 4.1. *Assume (H1)–(H3),(H4′)–(H7′) hold. Let* $\mathbf{x} = (x_1, \dots, x_m)^T$ *be the solution of* (1.14) *corresponding to initial function* $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_m)^T$*, and* $\mathbf{y} = (y_1, \dots, y_m)^T$ *be the solution of* (1.15) *corresponding to* $h > 0$ *and an initial function* $\boldsymbol{\psi} = (\psi_1, \dots, \psi_m)^T$*, and let* $\alpha_0 > 0$ *be the corresponding constant from Lemma* 2.2*. Then for every* $0 < \alpha < \alpha_0$ *there exist constants* $K_2 > 0$*,* $M > 0$ *and* $\bar{h} > 0$ *such that* (3.2) *holds. Hence, the solutions of* (1.14) *are approximated by that of* (1.15) *uniformly over* $[0, \infty)$.

5 An example

Now, we present an example to illustrate the applicability of our conditions.

Example 5.1. Consider the system

$$\begin{aligned} x_1'(t) &= -a_1(t)x_1(t) + \sum_{j=1}^{2} b_{1j}(t)g_j(x_j(t-\tau_j)) + c_1(t), \quad && t \neq t_n \\ x_2'(t) &= -a_2(t)x_2(t) + \sum_{j=1}^{2} b_{2j}(t)g_j(x_j(t-\tau_j)) + c_2(t), \quad && t \neq t_n \\ \Delta x_1(t_n) &= -p_{1,n}x_1(t_n^-) + e_{1,n} + J_{1,n}(x_1(t_n^-)), && n \in \mathbb{N} \\ \Delta x_2(t_n) &= -p_{2,n}x_2(t_n^-) + e_{2,n} + J_{2,n}(x_2(t_n^-)), && n \in \mathbb{N}, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} a_1(t) &= 2 + \sin(\sqrt{3}t), & a_2(t) &= 4 + \cos(t); \\ b_{11}(t) &= 0.5\sin(t), & b_{21}(t) &= 0.2\sin(t); \\ b_{12}(t) &= 0.3\cos(t), & b_{22}(t) &= 0.3\sin(t); \\ c_1(t) &= \exp(-t), & c_2(t) &= \exp(-2t); \\ p_{1,n} &= -0.15, \ n \in \mathbb{N}, & p_{2,n} &= -0.4, \ n \in \mathbb{N}; \\ e_{1,n} &= \exp(-3t_n), \ n \in \mathbb{N}, & e_{2,n} &= \exp(-4t_n), \ n \in \mathbb{N}; \\ g_1(x) &= \tanh(x), & g_2(x) &= \tanh(x); \\ J_{1,n}(x) &= \tfrac{1}{10}\tanh(x), \ n \in \mathbb{N}, & J_{2,n}(x) &= \tfrac{1}{10}\tanh(x), \ n \in \mathbb{N}. \end{aligned} \tag{5.2}$$

We suppose $\tau_1 = 1$, $\tau_2 = 2$, $\tau = \max\{\tau_1, \tau_2\} = 2$, the initial functions $\varphi_1(t), \varphi_2(t), \psi_1(t), \psi_2(t) : [-2, 0] \to \mathbb{R}$, defined as $\varphi_1(t) = \psi_1(t) = 0.5\sin(t)$ and $\varphi_2(t) = \psi_2(t) = \cos(t)$. Also, we consider $t_n = n$ for $n \in \mathbb{N}$.

System (5.1) is approximated by the following IDEPCA system

$$\begin{aligned} y_1'(t) &= -a_1(t)y_1(t) + \sum_{j=1}^{2} b_{1j}(t)g_j(y_j(\gamma(t) - \gamma(\tau_j))) + c_1(t), \quad && t \neq \gamma(t_n) \\ y_2'(t) &= -a_2(t)y_2(t) + \sum_{j=1}^{2} b_{2j}(t)g_j(y_j(\gamma(t) - \gamma(\tau_j))) + c_2(t), \quad && t \neq \gamma(t_n) \\ \Delta y_1(\gamma(t_n)) &= -p_{1,n}y_1(\gamma(t_n)^-) + e_{1,n} + I_{1,n}(y_1(\gamma(t_n)^-)), && n \in \mathbb{N} \\ \Delta y_2(\gamma(t_n)) &= -p_{2,n}y_2(\gamma(t_n)^-) + e_{2,n} + I_{2,n}(y_2(\gamma(t_n)^-)), && n \in \mathbb{N}, \end{aligned} \tag{5.3}$$

where $\gamma(t) = [t/h]h$ and all the coefficients are given in (5.2). Because $\tanh(x)$ is a Lipschitz-type function, with Lipschitz constant 1, we can conclude that $L_i = 1$ and $l_i^* = l_{i,n} = \frac{1}{10}$, $i = 1, 2,\ n \in \mathbb{N}$. We have $\underline{\delta} = 1$, $\underline{a}_1 = 1$, $\underline{a}_2 = 3$, $\underline{p}_1 = p_1^* = -0.15$ and $\underline{p}_2 = p_2^* = -0.4$, hence we get

$$\underline{a}_1 - \frac{1}{\underline{\delta}} \ln(1 - \underline{p}_1^-) \approx 0.86024, \qquad \underline{a}_2 - \frac{1}{\underline{\delta}} \ln(1 - \underline{p}_2^-) \approx 2.66353,$$

so $\sigma_1 = 0.86$ and $\sigma_2 = 2.66$ satisfy (H4′) (ii). Use $\varepsilon_0 = 0.01$, $b_{11}^* = 0.5$, $b_{12}^* = 0.3$, $b_{21}^* = 0.2$ and $b_{22}^* = 0.3$. Then

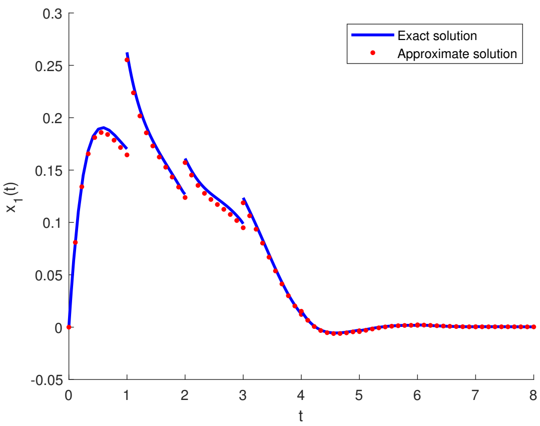
$$\frac{1 - \underline{q}_1^-}{\sigma_1 - \varepsilon_0} \sum_{j=1}^{m} b_{1j}^* L_j + \frac{(1 - \underline{q}_1^-) l_1^*}{(1 - q_1^*)(1 - e^{-\sigma_1 \underline{\delta}})} = \frac{1.15}{0.85}(0.5 + 0.3) + \frac{0.1}{1 - e^{-0.86}} \approx 0.91748,$$

and

$$\frac{1 - \underline{q}_2^-}{\sigma_2 - \varepsilon_0} \sum_{j=1}^{m} b_{2j}^* L_j + \frac{(1 - \underline{q}_2^-) l_1^*}{(1 - q_2^*)(1 - e^{-\sigma_2 \underline{\delta}})} = \frac{1.4}{0.85}(0.2 + 0.3) + \frac{0.1}{1 - e^{-2.66}} \approx 0.31884,$$

therefore (H5′) is satisfied. Therefore Corollary 4.1 yields that the solutions of (5.3) approximate that of (5.1) uniformly on $[0, \infty)$ as h goes to 0.

Figures 5.1–5.2 illustrate the solution of (5.1) and its approximation by the solution of (5.3) corresponding to the discretization parameter $h = 0.1$. Note that for this value of h and the definition of t_n, we have $\gamma(t_n) = t_n$ for all $n \in \mathbb{N}$. Both initial value problems are solved numerically using the function `ddesd` in *Matlab* on the consecutive intervals $[t_n, t_{n+1}]$. The blue curves are the graphs of $x_i(t)$, and the red dots are the values of the function $y_i(t)$ at the time values $t = 0.1n$, $n \in \mathbb{N}_0$. At the impulse time points, the left-hand limits of $y_i(t)$ are also displayed. Even though the discretization parameter is relatively large for this numerical experience, we see that the approximation error becomes smaller as time increases. This is a consequence of estimate (3.2).



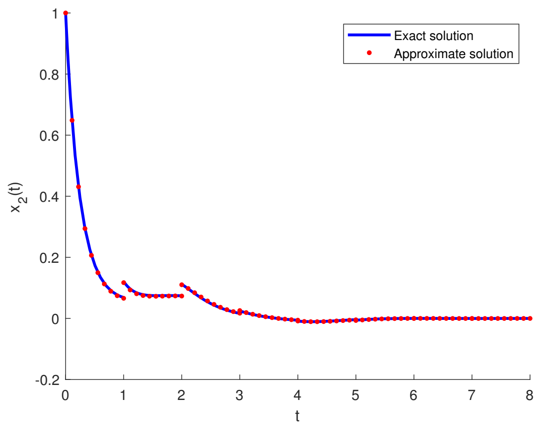


Figure 5.1: Graphs of $x_1(t), y_1(0.1n)$ (on the left) and $x_2(t), y_2(0.1n)$ (on the right) with $h = 0.1$

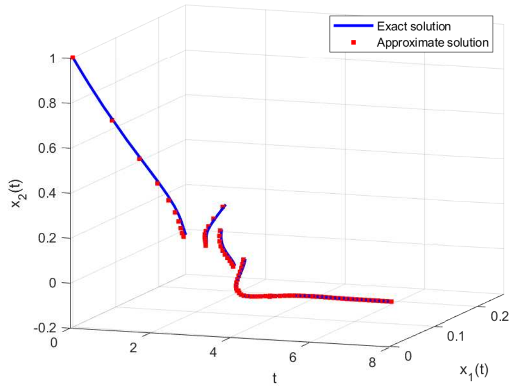


Figure 5.2: The graphs of $(t, x_1(t), x_2(t))$ and $(0.1n, z_1(0.1n), z_2(0.1n))$ with $h = 0.1$.

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Addendum to Higher order stroboscopic averaged functions: a general relationship with Melnikov functions

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Abstract. This addendum presents a relevant stronger consequence of the main theorem of the paper "Higher order stroboscopic averaged functions: a general relationship with Melnikov functions", *Electron. J. Qual. Theory Differ. Equ.* **2021**, No. 77.

Keywords: averaging theory, Melnikov method, averaged functions, Melnikov functions, higher order analysis.

2020 Mathematics Subject Classification: 34C29, 34E10, 34C25.

This addendum addresses the findings presented in the paper [1] titled "Higher order stroboscopic averaged functions: a general relationship with Melnikov functions" published in *Electron. J. Qual. Theory Differ. Equ.* **2021**, No. 77.

The main result of the referred paper, [1, Theorem A], establishes a general relationship between averaged functions $\mathbf{g}_i$ and Melnikov functions $\mathbf{f}_i$. As a direct consequence of this general relationship, [1, Corollary A] states that if, for some $\ell \in \{2,\dots,k\}$, either $\mathbf{f}_1 = \cdots = \mathbf{f}_{\ell-1} = 0$ or $\mathbf{g}_1 = \cdots = \mathbf{g}_{\ell-1} = 0$, then $\mathbf{f}_i = T\,\mathbf{g}_i$ for $i \in \{1,\dots,\ell\}$. This consequence was somewhat expected based on existing results in the literature within more restricted contexts. Here, we will demonstrate that under the same conditions, the relationship $\mathbf{f}_i = T\,\mathbf{g}_i$ actually holds for every $i \in \{1,\dots,2\ell-1\}$, which represents a more unexpected outcome. The expression for $\mathbf{g}_{2\ell}(z)$ will also be provided.

Proposition 1. *Let $\ell \in \{2,\dots,k\}$. If either $\mathbf{f}_1 = \cdots = \mathbf{f}_{\ell-1} = 0$ or $\mathbf{g}_1 = \cdots = \mathbf{g}_{\ell-1} = 0$, then $\mathbf{f}_i = T\,\mathbf{g}_i$ for $i \in \{1,\dots,2\ell-1\}$ and*

$$\mathbf{g}_{2\ell}(z) = \frac{1}{T}\left(\mathbf{f}_{2\ell}(z) - \frac{1}{2}d\mathbf{f}_\ell(z)\cdot\mathbf{f}_\ell(z)\right) \text{ or, equivalently, } \mathbf{f}_{2\ell}(z) = T\mathbf{g}_{2\ell}(z) + \frac{T^2}{2}d\mathbf{g}_\ell(z)\cdot\mathbf{g}_\ell(z).$$

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Proof. Given $\ell \in \{2,\dots,k\}$, assume that either $\mathbf{f}_1=\cdots=\mathbf{f}_{\ell-1}=0$ or $\mathbf{g}_1=\cdots=\mathbf{g}_{\ell-1}=0$. From [1, Corollary A], we have that

$$\mathbf{g}_i=\mathbf{f}_i=0,\quad \text{for } i\in\{1,\dots,\ell-1\},\quad \text{and}\quad \mathbf{g}_\ell=\frac{1}{T}\mathbf{f}_\ell. \tag{1}$$

For any i, [1, Theorem A] provides

$$\mathbf{g}_i(z)=\frac{1}{T}\left(\mathbf{f}_i(z)-\sum_{j=1}^{i-1}\sum_{m=1}^{j}\frac{1}{j!}d^m\mathbf{g}_{i-j}(z)\int_0^T B_{j,m}(\tilde{y}_1,\dots,\tilde{y}_{j-m+1})(s,z)ds\right), \tag{2}$$

where $\tilde{y}_i(t,z)$, for $i\in\{1,\dots,k\}$, are polynomial in the variable t recursively defined as follows:

$$\begin{aligned}
&\tilde{y}_1(t,z)=t\,\mathbf{g}_1(z)\\
&\tilde{y}_i(t,z)=i!t\,\mathbf{g}_i(z)+\sum_{j=1}^{i-1}\sum_{m=1}^{j}\frac{i!}{j!}d^m\mathbf{g}_{i-j}(z)\int_0^t B_{j,m}(\tilde{y}_1,\dots,\tilde{y}_{j-m+1})(s,z)ds.
\end{aligned} \tag{3}$$

Taking (1) into account, the function $\mathbf{g}_{i-j}$ in (2) vanishes for $i-j\le \ell-1$, that is, for $j\ge i-\ell+1$. Thus,

$$\mathbf{g}_i(z)=\frac{1}{T}\left(\mathbf{f}_i(z)-\sum_{j=1}^{i-\ell}\sum_{m=1}^{j}\frac{1}{j!}d^m\mathbf{g}_{i-j}(z)\int_0^T B_{j,m}(\tilde{y}_1,\dots,\tilde{y}_{j-m+1})(s,z)ds\right). \tag{4}$$

Also, from (1) and (3), one has

$$\tilde{y}_1=\cdots\tilde{y}_{\ell-1}=0\quad\text{and}\quad \tilde{y}_\ell(t,z)=\ell!t\mathbf{g}_\ell(z)=\frac{\ell!}{T}t\mathbf{f}_\ell(z). \tag{5}$$

Now, let $i\in\{\ell+1,\dots,2\ell-1\}$. Thus, for $j\le i-\ell$ and $m\ge 1$, one has that

$$j-m+1\le i-\ell\le 2\ell-1-\ell=\ell-1,$$

which implies, from (5), that $\tilde{y}_1=\cdots=\tilde{y}_{j-m+1}=0$ in (4). Consequently, $\mathbf{f}_i(z)=T\mathbf{g}_i(z)$.

Finally, from (4),

$$\mathbf{g}_{2\ell}(z)=\frac{1}{T}\left(\mathbf{f}_{2\ell}(z)-\sum_{j=1}^{\ell}\sum_{m=1}^{j}\frac{1}{j!}d^m\mathbf{g}_{2\ell-j}(z)\int_0^T B_{j,m}(\tilde{y}_1,\dots,\tilde{y}_{j-m+1})(s,z)ds\right). \tag{6}$$

Notice that, for $1\le j\le\ell$ and $1\le m\le j$, the relationship $j-m+1\ge\ell$ implies that $m=1$ and $j=\ell$, which are the only possible values for m and j for which $\tilde{y}_{j-m+1}$ in (6) is not vanishing. In this case, from (5), $\tilde{y}_1=\cdots=\tilde{y}_{j-m}=0$ and $\tilde{y}_{j-m+1}=\tilde{y}_\ell=\ell!t\mathbf{g}_\ell(z)$. Thus,

$$\begin{aligned}
\mathbf{g}_{2\ell}(z)&=\frac{1}{T}\left(\mathbf{f}_{2\ell}(z)-\frac{1}{\ell!}d\mathbf{g}_\ell(z)\int_0^T B_{\ell,1}(0,\dots,0,\ell!t\mathbf{g}_\ell(z))\,ds\right)\\
&=\frac{1}{T}\left(\mathbf{f}_{2\ell}(z)-\frac{1}{\ell!}d\mathbf{g}_\ell(z)\int_0^T \ell!t\mathbf{g}_\ell(z)\,ds\right)\\
&=\frac{1}{T}\left(\mathbf{f}_{2\ell}(z)-\frac{T^2}{2}d\mathbf{g}_\ell(z)\cdot\mathbf{g}_\ell(z)\right)=\frac{1}{T}\left(\mathbf{f}_{2\ell}(z)-\frac{1}{2}d\mathbf{f}_\ell(z)\cdot\mathbf{f}_\ell(z)\right).
\end{aligned}$$

Equivalently,

$$\mathbf{f}_{2\ell}(z)=T\mathbf{g}_{2\ell}(z)+\frac{T^2}{2}d\mathbf{g}_\ell(z)\cdot\mathbf{g}_\ell(z).$$

□

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Mirroring in lattice equations and a related functional equation

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Abstract. We use a functional form of the mirroring technique to fully characterize equivalence classes of unbounded stationary solutions of lattice reaction-diffusion equations with eventually negative and decreasing nonlinearities. We show that solutions which connect a stable fixed point of the nonlinearity with infinity can be characterized by a single parameter from a bounded interval. Within a two-dimensional parametric space, these solutions form a boundary to an existence region of solutions which diverge in both directions. Additionally, we reveal a natural relationship of lattice equations with an interesting functional equation which involves an unknown function and its inverse.

Keywords: Nagumo equation, lattice differential equation, patterns, unbounded solutions, equivalence class.

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1 Introduction

We study a special class of unbounded stationary solutions to reaction-diffusion lattice differential equations (LDE)

$$u_i'(t) = d(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)) + g(u_i(t)), \quad i \in \mathbb{Z}, \quad t > 0, \tag{1.1}$$

in which $d > 0$ is a diffusion rate and g is a reaction function. We assume that g is a C^1-function and satisfies the following assumptions:

(g_1) $g(\ell) = 0$ for some $\ell \in \mathbb{R}$,

(g_2) $g'(u) < 0$ for all $u \in [\ell, \infty)$.

Let us immediately note that general assumptions (g_1)–(g_2) cover well-known and widely studied prototypes of monostable and bistable dynamics – the Fisher lattice equation (with logistic reaction) and the Nagumo lattice equation (with cubic reaction) as well as many others reactions, their modifications, and caricatures which have been commonly used in numerous studies on lattice equations [2,5,12,15,21]. See Section 4 for detailed examples.

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The primary object of our interest is a class of unbounded stationary solutions which connect the stationary point ℓ with infinity. Therefore, we refer to them as onesided unbounded solutions.

Bounded stationary and traveling patterns

The LDE (1.1) serves as a discrete-space counterpart of the reaction-diffusion partial differential equation (PDE)

$$u_t(x,t) = du_{xx}(x,t) + g(u(x,t)), \quad x \in \mathbb{R}, \quad t > 0, \tag{1.2}$$

which occurs naturally as a model to many biological and chemical processes and has inspired many mathematical concepts and techniques – traveling wave solutions $u(x,t) = \Phi(x-ct)$, perturbation techniques, stability of waves, etc. [13].

Recent interest in the LDE (1.1) stems from its natural applications (see, e.g., [15]) and the fact that the discrete space provides new or richer dynamic phenomena in comparison with the PDE (1.2). The most notable among those is the pinning of traveling waves for sufficiently small diffusion parameters. The Nagumo lattice equation is a prototype of wave pinning [12,16,21]. In other words, for sufficiently small $d > 0$, solutions of the form $u_i(t) = \Phi(i-ct)$ of the LDE (1.1) with the bistable nonlinearity with a monotone profile Φ satisfy $c = 0$. The phenomenon is general and has been studied in various discrete-time models [11] and systems of lattice equations [4,6].

The presence of pinning regions in discrete-space models is naturally related to spatial topological chaos, the existence of large number of bounded heterogeneous stationary patterns of the LDE (1.1), [2,15]. Stationary solutions of the LDE (1.1) are double-sequences $u = (u_i)$, $i \in \mathbb{Z}$, satisfying difference equations

$$d(u_{i-1} - 2u_i + u_{i+1}) + g(u_i) = 0, \quad i \in \mathbb{Z}. \tag{1.3}$$

The structure of large number of solutions for small diffusion $0 < d \ll 1$ is still not fully understood. Partial results are related to localized pulses and their bifurcations [1] or the ordering and symmetry of exponential number of k-periodic patterns [9]. Explicit forms of specific solutions have been found for piecewise linear nonlinearities [3,5,18,20]. Connections of stable periodic patterns then lead to existence of nonmonotone waves, [8,10]. However, many fascinating open questions remain unanswered. These are related, for example, to bifurcations of pulses [1] finite-dimensional graph reaction-diffusion equations which are connected to k-periodic patterns [19] but also to the broader picture, e.g., coexistence of bounded and unbounded patterns and a related ambition to describe all types of nonnegative patterns of the LDE (1.1). The goal of this paper is to contribute by describing onesided unbounded patterns and as a by-product describe mirroring functional iterations and a relationship to a functional equation.

Unbounded stationary patterns

In [7] we fully characterized equivalence classes of generally asymmetric twosided unbounded stationary solutions of (1.1) with (g_1)–(g_2) being satisfied such that $u_i > \ell$ for every $i \in \mathbb{Z}$ and

$$\lim_{i \to \pm\infty} u_i = \infty, \tag{1.4}$$

see Figure 1.1. In contrast to twosided unbounded solutions of the PDE (1.2) we have shown that the twosided unbounded solutions (i) form a two-parametric family of equivalence classes,

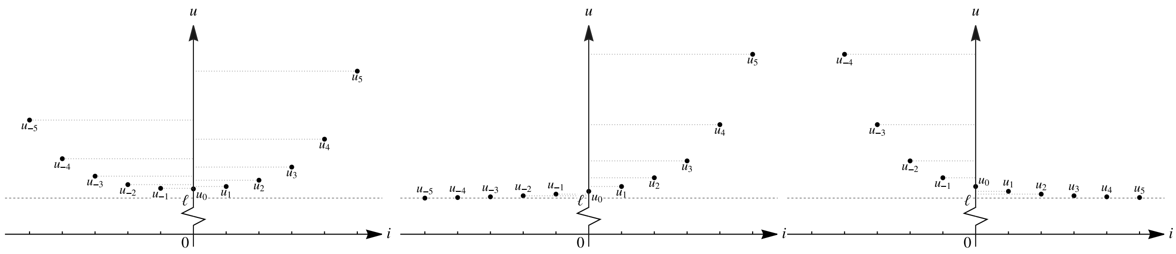

Figure 1.1: A twosided unbounded stationary solution of (1.1) satisfying (1.4) (left panel) and onesided unbounded stationary solutions satisfying either (1.5) or (1.6) (center and right panel).

(ii) are generally asymmetric, and (iii) exist on the whole unbounded integer lattice $\mathbb{Z}$ (i.e., they do not blow up at the ends of bounded spatial interval), see Theorem 3.1 below. Finally, in contrast to bounded patterns, twosided unbounded patterns of the LDE (1.1) exist for all diffusion values $d > 0$.

Onesided unbounded stationary solutions

Motivated by miscellaneous types of stationary solutions of (1.1), we primarily focus on the characterization of other type of unbounded stationary solutions, specifically, onesided unbounded stationary solutions which satisfy $u_i > \ell$ for all $i \in \mathbb{Z}$ and either (see Figure 1.1)

$$\lim_{i\to-\infty} u_i = \ell \quad \text{and} \quad \lim_{i\to\infty} u_i = \infty, \tag{1.5}$$

or

$$\lim_{i\to-\infty} u_i = \infty \quad \text{and} \quad \lim_{i\to\infty} u_i = \ell. \tag{1.6}$$

The equations (1.1) and (1.3) are autonomous in the spatial variable $i \in \mathbb{Z}$. Thus, every solution generates an equivalence class of another solutions which are only shifted in i. For this purpose, we say that stationary solutions u, u^* of (1.1) are *equivalent* (denoted $u \sim u^*$) if there exists an $s \in \mathbb{Z}$ such that $u_{i+s} = u_i^*$ for every $i \in \mathbb{Z}$. The equivalence class represented by a solution u^* is denoted by $[u^*] = \{u \in \mathbb{R}^{\mathbb{Z}} : \ u \sim u^*\}$.

The first main result of this manuscript characterizes the onesided stationary solutions of (1.1).

Theorem 1.1. *Let g be a C^1-function and satisfy (g1)–(g2). There exists a unique function $f : [\ell, \infty) \to [\ell, \infty)$ which is continuous, strictly increasing with $f(\ell) = \ell$, $\lim_{u\to\infty} f(u) = \infty$, and $f(u) > u$ for all $u > \ell$ such that for arbitrary $\xi > \ell$ every $\alpha \in [\xi, f(\xi))$ determines an equivalence class $[u^{\alpha,I}]$ of strictly increasing stationary solutions u of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.5); and an equivalence class $[u^{\alpha,D}]$ of strictly decreasing stationary solutions u of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.6). The representatives $u^{\alpha,I}$ and $u^{\alpha,D}$ satisfy*

$$u_0^{\alpha,I} = \alpha \quad \textit{and} \quad u_1^{\alpha,I} = f(\alpha); \tag{1.7}$$

and

$$u_0^{\alpha,D} = \alpha \quad \textit{and} \quad u_1^{\alpha,D} = f^{-1}(\alpha), \tag{1.8}$$

respectively. Moreover, for every $\tilde{\alpha} \in [\xi, f(\xi))$ there is $[u^{\alpha,I}] \neq [u^{\tilde{\alpha},I}]$ and $[u^{\alpha,D}] \neq [u^{\tilde{\alpha},D}]$ provided $\alpha \neq \tilde{\alpha}$.

On the contrary, every stationary solution u of (1.1) satisfying $u_i > \ell$ for every $i \in \mathbb{Z}$ and (1.5)

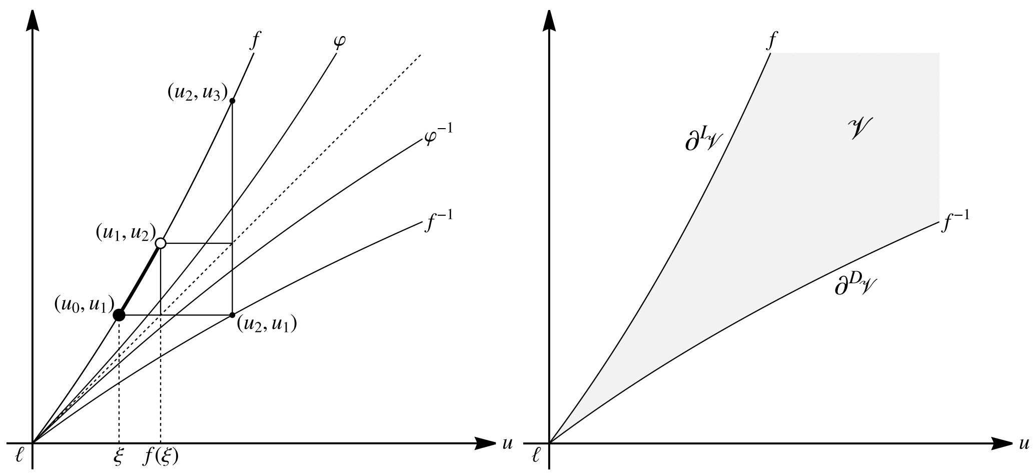

Figure 1.2: Illustration of two main results. Theorem 1.1 implies the existence of a function f and the corresponding interval $[\xi, f(\xi))$. Each value from this interval characterizes an equivalence class of increasing (1.5) or decreasing (1.6) onesided unbounded solutions. Mirroring symmetry via φ from (1.11) and cobwebbing which are used to construct the solutions are indicated (left panel). Theorem 1.2 then shows that the curves f and f^{-1} form a boundary to a set $\mathscr{V}$. All pairs $(u_i, u_{i+1}) \in \mathscr{V}$ generate twosided unbounded solutions with (1.4), whereas all pairs $(u_i, u_{i+1}) \in \partial^I\mathscr{V}$ or $(u_i, u_{i+1}) \in \partial^D\mathscr{V}$ lead to onesided unbounded solutions with (1.5) or (1.6) (right panel).

is strictly increasing and belongs into one of the above described equivalence classes $[u^{\alpha,I}]$ *for some* $\alpha \in [\xi, f(\xi))$; *and every stationary solution* u *of* (1.1) *satisfying* $u_i > \ell$ *for every* $i \in \mathbb{Z}$ *and* (1.6) *is strictly decreasing and belongs into one of the above described equivalence classes* $[u^{\alpha,D}]$ *for some* $\alpha \in [\xi, f(\xi))$.

In other words, we are able to characterize the equivalence classes by a single value $\alpha \in [\xi, f(\xi))$, see Figure 1.2. In Sections 2 and 3 we provide the proof of Theorem 1.1 which relies on an iterative construction of function f. The onesided unbounded stationary solutions can then also be iteratively constructed via mirroring or cobwebbing as indicated in Figure 1.2 as well.

Characterization of onesided and twosided unbounded stationary solutions

Combining Theorem 1.1 and results from [7] we obtain a full characterization of unbounded stationary solutions of (1.1) which satisfy $u_i > \ell$ for all $i \in \mathbb{Z}$. We define the following open set using the function f from Theorem 1.1:

$$\mathscr{V} = \left\{ (\xi, \zeta) \in \mathbb{R}^2 : \ \xi > \ell \ \text{ and } \ f^{-1}(\xi) < \zeta < f(\xi) \right\}, \tag{1.9}$$

and upper and lower parts of its boundary (see Figure 1.2)

$$\begin{aligned} \partial^I \mathscr{V} &= \left\{ (\xi, \zeta) \in \mathbb{R}^2 : \ \xi > \ell \ \text{ and } \ \zeta = f(\xi) \right\}, \\ \partial^D \mathscr{V} &= \left\{ (\xi, \zeta) \in \mathbb{R}^2 : \ \xi > \ell \ \text{ and } \ \zeta = f^{-1}(\xi) \right\}. \end{aligned} \tag{1.10}$$

Obviously, there is $\partial^I \mathscr{V} \cap \partial^D \mathscr{V} = \emptyset$ and the boundary $\partial \mathscr{V}$ of $\mathscr{V}$ satisfies

$$\partial \mathscr{V} = \partial^I \mathscr{V} \cup \partial^D \mathscr{V} \cup \{(\ell, \ell)\}.$$

Our second main result states that the sets $\mathscr{V}$, $\partial^I \mathscr{V}$, and $\partial^D \mathscr{V}$ fully describe all values of all twosided and onesided unbounded stationary solutions of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$, respectively.

Theorem 1.2. *Let g be a C^1-function, satisfy (g_1)–(g_2), and u be a stationary solution of* (1.1) *such that $u_i > \ell$ for all $i \in \mathbb{Z}$. Then:*

(i) (1.4) *holds if and only if $(u_i, u_{i+1}) \in \mathscr{V}$ for all $i \in \mathbb{Z}$,*

(ii) (1.5) *holds if and only if $(u_i, u_{i+1}) \in \partial^I \mathscr{V}$ for all $i \in \mathbb{Z}$,*

(iii) (1.6) *holds if and only if $(u_i, u_{i+1}) \in \partial^D \mathscr{V}$ for all $i \in \mathbb{Z}$.*

Mirroring

Our main tool to study asymmetric and symmetric twosided unbounded solutions in [7] was the mirroring technique. The second order difference equation (1.3) for finding stationary solutions of (1.1) can be transformed into

$$u_{i+1} - \left(u_i - \tfrac{1}{2d} g(u_i)\right) = \left(u_i - \tfrac{1}{2d} g(u_i)\right) - u_{i-1}, \quad i \in \mathbb{Z}.$$

If we define an auxiliary function (which we call a mirroring function)

$$\varphi(u) = u - \tfrac{1}{2d} g(u), \tag{1.11}$$

we obtain a mirroring symmetry with respect to φ for all stationary solutions (1.3), since

$$u_{i+1} - \varphi(u_i) = \varphi(u_i) - u_{i-1}, \quad i \in \mathbb{Z}. \tag{1.12}$$

In this paper we go one step further and study mirroring of functions and their sequences in Section 2.

Functional equation

The mirroring (1.12) in the proof of Theorem 1.1 closely relate the problem of finding stationary solutions (1.3) of the LDE (1.1) with an interesting functional equation

$$\frac{f(u) + f^{-1}(u)}{2} = \varphi(u), \quad u \in [0, \infty), \tag{1.13}$$

in which φ is a given function and f is an unknown function to be found, see Figure 1.2. To our best knowledge, this challenging problem has not been studied itself and its analysis can have deep consequences for other stationary patterns of the LDE (1.1), most notably classes of bounded patterns. See Section 5 for more details.

Paper structure

In Section 2 we generalize the mirroring technique (1.12) to a functional iterative scheme and study the monotonicity and convergence of generated function sequences. In Section 3 we then use these results to prove Theorems 1.1 and 1.2 and show that the onesided unbounded solutions (1.5) or (1.6) are generated by a value from a single interval and form a boundary to the two-parametric domain which generate twosided unbounded solutions satisfying (1.4). We then illustrate our results by several examples with different nonlinearities g in Section 4 and discuss the functional equation (1.13) and its solvability in a special case connected to our analysis in Section 5. We conclude in Section 6 by final remarks which connect our study to the solutions of the PDE (1.2), topological chaos of the LDE (1.1), and further possible applications of mirroring schemes and the functional equation (1.13).

2 Mirroring idea and functional iterative scheme

To describe the functional generalization of the mirroring (1.12) and establish that generated iterations are well-defined, we need some functions (for now specifically φ) satisfy some desired properties. For this purpose, we say that a function $p : [\ell, \infty) \to \mathbb{R}$ satisfies *(p_1)* or *(p_2)* provided:

(p_1) $p(\ell) = \ell$,

(p_2) $p'(u) > 1$ for all $u \in [\ell, \infty)$ (which also yields that $p(u)$ is strictly increasing and thus invertible),

respectively. The next lemma states that the function φ given by (1.11) satisfies *(p_1)*–*(p_2)* provided *(g_1)*–*(g_2)* hold.

Lemma 2.1. *Let (g_1)–(g_2) be satisfied. The function φ defined by* (1.11) *is of class C^1 and satisfies (p_1)–(p_2).*

Proof. The statements follow immediately from the definition (1.11) of φ. □

Now we are able to make the following considerations. We call the relation (1.12) the mirroring scheme, since for given initial conditions $u_0 > \ell$, $u_1 > \ell$ the point (u_2, u_1) as a point in the $\mathbb{R}^2$-plane is by (1.12) the horizontal mirror image of (u_0, u_1) with respect to the graph of φ^{-1}. Then, the point (u_2, u_3) is by (1.12) the vertical mirror image of (u_2, u_1) with respect to the graph of function φ, etc. (see Figure 1.2). Therefore, the forward solution u_i of (1.3) for $i = 2, 3, \dots$ can be generated from the initial conditions u_0, u_1 by mirroring of the points with respect to φ^{-1} horizontally and with respect to φ vertically, respectively, in the switching manner.

Analogically, the backward solution u_i of (1.3) for $i = -1, -2, \dots$ can be generated from the initial conditions u_0, u_1 by mirroring of the points with respect to φ vertically and with respect to φ^{-1} horizontally, respectively.

At this stage, we generalize the mirroring scheme (1.12) to functions. Let φ satisfy *(p_1)*–*(p_2)* and consider the following functional iterative scheme:

$$f_{n+1}(u) = 2\varphi(u) - f_n^{-1}(u), \quad n \in \mathbb{N}_0, \quad u \in [\ell, \infty). \tag{2.1}$$

Generally, the iterates do not have to be well-defined because of the inverses. This fundamentally depends on the properties of the initial function f_0. We focus on two special sequences given by specific initial iterates f_0 for which we establish that all iterates given by (2.1) are well-defined.

Definition 2.2. Let φ be a C^1-function and satisfy *(p_1)*–*(p_2)*. We define functional sequences $(\underline{f}_n)$ and $(\bar{f}_n)$, $n \in \mathbb{N}_0$, as follows:

(i) $(\underline{f}_n)$ are the iterates of (2.1) initiated by $\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u)$,

(ii) $(\bar{f}_n)$ are the iterates of (2.1) initiated by $\bar{f}_0(u) = 2\varphi(u) - \ell$.

Let us note that the inverse φ^{-1} is well-defined thanks to *(p₂)*. Further, let us verify that the iterates $\underline{f}_n$ and $\bar{f}_n$ in Definition 2.2 are correctly defined for every $n \in \mathbb{N}_0$ as well. This (besides others) follows from the following lemma.

Lemma 2.3. *Let φ be a C^1-function and satisfy (p_1)–(p_2). If f_n is a C^1-function and satisfies (p_1)–(p_2), then f_{n+1} defined by* (2.1) *is well-defined C^1-function and satisfies (p_1)–(p_2) as well.*

In particular, $\underline{f}_n$ and $\bar{f}_n$ are well-defined C^1-functions for every $n \in \mathbb{N}_0$ and all satisfy (p_1)–(p_2).

Proof. Since $f_n'(u) > 1$ for all $u \in [\ell, \infty)$ by (p_2), the inverse f_n^{-1} is well-defined and of class C^1 by the inverse function rule

$$(f_n^{-1})'(u) = \frac{1}{f_n'(f_n^{-1}(u))}. \tag{2.2}$$

Hence, f_{n+1} defined by (2.1) is also well-defined and of class C^1. If $f_n(\ell) = \ell$ by (p_1), then $f_{n+1}(\ell) = \ell$ immediately from (2.1) and thanks to $\varphi(\ell) = \ell$ (the function φ satisfies (p_1) as well by the assumption). Moreover, if $f_n'(u) > 1$ for all $u \in [\ell, \infty)$, then $(f_n^{-1})'(u) < 1$ for all $u \in [\ell, \infty)$ again by (2.2). Since also $\varphi'(u) > 1$ for all $u \in [\ell, \infty)$ (φ satisfies (p_2)), then

$$f_{n+1}'(u) = 2\varphi'(u) - (f_n^{-1})'(u) > 2 \cdot 1 - 1 = 1 \quad \text{for all} \quad u \in [\ell, \infty)\,.$$

Finally, one can similarly show that $\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u)$ and $\bar{f}_0(u) = 2\varphi(u) - \ell$ satisfy (p_1)–(p_2). Then (p_1)–(p_2) hold for all $\underline{f}_n$ and $\bar{f}_n$ as well by induction. □

In the next lemma we show that the sequence $(\underline{f}_n)$ is increasing, $(\bar{f}_n)$ is decreasing, and whole sequence $(\underline{f}_n)$ lies below $(\bar{f}_n)$, see Figure 2.1.

Lemma 2.4. *Let φ be a C^1-function and satisfy (p_1)–(p_2). Then for every $m, n \in \mathbb{N}_0$ and all $u \in [\ell, \infty)$ the following hold:*

(i) $\varphi(u) \le \underline{f}_n(u) \le \underline{f}_{n+1}(u)$,

(ii) $\bar{f}_{n+1}(u) \le \bar{f}_n(u)$,

(iii) $\underline{f}_n(u) \le \bar{f}_m(u)$.

Moreover, the equalities hold if and only if $u = \ell$.

Proof. Firstly, there is $u \le \varphi(u) \le \underline{f}_0(u) \le \bar{f}_0(u)$ for all $u \in [\ell, \infty)$. Indeed, the first and last inequalities follow immediately from (p_1)–(p_2) (Lemma 2.1). The middle one is verified again by (p_1)–(p_2) and by the following:

$$\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u) = \varphi(u) + (\varphi(u) - \varphi^{-1}(u)) \ge \varphi(u).$$

For the inverses, the reversed inequalities $\ell \le \bar{f}_0^{-1}(u) \le \underline{f}_0^{-1}(u) \le \varphi^{-1}(u)$ hold for all $u \in [\ell, \infty)$.

Let us prove *(i)*, i.e., that $\varphi(u) \le \underline{f}_n(u) \le \underline{f}_{n+1}(u)$ for all $n \in \mathbb{N}_0$ and $u \in [\ell, \infty)$ by induction. For $n = 0$ there is $\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u)$, i.e., $2\varphi(u) = \underline{f}_0(u) + \varphi^{-1}(u)$, and thus

$$\underline{f}_1(u) = 2\varphi(u) - \underline{f}_0^{-1}(u) = \underline{f}_0(u) + \varphi^{-1}(u) - \underline{f}_0^{-1}(u) \ge \underline{f}_0(u) \ge \varphi(u),$$

since $\underline{f}_0(u) \ge \varphi(u) \ge \varphi^{-1}(u)$ for all $u \in [\ell, \infty)$. Assume that $\varphi(u) \le \underline{f}_{n-1}(u) \le \underline{f}_n(u)$ for some $n \in \mathbb{N}$ and all $u \in [\ell, \infty)$. Then, $\ell \le \underline{f}_n^{-1}(u) \le \underline{f}_{n-1}^{-1}(u) \le \varphi^{-1}(u)$ for all $u \in [\ell, \infty)$ by inversion. Further, for $n + 1$ there is

$$\underline{f}_{n+1}(u) = 2\varphi(u) - \underline{f}_n^{-1}(u) \ge 2\varphi(u) - \underline{f}_{n-1}^{-1}(u) = \underline{f}_n(u) \ge \varphi(u),$$

which concludes the induction step.

The inequality in *(ii)*, i.e., $\bar{f}_{n+1}(u) \le \bar{f}_n(u)$ for every $n \in \mathbb{N}_0$ and $u \in [\ell, \infty)$, and also that

$$\underline{f}_n(u) \le \bar{f}_n(u) \quad \text{for every} \quad n \in \mathbb{N}_0 \quad \text{and all} \quad u \in [\ell, \infty) \tag{2.3}$$

can be proved similarly by induction.

Hence, let us finally show *(iii)*, i.e., that $\underline{f}_n(u) \le \bar{f}_m(u)$ for all $m, n \in \mathbb{N}$ and $u \in [\ell, \infty)$. Assume by contradiction that there are some $m_c, n_c \in \mathbb{N}$, $m_c \ne n_c$, such that $\underline{f}_{n_c}(u_c) > \bar{f}_{m_c}(u_c)$ for some $u_c > \ell$ (note that for $u = \ell$ there is $\underline{f}_n(\ell) = \bar{f}_m(\ell) = \ell$ for all $m_c, n_c \in \mathbb{N}_0$). If $n_c \ge m_c$ then by *(ii)*

$$\underline{f}_{n_c}(u_c) > \bar{f}_{m_c}(u_c) \ge \bar{f}_{m_c+1}(u_c) \ge \bar{f}_{m_c+2}(u_c) \ge \dots \ge \bar{f}_{n_c}(u_c),$$

a contradiction with (2.3). If otherwise $m_c \ge n_c$ then by *(i)*

$$\bar{f}_{m_c}(u_c) < \underline{f}_{n_c}(u_c) \le \underline{f}_{n_c+1}(u_c) \le \underline{f}_{n_c+2}(u_c) \le \dots \le \underline{f}_{m_c}(u_c),$$

again a contradiction with (2.3).

One can easily check that all the verified inequalities are strict if and only if $u > \ell$. □

As a by-product, Lemma 2.4 guarantees the existence of limit functions for both $(\underline{f}_n)$ and $(\bar{f}_n)$. We show their existence in the next corollary and provide several properties of these limit functions, see Figure 2.1.

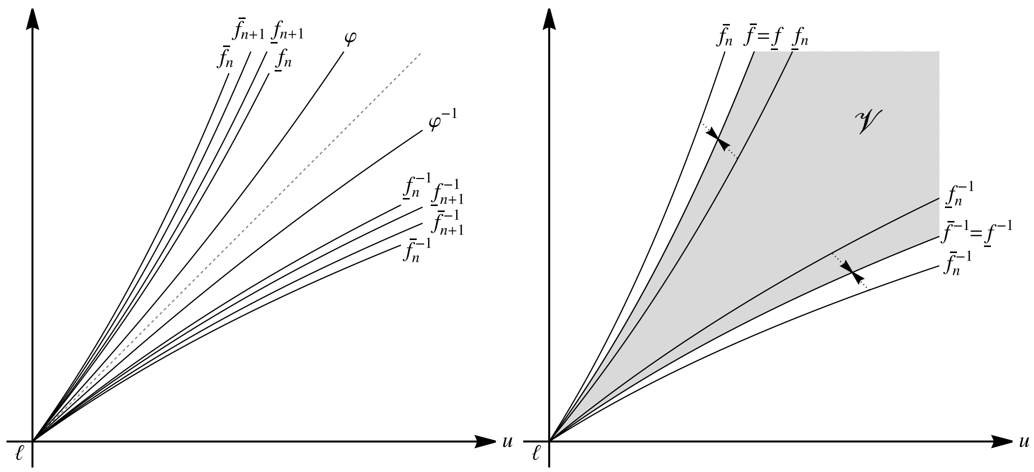

Figure 2.1: Illustration of the mirroring functional iterative scheme (2.1) and functional sequences $(\underline{f}_n)$ and $(\bar{f}_n)$ from Lemma 2.4 (left panel). Convergence of these sequences is implied by Corollaries 2.5 and 2.7 (right panel).

Corollary 2.5. *Let φ be a C^1-function and satisfy (p₁)–(p₂). There exist continuous limit functions*

$$\underline{f}(u) = \lim_{n\to\infty} \underline{f}_n(u) \quad \text{and} \quad \bar{f}(u) = \lim_{n\to\infty} \bar{f}_n(u), \qquad u \in [\ell, \infty),$$

which satisfy:

(i) $\varphi(u) \le \underline{f}_n(u) \le \underline{f}(u) \le \bar{f}(u) \le \bar{f}_n(u)$ *for all* $n \in \mathbb{N}_0$ *and* $u \in [\ell, \infty)$,

(ii) $\underline{f}$ *and* $\bar{f}$ *are strictly increasing, i.e., invertible, on* $[\ell, \infty)$ *and*

$$\underline{f}^{-1}(u) = \lim_{n\to\infty} \underline{f}_n^{-1}(u) \quad \text{and} \quad \bar{f}^{-1}(u) = \lim_{n\to\infty} \bar{f}_n^{-1}(u) \quad \text{for all} \quad u \in [\ell, \infty),$$

(iii) $\underline{f}(u) = 2\varphi(u) - \underline{f}^{-1}(u)$ *and* $\bar{f}(u) = 2\varphi(u) - \bar{f}^{-1}(u)$ *for all* $u \in [\ell, \infty)$.

Proof. The sequence $(\underline{f}_n(u))$, $n \in \mathbb{N}_0$, for given $u \in [\ell, \infty)$ is increasing and bounded from above by all $\bar{f}_n(u)$, $n \in \mathbb{N}_0$, (see Lemma 2.4) which guarantees the existence of pointwise limit $\underline{f}(u)$. The existence of $\bar{f}(u)$, $u \in [\ell, \infty)$, follows similarly.

Lemma 2.3 yields that the iterates $\underline{f}_n$ and $\bar{f}_n$ (and also their inverses $\underline{f}_n^{-1}$ and $\bar{f}_n^{-1}$), $n \in \mathbb{N}_0$, are strictly increasing C^1-functions and for all $n \in \mathbb{N}_0$ there is

$$1 < \underline{f}'_{n+1}(u) = 2\varphi'(u) - (\underline{f}_n^{-1})'(u) \le 2\varphi'(u) \quad \text{(analogically for } \bar{f}'_{n+1}(u)),$$

by (2.1). Since φ is a C^1-function, the functions $\underline{f}_n$ and $\bar{f}_n$ have uniformly bounded derivatives on every compact subinterval of $[\ell, \infty)$. This implies that they converge to their pointwise limits $\underline{f}$ and $\bar{f}$ uniformly on such intervals and thus, $\underline{f}$ and $\bar{f}$ are continuous on whole $[\ell, \infty)$.

The limits have to satisfy that $\varphi(u) \le \underline{f}_n(u) \le \underline{f}(u) \le \bar{f}(u) \le \bar{f}_n(u)$ for all $n \in \mathbb{N}_0$ and $u \in [\ell, \infty)$ by Lemma 2.4 again which proves *(i)*.

The limit functions $\underline{f}$ and $\bar{f}$ are strictly increasing and therefore invertible on $[\ell, \infty)$. Indeed, for every $u_1, u_2 \in [\ell, \infty)$ such that, e.g., $u_1 < u_2$ the mean value theorem and Lemma 2.3 implies

$$\underline{f}_n(u_2) - \underline{f}_n(u_1) = \underline{f}'_n(\xi)(u_2 - u_1) > u_2 - u_1.$$

Passing $n \to \infty$ we obtain

$$\underline{f}(u_2) - \underline{f}(u_1) \ge u_2 - u_1 > 0,$$

which implies that $\underline{f}$ (and analogously $\bar{f}$) is strictly increasing. By the reflection with respect to the axis of the first quadrant we obtain that the inverse functions $\underline{f}^{-1}$ and $\bar{f}^{-1}$ satisfy $\underline{f}^{-1}(u) = \lim_{n\to\infty} \underline{f}_n^{-1}(u)$ and $\bar{f}^{-1}(u) = \lim_{n\to\infty} \bar{f}_n^{-1}(u)$, $u \in [\ell, \infty)$, which proves *(ii)*.

Finally, both iterates $(\underline{f}_n)$ and $(\bar{f}_n)$, $n \in \mathbb{N}_0$, are consistent with the iterative scheme (2.1), specifically,

$$f_{n+1}(u) = 2\varphi(u) - f_n^{-1}(u), \quad u \in [\ell, \infty).$$

Taking $n \to \infty$ in this equality together with *(i)* and *(iii)* we obtain that both limit functions $\underline{f}$ and $\bar{f}$ satisfy

$$f(u) = 2\varphi(u) - f^{-1}(u), \quad u \in [\ell, \infty),$$

which concludes the proof of *(iii)*. □

In the rest of this section we show that $\underline{f} = \bar{f}$ on $[\ell, \infty)$, i.e., both sequences $(\underline{f}_n)$ and $(\bar{f}_n)$ converge to a common limit function (specifically, $(\underline{f}_n)$ from below and $(\bar{f}_n)$ from above). We build our argument on the following technical lemma.

Lemma 2.6. *Let φ be a C^1-function and satisfy (p_1)–(p_2). Then*

$$\bar{f}\left(\underline{f}^{-1}\left(\bar{f}(u)\right)\right) - \bar{f}(u) \ge \underline{f}^{-1}(\bar{f}(u)) - u \quad \textit{for all} \quad u \in [\ell, \infty).$$

Proof. Let $u \in [\ell, \infty)$ be arbitrary but fixed. Lemma 2.3 guarantees that $\bar{f}'_n(s) > 1$ for $s \in \left[u, \underline{f}^{-1}(\bar{f}(u))\right]$. Therefore, the mean value theorem yields that for some $\xi \in \left(u, \underline{f}^{-1}(\bar{f}(u))\right)$ there is

$$\bar{f}_n\left(\underline{f}^{-1}\left(\bar{f}(u)\right)\right) - \bar{f}_n(u) = \bar{f}'_n(\xi) \cdot (\underline{f}^{-1}(\bar{f}(u)) - u) > \underline{f}^{-1}(\bar{f}(u)) - u.$$

Taking $n \to \infty$ we obtain the statement of the lemma. □

Now we are able to show that the limit functions $\underline{f}$ and $\bar{f}$ of iterates $(\underline{f}_n)$ and $(\bar{f}_n)$ are the same, see again Figure 2.1.

Corollary 2.7. *Let φ be a C^1-function and satisfy (p₁)–(p₂). Then $\underline{f}(u) = \bar{f}(u)$ for all $u \in [\ell, \infty)$.*

Proof. Assume by contradiction that there exists $u_1 > \ell$ such that $\underline{f}(u_1) < \bar{f}(u_1)$, i.e., $\bar{f}(u_1) - \underline{f}(u_1) > 0$ (note that for $u = \ell$ there is $\underline{f}(\ell) = \bar{f}(\ell) = \ell$). Corollary 2.5 *(iii)* yields that

$$\begin{aligned}\bar{f}(u_1) - \underline{f}(u_1) &= (2\varphi(u_1) - \bar{f}^{-1}(u_1)) - (2\varphi(u_1) - \underline{f}^{-1}(u_1)) \\ &= \underline{f}^{-1}(u_1) - \bar{f}^{-1}(u_1).\end{aligned} \tag{2.4}$$

Let $u_1 = \bar{f}(\bar{u})$ for some $\bar{u} > \ell$ (recall that the limit functions are homeomorphisms of $[\ell, \infty)$, see Corollary 2.5 *(i)–(ii)*) and denote $u_2 = \underline{f}^{-1}(u_1) = \underline{f}^{-1}(\bar{f}(\bar{u}))$. Then

$$\begin{aligned}\bar{f}(u_2) - \underline{f}(u_2) &= \bar{f}(\underline{f}^{-1}(\bar{f}(\bar{u}))) - \underline{f}(\underline{f}^{-1}(\bar{f}(\bar{u}))) \\ &= \bar{f}(\underline{f}^{-1}(\bar{f}(\bar{u}))) - \bar{f}(\bar{u}) \\ &\geq \underline{f}^{-1}(\bar{f}(\bar{u})) - \bar{u} \\ &= \underline{f}^{-1}(\bar{f}(\bar{u})) - \bar{f}^{-1}(\bar{f}(\bar{u})) \\ &= \underline{f}^{-1}(u_1) - \bar{f}^{-1}(u_1) \\ &= \bar{f}(u_1) - \underline{f}(u_1),\end{aligned}$$

in which the inequality follows from Lemma 2.6 and the last equality from (2.4). Thus, there exists $u_2 = \underline{f}^{-1}(u_1) < u_1$ such that $\bar{f}(u_2) - \underline{f}(u_2) \geq \bar{f}(u_1) - \underline{f}(u_1) > 0$. By induction we construct a sequence (u_n), $n \in \mathbb{N}$, such that

$$\ell < u_{n+1} = \underline{f}^{-1}(u_n) < u_n$$

and

$$\bar{f}(u_{n+1}) - \underline{f}(u_{n+1}) \geq \bar{f}(u_n) - \underline{f}(u_n) > \bar{f}(u_1) - \underline{f}(u_1) > 0$$

for all $n \in \mathbb{N}$. Since (u_n) is decreasing, bounded, and satisfies $u_{n+1} = \underline{f}^{-1}(u_n)$, it has to converge to the unique fixed point of $\underline{f}^{-1}$, i.e., $u_n \to \ell$ for $n \to \infty$. The continuity of limit functions $\underline{f}$ and $\bar{f}$ (see Corollary 2.5) then yields

$$0 < \bar{f}(u_1) - \underline{f}(u_1) \leq \bar{f}(u_n) - \underline{f}(u_n) \to \bar{f}(\ell) - \underline{f}(\ell) = \ell - \ell = 0$$

for $n \to \infty$, a contradiction which concludes the proof. □

3 Proofs of main theorems

In this section we go back to the problem (1.3) for stationary solutions of (1.1) and prove our main result Theorem 1.1 with the help of the mirroring technique and related functional iterative scheme (2.1) for the specific mirroring function (1.11).

Firstly, let us note that using the mirroring scheme (1.12), the authors proved in [7] the following result on twosided unbounded stationary solutions of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.4). Specifically, we showed that these solutions are uniquely characterised and indexed by points of two-dimensional set

$$\mathscr{U} = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \ \alpha > \ell \ \text{ and } \ \varphi^{-1}(\alpha) \leq \beta \leq \varphi(\alpha)\right\}, \tag{3.1}$$

in contrast to Theorem 1.1 on onesided unbounded stationary solutions in which the characterizing set is one-dimensional and even bounded. Note that while $\mathscr{U}$ consists of all unique characteristic pairs generating twosided unbounded stationary solutions, the set $\mathscr{V}$ consists of all pairs generated by initial conditions from $\mathscr{U}$, see Figure 3.1. Theorem 1.1 implies the same relationship between the curve

$$\mathscr{C} = \{(\alpha, f(\alpha)) \in \mathbb{R}^2 : \xi \le \alpha < f(\xi)\}$$

and the boundaries $\partial^I \mathscr{V}$, $\partial^D \mathscr{V}$, see the left panel of Figure 3.1.

Theorem 3.1 ([7, Theorem 5]). *Let (g₁)–(g₂) be satisfied. Then every point $(\alpha, \beta) \in \mathscr{U}$ determines an equivalence class $[u^{\alpha,\beta}]$ of stationary solutions u of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.4) represented by a solution $u^{\alpha,\beta}$ such that $u_0^{\alpha,\beta} = \alpha$, $u_1^{\alpha,\beta} = \beta$, and:*

(i) if $\varphi^{-1}(\alpha) < \beta < \varphi(\alpha)$, then $[u^{\alpha,\beta}] \neq [u^{\tilde{\alpha},\tilde{\beta}}]$ for every $(\tilde{\alpha}, \tilde{\beta}) \neq (\alpha, \beta)$, $(\tilde{\alpha}, \tilde{\beta}) \in \mathscr{U}$,

(ii) if either $\varphi^{-1}(\alpha) = \beta$, or $\beta = \varphi(\alpha)$, then $[u^{\alpha,\beta}] = [u^{\beta,\alpha}]$.

Moreover, every stationary solution u of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.4) is an element of an equivalence class $[u^{\alpha,\beta}]$ determined by a point $(\alpha, \beta) \in \mathscr{U}$.

In order to establish relationship of twosided unbounded solutions characterized by Theorem 3.1 with iterative schemes from Section 2, we focus on initial conditions outside the set $\mathscr{U}$ given by (3.1). First, let us characterize solutions which eventually also have values $u_i \notin [\ell, \infty)$.

Lemma 3.2. *Let (g_1)–(g_2) be satisfied and u be a stationary solution of (1.1). If there exists $i_0 \in \mathbb{Z}$ such that*

$$u_{i_0+1} \ge \bar{f}_{n_0}(u_{i_0}) \quad \text{or} \quad u_{i_0} \ge \bar{f}_{n_0}(u_{i_0+1}) \quad \text{for some} \quad n_0 \in \mathbb{N}_0,$$

then there has to exists $j_0 \in \mathbb{Z}$ such that $u_{j_0} \le \ell$.

Proof. Let $u_{i_0+1} \ge \bar{f}_{n_0}(u_{i_0})$ for some $i_0 \in \mathbb{Z}$ and $n_0 \in \mathbb{N}_0$ (the other case $u_{i_0} \ge \bar{f}_{n_0}(u_{i_0+1})$ is similar). Then we obtain by (2.1) that $u_{i_0+1} \ge \bar{f}_{n_0}(u_{i_0}) = 2\varphi(u_{i_0}) - \underline{f}_{n_0-1}^{-1}(u_{i_0})$ and therefore, by (1.3) that

$$u_{i_0-1} = 2u_{i_0} - u_{i_0+1} - \tfrac{1}{d} g(u_{i_0}) = 2\varphi(u_{i_0}) - u_{i_0+1} \le \underline{f}_{n_0-1}^{-1}(u_{i_0}).$$

Applying the increasing function $\underline{f}_{n_0-1}$ to this inequality (note that Lemma 2.1 verifies that φ defined by (1.11) satisfies the needed hypotheses (p_1)–(p_2) from Section 2, i.e., Lemma 2.3 *(iii)* holds), we get

$$u_{i_0} \ge \underline{f}_{n_0-1}(u_{i_0-1}).$$

Repeating this procedure n_0-times and applying Definition 2.2 we obtain that

$$u_{i_0-n_0+1} \ge \underline{f}_0(u_{i_0-n_0}) = 2\varphi(u_{i_0-n_0}) - \ell.$$

Then (1.3) yields that

$$u_{i_0-n_0-1} = 2u_{i_0-n_0} - u_{i_0-n_0+1} - \tfrac{1}{d} g(u_{i_0-n_0}) = 2\varphi(u_{i_0-n_0}) - u_{i_0-n_0+1} \le \ell,$$

i.e., we obtain the statement for $j_0 = i_0 - n_0 - 1$. □

Our next auxiliary lemma characterize initial conditions which generate $u_i > \ell$ and do not lead to onesided unbounded solutions but to twosided ones from Theorem 3.1 and are thus part of solutions characterized by a pair $(\alpha, \beta) \in \mathscr{U}$.

Lemma 3.3. *Let (g_1)–(g_2) be satisfied and u be a stationary solution of (1.1). If there exists $i_0 \in \mathbb{Z}$ such that*

$$u_{i_0} \le u_{i_0+1} \le \underline{f}_{n_0}(u_{i_0}) \quad \text{or} \quad u_{i_0+1} \le u_{i_0} \le \underline{f}_{n_0}(u_{i_0+1}) \quad \text{for some} \quad n_0 \in \mathbb{N}_0,$$

then there has to exists $j_0 \in \mathbb{Z}$ such that $(u_{j_0}, u_{j_0+1}) \in \mathscr{U}$ or $(u_{j_0+1}, u_{j_0}) \in \mathscr{U}$, respectively, and thus, (1.4) holds.

Proof. If $u_{i_0} \leq u_{i_0+1} \leq \varphi(u_{i_0})$, then $(u_{i_0}, u_{i_0+1}) \in \mathscr{U}$ by the definition (3.1) of $\mathscr{U}$. Hence, (1.4) holds by Theorem 3.1. Otherwise, if $\varphi(u_{i_0}) < u_{i_0+1} \leq \underline{f}_{n_0}(u_{i_0})$, then one can proceed analogically as in the proof of Lemma 3.2 to verify the statement.

For $u_{i_0} \leq \underline{f}_{n_0}(u_{i_0+1})$ it is again similar. □

In other words, Lemma 3.2 characterizes initial conditions outside $\overline{\mathscr{V}}$ and Lemma 3.3 inside $\mathscr{V}$. We have thus collected all tools to prove the first main result of the manuscript, Theorem 1.1.

Proof of Theorem 1.1. Let us note that the function φ defined by (1.11) is of class C^1 and satisfies *(p₁)–(p₂)*, since g satisfies *(g₁)–(g₂)* (see Lemma 2.1). Thus, all results from Section 2 hold. Let $f(u) = \underline{f}(u) = \bar{f}(u)$, $u \in [\ell, \infty)$, be the limit function of iterative scheme (2.1) and $\xi > \ell$. Let us prove the existence of equivalence class $\left[u^{\alpha,I}\right]$ of strictly increasing solutions satisfying (1.7). Firstly, put $u_0^{\alpha,I} = \alpha \in [\xi, f(\xi))$ and for $i \neq 0$ define

$$u_{i+1}^{\alpha,I} = f(u_i^{\alpha,I}), \quad \text{or equivalently,} \quad u_{i-1}^{\alpha,I} = f^{-1}(u_i^{\alpha,I}), \tag{3.2}$$

since f is invertible by Corollary 2.5 *(ii)*. Then, $(u_i^{\alpha,I})$ is defined for all $i \in \mathbb{Z}$ and is strictly increasing, because $f(u) > u$, resp. $f^{-1}(u) < u$, for all $u > \ell$. Since $u = \ell$ is the only fixed point of the mapping f (and of f^{-1} as well) on $[\ell, \infty)$ and again $f(u) > u$, resp. $\ell < f^{-1}(u) < u$, for all $u \in [\ell, \infty)$, then

$$u_i^{\alpha,I} > \ell \quad \text{for all} \quad i \in \mathbb{Z}, \quad \lim_{i\to-\infty} u_i^{\alpha,I} = \ell, \quad \text{and} \quad \lim_{i\to\infty} u_i^{\alpha,I} = \infty.$$

Let us verify that such a sequence $(u_i^{\alpha,I})$, $i \in \mathbb{Z}$, complies with (1.3) and thus forms a stationary solution of (1.1). One can compute for arbitrary $i \in \mathbb{Z}$

$$\begin{aligned} d(u_{i-1}^{\alpha,I} - 2u_i^{\alpha,I} + u_{i+1}^{\alpha,I}) + g(u_i^{\alpha,I}) &= d\left(u_{i+1}^{\alpha,I} + u_{i-1}^{\alpha,I} - 2\left(u_i^{\alpha,I} - \tfrac{1}{2d}g(u_i^{\alpha,I})\right)\right) \\ &= d\left(f(u_i^{\alpha,I}) + f^{-1}(u_i^{\alpha,I}) - 2\varphi(u_i^{\alpha,I})\right) \\ &= 0, \end{aligned}$$

which is verified by Corollary 2.5 *(iii)*.

Let $\tilde{\alpha} \in [\xi, f(\xi))$ be such that $\tilde{\alpha} < \alpha$ (for $\tilde{\alpha} > \alpha$ it is similar) and assume that $\left[u^{\alpha,I}\right] = \left[u^{\tilde{\alpha},I}\right]$, i.e., there exists $s_0 \in \mathbb{N}$ such that $u_{s_0}^{\tilde{\alpha},I} = u_0^{\alpha,I} = \alpha$. Then Corollary 2.5 *(ii)* implies

$$f(\xi) \leq f(\tilde{\alpha}) \leq f^{s_0}(\tilde{\alpha}) = u_{s_0}^{\tilde{\alpha},I} = u_0^{\alpha,I} = \alpha$$

(the symbol f^{s_0} denotes s_0-multiple composition of f), which is a contradiction.

On the contrary, let u be a stationary solution of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$ and (1.5). This implies that (u_i), $i \in \mathbb{Z}$, is strictly increasing, since otherwise there exists $i \in \mathbb{Z}$ such that $u_{i-1} < u_i$ and $u_{i+1} \leq u_i$, i.e., $u_{i-1} - 2u_i + u_{i+1} < 0$. Therefore Eq. (1.3) implies $g(u_i) > 0$, a contradiction. Hence, there is $u_i < u_{i+1}$ for all $i \in \mathbb{Z}$. If there exists $i_0 \in \mathbb{Z}$ such that $u_{i_0+1} = f(u_{i_0})$, then the uniqueness of solution of (1.3) with given initial conditions $u_{i_0} = \alpha$ and $u_{i_0+1} = f(\alpha)$ yields that the solution lies in the equivalence class $\left[u^{\alpha,I}\right]$. If $\alpha \in [\xi, f(\xi))$, we are done. Thus, let us assume that $\alpha < \xi$ (for $\alpha \geq f(\xi)$ it is similar). Then there exists $s_0 \in \mathbb{N}$ such that $f^{s_0}(\alpha) \in [\xi, f(\xi))$ and thus, $\left[u^{\alpha,I}\right] = \left[u^{\gamma,I}\right]$ for $\gamma = f^{s_0}(\alpha)$. Indeed, since $f^s(\alpha) \to \infty$ for $s \to \infty$ there has to exist $s_0 \in \mathbb{Z}$ such that $f^{s_0-1}(\alpha) < \xi$ and $f^{s_0}(\alpha) \geq \xi$. Since f is strictly increasing on $[\ell, \infty)$ by Corollary 2.5 *(iii)*, there is

$$f^{s_0}(\alpha) = f(f^{s_0-1}(\alpha)) < f(\xi), \quad \text{i.e.,} \quad f^{s_0}(\alpha) \in [\xi, f(\xi))\,.$$

Finally, we show by Lemma 3.2 and Lemma 3.3 that other cases cannot occur. Indeed, if $u_{i+1} \neq f(u_i)$ for every $i \in \mathbb{Z}$, there is either $u_{i+1} > f(u_i)$, or $u_i < u_{i+1} < f(u_i)$ for

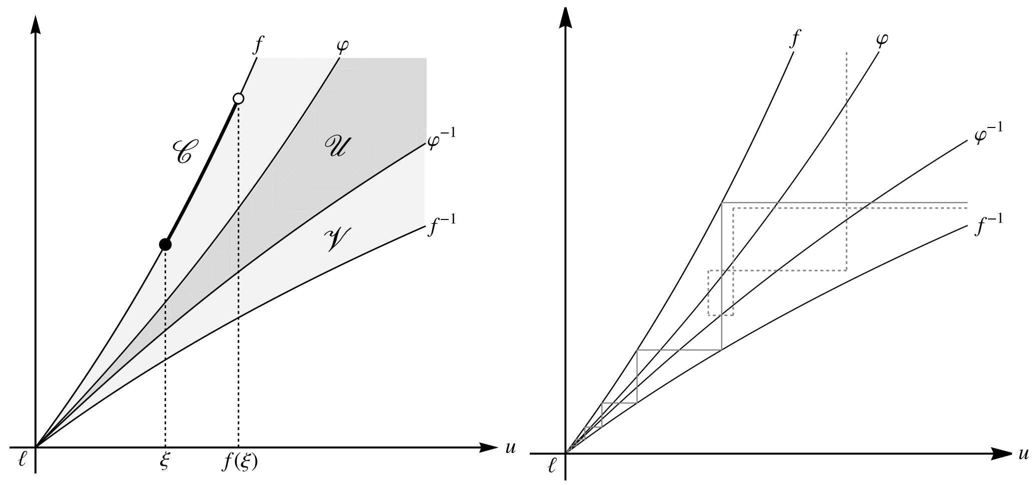

Figure 3.1: The left panel refines Figure 1.2 by including the set $\mathscr{U}$ from (3.1) which characterizes all twosided solutions (Theorem 3.1). The right panel then shows the mirroring scheme (1.12) for onesided (full line) and twosided unbounded solutions (dashed line).

some $i \in \mathbb{Z}$. If $u_{i+1} > f(u_i)$, then Corollary 2.5 *(i)* implies that there exists $n_0 \in \mathbb{N}_0$ such that $u_{i+1} > \bar{f}_{n_0}(u_i)$. Then Lemma 3.2 yields that there has to exists $j_0 \in \mathbb{Z}$ such that $u_{j_0} \leq \ell$, a contradiction. If $u_i < u_{i+1} < f(u_i)$, then Corollary 2.5 *(i)* implies that there exists $n_0 \in \mathbb{N}_0$ such that $u_i < u_{i+1} < \underline{f}_{n_0}(u_i)$. Now Lemma 3.3 implies that (1.4) holds, which is a contradiction with (1.5).

The statement of Theorem 1.1 for equivalence classes of decreasing solutions $\left[u^{\alpha,D}\right]$ can be proved similarly. □

To conclude, we prove the second main result – Theorem 1.2 which characterizes all unbounded stationary solutions of (1.1) satisfying $u_i > \ell$ for all $i \in \mathbb{Z}$.

Proof of Theorem 1.2. Let u be an unbounded stationary solution of (1.1) which satisfies $u_i > \ell$ for $i \in \mathbb{Z}$. Then $(u_i, u_{i+1}) \in \mathscr{V} \cup \partial^I\mathscr{V} \cup \partial^D\mathscr{V}$ for all $i \in \mathbb{Z}$. Indeed, assuming by contradiction that $u_{i_0+1} > f(u_{i_0})$ for some $i_0 \in \mathbb{N}_0$ (and similarly if $u_{i_0+1} < f^{-1}(u_{i_0})$) there has to exists an index $n_0 \in \mathbb{N}_0$ such that $u_{i_0+1} > \bar{f}_{n_0}(u_{i_0})$ ($\bar{f}_n \to f$). Then Lemma 3.2 yields that $u_{j_0} \leq \ell$ for some $j_0 \in \mathbb{N}_0$, a contradiction.

Let us prove *(i)* and assume that $(u_i, u_{i+1}) \in \mathscr{V}$ for all $i \in \mathbb{Z}$. Let $i_0 \in \mathbb{Z}$ be arbitrary, denote $\alpha = u_{i_0}$, and assume $u_{i_0} \leq u_{i_0+1}$ (for $u_{i_0} \geq u_{i_0+1}$ similarly). Since $(u_{i_0}, u_{i_0+1}) \in \mathscr{V}$, there is $u_{i_0} \leq u_{i_0+1} < f(u_{i_0})$ and thus $u_{i_0} \leq u_{i_0+1} \leq \underline{f}_{n_0}(u_{i_0})$ for an index $n_0 \in \mathbb{N}_0$. Consequently, Lemma 3.3 yields that (1.4) holds.

On the contrary, assuming (1.4) there has to be $(u_i, u_{i+1}) \in \mathscr{V}$ for all $i \in \mathbb{Z}$. Indeed, if $(u_{i_0}, u_{i_0+1}) \in \partial^I\mathscr{V}$ for some $i_0 \in \mathbb{N}_0$ (and similarly if $(u_{i_0}, u_{i_0+1}) \in \partial^D\mathscr{V}$), i.e., $u_{i_0+1} = f(u_{i_0})$ by definition of $\partial^I\mathscr{V}$, then Theorem 1.1 implies that $u \in \left[u^{\alpha,I}\right]$ with $\alpha = u_{i_0}$ and $\lim_{i\to-\infty} u_i = \ell$, a contradiction with (1.4).

Let us prove *(ii)* and suppose $(u_i, u_{i+1}) \in \partial^I\mathscr{V}$ for all $i \in \mathbb{Z}$. For arbitrary $i_0 \in \mathbb{Z}$ there is $u_{i_0+1} = f(u_{i_0})$ and Theorem 1.1 yields that $u \in \left[u^{\alpha,I}\right]$ with $\alpha = u_{i_0}$, i.e., (1.5) holds.

Assuming (1.5) there has to be $(u_i, u_{i+1}) \in \partial^I\mathscr{V}$ for all $i \in \mathbb{Z}$. Indeed, if $(u_{i_0}, u_{i_0+1}) \in \mathscr{V}$ for some $i_0 \in \mathbb{N}_0$ then $\lim_{i\to\pm\infty} u_i = \infty$ similarly as above which is a contradiction with (1.5).

In the same way, if $(u_{i_0}, u_{i_0+1}) \in \partial^D \mathscr{V}$, then $u_{i_0+1} = f^{-1}(u_{i_0})$ and Theorem 1.1 yields that $u \in [u^{\alpha,D}]$ and $\lim_{i\to+\infty} u_i = \ell$, again a contradiction with (1.5).

The third item *(iii)* can be proved similarly as *(ii)*. □

4 Examples of specific reaction-diffusion LDEs

In this section we illustrate Theorems 1.1 and 1.2 for specific reaction functions g in (1.1) satisfying the key assumptions *(g₁)*–*(g₂)*. Let us start with two most common reactions.

Example 4.1 (Fisher and Nagumo equation)**.** Considering the logistic (monostable) or cubic (bistable) reaction functions

$$g(u) = u(1-u) \qquad \text{or} \qquad g(u) = u(1-u)(u-a), \quad a \in (0,1), \tag{4.1}$$

the LDE (1.1) becomes the well-known Fisher or Nagumo lattice equation, respectively, [2, 15]. Both reaction functions in (4.1) satisfy (g_1)–(g_2) with $\ell = 1$. We can therefore apply Theorem 1.1 and 1.2 to characterize onesided and twosided unbounded stationary solutions u of the corresponding LDE (1.1) with $u_i > 1$ for all $i \in \mathbb{Z}$ via the function f and the set $\mathscr{V}$ (which are qualitatively the same in both cases, see Figure 3.1).

Our next example contains simple piecewise linear reaction functions for which we can analytically express boundary of $\mathscr{V}$ and explicit formulas for $u^{\alpha,I}$ and $u^{\alpha,D}$.

Example 4.2 (Sawtooth and McKean's caricatures of bistability)**.** For simplicity, let us consider the LDE (1.1) and piecewise linear caricatures of the standard cubic bistable nonlinearity

$$g(u) = \begin{cases} -u & \text{for } u \in \left[0, \frac{a}{2}\right), \\ u-a & \text{for } u \in \left[\frac{a}{2}, \frac{1+a}{2}\right], \\ 1-u & \text{for } u \in \left(\frac{1+a}{2}, \infty\right), \end{cases}$$

or

$$g(u) \begin{cases} = -u & \text{for } u \in [0,a), \\ \in [-a, 1-a] & \text{for } u = a, \\ = 1-u & \text{for } u \in (a,\infty), \end{cases} \qquad a \in (0,1),$$

(proposed by [17] and nicknamed as sawtooth and McKean's caricature, respectively, [5, 14, 20]). The functions are smooth on (a,∞) and the assumptions (g_1)–(g_2) are satisfied with $\ell = 1$ in both cases and the mirroring function φ is for $u \in [1,\infty)$ defined by

$$\varphi(u) = \tfrac{2d+1}{2d}(u-1) + 1$$

(i.e., $\varphi^{-1}(u) = \frac{2d}{2d+1}(u-1)+1$). Thus, all iterates $\underline{f}_n$ or $\bar{f}_n$, $n \in \mathbb{N}_0$, are linear functions which yields that the limit function f is linear as well, i.e., $f(u) = k(u-1)+1$ and $f^{-1}(u) = \frac{1}{k}(u-1)+1$ for some $k > 1$. Then Corollary 2.5 *(iii)* implies

$$k = k(d) = \frac{2d+1+\sqrt{4d+1}}{2d}.$$

The corresponding sets $\mathscr{U}$ and $\mathscr{V}$ are therefore cones in this case (see Figure 4.1) and

$$\lim_{d\to\infty} k(d) = 1 \qquad \text{and} \qquad \lim_{d\to 0+} k(d) = \infty.$$

Then, we obtain from (3.2) explicit formulas for the representatives $u^{\alpha,I}$ and $u^{\alpha,D}$ of equivalence

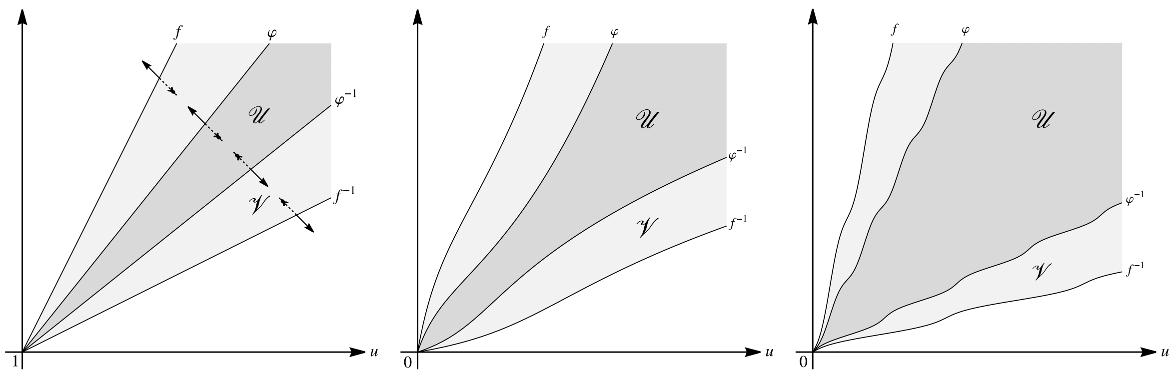

Figure 4.1: The left panel shows linear functions φ and f induced by piecewise linear bistable caricatures from Ex. 4.2. The arrows indicate widening (full arrows) and shrinking (dashed arrows) of conical sets $\mathscr{U}$ and $\mathscr{V}$ as d decreases or increases, respectively. The center and right panels then provide examples of nonconvex functions φ and f. We depict numerically obtained sets $\mathscr{U}$ and $\mathscr{V}$ for the Holling functional response of type II (4.2) with $a = 0.6$ and $b = 0.2$ from Ex. 4.3 (center panel) and the wavy reaction (4.4) with $a = 1.1$ from Ex. 4.4 (right panel).

classes $[u^{\alpha,I}]$, $[u^{\alpha,D}]$ of onesided unbounded stationary solutions, respectively, for $\alpha > 1$ as

$$u_i^{\alpha,I} = (\alpha - 1)k^i + 1 \qquad \text{and} \qquad u_i^{\alpha,D} = (\alpha - 1)k^{-i} + 1.$$

In contrast, our next example considers a reaction leading to more complicated sets $\mathscr{U}$ and $\mathscr{V}$, which we obtain only numerically. Logistic reaction with a predation term leads to sets $\mathscr{U}$ and $\mathscr{V}$ with nonconvex upper boundaries φ and f.

Example 4.3 (Holling functional response II)**.** Let us modify the Fisher equation and consider the LDE (1.1) with reaction function g consisting of the logistic term (describing the intraspecific competition) and of an external predation term determined by Holling functional response of type II (describing the interspecific competition), specifically,

$$g(u) = u(1-u) - \frac{au}{b+u}, \quad a, b > 0. \tag{4.2}$$

We discuss two distinct situations. The largest root of g is $\ell = \frac{1}{2}(1 - b + \sqrt{b^2 + 2b + 1 - 4a}) > 0$ provided

$$(a,b) \in \mathscr{P} = \left\{(a,b) \in \mathbb{R}_+^2 : \ (a \in (0,1) \wedge b > 2\sqrt{a} - 1) \ \vee \ (a \geq 1 \wedge b > a)\right\}.$$

It is possible to show that in this case

$$g''(u) < 0 \quad \text{for all} \quad u \in [\ell, \infty), \tag{4.3}$$

which implies that (g_1)–(g_2) hold for every such pair $(a,b) \in \mathscr{P}$. Thus, by application of Theorems 1.1 and 1.2 we obtain the function f and the set $\mathscr{V}$ describing onesided and twosided unbounded stationary solutions u of (1.1) with $u_i > \ell > 0$ for all $i \in \mathbb{Z}$. Moreover, (4.3) yields that $\varphi''(u) > 0$ for all $u \in [\ell, \infty)$ and hence, the function f satisfies $f''(u) > 0$ for all $u \in [\ell, \infty)$ from which we deduce that f and $\mathscr{V}$ have qualitatively same shape as in Ex. 4.1 and Figure 3.1.

For

$$(a,b) \in \mathscr{Z} = \left\{(a,b) \in \mathbb{R}_+^2 : \ (a \in (0,1) \wedge b \leq 2\sqrt{a} - 1) \ \vee \ (a \geq 1 \wedge b \leq a)\right\}$$

the largest root of g is $\ell = 0$. In this case the situation is more intricate and there are values of

$(a, b) \in \mathscr{X}$ for which (g_2) holds as well as values $(a, b) \in \mathscr{X}$ for which (g_2) is not satisfied. For example, (g_2) does not hold for $a = 0.4$ and $b = 0.2$. However, the assumption (g_2) is valid, e.g., for $a = 0.6$ and $b = 0.2$ and consequently, Theorems 1.1 and 1.2 provide the function f and the set $\mathscr{V}$ characterizing onesided and twosided unbounded stationary solutions u of (1.1) with $u_i > 0$ for all $i \in \mathbb{Z}$. Interestingly, the function φ is not convex for $u \in [\ell, \infty)$ in this case which implies that the limit function f has inflection as well, see Figure 4.1.

We conclude with an illustrative example which provides an interesting wavy shape of the corresponding function f and underlying sets $\mathscr{U}$ and $\mathscr{V}$.

Example 4.4. Considering the LDE (1.1) with

$$g(u) = \sin(u) - au, \quad a > 1, \tag{4.4}$$

the assumptions (g_1)–(g_2) are satisfied with $\ell = 0$ for all $a > 1$. The limit function f has infinitely many inflection points in this case. For the corresponding sets $\mathscr{U}$ and $\mathscr{V}$, see Figure 4.1.

5 Arithmetic mean of function with its inverse

The iterative scheme (2.1) is motivated by the mirroring form (1.12) of the equation (1.3). Focusing on Corollary 2.5 *(iii)*, we interestingly reveal a connection between stationary solutions of (1.1), iterative scheme (2.1), and the following functional equation with an unknown function f:

$$\frac{f(u) + f^{-1}(u)}{2} = \varphi(u), \quad u \in [0, \infty), \tag{5.1}$$

in which φ is a given C^1-function on $[0, \infty)$ which satisfies (p_1)–(p_2) with $\ell = 0$. In other words, the unknown function f should be such that the arithmetic mean of f and its inverse f^{-1} gives the prescribed function φ.

Remark 5.1. First of all, we point out that the functional equation (5.1) has in principle infinitely many solution pairs provided at least one exists. Indeed, let f be a solution of (5.1), $u_0 \in (0, \infty)$ be given, and (u_i), $i \in \mathbb{Z}$, be defined iteratively by

$$u_{i+1} = f(u_i) \quad \text{and} \quad u_{i-1} = f^{-1}(u_i).$$

Considering the following function:

$$g(u) = \begin{cases} f(u), & \text{if } u \neq u_i \text{ for all } i \in \mathbb{Z}, \\ f^{-1}(u), & \text{if } u = u_i \text{ for some } i \in \mathbb{Z}, \end{cases} \tag{5.2}$$

one can verify that g is also a solution of (5.1), different from f and f^{-1} (note that $f(u) \neq f^{-1}(u)$ for all $u \in (0, \infty)$ because of (p_1)–(p_2)), although it still uses only values of either f or f^{-1} (it only interchanges them at u_i, $i \in \mathbb{Z}$). This motivates the following definition.

Taking a solution f of (5.1), it has to satisfy for every $u \in [0, \infty)$ that either $f(u) > \varphi(u)$, or $f^{-1}(u) > \varphi(u)$, or $f(u) = f^{-1}(u) = \varphi(u)$. Define the mapping $P : f \mapsto P(f)$, where $P(f) : [0, \infty) \to \mathbb{R}$, as

$$P(f)(u) = \begin{cases} f(u), & \text{if } f(u) \geq \varphi(u), \\ f^{-1}(u), & \text{if } f^{-1}(u) > \varphi(u). \end{cases} \tag{5.3}$$

We immediately see that $P(f)(u) \geq \varphi(u)$ for all $u \in [0, \infty)$.

Consequently, we define an equivalence relation $f \backsim g$ between two solutions f and g of (5.1) saying that $f \backsim g$ provided $P(f) = P(g)$ (e.g., the function g defined by (5.2) is equivalent to the original solution f, also to its inverse f^{-1}, and also to its own inverse g^{-1}).

The following lemma claims that every equivalence class of solutions of (5.1) containing f and using the same values (as f and g above) has a unique representative $P(f)$ which satisfies $P(f)(u) \geq \varphi(u)$ for all $u \in [0, \infty)$.

Lemma 5.2. *Let φ be a C^1-function and satisfy (p₁)–(p₂) with $\ell = 0$. Let f be a solution of (5.1) and $P(f)$ be defined by (5.3). Then $P(f)$ is also a solution of (5.1).*

Proof. The function $P(f)$ is injective and thus invertible on $[0, \infty)$. Indeed, let us assume by contradiction that $P(f)(u_1) = P(f)(u_2)$ for some $u_1 < u_2$. If $f(u_1) \geq \varphi(u_1)$ and $f(u_2) \geq \varphi(u_2)$ (analogically for $f^{-1}(u_1) \geq \varphi(u_1)$ and $f^{-1}(u_2) \geq \varphi(u_2)$), then

$$f(u_1) = P(f)(u_1) = P(f)(u_2) = f(u_2),$$

a contradiction, since f is invertible. If $f(u_1) \geq \varphi(u_1)$ and $f^{-1}(u_2) \geq \varphi(u_2)$ (analogically for $f^{-1}(u_1) \geq \varphi(u_1)$ and $f(u_2) \geq \varphi(u_2)$), then

$$f(u_1) = P(f)(u_1) = P(f)(u_2) = f^{-1}(u_2) = w.$$

Since $w = f(u_1) = P(f)(u_1) \geq \varphi(u_1)$, $w = P(f)(u_2) = f^{-1}(u_2) \geq \varphi(u_2)$, and φ is strictly increasing and $\varphi^{-1}(u) \leq u \leq \varphi(u)$ for all $u \in [0, \infty)$ by *(p₁)–(p₂)*, then

$$f^{-1}(w) < f(w) \leq \varphi^{-1}(w) \leq \varphi(w),$$

a contradiction with (5.1). Thus, $P(f)$ is invertible, $(P(f))^{-1}(u) \leq \varphi^{-1}(u)$ for all $u \in [0, \infty)$, and by definition of $P(f)$ there has to be

$$(P(f))^{-1}(u) = \begin{cases} f^{-1}(u), & \text{if } f(u) \geq \varphi(u), \\ f(u), & \text{if } f^{-1}(u) > \varphi(u). \end{cases}$$

Therefore, $P(f)$ is also a solution of (5.1), since

$$\frac{P(f)(u) + (P(f))^{-1}(u)}{2} = \frac{f(u) + f^{-1}(u)}{2} = \varphi(u)$$

in both cases $f(u) \geq \varphi(u)$, or $f^{-1}(u) > \varphi(u)$ for a $u \in [0, \infty)$. □

Finally, as a byproduct of our previous considerations from Section 2 we obtain the following result which states that the functional equation (5.1) has a unique equivalence class of nonnegative solutions f and characterizes its representative $P(f)$.

Theorem 5.3. *Let φ be a C^1-function and satisfy (p₁)–(p₂) with $\ell = 0$. Then there exists a unique equivalence class of* nonnegative *solutions f of functional equation (5.1) (with respect to the relation $\backsim$) and its representative $P(f)$ is given by*

$$P(f)(u) = \lim_{n\to\infty} \underline{f}_n(u) = \lim_{n\to\infty} \bar{f}_n(u), \quad u \in [0, \infty),$$

in which $\underline{f}_n$ and $\bar{f}_n$ are given by the iterative scheme (2.1) with $\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u)$ and $\bar{f}_0(u) = 2\varphi(u)$, respectively (cf. Definition 2.2). In particular, the representative $P(f)$ is continuous.

Proof. The existence and properties of a common limit function $f(u) = \lim_{n\to\infty} \underline{f}_n(u) = \lim_{n\to\infty} \bar{f}_n(u)$, $u \in [0, \infty)$, of (2.1) follows from Corollary 2.5 *(i)–(ii)*. Moreover, f (and also f^{-1}) solves the functional equation (5.1) by Corollary 2.5 *(iii)*. Since the limit function f satisfies $f(u) \geq \varphi(u)$ for all $u \in [0, \infty)$, then $P(f)(u) = f(u)$ for all $u \in [0, \infty)$ which concludes the proof of existence and representation of the equivalence class.

Let g be another nonnegative solution of (5.1) such that $P(g) \neq P(f)$ and without loss of generality (see Lemma 5.2) assume that $P(g) = g$. Let us show that

$$\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u) \leq g(u) \leq 2\varphi(u) = \bar{f}_0(u) \quad \text{for all} \quad u \in [0, \infty). \tag{5.4}$$

For the former inequality, assume by contradiction that $g(u_c) < 2\varphi(u_c) - \varphi^{-1}(u_c)$ for some $u_c > 0$ (for $u = 0$ there has to be $g(0) = g^{-1}(0) = 0$, since both g and g^{-1} are assumed to be nonnegative and $\varphi(0) = 0$ by *(p_1)*). Then (5.1) yields that

$$2\varphi(u_c) - \varphi^{-1}(u_c) > g(u_c) = 2\varphi(u_c) - g^{-1}(u_c), \quad \text{i.e.,} \quad g^{-1}(u_c) > \varphi^{-1}(u_c).$$

Thus, since φ^{-1} is a strictly increasing function, there has to exist a $v_c > 0$ such that $\varphi(v_c) > g(v_c)$, which is a contradiction because $g(u) = P(g)(u) \geq \varphi(u)$ for all $u \in [0, \infty)$.

For the latter inequality in (5.4), assume again by contradiction that $g(u_c) > 2\varphi(u_c)$ for some $u_c > 0$. Then (5.1) implies that

$$2\varphi(u_c) < g(u_c) = 2\varphi(u_c) - g^{-1}(u_c), \quad \text{i.e.,} \quad g^{-1}(u_c) < 0,$$

which is a contradiction with the nonnegativeness of g^{-1}.

Finally, if two initial functions of the iterative scheme (2.1) are ordered for all $u \in [0, \infty)$, then all iterates of (2.1) are ordered in the same fashion for all $u \in [0, \infty)$, this can be proved similarly as in Lemma 2.4. Therefore, the inequalities (5.4) and the fact that g is the fixed element of (2.1) yield that

$$\underline{f}_n(u) \leq g(u) \leq \bar{f}_n(u) \quad \text{for all} \quad n \in \mathbb{N}_0 \quad \text{and} \quad u \in [0, \infty).$$

The squeeze argument then implies that

$$P(f)(u) \leftarrow \underline{f}_n(u) \leq g(u) \leq \bar{f}_n(u) \rightarrow P(f)(u) \quad \text{for all} \quad u \in [0, \infty) \quad \text{and} \quad n \to \infty,$$

i.e., $g(u) = P(g)(u) = P(f)(u)$ for all $u \in [0, \infty)$, a contradiction. This concludes the proof of the uniqueness of the equivalence class of nonnegative solutions of (5.1). □

Remark 5.4. Note that besides the existence, uniqueness, and several properties of the class of nonnegative solutions of (5.1), Theorem 5.3 presents the procedure how the continuous representative $P(f)$ can be approximated by the iterations (2.1) with $\underline{f}_0(u) = 2\varphi(u) - \varphi^{-1}(u)$ (from below) and $\bar{f}_0(u) = 2\varphi(u)$ (from above).

6 Discussion

In this paper we showed that onesided unbounded stationary solutions of the LDE (1.1) form a one-parametric family of equivalence classes, Theorem 1.1, and bound the region of unbounded twosided solutions in a two-parametric space, Theorem 1.2 and Figures 3.1 and 4.1.

Continuous counterpart

Let us emphasize the behaviour of corresponding solutions of the PDE (1.2). The simple phase plane analysis (e.g., [7, Section 4]) yields that there is only a unique equivalence class of strictly increasing solutions with (1.5) and a unique class of strictly decreasing onesided solutions satisfying (1.6). Moreover, these continuous solutions exist only on a bounded spatial interval and blow up to infinity at its ends.

Topological chaos and unbounded solutions

Let us also highlight the fact that both onesided and twosided lattice stationary solutions characterized by Theorem 1.2 exist for any diffusion parameters. This fact and a simple look at the white regions in Figure 3.1 lead to an intriguing problem. In the case of Nagumo lattice equation (1.1) with $g(u) = u(1-u)(u-a)$, $a \in (0,1)$, stationary solutions which are represented by

$$(u_i, u_{i+1}) \in \mathscr{W} = \left\{ (\xi, \zeta) \in \mathbb{R}^2 : \ \xi, \zeta > \ell \ \text{ and } \ (\xi, \zeta) \notin \overline{\mathscr{V}} \right\},$$

where $\mathscr{V}$ is defined in (1.9), can be very difficult to characterize fully because the iterations lead to the domain of topological chaos. For example, which initial conditions lead to positive stationary solutions? How does the set of such initial conditions depend on the value of $d > 0$ in the LDE (1.1)? Twosided and onesided lattice stationary solutions satisfying (1.4), (1.5), or (1.6) correspond to continuous counterparts in the PDE (1.2) and are more numerous, generally asymmetric in the twosided case (1.4) and do not blow up to infinity in finite spatial interval. Intuitively, solutions with $(u_i, u_{i+1}) \in \mathscr{W}$ may have richer behaviour and, most importantly, could generate qualitatively new types of stationary solutions.

Applications of mirroring

In this paper we generalized the mirroring technique to functional mirroring scheme and connected it to the functional equation (5.1). It is possible that this geometric approach could contribute to one of the many problems related to the topological chaos, e.g., explicit solutions for special reaction functions.

Functional equation

Our final remark is related to the functional equation (5.1). Note that φ in our case is the specific mirroring function defined by (1.11). The functional equation (5.1) represent an interesting problem itself once any function φ is considered. In principle, the solvability of functional equations is nontrivial and depend for example on the domain. Theorem 5.3 provides a specific existence and uniqueness result in the case in which φ satisfies *(p_1)*–*(p_2)*.

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On the preservation of Lyapunov exponents of integrally separated systems of differential equations under small nonlinear perturbations

Dedicated to Prof. Nguyen Huu Du on the occassion of his 70th birthday

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Abstract. This paper addresses the Lyapunov exponents of non-vanishing solutions to quasi-linear time-varying systems of differential equations. The linear part is not required to be regular but it is assumed to be integrally separated, which ensures that the associated Lyapunov exponents are distinct and stable. The nonlinear perturbations are assumed to be small in a certain sense, though less restrictive than the condition in Barreira and Valls' paper, *J. Differential Equations* **258**(2015), 339–361. The main result is a Perron-type theorem for upper and lower Lyapunov exponents, offering an alternative to Barreira and Valls' result. In addition, an analogous result holds for Bohl exponents.

Keywords: quasi-linear system, Lyapunov exponent, integral separation, nonlinear perturbation, Perron-type theorem.

2020 Mathematics Subject Classification: 34A09, 34D05, 34D08, 34E10.

1 Introduction

Asymptotic behaviour of solutions is a classical topic in the qualitative theory of differential equations. It has been discussed in well-known monographs such as [5–7]. In this paper, we study the asymptotic behavior of solutions of linear time-varying ordinary differential equations (ODEs) under nonlinear perturbations

$$x'(t) = A(t)x(t) + f(t, x(t)), \quad t \in \mathbb{I} = [t_0, \infty), \tag{1.1}$$

where the coefficient $A : \mathbb{I} \to \mathbb{C}^{n\times n}$ and the nonlinear term $f : \mathbb{I} \times \mathbb{C}^n \to \mathbb{C}^n$ are continuous. The question is that if the nonlinear term f is supposed to be sufficiently small in some sense, how certain solutions of the quasi-linear ODE (1.1) behave asymptotically comparing to those

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of the unperturbed linear ODE as t tends to infinity. In the case of constant matrix A, the result is known as Perron theorem, which was established long time ago, see [6, Theorem 5, p. 97].

Theorem 1.1. *Consider the equation (1.1) with a constant matrix A such that*

$$\|f(t, x(t))\| \leq \gamma(t)\|x(t)\|, \quad t \geq t_0,$$

where $\gamma(t)$ is a continuous nonnegative function satisfying

$$\int_t^{t+1} \gamma(s)ds \to 0, \quad t \to \infty. \tag{1.2}$$

If $x(t)$ is a bounded solution of (1.1) then either $x(t) = 0$ for all large t or the limit

$$\mu = \lim_{t\to\infty} \frac{1}{t} \ln \|x(t)\|$$

exists and is equal to the real part of one of the eigenvalues of A.

This is the classical version of Perron theorem. It is noted that actually Perron did prove a weaker form. This version is due to Lettenmeyer. Later Hartman and Wintner refined the proof. The number μ is called the (strict) Lyapunov exponent of the solution x [1]. Theorem 1.1 means that the Lyapunov exponent of the solution x exists and it is equal to one of the Lyapunov exponents of the linear system, i.e., no new Lyapunov exponent arises.

Recently, extensions of this result to functional differential equations [18], nonautonomous ODEs [4], and differential-algebraic equations [14] were obtained. Similar results were also obtained for difference equations [3, 17] and functional difference equations [16]. In the case of time-dependent coefficient matrix A, by using the regularity theory Barreira and Valls did prove a similar result but under a more restrictive assumption.

Theorem 1.2 ([4, Theorem 1])**.** *Consider the quasi-linear system (1.1), where $A(t)$ is supposed to be given in the block diagonal form. It is assumed further that the linear subsystems associated with each block have the same and sharp Lyapunov exponents, but the Lyapunov exponents belonging to different blocks are distinct. If $x(t)$ is a solution of (1.1) such that*

$$\|f(t, x(t))\| \leq \gamma(t)\|x(t)\|, \quad t \geq t_0,$$

where $\gamma(t)$ is a continuous nonnegative function satisfying

$$\int_t^{t+1} e^{\delta s}\gamma(s)ds \to 0, \quad t \to \infty, \tag{1.3}$$

for some $\delta > 0$ then either $x(t) = 0$ for all large t or the limit

$$\mu = \lim_{t\to\infty} \frac{1}{t} \ln \|x(t)\|$$

exists and it is equal to one of the Lyapunov exponents of the linear system.

One can see that the condition (1.3) on $\gamma(t)$ in Theorem 1.2 is much stronger than the condition (1.2) in Theorem 1.1. The variation of Lyapunov exponents under linear perturbations, i.e. $f(t, x(t)) = B(t)x(t)$ has been well investigated in the literature, see [1, Chapter 5]. The stability concept plays a key role in the preservation of Lyapunov exponents under small linear perturbations. Necessary and sufficient conditions for the stability of Lyapunov exponents have been discussed in details in [1, Chapter V]. We emphasize that neither the existence of sharp Lyapunov exponents nor the regularity does imply the stability. We note in addition that in the case of constant matrix A, the Lyapunov exponents of the linear systems are stable without any extra assumption. An analogue of the result in [4] was established for nonautonomous difference equations in [3]. Furthermore, an extension to the so-called μ-Lyapunov exponents, see [2], was obtained in [10] where more general growth rates of solutions are characterized. Some further discussions on the stability of Lyapunov exponents and computational consequences are given in [8, 9], where the numerical approximation of Lyapunov exponents is addressed.

In this paper, we present an alternative version of Perron theorem for the nonlinear system (1.1) with a time-varying coefficient A. Under an assumption that guarantees the stability of distinct (upper) Lyapunov exponents, we are able to relax the condition on the nonlinear term, i.e., the assumption on $\gamma(t)$ remains the same as in Theorem 1.1. Furthermore, we do not require the sharp Lyapunov exponents as in [4]. The proof, which differs from that in [4], relies on reducing the linear part to a diagonal system. Analogous statements are also obtained for lower Lyapunov exponents and Bohl exponents. This investigation is of particular interest because the quasi-linear system (1.1) may arise when linearizing a nonlinear system along a particular solution. As a consequence, we can obtain information about the rate at which nearby solutions converge or diverge to/from the particular solution.

The paper is organized as follows. In the next section, we provide a brief overview of the theory on Lyapunov and Bohl exponents, with a focus on the stability of Lyapunov exponents, the property of integral separation, and the relationship between them. In Section 3, we describe the asymptotic behavior of solutions under the assumption of integral separation. As the main result, Perron-type theorems that establish the exponential growth rates of solutions are then presented. In the last section, we discuss several open questions and conjectures.

2 Preliminaries

First, we recall the notion of Lyapunov exponents, which is used to characterize the asymptotic growth of functions. Then, we briefly summarize the results related to the stability of Lyapunov exponents. These results are given in details in [1].

Definition 2.1. For a non-vanishing function $f : [0,\infty) \longrightarrow \mathbb{R}^n$, the quantities $\chi^u(f) = \limsup_{t\to\infty} \frac{1}{t}\ln\|f(t)\|$, $\chi^\ell(f) = \liminf_{t\to\infty} \frac{1}{t}\ln\|f(t)\|$, are called *upper and lower Lyapunov exponents of f*, respectively. If the exact limit exists, i.e., the upper and the lower Lyapunov exponents coincide, then we say f has a sharp Lyapunov exponent.

Consider a linear system

$$x'(t) = A(t)x(t), \quad t \in \mathbb{I} = [0,\infty), \tag{2.1}$$

with a bounded and continuous matrix function A.

Definition 2.2. Given a fundamental solution matrix X of (2.1), we introduce

$$\lambda_i^u = \limsup_{t\to\infty} \frac{1}{t}\ln\|X(t)e_i\|, \qquad \lambda_i^\ell = \liminf_{t\to\infty} \frac{1}{t}\ln\|X(t)e_i\|,$$

where e_i denotes the i-th unit vector and $\|\cdot\|$ denotes the Euclidean norm. The columns of X form a *normal basis* if $\Sigma_{i=1}^n \lambda_i^u$ is minimal. The λ_i^u, $i = 1, 2, \ldots, n$ belonging to a normal basis are called *(upper) Lyapunov exponents* of (2.1).

We assume that the upper Lyapunov exponents are ordered

$$-\infty < \lambda_1^u \le \lambda_2^u \le \cdots \le \lambda_n^u < \infty.$$

The following result is known as Lyapunov's inequality.

Theorem 2.3 ([1, Theorem 2.5.1]). *Let* $\{\lambda_i^u\}_{i=1}^n$ *be the upper Lyapunov exponents of* (2.1)*. Then*

$$\sum_{i=1}^n \lambda_i^u \ge \limsup_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{trace} A(s)ds. \tag{2.2}$$

Here trace A *denotes the trace of the matrix function* A.

We say that the system (2.1) is *regular* if the inequality (2.2) becomes an equality and the exact limit exists, i.e.,

$$\sum_{i}^{n} \lambda_i^u = \liminf_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{trace} A(s)ds.$$

If (2.1) is regular, then for any nontrivial solution x, the sharp Lyapunov exponent exists. Hence, we have $\lambda_i^l = \lambda_i^u$ for $i = 1, \ldots, n$, i.e., the Lyapunov spectrum of (2.1) is a point spectrum. We apply the transformation $x = L(t)y$, where L is a nonsingular and continuously differentiable matrix function for $t \ge t_0$, to system (2.1), we obtain

$$y' = B(t)y, \quad B(t) = L^{-1}A(t)L(t) - L^{-1}(t)L'(t). \tag{2.3}$$

This transformation is called a kinematic similarity transformation.

Definition 2.4. The above transformation is called a Lyapunov transformation if $L(t)$, $L^{-1}(t)$ and $L'(t)$ are bounded for $t \ge t_0$.

If we apply a Lyapunov transformation to system (2.1) and obtain the new system (2.3), then we say (2.1) is reducible to (2.3). Lyapunov transformations form a group and they do not change the Lyapunov exponent.

One of the most important questions is the variation of Lyapunov exponents under small linear and nonlinear perturbations. Consider the perturbed system

$$y' = (A(t) + Q(t))y, \tag{2.4}$$

where Q is a continuous and bounded matrix function. Let the upper Lyapunov exponents of (2.4) be ordered and denoted as follows

$$-\infty < \gamma_1^u \le \gamma_2^u \le \cdots \le \gamma_n^u < \infty.$$

Definition 2.5. The upper Lyapunov exponents of system (2.1) are said to be stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the inequality $\sup_{t\ge t_0} \|Q(t)\| < \delta$ implies

$$|\lambda_i^u - \gamma_i^u| < \varepsilon, \quad i = 1, 2, \ldots, n.$$

It is known that regularity does not ensure the stability of exponents. In order to answer the question of stability, we need the property of integral separation.

Definition 2.6. The real, bounded and continuous functions $a_1(t)$, $a_2(t), \dots, a_n(t)$ are said to be separated on $\mathbb{R}^+$ if there exists a constant $a > 0$ such that

$$a_{k+1}(t) - a_k(t) \geq a, \quad k = 1, 2, \dots, n-1, \quad t \geq 0.$$

They are said to be integrally separated on $\mathbb{R}^+$ if there exist constants $a > 0$ and $d > 0$ such that

$$\int_s^t [a_{k+1}(\tau) - a_k(\tau)] d\tau \geq a(t-s) - d$$

for all $t \geq s \geq 0$, $k = 1, 2, \dots, n-1$.

Obviously, the condition for separateness implies integral separateness, but not vice versa.

Definition 2.7. A linear system is said to be system with integral separateness if it has solutions $x_1(t)$, $x_2(t)$, …, $x_n(t)$ such that the inequality

$$\frac{\|x_{i+1}(t)\|}{\|x_{i+1}(s)\|} : \frac{\|x_i(t)\|}{\|x_i(s)\|} \geq d e^{a(t-s)}, \quad i = 1, 2, \dots, n-1, \tag{2.5}$$

with some constants $a > 0$, $d \geq 1$, is valid for all $t \geq s$.

The definition of integral separateness implies some properties:

- integrally separated systems have different Lyapunov exponents;
- integral separateness is invariant under Lyapunov transformations;
- the solutions $x_1(t)$, $x_2(t)$, …, $x_n(t)$ in Definition 2.7 form a normal basis.

The following important property was proven by Bylov.

Theorem 2.8 ([1, Theorem 5.3.1 and Corollary 5.3.2]). *An integrally separated system is reducible to a real diagonal one by means of a Lyapunov transformation and the diagonal is integrally separated.*

Furthermore, by the use the Steklov function and the H-transformation, see [1, pp. 153-155], we have the following result.

Theorem 2.9 ([1, Theorem 5.4.1]). *A diagonal real system with an integrally separated diagonal is reducible to a diagonal system with a separated diagonal.*

By Millionshchikov's method of rotation, a necessary and sufficient condition for the stability of distinct Lyapunov exponents was obtained.

Theorem 2.10 ([1, Theorem 5.4.8]). *If system (2.1) has n distinct Lyapunov exponents $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, then they are stable if and only if the system is integrally separated, i.e. there exists an integrally separated fundamental solution matrix.*

In addition to Lyapunov exponents, another characteristic of the asymptotic behavior of the solutions of system (2.1) introduced by Bohl [7], has more natural properties.

Definition 2.11. Let x be a nontrivial solution of system (2.1). The *(upper) Bohl exponent* $\kappa_B^u(x)$ of this solution is the greatest lower bound of all those numbers ρ for which there exist numbers N_ρ such that

$$\|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\|, \quad t \geq s \geq 0.$$

If such numbers ρ do not exist, then one sets $\kappa_B^u(x) = +\infty$.

Similarly, the *lower Bohl exponent* $\kappa_B^\ell(x)$ is the least upper bound of all those numbers ρ' for which there exist numbers N'_ρ such that

$$\|x(t)\| \geq N'_\rho e^{\rho'(t-s)} \|x(s)\|, \quad 0 \leq s \leq t.$$

It is easy to verify the estimates

$$\kappa_B^\ell(x) \le \lambda^\ell(x) \le \lambda^u(x) \le \kappa_B^u(x)$$

as well as the formulas

$$\kappa_B^u(x) = \limsup_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}, \qquad \kappa_B^\ell(x) = \liminf_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}.$$

If $A(t)$ is *integrally bounded*, i.e., if

$$\sup_{t\ge 0} \int_t^{t+1} \|A(s)\|\, ds < \infty,$$

then the Bohl exponents are finite.

The relation between Lyapunov and Bohl exponents are as follows.

- Bohl exponents characterize the uniform growth rate of solutions, while Lyapunov exponents simply characterize the growth rate of solutions departing from $t = 0$.
- If the greatest bound of upper Lyapunov exponents for all solutions of (2.1) is negative, then the system is asymptotically stable. If the same holds for the greatest bound of the upper Bohl exponents then the system is uniformly exponentially stable.
- Unlike Lyapunov exponents, Bohl exponents are stable without any extra assumption, see [7, Theorem 4.6].

The following lemma will be used in the proof of the main theorem in Section 3.

Lemma 2.12 ([6, Lemma 1, p. 98])**.** *Let $\beta(v)$ be a continuous function at $v = v^*$ and let $\gamma(t)$ be a continuous nonnegative function such that (1.2) holds. If $v(t)$ is a solution of the differential inequality*

$$v' \ge \beta(v) - \gamma(t)$$

for $t \ge t_0$ and there exists a sequence $\tau_n \to \infty$ such that $v(\tau_n) \to v^*$, then $\beta(v^*) \le 0$. Moreover, the exact limit of $v(\tau)$ as $\tau \to \infty$ exists.

3 Perron-type theorems for integrally separated systems

First, we establish a generalization of the classical Perron theorem for time-varying systems (1.1).

Theorem 3.1. *Consider the quasi-linear system (1.1), where the associated linear system (2.1) is integrally separated and has finite exponents. If $x(t)$ is a solution of (1.1) such that*

$$\|f(t, x(t))\| \le \gamma(t)\|x(t)\|, \quad t \ge t_0,$$

where $\gamma(t)$ is a continuous nonnegative function satisfying condition (1.2), then either $x(t) = 0$ for all large t or

$$\limsup_{t\to\infty} \frac{1}{t} \ln\|x(t)\|$$

is equal to one of the upper Lyapunov exponents of the linear system.

The same statement holds true for the limit inferior and the lower Lyapunov exponents, respectively.

That is, here we assume the stability of distinct Lyapunov exponents instead of the regularity. The proof is quite similar to that in [6] for the case of a constant coefficient. While in the case of a constant A, the transformation to the Jordan canonical form is used, here thanks to the integral separation, the linear system can be transformed into a diagonal system with a separated diagonal.

Proof. Due to Theorem 2.8 and Theorem 2.9, without loss of generality, we assume that A is already of real diagonal form

$$A(t) = \operatorname{diag}\left(\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t)\right),$$

and

$$\lambda_i(t) - \lambda_{i+1}(t) \geq a > 0, \quad \forall t \geq t_0, \; i = 1, 2, \dots, n-1.$$

Then, the i-th equation ($i = 1, 2, \dots, n$) reads

$$x_i' = \lambda_i(t) x_i + f_i(t, x),$$

which implies

$$\frac{d}{dt}|x_i|^2 = 2\lambda_i |x_i|^2 + 2\bar{x}_i f_i.$$

Here, $\bar{x}_i$ and $|x_i|$ denote the complex conjugate and the modulus of x_i, respectively. Here, the argument t of the functions is omitted for brevity. Writing $r_i = |x_i|$, we have

$$\left| \frac{d}{dt} r_i^2 - 2\lambda_i r_i^2 \right| \leq 2 r_i |f_i(t, x)|, \quad i = 1, 2, \dots, n. \tag{3.1}$$

Putting

$$\limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \lambda_i(s) ds = \mu_i,$$

it clearly holds that μ_i, $i = 1, 2, \dots, n$, are the upper Lyapunov exponents of the linear system, and $\mu_i - \mu_{i+1} \geq a$. Let us denote

$$L_k = r_k^2, \qquad M_k = \sum_{i<k} r_i^2, \qquad N_k = \sum_{i \geq k} r_i^2,$$

we have so that $M_k + N_k = \sum_{i=1}^n r_i^2 = \|x\|^2$, where the Euclidean norm is used.

From (3.1), we obtain

$$\left| L_k' - 2\lambda_k L_k \right| \leq 2\gamma(t) L_k^{1/2} (M_k + N_k)^{1/2}, \tag{3.2}$$

$$M_k' \geq 2\lambda_{k-1} M_k - 2\gamma(t) M_k^{1/2} (M_k + N_k)^{1/2}, \tag{3.3}$$

$$N_k' \leq 2\lambda_k N_k + 2\gamma(t) N_k^{1/2} (M_k + N_k)^{1/2}. \tag{3.4}$$

Thus, $N_1 = \|x\|^2$ satisfies

$$2(\lambda_n(t) - \gamma(t)) N_1 \leq N_1' \leq 2(\lambda_1(t) + \gamma(t)) N_1.$$

By integration, we get for $t \geq t_1 \geq t_0$

$$e^{\int_{t_1}^t 2(\lambda_n(t) - \gamma(t)) ds} \|x(t_1)\| \leq \|x(t)\| \leq e^{\int_{t_1}^t 2(\lambda_1(t) + \gamma(t)) ds} \|x(t_1)\|.$$

This shows that if $x(t_1)$ for some $t_1 \geq t_0$, then $x(t) = 0$ for all $t \geq t_1$. Thus, we exclude this case from now on.

Consider the function

$$v(t) = v_k(t) = \frac{M_k}{M_k + N_k},$$

which is well defined for all $t \geq t_0$ and fulfills $0 \leq v \leq 1$. Furthermore, since

$$v' = \frac{M'N - MN'}{(M+N)^2} \quad \text{and} \quad v(1-v) = \frac{MN}{(M+N)^2},$$

from (3.3) and (3.4), we obtain

$$v' \geq bv(1-v) - \sqrt{2}\gamma(t), \tag{3.5}$$

where $b = b_k = 2(\lambda_{k-1} - \lambda_k) \geq 2a > 0$.

By Lemma 2.12, it follows that for each $k = 1, 2, \ldots, n$, the limit of $v_k(t)$ as $t \to \infty$ exists and equal either 0 or 1. Moreover, $v_1(t) \to 0$ as $t \to \infty$, since $M_1 \equiv 0$. Let m be the greatest value of k for which $\lim_{t\to\infty} v_k(t) = 0$. Since $M_k = L_1 + \cdots + L_{k-1}$, it follows that

$$\lim_{t\to\infty} \frac{L_k(t)}{M_k(t) + N_k(t)} = \begin{cases} 0 & \text{for } k \neq m, \\ 1 & \text{for } k = m. \end{cases} \tag{3.6}$$

Therefore, we have

$$\lim_{t\to\infty} \frac{L_k(t)}{L_m(t)} = 0 \quad \text{for } k \neq m \tag{3.7}$$

and by (3.2) with $k = m$

$$\left| \frac{d}{dt}(\ln L_m) - 2\lambda_m \right| \leq 2\gamma(t) \left(\frac{M_m + N_m}{L_m} \right)^{1/2}. \tag{3.8}$$

Note that

$$\lim_{t\to\infty} \frac{M_m(t) + N_m(t)}{L_m(t)} = 1$$

and assumption (1.2) implies $t^{-1} \int_{t_0}^t \gamma(s)ds \to 0$ as $t \to \infty$. Therefore, integrating both sides of (3.8) from t_0 to t and dividing by t, we get

$$\frac{\ln L_m(t)}{t} - \frac{2}{t} \int_{t_0}^t \lambda_m(s)ds = o(1), \quad t \to \infty.$$

This, together with the asymptotic relation

$$\frac{\ln \|x(t)\|^2}{t} = \frac{\ln L_m(t)}{t}(1 + o(1)), \quad t \to \infty,$$

implies that

$$\limsup_{t\to\infty} \frac{\ln \|x(t)\|}{t} = \limsup_{t\to\infty} \frac{\ln L_m(t)}{2t} = \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^t \lambda_m(s)ds = \mu_m,$$

which completes the proof.

If we take the limit inferior instead of the limit superior, the statement for the lower Lyapunov exponent is obtained, too. □

Remark 3.2. The condition on $f(t,x)$ in Theorem 3.1 is certainly satisfied by any solution $x(t)$ of (1.1) which tends to zero as $t \to \infty$ if

$$f(t,x) = o(\|x\|), \quad \text{for } t \to \infty,\ \|x\| \to 0.$$

In the special case of linear perturbation $f(t,x) = B(t)x$, the condition (1.2) holds in particular if $\|B(t)\| \to 0$ as $t \to \infty$ or $B(t) \in L_p[t_0,\infty)$, where $1 \le p < \infty$.

As a consequence, we obtain a theorem on the asymptotic stability of a non-stationary solution of a nonlinear system by using linearization.

Theorem 3.3. *Consider a nonlinear system*

$$y' = g(t,y), \quad t \ge t_0, \tag{3.9}$$

where g is continuous and continuously differentiable with respect to variable y. Suppose that $y^* = y^*(t)$ is a particular solution that exists on $[t_0,\infty)$. Consider the linearized system

$$x' = A(t)x, \quad A(t) = g_y(t, y^*(t)). \tag{3.10}$$

If the system (3.10) is integrally separated and all of its Lyapunov exponents are negative, then y^* is an asymptotically stable solution of (3.9).

We note the fact that all of its Lyapunov exponents are negative implies that the system (3.10) is exponentially stable, but it is not necessarily uniformly exponentially stable.

Analogously, we obtain a Perron theorem for Bohl exponents.

Theorem 3.4. *Consider the quasi-linear system (1.1), where the associated linear system (2.1) is integrally separated and has finite exponents. If $x(t)$ is a solution of (1.1) such that*

$$\|f(t,x(t))\| \le \gamma(t)\|x(t)\|, \quad t \ge t_0,$$

where $\gamma(t)$ is a continuous nonnegative function satisfying condition (1.2), then either $x(t) = 0$ for all large t or

$$\limsup_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}$$

is equal to one of the upper Bohl exponents of the linear system.

The same statement holds true for the limit inferior and the lower Bohl exponents, respectively.

Proof. We proceed as in the the proof of Theorem 3.1 and integrate both sides of (3.8) from s to t and dividing by $t-s$. Then, take the limit superior/inferior as s, $t-s$ tend to ∞. □

4 Discussion

In this paper, we have derived two extended versions of the classical Perron-type theorem. Assuming integral separation, which guarantees the stability of Lyapunov exponents, we have confirmed that the Perron theorem remains valid under the same smallness condition on the nonlinear part as in the constant coefficient case. Therefore, our version of the Perron theorem differs from the one obtained in [4]. Additionally, we have established a Perron theorem for Bohl exponents as well. Extensions of these results to different growth rates and the μ-Lyapunov exponents defined in [2,10] appear to be straightforward.

Several open problems and conjectures remain. First, our results can be extended to difference equations, providing an alternative to the Perron-type theorem presented in [3]. Second, it is known that the Lyapunov exponents of linear systems may be nondistinct but still stable, as characterized in [1, Theorem 5.4.9] and [9]. In such cases, the linear system (2.1) can be reduced to block-diagonal form with upper-triangular blocks subject to additional conditions. We conjecture that Theorem 3.1 still holds, i.e., the stability of Lyapunov exponents implies their preservation under small nonlinear perturbations satisfying (1.3). This would fully generalize the classical Perron theorem to the time-varying system (1.1).

Furthermore, it is well known that Bohl exponents are stable without any additional assumptions. Therefore, we also conjecture that the result of Theorem 3.4 holds without the integral separation assumption (see the related result in [7, Chapter VII, Section 3] and [9]). However, the arguments used in the proof of Theorem 3.1 are insufficient for the last two problems, as reducibility to diagonal form no longer holds. Addressing these questions would require overcoming further technical challenges.

Lastly, recent results on the asymptotic behavior of solutions and Lyapunov exponents have been extended from ODEs to DAEs (see [11–15]). Therefore, extending the Perron-type theorem to linear time-varying DAEs under small nonlinear perturbations would also be of interest.

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Structures and evolution of bifurcation diagrams for a multiparameter p-Laplacian Dirichlet problem

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Abstract. We study the multiparameter p-Laplacian Dirichlet problem

$$\begin{cases} \left(\varphi_p(u'(x))\right)' + \lambda(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}) - \mu \sum_{j=1}^{n} b_j u^{r_j} = 0, \ \ -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $p > 1$, $\varphi_p(y) = |y|^{p-2}y$, $\left(\varphi_p(u')\right)'$ is the one-dimensional p-Laplacian, $\lambda > 0$ and $\mu \geq 0$ are two bifurcation parameters. We assume that $k \geq 0$, $0 < p-1 < q_1 < q_2 < \cdots < q_m < r_1 < r_2 < \cdots < r_n$, $m, n \geq 1$, $a_1 = 1$, $a_i > 0$ for $i = 1, 2, \ldots, m$ and $b_1 = 1$, $b_j > 0$ for $j = 1, 2, \ldots, n$. We mainly prove that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation diagram consists of a strictly decreasing curve for $\mu = 0$, and always consists of a $\subset$-shaped curve for fixed $\mu > 0$. We then study the structures and evolution of the bifurcation diagrams with varying $\mu \geq 0$.

Keywords: bifurcation diagram, evolution, positive solution, p-Laplacian, $\subset$-shaped bifurcation curve, time map.

2020 Mathematics Subject Classification: 34B18, 74G35.

1 Introduction

In this paper we study the structures and evolution of bifurcation diagrams for the multiparameter p-Laplacian Dirichlet problem

$$\begin{cases} \left(\varphi_p(u'(x))\right)' + \lambda(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}) - \mu \sum_{j=1}^{n} b_j u^{r_j} = 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{1.1}$$

where $p > 1$, $\varphi_p(y) = |y|^{p-2}y$, $\left(\varphi_p(u')\right)'$ is the one-dimensional p-Laplacian, and $\lambda > 0$ and $\mu \geq 0$ are two bifurcation parameters. We assume that the nonlinearity

$$f_{k,\mu,\lambda}(u) \equiv \lambda(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}) - \mu \sum_{j=1}^{n} b_j u^{r_j} \tag{1.2}$$

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is a generalized polynomial (see [9]) satisfying

$$\begin{cases} k \geq 0,\ 0 < p-1 < q_1 < q_2 < \cdots < q_m < r_1 < r_2 < \cdots < r_n,\ m,n \geq 1, \\ a_1 = 1,\ a_i > 0 \text{ for } i = 1,2,\ldots,m \text{ and } b_1 = 1,\ b_j > 0 \text{ for } j = 1,2,\ldots,n. \end{cases} \tag{1.3}$$

This problem arises in the study of non-Newtonian fluids, nonlinear diffusion problems, and population dynamics of one species. The quantity p is a characteristic of the medium. Media with $1 < p < 2$ are called pseudoplastics fluids and those with $p > 2$ are called dilatant. If $p = 2$, they are Newtonian fluids (see, e.g., Díaz [3, 4] and their bibliographies). In population dynamics, in (1.1), the one-dimensional p-Laplacian operator $(\varphi_p(u'))'$ acts as the diffusive mechanism describing the migration of u throughout the habitat $(-1,1)$ which is assumed to be surrounded by a completely hostile boundary $\{\pm 1\}$. In (1.1), the reaction term $\lambda(ku^{p-1} + \sum_{i=1}^m a_i u^{q_i}) - \mu \sum_{j=1}^n b_j u^{r_j}$ is the growth rate of the population, which consists of a source term $\lambda(ku^{p-1} + \sum_{i=1}^m a_i u^{q_i})$ and an absorption term $\mu \sum_{j=1}^n b_j u^{r_j}$. Note that, by (1.3), if $\mu > 0$, the absorption term $\mu \sum_{j=1}^n b_j u^{r_j}$ is dominated by the source term when u near 0^+ and dominate the source term when u is large enough, and the domination of the absorption term over the source term is assumed to be strictly increasing on $(0,\infty)$. Murray [11] suggested using diffusion of the form p in the study of diffusion-kinetic enzymes problems. By a positive solution to p-Laplacian problem (1.1) with general $p > 1$, we mean a positive function $u \in C^1[-1,1]$ with $\varphi_p(u') \in C^1[-1,1]$ satisfying (1.1). Let $Z = \{x \in [-1,1] : u'(x) = 0\}$. We note that it is easy to show that, if u is a positive solution of (1.1), then $u \in C^2[-1,1]$ if $1 < p \leq 2$ and $u \in C^2([-1,1] \setminus Z)$ if $p > 2$. For the proof we refer to [1, Lemma 6].

To study bifurcation diagrams of positive solutions of (1.1), (1.3), it is important to study the shape of nonlinearity $f_{k,\mu,\lambda}(u)$ on $(0,\infty)$ in the beginning. We show that there exist three positive numbers $\beta_{\mu,\lambda} > \zeta_{\mu,\lambda} > \gamma_{\mu,\lambda}$ such that $f_{k,\mu,\lambda}(u)$ with $\lambda, \mu > 0$ satisfies (1.4), (1.9), and (1.11) stated behind. That is, positive numbers $\beta_{\mu,\lambda} > \zeta_{\mu,\lambda} > \gamma_{\mu,\lambda}$ are the unique positive zero, critical point, and p-inflection point of $f_{k,\mu,\lambda}(u)$ on $(0,\infty)$, respectively. First, we easily observe that, for $f_{k,\mu,\lambda}(u)$ with $\lambda, \mu > 0$ satisfying (1.3), the number of sign changes in the sequence of coefficients for the *generalized polynomial* $f_{k,\mu,\lambda}(u)$

$$(\lambda k, \lambda a_1, \lambda a_2, \ldots, \lambda a_m, -\mu b_1, -\mu b_2, \ldots, -\mu b_n)$$

is 1. Applying Laguerre's Theorem [10] (see also [9, Theorem 4.7]) on the number of positive zeros to the generalized polynomial $f_{k,\mu,\lambda}(u)$, we obtain that there exists a unique positive number $\beta_{\mu,\lambda}$ such that

$$\begin{cases} f_{k,\mu,\lambda}(u) > 0 \ \text{ on } (0, \beta_{\mu,\lambda}), \\ f_{k,\mu,\lambda}(0) = f_{k,\mu,\lambda}(\beta_{\mu,\lambda}) = 0, \\ f_{k,\mu,\lambda}(u) < 0 \ \text{ on } (\beta_{\mu,\lambda}, \infty). \end{cases} \tag{1.4}$$

We set $\beta_{\mu=0,\lambda} = \infty$ if $\mu = 0$. Notice that, by (1.3), it is easy to see that, for fixed $\lambda > 0$,

$$\lim_{\mu \to \infty} \beta_{\mu,\lambda} = 0. \tag{1.5}$$

In addition,

$$\text{for fixed } \mu > 0,\ \beta_{\mu,\lambda} \text{ is a continuous, strictly increasing function of } \lambda \text{ on } (0,\infty) \tag{1.6}$$

and

$$\text{for fixed } \lambda > 0,\ \beta_{\mu,\lambda} \text{ is a continuous, strictly decreasing function of } \mu \text{ on } (0,\infty). \tag{1.7}$$

Secondly, we compute that

$$f'_{k,\mu,\lambda}(u) = \lambda \left[(p-1)ku^{p-2} + \sum_{i=1}^{m} a_i q_i u^{q_i - 1} \right] - \mu \sum_{j=1}^{n} b_j r_j u^{r_j - 1}. \tag{1.8}$$

Thus again, similarly, applying (1.3) and Laguerre's Theorem [10] on the number of positive zeros to the *generalized polynomial* $f'_{k,\mu,\lambda}(u)$ in (1.8), we obtain that there exists a unique positive number $\zeta_{\mu,\lambda} < \beta_{\mu,\lambda}$ such that

$$\begin{cases} f'_{k,\mu,\lambda}(u) > 0 \ \text{ on } (0, \zeta_{\mu,\lambda}), \\ f'_{k,\mu,\lambda}(u)(\zeta_{\mu,\lambda}) = 0, \\ f'_{k,\mu,\lambda}(u) < 0 \ \text{ on } (\zeta_{\mu,\lambda}, \beta_{\mu,\lambda}). \end{cases} \tag{1.9}$$

So $f_{k,\mu,\lambda}(u)$ with $\lambda, \mu > 0$ is increasing-decreasing on $(0, \beta_{\mu,\lambda})$. Thirdly, we compute that

$$(p-2)f'_{k,\mu,\lambda}(u) - uf''_{k,\mu,\lambda}(u) = \lambda \sum_{i=1}^{m} a_i q_i (p-1-q_i) u^{q_i-1} - \mu \sum_{j=1}^{n} b_j r_j (p-1-r_j) u^{r_j-1}, \tag{1.10}$$

in which $p - 1 - q_i < 0$ for $i = 1, 2, \ldots, m$ and $p - 1 - r_j < 0$ for $j = 1, 2, \ldots, n$. Thus again, applying (1.3) and Laguerre's Theorem [10] on the number of positive zeros to the *generalized polynomial* $(p-2)f'_{k,\mu,\lambda}(u) - uf''_{k,\mu,\lambda}(u)$ in (1.10), we obtain that there exists a unique positive number $\gamma_{\mu,\lambda} < \zeta_{\mu,\lambda}$ such that

$$\begin{cases} (p-2)f'_{k,\mu,\lambda}(u) - uf''_{k,\mu,\lambda}(u) < 0 \ \text{ on } (0, \gamma_{\mu,\lambda}), \\ (p-2)f'_{k,\mu,\lambda}(\gamma_{\mu,\lambda}) - uf''_{k,\mu,\lambda}(\gamma_{\mu,\lambda}) = 0, \\ (p-2)f'_{k,\mu,\lambda}(u) - uf''_{k,\mu,\lambda}(u) > 0 \ \text{ on } (\gamma_{\mu,\lambda}, \beta_{\mu,\lambda}). \end{cases} \tag{1.11}$$

In this case $f_{k,\mu,\lambda}(u)$ with $\lambda, \mu > 0$ is said to be p-convex-concave on $(0, \beta_{\mu,\lambda})$.

Note that, in (1.1), $\lambda k u^{p-1}$ is the p-linear term for generalized polynomial nonlinearity $f_{k,\mu,\lambda}$ if bifurcation parameter $k > 0$. If $k = 0$, then $f_{k,\mu,\lambda}$ has no p-linear term. In this paper we are concerned only with positive solutions u of (1.1), (1.3) satisfying

$$0 < \|u\|_\infty < \beta_{\mu,\lambda} \begin{cases} = \infty & \text{if } \mu = 0, \\ < \infty & \text{if } \mu > 0. \end{cases} \tag{1.12}$$

Positive solutions u of (1.1), (1.3) satisfying (1.12) are called *classical* positive solutions. Note that positive solutions u of (1.1), (1.3) satisfying $\|u\|_\infty = \beta_{\mu,\lambda}$ are called *flat-core* positive solutions.

For problem (1.1), (1.3), we study evolutionary bifurcation diagrams $S_{p,k,\mu}$ on the $(\lambda, \|u\|_\infty)$-plane defined by:

$$S_{p,k,\mu} = \{(\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a (classical) positive solution of (1.1), (1.3)}\}, \ \mu \geq 0. \tag{1.13}$$

First, when $\mu = 0$ and $f_{k,\mu=0,\lambda}(u) \equiv \lambda(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i})$, we study $S_{p,k,\mu=0}$ on the $(\lambda, \|u\|_\infty)$-plane in the next proposition. We let

$$\bar{\lambda} \equiv \left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p \begin{cases} < \infty & \text{if } k > 0, \\ = \infty & \text{if } k = 0. \end{cases} \tag{1.14}$$

Proposition 1.1 (See Figs. 2.1–2.2 depicted behind)**.** *Let $p > 1$. Consider p-Laplacian problem (1.1), (1.3) with $\mu = 0$ and $f_{k,\mu=0,\lambda}(u) = \lambda(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}) > 0$ on $(0,\infty)$. Then the bifurcation diagram $S_{p,k,\mu=0}$ satisfies the following assertions (i)–(ii):*

(i) On the $(\lambda, \|u\|_\infty)$-plane, $S_{p,k,\mu=0}$ emanates from the positive $\|u\|_\infty$-axis as $\lambda \to 0^+$, tends to the point $(\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right)$ if $k > 0$ and tends to the positive λ-axis as $\lambda \to \infty$ if $k = 0$, and consists of a continuous, strictly decreasing curve.

(ii) Moreover, if $k = 0$, $m = 1$, $q \equiv q_1 > p - 1$ and $f_{k=0,\mu=0,\lambda}(u) \equiv \lambda u^q > 0$ on $(0,\infty)$, then

$$S_{p,k=0,\mu=0} = \left\{(\lambda, \|u_\lambda\|_\infty) = (c_{p,q}\alpha^{p-q-1}, \alpha),\ \alpha = \|u_\lambda\|_\infty > 0\right\},$$

where

$$c_{p,q} \equiv \left(\frac{p-1}{p}\right)(q+1)^{1-p}\left[\frac{\Gamma(\frac{p-1}{p})\Gamma(\frac{1}{q+1})}{\Gamma(\frac{pq+2p-q-1}{p(q+1)})}\right]^p > 0, \qquad (1.15)$$

and $\Gamma(t) \equiv \int_0^\infty x^{t-1}e^{-x}dx$ is the usual gamma function.

Proof. (I) We prove part (i). To study $S_{p,k,\mu=0}$ for p-Laplacian problem (1.1), (1.3) with $\mu = 0$, we apply the time-map method for which the time-map formula takes the form as follows:

$$\lambda^{1/p} = \left(\frac{p-1}{p}\right)^{1/p}\int_0^\alpha \frac{1}{[\bar{F}(\alpha) - \bar{F}(u)]^{1/p}}du \equiv T_{\bar{f}}(\alpha) \quad \text{for } \alpha = \|u\|_\infty > 0, \qquad (1.16)$$

where

$$\bar{f}(u) \equiv f_{k,\mu=0,\lambda=1}(u) = ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}$$

and $\bar{F}(u) \equiv \int_0^u \bar{f}(t)dt$; see, e.g., [2, Lemmas 2.1 and 2.2] for the derivation of the time map formula $T(\alpha)$ in (1.16). We have that positive solution $u_\lambda(x)$ of p-Laplacian problem (1.1), (1.3) with $\mu = 0$ corresponds to $\|u_\lambda\|_\infty = \alpha > 0$ satisfying (1.16), e.g., [13, p. 382]. It is easy to compute that, by (1.3),

$$\lim_{u\to 0^+}\frac{\bar{f}(u)}{u^{p-1}} = \frac{ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}}{u^{p-1}} = k \geq 0, \qquad \lim_{u\to\infty}\frac{\bar{f}(u)}{u^{p-1}} = \frac{ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}}{u^{p-1}} = \infty,$$

and

$$(p-1)\bar{f}(u) - u\bar{f}'(u) = \sum_{i=1}^{m} a_i(p-1-q_i)u^{q_i} < 0 \quad \text{on } (0,\infty).$$

Thus, by [13, (1.7), (1.9) and (4.4)], we have that $\lim_{\alpha\to 0^+} T_{\bar{f}}(\alpha) = \left(\frac{p-1}{k}\right)^{1/p}\frac{\pi}{p}\csc\frac{\pi}{p} \in (0,\infty]$, $\lim_{\alpha\to\infty} T_{\bar{f}}(\alpha) = 0$, and $T_{\bar{f}}(\alpha)$ is a strictly decreasing function on $(0,\infty)$. So part (i) directly follows from (1.13) and (1.16).

(II) We prove part (ii). We have that $\bar{f}(u) = u^q$, $q > p-1 > 0$ and $\bar{F}(u) \equiv \int_0^u \bar{f}(t)dt = \frac{1}{q+1}u^{q+1}$. It can be computed that

$$\begin{aligned} T_{\bar{f}}(\alpha) &= \left(\frac{p-1}{p}\right)^{1/p}\int_0^\alpha \frac{1}{[\bar{F}(\alpha) - \bar{F}(u)]^{1/p}}du \\ &= \left(\frac{p-1}{p}\right)^{1/p}(q+1)^{1/p}\int_0^\alpha \frac{1}{[\alpha^{q+1} - u^{q+1}]^{1/p}}du \\ &= \left(\frac{p-1}{p}\right)^{1/p}(q+1)^{(1-p)/p}\left[\frac{\Gamma(\frac{p-1}{p})\Gamma(\frac{1}{q+1})}{\Gamma(\frac{pq+2p-q-1}{p(q+1)})}\right]\alpha^{\frac{p-q-1}{p}} \end{aligned}$$

by [6, p. 212, formula 855.42] or using symbolic manipulator *Mathematica 11.0*. Thus by (1.15) and (1.16), we obtain that

$$\begin{aligned}\lambda = \left[T_{\bar{f}}(\alpha)\right]^p &= (\frac{p-1}{p})(q+1)^{1-p}\left[\frac{\Gamma(\frac{p-1}{p})\Gamma(\frac{1}{q+1})}{\Gamma(\frac{pq+2p-q-1}{p(q+1)})}\right]^p \alpha^{p-q-1} \\ &= c_{p,q}\alpha^{p-q-1}. \end{aligned} \tag{1.17}$$

So part (ii) holds.

The proof of Proposition 1.1 is now complete. □

2 Main results

The main results in this paper are next Theorem 2.1 and Theorem 2.2 for problem (1.1), (1.3) with $1 < p \leq 2$ and $p > 2$, respectively. In Theorems 2.1–2.2 with any fixed $\mu > 0$, we prove that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation diagram $S_{p,k,\mu}$ always consists of a continuous, ⊂-shaped curve with exactly one (right) turning point at some point $(\lambda^*, \|u_{\lambda^*}\|_\infty)$. While the upper branch of each ⊂-shaped bifurcation diagram $S_{p,k,\mu}$ is *unbounded* if $1 < p \leq 2$ and is *bounded* if $p > 2$. We then study the structures and evolution of bifurcation diagrams $S_{p,k,\mu}$ with varying $\mu \geq 0$; see Fig. 2.1 with $1 < p \leq 2$ and Fig. 2.2 with $p > 2$. Theorem 2.1 and Theorem 2.2 substantially improve [14, Corollary 2.2] and [14, Corollary 2.4], respectively. Cf. [14, Corollary 2.2] with $1 < p \leq 2$ and [14, Corollary 2.4] with $p > 2$ for details. Also see Remark 3.2 stated behind.

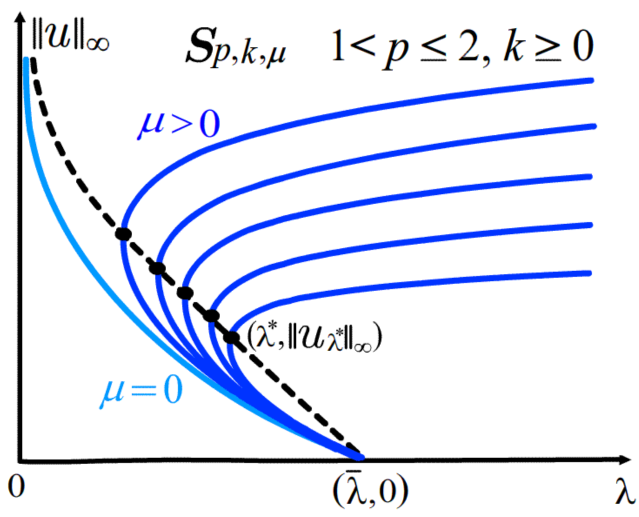

Figure 2.1: Evolutionary bifurcation diagrams $S_{p,k,\mu}$ for (1.1), (1.3) with fixed $p \in (1,2]$, $k \geq 0$ and varying $\mu \geq 0$.

Theorem 2.1 (See Fig. 2.1)**.** *Let $1 < p \leq 2$ and $k \geq 0$. Consider p-Laplacian problem* (1.1), (1.3) *with varying $\mu \geq 0$. Then the bifurcation diagram $S_{p,k,\mu}$ consists of a continuous curve on the $(\lambda, \|u\|_\infty)$-plane and the following assertions (i)–(v) hold:*

(i) For $\mu = 0$, $S_{p,k,\mu=0}$ emanates from the positive $\|u\|_\infty$-axis as $\lambda \to 0^+$, tends to the point $(\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right)$ if $k > 0$ and tends to the positive λ-axis as $\lambda \to \infty$ if $k = 0$, and consists of a strictly decreasing curve.

(ii) *For any fixed $\mu > 0$, $S_{p,k,\mu}$ always starts at the point $(\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right)$ if $k > 0$ and emanates from the positive λ-axis as $\lambda \to \infty$ if $k = 0$ (that is, $(\bar{\lambda}, 0) = (\infty, 0)$ if $k = 0$). $S_{p,k,\mu}$ is a $\subset$-shaped curve with exactly one turning point at some point $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ satisfying*

$$0 < \lambda^* < \bar{\lambda} \text{ and } 0 < \|u_{\lambda^*}\|_\infty < \beta_{\mu,\lambda^*}.$$

In addition, the upper branch of $S_{p,k,\mu}$ tends to infinity when $\lambda \to \infty$. Thus, (1.1), (1.3) has exactly two (classical) positive solutions for $\lambda^* < \lambda < \bar{\lambda}$, exactly one (classical) positive solution for $\lambda = \lambda^*$ and $\lambda \geq \bar{\lambda}$, and no (classical) positive solution for $0 < \lambda < \lambda^*$.

(iii) *For any nonnegative $\mu_1 < \mu_2$, S_{p,k,μ_2} lies on the right hand side of S_{p,k,μ_1}. (So S_{p,k,μ_1} and S_{p,k,μ_2} do not intersect.)*

(iv) *For the turning points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ of $S_{p,k,\mu}$ with $\mu > 0$, λ^* is a continuous, strictly increasing function of $\mu > 0$, $\|u_{\lambda^*}\|_\infty$ is a continuous function of $\mu > 0$,*

$$\lim_{\mu \to 0^+}(\lambda^*, \|u_{\lambda^*}\|_\infty) = (0, \infty) \text{ and } \lim_{\mu \to \infty}(\lambda^*, \|u_{\lambda^*}\|_\infty) = (\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right).$$

In particular, when $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p - 1$, $r \equiv r_1 > q$, and $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$, then $\|u_{\lambda^*}\|_\infty$ is a strictly decreasing function of $\mu > 0$.

(v) *When $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p - 1$, $r \equiv r_1 > q$, and $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$, then all points $(\lambda, \|u_\lambda\|_\infty) \in S_{p,k=0,\mu}$ satisfy*

$$0 < \left(\frac{c_{p,q}}{\lambda}\right)^{\frac{1}{q-p+1}} < \|u_\lambda\|_\infty < \beta_{\mu,\lambda} = \left(\frac{\lambda}{\mu}\right)^{\frac{1}{r-q}}, \tag{2.1}$$

where $c_{p,q}$ is defined in (1.15).

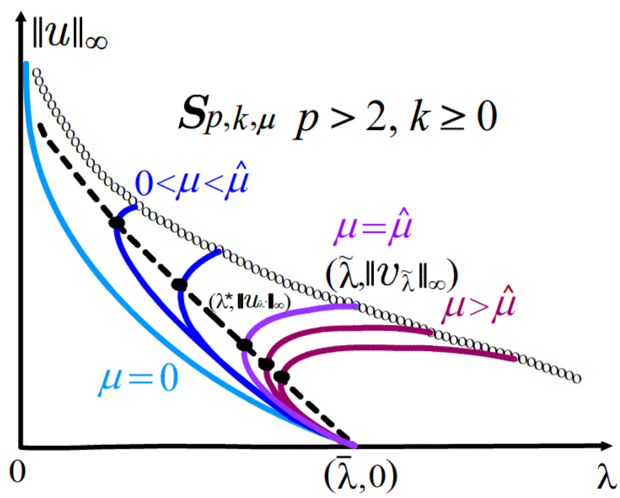

Figure 2.2: Evolutionary bifurcation diagrams $S_{p,k,\mu}$ for (1.1), (1.3) with fixed $p > 2$, $k \geq 0$ and varying $\mu \geq 0$.

Theorem 2.2 (See Fig. 2.2)**.** *Let $p > 2$ and $k \geq 0$. Consider one-dimensional p-Laplacian problem* (1.1), (1.3) *with varying $\mu \geq 0$. Then the bifurcation diagram $S_{p,k,\mu}$ consists of a continuous curve on the $(\lambda, \|u\|_\infty)$-plane and the following assertions (i)–(vi) hold:*

(i) *For* $\mu = 0$, $S_{p,k,\mu=0}$ *emanates from the positive* $\|u\|_\infty$*-axis as* $\lambda \to 0^+$, *tends to the point* $(\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right)$ *if* $k > 0$ *and tends to the positive* λ*-axis as* $\lambda \to \infty$ *if* $k = 0$, *and consists of a strictly decreasing curve.*

(ii) *For any fixed* $\mu > 0$, $S_{p,k,\mu}$ *starts at the same point* $(\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right)$ *if* $k > 0$ *and emanates from the positive* $\|u\|_\infty$*-axis as* $\lambda \to 0^+$ *if* $k = 0$ *(that is,* $(\bar{\lambda}, 0) = (\infty, 0)$ *if* $k = 0$*), ends at some point* $(\tilde{\lambda}, \|v_{\tilde{\lambda}}\|_\infty)$ *satisfying* $0 < \tilde{\lambda} < \infty$ *and* $0 < \|v_{\tilde{\lambda}}\|_\infty = v_{\tilde{\lambda}}(0) = \beta_{\mu,\tilde{\lambda}}$ *satisfying* $f_{k,\mu,\tilde{\lambda}}(\beta_{\mu,\tilde{\lambda}}) = 0$ *(that is,* $v_{\tilde{\lambda}}(x) \equiv \lim_{\lambda \to \tilde{\lambda}^-} v_\lambda(x)$ *is a* flat-core *positive solution of problem* (1.1), (1.3)*, see part (a) stated below for* (classical) *positive solutions* $v_\lambda(x)$ *with* $\lambda^* < \lambda < \tilde{\lambda}$*). Moreover,* $S_{p,k,\mu}$ *is a* $\subset$*-shaped curve with exactly one turning point at some point* $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ *satisfying*

$$0 < \lambda^* < \min(\bar{\lambda}, \tilde{\lambda}) \quad \text{and} \quad 0 < \|u_{\lambda^*}\|_\infty < \|v_{\tilde{\lambda}}\|_\infty = \beta_{\mu,\tilde{\lambda}}.$$

Moreover, there exists a unique positive $\hat{\mu} = \hat{\mu}(p, k, q_i, r_j, a_i, b_j) < \infty$ *if* $k > 0$ *and* $\hat{\mu} = \infty$ *if* $k = 0$ *such that:*

- *(a)* *If* $0 < \mu < \hat{\mu}$, *then* $(\lambda^* <)\ \tilde{\lambda} < \bar{\lambda}$ *such that problem* (1.1), (1.3) *has exactly two (classical) positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *satisfying* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda^* < \lambda < \tilde{\lambda}$, *exactly one (classical) positive solution* u_λ *satisfying* $\|u_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda = \lambda^*$ *and* $\tilde{\lambda} \le \lambda < \bar{\lambda}$, *and no (classical) positive solution for* $0 < \lambda < \lambda^*$ *and* $\lambda \ge \bar{\lambda}$. *In addition,* $\lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda \to \tilde{\lambda}^-} \|v_\lambda\|_\infty = \|v_{\tilde{\lambda}}\|_\infty = \beta_{\mu,\tilde{\lambda}}$.
- *(b)* *If* $\mu = \hat{\mu}$, *then* $(\lambda^* <)\ \tilde{\lambda} = \bar{\lambda}$ *such that problem* (1.1), (1.3) *has exactly two (classical) positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *satisfying* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda^* < \lambda < \bar{\lambda}$, *exactly one (classical) positive solution* u_λ *satisfying* $\|u_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda = \lambda^*$, *and no (classical) positive solution for* $0 < \lambda < \lambda^*$ *and* $\lambda \ge \bar{\lambda}$. *In addition,* $\lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda \to \tilde{\lambda}^-} \|v_\lambda\|_\infty = \|v_{\tilde{\lambda}}\|_\infty = \beta_{\mu,\tilde{\lambda}}$.
- *(c)* *If* $\mu > \hat{\mu}$, *then* $(\lambda^* <)\ \bar{\lambda} < \tilde{\lambda}$ *such that problem* (1.1), (1.3) *has exactly two (classical) positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *satisfying* $\|u_\lambda\|_\infty < \|v_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda^* < \lambda < \bar{\lambda}$, *exactly one (classical) positive solution* u_λ *satisfying* $\|u_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\lambda = \lambda^*$ *and exactly one (classical) positive solution* v_λ *satisfying* $\|v_\lambda\|_\infty < \beta_{\mu,\tilde{\lambda}}$ *for* $\bar{\lambda} \le \lambda < \tilde{\lambda}$, *and no (classical) positive solution for* $0 < \lambda < \lambda^*$ *and* $\lambda \ge \tilde{\lambda}$. *In addition,* $\lim_{\lambda \to \bar{\lambda}^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda \to \tilde{\lambda}^-} \|v_\lambda\|_\infty = \|v_{\tilde{\lambda}}\|_\infty = \beta_{\mu,\tilde{\lambda}}$.

(iii) *For any nonnegative* $\mu_1 < \mu_2$, S_{p,k,μ_2} *lies on the right hand side of* S_{p,k,μ_1}. *(So* S_{p,k,μ_1} *and* S_{p,k,μ_2} *do not intersect.)*

(iv) *For the ending points* $(\tilde{\lambda}, \|v_{\tilde{\lambda}}\|_\infty)$ *of* $S_{p,k,\mu}$ *with* $\mu > 0$, $\tilde{\lambda}$ *is a continuous, strictly increasing function of* $\mu > 0$, $\|v_{\tilde{\lambda}}\|_\infty$ *is a continuous, strictly decreasing function of* $\mu > 0$,

$$\lim_{\mu \to 0^+} (\tilde{\lambda}, \|v_{\tilde{\lambda}}\|_\infty) = (0, \infty) \ \text{ and } \ \lim_{\mu \to \infty} (\tilde{\lambda}, \|v_{\tilde{\lambda}}\|_\infty) = (\infty, 0). \tag{2.2}$$

(v) *For the turning points* $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ *of* $S_{p,k,\mu}$ *with* $\mu > 0$, λ^* *is a continuous, strictly increasing function of* $\mu > 0$, $\|u_{\lambda^*}\|_\infty$ *is a continuous function of* $\mu > 0$,

$$\lim_{\mu \to 0^+} (\lambda^*, \|u_{\lambda^*}\|_\infty) = (0, \infty) \ \text{ and } \ \lim_{\mu \to \infty} (\lambda^*, \|u_{\lambda^*}\|_\infty) = (\bar{\lambda}, 0) = \left(\left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p, 0\right).$$

(vi) When $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p - 1$, $r \equiv r_1 > q$, *and* $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$, *then all points* $(\lambda, \|u_\lambda\|_\infty) \in S_{p,k=0,\mu}$ *satisfy* (2.1).

Remark 2.3 (See Fig. 2.2)**.** By Theorem 2.2, for fixed $p \in (2, \infty)$ and $k > 0$, it is easy to see that, when $\mu \to \infty$, $S_{p,k,\mu}$ converges to the half-line $[\bar{\lambda}, \infty)$ on the positive λ-axis.

3 Lemmas

To prove Theorems 2.1–2.2 for p-Laplacian problem (1.1), (1.3), we need the following Lemmas 3.1 and 3.3–3.12. In particular, Theorems 2.1–2.2 is based on Lemma 3.1 which is due to Wang and Yeh [14]. Wang and Yeh [14] considered the p-Laplacian Dirichlet problem with one parameter λ:

$$\begin{cases} (\varphi_p(u'(x)))' + f_\lambda(u(x)) = 0, \ -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases} \tag{3.1}$$

They assumed that the nonlinearity

$$f_\lambda(u) \equiv \lambda g(u) - h(u), \tag{3.2}$$

where functions $g, h \in C[0, \infty) \cap C^2(0, \infty)$ satisfy hypotheses (H1)–(H4) if $1 < p \le 2$ and satisfy hypotheses (H1)–(H5) if $p > 2$:

(H1) $g(0) = h(0) = 0$, $g(u), h(u) > 0$ on $(0, \infty)$, and

$$0 = \lim_{u \to 0^+} \frac{h(u)}{u^{p-1}} \le m_0^g \equiv \lim_{u \to 0^+} \frac{g(u)}{u^{p-1}} < \infty.$$

(H2) The positive function $h(u)/g(u)$ is strictly increasing on $(0, \infty)$, and

$$\lim_{u \to 0^+} \frac{h(u)}{g(u)} = 0, \quad \lim_{u \to \infty} \frac{h(u)}{g(u)} = \infty.$$

(H3) $(p-2)g'(u) - ug''(u) < 0$ on $(0, \infty)$ and $(p-2)h'(u) - uh''(u) < 0$ on $(0, \infty)$.

(H4) The positive function $[(p-2)h'(u) - uh''(u)] \,/\, [(p-2)g'(u) - ug''(u)]$ is strictly increasing on $(0, \infty)$, and

$$\lim_{u \to 0^+} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = 0, \qquad \lim_{u \to \infty} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = \infty.$$

(H5) There exists a positive number $p^* > p - 1$ such that $g(u)/u^{p^*}$ is strictly decreasing on $(0, \infty)$ and $h(u)/u^{p^*}$ is strictly increasing on $(0, \infty)$. In addition, for each fixed $s \in (0, 1)$,

$$\frac{h(su)}{u^{p-1}} \left(\frac{h(u)g(su)}{g(u)h(su)} - 1 \right)$$

is a strictly increasing function of u on $(0, \infty)$, and

$$\lim_{u \to \infty} \frac{h(u)g(su)}{g(u)h(su)} \in (1, \infty].$$

Notice that, for p-Laplacian problem (3.1), hypotheses (H1)–(H2) imply that, for each fixed $\lambda > 0$, there exists a unique positive number β_λ such that

$$\begin{cases} f_\lambda(u) = \lambda g(u) - h(u) > 0 \text{ on } (0, \beta_\lambda), \\ f_\lambda(0) = \lambda g(0) - h(0) = 0 \text{ and } f_\lambda(\beta_\lambda) = \lambda g(\beta_\lambda) - h(\beta_\lambda) = 0, \\ f_\lambda(u) = \lambda g(u) - h(u) < 0 \text{ on } (\beta_\lambda, \infty). \end{cases}$$

Moreover, the number β_λ is a continuous, strictly increasing function of λ on $(0, \infty)$, $\lim_{\lambda\to 0^+} \beta_\lambda = 0$ and $\lim_{\lambda\to\infty} \beta_\lambda = \infty$. See [14, (1.4)–(1.5)]. Also, hypotheses (H1)–(H4) imply that, for each fixed $\lambda > 0$, the function $f_\lambda(u)$ with $\lambda > 0$ is p-convex-concave on $(0, \beta_\lambda)$. More precisely, there exists a unique positive number $\gamma_\lambda < \beta_\lambda$ such that

$$\begin{cases} (p-2)f'_\lambda(u) - uf''_\lambda(u) < 0 \text{ on } (0, \gamma_\lambda), \\ (p-2)f'_\lambda(\gamma_\lambda) - \gamma_\lambda f''_\lambda(\gamma_\lambda) = 0, \\ (p-2)f'_\lambda(u) - uf''_\lambda(u) > 0 \text{ on } (\gamma_\lambda, \beta_\lambda). \end{cases}$$

See [14, (1.6)]. In [14], Wang and Yeh are concerned only with positive solutions u of (3.1) satisfying $0 < \|u\|_\infty < \beta_\lambda$. Let

$$\hat{\lambda} \equiv \left(\frac{p-1}{m_0^g}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p \begin{cases} < \infty & \text{if } m_0^g > 0, \\ = \infty & \text{if } m_0^g = 0. \end{cases} \tag{3.3}$$

For $f_\lambda(u) = \lambda g(u) - h(u)$ in (3.2), we define $F_\lambda(u) = \int_0^u f_\lambda(t)dt$ and

$$T_\lambda(\alpha) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha [F_\lambda(\alpha) - F_\lambda(u)]^{-1/p} du \quad \text{for } 0 < \alpha < \beta_\lambda.$$

Lemma 3.1. *Consider p-Laplacian problem* (3.1) *with* $p > 1$*. Then the following assertions (i)–(ii) hold:*

(i) *([14, Theorem 2.1 and Fig. 1]) Let* $1 < p \le 2$*. If* $f_\lambda(u) = \lambda g(u) - h(u)$, $g, h \in C[0,\infty) \cap C^2(0,\infty)$ *satisfy (H1)–(H4). Then the bifurcation diagram consists of a continuous, ⊂-shaped curve on the* $(\lambda, \|u\|_\infty)$*-plane. More precisely, the there exists a positive number* $\lambda^* < \hat{\lambda}$ *such that* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda^* < \lambda < \hat{\lambda}$*, exactly one positive solution* v_λ *for* $\lambda = \lambda^*$ *and* $\lambda \ge \hat{\lambda}$*, and no positive solution for* $0 < \lambda < \lambda^*$*. Moreover,* $\lim_{\lambda\to\hat{\lambda}^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda\to\infty} \|v_\lambda\|_\infty = \infty$.

(ii) *([14, Theorem 2.3 and Fig. 3]) Let* $p > 2$*. If* $f_\lambda(u) = \lambda g(u) - h(u)$, $g, h \in C[0,\infty) \cap C^2(0,\infty)$ *satisfy (H1)–(H5). Then the bifurcation diagram consists of a continuous, ⊂-shaped curve on the* $(\lambda, \|u\|_\infty)$*-plane. More precisely, there exist three positive numbers* $\lambda^* < \tilde{\lambda}$ *and* $\beta_{\tilde{\lambda}}$ *satisfying* $\lambda^* < \hat{\lambda}$ $(\le \infty)$ *and* $f_{\tilde{\lambda}}(\beta_{\tilde{\lambda}}) = 0$ *and* $\lim_{\alpha\to\beta_{\tilde{\lambda}}^-} T_{\tilde{\lambda}}(\alpha) = 1$ *such that:*

  (a) *(See [14, Fig. 3(a)–(b)]) If* $\tilde{\lambda} < \hat{\lambda}$ $(\le \infty)$*, then* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda^* < \lambda < \tilde{\lambda}$*, exactly one positive solution* u_λ *for* $\lambda = \lambda^*$ *and* $\tilde{\lambda} \le \lambda < \hat{\lambda}$*, and no positive solution for* $0 < \lambda < \lambda^*$ *and for* $\lambda \ge \hat{\lambda}$ *(if* $\hat{\lambda} < \infty$*).*

  (b) *(See [14, Fig.3(c)–(d)]) If* $\hat{\lambda} \le \tilde{\lambda}$*, then* (1.1) *has exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $\lambda^* < \lambda < \hat{\lambda}$*, exactly one positive solution* v_λ *for* $\lambda = \lambda^*$ *and for* $\hat{\lambda} \le \lambda < \tilde{\lambda}$ *(if* $\tilde{\lambda} > \hat{\lambda}$*), and no positive solution for* $0 < \lambda < \lambda^*$ *and* $\lambda \ge \tilde{\lambda}$*. Moreover,* $\lim_{\lambda\to\hat{\lambda}^-} \|u_\lambda\|_\infty = 0$ *and* $\lim_{\lambda\to\tilde{\lambda}^-} \|v_\lambda\|_\infty = \beta_{\tilde{\lambda}}$.

Remark 3.2. To Lemma 3.1(i)–(ii), Wang and Yeh [14, Corollaries 2.2 and 2.4] gave examples of generalized polynomial nonlinearities for

$$f_\lambda(u) = \lambda g(u) - h(u) = \lambda(ku^{p-1} + u^q) - u^r$$

satisfying $r > q > p - 1 > 0$ and $k \geq 0$, which is a special case of

$$f_{k,\mu,\lambda}(u) = \lambda\left(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}\right) - \mu \sum_{j=1}^{n} b_j u^{r_j}$$

defined in (1.2) satisfying (1.3).

For p-Laplacian problem (1.1), (1.3) with two parameters μ and λ and $f_{k,\mu,\lambda}(u)$ defined in (1.2), we define the time map formula as follows:

$$T_{\mu,\lambda}(\alpha) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \frac{du}{\left[F_{k,\mu,\lambda}(\alpha) - F_{k,\mu,\lambda}(u)\right]^{1/p}} \quad \text{for } 0 < \alpha < \beta_{\mu,\lambda}, \tag{3.4}$$

where $\beta_{\mu,\lambda}$ is defined in (1.4) and

$$F_{k,\mu,\lambda}(u) = \int_0^u f_{k,\mu,\lambda}(t)dt. \tag{3.5}$$

We define $f_{k,\mu,\lambda}(u) = \lambda g(u) - \mu\tilde{h}(u)$ where $g(u) = ku^{p-1} + \sum_{i=1}^m a_i u^{q_i}$, $\tilde{h}(u) = \sum_{j=1}^n b_j u^{r_j}$, $G(u) = \int_0^u g(t)dt$ and $\tilde{H}(u) = \int_0^u \tilde{h}(t)dt$.

We suppose that $u_{\mu,\lambda}(x)$ is a (classical) positive solution of p-Laplacian problem (1.1), (1.3) satisfying (1.12). Then (classical) positive solution $u_{\mu,\lambda}(x)$ corresponds to $\|u_{\mu,\lambda}\|_\infty = \alpha$ and

$$T_{\mu,\lambda}(\alpha) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \frac{du}{\left[F_{k,\mu,\lambda}(\alpha) - F_{k,\mu,\lambda}(u)\right]^{1/p}} = 1 \quad \text{for } 0 < \alpha < \beta_{\mu,\lambda}. \tag{3.6}$$

See, e.g., [8, (3.9)].

Recall the number $\bar{\lambda} = \left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p$ defined in (1.14).

Lemma 3.3. *Consider p-Laplacian problem (1.1), (1.3) with $p > 1$, $\lambda > 0$ and $\mu > 0$. Then the following assertions (i)–(ii) hold:*

(i) $\lim_{\alpha\to 0^+} T_{\mu,\lambda}(\alpha) = (\frac{\bar{\lambda}}{\lambda})^{1/p}$ *and $T_{\mu,\lambda}(\alpha)$ has exactly an critical point at some $\alpha^*_{\mu,\lambda}$, a minimum, on $(0, \beta_{\mu,\lambda})$. Moreover,*

$$\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = \infty \quad \text{if } 1 \leq p < 2.$$

(ii) There exist two positive numbers $C < D < \beta_{\mu,\lambda}$ such that $C < \alpha^*_{\mu,\lambda} < D$ where $C = C(k, \mu, \lambda)$, $D = D(k, \mu, \lambda)$ satisfy

$$(p-1)f_{k,\mu,\lambda(\mu)}(C) - Cf'_{k,\mu,\lambda(\mu)}(C) = 0 \quad \text{and} \quad pF_{k,\mu,\lambda(\mu)}(D) - Df_{k,\mu,\lambda(\mu)}(D) = 0, \tag{3.7}$$

respectively. Cf. [14, (3.10) and (3.11)]. Then $T'_{\mu,\lambda}(\alpha) < 0$ for $\alpha \in (0, C]$ and $T'_{\mu,\lambda}(\alpha) > 0$ for $\alpha \in [D, \beta_{\mu,\lambda})$.

Proof. Parts (i) and (ii) simply follow by (1.4), (1.11), and slight modification of the proofs of [14, Lemmas 3.1 and 3.2]. We omit the detailed proofs here. □

We show comparison results for $T_{\mu,\lambda}(\alpha)$ in the next lemma; cf. [7, Lemma 3.3(i)–(ii)]. Notice that, for any fixed $\mu > 0$ and $0 < \lambda_1 < \lambda_2$, $\beta_{\mu,\lambda_1} < \beta_{\mu,\lambda_2}$ by (1.6), and for any fixed $\lambda > 0$ and $0 < \mu_1 < \mu_2$, $\beta_{\mu_2,\lambda} < \beta_{\mu_1,\lambda}$ by (1.7).

Lemma 3.4. *Consider p-Laplacian problem (1.1), (1.3) with $p > 1$. Then the following assertions (i)–(ii) hold:*

(i) For any fixed $\mu > 0$ and $0 < \lambda_1 < \lambda_2$, $T_{\mu,\lambda_1}(\alpha) > T_{\mu,\lambda_2}(\alpha)$ for $0 < \alpha < \beta_{\mu,\lambda_1}$.

(ii) For any fixed $\lambda > 0$ and $0 < \mu_1 < \mu_2$, $T_{\mu_1,\lambda}(\alpha) < T_{\mu_2,\lambda}(\alpha)$ for $0 < \alpha < \beta_{\mu_2,\lambda}$.

Proof. The proofs of parts (i)–(ii) follow by modification of those of [7, Lemma 3.3(i)–(ii)]. We omit them here. □

Lemma 3.5. *Consider p-Laplacian problem (1.1), (1.3) with $p > 1$. Then the following assertions (i)–(ii) hold:*

(i) For any fixed $\mu \geq 0$ and $0 < \lambda_1 < \lambda_2$, $T_{\mu,\lambda}(\alpha)$ is a continuous function of $\lambda \in [\lambda_1, \lambda_2]$ for $0 < \alpha < \beta_{\mu,\lambda_1}$.

(ii) For any fixed $\lambda > 0$ and $0 \leq \mu_1 < \mu_2$, $T_{\mu,\lambda}(\alpha)$ is a continuous function of $\mu \in [\mu_1, \mu_2]$ for $0 < \alpha < \beta_{\mu_2,\lambda}$.

Proof. The proofs of parts (i)–(ii) follow by modification of those of [7, Lemma 3.4(i)–(ii)]. We omit them here. □

By Lemma 3.3(i), $T_{\mu,\lambda}(\alpha)$ has exactly one critical point at some $\alpha^*_{\mu,\lambda}$, a minimum, on $(0, \beta_{\mu,\lambda})$. Let

$$m(\mu, \lambda) \equiv T_{\mu,\lambda}(\alpha^*_{\mu,\lambda}) = \min_{\alpha \in (0,\beta_{\mu,\lambda})} T_{\mu,\lambda}(\alpha).$$

Lemma 3.6. *Consider p-Laplacian problem (1.1), (1.3) with $p > 1$. Then the following assertions (i)–(ii) hold:*

(i) For any fixed $\mu \in (0, \infty)$, there exists a unique $\lambda^* > 0$ such that $m(\mu, \lambda^*) = 1$.

(ii) For any fixed $\lambda \in (0, \bar{\lambda})$, there exists a unique $\mu^* > 0$ such that $m(\mu^*, \lambda) = 1$.

Moreover, for any fixed $\lambda \geq \bar{\lambda}$ and $\mu > 0$, $m(\mu, \lambda) < 1$.

Proof. (I) We prove part (ii). We have that $\lim_{\alpha\to\infty} T_{\mu=0,\lambda}(\alpha) = 0$, which follows from Proposition 1.1(i) and since $\lim_{\mu\to 0^+} T_{\mu,\lambda}(\alpha) = T_{\mu=0,\lambda}(\alpha)$. So we can find a number $\mu_1 > 0$ such that $m(\mu_1, \lambda) = \min_{\alpha\in(0,\beta_{\mu_1,\lambda})} T_{\mu_1,\lambda}(\alpha) < 1$. In addition, we have that $\lim_{\alpha\to 0^+} T_{\mu,\lambda}(\alpha) > 1$, which follows from Lemma 3.3(i) for $0 < \lambda < \bar{\lambda}$. By (3.7), we compute and obtain that

$$\frac{\lambda}{\mu} = \frac{\sum_{j=1}^{n} b_j (r_j - p + 1) C^{r_j}}{\sum_{i=1}^{m} a_i (q_i - p + 1) C^{q_i}} = \frac{\sum_{j=1}^{n} b_j \frac{r_j - p + 1}{r_j + 1} D^{r_j}}{\sum_{i=1}^{m} a_i \frac{q_i - p + 1}{q_i + 1} D^{q_i}}.$$

So, for fixed $\lambda \in (0, \bar{\lambda})$, we have that $\lim_{\mu\to\infty} \alpha^*_{\mu,\lambda} = 0$ since $\lim_{\mu\to\infty} C = \lim_{\mu\to\infty} D = 0$ and $\alpha^*_{\mu,\lambda} \in (C, D)$ by Lemma 3.3(ii). Hence we can find a number $\mu_2 > 0$ such that $m(\mu_2, \lambda) = \min_{\alpha\in(0,\beta_{\mu_2,\lambda})} T_{\mu_2,\lambda}(\alpha) > 1$.

Next, we set positive numbers $\alpha_1 \equiv \inf_{\mu\in[\mu_1,\mu_2]} \alpha^*_{\mu,\lambda}$ and $\alpha_2 \equiv \sup_{\mu\in[\mu_1,\mu_2]} \alpha^*_{\mu,\lambda} \geq \alpha_1$. If $\alpha_1 = \alpha_2$, then $\alpha^*_{\mu,\lambda} = \alpha_1 = \alpha_2$ for all $\mu \in [\mu_1, \mu_2]$. Thus $T_{\mu_2,\lambda}(\alpha_1) = m(\mu_2, \lambda) > 1$ and $T_{\mu_1,\lambda}(\alpha_1) = m(\mu_1, \lambda) < 1$. So, by Lemma 3.5(ii) and the Intermediate Value Theorem, there exists $\mu^* \in (\mu_1, \mu_2)$ such that

$$m(\mu^*, \lambda) = T_{\mu^*,\lambda}(\alpha_1) = 1.$$

By Lemma 3.4(ii), $m(\mu, \lambda) = T_{\mu,\lambda}(\alpha_1)$ is strictly increasing in $\mu \in [\mu_1, \mu_2]$, and hence μ^* is unique.

While if $\alpha_1 < \alpha_2$, we first show that $m(\mu, \lambda)$ is a continuous function of μ on $[\mu_1, \mu_2]$ as follows. By Lemma 3.4(ii) and Lemma 3.5(ii), for each $\mu_1 < \mu_2$ and fixed $\alpha \in (0, \beta_{\mu_2,\lambda})$, $T_{\mu,\lambda}(\alpha)$ is a continuous, strictly increasing function of μ on (μ_1, μ_2). So for any fixed $\breve{\mu} \in [\mu_1, \mu_2]$, by the Dini Theorem [12, p. 195], it is easy to see that

$$\lim_{\mu\to\breve{\mu}} \left(\min_{\alpha\in[\alpha_1,\alpha_2]} T_{\mu,\lambda}(\alpha) \right) = \min_{\alpha\in[\alpha_1,\alpha_2]} T_{\breve{\mu},\lambda}(\alpha). \tag{3.8}$$

Since for any $\mu \in [\mu_1, \mu_2]$, the minimum of $T_{\mu,\lambda}(\alpha)$ occurs at $\alpha^*_{\mu,\lambda} \in [\alpha_1, \alpha_2]$. So

$$m(\mu, \lambda) = \min_{\alpha\in(0,\beta_{\mu,\lambda})} T_{\mu,\lambda}(\alpha) = \min_{\alpha\in[\alpha_1,\alpha_2]} T_{\mu,\lambda}(\alpha) \quad \text{for } \mu \in [\mu_1, \mu_2]. \tag{3.9}$$

By (3.8) and (3.9), $\lim_{\mu\to\breve{\mu}} m(\mu, \lambda) = m(\breve{\mu}, \lambda)$. Hence $m(\mu, \lambda)$ is a continuous function of μ on $[\mu_1, \mu_2]$. By the Intermediate Value Theorem, there exists $\mu^* \in (\mu_1, \mu_2)$ such that $m(\mu^*, \lambda) = 1$. Moreover, since $m(\mu, \lambda)$ is strictly increasing in $\mu \in [\mu_1, \mu_2]$, we obtain that μ^* is unique.

For any fixed $\lambda \geq \bar{\lambda}$ and $\mu > 0$. By Lemma 3.3, we have $\lim_{\alpha\to0^+} T_{\mu,\lambda}(\alpha) \leq 1$ and $m(\mu, \lambda) < 1$.

By above, part (ii) holds.

(II) The proof of part (i) is similar to that of part (ii). We omit it here.

The proof of Lemma 3.6 is now complete. □

Lemma 3.7 (See Theorems 2.1–2.2 and Figs. 2.1–2.2)**.** *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 1$. Then, for the turning points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ of $S_{p,k,\mu}$ with varying $\mu > 0$, λ^* is a continuous, strictly increasing function of $\mu > 0$. Moreover, $\lim_{\mu\to0^+} \lambda^* = 0$ and $\lim_{\mu\to\infty} \lambda^* = \bar{\lambda}$.*

Proof. For fixed $\mu_2 > \mu_1 > 0$, by Lemma 3.6(i), there exists $\lambda^*_2(\mu_2) > 0$ (resp. $\lambda^*_1(\mu_1) > 0$) such that $T_{\mu_2,\lambda^*_2}(\alpha)$ (resp. $T_{\mu_1,\lambda^*_1}(\alpha)$) has exactly one minimum point $\alpha_{\mu_2,\lambda^*_2} \in (0, \beta_{\mu_2,\lambda^*_2})$ (resp. $\alpha_{\mu_1,\lambda^*_1} \in (0, \beta_{\mu_1,\lambda^*_1})$) satisfying $T_{\mu_2,\lambda^*_2}(\alpha_{\mu_2,\lambda^*_2}) = 1$ (resp. $T_{\mu_1,\lambda^*_1}(\alpha_{\mu_1,\lambda^*_1}) = 1$). Observe that $\alpha_{\mu_2,\lambda^*_2} \in (0, \beta_{\mu_2,\lambda^*_2}) \subsetneq (0, \beta_{\mu_1,\lambda^*_2})$. So we obtain that

$$\begin{aligned}
T_{\mu_1,\lambda^*_2}(\alpha_{\mu_2,\lambda^*_2}) &= \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\alpha_{\mu_2,\lambda^*_2}} [\lambda^*_2(G(\alpha_{\mu_2,\lambda^*_2}) - G(u)) - \mu_1(\tilde{H}(\alpha_{\mu_2,\lambda^*_2}) - \tilde{H}(u))]^{-1/p} du \\
&< \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\alpha_{\mu_2,\lambda^*_2}} [\lambda^*_2(G(\alpha_{\mu_2,\lambda^*_2}) - G(u)) - \mu_2(\tilde{H}(\alpha_{\mu_2,\lambda^*_2}) - \tilde{H}(u))]^{-1/p} du \\
&= T_{\mu_2,\lambda^*_2}(\alpha_{\mu_2,\lambda^*_2}) \\
&= 1.
\end{aligned}$$

So

$$\min_{\alpha\in(0,\beta_{\mu_1,\lambda^*_2})} T_{\mu_1,\lambda^*_2}(\alpha) \leq T_{\mu_1,\lambda^*_2}(\alpha_{\mu_2,\lambda^*_2}) < 1. \tag{3.10}$$

For any $\alpha \in (0, \beta_{\mu_1,\lambda_1^*})$, we have that

$$T_{\mu_1,\lambda_1^*}(\alpha) \geq \min_{\alpha \in (0,\beta_{\mu_1,\lambda_1^*})} T_{\mu_1,\lambda_1^*}(\alpha) = T_{\mu_1,\lambda_1^*}(\alpha_{\mu_1,\lambda_1^*}) = 1. \tag{3.11}$$

By (3.10), (3.11) and Lemma 3.4(i), we obtain $\lambda_1^*(\mu_1) < \lambda_2^*(\mu_2)$. By Lemma 3.6, we obtain $\lambda^*((0,\infty)) = (0,\bar{\lambda})$. Hence $\lambda^*(\mu) : (0,\infty) \to (0,\bar{\lambda})$ is a continuous, strictly increasing function. Moreover, $\lim_{\mu \to 0^+} \lambda^*(\mu) = 0$ and $\lim_{\mu \to \infty} \lambda^*(\mu) = \bar{\lambda}$.

The proof of Lemma 3.7 is now complete. □

Lemma 3.8 (See Theorems 2.1–2.2 and Figs. 2.1–2.2)**.** *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 1$ and $k \geq 0$. Then, for the turning points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ of $S_{p,k,\mu}$ with varying $\mu > 0$, $\frac{\lambda^*(\mu)}{\mu}$ is a continuous, strictly decreasing function of $\mu > 0$. Moreover, $\lim_{\mu \to 0^+} \frac{\lambda^*(\mu)}{\mu} = \infty$ and $\lim_{\mu \to \infty} \frac{\lambda^*(\mu)}{\mu} = 0$.*

Proof. For fixed $\mu_2 > \mu_1 > 0$, we let $N_1 \equiv \frac{\lambda^*(\mu_1)}{\mu_1}$ and $N_2 \equiv \frac{\lambda^*(\mu_2)}{\mu_2}$. Then

$$\begin{aligned}
&T_{\mu_2,\lambda^*(\mu_2)}(\|u_{\lambda^*(\mu_2)}\|_\infty) = 1 = T_{\mu_1,\lambda^*(\mu_1)}(\|u_{\lambda^*(\mu_1)}\|_\infty) \\
&\quad = T_{\mu_1,N_1\mu_1}(\|u_{\lambda^*(\mu_1)}\|_\infty) \\
&\quad = \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\|u_{\lambda^*(\mu_1)}\|_\infty} [N_1\mu_1(G(\|u_{\lambda^*(\mu_1)}\|_\infty) - G(u)) - \mu_1(\tilde{H}(\|u_{\lambda^*(\mu_1)}\|_\infty) - \tilde{H}(u))]^{-1/p} du \\
&\quad = \left(\frac{\mu_2}{\mu_1}\right)^{1/p} \left(\frac{p-1}{p}\right)^{1/p} \\
&\qquad \times \int_0^{\|u_{\lambda^*(\mu_1)}\|_\infty} [N_1\mu_2(G(\|u_{\lambda^*(\mu_1)}\|_\infty) - G(u)) - \mu_2(\tilde{H}(\|u_{\lambda^*(\mu_1)}\|_\infty) - \tilde{H}(u))]^{-1/p} du \\
&\quad = \left(\frac{\mu_2}{\mu_1}\right)^{1/p} T_{\mu_2,N_1\mu_2}(\|u_{\lambda^*(\mu_1)}\|_\infty) \\
&\quad > T_{\mu_2,N_1\mu_2}(\|u_{\lambda^*(\mu_1)}\|_\infty)
\end{aligned}$$

since $\mu_2 > \mu_1 > 0$. If $\lambda^*(\mu_2) \geq N_1\mu_2$, then $T_{\mu_2,N_1\mu_2}(\|u_{\lambda^*(\mu_1)}\|_\infty) \geq T_{\mu_2,\lambda^*(\mu_2)}(\|u_{\lambda^*(\mu_1)}\|_\infty) \geq T_{\mu_2,\lambda^*(\mu_2)}(\|u_{\lambda^*(\mu_2)}\|_\infty)$ by Lemma 3.4(i) and Lemma 3.3(i), which leads to a contradiction since $T_{\mu_2,\lambda^*(\mu_2)}(\|u_{\lambda^*(\mu_2)}\|_\infty) > T_{\mu_2,N_1\mu_2}(\|u_{\lambda^*(\mu_1)}\|_\infty)$. Hence we obtain $\lambda^*(\mu_2) < N_1\mu_2$. So

$$N_2 = \frac{\lambda^*(\mu_2)}{\mu_2} < N_1 = \frac{\lambda^*(\mu_1)}{\mu_1}.$$

By Lemma 3.7, we obtain that $\lambda^*(\mu)$ is a continuous function of $\mu > 0$. Hence $\frac{\lambda^*(\mu)}{\mu}$ is a continuous, strictly decreasing function of $\mu > 0$.

We prove $\lim_{\mu \to 0^+} \frac{\lambda^*(\mu)}{\mu} = \infty$ by method of contradiction. Suppose $\lim_{\mu \to 0^+} \frac{\lambda^*(\mu)}{\mu} < \infty$, then there exists a positive $N_3 > \lim_{\mu \to 0^+} \frac{\lambda^*(\mu)}{\mu}$ such that

$$\begin{aligned}
&\lim_{\mu \to 0^+} \left(\frac{1}{\mu}\right)^{1/p} T_{\mu=1,\lambda=N_3}\left(\|u_{\lambda^*(\mu)}\|_\infty\right) \\
&\quad = \lim_{\mu \to 0^+} \left(\frac{1}{\mu}\right)^{1/p} \left(\frac{p-1}{p}\right)^{1/p} \\
&\qquad \times \int_0^{\|u_{\lambda^*(\mu)}\|_\infty} [N_3(G(\|u_{\lambda^*(\mu)}\|_\infty) - G(u)) - (\tilde{H}(\|u_{\lambda^*(\mu)}\|_\infty) - \tilde{H}(u))]^{-1/p} du
\end{aligned}$$

$$
\begin{aligned}
&< \lim_{\mu\to 0^+}\left(\frac{1}{\mu}\right)^{1/p}\left(\frac{p-1}{p}\right)^{1/p} \\
&\quad \times \int_0^{\|u_{\lambda^*(\mu)}\|_\infty} [\frac{\lambda^*(\mu)}{\mu}(G(\|u_{\lambda^*(\mu)}\|_\infty)-G(u))-(\tilde{H}(\|u_{\lambda^*(\mu)}\|_\infty)-\tilde{H}(u))]^{-1/p}du \\
&= \lim_{\mu\to 0^+}\left(\frac{p-1}{p}\right)^{1/p} \\
&\quad \times \int_0^{\|u_{\lambda^*(\mu)}\|_\infty} [\lambda^*(\mu)(G(\|u_{\lambda^*(\mu)}\|_\infty)-G(u))-\mu(\tilde{H}(\|u_{\lambda^*(\mu)}\|_\infty)-\tilde{H}(u))]^{-1/p}du \\
&= \lim_{\mu\to 0^+} T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty).
\end{aligned}
$$

By Lemma 3.3(i), there exist two fixed positive numbers α_{1,N_3} and L_{1,N_3} such that $T_{\mu=1,\lambda=N_3}(\alpha_{1,N_3}) = L_{1,N_3}$ is an absolute minimum on $(0,\beta_{1,N_3})$. So

$$
1 = \lim_{\mu\to 0^+} T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty) > \lim_{\mu\to 0^+}\left(\frac{1}{\mu}\right)^{1/p} T_{\mu=1,\lambda=N_3}(\|u_{\lambda^*(\mu)}\|_\infty) \geq \lim_{\mu\to 0^+}\left(\frac{1}{\mu}\right)^{1/p} L_{1,N_3} = \infty,
$$

which leads to a contradiction. Hence $\lim_{\mu\to 0^+}\frac{\lambda^*(\mu)}{\mu} = \infty$.

Similarly, we prove $\lim_{\mu\to\infty}\frac{\lambda^*(\mu)}{\mu} = 0$ by method of contradiction. Suppose $\lim_{\mu\to\infty}\frac{\lambda^*(\mu)}{\mu} > 0$, then there exists a positive N_4 such that $N_4 < \lim_{\mu\to\infty}\frac{\lambda^*(\mu)}{\mu}$. By Lemma 3.3(i), there exist two fixed positive numbers α_{1,N_4} and L_{1,N_4} such that $T_{\mu=1,\lambda=N_4}(\alpha_{1,N_4}) = L_{1,N_4}$ is an absolute minimum on $(0,\beta_{1,N_4})$. We then need the next claim.

Claim A. $\lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\alpha_{1,N_4}) \geq \lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty)$.

Proof of Claim A. Since $N_4 < \lim_{\mu\to\infty}\frac{\lambda^*(\mu)}{\mu}$, there exists a positive number μ_0 such that, for all $\mu > \mu_0$, $N_4 < \frac{\lambda^*(\mu)}{\mu}$. By (1.4) and (1.6), for all $\mu > \mu_0$, we have $\beta_{1,N_4} < \beta_{1,\frac{\lambda^*(\mu)}{\mu}}$. For $\mu > 0$, by (1.4), we have $\beta_{1,\frac{\lambda^*(\mu)}{\mu}} = \beta_{\mu,\lambda^*(\mu)}$ since $f_{k,1,\frac{\lambda^*(\mu)}{\mu}}(\beta_{\mu,\lambda^*(\mu)}) = \frac{1}{\mu} f_{k,\mu,\lambda^*(\mu)}(\beta_{\mu,\lambda^*(\mu)}) = 0$. Hence, for all $\mu > \mu_0$, $0 < \alpha_{1,N_4} < \beta_{1,N_4} < \beta_{\mu,\lambda^*(\mu)}$. By Lemma 3.3(i), Lemma 3.6(i) and (3.6), for all $\mu > \mu_0$, we have $T_{\mu,\lambda^*(\mu)}(\alpha_{1,N_4}) \geq T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty)$. Moreover, $\lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\alpha_{1,N_4}) \geq \lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty)$. So Claim A holds.

We thus have that

$$
\begin{aligned}
&\lim_{\mu\to\infty}\left(\frac{1}{\mu}\right)^{1/p} T_{\mu=1,\lambda=N_4}(\alpha_{1,N_4}) \\
&= \lim_{\mu\to\infty}\left(\frac{1}{\mu}\right)^{1/p}\left(\frac{p-1}{p}\right)^{1/p}\int_0^{\alpha_{1,N_4}}[N_4(G(\alpha_{1,N_4})-G(u))-(\tilde{H}(\alpha_{1,N_4})-\tilde{H}(u))]^{-1/p}du \\
&> \lim_{\mu\to\infty}\left(\frac{1}{\mu}\right)^{1/p}\left(\frac{p-1}{p}\right)^{1/p}\int_0^{\alpha_{1,N_4}}\left[\frac{\lambda^*(\mu)}{\mu}(G(\alpha_{1,N_4})-G(u))-(\tilde{H}(\alpha_{1,N_4})-\tilde{H}(u))\right]^{-1/p}du \\
&= \lim_{\mu\to\infty}\left(\frac{p-1}{p}\right)^{1/p}\int_0^{\alpha_{1,N_4}}[\lambda^*(\mu)(G(\alpha_{1,N_4})-G(u))-\mu(\tilde{H}(\alpha_{1,N_4})-\tilde{H}(u))]^{-1/p}du \\
&= \lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\alpha_{1,N_4}) \\
&\geq \lim_{\mu\to\infty} T_{\mu,\lambda^*(\mu)}(\|u_{\lambda^*(\mu)}\|_\infty) \quad \text{(by Claim A)} \\
&= 1,
\end{aligned}
$$

which leads to a contradiction since

$$\lim_{\mu\to\infty}\left(\frac{1}{\mu}\right)^{1/p} T_{\mu=1,\lambda=N_4}(\alpha_{1,N_4}) = \lim_{\mu\to\infty}\left(\frac{1}{\mu}\right)^{1/p} L_{1,N4} = 0.$$

Hence $\lim_{\mu\to\infty}\frac{\lambda^*(\mu)}{\mu} = 0$.

The proof of Lemma 3.8 is now complete. □

Lemma 3.9 (See Theorems 2.1–2.2 and Figs. 2.1–2.2)**.** *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 1$ and $k \geq 0$. Then, for the turning points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ of $S_{p,k,\mu}$ with varying $\mu > 0$,*

$$C < \|u_{\lambda^*}\|_\infty < D, \tag{3.12}$$

where $C = C(k,\mu,\lambda^*(\mu))$, $D = D(k,\mu,\lambda^*(\mu))$ satisfy

$$(p-1)f_{k,\mu,\lambda^*(\mu)}(C) - Cf'_{k,\mu,\lambda^*(\mu)}(C) = 0 \quad \text{and} \quad pF_{k,\mu,\lambda^*(\mu)}(D) - Df_{k,\mu,\lambda^*(\mu)}(D) = 0, \tag{3.13}$$

respectively. Moreover, $\lim_{\mu\to 0^+}\|u_{\lambda^*}\|_\infty = \infty$ and $\lim_{\mu\to\infty}\|u_{\lambda^*}\|_\infty = 0$.

Proof. By Lemma 3.3(ii) and (3.6), it is easy to see that $C < \|u_{\lambda^*}\|_\infty < D$. Equations in (3.13) follow by Eqs. (3.7) directly. By (3.13), (1.2) and (3.5), we then compute and obtain that

$$\frac{\lambda^*(\mu)}{\mu} = \frac{\sum_{j=1}^n b_j(r_j-p+1)C^{r_j}}{\sum_{i=1}^m a_i(q_i-p+1)C^{q_i}} = \frac{\sum_{j=1}^n b_j\frac{r_j-p+1}{r_j+1}D^{r_j}}{\sum_{i=1}^m a_i\frac{q_i-p+1}{q_i+1}D^{q_i}},$$

in which $r_j - p + 1 > 0$ for $j = 1,2,\ldots,n$ and $q_i - p + 1 > 0$ for $i = 1,2,\ldots,m$. Thus, by applying Lemma 3.8 and (1.3), we have that $\lim_{\mu\to 0^+} C = \lim_{\mu\to 0^+} D = \infty$ and $\lim_{\mu\to\infty} C = \lim_{\mu\to\infty} D = 0$. Hence $\lim_{\mu\to 0^+}\|u_{\lambda^*}\|_\infty = \infty$ and $\lim_{\mu\to\infty}\|u_{\lambda^*}\|_\infty = 0$ by (3.12).

The proof of Lemma 3.9 is now complete. □

Lemma 3.10. *Consider p-Laplacian problem* (1.1), (1.3) *with $p > 2$ and $k \geq 0$. Then the following assertions (i)–(ii) hold:*

(i) For any fixed $\mu > 0$, $\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha)$ is a continuous, strictly decreasing function of λ on $(0,\infty)$. Moreover,

$$\lim_{\lambda\to 0^+}\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = \infty \quad \text{and} \quad \lim_{\lambda\to\infty}\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = 0.$$

(ii) For any fixed $\lambda > 0$, $\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha)$ is a continuous, strictly increasing function of μ on $(0,\infty)$. Moreover,

$$\lim_{\mu\to 0^+}\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = 0 \quad \text{and} \quad \lim_{\mu\to\infty}\lim_{\alpha\to\beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = \infty.$$

Proof. (I) We prove part (i) where

$$g(u) = ku^{p-1} + \sum_{i=1}^m a_i u^{q_i} \quad \text{and} \quad h(u) = \mu\sum_{j=1}^n b_j u^{r_j} \quad (\text{with } m,n \geq 1 \text{ and } \mu > 0)$$

satisfy (1.3). We take $p^* = \frac{q_m + r_1}{2} > p - 1$. Then it is easy to see that $\frac{g(u)}{u^{p^*}}$ is strictly decreasing on $(0, \infty)$ and $\frac{h(u)}{u^{p^*}}$ is strictly increasing on $(0, \infty)$. For each fixed $s \in (0, 1)$,

$$\frac{h(su)}{u^{p-1}}\left[\frac{h(u)g(su)}{g(u)h(su)} - 1\right] = \frac{h(u)g(su) - h(su)g(u)}{u^{p-1}g(u)}$$
$$= \frac{\mu \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j (s^{q_i} - s^{r_j}) u^{q_i + r_j} + \mu k u^{p-1} \sum_{j=1}^{n} b_j (s^{p-1} - s^{r_j}) u^{r_j}}{u^{p-1}(k u^{p-1} + \sum_{i=1}^{m} a_i u^{q_i})}$$

is a strictly increasing function of u on $(0, \infty)$ and $\lim_{u\to\infty} \frac{h(u)g(su)}{g(u)h(su)} = s^{q_m - r_n} \in (1, \infty)$ since $s \in (0, 1)$ and $q_m < r_n$. So g, h satisfy (H5). For $\lambda \in (0, \infty)$, by (3.4), we obtain that

$$\lim_{\alpha \to \beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha)$$
$$= \lim_{\alpha \to \beta_{\mu,\lambda}^-} \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[\int_u^\alpha f_{k,\mu,\lambda}(t)dt\right]^{-1/p} du$$
$$= \lim_{\alpha \to \beta_{\mu,\lambda}^-} \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[\int_u^\alpha \frac{h(\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})} g(t) - h(t)dt\right]^{-1/p} du$$

(since $f_{k,\mu,\lambda}(u) = \lambda g(u) - h(u)$ and by (1.4))

$$= \lim_{\alpha \to \beta_{\mu,\lambda}^-} \left(\frac{p-1}{p}\right)^{1/p} \alpha^{(p-1)/p} \int_0^1 \left[\int_v^1 \frac{h(\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})} g(s\alpha) - h(s\alpha)ds\right]^{-1/p} dv$$

(let $u = \alpha v$ and $t = \alpha s$)

$$= \left(\frac{p-1}{p}\right)^{1/p} \beta_{\mu,\lambda}^{(p-1-p^*)/p} \lim_{\alpha \to \beta_{\mu,\lambda}^-} \int_0^1 \left[\int_v^1 s^{p^*} \left(\frac{h(\beta_{\mu,\lambda})g(s\alpha)}{g(\beta_{\mu,\lambda})\,(s\alpha)^{p^*}} - \frac{h(s\alpha)}{(s\alpha)^{p^*}}\right) ds\right]^{-1/p} dv$$
$$= \left(\frac{p-1}{p}\right)^{1/p} \beta_{\mu,\lambda}^{(p-1-p^*)/p} \int_0^1 \left[\int_v^1 \lim_{\alpha \to \beta_{\mu,\lambda}^-} s^{p^*} \left(\frac{h(\beta_{\mu,\lambda})g(s\alpha)}{g(\beta_{\mu,\lambda})\,(s\alpha)^{p^*}} - \frac{h(s\alpha)}{(s\alpha)^{p^*}}\right) ds\right]^{-1/p} dv$$

(by (H5) and the Monotone Convergence Theorem)

$$= \left(\frac{p-1}{p}\right)^{1/p} \int_0^1 \left[\int_v^1 \frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}} \left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})} - 1\right) ds\right]^{-1/p} dv. \tag{3.14}$$

By (H5) and since $\beta_{\mu,\lambda}$ is strictly increasing in $\lambda \in (0, \infty)$, we obtain that $\lim_{\alpha \to \beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha)$ is a strictly decreasing function of λ on $(0, \infty)$.

For any number $\lambda_0 \in (0,\infty)$, by (3.14), we obtain that

$$\begin{aligned}
\lim_{\lambda\to\lambda_0}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) &= \lim_{\lambda\to\lambda_0}\left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p} dv\\
&= \left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \lim_{\lambda\to\lambda_0}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p} dv\\
&\qquad \text{(by (H5) and the Monotone Convergence Theorem)}\\
&= \left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \frac{h(s\beta_{\mu,\lambda_0})}{\beta_{\mu,\lambda_0}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda_0})g(s\beta_{\mu,\lambda_0})}{g(\beta_{\mu,\lambda_0})h(s\beta_{\mu,\lambda_0})}-1\right)ds\right]^{-1/p} dv\\
&= \lim_{\alpha\to\beta_{\mu,\lambda_0}^-} T_{\mu,\lambda_0}(\alpha).
\end{aligned}$$

So we obtain that $\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha)$ is a continuous function of λ on $(0,\infty)$.

Finally, we prove $\lim_{\lambda\to 0^+}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) = \infty$ and $\lim_{\lambda\to\infty}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) = 0$. By (3.14), we obtain that

$$\begin{aligned}
\lim_{\lambda\to 0^+}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) &= \lim_{\lambda\to 0^+}\left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p} dv\\
&= \left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \lim_{\lambda\to 0^+}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p} dv
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\lambda\to \infty}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) &= \lim_{\lambda\to \infty}\left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p} dv\\
&= \left(\frac{p-1}{p}\right)^{1/p}\int_0^1\left[\int_v^1 \lim_{\lambda\to \infty}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right)ds\right]^{-1/p} dv
\end{aligned}$$

by (H5) and the Monotone Convergence Theorem. For each fixed $s \in (0,1)$, we have that, by (H5),

$$\lim_{\lambda\to 0^+}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right) = \lim_{\beta_{\mu,\lambda}\to 0^+}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right) = 0$$

and

$$\lim_{\lambda\to \infty}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right) = \lim_{\beta_{\mu,\lambda}\to \infty}\frac{h(s\beta_{\mu,\lambda})}{\beta_{\mu,\lambda}^{p-1}}\left(\frac{h(\beta_{\mu,\lambda})g(s\beta_{\mu,\lambda})}{g(\beta_{\mu,\lambda})h(s\beta_{\mu,\lambda})}-1\right) = \infty.$$

So $\lim_{\lambda\to 0^+}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) = \infty$ and $\lim_{\lambda\to\infty}\lim_{\alpha\to\beta_{\mu,\lambda}^-} T_{\mu,\lambda}(\alpha) = 0$.

(II) The proof of part (ii) is similar to that of part (i). We omit it here.

The proof of Lemma 3.10 is now complete. □

Lemma 3.11. *Consider p-Laplacian problem* (1.1), (1.3) *with* $p > 2$ *and* $k \geq 0$. *Then the following assertions (i)–(ii) hold:*

(i) For any fixed $\mu \in (0,\infty)$, *there exists a unique* $\tilde{\lambda} > 0$ *such that* $\lim_{\alpha\to\beta^-_{\mu,\tilde{\lambda}}} T_{\mu,\tilde{\lambda}}(\alpha) = 1$.

(ii) For any fixed $\lambda \in (0,\infty)$, *there exists a unique* $\tilde{\mu} > 0$ *such that* $\lim_{\alpha\to\beta^-_{\tilde{\mu},\lambda}} T_{\tilde{\mu},\lambda}(\alpha) = 1$.

Proof. (I) We prove part (i). For any fixed $\mu > 0$, by Lemma 3.10(i) and the Intermediate Value Theorem, there exists a unique $\tilde{\lambda} > 0$ such that $\lim_{\alpha\to\beta^-_{\mu,\tilde{\lambda}}} T_{\mu,\tilde{\lambda}}(\alpha) = 1$.

(II) We prove part (ii). For any fixed $\lambda > 0$, by Lemma 3.10(ii) and the Intermediate Value Theorem, there exists a unique $\tilde{\mu} > 0$ such that $\lim_{\alpha\to\beta^-_{\tilde{\mu},\lambda}} T_{\tilde{\mu},\lambda}(\alpha) = 1$.

The proof of Lemma 3.11 is now complete. □

Lemma 3.12 (See Theorem 2.2 and Fig. 2.2)**.** *Consider p-Laplacian problem* (1.1), (1.3) *with* $p > 2$ *and* $k \geq 0$. *Then, for the ending points* $(\tilde{\lambda}, \|v_{\tilde{\lambda}}\|_\infty)$ *of* $S_{p,k,\mu}$ *with varying* $\mu > 0$, $\tilde{\lambda}$ *is a continuous, strictly increasing function of* $\mu > 0$. *Moreover,* $\lim_{\mu\to 0^+} \tilde{\lambda} = 0$ *and* $\lim_{\mu\to\infty} \tilde{\lambda} = \infty$.

Proof. For fixed $\mu_2 > \mu_1 > 0$, by Lemma 3.10(ii), $\lim_{\alpha\to\beta^-_{\mu_2\tilde{\lambda}_1}} T_{\mu_2,\tilde{\lambda}_1}(\alpha) > \lim_{\alpha\to\beta^-_{\mu_1,\tilde{\lambda}_1}} T_{\mu_1,\tilde{\lambda}_1}(\alpha) = 1$. If $\tilde{\lambda}_1 \geq \tilde{\lambda}_2$, then by Lemma 3.10(i), $\lim_{\alpha\to\beta^-_{\mu_2,\tilde{\lambda}_1}} T_{\mu_2,\tilde{\lambda}_1}(\alpha) \leq \lim_{\alpha\to\beta^-_{\mu_2,\tilde{\lambda}_2}} T_{\mu_2,\tilde{\lambda}_2}(\alpha) = 1$, which leads to a contradiction. So we obtain $\tilde{\lambda}_1 < \tilde{\lambda}_2$. By Lemma 3.11, we obtain $\tilde{\lambda}((0,\infty)) = (0,\infty)$. Hence $\tilde{\lambda}(\mu) : (0,\infty) \to (0,\infty)$ is a continuous, strictly increasing function of $\mu > 0$. Moreover, $\lim_{\mu\to 0^+} \tilde{\lambda} = 0$ and $\lim_{\mu\to\infty} \tilde{\lambda} = \infty$.

The proof of Lemma 3.12 is now complete. □

4 Proofs of main results

Proof of Theorem 2.1. Let $1 < p \leq 2$ and $k \geq 0$.

(I)(a) We prove that, for any fixed $\mu > 0$, the bifurcation diagram $S_{p,k,\mu}$ consists of a continuous curve on the $(\lambda, \|u\|_\infty)$-plane. For any fixed $\mu > 0$ and $\lambda > 0$, it is easy to see that $T_{\mu,\lambda}(\alpha)$ defined in (3.4) is a continuous function of $\alpha \in (0, \beta_{\mu,\lambda})$. By Lemma 3.4(i) and Lemma 3.6(i), we have that, for any fixed $\mu > 0$, the set $\{\alpha \in (0,\beta_{\mu,\lambda}) : T_{\mu,\lambda}(\alpha) = 1 \text{ for all } \lambda > 0\}$ is connected. Thus, by Lemma 3.5(i), for any fixed $\mu > 0$, $S_{p,k,\mu}$ consists of a continuous curve on the $(\lambda, \|u\|_\infty)$-plane.

(I)(b) Part (i) follows from Proposition 1.1(i).

(II) We prove part (ii) where nonlinearities

$$g(u) = ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i} \quad \text{and} \quad h(u) = \mu \sum_{j=1}^{n} b_j u^{r_j} \quad (\text{with } \mu > 0)$$

satisfy (1.3). We prove that g, h satisfy (H1)–(H4). It is first easy to see that $g, h \in C[0,\infty) \cap C^2(0,\infty)$ satisfy (H1) with $m_0^g \equiv \lim_{u\to 0^+} \frac{g(u)}{u^{p-1}} = k \geq 0$. Hence, by (3.3), $\hat{\lambda} = \left(\frac{p-1}{k}\right)\left(\frac{\pi}{p} \csc \frac{\pi}{p}\right)^p = \bar{\lambda}$. By (1.3), it is easy to see that the function

$$\frac{h(u)}{g(u)} = \frac{\mu \sum_{j=1}^{n} b_j u^{r_j}}{ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i}} \quad (\mu > 0)$$

is positive and strictly increasing on $(0,\infty)$ and satisfies that

$$\lim_{u\to 0^+} \frac{h(u)}{g(u)} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{h(u)}{g(u)} = \infty.$$

Thus g, h satisfy (H2). It is clear that, by (1.3),

$$(p-2)g'(u) - ug''(u) = \sum_{i=1}^{m} a_i q_i (p-1-q_i) u^{q_i-1} < 0 \quad \text{on } (0,\infty),$$

$$(p-2)h'(u) - uh''(u) = \mu \sum_{j=1}^{n} b_j r_j (p-1-r_j) u^{r_j-1} < 0 \quad \text{on } (0,\infty).$$

Thus g, h satisfy (H3). Finally, by (1.3), we compute that

$$\begin{aligned}\frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} &= \frac{\mu \sum_{j=1}^{n} b_j r_j (p-1-r_j) u^{r_j-1}}{\sum_{i=1}^{m} a_i q_i (p-1-q_i) u^{q_i-1}} \quad (\mu > 0) \\ &= \frac{\mu \sum_{j=1}^{n} b_j r_j (p-1-r_j) u^{r_j}}{\sum_{i=1}^{m} a_i q_i (p-1-q_i) u^{q_i}}\end{aligned}$$

which is positive and strictly increasing on $(0,\infty)$ and satisfies that

$$\lim_{u\to 0^+} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = 0 \quad \text{and} \quad \lim_{u\to \infty} \frac{(p-2)h'(u) - uh''(u)}{(p-2)g'(u) - ug''(u)} = \infty.$$

So g, h satisfy (H4). By above, we conclude that g, h satisfy (H1)–(H4). So part (ii) follows from Lemma 3.1(i).

(III) We prove part (iii). Consider any nonnegative $\mu_1 < \mu_2$. If, on the $(\lambda, \|u\|_\infty)$-plane, bifurcation diagrams S_{p,k,μ_1} and S_{p,k,μ_2} attain a fixed number $\|u\|_\infty = \bar{\alpha}$ for any feasible $\bar{\alpha} > 0$ at $\lambda = \lambda_1 > 0$ and $\lambda = \lambda_2 > 0$, respectively. Then by (3.6), we have the following equalities:

$$T_{\mu_1,\lambda_1}(\bar{\alpha}) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\bar{\alpha}} [\lambda_1(G(\bar{\alpha}) - G(u)) - \mu_1(\tilde{H}(\bar{\alpha}) - \tilde{H}(u))]^{-1/p} du = 1, \tag{4.1}$$

$$T_{\mu_2,\lambda_2}(\bar{\alpha}) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^{\bar{\alpha}} [\lambda_2(G(\bar{\alpha}) - G(u)) - \mu_2(\tilde{H}(\bar{\alpha}) - \tilde{H}(u))]^{-1/p} du = 1. \tag{4.2}$$

Suppose that $\lambda_1 \geq \lambda_2$, since $0 < \mu_1 < \mu_2$ and $\lambda_1 \geq \lambda_2$, we have that

$$\lambda_1(G(\bar{\alpha}) - G(u)) - \mu_1(\tilde{H}(\bar{\alpha}) - \tilde{H}(u)) > \lambda_2(G(\bar{\alpha}) - G(u)) - \mu_2(\tilde{H}(\bar{\alpha}) - \tilde{H}(u)).$$

Thus $T_{\mu_1,\lambda_1}(\bar{\alpha}) < T_{\mu_2,\lambda_2}(\bar{\alpha})$, which leads to a contradiction since the above equality (4.1) for $T_{\mu_1,\lambda_1}(\bar{\alpha})$ and equality (4.2) for $T_{\mu_2,\lambda_2}(\bar{\alpha})$ are both equal to 1. So $\lambda_1 < \lambda_2$. Hence, for any nonnegative $\mu_1 < \mu_2$, on the $(\lambda, \|u\|_\infty)$-plane, S_{p,k,μ_2} lies on the right hand side of S_{p,k,μ_1}.

(IV) We prove part (iv). By Lemma 3.7, $\lambda^*(\mu) : (0,\infty) \to (0,\bar{\lambda})$ is a continuous, strictly increasing function. Moreover, $\lim_{\mu\to 0^+} \lambda^*(\mu) = 0$ and $\lim_{\mu\to\infty} \lambda^*(\mu) = \bar{\lambda}$. It is easy to show that $\|u_{\lambda^*}\|_\infty$ $(= \|u_{\mu,\lambda^*}\|_\infty)$ is a continuous function of $\mu > 0$. By Lemma 3.9, we have that $\lim_{\mu\to 0^+} \|u_{\lambda^*}\|_\infty = \infty$ and $\lim_{\mu\to\infty} \|u_{\lambda^*}\|_\infty = 0$.

In particular, when $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p-1$, $r \equiv r_1 > q$, and $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$, we let $u = u_{\mu,\lambda}$ be a (classical) positive solution of (1.1), (1.3). Then the change of variables

$$u_{\mu,\lambda}(x) = \left(\frac{\lambda}{\mu}\right)^{1/(r-q)} v(\mu^{\frac{p-q-1}{p(r-q)}} \lambda^{\frac{r-p+1}{p(r-q)}} x)$$

transforms $u_{\mu,\lambda}$ into a solution v of

$$\begin{cases} (\varphi_p(v'(x)))' + v^q - v^r = 0, & -L < x < L, \\ v(-L) = v(L) = 0, \end{cases} \tag{4.3}$$

with

$$L \equiv \mu^{\frac{p-q-1}{p(r-q)}} \lambda^{\frac{r-p+1}{p(r-q)}}. \tag{4.4}$$

Cf. [5, p. 463]. For p-Laplacian problem (4.3) with $1 < p \leq 2$ and $0 < p-1 < q < r$, we define the time map formula as follows:

$$\tilde{T}(\alpha) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \frac{dv}{\left[\tilde{F}(\alpha) - \tilde{F}(v)\right]^{1/p}} \quad \text{for } 0 < \alpha < 1, \tag{4.5}$$

where $\tilde{F}(v) \equiv \int_0^v \tilde{f}(t)dt$ and $\tilde{f}(v) \equiv v^q - v^r$. By [14, Lemma 3.1], there exist two fixed positive numbers $\|v^*\|_\infty$ and L^* such that $\tilde{T}(\alpha)$ has exactly one critical point, an absolute minimum $\tilde{T}(\|v^*\|_\infty) = L^*$, on $(0,1)$. Thus, by (4.4),

$$L^* = \mu^{\frac{p-q-1}{p(r-q)}} (\lambda^*)^{\frac{r-p+1}{p(r-q)}},$$

and hence

$$\left(\frac{\lambda^*}{\mu}\right)^{1/(r-q)} = (L^*)^{\frac{p}{r-p+1}} \mu^{\frac{1}{p-1-r}}.$$

So we get that

$$\|u_{\lambda^*}\|_\infty = \left(\frac{\lambda^*}{\mu}\right)^{1/(r-q)} \|v^*\|_\infty = (L^*)^{\frac{p}{r-p+1}} \mu^{\frac{1}{p-1-r}} \|v^*\|_\infty.$$

Since $r > p-1$, $\|u_{\lambda^*}\|_\infty$ is a continuous, strictly decreasing function of $\mu > 0$.

(V) We prove part (v). We consider $0 < \alpha < \beta_{\mu,\lambda}$ and have that

$$\begin{aligned} T_{\mu=0,\lambda}(\alpha) &= \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha [\lambda(G(\alpha) - G(u))]^{-1/p} du \\ &< \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha [\lambda(G(\alpha) - G(u)) - \mu(\tilde{H}(\alpha) - \tilde{H}(u))]^{-1/p} du = T_{\mu,\lambda}(\alpha). \end{aligned}$$

By (3.4), (1.16) and (1.17), we obtain that

$$\lambda^{1/p} T_{\mu=0,\lambda}(\alpha) = T_{\bar{f}}(\alpha) = (c_{p,q}\alpha^{p-1-q})^{1/p}.$$

If $T_{\mu=0,\lambda}(\alpha) = 1$, then $\alpha = \left(\frac{c_{p,q}}{\lambda}\right)^{\frac{1}{q-p+1}}$. Then by (3.6) and Proposition 1.1(ii), we obtain that

$$0 < \left(\frac{c_{p,q}}{\lambda}\right)^{\frac{1}{q-p+1}} < \|u_\lambda\|_\infty < \beta_{\mu,\lambda}$$

since $T_{\bar{f}}(\alpha)$ is a strictly decreasing function on $(0,\infty)$ and $T_{\mu=0,\lambda}\left(\left(\frac{c_{p,q}}{\lambda}\right)^{\frac{1}{q-p+1}}\right) = 1$. So (2.1) holds.

The proof of Theorem 2.1 is now complete. □

Proof of Theorem 2.2. Let $p > 2$ and $k \geq 0$.

(I)(a) We have that, for any fixed $\mu > 0$, the bifurcation diagram $S_{p,k,\mu}$ consists of a continuous curve on the $(\lambda, \|u\|_\infty)$-plane. The proof is exactly the same as that given in part (I)(a) of the proof of Theorem 2.1 with $1 < p \leq 2$. So we omit it here.

(I)(b) Part (i) follows from Proposition 1.1(i).

(II)(a) We prove part (ii) where

$$g(u) = ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i} \quad \text{and} \quad h(u) = \mu \sum_{j=1}^{n} b_j u^{r_j} \quad (\text{with } \mu > 0)$$

satisfy (1.3). We prove that g, h satisfy (H1)–(H5). We first notice that the proofs of g, h satisfying (H1)–(H4) when $p > 2$ are exactly the same as those of g, h satisfying (H1)–(H4) when $1 < p \le 2$, given in part (II) of the proof of Theorem 2.1. So we omit them here. We then show that g, h satisfy (H5). We take the number $p^* = \frac{q_m + r_1}{2} > p - 1$ by (1.3). It is easy to see that

$$\frac{g(u)}{u^{p^*}} = ku^{p-1-p^*} + \sum_{i=1}^{m} a_i u^{q_i - p^*} \quad \text{is strictly decreasing on } (0, \infty)$$

and

$$\frac{h(u)}{u^{p^*}} = \mu \sum_{j=1}^{n} b_j u^{r_j - p^*} \quad \text{is strictly increasing on } (0, \infty).$$

For each fixed $s \in (0, 1)$, we compute that

$$\begin{aligned} \frac{h(su)}{u^{p-1}} \left[\frac{h(u)g(su)}{g(u)h(su)} - 1 \right] &= \frac{h(u)g(su) - h(su)g(u)}{u^{p-1} g(u)} \\ &= \frac{\mu \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j (s^{q_i} - s^{r_j}) u^{q_i + r_j} + \mu k u^{p-1} \sum_{j=1}^{n} b_j (s^{p-1} - s^{r_j}) u^{r_j}}{u^{p-1}(ku^{p-1} + \sum_{i=1}^{m} a_i u^{q_i})} \end{aligned}$$

which is a strictly increasing function of u on $(0, \infty)$ and satisfies that

$$\lim_{u \to \infty} \frac{h(u)g(su)}{g(u)h(su)} = s^{q_m - r_n} \in (1, \infty)$$

since $s \in (0, 1)$ and $q_m < r_n$. So g, h satisfy (H5). By above, we conclude that g, h satisfy (H1)–(H5). So part (ii) follows from Lemma 3.1(ii).

(II)(b) By Lemma 3.10(ii), we have that

$$\lim_{\mu \to 0^+} \lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) = 0, \qquad \lim_{\mu \to \infty} \lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) = \infty$$

and $\lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha)$ is a continuous, strictly increasing function of μ on $(0, \infty)$. So by the Intermediate Value Theorem, there exists a positive number $\hat{\mu}$ such that $\lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) < 1$ if $0 < \mu < \hat{\mu}$, $\lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) = 1$ if $\mu = \hat{\mu}$, and $\lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) > 1$ if $\mu > \hat{\mu}$.

For each fixed $k > 0$ and $\mu > 0$, $\lim_{\alpha \to \beta^-_{\mu,\tilde{\lambda}}} T_{\mu,\tilde{\lambda}}(\alpha) = 1$ by Lemma 3.1(ii) and $\lim_{\alpha \to \beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha)$ is a continuous, strictly decreasing function of λ on $(0, \infty)$ by Lemma 3.10(i). Hence we obtain that $\tilde{\lambda} < \bar{\lambda}$ if $0 < \mu < \hat{\mu}$, $\tilde{\lambda} = \bar{\lambda}$ if $\mu = \hat{\mu}$, and $\bar{\lambda} < \tilde{\lambda}$ if $\mu > \hat{\mu}$.

For each $\mu > 0$ and $k = 0$, we have $\bar{\lambda} = \infty$. By Lemma 3.10(i), $\lim_{\lambda \to \infty} \lim_{\alpha \to \beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha) = 0$. Hence $\lim_{\alpha \to \beta^-_{\mu,\bar{\lambda}}} T_{\mu,\bar{\lambda}}(\alpha) = 0$. Since $\lim_{\alpha \to \beta^-_{\mu,\tilde{\lambda}}} T_{\mu,\tilde{\lambda}}(\alpha) = 1$ by Lemma 3.1(ii) and $\lim_{\alpha \to \beta^-_{\mu,\lambda}} T_{\mu,\lambda}(\alpha)$ is a continuous, strictly decreasing function of λ on $(0, \infty)$ by Lemma 3.10(i), we obtain that $\tilde{\lambda} < \bar{\lambda}$ and $\hat{\mu} = \infty$.

(III) The proof of part (iii) of Theorem 2.2 is exactly the same as that of part (iii) of Theorem 2.1. So we omit it here.

(IV) We prove part (iv). By Lemma 3.12, $\tilde{\lambda}(\mu) : (0, \infty) \to (0, \infty)$ is a continuous, strictly increasing function. Moreover, $\lim_{\mu \to 0^+} \tilde{\lambda} = 0$ and $\lim_{\mu \to \infty} \tilde{\lambda} = \infty$. We let $u = u_{\mu,\lambda}$ be a (classical) positive solution of (1.1), (1.3). Then the change of variables

$$u_{\mu,\lambda}(x) = v(\mu^{\frac{1}{p}} x)$$

transforms $u_{\mu,\lambda}$ into a solution v of

$$\begin{cases} \left(\varphi_p(v'(x))\right)' + \frac{\lambda}{\mu}(kv^{p-1} + \sum_{i=1}^m a_i v^{q_i}) - \sum_{j=1}^n b_j v^{r_j}, & -L < x < L, \\ v(-L) = v(L) = 0, \end{cases} \tag{4.6}$$

with

$$L \equiv \mu^{\frac{1}{p}}.$$

Cf. [5, p. 463]. By Lemma 3.11(i), for any fixed $\mu \in (0, \infty)$, there exists a unique $\tilde{\lambda}(\mu) > 0$ such that $\lim_{\alpha \to \beta^-_{\mu,\tilde{\lambda}(\mu)}} T_{\mu,\tilde{\lambda}(\mu)}(\alpha) = 1$. Then there exists a unique $\eta = \frac{\tilde{\lambda}(\mu)}{\mu} > 0$ such that $\lim_{\alpha \to \beta^-_{\mu,\tilde{\lambda}(\mu)}} T_{1,\eta}(\alpha) = \mu^{\frac{1}{p}}$ and $\beta_{\mu,\tilde{\lambda}(\mu)} = \beta_{1,\eta}$. For $0 < \mu_1 < \mu_2$,

$$\lim_{\alpha \to \beta^-_{\mu_1,\tilde{\lambda}(\mu_1)}} T_{1,\eta_1}(\alpha) = \mu_1^{\frac{1}{p}} < \mu_2^{\frac{1}{p}} = \lim_{\alpha \to \beta^-_{\mu_2,\tilde{\lambda}(\mu_2)}} T_{1,\eta_2}(\alpha).$$

Hence $\eta_1 > \eta_2$ by Lemma 3.10(i). Similarly, for any fixed $\eta \in (0, \infty)$, there exists a unique $\mu > 0$ such that $\lim_{\alpha \to \beta^-_{1,\eta}} T_{1,\eta}(\alpha) = \mu^{\frac{1}{p}}$ and $\beta_{\mu,\tilde{\lambda}(\mu)} = \beta_{1,\eta}$. It is clear that $\beta_{1,\eta}$ is a continuous, strictly increasing function of η on $(0, \infty)$. Hence

$$\|v_{\tilde{\lambda}(\mu_1)}\|_\infty = \beta_{\mu_1,\tilde{\lambda}(\mu_1)} = \beta_{1,\eta_1} > \beta_{1,\eta_2} = \beta_{\mu_2,\tilde{\lambda}(\mu_2)} = \|v_{\tilde{\lambda}(\mu_2)}\|_\infty.$$

So $\|v_{\tilde{\lambda}}\|_\infty$ is a continuous, strictly decreasing function of $\mu > 0$. By Lemma 3.10(i), $\lim_{\mu \to 0^+} \eta(\mu) = \infty$ and $\lim_{\mu \to \infty} \eta(\mu) = 0$ since $\lim_{\alpha \to \beta^-_{1,\eta}} T_{1,\eta}(\alpha) = \mu^{\frac{1}{p}}$. Hence

$$\lim_{\mu \to 0^+} \|v_{\tilde{\lambda}(\mu)}\|_\infty = \lim_{\mu \to 0^+} \beta_{\mu,\tilde{\lambda}(\mu)} = \lim_{\eta \to \infty} \beta_{1,\eta} = \infty$$

and

$$\lim_{\mu \to \infty} \|v_{\tilde{\lambda}(\mu)}\|_\infty = \lim_{\mu \to \infty} \beta_{\mu,\tilde{\lambda}(\mu)} = \lim_{\eta \to 0^+} \beta_{1,\eta} = 0.$$

(V) The proof of part (v) of Theorem 2.2 is exactly the same as that of part (iv) of Theorem 2.1. So we omit it here.

(VI) The proof of part (vi) of Theorem 2.2 is exactly the same as that of part (v) of Theorem 2.1. So we omit it here.

The proof of Theorem 2.2 is now complete. □

5 A final remark

For evolutionary bifurcation diagram $S_{p,k,\mu}$ on the $(\lambda, \|u\|_\infty)$-plane studied in Theorems 2.1–2.2, analogically, we also study evolutionary bifurcation diagrams $\Sigma_{p,k,\lambda}$ on the $(\mu, \|u\|_\infty)$-plane defined by:

$$\Sigma_{p,k,\lambda} = \left\{ (\mu, \|u_\mu\|_\infty) : \mu > 0 \text{ and } u_\mu \text{ is a (classical) positive solution of (1.1), (1.3)} \right\}, \ \lambda > 0.$$

Applying Theorems 2.1–2.2 and by modified analytic techniques used in the proof of [7, Theorem 2.2], we obtain the following Theorem 5.1 and Fig. 5.1 with $1 < p \leq 2$ and Theorem 5.2 and Fig. 5.2 with $p > 2$ for evolutionary bifurcation diagrams $\Sigma_{p,k,\lambda}$ on the $(\mu, \|u\|_\infty)$-plane. We omit the proofs here.

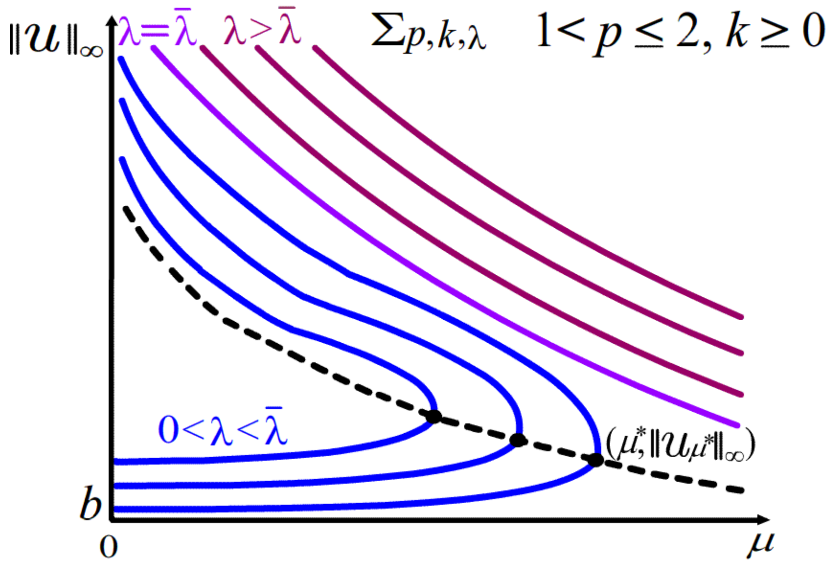

Figure 5.1: Evolutionary bifurcation diagrams $\Sigma_{p,k,\lambda}$ for (1.1), (1.3) with fixed $p \in (1, 2]$, $k \geq 0$ and varying $\lambda > 0$.

Theorem 5.1 (See Fig. 5.1)**.** *Let $1 < p \leq 2$ and $k \geq 0$. Consider p-Laplacian problem (1.1), (1.3) with varying $\lambda > 0$. Then the bifurcation diagram $\Sigma_{p,k,\lambda}$ consists of a continuous curve on the $(\mu, \|u\|_\infty)$-plane and the following assertions (i)–(iii) hold:*

(i) If

$$0 < \lambda < \bar{\lambda} = \left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p \begin{cases} < \infty & \text{if } k > 0, \\ = \infty & \text{if } k = 0, \end{cases}$$

then:

(a) $\Sigma_{p,k,\lambda}$ starts at some point $(0, b)$ where $b > 0$, tends to the positive $\|u\|_\infty$-axis as $\mu \to 0^+$, and is a reversed $\subset$-shaped curve with exactly one turning point at some point $(\mu^*, \|u_{\mu^*}\|_\infty)$ satisfying $\mu^* > 0$ and $\|u_{\mu^*}\|_\infty > b$. More precisely, problem (1.1), (1.3) has exactly two (classical) positive solutions u_μ, v_μ with $u_\mu < v_\mu$ for $0 < \mu < \mu^*$, exactly one (classical) positive solution u_{μ^*} for $\mu = \mu^*$, and no (classical) positive solution for $\mu > \mu^*$. In addition, $\lim_{\mu\to 0^+} \|u_\mu\|_\infty = b$ and $\lim_{\mu\to 0^+} \|v_\mu\|_\infty = \infty$.

(b) For the starting points $(0, b)$ of $\Sigma_{p,k,\lambda}$ with $0 < \lambda < \bar{\lambda}$, $b = b(\lambda)$ is a continuous, strictly decreasing function of $\lambda \in (0, \bar{\lambda})$, $\lim_{\lambda\to 0^+}(0, b) = (0, \infty)$ and $\lim_{\lambda\to\bar{\lambda}^-}(0, b) = (0, 0)$.

(c) For the turning points $(\mu^*, \|u_{\mu^*}\|_\infty)$ of $\Sigma_{p,k,\lambda}$ with $0 < \lambda < \bar{\lambda}$, μ^* is a continuous, strictly increasing function of $\lambda \in (0, \bar{\lambda})$, $\|u_{\mu^*}\|_\infty$ is a continuous function of $\lambda \in (0, \bar{\lambda})$,

$$\lim_{\lambda\to 0^+}(\mu^*, \|u_{\mu^*}\|_\infty) = (0, \infty) \quad \text{and} \quad \lim_{\lambda\to\bar{\lambda}^-}(\mu^*, \|u_{\mu^*}\|_\infty) = (\infty, 0).$$

In particular, when $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p - 1$, $r \equiv r_1 > q$, and $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$, then $\|u_{\mu^*}\|_\infty$ is a strictly decreasing function of $\lambda \in (0, \bar{\lambda})$.

(ii) *If $\lambda \geq \bar{\lambda}$, then $\Sigma_{p,k,\lambda}$ emanates from the positive $\|u\|_\infty$-axis as $\mu \to 0^+$, tends to the positive μ-axis as $\mu \to \infty$, and is a strictly monotone curve. More precisely, problem* (1.1), (1.3) *has exactly one (classical) positive solution for $\mu > 0$.*

(iii) *For any positive $\lambda_2 > \lambda_1$, Σ_{p,k,λ_2} lies on the right hand side of Σ_{p,k,λ_1}. (So Σ_{p,k,λ_1} and Σ_{p,k,λ_2} do not intersect.)*

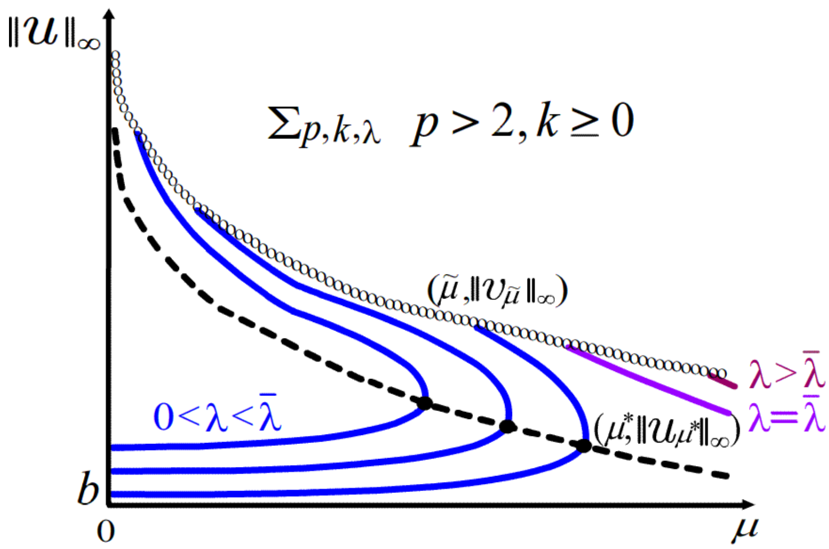

Figure 5.2: Evolutionary bifurcation diagrams $\Sigma_{p,k,\lambda}$ for (1.1), (1.3) with fixed $p > 2$, $k \geq 0$ and varying $\mu \geq 0$.

Theorem 5.2 (See Fig. 5.2)**.** *Let $p > 2$ and $k \geq 0$. Consider p-Laplacian problem (1.1), (1.3) with varying $\lambda > 0$. Then the bifurcation diagram $\Sigma_{p,k,\lambda}$ consists of a continuous curve on the $(\mu, \|u\|_\infty)$-plane and the following assertions (i)–(iv) hold:*

(i) *If*

$$0 < \lambda < \bar{\lambda} = \left(\frac{p-1}{k}\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p \begin{cases} < \infty & \text{if } k > 0, \\ = \infty & \text{if } k = 0, \end{cases}$$

then:

(a) *$\Sigma_{p,k,\lambda}$ starts at some point $(0, b)$ where $b > 0$, ends at some point $(\tilde{\mu}, \|v_{\tilde{\mu}}\|_\infty)$ satisfying $0 < \tilde{\mu} < \infty$ and $0 < \|v_{\tilde{\mu}}\|_\infty = v_{\tilde{\mu}}(0) = \beta_{\tilde{\mu},\lambda}$ satisfying $f_{k,\tilde{\mu},\lambda}(\beta_{\tilde{\mu},\lambda}) = 0$ (that is, $v_{\tilde{\mu}}(x) \equiv \lim_{\lambda \to \tilde{\lambda}^-} v_\lambda(x)$ is a* flat-core *positive solution of* (1.1), (1.3)*, see below for* (classical) *positive solutions $v_\lambda(x)$ with $\tilde{\mu} < \mu < \mu^*$). Moreover, $\Sigma_{p,k,\lambda}$ is a reverse $\subset$-shaped curve with exactly one turning point at some point $(\mu^*, \|u_{\mu^*}\|_\infty)$ satisfying*

$$0 < \tilde{\mu} < \mu^* \quad \text{and} \quad 0 < \|u_{\mu^*}\|_\infty < \|v_{\tilde{\mu}}\|_\infty = \beta_{\tilde{\mu},\lambda}.$$

More precisely, problem (1.1), (1.3) *has exactly two (classical) positive solutions u_μ, v_μ with $u_\mu < v_\mu$ for $\tilde{\mu} < \mu < \mu^*$, exactly one (classical) positive solution u_{μ^*} for $\mu = \mu^*$ and $0 < \mu \leq \tilde{\mu}$, and no (classical) positive solution for $\mu > \mu^*$. In addition,* $\lim_{\mu \to 0^+} \|u_\mu\|_\infty = b$ *and* $\lim_{\mu \to \tilde{\mu}^-} \|v_\mu\|_\infty = \|v_{\tilde{\mu}}\|_\infty = \beta_{\tilde{\mu},\lambda}$.

(b) *For the starting points* $(0, b)$ *of* $\Sigma_{p,k,\lambda}$ *with* $0 < \lambda < \bar{\lambda}$, $b = b(\lambda)$ *is a continuous, strictly decreasing function of* $\lambda \in (0, \bar{\lambda})$, $\lim_{\lambda \to 0^+}(0, b) = (0, \infty)$ *and* $\lim_{\lambda \to \bar{\lambda}^-}(0, b) = (0, 0)$.

(c) *For the turning points* $(\mu^*, \|u_{\mu^*}\|_\infty)$ *of* $\Sigma_{p,k,\lambda}$ *with* $0 < \lambda < \bar{\lambda}$, μ^* *is a continuous, strictly increasing function of* $\lambda \in (0, \bar{\lambda})$, $\|u_{\mu^*}\|_\infty$ *is a continuous function of* $\lambda \in (0, \bar{\lambda})$,

$$\lim_{\lambda \to 0^+} (\mu^*, \|u_{\mu^*}\|_\infty) = (0, \infty) \quad \textit{and} \quad \lim_{\lambda \to \bar{\lambda}^-} (\mu^*, \|u_{\mu^*}\|_\infty) = (\infty, 0).$$

In particular, when $k = 0$, $m = 1$, $n = 1$, $q \equiv q_1 > p - 1$, $r \equiv r_1 > q$, *and* $f_{k=0,\mu,\lambda}(u) = \lambda u^q - \mu u^r$, *then* $\|u_{\mu^*}\|_\infty$ *is a strictly decreasing function of* $\lambda \in (0, \bar{\lambda})$.

(ii) *If* $\lambda \geq \bar{\lambda}$, *then* $\Sigma_{p,k,\lambda}$ *emanates from the positive* μ*-axis as* $\mu \to \infty$, *and ends at some point* $(\tilde{\mu}, \|v_{\tilde{\mu}}\|_\infty)$ *in which* $v_{\tilde{\mu}}$ *is a* flat-core *positive solution. Moreover,* $\Sigma_{p,k,\lambda}$ *is a strictly monotone curve. More precisely, problem* (1.1), (1.3) *has exactly one (classical) positive solution for* $\mu > \tilde{\mu}$.

(iii) *For any positive* $\lambda_2 > \lambda_1$, Σ_{p,k,λ_2} *lies on the right hand side of* Σ_{p,k,λ_1}. *(So* Σ_{p,k,λ_1} *and* Σ_{p,k,λ_2} *do not intersect.)*

(iv) *For the ending points* $(\tilde{\mu}, \|v_{\tilde{\mu}}\|_\infty)$ *of* $\Sigma_{p,k,\lambda}$ *with* $\lambda > 0$, $\tilde{\mu}$ *is a continuous, strictly increasing function of* $\lambda > 0$, $\|v_{\tilde{\mu}}\|_\infty$ *is a continuous, strictly decreasing function of* $\lambda > 0$,

$$\lim_{\lambda \to 0^+} (\tilde{\mu}, \|v_{\tilde{\mu}}\|_\infty) = (0, \infty) \quad \textit{and} \quad \lim_{\lambda \to \infty} (\tilde{\mu}, \|v_{\tilde{\mu}}\|_\infty) = (\infty, 0).$$

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Existence of two infinite families of solutions to a singular superlinear equation on exterior domains

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Abstract. We are concerned with the radial solutions of the Dirichlet problem $-\Delta u = K(|x|)f(u)$ on the exterior of the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ with $N \geq 3$ where f is superlinear at ∞ and has a singularity at 0 with $f(u) \sim \frac{1}{|u|^{q-1}u}$ and $0 < q < 1$ for small u. We prove that if $K(|x|) \sim |x|^{-\alpha}$ with $\alpha > 2(N-1)$ then there exist two infinite families of sign-changing radial solutions.

Keywords: exterior domains, singular, superlinear, radial solution.

2020 Mathematics Subject Classification: 34B40, 35B05.

1 Introduction

In this paper we study the radial solutions of

$$-\Delta u = K(|x|)f(u) \text{ on } \mathbb{R}^N \backslash B_R(0) \tag{1.1}$$

$$u(x) = 0 \text{ on } \partial B_R(0), \quad \lim_{|x|\to\infty} u(x) = 0 \tag{1.2}$$

where $\Delta : C^k(\mathbb{R}^N) \to C^{k-2}(\mathbb{R}^N)$ denotes the N-dimensional Laplacian, $B_R(0)$ denotes the unit ball centered at the origin, $|x|$ denotes the Euclidean distance of x, and $u : \mathbb{R}^N \to \mathbb{R}$ with $N \geq 3$.

Numerous papers have proved the existence of *positive* solutions of these equations with $K(|x|) = 1$. See for example [4, 5, 10]. In [10], Miyamoto and Naito studied the problem in the domain $B_R(0) \setminus \{0\}$. Some other papers have dealt with the *positive* solutions of these equations with various nonlinearities $f(u)$ and $K(|x|) \sim |x|^{-\alpha}$ with $\alpha > 0$. (See [1, 9, 11]).

We prove the existence of sign-changing solutions of (1.1)–(1.2) and analyze their properties. The papers [2, 3, 7, 8] examined the case where the non-linear function $f(u)$ in (1.1) has a unique positive zero. We choose a superlinear function $f(u)$ that has no positive zeros.

Our study of the solutions of (1.1)–(1.2) is based on the following assumptions:

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(H1) $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is odd, locally Lipschitz, and $f > 0$ on $(0, \infty)$. (So, by the symmetry of f about the origin, $f < 0$ on $(-\infty, 0)$),

(H2) $f(u) = |u|^{p-1}u + g(u)$ with $p > 1$ for large u and $\lim_{u\to\infty} \frac{|g(u)|}{|u|^p} = 0$,

(H3) there exists a locally Lipschitz function $g_1 : \mathbb{R} \to \mathbb{R}$ such that $f(u) = \frac{1}{|u|^{q-1}u} + g_1(u)$ with $0 < q < 1$ and $g_1(0) = 0$,

(H4) $K(r)$, $K'(r)$ are continuous on $[R, \infty)$ with $K(r) > 0$ such that $2(N-1) + \frac{rK'}{K} < 0$ on $[R, \infty)$,

(H5) there exist a constant $k_0 > 0$ and $\alpha > 2(N-1)$ such that $\frac{k_0}{r^\alpha} \le K(r)$ on $[R, \infty)$.

Let $F(u) = \int_0^u f(t)\, dt$. From *(H3)* it follows that f is integrable at 0 and therefore F is continuous with $F(0) = 0$. Also, since f is odd and $f > 0$ on $(0, \infty)$, it follows that F is even and $F(u) > 0$ for $u \neq 0$.

Since we are studying the radial solutions of (1.1)–(1.2), we let $u(x) = u(|x|) = u(r)$ where $r = |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$. Denoting $\frac{\partial u}{\partial r}$ by u' and $\frac{\partial^2 u}{\partial r^2}$ by u'' then (1.1)–(1.2) becomes:

$$u''(r) + \frac{N-1}{r}u'(r) + K(r)f(u) = 0 \quad \text{for } R < r < \infty, \tag{1.3}$$

$$u(R) = 0, \quad \lim_{r\to\infty} u(r) = 0. \tag{1.4}$$

In this paper we prove the following:

Theorem 1.1. *Assume (H1)–(H5) hold and $N \ge 3$. There exist two infinite families of non-trivial radial solutions of (1.3)–(1.4). In addition, $\exists n_0 \ge 0$ such that for every $n \ge n_0$ then there are at least two solutions of (1.3)–(1.4) with exactly n zeros on (R, ∞).*

2 Preliminaries and behavior for large a

We prove the existence of a solution of (1.3)–(1.4) with

$$u(R) = 0, \quad u'(R) = a > 0 \tag{2.1}$$

on $[R, R+\epsilon)$ for some $\epsilon > 0$. We denote $u(r)$ by $u_a(r)$ to emphasize the dependence of u on the initial parameter a. We begin first by making the following change of variables

$$u_a(r) = v_a(r^{2-N}).$$

Let $r^{2-N} = t$ and denote R^{2-N} by R^*. We observe then that solving (1.3), (2.1) is equivalent to solving the following initial value problem

$$v_a'' + h(t)f(v_a) = 0 \quad \text{on } (0, R^*) \tag{2.2}$$

$$v_a(R^*) = 0, \quad v_a'(R^*) = -\frac{aR^{N-1}}{N-2} < 0 \tag{2.3}$$

where $h(t) = \frac{t^{\frac{2(N-1)}{2-N}} K(t^{\frac{1}{2-N}})}{(N-2)^2}$. We will then try to find values of a such that $v_a(0) = 0$. From $(H4)$, $(H5)$, and the definition of $h(t)$ it follows that

$$\begin{gathered} h(t) > 0,\ h'(t) > 0 \quad \text{on } (0, R^*] \\ \text{and } \exists h_1 > 0 \text{ such that } h_1 t^{\tilde{\alpha}} \leq h(t) \text{ on } (0, R^*] \text{ where } \tilde{\alpha} = \frac{\alpha - 2(N-1)}{N-2} > 0. \end{gathered} \tag{2.4}$$

We first prove the existence of a solution for (2.2)–(2.3) on $[R^* - \epsilon, R^*]$ for some $\epsilon > 0$. To do this, we transform this equation into an integral equation and use the contraction mapping principle to solve it. Let $t > 0$ and let v_a be a solution of (2.2)–(2.3). By integrating (2.2) over (t, R^*) and using (2.3) we obtain

$$v_a'(t) = -\frac{aR^{N-1}}{N-2} + \int_t^{R^*} h(x) f\left(v_a(x)\right) dx. \tag{2.5}$$

Now integrate (2.5) over (t, R^*) and use (2.3). This gives

$$v_a(t) = \frac{aR^{N-1}}{N-2}(R^* - t) - \int_t^{R^*} \left(\int_s^{R^*} h(x) f(v_a(x))\, dx \right) ds. \tag{2.6}$$

Letting $v_a(t) = (R^* - t)y(t)$ and $y(R^*) \equiv \lim_{t \to R^{*-}} \frac{v_a(t)}{R^* - t} = -v_a'(R^*) = \frac{aR^{N-1}}{N-2}$, we can rewrite the equation (2.6) in terms of $y(t)$ as

$$y(t) = \frac{aR^{N-1}}{N-2} - \frac{1}{R^* - t} \int_t^{R^*} \left(\int_s^{R^*} h(x) f\left((R^* - x)y(x)\right)\, dx \right) ds. \tag{2.7}$$

We now solve (2.7) by defining an operator on an appropriate space and showing that it has a fixed point. For this, let $a > 0$ and consider the Banach space

$$X = \left\{ y \in C[R^* - \epsilon, R^*] : y(R^*) = \frac{aR^{N-1}}{N-2}, \left| y(t) - \frac{aR^{N-1}}{N-2} \right| \leq \frac{aR^{N-1}}{2(N-2)} \text{ on } [R^* - \epsilon, R^*] \right\}$$

equipped with the supremum norm defined by

$$\|y\| = \sup_{x \in [R^* - \epsilon, R^*]} |y(x)|.$$

We define a map $T : X \to C[R^* - \epsilon, R^*]$ by

$$(Ty)(t) = \frac{aR^{N-1}}{N-2} - \frac{1}{R^* - t} \int_t^{R^*} \left(\int_s^{R^*} h(x) f\left((R^* - x)y(x)\right) dx \right) ds \quad \text{for } R^* - \epsilon \leq t < R^* \tag{2.8}$$

and $T(R^*) = \frac{aR^{N-1}}{N-2}$. Since $f = \frac{1}{|u|^{q-1}u} + g_1(u)$ by $(H3)$, we have from (2.8) that

$$(Ty)(t) = \frac{aR^{N-1}}{N-2} - \frac{1}{R^* - t} \int_t^{R^*} \left(\int_s^{R^*} h(x) \left(\frac{1}{(R^* - x)^q y^q(x)} + g_1\left((R^* - x)y(x)\right) \right) dx \right) ds. \tag{2.9}$$

Since $0 < q < 1$ by $(H3)$, it follows that $\frac{1}{(R^* - x)^q}$ is integrable on $[0, R^*]$. Using this fact together with that g_1 is locally Lipschitz, it can be shown that T is a contraction mapping from X into itself for sufficiently small ϵ (the details are carried out in [3]). Thus by the contraction mapping principle [6], there exists a unique element $y \in X$ such that $Ty = y$ on

$[R^* - \epsilon, R^*]$. Hence, we obtain a solution $v_a(t) = (R^* - t)y(t)$ of (2.2)–(2.3) on $[R^* - \epsilon, R^*]$ if $a > 0$ and $\epsilon > 0$ is sufficiently small.

Next let $(R_1, R^*]$ be the maximal half-open interval of existence of the solution to (2.2)–(2.3). Now we define the energy of the solution

$$E_a = \frac{1}{2}\frac{v_a'^2}{h(t)} + F(v_a) \quad \text{for } R_1 < t \le R^*. \tag{2.10}$$

Then it follows from (2.2) and (2.4) that

$$E_a' = -\frac{v_a'^2 h'}{2h^2} \le 0 \quad \text{on } (R_1, R^*]. \tag{2.11}$$

Thus, E_a is non-increasing on $(R_1, R^*]$ and hence for $R_1 < t \le R^*$ we have

$$0 < \frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2 h(R^*)} = \frac{1}{2}\frac{v_a'^2(R^*)}{h(R^*)} = E_a(R^*) \le E_a = \frac{1}{2}\frac{v_a'^2}{h(t)} + F(v_a) \quad \text{on } (R_1, R^*]. \tag{2.12}$$

So $E_a > 0$ on $(R_1, R^*]$.

We next claim that the solution of (2.2)–(2.3) exists on $[0, R^*]$ and analyze the properties of the solution in several lemmas.

Lemma 2.1. *Assume (H1)–(H5) hold, $N \ge 3$ and $a > 0$. Let v_a be the solution of (2.2)–(2.3). Then v_a can be extended to the maximal interval $[0, R^*]$.*

Proof. Let v_a be the unique solution of (2.2)–(2.3) on the maximal half-open interval of existence $(R_1, R^*]$. We show that $R_1 = 0$. Suppose on the contrary that $R_1 > 0$. Using (2.2), (2.4) and that $F(v_a) \ge 0$ we obtain

$$\left(\frac{1}{2}v_a'^2 + h(t)F(v_a)\right)' = h'(t)F(v_a) \ge 0 \quad \text{on } (R_1, R^*]. \tag{2.13}$$

Let $0 < t < R_1$. Now by integrating (2.13) over (t, R^*), using (2.3) and that $h(t) > 0$, $F(v_a) \ge 0$ we obtain

$$\frac{1}{2}v_a'^2 \le \frac{1}{2}v_a'^2 + h(t)F(v_a) \le \frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2} \quad \text{on } (R_1, R^*]. \tag{2.14}$$

Therefore,

$$|v_a'| \le \frac{aR^{N-1}}{N-2} \quad \text{on } (R_1, R^*]. \tag{2.15}$$

Also, we have

$$|v_a| = \left|\int_t^{R^*} v_a' \, ds\right| \le \int_t^{R^*} |v_a'| \, ds \le \frac{aR^{N-1}}{N-2}(R^* - t) \le \frac{aR^{N-1}}{N-2}R^* = \frac{aR}{N-2} \quad \text{on } (R_1, R^*]. \tag{2.16}$$

Now let $(t_n) \subset (R_1, R^*]$ such that $t_n \to R_1^+$. Then by the mean value theorem and (2.15) we obtain

$$|v_a(t_n) - v_a(t_m)| = |v_a'(c_{n,m})||t_n - t_m| \le \frac{aR^{N-1}}{N-2}|t_n - t_m| \to 0 \quad \text{as } m, n \to \infty.$$

This shows that $(v_a(t_n))$ is a Cauchy sequence on $(R_1, R^*]$ and so $\exists L \in \mathbb{R}$ such that $\lim_{t\to R_1^+} v_a(t) = L$. Also since $h(t)F(v_a)$ and $h'(t)F(v_a)$ are continuous on $(R_1, R^*]$, integrating (2.13) on (t, R^*) we see that $\lim_{t\to R_1^+} v_a'(t) = L_1$ exists. From (2.12) we see $0 < E_a \le \frac{1}{2}\frac{L_1^2}{h(R^*)} + F(L)$ on $(R_1, R^*]$ which shows that L and L_1 cannot both be zero. Now if $L = 0$ then $L_1 \neq 0$ and we can use the contraction mapping principle as we did earlier to extend our solution to $(R_1 - \delta, R^*]$ for some $\delta > 0$. On the other hand, if $L \neq 0$, then we can use the standard existence theorem for ordinary differential equations to obtain a solution on $(R_1 - \delta, R^*]$ for some $\delta > 0$. Therefore in both cases the solution of (2.2)-(2.3) can be extended to $(R_1 - \delta, R^*]$ for some $\delta > 0$, contradicting the maximality of $(R_1, R^*]$. Hence $R_1 = 0$. It then follows from (2.15) and (2.16) that v_a and v_a' are bounded on $(0, R^*]$ and so in a similar way to earlier we see that the limits $\lim_{t\to 0^+} v_a(t)$ and $\lim_{t\to 0^+} v_a'(t)$ exist. Thus v_a and v_a' are defined and continuous $[0, R^*]$. □

Remark 2.2. If v_a solves (2.2)–(2.3) and $z \in (0, R^*)$ is such that $v_a(z) = 0$ then by (2.12), $0 < E_a(z_a) = \frac{1}{2}\frac{v_a'^2(z)}{h(z)}$ and hence $v_a'(z) \neq 0$. Thus the zeros of v_a on $(0, R^*)$ are simple. Also, since $\lim_{u\to 0}|f(u)| = \infty$, by (H3) it follows that the solution to (2.2)–(2.3) is twice differentiable except at points where $v_a(t_0) = 0$. Therefore, by a solution v_a of (2.2)–(2.3) we mean a continuously differentiable function v_a on $[0, R^*]$ that satisfies the equation (2.6) with (2.3).

Lemma 2.3. *Assume (H1)–(H5) hold, $N \ge 3$ and $a > 0$. Let v_a solve (2.2)–(2.3) on $[0, R^*]$. Then v_a depends continuously on the initial parameter a on $[0, R^*]$.*

Proof. Let $0 < a_1 < a < a_2$. Then from (2.15) we have

$$|v_a'| \le \frac{aR^{N-1}}{N-2} \le a_2 c_1 \quad \text{for all } a \text{ such that } 0 < a_1 \le a \le a_2 \tag{2.17}$$

where $c_1 = \frac{R^{N-1}}{N-2}$. And from (2.16) we have

$$|v_a| = \frac{aR}{N-2} \le a_2 c_2 \quad \text{for all } a \text{ such that } 0 < a_1 \le a \le a_2 \tag{2.18}$$

where $c_2 = \frac{R}{N-2}$. Thus, (2.17) and (2.18) show that the upper bounds for $|v_a|, |v_a'|$ can be chosen to be independent of a on $[0, R^*]$ for all a such that $0 < a_1 \le a \le a_2$.

Now let $\tilde{a} > 0$ and suppose $a \to \tilde{a}$. Then, we want to show that $v_a \to v_{\tilde{a}}$ uniformly on $[0, R^*]$. Suppose on the contrary, that there is a subsequence $(a_j) \subset \mathbb{R}$ such that $a_j \to \tilde{a}$ as $j \to \infty$ and $\epsilon_0 > 0$ such that

$$|v_{a_j}(t_j) - v_{\tilde{a}}(t_j)| \ge \epsilon_0 \quad \text{for some sequence } t_j \in [0, R^*]. \tag{2.19}$$

Since $a_j \to \tilde{a}$, there exists $N_0 \in \mathbb{N}$ such that for all $j \ge N_0$ $|a_j| \le \tilde{a} + 1$. From (2.15) and (2.16) we know that v_a and v_a' are uniformly bounded on the compact domain $[0, R^*]$. Hence, by the Arzelà–Ascoli theorem, there exists a subsequence $(v_{a_{j_k}}) \subset (v_{a_j})$ such that $v_{a_{j_k}} \to v_{\tilde{a}}$ uniformly on $[0, R^*]$ as $k \to \infty$. Therefore, as $k \to \infty$ from (2.19) we obtain

$$0 \leftarrow |v_{a_{j_k}}(t_{j_k}) - v_{\tilde{a}}(t_{j_k})| \ge \epsilon_0$$

which is a contradiction. Thus, $v_a \to v_{\tilde{a}}$ uniformly on $[0, R^*]$ and this completes the proof of the lemma. □

Lemma 2.4. *Assume (H1)–(H5) hold and $N \geq 3$. If $a > 0$ and v_a is a solution of (2.2)–(2.3), then v_a has at most finitely many zeros on $(0, R^*)$.*

Proof. Suppose on the contrary that $\exists$ a sequence $(z_{k,a}) \subset (0, R^*)$ with $0 < \cdots < z_{2,a} < z_{1,a}$ such that $v_a(z_{k,a}) = 0$. Then $z_{k,a}$ converges to some z_a^* on $[0, R^*]$. Since v_a has infinitely many zeros, $z_{k,a}$, and $v_a'(z_{k,a}) \neq 0$ by the Remark 2.2, it follows that v_a has infinitely many local extrema, $\{M_{k,a}\}_{k=1}^{\infty}$, with $z_{k+1,a} < M_{k,a} < z_{k,a}$ and so $\lim_{k\to\infty} M_{k,a} = z_a^*$. Since $E_a(t) > 0$ on $(0, R^*]$ and E is non-increasing by (2.12) we have $F(v_a(M_{k,a})) = E_a(M_{k,a}) \geq \frac{1}{2}\frac{a^2R^{2(N-1)}}{(N-2)^2h(R^*)} > 0$. So $\exists\beta_a > 0$ such that $|v_a(M_{k,a})| \geq \beta_a$ for all k. Now by the mean value theorem and (2.15) $\exists t_{k,a} \in (M_{k,a}, z_{k,a})$ such that

$$0 < \beta_a \leq |v_a(M_{k,a})| = |v_a(M_{k,a}) - v_a(z_{k,a})| = |v_a'(t_{k,a})||M_{k,a} - z_{k,a}| \leq \frac{aR^{N-1}}{(N-2)}|M_{k,a} - z_{k,a}|. \quad (2.20)$$

Since $M_{k,a} \to z_a^*$ and $z_{k,a} \to z_a^*$ as $k \to \infty$, the right-hand side of (2.20) goes to 0 as $k \to \infty$ which gives a contradiction. Therefore v_a has at most finitely many zeros on $(0, R^*)$ for $a > 0$. □

Lemma 2.5. *Assume (H1)–(H5) hold, $N \geq 3$ and let v_a solve (2.5). Then for $a > 0$ sufficiently large v_a has a local maximum, M_a. In addition, $v_a(M_a) \to \infty$ and $M_a \to R^*$ as $a \to \infty$.*

Proof. First we show for any $0 \leq t_0 < R^*$ that $\max_{[t_0,R^*)} |v_a(t)| \to \infty$ as $a \to \infty$.

If v_a has a local maximum $M_a \in [t_0, R^*)$, then $v_a'(M_a) = 0$. So, by letting $t = M_a$ in (2.12) we obtain

$$F(v_a(M_a)) \geq \frac{1}{2}\frac{a^2R^{2(N-1)}}{h(R^*)(N-2)^2}. \quad (2.21)$$

Since $h(R^*) > 0$, it follows that the right-hand side of (2.21) approaches infinity as $a \to \infty$ and hence from the definition of F we see that

$$v_a(M_a) \to \infty \quad \text{as } a \to \infty. \quad (2.22)$$

On the other hand, if v_a has no local maximum on (t_0, R^*) then v_a is decreasing on (t_0, R^*). We want to show that $\max_{[t_0,R^*)} |v_a(t)| \to \infty$ as $a \to \infty$. Suppose on the contrary that this is false. Then there exists a constant $c_3 > 0$ independent of a such that $|v_a(t)| \leq c_3$ on $[t_0, R^*]$. Then by the continuity of F there exists $c_4 > 0$ such that $F(v_a(t)) \leq c_4$. Using this and (2.3), it follows from (2.12) that

$$\frac{1}{2}\frac{v_a'^2(t)}{h(t)} + c_4 \geq \frac{1}{2}\frac{v_a'^2(t)}{h(t)} + F(v_a(t)) \geq \frac{1}{2}\frac{v_a'^2(R^*)}{h(R^*)} = \frac{1}{2}a^2c_5^2 \quad \text{on } [t_0, R^*] \quad (2.23)$$

where $c_5 = \frac{R^{N-1}}{(N-2)\sqrt{h(R^*)}}$. Rewriting (2.23) we obtain

$$|v_a'(t)| \geq \sqrt{a^2c_5^2 - 2c_4}\,\sqrt{h(t)}. \quad (2.24)$$

By (2.4) there exists $h_1 > 0$ such that $h(t) \geq h_1 t^{\tilde{\alpha}}$ on $[t_0, R^*]$. By using this and choosing a sufficiently large we can ensure that

$$|v_a'(t)| \geq \frac{ac_5}{2}\sqrt{h(t)} \geq \frac{ac_5}{2}\sqrt{h_1}t^{\frac{\tilde{\alpha}}{2}}. \quad (2.25)$$

Since v_a is decreasing, then by (2.25) we have $v_a' < 0$ on $[t_0, R^*]$. Now integrating (2.25) over (t_0, R^*) yields

$$c_3 \geq v_a(t_0) = \int_{t_0}^{R^*} -v_a'(t)\, dt \geq \frac{ac_5}{2}\sqrt{h_1}\int_{t_0}^{R^*} t^{\frac{\tilde{\alpha}}{2}}\, dt = \frac{ac_5}{2}\sqrt{h_1}\left(\frac{(R^*)^{\frac{\tilde{\alpha}}{2}+1} - t_0^{\frac{\tilde{\alpha}}{2}+1}}{\tilde{\alpha}+2}\right). \tag{2.26}$$

The left hand side of (2.26) is a constant while the right-hand side approaches ∞ as $a \to \infty$ which is a contradiction. Thus we conclude that for any $t_0 \in [0, R^*)$

$$\max_{[t_0,R^*)} |v_a(t)| \to \infty \quad \text{as } a \to \infty. \tag{2.27}$$

We claim next that v_a has a local max, M_a, and $\frac{1}{2}R^* < M_a < R^*$ if a is sufficiently large. Suppose on the contrary that v_a is decreasing on $[\frac{1}{2}R^*, R^*]$. Let

$$C_a = \frac{1}{2}\min_{[\frac{1}{2}R^*,\frac{3}{4}R^*]} \frac{h(t)f(v_a)}{v_a}. \tag{2.28}$$

By letting $t_0 = \frac{3}{4}R^*$ in (2.27), we obtain $v_a(\frac{3}{4}R^*) \to \infty$ as $a \to \infty$. Since v_a is decreasing on the interval $[\frac{1}{2}R^*, \frac{3}{4}R^*]$ we see that $v_a \to \infty$ uniformly as $a \to \infty$ on the interval $[\frac{1}{2}R^*, \frac{3}{4}R^*]$. By (2.4) $h_1 t^{\tilde{\alpha}} \leq h(t)$ on $(0, R^*]$ for some constant $h_1 > 0$ from which it follows that $h(t)$ is bounded from below on $[\frac{1}{2}R^*, \frac{3}{4}R^*]$. Also we have $f(v_a) = |v_a|^{p-1}v_a + g(v_a)$ by $(H2)$ and so it follows that if v_a is large then $f(v_a) \geq \frac{1}{2}v_a^p$. It then follows from this that $\frac{f(v_a)}{v_a} \geq \frac{1}{2}v_a^{p-1}(t) \geq \frac{1}{2}v_a^{p-1}\left(\frac{3}{4}R^*\right)$ on $[\frac{1}{2}R^*, \frac{3}{4}R^*]$. Since $p-1 > 0$ and $v_a(\frac{3}{4}R^*) \to \infty$ as $a \to \infty$, then we see $\frac{f(v_a)}{v_a} \to \infty$ on $[\frac{1}{2}R^*, \frac{3}{4}R^*]$ as $a \to \infty$. And since h is bounded from below on $[\frac{1}{2}R^*, \frac{3}{4}R^*]$, it follows from this and (2.28) that

$$C_a \to \infty \quad \text{as } a \to \infty.$$

Now we consider the differential equation

$$w_a'' + C_a w_a = 0 \tag{2.29}$$

with

$$\begin{aligned} w_a\left(\frac{3}{4}R^*\right) &= v_a\left(\frac{3}{4}R^*\right) > 0, \\ w_a'\left(\frac{3}{4}R^*\right) &= v_a'\left(\frac{3}{4}R^*\right) < 0. \end{aligned} \tag{2.30}$$

Clearly, $\{\cos\sqrt{C_a}(t - \frac{3}{4}R^*), \sin\sqrt{C_a}(t - \frac{3}{4}R^*)\}$ is a fundamental set of solutions of (2.29). So, $w_a = \alpha_1 \cos\sqrt{C_a}(t - \frac{3}{4}R^*) + \alpha_2 \sin\sqrt{C_a}(t - \frac{3}{4}R^*)$ for some constants α_1 and α_2. We also know that the distance between two consecutive zeros of w_a is $\frac{\pi}{\sqrt{C_a}} \to 0$ as $a \to \infty$. So, for $a > 0$ sufficiently large we have $\frac{1}{2}R^* < \frac{3}{4}R^* - \frac{\pi}{\sqrt{C_a}}$. Therefore, for $a > 0$ sufficiently large w_a has a zero on $[\frac{1}{2}R^*, \frac{3}{4}R^*]$ and hence has a local maximum $\tilde{M}$ on this interval with $w_a' < 0$ on $(\tilde{M}, \frac{3}{4}R^*]$.

Next, we rewrite equation (2.2) and consider

$$v_a'' + \left(\frac{h(t)f(v_a)}{v_a}\right)v_a = 0. \tag{2.31}$$

Multiplying (2.29) by v_a, (2.31) by w_a, and subtracting we obtain

$$(w_a' v_a - w_a v_a')' + \left(C_a - \frac{h(t)f(v_a)}{v_a} \right) w_a v_a = 0.$$

Integrating this on $(\tilde{M}, \frac{3}{4}R^*)$ and using (2.30) gives

$$w_a(\tilde{M}) v_a'(\tilde{M}) = \int_{\tilde{M}}^{\frac{3}{4}R^*} \left(\frac{h(t)f(v_a)}{v_a} - C_a \right) w_a v_a \, dt. \tag{2.32}$$

Since $w_a(\tilde{M}) > 0$, $C_a < \frac{h(t)f(v_a)}{v_a}$ on $[0, \frac{3}{4}R^*]$, and w_a, v_a stay positive on $[\tilde{M}, \frac{3}{4}R^*]$ it follows from (2.32) that $v_a'(\tilde{M}) > 0$, contradicting our assumption that v_a is decreasing on $[\frac{1}{2}R^*, R^*]$. Thus v_a has a local maximum, M_a, and $\frac{1}{2}R^* < M_a < R^*$ with v_a decreasing on $[M_a, R^*]$ for $a > 0$ sufficiently large. It also follows immediately from (2.22) that $v_a(M_a) \to \infty$ as $a \to \infty$.

Next we show that $M_a \to R^*$ as $a \to \infty$. Since v_a is decreasing on $[M_a, R^*)$ and $v_a(R^*) = 0$ so we see $v_a > 0$ on $[M_a, R^*)$. But then from (2.2) we know $v_a'' = -h(t)f(v_a) < 0$ on $[M_a, R^*)$ and so v_a is concave down on $[M_a, R^*)$. This implies

$$v_a \left(\lambda M_a + (1-\lambda) R^* \right) \geq \lambda v_a(M_a) + (1-\lambda) v_a(R^*) \quad \text{for } 0 \leq \lambda \leq 1.$$

So by letting $\lambda = \frac{1}{2}$ we obtain

$$v_a \left(\frac{M_a + R^*}{2} \right) \geq \frac{v_a(M_a) + v_a(R^*)}{2} = \frac{v_a(M_a)}{2} \to \infty \quad \text{as } a \to \infty. \tag{2.33}$$

By the superlinearity of f it follows that $f(v_a(t)) \geq \frac{1}{2} v_a^p(t)$ on $[M_a, \frac{M_a+R^*}{2}]$ if a is sufficiently large. By using this in (2.2) we obtain

$$v_a'' = -h(t)f(v_a(t)) \leq -\frac{1}{2} v_a^p(t).$$

Now integrating this on $[M_a, t]$ where $M_a \leq t \leq \frac{M_a+R^*}{2}$ and recalling that M_a is a local maximum of v_a with v_a decreasing on $[M_a, R^*]$ yields

$$v_a'(t) \leq -\frac{1}{2} \int_{M_a}^{t} v_a^p(x) \, dx \leq -\frac{1}{2} v_a^p(t) \int_{M_a}^{t} h(x) \, dx.$$

Rewriting the above gives

$$\frac{-v_a'}{v_a^p} \geq \frac{1}{2} \int_{M_a}^{t} h(x) \, dx.$$

Integrating again on (M_a, t) gives,

$$\frac{1}{(p-1)v_a^{p-1}(t)} \geq \frac{1}{p-1} [v_a^{1-p}(t) - v_a^{1-p}(M_a)] \geq \frac{1}{2} \int_{M_a}^{t} \int_{M_a}^{s} h(x) \, dx \, ds.$$

Evaluating at $t = \frac{M_a+R^*}{2}$ we obtain

$$\frac{1}{(p-1)v_a^{p-1}(\frac{M_a+R^*}{2})} \geq \frac{1}{2} \int_{M_a}^{\frac{M_a+R^*}{2}} \int_{M_a}^{s} h(x) \, dx \, ds. \tag{2.34}$$

Since $p - 1 > 0$, it follows from (2.33) that the left-hand side of (2.34) goes to zero as $a \to \infty$. Thus, since $h(x) > 0$ and h is continuous on $[M_a, R^*]$, it follows from (2.34) that $M_a \to R^*$ as $a \to \infty$. This completes the lemma. □

Lemma 2.6. *Assume (H1)–(H5) hold, $N \geq 3$ and let v_a solve (2.5). Then for $a > 0$ sufficiently large v_a has a zero, z_a, with $0 < z_a < M_a < R^*$ where $z_a \to R^*$ and $|v_a'(z_a)| \to \infty$ as $a \to \infty$. In addition, if a is sufficiently large and $n \geq 1$, then v_a has n zeros on $(0, R^*)$.*

Proof. First we show that $\exists z_a \in (0, M_a)$ such that $v_a(z_a) = 0$. Suppose on the contrary that v_a stays positive on $(0, M_a)$. We note that v_a cannot have a positive critical point on $(0, M_a)$. If it has a positive critical point c_a with $v_a' > 0$ on (c_a, M_a), then $v_a(c_a) > 0$ and $v_a''(c_a) \geq 0$. So by (2.2) $f(v_a(c_a)) \leq 0$ but then $v_a(c_a) \leq 0$ contradicting that $v_a > 0$ on $(0, M_a)$. Thus v_a is increasing on $(0, R^*)$. Next recall from (2.11) that $E_a' \leq 0$ on $(0, R^*]$. So we have

$$\frac{1}{2}\frac{v_a'^2}{h(t)} + F(v_a) \geq F(v_a(M_a)) \quad \text{on } (0, M_a]. \tag{2.35}$$

Rewriting (2.35) and integrating on $(0, M_a)$ by making the change of variable $s = v_a(t)$ gives

$$\begin{aligned} \int_0^{M_a} \sqrt{2h(t)}\, dt \leq \int_0^{M_a} \frac{v_a'(t)\, dt}{\sqrt{F(V_a(M_a) - F(v_a(t))}} &= \int_{v_a(0)}^{v_a(M_a)} \frac{ds}{\sqrt{F(v_a(M_a)) - F(s)}} \\ &\leq \int_0^{v_a(M_a)} \frac{ds}{\sqrt{F(v_a(M_a)) - F(s)}}. \end{aligned} \tag{2.36}$$

We now estimate the integral on the right-hand side of (2.36). Letting $s = v_a(M_a)x$, we obtain

$$\int_0^{v_a(M_a)} \frac{ds}{\sqrt{F(v_a(M_a)) - F(s)}} = \frac{v_a(M_a)}{\sqrt{F(v_a(M_a))}} \int_0^1 \frac{dx}{\sqrt{1 - \frac{F(v_a(M_a)x)}{F(v_a(M_a))}}}. \tag{2.37}$$

Let $G(u) = \int_0^u g(s)\, ds$. Then by $(H2)$ it follows that

$$\begin{aligned} \frac{F(v_a(M_a)x)}{F(v_a(M_a))} &= \frac{v_a^{p+1}(M_a)x^{p+1} + G(v_a(M_a)x)}{v_a^{p+1}(M_a) + G(v_a(M_a))} \\ &= \frac{x^{p+1} + \frac{G(v_a(M_a)x)}{v_a^{p+1}(M_a)}}{1 + \frac{G(v_a(M_a))}{v_a^{p+1}(M_a)}}. \end{aligned} \tag{2.38}$$

By $(H2)$ and L'Hôpital's rule it follows that $\frac{|G(u)|}{|u^{p+1}|} \to 0$ as $u \to \infty$. This implies that given $\epsilon > 0$ there exists U such that $|G(u)| \leq \epsilon |u|^{p+1}$ for $|u| \geq U$. Also the continuity of G implies that there exists $c_6 > 0$ such that $|G(u)| \leq c_6$ for $|u| \leq U$. Therefore

$$|G(u)| \leq c_6 + \epsilon |u|^{p+1} \quad \text{for all } u.$$

Letting $u = v_a(M_a)x$ in the above inequality and using (2.22) we obtain

$$\begin{aligned} \frac{|G(v_a(M_a)x)|}{v_a^{p+1}(M_a)} &\leq \frac{c_6}{v_a^{p+1}(M_a)} + \epsilon x^{p+1} \\ &\leq \frac{c_6}{v_a^{p+1}(M_a)} + \epsilon (R^*)^{p+1} \\ &\leq 2(R^*)^{p+1}\epsilon \quad \text{for } a \text{ sufficiently large.} \end{aligned}$$

Therefore $\lim_{a\to\infty} \frac{G(v_a(M_a)x)}{v_a^{p+1}(M_a)} = 0$ uniformly on $[0, 1]$. In particular it follows that $\lim_{a\to\infty} \frac{G(v_a(M_a))}{v_a^{p+1}(M_a)} = 0$. Thus it follows from (2.38) that $\frac{F(v_a(M_a)x)}{F(v_a(M_a))} \to x^{p+1}$ uniformly as $a \to \infty$.

Also we know that $\int_0^1 \frac{dx}{\sqrt{1-x^{p+1}}} < \infty$ since $p > 1$. So it follows from this and the fact that f is superlinear that $\frac{v_a(M_a)}{\sqrt{F(v_a(M_a)}} \to 0$ as $a \to \infty$. Therefore it follows from (2.37) that

$$\lim_{a\to\infty} \int_0^{v_a(M_a)} \frac{ds}{\sqrt{F(v_a(M_a)) - F(s)}} = 0.$$

Hence, the right-hand side of (2.36) goes to 0 as $a \to \infty$. However, we know $h(t) > 0$ on $(0, R^*)$ and $M_a \to R^*$ as $a \to \infty$ (by Lemma 2.4), so the integral on the left-hand side of (2.36) goes to $\int_0^{R^*} \sqrt{2h(t)}\, dt > 0$ which gives a contradiction. Therefore v_a has a zero, z_a, with $0 < z_a < M_a < R^*$. Now we show that $z_a \to R^*$ as $a \to \infty$. Rewriting (2.35) and integrating on (z_a, M_a) by letting $x = v_a(t)$ we obtain

$$\int_0^{v_a(M_a)} \frac{dx}{\sqrt{F(v_a(M_a)) - F(x)}} \geq \int_{z_a}^{M_a} \sqrt{2h(t)}\, dt. \tag{2.39}$$

As we have just proved above that the left-hand side of (2.39) goes to 0 as $a \to \infty$. Thus since $h > 0$ is continuous we must have $(M_a - z_a) \to 0$ as $a \to \infty$. Since we know from Lemma 2.4 that $M_a \to R^*$ as $a \to \infty$, it follows that $z_a \to R^*$ as $a \to \infty$.

Next we show that $|v_a'(z_a)| \to \infty$ as $a \to \infty$. Since $0 < z_a < M_a$ and E_a is non-increasing we have

$$\frac{1}{2}\frac{v_a'^2(z_a)}{h(z_a)} = E_a(z_a) \geq E_a(M_a) = F(v_a(M_a)).$$

So by rewriting this we obtain

$$2h(z_a)F(v_a(M_a)) \leq v_a'^2(z_a). \tag{2.40}$$

Since $z_a \to R^*$ as $a \to \infty$ and h is continuous then $h(z_a) \to h(R^*) > 0$ as $a \to \infty$. Also, in Lemma 2.4 we saw that $v_a(M_a) \to \infty$ as $a \to \infty$ and thus since F is continuous, it follows that $F(v_a(M_a)) \to \infty$ as $a \to \infty$. Thus, from (2.40) we see that $v_a'^2(z_a) \to \infty$ as $a \to \infty$ which then implies $|v_a'(z_a)| \to \infty$ as $a \to \infty$.

Finally, we denote the largest zero of v_a on $(0, R^*)$ as $z_{1,a}$. Using a similar argument as in Lemma 2.5, it can be shown that v_a has a local minimum, $m_a \in (0, z_{1,a})$ if a is sufficiently large. And by following a similar argument as above we can show that there exists a second zero, $z_{2,a} \in (0, m_a)$ of v_a, $z_{2,a} \to R^*$ as $a \to \infty$, and $|v_a'(z_{2,a})| \to \infty$ as $a \to \infty$. Continuing in this way if a is sufficiently large and n is a given non-negative integer, then v_a has n zeros on $(0, R^*)$ if a is sufficiently large. □

3 Behavior for small $a > 0$

Lemma 3.1. *Assume (H1)–(H5) hold and let v_a solve (2.2)–(2.3). Suppose a is sufficiently small. Then v_a has a zero, z_a, and a local maximum, M_a, with $0 < z_a < M_a < R^*$. In addition, $z_a \to R^*$, $M_a \to R^*$, $|v_a'(z_a)| \to 0$, and $v_a(M_a) \to 0$ as $a \to 0^+$. Furthermore, given $n \geq 1$, if a is sufficiently small then v_a has n zeros on $(0, R^*)$.*

Proof. First we want to show that v_a has a zero on $(0, R^*)$ if a is sufficiently small. Suppose on the contrary that $v_a > 0$ on $(0, R^*)$ for all $a > 0$. By (2.6) we have

$$v_a(t) = \frac{aR^{N-1}}{N-2}(R^* - t) - \int_t^{R^*} \left(\int_s^{R^*} h(x) f(v_a(x))\, dx \right) ds. \tag{3.1}$$

Since $v_a > 0$ near R^* it follows from (2.2) that $v_a'' < 0$ near R^* so by integrating this inequality twice we obtain

$$0 < v_a < \frac{aR^{N-1}}{N-2}(R^* - t). \tag{3.2}$$

From $(H1)$ and $(H3)$ there exists $f_1 > 0$ such that $f(v_a) \geq f_1 v_a^{-q}$. Substituting this into (3.1) gives

$$v_a(t) \leq ac_7(R^* - t) - f_1 \int_t^{R^*} \left(\int_s^{R^*} h(x) v_a^{-q}(x)\, dx \right) ds \tag{3.3}$$

where $c_7 = \frac{R^{N-1}}{N-2}$. Since h is increasing on $[0, R^*]$ then from (3.2) and (3.3) we obtain

$$v_a(t) \leq ac_7(R^* - t) - f_1 h(t) \int_t^{R^*} \left(\int_s^{R^*} v_a^{-q}(x)\, dx \right) ds = ac_7(R^* - t) - \frac{f_1 h(t)(R^* - t)^{2-q}}{a^q c_7^q (1-q)(2-q)}. \tag{3.4}$$

Therefore if $v_a > 0$ on $[\frac{R^*}{2}, R^*]$, then from (3.4) we obtain

$$\frac{f_1 h(t)(R^* - t)^{1-q}}{c_7^{q+1}(1-q)(2-q)} \leq a^{q+1}. \tag{3.5}$$

Letting $t = \frac{R^*}{2}$ in (3.5) we obtain

$$\frac{f_1 h(\frac{R^*}{2})(R^*)^{1-q}}{c_7^{q+1} 2^{1-q}(1-q)(2-q)} \leq a^{q+1}. \tag{3.6}$$

The left-hand side of (3.6) is a positive constant but the right-hand side goes to 0 as $a \to 0^+$. Thus we obtain a contradiction if a is sufficiently small. Hence v_a has a zero, z_a, on $[\frac{R^*}{2}, R^*]$ if $a > 0$ is sufficiently small and $v_a > 0$ on (z_a, R^*). Since $v_a(z_a) = 0 = v_a(R^*)$ and $v_a'(R^*) < 0$, it follows that v_a has a local maximum, M_a, with $0 < z_a < M_a < R^*$.

Next by letting $t = z_a$ in (3.5) we obtain

$$\frac{f_1 h(z_a)(R^* - z_a)^{1-q}}{c_7^{q+1}(1-q)(2-q)} \leq a^{q+1}. \tag{3.7}$$

Since the right-hand side of (3.7) goes to 0 as $a \to 0^+$ it follows that $z_a \to R^*$ as $a \to 0^+$. Since $z_a < M_a < R^*$ it then follows that $M_a \to R^*$ as $a \to 0^+$.

Next we know that $\frac{1}{2}v_a'^2 + h(t)F(v_a)$ is increasing by (2.13). So it follows that

$$\frac{1}{2}v_a'^2(z_a) = \frac{1}{2}v_a'^2(z_a) + h(z_a)F(v_a(z_a)) \leq \frac{1}{2}v_a'^2(R^*) + h(R^*)F(v_a(R^*)) = \frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2}. \tag{3.8}$$

The right-hand side of (3.8) goes to 0 as $a \to 0^+$ which implies that $|v_a'(z_a)| \to 0$ as $a \to 0^+$.

Now we show that $v_a(M_a) \to 0$ as $a \to 0^+$. From (2.16) we have $|v_a| \leq \frac{aR}{N-2}$ on $(0, R^*)$. Since $v_a(M_a) \geq 0$ it then follows that

$$0 \leq v_a(M_a) \leq \frac{aR}{N-2} \to 0 \quad \text{as } a \to 0^+.$$

Now if we denote the largest zero of v_a on $(0, R^*)$ as $z_{1,a}$ then by using a similar argument as above we can show that v_a has a local minimum, m_a, on $(0, z_{1,a})$ if a is sufficiently small. Also, it can be shown that there exists a zero, $z_{2,a} \in (0, m_a)$ of v_a and $z_{2,a} \to R^*$ as $a \to 0^+$. Continuing in this way, given $n \geq 1$ then v_a has n zeros on $(0, R^*)$ if a is sufficiently small . □

4 Proof of Theorem 1.1

Let $n \geq 0$ and consider the set

$$S_n = \{a > 0 \mid v_a \text{ solves (2.2)–(2.3) and } v_a \text{ has exactly } n \text{ zeros on } (0, R^*)\}.$$

By Lemma 2.4 we observe that if $a > 0$ then $S_n \neq \emptyset$ for some n. Let $n_0 \geq 0$ be the least integer n such that $S_n \neq \emptyset$ (i.e, $S_{n_0} \neq \emptyset$ and $S_n = \emptyset$ for all $0 \leq n < n_0$). Also it follows from Lemma 2.6 that S_{n_0} is bounded from above. So let

$$a_{n_0}^+ = \sup S_{n_0}.$$

Lemma 4.1. *$v_{a_n^+}$ has exactly n zeros, $v_{a_n^+}(0) = 0$, and $v'_{a_n^+}(0) \neq 0$ for all $n \geq n_0$.*

Proof. It follows from the definition of S_{n_0} that $v_{a_{n_0}^+}$ has at least n_0 zeros on $(0, R^*)$. Suppose that $v_{a_{n_0}^+}$ has an $(n_0 + 1)$st zero. Then by the continuous dependence of v_a on a it follows that v_a has an $(n_0 + 1)$st zero if a is sufficiently close to a_{n_0}. But if we choose $a \in S_{n_0}$ such that $a < a_{n_0}$ and a is sufficiently close to a_{n_0}, then v_a has only n_0 zeros on $(0, R^*)$ which gives a contradiction. Thus $v_{a_{n_0}^+}$ has exactly n_0 zeros on $(0, R^*)$. Now we want to show that $v_{a_{n_0}^+}(0) = 0$. Assume without the loss of generality that $v_{a_{n_0}^+} > 0$ on $(0, z_{a_{n_0}})$. Then by the continuity of $v_{a_{n_0}^+}$ we have $v_{a_{n_0}^+}(0) \geq 0$. Suppose $v_{a_{n_0}^+}(0) > 0$. Since the zeros of v_a are simple and $v_a(0) > 0$ it follows that v_a has exactly n_0 zeros on $(0, R^*)$ if a is close to a_{n_0}. But if $a > a_{n_0}$ then v_a has at least $n_0 + 1$ zeros on $(0, R^*)$ which is a contradiction. Therefore, we must have $v_{a_{n_0}^+}(0) = 0$.

Next we want to show that $v'_{a_{n_0}^+}(0) \neq 0$. Assume without loss of generality that $v_{a_{n_0}^+} > 0$ on $(0, z_{n_0})$ where z_{n_0} is the n_0^{th} zero of $a_{n_0}^+$ on $(0, R^*)$. Since $v_{a_{n_0}^+}$ solves (2.2) we have

$$v''_{a_{n_0}^+} + h(t) f(v_{a_{n_0}^+}) = 0.$$

From the above equation it follows that

$$(t v'_{a_{n_0}^+} - v_{a_{n_0}^+})' = t v''_{a_{n_0}^+} = -t h(t) f(v_{a_{n_0}^+}) < 0.$$

Thus, $t v'_{a_{n_0}^+} - v_{a_{n_0}^+}$ is decreasing. Also, since $\lim_{t \to 0^+}(t v'_{a_{n_0}^+} - v_{a_{n_0}^+})$=0 we have that $(t v'_{a_{n_0}^+} - v_{a_{n_0}^+}) \leq 0$ on $(0, z_{n_0})$. It then follows that

$$\left(\frac{v_{a_{n_0}^+}}{t}\right)' \leq 0. \tag{4.1}$$

Since $v_{a_{n_0}^+} > 0$ on $(0, z_{a_{n_0}})$, we see from (4.1) that $\lim_{t \to 0^+} \frac{v_{a_{n_0}^+}}{t}$ exists. Integrating (4.1) on (t, t_0) we obtain

$$0 < \frac{v_{a_{n_0}^+}(t_0)}{t_0} \leq \lim_{t \to 0^+} \frac{v_{a_{n_0}^+}(t)}{t} = v'_{a_{n_0}^+}(0).$$

Therefore, $v'_{a_{n_0}^+}(0) > 0$. □

Next let

$$S_{n_0+1} = \{a > 0 \mid v_a \text{ solves (2.2)–(2.3) and } v_a \text{ has exactly } (n_0 + 1) \text{ zeros on } (0, R^*)\}.$$

If a is sufficiently close to $a_{n_0}^+$ with $a > a_{n_0}^+$, then by the definition of $a_{n_0}^+$ it follows that v_a has an (n_0+1)st zero, $z_{a_{n_0}+1} \in (0, R^*)$. By integrating (2.13) on (t, R^*) we obtain

$$\frac{1}{2}v_a'^2 = \frac{1}{2}\frac{a^2 R^{2(N-1)}}{(N-2)^2} - \int_t^{R^*} h' F(v_a). \tag{4.2}$$

Similarly, we have

$$\frac{1}{2}v_{a_{n_0}^+}'^2 = \frac{1}{2}\frac{{a_{n_0}^+}^2 R^{2(N-1)}}{(N-2)^2} - \int_t^{R^*} h' F(v_{a_{n_0}^+}). \tag{4.3}$$

Since $v_a \to v_{a_{n_0}^+}$ uniformly as $a \to a_{n_0}^+$ it follows from (4.2) and (4.3) that

$$\lim_{a \to a_{n_0}^+} v_a'^2 = v_{a_{n_0}^+}'^2 \text{ uniformly on } [0, t_0] \text{ for } t_0 > 0. \tag{4.4}$$

Since $v_{a_{n_0}^+}'^2(0) > 0$ it follows from (4.4) that $v_a'(t) \neq 0$ if $a > a_{n_0}^+$ and a close to $a_{n_0}^+$ and t is close to 0. Hence, v_a has at most (n_0+1) zeros and therefore v_a has exactly (n_0+1) zeros if a is sufficiently close to $a_{n_0}^+$ and $a > a_{n_0}^+$. Thus, $S_{n_0+1} \neq \emptyset$. Also it follows from Lemma 2.6 that S_{n_0+1} is bounded above.

Now let

$$a_{n_0+1}^+ = \sup S_{n_0+1}.$$

Then by using a similar argument as above we can show that $v_{a_{n_0}^+ + 1}$ has exactly (n_0+1) zeros on $(0, R^*)$ and that $v_{a_{n_0}^+ + 1}(0) = 0$. Continuation of this process will generate an infinite family of solutions $\{v_{a_n^+}\}_{n \geq n_0}$ of (2.2)–(2.3) where $v_{a_n^+}$ has exactly n zeros on $(0, R^*)$ and $v_{a_{n_0}^+}(0) = 0$.

To complete the proof we again consider the set S_{n_0} as above which is non-empty. By Lemma 3.1 it follows that S_{n_0} is bounded from below by a positive real number. So we define

$$a_{n_0}^- = \inf S_{n_0}.$$

Then by using the continuous dependence of the solution v_a on a as above we can show that $v_{a_{n_0}^-}$ has exactly n_0 zeros and $v_{a_{n_0}^-}(0) = 0$ and $v'_{a_{n_0}^-}(0) \neq 0$. Now it may be possible that S_{n_0} is a singleton set. Then we have $a_{n_0}^- = a_{n_0}^+$. In this case there is only one solution with n_0 zeros. But we know that if $a > a_{n_0}^+$ then $S_{n_0+1} \neq \emptyset$. Also if $a < a_{n_0}^- = a_{n_0}^+$ and a is close to $a_{n_0}^-$, then v_a has exactly (n_0+1) zeros. Thus S_{n_0+1} has at least two points. Next let

$$a_{n_0+1}^- = \inf S_{n_0+1}.$$

Then $a_{n_0+1}^- < a_{n_0+1}^+$ and we can also show that $v_{a_{n_0}^- + 1}$ has exactly (n_0+1) solutions and $v_{a_{n_0}^- + 1}(0) = 0$. Thus, $v_{a_{n_0}^+ + 1}$ and $v_{a_{n_0}^- + 1}$ are two solutions with exactly (n_0+1) zeros on $(0, R^*)$. Continuation of this process will generate a second infinite family of solutions $\{v_{a_n^-}\}_{n \geq n_0}$ of (2.2)–(2.3) where $v_{a_n^-}$ has exactly n zeros on $(0, R^*)$ and $v_{a_{n_0}^-}(0) = 0$.

Finally, by letting $u_n^+(t) = v_{a_n}^+(t^{\frac{1}{2-N}})$ and $u_n^-(t) = v_{a_n}^-(t^{\frac{1}{2-N}})$ we obtain two infinite families of solutions of (1.3)–(1.4) with prescribed number of zeros. This ends the proof of Theorem 1.1. □

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Weak solutions for a class of quasilinear elliptic equations containing the $p(\cdot)$-Laplacian and the mean curvature operator in a variable exponent Sobolev space

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Abstract. In this paper, we consider the equation for a class of nonlinear operators containing $p(\cdot)$-Laplacian and mean curvature operator with mixed boundary conditions in a bounded domain Ω of $\mathbb{R}^N$, under the hypothesis $p(x) > 1$ in $\overline{\Omega}$. More precisely, we are concerned with the problem under the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of one, two and infinitely many nontrivial weak solutions of the equation according to the conditions on given functions.

Keywords: $p(\cdot)$-Laplacian type operator, mean curvature operator, mixed boundary value problem, variable exponent Sobolev space.

2020 Mathematics Subject Classification: 35D30, 35A01, 35J62, 35J57.

1 Introduction

In this paper, we consider the following equation

$$\begin{cases} -\mathrm{div}\,[\boldsymbol{a}(x, \nabla u(x))] = f(x, u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma_1, \\ \boldsymbol{n}(x) \cdot \boldsymbol{a}(x, \nabla u(x)) = g(x, u(x)) & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz-continuous ($C^{0,1}$ for short) boundary $\partial\Omega = \Gamma$ satisfying that

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint open subsets of } \Gamma \text{ such that } \overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset, \tag{1.2}$$

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and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. The function $\boldsymbol{a}(x,\boldsymbol{\xi}) = \nabla_{\xi}A(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$ satisfying some structure conditions associated with an anisotropic exponent function $p \in C(\overline{\Omega})$ with $1 < p(x)$ for $x \in \overline{\Omega}$. Then the operator div $[\boldsymbol{a}(x, \nabla u(x))]$ is more general than the $p(\cdot)$-Laplacian

$$\Delta_{p(x)}u(x) = \mathrm{div}\,[|\nabla u(x)|^{p(x)-2}\nabla u(x)]$$

and the mean curvature operator

$$\mathrm{div}\,[(1 + |\nabla u(x)|^2)^{(p(x)-2)/2}\nabla u(x)].$$

These generalities bring about difficulties and requires some conditions.

We impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions satisfying some conditions.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [31]), in electrorheological fluids (Diening [10], Halsey [19], Mihăilescu and Rădulescu [22], Růžička [24]).

Since we can only find a few of papers associate with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1). See Aramaki [2, 5]. We are convinced of the reason for existence of this paper.

Fan [13] considered the problem (1.1) when $A(x,\boldsymbol{\xi}) = \frac{1}{p(x)}|\boldsymbol{\xi}|^{p(x)}$ and $\Gamma_2 = \emptyset$, and derived the existence of a nontrivial weak solution to (1.1). Yücedağ [29] and Mashiyev et al. [21] and many authors extended the result to the case where $A(x,\boldsymbol{\xi})$ satisfies the $p(\cdot)$-uniform convexity. In Aramaki [3] and Dai and Hao [8], the authors treated the Kirchhoff-type operator in the case where $A(x,\boldsymbol{\xi})$ satisfies the $p(\cdot)$-uniform convexity. Here the $p(\cdot)$-uniform convexity of $A(x,\boldsymbol{\xi})$ means that

$$A\left(x, \frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right) + c|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)} \leq \frac{1}{2}A(x,\boldsymbol{\xi}) + \frac{1}{2}A(x,\boldsymbol{\eta}) \tag{1.3}$$

for a.e. $x \in \Omega$ and all $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^N$ with some constant $c > 0$. However, even in the case where $A(x,\boldsymbol{\xi}) = \frac{1}{p(x)}|\boldsymbol{\xi}|^{p(x)}$, in general, if $1 < p(x) < 2$ in a non-empty subset of Ω, then this $p(\cdot)$-uniform convexity does not hold. Of course, if $p(x) \geq 2$ in Ω, then (1.3) holds.

In this paper, we give up this condition, but we assume that $\boldsymbol{a}(x,\boldsymbol{\xi})$ is uniformly monotone (see (A.2) below in Section 3), because we think that this hypothesis is more natural for the $p(\cdot)$-Laplacian and the mean curvature operator, and allow not only the case $2 \leq p(x)$ in $\overline{\Omega}$, but also the case $1 < p(x)$ in $\overline{\Omega}$. To overcome this, if we apply a version of the idea of Glowinski and A. Marroco [18] who treated the case $p(x) = p =$ const., then we get Proposition 3.7 below. So our results are new, because the results contain the case $1 < p(x)$ in $\overline{\Omega}$.

We derive that there exist one, two and infinitely many nontrivial weak solutions. We use the standard Mountain-Pass Theorem, Ekeland variational principle and the Symmetric Mountain-Pass Theorem, respectively (cf. Aramaki [4, 6], [21]).

This paper is also an extension of the articles [13] to the case of mixed boundary value problem and of a class of operators containing the $p(\cdot)$-Laplacian and the mean curvature operator with the case where $p(x) > 1$ in $\overline{\Omega}$.

The paper is organized as follows. In Section 2, we recall some well-known results on variable exponent Lebesgue-Sobolev spaces. In Section 3, we give the assumptions to the main theorems. In Section 4, we state the main theorems (Theorem 4.3, 4.5 and 4.6) on the existence of at least one, two and infinitely many nontrivial weak solutions according to the hypotheses on given functions f and g. The proofs of these main theorems are given in Section 5.

2 Preliminaries

Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ and Ω is locally on the same side of Γ. Moreover, we assume that Γ satisfies (1.2).

In the present paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^N by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \ldots, a_N)$ and $\boldsymbol{b} = (b_1, \ldots, b_N)$ in $\mathbb{R}^N$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^N a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle \cdot, \cdot \rangle_{B^*,B}$.

We recall some well-known results on variable exponent Lebesgue and Sobolev spaces. See Fan and Zhang [15], Kováčik and Rácosník [20] and references therein for more detail. Furthermore, we consider some new properties on variable exponent Lebesgue space. Define $C(\overline{\Omega}) = \{p; p \text{ is a continuous function on } \overline{\Omega}\}$, and for any $p \in C(\overline{\Omega})$, put

$$p^+ = p^+(\Omega) = \sup_{x \in \Omega} p(x) \text{ and } p^- = p^-(\Omega) = \inf_{x \in \Omega} p(x).$$

For any $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and for any measurable function u on Ω, a modular $\rho_{p(\cdot)} = \rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\}$$

equipped with the (Luxemburg) norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0; \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

where ∇u is the gradient of u, that is, $\nabla u = (\partial_1 u, \ldots, \partial_N u)$, $\partial_i = \partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \||\nabla u|\|_{L^{p(\cdot)}(\Omega)}.$$

The following three propositions are well known (see Fan et al. [16], Fan and Zhao [17], Zhao et al. [30]).

Proposition 2.1. *Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$, and let $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, \ldots)$. Then we have the following properties.*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1(= 1, > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1(= 1, > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{p(\cdot)}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let* $p \in C_+(\overline{\Omega})$, *where*

$$C_+(\overline{\Omega}) := \{p \in C(\overline{\Omega}); p^- > 1\}.$$

For any $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$, *we have*

$$\int_\Omega |u(x)v(x)|dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Here and from now on, for any $p \in C_+(\overline{\Omega})$, $p'(\cdot)$ *denote the conjugate exponent of* $p(\cdot)$, *that is,* $p'(x) = p(x)/(p(x) - 1)$.

For $p \in C_+(\overline{\Omega})$, define for $x \in \overline{\Omega}$,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

Proposition 2.3. *Let* Ω *be a bounded domain of* $\mathbb{R}^N$ *with* $C^{0,1}$*-boundary and let* $p \in C_+(\overline{\Omega})$. *Then we have the following properties.*

(i) *The spaces* $L^{p(\cdot)}(\Omega)$ *and* $W^{1,p(\cdot)}(\Omega)$ *are separable, reflexive and uniformly convex Banach spaces.*

(ii) *If* $q(x) \in C(\overline{\Omega})$ *with* $q^- \ge 1$ *satisfies that* $q(x) \le p^*(x)$ *for all* $x \in \Omega$, *then the embedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, *where* $\hookrightarrow$ *means that the embedding is continuous.*

(iii) *If* $q(x) \in C(\overline{\Omega})$ *with* $q^- \ge 1$ *satisfies that* $q(x) < p^*(x)$ *for all* $x \in \Omega$, *then the embedding* $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *is compact.*

Next we consider the trace (cf. Fan [14]). Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and $p \in C(\overline{\Omega})$ with $p^- \ge 1$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}(\Omega)$, the trace $\gamma(u) = u\big|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1(\Gamma)$. We define

$$(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma) = \{f; f \text{ is the trace to } \Gamma \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|f\|_{(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)} = \inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)}; F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F\big|_\Gamma = f\}$$

for $f \in (\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)$, where the infimum can be achieved. Then we can see that $(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)$ is a Banach space. In the later we also write $F\big|_\Gamma = g$ by $F = g$ on Γ. Moreover, for $i = 1, 2$, we denote

$$(\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma_i) = \{f\big|_{\Gamma_i}; f \in (\mathrm{Tr}\, W^{1,p(\cdot)})(\Gamma)\}$$

equipped with the norm

$$\|g\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma_i)} = \inf\{\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}; f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma) \text{ satisfying } f|_{\Gamma_i} = g\},$$

where the infimum can also be achieved, so for any $g \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma_i)$, there exists $F \in W^{1,p(\cdot)}(\Omega)$ such that $F|_{\Gamma_i} = g$ and $\|F\|_{W^{1,p(\cdot)}(\Omega)} = \|g\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma_i)}$.

Let $q \in C_+(\Gamma) := \{q \in C(\Gamma); q^- > 1\}$ and denote the surface measure on Γ induced from the Lebesgue measure dx on Ω by $d\sigma_x$. We define

$$L^{q(\cdot)}(\Gamma) = \Big\{u; u : \Gamma \to \mathbb{R} \text{ is a measurable function with respect to } d\sigma_x \text{ satisfying } \int_\Gamma |u(x)|^{q(x)} d\sigma_x < \infty\Big\}$$

and the norm is defined by

$$\|u\|_{L^{q(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0; \int_\Gamma \left|\frac{u(x)}{\lambda}\right|^{q(x)} d\sigma_x \le 1\right\},$$

and we also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u) = \int_\Gamma |u(x)|^{q(x)} d\sigma_x.$$

Similarly as Proposition 2.1, we have the following proposition.

Proposition 2.4. *Let $q \in C(\Gamma)$ with $q^- \ge 1$, and let $u, u_n \in L^{q(\cdot)}(\Gamma)$. Then we have the following properties.*

(i) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1(= 1, > 1) \iff \rho_{q(\cdot),\Gamma}(u) < 1(= 1, > 1)$.

(ii) $\|u\|_{L^{q(\cdot)}(\Gamma)} > 1 \implies \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.

(iii) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 \implies \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.

(iv) $\|u_n\|_{L^{q(\cdot)}(\Gamma)} \to 0 \iff \rho_{q(\cdot),\Gamma}(u_n) \to 0$.

(v) $\|u_n\|_{L^{q(\cdot)}(\Gamma)} \to \infty \iff \rho_{q(\cdot),\Gamma}(u_n) \to \infty$.

The Hölder inequality also holds for functions on Γ.

Proposition 2.5. *Let $q \in C(\Gamma)$ with $q^- > 1$. Then the following inequality holds.*

$$\int_\Gamma |f(x)g(x)| d\sigma_x \le 2\|f\|_{L^{q(\cdot)}(\Gamma)} \|g\|_{L^{q'(\cdot)}(\Gamma)} \quad \textit{for all } f \in L^{q(\cdot)}(\Gamma), g \in L^{q'(\cdot)}(\Gamma).$$

Proposition 2.6. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and let $p \in C_+(\overline{\Omega})$. If $f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma)$ and there exists a constant $C > 0$ such that*

$$\|f\|_{L^{p(\cdot)}(\Gamma)} \le C\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}.$$

In particular, If $f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma_i)$ and $\|f\|_{L^{p(\cdot)}(\Gamma_i)} \le C\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}$ for $i = 1, 2$.

For $p \in C_+(\overline{\Omega})$, define for $x \in \overline{\Omega}$,

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

The following proposition follows from Yao [28, Proposition 2.6].

Proposition 2.7. *Let $p \in C_+(\overline{\Omega})$. Then if $q \in C_+(\Gamma)$ satisfies $q(x) \leq p^\partial(x)$ for all $x \in \Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is well-defined, continuous and*

$$\|u\|_{L^{q(\cdot)}(\Gamma)} \leq C\|u\|_{W^{1,p(\cdot)}(\Omega)} \text{ for } u \in W^{1,p(\cdot)}(\Omega)$$

for some constant $C > 0$.

In particular, if $q(x) < p^\partial(x)$ for all $x \in \Gamma_2$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is compact.

Now we consider the weighted variable exponent Lebesgue space. Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$ and let $a(x)$ be a measurable function on Ω with $a(x) > 0$ a.e. $x \in \Omega$. We define a modular

$$\rho_{(p(\cdot),a(\cdot))}(u) = \int_\Omega a(x)|u(x)|^{p(x)}dx \text{ for any measurable function } u \text{ in } \Omega.$$

Then the weighted Lebesgue space is defined by

$$L^{p(\cdot)}_{a(\cdot)}(\Omega) = \left\{u; u \text{ is a measurable function on } \Omega \text{ satisfying } \rho_{(p(\cdot),a(\cdot))}(u) < \infty\right\}$$

equipped with the norm

$$\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = \inf\left\{\lambda > 0; \int_\Omega a(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

Then $L^{p(\cdot)}_{a(\cdot)}(\Omega)$ is a Banach space.

We have the following proposition (cf. [13, Proposition 2.5]).

Proposition 2.8. *Let $p \in C(\overline{\Omega})$ with $p^- \geq 1$. For $u, u_n \in L^{p(\cdot)}_{a(\cdot)}(\Omega)$, we have the following.*

(i) *For $u \neq 0$, $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = \lambda \Longleftrightarrow \rho_{(p(\cdot),a(\cdot))}\left(\frac{u}{\lambda}\right) = 1$.*

(ii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} < 1\ (= 1, > 1) \Longleftrightarrow \rho_{(p(\cdot),a(\cdot))}(u) < 1\ (= 1, > 1)$.

(iii) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \leq \rho_{(p(\cdot),a(\cdot))}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(iv) $\|u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \leq \rho_{(p(\cdot),a(\cdot))}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}_{a(\cdot)}(\Omega)}$.

(v) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{(p(\cdot),a(\cdot))}(u_n - u) = 0$.

(vi) $\|u_n\|_{L^{p(\cdot)}_{a(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{(p(\cdot),a(\cdot))}(u_n) \to \infty$ *as* $n \to \infty$.

The author of [13] also derived the following proposition (cf. [13, Theorem 2.1]).

Proposition 2.9. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary and $p \in C_+(\overline{\Omega})$. Moreover, let $a \in L^{\alpha(\cdot)}(\Omega)$ satisfy $a(x) > 0$ a.e. $x \in \Omega$ and $\alpha \in C_+(\overline{\Omega})$. If $q \in C(\overline{\Omega})$ satisfies*

$$1 \leq q(x) < \frac{\alpha(x) - 1}{\alpha(x)} p^*(x) \quad \text{for all } x \in \overline{\Omega},$$

then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}_{a(\cdot)}(\Omega)$ is compact.

Similarly, let $q \in C(\Gamma)$ with $q^- \geq 1$ and let $b(x)$ be a measurable function with respect to σ on Γ with $b(x) > 0$ σ-a.e. $x \in \Gamma$. We define a modular

$$\rho_{(q(\cdot),b(\cdot)),\Gamma}(u) = \int_\Gamma b(x)|u(x)|^{q(x)} d\sigma_x.$$

Then the weighted Lebesgue space on Γ is defined by

$$L^{q(\cdot)}_{b(\cdot)}(\Gamma) = \{u; u \text{ is a } \sigma\text{-measurable function on } \Gamma \text{ satisfying } \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) < \infty\}$$

equipped with the norm

$$\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0; \int_\Gamma b(x)\left|\frac{u(x)}{\lambda}\right|^{q(x)} d\sigma_x \leq 1\right\}.$$

Then $L^{q(\cdot)}_{b(\cdot)}(\Gamma)$ is a Banach space.

Then we have the following proposition.

Proposition 2.10. *Let $q \in C(\Gamma)$ with $q^- \geq 1$. For $u, u_n \in L^{q(\cdot)}_{b(\cdot)}(\Gamma)$, we have the following.*

(i) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} < 1\ (= 1, > 1) \Longleftrightarrow \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) < 1\ (= 1, > 1)$.

(ii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} > 1 \Longrightarrow \|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \leq \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) \leq \|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} < 1 \Longrightarrow \|u\|^{q^+}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \leq \rho_{(q(\cdot),b(\cdot)),\Gamma}(u) \leq \|u\|^{q^-}_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{q(\cdot)}_{b(\cdot)}(\Gamma)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{(q(\cdot),b(\cdot)),\Gamma}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition plays an important role in the present paper.

Proposition 2.11. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary Γ and let $p \in C_+(\overline{\Omega})$. Assume that $0 < b \in L^{\beta(\cdot)}(\Gamma)$, $\beta \in C_+(\Gamma)$. If $r \in C(\Gamma)$ satisfies*

$$1 \leq r(x) < \frac{\beta(x) - 1}{\beta(x)} p^\partial(x) \quad \text{for all } x \in \Gamma,$$

then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}_{b(\cdot)}(\Gamma)$ is compact.

The following proposition is due to Edmunds and Rákosník [11, Lemma 2.1].

Proposition 2.12. *Let $q \in L^\infty(\Omega)$ and p be a measurable function on Ω such that $1 \le p(x) \le \infty$ and $1 \le q(x)p(x) \le \infty$. Assume that $f \in L^{p(\cdot)}(\Omega)$ with $f \neq 0$. Then we have the following.*

$$\|f\|_{L^{q(\cdot)p(\cdot)}(\Omega)} \le 1 \Longrightarrow \|f\|^{q^+}_{L^{q(\cdot)p(\cdot)}(\Omega)} \le \||f|^{q(\cdot)}\|_{L^{p(\cdot)}(\Omega)} \le \|f\|^{q^-}_{L^{q(\cdot)p(\cdot)}(\Omega)}.$$
$$\|f\|_{L^{q(\cdot)p(\cdot)}(\Omega)} \ge 1 \Longrightarrow \|f\|^{q^-}_{L^{q(\cdot)p(\cdot)}(\Omega)} \le \||f|^{q(\cdot)}\|_{L^{p(\cdot)}(\Omega)} \le \|f\|^{q^+}_{L^{q(\cdot)p(\cdot)}(\Omega)}.$$

In particular, if $q(x) = q = \text{const.}$, then $\||f|^q\|_{L^{p(\cdot)}(\Omega)} = \|f\|^q_{L^{qp(\cdot)}(\Omega)}$.

Define a space by

$$X = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma_1\}. \tag{2.1}$$

Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable Banach space. We get the following Poincaré-type inequality (cf. Ciarlet and Dinca [7]).

Proposition 2.13. *Let Ω be a bounded domain of $\mathbb{R}^N$ with a $C^{0,1}$-boundary and let $p \in C_+(\overline{\Omega})$. Then there exists a constant $C = C(\Omega, N, p) > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \quad \textit{for all } u \in X.$$

In particular, $\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$ is equivalent to $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ for $u \in X$.

For the direct proof, see Aramaki [1, Lemma 2.5].

Thus we can define the norm on X so that

$$\|v\|_X = \|\nabla v\|_{L^{p(\cdot)}(\Omega)} \quad \text{for } v \in X, \tag{2.2}$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ from Proposition 2.13.

3 Assumptions to the main theorems

In this section, we state the assumptions to the main theorems. Let $p \in C_+(\overline{\Omega})$ be fixed.

Throughout this paper, we assume the following.

(A.0) Let $A : \Omega \times \mathbb{R}^N \to [0, \infty)$ be a function satisfying that for a.e. $x \in \Omega$ the function $A(x, \cdot) : \mathbb{R}^N \ni \boldsymbol{\xi} \mapsto A(x, \boldsymbol{\xi})$ is of C^1-class, and for all $\boldsymbol{\xi} \in \mathbb{R}^N$ the function $A(\cdot, \boldsymbol{\xi}) : \Omega \ni x \mapsto A(x, \boldsymbol{\xi})$ is measurable. Moreover, suppose that $A(x, \mathbf{0}) = 0$ and put $\boldsymbol{a}(x, \boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} A(x, \boldsymbol{\xi})$. Then $\boldsymbol{a}(x, \boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^N$.

Moreover, we assume the following structure conditions. There exist constants $C_0, k_0 > 0$, nonnegative functions $h_0 \in L^{p'(\cdot)}(\Omega)$ and $h_1 \in L^1(\Omega)$ with $h_1(x) \ge 1$ for a.e. $x \in \Omega$ such that the following conditions hold.

(A.1) $|\boldsymbol{a}(x, \boldsymbol{\xi})| \le C_0(h_0(x) + h_1(x)|\boldsymbol{\xi}|^{p(x)-1})$ for all $\boldsymbol{\xi} \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A.2) $\boldsymbol{a}(x, \mathbf{0}) = \mathbf{0}$ for a.e. $x \in \Omega$ and

$$(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \ge \begin{cases} k_0 h_1(x)|\boldsymbol{\xi} - \boldsymbol{\eta}|^{p(x)} & \text{if } p(x) \ge 2, \\ k_0 h_1(x)(1 + |\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{p(x)-2}|\boldsymbol{\xi} - \boldsymbol{\eta}|^2 & \text{if } p(x) < 2 \end{cases}$$

for a.e. $x \in \Omega$ and all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$.

(A.3) A is $p(\cdot)$-subhomogeneous in the sense of

$$\boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi} \le p(x)A(x,\boldsymbol{\xi}) + h_1(x) \text{ for all } \boldsymbol{\xi}\in\mathbb{R}^N \text{ and a.e. } x\in\Omega.$$

Lemma 3.1. *Under (A.0) and (A.2), there exists a constant $c > 0$ such that*

$$\frac{1}{2}A(x,\boldsymbol{\xi}) + \frac{1}{2}A(x,\boldsymbol{\eta}) - A\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right) \ge \begin{cases} c\,h_1(x)|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)} & \text{if } p(x)\ge 2,\\ c\,h_1(x)(1+|\boldsymbol{\xi}|+|\boldsymbol{\eta}|)^{p(x)-2}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2 & \text{if } p(x) < 2\end{cases}$$

for a.e. $x\in\Omega$ and all $\boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{R}^N$.

In particular, $A(x,\boldsymbol{\xi})$ is convex with respect to $\boldsymbol{\xi}$.

Proof. Since

$$A(x,\boldsymbol{\eta}) - A\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right) = \int_0^1 \boldsymbol{a}\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2} + s\left(\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right)\right)\cdot\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}ds,$$

and

$$A\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right) - A(x,\boldsymbol{\xi}) = \int_0^1 \boldsymbol{a}\left(x,\boldsymbol{\xi} + s\left(\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right)\right)\cdot\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}ds,$$

it follows from (A.0) and (A.2) that

$$\begin{aligned}
&\frac{1}{2}A(x,\boldsymbol{\xi}) + \frac{1}{2}A(x,\boldsymbol{\eta}) - A\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}\right)\\
&\quad= \frac{1}{2}\int_0^1\left(\boldsymbol{a}\left(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}+s\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right) - \boldsymbol{a}\left(x,\boldsymbol{\xi}+s\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right)\right)\cdot\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}ds\\
&\quad\ge \begin{cases}\frac{1}{2}k_0h_1(x)\left|\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right|^{p(x)} & \text{if } p(x)\ge 2,\\ \frac{1}{2}k_0h_1(x)\int_0^1\left(1+\left|\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}+s\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right| + \left|\boldsymbol{\xi}+s\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right|\right)^{p(x)-2}\left|\frac{\boldsymbol{\eta}-\boldsymbol{\xi}}{2}\right|^2 ds & \text{if } p(x)<2\end{cases}\\
&\quad\ge \begin{cases}\left(\frac{1}{2}\right)^{p^++1}k_0h_1(x)|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)} & \text{if } p(x)\ge 2,\\ \frac{1}{4}k_0h_1(x)(1+|\boldsymbol{\xi}|+|\boldsymbol{\eta}|)^{p(x)-2}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2 & \text{if } p(x)<2.\end{cases}
\end{aligned}$$

In particular, since $A(x,\frac{\boldsymbol{\xi}+\boldsymbol{\eta}}{2}) \le \frac{1}{2}A(x,\boldsymbol{\xi}) + \frac{1}{2}A(x,\boldsymbol{\eta})$ and $A(x,\boldsymbol{\xi})$ is continuous with respect to $\boldsymbol{\xi}$, it is well known that $A(x,\boldsymbol{\xi})$ is convex. □

Example 3.2.

(i) $A(x,\boldsymbol{\xi}) = \frac{h(x)}{p(x)}|\boldsymbol{\xi}|^{p(x)}$ with $h\in L^1(\Omega)$ satisfying $h(x)\ge 1$ for a.e. $x\in\Omega$.

(ii) $A(x,\boldsymbol{\xi}) = \frac{h(x)}{p(x)}((1+|\boldsymbol{\xi}|^2)^{p(x)/2}-1)$ with $h\in L^{p'(\cdot)}(\Omega)$ satisfying $h(x)\ge 1$ for a.e. $x\in\Omega$.

Then $A(x,\boldsymbol{\xi})$ and $\boldsymbol{a}(x,\boldsymbol{\xi}) = \boldsymbol{\nabla}_{\boldsymbol{\xi}}A(x,\boldsymbol{\xi})$ satisfy the above assumptions (A.0)–(A.3).

Proof. In the case (i), $A(x,\boldsymbol{\xi})$ is clearly differentiable with respect to $\boldsymbol{\xi}$ for $\boldsymbol{\xi}\ne\mathbf{0}$ and $\boldsymbol{a}(x,\boldsymbol{\xi}) = h(x)|\boldsymbol{\xi}|^{p(x)-2}\boldsymbol{\xi}$ for $\boldsymbol{\xi}\ne\mathbf{0}$. Since $p(x)>1$, if we define $\boldsymbol{a}(x,\mathbf{0}) = \mathbf{0}$, then we see that $A(x,\boldsymbol{\xi})$ is of C^1-class with respect to $\boldsymbol{\xi}$, so (A.0) holds. (A.1) easily holds. If we use the well-known inequality (cf. Thelin [25]): there exists a constant $k_0 > 0$ such that

$$(|\boldsymbol{\xi}|^{p(x)-2}\boldsymbol{\xi} - |\boldsymbol{\eta}|^{p(x)-2}\boldsymbol{\eta})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) \ge \begin{cases}k_0|\boldsymbol{\xi}-\boldsymbol{\eta}|^{p(x)} & \text{if } p(x)\ge 2,\\ k_0(1+|\boldsymbol{\xi}|+|\boldsymbol{\eta}|)^{p(x)-2}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2 & \text{if } p(x)<2,\end{cases}$$

for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^N$, then we see that (A.2) holds. We can easily see that (A.3) holds.

In the case (ii), clearly $A(x, \boldsymbol{\xi})$ is of C^1-class with respect to $\boldsymbol{\xi}$ and $\boldsymbol{a}(x, \boldsymbol{\xi}) = h(x)(1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}\boldsymbol{\xi}$.

If $p(x) \geq 2$, since $|\boldsymbol{\xi}| \leq 1 + |\boldsymbol{\xi}|^{p(x)-1}$, we have

$$|\boldsymbol{a}(x, \boldsymbol{\xi})| \leq h(x)2^{(p^+-2)/2}(1 + |\boldsymbol{\xi}|^{p(x)-2})|\boldsymbol{\xi}| \leq 2^{p^+/2}(h(x) + h(x)|\boldsymbol{\xi}|^{p(x)-1}).$$

If $p(x) < 2$,

$$|\boldsymbol{a}(x, \boldsymbol{\xi})| \leq h(x)|\boldsymbol{\xi}|^{p(x)-2}|\boldsymbol{\xi}| = h(x)|\boldsymbol{\xi}|^{p(x)-1}.$$

Thus (A.1) with $h_0 = h_1 = h$ holds. We show that (A.2) holds. We have

$$\begin{aligned}
&(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \\
&\quad = h(x) \int_0^1 \frac{d}{ds} \Big[(1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-2)/2}(s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta})\Big]\, ds \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \\
&\quad = h(x) \int_0^1 (1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-2)/2} ds |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \\
&\qquad + h(x)(p(x) - 2) \int_0^1 (1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-4)/2} |(s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta})|^2 ds.
\end{aligned}$$

If $p(x) \geq 2$, it follows from DiBenedetto [9, p. 14] that

$$(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \geq h(x) \int_0^1 |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^{p(x)-2} ds |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \geq k_0 h(x) |\boldsymbol{\xi} - \boldsymbol{\eta}|^{p(x)}.$$

If $p(x) < 2$, we have

$$\begin{aligned}
&(\boldsymbol{a}(x, \boldsymbol{\xi}) - \boldsymbol{a}(x, \boldsymbol{\eta})) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) \\
&\quad \geq h(x) \int_0^1 (1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-2)/2} ds |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \\
&\qquad + h(x)(p(x) - 2) \int_0^1 (1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-4)/2} |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2 |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 ds \\
&\quad \geq h(x)(p(x) - 1) \int_0^1 (1 + |s\boldsymbol{\xi} + (1-s)\boldsymbol{\eta}|^2)^{(p(x)-2)/2} ds |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \\
&\quad \geq (p^- - 1) h(x) (1 + |\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{p(x)-2} |\boldsymbol{\xi} - \boldsymbol{\eta}|^2.
\end{aligned}$$

Thus (A.2) holds. We show that (A.3) holds.

$$\begin{aligned}
\boldsymbol{a}(x, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} &= h(x)(1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2} |\boldsymbol{\xi}|^2 \\
&= h(x)(1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}(1 + |\boldsymbol{\xi}|^2 - 1) \\
&= h(x)(1 + |\boldsymbol{\xi}|^2)^{p(x)/2} - h(x)(1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2} \\
&= p(x)A(x, \boldsymbol{\xi}) + h(x)(1 - (1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}) \\
&\leq p(x)A(x, \boldsymbol{\xi}) + h(x).
\end{aligned}$$

If $p(x) \geq 2$, then we can delete the last term $h(x)$, however if $p(x) < 2$, then we can not delete the last term $h(x)$ since $\{(1 + |\boldsymbol{\xi}|^2)^{(p(x)-2)/2}; \boldsymbol{\xi} \in \mathbb{R}^N\} = [0, 1]$. □

Remark 3.3.

(i) When $h(x) \equiv 1$, (i) corresponds to the $p(\cdot)$-Laplacian and (ii) corresponds to the prescribed mean curvature operator for nonparametric surface.

(ii) In many papers (for example, [29], [21], [6], [4]), the authors assume that $\boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi} \le p(x)A(x,\boldsymbol{\xi})$ instead of (A.3). However, in the above Example 3.2 we saw that if the example (ii) satisfies $1 < p(x) < 2$ in a subset of Ω with positive measure, then we have to assume (A.3).

Lemma 3.4. *Under (A.0)–(A.2), we have the following.*

(i) $|A(x,\boldsymbol{\xi})| \le C_0(h_0(x)|\boldsymbol{\xi}| + h_1(x)|\boldsymbol{\xi}|^{p(x)})$ *for a.e.* $x\in\Omega$ *and all* $\boldsymbol{\xi}\in\mathbb{R}^N$.

(ii) *There exist constants* $c>0$ *and* $C\ge 0$ *such that*

$$\boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi} \ge ch_1(x)|\boldsymbol{\xi}|^{p(x)} - Ch_1(x) \text{ for a.e. } x\in\Omega \text{ and all } \boldsymbol{\xi}\in\mathbb{R}^N.$$

In particular, if $p^- \ge 2$, *then we can take* $C=0$.

Proof. (i) From (A.0) and (A.1), we have

$$|A(x,\boldsymbol{\xi})| = |A(x,\boldsymbol{\xi}) - A(x,\mathbf{0})| = \left|\int_0^1 \frac{d}{dt}A(x,t\boldsymbol{\xi})dt\right| = \left|\int_0^1 \boldsymbol{a}(x,t\boldsymbol{\xi})\cdot\boldsymbol{\xi}dt\right| \\ \le C_0(h_0(x)|\boldsymbol{\xi}| + h_1(x)|\boldsymbol{\xi}|^{p(x)}).$$

(ii) Since it follows from (A.2) with $\boldsymbol{\eta}=\mathbf{0}$ that

$$\boldsymbol{a}(x,\boldsymbol{\xi})\cdot\boldsymbol{\xi} \ge \begin{cases} k_0h_1(x)|\boldsymbol{\xi}|^{p(x)} & \text{if } p(x)\ge 2,\\ k_0h_1(x)(1+|\boldsymbol{\xi}|)^{p(x)-2})|\boldsymbol{\xi}|^2 & \text{if } p(x)<2,\end{cases}$$

it suffices to show that when $p(x)<2$, we have $(1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^2 \ge c'|\boldsymbol{\xi}|^{p(x)} - C'$ for some constant $c',C'>0$. Using an elementary inequality $(a+b)^q \le 2^q(a^q+b^q)$ for real numbers $a,b\ge 0$ and $q>0$, we have

$$(1+|\boldsymbol{\xi}|)^{2-p(x)} \le 2^{2-p(x)}(|\boldsymbol{\xi}|^{2-p(x)}+1) \le 2^{2-p^-}|\boldsymbol{\xi}|^{2-p(x)} + 2^{2-p^-}.$$

Thereby, $|\boldsymbol{\xi}|^{2-p(x)} \ge 2^{p^--2}(1+|\boldsymbol{\xi}|)^{2-p(x)} - 1$. When $|\boldsymbol{\xi}|\le 1$, since $p(x)-1>0$, we have

$$\begin{aligned}(1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^2 &= (1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^{2-p(x)}|\boldsymbol{\xi}|^{p(x)}\\ &\ge (1+|\boldsymbol{\xi}|)^{p(x)-2}(2^{p^--2}(1+|\boldsymbol{\xi}|)^{2-p(x)}-1)|\boldsymbol{\xi}|^{p(x)}\\ &= 2^{p^--2}|\boldsymbol{\xi}|^{p(x)} - (1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^{p(x)}\\ &\ge 2^{p^--2}|\boldsymbol{\xi}|^{p(x)} - (2|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^{p(x)}\\ &\ge 2^{p^--2}|\boldsymbol{\xi}|^{p(x)} - 2^{p^+-2}|\boldsymbol{\xi}|^{2(p(x)-1)}\\ &\ge 2^{p^--2}|\boldsymbol{\xi}|^{p(x)} - 2^{p^+-2}.\end{aligned}$$

When $|\boldsymbol{\xi}|\ge 1$, we have $(1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^2 \ge (2|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^2 \ge 2^{p^--2}|\boldsymbol{\xi}|^{p(x)}$. Therefore, we have $(1+|\boldsymbol{\xi}|)^{p(x)-2}|\boldsymbol{\xi}|^2 \ge 2^{p^--2}|\boldsymbol{\xi}|^{p(x)} - 2^{p^+-2}$ for all $\boldsymbol{\xi}\in\mathbb{R}^N$. □

For the function $h_1\in L^1(\Omega)$ with $h_1(x)\ge 1$ for a.e. $x\in\Omega$, we define a modular

$$\rho_{p(\cdot),h_1(\cdot)}(v) = \rho_{p(\cdot),h_1(\cdot),\Omega}(v) = \int_\Omega h_1(x)|\nabla v(x)|^{p(x)}dx \quad \text{for } v\in Y,$$

where Y is our basic space defined by

$$Y = Y(\Omega) = \{v \in X; \rho_{p(\cdot),h_1(\cdot)}(v) < \infty\}, \tag{3.1}$$

the space X is defined by (2.1), equipped with the norm

$$\|v\|_Y = \inf\left\{\lambda > 0; \rho_{p(\cdot),h_1(\cdot)}\left(\frac{v}{\lambda}\right) \le 1\right\}.$$

Then Y is a Banach space (see Proposition 3.5 below). We note that $C_0^\infty(\Omega) \subset Y$. Since

$$\rho_{p(\cdot),h_1(\cdot)}(v) = \rho_{p(\cdot)}(h_1^{1/p(\cdot)}\nabla v),$$

we have

$$\|v\|_Y = \|h_1^{1/p(\cdot)}\nabla v\|_{L^{p(\cdot)}(\Omega)}. \tag{3.2}$$

We have the following propositions.

Proposition 3.5. *The space $(Y, \|\cdot\|_Y)$ is a separable and reflexive Banach space.*

For the proof, see [4, Lemma 2.12].

Proposition 3.6. *Let Y be the above Banach space defined by (3.1) and X be the space defined by (2.1). Then we have the following properties.*

(i) *$Y \hookrightarrow X$ and $\|v\|_X \le \|v\|_Y$ for all $v \in Y$.*

(ii) *Let $v \in Y$. Then $\|v\|_Y > 1 (= 1, < 1) \Longleftrightarrow \rho_{p(\cdot),h_1(\cdot)}(v) > 1 (= 1, < 1)$.*

(iii) *Let $v \in Y$. Then $\|v\|_Y > 1 \Longrightarrow \|v\|_Y^{p^-} \le \rho_{p(\cdot),h_1(\cdot)}(v) \le \|v\|_Y^{p^+}$.*

(iv) *Let $v \in Y$. Then $\|v\|_Y < 1 \Longrightarrow \|v\|_Y^{p^+} \le \rho_{p(\cdot),h_1(\cdot)}(v) \le \|v\|_Y^{p^-}$.*

(v) *Let $u_n, u \in Y$. Then $\lim_{n\to\infty}\|u_n - u\|_Y = 0 \Longleftrightarrow \lim_{n\to\infty}\rho_{p(\cdot),h_1(\cdot)}(u_n - u) = 0$.*

(vi) *Let $u_n \in Y$. Then $\|u_n\|_Y \to \infty$ as $n \to \infty \Longleftrightarrow \rho_{p(\cdot),h_1(\cdot)}(u_n) \to \infty$ as $n \to \infty$.*

The following proposition fulfills an important role in this paper. In the following, we denote positive constants by c, c', C, C' which may vary from line to line, and put $\Omega_1 = \{x \in \Omega; p(x) \ge 2\}$, $\Omega_2 = \{x \in \Omega; p(x) < 2\}$.

Proposition 3.7. *Under (A.0)–(A.2), there exist positive constants c and C such that*

$$\begin{aligned}
\int_\Omega (\boldsymbol{a}(x, \nabla u(x)) - \boldsymbol{a}(x, \nabla v(x))) \cdot (\nabla u(x) - \nabla v(x))dx \ge c\rho_{h_1(\cdot),p(\cdot),\Omega_1}(u - v) \\
+ \left\{c(C + \|u\|_Y + \|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2}\rho_{h_1(\cdot),p(\cdot),\Omega_2}(u - v)\right\}^{2/p^+(\Omega_2)} \\
\wedge \left\{c(C + \|u\|_Y + \|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2}\rho_{h_1(\cdot),p(\cdot),\Omega_2}(u - v)\right\}^{2/p^-(\Omega_2)}
\end{aligned}$$

for $u, v \in Y$. Here and from now on, we denote $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for real numbers a and b.

In particular, if $v = 0$ and $\|u\|_Y < 1$, then we have

$$\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla u(x)dx \ge c_1(\rho_{h_1(\cdot),p(\cdot),\Omega_1}(u) + \rho_{h_1(\cdot),p(\cdot),\Omega_2}(u)^{2/p^-})$$

for some constant $c_1 > 0$. We also get the following estimate.

$$\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla u(x)dx \ge c\|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-} - C\|h_1\|_{L^1(\Omega)} \quad \text{for all } u \in Y. \tag{3.3}$$

Proof. For brevity of notation, for $u, v \in Y$, we put

$$J(u(x); v(x)) = (\boldsymbol{a}(x, \nabla u(x)) - \boldsymbol{a}(x, \nabla v(x))) \cdot (\nabla u(x) - \nabla v(x)).$$

We decompose the integral of $J(u(x); v(x))$ over Ω as follows.

$$\int_\Omega J(u(x); v(x))dx = \int_{\Omega_1} J(u(x); v(x))dx + \int_{\Omega_2} J(u(x); v(x))dx.$$

We can easily see that when $|\Omega_1| > 0$, it follows from (A.2) that

$$\int_{\Omega_1} J(u(x); v(x))dx \geq k_0 \int_{\Omega_1} h_1(x)|\nabla u(x) - \nabla v(x)|^{p(x)}dx.$$

When $|\Omega_2| > 0$, it follows from (A.2) that

$$(h_1(x)^{1/p(x)} + h_1(x)^{1/p(x)}|\nabla u(x)| + h_1(x)^{1/p(x)}|\nabla v(x)|)^{2-p(x)} J(u(x); v(x)) \\ \geq k_0|h_1(x)^{1/p(x)}\nabla u(x) - h_1(x)^{1/p(x)}\nabla v(x)|^2.$$

By integrating $p(x)/2$-powers of the above inequality over Ω_2, we have

$$\int_{\Omega_2} k_0^{p(x)/2}|h_1(x)^{1/p(x)}\nabla u(x) - h_1(x)^{1/p(x)}\nabla v(x)|^{p(x)}dx \\ \leq \int_{\Omega_2} (h_1(x)^{1/p(x)} + h_1(x)^{1/p(x)}|\nabla u(x)| + h_1(x)^{1/p(x)}|\nabla v(x)|)^{(2-p(x))p(x)/2} \\ \times J(u(x); v(x))^{p(x)/2}dx.$$

We note that

$$(h_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|)^{(2-p(\cdot))p(\cdot)/2} \in L^{2/(2-p(\cdot))}(\Omega_2),$$

and $(J(u(\cdot); v(\cdot))^{p(\cdot)/2} \in L^{2/p(\cdot)}(\Omega_2)$, and $(2 - p(x))/2 + p(x)/2 = 1$. By the Hölder inequality (Proposition 2.2), we have

$$k_1 \int_{\Omega_2} h_1(x)^{1/p(x)}|\nabla u(x) - h_1(x)^{1/p(x)}\nabla v(x)|^{p(x)}dx \\ \leq 2\|(h_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|)^{(2-p(\cdot))p(\cdot)/2}\|_{L^{2/(2-p(\cdot))}(\Omega_2)} \\ \times \|J(u(\cdot); v(\cdot))^{p(\cdot)/2}\|_{L^{2/p(\cdot)}(\Omega_2)},$$

where $k_1 = k_0^{p^+(\Omega_2)/2} \wedge k_0^{p^-(\Omega_2)/2}$. We choose $C > 1$ so that $C\|h_1(\cdot)^{1/p(\cdot)}\|_{L^{p(\cdot)}(\Omega_2)} \geq 1$. Then $\|Ch_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|\|_{L^{p(\cdot)}(\Omega_2)} \geq 1$ by the definition of $L^{p(\cdot)}$-norm. By Proposition 2.12,

$$\|(Ch_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|)^{(2-p(\cdot))p(\cdot)/2}\|_{L^{2/(2-p(\cdot))}(\Omega_2)} \\ \leq \|Ch_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|\|_{L^{p(\cdot)}(\Omega_2)}^{((2-p(\cdot))p(\cdot)/2)^+(\Omega_2)}.$$

Here since $(2 - p(x))p(x)/2 = -\frac{1}{2}(p(x) - 1)^2 + \frac{1}{2}$, we see that $(2 - p(x))p(x)/2)^+(\Omega_2) = (2 - p^-(\Omega_2))p^-(\Omega_2)/2$. Since it follows from Proposition 2.12 that

$$\|Ch_1(\cdot)^{1/p(\cdot)}\|_{L^{p(\cdot)}(\Omega_2)} \leq C\|h_1\|_{L^1(\Omega)}^{1/p^+(\Omega_2)} \vee \|h_1\|_{L^1(\Omega)}^{1/p^-(\Omega_2)} =: C_1,$$

and $\|h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)|\|_{L^{p(\cdot)}(\Omega_2)} \le \|u\|_Y$ and $\|h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|\|_{L^{p(\cdot)}(\Omega_2)} \le \|v\|_Y$, we have

$$\|(Ch_1(\cdot)^{1/p(\cdot)} + h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)| + h_1(\cdot)^{1/p(\cdot)}|\nabla v(\cdot)|)^{(2-p(\cdot))p(\cdot)/2}\|_{L^{2/(2-p(\cdot))}(\Omega_2)} \le (C_1 + \|u\|_Y + \|v\|_Y)^{(2-p^-(\Omega_2))p^-(\Omega_2)/2}.$$

Using Proposition 2.12,

$$\|J(u(\cdot);v(\cdot))^{p(\cdot)/2}\|_{L^{2/p(\cdot)}(\Omega_2)} \le \|J(u(\cdot);v(\cdot))\|_{L^1(\Omega_2)}^{p^+(\Omega_2)/2} \vee \|J(u(\cdot);v(\cdot))\|_{L^1(\Omega_2)}^{p^-(\Omega_2)/2}.$$

Hence we have

$$\begin{aligned}\int_{\Omega_2} J(u(x);v(x))dx &= \|J(u(\cdot);v(\cdot))\|_{L^1(\Omega_2)} \\ &\ge \left\{(C_1 + \|u\|_Y + \|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2} k_1 \int_{\Omega_2} h_1(x)|\nabla u(x) - \nabla v(x)|^{p(x)}dx\right\}^{2/p^+(\Omega_2)} \\ &\quad \wedge \left\{(C_1 + \|u\|_Y + \|v\|_Y)^{(p^-(\Omega_2)-2)p^-(\Omega_2)/2} k_1 \int_{\Omega_2} h_1(x)|\nabla u(x) - \nabla v(x)|^{p(x)}dx\right\}^{2/p^-(\Omega_2)}.\end{aligned}$$

In particular case where $v = 0$ and $\|u\|_Y < 1$,

$$\begin{aligned}\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla u(x)dx &\ge k_0 \int_{\Omega_1} h_1(x)|\nabla u(x)|^{p(x)}dx \\ &\quad + (C_1+1)^{p^-(\Omega_2)-2} k_1^{2/p^+(\Omega_2)} \wedge k_1^{2/p^-(\Omega_2)} \left\{\int_{\Omega_2} h_1(x)|\nabla u(x)|^{p(x)}dx\right\}^{2/p^-(\Omega_2)}.\end{aligned}$$

For the estimate (3.3), it suffices to use Lemma 3.4 (ii). This completes the proof. □

Here we state the assumptions on functions f and g in (1.1).

(f.1) $f = f(x,t)$ is a real Carathéodory function on $\Omega \times \mathbb{R}$ and there exist $1 \le a \in L^{\alpha(\cdot)}(\Omega)$ with $\alpha \in C_+(\overline{\Omega})$, and $q \in C_+(\overline{\Omega})$ with

$$q(x) < \frac{\alpha(x)-1}{\alpha(x)} p^*(x) \quad \text{for all } x \in \overline{\Omega}$$

such that

$$|f(x,t)| \le C_1(1 + a(x)|t|^{q(x)-1}) \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

where C_1 is a positive constant and $p^+ < q^-$.

(f.2) There exist $\theta > p^+$ and $t_0 > 0$ such that

$$0 < \theta F(x,t) \le f(x,t)t \quad \text{for all } t \in \mathbb{R} \setminus (-t_0, t_0) \text{ and a.e. } x \in \Omega,$$

where

$$F(x,t) = \int_0^t f(x,s)ds. \tag{3.4}$$

(f.3) $f(x,t) = o(|t|^{p^+-1})$ uniformly as $t \to 0$.

(g.1) $g = g(x,t)$ is a real Carathéodory function on $\Gamma_2 \times \mathbb{R}$ and there exist $1 \le b \in L^{\beta\cdot)}(\Gamma_2)$ with $\beta \in C_+(\overline{\Gamma_2})$, and $r \in C_+(\overline{\Gamma_2})$ with

$$r(x) < \frac{\beta(x)-1}{\beta(x)} p^{\partial}(x) \quad \text{for all } x \in \overline{\Gamma_2}$$

such that

$$|g(x,t)| \le C_2(1 + b(x)|t|^{r(x)-1}) \quad \text{for all } t \in \mathbb{R} \text{ and } \sigma\text{-a.e. } x \in \Gamma_2,$$

where C_2 is a positive constant and $p^+ < r^-$.

(g.2) Let θ and t_0 be as in (f.2). That is, there exist $\theta > p^+(\Omega_1) \vee 2p^+(\Omega_2)/p^-(\Omega_2)$ and $t_0 > 0$ such that

$$0 < \theta G(x,t) \le g(x,t)t \quad \text{for all } t \in \mathbb{R} \setminus (-t_0, t_0) \text{ and a.e. } x \in \Gamma_2,$$

where

$$G(x,t) = \int_0^t g(x,s)ds. \tag{3.5}$$

(g.3) $g(x,t) = o(|t|^{p^+-1})$ uniformly as $t \to 0$.

Lemma 3.8. *Under (f.1)–(f.3) and (g.1)–(g.3), we have the following.*

(i) *For any $\lambda > 0$, there exists a constant $C_1' > 0$ such that*

$$|F(x,t)| \le \frac{\lambda}{p^+}|t|^{p^+} + C_1' a(x)|t|^{q(x)} - \frac{1}{|\Omega|}\|h_1/p\|_{L^1(\Omega)} \quad \textit{for a.e. } x \in \Omega, t \in \mathbb{R}.$$

(ii) *For any $\lambda > 0$, there exists a constant $C_2' > 0$ such that*

$$|G(x,t)| \le \frac{\lambda}{p^+}|t|^{p^+} + C_2' b(x)|t|^{r(x)} \quad \textit{for } \sigma\textit{-a.e. } x \in \Gamma_2, t \in \mathbb{R}.$$

Proof. From (f.3), for any $\lambda > 0$, there exists $\delta \in (0,1)$ such that

$$|f(x,t)| \le \lambda |t|^{p^+-1} \quad \text{for a.e. } x \in \Omega, t \in (-\delta, \delta).$$

Hence we have

$$|F(x,t)| \le \frac{\lambda}{p^+}|t|^{p^+} \quad \text{for a.e. } x \in \Omega, t \in (-\delta, \delta).$$

On the other hand, from (f.1), we have

$$|F(x,t)| \le C_1(|t| + \frac{a(x)}{q(x)}|t|^{q(x)}) \le C_2' a(x)|t|^{q(x)} \text{ for a.e. } x \in \Omega, |t| \ge \delta.$$

If we choose $C_2'' > 0$ so that $C_2'' \delta^{q^+} \ge \frac{1}{|\Omega|}\|h_1/p\|_{L^1(\Omega)}$, then

$$C_2'' a(x)|t|^{q(x)} - \frac{1}{|\Omega|}\|h_1/p\|_{L^1(\Omega)} \ge C_2''\delta^{q^+} - \frac{1}{|\Omega|}\|h_1/p\|_{L^1(\Omega)} \ge 0$$

for a.e. $x \in \Omega$ and $|t| \ge \delta$. Hence $|F(x,t)| \le (C_2' + C_2'')a(x)|t|^{q(x)} - \frac{1}{|\Omega|}\|h_1/p\|_{L^1(\Omega)}$ for a.e. $x \in \Omega$ and $|t| \ge \delta$. It suffices to put $C_1' = C_2' + C_2''$.

Similarly (ii) holds. □

Define a functional on Y by

$$I(u) = \Phi(u) - J(u) - K(u) \quad \text{for } u \in Y, \tag{3.6}$$

where

$$\Phi(u) = \int_\Omega A(x, \nabla u(x))dx, \tag{3.7}$$

$$J(u) = \int_\Omega F(x, u(x))dx, \; F(x,t) \quad \text{is defined by (3.4)}, \tag{3.8}$$

$$K(u) = \int_{\Gamma_2} G(x, u(x))d\sigma_x, \; G(x,t) \quad \text{is defined by (3.5)}. \tag{3.9}$$

Proposition 3.9. *Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold. Then the functionals* $\Phi, J, K \in C^1(Y, \mathbb{R})$ *and the Fréchet derivatives* Φ', J' *and* K' *satisfy the following equalities.*

$$\langle \Phi'(u), v \rangle = \int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla v(x)dx, \tag{3.10}$$

$$\langle J'(u), v \rangle = \int_\Omega f(x, u(x))v(x)dx, \tag{3.11}$$

$$\langle K'(u), v \rangle = \int_{\Gamma_2} g(x, u(x))v(x)d\sigma_x \tag{3.12}$$

for all $u, v \in Y$. *Here and hereafter, we write the duality* $\langle \cdot, \cdot \rangle_{Y^*,Y}$ *by simply* $\langle \cdot, \cdot \rangle$.

For the proof, see [4, Proposition 4.2].

Proposition 3.10. *Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold. Then we have the following.*

(i) *The functionals* J *and* K *are weakly continuous in* Y, *that is, if* $u_n \to u$ *weakly in* Y *as* $n \to \infty$, *then* $J(u_n) \to J(u)$ *and* $K(u_n) \to K(u)$ *as* $n \to \infty$.

(ii) *The functional* Φ *is sequentially weakly lower semi-continuous in* Y, *that is, if* $u_n \to u$ *weakly in* Y *as* $n \to \infty$, *then* $\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$.

(iii) $\Phi(u) - \Phi(v) \ge \langle \Phi'(v), u - v \rangle$ *for all* $u, v \in Y$.

For the proof, see [4, Proposition 4.4].

Lemma 3.11. *Under (f.1)–(f.3) and (g.1)–(g.3), there exist constants* c_1, c_2, C_3 *and* C_4 *such that for* $u \in Y$ *with* $\|u\|_Y < 1$, *the following inequalities hold.*

(i) *We have*

$$J(u) \le \frac{\lambda}{p^+} c_1 \|u\|_Y^{p^+} + C_3 \|u\|_Y^{q^-} - \|h_1/p\|_{L^1(\Omega)}.$$

(ii) *We have*

$$K(u) \le \frac{\lambda}{p^+} c_2 \|u\|_Y^{p^+} + C_4 \|u\|_Y^{r^-}.$$

Proof. From Lemma 3.8,

$$J(u) \leq \frac{\lambda}{p^+} \int_\Omega |u(x)|^{p^+} dx + C_3 \int_\Omega a(x)|u(x)|^{q(x)} dx - \|h_1/p\|_{L^1(\Omega)}.$$

Here it suffices to note that

$$\int_\Omega |u(x)|^{p^+} dx \leq C\|u\|_Y^{p^+}$$

with some constant $C > 0$, and

$$\int_\Omega a(x)|u(x)|^{q(x)} dx \leq C'\|u\|_Y^{q^-}.$$

(ii) follows from the similar arguments as (i). □

Proposition 3.12. *Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold. Then there exist constants $c, c_1, c_2 > 0$ and $C_1', C_2' > 0$ such that for $u \in Y$ with $\|u\|_Y < 1$,*

$$I(u) \geq (c - \lambda c_1 - \lambda c_2)\, \|u\|_Y^{p^+} - C_1'\|u\|_Y^{q^-} - C_2'\|u\|_Y^{r^-}.$$

In particular, there exists $\rho \in (0, 1)$ such that

$$\inf_{\|u\|_Y = \rho} I(u) > 0. \tag{3.13}$$

Proof. Let $\|u\|_Y < 1$. It follows from (A.3) and Proposition 3.7 that

$$\begin{aligned}\Phi(u) = \int_\Omega A(x, \nabla u(x))dx &\geq \int_\Omega \frac{1}{p(x)} \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla u(x)dx - \|h_1/p\|_{L^1(\Omega)} \\ &\geq c\|u\|_Y^{p^+} - \|h_1/p\|_{L^1(\Omega)}.\end{aligned}$$

From Lemma 3.11,

$$I(u) = \Phi(u) - J(u) - K(u) \geq (c - \lambda c_1 - \lambda c_2)\|u\|_Y^{p^+} - C_1'\|u\|_Y^{q^-} - C_2'\|u\|_Y^{r^-}.$$

If we choose $\lambda > 0$ small enough so that $c'' := c - \lambda c_1 - \lambda c_2 > 0$, then we have

$$I(u) \geq \|u\|_Y^{p^+} (c'' - C_1'\|u\|_Y^{q^- - p^+} - C_2'\|u\|_Y^{r^- - p^+}).$$

Since $q^- > p^+$ and $r^- > p^+$, if $\|u\|_Y = \rho > 0$ is small, then we have $\inf_{\|u\|_Y = \rho} I(u) > 0$. □

Proposition 3.13. *Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold. Then there exists a constant $C_4 > 0$ such that*

$$I(u) \geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right) c\|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-} + \frac{1}{\theta}\langle I'(u), u\rangle - C_4 \quad \textit{for all } u \in Y.$$

Proof. From (A.3) and Lemma 3.4 (ii), for $u \in Y$, we have

$$\begin{aligned}\Phi(u) - \frac{1}{\theta}\langle \Phi'(u), u\rangle &= \int_\Omega A(x, \nabla u(x))dx - \frac{1}{\theta}\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla u(x)dx \\ &\geq \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\theta}\right) \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla u(x)dx - \|h_1/p\|_{L^1(\Omega)} \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\left(c\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}dx - C\int_\Omega h_1(x)dx\right) - \|h_1/p\|_{L^1(\Omega)} \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right) c\|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-} - C_1\|h_1\|_{L^1(\Omega)}\end{aligned}$$

for some constant $C_1 > 0$.

On the other hand, it follows from (f.2) that

$$0 < \theta F(x,t) \le f(x,t)t \text{ for a.e. } x \in \Omega, t \in \mathbb{R} \setminus (-t_0, t_0).$$

Put $\Omega_u = \{x \in \Omega; |u(x)| > t_0\}$. Then $\frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x)) \ge 0$ for a.e. $x \in \Omega_u$. For $x \in \Omega \setminus \Omega_u$, we have

$$\left|\frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x))\right| \le C_2(t_0 + a(x)t_0^{q^+} \vee t_0^{q^-}).$$

Hence we have

$$\begin{aligned}\frac{1}{\theta}\langle J'(u),u\rangle - J(u) &= \int_{\Omega_u} \left(\frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x))\right) dx \\ &\quad + \int_{\Omega\setminus\Omega_u} \left(\frac{1}{\theta} f(x,u(x))u(x) - F(x,u(x))\right) dx \\ &\ge -C_2 \int_{\Omega\setminus\Omega_u} (t_0 + a(x)t_0^{q^+} \vee t_0^{q^-}) dx \\ &\ge -C_2 t_0 |\Omega| - C_2 t_0^{q^+} \vee t_0^{q^-} \|a\|_{L^1(\Omega)}.\end{aligned}$$

Similarly we have

$$\frac{1}{\theta}\langle K'(u),u\rangle - K(u) \ge -C_3 t_0 |\Gamma_2| - C_3 t_0^{r^+} \vee t_0^{r^-} \|b\|_{L^1(\Gamma_2)}.$$

Thus we have

$$\begin{aligned}I(u) - \frac{1}{\theta}\langle I'(u),u\rangle &= \Phi(u) - \frac{1}{\theta}\langle \Phi'(u),u\rangle - \left(J(u) - \frac{1}{\theta}\langle J'(u),u\rangle\right) \\ &\quad - \left(K(u) - \frac{1}{\theta}\langle K'(u),u\rangle\right) \ge \left(\frac{1}{p^+} - \frac{1}{\theta}\right) c\|u\|_Y^{p^+} \wedge \|u\|_Y^{p^-} - C_4\end{aligned}$$

for some constant C_4. □

Proposition 3.14. *Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold. Then the functional I satisfies the Palais–Smale condition, that is, if a sequence $\{u_n\} \subset Y$ satisfies that $\lim_{n\to\infty} I(u_n) = \gamma \in \mathbb{R}$ exists and $\lim_{n\to\infty} \|I'(u_n)\|_{Y^*} = 0$, then $\{u_n\}$ has a convergent subsequence.*

Proof. Let $\{u_n\} \subset Y$ satisfy that $\lim_{n\to\infty} I(u_n) = \gamma \in \mathbb{R}$ exists and $\lim_{n\to\infty} \|I'(u_n)\|_{Y^*} = 0$.

Step 1. $\{u_n\}$ is bounded in Y. Indeed, if it is false, then passing to a subsequence, we can assume that $\lim_{n\to\infty} \|u_n\|_Y = \infty$. By proposition 3.13, we have

$$I(u_n) \ge \left(\frac{1}{p^+} - \frac{1}{\theta}\right) k_0 \|u_n\|_Y^{p^-} - \frac{1}{\theta}\|I'(u_n)\|_{Y^*}\|u_n\|_Y - C_4$$

for large n. Since $\frac{1}{p^+} - \frac{1}{\theta} > 0$ and $p^- > 1$ and $\lim_{n\to\infty} \|I'(u_n)\|_{Y^*} = 0$, we have $I(u_n) \to \infty$ as $n \to \infty$. This is a contradiction.

Step 2. Since Y is a reflexive Banach space from Proposition 3.5, there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u \in Y$ such that $u_{n'} \to u$ weakly in Y as $n' \to \infty$. Since $\{u_{n'} - u\}$ is bounded in Y and $\lim_{n'\to\infty} \|I'(u_{n'})\|_{Y^*} = 0$, we see that

$$\langle I'(u_{n'}), u_{n'} - u\rangle \to 0 \text{ as } n' \to \infty.$$

By Proposition 2.9, $u_{n'} \to u$ strongly in $L^{q(\cdot)}_{a(\cdot)}(\Omega)$ and $L^{r(\cdot)}_{b(\cdot)}(\Gamma_2)$ as $n' \to \infty$. From (f.1), using the Hölder inequality,

$$\begin{aligned}
&\left| \int_\Omega f(x, u_{n'}(x))(u_{n'}(x) - u(x))dx \right| \\
&\quad \le \int_\Omega C_1(1 + a(x)|u_{n'}(x)|^{q(x)-1})|u_{n'}(x) - u(x)|dx \\
&\quad \le C_1 \int_\Omega (a(x)^{1/q(x)}|u_{n'}(x) - u(x)| + a(x)^{1/q'(x)}|u_{n'}(x)|^{q(x)-1}a(x)^{1/q(x)}|u_{n'}(x) - u(x)|)dx \\
&\quad \le 2C_1\|1\|_{L^{q'(\cdot)}(\Omega)}\|a^{1/q(\cdot)}|u_{n'} - u|\|_{L^{q(\cdot)}(\Omega)} \\
&\qquad + 2C_1\|a^{1/q'(\cdot)}|u_{n'}(\cdot)|^{q(\cdot)-1}\|_{L^{q'(\cdot)}(\Omega)}\|a^{1/q(\cdot)}|u_{n'} - u|\|_{L^{q(\cdot)}(\Omega)}.
\end{aligned}$$

Since

$$\rho_{q'(\cdot)}(a^{1/q'(\cdot)}|u_{n'}|^{q(\cdot)-1}) = \int_\Omega a(x)|u_{n'}(x)|^{q(x)}dx$$

is bounded, we see that $\|a^{1/q'(\cdot)}|u_{n'}|^{q(\cdot)-1}\|_{L^{q'(\cdot)}(\Omega)}$ is bounded. Since $\|u_{n'} - u\|_{L^{q(\cdot)}_{a(\cdot)}(\Omega)} \to 0$ as $n' \to \infty$, we see that

$$\lim_{n'\to\infty} \langle J'(u_{n'}), u_{n'} - u\rangle = \lim_{n'\to\infty} \int_\Omega f(x, u_{n'}(x))(u_{n'}(x) - u(x))dx = 0.$$

Similarly, we have

$$\lim_{n'\to\infty} \langle K'(u_{n'}), u_{n'} - u\rangle = \lim_{n'\to\infty} \int_{\Gamma_2} g(x, u_{n'}(x))(u_{n'}(x) - u(x))d\sigma_x = 0.$$

Thus we have

$$\lim_{n'\to\infty} \langle \Phi'(u_{n'}), u_{n'} - u\rangle = \lim_{n'\to\infty} (\langle J'(u_{n'}), u_{n'} - u\rangle + \langle K'(u_{n'}), u_{n'} - u\rangle + \langle I'(u_{n'}), u_{n'} - u\rangle) = 0.$$

Since $u_{n'} \to u$ weakly in Y, we have $\lim_{n'\to\infty}\langle \Phi'(u), u_{n'} - u\rangle = 0$, so

$$\lim_{n'\to\infty} \langle \Phi'(u_{n'}) - \Phi'(u), u_{n'} - u\rangle = 0.$$

Since $\{u_{n'}\}$ is bounded in Y, it follows from Proposition 3.7 that

$$\int_\Omega h_1(x)|\nabla u_{n'}(x) - \nabla u(x)|^{p(x)}dx \to 0 \quad \text{as } n' \to \infty,$$

so $u_{n'} \to u$ strongly in Y. □

4 Main theorems

In this section, we state the main theorems (Theorem 4.3, 4.5 and 4.6).

Definition 4.1. We say $u \in Y$ is a weak solution of (1.1) if u satisfies that

$$\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla v(x)dx = \int_\Omega f(x, u(x))v(x)dx + \int_{\Gamma_2} g(x, u(x))v(x)d\sigma_x \tag{4.1}$$

for all $v \in Y$.

Remark 4.2. Since $C_0^\infty(\Omega) \subset Y$, if $u \in Y$ satisfies (4.1), then the equation (1.1) holds in the distribution sense.

Now we obtain the following three theorems.

Theorem 4.3. *Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ satisfying* (1.2). *Under the hypotheses (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3), the problem* (1.1) *has a nontrivial weak solution.*

Remark 4.4. This theorem extends the result of [8] in which the authors considered the case where $A(x,\xi) = \frac{1}{p(x)}|\xi|^{p(x)}$, $\Gamma_2 = \emptyset$ and $p^- \geq 2$. This theorem is new and also an extension to the case $p^- > 1$.

We impose one more assumption.

(f.4) For any $\delta' \in (0,1)$, the function $f(x,t)$ satisfies the following inequality.

$$f(x,t) \geq \begin{cases} ct^{m-1} & \text{for } t \in [\delta', 1], \\ 0 & \text{for } t \in [0,\infty) \setminus [\delta', 1], \end{cases}$$

where $c > 0$ and $0 < m < 1$.

For example, A function $f(x,t) = \chi_{\delta'}(t)|t|^{m-2}t + a(x)|t|^{q(x)-2}t$, where $\chi_{\delta'} \in C_0(\mathbb{R})$ satisfying that $0 \leq \chi_{\delta'} \leq 1$,

$$\chi_{\delta'}(t) = \begin{cases} 0 & \text{for } |t| \leq \delta'/2 \\ 1 & \text{for } \delta' \leq |t| \leq 1 \end{cases}$$

and that a function a is as in (f.1) verifies (f.1)–(f.4).

Theorem 4.5. *In addition to the hypotheses of Theorem 4.3, assume that (f.4) also holds. Then the problem* (1.1) *has at least two nontrivial weak solutions.*

Finally, in addition to the hypotheses of Theorem 4.3, we assume the following hypotheses.

(A.4) $A(x,\xi)$ is even with respect to ξ, that is, $A(x,-\xi) = A(x,\xi)$ for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

(f.5) $f(x,t)$ is odd with respect to t, that is, $f(x,-t) = -f(x,t)$ for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$.

(g.4) $g(x,t)$ is odd with respect to t, that is, $g(x,-t) = -g(x,t)$ for σ-a.e. $x \in \Gamma_2$ and all $t \in \mathbb{R}$.

Then we can derive that there exist infinitely many weak solutions of (1.1).

Theorem 4.6. *In addition to the hypotheses of Theorem 4.3, assume that (A.4), (f.5) and (g.4) also hold. Then the problem* (1.1) *has infinitely many nontrivial weak solutions.*

5 Proofs of Theorem 4.3, 4.5 and 4.6

In this section, we give proofs of Theorem 4.3, 4.5 and 4.6. Assume that (A.0)–(A.3), (f.1)–(f.3) and (g.1)–(g.3) hold.

The proofs of Theorem 4.3, 4.5 and 4.6 consist of some lemma and propositions.

Lemma 5.1. *Under the hypotheses of Theorem 4.3, we have the following.*

(i) $|F(x,t)| \le C_1'(1 + a(x)|t|^{q(x)})$ *for a.e.* $x \in \Omega$ *and all* $t \in \mathbb{R}$ *with some constant* $C_1' > 0$.

(ii) *There exists* $\gamma \in L^{\alpha(\cdot)}(\Omega)$ *such that* $\gamma(x) > 0$ *a.e.* $x \in \Omega$ *and* $F(x,t) \ge \gamma(x)t^\theta$ *for all* $t \in [t_0, \infty)$ *and a.e.* $x \in \Omega$, *where* $\alpha(\cdot)$ *and* t_0 *are as in (f.1) and (f.2), respectively.*

(iii) $|G(x,t)| \le C_2'(1 + b(x)|t|^{r(x)})$ *for* σ*-a.e.* $x \in \Gamma_2$ *and all* $t \in \mathbb{R}$ *with some constant* $C_2' > 0$.

(iv) *There exists* $\delta \in L^{\beta(\cdot)}(\Gamma_2)$ *such that* $\delta(x) > 0$ σ*-a.e.* $x \in \Gamma_2$ *and* $G(x,t) \ge \delta(x)t^\theta$ *for all* $t \in [t_0, \infty)$ *and* σ*-a.e.* $x \in \Gamma_2$, *where* $\beta(\cdot)$ *and* t_0 *are as in (g.1) and (g.2), respectively.*

Proof. (i) easily follows from (f.1).

(ii) From (f.2), for $t \ge t_0$,

$$0 < \theta F(x,t) \le f(x,t)t. \tag{5.1}$$

Put $\gamma(x) = F(x,t_0)t_0^{-\theta}$. Then $\gamma(x) > 0$ for a.e. $x \in \Omega$ and it follows from (ii) that

$$\gamma(x) \le C_1'(1 + a(x)t_0^{q(x)})t_0^{-\theta} \le C_1'(1 + a(x)t_0^{q^+} \vee t_0^{q^-})t_0^{-\theta}.$$

So $\gamma \in L^{\alpha(\cdot)}(\Omega)$. From (5.1),

$$\frac{\theta}{\tau} \le \frac{f(x,\tau)}{F(x,\tau)} = \frac{\frac{\partial F}{\partial \tau}(x,\tau)}{F(x,\tau)}.$$

Integrating this inequality over (t_0, t), we have

$$\theta \log \frac{t}{t_0} \le \log \frac{F(x,t)}{F(x,t_0)} \quad \text{for all } t \ge t_0.$$

This implies that $F(x,t) \ge \gamma(x)t^\theta$ for all $t \ge t_0$.

(iii) and (iv) follow from the similar argument as (i) and (ii) using (g.1) and (g.2), respectively. □

5.1 Proof of Theorem 4.3

For a proof of Theorem 4.3, we apply the following standard Mountain-Pass Theorem (cf. Willem [26]).

Proposition 5.2. *Let* $(V, \|\cdot\|_V)$ *be a Banach space and* $I \in C^1(V, \mathbb{R})$ *be a functional satisfying the Palais–Smale condition. Assume that* $I(0) = 0$, *and there exist* $\rho > 0$ *and* $z_0 \in V$ *such that* $\|z_0\|_V > \rho$, $I(z_0) \le I(0) = 0$ *and*

$$\alpha := \inf\{I(u); u \in V \text{ with } \|u\|_V = \rho\} > 0.$$

Let $G := \{\varphi \in C([0,1], V); \varphi(0) = 0, \varphi(1) = z_0\} \ne \emptyset$ *and* $\beta = \inf\{\max I(\varphi([0,1])); \varphi \in G\}$. *Then* $\beta \ge \alpha$ *and* β *is a critical value of* I.

We apply Proposition 5.2 with $(V, \|\cdot\|_V) = (Y, \|\cdot\|_Y)$. By Proposition 3.9 and Proposition 3.14, the functional I satisfies that $I \in C^1(Y, \mathbb{R})$ and the Palais–Smale condition holds. Since $\Phi(0) = J(0) = K(0) = 0$, we have $I(0) = 0$. According to (3.13),

$$\alpha = \inf_{\|v\|_Y = \rho} I(v) > 0. \tag{5.2}$$

We show that there exists $u_0 \in Y$ such that $\|u_0\|_Y > \rho$ and $I(u_0) \le 0$. Choose $v_0 \in C_0^\infty(\Omega)$ such that $v_0 \ge 0$ and $W = \{x \in \Omega; v_0(x) \ge t_0\}$ has a positive measure, where t_0 is as in (f.2). We see that $F(x, v_0(x)) > 0$ for a.e. $x \in W$ from (f.2). Let $t > 1$ and define $W_t = \{x \in \Omega; tv_0(x) \ge t_0\}$, then $W \subset W_t$. By Lemma 5.1 (ii), there exists $\gamma \in L^{\alpha(\cdot)}(\Omega)(\subset L^1(\Omega))$ such that $\gamma(x) > 0$ a.e. $x \in \Omega$ and $F(x,t) \ge \gamma(x)t^\theta$ for $t \in [t_0, \infty)$. Thereby,

$$\int_{W_t} F(x, tv_0(x))dx \ge \int_{W_t} \gamma(x)t^\theta v_0(x)^\theta dx \ge t^\theta L(v_0),$$

where $L(v_0) = \int_W \gamma(x)v_0(x)^\theta dx > 0$. For $t \in [0, t_0]$, it follows from Lemma 5.1 (i) that

$$|F(x,t)| \le C_1'(1 + a(x)t^{q(x)}) \le C_1'(1 + a(x)t_0^{q^+} \vee t_0^{q^-}).$$

By (f.2), $F(x, st) \ge F(x,t)s^\theta$ for $t \in \mathbb{R} \setminus (-t_0, t_0)$ and $s > 1$. Indeed, if we define $h(s) = F(x, st)$, then

$$h'(s) = f(x, st)t = \frac{1}{s}f(x, st)st \ge \frac{\theta}{s}F(x, st) = \frac{\theta}{s}h(s).$$

Thus $h'(s)/h(s) \ge \theta/s$, so $\log h(s)/h(1) \ge \theta \log s$ for $s > 1$. This implies $h(s) \ge h(1)s^\theta$.

(A.3) implies that

$$A(x, s\boldsymbol{\xi}) + \frac{h_1(x)}{p(x)} \le s^{p(x)}\left(A(x, \boldsymbol{\xi}) + \frac{h_1(x)}{p(x)}\right) \quad \text{for } s > 1.$$

Indeed, if we define $k(s) = A(x, s\boldsymbol{\xi}) + h_1(x)/p(x)$, then we see that $k'(s) \le \frac{1}{s}p(x)k(s)$. Hence we obtain the inequality. Thus we see that, for $t > 1$, $\Phi(su) + \|h_1/p\|_{L^1(\Omega)} \le s^{p(x)}(\Phi(u) + \|h_1/p\|_{L^1(\Omega)}$ for $u \in Y$ and $s > 1$. Thereby we see that, for $t > 1$,

$$\begin{aligned}
I(tv_0) &= \Phi(tv_0) - J(tv_0) \\
&\le \Phi(tv_0) - \int_{W_t} F(x, tv_0(x))dx - \int_{\Omega\setminus W_t} F(x, tv_0(x))dx \\
&\le t^{p^+}\Phi(v_0) + t^{p^+}\|h_1/p\|_{L^1(\Omega)} - \|h_1/p\|_{L^1(\Omega)} - t^\theta L(v_0) + C_1'|\Omega| + t_0^{q^+} \vee t_0^{q^-}\|a\|_{L^1(\Omega)}.
\end{aligned}$$

Since $\theta > p^+$ and $L(v_0) > 0$, we can see that $I(tv_0) \to -\infty$ as $t \to \infty$. Hence there exists $t_1 > 1$ such that $\|t_1v_0\|_Y > \rho$ and $I(t_1v_0) \le 0$. Put $u_0 = t_1v_0$.

If we define $\varphi(t) = tu_0$, then $\varphi \in G$, so $G \ne \emptyset$. Hence all the hypotheses of Proposition 5.2 hold. Therefore, $\beta = \inf\{\max I(\varphi([0,1]); \varphi \in G\}$ satisfies that $\beta \ge \alpha > 0$ and β is a critical value of I, that is, there exists $u_1 \in Y$ such that $I(u_1) = \beta$ and $I'(u_1) = 0$. Thus u_1 is a weak solution of (1.1). Since $I(u_1) = \beta \ge \alpha > 0 = I(0)$, u_1 is nontrivial weak solution of (1.1). This completes the proof of Theorem 4.3.

5.2 Proof of Theorem 4.5.

It follows from (f.4) that for $0 \le t \le 1$,

$$F(x,t) \ge \begin{cases} \int_{\delta'}^t f(x,s)ds & \text{if } t \ge \delta', \\ 0 & \text{if } t < \delta' \end{cases} \ge \begin{cases} \frac{c}{m}(t^m - (\delta')^m) & \text{if } t \ge \delta', \\ 0 & \text{if } t < \delta'. \end{cases}$$

Fix $t_1 \in (0,1)$ small enough and choose $\delta' \in (0,1)$ such that $(\delta')^m \le t_1$. If $(\delta')^m \le t$, then $F(x,t) \ge \frac{c}{m}(t^m - t)$ since $(\delta')^m \ge \delta'$. Choose $\varphi \in C_0^\infty(\Omega)$ so that $0 \le \varphi \le 1$ and $\varphi \not\equiv 0$. Put

$\Omega_{\delta'} = \{x \in \Omega; (\delta')^m \le t_1\varphi(x)\}$. Then we note that $|\Omega \setminus \Omega_{\delta'}| \to 0$ as $\delta' \to 0$, where $|A|$ denotes the measure of a measurable set A. Thus we have

$$\begin{aligned} J(t_1\varphi) &= \int_\Omega F(x, t_1\varphi(x))dx \\ &\ge \int_{\Omega_{\delta'}} F(x, t_1\varphi(x))dx \\ &\ge \frac{c}{m}\int_{\Omega_{\delta'}} ((t_1\varphi(x))^m - t_1\varphi(x))dx \\ &\ge \frac{c}{m}t_1^m\left(\int_\Omega \varphi(x)^m dx - \int_{\Omega\setminus\Omega_{\delta'}} \varphi(x)^m dx\right) - \frac{c}{m}t_1\int_{\Omega_{\delta'}}\varphi(x)dx \\ &\ge \frac{c}{m}t_1^m\left(\int_\Omega \varphi(x))^m dx - |\Omega\setminus\Omega_{\delta'}|\right) - \frac{c}{m}t_1|\Omega|. \end{aligned}$$

If we replace δ' with smaller one, if necessary, we may assume that $\int_\Omega \varphi(x)^m dx - |\Omega \setminus \Omega_{\delta'}| > 0$.

On the other hand, since $A(x, \boldsymbol{\xi})$ is convex with respect to $\boldsymbol{\xi}$ and $A(x, \mathbf{0}) = 0$, we have $A(x, t_1\boldsymbol{\xi}) = A(x, t_1\boldsymbol{\xi} + (1 - t_1)\mathbf{0}) \le t_1 A(x, \boldsymbol{\xi})$. Thus

$$\Phi(t_1\varphi) = \int_\Omega A(x, t_1\nabla\varphi(x))dx \le t_1\Phi(\varphi).$$

Therefore, we have

$$I(t_1\varphi) = \Phi(t_1\varphi) - J(t_1\varphi) \le t_1\left(\Phi(\varphi) + \frac{c}{m}|\Omega|\right) - \frac{c}{m}t_1^m\left(\int_\Omega \varphi(x)^m dx - |\Omega\setminus\Omega_{\delta'}|\right).$$

Since $0 < m < 1$, if $t_1 > 0$ is small enough, then we see that $I(t_1\varphi) < 0$. By Proposition 3.12, I is bounded from below on $\overline{B_\rho(0)}$, where $B_\rho(0) = \{v \in Y; \|v\|_Y < \rho\}$, ρ is as in (3.13). Hence

$$-\infty < \underline{c} := \inf_{v\in\overline{B_\rho(0)}} I(v) < 0.$$

Let $0 < \varepsilon < \inf_{v\in\partial B_\rho(0)} I(v) - \inf_{v\in\overline{B_\rho(0)}} I(v)$. Here we note that $\inf_{v\in\partial B_\rho(0)} I(v) > 0$. Then there exists $u \in \overline{B_\rho(0)}$ such that

$$\inf_{v\in\overline{B_\rho(0)}} I(v) \le I(u) \le \inf_{v\in\overline{B_\rho(0)}} I(v) + \varepsilon^2.$$

Since $\inf_{v\in\overline{B_\rho(0)}} I(v) < 0$, we can choose $u \in \overline{B_\rho(0)}$ so that $I(u) < 0$. By applying the Ekeland variational principle (cf. Ekeland [12, Theorem 1.1]) in the complete metric space $\overline{B_\rho(0)}$, there exists $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$I(u_\varepsilon) \le I(u), \tag{5.3}$$

$$I(u_\varepsilon) \le I(v) + \varepsilon\|v - u_\varepsilon\|_Y \text{ for all } v \in \overline{B_\rho(0)}, \tag{5.4}$$

$$\|u - u_\varepsilon\|_Y \le \varepsilon. \tag{5.5}$$

Define a functional $\widehat{I} : \overline{B_\rho(0)} \to \mathbb{R}$ by $\widehat{I}(v) = I(v) + \varepsilon\|v - u_\varepsilon\|_Y$ for $v \in \overline{B_\rho(0)}$. Since $I(u_\varepsilon) \le I(u) < 0$ from (5.3) and $I(v) > 0$ for all $v \in \partial B_\rho(0)$, we have $u_\varepsilon \in B_\rho(0)$. Choose $\rho' > 0$ small enough so that $u_\varepsilon + w \in \overline{B_\rho(0)}$ for $w \in \overline{B_{\rho'}(0)}$. From (5.4), since $\widehat{I}(u_\varepsilon) \le \widehat{I}(u_\varepsilon + w)$ for all $w \in \overline{B_{\rho'}(0)}$, we have

$$\begin{aligned} &\frac{\langle I'(u_\varepsilon), w\rangle + \varepsilon\|w\|_Y}{\|w\|_Y} \\ &\quad = \frac{\langle I'(u_\varepsilon), tw\rangle + \varepsilon t\|w\|_Y - (\widehat{I}(u_\varepsilon + tw) - \widehat{I}(u_\varepsilon))}{t\|w\|_Y} + \frac{\widehat{I}(u_\varepsilon + tw) - \widehat{I}(u_\varepsilon)}{t\|w\|_Y}. \end{aligned}$$

Here we note that from (5.4),

$$\widehat{I}(u_\varepsilon + tw) - \widehat{I}(u_\varepsilon) = I(u_\varepsilon + tw) + \varepsilon\|tw\|_Y - I(u_\varepsilon) \geq 0$$

for $t \in (0,1)$. Hence

$$\frac{\langle I'(u_\varepsilon), w\rangle + \varepsilon\|w\|_Y}{\|w\|_Y} \geq \frac{\langle I'(u_\varepsilon), tw\rangle - (I(u_\varepsilon + tw) - I(u_\varepsilon))}{t\|w\|_Y} \to 0 \quad \text{as } t \to +0.$$

So $\langle I'(u_\varepsilon), w\rangle + \varepsilon\|w\|_Y \geq 0$ for all $w \in \overline{B_{\rho'}(0)}$, so $\langle I'(u_\varepsilon), w\rangle \geq -\varepsilon\|w\|_Y$. Replacing w with $-w$, we have $|\langle I'(u_\varepsilon), w\rangle| \leq \varepsilon\|w\|_Y$ for all $w \in \overline{B_{\rho'}(0)}$. Thus $\|I'(u_\varepsilon)\|_{Y^*} \leq \varepsilon$. Letting $\varepsilon \to 0$, we see that $I(u_\varepsilon) \to \underline{c}$ and $I'(u_\varepsilon) \to 0$ in Y^*. Since I satisfies the Palais–Smale condition in Y, there exist a subsequence $\{u_n\}$ of $\{u_\varepsilon\}$ and $u_2 \in \overline{B_\rho(0)}$ such that $u_n \to u_2$ in Y and $I'(u_2) = 0$. Therefore, u_2 is a weak solution of (1.1). Since $I(u_2) = \underline{c} < 0 = I(0)$, u_2 is a nontrivial weak solution of (1.1). Since $I(u_2) = \underline{c} < 0 < I(u_1)$, we have $u_1 \neq u_2$. This completes the proof of Theorem 4.5.

5.3 Proof of Theorem 4.6

We apply the following Symmetric Mountain-Pass Theorem due to the Rabinowitz [23, Theorem 9.12] (cf. Xie and Xiao [27, Proposition 2.1]).

Proposition 5.3. *Let V be an infinite-dimensional real Banach space. A functional $I : V \to \mathbb{R}$ is of C^1-class and satisfies the Palais–Smale condition. Furthermore, assume that*

(I.1) *$I(0) = 0$ and I is an even functional, that is, $I(-u) = I(u)$ for all $u \in V$.*

(I.2) *There exist positive constants α and ρ such that*

$$\inf_{u\in\partial B_\rho(0)} I(u) \geq \alpha.$$

(I.3) *For each finite-dimensional linear subspace $V_1 \subset V$, the set $\{u \in V_1; I(u) \geq 0\}$ is bounded.*

Then I has an unbounded sequence of critical values.

We apply Proposition 5.3 with $V = Y$. Note that the functional I defined by (3.6) is of class C^1 (Proposition 3.9) and satisfies the Palais–Smale condition (Proposition 3.14). From (A.4), (f.5) and (g.4), (I.1) is trivial. (I.2) follows from (3.13). Thus it suffices to derive (I.3).

Let $u \in Y$ with $\|u\|_Y > 1$. Since $\Phi(u) \leq c_1\|h_0\|_{L^{p'(\cdot)}(\Omega)}\|u\|_Y + C_1\|u\|_Y^{p^+}$ from Lemma 3.4 and $p^+ > 1$, we have

$$\Phi(u) \leq C_5\|u\|_Y^{p^+} \quad \text{for some constant } C_5 > 0. \tag{5.6}$$

Since $F(x,t)$ is an even function with respect to t, it follows from Lemma 5.1 (ii) that $F(x,t) \geq \gamma(x)|t|^\theta$ for $|t| \geq t_0$. Define $\Omega_{t_0} = \{x \in \Omega; |u(x)| \geq t_0\}$. Then

$$J(u) = \int_\Omega F(x,u(x))dx = \int_{\Omega_{t_0}} F(x,u(x))dx + \int_{\Omega\setminus\Omega_{t_0}} F(x,u(x))dx.$$

From (f.1),

$$\int_{\Omega\setminus\Omega_{t_0}} |F(x,u(x))|dx \leq C_1'|\Omega| + t_0^{q^+} \vee t_0^{q^-} \|a\|_{L^1(\Omega)}.$$

Hence we have

$$\begin{aligned} J(u) &\ge \int_{\Omega_{t_0}} \gamma(x)|u(x)|^\theta dx - C_6 \\ &= \int_\Omega \gamma(x)|u(x)|^\theta dx - \int_{\Omega\setminus\Omega_{t_0}} \gamma(x)|u(x)|^\theta dx - C_6 \\ &\ge \int_\Omega \gamma(x)|u(x)|^\theta dx - C_7, \end{aligned} \tag{5.7}$$

where C_7 is a constant. Similarly we have

$$K(u) \ge \int_{\Gamma_2} \delta(x)|u(x)|^\theta d\sigma_x - C_8, \tag{5.8}$$

where C_8 is a constant.

We note that

$$\left(\int_\Omega \gamma(x)|u(x)|^\theta dx + \int_{\Gamma_2} \delta(x)|u(x)|^\theta d\sigma\right)^{1/\theta}$$

is a norm in Y.

Let Y_1 be any finite-dimensional linear subspace of Y. Since Y_1 is of finite-dimensional, the above norm is equivalent to the norm $\|u\|_Y$ in Y_1, so there exists $C_9 > 0$ such that

$$C_9\|u\|_Y^\theta \le \int_\Omega \gamma(x)|u(x)|^\theta dx + \int_{\Gamma_2} \delta(x)|u(x)|^\theta d\sigma_x.$$

Therefore, for $u \in Y_1$ with $\|u\|_Y > 1$, it follows from (5.6), (5.7) and (5.8) that

$$I(u) \le C_5\|u\|_Y^{p^+} - C_9\|u\|_Y^\theta + C_7 + C_8.$$

If $u \in Y_1$ with $\|u\|_Y > 1$ satisfies $I(u) \ge 0$, then we have $C_9\|u\|_Y^\theta \le C_5\|u\|_Y^{p^+} + C_7 + C_8$. Since $\theta > p^+$, the set $\{u \in Y_1; \|u\|_Y > 1, I(u) \ge 0\}$ is bounded, so $\{u \in Y_1; I(u) \ge 0\}$ is bounded. Since all the assumptions of Proposition 5.3 hold, I has an unbounded sequence of critical values, so problem (1.1) has infinitely many weak solutions. This completes the proof of Theorem 4.6.

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New oscillation criteria for third order nonlinear functional differential equations

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Abstract. The authors consider the general third order functional differential equation

$$\left(a_2(\nu)\left[\left(a_1(\nu)\left(x'(\nu)\right)^{\alpha_1}\right)'\right]^{\alpha_2}\right)' + q(\nu)x^{\beta}(\tau(\nu)) = 0, \qquad \nu \geq \nu_0,$$

and obtain sufficient conditions for the oscillation of all solutions. It is important to note that α_i for $i = 1, 2$, and β are somewhat independent of each other. The results obtained are illustrated with examples.

Keywords: oscillation, nonoscillation, delay differential equation, comparison method.

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1 Introduction

The primary objective of this work is to study the oscillatory behavior of solutions of the nonlinear third order differential equation

$$\left(a_2(\nu)\left[\left(a_1(\nu)\left(x'(\nu)\right)^{\alpha_1}\right)'\right]^{\alpha_2}\right)' + q(\nu)x^{\beta}(\tau(\nu)) = 0, \qquad \nu \geq \nu_0, \tag{1.1}$$

where α_i, $i = 1, 2$, and β are quotients of odd positive integers. A *solution* x of (1.1) is a continuous function on $[T_x, \infty)$, $T_x \geq \nu_0$ that satisfies (1.1) on $[T_x, \infty)$. We consider only those solutions $x(\nu)$ of (1.1) that are continuable, i.e., they satisfy $\sup\{|x(\nu)| : \nu \geq T\} > 0$ for all $T > T_x \geq \nu_0$. Such a solution is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and to be *nonoscillatory* otherwise.

Throughout, we always assume that

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($\mathcal{A}_1$) $a_i(\nu), q(\nu) \in C([\nu_0,\infty),\mathbb{R}_+)$ for $i = 1,2$, with $q(\nu) \not\equiv 0$ and

$$\int_{\nu_0}^{\infty} a_1^{-\frac{1}{\alpha_1}}(s)ds = \infty = \int_{\nu_0}^{\infty} a_2^{-\frac{1}{\alpha_2}}(s)ds; \tag{1.2}$$

($\mathcal{A}_2$) $\tau \in C^1([\nu_0,\infty),\mathbb{R})$ with $\tau(\nu) \le \nu$, $\tau'(\nu) \ge 0$, and $\lim_{\nu\to\infty} \tau(\nu) = \infty$.

As equation (1.1) is regarded as a useful instrument for simulating processes in various fields of applied mathematics, physics, and chemistry (see the monographs [6,22,24]), it is important to analyze the qualitative properties of equation (1.1). For several years now, there has been a growing interest in the asymptotic behavior of solutions of various forms of linear and nonlinear third order differential equations and their applications; see, e.g., [1–5,7–16,18,21] and the references therein.

In particular, Baculíková and Džurina [4] considered the third-order nonlinear delay differential equation of the form

$$\left(a_1(\nu)\left[x''(\nu)\right]^{\alpha_1}\right)' + q(\nu)x^{\beta}(\tau(\nu)) = 0. \tag{1.3}$$

They used a comparison theorem with appropriate lower-order equations to derive sufficient condition for the asymptotic and oscillatory behaviour of Eq. (1.3). This work allows us to note the following:

(1) Eq. (1.3) is a particular case of Eq. (1.1);

(2) There is no general rule to choose the function $\xi(\nu)$ that plays a very important role in deriving the oscillation of Eq. (1.1).

Chatzarakis et al. [9] considered the third-order linear differential equation of the form

$$\left(a_2(\nu)\left[\left(a_1(\nu)\left(x'(\nu)\right)\right)'\right]\right)' + q(\nu)x(\tau(\nu)) = 0, \tag{1.4}$$

and using the integral technique, comparison method, and Gronwall inequality, they improved the results reported in [4] by relaxing the above mentioned second observation. Inspired by the papers referenced here, we wish to the study of the general equation (1.1) and derive some easily verifiable sufficient conditions for the oscillation of all it solutions.

2 Basic lemmas

In view of (1.2), we introduce the following notation:

$$A(\nu,\nu_0) = \int_{\nu_0}^{\nu} a_2^{-\frac{1}{\alpha_2}}(s)ds \quad \text{and} \quad A^*(\nu,\nu_0) = \int_{\nu_0}^{\nu} \left(\frac{A(s,\nu_0)}{a_1(s)}\right)^{\frac{1}{\alpha_1}} ds.$$

Setting $G_1(x(\nu)) = (x'(\nu))^{\alpha_1}$ and $G_2(x(\nu)) = \left[(a_1(\nu)G_1(x(\nu)))'\right]^{\alpha_2}$, we can write equation (1.1) as the equivalent equation

$$(a_2(\nu)G_2(x(\nu)))' + q(\nu)x^{\beta}(\tau(\nu)) = 0 \quad \text{for } \nu \ge \nu_0. \tag{2.1}$$

To obtain our main results, we will utilize the following lemmas, the first of which is well known.

Lemma 2.1. *Let* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *hold. If* x *is an eventually positive solution of* (1.1) *for* $\nu \geq \nu_0$*, then there exists* $\nu_1 > \nu_0$ *such that either*

$$\text{(I)} \quad G_1(x(\nu)) \geq 0 \text{ and } G_2(x(\nu)) \geq 0, \quad \text{or} \quad \text{(II)} \quad G_1(x(\nu)) \leq 0 \text{ and } G_2(x(\nu)) \geq 0$$

for $\nu \geq \nu_1$.

Lemma 2.2. *Let* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *hold. If* x *is a positive solution of* (1.1) *such that Case I of Lemma 2.1 holds for* $\nu \geq \nu_1$*, then*

$$x(\nu) \geq A^*(\nu, \nu_1)\left((a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}}\right) \tag{2.2}$$

for $\nu \geq \nu_2 > \nu_1$.

Proof. Let x be a nonoscillatory solution of Eq. (1.1) such that $x(\nu) > 0$, $x(\tau(\nu)) > 0$, and which satisfies Case I of Lemma 2.1 for $\nu \geq \nu_1$ for some $\nu_1 > \nu_0$. Then,

$$a_1(\nu)G_1(x(\nu)) \geq \int_{\nu_1}^{\nu} (a_1(s)G_1(x(s)))'\, ds = \int_{\nu_1}^{\nu} \frac{a_2^{\frac{1}{\alpha_2}}(s)G_2^{\frac{1}{\alpha_2}}(x(s))}{a_2^{\frac{1}{\alpha_2}}(s)} ds,$$

that is,

$$a_1(\nu)(x'(\nu))^{\alpha_1} \geq A(\nu, \nu_1)a_2^{\frac{1}{\alpha_2}}(\nu)G_2^{\frac{1}{\alpha_2}}(x(\nu)),$$

so

$$x'(\nu) \geq \left(\frac{A(\nu,\nu_1)}{a_1(\nu)}\right)^{\frac{1}{\alpha_1}} (a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}}. \tag{2.3}$$

Integrating from ν_1 to ν gives

$$x(\nu) \geq (a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}} \int_{\nu_1}^{\nu} \left(\frac{A(s,\nu_1)}{a_1(s)}\right)^{\frac{1}{\alpha_1}} ds = A^*(\nu,\nu_1)\left((a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}}\right),$$

which completes the proof. □

For convenience, we let

$$B(\nu, s) = \left(\frac{A(\nu,s)}{a_1(s)}\right)^{\frac{1}{\alpha_1}}$$

and

$$\widehat{A}^*(\nu, \tau(\nu)) = \int_{\tau(\nu)}^{\nu} B(\nu,s)ds.$$

Lemma 2.3. *Let* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *hold. If* x *is a positive solution of* (1.1) *such that Case II of Lemma 2.1 holds for* $\nu \geq \nu_1$*, then*

$$x(\tau(\nu)) \geq \widehat{A}^*(\nu, \tau(\nu))\Big(a_2(\nu)G_2(x(\nu))\Big)^{\frac{1}{\alpha_1\alpha_2}} \tag{2.4}$$

for $\nu \geq \nu_2 > \nu_1$.

Proof. Let x be a nonoscillatory solution of Eq. (1.1) such that $x(\nu) > 0$, $x(\tau(\nu)) > 0$, and Case II of Lemma 2.1 is satisfied for $\nu \geq \nu_1$ for some $\nu_1 > \nu_0$. For $\nu \geq s > \nu_1$, we have

$$a_1(\nu)G_1(x(\nu)) - a_1(s)G_1(x(s)) = \int_s^{\nu} (a_1(u)G_1(x(u)))'\, du = \int_s^{\nu} \frac{a_2^{\frac{1}{\alpha_2}}(u)G_2^{\frac{1}{\alpha_2}}(x(u))}{a_2^{\frac{1}{\alpha_2}}(s)} du.$$

That is,

$$-a_1(s)(x'(s))^{\alpha_1} \geq A(\nu,s)a_2^{\frac{1}{\alpha_2}}(\nu)G_2^{\frac{1}{\alpha_2}}(x(\nu)),$$

so

$$-x'(s) \geq \left(\frac{A(\nu,s)}{a_1(\nu)}\right)^{\frac{1}{\alpha_1}} (a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}} \geq B(\nu,s)\,(a_2(\nu)G_2(x(\nu)))^{\frac{1}{\alpha_1\alpha_2}}. \tag{2.5}$$

Integrating from $\tau(\nu)$ to ν, we obtain

$$-x(\nu)+x(\tau(\nu)) \geq \Big(a_2(\nu)G_2(x(\nu))\Big)^{\frac{1}{\alpha_1\alpha_2}} \int_{\tau(\nu)}^{\nu} B(\nu,s)ds,$$

or

$$x(\tau(\nu)) \geq \widehat{A}^*(\nu,\tau(\nu))\Big(a_2(\nu)G_2(x(\nu))\Big)^{\frac{1}{\alpha_1\alpha_2}}.$$

This proves the lemma. □

Remark 2.4. In view of Lemma 2.3, from (1.1) and (2.4), we see that

$$-\left(a_2(\nu)G_2(x(\nu))\right)' = q(\nu)x^{\beta}(\tau(\nu)) \geq q(\nu)\left(\widehat{A}^*(\nu,\tau(\nu))\right)^{\beta}\left(a_2(\nu)G_2(x(\nu))\right)^{\frac{\beta}{\alpha_1\alpha_2}}.$$

Integrating this inequality from $\tau(\nu)$ to ν, we have

$$\limsup_{\nu\to\infty}\int_{\tau(\nu)}^{\nu} q(u)\left(\widehat{A}^*(u,\tau(u))\right)^{\beta} du > 1$$

in the case where $\frac{\beta}{\alpha_1\alpha_2}=1$.

We also have the following lemma.

Lemma 2.5. *In addition to the hypotheses of Lemma 2.3, assume that there exists a constant $\gamma > 1$ such that $\gamma\tau(\nu) \leq \nu$ for $\nu \geq \nu_2 > \nu_1$. Then*

$$x(\tau(\nu)) \geq \widehat{A}^*(\gamma\tau(\nu),\tau(\nu))\Big(a_2(\gamma\tau(\nu))G_2(x(\gamma\tau(\nu)))\Big)^{\frac{1}{\alpha_1\alpha_2}} \tag{2.6}$$

for $\nu \geq \nu_2 > \nu_1$.

Proof. If we integrate (2.5) from $\tau(\nu)$ to $\gamma\tau(\nu)$, we can obtain (2.6). □

3 Oscillation results

Our first oscillation result is as follows.

Theorem 3.1. *Let $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold and assume that there exists a constant $\gamma > 1$ such that $\gamma\tau(\nu) \leq \nu$ for $\nu \geq \nu_2 > \nu_1$. If the first-order delay equations*

$$Y'(\nu) + q(\nu)\left(A^*(\tau(\nu),\nu_1)\right)^{\beta}\left(Y(\tau(\nu))\right)^{\frac{\beta}{\alpha_1\alpha_2}} = 0 \tag{3.1}$$

and

$$Z'(\nu) + q(\nu)\left(\widehat{A}^*(\gamma\tau(\nu),\tau(\nu))\right)^{\beta}\left(Z(\gamma\tau(\nu))\right)^{\frac{\beta}{\alpha_1\alpha_2}} = 0 \tag{3.2}$$

are oscillatory, then Eq. (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1) such that $x(\nu) > 0$ and $x(\tau(\nu)) > 0$ for $\nu \geq \nu_1 > \nu_0$. According to Lemma 2.1, we distinguish the following two cases.

Case I. Using (2.2) in (2.1), we obtain

$$\begin{aligned} -(a_2(\nu)G_2(x(\nu)))' &= q(\nu)x^\beta(\tau(\nu)) \\ &\geq q(\nu)\left(A^*(\tau(\nu),\nu_1)\right)^\beta \left(\left(a_2(\tau(\nu))G_2(x(\tau(\nu)))\right)^{\frac{1}{\alpha_1\alpha_2}}\right)^\beta. \end{aligned}$$

Setting $Y(\nu) = a_2(\nu)G_2(x(\nu))$, this becomes

$$Y'(\nu) + q(\nu)\left(A^*(\tau(\nu),\nu_1)\right)^\beta (Y(\tau(\nu)))^{\frac{\beta}{\alpha_1\alpha_2}} \leq 0.$$

By [3, Lemma 2.1(I)], the related differential equation (3.1) also has a positive solution, which is a contradiction.

Case II. Using (2.6) in Eq. (2.1), we obtain

$$\begin{aligned} -(a_2(\nu)G_2(x(\nu)))' &= q(\nu)x^\beta(\tau(\nu)) \\ &\geq q(\nu)\left(\widehat{A}^*(\gamma\tau(\nu),\tau(\nu))\left((a_2(\gamma\tau(\nu))G_2(x(\gamma\tau(\nu))))^{\frac{1}{\alpha_1\alpha_2}}\right)\right)^\beta. \end{aligned}$$

Setting $Z(\nu) = a_2(\nu)G_2(x(\nu))$, this becomes

$$Z'(\nu) + q(\nu)\left(\widehat{A}^*(\gamma\tau(\nu),\tau(\nu))\right)^\beta (Z(\gamma\tau(\nu)))^{\frac{\beta}{\alpha_1\alpha_2}} \leq 0.$$

Again by [3, Lemma 2.1(I)], the corresponding differential equation (3.2) must have a positive solution. This contradiction proves the theorem. □

Example 3.2. Consider the third-order delay equation

$$\left(\nu\left[\left(\frac{1}{\nu^2}\left(x'(\nu)\right)\right)'\right]^3\right)' + \frac{c}{\nu^2}x^{\frac{1}{3}}\left(\frac{\nu}{3}\right) = 0, \qquad \nu \geq 1, \tag{3.3}$$

where $c > 0$ is a constant, $\alpha_1 = 1$, $\alpha_2 = 3$, $a_1(\nu) = \frac{1}{\nu^2}$, $a_2(\nu) = \nu$, $q(\nu) = \frac{c}{\nu^2}$, $\beta = \frac{1}{3}$, and $\tau(\nu) = \frac{\nu}{3}$. Clearly, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and (1.2) hold. Using

$$A(\nu,1) = \int_1^\nu a_2^{-\frac{1}{\alpha_2}}(s)ds = \int_1^\nu s^{-\frac{1}{3}}ds = \frac{3\nu^{\frac{2}{3}} - 3}{2}$$

and

$$A^*(\tau(\nu),1) = \int_1^{\tau(\nu)} \left(\frac{A(s,1)}{a_1(s)}\right)^{\frac{1}{\alpha_1}} ds = \int_1^{\frac{\nu}{3}} \left(\frac{s^2\left(3s^{\frac{2}{3}} - 3\right)}{2}\right) ds = \frac{1}{2}\left(\frac{\nu^{\frac{11}{3}}}{33 \cdot 3^{\frac{2}{3}}} - \frac{\nu^3}{27} + \frac{2}{11}\right),$$

it is not difficult to see that equation (3.1) becomes

$$Y'(\nu) + \frac{c}{2\nu^2}\left(\frac{\nu^{\frac{11}{3}}}{33 \cdot 3^{\frac{2}{3}}} - \frac{\nu^3}{27} + \frac{2}{11}\right)^{\frac{1}{3}} Y^{\frac{1}{9}}\left(\frac{\nu}{3}\right) = 0. \tag{3.4}$$

Also, using $\gamma = 2$ and

$$B(\nu,s) = \left(\frac{A(\nu,s)}{a_1(s)}\right)^{\frac{1}{\alpha_1}} = \frac{\int_s^\nu u^{-\frac{1}{3}}du}{\frac{1}{\nu^2}} = \frac{3\nu^2(\nu^{\frac{2}{3}} - s^{\frac{2}{3}})}{2},$$

we see that

$$\widehat{A}^*(\gamma\tau(\nu),\tau(\nu)) = \int_{\tau(\nu)}^{\gamma\tau(\nu)} B(\nu,s)ds = \int_{\frac{\nu}{3}}^{\frac{2\nu}{3}} \frac{3\nu^2(\nu^{\frac{2}{3}} - s^{\frac{2}{3}})}{2}ds = \frac{\nu^{\frac{11}{3}}}{2} - \frac{2^{\frac{5}{3}}\nu^{\frac{11}{3}} - \nu^{\frac{11}{3}}}{3^{\frac{5}{3}}},$$

and so equation (3.2) becomes

$$Z'(\nu) + \frac{c}{2\nu^2}\left(\frac{\nu^{\frac{11}{3}}}{2} - \frac{2^{\frac{5}{3}}\nu^{\frac{11}{3}} - \nu^{\frac{11}{3}}}{3^{\frac{5}{3}}}\right)^{\frac{1}{3}} Z^{\frac{1}{9}}\left(\frac{2\nu}{3}\right) = 0. \tag{3.5}$$

Clearly, [19, Theorem 5] guarantee that all solutions of Eqs. (3.4) and (3.5) are oscillatory. Thus, every solution of Eq. (3.3) oscillates.

Theorem 3.3. *Let $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold. If the first-order delay equation (3.1) is oscillatory and*

$$\limsup_{\nu\to\infty} \int_{\tau(\nu)}^{\nu} q(u)\left(A^*(\tau(\nu),\tau(s))\right)^\beta ds > 1 \tag{3.6}$$

for $\beta = \alpha_1\alpha_2$, then Eq. (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of Eq. (1.1) such that $x(\nu) > 0$ and $x(\tau(\nu)) > 0$ for $\nu \ge \nu_1 > \nu_0$. We again consider the two cases in Lemma 2.1.

Case I. Proceeding as in the proof of Theorem 3.1, we again obtain a contradiction.

Case II. Clearly, for $v \ge u > \nu_1$,

$$a_1(v)G_1(x(v)) - a_1(u)G_1(x(u)) = \int_u^v (a_1(s)G_1(x(s)))'ds = \int_u^v \frac{a_2^{\frac{1}{\alpha_2}}(s)G_2^{\frac{1}{\alpha_2}}(x(s))}{a_2^{\frac{1}{\alpha_2}}(s)}ds,$$

that is,

$$-a_1(u)G_1(x(u)) \ge a_2^{\frac{1}{\alpha_2}}(v)G_2^{\frac{1}{\alpha_2}}(x(v))\int_u^v \frac{1}{a_2^{\frac{1}{\alpha_2}}(s)}ds,$$

and so

$$-a_1(u)(x'(u))^{\alpha_1} \ge a_2^{\frac{1}{\alpha_2}}(v)G_2^{\frac{1}{\alpha_2}}(x(v))\int_u^v \frac{1}{a_2^{\frac{1}{\alpha_2}}(s)}ds,$$

Hence,

$$-x'(u) \ge (a_2(v)G_2(x(v)))^{\frac{1}{\alpha_1\alpha_2}}\left(\frac{1}{a_1(u)}\int_u^v \frac{1}{a_2^{\frac{1}{\alpha_2}}(s)}ds\right)^{\frac{1}{\alpha_1}},$$

and integrating from u to v gives

$$x(u) - x(v) \ge (a_2(v)G_2(x(v)))^{\frac{1}{\alpha_1\alpha_2}}\int_u^v \left(\frac{1}{a_1(y)}\int_y^v \frac{1}{a_2^{\frac{1}{\alpha_2}}(s)}ds\right)^{\frac{1}{\alpha_1}} dy,$$

or

$$x(u) \geq (a_2(v)G_2(x(v)))^{\frac{1}{\alpha_1\alpha_2}} A^*(v,u).$$

Now, for any $\nu \geq s > \nu_2$, for some $\nu_2 > \nu_1$, if we set $u = \tau(s)$ and $v = \tau(\nu)$ in the preceding inequality, gives

$$x(\tau(s)) \geq \Big(a_2(\tau(\nu))G_2(x(\tau(\nu)))\Big)^{\frac{1}{\alpha_1\alpha_2}} A^*(\tau(\nu),\tau(s)). \tag{3.7}$$

Integrating Eq. (1.1) from $\tau(\nu)$ to ν and then applying (3.7),

$$\begin{aligned} a_2(\tau(\nu))G_2(x(\tau(\nu))) &\geq \int_{\tau(\nu)}^{\nu} q(s)x^{\beta}(\tau(s))ds \\ &\geq \Big(a_2(\tau(\nu))G_2(x(\tau(\nu)))\Big)^{\frac{\beta}{\alpha_1\alpha_2}} \int_{\tau(\nu)}^{\nu} q(s)(A^*(\tau(\nu),\tau(s)))^{\beta} ds, \end{aligned}$$

which implies

$$\int_{\tau(\nu)}^{\nu} q(s)\,(A^*(\tau(\nu),\tau(s)))^{\beta}\, ds \leq 1,$$

and contradicts (3.6). □

Example 3.4. Consider the equation

$$\left(\frac{1}{\nu^2}\left[\left(\frac{1}{9\nu^2}\,(x'(\nu))\right)'\right]^3\right)' + \frac{\delta}{\nu^7}x^3\left(\frac{\nu}{2}\right) = 0, \qquad \nu \geq 1, \tag{3.8}$$

where we have $\alpha_1 = 1$, $\alpha_2 = 3$, $a_1(\nu) = \frac{1}{9\nu^2}$, $a_2(\nu) = \frac{1}{\nu^2}$, $q(\nu) = \frac{\delta}{\nu^7}$ for $\delta > 0$, $\beta = 3$ and $\tau(\nu) = \frac{\nu}{2}$. Clearly, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and (1.2) hold. Using

$$A(\nu,1) = \int_1^{\nu} a_2^{-\frac{1}{\alpha_2}}(s)ds = \int_1^{\nu}\left(\frac{1}{s^2}\right)^{-\frac{1}{3}} ds = \frac{\left(3\nu^{\frac{5}{3}} - 3\right)}{5}$$

and

$$\begin{aligned} A^*(\tau(\nu),1) = \int_1^{\tau(\nu)}\left(\frac{A(s,1)}{a_1(s)}\right)^{\frac{1}{\alpha_1}} ds &= \int_1^{\frac{\nu}{2}} \frac{s^2\left(3s^{\frac{5}{3}} - 3\right)}{5}ds \\ &= \frac{1}{5}\left(\frac{9\nu^{\frac{14}{3}}}{224\cdot 2^{\frac{2}{3}}} - \frac{\nu^3}{8} - \frac{5}{14}\right), \end{aligned}$$

it is not difficult to see that (3.1) becomes

$$Y'(\nu) + \frac{42}{125\cdot\nu^7}\left(\frac{9\nu^{\frac{14}{3}}}{7\cdot 2^{\frac{17}{3}}} - \frac{\nu^3}{8} - \frac{5}{14}\right)^3 Y\left(\frac{\nu}{2}\right) = 0. \tag{3.9}$$

Indeed, following [20, Theorem 2.1.1], Eq. (3.9) is oscillatory if

$$\lim_{\nu\to\infty}\int_{\frac{\nu}{2}}^{\nu} \frac{\delta}{125\cdot s^7}\left(\frac{9s^{\frac{14}{3}}}{7\cdot 2^{\frac{17}{3}}} - \frac{s^3}{8} - \frac{5}{14}\right)^3 ds > \frac{1}{e}.$$

And using

$$A(\nu,u)=\int_u^\nu a_2^{-\frac{1}{\alpha_2}}(s)ds=\int_u^\nu\left(\frac{1}{s^2}\right)^{\frac{-1}{3}}ds=\frac{3\nu^{\frac{5}{3}}-3u^{\frac{5}{3}}}{5}.$$

$$\begin{aligned}A^*(\tau(\nu),\tau(s))=\int_{\tau(s)}^{\tau(\nu)}\left(\frac{A(\nu,y)}{a_1(y)}\right)^{\frac{1}{\alpha_1}}dy&=\int_{\frac{s}{2}}^{\frac{\nu}{2}}\frac{27y^2\left(\nu^{\frac{5}{3}}-y^{\frac{5}{3}}\right)}{5}dy\\&=\frac{27}{25}\left(\frac{\nu^{\frac{5}{3}}\left(\nu^3-s^3\right)}{8}-\frac{3\nu^{\frac{14}{3}}-3s^{\frac{14}{3}}}{7\cdot 2^{\frac{17}{3}}}\right).\end{aligned}$$

Eq. (3.6) becomes

$$\begin{aligned}\int_{\tau(\nu)}^{\nu}q(s)\left(A^*(\tau(\nu),\tau(s))\right)^{\beta}ds&=\int_{\frac{\nu}{2}}^{\nu}\frac{\delta}{s^7}\left(\frac{27}{25}\left(\frac{\nu^{\frac{5}{3}}\left(\nu^3-s^3\right)}{8}-\frac{3\nu^{\frac{14}{3}}-3s^{\frac{14}{3}}}{7\cdot 2^{\frac{17}{3}}}\right)\right)^3ds\\&>1.\end{aligned}$$

By Theorem 3.3, every solution of (3.8) oscillates.

Theorem 3.5. *Let* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *hold. If* $\beta=\alpha_1\alpha_2$ *and there is a nondecreasing function* $\phi\in C^1([\nu_0,\infty),(0,\infty)$ *such that* (3.6) *and*

$$\limsup_{\nu\to\infty}\int_{\nu_1}^{\nu}\left[\phi(s)q(s)-\frac{(\phi'(s))^2(\phi(s))^{\frac{1}{\alpha_1\alpha_2}-2}}{4\beta\tau'(s)}\left(\frac{A(\tau(s),\nu_1)}{a_1(s)}\right)^{\frac{-1}{\alpha_1}}\right]ds=\infty \tag{3.10}$$

hold, then equation (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of Eq. (1.1) such that $x(\nu)>0$ and $x(\tau(\nu))>0$ for $\nu\geq\nu_1>\nu_0$. We again consider cases.

Case I. Define

$$\mathcal{W}(\nu)=\phi(\nu)\frac{a_2(\nu)G_2(x(\nu))}{x^{\beta}(\tau(\nu))}.$$

Then $\mathcal{W}(\nu)>0$, and using Lemma 2.2, the decreasing nature of $a_2(\nu)G_2(x(\nu))$, and (2.3)

$$\begin{aligned}\mathcal{W}'(\nu)&=\frac{\phi(\nu)(a_2(\nu)G_2(x(\nu)))'}{x^{\beta}(\tau(\nu))}+\frac{a_2(\nu)G_2(x(\nu))\phi'(\nu)}{x^{\beta}(\tau(\nu))}-\beta\frac{\phi(\nu)(a_2(\nu)G_2(x(\nu)))x'(\tau(\nu))\tau'(\nu)}{x^{\beta+1}(\tau(\nu))}\\&\leq-\phi(\nu)q(\nu)+\frac{\phi'(\nu)}{\phi(\nu)}\mathcal{W}(\nu)-\beta\tau'(\nu)\left(\frac{A(\tau(\nu),\nu_1)}{a_1(\nu)}\right)^{\frac{1}{\alpha_1}}\frac{\phi(\nu)(a_2(\nu)G_2(x(\nu)))^{1+\frac{1}{\alpha_1\alpha_2}}}{x^{\beta+1}(\tau(\nu))}\\&\leq-\phi(\nu)q(\nu)+\frac{\phi'(\nu)}{\phi(\nu)}\mathcal{W}(\nu)-\frac{\beta\tau'(\nu)}{\phi^{\frac{1}{\alpha_1\alpha_2}}(\nu)}\left(\frac{A(\tau(\nu),\nu_1)}{a_1(\nu)}\right)^{\frac{1}{\alpha_1}}\mathcal{W}^2(\nu).\end{aligned}$$

If we complete the square on the right hand side, we find that

$$\mathcal{W}'(\nu)\leq-\phi(\nu)q(\nu)+\frac{(\phi'(\nu))^2}{4\beta\tau'(\nu)}(\phi(\nu))^{\frac{1}{\alpha_1\alpha_2}-2}\left(\frac{A(\tau(\nu),\nu_1)}{a_1(\nu)}\right)^{\frac{-1}{\alpha_1}}.$$

Integrating the preceding inequality from ν_1 to ν, we see that (3.10) gives a contradiction to the fact that $W(\nu)\geq 0$.

Case II. Proceeding as in the proof of Theorem 3.3, leads to a contradiction in this case. □

Example 3.6. Consider the equation

$$\left(\frac{1}{\nu}\left[\left(\frac{1}{\nu}\left(x'(\nu)\right)^{\frac{1}{3}}\right)'\right]^3\right)' + \frac{\delta}{\nu^3}\, x\left(\frac{\nu}{3}\right) = 0, \qquad \nu \geq 1, \tag{3.11}$$

where we have $\alpha_1 = \frac{1}{3}$, $\alpha_2 = 3$, $a_1(\nu) = \frac{1}{\nu}$, $a_2(\nu) = \frac{1}{\nu}$, $q(\nu) = \frac{\delta}{\nu^3}$ for $\delta > 0$, $\beta = 1$ and $\tau(\nu) = \frac{\nu}{3}$. Clearly, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and (1.2) hold. Using $\phi(\nu) = \nu^4$ and $A(\tau(\nu), \nu_1) = \frac{3}{4}\left[\left(\frac{\nu}{3}\right)^{\frac{4}{3}} - 1\right]$ in Eq. (3.10), we have

$$\limsup_{\nu\to\infty} \int_1^{\nu} \left[\phi(s)q(s) - \frac{(\phi'(s))^2(\phi(s))^{\frac{1}{\alpha_1\alpha_2}-2}}{4\beta\tau'(s)} \left(\frac{A(\tau(s),1)}{a_1(s)}\right)^{\frac{-1}{\alpha_1}}\right] ds$$
$$= \limsup_{\nu\to\infty} \int_1^{\nu} \left[\delta s - \frac{3s^6}{s^4}\left(\frac{3s}{4}(s^{\frac{4}{3}} - 1)\right)^{-3}\right] ds = \infty.$$

It is not difficult to see that (3.6) holds, so by Theorem 3.5, every solution of (3.11) oscillates.

4 Concluding remark

Employing the methods of comparison, Riccati substitution, and the integral method, we introduced three novel conditions for the oscillation of a general third-order nonlinear delay differential equation. Interestingly, our results are applicable to linear, sublinear, and superlinear equations. Some illustrative examples are given to show the applicability of our results.

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Global solution for a system of semilinear diffusion-reaction equations with distinct diffusion coefficients

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Abstract. In this paper, we show the existence of solution for a relatively general system of semilinear parabolic equations with nonlinear reaction rate terms and inflow-outflow boundary conditions. Generally, to show the existence of global solution, it has been seen in the literature that either mass conservation or some growth condition on the source term is needed. Also, in several recent works only the nonlinearity up to certain order or of certain structure is allowed. However, our work considerably weakens the ones previously made by several authors on the coefficients of the elliptic operator, on the source (reaction rate) terms as well as on the boundary conditions. Our proof is also rather small and uses an argument based on implicit function theorem.

Keywords: semilinear parabolic equations, system of PDEs, equilibrium reaction rates, non-identical diffusion coefficients, global solution, mass action kinetics.

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1 Introduction

Even a very simple semilinear parabolic equation (e.g., diffusion-reaction equation) can have merely local solutions, i.e. solutions in some perhaps small neighbourhood of the initial time t_0. The same applies to coupled systems of more than one variable. Therefore, it takes some special structure of the nonlinearities to guarantee the existence of the *global* solutions, i.e. solutions on any given time interval $[0, T)$, $T < \infty$. In this regard, nonlinearities which are obtained by modelling *equilibrium (reversible)* reactions amongst chemical species via mass-action kinetics bear some potential for producing global solutions since positive productions are accompanied by negative ones. Kräutle [11] and Mahato et al. [12] showed that the production rates of a large class of J number of equilibrium (reversible) reactions of I number of chemical species can be reduced to the following setting: let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$, $S := [0, T)$ be the time interval for some $T < \infty$ and we denote by

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$\mathcal{S}$ the stoichiometric matrix such that $\mathcal{S} = (s_{ij}) \in M_{I\times J}$, the set of all $I \times J$ matrices, with $\text{rank}(\mathcal{S}) = J$ and $|s_{ij}| \in \{0\} \cup [1, \infty)$. The entries $s_{ij} = \nu_{ij} - \tau_{ij}$ for $1 \le i \le I$ and $1 \le j \le J$, where $-\tau_{ij} \in \mathbb{Z}_0^-$ and $\nu_{ij} \in \mathbb{Z}_0^+$ are the stoichiometric coefficients for reversible reactions given by

$$\tau_{1j}X_1 + \tau_{2j}X_2 + \cdots + \tau_{Ij}X_I \rightleftharpoons \nu_{1j}X_1 + \nu_{2j}X_2 + \cdots + \nu_{Ij}X_I, \tag{1.1}$$

where X_i, $1 \le i \le I$, denotes the chemical species involved in J reactions. Let $u = (u_1, \ldots, u_I)$ be the unknown concentration vector of I chemical species. For $k_j^f, k_j^b > 0$ and $j = 1, 2, \ldots, J$, we set

$$R_j^f(u) = k_j^f \prod_{\substack{m=1\\ s_{mj}<0}}^{I} u_m^{-s_{mj}}, R_j^b(u) = k_j^b \prod_{\substack{m=1\\ s_{mj}>0}}^{I} u_m^{s_{mj}} \tag{1.2a}$$

$$R_j(u) = \left(R_j^f(u) - R_j^b(u)\right), \quad R = (R_1, \ldots, R_J)^T, \quad \hat{f}(u) = \mathcal{S}R(u), \tag{1.2b}$$

where s_{mj}^- and s_{mj}^+ denote the negative and positive parts of s_{mj}, respectively, such that $s_{mj} = s_{mj}^+ - s_{mj}^-$. For the $i-th$ species,

$$(\mathcal{S}R(u))_i = \sum_{j=1}^{J} s_{ij} R_j(u). \tag{1.3}$$

Furthermore, let $\mathbb{D} = (D_{0_1}, \ldots, D_{0_I})$, where D_{0_i} denotes the symmetric $N \times N$ diffusive matrix of the i-th chemical species such that $D_{0_i} \in L^\infty(\Omega; \text{Sym}(N))$. Let $\boldsymbol{q} : S \times \Omega \to \mathbb{R}^N$ denote the velocity vector such that $\nabla \cdot \boldsymbol{q} = 0$. For $i = 1, 2, \ldots, I$, we now define the following operators:

$$A := \text{diag}(A_1, \ldots, A_I), \quad B := \text{diag}(B_1, \ldots, B_I), \quad A_i(\mathbb{D}, u) := A_i := \text{div}(\boldsymbol{j}_i) \tag{1.4a}$$

$$\boldsymbol{j}_i := -a_i D_{0_i} \nabla u_i + \mathbf{q} u_i \quad (= \text{diffusive + advective flux}) \tag{1.4b}$$

$$B_i(\mathbb{D}, u) := B_i := -(a_i D_{0i} \nabla u_i - \mathbf{q} u_i) \cdot \vec{n}, \tag{1.4c}$$

where $\vec{n} = \vec{n}(x)$ is the unit outward normal on $\partial\Omega$. The coefficients $a_i = a_i(x) \in \{0, 1\}$ and $\theta \in (0, 1]$ are explained below. The semilinear problem is: let g and h be given, then find a $u : \overline{S} \times \overline{\Omega} \to \mathbb{R}^I$ such that

$$\theta \frac{\partial u}{\partial t} + A(\mathbb{D}, u) = \hat{f}(u) \text{ in } S \times \Omega, \quad u(0, \cdot) = g \quad \text{in } \Omega \text{ and} \tag{1.5}$$

$$B(\mathbb{D}, u) = h \quad \text{on } S \times \partial\Omega. \tag{1.6}$$

For identical diffusion coefficients and $p > N + 1$, in [11] Kräutle's showed that the solution $u(t, \cdot)$ belongs in $H^{2,p}(\Omega)^I$ for a.a. t. Since he deals with diffusion and reaction in porous media, in his setting the porosity θ might be different from one whereas, Mahato et. al. in [12] considers a free flow and thus in his setting $\theta = 1$. Kräutle in [11] splits $\partial\Omega$ in two disjoint parts Γ_{in} and Γ_{out}, the inflow and outflow boundary parts, respectively, and specifies (1.5) as

$$-\mathbf{q}\cdot\vec{n} = 0 \quad \text{on } S \times \Gamma_{\text{in}}, \qquad -\mathbf{q}\cdot\vec{n} \le 0 \quad \text{on } S \times \Gamma_{\text{out}} \quad \text{and} \tag{1.7}$$

$$-D_{0i}\frac{\partial u_i}{\partial \vec{n}} = 0 \quad \text{on } S \times \Gamma_{\text{out}}, \qquad h \le 0 \quad \text{on } S \times \Gamma_{\text{in}} \quad \text{and} \quad h \ge 0 \quad \text{on } S \times \Gamma_{\text{out}}, \tag{1.8}$$

i.e. the outflow is entirely advective (cf. [11]). Let us denote the problem (1.5)–(1.6) by (P). In order to model the conditions in (1.6) and (1.7), we choose $a_i(x) := 1$ on (1.5)–(1.7) and $a_i(x) := 0$ on Γ_{out} whereas in Ω, $a_i(x) = 1$. The existence of solution of (1.5)–(1.7) in [11] is based on L^∞-estimates obtained via a Lyapunov functional, a fixed-point argument and classical $H^{2,p}$-theory for linear parabolic systems. An essential drawback of their approaches is that the diffusion coefficients need to be the same for all species. For a diffusion setting in a porous medium this can be justified by the observation that, usually, the advective flux dominates the diffusive one by order of magnitudes. In this note we address the issue of non-identical diffusion coefficients and show that the unique existence of weak solutions can still be guaranteed under certain assumptions which are given in next section.

1.1 Literature survey

In regards to the global in time solution, the authors in [18] showed that only mass control and positivity of the solutions are not sufficient to prevent the blowup in the solution and therefore we need a growth control condition. In the survey paper [16], the author has summarized the conditions (and limitations) under which the global solution can be guaranteed. The existence of weak solutions for (P) is shown in [17] under the assumption that the nonlinearities belong to $L^1(S \times \Omega)$. For quadratic nonlinearities, the existence of weak solutions is shown in [4] via a duality method. The authors in [8,20] showed the existence of a global renormalized solution if nonlinearities satisfy the entropy condition: $\sum_{i=1}^{I} \hat{f}_i(u)(\log u_i + \alpha_i) \leq 0$ for all $u \in (0,\infty)^I$ for some $\alpha_1, \alpha_2, \ldots, \alpha_I \in \mathbb{R}$. In [9], the global classical solutions for (P) with homogeneous Neumann boundary condition is shown under the assumption of mass conservation and entropy condition. In [9], the authors have assumed that $\hat{f}_i(u)$ has the cubic growth for $n = 1$ and $\hat{f}_i(u)$ has the quadratic growth for $n = 2$. The result later improved in [24] by utilizing a modified Gagliardo–Nirenbarg inequality. For higher-order nonlinearities in any space dimension if the diffusion coefficients are close to each other, i.e. if they are quasi-uniform, then the global existence of classical solutions is proved in [3, 5]. In [1], the authors have shown that under the polynomial growth condition, the L^∞-norm of the classical solution can be obtained, however, later on this growth condition is removed in [3]. The authors in [19] proved the global existence and uniform boundedness for quadratic growth and dimension $n = 2$ by relaxing the mass conservation to mass dissipation. This result is improved in [15] by replacing the mass dissipation assumption with a weaker intermediate sum condition. In higher dimensions, the existence of global classical solutions for nonlinearities with quadratic growth has been proved in [2,6,23] and for the case of $\Omega = \mathbb{R}^n$ is deduced in [10]. The work in [2] is based on mass conservation assumption together with the entropy condition, whereas in [23] the mass conservation condition is replaced by the mass dissipation assumption. A more general work is done in [6] under the mass control assumption. The uniform in time bound for the solutions is shown in [7]. Thus, in the previous works the global classical solutions in any space dimensions and for the higher order nonlinearities is shown under the restriction that the diffusion coefficients are close to each other and on some particular structure of the nonlinearities. Our work shows the existence of weak solution in a $H^{1,p}$ setting and far less assumptions on the nonlinearities. Our argument to prove the existence of global solution is rather small and involves less calculations. We have also incorporated inflow-outflow boundary conditions which in turn do not disturb the global existence of the solution.

2 Mathematical preliminaries

2.1 Function spaces

Let $p,q \in [1,\infty]$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda \in [0,1]$, $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. $\mathrm{Sym}(N)$ is the set of all real symmetric matrices $A=(A_{ij})$ normed by $|A|_{\mathrm{Sym}(N)} := \max_{i,j=1,\dots,N} |A_{ij}|$. $(\cdot,\cdot)_{p,\lambda}$ and $[\cdot,\cdot]_\lambda$ stand for the real- and complex-interpolation functor, respectively (cf. [25]). Likewise $L^p(\Omega)$, $H^{\alpha,p}(\Omega), \alpha \in \mathbb{N}$ and $C^\lambda(\overline{\Omega})$ denote the Lebesgue, Sobolev and Hölder spaces, respectively, with their usual norms (cf. [25]). " $\hookrightarrow$ " denotes a continuous imbedding. For a normed space Y, $L^p(S;Y)$ and $H^{1,p}(S;Y)$ are the (standard) Bochner and Sobolev–Bochner spaces (cf. [25]). Y^* stands for the dual of Y. $\langle\cdot,\cdot\rangle_I$ denotes the inner product on $\mathbb{R}^I$ and $|\cdot|_I$ be the corresponding norm. If X and Y are normed spaces, then $\mathfrak{L}(X,Y)$ represents the set of all bounded linear operators from X to Y and $\mathrm{Iso}(X;Y)$ stands for the set of all linear isomorphisms of $\mathfrak{L}(X;Y)$. From here on we assume $p>N+2$ is fixed. The imbedding $L^p(\Omega)^I \hookrightarrow (H^{1,q}(\Omega)^*)^I$ is given by $L^p(\Omega)^I \ni h_0 \mapsto L_h : \langle L_h, w\rangle := \sum_{i=1}^I \int_\Omega h_{0i}(x) w_i(x)dx,\ w \in (H^{1,q}(\Omega))^I$. We set $F_p := F_p(S,\Omega) := L^p(S;H^{1,p}(\Omega)) \cap H^{1,p}(S;H^{1,q}(\Omega)^*)$ normed by

$$\|\psi\|_{F_p} := \|\psi\|_{L^p(S;H^{1,p}(\Omega))} + \|\psi'\|_{L^p(S;H^{1,q}(\Omega)^*)}, \tag{2.1}$$

where u' is the distributional derivative of u. The solution space of the system under consideration is $F_p^I = \underbrace{F_p \times F_p \times \cdots \times F_p}_{I\text{-times}}$ and its norm is defined by $\|u\|_{F_p^I} := \left[\sum_{i=1}^I \|u_i\|_{F_p}^p\right]^{\frac{1}{p}}$. Note that for $p>N+2$, $F_p \subset C(\overline{S};(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p})$ and $F_p \subset C(\overline{S};C(\overline{\Omega}))$ (cf. [12]). For abbreviation, we set

$$V := H^{1,p}(\Omega)^I,\ W := H^{1,q}(\Omega)^I,\ W^* := [H^{1,q}(\Omega)^*]^I,\ V_0 := (H^{1,q}(\Omega)^*, H^{1,p}(\Omega))^I_{1-\frac{1}{p},p}, \tag{2.2a}$$

$$V_{\partial\Omega} := L^p(\partial\Omega)^I,\ P_0 = L^p(S;V_0),\ P_1 = F_p^I,\ P_2 := L^p(S;H^{1,q}(\Omega)^*)^I,\ P := P_2 \times V_0, \tag{2.2b}$$

$$Q_1 := C(S;C(\overline{\Omega}))^I,\ Q_2 := L^p(S;C(\overline{\Omega}))^I,\ E := Q_0 \times P_1 \text{ and } Q_0 := L^\infty(\Omega;\mathrm{Sym}(N))^I. \tag{2.2c}$$

Then, for $p>N+2$, a simple embedding result from [12] yields

$$L^p(S;V) \cap H^{1,p}(S;W^*) \hookrightarrow C(\overline{S};V_0) \hookrightarrow C(\bar{S};C(\bar{\Omega}))^I \hookrightarrow C(\bar{S};L^\infty(\Omega))^I. \tag{2.3}$$

Remark 2.1. If $\mathbb{D} = (D_{0_1}, D_{0_2}, \dots, D_{0_I}) \in L^\infty(\Omega;\mathrm{Sym}(N))^I$, i.e. $\forall i\ D_{0_i} \in L^\infty(\Omega;\mathrm{Sym}(N))$, then $|||\mathbb{D}|||_{L^\infty(\Omega;\mathrm{Sym}(N))^I} := \max_{i,j,k} \|D_{0_{ikj}}\|_{L^\infty(\Omega)} = \max_{i,j,k} \operatorname{ess\,sup}_{x\in\Omega} |D_{0_{ikj}}(x)|_{L^\infty(\Omega)}$. However, in our case, we only have $D_{0_{ikj}}(x) = D_0 = $ constant $\forall i,j,k$ and $x \in \Omega$, then $|||\mathbb{D}|||_{L^\infty(\Omega;Sym(N))^I} := \max_{i,j,k} \|D_{0_{ikj}}\|_{L^\infty(\Omega)} = \max_{i,j,k} \operatorname{ess\,sup}_{x\in\Omega} |D_{0_{ikj}}(x)| = D_0$.

To state the main theorem of the paper, we would require the following assumptions:

A1. let D_0 be a positive constant. For each $i=1,2,\dots,I$, $D_{0_i} := \mathrm{diag}(D_0,\dots,D_0) \in \mathrm{Sym}(N) \subset \mathbb{R}^{N\times N}$ be a diagonal matrix such that $\mathbb{D} := (D_{0_1},\dots,D_{0_I}) \in \mathrm{Sym}(N)^I$, where $I \in \mathbb{N}$.

A2. let $h \in L^p(\partial\Omega)^I$ and let $g \in V_0$ such that $g_i \geq 0$ for each i.

A3. let (1.6) and (1.7) hold true and $\vec{q} \in L^\infty(S\times\Omega)$ be such that $Q := \|\vec{q}\|_{L^\infty(S\times\Omega)} < \infty$ and $\vec{q}\cdot\vec{n} \in L^\infty(S\times\Gamma_{\mathrm{out}})$.

Definition 2.2 (Weak formulation). Let the assumptions A1–A3 hold true. Then, a vector $u \in F_p^I$ is said to be a weak solution of the problem (P) if $u(0) = g$ and

$$\int_S \langle \partial_t u_i, \phi_i \rangle_{W^* \times W} \, dt + \int_S \int_\Omega (D_{0_i} \nabla u_i - \vec{q} u_i) \nabla \phi_i \, dx \, dt - \int_{S \times \Gamma_i n} h\phi_i \, ds \, dt$$
$$+ \int_{S \times \Gamma_o ut} \vec{q} \cdot \vec{n} \phi_i \, ds \, dt = \int_S \langle \hat{f}_i, \phi_i \rangle_{W^* \times W} \, dt, \quad \forall \phi \in L^q(S; W). \quad (2.4)$$

Next, we state the existence theorem from [13] which addresses the question of same diffusion coefficients in the system.

Theorem 2.3. *Let the assumptions A1–A3 hold true. Then, there exists a unique global positive weak solution (in the sense of Definition 2.2) $u \in F_p^I$ of the problem (P).*

For Theorem 2.3, the global existence of solution follows from the construction of a particular type of Lyapunov function and exploiting the dissipative property of the reaction rate term and an application of Schaefer's fixed point theorem. The uniqueness and positivity follow from Gronwall's inequality.

Now, we shall state the main theorem of this paper which is existence of solution for different diffusion coefficients.

Theorem 2.4. *Let the assumptions A1–A3 hold true. Then, there is a neighborhood $U = U(D_0)$ in $L^\infty(\Omega; \mathrm{Sym}(N))^I$ such that (1.5)–(1.7) is solvable for all $\mathbb{D} \in U$. Moreover, the components of the solutions are non-negative.*

Remark 2.5. The proof of the implicit function theorem provides estimates for the size of $U(D_0)$. Here we do not yet go into detail, however, in [5,6] it has been shown that the if the diffusion co-efficients are very close to one and another, then there exists a classical solution.

Remark 2.6. In this note we do not directly employ the particular structure (1.5) of the reaction rates incorpoated into $\hat{f}(u)$, rather we use $\hat{f}$ is locally Lipschitz, then by Rademacher's theorem, we have $\hat{f} \in C^1(\mathbb{R}^I)^I$ in Theorem 2.4.

2.2 Operators

Let U, V be two Banach spaces and $x \in U$, then a continuous linear operator $\Psi : U \to V$ is called the Fréchet derivative of the operator $T : U \to V$ at x if $T(x+\theta) - T(x) = \Psi(\theta) + \phi(x,\theta)$ and $\lim_{\|\theta\|_U \to 0} \frac{\|\phi(x,\theta)\|_V}{\|\theta\|_U} = 0$ or, equivalently $\lim_{\|\theta\|_U \to 0} \frac{\|T(x+\theta) - T(x) - \Psi(\theta)\|_V}{\|\theta\|_U} = 0$. For a function $v = v(t,x)$, $t \in \overline{S}, x \in \overline{\Omega}$, we set $v(t) := v(t,\cdot)$. Let $H : U \to V$ and $G : \prod_{i=1}^n U_i \to V$, where U, U_i and V are Banach spaces. For functions $\xi \in U, \xi_i \in U_i \; \forall i$, $\mathcal{D}_{\xi^*} H(\xi)$ is the Fréchet derivative of $H = H(\xi)$ at ξ^* and $\partial_i G := \partial_{\xi_i} G := \frac{\partial G}{\partial \xi_i}$ is the partial Fréchet derivative of $G = G(\xi_1, \ldots, \xi_n)$.

We will now define the following operators:

By Remark 2.6, $\widehat{f} \in C^1(\mathbb{R}^I, \mathbb{R}^I)$ (production-rate vector). Clearly, $\widehat{f} : L^p(S;V) \to L^p(S;W^*)$ and we define $\left\langle \widehat{f}(u), v \right\rangle := \int_\Omega \widehat{f}(u(x))^T v(x) dx$ (the V-realisation of $\widehat{f}$). We then introduce the operator $F : F_p^I \to L^p(S;W^*)$ via

$$\langle \mathcal{F}(u)(t), v \rangle := \int_\Omega \langle \widehat{f}(u(t,x)), v(x) \rangle_I dx \quad a.e. \; t \in S, v \in W. \quad (2.5)$$

We further define $A(\mathbb{D},u) := A_1(\mathbb{D},u) + A_2(\mathbb{D},u)$, where $A_1(\mathbb{D},u) := -(\text{div}(D_{0_1}\nabla u_1),\ldots,\text{div}(D_{0_I}\nabla u_I))^T$, $A_2(\mathbb{D},u) := (\text{div}\, u_1\mathbf{q},\ldots,\text{div}\, u_I\mathbf{q})^T$, where

$$A(\mathbb{D},u) := A_1(\mathbb{D},u) + A_2(\mathbb{D},u),$$
$$\langle A_1(\mathbb{D},u),v\rangle := \textstyle\sum_{i=1}^{I} \int_\Omega D_{0_i}\nabla u_i \cdot \nabla v_k\, dx, \quad u \in V, v \in W,$$
$$\langle A_2(\mathbb{D},u),v\rangle := \textstyle\sum_{i=1}^{I} \int_\Omega u_i\mathbf{q} \cdot \nabla v_i\, dx, \quad u \in V, v \in W \quad \text{and}$$
$$\langle B(h),v\rangle := \textstyle\sum_{i=1}^{I} \int_{\partial\Omega} h_i v_i\, d\sigma, v \in V.$$

We note that $A : E \to P_2$. The corresponding extensions for time dependent $u = u(t)$ is, with the same notation, given by $A(\mathbb{D},u)(t) := A(\mathbb{D},u(t))$. Similarly, we proceed with A_1, A_2 and B and obtain

$$A, A_1, A_2 : E \to P_2, \quad B : L^p(S;V_{\partial\Omega}) \to P_2.$$

Also, note that $h \neq 0$ corresponds to non-homogeneous flux boundary conditions. Finally, we set

$$G_1(\mathbb{D},u) := u' + A(\mathbb{D},u) + B(h) - \mathcal{F}(u), \tag{2.6}$$
$$G_2(\mathbb{D},u) := u(0) - g, \tag{2.7}$$
$$G(\mathbb{D},u) := (G_1(\mathbb{D},u), G_2(\mathbb{D},u))^T. \tag{2.8}$$

Therefore, the problem (1.5)–(1.7) has now been formulated into an abstract evolution equation (2.6)–(2.8) and its weak formulation can be given by

Definition 2.7. Let the assumptions **A1–A3** be true. A function $u \in F_p^I$ is called a weak solution of problem (2.6)–(2.8), if $u(0) = g$ and $u'(t) + A(\mathbb{D},u(t)) + B(h(t)) = \mathcal{F}(u(t))$ in P_2. Alternatively, a function $u \in F_p^I$ is a weak solution of (2.6)–(2.8) if $G(\mathbb{D}),u) = 0$ in P.

In order to prove Theorem 2.4, we will first look in to following lemmas:

Lemma 2.8. *Let $p > N+2$. For $k = 1,2$, let P_k and Q_k be the normed spaces, defined as in (2.2a)–(2.2c), such that $P_1 \hookrightarrow Q_1$, $Q_2 \hookrightarrow P_2$. Assume further that $M : Q_1 \to Q_2$ be a Fréchet differentiable operator and set $\overline{M} := M|_{P_1}$ = restriction of M on P_1. Then,*

$$\mathfrak{L}(Q_1;Q_2) \hookrightarrow \mathfrak{L}(P_1;P_2), \quad \mathcal{D}_u\overline{M} = \mathcal{D}_u M|_{P_1} \in \mathfrak{L}(P_1;P_2). \tag{2.9}$$

Proof. Let $v \in Q_2$, then $\|v\|_{Q_2} \leq C\|v\|_{Q_1}$. Since $P_1 \hookrightarrow Q_1$ and $Q_2 \hookrightarrow P_2$, $\|v\|_{P_2} \leq C\|v\|_{Q_2} \leq C\|v\|_{Q_1} \leq C\|v\|_{P_1}$. This concludes (2.9). Now, we shall prove (2.9). We note that $M : Q_1 \to Q_2$ is a Fréchet differentiable operator, i.e. $Q_1 \ni l \mapsto \mathcal{D}_l M(l) \in Q_2$ is a bounded linear operator from Q_1 to Q_2. Then, by the definition of Fréchet derivative, we have for $\varepsilon > 0$

$$\|M(u+l) - M(u) - \mathcal{D}_l M(l)\|_{Q_2} < \varepsilon\|l\|_{Q_1}.$$

Then, by $P_1 \hookrightarrow Q_1$, $Q_2 \hookrightarrow P_2$, we obtain

$$\|M(u+l) - M(u) - \mathcal{D}_l M(l)\|_{P_2} < \varepsilon\|l\|_{P_1}. \tag{2.10}$$

(2.10) implies that $\mathcal{D}_l M : P_1 \to P_2$ is a bounded linear operator, i.e. $M : P_1 \to P_2$ is a Fréchet differentiable operator. Now, for $u,l \in P_1$, $\overline{M}(u+l) = M(u+l)$, $\overline{M}(u) = M(u)$, therefore from (2.10), we have $\|\overline{M}(u+l) - \overline{M}(u) - \mathcal{D}_l\overline{M}(l)\|_{P_2} = \|M(u+l) - M(u) - \mathcal{D}_l M(l)\|_{P_2} < \varepsilon\|l\|_{P_1}$ which implies $\mathcal{D}_l\overline{M}|_{P_1} \in \mathfrak{L}(P_1;P_2)$. □

Lemma 2.9 (Implicit Function Theorem, cf. [22]). *Suppose that X, Y, Z are Banach spaces, C is an open subset of $X \times Y$ and $T : C \to Z$ is continuous operator. Suppose further that for some $(x_0, y_0) \in C$,*

(i) $T(x_0, y_0) = 0$,

(ii) The Fréchet derivative of $T(\cdot, \cdot)$, when x is fixed is denoted by $T_y(x, y)$, is called the partial Fréchet derivative w.r.t. y which exists at each point (x, y) in a neighbourhood of the point (x_0, y_0) and is continuous at (x, y).

(iii) $[T_y(x_0, y_0)]^{-1} \in \mathfrak{L}(Z, Y)$.

Then there is an open subset U of X containing x_0 and a unique continuous mapping $y : U \to Y$ such that $T(x, y(x)) = 0$ and $y(x_0) = y_0$.

3 Proof of Theorem 2.4

Before we prove Theorem 2.4, we recall Theorem 2.4 since it deals with the system of semilinear parabolic PDEs with identical diffusion coefficients. It states that under the assumptions A1–A3, there exists a unique positive global weak solution $u \in P_1$ of the problem (1.5)–(1.7).

In other words, for a fixed $\mathbb{D} \in Q_0$ there exists a unique $u^* \in P_1$ such that we have

$$G(\mathbb{D}, u^*) = 0. \tag{3.1}$$

Now, in the spirit of Lemma 2.9, we denote $T = G, X = Q_0, Y = P_1$ and $Z = P$. Next, we will show that the following equations holds true:

$$\langle \mathcal{D}_{u^*} \mathcal{F}(u), v \rangle = \int_\Omega \langle \mathcal{D}_{u^*} \hat{f}(u(t,x)), v(x) \rangle_I dx \text{ for a.a. } t \in S,\ u \in P_1,\ v \in W, \tag{3.2}$$

$$\langle \mathcal{D}_{u^*} G_1(\mathbb{D}, u), v \rangle = \langle \partial_t u^* + A(\mathbb{D}, u^*) - \mathcal{D}_{u^*} F(u), v \rangle \text{ for } u \in P_1, \text{ and } v \in W, \tag{3.3}$$

$$\langle \mathcal{D}_{u^*} G_2(\mathbb{D}, u), v \rangle = \langle u^*(0) - g, v \rangle, \tag{3.4}$$

$$G_1 \in P_2 \text{ is Fréchet differentiable on } Q_0 \times P_1, \tag{3.5}$$

For fixed $(\mathbb{D}, u^*)$,

$$L := \mathcal{D}_{u^*} G(\mathbb{D}, u) \in \mathfrak{L}(P, P_1), \text{ i.e. } L = (\mathcal{D}_{u^*} G_1(\mathbb{D}, u), \mathcal{D}_{u^*} G_2(\mathbb{D}, u)) \in \text{Iso}(P_1, P). \tag{3.6}$$

We shall prove (3.2)–(3.6) in several steps.

Step 1: At first, we show that $G : Q_0 \times P_1 \to P_2 \times P_0$ is a continuous operator. We note that $G_1(\mathbb{D}, u) := u' + A_0(\mathbb{D}, u) + B(h) - F(u), G_2(\mathbb{D}, u) = u(0) - g$, Then, for a $\phi \in \mathfrak{D} := C_0^\infty(S \times \Omega)^I$, we have

$$\begin{aligned}\langle G_1(\mathbb{D}, u), \phi \rangle &= \langle u' + A(\mathbb{D}, u) + B(h) - \mathcal{F}(u), \phi \rangle \\ &= -\langle u, \partial_t \phi \rangle + \int_{S\times\Omega} \langle \mathbb{D} \nabla u, \nabla v \rangle_I + \int_{S\times\Omega} \langle u\mathbf{q}, \nabla \phi \rangle_I + \int_{S\times\partial\Omega} \langle h, v \rangle_I - \int_{S\times\Omega} \langle \hat{f}, \phi \rangle_I.\end{aligned}$$

By Hölder's inequality and $\mathfrak{D} \hookrightarrow P_1 \hookrightarrow P_0 \hookrightarrow P_2$, it follows that $\|G_1(\mathbb{D}, u)\|_{P_2} < \infty$. We also note that $\|G_2(\mathbb{D}, u)\|_{P_0} < \infty$. Altogether these two estimates imply that $\|G(\mathbb{D}, u)\|_{P_2 \times P_0} < \infty$, i.e. $G : Q_0 \times P_1 \to P_2 \times P_0$ is a continuous operator. Since $\hat{f} \in C^1(\mathbb{R}^I)^I$, then by definition of $\mathcal{F}(u)$ and Fréchet derivative, we obtain

$$\begin{aligned}\langle \mathcal{D}_{u^*}F(u), v\rangle &= \int_\Omega \Big\langle \Big[\hat{f}(u^*(t,x)+\theta(t,x)) - \hat{f}(u^*(t,x)) - \Psi(u^*(t,x),\theta(t,x))\Big], v(x)\Big\rangle_I \, dx \\ &= \int_\Omega \langle \mathcal{D}_{u^*}\hat{f}(u(t,x)), v(x)\rangle_I \, dx, \end{aligned} \tag{3.7}$$

for a.e. $t \in S$, $u, \theta \in V$ and $v \in W$. By (3.7), it follows that $\mathcal{D}_u F \in \mathfrak{L}(V;W)$ exists as $C(S) \hookrightarrow L^p(S)$. Now, since $H^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \hookrightarrow H^{1,q}(\Omega)^*$, the restriction $F|_{P_1}$ is Fréchet differentiable with $\mathcal{D}_u F|_{P_1} \in \mathfrak{L}(P_1; P_2)$ for all $u \in P_1$. Therefore, $\langle \mathcal{D}_u F, v\rangle$ exists in (3.2).

Step 2: We note from definition 2.1.1 that the Fréchet derivative of a linear operator T is T itself. Since, $\partial_t, A : F_p^I \to P_2$ are linear and B can be treated as a constant w.r.t. $u \in F_p^I$, therefore the Fréchet derivative of G_1 will yield

$$\mathcal{D}_{u^*}G_1(\mathbb{D}, u) = \partial_t u^* + A(\mathbb{D}, u^*) - \mathcal{D}_{u^*}\mathcal{F}(u) \in P_2.$$

This concludes (3.3). Likewise, (3.4) follows with similar arguments. Furthermore, by step 1, we know $\mathcal{D}_u F|_{P_1} \in \mathfrak{L}(P_1; P_2)$. This implies $\mathcal{D}_u G_1 \in \mathfrak{L}(Q_0 \times P_1; P_2)$, i.e. $\mathcal{D}_u G_1 : Q_0 \times P_1 \to P_2$ exists, i.e. $G_1(\mathbb{D}, u^*) \in P_2$ is Fréchet differentiable on $Q_0 \times P_1$.

Step 3.: We have obtained the continuity of one of the partial derivative $(\mathcal{D}_u G_1(\mathbb{D}, .))$ of a total of two and the existence of $\mathcal{D}_u G_2(\mathbb{D}, u^*)$, we obtain the existence of $\mathcal{D}G(\cdot,\cdot)$. Now the estimate

$$\|\mathcal{D}_u G_2(\mathbb{D}, u^*)\|_{P_2} = \|u_0^*\|_{P_2} \le C\|u_0^*\|_{L^p(S;V_0)} \le C\|u_0^*\|_{P_1} < \infty,$$

by the definition of function spaces and a straightforward imbedding of $P_1 \hookrightarrow P_0 \hookrightarrow P_2$. This implies the continuity of $\mathcal{D}_u G_2$. Hence, the continuity of both $\mathcal{D}_u G_1$ and $\mathcal{D}_u G_2$ imply the continuity of $\mathcal{D}G$.

Step 4: Let $(f, g) \in P_2 \times V_0$. In order to verify (3.6) it remains to show that the problem:

$$\text{Find } (\mathbb{D}, u^*) \in Q_0 \times P_1 \text{ with} \tag{3.8}$$

$$\partial_t u^* + A(\mathbb{D}, u^*) = \mathcal{D}_{u^*}\mathcal{F}(u), \quad u^*(0) = g, \tag{3.9}$$

has a unique solution. The operator $A(\mathbb{D}, u^*)$ possesses the maximal parabolic regularity property on P_1 in the L^p-sense (cf. [21]). From [21], it follows that (3.8)–(3.9) is uniquely solvable. Upon combining the steps 1 to 4, all the three conditions of implicit function theorem (Lemma 2.9) satisfied. Therefore, there exists an open neighbourhood of $\mathbb{D}$, $U(\mathbb{D}) \subset Q_0$ such that $G(D, u) = 0$ for all $D \in U(\mathbb{D})$. Moreover, the size of the neighbourhood $U(\mathbb{D})$ can be estimated, however this will be addressed somewhere else. Now, for the positivity, we multiply the PDE (1.5) with $-u_i^-$ ($-1\times$ negative part of u_i) and integrate over Ω_i^- (support of u_i^-) for all $i = 1, 2, \ldots, I$ and for a.e. $t \in S$. We use the fact that $u(0) \ge 0$ which eventually yields the positivity of the solutions via Gronwall's inequality.

Remark 3.1. Although we did not show the size of this neighbourhood in which the diffusion coefficient must lie, in [6] an idea regarding that is mentioned and recently in [14] has been shown that this can be further refined and a rather general neighbourhood can be chosen.

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Homoclinic solution to zero of a non-autonomous, nonlinear, second order differential equation with quadratic growth on the derivative

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Abstract. This work aims to obtain a positive, smooth, even, and homoclinic to zero (i.e. zero at infinity) solution to a non-autonomous, second-order, nonlinear differential equation involving quadratic growth on the derivative. We apply Galerkin's method combined with Strauss' approximation on the term involving the first derivative to obtain weak solutions. We also study the regularity of the solutions and the dependence on their existence with a parameter.

Keywords: Galerkin method, homoclinic solution, quadratic growth on the derivative, differential equation.

2020 Mathematics Subject Classification: 34A34, 34C37, 34D10, 34K12, 34L30.

1 Introduction

The existence of homoclinic solutions for autonomous and non-autonomous differential equations and Hamiltonian systems is a crucial subject in qualitative theory (see [19]).

In this work, the second-order equation in the real line considered is

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), & \text{in } \mathbb{R} \\ u(t) > 0, & \text{in } \mathbb{R} \\ \lim_{t\to\pm\infty} u(t) = 0, \end{cases} \tag{1.1}$$

with

(H_1) $1 < q < 2 < p < +\infty$ and $a_1 \in L^{\mathfrak{s}}(\mathbb{R}) \cap C(\mathbb{R})$, $\mathfrak{s} = \frac{2}{2-q}$, a positive even function;

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(H_2) $A : \mathbb{R} \to \mathbb{R}$ a Lipschitz, smooth (at least $C^1(\mathbb{R})$), non-decreasing function satisfying:

$$\exists\, \gamma \in (0,1) \quad \text{such that} \quad 0 < \gamma \leq A(t) \quad \forall t \in \mathbb{R};$$

(H_3) $g : \mathbb{R} \to \mathbb{R}$ a continuous function satisfying:

$$0 \leq sg(s) \leq |s|^{\theta} \quad \text{for all } s \in \mathbb{R}, \text{ where } 2 < \theta \leq 3. \tag{1.2}$$

The equation in (1.1) arises in several real phenomena, for instance, as the study of traveling wavefronts for parabolic reaction-diffusion equations with a local reaction term, chemical models, and others, as mentioned in [13, 14], and generalizes several classical equations such as Duffing-type equations [3, 10, 16] or Liénard-like systems [18].

Now, we state our main result.

Theorem 1.1. *There exists $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*]$, problem (1.1) has an even, positive and $C^2(\mathbb{R})$ homoclinic solution to the origin. Moreover, as $\lambda \to 0$, this solution goes to 0 in $C^0(\mathbb{R})$.*

We also find an additional result with respect to appropriate ranges of λ in order to guarantee the existence of solutions.

Proposition 1.2. *Assume the hypotheses of Theorem 1.1. If $\lambda > 0$ is sufficiently large, then equation (1.1) has no (positive) solution in $H^1(\mathbb{R})$.*

The idea to consider problem (1.1) came from article [3], where the authors considered a similar equation but with a different set of hypotheses; namely, their formulation was focused on the study of the equation

$$\begin{cases} -(A(u)u')' + u(t) = h(t, u(t)) + g(t, u'(t)) \text{ in } \mathbb{R} \\ u(\pm\infty) = u'(\pm\infty) = 0, \end{cases}$$

with

$(\widetilde{H}_1)$ $h, g : \mathbb{R}^2 \to \mathbb{R}$ locally Hölder continuous, even in the first variable and $h(t, 0) = g(t, 0) = 0$;

$(\widetilde{H}_2)$ there exist constants $0 < r_1, r_2 < 1$ and smooth functions $b \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $b(t) > 0$ for all $t \in \mathbb{R}$, $a_1 \in L^2(\mathbb{R})$ and $a_2 \in L^{\frac{2}{1-r_2}}(\mathbb{R})$, satisfying

$$b(t)|\mu|^{r_1} \leq h(t, \mu) \leq a_1(t) + a_2(t)|\mu|^{r_2}, \quad \forall (t, \mu) \in \mathbb{R}^2;$$

$(\widetilde{H}_3)$ there exist a constant $0 < r_3 < 1$ and smooth functions $a_3 \in L^{\frac{2}{1-r_3}}(\mathbb{R})$ and $a_4 \in L^2(\mathbb{R})$ satisfying

$$0 \leq g(t, \eta) \leq a_4(t) + a_3(t)|\eta|^{r_3} \quad \forall (t, \eta) \in \mathbb{R}^2;$$

$(\widetilde{H}_4)$ the function A is smooth, nondecreasing and there exists $\gamma \in (0,1)$ satisfying

$$0 < \gamma \leq A(t) \quad \forall t \in \mathbb{R}.$$

By comparison with our work, we considered sup-linear growth on u and u', terms involving this type of growth are not covered in [3]. Another aspect that we would like to emphasize is the weakening of the hypothesis over g: comparing with [3], we asked only for continuity over g, instead of Hölder continuity. Although the formulation presented here is not an immediate consequence of [3], some techniques therein proved to be quite solid and very useful in the study of this type of problem, transcending the circumstances framed by the authors.

Our formulation presented some interesting challenges, for instance, the problem is not variational. Among the non-variational techniques, we chose the Galerkin method as a tool to gather information about the existence of weak solutions. Although proving itself beneficial, the Galerkin method presented us with other types of challenges to circumvent. For example, the nonlinear term $g(|u'(t)|)$ with $0 \leq sg(s) \leq |s|^\theta$ and $2 < \theta \leq 3$ enables us to take $g(s) \equiv \operatorname{sign}(s)|s|^2$. Thus estimations involving $\int_\Omega |u'|^2$ become essential to the calculations but, at the same time, we cannot say much about it *a priori*: this is due to the lack of information about u', since the embedding theorems of $H_0^1(\Omega)$ do not provide substation information about u' as they do for u.

We consider the case $\theta = 3$ as the *critical* one and treat it separately in our estimations. For $\theta > 3$ we would get expressions involving $\int |u'|^{\theta-1}$ that we could not control, because $\theta - 1 > 2$ and we only know that $u' \in L^2$; for this reason we limited $\theta \leq 3$, and $\theta > 2$ was required because we wanted to focus on the sup-linear case.

There is some literature about equations on domains in $\mathbb{R}^n$ involving the term $|\nabla u|^2$ in the nonlinearity (see [1, 2, 9, 15]), some authors call this type of growth: "*critical growth on the gradient*". Simple changes on how this term appears in the equation can have dramatic effects on the outcome. For instance, a simple change in the sign of $|\nabla u|^2$ can lead to a total failure to obtain a solution (even in the weak sense), see the article [2] for more information. We also would like to emphasize article [9] for its results and broader discussion about PDE with quadratic growth on the gradient: the model problem studied by the authors is

$$-\operatorname{div}(A(x)\nabla u) = c_0(x)u + \mu(x)|\nabla u|^2 + f(x),$$

with suitable hypothesis. In this context, our problem (1.1) presents a similar structure that was not covered before, thus we believe it contributes to the discussion previously mentioned.

The methods applied in our work require certain symmetry, which is due mainly to a lack of a comparison principle (known to the authors) to guarantee that some limit-functions are not zero almost everywhere (a.e.), (see Proposition 2.24). To overcome this obstacle, we founded this work focusing on the set $\mathbb{E}_0^1(I) = \{u \in H_0^1(I); u(t) = u(-t) \text{ a.e.}\}$, $I \subset \mathbb{R}$ a symmetric interval, which is the subset of $H_0^1(I)$ consisting of even (or radial symmetric) functions.

In order to develop our study, in Section 2, we started by analyzing our equation on a bounded interval:

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), \text{ in } (-n, n) \\ u(n) = u(-n) = 0. \end{cases} \tag{P_n}$$

This restriction was essential to realize our estimations and to obtain upper bound constants that were crucial to construct the solution in $\mathbb{R}$ to the problem (1.1). The process developed in Section 2 consisted mainly of two steps:

First Approximate g by a sequence of Lipschitz functions (f_k) using the *Strauss Approximation*; this sequence received this name after its first appearance in the article [17].

This approximation was useful because it helped us to work with the necessary estimations without extra hypotheses over g. We followed [5] in the definition and presentation of the properties of the sequence (f_k). In this article, the authors used this approximation to avoid the usage of the Ambrosetti–Rabinowitz condition and were able to obtain a positive solution to the equation

$$\begin{cases} -\Delta u = \lambda u^{q(r)-1} + f(r,u) & \text{in } B(0,1) \\ u > 0 & \text{in } B(0,1) \\ u = 0 & \text{on } \partial B, \end{cases}$$

see [5] for more information.

We would like to emphasize that, in [5], the authors used this approximation in a term involving u; namely they used it to approximate $f(r,u)$. In our work, we used it in u'.

Second We used the sequence (f_k) to define an approximate problem in $(-n,n)$ and used the Galerkin method to obtain a weak solution. Then, using the work done by Gary M. Liberman [12], we obtained the *a priori estimation* summarized in Proposition 2.21. Thus we obtained a strong solution to this problem. Afterward, we were able to construct a strong solution to the problem (P_n).

In Section 4 we used the pieces of information gathered in Section 2 to construct a solution to the problem (1.1), thus proving Theorem 1.1. We also prove Proposition 1.2 in Section 4. We would like to point out the role of Section 3: there we study the asymptotic behavior, in respect to λ, of the solution to the problem (P_n) – the arguments presented were inspired by the article [8]. This was useful to tackle the last assertion of Theorem 1.1.

2 Solution in a bounded interval

First, we will obtain a solution to a problem related to (1.1); namely, we will study

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|), \text{ in } (-n,n) \\ u(n) = u(-n) = 0, \end{cases} \tag{P_n}$$

with the same set of hypothesis $(H_1), (H_2)$ and (H_3). The motivation for this approach is to construct a solution to the problem (1.1) using the solutions of (P_n). Although the analysis of (P_n) is easier since it is defined over $(-n,n)$, rather than $\mathbb{R}$, the lack of hypothesis over g creates a difficult situation for our estimations. To overcome this matter, we will utilize the *Strauss Approximation* on g at the same time that we approximate the problem (P_n). Let us define the sequence of functions that will approximate g.

Define $G(s) = \int_0^s g(t)dt$ so that G is differentiable and $G'(s) = g(s)$. By means of G we shall construct a sequence of approximations of g by Lipschitz functions $f_k : \mathbb{R} \longrightarrow \mathbb{R}$. Let

$$f_k(s) = \begin{cases} -k[G(-k-\frac{1}{k}) - G(-k)], & \text{if } s \le -k \\ -k[G(s-\frac{1}{k}) - G(s)], & \text{if } -k \le s \le \frac{-1}{k} \\ k^2 s[G(\frac{-2}{k}) - G(\frac{-1}{k})], & \text{if } \frac{-1}{k} \le s \le 0 \\ k^2 s[G(\frac{2}{k}) - G(\frac{1}{k})], & \text{if } 0 \le s \le \frac{1}{k} \\ k[G(s+\frac{1}{k}) - G(s)], & \text{if } \frac{1}{k} \le s \le k \\ k[G(k+\frac{1}{k}) - G(k)], & \text{if } s \ge k. \end{cases} \tag{2.1}$$

Remark 2.1. The construction of the sequence (f_k) is due to [17].

The advantage of this sequence lies in the properties that one can obtain from it:

Theorem 2.2 ([5, Lemma 1])**.** *The sequence f_k as defined above satisfies:*

1. $sf_k(s) \geq 0$ *for all* $s \in \mathbb{R}$*;*
2. *for all* $k \in \mathbb{N}$ *there is a constant* $c(k)$ *such that* $|f_k(\xi) - f_k(\eta)| \leq c(k)|\xi - \eta|$*, for all* $\xi, \eta \in \mathbb{R}$*;*
3. f_k *converges uniformly to* g *in bounded sets.*

Remark 2.3. From the definition of the sequence f_k, and the fact that $\text{sign}(g(s)) = \text{sign}(s)$ for all $s \in \mathbb{R}$, it follows without difficulties that *1* is true. In [5, p. 6, Prop. 5] one can find a detailed proof of *2*, so we will only prove *3* by an alternative argumentation.

Proof. Given a bounded set $J \subset \mathbb{R}$, there exists $m_0 \in \mathbb{N}$ such that $J \subset (-m_0, m_0)$; so to prove *3* we only need to prove that it holds in intervals such as $(-m, m)$, $m \in \mathbb{N}$. We may also assume that $k > m$. Given $\epsilon > 0$, for $s \in (-m, m)$ there are four possible cases:

Case I. $-m < s \leq \frac{-1}{k}$ Here we have that

$$|f_k(s) - g(s)| = \left| -k \left[G\left(s - \frac{1}{k}\right) - G(s) \right] - g(s) \right| = \left| \frac{[G(s - \frac{1}{k}) - G(s)]}{\frac{-1}{k}} - g(s) \right|.$$

Then, since $G'(s) = g(s)$, there exists $\delta(s) > 0$ such that $0 < |h| < \delta(s)$ implies

$$\left| \frac{[G(s+h) - G(s)]}{h} - g(s) \right| < \epsilon$$

From the family of open sets $\{(s - \delta(s), s + \delta(s)); s \in [-m, 0]\}$ we extract a finite subcover $\{(s_i - \delta(s_i), s_i + \delta(s_i)); i = 1 \dots l\}$ of the compact set $[-m, 0]$ and take $\delta = \min\{\delta(s_i); i = 1, \dots l\}$. Thus, for $k > \frac{1}{\delta}$ we get $|f_k(s) - g(s)| < \epsilon$.

Case II. $\frac{-1}{k} \leq s \leq 0$

Since $g(0) = 0$ and g is continuous, for the given $\epsilon > 0$ there exists $\delta > 0$ such that $|t| < \delta$ implies $|g(t)| < \epsilon/2$. Let $k_0 \in \mathbb{N}$ be such that $k_0 > m$ and $k_0 > 2/\delta$. Then, for $k > k_0$

$$\begin{aligned} |f_k(s) - g(s)| &= \left| k^2 s \left[G\left(\frac{-2}{k}\right) - G\left(\frac{-1}{k}\right) \right] - g(s) \right| \\ &\leq k^2|s| \left| \int_{-1/k}^{-2/k} |g(t)| dt \right| + |g(s)| \\ &\leq k^2 (\frac{1}{k^2}) \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall s \in [\frac{-1}{k}, 0]. \end{aligned}$$

The cases $0 \leq s \leq \frac{1}{k}$ and $\frac{1}{k} \leq s < m$ can be analyzed in a similar fashion. Thus we see that, for $\epsilon > 0$, we can take $k \in \mathbb{N}$ big enough such that $|f_k(s) - g(s)| < \epsilon$ *independently* of $s \in (-m, m)$. □

Lemma 2.4 ([5, Lemma 2])**.** *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* (1.2)*. Then the sequence* f_k *of Theorem* 2.2 *satisfies*

1. *For all* $k \in \mathbb{N}$, $0 \leq sf_k(s) \leq C_1|s|^\theta$ *for every* $|s| \geq \frac{1}{k}$*;*

2. *for all $k \in \mathbb{N}$, $0 \le s f_k(s) \le C_1|s|^2$ for every $|s| \le \frac{1}{k}$;*

where C_1 is a constant independent of k.

Proof. See [5, Page 8, Lemma 2]. □

Now we are in condition – using (f_k) – to define a problem that approximates problem (P_n). Let $\psi \in L^2(-n,n) \cap C(-n,n)$ be a positive, even function. We define our *approximate problem* by:

$$\begin{cases} -(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + f_k(|u'(t)|) + \frac{\psi}{k}, \text{ in } (-n,n) \\ u(n) = u(-n) = 0. \end{cases} \tag{P_n^k}$$

In the next subsection we will utilize the *Galerkin method* to obtain a solution to (P_n^k); afterward, we will let k vary and thus, as $k \to \infty$, obtain a solution to (P_n). Before jumping into the next subsection, let us define what we understand as *weak solution* to problem (P_n):

Definition 2.5. We will call $w \in H_0^1(-n,n)$ a *weak solution* of (P_n) if

$$\int_{-n}^{n} A(w)w'v' + \int_{-n}^{n} wv = \int_{-n}^{n} \lambda a_1|w|^{q-1}v + \int_{-n}^{n} |w|^{p-1}v + \int_{-n}^{n} g(|w'|)v$$

for all $v \in H_0^1(-n,n)$.

Remark 2.6. We will use, for the sake of clarity, the notation $\|\cdot\|_{W^{1,2}}$ for the usual norm of H_0^1 and for $(\|u\|_{L^2} + \|u'\|_{L^2})$ or $(\|u\|_{L^2}^2 + \|u'\|_{L^2}^2)^{1/2}$. Since these norms are equivalent the results will not change but the constants may. In most of the cases $\|u\|_{W^{1,2}} = \|u\|_{L^2} + \|u'\|_{L^2}$. We also emphasize that, when the context is clear, we will omit the domain in norms such as those from the spaces $L^p(-n,n)$.

Remark 2.7. The integrals in the definition above are well defined, see for instance the estimations of Proposition 2.16. The same is true for the definitions given in the next subsection.

2.1 Solution to the approximate problem

Our main goal in this subsection is to prove the following theorem:

Theorem 2.8. *There exist $\lambda^* > 0$, $\beta \in (0,1)$ and $k^* \in \mathbb{N}$ for which the problem (P_n^k) admits a nontrivial, even, non-negative $C^{1,\beta}[-n,n] \cap C^2(-n,n)$ solution for every $\lambda \in (0,\lambda^*)$ and $k \ge k^*$.*

As mentioned, we will utilize the *Galerkin method*; thus we will start by presenting the foundations that this method requires. The next lemma is a well-known result, but it plays a central role in all arguments involving the Galerkin method.

Lemma 2.9. *Let $\mathfrak{F} : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function such that $\langle \mathfrak{F}(x), x\rangle \ge 0$ for all $x \in \mathbb{R}^N$ with $\|x\|_{\mathbb{R}^N} = r$. Then there exists x_0 in the closed ball $B[0,r]$ such that $\mathfrak{F}(x_0) = 0$.*

Proof. See [11, Chap. 5, Theorem 5.2.5]. □

Now we will define an entity called *E-weak solution*. It is well known that the main focus of the Galerkin method is to obtain a weak solution, but we will utilize it to obtain an E-weak solution first.

Definition 2.10. A function $w \in H_0^1(-n,n)$ is called an *E-weak solution* of (P_n^k) if w is an *even function* satisfying

$$\int_{-n}^{n} A(w)w'\varphi' + \int_{-n}^{n} w\varphi = \int_{-n}^{n} \lambda a_1|w|^{q-1}\varphi + \int_{-n}^{n} |w|^{p-1}\varphi + \int_{-n}^{n} f_k(|w'|)\varphi + \int_{-n}^{n} \frac{\psi}{k}\varphi$$

for all $\varphi \in \mathbb{E}_0^1(-n,n) = \{u \in H_0^1(-n,n); u(t) = u(-t) \text{ a.e.}\}$.

The use of the E-weak solution will be central in our argumentation to obtain an *even* solution to the problem (P_n^k). This symmetry – being even – will also be beneficial in the use of our comparison principle, which is stated as follows:

Theorem 2.11 ([3, Theorem 3.1])**.** *Let $\sigma : (0,+\infty) \to \mathbb{R}$ be a continuous function such that the mapping $(0,+\infty) \ni s \mapsto \frac{\sigma(s)}{s}$ is strictly decreasing and $\rho > 0$. Suppose that there exist* even functions $v, w \in C^2(-\rho,\rho) \cap C[-\rho,\rho]$ *such that:*

1. $(A(w)w')' - w + \sigma(w) \le 0 \le (A(v)v') - v + \sigma(v)$ *in* $(-\rho,\rho)$*;*
2. $v, w \ge 0$ *in* $(-\rho,\rho)$ *and* $v(\rho) \le w(\rho)$*;*
3. $\{x \in (-\rho,\rho); v(x) = 0\}$ *and* $\{x \in (-\rho,\rho); w(x) = 0\}$ *have null measure in* $\mathbb{R}$*;*
4. $v' \cdot w' \ge 0$ *in* $(-\rho,\rho)$*;*
5. $v', w' \in L^\infty(-\rho,\rho)$.

Then $v \le w$ *in* $(-\rho,\rho)$.

Proof. The same as [3, p. 2419, Thm 3.1]. □

Turns out that, obtaining an E-weak solution enables us to recuperate a weak solution in the usual sense:

Definition 2.12. We will call $w \in H_0^1(-n,n)$ a *weak solution* of (P_n^k) if

$$\int_{-n}^{n} A(w)w'v' + \int_{-n}^{n} wv = \int_{-n}^{n} \lambda a_1|w|^{q-1}v + \int_{-n}^{n} |w|^{p-1}v + \int_{-n}^{n} f_k(|w'|)v + \int_{-n}^{n} \frac{\psi}{k}v$$

for all $v \in H_0^1(-n,n)$.

This is achieved by

Lemma 2.13 ([3, Lemma 4.1])**.** *Let $w \in H_0^1(-n,n)$ be an E-weak solution of (P_n^k). Then w is a weak solution of (P_n^k).*

Proof. See [3, p. 2421, Lemma 4.1]. □

The subset $\mathbb{E}_0^1(-n,n) \subset H_0^1(-n,n)$ can be understood as the set of radial symmetric functions in $\mathbb{R}$. One can prove without difficulties the following properties of $\mathbb{E}_0^1(-n,n)$:

i) it is a Hilbert space;

ii) it is separable;

iii) it has an orthonormal basis.

Let $\mathbb{E}_0^1(-n,n) = \{u \in H_0^1(-n,n); u(t) = u(-t) \text{ a.e.}\}$ and $(e_l)_{l=1}^{\infty}$ be an orthonormal basis of $\mathbb{E}_0^1(-n,n)$.

Define $V_M = \operatorname{span}\{e_1,\dots,e_M\}$; then for every $u \in V_M$ there exist $\xi_1,\dots,\xi_M$ in $\mathbb{R}$ such that $u = \sum_{i=1}^M \xi_i e_i$. By means of $T : V_M \to \mathbb{R}^M$, $T(u) = T(\sum_{i=1}^M \xi_i e_i) = (\xi_1,\dots,\xi_M)$, which is a linear isomorphism and preserve norm, we may define $\mathfrak{F} : \mathbb{R}^M \to \mathbb{R}^M$ such that

$$\mathfrak{F}(\xi) = (\mathfrak{F}_1(\xi),\dots,\mathfrak{F}_M(\xi)) \tag{2.2}$$

and

$$\mathfrak{F}_j(\xi) = \int_{-n}^{n} A(u)u'e_j' + \int_{-n}^{n} ue_j - \int_{-n}^{n} \lambda a_1|u|^{q-1}e_j - \int_{-n}^{n} |u|^{p-1}e_j - \int_{-n}^{n} f_k(|u'|)e_j - \int_{-n}^{n} \frac{\psi}{k}e_j,$$

where $j \in \{1,\dots,M\}$ and $u = T^{-1}(\xi)$, for all $\xi \in \mathbb{R}^M$.

Lemma 2.14. *The function $\mathfrak{F}$ is continuous.*

Remark 2.15. Our proof will use the fact that, if (x_n) is a sequence that converges to x and, for all subsequence (x_{n_l}) of (x_n), there exist a subsequence $(x_{n_{l_k}})$ of (x_{n_l}) such that $\mathfrak{F}(x_{n_{l_k}})$ converges to $\mathfrak{F}(x)$, then $\mathfrak{F}(x_n)$ converges to $\mathfrak{F}(x)$.

Proof. Given $\xi = (\xi_1,\dots,\xi_M) \in \mathbb{R}^M$, let $(\xi_l)_{l=1}^{\infty}$ be a sequence in $\mathbb{R}^M$ such that $\|\xi_l - \xi\|_{\mathbb{R}^M} \to 0$. By means of T we can identify $T^{-1}(\xi) = u = \sum_{i=1}^M e_i\xi_i$ and $T^{-1}(\xi_l) = u_l = \sum_{i=1}^M e_i\xi_i^l$. Since T is isometry we have that $\|u_l - u\|_{W^{1,2}} \to 0$. That is, $\|u_l - u\|_{L^2} \to 0$ and $\|u_l' - u'\|_{L^2} \to 0$. Taking a subsequence, if necessary, we may assume that

$$u_l(x) \to u(x) \quad \text{a.e. on } (-n,n),$$
$$u_l'(x) \to u'(x) \quad \text{a.e. on } (-n,n),$$

and $|u_l(x)| \le h_1(x)$, $|u_l'(x)| \le h_2(x)$ a.e. on $(-n,n)$, with $h_1,h_2 \in L^2(-n,n)$. Let $j \in \{1,2,\dots,M\}$, we will prove that $\mathfrak{F}_j(\xi_l) \to \mathfrak{F}_j(\xi)$.

$$\left|\int_{-n}^{n} A(u_l)u_l'e_j' - \int_{-n}^{n} A(u)u'e_j'\right| \le \int_{-n}^{n} \left(|u_l'||A(u_l) - A(u)| + |A(u)||u_l' - u'|\right)|e_j'|, \tag{2.3}$$

since $|u_l'(x)||A(u_l(x)) - A(u(x))||e_j'(x)| \to 0$ a.e. and $|A(u(x))||u_l'(x) - u'(x)||e_j'(x)| \to 0$ a.e., by the Dominated Convergence Theorem (D.C.T) the left side of (2.3) tends to zero as $l \to +\infty$.

$$\left|\int_{-n}^{n} u_le_j - \int_{-n}^{n} ue_j\right| \le \int_{-n}^{n} |u_l - u||e_j| \to 0 \quad \text{by (D.C.T).} \tag{2.4}$$

$$\begin{aligned}
&\left|\int_{-n}^{n} [\lambda a_1(|u_l|^{q-1} - |u|^{q-1}) + (|u_l|^{p-1} - |u|^{p-1}) + (f_k(u_l') - f_k(u'))]e_j\right| \\
&\quad\le \int_{-n}^{n} \lambda|a_1|\left||u_l|^{q-1} - |u|^{q-1}\right||e_j| + \int_{-n}^{n} \left||u_l|^{p-1} - |u|^{p-1}\right||e_j| \\
&\qquad + \int_{-n}^{n} \left|f_k(|u_l'|) - f_k(|u'|)\right||e_j|,
\end{aligned} \tag{2.5}$$

since that $|u_l|^{q-1} \to |u|^{q-1}$ a.e. and $|u_l|^{p-1} \to |u|^{p-1}$ a.e., (D.C.T) implies that the first two integrals above converge to zero. Using the second item of Theorem 2.2, we have

$$\int_{-n}^{n} \left|f_k(|u_l'|) - f_k(|u'|)\right||e_j| \le \int_{-n}^{n} c(k)|u_l' - u'||e_j|. \tag{2.6}$$

Then, by (D.C.T), (2.6) converges to 0 as $l \to +\infty$.

These estimations show us that for every subsequence (ξ_{l_k}) of (ξ_l), there exists a subsequence $(\xi_{l_{k_n}})$ of (ξ_{l_k}) that $\mathfrak{F}_j(\xi_{l_{k_n}}) \to \mathfrak{F}_j(\xi)$. Therefore $\mathfrak{F}_j(\xi_l) \to \mathfrak{F}_j(\xi)$. □

Proposition 2.16. *There exist $\lambda^* > 0$ and $k^* \in \mathbb{N}$ for which the problem (P_n^k) admits a nontrivial weak solution for every $\lambda \in (0, \lambda^*)$ and $k \geq k^*$.*

Remark 2.17. We will, in fact, search for an E-weak solution; but as seen in Lemma 2.13 this will also be a weak solution.

Proof. Our aim is to use Lemma 2.9, with the function $\mathfrak{F}$ defined in (2.2). Given $\xi \in \mathbb{R}^M$, we have that

$$\langle \mathfrak{F}(\xi), \xi \rangle = \int_{-n}^{n} A(u)|u'|^2 + \int_{-n}^{n} |u|^2 - \int_{-n}^{n} \lambda a_1 |u|^{q-1} u - \int_{-n}^{n} |u|^{p-1} u - \int_{-n}^{n} f_k(|u'|) u - \int_{-n}^{n} \frac{\psi}{k} u. \quad (2.7)$$

In the following, we will estimate the above integrals. We have that

$$\int_{-n}^{n} \lambda a_1 |u|^{q-1} u \leq \lambda \|a_1\|_{L^s(\mathbb{R})} \|u\|_{L^2}^q \leq \lambda C_2 \|u\|_{W^{1,2}}^q, \quad (2.8)$$

$$\int_{-n}^{n} \frac{\psi}{k} u \leq \frac{\|\psi\|_{L^2(-n,n)} \|u\|_{W^{1,2}}}{k}. \quad (2.9)$$

Now let $\tilde{u} : \mathbb{R} \to \mathbb{R}$ be the extension by zero of u, then

$$\int_{-n}^{n} |u|^{p-1} u \leq \int_{-n}^{n} |u|^p = \int_{-n}^{n} |u|^2 |u|^{p-2} \quad (2.10)$$

$$\leq \|\tilde{u}\|_{L^\infty(\mathbb{R})}^{p-2} \int_{-n}^{n} |u|^2 \quad (2.11)$$

$$= \|\tilde{u}\|_{L^\infty(\mathbb{R})}^{p-2} \|u\|_{L^2}^2 \quad (2.12)$$

$$\leq C^{p-2} \|u\|_{W^{1,2}}^{p-2} \|u\|_{W^{1,2}}^2 = C^{p-2} \|u\|_{W^{1,2}}^p. \quad (2.13)$$

Where C is the constant for the embedding $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

Define

$$\Omega_{\geq} = \left\{ s \in (-n, n) : |u'(s)| \geq \frac{1}{k} \right\} \quad \text{and} \quad \Omega_{\leq} = \left\{ s \in (-n, n) : 0 < |u'(s)| \leq \frac{1}{k} \right\}.$$

Then

$$\int_{-n}^{n} f_k(|u'|) u = \int_{\Omega_{\geq}} f_k(|u'|) u + \int_{\Omega_{\leq}} f_k(|u'|) u.$$

Notice that by Lemma 2.4,

$$\int_{\Omega_{\leq}} f_k(|u'|) u \leq \int_{\Omega_{\leq}} C_1 |u'||u| \leq \int_{\Omega_{\leq}} \frac{C_1}{k} |u|$$
$$\leq \frac{C_1}{k} \int_{-n}^{n} |u| \leq \frac{C_1 (2n)^{1/2}}{k} \|u\|_{L^2}$$
$$\leq \frac{C_1 (2n)^{1/2}}{k} \|u\|_{W^{1,2}}.$$

To estimate the integral over $\Omega_{\geq}$, consider the following cases :

Case 1. $2 < \theta < 3$.

Using Lemma 2.4, we have

$$\begin{aligned}\int_{\Omega_\geq} f_k(|u'|)u &\leq \int_{\Omega_\geq} C_1|u'|^{\theta-1}|u| \leq \int_{-n}^{n} C_1|u'|^{\theta-1}|u|\\ &\leq C_1\left(\int_{-n}^{n}|u|^w\right)^{\frac{1}{w}}\left(\int_{-n}^{n}|u'|^2\right)^{\frac{\theta-1}{2}}\\ &\leq C_1\left(\int_{\mathbb{R}}|\tilde{u}|^2|\tilde{u}|^{w-2}\right)^{\frac{1}{w}}\|u'\|_{L^2}^{\theta-1}\\ &\leq C_1\|\tilde{u}\|_{L^\infty(\mathbb{R})}^{\frac{w-2}{w}}\|u\|_{L^2}^{\frac{2}{w}}\|u'\|_{L^2}^{\theta-1}\\ &\leq C_1 C^{\frac{w-2}{w}}\|u\|_{W^{1,2}}^{\frac{w-2}{w}}\|u\|_{W^{1,2}}^{\frac{2}{w}}\|u\|_{W^{1,2}}^{\theta-1} = C_1 C^{\frac{w-2}{w}}\|u\|_{W^{1,2}}^{\theta}.\end{aligned}$$

Where $w = \left(\frac{2}{\theta-1}\right)' = \frac{2}{3-\theta} > 2$.

Case 2. $\theta = 3$.

$$\begin{aligned}\int_{\Omega_\geq} f_k(|u'|)u &\leq \int_{\Omega_\geq} C_1|u'|^{2}|u| \leq \int_{-n}^{n} C_1|u'|^{2}|u|\\ &\leq C_1\|\tilde{u}\|_{L^\infty(\mathbb{R})}\|u'\|_{L^2}^2 \leq C_1 C\|u\|_{W^{1,2}}\|u\|_{W^{1,2}}^2\\ &= C_1 C\|u\|_{W^{1,2}}^3.\end{aligned}$$

Now we are able to estimate (2.7). Notice that $\frac{w-2}{w} = \theta - 2$.

$$\begin{aligned}\langle\mathfrak{F}(\xi),\xi\rangle \geq{}& \gamma\|u\|_{W^{1,2}}^2 - \lambda C_2\|u\|_{W^{1,2}}^q - C^{p-2}\|u\|_{W^{1,2}}^p\\ &- C_1\max\{C^{\theta-2},C\}\|u\|_{W^{1,2}}^\theta - \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)\|u\|_{W^{1,2}}.\end{aligned}$$

Define $Z_k : \mathbb{R}^+ \to \mathbb{R}$ by

$$Z_k(x) = \gamma x^2 - \lambda C_2 x^q - C^{p-2}x^p - C_1\max\{C^{\theta-2},C\}x^\theta - \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)x.$$

We would like to find $r > 0$ such that

$$\gamma r^2 - C^{p-2}r^p - C_1\max\{C^{\theta-2},C\}r^\theta > \frac{r^2}{2}\gamma \tag{2.14}$$

or equivalently,

$$\frac{\gamma}{2} > C^{p-2}r^{p-2} + C_1\max\{C^{\theta-2},C\}r^{\theta-2}.$$

For this, if we take

$$\delta_1 = \min\left\{\left(\frac{\gamma}{4C^{p-2}}\right)^{1/(p-2)}, \left(\frac{\gamma}{4C_1\max\{C^{\theta-2},C\}}\right)^{1/(\theta-2)}\right\},$$

then for $0 < r < \delta_1$ (2.14) is true. Consequently,

$$Z_k(r) \geq \frac{r^2}{2}\gamma - \lambda C_2\delta_1^q - \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)\delta_1.$$

Define $\rho_1 = \frac{r^2}{2}\gamma - \lambda C_2 \delta_1^q$. We will adjust $\lambda > 0$ so that $\rho_1 > 0$; for this if $\rho_1 > 0$ it would imply that

$$\frac{r^2}{2}\gamma - \lambda C_2 \delta_1^q > 0 \Leftrightarrow \frac{r^2\gamma}{2C_2\delta_1^q} > \lambda.$$

Take $\lambda^* = \frac{r^2\gamma}{2C_2\delta_1^q}$ and $0 < \lambda < \lambda^*$. Thus, $\rho_1 > 0$ and we can find $k^* \in \mathbb{N}$ such that for $k > k^*$, $\rho_1 > \left(\frac{C_1(2n)^{1/2}}{k} + \frac{\|\psi\|_{L^2(-n,n)}}{k}\right)\delta_1 > 0$. Therefore, for $0 < r < \delta_1$, $0 < \lambda < \lambda^*$ and $k > k^*$

$$Z_k(r) > 0,$$

and so, with $\|u\|_{W^{1,2}} = r$,

$$\langle \mathfrak{F}(\xi), \xi \rangle > 0. \tag{2.15}$$

By Lemma 2.9, there exists $y_M \in B[0,r]$ such that $\mathfrak{F}(y_M) = 0$ that is, identifying $v_M = T^{-1}(y_M)$, for all $j \in \{1, \ldots, M\}$

$$\begin{aligned}\int_{-n}^{n} A(v_M)v_M' e_j' + \int_{-n}^{n} v_M e_j \\ = \int_{-n}^{n} \lambda a_1 |v_M|^{q-1} e_j + \int_{-n}^{n} |v_M|^{p-1} e_j + \int_{-n}^{n} f_k(|v_M'|) e_j + \int_{-n}^{n} \frac{\psi}{k} e_j.\end{aligned} \tag{2.16}$$

Therefore (2.16) holds for all $\varphi \in V_M$, because $\{e_1, \ldots, e_M\}$ is a basis of V_M. Notice that

$$\|v_M\|_{W^{1,2}} \le r \quad \text{for all } M \in \mathbb{N}. \tag{2.17}$$

Remark 2.18. Our choice of r does not depend on M,n, λ or k. This free determination of r will be useful further down in the argumentation, because using the embedding $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ we will be able to obtain a uniform upper bound, in the norm of $L^\infty(\mathbb{R})$, for the sequence of solutions of the problem (P_n^k). Then this upper bound will naturally be transferred to also bound the sequence of solution of (P_n).

Since $\|v_M\|_{W^{1,2}} \le r$ there is $v_0 \in \mathbb{E}_0^1(-n,n)$ such that $v_M \rightharpoonup v_0$ in $H_0^1(-n,n)$. By the compact embedding $W^{1,2}(-n,n) \hookrightarrow L^2(-n,n)$ we conclude $v_M \to v_0$ in $L^2(-n,n)$. Our goal is to show that v_0 is a weak solution of (P_n^k). Let $\Gamma_M : V_M \to V_M^*$ and $B_M : V_M \to V_M^*$ be defined by

$$\langle \Gamma_M(v), \varphi \rangle = \int_{-n}^{n} A(v)v'\varphi' \tag{2.18}$$

and

$$\langle B_M(v), \varphi \rangle = \int_{-n}^{n} \left(-v + \lambda a_1 |v|^{q-1} + |v|^{p-1} + f_k(|v'|) + \frac{\psi}{k} \right) \varphi. \tag{2.19}$$

Hence, $\langle \Gamma_M(v_M) - B_M(v_M), \varphi \rangle = 0$ for all $\varphi \in V_M$.

Denoting $P_M : \mathbb{E}_0^1(-n,n) \to V_M$ the projection of $\mathbb{E}_0^1(-n,n)$ onto V_M, (that is, if $u = \sum_{i=1}^{\infty} \alpha_i e_i$ then $P_M(u) = \sum_{i=1}^{M} \alpha_i e_i$) we have

$$\langle \Gamma_M(v_M) - B_M(v_M), v_M - P_M v_0 \rangle = 0,$$

so

$$\begin{aligned}\langle \Gamma_M(v_M), v_M - P_M v_0 \rangle &= \langle B_M(v_M), v_M - P_M v_0 \rangle \\ &= \int_{-n}^{n} \left(-v_M + \lambda a_1 |v_M|^{q-1} + |v_M|^{p-1} + f_k(|v_M'|) + \frac{\psi}{k} \right) (v_M - P_M v_0).\end{aligned} \tag{2.20}$$

Letting $M \to \infty$ one can see without difficulties that $\langle \Gamma_M(v_M), v_M - P_M v_0 \rangle \to 0$. This convergence allows us to prove the following

Lemma 2.19. *$v_M \to v_0$ strongly, i.e. in the norm of $H^1_0(-n,n)$.*

Remark 2.20. The idea to consider the operators Γ_M and B_M was an inspiration from the arguments presented in [7].

Proof. The limit $\|v_M - v_0\|_{L^2(-n,n)} \to 0$ has been established before, thus we will focus our efforts demonstrating the same for $\|v'_M - v'_0\|_{L^2(-n,n)}$. Let $\Phi_M, \Phi, \Psi_M, \zeta_M \in (\mathbb{E}^1_0(-n,n))^*$ be given by

$$\Phi_M(w) = \int_{-n}^{n} A(v_M) v'_0 w' \tag{2.21}$$

$$\Phi(w) = \int_{-n}^{n} A(v_0) v'_0 w' \tag{2.22}$$

$$\Psi_M(w) = \int_{-n}^{n} A(v_M) P_M v'_0 w' \tag{2.23}$$

$$\zeta_M(w) = \int_{-n}^{n} A(v_0) P_M v'_0 w'. \tag{2.24}$$

Then, by a straightforward calculation, $|\Phi_M - \Phi| \to 0$,$|\Psi_M - \Phi_M| \to 0$ and $|\zeta_M - \Phi| \to 0$ in $(\mathbb{E}^1_0(-n,n))^*$. Thus, $|\Psi_M - \zeta_M| \to 0$ in $(\mathbb{E}^1_0(-n,n))^*$, since $|\Psi_M - \zeta_M| \le |\Psi_M - \Phi_M| + |\Phi_M - \Phi| + |\Phi - \zeta_M|$. Writing $\Psi_M = (\Psi_M - \zeta_M) + \zeta_M$ yields that $\Psi_M \to \Phi$ in $(\mathbb{E}^1_0(-n,n))^*$. Remembering the weak convergence $v_M \rightharpoonup v_0$ one can conclude $(v_M - P_M v_0) \rightharpoonup 0$ in $\mathbb{E}^1_0(-n,n)$ because for all $f \in (\mathbb{E}^1_0(-n,n))^*$

$$|f(v_M) - f(P_M v_0)| \le |f(v_M) - f(v_0)| + \|f\| \|v_0 - P_M v_0\|_{W^{1,2}}.$$

Consequently, letting $M \to \infty$, $\Psi_M(v_M - P_M v_0) \to \Phi(0) = 0$. This means that

$$\int_{-n}^{n} A(v_M) P_M v'_0 (v'_M - P_M v'_0) \to 0. \tag{2.25}$$

Also, rewriting (2.20)

$$\int_{-n}^{n} A(v_M) v'_M (v'_M - P_M v'_0) \to 0 \quad \text{as } M \to \infty. \tag{2.26}$$

Therefore, from (2.26)–(2.25)

$$\int_{-n}^{n} A(v_M) (v'_M - P_M v'_0)^2 \to 0 \quad \text{as } M \to \infty. \tag{2.27}$$

Since $A(x) \ge \gamma > 0$ for all $x \in \mathbb{R}$ we conclude $\|v'_M - P_M v'_0\|_{L^2(-n,n)} \to 0$ as $M \to \infty$. Then $\|v'_M - v'_0\|_{L^2(-n,n)} \to 0$ as result of $\|v'_M - v'_0\|_{L^2(-n,n)} \le \|v'_M - P_M v'_0\|_{L^2(-n,n)} + \|v'_0 - P_M v'_0\|_{L^2(-n,n)}$, proving the lemma. □

We know that for every $\varphi \in V_M$

$$\int_{-n}^{n} A(v_M) v'_M \varphi' + \int_{-n}^{n} v_M \varphi = \int_{-n}^{n} \lambda a_1 |v_M|^{q-1} \varphi + \int_{-n}^{n} |v_M|^{p-1} \varphi + \int_{-n}^{n} f_k(|v'_M|) \varphi + \int_{-n}^{n} \frac{\psi}{k} \varphi. \tag{2.28}$$

By the previous lemma, taking a subsequence if necessary, we may assume that $v_M'(x)$ converges a.e. to $v_0'(x)$ and there exists $h \in L^2(-n,n)$ such that $|v_M'(x)| \le h(x)$ a.e. Then notice that

$$\left| \int_{-n}^{n} \left(A(v_M)v_M' - A(v_0)v_0'\right) \varphi' \right| \le \left(\int_{-n}^{n} |A(v_M)v_M' - A(v_0)v_0'|^2 \right)^{1/2} \|\varphi'\|_{L^2} \tag{2.29}$$

and exists $Q > 0$ such that $\|v_M\|_\infty < Q$ for all $M \in \mathbb{N}$, because v_M converges to v_0 in $C^0[-n,n]$ due to the embedding $W^{1,2}(-n,n) \hookrightarrow C^0[-n,n]$. We can suppose Q big enough so that $Q > 2\,(r + A(0))$ and we take $\tilde{A} = \sup_{x\in[-Q,Q]} A(x)$. Since

$$|A(v_M(x))v_M'(x) - A(v_0(x))v_0'(x)| \to 0 \quad \text{a.e.} \tag{2.30}$$

and

$$\begin{aligned} |A(v_M(x))v_M'(x) - A(v_0(x))v_0'(x)|^2 &\le \left(|A(v_M(x))v_M'(x)| + |A(v_0(x))v_0'(x)|\right)^2 \\ &= |A(v_M(x))|^2 |v_M'(x)|^2 \\ &\quad + 2|A(v_M(x))||A(v_0(x))||v_M'(x)||v_0'(x)| \\ &\quad + |A(v_0(x))|^2 |v_0'(x)|^2 \\ &\le \tilde{A}^2 Q^2 h^2(x) + 2\tilde{A}^2 Q^2 |v_0'(x)| h(x) \\ &\quad + \tilde{A}^2 Q^2 |v_0'(x)|^2 \end{aligned}$$

almost everywhere, we conclude by (D.C.T) that

$$\int_{-n}^{n} A(v_M)v_M'\varphi' \to \int_{-n}^{n} A(v_0)v_0'\varphi' \quad \text{as } M \to \infty. \tag{2.31}$$

Also, by direct calculation, the following convergences are true

$$\int_{-n}^{n} v_M\varphi \to \int_{-n}^{n} v_0\varphi \tag{2.32}$$

$$\int_{-n}^{n} \lambda a_1 |v_M|^{q-1}\varphi \to \int_{-n}^{n} \lambda a_1 |v_0|^{q-1}\varphi \tag{2.33}$$

$$\int_{-n}^{n} |v_M|^{p-1}\varphi \to \int_{-n}^{n} |v_0|^{p-1}\varphi \tag{2.34}$$

$$\int_{-n}^{n} f_k(|v_M'|)\varphi \to \int_{-n}^{n} f_k(|v_0'|)\varphi \tag{2.35}$$

as $M \to \infty$. Thus, for every $\varphi \in V_M$

$$\int_{-n}^{n} A(v_0)v_0'\varphi' + \int_{-n}^{n} v_0\varphi = \int_{-n}^{n} \lambda a_1 |v_0|^{q-1}\varphi + \int_{-n}^{n} |v_0|^{p-1}\varphi + \int_{-n}^{n} f_k(|v_0'|)\varphi + \int_{-n}^{n} \frac{\psi}{k}\varphi. \tag{2.36}$$

Furthermore, for every $u \in \mathbb{E}_0^1(-n,n)$, it follows that

$$\int_{-n}^{n} A(v_0)v_0' P_M u' \to \int_{-n}^{n} A(v_0)v_0' u' \tag{2.37}$$

$$\int_{-n}^{n} v_0 P_M u \to \int_{-n}^{n} v_0 u \tag{2.38}$$

$$\int_{-n}^{n} \lambda a_1 |v_0|^{q-1} P_M u \to \int_{-n}^{n} \lambda a_1 |v_0|^{q-1} u \tag{2.39}$$

$$\int_{-n}^{n} |v_0|^{p-1} P_M u \to \int_{-n}^{n} |v_0|^{p-1} u \tag{2.40}$$

$$\int_{-n}^{n} f_k(|v_0'|) P_M u \to \int_{-n}^{n} f_k(|v_0'|) u \tag{2.41}$$

as $M \to \infty$. Thus, for every $u \in \mathbb{E}_0^1(-n,n)$

$$\int_{-n}^{n} A(v_0)v_0'u' + \int_{-n}^{n} v_0 u = \int_{-n}^{n} \lambda a_1 |v_0|^{q-1} u + \int_{-n}^{n} |v_0|^{p-1} u + \int_{-n}^{n} f_k(|v_0'|) u + \int_{-n}^{n} \frac{\psi}{k} u. \quad (2.42)$$

So v_0 is an *E-weak solution* of (P_n^k); by Lemma 2.13 v_0 is also a weak solution. Notice that

$$\|v_0\|_{W^{1,2}} \le r,$$

and our choice of r does not depend on n, λ or k. This finishes the proof of Proposition 2.16. □

In what follows we will make $k \to \infty$ thus we can consider $\psi \equiv 1$, because the term $\frac{\psi}{k}$ will converge to 0 as $k \to \infty$.

Proposition 2.21. *The above weak solution v_0 satisfies:*

1. *there exist $\beta(L/\gamma)$ and $\hat{C}(L/\gamma, n)$, such that $v_0 \in C^{1,\beta}[-n,n] \cap C^2(-n,n)$ and*

$$|v_0|_{1+\beta} \le \hat{C},$$

where

$$L > 2\max\{Cr + \lambda a_1(n)|Cr|^{q-1} + |Cr|^{p-1} + 1, 2C_1, A(Cr), \tilde{A}\};$$

2. $v_0(t) \ge 0$.

Proof. To prove *1*, we will use [12, Theorem 1]. Let $F : [-n,n] \times [-Cr, Cr] \times \mathbb{R} \to \mathbb{R}$ be defined by $F(x,z,p) = A(z)p$, where C is the embedding constant for $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, and $B(x,z,p) = z - (\lambda a_1(x)|z|^{q-1} + |z|^{p-1} + f_k(|p|) + \frac{1}{k})$ be defined in the same domain. Then, problem (P_n^k) may be rewritten as

$$\operatorname{div}_x F(x, u(x), u'(x)) + B(x, u(x), u'(x)) = 0.$$

In order to use [12, Theorem 1] we must verify the existence of nonnegative constants l, L, M_0, m, κ with $l \le L$ such that

$$\frac{\partial F}{\partial p}(x,z,p)\xi^2 \ge l(\kappa + |p|)^m \xi^2, \quad (2.43)$$

$$\left|\frac{\partial F}{\partial p}(x,z,p)\right| \le L(\kappa + |p|)^m, \quad (2.44)$$

$$|F(x,z,p) - F(y,w,p)| \le L(1+|p|)^{m+1} \cdot |z-w|, \quad (2.45)$$

$$|B(x,z,p)| \le L(1+|p|)^{m+2}, \quad (2.46)$$

for all $(x,z,p) \in \{-n,n\} \times [-M_0, M_0] \times \mathbb{R}$, $w \in [-M_0, M_0]$ and $\xi \in \mathbb{R}$. Since $\frac{\partial F}{\partial p}(x,z,p) = A(z)$, inequality (2.43) follows from $A(z)\xi^2 \ge \gamma\xi^2$, that is, $l = \gamma$.

To prove the remaining inequalities take $M_0 = Cr$,

$$T > \max\{Cr + \lambda a_1(n)|Cr|^{q-1} + |Cr|^{p-1} + 1, 2C_1, A(Cr), \tilde{A}\},$$

$L = 2T, \kappa = 0$ and $m = 0$, where $\tilde{A}$ is the Lipchitz constant of A. Then:

(2.44)

$$\left|\frac{\partial F}{\partial p}(x,z,p)\right| = A(z) \leq A(Cr) < L;$$

(2.45)

$$|F(x,z,p) - F(y,w,p)| = |A(z)p - A(w)p| \leq \tilde{A}|p||z-w| \leq L(1+|p|)|z-w|;$$

(2.46)

$$\begin{aligned} |B(x,z,p)| &= \left| z - \left(\lambda a_1(x)|z|^{q-1} + |z|^{p-1} + f_k(|p|) + \frac{1}{k} \right) \right| \\ &\leq Cr + \lambda a_1(n)|Cr|^{q-1} \\ &\quad + |Cr|^{p-1} + 1/k + C_1(1+|p|^{\theta-1}) \\ &\leq T + C_1(1+(1+|p|)^{\theta-1}) \\ &\leq T + 2C_1(1+|p|)^2 \\ &\leq T(1+(1+|p|)^2) \\ &\leq 2T(1+|p|)^2 = L(1+|p|)^2. \end{aligned} \tag{2.47}$$

Therefore, by [12, Theorem 1] there exists $\beta \in (0,1)$ and a constant $\hat{C}$, independent of k, such that $v_0 \in C^{1,\beta}([-n,n])$ and

$$|v_0|_{1+\beta} \leq \hat{C}. \tag{2.48}$$

It follows from [6, p. 317, Chap. 6, Theorem 4] that $v_0 \in W^{2,2}(-n,n)$ and since v_0 is a weak solution of (P_n^k) we have

$$v_0'' = \frac{v_0 - \lambda a_1|v_0|^{q-1} - |v_0|^{p-1} - f_k(|v_0'|) - 1/k - A'(v_0)|v_0'|^2}{A(v_0)} \tag{2.49}$$

showing that v_0'' is continuous, thus $v_0 \in C^2(-n,n)$.

To prove that $v_0(t) \geq 0$ for all $t \in (-n,n)$ we first notice that $v_0^-(t) = \max\{0, -v_0(t)\} \in H_0^1(-n,n)$. Using $v_0^-(t)$ as a test function in the definition of weak solution provides

$$-\int_{-n}^{n} A(v_0)|v_0^-|^2 - \int_{-n}^{n} |v_0^-|^2 = \int_{-n}^{n} \lambda a_1|v_0|^{q-1}v_0^- + \int_{-n}^{n} |v_0|^{p-1}v_0^- + \int_{-n}^{n} f_k(|v_0'|)v_0^- + \int_{-n}^{n} \frac{1}{k}v_0^-. \tag{2.50}$$

Then $-\gamma\|v_0^-\|_{W^{1,2}}^2 \geq 0$, thus $\|v_0^-\|_{W^{1,2}} = 0$ implying $v_0^- \equiv 0$ a.e. Since v_0 is continuous, $v_0(t) \geq 0$ for all $t \in (-n,n)$. This finishes the proof of Proposition 2.21. □

Thus, by Proposition 2.16 and Proposition 2.21 we obtain the proof of Theorem 2.8.

2.2 Constructing a solution to problem (P_n)

Let v_k be the (strong) solution of problem (P_n^k), obtained just above, with k varying. By the previous constructions, we have that $\|v_k\|_{W^{1,2}(-n,n)} \leq r$ independent of k, as noticed in Remark 2.18. Then there exists $u_n \in H_0^1(-n,n)$, $\|u_n\|_{W^{1,2}(-n,n)} \leq r$, so that v_k has a subsequence converging weakly in $H_0^1(-n,n)$ to u_n. From now on v_k will denote this subsequence. Since the function

$$H_0^1(-n,n) \ni w \mapsto \int_{-n}^{n} A(u_n)u_n' w'$$

belongs to $(H_0^1(-n,n))^*$, we have, by the weak convergence, that

$$\int_{-n}^{n} A(u_n)u_n'(v_k-u_n)' \to 0 \quad \text{as } k\to\infty.$$

This convergence will be useful in our next task: to prove that $v_k \to u_n$ strongly in $H_0^1(-n,n)$.

Lemma 2.22. *The following convergence is true*

$$\int_{-n}^{n} A(u_n)v_k'(v_k-u_n)' \to 0 \quad \text{as } k\to\infty.$$

Proof. We might write

$$\begin{aligned}\int_{-n}^{n} A(u_n)v_k'(v_k-u_n)' &= \int_{-n}^{n}[A(u_n)-A(v_k)+A(v_k)]v_k'(v_k'-u_n')\\ &= \underbrace{\int_{-n}^{n}[A(u_n)-A(v_k)]v_k'(v_k'-u_n')}_{I_1} + \underbrace{\int_{-n}^{n} A(v_k)v_k'(v_k'-u_n')}_{I_2}\end{aligned}$$

and analyze I_1 and I_2 separately.

Analysis of I_2 By the weak formulation of (P_n^k)

$$\begin{aligned}&\int_{-n}^{n} A(v_k)v_k'(v_k'-u_n')\\ &\quad=\underbrace{\int_{-n}^{n} -v_k(v_k-u_n)+\lambda a_1|v_k|^{q-1}(v_k-u_n)+|v_k|^{p-1}(v_k-u_n)+\frac{(v_k-u_n)}{k}}_{E_1}\\ &\quad\underbrace{+\int_{-n}^{n} f_k(|v_k'|)(v_k-u_n)}_{E_2}.\end{aligned}$$

Since we have compact injection of $H_0^1(-n,n)$ onto $L^2(-n,n)$, the weak convergence of v_k to u_n in $H_0^1(-n,n)$ implies $\|v_k-u_n\|_{L^2}\to 0$. Thus, it is straightforward to see that (E_1) converges to 0 as $k\to\infty$. Remains to verify what happens with (E_2) in the limit. We have that

$$\int_{-n}^{n} f_k(|v_k'|)(v_k-u_n) \le \int_{-n}^{n} C_1(|v_k'|^{\theta-1}+|v_k'|)|v_k-u_n|$$

by Lemma 2.4. Using Proposition 2.21, item *1*, that is, the estimation $|v_k|_{1,\beta}\le \hat{C}$ which is independent of k, we have that

$$|v_k'|^{\theta-1}+|v_k'| \le (\hat{C})^{\theta-1}+\hat{C}.$$

Then,

$$\begin{aligned}\int_{-n}^{n} f_k(|v_k'|)(v_k-u_n) &\le C_1[(\hat{C})^{\theta-1}+\hat{C}]\int_{-n}^{n}|v_k-u_n|\\ &\le \underbrace{(2n)^{1/2}C_1[(\hat{C})^{\theta-1}+\hat{C}]\|v_k-u_n\|_{L^2}}_{\to 0 \text{ as } k\to\infty}.\end{aligned}$$

Thus, $\lim_{k\to\infty} I_2(k)=0$.

Analysis of I_1 We also have that $\lim_{k\to\infty} I_1(k) = 0$, as one can see through

$$\begin{aligned}\left|\int_{-n}^{n}[A(u_n)-A(v_k)]v_k'(v_k'-u_n')\right| &\le \int_{-n}^{n}|A(u_n)-A(v_k)||v_k'||v_k'-u_n'| \\ &\le \hat{C}\int_{-n}^{n}|A(u_n)-A(v_k)||v_k'-u_n'| \\ &\le \hat{C}\tilde{A}\int_{-n}^{n}|u_n-v_k||v_k'-u_n'| \\ &\le \hat{C}\tilde{A}\|u_n-v_k\|_{L^2}\|v_k'-u_n'\|_{L^2} \\ &\le \hat{C}\tilde{A}2r\|u_n-v_k\|_{L^2}.\end{aligned}$$

Thus,

$$\int_{-n}^{n} A(u_n)u_n'(v_k-u_n)' \to 0 \quad \text{as } k\to\infty \tag{2.51}$$

$$\int_{-n}^{n} A(u_n)v_k'(v_k-u_n)' \to 0 \quad \text{as } k\to\infty. \tag{2.52}$$

Subtracting (2.52) from (2.51) we have

$$\int_{-n}^{n} A(u_n)(v_k'-u_n')^2 \to 0 \quad \text{as } k\to\infty \tag{2.53}$$

implying that $v_k' \to u_n'$ in $L^2(-n,n)$, since γ is a uniform lower-bound for A. Hence $v_k \to u_n$ in $H_0^1(-n,n)$. □

Remark 2.23. Since $v_k \to u_n$ in $H_0^1(-n,n)$ we conclude that u_n is also an even function; due to the embedding $W^{1,2}(-n,n) \hookrightarrow C[-n,n]$.

Proposition 2.24. *The function u_n satisfies:*

1. *u_n is strictly positive in $(-n,n)$;*
2. *u_n is a solution to (P_n).*

Proof.

Item 1. Let $\tilde{a} := \inf_{x\in[-n,n]} a_1(x)$. We will divide our argument into two cases:

Remark 2.25. This division of cases is a geometric argument that we borrowed from [3].

Case 1. There exists a subsequence $(v_{k_i})_{i\in\mathbb{N}}$ of (v_k) such that $v_{k_i}' \ge 0$ in $(-n,0)$ for all i. Consider the problem

$$\begin{cases} -(A(u)u')' + u = \lambda\tilde{a}|u|^{q-1} & \text{in } (-n,n) \\ u > 0 & \text{in } (-n,n) \\ u(-n) = u(n) = 0. \end{cases} \tag{2.54}$$

Since $v_{k_i}' \ge 0$ in $(-n,0)$ we get that $v_{k_i} > 0$ in $(-n,0)$, because, due to Proposition 2.21, v_{k_i} is an even solution of (P_n^k), thus *it can not be identically zero in an interval* and $v_{k_i} \ge 0$; i.e., supposing the existence of $x_i \in (-n,0)$ such that $v_{k_i}(x_i) = 0$ implies the existence of $y_i \in (-n,0)$ such that $v'_{k_i}(y_i) < 0$, which would be a contradiction. Thus, we see that v_{k_i} is a sup-solution for equation (2.54). Let ϕ_1 be an even and positive eigenfunction for the eigenvalue problem

$$\begin{cases} -u'' = \lambda_1 u & \text{in } (-n,n) \\ u(-n) = u(n) = 0 \end{cases} \tag{2.55}$$

where $\lambda_1 = \frac{\pi^2}{(2n)^2}$. Thus, choosing τ such that

$$\frac{\tau^{2-q}(1+\gamma\lambda_1)}{\lambda\tilde{a}} \leq \phi_1^{q-2}$$

we have that $\tau\phi_1$ is as sub-solution of (2.54). By Theorem 2.11

$$v_{k_i}(t) \geq \tau\phi_1(t) \qquad \forall t \in (-n,n),$$

therefore, in the limit,

$$u_n(t) \geq \tau\phi_1(t) > 0 \qquad \forall t \in (-n,n).$$

Case 2. There exists a subsequence $(v_{k_i})_{i\in\mathbb{N}}$ of (v_k) and there exists a sequence $(z_i)_{i\in\mathbb{N}} \subset (-n,0)$ such that $v'_{k_i}(z_i) < 0$.

Remark 2.26. Although the geometric argument is an inspiration from [3], we still need to adjust it to our necessity. Lemma 2.27 is one such adjustment.

Lemma 2.27. *Let $x \in (-n,n)$ such that $v''_{k_i}(x) \geq 0$, then $v_{k_i}(x) > (\lambda\tilde{a})^{\frac{1}{2-q}}$.*

Proof. Since v_{k_i} is a solution for the problem (P_n^k), with $k = k_i$, for all t

$$\begin{aligned} &- A'(v_{k_i}(t))|v'_{k_i}(t)|^2 - A(v_{k_i}(t))v''_{k_i}(t) + v_{k_i}(t) \\ &\quad = \lambda a_1(t)|v_{k_i}(t)|^{q-1} + |v_{k_i}(t)|^{p-1} + f_{k_i}(|v'_{k_i}(t)|) + \frac{1}{k_i} \\ &\quad \geq \lambda a_1(t)|v_{k_i}(t)|^{q-1} + |v_{k_i}(t)|^{p-1} + \frac{1}{k_i}. \end{aligned}$$

Here we used that $\operatorname{sign}(f_{k_i}(s)) = \operatorname{sign}(s)$, thus $f_{k_i}(|v'_{k_i}|) \geq 0$. Using that $|v_{k_i}|^{p-1} \geq 0$ and $a_1(t) \geq \tilde{a} > 0$ we obtain:

$$-A'(v_{k_i}(t))|v'_{k_i}(t)|^2 - A(v_{k_i}(t))v''_{k_i}(t) + v_{k_i}(t) \geq \lambda\tilde{a}|v_{k_i}(t)|^{q-1} + \frac{1}{k_i}. \tag{2.56}$$

Then, with $t = x$,

$$\begin{aligned} -A(v_{k_i}(x))v''_{k_i}(x) &\geq \lambda\tilde{a}|v_{k_i}(x)|^{q-1} - v_{k_i}(x) + A'(v_{k_i}(x))|v'_{k_i}(x)|^2 + \frac{1}{k_i} \\ &> \lambda\tilde{a}|v_{k_i}(x)|^{q-1} - v_{k_i}(x). \end{aligned}$$

Where, in the last inequality, we used that A is non-decreasing, $|v'_{k_i}| \geq 0$ and $1/k_i > 0$. Notice that the resulting estimation is strict because $1/k_i > 0$. By hypotheses $v''_{k_i}(x) \geq 0$, then $-A(v_{k_i}(x))v''_{k_i}(x) \leq 0$. Using the previous inequality,

$$v_{k_i}(x) > \lambda\tilde{a}|v_{k_i}(x)|^{q-1},$$

thus $v_{k_i}(x) \neq 0$ and $v_{k_i}(x) > (\lambda\tilde{a})^{\frac{1}{2-q}}$. □

Now, in order to use this lemma, we ought to find a $x_i \in (-n,n)$ such that $v''_{k_i}(x_i) \geq 0$. Using the fact that v_{k_i} is even and $v'_{k_i}(z_i) < 0$ we have that $v'_{k_i}(-z_i) > 0$. Let $x_i = \min_{x\in[z_i,-z_i]} v_{k_i}(x)$ and notice that $x_i \neq z_i$ and $x_i \neq -z_i$; indeed, there exist $\delta > 0$ such that, if $x \in (z_i, z_i+\delta) \cup (-z_i-\delta, -z_i)$, then $v_{k_i}(x) < v_{k_i}(z_i) = v_{k_i}(-z_i)$. Hence $x_i \in (z_i, -z_i)$ and $v'_{k_i}(x_i) = 0$; therefore $v''_{k_i}(x_i)$ must be *greater or equal than 0*, because if $v''_{k_i}(x_i) < 0$ there would be $\xi > 0$ such that, for $x \in (x_i, x_i+\xi) \subset (z_i, -z_i)$, $v'_{k_i}(x) < 0$; and for this neighborhood $v_{k_i}(x) < v_{k_i}(x_i)$ – a contradiction with the minimality of x_i. Thus, $v''_{k_i}(x_i) \geq 0$. By Lemma 2.27, we obtain for all i

$$v_{k_i}(x_i) > (\lambda\tilde{a})^{\frac{1}{2-q}}.$$

From the compactness of $[-n,n]$, there exist $x_0 \in [-n,n]$ such that $x_i \to x_0$ when $i \to \infty$; taking a subsequence if necessary. Then

$$u_n(x_0) = \lim_{i\to\infty} v_{k_i}(x_i) \geq (\lambda\tilde{a})^{\frac{1}{2-q}} > 0.$$

Finally, we will conclude *item 1* showing that, also in this case, u_n is strictly positive in $(-n,n)$.

Suppose by contradiction that there exists $y \in (-n,n)$ such that $u_n(y) = 0$. Let $(d,s) \subset (-n,n)$ be the biggest interval containing y satisfying the property: if $x \in (d,s)$ then $u_n(x) < \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{2}$. Since $u_n(x_0) = u_n(-x_0) > \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{2}$ we have that $d \neq -n$ or $s \neq n$. Thus we can suppose without loss of generality that $d > -n$, because on the contrary, we would apply our following arguments using the interval (d', s'), where $d' = -s$ and $s' = -d$, and the point $y' = -y$.

By the maximality of (d,s) and the continuity of u_n we have that $u_n(d) = \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{2}$. Since $u_n(x) < \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{2}$ for all $x \in (d,s)$ and v_{k_i} converges uniformly to u_n, there exists $i_1 \in \mathbb{N}$ such that, for $i > i_1$ and $x \in (d,s)$,

$$v_{k_i}(x) < (\lambda\tilde{a})^{\frac{1}{2-q}}.$$

Then, by Lemma 2.27 $v''_{k_i}(x) < 0$. Using that $u_n(d) = \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{2}$, there exist $i_2 \in \mathbb{N}$ such that $i > i_2$ implies

$$v_{k_i}(d) > \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{4}.$$

Let $i_0 > \max\{i_1, i_2\}$ and define $f : (d,s) \to \mathbb{R}$ by

$$f(x) = \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{4} \cdot \frac{x-s}{d-s}.$$

We have that $f(d) = \frac{(\lambda\tilde{a})^{\frac{1}{2-q}}}{4}$ and $f(s) = 0$. Let $U_i(x) = v_{k_i}(x) - f(x)$ for $i \geq i_0$, then

$$\begin{cases} U''_i(x) < 0, & \text{for } x \in (d,s) \\ U_i(d) > 0, U_i(s) = v_{k_i}(s) \geq 0. \end{cases} \tag{2.57}$$

By the maximum principle, the minimum of U_i is reached on the border of the interval (d,s), implying that $U_i(x) > 0$ for all $x \in (d,s)$, that is, $v_{k_i}(x) > f(x)$ for all $x \in (d,s)$ and $i \geq i_0$. Thus, taking $x = y$ and making $i \to \infty$, we obtain

$$u_n(y) \geq f(y) > 0,$$

which is a contradiction.

Item 2. Since the estimation from Proposition 2.21 item *1* holds, that is,

$$|v_k|_{1+\beta} \leq \hat{C}$$

for all $k \in \mathbb{N}$; and, for all $1 < \alpha < \beta$, we have compact embedding $C^{1,\beta}[-n,n] \hookrightarrow C^{1,\alpha}[-n,n]$, we may assume, taking a subsequence, if necessary, that there exist $\widetilde{u_n} \in C^{1,\alpha}[-n,n]$ such that $v_k \to \widetilde{u_n}$ in $C^{1,\alpha}[-n,n]$ as $k \to \infty$. Thus,

$$\begin{aligned} v_k &\to u_n \quad \text{in } C^0[-n,n] \text{ as } k \to \infty \\ v_k &\to \widetilde{u_n} \quad \text{in } C^{1,\alpha}[-n,n] \text{ as } k \to \infty. \end{aligned}$$

Then for all $x \in [-n,n]$ we have

$$u_n(x) = \lim_{k\to\infty} v_k(x) = \widetilde{u_n}(x),$$

i.e., $u_n = \widetilde{u_n} \in C^{1,\alpha}[-n,n]$.

Considering the definition of weak solution, for all $\varphi \in H_0^1(-n,n)$

$$\int_{-n}^{n} A(v_k)v_k'\varphi' + \int_{-n}^{n} v_k\varphi = \int_{-n}^{n} (\lambda a_1|v_k|^{q-1} + |v_k|^{p-1})\varphi + \int_{-n}^{n} f_k(|v_k'|)\varphi + \int_{-n}^{n} \frac{\varphi}{k}.$$

By (D.C.T), it is straightforward to see that the following convergences are true:

$$\begin{aligned} \int_{-n}^{n} A(v_k)v_k'\varphi' &\to \int_{-n}^{n} A(u_n)u_n'\varphi', \\ \int_{-n}^{n} v_k\varphi &\to \int_{-n}^{n} u_n\varphi, \\ \int_{-n}^{n} (\lambda a_1|v_k|^{q-1} + |v_k|^{p-1})\varphi &\to \int_{-n}^{n} (\lambda a_1|u_n|^{q-1} + |u_n|^{p-1})\varphi, \\ \int_{-n}^{n} \frac{\varphi}{k} &\to 0, \end{aligned}$$

as $k \to \infty$. Let us examine the remaining integral. First notice that $f_k(|v_k'|)$ converges uniformly to $g(|u_n'|)$; indeed, g is uniformly continuous in compacts, then for the compact $[-\hat{C},\hat{C}]$ given $\epsilon > 0$ there exists $\delta > 0$ such that if $|x-y| < \delta$, then

$$|g(x) - g(y)| < \frac{\epsilon}{2}. \tag{2.58}$$

Also there exists $k_0 \in \mathbb{N}$ such that $k > k_0$ implies

$$||v_k'(x)| - |u_n'(x)|| < \delta \qquad \forall x \in [-n,n]. \tag{2.59}$$

thus for $k > k_0$

$$|g(|v_k'(x)|) - g(|u_n'(x)|)| < \frac{\epsilon}{2} \qquad \forall x \in [-n,n]. \tag{2.60}$$

In the perspective of Theorem 2.2, f_k converges to g uniformly in bounded sets; since $\|u_n'\|_\infty \leq \hat{C}$, for $x \in [-\hat{C},\hat{C}]$ there exist $k_1 \in \mathbb{N}$ such that $k > k_1$ implies

$$|f_k(x) - g(x)| < \frac{\epsilon}{2} \qquad \forall x \in [-\hat{C},\hat{C}] \tag{2.61}$$

and with all these ingredients we obtain the uniform convergence, because for $k > \max\{k_0,k_1\}$

$$\begin{aligned} |f_k(|v_k'(x)|) - g(|u_n'(x)|)| &\leq |f_k(|v_k'(x)|) - g(|v_k'(x)|)| + |g(|v_k'(x)|) - g(|u_n'(x)|)| \\ &< \epsilon \qquad \forall x \in [-n,n]. \end{aligned}$$

Thus, by (D.C.T)

$$\int_{-n}^{n} f_k(|v_k'|)\varphi \to \int_{-n}^{n} g(|u_n'|)\varphi$$

as $k \to \infty$. All these convergences together show that u_n is a weak solution for the problem (P_n).

From [6, Pag. 317, Chap. 6, Theorem 4] we conclude that $u_n \in W^{2,2}(-n,n)$; and similarly, to the argument showed in equation (2.49) we obtain that $u_n'' \in C^0(-n,n)$. Thus, u_n is a strong solution to the problem (P_n). □

3 Asymptotic solution to problem (P_n)

In this section we will briefly study the solution's behavior of problem (P_n) when $\lambda \to 0$ or $\lambda \to \lambda^*$. Our main result is the following:

Theorem 3.1. *Denoting by u_λ the solution of (P_n):*

1. *as $\lambda \to 0$, we get that $\|u_\lambda\|_{W^{1,2}} \to 0$;*
2. *one can take $\lambda = \lambda^*$ and still obtain a solution to problem (P_n).*

Proof. Let v_k be the strong solution of problem (P_n^k), with $\psi \equiv 1$. In the previous section, we proved, among other things, that $v_k \to u_\lambda$ as $k \to \infty$, (taking a subsequence, if necessary). Using v_k as test function in the Definition 2.12, we get:

$$\int_{-n}^{n} A(v_k)|v_k'|^2 + \int_{-n}^{n} |v_k|^2 = \int_{-n}^{n} \lambda a_1 |v_k|^{q-1} v_k + \int_{-n}^{n} |v_k|^{p-1} v_k + \int_{-n}^{n} f_k(|v_k'|) v_k + \int_{-n}^{n} \frac{v_k}{k}.$$

Thus, following the estimations done in Proposition 2.16, we can estimate these integrals to obtain

$$\begin{aligned}\gamma \|v_k\|_{W^{1,2}}^2 \le \lambda C_2 \|v_k\|_{W^{1,2}}^q + C^{p-2}\|v_k\|_{W^{1,2}}^p + C_1 \max\{C^{\theta-2}, C\}\|v_k\|_{W^{1,2}}^\theta \\ + \left(\frac{C_1(2n)^{1/2}}{k} + \frac{(2n)^{1/2}}{k}\right)\|v_k\|_{W^{1,2}}.\end{aligned}$$

Rearranging we obtain

$$\begin{aligned}\|v_k\|_{W^{1,2}}^2 \left(\gamma - C^{p-2}\|v_k\|_{W^{1,2}}^{p-2} - C_1 \max\{C^{\theta-2}, C\}\|v_k\|_{W^{1,2}}^{\theta-2}\right) \\ \le \lambda C_2 \|v_k\|_{W^{1,2}}^q + \left(\frac{C_1(2n)^{1/2}}{k} + \frac{(2n)^{1/2}}{k}\right)\|v_k\|_{W^{1,2}}. \quad (3.1)\end{aligned}$$

Notice that $\|v_k\|_{W^{1,2}} \le r$ independent of k and

$$r < \min\left\{\left(\frac{\gamma}{4C^{p-2}}\right)^{1/(p-2)}, \left(\frac{\gamma}{4C_1 \max\{C^{\theta-2}, C\}}\right)^{1/(\theta-2)}\right\}.$$

Thus,

$$\|v_k\|_{W^{1,2}}^2 \le \frac{2}{\gamma}\left(\lambda C_2 r^q + \left(\frac{C_1(2n)^{1/2}}{k} + \frac{(2n)^{1/2}}{k}\right) r\right). \quad (3.2)$$

Making $k \to \infty$ we end up with

$$\|u_\lambda\|^2_{W^{1,2}} \leq \lambda \cdot \left(\frac{2C_2 r^q}{\gamma}\right). \tag{3.3}$$

Then, as $\lambda \to 0$ we see that $\|u_\lambda\|^2_{W^{1,2}} \to 0$. This proves item *1*.

To prove item *2*, one can take a sequence (λ_n) in $(0, \lambda^*)$ such that $\lambda_n \to \lambda^*$. Noticing that $\|u_{\lambda_n}\|_{W^{1,2}} \leq r$ independent of λ_n, one can obtain a candidate $u_{\lambda^*} \in H^1_0(-n,n)$ such that u_{λ_n} converges weakly to u_{λ^*} in $H^1_0(-n,n)$. Then, following similar argumentation as exposed in Section 2, one can prove that u_{λ^*} is an even, positive solution to (P_n) with $\lambda = \lambda^*$. □

Remark 3.2. Let

$$\tilde{r} = \min\left\{r, \lambda^{1/2}\left(\frac{2C_2 r^q}{\gamma}\right)^{1/2}\right\}.$$

Then, by inequality (3.3) and the estimations from Proposition 2.16, we get that $\|u_\lambda\|_{W^{1,2}} \leq \tilde{r}$. Notice that $\tilde{r} \to 0$ as $\lambda \to 0$.

Remark 3.3. For now, on we will resume the previous notation, that is, we will call the strong solution of problem (P_n) by u_n. This will be useful in the next section.

4 Solution in $\mathbb{R}$

To obtain the homoclinic solution, we can proceed similarly as in [3]. However, we present a slightly different approach.

To obtain a solution defined in $\mathbb{R}$ we will utilize a subsequence construction wrapping it up with the arguments presented in Section 2. The reader should notice that the notation "u_n" used for the solution of (P_n), previously obtained, in $(-n,n)$ is not accidental: extending u_n by zero out of $[-n,n]$ we obtain a sequence (u_n) in $H^1(\mathbb{R})$. Throughout this section, we will use u_n to denote the solution "u_n" and its extension. Also, one can see that $\|u_n\|_{H^1(\mathbb{R})} = \|u_n\|_{W^{1,2}(-n,n)} \leq \tilde{r}$ for all n.

Let $K_1 = [-1,1]$; then for all $n \geq 1$ we have that $u^1_n := u_n|_{K_1}$ is well defined and $u^1_n \in H^1(-1,1)$. By the limitation $\|u^1_n\|_{W^{1,2}(-1,1)} \leq \tilde{r}$ there exists a subsequence $u_{n,1}$ and $s_1 \in H^1(-1,1)$ such that $u_{n,1} \rightharpoonup s_1$ in $H^1(-1,1)$. Notice that the compact injection $H^1(-1,1) \hookrightarrow C^0[-1,1]$ implies that, passing to a subsequence, $u_{n,1} \to s_1$ in $C^0[-1,1]$.

Let $K_2 = [-2,2]$. Taking n in the set of indices of the subsequence $u_{n,1}$, for $n \geq 2$ we have that $u^2_n := u_n|_{K_2}$ is well defined and $\|u^2_n\|_{W^{1,2}(-2,2)} \leq \tilde{r}$. Thus, there exists a subsequence $u_{n,2}$ of u^2_n and $s_2 \in H^1(-2,2)$ such that $u_{n,2} \rightharpoonup s_2$ in $H^1(-2,2)$.

Repeating the same argument, by induction we get that for all $j \in \mathbb{N}$ there exists a subsequence $u_{n,j}$ of $u_{n,j-1}$ and $s_j \in H^1(-j,j)$ such that $u_{n,j} \rightharpoonup s_j$ in $H^1(-j,j)$. Notice that $\|s_j\|_{H^1(-j,j)} \leq \tilde{r}$ for all $j \in \mathbb{N}$.

Remark 4.1. $u_{n,j}$ is the subsequence of u_n that converges weakly in $H^1(-j,j)$ to s_j. As mentioned, this weak convergence implies convergence in $C^0[-j,j]$ which gives us, in particular, that s_j is an even function, since $u_{n,j}$ is even for all $n \in \mathbb{N}$.

Lemma 4.2. $s_j|_{[1-j,j-1]} = s_{j-1}$

Proof. Given $x \in [1-j, j-1]$ we have that

$$s_{j-1}(x) = \lim_{n\to\infty} u_{n,j}(x) = s_j(x),$$

because $u_{n,j}$ is a subsequence of $u_{n,j-1}$. □

Fix $j \in \mathbb{N}$; from here and forward we will focus our attention on proving that, in fact, s_j *is smooth and positive.*

Define $W : [-n,n] \times [-Cr, Cr] \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$W(x,z,p) = z - (\lambda a_1(x)|z|^{q-1} + |z|^{p-1} + g(|p|))$$

one can see that the estimation (2.47) also holds, with W taking part as B (remember that C is the constant for the embedding $W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$). Then, for any $n \geq j$, by Theorem 1 from [12] there exist $\hat{C}(j) > 0$ and $0 < \beta \leq 1$ such that

$$|u_n|_{1+\beta} \leq \hat{C}(j) \quad \text{in } C^{1,\beta}[-j,j].$$

Taking $0 < \alpha < \beta \leq 1$ we get (see the argumentation on *item 2* Proposition 2.24)

$$u_{n,j} \to s_j \quad \text{in } C^{1,\alpha}[-j,j].$$

Let $\widetilde{a}_j := \inf_{x\in[-j,j]} a_1(x)$. We will use the arguments presented on *Item 1* from Proposition 2.24 to prove that s_j *is strictly positive on the interval* $[-j,j]$.

Case 1. There exists a subsequence $(u_{n_i,j})_{i\in\mathbb{N}}$ of $(u_{n,j})$ such that $u'_{n_i,j} \geq 0$ in $(-j,0)$ for all i.

The analysis of this case follows exactly the same parameters of **Case 1** from *Item 1*, Proposition 2.24. The main difference is the change of $\tilde{a}$ to $\widetilde{a}_j$.

Case 2. For all subsequence of $(u_{n,j})$ there exists a sub-subsequence $(u_{n_i,j})_{i\in\mathbb{N}}$ and exists a sequence $(z_i)_{i\in\mathbb{N}} \subset (-j,0)$ such that $u'_{n_i,j}(z_i) < 0$.

For this case we can reformulate Lemma 2.27 as follows: If $x \in (-j,j)$ and $u''_{n_i,j}(x) \geq 0$, then $u_{n_i,j}(x) > (\lambda\widetilde{a}_j)^{\frac{1}{2-q}}$. This is true because we already know that $u_{n_i,j}$ is strictly positive in $(-j,j)$, then the estimation

$$-A'(u_{n_i,j}(t))|u'_{n_i,j}(t)|^2 - A(u_{n_i,j}(t))u''_{n_i,j}(t) + u_{n_i,j}(t) > \lambda\widetilde{a}_j|u_{n_i,j}(t)|^{q-1} \tag{4.1}$$

is immediately established. The remaining argumentation is similar.

Thus, we conclude that $s_j > 0$, as in Proposition 2.24. At last, let $\varphi \in C_0^\infty(-j,j)$. Then

$$\int_{-j}^{j} A(u_{n,j})u'_{n,j}\varphi' + \int_{-j}^{j} u_{n,j}\varphi = \int_{-j}^{j} (\lambda a_1|u_{n,j}|^{q-1} + |u_{n,j}|^{p-1})\varphi + \int_{-j}^{j} g(|u'_{n,j}|)\varphi.$$

When $n \to \infty$ we get

$$\int_{-j}^{j} A(s_j)s'_j\varphi' + \int_{-j}^{j} s_j\varphi = \int_{-j}^{j} (\lambda a_1|s_j|^{q-1} + |v|^{p-1})\varphi + \int_{-j}^{j} g(|s'_j|)\varphi.$$

Since $\varphi \in C_0^\infty(-j,j)$ is arbitrary, we conclude that s_j is a weak solution for the problem

$$-(A(u)u')'(t) + u(t) = \lambda a_1(t)|u(t)|^{q-1} + |u(t)|^{p-1} + g(|u'(t)|) \tag{4.2}$$

in $(-j, j)$; by [6, sect. 6.3, Theorem 1] we have that $s_j \in H^2_{\text{loc}}(-j, j)$, thus, using the same arguments as in (2.49), $s_j \in C^2(-j, j)$.

Now we will construct our candidate solution of problem (1.1). Define $w_n = u_{n,n}$, that is, w_n is the diagonal sequence. Notice that, for $n \geq j$, w_n is a subsequence of $u_{n,j}$; thus $w_n|_{[-j,j]} \to s_j$ in $C^{1,\alpha}[-j, j]$ for all $j \in \mathbb{N}$. Let $v(x)$ be defined by

$$v(x) = \lim_{n\to\infty} w_n(x).$$

Then, $v(x) = s_j(x)$ for $x \in [-j, j]$. Since $\mathbb{R} = \bigcup_{j\in\mathbb{N}}[-j, j]$, by Lemma 4.2, v is well defined in $\mathbb{R}$. Using the properties of s_j obtained just above, and the fact that $v|_{[-j,j]} = s_j$ for all $j \in \mathbb{N}$, we conclude that $v \in C^2(\mathbb{R})$, $\|v\|_{H^1(\mathbb{R})} \leq \tilde{r}$ and v is a positive, even solution of problem (1.1). From [4, p. 214, Corol. 8.9] we get the homoclinic condition.

4.1 Asymptotic solution

Throughout our previous argumentation, we fixed $\lambda \in (0, \lambda^*]$. Denote by v_λ the strong solution – obtained above – to the problem (1.1). We will analyze the behavior of v_λ as $\lambda \to 0$.

Proposition 4.3. *As $\lambda \to 0$, $v_\lambda \to 0$ in $C^0(\mathbb{R})$.*

Proof. By [4, Theorem 8.8.], we obtain

$$\|v_\lambda\|_{L^\infty(\mathbb{R})} \leq C\|v_\lambda\|_{H^1(\mathbb{R})} \leq C\tilde{r}. \tag{4.3}$$

But remember that

$$\tilde{r} = \min\left\{r, \lambda^{1/2}\left(\frac{2C_2 r^q}{\gamma}\right)^{1/2}\right\}.$$

Then, as $\lambda \to 0$, we obtain that $\|v_\lambda\|_{L^\infty(\mathbb{R})} \to 0$, completing the proof of Theorem 1.1 . □

4.2 Proof of Proposition 1.2

We proceed to prove Proposition 1.2 that says that there is no solution of (1.1) for λ large.

Proof. Suppose on the contrary that $\lambda^* = \infty$. In this way there is a sequence $\lambda_n \to \infty$ and corresponding solutions $v_{\lambda_n} > 0$ in $\mathbb{R}$ given by Theorem 1.1.

Fix $R > 0$ and define $\mathcal{P}(t, s) = \lambda a_1(t)s^{q-1} + s^{p-1}$ and $\tilde{a}_R = \inf_{(-R,R)} a_1(t)$. Define also

$$\Lambda = \lambda\, \tilde{a}_R.$$

We claim that there is a constant $C_\Lambda > 0$ such that

$$\mathcal{P}(t, s) \geq \Lambda s^{q-1} + s^{p-1} \geq C_\Lambda s \quad \text{for } s > 0, \ \ t \in (-R, R).$$

Consider the function $\mathcal{Q}(s) = (\Lambda s^{q-1} + s^{p-1})s^{-1}$. Then $\mathcal{Q}(s) \to \infty$ as $s \to 0^+$ and as $s \to \infty$. The minimum value of $\mathcal{Q}$ is achieved at the unique point

$$m = \left(\Lambda\, \frac{2-q}{p-2}\right)^{\frac{1}{p-q}}.$$

Thus $C_\Lambda = \mathcal{Q}(m)$.

Let $\sigma_1 > 0$ and $\varphi_1 > 0$, respectively, the first eigenvalue and the first eigenfunction satisfying

$$\begin{cases} -\varphi_1'' = \sigma_1 \varphi_1 & \text{in } (-R, R) \\ \varphi_1(-R) = \varphi_1(R) = 0. \end{cases}$$

Since C_Λ increases as λ_n increases, there is λ_0 such that the corresponding constant satisfies $C_{\Lambda_0} \geq A(C\tilde{r})\sigma_1 + A(C\tilde{r})\delta + 1$, for all $\delta \in (0,1)$. Hence, by (H_2)–(H_3) and (4.3), the solution $v_{\lambda_0} > 0$ in $\mathbb{R}$ of (1.1) associated to λ_0 satisfies

$$\begin{cases} -v_{\lambda_0}'' \geq \frac{(C_{\Lambda_0}-1)}{A(v_{\lambda_0})} v_{\lambda_0} \geq \frac{(C_{\Lambda_0}-1)}{A(C\tilde{r})} v_{\lambda_0} \geq (\sigma_1 + \delta) v_{\lambda_0} & \text{in } (-R, R) \\ v_{\lambda_0}(-R), v_{\lambda_0}(R) \geq 0. \end{cases}$$

Otherwise, taking $\varepsilon > 0$ small enough we obtain $\varepsilon\varphi_1 < v_0$ in $(-R, R)$ and

$$\begin{cases} -(\varepsilon\varphi_1)'' = (\varepsilon\sigma_1)\varphi_1 \leq (\sigma_1 + \delta)\varphi_1 & \text{in } (-R, R) \\ \varphi_1(-R) = \varphi_1(R) = 0. \end{cases}$$

By the method of subsolution and supersolution, there is a solution $\varepsilon\varphi_1 < \omega < v_0$ in $B_R(0)$ of

$$\begin{cases} -\omega'' = (\sigma_1 + \delta)\omega & \text{in } (-R, R) \\ \omega(-R) = \omega(R) = 0. \end{cases}$$

Hence there is a contradiction to the fact that σ_1 is isolated. Therefore, $\lambda^* < \infty$, indeed. □

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Quasilinear Schrödinger equations with general sublinear conditions

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Abstract. In this paper, we study the quasilinear Schrödinger equations

$$-\Delta u + V(x)u + \Delta(u^2)u = f(x,u), \qquad \forall x \in \mathbb{R}^N,$$

where $V \in C(\mathbb{R}^N;\mathbb{R})$ may change sign and f is only locally defined for $|u|$ small. Under some new assumptions on V and f, we show that the above equation has a sequence of solutions converging to zero. Some recent results in the literature are generalized and significantly improved and some examples are also given to illustrate our main theoretical results.

Keywords: variational methods, critical points, quasilinear Schrödinger equations.

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1 Introduction

The aim of this paper is to establish the existence of multiple small solutions for the following quasilinear Schrödinger equations

$$-\Delta u + V(x)u + \Delta(u^2)u = f(x,u), \quad \forall x \in \mathbb{R}^N, \tag{QSE}$$

where $V \in C(\mathbb{R}^N;\mathbb{R})$ may change sign and f is only locally defined near the origin with respect to u and satisfies some weak and general sublinear assumptions.

Quasilinear Schrödinger equations (QSE) are widely used in non-Newtonian fluids, reaction-diffusion problems and other physical phenomena. More information on the physical background of these equations can be found in [6].

In recent years, with the aid of variational methods, the existence, nonexistence and multiplicity results of various solutions for (QSE) have been extensively investigated in the literature see [1, 5, 8, 10, 13] and the references therein. Here we emphasize that in all these papers V is a positive constant or possesses some kind of periodicity or radially symmetric, and the

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nonlinear term $f(x,u)$ is always required to satisfied various growth conditions at infinity with respect to u.

Recently, Chong et al. in [8] studied the equation (QSE) and proved the existence of multiple small solutions under the following conditions:

(C_1) There exist $\delta > 0$ and $C > 0$ such that $f \in C(\mathbb{R}^N \times [-\delta, \delta], \mathbb{R}^N)$, f is odd in x and

$$|f(x,u)| \leq C|u|, \quad \text{uniformly in } x \in \mathbb{R}^N;$$

(C_2) There exist $x_0 \in \mathbb{R}^N$ and $r_0 > 0$ such that

$$\liminf_{u\to 0}\left(\inf_{x\in B_{r_0}(x_0)} \frac{F(x,u)}{|u|^2}\right) > -\infty$$

and

$$\limsup_{u\to 0}\left(\inf_{x\in B_{r_0}(x_0)} \frac{F(x,u)}{|t|^2}\right) = +\infty,$$

where

$$F(x,u) = \int_0^u f(x,s)ds.$$

(V) For all $x \in \mathbb{R}^N$, $0 < V(x)$.

Motivated by the work of Chong et al. [8] and the [17, Lemma 2.3], in [5] the authors replaced the Condition (C_2) by a weak condition and proved the existence of multiple small solutions. Precisely, they supposed the following assumption:

(C_2') There exist $x_0 \in \mathbb{R}^{\mathbb{N}}$, two sequences (δ_n), (M_n) and constants α, $r_0 > 0$ such that δ_n, $M_n > 0$ and

$$\lim_{n\to\infty} \delta_n = 0, \quad \lim_{n\to\infty} M_n = +\infty,$$

$$\frac{F(x,u)}{{\delta_n}^2} \geq M_n \quad \text{for } |x - x_0| \leq r_0 \text{ and } |u| = \delta_n,$$

$$F(x,u) \geq -\alpha u^2 \quad \text{for } |x - x_0| \leq r_0 \text{ and } |u| \leq \delta.$$

In the present paper, different from the references mentioned above, we are going to study the existence of infinitely many solutions for (QSE) without any growth condition assumed on $f(x,u)$ at infinity with respect to u and the potential $V \in C(\mathbb{R}^N;\mathbb{R})$ may change sign. In fact, we will only require that $f(x,u)$ is locally defined for u small and satisfies some general and weak sufficient sublinear condition in u and V is neither of constant sign nor periodic. More precisely, we make the following assumptions:

(V_0) There exists a constant $a_0 > 0$ such that

$$V(x) + a_0 \geq 1, \quad \forall x \in \mathbb{R}^N,$$

$$\int_{\mathbb{R}^N} (V(x) + a_0)^{-1} dx < \infty,$$

and $\{x \in \mathbb{R}^N / V(x) \equiv 0\} \supset B(0,1)$, where $B(0,1)$ is the unit ball in $\mathbb{R}^N$.

(F_1) $F \in C^1(\mathbb{R}^N \times (-\delta, \delta))$ is even, and there exists a constant $a_1 > 0$ such that

$$|f(x,u)| \leq a_1, \quad \forall (x,u) \in \mathbb{R}^N \times (-\delta, \delta),$$

where $\delta > 0$.

For $\rho > 0$, $x \in B(0,1)$ satisfying $B(x,\rho) \subset B(0,1)$ and for $u \in (0,\delta)$, we define

$$\overline{F}(x,u,\rho) := \inf \left\{ \frac{F(y,u)}{u^2}\rho^2 : y \in B(x,\rho) \right\}, \tag{1.1}$$

$$\underline{F}(x,u,\rho) := \inf \left\{ \frac{F(y,mu)}{u^2}\rho^2 : y \in B(x,\rho), 0 \leq m \leq 1 \right\}. \tag{1.2}$$

Substituting $m = 0$ into $\frac{F(y,mu)}{u^2}\rho^2$, we see that $\underline{F}(x,u,\rho) \leq 0$. We assume:

(F_2) There exists a positive integer k_0 satisfying the following condition:
For each $k \geq k_0$, there exist $\mu_k \in (-\frac{\delta}{2}, 0) \cup (0, \frac{\delta}{2})$, $x_{k,i} \in B(0,1)$, with $1 \leq i \leq 2k$ and $\rho_k > 0$ such that $B(x_{k,i}, \rho_k) \subset B(0,1)$, $B(x_{k,i}, \rho_k) \cap B(x_{k,j}, \rho_k) = \emptyset$ for $i \neq j$ and

$$\min_{1 \leq i \leq 2k} \overline{F}(x_{k,i}, \mu_k, \rho_k) + (2^{N+1} - 1) \min_{1 \leq i \leq 2k} \underline{F}(x_{k,i}, \mu_k, \rho_k) > 2^{N+2}. \tag{1.3}$$

In (1.3), N is the dimension of the domain $\mathbb{R}^N$.

Our main results reads as follows.

Theorem 1.1. *Suppose that (V_0) and (F_1), (F_2) are satisfied. Then, equation (QSE) possesses a sequence of solutions $\{u_k\}$ such that $u_k(x) \to 0$ in L^∞ as $k \to \infty$.*

Remark 1.2.

- We insist on the fact that in the hypotheses (F_1)–(F_2), the conditions on the nonlinearity $F(x,u)$ are supposed only near $u = 0$ and there are no conditions for large $|u|$. This is essential and important. Indeed, this assumptions allows us to study equations having singularity or supercritical terms as $|u| \to \infty$.
- Under (F_1)–(F_2), $F(x,u)$ can be subquadratic, superquadratic or asymptotically quadratic at infinity. Our Theorem 1.1 is in some sense an improvement for some related results in the existing literature.
- To the best of our knowledge, there is no result concerning the existence and multiplicity of solutions for the equation (QSE) with the conditions.

Corollary 1.3. *Suppose that (V_0) and (F_1) are satisfied and $\delta > 0$ be as in (F_1). We assume that there exist sequences $M_n \to \infty$ as $n \to \infty$, $u_n \in (-\frac{\delta}{2}, 0) \cup (0, \frac{\delta}{2})$ and $\rho_n > 0$, $v_n \in B(0,1)$ such that $B(v_n, \rho_n) \subset B(0,1)$ and a constant $c \geq 0$, satisfy*

$$F(x,u_n)\rho_n^2 \geq M_n u_n^2, \; F(x, l u_n)\rho_n^2 \geq -c u_n^2 \quad \text{for } x \in B(v_n, \rho_n), \; 0 \leq l \leq 1. \tag{1.4}$$

Then, equation (QSE) possesses a sequence of solutions $\{u_k\}$ such that $u_k(x) \to 0$ in L^∞ as $k \to \infty$.

Corollary 1.4. *Suppose that (V_0) and (F_1) are satisfied and $\delta > 0$ be as in (F_1). We assume that there exist sequences $u_n \in (0, \frac{\delta}{2})$, $\rho_n > 0$ and $v_n \in B(0,1)$ such that $B(v_n, \rho_n) \subset B(0,1)$, and they satisfy*

$$\lim_{n \to \infty} \overline{F}(v_n, u_n, \rho_n) = \infty, \tag{1.5}$$

$$\liminf_{n \to \infty} \underline{F}(v_n, u_n, \rho_n) > -\infty. \tag{1.6}$$

Then, equation (QSE) possesses a sequence of solutions $\{u_k\}$ such that $u_k(x) \to 0$ in L^∞ as $k \to \infty$.

Corollary 1.5. *Suppose that* (V_0), (F) *and* (F_1) *are satisfied. Then, equation* (*QSE*) *possesses a sequence of solutions* $\{u_k\}$ *such that* $u_k(x) \to 0$ *in* L^∞ *as* $k \to \infty$.

Corollary 1.6. *Suppose that* $(V_0), (F_1)$ *and*

$$\inf_{x \in B(x_0, r_0)} u^{-2} F(x,u) \to \infty \quad \text{as } u \to 0, \tag{1.7}$$

are satisfied. Then, equation (*QSE*) *possesses a sequence of solutions* $\{u_k\}$ *such that* $u_k(x) \to 0$ *in* L^∞ *as* $k \to \infty$.

2 Preliminary results and variational setting

We employ an argument inspired by the work of Costa, Wang [11], the quasilinear problem was can be established:

$$-\operatorname{div}(h^2(u)\nabla u) + h(u)h'(u)|\nabla u|^2 + V(x)u = f(x,u), \qquad x \in \mathbb{R}^N, \tag{2.1}$$

where $h : [0, +\infty) \to \mathbb{R}$ satisfying

$$h(t) = \begin{cases} \sqrt{1-2t^2} & \text{if } 0 \le t < \frac{1}{\sqrt{6}}, \\ \frac{1}{6t} + \frac{1}{\sqrt{6}} & \text{if } t \ge \frac{1}{\sqrt{6}}, \end{cases}$$

and $h(t) = h(-t)$ for $t < 0$. It deduces that $h \in C^1(\mathbb{R}, (\left(\frac{1}{\sqrt{6}}\right), 1))$ and is increasing in $(-\infty, 0)$ and decreasing in $[0, +\infty)$. Then, we define

$$H(t) := \int_0^t h(s)ds.$$

It is well known that $H(t)$ is an odd function and inverse function $H^{-1}(t)$ exists. We now summarize some properties of $H^{-1}(t)$ as follow.

Lemma 2.1 ([1])**.** *We have:*

1. $|t| \le |H^{-1}(t)| \le \sqrt{6}|t|$ *for all* $t \in \mathbb{R}$;
2. $|H(t)| \le |t|$ *for all* $t \in \mathbb{R}$;
3. $-\frac{1}{2} \le \frac{t}{h(t)} h'(t) \le 0$ *for all* $t \ge 0$.

As in [11], in the present paper we are concerned to provide that the problem (2.1) has a sequence of weak solution $\{u_n\}$ satisfying $\|u_n\|_{L^\infty} < \min\{\delta/2, \frac{1}{\sqrt{6}}\}$, in this situation

$$h(u_n) = \left(1 - 2|u_n|^2\right)^{1/2}.$$

In order to prove our main result via the critical point theory, we need to establish the variational setting for (*QSE*). Before this, we have the following remark:

Remark 2.2. Let $V_0(x) = V(x) + a_0$, $F_0(x, H^{-1}(v)) = F(x, H^{-1}(v)) + \frac{a_0}{2}(H^{-1}(v))^2$ and $F_0(x,u) := \int_0^u f_0(x,s)ds$. Consider the following equation

$$-\Delta v + V_0(x)\frac{H^{-1}(v)}{h(H^{-1}(v))} = \frac{f_0(x, H^{-1}(v))}{h(H^{-1}(v))}, \qquad \forall x \in \mathbb{R}^N. \tag{2.2}$$

Then, equation (2.2) is equivalent to equation (QSE). It is easy to check that the hypotheses (V_0) and (F_1), (F_2) still hold for V_0 and F_0 provided that those hold for V and F. Hence, in what follows, we always assume without loss of generality that $V(x) \geq 1$ for all $x \in \mathbb{R}^N$ and $\int_{\mathbb{R}^N}(V(x))^{-1}dx < \infty$.

In view of Remark 2.2, we consider the space $E := \{u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x)u^2dx < \infty\}$ equipped with the following inner product

$$(u,v) := \int_{\mathbb{R}^N}(\nabla u \cdot \nabla v + V(x)uv)dx.$$

Then E is a Hilbert space and we denote by $\|\cdot\|$ the associated norm. In what follows, E becomes our working space. Moreover, we write E^* for the topological dual of E, and $\langle\cdot,\cdot\rangle$: $E^* \times E \to \mathbb{R}$ for the dual pairing. Evidently, E is continuously embedded into $H^1(\mathbb{R}^N)$. Using the Sobolev embedding theorem, we immediately get the following lemma.

Lemma 2.3. *If V satisfies (V_0), then E is continuously embedded in L^1.*

Proof. By (V_0) and Hölder inequality, we have for all $u \in E$

$$\begin{aligned}\int_{\mathbb{R}^N} |u|\,dx &= \int_{\mathbb{R}^N} \left|(V(x))^{\frac{-1}{2}}(V(x))^{\frac{1}{2}}u\right| dx \\ &\leq \int_{\mathbb{R}^N} (V(x))^{\frac{-1}{2}} \left|(V(x))^{\frac{1}{2}}u\right| dx \\ &\leq \left(\int_{\mathbb{R}^N}(V(x))^{-1}dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}V(x)u^2dx\right)^{\frac{1}{2}} \\ &\leq \left(\int_{\mathbb{R}^N}(V(x))^{-1}dx\right)^{\frac{1}{2}}\|u\|.\end{aligned} \tag{2.3}$$

□

Lemma 2.4. *If V satisfies (V_0) then E is compactly embedded into L^1.*

Proof. Let $(u_n) \subset E$ be a bounded sequence such that $u_n \rightharpoonup u$ in E. We will show that $u_n \to u$ in L^1. By Hölder's inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|u_n - u|\,dx \\ &= \int_{|x|\leq R}|u_n-u|\,dx + \int_{|x|>R}|u_n-u|\,dx \\ &\leq \omega R^N\left(\int_{|x|\leq R}|u_n-u|^2dx\right)^{\frac{1}{2}} + \int_{|x|>R}\left|(V(x))^{\frac{-1}{2}}(V(x))^{\frac{1}{2}}(u_n-u)\right|dx \\ &\leq \omega R^N\left(\int_{|x|\leq R}|u_n-u|^2dx\right)^{\frac{1}{2}} + \int_{|x|>R}(V(x))^{-\frac{1}{2}}\left|(V(x))^{\frac{1}{2}}(u_n-u)\right|dx \\ &\leq \omega R^N\left(\int_{|x|\leq R}|u_n-u|^2dx\right)^{\frac{1}{2}} + \left(\int_{|x|>R}(V(x))^{-1}dx\right)^{\frac{1}{2}}\left(\int_{|x|>R}V(x)(u_n-u)^2dx\right)^{\frac{1}{2}} \\ &\leq \omega R^N\left(\int_{|x|\leq R}|u_n-u|^2dx\right)^{\frac{1}{2}} + \left(\int_{|x|>R}(V(x))^{-1}dx\right)^{\frac{1}{2}}\|u_n-u\|,\end{aligned} \tag{2.4}$$

where $R>0$, ω the volume of the unit ball in $\mathbb{R}^N$. Then by (V_0) and the Sobolev embedding Theorem, for any $\varepsilon>0$ there exits $R_0>0$ such that for $R>R_0$, we have

$$\int_{\mathbb{R}^N} |u_n-u|\,dx \le \varepsilon. \qquad \square$$

Lemma 2.5 ([2]). *E is continuously embedded into $L^p(\mathbb{R}^N)$ for all $p\in[2,6]$, and hence there exists $\tau_p>0$ such that*

$$\|v\|_{L^p(\mathbb{R}^N)} \le \tau_p\|u\|, \qquad \forall u\in E \text{ and } p\in[2,6]. \tag{2.5}$$

3 Proofs of main results

In order to define the corresponding variational functional on our working space E, we need modify $f(x,u)$ for u outside a neighborhood of the origin to get a globally defined $\widetilde{f}(x,u)$ as follows: Choose a constant $b\in(0,\frac{\delta}{2})$ and define a cut-off function $\chi\in C(\mathbb{R},\mathbb{R})$ satisfying

$$\chi(t) := \begin{cases} 1 \ \text{ if } -b\le t\le b \\ 0 \ \text{ if } t\ge 2b \end{cases} \quad \text{and,} \quad -\frac{2}{b}\le \chi'(t)<0 \quad \text{for } b<|t|<2b. \tag{3.1}$$

Let $\widetilde{f}(x,u):=\chi(u)f(t,u)$, for all $(x,u)\in\mathbb{R}^N\times\mathbb{R}$, and $\widetilde{F}(x,u):=\int_0^u\widetilde{f}(x,s)ds$, for all $(x,u)\in\mathbb{R}^N\times\mathbb{R}$. By (3.1) and assumption (F_1) we have, for all $(x,u)\in\mathbb{R}^N\times\mathbb{R}$,

$$\left|\widetilde{F}(x,u)\right| \le a_1\,|u| \quad \text{and} \quad \left|\widetilde{f}(x,u)\right| \le a_2, \tag{3.2}$$

where a_1 is the constant given in assumption (F_1) and a_2 is a positive constant.

Remark 3.1. As we have mentioned above, it is easy to verify that the equation (3.2) becomes

$$\left|\widetilde{F}(x,H^{-1}(v))\right| \le a_1\left|H^{-1}(v)\right| \quad \text{and} \quad \left|\widetilde{f}(x,H^{-1}(v))\right| \le a_2\left|h(H^{-1}(v))\right|. \tag{3.3}$$

Now, we consider the following modified equation

$$-\Delta v+V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))} = \frac{\widetilde{f}(x,H^{-1}(v))}{h(H^{-1}(v))}, \qquad \forall x\in\mathbb{R}^N. \tag{$\widetilde{QSE}$}$$

To find the weak solutions of ($\widetilde{QSE}$) with desired properties, we focus on a Lagrangian functional defined by

$$\Phi(v) := \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2+V(x)|H^{-1}(v)|^2\right)dx-\Psi(H^{-1}(v)), \tag{3.4}$$

with the change of variable $v=H(u)$ and $\Psi(v)=\int_{\mathbb{R}^N}\widetilde{F}(x,H^{-1}(v))dx$.

Lemma 3.2. *Suppose that conditions (V_0) and (F_1) are satisfied. If $v\in E$ is a critical point of Φ, then $u=H^{-1}(v)\in E$ and this u is a weak solution for ($\widetilde{QSE}$).*

Proof. Since $v\in E$ and by Lemma 2.1, we can conclude that $u=H^{-1}(v)\in E$. Furthermore, v is a critical point for Φ, it follows that

$$\int_{\mathbb{R}^N}\nabla v\nabla\varphi dx+\int_{\mathbb{R}^N}V(x)\frac{H^{-1}(v)}{h(H^{-1}(v))}\varphi dx = \int_{\mathbb{R}^N}\frac{\widetilde{f}(x,H^{-1}(v))}{h(H^{-1}(v))}\varphi dx, \quad \text{for all } \varphi\in E.$$

If we take the function $\varphi = h(u)\psi$, where $u = H^{-1}(v)$ and $\psi \in C_0^\infty(\mathbb{R}^N)$, then we can obtain

$$\int_{\mathbb{R}^N} \nabla v \nabla u h'(u)\psi dx + \int_{\mathbb{R}^N} \nabla v \nabla \psi h(u) dx + \int_{\mathbb{R}^N} V(x) u\psi dx - \int_{\mathbb{R}^N} \widetilde{f}(x,u)\psi dx = 0.$$

Then, we get

$$\int_{\mathbb{R}^N} \Big(-\operatorname{div}(h^2(u)\nabla u) + h(u)h'(u)|\nabla u|^2 + V(x)u - \widetilde{f}(x,u) \Big)\psi dx = 0. \qquad \square$$

According to [8], we know that in order to find solutions of ($\widetilde{QSE}$) it suffices to obtain the critical points of Φ. For this purpose we recall the following definitions and results (see [14, 15]).

Definition 3.3 ([15])**.** Let E be a real Banach space and $\phi \in C^1(E, \mathbb{R})$.

- ϕ is said to satisfy (PS) condition if any sequence $(u_k) \subset E$ for which $(\phi(u_k))$ is bounded and $\phi'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in E. Here $\phi'(u)$ denotes the Fréchet derivative of $\phi(u)$.
- Set $\Gamma := \{A \subset E \backslash \{0\} : A \text{ is closed and symmetric with respect to the origin}\}$. For $A \in \Gamma$, we say genus of A is n (denoted by $\sigma(A) = n$), if there is an odd mapping $\varphi \in C(A, \mathbb{R}^n \backslash \{0\})$, and n is the smallest integer with this property.

Theorem 3.4 ([14, Theorem 1])**.** *Let ϕ be an even C^1 functional on E with $\phi(0) = 0$. Suppose that ϕ satisfies the (PS) condition and*

(1) ϕ is bounded from below.

(2) For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma$ such that $sup_{u \in A_k}\phi(u) < 0$, where $\Gamma_k = \{A \in \Gamma : \sigma(A) \geq k\}$.

Then either (i) or (ii) below holds.

(i) There exists a critical point sequence (u_k) such that $\phi(u_k) < 0$ and $\lim_{k\to\infty} u_k = 0$.

(ii) There exist two critical point sequences (u_k) and (v_k) such that $\phi(u_k) = 0, u_k \neq 0, \lim_{k\to\infty} u_k = 0, \phi(v_k) < 0, \lim_{k\to\infty} \phi(v_k) = 0$, and (v_k) converges to a non-zero limit.

Lemma 3.5. *Let (V_0) and (F_1) be satisfied. Then $\Psi \in C^1(E, \mathbb{R})$, and hence $\Phi \in C^1(E, \mathbb{R})$. Moreover,*

$$\langle \Psi'(v), \varphi \rangle = \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi \, dx, \tag{3.5}$$

and

$$\begin{aligned} \langle \Phi'(v), \varphi \rangle &= \int_{\mathbb{R}^N} \Big(\nabla v \nabla \varphi + V(x) \frac{H^{-1}(v)}{h(H^{-1}(v))} \varphi \Big) dx - \langle \Psi'(v), \varphi \rangle, \\ &= \int_{\mathbb{R}^N} \Big(\nabla v \nabla \varphi + V(x) \frac{H^{-1}(v)}{h(H^{-1}(v))} \varphi \Big) dx - \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi \, dx, \end{aligned} \tag{3.6}$$

for all $v, \varphi \in E$, and nontrivial critical points of Φ on E are solutions of equation ($\widetilde{QSE}$).

Proof. First, we show that Φ and Ψ are both well defined. For any $v \in E$, by (2.3) and (3.2), we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\widetilde{F}(x, H^{-1}(v))|dx &\le a_1 \int_{\mathbb{R}^N} |H^{-1}(v)|\ dx \\ &\le a_1 \int_{\mathbb{R}^N} |v|\ dx \\ &\le a_1 \left(\int_{\mathbb{R}^N} (V(x))^{-1} dx\right)^{\frac{1}{2}} \|v\|.\end{aligned}$$

This implies that Φ and Ψ are both well defined.

Next, we prove $\Psi \in C^1(E, \mathbb{R})$. For any given $v \in E$, define an associated linear operator $J(v) : E \to \mathbb{R}$ by

$$\langle J(v), \varphi\rangle = \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi\ dx, \quad \forall \varphi \in E.$$

By (2.3) and (3.2), there holds

$$\begin{aligned}|\langle J(v), \varphi\rangle| &= \int_{\mathbb{R}^N} \left|\frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}\right| |\varphi|\ dx \\ &\le a_2 \int_{\mathbb{R}^N} |\varphi|\ dx \\ &\le a_2 \left(\int_{\mathbb{R}^N} (V(x))^{-1}\ dx\right)^{\frac{1}{2}} \|\varphi\|.\end{aligned}$$

This implies that $J(v)$ is well defined and bounded. Observing (2.3) and (3.2), for any $v, \varphi \in E$, by the Mean Value Theorem and Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{s\to 0} \frac{\Psi(H^{-1}(v) + s\varphi) - \Psi(H^{-1}(v))}{s} &= \lim_{s\to 0} \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v) + \theta(x)s\varphi)}{h(H^{-1}(v) + \theta(x)s\varphi)} \varphi\ dx \\ &= \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi\ dx \\ &= \langle J(v), \varphi\rangle,\end{aligned} \tag{3.7}$$

where $\theta(x) \in [0, 1]$ depends on v, φ, s. This implies that Ψ is Gâteaux differentiable on E and the Gâteaux derivative of Ψ at $v \in E$ is $J(v)$. Now for any $\epsilon > 0$, by (V_0), there exists $R_\epsilon > 0$ such that

$$\left(\int_{|x|>R_\epsilon} (V(x))^{-1} dx\right)^{\frac{1}{2}} < \frac{\epsilon}{4a_2}. \tag{3.8}$$

For this end, we claim that if $H^{-1}(v_n) \rightharpoonup H^{-1}(v)$ in E, then for any $R > 0$, $\frac{\widetilde{f}(x,H^{-1}(v_n))}{h(H^{-1}(v_n))} \to \frac{\widetilde{f}(x,H^{-1}(v))}{h(H^{-1}(v))}$ in $L^2(B_R)$, where B_R denotes the ball in $\mathbb{R}^N$ centered at 0 with radius R. Arguing indirectly, by Lemma 2.5, we assume that there exist constants $R_\epsilon, \epsilon > 0$ and a subsequence $\{H^{-1}(v_{n_k})\}_{k\in\mathbb{N}}$ such that

$$H^{-1}(v_{n_k}) \to H^{-1}(v) \quad \text{in } L^2(B_{R_\epsilon}) \quad \text{and} \quad H^{-1}(v_{n_k}) \to H^{-1}(v) \quad \text{a.e. in } B_{R_\epsilon} \quad \text{as } k \to \infty, \tag{3.9}$$

but using (F_1), we have

$$\int_{|x|\le R_\epsilon} \left|\frac{\widetilde{f}(x, H^{-1}(v_{n_k}))}{h(H^{-1}(v_{n_k}))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))}\right|^2 dx \ge \epsilon, \quad \forall k \in \mathbb{N}. \tag{3.10}$$

By (3.9), passing to a subsequence if necessary, we can assume that

$$\sum_{k=1}^{\infty} \|H^{-1}(v_{n_k}) - H^{-1}(v)\|_{L^2(B_{R_\epsilon})} < +\infty.$$

By virtue of (3.3), we get

$$\int_{|x|\le R_\epsilon} \left| \frac{\widetilde{f}(x, H^{-1}(v_{n_k}))}{h(H^{-1}(v_{n_k}))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right|^2 dx < +\infty. \tag{3.11}$$

For the R_ϵ given above, combining (3.9), (3.11) and Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{k\to\infty} \int_{|x|\le R_\epsilon} \left| \frac{\widetilde{f}(x, H^{-1}(v_{n_k}))}{h(H^{-1}(v_{n_k}))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right|^2 dx = 0,$$

which contradicts (3.10). Thus the claim is true. Consequently, there exists $N_\epsilon \in \mathbb{N}$ such that

$$\int_{|x|\le R_\epsilon} \left| \frac{\widetilde{f}(x, H^{-1}(v_{n_k}))}{h(H^{-1}(v_{n_k}))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right|^2 dx < \frac{\epsilon}{2}, \ \ \forall n \ge N_\epsilon. \tag{3.12}$$

Combining (3.3), (3.8), (3.12) and the Hölder inequality, for each $n \ge N_\epsilon$, we have

$$\begin{aligned}
\|J(v_n) - J(v)\|_{E^*} &= \sup_{\|H^{-1}(v)\|=1} |\langle J(v_n) - J(v), \varphi\rangle| \\
&\le \sup_{\|H^{-1}(v)\|=1} \left| \int_{\mathbb{R}^N} \left[\frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right] \varphi dx \right| \\
&\le \sup_{\|H^{-1}(v)\|=1} \left| \int_{|x|\le R_\epsilon} \left[\frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right] \varphi dx \right| \\
&\quad + \sup_{\|H^{-1}(v)\|=1} \left| \int_{|x|> R_\epsilon} \left[\frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \right] \varphi dx \right| \\
&\le \sup_{\|H^{-1}(v)\|=1} \left(\int_{|x|\le R_\epsilon} \left| \frac{\widetilde{f}(x, H^{-1}(v_n))}{h(H^{-1}(v_n))} - \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} dx \right|^2 \right)^{\frac{1}{2}} \left(\int_{|x|\le R_\epsilon} |\varphi|^2 dx \right)^{\frac{1}{2}} \\
&\quad + 2a_2 \sup_{\|H^{-1}(v)\|=1} \left(\int_{|x|>R_\epsilon} (V(x))^{-1} dx \right)^{\frac{1}{2}} \left(\int_{|x|>R_\epsilon} V(x)\varphi^2 dx \right)^{\frac{1}{2}} \\
&\le \frac{\epsilon}{2} + \frac{2a_2\epsilon}{4a_2} = \epsilon.
\end{aligned}$$

This, means that J is continuous in u. Thus, $\Psi \in C^1(E, \mathbb{R})$ and (3.5) holds. Due to the form of ϕ, we know that $\Phi \in C^1(E, \mathbb{R})$ and (3.6) also holds.

Finally, a standard argument shows that nontrivial critical points of Φ on E are solutions of ($\widetilde{QSE}$) (see, e.g., [8]). The proof is completed. □

Lemma 3.6. *Let (V_0) and (F_1) be satisfied. Then Φ is bounded from below and satisfies (PS) condition.*

Proof. We first prove that Φ is bounded from below. Combining $(F1)$, (2.3), (3.2) and the Hölder inequality, we have

$$\begin{aligned}\Phi(v) &\geq \frac{1}{2}\|v\|^2 - a_1 \int_{\mathbb{R}^N} |H^{-1}(v)|dx \\ &\geq \frac{1}{2}\|v\|^2 - a_1 \left(\int_{\mathbb{R}^N} (V(x))^{-1}dx\right)^{\frac{1}{2}} \|v\|, \quad \forall v \in E,\end{aligned} \tag{3.13}$$

where a_2 is the constant given in (3.2). Then it follows that Φ is bounded from below.

Next, we show that Φ satisfies (PS)-condition.

Let $\{v_n\} \subset E$ be a (PS)-sequence, i.e.,

$$|\Phi(v_n)| \leq D_2 \quad \text{and} \quad \Phi'(v_n) \to 0 \quad \text{as } n \to \infty \tag{3.14}$$

for some $D_2 > 0$. By (3.13) and (3.14), we have

$$D_2 \geq \frac{1}{2}\|v_n\|^2 - a_2\left(\int_{\mathbb{R}^N} (V(x))^{-1}dx\right)^{\frac{1}{2}} \|v_n\|, \quad \forall n \in \mathbb{N}.$$

This implies that $\{v_n\}$ is bounded in E. Thus, there exists a subsequence $\{H^{-1}(v)_{n_k}\}$ such that

$$H^{-1}(v_{n_k}) \rightharpoonup H^{-1}(v_0) \quad \text{as } k \to \infty \tag{3.15}$$

for some $v_0 \in E$. By Lemma 2.4, it holds that

$$H^{-1}(v_{n_k}) \to H^{-1}(v_0) \quad \text{in } L^1 \text{ as } k \to \infty. \tag{3.16}$$

This together with (3.3) yields

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} \left[\frac{\widetilde{f}(x, H^{-1}(v_{n_k}))}{h(H^{-1}(v_{n_k}))} - \frac{\widetilde{f}(x, H^{-1}(v_0))}{h(H^{-1}(v_0))}\right] (H^{-1}(v_{n_k}) - H^{-1}(v_0))dx\right| \\ &\qquad \leq 2a_2 \int_{\mathbb{R}^N} |H^{-1}(v_{n_k}) - H^{-1}(v_0)|dx \to 0 \quad \text{as } k \to \infty.\end{aligned} \tag{3.17}$$

Noting that $\{\xi_n\}$ is bounded in E, we infer from (3.14) and (3.15) that

$$\langle \Phi'(\xi_{n_k}) - \Phi'(\xi_0), H^{-1}(\xi_{n_k}) - H^{-1}(\xi_0)\rangle \to 0 \quad \text{as } k \to \infty. \tag{3.18}$$

Combining (3.6), (3.17) and (3.18), we have

$$\begin{aligned}&\|H^{-1}(\xi_{n_k}) - H^{-1}(\xi_0)\|^2 \\ &\quad = \langle \Phi'(\xi_{n_k}) - \Phi'(\xi_0), H^{-1}(\xi_{n_k}) - H^{-1}(\xi_0)\rangle \\ &\qquad + \int_{\mathbb{R}N} \left(\frac{\widetilde{f}(x, \xi_{n_k})}{h(H^{-1}(\xi_{n_k}))} - \frac{\widetilde{f}(x, \xi_0)}{h(H^{-1}(\xi_0))}\right) (H^{-1}(\xi_{n_k}) - H^{-1}(\xi_0))dx \to 0 \quad \text{as } k \to \infty.\end{aligned} \tag{3.19}$$

This means that $H^{-1}(\xi_{n_k}) \to H^{-1}(\xi_0)$ in E as $k \to \infty$. Thus Φ satisfies (PS)-condition. □

We introduce a closed symmetric set V_k as below:

$$V_k \equiv \{(l_1, l_2, \ldots, l_{2k}) \in \mathbb{R}^{2k}; |l_i| \leq 1 \text{ for all } i, \text{ card}\{i : |l_i| = 1\} \geq k\}. \tag{3.20}$$

Lemma 3.7 ([15, Lemma 4.5]). *V_k has the genus of $k+1$.*

Lemma 3.8. *Let (V_0), (F_1) and (F_2) be satisfied. Then for each $k \in \mathbb{N}$, there exists an $A_k \subseteq E$ with genus $\sigma(A_k) = k+1$ such that $\sup_{u \in A_k} \Phi(v) < 0$.*

Proof. Let $\mu_k, x_{k,i}$ and ρ_k with $k \geq k_0$ be given in assumption (F_2). Since $\Gamma_k \subset \Gamma_{k-1}$ by definition, it is enough to construct an $A_k \in \Gamma_k$ for $k \geq k_0$ such that $\sup_{u \in A_k} \Phi(u) < 0$. Fix $k \geq k_0$. Instead of $\mu_k, x_{k,i}$ and ρ_k we write μ, x_i and ρ for simplicity. Using $\overline{F}$ and $\underline{F}$ given by (1.1) and (1.2) respectively, we define

$$\overline{F}_i := \overline{F}(x_i, \mu, \rho), \qquad \underline{F}_i := \underline{F}(x_i, \mu, \rho),\ 1 \leq i \leq 2k.$$

It follows from (1.1) and (1.2) and for $x \in B(x_i, \rho)$, that

$$F(x,\mu) \geq \frac{1}{\rho^2}\overline{F}_i (H^{-1}(\mu))^2 \geq \frac{1}{\rho^2}\overline{F}_i \mu^2, \tag{3.21}$$

$$F(x, l(\mu)) \geq \frac{1}{\rho^2}\underline{F}_i (H^{-1}(\mu))^2 \geq \frac{1}{\rho^2}\underline{F}_i \mu^2, \qquad |l| \leq 1. \tag{3.22}$$

We define a function $\varphi(t)$ on $\mathbb{R}$ by $\varphi(t) = 1$ for $|t| \leq 1/2$, $\varphi(t) = 2(1-|t|)$ for $1/2 \leq |t| \leq 1$, $\varphi(t) = 0$ for $|t| \geq 1$. Put $\varphi_i(x) = \varphi(|x - x_i|/\rho)$ for $x \in \mathbb{R}^N$. Then $\varphi_i \in W^{1,\infty}(\mathbb{R}^N)$. Define $B_i := B(x_i, \rho)$ and $D_i := B(x_i, \rho/2)$. Then $0 \leq \varphi_i(x) \leq 1$ in $\mathbb{R}^N$, $\varphi_i(x) = 0$ for $x \in \mathbb{R}^N \backslash B_i$ and

$$\varphi_i(x) = 1 \quad \text{for } x \in D_i, \quad |\nabla \varphi_i(x)| \leq \frac{2}{\rho} \quad \text{for } x \in \mathbb{R}^N. \tag{3.23}$$

Let V_k be given by (3.20). We define

$$A_k := \left\{ \mu \sum_{i=1}^{2k} l_i \varphi_i(x) :\ (l_1, \dots, l_{2k}) \in V_k \right\}.$$

Since all the supports of φ_i $(1 \leq i \leq 2k)$ are disjoint, they are linearly independent. Define $g(l_1, \dots, l_{2k}) := \mu \sum_{i=1}^{2k} l_i \varphi_i(x)$. Then g is a mapping from V_k onto A_k and it is an odd homeomorphism. By Lemma 3.7, the genus of V_k is $k+1$ and so is A_k. Thus $A_k \in \Gamma_k$.

We shall show that $\sup_{A_k} \Phi(v) < 0$. Fix $(l_1, \dots, l_{2k}) \in V_k$ arbitrary. Let $v := \mu \sum_{i=1}^{2k} l_i \varphi_i(x) \in A_k$ and $\mu \in (0, \frac{1}{2\sqrt{6}}\delta)$ be arbitrary. Since the support of φ_i is $\overline{B_i}$ and $B_i \cap B_j = \emptyset$ for $i \neq j$, we have

$$\begin{aligned}
\Phi(v) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V_0(x)(H^{-1}(v))^2)dx - \int_{\mathbb{R}^N} \widetilde{F}_0(x, H^{-1}(v))dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)(H^{-1}(v))^2)dx - \int_{\mathbb{R}^N} \widetilde{F}(x, H^{-1}(v))dx \\
&= \sum_{i=1}^{2k} \int_{B_i} \frac{1}{2}\mu^2 |l_i|^2 |\nabla \varphi_i|^2 dx - \sum_{i=1}^{2k} \int_{B_i} F(x, H^{-1}(\mu l_i \varphi_i))dx.
\end{aligned}$$

By the assumption (V_0) and (3.23), we have

$$\Phi(v) \leq 4k\omega\mu^2\rho^{N-2} - \sum_{i=1}^{2k} \int_{B_i} F(x, H^{-1}(\mu l_i \varphi_i))dx. \tag{3.24}$$

To estimate the second term, we define

$$\Lambda_1 := \{i \in \{1, \dots, 2k\} : |l_i| = 1\},$$
$$\Lambda_2 := \{i \in \{1, \dots, 2k\} : |l_i| < 1\}.$$

By the definition of V_k, the cardinal number of Λ_1 greater than or equal to k. We compute the integral of F on B_i for $i \in \Lambda_1$, and for $i \in \Lambda_2$, separately. Recall that $F(x, v)$ is even with respect to v and $\varphi_i(x) = 1$ on D_i. Clearly, the volume of D_i is $2^{-N}\omega\rho^N$. By (3.21) and (3.22), we obtain, for $i \in \Lambda_1$,

$$\begin{aligned}\int_{B_i} F(x, H^{-1}(\mu l_i\varphi_i))dx &= \int_{D_i} F(x, H^{-1}(\mu))dx + \int_{B_i\setminus D_i} F(x, H^{-1}(\mu l_i\varphi_i))dx \\ &\geq 2^{-N}\omega\mu^2\rho^{N-2}\overline{F}_i + (1 - 2^{-N})\omega\mu^2\rho^{N-2}\underline{F}_i.\end{aligned} \tag{3.25}$$

We define

$$\alpha := \min_{1\leq i\leq 2k} \overline{F}_i, \quad \beta := \min_{1\leq i\leq 2k} \underline{F}_i.$$

As stated after (1.2), it holds that $\underline{F}_i \leq 0$, and hence $\beta \leq 0$. We rewrite (1.3) as

$$\alpha + (2^{N+1} - 1)\beta > 2^{N+2}. \tag{3.26}$$

We reduce (3.25) to

$$\int_{B_i} F(x, \mu l_i\varphi_i)dx \geq \left[2^{-N}\alpha + (1 - 2^{-N})\beta\right]\omega\mu^2\rho^{N-2}.$$

The right hand side is positive because of (3.26) with $\beta \leq 0$. Recall that the cardinal number of Λ_1 is greather than ou equal to k. Summing up both sides of the inequality above over $i \in \Lambda_1$, we obtain

$$\sum_{i\in\Lambda_1} \int_{B_i} F(x, \mu l_i\varphi_i)dx \geq \left[2^{-N}\alpha + (1 - 2^{-N})\beta\right]k\omega\mu^2\rho^{N-2}. \tag{3.27}$$

Next, by (3.22), for $i \in \Lambda_2$, we have

$$\int_{B_i} F(x, \mu l_i\varphi_i)dx \geq \omega\mu^2\rho^{N-2}\underline{F}_i \geq \beta\omega\mu^2\rho^{N-2}. \tag{3.28}$$

Recall that the cardinal number of Λ_2 is less than or equal to k. Summing up both sides over $i \in \Lambda_2$ and using $\beta \leq 0$, we find

$$\sum_{i\in\Lambda_2} \int_{B_i} F(x, \mu l_i\varphi_i)dx \geq k\beta\omega\mu^2\rho^{N-2}. \tag{3.29}$$

The set Λ_2 may be empty. In this case, we consider the left hand side to be zero. Then the inequality above is still valid because $\beta \leq 0$. Substituting (3.27) and (3.29) into (3.24) and using (3.26), we obtain

$$\Phi(v) \leq -\left[\alpha(2^{N+1} - 1) + \beta - 2^{N+2}\right]k\omega\mu^2\rho^{N-2} < 0,$$

which implies that $\sup_{v\in A_k} \Phi(v) < 0$. □

In order to prove our main results, we further need the following lemma.

Lemma 3.9. *If $\{v_k\}$ is a critical point sequence of Φ satisfying $v_k \to 0$ in E as $k \to \infty$, then $v_k \to 0$ in $L^\infty(\mathbb{R}^N)$ as $k \to \infty$.*

Proof. Let $v \in E$ be a weak solution of ($\widetilde{QSE}$), i.e.,

$$\int_{\mathbb{R}^N} \nabla v \nabla \varphi dx + \int_{\mathbb{R}^N} V(x) \frac{H^{-1}(v)}{h(H^{-1}(v))} \varphi dx - \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} \varphi dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{3.30}$$

Set $T > 0$, and denote

$$v_T := \begin{cases} -T, & \text{if } v \le T, \\ v, & \text{if } -T < v < T, \\ T, & \text{if } v \ge T. \end{cases} \tag{3.31}$$

Taking $\varphi = |v_T|^{2(\eta-1)} v_T$ as the text function, where $\eta > 1$ to be determined later, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} |v_T|^{2(\eta-1)} \nabla v \nabla v_T dx + 2(\eta - 1) \int_{\mathbb{R}^N} |v_T|^{2(\eta-1)-1} \nabla v \nabla v_T dx \\ &\quad + \int_{\mathbb{R}^N} V(x) \frac{H^{-1}(v)}{h(H^{-1}(v))} |v_T|^{2(\eta-1)} v_T dx \\ &= \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} |v_T|^{2(\eta-1)} v_T dx. \end{aligned} \tag{3.32}$$

By using the facts

$$(\eta - 1) \int_{\mathbb{R}^N} |v_T|^{2(\eta-1)-1} \nabla v \nabla v_T dx \ge 0,$$
$$\int_{\mathbb{R}^N} V(x) \frac{H^{-1}(v)}{h(H^{-1}(v))} |v_T|^{2(\eta-1)} v_T dx \ge 0$$

and Lemma 2.1, we have

$$\frac{1}{\eta^2} \int_{\mathbb{R}^N} |\nabla |v_T|^\eta|^2 dx \le \int_{\mathbb{R}^N} \frac{\widetilde{f}(x, H^{-1}(v))}{h(H^{-1}(v))} |v_T|^{2\eta-1} dx \le a_2 \int_{\mathbb{R}^N} |v|^{2\eta-1} dx. \tag{3.33}$$

On the other hand, it follows from the Sobolev inequality that

$$\frac{S}{\eta^2} \|v_T\|_{2^*\eta}^{2\eta} \le \frac{1}{\eta^2} \int_{\mathbb{R}^N} |\nabla |v_T|^\eta|^2 dx, \tag{3.34}$$

where $S = \inf\{\int_{\mathbb{R}^N} |\nabla v|^2 dx \setminus \int_{\mathbb{R}^N} |v|^{2^*} dx = 1\}$ and $2^* = 2N/(N-2)$. In what follows, by (3.33) and (3.34), we get

$$\frac{1}{\eta^2} \|v_T\|_{2^*\eta}^{2\eta} \le a_2 \int_{\mathbb{R}^N} |v|^{2\eta-1} dx. \tag{3.35}$$

From Fatou's lemma, sending $T \to \infty$ in (3.35), it follows that

$$\|v\|_{2^*\eta} \le (c\eta)^{1/\eta} \|v\|_{2\eta-1}^{(2\eta-1)/2\eta}. \tag{3.36}$$

Let us define $\eta_k := \frac{2^*\eta_k - 1}{2}$, where $k = 1, 2, \ldots.$ and $\eta_0 = \frac{2^*-1}{2}$. Next, we present the first step of Moser's iteration, which is shown below:

$$\|v\|_{\eta_1 2^*} \le (C\eta_1)^{1/\eta_1} \|v\|_{2\eta_1-1}^{(2\eta_1-1)/2\eta_1} \tag{3.37}$$
$$\le (C\eta_1)^{1/\eta_1} (C\eta_0)^{1/\eta_0(2\eta_1-1)/2\eta_1} \|v\|_{2\eta_0-1}^{(2\eta_0-1)/2\eta_0(2\eta_1-1)/2\eta_1}. \tag{3.38}$$

We can assume, without loss of generality, that $C > 1$. Moreover, for any $i < j$, we we have the inequality given by equation

$$(C\eta_i)^{(2\eta_j-1)/2\eta_j} \le C\eta_j. \tag{3.39}$$

Using equations (3.37) and (3.39), we obtain the inequality

$$\|v\|_{\eta_1 2^*} \le (C\eta_1)^{1/\eta_1}(C\eta_0)^{1/\eta_0}\|v\|_{2\eta_1-1}^{(2\eta_0-1)/p\eta_0(2\eta_1-1)/2\eta_1}.$$

Applying Moser's iteration method, we can now derive the following result.

$$\|v\|_{2\eta_{k+1}-1} \le \exp\left(\sum_{i=0}^{k}\frac{\ln(C\eta_i)}{\eta_i}\right)\|v\|_{2^*}^{\mu_k},$$

where $\mu_k = \Pi_{i=0}^{k}\frac{2\eta_i-1}{2\eta_i}$. Taking the limit as $k\to\infty$, we obtain the following result.

$$\|v\|_\infty \le \exp\left(\sum_{i=0}^{k}\frac{\ln(C\eta_i)}{\eta_i}\right)\|v\|_{2^*}^{\mu},$$

where $\mu = \Pi_{i=0}^{k}\frac{2\eta_i-1}{2\eta_i}$ $(0<\mu<1)$ and $\exp\left(\sum_{i=0}^{k}\frac{\ln(C\eta_i)}{\eta_i}\right)$ is a positive constant. This, together with the Sobolev embedding theorem, we can conclude that if v_k is a sequence of critical points of Φ such that $v_k \to 0$ strongly in E as $k\to\infty$, then v_k converges strongly to zero in $L^\infty(\mathbb{R}^N)$. □

Now we are in the position to give the proofs of our main results.

4 Proofs of Theorem 1.1 and Corollaries 1.3–1.6

The aim of this section is to establish the proofs of Theorem 1.1 and Corollaries 1.3–1.6.

4.1 Proof of Theorem 1.1

Lemmas 3.6, 3.7 and 3.8 shows that the functional Φ satisfies conditions (1) and (2) in Theorem 3.4. Therefore, there exist a sequence of nontrivial critical points (u_k) of Φ such that $\Phi(u_k) \le 0$ for all $k\in\mathbb{N}$ and $u_k \to 0$ in E as $k\to\infty$. By virtue of Lemma 3.5, $\{u_k\}$ is a sequence of solutions of $(\widetilde{QSE})$ with $u_k\to 0$ in E as $k\to\infty$. Hence, there exists $k_0\in\mathbb{N}$ such that u_k is a solution of (QSE) for each $k \ge k_0$.

4.2 Proof of Corollary 1.3 and 1.4

It is enough to show that (1.5) and (1.6) $\Rightarrow$ (1.4) $\Rightarrow$ (1.3). Impose (1.5) and (1.6). Then we shall construct $\mu_k, x_{k,i}$ and ρ_k satisfying (1.3). Fix k arbitrarily. Let C_n be the inscribed cube in $B(v_n,\rho_n)$. Then its edge has the length of $2\rho_n/\sqrt{N}$. Let q be the smallest positive integer satisfying $q^N \ge 2k$. We divide the cube $\mathcal{C}_n$ equally into q^N small cubes by planes parallel to each face of $\mathcal{C}_n$ and denote them by $\mathcal{C}_{n,i}$ with $1\le i\le q^N$. More precisely, denote C_n by

$$C_n := [0,a]\times\cdots\times[0,a] \quad \text{with } a := 2\rho_n/\sqrt{N}.$$

Put $I_j := [a(j-1)/q, aj/q]$ with $1\le j\le q$ and define

$$I(j_1,\ldots,j_N) := I_{j_1}\times\cdots\times I_{j_N} \quad \text{with } 1\le j_1,\ldots,j_N\le q.$$

This, is a cube in $\mathbb{R}^N$ and C_n is the union of all these cubes. We rename all $I(j_1, \ldots, j_N)$ to $C_{n,i}$ with $1 \le i \le q^N$. Then the edge of each $C_{n,i}$ has the length of $2\rho_n/q\sqrt{N}$. Denote the inscribed ball in $C_{n,i}$ by $B(x_{n,i}, r_n)$. Then $r_n = \rho/q\sqrt{N}$. Since $q^N \ge 2k$, $x_{n,i}$ is defined for all $1 \le i \le 2k$.

We shall show that assumption (F_2) is fulfilled with μ_k, $x_{k,i}$ and ρ_k replaced by $u_n, x_{n,i}$ and r_n, respectively, if n is large enough. It is clear that $B(x_{n,i}, r_n) \subset B(0,1)$ and $B(x_{n,i}, r_n) \cap B(x_{n,j}, r_n) = \emptyset$ when $i \ne j$. Define $M_n := \overline{F}(v_n, u_n, \rho_n)$, which implies that

$$\frac{F(x,u_n)}{u_n^2}\rho_n^2 \ge M_n \quad \text{for } x \in B(v_n, \rho_n).$$

By (1.6), there exists a $c \ge 0$ such that

$$\frac{F(x,lu_n)}{u_n^2}\rho_n^2 \ge -c \quad \text{for } x \in B(v_n, \rho_n),\ 0 \le l \le 1.$$

Then we obtain (1.4). On the other hand, substituting $\rho_n = q\sqrt{N}\,r_n$ in the two inequalities above, we have

$$\frac{NF(x,u_n)}{u_n^2}q^2r_n^2 \ge M_n, \qquad \frac{NF(x,lu_n)}{u_n^2}q^2r_n^2 \ge -c,$$

for $x \in B(v_n, \rho_n)$ and $0 \le l \le 1$. Since $B(x_{n,i}, r_n) \subset B(v_n, \rho_n)$, the inequalities above are valid for $x \in B(x_{n,i}, r_n)$ also. Taking the infimum on $B(x_{n,i}, r_n)$, we have

$$\overline{F}(x_{n,i}, u_n, r_n) \ge \frac{M_n}{Nq^2}, \qquad \underline{F}(x_{n,i}, u_n, r_n) \ge -\frac{c}{Nq^2}.$$

Then we get

$$\min_{1\le i\le 2k} \overline{F}(x_{n,i}, u_n, r_n) + (2^{N+1}-1) \min_{1\le i\le 2k} \underline{F}(x_{n,i}, u_n, r_n) \ge \frac{1}{Nq^2}\left(M_n - (2^{N+1}-1)c\right).$$

Since $\lim_{n\to\infty} M_n = \infty$ by (1.5), the right hand side is larger than 2^{N+2} for n large enough.

4.3 Proof of Corollary 1.5

To prove this corollary, it is enough to show that the assumption (F) implies (1.5) and (1.6). By (F) there exists a sequence u_n converging to zero such that

$$\inf_{x\in B(x_0,r_0)} u_n^{-2}F(x,u_n) \to \infty \quad \text{as } n \to \infty.$$

Put $B(x_n, r_n) := B(x_0, r_0)$ for all n. Then the above inequality shows (1.5). Also, by (F), there exists a constant $c \ge 0$ such that

$$\inf_{x\in B(x_0,r_0)} u^{-2}F(x,u) \ge -c \quad \text{for } 0 < |u| \le 1.$$

Putting $u := lu_n$, we find

$$\inf_{x\in B(x_0,r_0)} (lu_n)^{-2}F(x,lu_n) \ge -c \quad \text{for all large } n \text{ and } 0 < l \le 1,$$

which leads to

$$\inf_{x\in B(x_0,r_0)} u_n^{-2}F(x,lu_n) \ge -cl^2 \ge -c.$$

Therefore (1.6) holds.

4.4 Proof of Corollary 1.6

We observe that (1.7) implies (F). Therefore, Corollary 1.5 yields Corollary 1.6.

5 Example

For the reader's convenience, we present one example to illustrate our main results.

Let

$$V(x) = \begin{cases} 0 \text{ if } |x| \leq p, \\ (p^2+1)^2(|x|-p), \text{ if } p \leq |x| < p + \frac{1}{p^2+1}, \\ p^2+1, \text{ if } p + \frac{1}{p^2+1} \leq |x| < p + \frac{p^2}{p^2+1}, \\ (p^2+1)^2(p+1-|x|), \text{ if } p + \frac{p^2}{p^2+1} \leq |x| < p+1, \end{cases}$$

and

$$F(x,u) = \frac{a}{s}|u|^s - \frac{d(x)}{r}|u|^r, \tag{5.1}$$

where $p \in \mathbb{N}^*$, and s, r, a are constants satisfying $1 < r < 2,\ 1 < s < \frac{2}{3}(r+1),\ a > 0$ and

$$d(x) := \inf\{|x-y| : y \in \partial B(0,1)\}.$$

Then V is neither of constant sign nor periodic. Moreover, we have

$$\inf_{x \in B(x_0,r_0)} \frac{F(x,u)}{u^2} = \frac{a}{s}|u|^{-(2-s)} - \frac{D}{r}|u|^{-(2-r)} \to -\infty \quad \text{as } u \to 0,$$

for any $B(x_0,r_0) \subset B(0,1)$, where $D := \max_{|x-x_0| \leq r_0} d(x) > 0$. Which implies that the assumption (C_2) and (C_2') are not satisfied. Now, we show that V and F match Theorem 1.1. Indeed, it is clear that $V(x)$ and $F(x,u)$ satisfy (V_0) and (F_1) respectively. It remains to check that $F(x,u)$ satisfies (F_2). For this purpose we assume that there exists a $\delta > 0$ such that for each $k \in \mathbb{N}$, there exist points $\xi_i \in \partial B(0,1)$ with $1 \leq i \leq 2k$ which satisfy $|\xi_i - \xi_j| \geq 4\delta/k$ for $i \neq j$, and δ is independent of k. Indeed, for example, choose a smooth curve on $\partial B(0,1)$ such that $g : [0,1] \to \partial B(0,1)$ is a C^1-diffeomorphism from $[0,1]$ onto $g([0,1])$. Since g^{-1} is Lipschitz continuous, there exists a $c_0 > 0$ such that $|g(t) - g(s)| \geq c_0|t-s|$ for $t, s \in [0,1]$. Put $\xi_i := g(i/2k)$ with $1 \leq i \leq 2k$. Then we have for $i \neq j$,

$$|\xi_i - \xi_j| = |g(i/2k) - g(j/2k)| \geq c_0|(i-j)/2k| \geq c_0/2k.$$

Define $\delta := c_0/8$. Then $|\xi_i - \xi_j| \geq 4\delta/k$ for $i \neq j$ and δ is independent of k.

Put $\rho_k := \delta/k$. For each $1 \leq i \leq 2k$, there exists a unique point $x_i \in B(0,1)$ such that $B(x_i,\rho_k) \subset B(0,1)$ and $\partial B(x_i,\rho_k) \cap \partial B(0,1) = \{\xi_i\}$, after replacing δ by a small constant if necessary. Since $|\xi_i - \xi_j| \geq 4\delta/k$ for $i \neq j$, $B(x_i,\rho_k) \cap B(x_j,\rho_k) = \emptyset$ for $i \neq j$. Since $d(x) \leq 2\rho_k$ in $B(x_i,\rho_k)$, we have

$$F(x,u) \geq \frac{a}{s}|u|^s - \frac{2}{r}|u|^r\rho_k \quad \text{for } x \in B(x_i,\rho_k). \tag{5.2}$$

Define θ as follows

$$\frac{2}{2-s} < \theta < \frac{s}{2(s-r)} + 1 \quad \text{when } s > r, \tag{5.3}$$

$$\frac{2}{2-s} < \theta \quad \text{when } s \le r. \tag{5.4}$$

It follows from (5.3) and (5.4) and $1 < s < 2(r+1)/3$ that

$$-(2-s)\theta + 2 < 0, \qquad -(2-s)\theta + 2 < -(2-r)\theta + 3. \tag{5.5}$$

We define $\mu_k := \rho_k^\theta$. Let us compute $\overline{F}$ defined by (1.1). Using (5.2), we have

$$\overline{F}(x_i, \mu_k, \rho_k) \ge \frac{a}{s}\rho_k^{-(2-s)\theta+2} - \frac{2}{r}\rho_k^{-(2-r)\theta+3} \to \infty, \tag{5.6}$$

as $k \to \infty$ by (5.5). Using (5.2) and $\mu_k := \rho_k^\theta$, we compute

$$\frac{F(x, m\mu_k)}{\mu_k^2}\rho_k^2 \ge \frac{am^s}{s}\rho_k^{-(2-s)\theta+2} - \frac{2m^r}{r}\rho^{-(2-r)\theta+3}, \tag{5.7}$$

for $x \in B(x_i, \rho_k)$ and $0 \le m \le 1$. We put

$$\alpha_k := a\rho_k^{-(2-s)\theta+2}, \qquad \beta_k := 2\rho_k^{-(2-r)\theta+3}$$

and denote the right hand side of (5.7) by

$$g_k(m) := \frac{\alpha_k}{s}m^s - \frac{\beta_k}{r}m^r \quad \text{for } m \in [0,1].$$

We shall show that $g_k(m)$ is bounded from below by a constant independent of k and $m \in [0,1]$. By (5.6), $g_k(1) > 0$ for $k \ge k_0$ with a large k_0. We divide the proof into two cases.

- $s > r$. Then $g_k(m)$ achieves a negative minimum in $[0,1]$, which is computed as

  $$\min_{0\le m\le 1} g_k(m) = -\frac{s-r}{sr}\alpha_k^{-\frac{r}{s-r}}\beta_k^{\frac{s}{s-r}} = -\frac{s-r}{sr}2^{\frac{s}{s-r}}a^{-\frac{r}{s-r}}\rho_k^\nu,$$

  where
  $$\nu = \frac{1}{s-r}\Big(-2(s-r)\theta + 3s - 2r\Big).$$
  Then $\nu > 0$ because of (5.3). Thus, the minimum of g_k converges to zero as $k \to \infty$.

- $s \le r$. Since $m^s \ge m^r$, we have $g_k(m) \ge \Big((\alpha_k/s) - (\beta_k/r)\Big)m^s \ge 0$ for $k \ge k_0$ and $m \in [0,1]$.

By Cases 1 and 2, we have the inequality $g_k(m) \ge -c$ with some $c \ge 0$ independent of k and $m \in [0,1]$, which shows that $\underline{F}(x_i, \mu_k, \rho_k) \ge -c$ for all $1 \le i \le 2k$ and $k \in \mathbb{N}$. This estimate with (5.6) shows (1.3) for all large k.

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New results concerning a Schrödinger equation involving logarithmic nonlinearity

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Abstract. In this paper, we investigate the existence of ground state solution to a class of Schrödinger equation involving logarithmic nonlinearity. To overcome the lack of smoothness, the corresponding functional J is first decomposed into the sum of a C^1 functional and a convex lower semicontinuous functional by adapting to the approach of Squassina–Szulkin in [*Calc. Var. Partial Differential Equations* **54**(2015), 585–597]. Secondly, the existence of a ground state solution to the studied equation is proved by using the Mountain Pass Theorem under the weakened Ambrosetti–Rabinowitz conditions.

Keywords: Schrödinger equations, ground state solution, logarithmic nonlinearity, Mountain Pass Theorem.

Mathematics Subject Classification 35J35, 35J62, 35J75, 35D30.

1 Introduction

In this paper, we consider the following Schrödinger equation involving logarithmic nonlinearity

$$\begin{cases} -\Delta u + V(x)u = Q(x)u\log u^2 + f(x,u), & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 1$, the external potential $V(x)$, the term $Q(x)$ and $f(x,u)$ are continuous functions and satisfy certain properties given later.

The Schrödinger equation was first proposed by the Austrian physicist E. Schrödinger. As a more complex nonlinear Schrödinger equation, it is derived from the following classical model

$$i\partial_t\Psi + \Delta\Psi - (V(x) + w)\Psi + f(|\Psi|) = 0. \tag{1.2}$$

The solution Ψ is called the standing wave solution of (1.2). Standing wave phenomenon refers to the phenomenon that electromagnetic waves can stay in a fixed position in some media without propagating and forming a resident electromagnetic field. Its application is widely reflected in our daily lives, such as magnetic resonance imaging to scan and image the

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internal structure of the human body. Applied to optical filters that allow specific wavelengths of light to pass through for spectral analysis and filtering. It is used in the field of acoustic standing wave cancellers to reduce noise and quantum mechanics theory. The interference function f in (1.2) is a nonlinear term, it can also be used to describe a variety of nonlinear waves in quantum physics, such as laser beam propagation in the medium with refractive index and wave amplitude, ionic sound wave in plasma, etc, see [4, 23] and the references therein. At present, many scholars have studied the existence and multiplicity of solutions to Schrödinger equation, see [2, 5, 6, 8–11, 14, 18, 19, 21, 22, 25, 26, 28–31] and the references therein for an overview on the related topic.

The logarithmic Schrödinger equation

$$i\partial_t \Psi + \Delta\Psi + \Psi \log |\Psi|^2 = 0, \quad \Psi : [0, +\infty) \times \mathbb{R}^N \to \mathbb{C}, \quad N \geq 3, \tag{1.3}$$

possesses wide applications to quantum mechanics, nuclear physics, open quantum systems, effective quantum gravity, transport and diffusion phenomena, theory of superfluidity and Bose–Einstein condensation, see [32] and the references therein. We refer to [7, 12] for a study of the existence and uniqueness of the solutions of the associated Cauchy problem in some suitable condition as well as the global existence and blow-up of the solutions. On the other hand, many researchers are interested in the existence, multiplicity and qualitative properties of the standing waves solution of problem (1.3). Consider the following Schrödinger equation with logarithmic nonlinear terms

$$-\Delta u + V(x)u = Q(x)u \log u^2, \qquad x \in \mathbb{R}^N. \tag{1.4}$$

When $V(x) = Q(x) = 1$, Avenia–Montefusco–Squassina [13] proved that there are infinitely many solutions to equation (1.4) by introducing weak slope and using non-smooth critical point theory. Ji–Szulkin [16] proved the existence of infinitely many solutions to equation (1.4) by Fountain theorem in the case which $Q(x) = 1$ and potential function $V(x)$ satisfies the mandatory condition, that is $\lim_{|x|\to+\infty} V(x) = +\infty$. In addition, Shuai [20] used the constraint minimization method to obtain the existence of positive solutions and node solutions of equation (1.4) under different assumptions of potential function $V(x)$. When $V(x)$ and $Q(x)$ are 1-period, Squassina–Szulkin [22] investigated the existence of infinitely many different solutions to problem (1.4) by applying non-smooth critical point theory and $\mathbb{Z}_2$ index theory, and the existence of ground state solutions of problem (1.4) was proved .

Through the analysis of the above mentioned results, as far as we know, there is few corresponding result if $V(x) \neq 1$, $Q(x) \neq 1$ and $V(x)$, $Q(x)$ are not 1-period in equation (1.4). Therefore, **a natural question is whether the equation (1.4) plus the disturbance term $f(x, u)$ can also obtain the ground state solution through the Mountain Pass Theorem with the weakened Ambrosetti–Rabinowitz conditions?** There are two key difficulties in the proof process, one is that the energy functional is not well defined and not C^1 smooth, the other is to prove the existence of the ground state solution when the Ambrosetti–Rabinowitz condition (for short AR-condition) is not satisfied.

For convenience, in this paper, it is assumed that the potential function V and the disturbance term f satisfy the following conditions:

(V1) The potential function $V(x) \in \mathcal{C}(\mathbb{R}^N, [0, +\infty))$, and there exists a constant $a_0 > 0$ such that $|\{x \in \mathbb{R}^N : V(x) \leq a_0\}| < +\infty$;

(V2) $\mathrm{int}V^{-1}(0) \neq \emptyset, Q(x) \in \mathcal{C}'(\mathbb{R}^N, [0, +\infty)), \min(V(x) + Q(x)) \geq 1.$

(f1) The function $f(x,t) \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and for any $\varepsilon > 0$ there exist constants $C(\varepsilon) > 0$ and $p \in (2, 2^*)$ such that $|f(x,t)| \leq \varepsilon|t| + C(\varepsilon)|t|^{p-1}$, where $2^* := 2N/(N-2)$ if $N \geq 3$, $2^* := \infty$ if $N = 1$ or 2;

(f2) $\lim_{|t|\to+\infty} \frac{F(x,t)}{t^2} = +\infty$, where $F(x,t) = \int_0^t f(x,\tau)d\tau$;

(f3) There exist constants $\alpha > 2$ and θ with $0 < \theta < \frac{S(\alpha-2)}{4}$ (S be given in (2.9) below), such that
$$\lim_{|t|\to\infty} \inf \frac{f(x,t)t - \alpha F(x,t)}{|t|^2} > -\theta, \quad \text{uniformly for a.e. } x \in \mathbb{R}^N.$$

Our main result states as follows.

Theorem 1.1. *Assume that (V1)–(V2) and (f1)–(f3) hold. Then problem (1.1) admits a ground state solution.*

Remark 1.2. *By (f3), there exist constants $M > 0, \alpha > 2$ such that $uf(x,u) \geq \alpha F(x,u)$ for $|u| \geq M, (x,u) \in \mathbb{R}^N \times \mathbb{R}$. Thus,*
$$uf\,(x,u) - aF\,(x,u) \geq 0 \geq -\theta|u|^2.$$
Obviously, we can show that the AR-condition implies (f3), while the inverse implication fails.

Remark 1.3. *There exists extensively the disturbance term $f(x,u)$ which satisfies conditions (f1)–(f3) of Theorem 1.1. Such as, taking $N = 3$, $2^* = 6$, then $p \in (2,6)$, if $p = 4$ for all $x \in \mathbb{R}^3$ and*
$$f(x,t) = \frac{S}{3}|\sin x|\left(|t|t + \frac{1}{2}t\sin 2t\right),$$
where S be given by formula (2.9) below, then
$$F(x,t) = \frac{S}{3}|\sin x|\left(\frac{1}{3}|t|^3 - \frac{1}{4}t\cos 2t + \frac{1}{8}\sin 2t\right).$$
Set $\alpha = 3$, $\theta = \frac{S}{5}$, then
$$\begin{aligned} f(x,t)t - \alpha F(x,t) &= \frac{S}{3}|\sin x|\left(|t|^3 + \frac{1}{2}t^2\sin 2t - \frac{\alpha}{3}|t|^3 + \frac{\alpha}{4}t\cos 2t - \frac{\alpha}{8}\sin 2t\right) \\ &\geq \frac{S}{3}|\sin x|\left((1-\frac{\alpha}{3})|t|^3 - \frac{1}{2}|t|^2 - \frac{\alpha}{4}t - \frac{\alpha}{8}\right) \\ &= -\frac{S}{3}|\sin x|\left(\frac{1}{2}|t|^2 + \frac{3}{4}t + \frac{3}{8}\right) \\ &\geq -\frac{S}{3}\left(\frac{1}{2}|t|^2 + \frac{3}{4}t + \frac{3}{8}\right), \end{aligned}$$
which implies
$$\lim_{|t|\to\infty} \inf \frac{f(x,t)t - 3F(x,t)}{|t|^2} = \lim_{|t|\to\infty} \frac{-\frac{S}{3}\left(\frac{1}{2}|t|^2 + \frac{3}{4}t + \frac{3}{8}\right)}{|t|^2} = -\frac{S}{6} > -\theta.$$
Obviously $f(x,t)$ satisfies all the conditions of Theorem 1.1.

This paper is organized as follows. In Section 2, we present some preliminary results that will be used later. Section 3 is devoted to proving the existence of ground state solutions to problem (1.1).

Notation. From now on, otherwise mentioned, we use the following notations:

- C, C_1, C_2, C_3, etc. will denote positive constants, whose exact values are not relevant.
- $\|\cdot\|_k$ denotes the usual norm of the Lebesgue space $L^k(\mathbb{R}^N)$, for $k \in [1, +\infty]$.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.

2 Preliminaries

To prove Theorem 1.1, we need to present some notation and auxiliary lemmas, which will be crucial in dealing with ground state solutions of problem (1.1).

Towards problem (1.1), we define the space E as follows:

$$E := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (V(x) + Q(x))u^2 dx < +\infty \right\},$$

endowed with the following norm:

$$\|u\|_E := \left(\int_{\mathbb{R}^N} |\nabla u|^2 + (V(x) + Q(x))u^2 dx \right)^{\frac{1}{2}}, \qquad u \in E,$$

and the space E is a Hilbert space. Problem (1.1) has a variational structure and is properly associated with the energy functional $J : E \to \mathbb{R} \cup \{+\infty\}$ defined by

$$J(u) = \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} F(x,u)dx - \frac{1}{2}\int_{\mathbb{R}^N} Q(x)u^2 \log u^2 dx, \qquad u \in E. \tag{2.1}$$

One premise, the critical point of the energy functional J is the weak solution of the corresponding equation, is that the energy functional J can be well defined and smooth. But, in general, the logarithmic term may cause that the energy functional J fails to be finite and $\mathcal{C}^1$ smooth in $H^1(\mathbb{R}^N)$. Concretely, from [17], we have the following simple modification of the standard logarithmic Sobolev inequality:

$$\int_{\mathbb{R}^N} u^2 \log u^2 dx \le \frac{a^2}{\pi}\|\nabla u\|_2^2 + (\log \|u\|_2^2 - N(1 + \log a))\|u\|_2^2, \tag{2.2}$$

for any $u \in H^1(\mathbb{R}^N)$ and $a > 0$. It follows from (2.2) that $J(u) > -\infty$ for all $u \in H^1(\mathbb{R}^N)$, but there exists $u_* \in H^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u_*^2 \log u_*^2 dx = -\infty$.

Recently, many scholars have tried to find some different techniques and methods to overcome the above given difficulty. For instance, in [7], Cazenave's main idea was to find a suitable Banach space endowed with a Luxemburg type norm. On account of the definition of Luxemburg type norm and some special properties, the functional $J(u)$ is finally well defined and $\mathcal{C}^1$ smooth. Another way to overcome it comes from [15], the authors penalize the nonlinearity term around the origin and obtain a priori estimates to get a nontrivial solution at the limit.

In what follows, by *a solution* to problem (1.1) we shall usually indicate a function $u \in H^1(\mathbb{R}^N)$ such that $u^2 \log u^2 \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)dx = \int_{\mathbb{R}^N} Q(x)uv \log u^2 dx + \int_{\mathbb{R}^N} f(x,u)v dx, \quad \text{for any } v \in \mathcal{C}_0^\infty(\mathbb{R}^N).$$

In this paper, we take the approach from [22], although $J(u)$ is not smooth, we split the functional $J(u)$ into the sum of a $\mathcal{C}^1$ functional and a convex lower semicontinuous functional. In order to split $J(u)$, we define the following two functions for $\delta > 0$:

$$F_1(s) = \begin{cases} 0, & s = 0, \\ -\frac{1}{2}s^2 \log s^2, & 0 < |s| \le \delta, \\ -\frac{1}{2}s^2(\log \delta^2 + 3) + 2\delta|s| - \frac{1}{2}\delta^2, & |s| > \delta, \end{cases}$$

and

$$F_2(s) = \begin{cases} 0, & |s| \le \delta, \\ -\frac{1}{2}s^2 \log(s^2/\delta^2) + 2\delta|s| - \frac{3}{2}s^2 - \frac{1}{2}\delta^2, & |s| > \delta. \end{cases}$$

Then $F_2(s) - F_1(s) = \frac{1}{2}s^2 \log s^2$ for all $s \in \mathbb{R}$. Thus, the functional $J : E \to (-\infty, +\infty]$ can be rewritten as

$$J(u) = \tilde{\Phi}(u) + \Psi(u), u \in E$$

where

$$\tilde{\Phi}(u) = \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} Q(x)F_2(u)dx - \int_{\mathbb{R}^N} F(x,u)dx \tag{2.3}$$

and

$$\Psi(u) = \int_{\mathbb{R}^N} Q(x)F_1(u)dx. \tag{2.4}$$

In the sequel, we list some properties about $F_1(s), F_2(s)$ that shall be useful for our proofs later, which were proved in [16,22].

Proposition 2.1. *If $\delta > 0$ is sufficiently small, then the following results are true:*

(i) $F_1(s), F_2(s) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$.

(ii) The function $F_1(s)$ is convex, even, and

$$F_1(s) \ge 0, F_1'(s)s \ge 0, \qquad \forall s \in \mathbb{R}. \tag{2.5}$$

(iii) For each fixed $q \in (2, 2^*)$, there is a constant $C > 0$ such that

$$|F_2'(s)| \le C|s|^{q-1}, \qquad \forall s \in \mathbb{R}. \tag{2.6}$$

Hereafter, $\delta > 0$ is fixed and sufficiently small such that the above properties of F_1, F_2 hold.

Corollary 2.1. *Assume that (V1)–(V2) and (f1) hold, then $\tilde{\Phi} \in \mathcal{C}^1(E, \mathbb{R})$.*

Proof. Let $\Phi(u) = \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} Q(x)F_2(u)dx$, according to Lemma 3.10 in [27] and (2.5), it is easy to show that $\Phi \in \mathcal{C}^1(E, \mathbb{R})$. Since

$$\tilde{\Phi}(u) = \Phi(u) - \int_{\mathbb{R}^N} F(x,u)dx,$$

we just need to prove $\int_{\mathbb{R}^N} F(x,u)dx \in \mathcal{C}^1(E, \mathbb{R})$. By condition (f1), there holds

$$|F(x,t)| \le \frac{1}{2}\varepsilon|t|^2 + \frac{1}{p}C(\varepsilon)|t|^p, \tag{2.7}$$

which implies $\int_{\mathbb{R}^N} F(x,u)dx \in \mathcal{C}^1(E, \mathbb{R})$. Thus $\tilde{\Phi} \in \mathcal{C}^1(E, \mathbb{R})$. □

Corollary 2.2. *The functional Ψ admits the following properties:*

(i) The functional $\Psi(u)$ is convex, $\Psi \geq 0$ for all $u \in E$, and $\Psi(u) = +\infty$ for certain $u \in E$. Furthermore, Ψ (and hence J) is lower semicontinuous.

(ii) If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the functional Ψ (and therefore J) is of class $\mathcal{C}^1$ in $H^1(\Omega)$.

Proof. (i) By the definition of the function $F_1(s)$, we get $\Psi \geq 0$. Since $Q(x) > 0$ and $F_1(s)$ is convex, then for all $\lambda \in [0,1]$ and u_1, $u_2 \in E$, there holds

$$\begin{aligned}\Psi(\lambda u_1 + (1-\lambda)u_2) &= \int_{\mathbb{R}^N} Q(x)F_1(\lambda u_1 + (1-\lambda)u_2)dx \\ &\leq \lambda \int_{\mathbb{R}^N} Q(x)F_1(u_1)dx + (1-\lambda)\int_{\mathbb{R}^N} Q(x)F_1(u_2)dx \\ &= \lambda\Psi(u_1) + (1-\lambda)\Psi(u_2).\end{aligned}$$

This implies that Ψ is convex, and $\Psi(u) = +\infty$ for certain $u \in E$. Moreover, there exists $s_0 \in E$, for all sequence $\{s_n\}$ with $s_n \to s_0$ as $n \to +\infty$. It follows naturally from the Fatou's lemma that

$$\begin{aligned}\lim_{n\to+\infty} \inf \Psi(s_n) &= \lim_{n\to+\infty} \inf \left(\int_{\mathbb{R}^N} Q(x)F_1(s_n)\mathrm{d}x\right) \\ &\geq \int_{\mathbb{R}^N} \lim_{n\to+\infty} \inf Q(x)F_1(s_n)dx = \Psi(s_0).\end{aligned}$$

Thus Ψ is lower semicontinuous, and the functional J is also lower semicontinuous.

(ii) By the definition of the function $F_1(s)$, we have $|F_1'(s)| \leq C(1+|s|^{p-1})$ for $p \in (2, 2^*)$. Then it follows from Lemma 2.16 of [27] that the conclusion is true in $H_0^1(\Omega)$ but the argument remains valid in $H^1(\Omega)$. □

According to some arguments in [24] and by Corollary 2.1 and Corollary 2.2, we also give the following definition:

Definition 2.3. *Let E be a Banach space, E' be the dual space of E, and $\langle\cdot,\cdot\rangle$ be the duality pairing between E' and E. Let $J : E \to \mathbb{R}$ be a functional of the form $J(u) = \tilde{\Phi}(u) + \Psi(u)$, where $\tilde{\Phi} \in \mathcal{C}^1(E, \mathbb{R})$ and $\Psi(u)$ is convex and lower semicontinuous. Let us list some definitions:*

(1) The sub-differential $\partial J(u)$ of the functional J at a point $u \in E$ is the following set

$$\{w \in E' : \langle\tilde{\Phi}'(u), v-u\rangle + \Psi(v) - \Psi(u) \geq \langle w, v-u\rangle,\ \forall v \in E\}.$$

(2) A critical point of J is a point $u \in E$ such that $J(u) < +\infty$ and $0 \in \partial J(u)$, i.e.

$$\langle\tilde{\Phi}'(u), v-u\rangle + \Psi(v) - \Psi(u) \geq 0, \qquad \forall v \in E.$$

(3) A Palais–Smale sequence at level c for J is a sequence $\{u_n\} \subset E$ such that $J(u_n) \to c$ and there is a numerical sequence $\tau_n \to 0^+$ with

$$\langle\tilde{\Phi}'(u_n), v-u_n\rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n\|v-u_n\|_E, \qquad \forall v \in E.$$

(4) The functional J satisfies the Palais–Smale condition at level c ($(PS)_c$ condition, for short) if all Palais–Smale sequences at level c have a convergent subsequence.

(5) The effective domain of J is the set $D(J) = \{u \in E : J(u) < +\infty\}$.

In the sequel, for each $u \in D(J)$, we set the following functional $J'(u) : H_c^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\langle J'(u), z\rangle = \langle \tilde{\Phi}'(u), z\rangle + \int_{\mathbb{R}^N} Q(x)F_1'(u)zdx \quad \text{for any } z \in H_c^1(\mathbb{R}^N),$$

where $H_c^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : \ u \text{ has compact support}\}$, and define

$$\|J'(u)\| = \sup\{\langle J'(u), z\rangle : z \in H_c^1(\mathbb{R}^N) \text{ and } \|z\|_E \leq 1\}.$$

Hence, $J'(u)$ can be extended to a bounded operator in E when $\|J'(u)\|$ is finite, and it may be seen as an element of E'.

In order to prove Theorem 1.1, we will use the following Lemma 2.4, whose proof can be found in Lemma 2.4 of [22] .

Lemma 2.4. *If $u \in D(J)$, then $\partial J(u) \neq \emptyset$, i.e. there exists $w \in E'$ such that*

$$\langle \Phi'(u), v-u\rangle + \Psi(v) - \Psi(u) \geq \langle w, v-u\rangle, \quad \textit{for all } v \in E.$$

Moreover, this w is unique and satisfies

$$\langle \Phi'(u), z\rangle + \int_{\mathbb{R}^N} Q(x)F_1'(u)zdx = \langle w, z\rangle, \quad \textit{for all } z \in E \textit{ such that } F_1'(u)z \in L^1(\mathbb{R}^N).$$

As an immediate consequence, we know that the unique element $w \in E'$ introduced in Lemma 2.4 will be denoted by $J'(u)$. Moreover, there holds

$$\langle J'(u), u\rangle = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx - \int_{\mathbb{R}^N} Q(x)u^2 \log u^2 dx - \int_{\mathbb{R}^N} f(x,u)udx, \tag{2.8}$$

for each $u \in D(J)$ with $\|J'(u)\| < +\infty$. Thus solution of problem (1.1) is equivalent to a nontrivial critical point of the functional J.

Theorem 2.5 ([27, Theorem 1.8], Sobolev imbedding theorem)**.** *The following imbeddings are continuous:*

$$\begin{aligned} H^1(\mathbb{R}^N) &\hookrightarrow L^p(\mathbb{R}^N), \qquad 2 \leq p < \infty, N = 1,2,\\ H^1(\mathbb{R}^N) &\hookrightarrow L^p(\mathbb{R}^N), \qquad 2 \leq p \leq 2^*, N \geq 3,\\ D^{1,2}(\mathbb{R}^N) &\hookrightarrow L^{2^*}(\mathbb{R}^N), \qquad N \geq 3. \end{aligned}$$

According to the norm in the space E and $L^2(\mathbb{R}^N)$ respectively, we have $\|u\|_2^2 \leq \|u\|_E^2$. In particular, the best constant for the Sobolev embedding $E \hookrightarrow L^2(\mathbb{R}^N)$ is given by

$$S := \inf_{u \in E\setminus\{0\}} \frac{\|u\|_E^2}{\|u\|_2^2}. \tag{2.9}$$

3 Proof of Theorem 1.1

In what follows, we will show that the functional J satisfies the Mountain pass geometry. The following two conclusions can be found in Theorem 3.1 and Corollary 3.1 of [1], which are crucial in our approach.

Theorem 3.1 (Mountain Pass Theorem without (PS) condition)**.** *Let X be a real Banach space and $J : X \to \mathbb{R}$ be a functional such that:*

(i) $J(u) = \tilde{\Phi}(u) + \Psi_(u), u \in X$ *with* $\tilde{\Phi}(u) \in \mathcal{C}^1(X, \mathbb{R})$*, and* $\Psi : X \to \mathbb{R}$ *is convex,* $\Psi \not\equiv +\infty$ *and is lower semicontinuous (l.s.c);*

(ii) $J(0) = 0$ *and* $J|_{\partial B_\rho} \geq \alpha_0$*, for real constants* $\rho, \alpha_0 > 0$*;*

(iii) $J(e) \leq 0$*, for some* $e \notin \overline{B_\rho}(0)$*.*

If

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)) \geq \alpha_0 > 0, \qquad \Gamma = \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

Then, for a given $\epsilon > 0$*, there is* $u_\epsilon \in X$ *such that*

$$\langle \tilde{\Phi}'(u_\epsilon), v - u_\epsilon \rangle + \Psi(v) - \Psi(u_\epsilon) \geq -3\epsilon \|v - u_\epsilon\|, \qquad \forall v \in X, \tag{3.1}$$

and $J(u_\epsilon) \in [c - \epsilon, c + \epsilon]$*.*

Corollary 3.2. *Under the conditions of Theorem 3.1, there is a* $(PS)_c$ *sequence* $\{u_n\}$ *of the functional* $J(u)$*, that is,* $J(u_n) \to c$ *and*

$$\langle \tilde{\Phi}'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n \|v - u_n\|, \qquad \forall v \in X,$$

with $\tau_n \to 0^+$*.*

Lemma 3.3. *Assume that (V1)–(V2) and (f1)–(f2) hold. Then the functional* $J(u)$*, defined with* $\tilde{\Phi}(u)$ *and* $\Psi(u)$ *in* (2.3) *and* (2.4)*, has the Mountain pass geometry.*

Proof. Now we will show that the functional J satisfies (i), (ii) and (iii) of the Theorem 3.1.

(i) For each $u \in D(J)$, the functional $J(u)$ defined with $\tilde{\Phi}(u)$ and $\Psi(u)$ in (2.3)–(2.4), respectively. According to Corollary 2.1–2.2, then $\tilde{\Phi}(u) \in \mathcal{C}^1$, $\Psi(u)$ is a convex lower semicontinuous function and $\Psi \not\equiv +\infty$.

(ii) Obviously, $J(0) = 0$. It follows from (2.5)–(2.7), and embedding theorem that there holds

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} Q(x)F_2(u)dx + \int_{\mathbb{R}^N} Q(x)F_1(u)dx \\
&\geq \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} Q(x)F_2(u)dx \\
&\geq \frac{1}{2}\|u\|_E^2 - \frac{1}{2}\varepsilon \int_{R^N} |u|^2 dx - \frac{1}{p}C(\varepsilon) \int_{R^N} |u|^p dx - \int_{R^N} Q(x)F_2(u)dx. \\
&\geq \frac{1}{2}\|u\|_E^2 - \frac{1}{2}\varepsilon\|u\|_2^2 - \frac{1}{p}C(\varepsilon)\|u\|_p^p - CQ_\infty\|u\|_q^q \\
&\geq \frac{1}{2}\|u\|_E^2 - \frac{\varepsilon}{2S}\|u\|_E^2 - \frac{C(\varepsilon)}{pS^{\frac{p}{2}}}\|u\|_E^p - C_1Q_\infty\|u\|_E^q \\
&= \left(\frac{1}{2} - \frac{\varepsilon}{2S} - \frac{C(\varepsilon)}{pS^{\frac{p}{2}}}\|u\|_E^{p-2} - C_1Q_\infty\|u\|_E^{q-2}\right)\|u\|_E^2,
\end{aligned}$$

where $Q_\infty := |Q(x)|_\infty$ and C_1 is a positive constant. We may choose $\varepsilon = \frac{S}{2}$ and ρ sufficiently small (i.e. ρ is such that $\frac{1}{4} - \frac{C(\varepsilon)}{pS^{\frac{p}{2}}}\rho^{p-2} - C_1Q_\infty\rho^{q-2} > 0$), thus

$$J(u) \geq \left(\frac{1}{4} - \frac{C(\varepsilon)}{pS^{\frac{p}{2}}}\rho^{p-2} - C_1Q_\infty\rho^{q-2}\right)\rho^2 : = \alpha_0 > 0, \quad \text{for any } u \in \partial B_\rho.$$

(iii) We may choose $u_* \in D(J)$ with $u_* \geq 0$ and $\text{supp}(\phi) \subset B_R(0)$ for some $R > 0$. By the condition (f2) we know that there exist constants $C_2, C_3 > 0$, such that $|F(x,u)| \geq C_2|u|^2 - C_3$ for any $u \in \mathbb{R}^+$. Then let $e := tu_*$ for any $t > 0$, there holds

$$\begin{aligned} J(e) = J(tu_*) &= \frac{1}{2}\|tu_*\|_E^2 - \frac{1}{2}\int_{\mathbb{R}^N} Q(x)t^2u_*^2 \log(tu_*)^2 dx - \int_{\mathbb{R}^N} F(x,tu_*)dx \\ &\leq t^2\left(I(u_*) - \frac{1}{2}\int_{\mathbb{R}^N} Q(x)u_*^2 \log t^2 dx\right) - C_2t^2\left(\int_{B_R(0)} u_*^2 dx + \int_{\mathbb{R}^N\setminus B_R(0)} u_*^2 dx\right) + C_4 \\ &\leq t^2\left(I(u_*) - \log t\int_{\mathbb{R}^N} Q(x)u_*^2 dx - C_2\int_{B_R(0)} u_*^2 dx\right) + C_4 \\ &\to -\infty \quad \text{as } t \to +\infty, \end{aligned}$$

where $I(u_*) = \frac{1}{2}\|u_*\|_E^2 - \frac{1}{2}\int_{\mathbb{R}^N} Q(x)u_*^2 \log u_*^2 dx$ is the energy functional of (1.4). Therefore there exists enough large $t_0 > 0$ with $\|e\|_E = \|t_0u_*\|_E > \rho$, i.e. $e \notin \bar{B}_\rho(0)$ such that

$$J(e) = J(t_0u_*) \leq 0.$$

So the proof of Lemma 3.3 is now completed. □

By Theorem 3.1 and Lemma 3.3, $J(u)$ admits a $(PS)_c$ sequence, where c is the Mountain level of $J(u)$.

Lemma 3.4. *Assume that (V1)–(V2), and (f1)–(f3) hold, then all $(PS)_c$ sequence $\{u_n\}$ are bounded in E.*

Proof. If $\{u_n\}$ is unbounded in E, then we can take, passing to a subsequence if necessary, that $\|u_n\|_E > 1$. Since $\{u_n\} \subset E$ is a $(PS)_c$ sequence, then $\{J(u_n)\}$ is bounded above and $\langle J'(u_n), u_n\rangle \to 0$ as $n \to +\infty$. Thus

$$\langle J'(u_n), z\rangle = o_n(1)\|z\|_E, \qquad \forall z \in E.$$

Since $Q(x) > 0$, then take $\sqrt{Q(x)}u$ instead of u in (2.2), there yields

$$\begin{aligned} &\int_{\mathbb{R}^N} Q(x)u^2 \log(Q(x)u^2)dx \\ &\qquad \leq \frac{a^2}{\pi}\|\nabla(\sqrt{Q(x)}u)\|_2^2 + \left(\log\|\sqrt{Q(x)}u\|_2^2 - N(1+\log a)\right)\|\sqrt{Q(x)}u\|_2^2. \end{aligned} \tag{3.2}$$

From Lemma 2.2 of [3], there exists a positive constant C_5 such that

$$N(1+\log a)\|u\|_2^2 \leq (\log\|u\|_2^2)\|u\|_2^2 + C_5\|u\|_2^2. \tag{3.3}$$

Then taking $a > 0$ enough small, there exists a positive constant C_6 such that

$$\int_{\mathbb{R}^N} Q(x)u^2 \log u^2 dx \leq \frac{1}{2}\|\nabla u\|_2^2 + C_6(\log\|u\|_2^2 + 1)\|u\|_2^2. \tag{3.4}$$

We have exploited the fact that the function $t \to \log t$ $(t > 0)$ is increasing. Then for $r \in (0,1)$, there is a constant $C_7 > 0$ satisfying

$$|t\log t| \leq C_7(1 + |t|^{1+r}), \quad \text{for any } t > 0.$$

Therefore there exists a positive constant C_8 such that

$$\|u_n\|_2^2 \log(\|u_n\|_2^2) \le C_7(1 + (\|u_n\|_2^2)^{1+r}) \le C_8(1 + \|u_n\|_E)^{1+r}. \tag{3.5}$$

From (3.3)–(3.5) and Theorem 2.5, there holds

$$\int_{\mathbb{R}^N} Q(x)u^2 \log u^2 dx \le \frac{1}{2}\|\nabla u\|_2^2 + C_9(1 + \|u_n\|_E)^{1+r}, \tag{3.6}$$

where C_9 is a positive constant. By the condition (f3), there exists a constant $M_0 > 0$ with $|u| > M_0$ such that $f(x,t)t - \alpha F(x,t) \ge -\theta|t|^2$ for all $|t| > M_0$, $x \in \mathbb{R}^N$. Let $\Omega_n = \{x \in \mathbb{R}^N : |u| > M_0\}$. It follows from (2.1), (2.8), (3.2) and (3.6) that

$$\begin{aligned}
c + \|u_n\|_E &\ge J(u_n) - \frac{1}{\alpha}\langle J'(u_n), u_n\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + (V(x) + Q(x))u_n^2\right) dx - \frac{1}{\alpha}\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 dx + V(x)u_n^2\right) dx \\
&\quad + \left(\frac{1}{\alpha} - \frac{1}{2}\right)\int_{\mathbb{R}^N} Q(x)u_n^2 \log u_n^2 dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} \left(f(x,u_n)u_n - \alpha F(x,u_n)\right) dx \\
&\ge \left(\frac{1}{2} - \frac{1}{\alpha}\right)||u_n||_E^2 + \left(\frac{1}{\alpha} - \frac{1}{2}\right)\left(\frac{1}{2}\|\nabla u_n\|_2^2 + C_9(1 + \|u_n\|_E)^{1+r}\right) \\
&\quad - \frac{\theta}{\alpha}\left(\int_{\Omega_n} + \int_{\mathbb{R}^N\setminus\Omega_n}\right)|u_n|^2 dx \\
&\ge \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|_E^2 + \left(\frac{1}{\alpha} - \frac{1}{2}\right)\left(\frac{1}{2}\|u_n\|_E^2 + C_9\left(1 + \|u_n\|_E\right)^{1+r}\right) - \frac{\theta}{\alpha}\int_{\Omega_n} |u_n|^2 dx - C_{10} \\
&\ge \left(\frac{1}{4} - \frac{1}{2\alpha} - \frac{\theta}{S\alpha}\right)\|u_n\|_E^2 + \left(\frac{1}{\alpha} - \frac{1}{2}\right) C_9\left(1 + \|u_n\|_E\right)^{1+r} - C_{10},
\end{aligned}$$

where C_{10} is a positive constant. Divide both sides of this inequality by the norm $\|u_n\|_E^2$, then this leads to the following contradiction:

$$0 > \frac{S(\alpha - 2) - 4\theta}{4S\alpha} > 0,$$

and the proof of Lemma 3.4 is completed. □

Since the sequence $\{u_n\}$ is bounded in E, it has a weakly convergent subsequence in E. Without loss of generality we can assume that there exist $u \in E$ and a subsequence of $\{u_n\}$, still denoted by itself, such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } E, \\ u_n \to u & \text{in } L^p_{\text{loc}}(\mathbb{R}^N),\ p \in (2, 2^*), \\ u_n \to u & \text{a.e. } x \in \mathbb{R}^N, \end{cases}$$

as $n \to \infty$.

Lemma 3.5. *Assume that (f1) satisfies, then there hold*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} F_2'(u_n)u_n dx = \int_{\mathbb{R}^N} F_2'(u)u dx,$$
$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} f(x,u_n)u_n dx = \int_{\mathbb{R}^N} f(x,u)u dx.$$

Proof. It follows from (2.6) that we know $|F_2'(s)| \leq C|s|^{q-1}$, then

$$\left| \int_{\mathbb{R}^N} F_2'(u_n)u_n dx - \int_{\mathbb{R}^N} F_2'(u)udx \right| \leq \int_{\mathbb{R}^N} C \left||u_n|^q - |u|^q\right| dx \to 0, \quad as\ n \to +\infty.$$

By the condition (f1), there holds

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (f(x,u_n)u_n - f(x,u)u) \, dx \right| &\leq \left| \int_{\mathbb{R}^N} \left(\varepsilon|u_n|^2 + C(\varepsilon)|u_n|^p - \varepsilon|u|^2 - C(\varepsilon)|u|^p\right) dx \right| \\ &\leq \varepsilon \int_{\mathbb{R}^N} \left||u_n|^2 - |u|^2\right| dx + C(\varepsilon) \int_{\mathbb{R}^N} \left||u_n|^p - |u|^p\right| dx \to 0, \end{aligned}$$

as $n \to +\infty$. The proof of Lemma 3.5 is completed. □

Lemma 3.6. *Let $\{u_n\}$ be a $(PS)_c$ sequence of J in E, then $u_n \to u$ in E.*

Proof. By Lemma 3.4, the sequence $\{u_n\}$ is bounded in E. Then without loss of generality we can assume that $u_n \rightharpoonup u$ in E, recalling that $\langle J'(u_n), u_n \rangle = o_n(1)\|u_n\|_E$ yields

$$\begin{aligned} &\int_{\mathbb{R}^N} \left[|\nabla u_n|^2 + (V(x) + Q(x))u_n^2\right] dx + \int_{\mathbb{R}^N} Q(x)F_1'(u_n)u_n dx \\ &\quad = \int_{\mathbb{R}^N} f(x,u_n)u_n dx + \int_{\mathbb{R}^N} Q(x)F_2'(u_n)u_n dx + o_n(1). \end{aligned} \tag{3.7}$$

Moreover, $\lim_{n\to\infty} \langle J'(u_n), u_n \rangle = 0$, i.e.

$$\begin{aligned} &\int_{\mathbb{R}^N} \left[|\nabla u|^2 + (V(x) + Q(x))u^2\right] dx + \int_{\mathbb{R}^N} Q(x)F_1'(u)udx \\ &\quad = \int_{\mathbb{R}^N} f(x,u)udx + \int_{\mathbb{R}^N} Q(x)F_2'(u)udx. \end{aligned} \tag{3.8}$$

By the Lemma 3.5, the right-hand side of (3.7) and (3.8) are equal. Therefore, there holds

$$\|u_n\|_E^2 + \int_{\mathbb{R}^N} Q(x)F_1'(u_n)u_n dx + o_n(1) = \|u\|_E^2 + \int_{\mathbb{R}^N} Q(x)F_1'(u)udx.$$

Without loss of generality we have $\int_{\mathbb{R}^N} F_1'(u_n)u_n dx \to \int_{\mathbb{R}^N} F_1'(u)udx$, and

$$\|u_n\|_E^2 \to \|u\|_E^2,$$

as $n \to +\infty$. Thus we can conclude that $u_n \to u$ in E. □

Because $\{u_n\} \subset E$ is the $(PS)_c$ sequence of the functional $J(u)$, and by (3) of Definition 2.3, then there exists a function $v \in C_0^\infty(\mathbb{R}^N)$, for $\tau_n \to 0^+$ such that

$$\begin{aligned} -\tau_n\|v - u_n\|_E \leq &\int_{\mathbb{R}^N} \left[\nabla u_n \nabla(v - u_n) + (V(x) + Q(x))u_n(v - u_n)\right] dx - \int_{\mathbb{R}^N} f(x,u_n)(v - u_n)dx \\ &- \int_{\mathbb{R}^N} Q(x)F_2'(u_n)(v - u_n)dx + \int_{\mathbb{R}^N} Q(x)F_1(v)dx - \int_{\mathbb{R}^N} Q(x)F_1(u_n)dx. \end{aligned}$$

Since Ψ is lower semicontinuous, then

$$\Psi(u_n) \geq \liminf_{n\to\infty} \Psi(u_n) \geq \Psi(u).$$

It follows from Lemmas 3.5–3.6 that there holds

$$\lim_{n\to\infty}(-\tau_n\|v-u\|_E) \le \int_{\mathbb{R}^N}[\nabla u\nabla(v-u)+(V(x)+Q(x))u(v-u)]\,dx - \int_{\mathbb{R}^N} f(x,u)(v-u)dx - \int_{\mathbb{R}^N} Q(x)F_2'(u)(v-u)dx + \int_{\mathbb{R}^N} Q(x)F_1(v)dx - \int_{\mathbb{R}^N} Q(x)F_1(u)dx. \tag{3.9}$$

The above formula (3.9) is equivalent to

$$\langle \Phi'(u), v-u\rangle + \Psi(v) - \Psi(u) \ge 0.$$

It can be seen that it satisfies (2) of Definition 2.3. This implies that u is the critical point of the functional $J(u)$. Therefore, u is the solution of problem (1.1).

In what follows, we mainly prove that the solution u is nontrivial and reachable.

Lemma 3.7. *Assume that (V1)–(V2) and (f1)–(f3) hold, then the functional $J(u)$ satisfies $(PS)_c$ condition, and u is a critical point of J. Furthermore, u is nontrivial solution of equation* (1.1).

Proof. If $u = 0$, then $u_n \rightharpoonup 0$ in E. One of the following two cases is always true.

(i) $u_n \to 0$, as $n \to +\infty$.

(ii) $\liminf_{n\to+\infty}\sup_{y\in\mathbb{R}^N\setminus\{0\}}\int_{B_r(y)}|u_n|^2dx > 0$.

Now we next will prove that neither (i) nor (ii) is true:

If (i) is true, then $J(u) \to 0$, but $J(u) \to c$ $(c > 0)$, which leads to a contradiction. Therefore (ii) is true. Let $\beta := \liminf_{n\to+\infty}\sup_{y\in\mathbb{R}^N\setminus\{0\}}\int_{B_r(y)}|u_n|^2dx > 0$. Choose $A_r := \{x \in \mathbb{R}^N \setminus B_r(0) : V(x) < a_0\}$, as $r \to +\infty$. It follows from (V1) that $\mathrm{meas}(A_r) \to 0$. There exists a constant r^* such that if $r > r^*$ and $q^* \in (2, 2^*)$, and according to the Hölder inequality and Sobolev imbedding inequality, we have

$$\int_{A_r}|u_n|^2dx \le \left(\int_{A_r}|u_n|^{q^*}dx\right)^{\frac{2}{q^*}}\left(\int_{A_r}1dx\right)^{\frac{q^*-2}{q^*}} \le C\|u_n\|_E^2\,(\mathrm{meas}(A_r))^{\frac{q^*-2}{q^*}} \le \frac{\beta}{4}. \tag{3.10}$$

Take $D_r := \{x \in \mathbb{R}^N \setminus B_r(0) : V(x) \ge a_0\}$, as a_0 is enough large. Then there holds

$$\int_{D_r}|u_n|^2dx \le \frac{1}{1+a_0}\int_{\mathbb{R}^N}(1+V(x))|u_n|^2dx \le \frac{C}{1+a_0} \le \frac{\beta}{4}. \tag{3.11}$$

It follows from (3.10) and (3.11) that

$$\begin{aligned}\beta &= \liminf_{n\to+\infty}\sup_{y\in\mathbb{R}^N\setminus\{0\}}\int_{B_r(y)}|u_n|^2dx \\ &\le \liminf_{n\to+\infty}\int_{\mathbb{R}^N\setminus B_r(0)}|u_n|^2dx \\ &= \liminf_{n\to+\infty}\left(\int_{D_r}|u_n|^2dx + \int_{A_r}|u_n|^2dx\right) \le \frac{\beta}{2},\end{aligned}$$

which leads to a contradiction. Thus we have $u \ne 0$. The proof of Lemma 3.7 is completed. □

Proof of Theorem 1.1. By the statement in Section 3, we admit that the sequence $\{u_n\} \subset E$ is the $(PS)_c$ sequence of the functional $J(u)$. From Lemma 3.7, the weak limit of $(PS)_c$ sequence is nontrivial, we easily infer that the weak limit is the desired ground state. On the one hand, since $u_n \to u$ $(n \to \infty)$, the norm $\|u\|_E$ and $\Psi(u)$ are lower semicontinuous, it follows from (2.1), (2.3), (2.4) that

$$\begin{aligned}
J(u) &= \tilde{\Phi}(u) + \Psi(u) \\
&= \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} Q(x)F_2(u)dx + \int_{\mathbb{R}^N} Q(x)F_1(u)dx \\
&\leq \frac{1}{2}\|u\|_E^2 - \inf\int_{\mathbb{R}^N} F(x,u)dx - \inf\int_{\mathbb{R}^N} Q(x)F_2(u)dx + \int_{\mathbb{R}^N} Q(x)F_1(u)dx \\
&\leq \liminf_{n\to+\infty}\frac{1}{2}\|u_n\|_E^2 - \liminf_{n\to+\infty}\int_{\mathbb{R}^N} F(x,u_n)dx - \liminf_{n\to+\infty}\int_{\mathbb{R}^N} Q(x)F_2(u_n)dx \\
&\quad + \liminf_{n\to+\infty}\int_{\mathbb{R}^N} Q(x)F_1(u_n)dx \\
&= \liminf_{n\to+\infty} J(u_n) = c.
\end{aligned}$$

Then we can get $J(u) \leq c$. On the other hand, by the definition of c, we have $J(u) \geq c$. Hence $J(u) = c$. In a word, we deduce that c is attained and the corresponding minimizer is a ground state solution of problem (1.1). The proof of Theorem 1.1 is completed. □

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Normalized solutions for Kirchhoff-type equations with combined nonlinearities: the L^2-critical case

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Abstract. In this paper, we consider the existence of normalized solutions for the following Kirchhoff-type problem:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=\lambda u+|u|^{p-2}u+\mu|u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

with prescribed L^2-norm:

$$\int_{\mathbb{R}^N}|u|^2dx=c^2,$$

where $N=2,3$, $a\geq 0$, $b>0$ and $c>0$ are constants, $\lambda\in\mathbb{R}$, $2<q<p=2+\frac{8}{N}$ and $\mu>0$. The number $2+\frac{8}{N}$ behaves as the L^2 -critical exponent for the above problem. We prove the multiplicity of normalized solutions for the above Kirchhoff-type problem with L^2-critical nonlinearity (that is, $p=2+\frac{8}{N}$) in the two cases: $2<q<2+\frac{4}{N}$ and $2+\frac{4}{N}<q<2+\frac{8}{N}$.

Keywords: Kirchhoff equation, constrained minimization, variational method, Pohozaev manifold.

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1 Introduction and main results

In this paper, we investigate the multiplicity of normalized solutions for the following Kirchhoff-type problem

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=\lambda u+|u|^{p-2}u+\mu|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

with prescribed mass

$$\int_{\mathbb{R}^N}|u|^2dx=c^2,$$

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where $N = 2,3$, $a \geq 0$, $b,c > 0$, $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier, $2 < q < p = 2 + \frac{8}{N}$ and $\mu > 0$. Let $L^s(\mathbb{R}^N)(1 \leq s < +\infty)$ be the Lebesgue space with norm $|u|_s = (\int_{\mathbb{R}^N} |u|^s dx)^{1/s}$, $H^1(\mathbb{R}^N)$ be the Hilbert space with the norm $\|u\| = \big(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) dx\big)^{\frac{1}{2}}$.

Problem (1.1) is a special form of the following Kirchhoff problem

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u = f(x,u) \quad \text{in } \mathbb{R}^N,$$

which is also a variant of Dirichlet problem

$$\begin{cases} -\left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. It is well-known that problem (1.2) appears naturally in the context of physics. Problem (1.2) is the stationary case of a nonlinear wave equation

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = f(x,u), \tag{1.3}$$

first proposed by Kirchhoff [9] in 1883. Problem (1.3) is a generalization of the classical D'Alembert's wave equation which describes free vibrations of elastic strings. The parameters in problem (1.3) have specific physical meaning: f is the external force, a is related to the intrinsic properties of the string, and u means the displacement while b denotes the initial tension. Since then, problem (1.3) has received much attention, see [1, 11, 12, 14, 15] and the references therein. Since Lions in [11] proposed an abstract functional analysis framework, Kirchhoff type problem has been intensively studied during the last decades. From a mathematical perspective, problem (1.2) is not a pointwise identity as the appearance of the nonlocal term $\int_\Omega |\nabla u|^2 dx$. The nonlocal term causes some mathematical difficulties and the investigation of problem (1.2) is more interesting and challenging. Such a nonlocal model also appears in other fields as biological systems describing a process depending on the average of itself, for example one species' population density.

A way to study problem (1.1) is to search for solutions with L^2-norm constraint, and such solutions are known as normalized solutions and $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. In addition, the study of L^2-norm constraint problem can give a better insight of the dynamical properties, like orbital stability or instability, and can describe attractive Bose-Einstein condensate. Normalized solutions of problem (1.1) can be obtained by looking for critical points of the energy functional $E_{a,\mu}(u)$ constrained on S_c, where

$$E_{a,\mu}(u) := \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q dx,$$

and

$$S_c := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = c^2\right\}.$$

Many interesting results on the normalized solutions of Kirchhoff problem are also obtained not long ago, see [2,3,5,6,8,13,17,23]. Especially, many experts considered the existence of normalized solutions for problem (1.1) with combined nonlinearities. For the case $\mu > 0$, under different ranges of p and q, Li and Lou in [10] proved a multiplicity result for problem (1.1). In detail, if $2 < q < \frac{10}{3}$, $\frac{14}{3} < p < 6$ and $\mu < \min\{\mu', \mu''\}$, two solutions for problem (1.1) were obtained. If $\frac{14}{3} < q < p < 6$, problem (1.1) has a mountain pass type solution. Hu and Mao in [6] considered the following minimization problem

$$m_{a,c} = \inf_{u \in S_c} E_{a,\mu}(u), \tag{1.4}$$

and they proved that if $2 < q < \frac{10}{3}$ and $2 < q < p \leq \frac{14}{3}$, problem (1.4) has a minimizer for every $c \in (0, c_p^*)$. At the same time, when c satisfies the suitable conditions, the nonexistence of minimizers for problem (1.4) was considered in the following four cases: (i) $q = \frac{10}{3}$ and $p = \frac{14}{3}$; (ii) $\frac{10}{3} = q < p < \frac{14}{3}$; (iii) $2 < q < p = \frac{14}{3}$; (iv) $2 < q < p, \frac{14}{3} < p < 6$. Moreover, if $\frac{14}{3} < q < p < 6$, they also obtained the existence of normalized solutions for problem (1.1) by using constraint minimization on a suitable submanifold of S_c. For the Sobolev critical case (that is, $p = 6$), Feng, Liu and Zhang in [3] proved the existence and multiplicity of normalized solutions for problem (1.1) under suitable assumptions on μ and c for the following four cases: $2 < q < \frac{10}{3}$, $q = \frac{10}{3}$, $\frac{10}{3} < q < \frac{14}{3}$, $\frac{14}{3} \leq q < p = 6$. Some similar results were also obtained in [10,23]. For the case $\mu = 0$, the existence, multiplicity and uniqueness of normalized solutions for problem (1.1) have been considered in [13,19–22]. For the case $\mu < 0$, we refer to [2,6], and for the nonlinear Kirchhoff-type equations in high dimensions see [8].

As far as we known, there are few papers to consider the existence and multiplicity of normalized solutions for problem (1.1) with L^2-critical nonlinearity (that is $p = 2 + \frac{8}{N}$) in the two cases: $2 < q < 2 + \frac{4}{N}$ and $2 + \frac{4}{N} < q < 2 + \frac{8}{N}$. The object of this paper is to prove the existence and multiplicity of normalized solutions for problem (1.1) in those cases under suitable assumptions on μ and a.

Before stating the main results of this paper, let us recall the Gagliardo–Nirenberg inequality (see [18]): for any $s \in [2, \frac{2N}{N-2})$ if $N \geq 3$ and $s \geq 2$ if $N = 1, 2$, we have

$$\frac{1}{s}|u|_s^s \leq \frac{1}{2|Q_s|_2^{s-2}} |\nabla u|_2^{s\gamma_s} |u|_2^{s - s\gamma_s}, \tag{1.5}$$

where $\gamma_s := \frac{N(s-2)}{2s}$ and with equality only for $u = Q_s$, and up to translations, Q_s is the unique positive solution of

$$-\frac{N(s-2)}{4}\Delta u + \left(1 + \frac{s-2}{4}(2-N)\right) u = |u|^{s-2}u \quad \text{in } \mathbb{R}^N,$$

and satisfies

$$|\nabla Q_s|_2^2 = |Q_s|_2^2 = \frac{2}{s}|Q_s|_s^s.$$

Especially, let $p = 2 + \frac{8}{N}$, define

$$c^* := \left(\frac{b|Q_p|_2^{\frac{8}{N}}}{2}\right)^{\frac{N}{8-2N}}.$$

For $s = p = 2 + \frac{8}{N}$ and for any $u \in S_c$, we have

$$\frac{1}{p}|u|_p^p \leq \frac{1}{4}\frac{2}{c^2}\left(\frac{c}{|Q_p|_2}\right)^{\frac{8}{N}} |\nabla u|_2^4 = \frac{b}{4}\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}} |\nabla u|_2^4. \tag{1.6}$$

Set

$$\mu_* := \frac{2a|Q_q|_2^{q-2}}{(4-q\gamma_q)c^{q-q\gamma_q}} \left(\frac{2a(2-q\gamma_q)}{b(4-q\gamma_q))\left((\frac{c}{c^*})^{\frac{8-2N}{N}}-1\right)} \right)^{\frac{2-q\gamma_q}{2}}. \tag{1.7}$$

Now, our main results are following.

Theorem 1.1. *Let $2 < q < 2 + \frac{4}{N}$, $p = 2 + \frac{8}{N}$, $c > c^*$ and $0 < \mu < \mu_*$. Then problem (1.1) has two radial solutions, denoted by $\tilde{u}_{c,\mu}$ and $\hat{u}_{c,\mu}$. Moreover, $\tilde{u}_{c,\mu}$ is a local minimizer of the functional $E_{a,\mu}$ on the set*

$$\mathcal{A}_{R_0} := \{u \in S_{c,r} : |\nabla u|_2^2 < R_0\}$$

for a suitable $R_0 = R_0(c,\mu) > 0$ with $E_{a,\mu}(\tilde{u}_{c,\mu}) < 0$ and $\tilde{u}_{c,\mu}$ solves problem (1.1) for some $\tilde{\lambda}_{c,\mu} < 0$, and $\hat{u}_{c,\mu}$ is a critical point of mountain pass type for $E_{a,\mu}$ with $E_{a,\mu}(\hat{u}_{c,\mu}) > 0$ and $\hat{u}_{c,\mu}$ solves problem (1.1) for some $\hat{\lambda}_{c,\mu} < 0$.

Theorem 1.2. *If $2 + \frac{4}{N} < q < p = 2 + \frac{8}{N}$, $\mu > 0$ and $c < c^*$, we have the following results:*

(i) if $a = 0$, $m_{0,c} := \inf_{u \in S_c} E_{0,\mu}(u)$ has a radial minimizer $\tilde{u}$, and $\tilde{u}$ solves problem (1.1) for some $\tilde{\lambda} < 0$.

(ii) let $\bar{a} = \frac{b}{2}\left(1 - (\frac{c}{c^*})^{\frac{8-2N}{N}}\right)\left(\frac{2}{N(q-2)} - \frac{1}{4}\right)|\nabla \tilde{u}|_2^2 > 0$, for any $a \in (0, \bar{a})$, problem (1.1) has two radial solutions, the one is a global minimizer $\tilde{u}_{c,a}$ with $\tilde{\lambda}_{c,a} < 0$, and the other is the mountain pass type solution $u_{c,a}$ with $\lambda_{c,a} < 0$.

Remark 1.3. Theorem 1.1 complements [6, Theorem 1.2], where Hu and Mao considered the case $c \in (0, c^*)$ and obtained a minimizer of the functional $E_{a,\mu}$ on S_c. However, we deal with the case $c > c^*$ and obtain two solutions for problem (1.1) under suitable assumptions on the constant $\mu > 0$. In the proof of Theorem 1.1, since the functional $E_{a,\mu}$ is not bounded from below on S_c for $c > c^*$, we will restrict the functional $E_{a,\mu}$ on the Pohozaev set $\mathcal{P}_{c,\mu}$. We can get a local minimizer for $E_{a,\mu}|_{\mathcal{P}_{c,\mu}}$ and use mountain pass theorem to get the second critical point. We emphasis that (1.7) has been used to ensure that $\mathcal{P}_{c,\mu}$ is a smooth manifold and the existence of mountain pass type solution.

To the best of our knowledge, Hu and Mao in [6] proved that if $N = 3$, $\frac{10}{3} < q < \frac{14}{3}$, $p = \frac{14}{3}$, $c < c^*$ and $\mu > 0$ satisfy appropriate condition, problem (1.1) with has no minimizer. However, we try to prove the existence of normalized solution for problem (1.1) with $2 + \frac{4}{N} < q < p = 2 + \frac{8}{N}$ for the suitable constant $a > 0$. Furthermore, there are few results about the existence of normalized solutions to degenerate Kirchhoff equations, that is, $a = 0$, so we first establish the existence of minimizer $m_{0,c} = \inf_{u \in S_c} E_{0,\mu}(u) < 0$, which is a normalized solution of the degenerate Kirchhoff equation. And then, we establish $m_{a,c} := \inf_{u \in S_{c,r}} E_{a,\mu}(u) < 0$ with the help of the minimizer of $m_{0,c}$. At last, we will prove the existence of the second solution with the mountain pass type for problem (1.1).

To overcome the lack of compactness, we work in $H_r^1(\mathbb{R}^N)$. Although the energy functional $E_{a,\mu}$ has a bounded Palais–Smale sequence on the mass constraint set $S_{c,r}$, unfortunately, we can not deduce whether $E_{a,\mu}$ satisfies the Palais–Smale condition. To overcome this difficulty, in the proof of Theorem 1.1, we will constrain the energy functional $E_{a,\mu}$ on a submanifold of $S_{c,r}$ corresponding to the Pohozaev identity. In the proof of (2) of Theorem 1.2, we use Jeanjean's method in [7] and construct an auxiliary map $I_{a,\mu}(u,\tau) := E_{a,\mu}(\tau \star u)$, which has the same type of geometric structure on $S_{c,r} \times \mathbb{R}$ as $E_{a,\mu}$ on $S_{c,r}$.

2 Preliminaries

In this section, we will introduce some notations, then we recall a version of linking theorem. Finally, we give the compactness analysis of Palais–Smale sequences for $E_{a,\mu}|_{S_{c,r}}$. Let

$$H_r^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u(|x|) = u(x)\},$$
$$S_{c,r} := S_c \cap H_r^1(\mathbb{R}^N) = \{u \in S_c : u(x) = u(|x|)\}.$$

For $u \in S_c$, and $\tau \in \mathbb{R}$, define the fiber map preserving the L^2-norm

$$(\tau \star u)(x) := e^{\frac{N}{2}\tau} u(e^\tau x) \quad \text{for any } x \in \mathbb{R}^N.$$

We introduce the auxiliary functional $I_{a,\mu} : H^1(\mathbb{R}^N) \times \mathbb{R}^+ \to \mathbb{R}$ by

$$I_{a,\mu}(u,\tau) := E_{a,\mu}(\tau \star u) = \frac{e^{2\tau}a}{2}|\nabla u|_2^2 + \frac{e^{4\tau}b}{4}|\nabla u|_2^4 - \frac{e^{4\tau}}{p}|u|_p^p - \mu\frac{e^{\gamma_q q\tau}}{q}|u|_q^q, \tag{2.1}$$

then we easily see that the functional $I_{a,\mu}$ is of class C^1. In addition, we define the Pohozaev set by

$$\mathcal{P}_{c,\mu} = \{u \in S_{c,r} : P_\mu(u) = 0\}$$

with

$$P_\mu(u) = a|\nabla u|_2^2 + b|\nabla u|_2^4 - \frac{4}{p}|u|_p^p - \mu\gamma_q|u|_q^q.$$

Lemma 2.1 ([4, Theorem 2.7]). *Let φ be a C^1-functional on a complete connected C^1-Finsler manifold X and consider a homotopy-stable family $\mathcal{F}$ with extended boundary B. Set*

$$c = c(\varphi, \mathcal{F}) = \inf_{A\in\mathcal{F}} \max_{x\in A} \varphi(x)$$

and let F be a closed subset of X satisfying

$$A \cap F \backslash B \neq \emptyset \quad \text{for every } A \in \mathcal{F} \tag{2.2}$$

and

$$\sup_{x\in B} \varphi(x) \le c \le \inf_{x\in F} \varphi(x). \tag{2.3}$$

Then, for any sequence of sets $(A_n)_n \in \mathcal{F}$ such that $\lim_{n\to\infty}\sup_{A_n}\varphi = c$, there exists a sequence $(x_n)_n$ in $X\backslash B$ such that

$$\lim_{n\to\infty} \varphi(x_n) = c, \quad \lim_{n\to\infty} \|d\varphi(x_n)\| = 0, \quad \lim_{n\to\infty} \operatorname{dist}(x_n, F) = 0, \quad \lim_{n\to\infty} \operatorname{dist}(x_n, A_n) = 0.$$

Lemma 2.2. *Let $a > 0$, $b > 0$, $c > 0$, $\mu > 0$, $2 < q < p = 2 + \frac{8}{N}$. Let $\{u_n\} \subset S_{c,r}$ be a bounded Palais–Smale sequence for $E_{a,\mu}|_{S_{c,r}}$ at energy level $m \neq 0$ with $P_\mu(u_n) \to 0$ as $n \to \infty$. Then up to a subsequence $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$. Moreover, $u \in S_{c,r}$ and u is a radial solution for problem (1.1) for some $\lambda < 0$.*

Proof. The proof is divided into three steps.

Step 1: Lagrange multipliers $\lambda_n \to \lambda$ in $\mathbb{R}$. Since $H_r^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s \in (2, \frac{2N}{N-2})$, from the boundedness of Palais–Smale sequence $\{u_n\}$, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $u \in H_r^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{in } H_r^1(\mathbb{R}^N), \qquad u_n \to u \quad \text{in } L^s(\mathbb{R}^N), \qquad u_n \to u \quad \text{a.e. on } \mathbb{R}^N.$$

Because $\{u_n\}$ is a Palais–Smale sequence of $E_{a,\mu}|_{S_{c,r}}$, by the Lagrange multipliers rule, there exists $\lambda_n \in \mathbb{R}$ such that

$$(a+b|\nabla u_n|_2^2)\int_{\mathbb{R}^N}\nabla u_n\nabla\varphi dx-\mu\int_{\mathbb{R}^N}|u_n|^{q-2}u_n\varphi dx \\ -\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\varphi dx-\lambda_n\int_{\mathbb{R}^N}u_n\varphi dx=o_n(1) \quad (2.4)$$

for every $\varphi \in H^1(\mathbb{R}^N)$, where $o_n(1)\to 0$ as $n\to\infty$. In particular, taking $\varphi=u_n$ in (2.4), we have

$$\lambda_n c^2=a|\nabla u_n|_2^2+b|\nabla u_n|_2^4-\mu|u_n|_q^q-|u_n|_p^p+o_n(1).$$

The boundedness of $\{u_n\}$ in $H_r^1(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)\cap L^p(\mathbb{R}^N)$ implies that $\{\lambda_n\}$ is bounded as well. Hence, up to a subsequence, we have $\lambda_n\to\lambda\in\mathbb{R}$.

Step 2: $\lambda<0$ and $u\not\equiv 0$. Recalling that $P_\mu(u_n)\to 0$, we have

$$\lambda_n c^2=\mu(\gamma_q-1)|u_n|_q^q+(\gamma_p-1)|u_n|_p^p+o_n(1),$$

hence, let $n\to\infty$, we have

$$\lambda c^2=\mu(\gamma_q-1)|u|_q^q+(\gamma_p-1)|u|_p^p.$$

Since $\mu>0$ and $0<\gamma_q,\gamma_p<1$, we deduce that $\lambda\le 0$, with "=" if and only if $u\equiv 0$. If $\lambda_n\to 0$, we have $\lim_{n\to\infty}|u_n|_p^p=0=\lim_{n\to\infty}|u_n|_q^q$. Using again $P_\mu(u_n)\to 0$, we have $E_{a,\mu}(u_n)\to 0$, which is a contradiction with $E_{a,\mu}(u_n)\to m\neq 0$ and thus $\lambda_n\to\lambda<0$ and $u\not\equiv 0$.

Step 3: $u_n\to u$ in $H^1(\mathbb{R}^N)$. Since $u_n\rightharpoonup u\not\equiv 0$ in $H^1(\mathbb{R}^N)$, we get $B:=\lim_{n\to\infty}|\nabla u_n|_2^2\ge|\nabla u|_2^2>0$. Then, (2.4) implies that

$$(a+bB)\int_{\mathbb{R}^N}\nabla u\nabla\varphi dx-\mu\int_{\mathbb{R}^N}|u|^{q-2}u\varphi dx-\int_{\mathbb{R}^N}|u|^{p-2}u\varphi dx-\lambda\int_{\mathbb{R}^N}u\varphi dx=0 \quad (2.5)$$

for any $\varphi\in H^1(\mathbb{R}^N)$. Combining (2.4) with (2.5) and taking $\varphi=u_n-u$, we obtain

$$(a+bB)|\nabla(u_n-u)|_2^2-\lambda|u_n-u|_2^2\to 0 \quad \text{as } n\to\infty.$$

Since $\lambda<0$, we conclude that $\{u_n\}$ converges strongly in $H^1(\mathbb{R}^N)$. □

3 Proof of Theorem 1.1

In this section, we deal with the case $2<q<2+\frac{4}{N}$, $p=2+\frac{8}{N}$, $c>c^*$, $\mu>0$ and prove Theorem 1.1. First of all, it is well known that any critical point of the functional $E_{a,\mu}$ belongs to $\mathcal{P}_{c,\mu}$. Conversely, if $u\in\mathcal{P}_{c,\mu}$, we get $\partial_\tau I_{a,\mu}(u,0)=0$. Now, we consider the decomposition of $\mathcal{P}_{c,\mu}$ into the disjoint union $\mathcal{P}_{c,\mu}=\mathcal{P}_{c,\mu}^+\cup\mathcal{P}_{c,\mu}^0\cup\mathcal{P}_{c,\mu}^-$, where

$$\mathcal{P}_{c,\mu}^+:=\{u\in\mathcal{P}_{c,\mu}:2a|\nabla u|_2^2+4b|\nabla u|_2^4>\mu q\gamma_q^2|u|_q^q+p\gamma_p^2|u|_p^p\}=\{u\in\mathcal{P}_{c,\mu}:\partial_{\tau\tau}I_{a,\mu}(u,0)>0\},$$

$$\mathcal{P}_{c,\mu}^0:=\{u\in\mathcal{P}_{c,\mu}:2a|\nabla u|_2^2+4b|\nabla u|_2^4=\mu q\gamma_q^2|u|_q^q+p\gamma_p^2|u|_p^p\}=\{u\in\mathcal{P}_{c,\mu}:\partial_{\tau\tau}I_{a,\mu}(u,0)=0\},$$

$$\mathcal{P}_{c,\mu}^-:=\{u\in\mathcal{P}_{c,\mu}:2a|\nabla u|_2^2+4b|\nabla u|_2^4<\mu q\gamma_q^2|u|_q^q+p\gamma_p^2|u|_p^p\}=\{u\in\mathcal{P}_{c,\mu}:\partial_{\tau\tau}I_{a,\mu}(u,0)<0\}.$$

By (1.5) and (1.6), we have

$$E_{a,\mu}(u) \geq \frac{a}{2}|\nabla u|_2^2 + \frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)|\nabla u|_2^4 - \mu\frac{c^{q-\gamma_q q}}{2|Q_q|_2^{q-2}}|\nabla u|_2^{\gamma_q q} \tag{3.1}$$

for every $u \in S_{c,r}$. Therefore, to understand the geometry of the functional $E_{a,\mu}|_{S_{c,r}}$, it is useful to consider the function $h : \mathbb{R}^+ \to \mathbb{R}$:

$$h(t) := \frac{a}{2}t + \frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)t^2 - \mu\frac{c^{q-\gamma_q}}{2|Q_q|_2^{q-2}}t^{\frac{\gamma_q q}{2}}.$$

Now, we study the properties of $h(t)$.

Lemma 3.1. *Let $c > c^*$, $2 < q < 2+\frac{4}{N}$, $p = 2+\frac{8}{N}$, $0 < \mu < \mu_*$, where μ_* is defined in (1.7), the function h has a local strict minimum at negative level and a global strict maximum at positive level. Moreover, there exist $0 < R_0 < R_1$, both depending on c and μ, such that $h(R_0) = 0 = h(R_1)$ and $h(t) > 0$ for any $t \in (R_0, R_1)$.*

Proof. Since

$$h(t) = t^{\frac{\gamma_q q}{2}}\left(\frac{a}{2}t^{1-\frac{\gamma_q q}{2}} + \frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)t^{2-\frac{\gamma_q q}{2}} - \mu\frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}\right)$$

for $t > 0$, we have $h(t) > 0$ if and only if

$$\varphi(t) > \mu\frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}, \quad \text{with } \varphi(t) := \frac{a}{2}t^{1-\frac{\gamma_q q}{2}} + \frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)t^{2-\frac{\gamma_q q}{2}}.$$

It is not difficult to check that φ has a unique critical point $\bar{t}$ on $(0,\infty)$, which is a global maximum point at positive level:

$$\bar{t} := \frac{2a(2-q\gamma_q)}{b(4-q\gamma_q)\left((\frac{c}{c^*})^{\frac{8-2N}{N}}-1\right)},$$

and the maximum level is

$$\varphi(\bar{t}) = \frac{a}{(4-q\gamma_q)}\left(\frac{2a(2-q\gamma_q)}{b(4-q\gamma_q)\left((\frac{c}{c^*})^{\frac{8-2N}{N}}-1\right)}\right)^{1-\frac{q\gamma_q}{2}} > 0.$$

From $0 < \frac{\gamma_q q}{2} < 1$, $\mu > 0$ and $c > c^*$, it is obvious that $\lim_{t\to 0^+} h(t) = 0^-$ and $\lim_{t\to+\infty} h(t) = -\infty$. Therefore, h is positive on an open interval (R_0, R_1) if $\varphi(\bar{t}) > \mu\frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}$, which is ensured by

$$0 < \mu < \mu_* := \frac{2a|Q_q|_2^{q-2}}{(4-q\gamma_q)c^{q-q\gamma_q}}\left(\frac{2a(2-q\gamma_q)}{b(4-q\gamma_q)\left((\frac{c}{c^*})^{\frac{8-2N}{N}}-1\right)}\right)^{1-\frac{q\gamma_q}{2}}.$$

It follows immediately that h has a global maximum at positive level in (R_0, R_1). Moreover, since $\lim_{t\to 0^+} h(t) = 0^-$, there exists a local minimum point at negative level in $(0, R_0)$. The fact that h has no other critical points can be verified observing that $h'(t) = 0$ if and only if

$$\psi(t) = \mu\frac{\gamma_q q c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}} \quad \text{with } \psi(t) := at^{\frac{2-q\gamma_q}{2}} + b\left(1-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)t^{\frac{4-q\gamma_q}{2}}.$$

Clearly ψ has only one critical point, which is a strict maximum, and hence the above equation has at most two solutions, which necessarily are the local minimum and the global maximum of h previously found. □

We now study the structure of the Pohozaev manifold $\mathcal{P}_{c,\mu}$. Recalling the decomposition of $\mathcal{P}_{c,\mu} = \mathcal{P}^+_{c,\mu} \cup \mathcal{P}^0_{c,\mu} \cup \mathcal{P}^-_{c,\mu}$.

Lemma 3.2. *If $2 < q < 2 + \frac{4}{N}$, $p = 2 + \frac{8}{N}$ and $0 < \mu < \mu_*$, then $\mathcal{P}^0_{c,\mu} = \emptyset$ and $\mathcal{P}_{c,\mu}$ is a smooth manifold of codimension 2 in $H^1(\mathbb{R}^N)$.*

Proof. Otherwise, let $u \in \mathcal{P}^0_{c,\mu}$, from $P_{c,\mu}(u) = 0$ and $\partial_{\tau\tau} I_{a,\mu}(u,0) = 0$, we have

$$a|\nabla u|_2^2 + b|\nabla u|_2^4 - \mu\gamma_q |u|_q^q - \frac{4}{p}|u|_p^p = 0,$$

$$2a|\nabla u|_2^2 + 4b|\nabla u|_2^4 - \mu q\gamma_q^2 |u|_q^q - p\gamma_p^2 |u|_p^p = 0.$$

By (1.5), we obtain

$$(2-q\gamma_q)a|\nabla u|_2^2 + (4-q\gamma_q)b|\nabla u|_2^4 = \gamma_p(p\gamma_p - q\gamma_q)|u|_p^p \leq (4-q\gamma_q)b\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}|\nabla u|_2^4,$$

$$2a|\nabla u|_2^2 = \mu\gamma_q(4-q\gamma_q)|u|_q^q \leq \mu q\gamma_q(4-q\gamma_q)\frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}|\nabla u|_2^{q\gamma_q}.$$

Then, the lower and upper bounds of $|\nabla u|_2$ are given by

$$\left(\frac{a(2-q\gamma_q)}{b(4-q\gamma_q)\left((\frac{c}{c_*})^{\frac{8-2N}{N}}-1\right)}\right)^{\frac{1}{2}} \leq |\nabla u|_2 \leq \left(\frac{\mu q\gamma_q(4-q\gamma_q)c^{q-q\gamma_q}}{4a|Q_q|_2^{q-2}}\right)^{\frac{1}{2-q\gamma_q}},$$

which leads to

$$\mu > \frac{4a|Q_q|_2^{q-2}}{q\gamma_q(4-q\gamma_q)c^{q-q\gamma_q}}\left(\frac{a(2-q\gamma_q)}{b(4-q\gamma_q)\left((\frac{c}{c_*})^{\frac{8-2N}{N}}-1\right)}\right)^{\frac{2-q\gamma_q}{2}} > \mu_*,$$

which contradicts to $0 < \mu < \mu_*$, hence, $\mathcal{P}^0_{c,\mu} = \emptyset$. $\mathcal{P}_{c,\mu}$ is a smooth manifold of codimension 2 in $H^1(\mathbb{R}^N)$, see proof of [16, Lemma 5.2]. □

Lemma 3.3. *Let $a > 0, b > 0, 2 < q < 2 + \frac{4}{N}$, $p = 2 + \frac{8}{N}$, $0 < \mu < \mu^*$, if $u \in P_{c,\mu}$ is a critical point for $E_{a,\mu}|_{P_{c,\mu}}$, then u is a critical point for $E_{a,\mu}|_{S_{c,r}}$, where μ^* is defined in (1.7).*

Proof. From Lemma 3.2, we deduce that $\mathcal{P}_{c,\mu}$ is a smooth manifold of codimension 2 in $H^1(\mathbb{R}^N)$ and $\mathcal{P}^0_{c,\mu} = \emptyset$. If $u \in \mathcal{P}_{c,\mu}$ is a critical point for $E_{a,\mu}|_{\mathcal{P}_{c,\mu}}$, then by the Lagrange multipliers rule, there exists $\lambda, \xi \in \mathbb{R}$ such that

$$\left\langle E'_{a,\mu}(u), \varphi\right\rangle - \lambda\int_{\mathbb{R}^N} u\varphi dx - \xi\left\langle P'_\mu(u), \varphi\right\rangle = 0, \quad \forall \varphi \in H^1\left(\mathbb{R}^N\right).$$

So u solves

$$-((1-2\xi)a + (1-4\xi)b|\nabla u|_2^2)\Delta u - \lambda u + \mu(\xi q\delta_q - 1)|u|^{q-2}u + (p\xi\gamma_p - 1)|u|^{p-2}u = 0.$$

Combining with the Pohozaev identity, we have

$$(1-2\xi)a|\nabla u|_2^2 + (1-4\xi)b|\nabla u|_2^4 + \mu\gamma_q(\xi q\gamma_q - 1)|u|_q^q + \gamma_p(p\xi\gamma_p - 1)|u|_p^p = 0.$$

Since $u \in \mathcal{P}_{c,\mu}$ and $u \notin \mathcal{P}^0_{c,\mu}$, we deduce from $\xi(2a|\nabla u|_2^2 + 4b|\nabla u|_2^4 - \mu q\gamma_q^2|u|_q^q - \gamma_p^2 p|u|_p^p) = 0$ that $\xi = 0$. □

The manifold $\mathcal{P}_{c,\mu}$ is then divided into two components $\mathcal{P}^+_{c,\mu}$ and $\mathcal{P}^-_{c,\mu}$, having disjoint closure.

Lemma 3.4. *For every $u \in S_{c,r}$, we have*

(i) if $\frac{b}{4}|\nabla u|_2^4 \geq \frac{1}{p}|u|_p^p$, the function $I_{a,\mu}(u,\cdot)$ has a critical point $s_u \in \mathbb{R}$ and a zero $c_u \in \mathbb{R}$, with $s_u < c_u$;

(ii) if $\frac{b}{4}|\nabla u|_2^4 < \frac{1}{p}|u|_p^p$, the function $I_{a,\mu}(u,\cdot)$ has exactly two critical points $s_u < t_u \in \mathbb{R}$ and two zeros $c_u < d_u \in \mathbb{R}$, with $s_u < c_u < t_u < d_u$;

(iii) $\int_{\mathbb{R}^N} |\nabla(\tau \star u)|^2 \leq R_0$ for every $\tau \leq c_u$, and

$$E_{a,\mu}(s_u \star u) = \min\left\{E_{a,\mu}(\tau\star u) : \tau \in \mathbb{R} \text{ and } \int_{\mathbb{R}^N} |\nabla(\tau\star u)|^2 dx < R_0\right\} < 0; \tag{3.2}$$

(iv) For any $u \in S_{c,r}$ with $\frac{b}{4}|\nabla u|_2^4 < \frac{1}{p}|u|_p^p$, we have

$$E_{a,\mu}(t_u \star u) = \max\{E_{a,\mu}(\tau\star u) : \tau \in \mathbb{R}\} > 0, \tag{3.3}$$

and $I_{a,\mu}$ is strictly decreasing and concave on $\tau \in (t_u, +\infty)$;

(v) The maps $u \in S_{c,r} \mapsto s_u \in \mathbb{R}$ and $u \in S_{c,r} \mapsto t_u \in \mathbb{R}$ are of class C^1.

Proof. We recall that by (3.1)

$$I_{a,\mu}(u,\tau) = E_{a,\mu}(\tau\star u) \geq h\left(\int_{\mathbb{R}^N} |\nabla(\tau\star u)|^2 dx\right) = h\left(e^{2\tau}\int_{\mathbb{R}^N} |\nabla u|^2 dx\right).$$

Thus, the function $I_{a,\mu}(u,\cdot)$ is positive on $(C(R_0), C(R_1))$ with

$$(C(R_0), C(R_1)) := \left(\frac{1}{2}\ln\left(R_0/\int_{\mathbb{R}^N}|\nabla u|^2 dx\right), \frac{1}{2}\ln\left(R_1/\int_{\mathbb{R}^N}|\nabla u|^2 dx\right)\right).$$

If $\frac{b}{4}|\nabla u|_2^4 \geq \frac{1}{p}|u|_p^p$, from (2.1), $I_{a,\mu}(u,\tau) \to +\infty$ as $\tau \to +\infty$, and $I_{a,\mu}(u,\tau) \to 0^-$ as $\tau \to -\infty$. Hence, it follows that $I_{a,\mu}$ has at least a critical point s_u, with s_u local minimum point on $(-\infty, C(R_0))$ at negative level, and $I_{a,\mu}$ has at least a zero point c_u with $s_u < c_u < C(R_0)$. Note that $\partial_\tau I_{a,\mu}(u,\tau) = 0$ reads

$$\phi(\tau) = \mu\gamma_q|u|_q^q \quad \text{with } \phi(\tau) := ae^{\frac{4-N(q-2)}{2}\tau}|\nabla u|_2^2 + be^{\frac{8-N(q-2)}{2}\tau}|\nabla u|_2^4 - \frac{4}{p}e^{\frac{8-N(q-2)}{2}\tau}|u|_p^p. \tag{3.4}$$

But $\phi(\tau)$ is increasing on $(-\infty, +\infty)$, hence, $I_{a,\mu}$ has exactly a critical point s_u and a zero point c_u.

If $\frac{b}{4}|\nabla u|_2^4 < \frac{1}{p}|u|_p^p$, $I_{a,\mu}(u,\tau) \to -\infty$ as $\tau \to +\infty$ and ϕ has a unique maximum point, and $I_{a,\mu}(u,\tau) \to 0^-$ as $\tau \to -\infty$. Therefore, we conclude that $I_{a,\mu}$ has exactly two critical points:

s_u, local minimum on $(-\infty, C(R_0))$ at negative level, and t_u, global maximum at positive level, which also gives (3.3).

From $s_u < C(R_0)$, then it holds that

$$\int_{\mathbb{R}^N} |\nabla(s_u \star u)|^2 dx = e^{2s_u} \int_{\mathbb{R}^N} |\nabla u|^2 dx < R_0.$$

In addition, we have $s_u \star u \in \mathcal{P}_{c,\mu}$, $t_u \star u \in \mathcal{P}_{c,\mu}$, and $\tau \star u \in \mathcal{P}_{c,\mu}$ implies $\tau \in \{s_u, t_u\}$. By minimality and $\mathcal{P}^0_{c,\mu} = \emptyset$, we have $\partial_{\tau\tau} I_{a,\mu}(u, s_u) > 0$, that is, $s_u \star u \in \mathcal{P}^+_{c,\mu}$. In the same way, $t_u \star u \in \mathcal{P}^-_{c,\mu}$. In particular, $I_{a,\mu}(u, \cdot)$ is concave on $[t_u, +\infty)$.

Finally, we show that $u \mapsto s_u$ and $u \mapsto t_u$ are of class C^1. To this end, we apply the implicit function theorem on the C^1 function $\Phi(u, \tau) := \partial_\tau I_{a,\mu}(u, \tau)$. We see $\Phi(u, s_u) = 0$ and $\partial_\tau \Phi(u, s_u) = \partial_{\tau\tau} I_{a,\mu}(u, s_u) > 0$, and the fact that it is not possible to pass with continuity from $\mathcal{P}^+_{c,\mu}$ to $\mathcal{P}^-_{c,\mu}$ (since $\mathcal{P}^0_{c,\mu} = \emptyset$). By the same argument, we have that $u \mapsto t_u$ is of C^1. □

From the proof of Lemma 3.4, we see that $s_u < C(R_0) < t_u$ and

$$\int_{\mathbb{R}^N} |\nabla(s_u \star u)|^2 dx < R_0 < \int_{\mathbb{R}^N} |\nabla(t_u \star u)|^2 dx,$$

which implies

$$\mathcal{P}^+_{c,\mu} \subseteq \{u \in S_{c,r} : |\nabla u|_2^2 < R_0\}$$

and

$$\mathcal{P}^-_{c,\mu} \subseteq \{u \in S_{c,r} : |\nabla u|_2^2 > R_0\}.$$

For $k > 0$, let us set

$$\mathcal{A}_k := \{u \in S_{a,r} : |\nabla u|_2^2 < k\},$$

and

$$M_{c,\mu} := \inf_{u \in \mathcal{A}_{R_0}} E_{a,\mu}(u).$$

As an immediate lemma, we have:

Lemma 3.5. $\sup_{\mathcal{P}^+_{c,\mu}} E_{a,\mu} \le 0 \le \inf_{\mathcal{P}^-_{c,\mu}} E_{a,\mu}$.

Lemma 3.6. *It results that $M_{c,\mu} \in (-\infty, 0)$, that*

$$M_{c,\mu} = \inf_{\mathcal{P}_{c,\mu}} E_{a,\mu} = \inf_{\mathcal{P}^+_{c,\mu}} E_{a,\mu}, \quad \textit{and that} \quad M_{c,\mu} < \inf_{\mathcal{A}_{R_0} \setminus \mathcal{A}_{R_0 - \rho}} E_{a,\mu}$$

for $\rho > 0$ small enough.

Proof. For any $u \in \mathcal{A}_{R_0}$, we have

$$E_{a,\mu}(u) \ge h(|\nabla u|_2^2) \ge \min_{t \in [0, R_0]} h(t) > -\infty,$$

and hence $M_{c,\mu} > -\infty$. Moreover, for any $u \in S_{c,r}$, we have $|\nabla(\tau \star u)|_2^2 < R_0$ and $E_{a,\mu}(\tau \star u) < 0$ for $\tau \ll -1$, and hence $M_{c,\mu} < 0$.

Now, $M_{c,\mu} \le \inf_{\mathcal{P}^+_{c,\mu}} E_{a,\mu}$ from $\mathcal{P}^+_{c,\mu} \subset \mathcal{A}_{R_0}$. On the other hand, if $u \in \mathcal{A}_{R_0}$, then $s_u \star u \in \mathcal{P}^+_{c,\mu} \subset \mathcal{A}_{R_0}$, and

$$E_{a,\mu}(s_u \star u) = \min \left\{ E_{a,\mu}(\tau \star u) : \tau \in \mathbb{R} \text{ and } \int_{\mathbb{R}^N} |\nabla(\tau \star u)|^2 dx < R_0 \right\} \le E_{a,\mu}(u),$$

which implies that $\inf_{\mathcal{P}^+_{c,\mu}} E_{a,\mu} \le M_{c,\mu}$. To prove that $\inf_{\mathcal{P}^+_{c,\mu}} E_{a,\mu} = \inf_{\mathcal{P}_{c,\mu}} E_{a,\mu}$, it is sufficient to recall that $E_{a,\mu}(u) > 0$ on $\mathcal{P}^-_{c,\mu}$.

Finally, by the continuity of h, there exists $\rho > 0$ such that $h(t) \ge \frac{M_{c,\mu}}{2}$ for any $t \in [R_0 - \rho, R_0]$. Therefore, we have

$$E_{a,\mu}(u) \ge h(|\nabla u|_2^2) \ge \frac{M_{c,\mu}}{2} > M_{c,\mu}$$

for every $u \in S_{c,r}$ with $R_0 - \rho \le |\nabla u|_2^2 \le R_0$. □

Lemma 3.7. *$M_{c,\mu}$ can be achieved by some $\tilde{u}_{c,\mu} \in S_{c,r}$. Moreover, $\tilde{u}_{c,\mu}$ is an interior local minimizer for $E_{a,\mu}|_{\mathcal{A}_{R_0}}$, and $\tilde{u}_{c,\mu}$ solves problem (1.1) for some $\tilde{\lambda}_{c,\mu} < 0$. Moreover, $\tilde{u}_{c,\mu}$ is a ground state of $E_{a,\mu}|_{S_{c,r}}$, any ground state of $E_{a,\mu}|_{S_{c,r}}$ is a local minimizer of $E_{a,\mu}$ on A_{R_0}.*

Proof. Let us consider a minimizing sequence $\{v_n\}$ for $E_{a,\mu}|_{\mathcal{A}_{R_0}}$. By Lemma 3.4, there exists a sequence $\{s_{v_n}\}$ such that $s_{v_n} \star v_n \in \mathcal{P}^+_{c,\mu}$ and

$$E_{a,\mu}(s_{v_n} \star v_n) = \min\left\{E_{a,\mu}(\tau \star s_{v_n}) : \tau \in \mathbb{R} \text{ and } \int_{\mathbb{R}^N} |\nabla(\tau \star s_{v_n})|^2 dx < R_0\right\} < E_{a,\mu}(v_n),$$

where the last inequality follows from $v_n \in \mathcal{A}_{R_0}$. Besides, we also see that

$$\int_{\mathbb{R}^N} |\nabla(s_{v_n} \star v_n)|^2 dx < R_0,$$

furthermore, by Lemma 3.6, we have

$$\int_{\mathbb{R}^N} |\nabla(s_{v_n} \star v_n)|^2 dx < R_0 - \rho.$$

Once again by Lemma 3.6, it holds that

$$M_{c,\mu} = \inf_{\mathcal{P}_{c,\mu}} E_{a,\mu} = \inf_{\mathcal{P}^+_{a,\mu}} E_{a,\mu}.$$

Setting $u_n = s_{v_n} \star v_n$ and using the Ekeland's variational principle, we may assume that $\{u_n\}$ is a Palais–Smale sequence for $E_{a,\mu}$ on $S_{c,r}$ and $P_\mu(u_n) = 0$. Hence, we have

$$E_{a,\mu}(u_n) = \frac{a}{4}|\nabla u_n|_2^2 - \frac{\mu}{q}\left(1 - \frac{N(q-2)}{8}\right)|u_n|_q^q = M_{c,\mu} + o_n(1).$$

It results to

$$\frac{a}{4}|\nabla u_n|_2^2 \le (M_{c,\mu} + 1) + \frac{\mu}{q}\left(1 - \frac{N(q-2)}{8}\right)\frac{c^{q-\frac{N(q-2)}{2}}}{2|Q_q|_2^{q-2}}|\nabla u_n|_2^{\frac{N(q-2)}{2}}, \tag{3.5}$$

which gives $\{|\nabla u_n|_2\}$ is bounded, hence, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. From Lemma 2.2, up to a subsequence, $u_n \to \tilde{u}_{c,\mu}$ strongly in $H^1(\mathbb{R}^N)$, and $\tilde{u}_{c,\mu}$ solves problem (1.1) for some $\tilde{\lambda}_{c,\mu} < 0$. Moreover, we have $\int_{\mathbb{R}^N} |\nabla\tilde{u}_{c,\mu}|^2 dx < R_0 - \rho$ and $\tilde{u}_{c,\mu}$ is an interior local minimizer for $M_{c,\mu}$.

Since any critical point of $E_{a,\mu}|_{S_{c,r}}$ lies in $\mathcal{P}_{c,\mu}$ and $M_{c,\mu} = \inf_{\mathcal{P}_{c,\mu}} E_{a,\mu}$, we see that $\tilde{u}_{c,\mu}$ is a ground state for $E_{a,\mu}|_{S_{c,r}}$. It only remains to prove that any ground state of $E_{a,\mu}|_{S_{c,r}}$ is a local minimizer of $E_{a,\mu}$ in A_{R_0}. Let u be a critical point of $E_{a,\mu}|_{S_{c,r}}$ with $E_{a,\mu}(u) = M_{c,\mu} = \inf_{\mathcal{P}_{c,\mu}} E_{a,\mu}$. Since $E_{a,\mu}(u) < 0 < \inf_{\mathcal{P}^-_{c,\mu}} E_{a,\mu}$, necessarily $u \in \mathcal{P}^+_{c,\mu}$. Then Lemma 3.6 implies that $\mathcal{P}^+_{c,\mu} \subset A_{R_0}$. This leads to $|\nabla u|_2 < R_0$, and as a consequence u is a local minimizer for $E_{a,\mu}|_{A_{R_0}}$. Lemma 3.4 implies that $E_{a,\mu}(u) \le 0$ for any $u \in \mathcal{P}^+_{c,\mu}$, and $|\nabla u|_2^2 < R_0$. Hence, u is a local minimizer for $E_{a,\mu}|_{A_{R_0}}$. □

In the following, we focus on the existence of a second critical point for $E_{a,\mu}|_{S_{c,r}}$. Let

$$\widetilde{Q}_p(x) := c\frac{Q_p(x)}{|Q_p|_2}, \qquad Q_p^\tau(x) := c\frac{e^{\frac{N}{2}\tau}Q_p(e^\tau x)}{|Q_p|_2} \quad \text{for any } \tau > 0,$$

we have $\widetilde{Q}_p(x), Q_p^\tau(x) \in S_{c,r}$.

Lemma 3.8. *If $2 < q < 2 + \frac{4}{N}$, $p = 2 + \frac{8}{N}$, and $c > c^*$, we have $\int_{\mathbb{R}^N} |\nabla Q_p^\tau|^2 dx \to +\infty$ and $I_{a,\mu}(\widetilde{Q}_p, \tau) \to -\infty$ as $\tau \to +\infty$.*

Proof. A straightforward calculation shows that

$$\int_{\mathbb{R}^N} |\nabla Q_p^\tau|^2 dx = e^{2\tau} \int_{\mathbb{R}^N} |\nabla \widetilde{Q}_p|^2 dx.$$

From (1.5) with $s = p$ and (2.1), we have

$$\begin{aligned}
&I_{a,\mu}(\widetilde{Q}_p, \tau) \\
&= \frac{ae^{2\tau}}{2}\int_{\mathbb{R}^N} |\nabla \widetilde{Q}_p|^2 dx + \frac{be^{4\tau}}{4}\left(\int_{\mathbb{R}^N} |\nabla \widetilde{Q}_p|^2 dx\right)^2 - \frac{e^{4\tau}}{p}\int_{\mathbb{R}^N} |\widetilde{Q}_p|^p dx - \mu\frac{e^{\gamma_q q\tau}}{q}\int_{\mathbb{R}^N} |\widetilde{Q}_p|^q dx \\
&= \frac{ae^{2\tau}}{2}\frac{c^2|\nabla Q_p|_2^2}{|Q_p|_2^2} - \mu\frac{e^{\gamma_q q\tau}}{q}\frac{c^q}{|Q_p|_2^q}|Q_p|_q^q + c^4 e^{4\tau}\left(\frac{b}{4}\frac{|\nabla Q_p|_2^4}{|Q_p|_2^4} - \frac{1}{4}\frac{2}{c^2}\left(\frac{c}{|Q_p|_2}\right)^{\frac{8}{N}}\frac{2|Q_p|_p^p}{q|Q_p|_2^2}\right) \\
&= \frac{ac^2e^{2\tau}}{2} - \mu\frac{e^{\gamma_q q\tau}}{q}\frac{c^q}{|Q_p|_2^q}|Q_p|_q^q + \frac{bc^4e^{4\tau}}{4}\left(1 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right),
\end{aligned}$$

from $c > c^*$, we have $I_{a,\mu}(\widetilde{Q}_p, \tau) \to -\infty$ as $\tau \to +\infty$. □

Lemma 3.9. *Suppose that $E_{a,\mu}(u) < M_{c,\mu}$. Then the value t_u defined by Lemma 3.4 is negative.*

Lemma 3.10. *It results that*

$$\tilde{\sigma}_{c,\mu} = \inf_{u \in \mathcal{P}_{c,\mu}^-} E_{a,\mu}(u) > 0.$$

We introduce the minimax class

$$\Gamma := \left\{\gamma \in C([0,1], S_{c,r}) : \gamma(0) \in \mathcal{P}_{c,\mu}^+ \text{ with } \frac{b}{4}|\nabla\gamma(0)|_2^4 < \frac{1}{p}|\gamma(0)|_p^p,\ E_{a,\mu}(\gamma(1)) \le 2M_{c,\mu}\right\},$$

then $\Gamma \neq \emptyset$. In fact, we have $s_{\widetilde{Q}_p} \star \widetilde{Q}_p \in \mathcal{P}_{c,\mu}^+$ by Lemma 3.4 and $E_{a,\mu}(\tau \star \widetilde{Q}_p) \to -\infty$ as $\tau \to +\infty$ by Lemma 3.8, and $\tau \mapsto \tau \star \widetilde{Q}_p$ is continuous. Thus, we can define the minimax value

$$\sigma_{c,\mu} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_{a,\mu}(\gamma(t)).$$

Lemma 3.11. *$\sigma_{c,\mu} > 0$ can be achieved by some $\hat{u}_{c,\mu} \in S_{c,r}$, and $\hat{u}_{c,\mu}$ solves problem (1.1) for some $\hat{\lambda}_{c,\mu} < 0$.*

Proof. Since we want to use Lemma 2.1, next we verify the conditions of Lemma 2.1 one by one. Let us set

$$\mathcal{F} := \Gamma, \quad A := \gamma([0,1]), \quad F := \mathcal{P}_{c,\mu}^- \quad \text{and} \quad B := \mathcal{P}_{c,\mu}^+ \cup E_{a,\mu}^{2M_{c,\mu}},$$

where $E_{a,\mu}^c := \{u \in S_{c,r} : E_{a,\mu}(u) \le c\}$.

We first show that $\mathcal{F}$ is homotopy-stable family with extended boundary B: for any $\gamma \in \Gamma$ and any $\eta \in C([0,1] \times S_{c,r}; S_{c,r})$ satisfying $\eta(t,u) = u, (t,u) \in (0 \times S_{c,r}) \cup ([0,1] \times B)$, we want to get $\eta(1,\gamma(t)) \in \Gamma$. In fact, let $\tilde{\gamma}(t) = \eta(1,\gamma(t))$, then $\tilde{\gamma}(0) = \eta(1,\gamma(0)) = \gamma(0) \in \mathcal{P}^+_{c,\mu}$. Besides, $\tilde{\gamma}(1) = \eta(1,\gamma(1)) = \gamma(1) \in E^{2M_{c,\mu}}_{a,\mu}$. Therefore, we have $\eta(1,\gamma(t)) \in \Gamma$.

Next we verify the condition (2.2): by Lemma 3.5 and Lemma 3.9, we know $F \cap B = \emptyset$ and hence $F \backslash B = F$. We claim that

$$A \cap (F \backslash B) = A \cap F = \gamma([0,1]) \cap \mathcal{P}^-_{c,\mu} \neq \emptyset, \quad \forall \gamma \in \Gamma. \tag{3.6}$$

Indeed, since $\gamma(0) \in \mathcal{P}^+_{c,\mu}$ with $\frac{b}{4}|\nabla \gamma(0)|_2^4 < \frac{1}{p}|\gamma(0)|_p^p$, we know $s_{\gamma(0)} = 0$ (see the definition of s_u in Lemma 3.4) and hence $t_{\gamma(0)} > s_{\gamma(0)} = 0$. On the other hand, since $E_{a,\mu}(\gamma(1)) \leq 2M_{c,\mu} < M_{c,\mu}$ (see Lemma 3.6), we by Lemma 3.8 have $t_{\gamma(1)} < 0$. By Lemma 3.4, we know $t_{\gamma(\tau)}$ is continuous in τ. It follows that for every $\gamma \in \Gamma$ there exists $\tau_\gamma \in (0,1)$ such that $t_{\gamma(\tau_\gamma)} = 0$, that is, $\gamma(\tau_\gamma) \in \mathcal{P}^-_{c,\mu}$, and hence $A \cap (F \backslash B) \neq \emptyset$.

Finally, we verify the condition (2.3), that is, we need to show

$$\inf_{\mathcal{P}^-_{c,\mu}} E_{a,\mu} \geq \sigma_{c,\mu} \geq \sup_{\mathcal{P}^+_{c,\mu} \cup E^{2M_{c,\mu}}_{a,\mu}} E_{a,\mu}.$$

By (3.6), for every $\gamma \in \Gamma$, we have

$$\max_{t \in [0,1]} E_{a,\mu}(\gamma(t)) \geq \inf_{\mathcal{P}^-_{c,\mu}} E_{a,\mu},$$

so that $\sigma_{c,\mu} \geq \tilde{\sigma}_{c,\mu}$. On the other hand, if $u \in \mathcal{P}^-_{c,\mu}$ with $\frac{b}{4}|\nabla u|_2^4 < \frac{1}{p}|u|_p^p$, then for $s_1 \gg 1$ large enough

$$\gamma_u : \tau \in [0,1] \mapsto ((1-\tau)s_u + \tau s_1) \star u \in S_{c,r}$$

is a path in Γ. Since $u \in \mathcal{P}^-_{c,\mu}$, we know $t_u = 0$ is a global maximum point for $I_{a,\mu}$, and deduce that

$$E_{a,\mu}(u) \geq \max_{t \in [0,1]} E_{a,\mu}\left(\gamma_u(t)\right) \geq \sigma_{c,\mu},$$

which implies that $\tilde{\sigma}_{c,\mu} \geq \sigma_{c,\mu}$. Thus, we get $\sigma_{c,\mu} = \tilde{\sigma}_{c,\mu} > 0$. By Lemma 3.5, we know $E_{a,\mu}(u) \leq 0$ for any $u \in \mathcal{P}^+_{c,\mu} \cup E^{2M_{c,\mu}}_{a,\mu}$, hence we get (2.3). From Lemma 2.1, we obtain a Palais–Smale sequence $\{u_n\}$ for the functional $E_{a,\mu}$ on $S_{c,r}$ and $P_\mu\left(u_n\right) \to 0$. Similar to (3.5), $\{u_n\}$ is bounded. Hence, from Lemma 2.2, up to a subsequence, $u_n \to \hat{u}_{c,\mu}$ strongly in $H^1(\mathbb{R}^N)$, and $\hat{u}_{c,\mu}$ solves problem (1.1) for some $\hat{\lambda}_{c,\mu} < 0$. □

Proof of Theorem 1.1. Theorem 1.1 comes from Lemma 3.7 and Lemma 3.11. □

4 Proof of Theorem 1.2

In this section, we deal with the case $2 + \frac{4}{N} < q < p = 2 + \frac{8}{N}$, $\mu > 0$, $a \geq 0$ and prove Theorem 1.2. We first consider the existence of normalized ground state solution for the degenerate Kirchhoff-type equations, that is, $a = 0$, by the following minimization problem:

$$m_{0,c} = \inf_{u \in S_c} E_{a,\mu}(u).$$

And then, we discuss the the existence of normalized solutions for the nondegenerate Kirchhoff-type equations, that is, $a > 0$.

Lemma 4.1. *If $a \geq 0$, $2+\frac{4}{N} < q < p = 2+\frac{8}{N}$ and $c < c^*$, the functional $E_{a,\mu}$ is coercive on S_c. Moreover, $m_{0,c} < 0$.*

Proof. Utilizing (1.5) and (1.6), we see that for any $u \in S_c$,

$$E_{a,\mu}(u) \geq \frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)|\nabla u|_2^4 - \mu\frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}|\nabla u|_2^{q\gamma_q},$$

hence, from $2 < \gamma_q q < 4$ and $c < c^*$, we obtain that the functional $E_{a,\mu}$ is coercive on S_c.

For any $u \in S_c$, set $u^t(x) = t^{\frac{N}{2}}u(tx)$ for any $t > 0$, then $u^t \in S_c$ and

$$m_{0,c} \leq E_{0,\mu}(u^t) = \frac{b}{4}|\nabla u|_2^4 t^4 - \frac{1}{p}|u|_p^p t^4 - \frac{\mu}{q}|u|_q^q t^{\gamma_q q} \to 0^- \quad \text{as } t \to 0^+,$$

hence, from $\mu > 0$ and $2 < \gamma_q q < 4$, we obtain $m_{0,c} < 0$. □

In order to prove that the minimizer of $m_{a,c}$ can be obtained, we now give two lemmas.

Lemma 4.2. *If $m_{a,c} < 0$, we have $m_{a,c} < m_{a,\gamma} + m_{a,c-\gamma}$ for any $0 < \gamma < c$.*

Proof. The proof is similar to [19, Lemma 2.5], so we omit it. □

Corollary 4.3. *$m_{a,c}$ is strictly decreasing in $c \in (0, +\infty)$.*

Lemma 4.4. *Let $c < c^*$, $m_{0,c} := \inf u \in S_c E_{0,\mu}(u)$ has a radial minimizer $\tilde{u}$, and $\tilde{u}$ solves problem (1.1) for some $\tilde{\lambda} < 0$.*

Proof. Let $\{u_n\} \subset S_c$ be a minimizing sequence of $m_{0,c} < 0$, it can easily see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ by Lemma 4.1. Since $E_{0,\mu}$ is even, we can suppose that $u_n \geq 0$. Moreover, let u_n^* be the symmetric radial decreasing rearrangement of u_n, up to subsequence, we may assume that there exists $\tilde{u} \in H_r^1(\mathbb{R}^N)$ such that

$$u_n^* \rightharpoonup \tilde{u} \text{ in } H^1(\mathbb{R}^N), \quad u_n^* \to \tilde{u} \text{ in } L^s(\mathbb{R}^N), \quad s \in (2, 2^*), \quad u_n^*(x) \to \tilde{u}(x) \text{ a.e. in } \mathbb{R}^N. \tag{4.1}$$

Hence, we have

$$E_{0,\mu}(\tilde{u}) \leq \liminf_{n\to\infty} E_{0,\mu}(u_n^*) \leq \liminf_{n\to\infty} E_{0,\mu}(u_n) = m_{0,c}, \qquad |\tilde{u}|_2^2 \leq c^2.$$

From $E_{0,\mu}(\tilde{u}) \leq m_{0,c} < 0$, it follows that $\tilde{u} \not\equiv 0$. By Corollary 4.3, it must hold that

$$E_{0,\mu}(\tilde{u}) = m_{0,c}, \qquad |\tilde{u}|_2^2 = c^2.$$

By the Lagrange multiplier rule, there is $\tilde{\lambda} \in \mathbb{R}$ such that

$$-b|\nabla\tilde{u}|_2^2\Delta\tilde{u} = \tilde{\lambda}\tilde{u} + |\tilde{u}|^{p-2}\tilde{u} + \mu|\tilde{u}|^{q-2}\tilde{u},$$

and then, combining with the Pohozaev identity, we have

$$\tilde{\lambda}|\tilde{u}|_2^2 = \frac{4-p}{p}|\tilde{u}|_p^p + \frac{\mu}{2q}(N(q-2)-2q)|\tilde{u}|_q^q,$$

which implies $\tilde{\lambda} < 0$ from $2+\frac{4}{N} < q < p = 2+\frac{8}{N}$. □

Lemma 4.5. *Let $2 + \frac{4}{N} < q < p = 2 + \frac{8}{N}$, there is a constant $\bar{a} = \bar{a}(b, c, q) > 0$ such that for any $a \in (0, \bar{a})$, we have*

$$m_{a,c} := \inf_{u \in S_c} E_{a,\mu}(u) < 0.$$

Proof. From [21, Lemma 2.1], $\tilde{u}$ satisfies the following Pohozeav identity:

$$b|\nabla \tilde{u}|_2^4 - \frac{4}{p}|\tilde{u}|_p^p - \mu \frac{N(q-2)}{2q}|\tilde{u}|_q^q = 0,$$

and it follows that

$$\begin{aligned} m_{0,c} &= E_{0,\mu}(\tilde{u}) \\ &= \frac{b}{4}|\nabla \tilde{u}|_2^4 - \frac{1}{p}|\tilde{u}|_p^p - \mu \frac{1}{q}|\tilde{u}|_q^q \\ &= \left(\frac{1}{4} - \frac{2}{N(q-2)}\right) b|\nabla \tilde{u}|_2^4 + \left(\frac{8}{N(q-2)} - 1\right) \frac{1}{p}|\tilde{u}|_p^p \\ &< 0. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} E_{a,\mu}(\tilde{u}) &= \frac{a}{2}|\nabla \tilde{u}|_2^2 + E_{0,\mu}(\tilde{u}) \\ &= \frac{a}{2}|\nabla \tilde{u}|_2^2 + \left(\frac{1}{4} - \frac{2}{N(q-2)}\right) b|\nabla \tilde{u}|_2^4 + \left(\frac{8}{N(q-2)} - 1\right) \frac{1}{p}|\tilde{u}|_p^p \\ &\leq \frac{a}{2}|\nabla \tilde{u}|_2^2 + \frac{b}{4}\left(1 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)\left(\frac{1}{4} - \frac{2}{N(q-2)}\right)|\nabla \tilde{u}|_2^4. \end{aligned}$$

Let

$$\bar{a} = \frac{b}{2}\left(1 - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)\left(\frac{2}{N(q-2)} - \frac{1}{4}\right)|\nabla \tilde{u}|_2^2,$$

for any $a \in (0, \bar{a})$, we have $E_{a,\mu}(\tilde{u}) < 0$, and hence $m_{a,c} \leqslant E_{a,\mu}(\tilde{u}) < 0$. □

Lemma 4.6. *Let $0 < a < \bar{a}$ and $c < c^*$, $m_{a,c} := \inf_{u \in S_c} E_{a,\mu}(u)$ has a radial minimizer $\tilde{u}_{c,a}$, and $\tilde{u}_{c,a}$ solves problem (1.1) for some $\tilde{\lambda}_{c,a} < 0$.*

Proof. The proof is similar with that of Lemma 4.4, and we omit it. □

Lemma 4.7. *Let $0 < a < \bar{a}$, $2 + \frac{4}{N} < q < p = 2 + \frac{8}{N}$ and $c < c^*$, there exists $0 < K_{c,a} < \frac{|\nabla \tilde{u}_{c,a}|_2^2}{2}$ small enough such that*

$$0 < \sup_{u \in \mathcal{A}} E_{a,\mu}(u) < \inf_{u \in \mathcal{B}} E_{a,\mu}(u),$$

where $\mathcal{A} = \{u \in S_{c,r} : |\nabla u|_2^2 < K_{c,a}\}$, $\mathcal{B} = \{u \in S_{c,r} : |\nabla u|_2^2 = 2K_{c,a}\}$.

Proof. Let $K > 0$ be arbitrary but fixed and suppose that $u, v \in S_{c,r}$ satisfies

$$|\nabla u|_2^2 < K \quad \text{and} \quad |\nabla v|_2^2 = 2K.$$

From (1.5), we have

$$\begin{aligned} E_{a,\mu}(v) - E_{a,\mu}(u) &\geq E_{a,\mu}(v) - \frac{a}{2}|\nabla u|_2^2 - \frac{b}{4}|\nabla u|_2^4 \\ &\geq \frac{aK}{2} + \frac{3bK^2}{4} - b\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}} K^2 - \mu \frac{c^{q-q\gamma_q}}{|Q_q|_2^{q-2}}(2K)^{\frac{N(q-2)-4}{4}} \\ &= K\left(\frac{a}{2} + \left(\frac{3}{4} - \left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right) bK - \mu \frac{c^{q-q\gamma_q}}{|Q_q|_2^{q-2}}(2K)^{\frac{N(q-2)-4}{4}}\right), \end{aligned}$$

and
$$E_{a,\mu}(u) \geq \frac{a}{2}|\nabla u|_2^2 + \frac{b}{4}\left(1-\left(\frac{c}{c^*}\right)^{\frac{8-2N}{N}}\right)|\nabla u|_2^4 - \mu\frac{c^{q-q\gamma_q}}{2|Q_q|_2^{q-2}}|\nabla u|_2^{q\gamma_q}.$$

In summary, we can choose sufficiently small constant $0 < K_{c,a} < \frac{|\nabla \tilde{u}_{c,a}|_2^2}{2}$ such that
$$0 < \sup_{u\in\mathcal{A}} E_{a,\mu}(u) < \inf_{u\in\mathcal{B}} E_{a,\mu}(u)$$

where $\mathcal{A} = \{u \in S_{c,r} : |\nabla u|_2^2 < K_{c,a}\}$, $\mathcal{B} = \{u \in S_{c,r} : |\nabla u|_2^2 = 2K_{c,a}\}$. □

Let $u \in S_{c,r}$ be arbitrary and fixed, it is easy to see that $|\nabla(\tau \star u)|_2^2 \to 0$ and $I_{a,\mu}(u,\tau) \to 0^+$ as $\tau \to 0^+$. Hence, there exists $\hat{u}_{c,a} \in S_{c,r}$ such that $|\nabla \hat{u}_{c,a}|_2^2 < K_{c,a}$ and $E_{a,\mu}(\hat{u}_{c,a}) > 0$. Combining with Lemma 4.7, we can construct the minimax value for the functionals $E_{a,\mu}$ and $I_{a,\mu}$:
$$\widetilde{\gamma}_c = \inf_{\widetilde{h}\in\widetilde{\Gamma}_c}\max_{t\in[0,1]} I_{a,\mu}(\widetilde{h}(t))$$

with $\widetilde{\Gamma}_c = \{\widetilde{h} \in C([0,1], S_{c,r}\times\mathbb{R}) : \widetilde{h}(0) = (\hat{u}_{c,a},0),\ \widetilde{h}(1) = (\tilde{u}_{c,a},0)\}$, and
$$\gamma_c = \inf_{h\in\Gamma_c}\max_{t\in[0,1]} E_{a,\mu}(h(t))$$

with $\Gamma_c = \{h \in C([0,1], S_{c,r}) : h(0) = \hat{u}_{c,a},\ h(1) = \tilde{u}_{c,a}\}$, where $\tilde{u}_{c,a}$ is obtained in Lemma 4.6. We have the following lemma.

Lemma 4.8. *If $0 < a < \bar{a}$, $2 + \frac{4}{N} < q < p = 2 + \frac{8}{N}$ and $c < c^*$, we have*
$$\widetilde{\gamma}_c = \gamma_c \geq \max\{E_{a,\mu}(\hat{u}_{c,a}),\ E_{a,\mu}(\tilde{u}_{c,a})\} := \delta_c > 0.$$

Proof. For any $\widetilde{h} \in \widetilde{\Gamma}_c$, we can write it into
$$\widetilde{h}(t) = (\widetilde{h}_1(t), \widetilde{h}_2(t)) \in S_{c,r}\times\mathbb{R}.$$

Setting $h(t) = \widetilde{h}_2(t) \star \widetilde{h}_1(t)$, we have $h(t) \in \Gamma_c$ and
$$\max_{t\in[0,1]} I_{a,\mu}(\widetilde{h}(t)) = \max_{t\in[0,1]} E_{a,\mu}(\widetilde{h}_2(t)\star\widetilde{h}_1(t)) = \max_{t\in[0,1]} E_{a,\mu}(h(t)),$$

which implies $\widetilde{\gamma}_c \geq \gamma_c$. On the other hand, for any $h \in \Gamma_c$, set $\widetilde{h}(t) = (h(t),0)$, we get $\widetilde{h} \in \widetilde{\Gamma}_c$ and
$$\max_{t\in[0,1]} I_{a,\mu}(\widetilde{h}(t)) = \max_{t\in[0,1]} E_{a,\mu}(h(t)),$$

which provides that $\gamma_c \geq \widetilde{\gamma}_c$. Thus, we have $\widetilde{\gamma}_c = \gamma_c$. Finally, $\gamma_c \geq \max\{E_{a,\mu}(\hat{u}_{c,a}), E_{a,\mu}(\tilde{u}_{c,a})\} > 0$ follows from the definition of γ_c. □

In what follows, we give the relationship between the Palais–Smale sequence for the functional $I_{a,\mu}$ and that of the functional $E_{a,\mu}$.

Lemma 4.9. *There exists a sequence $\{(v_n, \tau_n)\} \subset S_{c,r}\times\mathbb{R}^+$ such that for $n \to \infty$, we have*

(1) $I_{a,\mu}(v_n,\tau_n) \to \widetilde{\gamma}_c$,

(2) $I'_{a,\mu}|_{S_{c,r}\times\mathbb{R}}(v_n,\tau_n)\to 0$, *i.e., it holds that*

$$\partial_\tau I_{a,\mu}(v_n,\tau_n)\to 0 \quad \textit{and} \quad \langle \partial_u I_{a,\mu}(v_n,\tau_n),\widetilde{\varphi}\rangle \to 0$$

for any

$$\widetilde{\varphi}\in T_{v_n}=\left\{\widetilde{\varphi}\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N} v_n\widetilde{\varphi}dx=0\right\}.$$

In addition, setting $u_n(x)=\tau_n\star v_n(x)$, *then for* $n\to\infty$ *we get*

(i) $E_{a,\mu}(u_n)\to\gamma_c$,

(ii) $P_\mu(u_n)\to 0$,

(iii) $E'_{a,\mu}|_{S_{c,r}}(u_n)\to 0$, *i.e., it holds that* $\langle E'_{a,\mu}(u_n),\varphi\rangle\to 0$ *for any*

$$\varphi\in T_{u_n}=\left\{\varphi\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N} u_n\varphi dx=0\right\}.$$

Proof. According to the construction of $\widetilde{\gamma}_c$, we know that the conclusions (1) and (2) follow directly from Ekeland's Variational Principle. Next we mainly prove (i)–(iii).

For (i), it is obvious from

$$E_{a,\mu}(u_n)=E_{a,\mu}(\tau_n\star v_n)=I_{a,\mu}(v_n,\tau_n)$$

and $\widetilde{\gamma}_c=\gamma_c$.

For (ii), we first have

$$\begin{aligned}
\partial_\tau I_{a,\mu}(v_n,\tau_n)&=e^{2\tau_n}a|\nabla v_n|_2^2+e^{4\tau_n}b|\nabla v_n|_2^4-\mu e^{\gamma_q q\tau_n}\gamma_q|v_n|_q^q-e^{4\tau_n}\frac{4}{p}|v_n|_p^p\\
&=a|\nabla(\tau_n\star v_n)|_2^2+b|\nabla(\tau_n\star v_n)|_2^4-\mu\gamma_q|\tau_n\star v_n|_q^q-\frac{4}{p}|\tau_n\star v_n|_p^p\\
&=a|\nabla u_n|_2^2+b|\nabla u_n|_2^4-\mu\gamma_q|u_n|_q^q-\frac{4}{p}|u_n|_p^p\\
&=P_\mu(u_n).
\end{aligned}$$

Thus, (ii) is a consequence of $\partial_\tau I_{a,\mu}(v_n,\tau_n)\to 0$ as $n\to\infty$.

For (iii), by the definition of the functional $I_{a,\mu}$, we have

$$\begin{aligned}
\langle \partial_u I_{a,\mu}(v_n,\tau_n),\widetilde{\varphi}\rangle&=e^{2\tau_n}a\int_{\mathbb{R}^N}\nabla v_n\nabla\widetilde{\varphi}dx+e^{4\tau_n}b|\nabla v_n|_2^2\int_{\mathbb{R}^N}\nabla v_n\nabla\widetilde{\varphi}dx\\
&\quad-\mu e^{\gamma_q q\tau_n}\int_{\mathbb{R}^N}|v_n|^{q-2}v_n\widetilde{\varphi}dx-e^{4\tau_n}\int_{\mathbb{R}^N}|v_n|^{p-2}v_n\widetilde{\varphi}dx,
\end{aligned}$$

where

$$\widetilde{\varphi}\in T_{v_n}=\left\{\widetilde{\varphi}\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N} v_n\widetilde{\varphi}dx=0\right\}.$$

On the other hand, for any

$$\varphi\in T_{u_n}=\left\{\varphi\in H^1(\mathbb{R}^N):\int_{\mathbb{R}^N} u_n\varphi dx=0\right\},$$

from $u_n(x) = \tau_n \star v_n(x)$, we have

$$\begin{aligned}
&\langle E'_{a,\mu}(u_n), \varphi\rangle \\
&\quad = a\int_{\mathbb{R}^N} \nabla u_n \nabla \varphi dx + b|\nabla u_n|_2^2 \int_{\mathbb{R}^N} \nabla u_n \nabla \widetilde{\varphi} dx - \mu \int_{\mathbb{R}^N} |u_n|^{q-2} u_n \varphi dx - \int_{\mathbb{R}^N} |u_n|^{p-2} u_n \varphi dx \\
&\quad = e^{2\tau_n} a \int_{\mathbb{R}^N} \nabla v_n e^{-\frac{N\tau_n}{2}} \nabla \varphi(e^{-\tau_n} x) dx + e^{4\tau_n} b |\nabla v_n|_2^2 \int_{\mathbb{R}^N} \nabla v_n e^{-\frac{N\tau_n}{2}} \nabla \varphi(e^{-\tau_n} x) dx \\
&\qquad - \mu e^{\gamma_q q \tau_n} \int_{\mathbb{R}^N} |v_n(x)|^{q-2} v_n(x) e^{-\frac{N\tau_n}{2}} \varphi(e^{-\tau_n} x) dx \\
&\qquad - e^{4\tau_n} \int_{\mathbb{R}^N} |v_n(x)|^{p-2} v_n(x) e^{-\frac{N\tau_n}{2}} \varphi(e^{-\tau_n} x) dx.
\end{aligned}$$

Setting

$$\widetilde{\varphi}(x) = e^{-\frac{N\tau_n}{2}} \varphi(e^{-\tau_n} x),$$

we get (iii) if we could show $\widetilde{\varphi} \in T_{v_n}$. In fact, $\widetilde{\varphi} \in T_{v_n}$ comes from the following equalities:

$$0 = \int_{\mathbb{R}^N} u_n \varphi dx = \int_{\mathbb{R}^N} e^{\frac{N\tau_n}{2}} v_n(e^{\tau_n} x) \varphi(x) dx = \int_{\mathbb{R}^N} v_n(x) e^{-\frac{N\tau_n}{2}} \varphi(e^{-\tau_n} x) dx = \int_{\mathbb{R}^N} v_n \widetilde{\varphi} dx. \qquad \square$$

Lemma 4.10. *$\gamma_c > 0$ can be achieved by some $u_{c,a} \in S_{c,r}$, and $u_{c,a}$ is a radial solution of problem (1.1) for some $\lambda_c < 0$.*

Proof. By Lemma 4.1 and Lemma 4.9, we obtain a bounded Palais–Smale sequence $\{u_n\} \subset S_{c,r}$ for $E_{a,\mu}|_{S_{c,r}}$ at level $\gamma_c > 0$ such that $P_\mu(u_n) \to 0$ as $n \to \infty$. By Lemma 2.2, we have $u_n \to u_{c,a}$ in $H^1_r(\mathbb{R}^N)$, and $u_{c,a} \in S_{c,r}$ is a radial solution of problem (1.1) for some $\lambda_c < 0$. □

Proof of Theorem 1.2. Theorem 1.2 comes from Lemma 4.4, Lemma 4.6 and Lemma 4.10. □

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Multiplicity of solutions for $p(x)$-curl systems arising in electromagnetism

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Abstract. We are interested in the existence of multiple solutions for a class of $p(x)$-curl systems arising in electromagnetism. We work on variable exponent Sobolev spaces and by using critical point theory and the variational method, we investigate the existence of at least one, two, and three solutions to the problem.

Keywords: $p(x)$-curl systems, electromagnetic problems, variational methods, variable exponent Sobolev spaces.

2020 Mathematics Subject Classification: 35J50, 47J20, 78A30, 78M30.

1 Introduction

The study of partial differential equations or systems with variable exponents is a recent research topic that developed quickly. It started when it was understood that variable exponents give better descriptions of the behavior of certain materials or phenomena.

Let $\Omega \subset \mathbb{R}^3$, is a bounded simply connected domain with a $C^{1,1}$ boundary denoted by $\partial\Omega$. In what follows, vector functions and spaces of vector functions will be denoted by boldface symbols. We will use n to denote the outward unitary normal vector to $\partial\Omega$ and ∂_x to denote the partial derivative of a function with respect to the variable x.

The divergence of a vector function $\mathbf{v} = (v_1, v_2, v_3)$ is denoted by

$$\nabla.\mathbf{v} = \partial_{x_1} v_1 + \partial_{x_2} v_2 + \partial_{x_3} v_3$$

and the curl of $\mathbf{v}$ by

$$\nabla \times \mathbf{v} = (\partial_{x_2} v_3 - \partial_{x_3} v_2, \partial_{x_3} v_1 - \partial_{x_1} v_3, \partial_{x_1} v_2 - \partial_{x_2} v_1).$$

We recall the identity

$$-\Delta \mathbf{v} = \nabla \times (\nabla \times \mathbf{v}) - \nabla.(\nabla.\mathbf{v}),$$

where $\Delta \mathbf{v} = (\Delta v_1, \Delta v_2, \Delta v_3)$ and $\Delta v_i = \nabla.(\nabla v_i), i = 1, 2, 3.$

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In this article, We are interested in the existence of multiple solutions for the following intriguing system

$$\begin{cases} \nabla \times (|\nabla \times u|^{p(x)-2}\nabla \times u) + a(x)|u|^{p(x)-2}u = \lambda f(x,u), \ \nabla . u = 0 & \text{in } \Omega, \\ |\nabla \times u|^{p(x)-2}\nabla \times u \times n = 0, \ u.n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \in (0,+\infty)$, $\Omega \subset \mathbb{R}^3$, is a bounded simply connected domain with a $C^{1,1}$ boundary denoted by $\partial\Omega$. We will use n to denote the outward unitary normal vector to $\partial\Omega$. a is a functional in L^∞ and there exist $a_0, a_1 > 0$ such that

$$a_0 < a(x) < a_1, \qquad \forall x \in \Omega.$$

$p \in C(\bar{\Omega})$, with

$$3 < p^- = \min_{x\in\Omega} p(x) \le p^+ = \max_{x\in\Omega} p(x) < \infty,$$

and $p(x)$ satisfies logarithmic continuity: there exists a function $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\forall x, y \in \bar{\Omega}, |x-y| < 1, \quad |p(x) - p(y)| \le \omega(|x-y|), \quad \text{and} \quad \lim_{\tau\to 0^+} \omega(\tau)\log\frac{1}{\tau} = C < \infty. \tag{1.2}$$

The interest in transposing the problems into new problems with variable exponents is linked to a large scale of applications that are involving some nonhomogeneous materials. It was established that for appropriate treatment of these materials, we can not rely on the classical Sobolev space and that we have to allow the exponent to vary instead. Working with variable exponents, hence working in the framework of variable exponent spaces, opens the door for multiple applications. The variable exponent problems arise in many different applications, such as nonlinear elastic [26], electrorheological fluids [22], image processing [13] and other physics phenomena [2, 27]. The literature on variable exponent Sobolev spaces and their applications is quite large, here we just quote a few, see [5,6,12,13,19,20,23] and the references therein. For the basic properties of variable exponent Sobolev spaces and their applications to partial differential equations, we refer the readers to [14,21].

In [4], Antontsev, Miranda, and Santos studied the qualitative properties of solutions for the following $p(x,t)$-curl systems:

$$\begin{cases} \partial_t u + \nabla \times (|\nabla \times u|^{p(x,t)-2}\nabla \times u) = \lambda f(u), \ \nabla . u = 0 & \text{in } \Omega \times (0,T), \\ |\nabla \times u|^{p(x,t)-2}\nabla \times u \times n = 0, \ u.n = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $\nabla \times (|\nabla \times u|^{p(x,t)-2}\nabla \times u)$ is the $p(x,t)$-curl operator, $f(u) = \lambda u(\int_\Omega |u|^2 dx)^{\frac{\rho-2}{2}}$ with $\lambda \in \{-1,0,1\}$ and $\rho > 0$ is constant. The authors introduced a suitable variable exponent Sobolev space and obtained the existence of local or global weak solutions for system (1.3) by using Galerkin's method. The authors also studied the blow-up and finite-time extinction properties of solutions. When $p(x,t) \equiv p$, then problem (1.3) turns into a model from the generalized Maxwell's equations in the electromagnetic field theory. More precisely, u denotes the magnetic field, $\nabla \times u$ denotes the total current density, f denotes an internalmagnetic current, and $\nabla \times (|\nabla \times u|^{p-2}\nabla \times u)$ denotes the electric field.

Motivated by the above works, we study the existence and multiplicity of solutions for systems (1.1) with general nonlinearities. To the best of our knowledge, this is the first time to deal with the existence of steady-state solutions for systems (1.1) involving the $p(x)$-curl operator by applying variational methods different from that used in [24].

Xianga, Wanga, and Zhangc [24] investigated the existence and multiplicity of solutions to problem (1.1) in the case $\lambda = 1$. They studied the existence of ground state solutions and infinitely many solutions for (1.1) in the case $\lambda = 1$ with the nonlinearity f satisfying superlinear growth conditions is obtained by combining the mountain pass theorem with the Nehari manifold method, and a variant of the mountain pass theorem.

In this paper, we obtain three different results about the existence of weak solutions to the problem (1.1) by using critical point theorems established in [8, 9, 11]. The first aim of this paper is to provide an estimate of the positive interval for the parameter λ in which the problem (1.1) possesses at least one nontrivial weak solution. We also wish to consider the existence of two solutions to our problem by using a result of Bonanno [9, Theorem 3.2]. In a recent paper, Bonanno and Chinnì [10] studied the existence of at least two distinct weak solutions to a problem involving a p(x)-Laplacian by applying critical point theory. Our first main result will require the $(P.S.)^{[r]}$ condition, while in our second one, we will ask that the (AR)-condition holds and use it to ensure that the (usual) (PS)-condition is satisfied. We refer the reader to the papers [7, 10, 17, 18] where this approach was applied successfully. Finally, our third goal is to obtain the existence of three solutions to (1.1); this problem is less studied by researchers. In this case, we consider problem (1.1) where the nonlinearity f has subcritical growth, and we apply variational methods and critical point theory. The main tool used is the critical point theorem of Bonanno and Marano [11, Theorem 3.6].

The remainder of this paper is organized as follows. First, in Section 2, we recall briefly some basic results for fractional Sobolev spaces. In Section 3, we obtain the existence of at least one, two, or three nontrivial weak solutions to the problem (1.1) provided the parameter λ belongs to a positive interval to be determined.

2 Preliminaries

In this section, we introduce some definitions and results of Sobolev spaces with variable exponents.

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with a $C^{1,1}$ boundary denoted by $\partial\Omega$. Let $p \in C(\bar{\Omega})$. Set

$$p^- = \min_{x\in\bar{\Omega}} p(x), \quad \text{and} \quad p^+ = \max_{x\in\bar{\Omega}} p(x), \quad \text{with } 1 < p^- \le p^+ < \infty.$$

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx,$$

is finite. We define a norm, the so-called Luxemburg norm, on this space by the formula

$$\|u\|_{p(\cdot)} = \inf\left\{\gamma > 0 : \rho_{p(\cdot)}\left(\frac{u}{\gamma}\right) \le 1\right\}.$$

The space $(L^{p(\cdot)}(\Omega), \|.\|_{L^{p(\cdot)}(\Omega)})$ is a separable and reflexive Banach space. Moreover, the space $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Finally, we have the Hölder type inequality:

$$\left|\int_{\Omega} uvdx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$. An important role in manipulating the generalized Lebesgue spaces is played by the $\rho_{p(\cdot)}$-modular of the space $L^{p(\cdot)}(\Omega)$, we have the following result.

Proposition 2.1 (See [16]). *If $u \in L^{p(\cdot)}(\Omega)$, $u_n \in L^{p(\cdot)}(\Omega)$ and $p^+ < \infty$, then*

(i) if $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$, then $\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$;

(ii) if $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$, then $\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$;

(iii) $\lim_{n\to\infty} \|u_n - u\|_{p(\cdot)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0.$

Define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ by

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|.\|_{W^{1,p(\cdot)}(\Omega)})$ is a separable and reflexive Banach space. We consider also

$$W_0^{1,p(\cdot)}(\Omega) = \left\{u \in W^{1,p(\cdot)}(\Omega) : u \mid_{\partial\Omega)} = 0\right\},$$

with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

Remark 2.2. Assuming (1.2), we have $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ and this last space can be defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_{W^{1,p(\cdot)}(\Omega)}$. The density of smooth functions in the space $W_0^{1,p(\cdot)}(\Omega)$ is crucial for the understanding of these spaces. The condition of log-continuity of $p(\cdot)$ is the best known and the most frequently used sufficient condition for the density of $C_0^\infty(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$ (see [3, 14]). Although this condition is not necessary and can be substituted by other conditions (see [14, Chapter 9] for a discussion of this question) we keep it throughout the paper for the sake of simplicity of presentation.

Also, we observe that $W_0^{1,p(\cdot)}(\Omega) \subseteq W_0^{1,p^-}(\Omega)$, the Sobolev inequality

$$\|u\|_{L^{q(\cdot)}(\Omega)} \leq C\|u\|_{W^{1,p(\cdot)}(\Omega)},$$

holds, with $1 \leq q(x) < \frac{3p(x)}{3-p(x)}$ if $p^- < 3$, any q if $p^- = 3$, and $q = \infty$ if $p^- > 3$ Here $C = C(p^-, \Omega)$ is a positive constant.

Now, we define the space $\mathbf{w}^{p(x)}(\Omega)$

Let $\mathbf{L}^{p(x)}(\Omega) = L^{p(x)}(\Omega) \times L^{p(x)}(\Omega) \times L^{p(x)}\Omega$ and define

$$\mathbf{W}^{p(x)}(\Omega) = \{\mathbf{v} \in \mathbf{L}^{p(x)}(\Omega) : \nabla \times \mathbf{v} \in \mathbf{L}^{p(x)}(\Omega), \nabla.\mathbf{v}, \mathbf{v}.\mathbf{n}|_{\partial\Omega} = 0\}$$

where $\mathbf{n}$ denotes the outward unitary normal vector to $\partial\Omega$. Equip $\mathbf{W}^{p(x)}(\Omega)$ with the norm

$$\|\mathbf{v}\|_{\mathbf{W}^{p(x)}(\Omega)} = \|\mathbf{v}\|_{\mathbf{L}^{p(x)}(\Omega)} + \|\nabla \times \mathbf{v}\|_{\mathbf{L}^{p(x)}(\Omega)}.$$

If $p^- > 1$, by [4, Theorem 2.1], $\mathbf{W}^{p(x)}(\Omega)$ is a closed subspace of $\mathbf{W}_n^{1,p(x)}(\Omega)$, where

$$\mathbf{W}_n^{1,p(x)}(\Omega) = \{\mathbf{v} \in \mathbf{W}^{p(x)}(\Omega) : \mathbf{v}.\mathbf{n}|_{\partial\Omega} = 0\}$$

and

$$\mathbf{W}^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega) \times W^{1,p(x)}(\Omega) \times W^{1,p(x)}\Omega.$$

Thus, we have the following theorem.

Theorem 2.3 (see [4, Theorem 2.1])**.** *Assume that $1 < p^- \leq p^+ < \infty$ and p satisfies (1.2). Then $\mathbf{W}^{p(x)}(\Omega)$ is a closed subspace of $\mathbf{W}_n^{1,p(x)}(\Omega)$. Moreover, if $p^- > \frac{6}{5}$, then $\|\nabla \times .\|_{\mathbf{L}^{p(x)}(\Omega)}$ is a norm in $\mathbf{W}^{p(x)}(\Omega)$ and there exists $C = C(N, p^-, p^+) > 0$ such that*

$$\|\mathbf{v}\|_{\mathbf{W}^{p(x)}(\Omega)} \leq C\|\nabla \times \mathbf{v}\|_{\mathbf{L}^{p(x)}(\Omega)}.$$

Remark 2.4. By Remark 2.2 and Theorem 2.3, we know the embedding $\mathbf{W}^{p(x)}(\Omega) \hookrightarrow C_0^\infty(\Omega)$ is compact, with $3 < p^- \leq p^+ < \infty$, for all $x \in \bar{\Omega}$. Moreover, $(\mathbf{W}^{p(x)}(\Omega), \|.\|_{\mathbf{W}^{p(x)}(\Omega)})$ is a reflexive Banach space. We set

$$c_0 = \sup_{u \in \mathbf{W}^{p(x)}(\Omega)} \frac{\|u\|_\infty}{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}}.$$

Definition 2.5 ([8, p. 2993], [9, p. 210])**.** Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on a real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais-Smale condition cut off upper at r, denoted by $(P.S.)^{[r]}$ if any sequence $\{u_n\}_{n \in \mathbb{N}}$ in X such that

(1) $\{I(u_n)\}$ is bounded;

(2) $\lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$;

(3) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$

has a convergent subsequence.

If only conditions (1) and (2) hold, then $I = \Phi - \Psi$ is said to satisfy the (usual) Palais–Smale $(P.S.)$ condition.

We next wish to define what is meant by a weak solution to our problem.

Definition 2.6. We say that a function $u \in \mathbf{W}^{p(x)}(\Omega)$ is a weak solution of the problem (1.1) if

$$\int_\Omega |\nabla \times u|^{p(x)-2} \nabla \times u . \nabla \times v dx + \int_\Omega a(x)|u|^{p(x)-2} u.v dx = \int_\Omega f(x, u).v dx,$$

holds for all $v \in \mathbf{W}^{p(x)}(\Omega)$.

Remark 2.7. Let u be a classical solution of (1.1). Let $e = |\nabla \times u|^{p(x)-2}\nabla \times u$ and v be a smooth function in Ω, then we obtain

$$\nabla(e \times v) = v.\nabla \times e - e.\nabla \times v. \tag{2.1}$$

Multiplying the first equation of (1.1) by v and integrating over Ω, we get

$$\int_\Omega \nabla \times e.vdx + \int_\Omega a(x)|u|^{p(x)-2}u.vdx = \int_\Omega f(x,u).vdx.$$

Using (2.1) and the boundary conditions in (1.1) and integrating by parts, we have

$$\int_\Omega e.\nabla \times vdx + \int_\Omega a(x)|u|^{p(x)-2}u.vdx = \int_\Omega f(x,u).vdx,$$

which means that Definition 2.6 is correct.

Assume that $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ is a Carathédory function. We set

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}^3.$$

The variational structure of this problem leads us to introduce We define the functionals $\Phi, \Psi : \mathbf{W}^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\Phi(u) := \int_\Omega \frac{|\nabla \times u|^{p(x)} + a(x)|u|^{p(x)}}{p(x)}dx \tag{2.2}$$

and

$$\Psi(u) := \int_\Omega F(x,u)dx. \tag{2.3}$$

Lemma 2.8 ([24, Lemmas 3.1. and 3.2.]). *The functional Φ is of class C^1 and*

$$\langle \Phi'(u), v\rangle = \int_\Omega |\nabla \times u|^{p(x)-2}\nabla \times u.\nabla \times vdx + \int_\Omega a(x)|u|^{p(x)-2}u.vdx$$

for every $u, v \in \mathbf{W}^{p(x)}(\Omega)$. For each $u \in \mathbf{W}^{p(x)}(\Omega)$, $\Phi'(u) \in (\mathbf{W}^{p(x)}(\Omega))^*$, where $(\mathbf{W}^{p(x)}(\Omega))^*$ is the dual space of $\mathbf{W}^{p(x)}(\Omega)$. Moreover, Φ is a convex functional in $\mathbf{W}^{p(x)}(\Omega)$.

The functional Ψ is of class C^1 and

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x,u).vdx$$

for every $u, v \in \mathbf{W}^{p(x)}(\Omega)$.

By Lemma 2.8, we know that $I_\lambda = \Phi - \lambda\Psi$ is of class C^1 and

$$\langle I_\lambda'(u), v\rangle = \int_\Omega |\nabla \times u|^{p(x)-2}\nabla \times u.\nabla \times vdx + \int_\Omega a(x)|u|^{p(x)-2}u.vdx - \lambda \int_\Omega f(x,u).vdx,$$

for all $u, v \in \mathbf{W}^{p(x)}(\Omega)$. Hence a critical point of I_λ is a (weak) solution of (1.1).

3 Main results

We begin by presenting a result that guarantees the existence of at least one solution to problem (1.1).

Theorem 3.1. *Let $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ be a Carathédory function and assume that there exist two positive constants τ and δ, such that:*

(H_1) $\quad p^+ c_0^{p^-} a_1 \operatorname{meas}(\Omega) \max\{\delta^{p^-}, \delta^{p^+}\} < p^- \min\{1, a_0\} \tau^{p^-};$

(H_2) $\quad \frac{\int_\Omega \sup_{|t|\le\tau} F(x,t)dx}{\tau^{\underline{p}}} < \frac{p^- \min\{1,a_0\} \int_\Omega F(x,\delta)dx}{p^+ c_0^{p^-} a_1 \operatorname{meas}(\Omega) \max\{\delta^{p^-},\delta^{p^+}\}};$

(H_3) $\quad \inf_{x\in\Omega, t\in\mathbb{R}^3, |t|=1} F(x,t) > 0.$

Then, for each

$$\lambda \in \Lambda_w := \left] \frac{a_1 \operatorname{meas}(\Omega) \max\{\delta^{p^-}, \delta^{p^+}\}}{p^- \int_\Omega F(x,\delta)dx}, \frac{\min\{1, a_0\}\tau^{p^-}}{p^+ c_0^{p^-} \int_\Omega \sup_{|t|\le\tau} F(x,t)dx} \right[, \tag{3.1}$$

problem (1.1) *admits at least one nontrivial solution $u_\lambda \in \mathbf{W}^{p(x)}(\Omega)$ such that $\|u_\lambda\|_\infty \le \tau$.*

Proof. Our goal is to apply [9, Theorem 2.3] to problem (1.1). To this end, take the real Banach space $\mathbf{W}^{p(x)}(\Omega)$ with the norm as defined in Section 2, Φ, Ψ be the functionals defined in (2.2) and (2.3). We can see Φ, Ψ are of C^1 in Lemma 2.8. For each $u \in \mathbf{W}^{p(x)}(\Omega)$ we have

$$\frac{\min\{1, a_0\}}{p^+} \|u\|^{p(x)}_{\mathbf{W}^{p(x)}(\Omega)} \le \Phi(u) \le \frac{\max\{1, a_1\}}{p^-} \|u\|^{p(x)}_{\mathbf{W}^{p(x)}(\Omega)}. \tag{3.2}$$

From the first inequality in (3.2), it follows that Φ is coercive. To show that Φ' admits a continuous inverse, in view of [25, Theorem 26.A(d)], it suffices to show that Φ' is coercive, hemicontinuous, and uniformly monotone. For any $u \in \mathbf{W}^{p(x)}(\Omega)$ we have

$$\frac{\langle \Phi'(u), u\rangle}{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}} = \frac{\int_\Omega |\nabla \times u|^{p(x)}dx + \int_\Omega a(x)|u|^{p(x)}dx}{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}} \ge \frac{\min\{1, a_0\}\|u\|^{p(x)}_{\mathbf{W}^{p(x)}(\Omega)}}{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}}.$$

By Proposition 2.1 for any $u \in \mathbf{W}^{p(x)}(\Omega)$

$$\lim_{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}\to\infty} \frac{\langle \Phi'(u), u\rangle}{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}} \ge \lim_{\|u\|_{\mathbf{W}^{p(x)}(\Omega)}\to\infty} (\min\{1, a_0\}\|u\|^{p^- -1}_{\mathbf{W}^{p(x)}(\Omega)}) = \infty,$$

i.e. Φ' is coercive. The fact that Φ' is hemicontinuous can be verified using standard arguments. (see, for example, [18]).

Finally, we show that Φ' is uniformly monotone. First, recall the inequality that for any $\xi, \psi \in \mathbb{R}$,

$$(|\xi|^{r-2}\xi - |\psi|^{r-2}\psi)(\xi - \psi) \ge 2^{-r}|\xi - \psi|^r, \quad \text{for all } r > 2. \tag{3.3}$$

Thus, for every $u, v \in X$, we deduce that

$$
\begin{aligned}
&\langle \Phi'(u) - \Phi'(v), u - v\rangle \\
&= \int_\Omega (|\nabla \times u|^{p(x)-2}\nabla \times u - |\nabla \times v|^{p(x)-2}\nabla \times v)(\nabla \times u - \nabla \times v)dx \\
&\quad + \int_\Omega a(x)(|u|^{p(x)-2}u - |v|^{p(x)-2}v)(u-v)dx \\
&\geq 2^{-p^+}(\int_\Omega |\nabla \times u - \nabla \times v|^{p(x)}dx + \int_\Omega a(x)|u-v|^{p(x)}dx) \\
&\geq \min\{2^{-p^+}, a_0 2^{-p^+}\}(\int_\Omega |\nabla \times u - \nabla \times v|^{p(x)}dx + \int_\Omega |u-v|^{p(x)}dx) \\
&\geq \begin{cases} c_1\|u-v\|^{p^-}_{\mathbf{W}^{p(x)}(\Omega)} & \text{if } \|u-v\|_{\mathbf{L}^{p(x)}(\Omega)}, \|\nabla \times (u-v)\|_{\mathbf{L}^{p(x)}(\Omega)} > 1, \\ c_2\|u-v\|^{p^+}_{\mathbf{W}^{p(x)}(\Omega)} & \text{if } \|u-v\|_{\mathbf{L}^{p(x)}(\Omega)}, \|\nabla \times (u-v)\|_{\mathbf{L}^{p(x)}(\Omega)} < 1, \end{cases}
\end{aligned}
$$

the last inequality is obtained from Proposition 2.1. It is easy to check that Φ' is uniformly monotone. Moreover, Ψ' is a compact operator. Indeed, it is enough to show that Ψ' is strongly continuous on $\mathbf{W}^{p(x)}(\Omega)$. For this end, for fixed $u \in \mathbf{W}^{p(x)}(\Omega)$, let $u_n \to u$ weakly in $\mathbf{W}^{p(x)}(\Omega)$ as $n \to \infty$, then $u_n(x)$ converges uniformly to $u(x)$ on Ω as $n \to \infty$; see [25]. Since f is continuous in $\mathbb{R}^3$ for every $x \in \Omega$, so

$$f(x, u_n) \to f(x, u),$$

as $n \to \infty$. Thus $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Hence we proved that Ψ' is a compact operator by [25, Proposition 26.2]. This ensures that the functional $I_\lambda = \Phi - \lambda\Psi$ verifies $(P.S.)^{[r]}$ condition for each $r > 0$ (see [8, Proposition 2.1]). To apply [9, Theorem 2.3] to the functional I_λ, first note that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. We need to show that there is an $r > 0$ and $w \in X$ with $0 < \Phi(w) < r$ such that $\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} < \frac{\Psi(w)}{\Phi(w)}$. To this end, set

$$r := \frac{\min\{1, a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-},$$

and define $w \in \mathbf{W}^{p(x)}(\Omega)$ by

$$w(x) = \begin{cases} \delta, & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

One has

$$\Phi(w) = \int_\Omega \frac{a(x)}{p(x)}|w(x)|^{p(x)}dx \leq \frac{\text{meas}(\Omega)a_1}{p^-}\max\{\delta^{p^+}, \delta^{p^-}\}. \tag{3.5}$$

Hence, it follows from (H_1) that $0 < \Phi(w) < r$. If $u \in \Phi^{-1}([0, r])$, by Proposition 2.1 (*i*) and (3.2), for any $u \in \mathbf{W}^{p(x)}(\Omega)$ with $\|u\|_{\mathbf{W}^{p(x)}(\Omega)} > 1$, we obtain

$$\frac{\min\{1, a_0\}}{p^+}\|u\|^{p^-}_{\mathbf{W}^{p(x)}(\Omega)} \leq \Phi(u) \leq \frac{\min\{1, a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-}.$$

Similarly, by Proposition 2.1 (*ii*) and (3.2), for any $u \in \mathbf{W}^{p(x)}(\Omega)$ with $\|u\|_{\mathbf{W}^{p(x)}(\Omega)} < 1$, we obtain

$$\frac{\min\{1, a_0\}}{p^+}\|u\|^{p^+}_{\mathbf{W}^{p(x)}(\Omega)} \leq \Phi(u) \leq \frac{\min\{1, a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-}.$$

Then

$$\|u\|_{\mathbf{W}^{p(x)}(\Omega)} \le \frac{\tau}{c_0}.$$

Hence, we obtain

$$|u(x)| \le \|u\|_{L^\infty(\Omega)} \le c_0 \|u\|_{\mathbf{W}^{p(x)}(\Omega)} \le \tau \qquad \forall x \in \Omega.$$

Hence, for each $u \in \Phi^{-1}((-\infty, r])$

$$\frac{\sup_{\Phi(u)\le r} \Psi(u)}{r} = \frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \int_\Omega F(x,u)dx}{\frac{\min\{1,a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-}} \le \frac{c_0^{p^-}\, p^+ \int_\Omega \sup_{|t|\le\tau} F(x,t)dx}{\min\{1,a_0\}\tau^{p^-}}. \tag{3.6}$$

Moreover, thanks to (H_2) and (3.5), one has

$$\begin{aligned} \frac{\Psi(w)}{\Phi(w)} &\ge \frac{p^- \int_\Omega F(x,\delta)dx}{a_1 \operatorname{meas}(\Omega) \max\{\delta^{p^-}, \delta^{p^+}\}} \\ &\ge \frac{c_0^{p^-}\, p^+ \int_\Omega \sup_{|t|\le\tau} F(x,t)dx}{\min\{1,a_0\}\tau^{p^-}} \\ &\ge \frac{\sup_{\Phi(u)\le r} \Psi(u)}{r}, \end{aligned}$$

which means that $\frac{\Phi(\bar v)}{\Psi(\bar v)} \ge \frac{r}{\sup_{\Phi(x)\le r} \Psi(u)}$ holds for some $\bar v \in \mathbf{W}^{p(x)}(\Omega)$. Hence, for each $\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(u)}\right[$, the functional I_λ admits at least one critical point u_λ with

$$0 < \Phi(u_\lambda) < r$$

which in turn is a nontrivial solution of problem (1.1) such that $\|u_\lambda\|_\infty < \tau$. □

Our second aim in this paper is to obtain a result on the existence of two distinct solutions to problem (1.1). The following theorem is obtained by applying [9, Theorem 3.2].

Theorem 3.2. *Let $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ be a Carathédory function, and assume that*

(H_4) *(Ambrosetti–Rabinowitz Condition) there exist $\mu > p^+$ and $R > 0$ such that*

$$0 < \mu F(x,t) \le t f(x,t) \qquad \forall x \in \Omega \text{ and } t \in \mathbb{R}^3 \backslash \{0\}, \text{ with } |t| \ge R.$$

Then, for each

$$\lambda \in \Lambda_r := \left]0, \frac{\min\{1,a_0\}\tau^{p^-}}{p^+ c_0^{p^-} \int_\Omega \sup_{|t|\le\tau} F(x,t)dx}\right[,$$

the problem (1.1) admits at least two nontrivial solutions.

Proof. Let Φ, Ψ be the functionals defined in Theorem (2.2) and (2.3). Notice that they satisfy all regularity assumptions required in [9, Theorem 3.2]). Arguing as in the proof of Theorem 3.1, choosing

$$r = \frac{\min\{1,a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-},$$

for each $\lambda \in \Lambda_r$ we obtain

$$\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} \le \frac{c_0^{p^-}\, p^+ \int_\Omega \sup_{|t|\le\tau} F(x,t)dx}{\min\{1,a_0\}\tau^{p^-}} < \frac{1}{\lambda}$$

(see (3.6)). Now, from condition (H$_4$), a straight forward calculation shows that there are positive constants m and C such that

$$F(x,t) \ge m|t|^\mu - C \quad \text{for all } x\in\Omega, t\in\mathbb{R}^3.$$

Hence, for every $\lambda \in \Lambda_r$, $u \in \mathbf{W}^{p(x)}(\Omega)\setminus\{0\}$ and $t > 1$, we obtain

$$\begin{aligned} I_\lambda(tu) &= \Phi(tu) - \lambda\int_\Omega F(x,tu)dx \\ &\le \frac{t^{p^+}}{p^-}\int_\Omega (|\nabla\times u|^{p(x)} + a(x)|u|^{p(x)})dx \\ &\quad - m\lambda t^\mu \int_\Omega |u|^\mu dx + C\lambda\,\mathrm{meas}(\Omega). \end{aligned}$$

Since $\mu > p^+$, this condition guarantees that I_λ is unbounded from below. To show that I_λ satisfies the (PS)-condition, let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbf{W}^{p(x)}(\Omega)$ such that $\{I_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'_{\lambda,\mu}(u_n) \to 0$ in $(\mathbf{W}^{p(x)}(\Omega))^*$ as $n\to+\infty$. Then, there exists a positive constant s_0 such that

$$|I_\lambda(u_n)| \le s_0, \qquad \|I'_\lambda(u_n)\| \le s_0 \qquad \forall n\in\mathbb{N}.$$

Using also the condition (H$_4$), and the definition of I'_λ, we see that, for all $n\in\mathbb{N}$, there exists $D > 0$ such that

$$\begin{aligned} \mu s_0 + s_0\|u_n\|_{\mathbf{W}^{p(x)}(\Omega)} &\ge \mu I_\lambda(u_n) - I'_\lambda(u_n)u_n \\ &\ge \left(\frac{\mu}{p^+}-1\right)\int_\Omega |\nabla\times u|^{p(x)} + a_0\left(\frac{\mu}{p^+}-1\right)\int_\Omega |u|^{p(x)}dx \\ &\quad + \lambda\int_\Omega (f(x,u_n)u_n - \mu F(x,u_n))\,dx \\ &\ge \left(\frac{\mu}{p^+}-1\right)\min\{1,a_0\}\|u_n\|^{p(x)}_{\mathbf{W}^{p(x)}(\Omega)} - D. \end{aligned}$$

Since $\mu > p^+$ it follows $\{u_n\}_{n\in\mathbb{N}}$ is bounded. Consequently, since $\mathbf{W}^{p(x)}(\Omega)$ is a reflexive Banach space we have, up to taking a subsequence if necessary,

$$u_n \rightharpoonup u \quad \text{in } \mathbf{W}^{p(x)}(\Omega).$$

By the fact that $I'_\lambda(u_n) \to 0$ and $u_n \rightharpoonup u$ in $\mathbf{W}^{p(x)}(\Omega)$, we obtain

$$(I'_\lambda(u_n) - I'_\lambda(u))(u_n - u) \to 0.$$

Furthermore,

$$\int_\Omega (f(x,u_n) - f(x,u))(u_n-u)dx \to 0, \quad \text{as } n\to+\infty.$$

An easy computation shows that

$$\begin{aligned}
&\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \\
&= \int_\Omega (|\nabla \times u_n|^{p(x)-2}\nabla \times u_n - |\nabla \times u|^{p(x)-2}\nabla \times u)(\nabla \times u_n - \nabla \times u)dx \\
&\quad + \int_\Omega a(x)(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx \\
&\quad - \lambda \int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx \\
&\geq 2^{-p^+}\|\nabla \times (u_n - u)\|^{p(x)}_{\mathbf{L}^{p(x)}(\Omega)} + a_0 2^{-p^+}\|u_n - u\|^{p(x)}_{\mathbf{L}^{p(x)}(\Omega)} \\
&\quad - \lambda \int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx \\
&\geq \min\{2^{-p^+}, a_0 2^{-p^+}\}\|u_n - u\|^{p(x)}_{\mathbf{W}^{p(x)}(\Omega)} - \lambda \int_\Omega (f(x,u_n) - f(x,u))(u_n - u)dx.
\end{aligned}$$

The last of the above inequality is obtained by using (3.3). Combining the last relation with Proposition 2.1 (iii), we find that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges strongly to u in $\mathbf{W}^{p(x)}(\Omega)$. Therefore, $I_{\lambda,\mu}$ satisfies the (PS)-condition and so all hypotheses of [9, Theorem 3.2], are verified. Hence, for each $\lambda \in \Lambda_r$ the function I_λ admits at least two distinct critical points that are solutions of the problem (1.1). □

In our final result, we discuss the existence of at least three solutions to the problem (1.1).

Theorem 3.3. *Let $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ be a Carathédory function, and let (H_2), (H_3) in Theorem 3.1 hold. Moreover, assume that there exist two positive constants τ and δ, such that*

(H_5) $a_0 c_0^{p^-} \operatorname{meas}(\Omega)\min\{\delta^{p^-}, \delta^{p^+}\} > \tau^{p^-}\min\{1, a_0\}$,

(H_6) *there exist constants $c > 0$, $q \in C(\bar{\Omega})$ and $1 < q(x) \leq q^+ < p^-$ in $\bar{\Omega}$ such that*

$$|F(x,t)| \leq c(1 + |t|^{q(x)}) \quad \forall (x,t) \in \Omega \times \mathbb{R}^3.$$

Then for every $\lambda \in \Lambda_w$ as in (3.1), the problem (1.1) admits at least three distinct solutions.

Proof. Our aim is to apply [11, Theorem 3.6]. We consider the functionals Φ and Ψ, defined in (2.2) and (2.3). Once again, they satisfy the regularity assumptions needed in [11, Theorem 3.6]. Now, we argue as in the proof of Theorem 3.1 with $w(x)$ defined in (3.4), and

$$r = \frac{\min\{1, a_0\}}{p^+}\left(\frac{\tau}{c_0}\right)^{p^-}.$$

In view of (H_5) we have $\Phi(w) > r > 0$. Therefore, from (H_2), inequality (3.6) holds, and so

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$$

holds for some $\bar{v} \in \mathbf{W}^{p(x)}(\Omega)$.

Now, we prove that, for each $\lambda \in \Lambda_w$ the functional I_λ is coercive. By using inequality (3.2), conditions (H_6), and Sobolev embedding theorem, we easily obtain for all $u \in \mathbf{W}^{p(x)}(\Omega)$:

$$\begin{aligned}
I_\lambda(u) &\geq \frac{\min\{1, a_0\}}{p^+}\|u\|^{p^-}_{\mathbf{W}^{p(x)}(\Omega)} - \lambda \int_\Omega F(x,u)dx \\
&\geq \frac{\min\{1, a_0\}}{p^+}\|u\|^{p^-}_{\mathbf{W}^{p(x)}(\Omega)} - \lambda c\|u\|^{q^+}_{\mathbf{W}^{p(x)}(\Omega)} - \lambda c \operatorname{meas}(\Omega).
\end{aligned}$$

Since $p^- > q^+$ we see that $I_\lambda \to +\infty$ as $\|u\|_{\mathbf{W}^{p(x)}(\Omega)} \to +\infty$, so the functional I_λ is coercive. Thus, for each $\lambda \in \Lambda_w$, [11, Theorem 3.6] implies that the functional I_λ admits at least three critical points in $\mathbf{W}^{p(x)}(\Omega)$ that are solutions of the problem (1.1). □

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Normalized solutions for a fractional coupled critical Hartree system

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Abstract. We consider the existence of normalized solutions for a fractional coupled Hartree system, with the upper critical exponent in the sense of the Hardy–Littelwood–Sobolev inequality. Particularly, in an L^2-subcritical regime or an L^2-supercritical regime, we establish the existence of positive normalized solutions for the two cases, respectively. Furthermore, we prove the nonexistence of positive normalized solutions, under the nonlinearities satisfying the Sobolev critical growth.

Keywords: fractional Hartree system, normalized solutions, Hardy–Littlewood–Sobolev critical exponent.

2020 Mathematics Subject Classification: 35J50, 35B33, 35R11, 58E05.

1 Introduction

This paper is concerned with the existence of solutions $(\lambda_1, \lambda_2, u, v) \in \mathbb{R}^2 \times H^s(\mathbb{R}^N, \mathbb{R}^2)$ to the following fractional critical Hartree system:

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2} u + \beta r_1 |u|^{r_1-2} u |v|^{r_2} + (I_\alpha * |v|^{2^*_{\alpha,s}}) |u|^{2^*_{\alpha,s}-2} u, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{p-2} v + \beta r_2 |v|^{r_2-2} v |u|^{r_1} + (I_\alpha * |u|^{2^*_{\alpha,s}}) |v|^{2^*_{\alpha,s}-2} v, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

satisfying the additional conditions

$$\int_{\mathbb{R}^N} u^2 dx = a^2, \quad \text{and} \quad \int_{\mathbb{R}^N} v^2 dx = b^2. \tag{1.2}$$

The masses $a, b > 0$ are prescribed and the parameters $\mu_1, \mu_2, \beta > 0$. Here $(-\Delta)^s$ is the fractional Laplacian, $s \in (0,1)$, $2s < N \leq 4s$, $\alpha \in (0, N)$, $2^*_{\alpha,s} = \frac{2N-\alpha}{N-2s}$ is the upper critical exponent due to the Hardy–Littlewood–Sobolev inequality, $2^*_s = \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent, $r_1, r_2 > 1$, $p, r_1 + r_2 \in (2, 2^*_s]$ with $p < r_1 + r_2$ and $*$ stands for the convolution on $\mathbb{R}^N$ with $I_\alpha : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is the Riesz potential,

$$I_\alpha(x) = \frac{A_{N,\alpha}}{|x|^\alpha}, \quad \text{with } A_{N,\alpha} = \frac{\Gamma(\frac{\alpha}{2})}{2^{N-\alpha} \pi^{\frac{N}{2}} \Gamma(\frac{N-\alpha}{2})}.$$

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The fractional Laplacian operator $(-\Delta)^s$ is defined for any $u : \mathbb{R}^N \to \mathbb{R}$ sufficiently smooth by

$$(-\Delta)^s u(x) = C(N,s)P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

where $P.V.$ stands for the Cauchy principal value and $C(N,s)$ is a positive constant depending only on N and s. Recently, a great attention has been devoted to study the nonlinear problems involving fractional elliptic operators, both for the pure mathematical research and applications. We refer to [3,11,16,37] for a simple introduction to basic properties of the fractional Laplacian operator and concrete applications based on variational methods. Moreover, fractional Choquard type equation with critical growth has been studied by many researchers, see [1,22,23,35,36] and references therein.

The problem under investigation comes from the research of solitary waves for the following physical model:

$$\begin{cases} (-\Delta)^s\phi_1 = -i\frac{\partial\phi_1}{\partial t} + \mu_1|\phi_1|^{p-2}\phi_1 + \beta r_1|\phi_1|^{r_1-2}\phi_1|\phi_2|^{r_2} + (I_\alpha * |\phi_2|^{2^*_{\alpha,s}})|\phi_1|^{2^*_{\alpha,s}-2}\phi_1, \\ (-\Delta)^s\phi_2 = -i\frac{\partial\phi_2}{\partial t} + \mu_2|\phi_2|^{p-2}\phi_2 + \beta r_2|\phi_2|^{r_2-2}\phi_2|\phi_1|^{r_1} + (I_\alpha * |\phi_1|^{2^*_{\alpha,s}})|\phi_2|^{2^*_{\alpha,s}-2}\phi_2, \end{cases} \tag{1.3}$$

where $i^2 = -1$ and $\phi_j (j = 1,2)$ is the wave function of the j_{th} component, and μ_j, β denote the intra-species and intra-species scattering lengths. In particular, the interaction of states is attractive if $\beta > 0$, while the interaction of states is repulsive when $\beta < 0$. Solitary wave solutions of system (1.3) are solutions having the form

$$\phi_1(x,t) = e^{i\lambda_1 t}u(x), \qquad \phi_2(x,t) = e^{i\lambda_1 t}v(x),$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ are the chemical potentials and (u,v) solves (1.1). Since $\phi_1(x,t), \phi_2(x,t)$ retain their masses over time, we consider this problem from two aspects: one can either regard the frequencies λ_1, λ_2 as fixed, or include them in the unknown and prescribe the masses.

Fixing the parameters λ_1, λ_2 in (1.1), we call it the fixed frequency problem. The two-component system with Hartree-type nonlinearities describes the boson stars in mean-field theory [18, 27], which appears naturally in optical systems [30] and is known to influence the propagation of electromagnetic waves in plasmas [7]. Moreover, the non-locality of the critical term also plays an important role in the theory of Bose-Einstein condensation, where it accounts for the finite-range many-body interaction [15]. The Hartree type systems, mainly on λ_1, λ_2 are prescribed, have been widely studied. We refer to [20] and references therein. However, much less is known when the masses are prior prescribed. In this case, $\lambda_1, \lambda_2 \in \mathbb{R}$ are unknown quantities arising as Lagrange multipliers. In recent years, since physicists are interested in normalized solutions (which L^2-norms of solutions are prescribed), mathematical researchers began to investigate the solutions of various classes of Schrödinger equations or systems having a prescribed L^2-norm, that is a solution which satisfies $\int_{\mathbb{R}^N} |u|^2 dx = c$ for a priori given c.

When $s = 1$, i.e. the fractional Laplace operator $(-\Delta)^s$ reduces to the local differential operator $-\Delta$, the literature for the normalized solutions of Schrödinger equations or systems is abundant. Starting from the seminal paper by Jeanjean in [25], he firstly studied L^2-supercritical case, and dealt with the existence of normalized solutions when the energy functional is unbounded from below, by using the mountain pass lemma and a skillful compactness argument. Furthermore, for the particular case of a combined nonlinearity of power type, in [38], Soave considered the existence of normalized solutions and orbitally stable for the following problem:

$$\begin{cases} -\Delta u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

where $N \geq 1$, $q, p \in (2, 2^*)$ and $q < p$. Moreover, when $p = 2^*$ in (1.4), in [39], the Sobolev critical case was studied by Soave, where he considered the energy level less than a certain number to get the compactness, and obtained the existence and nonexistence of normalized solutions. For the system case, Bartsch, Jeanjean and Soave investigated the following elliptic system

$$\begin{cases} -\Delta u = \lambda_1 u + \mu_1 u^3 + \beta u v^2, & \text{in } \mathbb{R}^3, \\ -\Delta v = \lambda_2 v + \mu_2 u^3 + \beta v u^2, & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = a, \int_{\mathbb{R}^3} |v|^2 dx = b, \end{cases} \tag{1.5}$$

where $\mu_1, \mu_2, a, b > 0$. In [4], Bartsch, Jeanjean and Soave obtained the existence results for different ranges of $\beta > 0$ and stability properties of (1.5). Furthermore, in [6], Bartsch and Soave considered the case $\beta < 0$ of (1.5) and showed phase separation occurs for the solutions as $\beta \to -\infty$. In particular, Bartsch, Li and Zou [5] studied the normalized solutions for a Schrödinger systems with Sobolev critical nonlinearities. Specifically, in [5], they proved the existence and nonexistence results and obtained the asymptotic behavior as $\beta \to 0^+$ or $\beta \to +\infty$. When $3 \leq N \leq 4$, in [29], Li and Zou obtained the existence of positive normalized ground state for (1.5). For more researches of the normalized solutions of the Laplacian systems, we refer to [31, 34] and references therein.

The situation is different when $s \in (0, 1)$, and few results are available. We note that the L^2-critical exponent for fractional case is $\bar{p} := 2 + \frac{4s}{N}$. In [32], Luo and Zhang studied the existence and nonexistence of normalized solutions for the following fractional problem

$$\begin{cases} (-\Delta)^s u = \lambda u + \mu |u|^{q-2}u + |u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

where $q, p \in (2, 2_s^*)$, $q < p$ and $\mu \in \mathbb{R}$. Moreover, when $p = 2_s^*$ in (1.6), Zhen and Zhang [44] proved the existence and nonexistence results of the normalized solutions by using the Jeanjean's skill in [25], and they also considered the behavior of the ground state obtained as $\mu \to 0^+$. Furthermore, in [24], He, Rădulescu and Zou showed the existence and nonexistence of solutions for a fractional equation with the upper critical exponent, among 3 cases: L^2-subcritical, L^2-critical and L^2-supercritical. In the case of fractional systems, Zuo and Rădulescu studied the following problem

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2}u + |u|^{2_s^*-2}u + \gamma\alpha |u|^{\alpha-2}u|v|^{\beta}, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{q-2}v + |v|^{2_s^*-2}v + \gamma\beta |v|^{\alpha-2}v|u|^{\beta}, & \text{in } \mathbb{R}^N. \\ \int_{\mathbb{R}^N} |u|^2 dx = a, \int_{\mathbb{R}^N} |v|^2 dx = b, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $s \in (0, 1)$, $p, q, \alpha + \beta \in (\bar{p}, 2_s^*)$. In [45], Zuo and Rădulescu showed the existence of positive normalized solutions when γ is big enough, and obtained the nonexistence of positive normalized solutions if $p = q = \alpha + \beta = 2_s^*$. Li [28] studied the existence of positive radial solutions for a fractional Hartree–Fock type system in L^2-subcritical case, L^2-critical case and L^2-supercritical case, but without the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality (see Lemma 2.1).

Inspired by the above mentioned works, in the present paper, our goal is two-fold. On one hand, we show the existence of normalized ground states for $p \in (2, 2_s^*)$, and $r_1 + r_2 \in (p, 2_s^*)$; on the other hand, we obtain the nonexistence result for $p = r_1 + r_2 = 2_s^*$. Compared to the Laplace operator, the fractional Laplacian problems are nonlocal and more challenging. Moreover, since the compactness of system is closely related to (see Proposition 4.9) the following problem

$$\begin{cases} (-\Delta)^s u = \lambda u + \mu |u|^{p-2} u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = c^2, \end{cases} \tag{1.8}$$

we may be more careful to the energy level and solutions of (1.8). However, for $p = \bar{p}$ and $c > 0$, the Pohožaev manifold related to (1.8) is indefinite (see Lemma 5.2), which makes it difficult to construct the geometry for the related energy functional.

Before we state our main results, we introduce some notations for the fractional Sobolev space $H^s(\mathbb{R}^N)$. Let $s \in (0,1)$. We denote by $D^s(\mathbb{R}^N)$ the completion of $C_c^\infty(\mathbb{R}^N)$ with

$$[u]^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy.$$

The fractional Sobolev space is defined by

$$H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \ [u] < \infty\},$$

with the standard norm and inner product

$$\|u\|^2 = [u]^2 + \int_{\mathbb{R}^N} |u|^2 dx, \quad \text{and} \quad \langle u, \varphi \rangle = \int_{\mathbb{R}^N} \Big((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi + u\varphi \Big) dx.$$

It is well known (see [2]) that the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for all $q \in [2, 2_s^*]$, locally compact for all $q \in [1, 2_s^*)$ and $D^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is continuous. Then we define the working space H as

$$H := \{(u,v) : u \in H^s(\mathbb{R}^N), v \in H^s(\mathbb{R}^N)\},$$

endowed with the norm

$$\|(u,v)\|_H^2 := \|u\|^2 + \|v\|^2,$$

and related inner product is, for any $(\varphi, \psi) \in H$:

$$\langle (u,v), (\varphi, \psi) \rangle_H := \langle u, \varphi \rangle + \langle v, \psi \rangle.$$

By using the variational methods, a classical way for studying the normalized solutions of system (1.1) is to look for critical points of the following C^1-functional

$$J(u,v) = \frac{1}{2}([u]^2 + [v]^2) - \frac{1}{p}(\mu_1 |u|_p^p + \mu_2 |v|_p^p) - \beta \int_{\mathbb{R}^N} |u|^{r_1} |v|^{r_2} dx - \frac{1}{2_{\alpha,s}^*} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2_{\alpha,s}^*}) |v|^{2_{\alpha,s}^*} dx,$$

constrained on the set

$$S := \{(u,v) \in H : (u,v) \in S_a \times S_b\},$$

where $|u|_r = \big(\int_{\mathbb{R}^N} |u|^r dx \big)^{\frac{1}{r}}$ and $S_a := \{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = a^2\}$. The main results of this paper can be stated as follows:

Theorem 1.1. *When $2s < N \leq 4s$, $p \in (2, \bar{p})$ and $r_1 + r_2 \in (p, 2_s^*)$, there exists $\beta^* > 0$ such that for $0 < \beta < \beta^*$, there exist $\mu_1^* = \mu_1^*(\beta), \mu_2^* = \mu_2^*(\beta)$, such that for any $\mu_1 \in (0, \mu_1^*), \mu_2 \in (0, \mu_2^*)$, (1.1)–(1.2) has a normalized ground state (u, v), which is a positive and radially symmetric function, for some $\lambda_1, \lambda_2 < 0$. Moreover, (u, v) is an interior local minimizer on the set*

$$B_r(a, b) := \{(u, v) \in S : ([u]^2 + [v]^2)^{\frac{1}{2}} < r\},$$

for a suitable $r > 0$ small enough; and any other ground state solution of $J(u, v)|_S$ is a local minimizer of $J(u, v)$ on $B_r(a, b)$.

Theorem 1.2. *When $2s < N \leq 4s$, $p \in (\bar{p}, 2_s^*)$ and $r_1 + r_2 \in (p, 2_s^*)$, there exists $\beta_0 > 0$, such that for any $\beta > \beta_0$, (1.1)–(1.2) has a normalized ground state (u, v), which is a positive and radially symmetric function, for some $\lambda_1, \lambda_2 < 0$, and (u, v) is a Mountain Pass type solution.*

Theorem 1.3. *When $2s < N \leq 4s$, suppose $p = r_1 + r_2 = 2_s^*$, then the system (1.1)–(1.2) has no positive normalized solutions.*

Remark 1.4.

(I) In Theorem 1.1, we consider 3 cases: $r_1 + r_2 \in (2, \bar{p})$, $r_1 + r_2 = \bar{p}$ and $r_1 + r_2 \in (\bar{p}, 2_s^*)$. These different situations are mainly reflected in Lemmas 4.3 and 5.3.

(II) From the processes in our proof, one difference between Theorems 1.1 and 1.2 lies in their respective geometric structures. In fact, when p changes from L^2-subcritical to L^2-supercritical, it changes the geometry of $J(u, v)|_S$ and prevents the existence of a local minimizer in Theorem 1.2.

(III) Compared with the result in [24], we need an elementary inequality (see Proposition 4.9), which combined the single case (1.8) with the coupling case (1.1), to ensure compactness result. Theorems 1.1, 1.2, 1.3 seem to be the first results of normalized solutions for a fractional coupling systems with the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality.

The paper is organized as follows. In Section 2, we give some preliminaries for the functional space. In Section 3, we will briefly introduce the properties of a single case (1.8), which plays an important role to the proof of Palais–Smale condition in our problem. In Section 4, we prove Theorem 1.1. In Section 5, we obtain Theorem 1.2. At last, we show the nonexistence for Theorem 1.3 in Section 6.

2 Preliminaries

Following, for the convenience of the reader, we recall some basic properties, which we shall need in the sequel. Let us first recall the well-known Hardy–Littlewood–Sobolev inequality.

Lemma 2.1 ([30])**.** *Let $t, r > 1$, $0 < \alpha < N$, with $\frac{1}{t} + \frac{\alpha}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(N, t, \alpha, r)$ independent of f and h, such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x - y|^\alpha} \leq C(N, t, \alpha, r)|f|_t |h|_r, \tag{2.1}$$

where $|\cdot|_q$ *stands for the* $L^q(\mathbb{R}^N)$*-norm for* $q \in [1, +\infty)$. *If* $t = r = \frac{2N}{2N-\alpha}$, *then*

$$C(N, t, \alpha, r) = C(N, \alpha) = \pi^{\frac{\alpha}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\alpha}{2})}{\Gamma(N - \frac{\alpha}{2})}.$$

Besides, there is a equality in (2.1) *if and only if* $f \equiv (constant.)h$ *and*

$$h(x) = C(\gamma^2 + |x - a|^2)^{-\frac{2N-\alpha}{2}},$$

for some $C \in \mathbb{C}$, $\gamma \neq 0$ *and* $a \in \mathbb{R}^N$.

According to Lemma 2.1, the functional

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^p(x) u^p(y)}{|x - y|^\alpha} dy dx,$$

is well defined in $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ if $\frac{2N-\alpha}{N} \leq p \leq \frac{2N-\alpha}{N-2s}$. We often call $\frac{2N-\alpha}{N}$ is the lower Hardy–Littlewood–Sobolev critical exponent and $\frac{2N-\alpha}{N-2s}$ is the upper Hardy–Littlewood–Sobolev critical exponent. From Lemma 2.1, we define the best constant

$$S_{h,l} = \inf_{D^s(\mathbb{R}^N \setminus \{0\})} \frac{[u]^2}{\left(\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}}) |u|^{2^*_{\alpha,s}} dx\right)^{\frac{1}{2^*_{\alpha,s}}}},$$

and from [23], we know $S_{h,l}$ is attained by the function

$$\tilde{u}_{\varepsilon,y} = \tilde{C}_{N,\alpha,s} u_{\varepsilon,y}, \quad x, y \in \mathbb{R}^N, \quad \text{and } \varepsilon > 0,$$

such that

$$[\tilde{u}_{\varepsilon,y}]^2 = S_{h,l}^{\frac{2N-\alpha}{N-\alpha+2s}},$$

with $\tilde{u}_{\varepsilon,y}$ satisfying this equation

$$(-\Delta)^s u = (I_\alpha * |u|^{2^*_{\alpha,s}}) |u|^{2^*_{\alpha,s}-2} u, \quad x \in \mathbb{R}^N.$$

The function $u_{\varepsilon,y} = \kappa(\varepsilon^2 + |x - y|^2)^{-\frac{N-2s}{2}}$ solves

$$(-\Delta)^s u = |u|^{2^*_s - 2} u, \quad \text{in } \mathbb{R}^N,$$

and achieves the infimum of

$$S := \inf_{D^s(\mathbb{R}^N \setminus \{0\})} \frac{[u]^2}{|u|^2_{2^*_s}},$$

with

$$S_{h,l} = S C_{N,\alpha,s}^{-\frac{1}{2^*_{\alpha,s}}} \quad \text{and} \quad \kappa = \left(\frac{S^{\frac{N}{2s}} \Gamma(N)}{\pi^{\frac{N}{2}} \Gamma(\frac{N}{2})} \right)^{\frac{N-2s}{2N}}.$$

In order to prove our problem, we shall make use of the following infimum

$$S^* := \inf_{(u,v) \in D^s(\mathbb{R}^N) \times D^s(\mathbb{R}^N)} \frac{[u]^2 + [v]^2}{\left(\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}}) |v|^{2^*_{\alpha,s}} dx \right)^{\frac{1}{2^*_{\alpha,s}}}}, \tag{2.2}$$

and from [43, Lemma 2.2], we know

Lemma 2.2. *We have*

$$S^* = 2S_{h,l},$$

and S^* is achieved if and only if, for $C > 0$,

$$u = v = Cu_{\varepsilon,y}.$$

Then we recall the fractional Gagliardo–Nirenberg–Sobolev inequality, which can be seen in [19].

Lemma 2.3. *Let $N > 2s$ and $p \in (2, 2_s^*)$, then there exists a constant $C(N,p,s) > 0$, such that for all $u \in H^s(\mathbb{R}^N)$,*

$$|u|_p^p \le C(N,p,s)|(-\Delta)^{\frac{s}{2}}u|_2^{\frac{N(p-2)}{2s}}|u|_2^{p-\frac{N(p-2)}{2s}}. \tag{2.3}$$

Defining $\gamma_p := \frac{N(p-2)}{2ps}$, it is easy to see

$$p\gamma_p \begin{cases} < 2, & \text{if } 2 < p < \bar{p}, \\ = 2, & \text{if } p = \bar{p}, \text{ and } \gamma_{2_s^*} = 1, \\ > 2, & \text{if } \bar{p} < p < 2_s^*. \end{cases}$$

and

$$|u|_p^p \le C(N,p,s)|(-\Delta)^{\frac{s}{2}}u|_2^{p\gamma_p}|u|_2^{p(1-\gamma_p)}. \tag{2.4}$$

Following, we obtain the corresponding Pohožaev type identity for system (1.1). Before the statement of this result, we introduce the s-harmonic extension (see [11]) techniques. Denote $\mathbb{R}^{N+1} = \{(x,y) : x \in \mathbb{R}^N, y \in \mathbb{R}\}$ and define $X = X^s(\mathbb{R}_+^{N+1}) \times X^s(\mathbb{R}_+^{N+1})$ under the norms

$$\|(U,V)\|_X = \left(\kappa_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla U|^2 dxdy + \kappa_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla V|^2 dxdy\right)^{\frac{1}{2}},$$

where $X^s(\mathbb{R}_+^{N+1})$ is the completion of $C_0^\infty(\mathbb{R}_+^{N+1})$ with the norm

$$\|U\|_{X^s(\mathbb{R}_+^{N+1})} = \left(\kappa_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s}|\nabla U|^2 dxdy\right)^{\frac{1}{2}}.$$

Let $(u,v) \in H$ be a solution of (1.1) and define $(U,V) \in X$ be its s-harmonic extension to the upper half space $\mathbb{R}_+^{N+1}$, then $u = U(x,0)$, $v = V(x,0)$ and (U,V) is a solution to the following problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla U) = 0; -\mathrm{div}(y^{1-2s}\nabla V) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ -\frac{\partial U}{\partial y^{1-2s}} = \lambda_1 u + \mu_1|u|^{p-2}u + \beta r_1|u|^{r_1-2}u|v|^{r_2} + (I_\alpha * |v|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^*-2}u, & \text{on } \mathbb{R}^N, \\ -\frac{\partial V}{\partial y^{1-2s}} = \lambda_2 v + \mu_2|v|^{p-2}v + \beta r_2|v|^{r_2-2}v|u|^{r_1} + (I_\alpha * |u|^{2_{\alpha,s}^*})|v|^{2_{\alpha,s}^*-2}v, & \text{on } \mathbb{R}^N. \end{cases} \tag{2.5}$$

From [8, Proposition A.1] and [42, Lemma 4.1], we have the following result.

Proposition 2.4. *Let $(u,v) \in H$ be a weak solution of (1.1), that is (u,v) satisfies:*

$$\begin{aligned} 0 = \langle u, \varphi\rangle + \langle v, \psi\rangle - \int_{\mathbb{R}^N}(\lambda_1 u\varphi + \lambda_2 v\psi)dx \\ - \int_{\mathbb{R}^N}(\mu_1|u|^{p-2}u\varphi + \mu_2|v|^{p-2}v\varphi)dx - \beta\int_{\mathbb{R}^N}(r_1|u|^{r_1-2}u|v|^{r_2}\varphi + r_2|u|^{r_1}|v|^{r_2-2}v\psi)dx \\ - \int_{\mathbb{R}^N}[(I_\alpha * |v|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^*-2}u\varphi + (I_\alpha * |u|^{2_{\alpha,s}^*})|v|^{2_{\alpha,s}^*-2}v\psi]dx, \quad \forall(\varphi,\psi) \in H, \end{aligned}$$

then we have (u,v) *satisfies*

$$\begin{aligned}&\frac{N-2s}{2}\int_{\mathbb{R}^N}(|(-\Delta)^{\frac{s}{2}}u|^2+|(-\Delta)^{\frac{s}{2}}v|^2)dx\\&=\frac{N}{2}\int_{\mathbb{R}^N}(\lambda_1|u|^2+\lambda_2|v|^2)dx+\frac{N}{p}\int_{\mathbb{R}^N}(\mu_1|u|^p+\mu_2|v|^p)dx\\&\quad+\beta N\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx+\frac{2N-\alpha}{2^*_{\alpha,s}}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx.\end{aligned}$$

Proof. If $(u,v)\in H$ is a weak solution of (1.1), from [2, Proposition 3.2.14], we have $(u,v)\in L^\infty(\mathbb{R}^N)\times L^\infty(\mathbb{R}^N)$. Using the same arguments as in [13, Proposition 4.1], we get $(u,v)\in C^{2,\tau}(\mathbb{R}^N)\times C^{2,\tau}(\mathbb{R}^N)$ with τ depending on s. Let (U,V) be its s-harmonic extension and satisfy (2.5), then $(U,V)\in C^2(\mathbb{R}^{N+1}_+)\times C^2(\mathbb{R}^{N+1}_+)$.

Set $D_m:=\{(x,y)\in\mathbb{R}^{N+1}:|(x,y)|\le m\}$ and $Q_r=D_r^+\cup(D_r\cap(\mathbb{R}^N\times\{0\}))$, where $D_r^+=D_r\cap\mathbb{R}^{N+1}_+$. Let $\varphi\in C_0^\infty(\mathbb{R}^{N+1})$ with $0\le\varphi\le1$, $\varphi=1$ in D_1, $\varphi=0$ outside D_2 and $|\nabla\varphi|\le2$. For $R>0$, define

$$\psi_R(x,y)=\psi\left(\frac{(x,y)}{R}\right),\quad \text{where } \psi=\varphi|_{\overline{\mathbb{R}^{N+1}_+}}.$$

Multiplying (2.5) by $((x,y)\cdot\nabla U)\psi_R$ and $((x,y)\cdot\nabla V)\psi_R$ respectively, we obtain from [8, Proposition A.1],

$$\begin{aligned}\lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}\times\{0\})}|u|^{p-2}u\cdot(x,y)\cdot\nabla U\psi_R dx&=-\frac{N}{p}\int_{\mathbb{R}^N}|u|^pdx.\\\lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}\times\{0\})}u\cdot(x,y)\cdot\nabla U\psi_R dx&=-\frac{N}{2}\int_{\mathbb{R}^N}|u|^2dx.\end{aligned}$$

and

$$\begin{aligned}\lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}^N\times\{0\})}|v|^{p-2}v\cdot(x,y)\cdot\nabla V\psi_R dx&=-\frac{N}{p}\int_{\mathbb{R}^N}|v|^pdx.\\\lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}^N\times\{0\})}v\cdot(x,y)\cdot\nabla V\psi_R dx&=-\frac{N}{2}\int_{\mathbb{R}^N}|v|^2dx.\end{aligned}$$

Moreover,

$$\begin{aligned}&\lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}^N\times\{0\})}\left(r_1|v|^{r_2}|u|^{r_1-2}u\cdot(x,y)\cdot\nabla U\psi_R+r_2|u|^{r_1}|v|^{r_2-2}v\cdot(x,y)\cdot\nabla V\psi_R\right)dx\\&=-N\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx.\end{aligned}$$

and

$$\begin{aligned}\lim_{R\to\infty}\int_{Q_{2R}}y^{1-2s}\nabla U\nabla[((x,y)\cdot\nabla U)\psi_R]dxdy&=-\frac{N-2s}{2}\int_{\mathbb{R}^{N+1}_+}y^{1-2s}|\nabla U|^2dxdy,\\\lim_{R\to\infty}\int_{Q_{2R}}y^{1-2s}\nabla V\nabla[((x,y)\cdot\nabla V)\psi_R]dxdy&=-\frac{N-2s}{2}\int_{\mathbb{R}^{N+1}_+}y^{1-2s}|\nabla V|^2dxdy.\end{aligned}$$

Furthermore, combining with [42, Lemma 4.1], we have

$$\begin{aligned}&\lim_{R\to\infty}\int_{D_{2R}\cap(\mathbb{R}^N\times\{0\})}\big((I_\alpha*|v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}\cdot(x,y)\cdot\nabla U\psi_R\\&\qquad+(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}\cdot(x,y)\cdot\nabla V\psi_R\big)dx\\&=\frac{\alpha-2N}{2^*_{\alpha,s}}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx.\end{aligned}$$

Multiplying (2.5) by $U\psi_R$ and $V\psi_R$ respectively, and using the same techniques of [8, Proposition A.1], we firstly obtain

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla U|^2 dxdy = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx,$$
$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla V|^2 dxdy = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx,$$

and then we finish this proof. □

Lemma 2.5. *Let $(u,v) \in H$ be a weak solution of (1.1), then we have Pohožaev manifold*

$$\mathcal{P}_{\mu_1,\mu_2} = \{(u,v) \in S : P_{\mu_1,\mu_2}(u,v) = 0\},$$

where

$$\begin{aligned} P_{\mu_1,\mu_2}(u,v) = s([u]^2 + [v]^2) - s\gamma_p(\mu_1|u|_p^p + \mu_2|v|_p^p) - s\beta(r_1+r_2)\gamma_{(r_1+r_2)} \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx \\ - 2s\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx. \end{aligned} \tag{2.6}$$

Proof. Since Proposition 2.4, we have (u,v) satisfies

$$\begin{aligned} &\frac{N-2s}{2}\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) dx \\ &\quad = \frac{N}{2}\int_{\mathbb{R}^N} (\lambda_1|u|^2 + \lambda_2|v|^2) dx + \frac{N}{p}\int_{\mathbb{R}^N} (\mu_1|u|^p + \mu_2|v|^p) dx \\ &\quad\quad + \beta N \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx + \frac{2N-\alpha}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) dx = &\int_{\mathbb{R}^N} (\lambda_1|u|^2 + \lambda_2|v|^2) dx + \int_{\mathbb{R}^N} (\mu_1|u|^p + \mu_2|v|^p) dx \\ &+ \beta(r_1+r_2)\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx + 2\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx. \end{aligned}$$

Thus,

$$\begin{aligned} &s\int_{\mathbb{R}^N} (|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2) dx \\ &\quad = s\gamma_p \int_{\mathbb{R}^N} (\mu_1|u|^p + \mu_2|v|^p) dx + \beta s(r_1+r_2)\gamma_{(r_1+r_2)} \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx \\ &\quad\quad + 2s\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx, \end{aligned}$$

and the conclusions follows. □

Under the L^2-invariant scaling introduced by Jeanjean in [25],

$$t * u := e^{\frac{Nt}{2}} u(e^t x), \quad \text{and} \quad t * (u,v) := (t * u, t * v),$$

it is natural to study the fiber maps

$$\begin{aligned} \Psi_{\mu_1,\mu_2}(t) := J(t*(u,v)) = &\frac{e^{2st}}{2}([u]^2 + [v]^2) - \frac{e^{sp\gamma_p t}}{p}(\mu_1|u|_p^p + \mu_2|v|_p^p) \\ &- \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)} t}\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx \\ &- \frac{e^{22^*_{\alpha,s} st}}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx, \end{aligned} \tag{2.7}$$

satisfying $\Psi'_{\mu_1,\mu_2}(t) = P_{\mu_1,\mu_2}(t * u, t * v)$, that is

$$\mathcal{P}_{\mu_1,\mu_2} = \{(u,v) \in S : \Psi'_{\mu_1,\mu_2}(0) = 0\}.$$

We decompose $\mathcal{P}_{\mu_1,\mu_2}$ into 3 disjoint unions $\mathcal{P}_{\mu_1,\mu_2} = \mathcal{P}^+_{\mu_1,\mu_2} \cup \mathcal{P}^0_{\mu_1,\mu_2} \cup \mathcal{P}^-_{\mu_1,\mu_2}$, defined by

$$\begin{aligned}\mathcal{P}^+_{\mu_1,\mu_2} &:= \{u \in \mathcal{P}_{\mu_1,\mu_2} : \Psi''_{\mu_1,\mu_2}(0) > 0\};\\ \mathcal{P}^0_{\mu_1,\mu_2} &:= \{u \in \mathcal{P}_{\mu_1,\mu_2} : \Psi''_{\mu_1,\mu_2}(0) = 0\};\\ \mathcal{P}^-_{\mu_1,\mu_2} &:= \{u \in \mathcal{P}_{\mu_1,\mu_2} : \Psi''_{\mu_1,\mu_2}(0) < 0\}.\end{aligned}$$

Set $m(a,b) = \inf_{\mathcal{P}_{\mu_1,\mu_2}} J(u,v)$ and $m^{\pm}(a,b) = \inf_{(u,v)\in\mathcal{P}^{\pm}_{\mu_1,\mu_2}} J(u,v)$, respectively. The main idea of this paper is to show whether $m(a,b)$ is achieved.

3 The relevant results

Before solving problem (1.1) and (1.2), we study the following problem:

$$\begin{cases}(-\Delta)^s u = \lambda u + \mu|u|^{p-2}u, & \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N} u^2 dx = c^2, & u \in H^s(\mathbb{R}^N),\end{cases} \tag{3.1}$$

where $\mu, c > 0$, $p \in (2, 2^*_s) \setminus \{\bar{p}\}$. The standard method obtaining the normalized solutions of (3.1) is to search for the critical points of

$$I_{\mu,c}(u) = \frac{1}{2}[u]^2 - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p dx,$$

constrained on $S_c := \{u \in H^s(\mathbb{R}^N) : |u|_2^2 = c^2\}$. By the same arguments as in Section 2, the Pohožaev identity related to (3.1) is

$$P_{\mu,c}(u) = s[u]^2 - \mu\gamma_p s|u|_p^p,$$

and the corresponding Pohožaev manifold is

$$\mathcal{P}_{\mu,c} := \{u \in S_c : [u]^2 = \mu\gamma_p|u|_p^p\}.$$

Moreover, we have

$$\Psi_{\mu,c}(t) := I_{\mu,c}(t * u) = \frac{e^{2st}}{2}[u]^2 - \frac{\mu e^{p\gamma_p st}}{p}\int_{\mathbb{R}^N} |u|^p dx,$$

and $\mathcal{P}_{\mu,c}$ can also be divided into 3 disjoint unions $\mathcal{P}_{\mu,c} = \mathcal{P}^+_{\mu,c} \cup \mathcal{P}^0_{\mu,c} \cup \mathcal{P}^-_{\mu,c}$, where

$$\begin{aligned}\mathcal{P}^+_{\mu,c} &:= \{u \in \mathcal{P}_{\mu,c} : \Psi''_{\mu,c}(0) > 0\};\\ \mathcal{P}^0_{\mu,c} &:= \{u \in \mathcal{P}_{\mu,c} : \Psi''_{\mu,c}(0) = 0\};\\ \mathcal{P}^-_{\mu,c} &:= \{u \in \mathcal{P}_{\mu,c} : \Psi''_{\mu,c}(0) < 0\}.\end{aligned}$$

Define $m_\mu(c) = \inf_{u\in\mathcal{P}_{\mu,c}} I_{\mu,c}(u)$ and let $m(a,0) = m_{\mu_1}(a)$, $m(0,b) = m_{\mu_2}(b)$. From Lemma 2.3, for any $u \in S_a$, there is $C_1 := C_1(N,p,a,s) > 0$, such that

$$\int_{\mathbb{R}^N} |u|^p dx \le C(N,p,s)|u|_2^{p(1-\gamma_p)}[u]^{p\gamma_p} = C_1[u]^{p\gamma_p} \le C_1([u]^2 + [v]^2)^{\frac{p\gamma_p}{2}}. \tag{3.2}$$

In particular, when $p < \bar{p}$, from (3.2) we get

$$I_{\mu_1,a}(u) = \frac{1}{2}[u]^2 - \frac{\mu_1}{p}\int_{\mathbb{R}^N} |u|^p dx \geq \frac{1}{2}[u]^2 - \frac{C_1\mu_1}{p}[u]^{p\gamma_p} =: h([u]),$$

where

$$h(\rho) := \frac{1}{2}\rho^2 - \frac{C_1\mu_1}{p}\rho^{p\gamma_p}. \tag{3.3}$$

Setting

$$\rho_* := (C_1\mu_1\gamma_p)^{\frac{1}{2-p\gamma_p}},$$

we have that $h(\rho_*) < 0$, $h(\rho)$ is strictly decreasing in $(0, \rho_*)$, and is strictly increasing in (ρ_*, ∞). If we denote $R_0 = (\frac{2C_1\mu_1}{p})^{\frac{1}{2-p\gamma_p}}$, then $h(R_0) = 0$ and $h(\rho) < 0$ iff $\rho \in (0, R_0)$.

From [44], we have the following already known results. For the mass subcritical case:

Theorem 3.1 ([44, Theorem 1.1]). *When $2s < N \leq 4s$, $p \in (2, \bar{p})$ and $\mu, c > 0$ in (3.1), there is $\hat{\mu} > 0$, for any $\mu \in (0, \hat{\mu})$, then $I_{\mu,c}|_{S_c}$ has a ground state solution u_μ for some $\lambda < 0$. Moreover,*

$$m_\mu(c) = \inf_{u\in S_c} I_{\mu,c}(u) = I_{\mu,c}(u_\mu) < 0,$$

and u_μ is an interior local minimizer of $I_{\mu,c}$ on the set

$$\hat{B}_{R_0} := \{u \in S_c : [u] < R_0\}.$$

Besides, any other normalized ground state solution is a minimizer of $I_{\mu,c}$ on B_{R_0}.

Remark 3.2. We set $\hat{\mu}_1, \hat{\mu}_2$ to obtain Theorem 3.1, under $\mu = \mu_1, c = a$ and $\mu = \mu_2, c = b$ in (3.1), respectively.

For the mass supercritical case:

Theorem 3.3 ([44, Theorem 1.3]). *When $2s < N \leq 4s$, $p \in (\bar{p}, 2_s^*)$ and $\mu, c > 0$ in (3.1), then $I_{\mu,c}|_{S_c}$ has a ground state solution u_μ for some $\lambda < 0$. Moreover u_μ is a critical point of Mountain Pass type and*

$$m_\mu(c) = \inf_{u\in S_c}\max_{t\in\mathbb{R}} I_{\mu,c}(t * u) = \max_{t\in\mathbb{R}} I_{\mu,c}(t * u_\mu) = I_{\mu,c}(u_\mu) > 0.$$

In order to proceed our proof, we also need the following monotonicity result which is essential for Lemmas 4.6 and 5.7.

Lemma 3.4. *$m_{\mu_1}(a)$ is non-increasing with respect to a, that is*

$$m_{\mu_1}(a) \leq m_{\mu_1}(a_1), \quad \textit{for any } 0 < a_1 \leq a.$$

Proof. We will prove for any $0 < a_1 \leq a$ and an arbitrary $\varepsilon > 0$,

$$m_{\mu_1}(a) \leq m_{\mu_1}(a_1) + \varepsilon.$$

We divide this proof into two cases.

Case 1: $2 < p < \bar{p}$. For this case, from the definition of R_0 in (3.3), we see R_0 is increasing as a is increasing. Hence, by Theorem 3.1 and $a_1 \leq a$, there exists a $\hat{R}_0$ with $\hat{R}_0 < R_0$, such that

$$m_{\mu_1}(a_1) = \inf_{u\in\hat{B}_{\hat{R}_0}} I_{\mu_1}(u).$$

Let $u \in \hat{B}_{\hat{R}_0} \subset \hat{B}_{R_0}$ be such that $I_{\mu_1}(u) \leq m_{\mu_1}(a_1) + \frac{\varepsilon}{2}$. Setting $\phi \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function satisfies $0 \leq \phi \leq 1$ and

$$\phi(x) = \begin{cases} 0, & \text{if } |x| \geq 2, \\ 1, & \text{if } |x| \leq 1. \end{cases}$$

For $\delta > 0$, defined $u_\delta(x) = u(x)\phi(\delta x)$, we get $u_\delta \to u$ in $H^s(\mathbb{R}^N)$ as $\delta \to 0$. Thus, for $\eta = \frac{\varepsilon}{6} > 0$, there exists $\delta > 0$ such that

$$I_{\mu_1}(u_\delta) \leq I_{\mu_1}(u) + \frac{\varepsilon}{6}, \quad \text{and} \quad [u_\delta] < R_0 - \frac{\eta}{R_0}. \tag{3.4}$$

Taking $\varphi \in C_0^\infty(\mathbb{R}^N)$ satisfies $\operatorname{supp}(\varphi) \subset O_{1+\frac{3}{\delta}}(0) \setminus O_{\frac{3}{\delta}}(0)$, where $O_m(n)$ means a ball in $\mathbb{R}^N$ with radius m and centered at n. Let

$$w(x) = \frac{(a^2 - |u_\delta|_2^2)^{\frac{1}{2}}}{|\varphi|_2}\varphi,$$

then for $t < 0$,

$$\operatorname{supp}(u_\delta) \cap \operatorname{supp}(t * w) = \emptyset.$$

Therefore, we get $u_\delta + t * w \in S_a$. Moreover, as $t \to -\infty$, we have

$$I_{\mu_1}(t * w) \leq \frac{\varepsilon}{6}, \quad \text{and} \quad [t * w] \leq \frac{\eta}{R_0}. \tag{3.5}$$

By the Hölder inequality, we obtain

$$\begin{aligned}
[u_\delta + t * w]^2 &= \iint_{\mathbb{R}^{2N}} \frac{|(u_\delta + t * w)(x) - (u_\delta + t * w)(y)|^2}{|x-y|^{N+2s}} dxdy \\
&= \iint_{\mathbb{R}^{2N}} \frac{|u_\delta(x) - u_\delta(y)|^2}{|x-y|^{N+2s}} dxdy + \iint_{\mathbb{R}^{2N}} \frac{|(t * w)(x) - (t * w)(y)|^2}{|x-y|^{N+2s}} dxdy \\
&\quad + 2\iint_{\mathbb{R}^{2N}} \frac{(u_\delta(x) - u_\delta(y))((t * w)(x) - (t * w)(y))}{|x-y|^{N+2s}} dxdy \\
&\leq [u_\delta]^2 + [t * w]^2 + 2[u_\delta][t * w] \\
&= ([u_\delta] + [t * w])^2,
\end{aligned}$$

then $[u_\delta + t * w] < R_0$. Now from Theorem 3.1, $m_{\mu_1}(a) = \inf_{u \in \hat{B}_{R_0}} I_{\mu_1}(u)$, by (3.4)–(3.5), we obtain

$$\begin{aligned}
m_{\mu_1}(a) &\leq I_{\mu_1}(u_\delta + t * w) \leq I_{\mu_1}(u_\delta) + I_{\mu_1}(t * w) + [u_\delta][t * w] \\
&\leq m_{\mu_1}(a_1) + \frac{\varepsilon}{2} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \leq m_{\mu_1}(a_1) + \varepsilon.
\end{aligned}$$

Case 2: $\bar{p} < p < 2_s^*$. In this case, $p\gamma_p > 2$, and by the definition of $m_{\mu_1}(a_1)$, there exists $u \in \mathcal{P}_{\mu_1, a_1}$, such that

$$I_{\mu_1}(u) \leq m_{\mu_1}(a_1) + \frac{\varepsilon}{2}.$$

From Theorem 3.3, we have u is bounded in $H^s(\mathbb{R}^N)$ and

$$[u]^2 = \mu_1 \gamma_p |u|_p^p.$$

Since (3.2) and $a_1 \leq a$, we get

$$[u] \geq \left(\frac{1}{\mu_1 \gamma_p C(N,p,s) a_1^{p(1-\gamma_p)}} \right)^{\frac{1}{p\gamma_p - 2}} \geq \left(\frac{1}{\mu_1 \gamma_p C(N,p,s) a^{p(1-\gamma_p)}} \right)^{\frac{1}{p\gamma_p - 2}}.$$

Hence there are $\hat{C}, \tilde{C} > 0$, which are independent with a_1, such that $[u] \geq \hat{C}$, and $|u|_p^p \geq \tilde{C}$. Later we may assume $\varepsilon < \tilde{C}$. Same definitions as in Case 1, we have $u_\delta \to u$ in $H^s(\mathbb{R}^N)$ as $\delta \to 0$ and from Theorem 3.3, $t_{u_\delta} * u_\delta \to t_u * u$ in $H^s(\mathbb{R}^N)$ as $\delta \to 0$, where t_u means strict maximum point of $\Psi_{\mu_1,a_1}(t)$ and the map $u \to t_u$ is of C^1 class. Then, for fixed $\delta > 0$, there exists $C > 0$ such that

$$I_{\mu_1}(t_{u_\delta} * u_\delta) \leq I_{\mu_1}(u) + \frac{\varepsilon}{6}, \quad [u_\delta] \leq C, \quad \text{and} \quad |u_\delta|_p^p \geq \tilde{C} - \frac{\varepsilon}{2}. \tag{3.6}$$

Choose $\psi \in C_0^\infty(\mathbb{R}^N)$ with $\text{supp}(\psi) \subset O_{1+\frac{3}{\delta}}(0) \setminus O_{\frac{3}{\delta}}(0)$, where $O_m(n)$ means a ball as defined on the previous page. Set

$$\kappa = \frac{(a^2 - |u_\delta|_2^2)^{\frac{1}{2}}}{|\psi|_2}\psi.$$

Then for $\tau < 0$, we have

$$\text{supp}(u_\delta) \cap \text{supp}(\tau * \kappa) = \emptyset.$$

Let $u_\tau := u_\delta + \tau * \kappa \in S_a$, and as $\tau \to -\infty$,

$$\begin{aligned}\int_{\mathbb{R}^N} |u_\tau|^p dx &= \int_{\mathbb{R}^N} |u_\delta|^p dx + \int_{\mathbb{R}^N} |\tau * \kappa|^p dx \\ &= \int_{\mathbb{R}^N} |u_\delta|^p dx + e^{p\gamma_p s\tau} \int_{\mathbb{R}^N} |\kappa|_p^p dx \to |u_\delta|_p^p.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}[u_\tau]^2 &\leq [u_\delta]^2 + [\tau * \kappa]^2 + 2[u_\delta][\tau * \kappa] \\ &= [u_\delta]^2 + e^{2s\tau}[\kappa]^2 + 2e^{s\tau}[u_\delta][\kappa] \to [u_\delta]^2.\end{aligned}$$

From Theorem 3.3, there exists t_τ such that $P_{\mu_1,a}(t_\tau * u_\tau) = 0$, i.e.

$$\frac{1}{e^{(p\gamma_p - 2)st_\tau}}[u_\tau]^2 = \gamma_p \mu_1 |u_\tau|_p^p.$$

Then as $\tau \to -\infty$,

$$e^{(p\gamma_p-2)st_\tau} = \frac{[u_\tau]^2}{\gamma_p \mu_1 |u_\tau|_p^p} \leq \frac{[u_\delta]^2}{\gamma_p \mu_1 |u_\delta|_p^p}.$$

Combining with (3.6), we get t_τ is bounded from above as $\tau \to -\infty$. Hence, for $\tau < -1$ sufficiently small, there exists $C^* > 0$ such that

$$[t_\tau * u_\delta] \leq C^*, \quad I_{\mu_1}((t_\tau + \tau) * \kappa) \leq \frac{\varepsilon}{6}, \quad \text{and} \quad [(t_\tau + \tau) * \kappa] < \frac{\varepsilon}{6C^*}. \tag{3.7}$$

Thus from (3.6) and (3.7), we obtain

$$\begin{aligned}m_{\mu_1}(a) &\leq I_{\mu_1}(t_\tau * u_\tau) \leq I_{\mu_1}(t_\tau * u_\delta) + I_{\mu_1}((t_\tau + \tau) * \kappa) + [t_\tau * u_\delta][(t_\tau + \tau) * \kappa] \\ &\leq I_{\mu_1}(t_{u_\delta} * u_\delta) + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \\ &\leq I_{\mu_1}(u) + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \leq m_{\mu_1}(a_1) + \varepsilon.\end{aligned}$$

Then, we complete this proof. □

4 The case: $2 < p < \bar{p}$, $p < (r_1 + r_2) < 2_s^*$

In this section, we consider the mixed exponent case. For any $(u, v) \in S$, from (2.2), the Hölder inequality and (2.4), there are $C_2 = C_2(N, p, b, s) > 0$, $C_3 = C_3(N, (r_1 + r_2), s, a, b)$ and $C_4 = (S^*)^{-2^*_{\alpha,s}}$, such that

$$\int_{\mathbb{R}^N} |v|^p dx \leq C(N, p, s)|v|_2^{p(1-\gamma_p)}[v]^{p\gamma_p} = C_2[v]^{p\gamma_p} \leq C_2([v]^2 + [u]^2)^{\frac{p\gamma_p}{2}}, \tag{4.1}$$

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}dx &\leq |u|^{r_1}_{(r_1+r_2)}|v|^{r_2}_{(r_1+r_2)} \\ &\leq C(N, (r_1 + r_2), s)|u|_2^{r_1(1-\gamma_{(r_1+r_2)})}[u]^{r_1\gamma_{(r_1+r_2)}}|v|_2^{r_2(1-\gamma_{(r_1+r_2)})}[v]^{r_2\gamma_{(r_1+r_2)}} \\ &\leq C_3([u]^2 + [v]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}},\end{aligned} \tag{4.2}$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}}dx \leq (S^*)^{-2^*_{\alpha,s}}([u]^2 + [v]^2)^{2^*_{\alpha,s}} = C_4([u]^2 + [v]^2)^{2^*_{\alpha,s}}. \tag{4.3}$$

Hence, substituting (3.2), (4.1)–(4.3) into $J(u, v)$, we obtain

$$\begin{aligned}J(u, v) \geq &\frac{1}{2}([u]^2 + [v]^2) - \frac{\mu_1 C_1 + \mu_2 C_2}{p}([u]^2 + [v]^2)^{\frac{p\gamma_p}{2}} - \beta C_3([u]^2 + [v]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}} \\ &- \frac{C_4}{2^*_{\alpha,s}}([u]^2 + [v]^2)^{2^*_{\alpha,s}}.\end{aligned} \tag{4.4}$$

Then we introduce the function $k : \mathbb{R}^+ \to \mathbb{R}$ by

$$k(t) := \frac{1}{2}t^2 - \frac{\mu_1 C_1 + \mu_2 C_2}{p}t^{p\gamma_p} - \beta C_3 t^{(r_1+r_2)\gamma_{(r_1+r_2)}} - \frac{C_4}{2^*_{\alpha,s}}t^{22^*_{\alpha,s}}, \tag{4.5}$$

and $k(0^+) = 0^-$, and $k(+\infty) = -\infty$.

Lemma 4.1. *There exists $\beta_* > 0$, such that for any $\beta \in (0, \beta_*)$, there exist $\mu_{1,*} = \mu_{1,*}(\beta) > 0$ and $\mu_{2,*} = \mu_{2,*}(\beta) > 0$, for any $\mu_1 \in (0, \mu_{1,*})$, $\mu_2 \in (0, \mu_{2,*})$, the function $k(t)$ has exactly two critical points, one is a local strict minimum at a negative level, and the other one is a global maximum at a positive level. Further, there exist $0 < R_2 < R_3$ such that $k(R_2) = k(R_3) = 0$, $k(t) > 0$ if and only if $t \in (R_2, R_3)$.*

Proof. Since the monotonicity of $k(t)$ will be strongly affected by the comparison of p and $r_1 + r_2$, we may divide this proof into 3 different situations.

Case 1: $2 < p < (r_1 + r_2) < \bar{p}$. In this case, we have $p\gamma_p < (r_1 + r_2)\gamma_{(r_1+r_2)} < 2$ and

$$k'(t) = t^{p\gamma_p-1}[t^{2-p\gamma_p} - C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}t^{(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p} - 2C_4 t^{22^*_{\alpha,s}-p\gamma_p} - \gamma_p(\mu_1 C_1 + \mu_2 C_2)].$$

Denote

$$\tilde{k}(t) := t^{2-p\gamma_p} - C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}t^{(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p} - 2C_4 t^{22^*_{\alpha,s}-p\gamma_p},$$

then

$$\begin{aligned}\tilde{k}'(t) = &t^{(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p-1}[(2 - p\gamma_p)t^{2-(r_1+r_2)\gamma_{(r_1+r_2)}} - 2C_4(22^*_{\alpha,s} - p\gamma_p)t^{22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)}} \\ &- C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}((r_1 + r_2)\gamma_{(r_1+r_2)} - p\gamma_p)].\end{aligned}$$

Let

$$\hat{k}(t) := (2-p\gamma_p)t^{2-(r_1+r_2)\gamma_{(r_1+r_2)}} - 2C_4(22^*_{\alpha,s} - p\gamma_p)t^{22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)}}, \tag{4.6}$$

then

$$\begin{aligned}\hat{k}'(t) = t^{1-(r_1+r_2)\gamma_{(r_1+r_2)}}[&(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)}) \\ &- 2C_4(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1+r_2)\gamma_{(r_1+r_2)})t^{22^*_{\alpha,s}-2}].\end{aligned}$$

We see from the definition of $\hat{k}'(t)$ that $\hat{k}(t)$ has a unique critical point t_0 in $(0,+\infty)$ satisfying

$$t_0^{22^*_{\alpha,s}-2} = \frac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1+r_2)\gamma_{(r_1+r_2)})}.$$

Moreover, since $p\gamma_p < (r_1+r_2)\gamma_{(r_1+r_2)} < 2$, we have $\tilde{k}(+\infty) = -\infty$, $\tilde{k}(0^+) = 0^-$. If

$$\begin{aligned}&\hat{k}(t_0) > C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}[(r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p]; \\ &\tilde{k}(t_0) > \gamma_p(\mu_1 C_1 + \mu_2 C_2), \quad \text{and} \quad k(t_0) > 0,\end{aligned} \tag{4.7}$$

i.e.

$$\begin{cases} \left[\dfrac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{2-(r_1+r_2)\gamma_{(r_1+r_2)}}{22^*_{\alpha,s}-2}} \dfrac{(2-p\gamma_p)(22^*_{\alpha,s}-2)}{22^*_{\alpha,s} - (r_1+r_2)\gamma_{(r_1+r_2)}} \\ \quad > C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}((r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p); \\ \left[\dfrac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{p\gamma_p}{22^*_{\alpha,s}-2}} \left[1 - \dfrac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1+r_2)\gamma_{(r_1+r_2)})}\right] \\ \quad > \gamma_p(\mu_1 C_1 + \mu_2 C_2) + C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)} \\ \qquad \times \left[\dfrac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p}{22^*_{\alpha,s}-2}}; \\ \left[\dfrac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{22^*_{\alpha,s}}{22^*_{\alpha,s}-2}} C_4\left[\dfrac{(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1+r_2)\gamma_p)}{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})} - \dfrac{1}{2^*_{\alpha,s}}\right] \\ \quad > \dfrac{\mu_1 C_1 + \mu_2 C_2}{p} \times \left[\dfrac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{p\gamma_p}{22^*_{\alpha,s}-2}} \\ \qquad + \beta C_3 \left[\dfrac{(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{22^*_{\alpha,s}-2}}, \end{cases} \tag{4.8}$$

then the function $k(t)$ has exactly two critical points, one is a local minimum at a negative level, the other one is a global maximum at a positive level. Therefore, there exist R_2, R_3 with $0 < R_2 < R_3$ such that $k(R_2) = k(R_3) = 0$, $k(t) > 0$ if and only if $t \in (R_2, R_3)$.

Case 2: $2 < p < r_1 + r_2 = \bar{p}$. This implies $p\gamma_p < (r_1+r_2)\gamma_{(r_1+r_2)} = 2$. We choose β such that $C_3\beta < \frac{1}{2}$ and $k(t)$ turns to be

$$k(t) = \left(\frac{1}{2} - C_3\beta\right)t^2 - \frac{\mu_1 C_1 + \mu_2 C_2}{p}t^{p\gamma_p} - \frac{C_4}{2^*_{\alpha,s}}t^{22^*_{\alpha,s}}.$$

Taking a similar argument as in *Case 1*, first we have

$$k'(t) = t^{p\gamma_p-1}[(1-2C_3\beta)t^{2-p\gamma_p} - 2C_4 t^{22^*_{\alpha,s}-p\gamma_p} - \gamma_p(\mu_1 C_1 + \mu_2 C_2)].$$

Denote

$$\tilde{k}(t) = (1-2C_3\beta)t^{2-p\gamma_p} - 2C_4 t^{22^*_{\alpha,s}-p\gamma_p},$$

and

$$\tilde{k}'(t) = t^{1-p\gamma_p}[(1-2C_3\beta)(2-p\gamma_p) - 2C_4(22^*_{\alpha,s}-p\gamma_p)t^{22^*_{\alpha,s}-2}].$$

Thus there exists $t_1 \in (0,+\infty)$ satisfying

$$t_1^{22^*_{\alpha,s}-2} = \frac{(1-2C_3\beta)(2-p\gamma_p)}{2C_4(22^*_{\alpha,s}-p\gamma_p)},$$

and if

$$\tilde{k}(t_1) > \gamma_p(\mu_1 C_1 + \mu_2 C_2), \quad \text{and} \quad k(t_1) > 0, \tag{4.9}$$

that is

$$\begin{cases} \left(\dfrac{2-p\gamma_p}{2C_4}\right)^{\frac{(2-p\gamma_p)}{22^*_{\alpha,s}-2}} (22^*_{\alpha,s}-2)(22^*_{\alpha,s}-p\gamma_p)^{\frac{p\gamma_p-22^*_{\alpha,s}}{22^*_{\alpha,s}-2}} > \gamma_p(\mu_1C_1+\mu_2C_2)(1-2C_3\beta)^{\frac{p\gamma_p-22^*_{\alpha,s}}{22^*_{\alpha,s}-2}}; \\ \left(\dfrac{22^*_{\alpha,s}-p\gamma_p}{2-p\gamma_p} - \dfrac{1}{2^*_{\alpha,s}}\right)C_4 > \dfrac{\mu_1C_1+\mu_2C_2}{p}\left[\dfrac{(1-2C_3\beta)(2-p\gamma_p)}{2C_4(22^*_{\alpha,s}-p\gamma_p)}\right]^{\frac{p\gamma_p-22^*_{\alpha,s}}{22^*_{\alpha,s}-2}}, \end{cases} \tag{4.10}$$

then we get the same conclusions as *Case 1*.

Case 3: $2 < p < \bar{p} < r_1 + r_2 < 2^*_s$. In this case, $p\gamma_p < 2 < (r_1+r_2)\gamma_{(r_1+r_2)}$. Similarily, we have

$$\tilde{k}(t) := t^{2-p\gamma_p} - C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)} t^{(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p} - 2C_4 t^{22^*_{\alpha,s}-p\gamma_p},$$

and

$$\begin{aligned} \tilde{k}'(t) = t^{1-p\gamma_p}[&(2-p\gamma_p) - C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}((r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p)t^{(r_1+r_2)\gamma_{(r_1+r_2)}-2} \\ &- 2C_4(22^*_{\alpha,s}-p\gamma_p)t^{22^*_{\alpha,s}-2}]. \end{aligned}$$

Therefore, $\tilde{k}(t)$ has a unique critical point $t_2 \in (0,+\infty)$. If

$$\tilde{k}(t_2) > \gamma_p(\mu_1 C_1 + \mu_2 C_2), \quad \text{and} \quad k(t_2) > 0, \tag{4.11}$$

we obtain the same conclusions as *Case 1*. Following, we get an estimate at t_2. Let

$$t_* = \left[\frac{(\mu_1C_1+\mu_2C_2)d((r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p)}{2-p\gamma_p}\right]^{\frac{1}{2-p\gamma_p}},$$

where d will be fixed later. If $t_2 > t_*$ and $d > \frac{\gamma_p(2-p\gamma_p)}{(r_1+r_2)\gamma_{(r_1+r_2)}-2}$, we get

$$\begin{aligned} &(\mu_1C_1+\mu_2C_2)\gamma_p t_2^{p\gamma_p} + C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)} t_2^{(r_1+r_2)\gamma_{(r_1+r_2)}} + 2C_4 t_2^{22^*_{\alpha,s}} \\ &\quad \le (\mu_1C_1+\mu_2C_2)\gamma_p t_*^{p\gamma_p-2} t_2^2 + \frac{2-p\gamma_p}{(r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p} t_2^2 < t_2^2, \end{aligned}$$

and if $d > \frac{2(r_1+r_2)\gamma_{(r_1+r_2)}}{p[(r_1+r_2)\gamma_{(r_1+r_2)}-2]}$,

$$\begin{aligned} &\frac{\mu_1C_1+\mu_2C_2}{p} t_2^{p\gamma_p} + C_3\beta t_2^{(r_1+r_2)\gamma_{(r_1+r_2)}} + \frac{C_4}{2^*_{\alpha,s}} t_2^{22^*_{\alpha,s}} \\ &\quad \le \frac{2-p\gamma_p}{(r_1+r_2)\gamma_{(r_1+r_2)}[(r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p]} t_2^2 + \frac{\mu_1C_1+\mu_2C_2}{p} t_*^{p\gamma_p-2} t_2^2 < \frac{1}{2} t_2^2. \end{aligned}$$

Therefore, if we choose $d > \frac{2(r_1+r_2)\gamma_{(r_1+r_2)}}{p((r_1+r_2)\gamma_{(r_1+r_2)}-2)}$, we get (4.11). Hence we only need $t_2 > t_*$. By the definition of t_2, we need

$$
\begin{aligned}
(2-p\gamma_p) > C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}((r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p)t_*^{(r_1+r_2)\gamma_{(r_1+r_2)}-2} \\
+2C_4(22^*_{\alpha,s}-p\gamma_p)t_*^{22^*_{\alpha,s}-2},
\end{aligned}
$$

that is,

$$
\begin{aligned}
(2-p\gamma_p) > & \left[\frac{(\mu_1C_1+\mu_2C_2)d((r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p)}{2-p\gamma_p}\right]^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}-2}{2-p\gamma_p}} \\
& \times \Bigg[C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}((r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p) \\
& +2C_4(22^*_{\alpha,s}-p\gamma_p)\left(\frac{(\mu_1C_1+\mu_2C_2)d((r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p)}{2-p\gamma_p}\right)^{\frac{22^*_{\alpha,s}-2}{2-p\gamma_p}}\Bigg].
\end{aligned} \tag{4.12}
$$

To sum up, there exists $\beta_* > 0$, such that for any $\beta \in (0,\beta_*)$, there exist $\mu_{1,*} = \mu_{1,*}(\beta) > 0$ and $\mu_{2,*} = \mu_{2,*}(\beta) > 0$, for any $\mu_1 \in (0,\mu_{1,*})$, $\mu_2 \in (0,\mu_{2,*})$, then (4.8), (4.10) and (4.12) are satisfied. We complete this lemma. □

We now study the structure of Pohožaev manifold. Recalling the decomposition of $\mathcal{P}_{\mu_1,\mu_2} = \mathcal{P}^+_{\mu_1,\mu_2} \cup \mathcal{P}^0_{\mu_1,\mu_2} \cup \mathcal{P}^-_{\mu_1,\mu_2}$, we have:

Lemma 4.2. *There exists $\tilde{\beta}_* > 0$, such that for any $\beta \in (0,\tilde{\beta}_*)$, there exist $\tilde{\mu}_{1,*} = \tilde{\mu}_{1,*}(\beta) > 0$ and $\tilde{\mu}_{2,*} = \tilde{\mu}_{2,*}(\beta) > 0$, for every $\mu_1 \in (0,\tilde{\mu}_{1,*})$, $\mu_2 \in (0,\tilde{\mu}_{2,*})$, then $\mathcal{P}^0_{\mu_1,\mu_2} = \emptyset$ and $\mathcal{P}_{\mu_1,\mu_2}$ is a C^1-submanifold in H with codimension 3.*

Proof. Firstly, assume by contradiction that there exists a $(u,v) \in \mathcal{P}^0_{\mu_1,\mu_2}$ satisfying

$$
\begin{aligned}
([u]^2+[v]^2) = \gamma_p(\mu_1|u|_p^p+\mu_2|v|_p^p) + \beta(r_1+r_2)\gamma_{(r_1+r_2)}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx \\
+2\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx,
\end{aligned} \tag{4.13}
$$

and

$$
\begin{aligned}
2([u]^2+[v]^2) = p\gamma_p^2(\mu_1|u|_p^p+\mu_2|v|_p^p) + \beta(r_1+r_2)^2\gamma^2_{(r_1+r_2)}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx \\
+42^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx.
\end{aligned} \tag{4.14}
$$

Following we define

$$
\begin{aligned}
\hbar(\rho) :=\ & \rho\Psi'_{\mu_1,\mu_2}(0) - \Psi''_{\mu_1,\mu_2}(0) \\
=\ & (\rho-2)([u]^2+[v]^2) - \gamma_p(\rho-p\gamma_p)(\mu_1|u|_p^p+\mu_2|v|_p^p) \\
& -2(\rho-22^*_{\alpha,s})\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx \\
& -\beta(r_1+r_2)\gamma_{(r_1+r_2)}(\rho-(r_1+r_2)\gamma_{(r_1+r_2)})\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx = 0,
\end{aligned} \tag{4.15}
$$

and let

$$\eta := ([u]^2 + [v]^2)^{\frac{1}{2}}.$$

Case 1: When $p < r_1 + r_2 < \bar{p}$, we have $p\gamma_p < (r_1 + r_2)\gamma_{(r_1+r_2)} < 2$. From (4.15) and (4.3), we have $\hbar((r_1 + r_2)\gamma_{(r_1+r_2)}) = 0$ and

$$\begin{aligned} \left[2 - (r_1 + r_2)\gamma_{(r_1+r_2)}\right]\eta^2 &\leq 2\left[22^*_{\alpha,s} - (r_1 + r_2)\gamma_{(r_1+r_2)}\right] \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx \\ &\leq 2C_4\left[22^*_{\alpha,s} - (r_1 + r_2)\gamma_{(r_1+r_2)}\right]\eta^{22^*_{\alpha,s}}. \end{aligned} \tag{4.16}$$

It follows $\eta \geq \left[\frac{2-(r_1+r_2)\gamma_{(r_1+r_2)}}{2C_4(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{1}{22^*_{\alpha,s}-2}}$. Moreover, by $\hbar(22^*_{\alpha,s}) = 0$, from (3.2), (4.1) and (4.2),we obtain

$$\begin{aligned} (22^*_{\alpha,s} - 2)\eta^2 &= \gamma_p(22^*_{\alpha,s} - p\gamma_p)(\mu_1|u|_p^p + \mu_2|v|_p^p) \\ &\quad + \beta(r_1 + r_2)\gamma_{(r_1+r_2)}\left[22^*_{\alpha,s} - (r_1 + r_2)\gamma_{(r_1+r_2)}\right] \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx \\ &\leq \gamma_p(22^*_{\alpha,s} - p\gamma_p)(\mu_1 C_1 + \mu_2 C_2)t^{p\gamma_p} \\ &\quad + C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}\left[22^*_{\alpha,s} - (r_1 + r_2)\gamma_{(r_1+r_2)}\right]t^{(r_1+r_2)\gamma_{(r_1+r_2)}}, \end{aligned}$$

that is

$$\begin{aligned} 22^*_{\alpha,s} - 2 \leq{}& \gamma_p(22^*_{\alpha,s} - p\gamma_p)(\mu_1 C_1 + \mu_2 C_2)\left[\frac{2 - (r_1 + r_2)\gamma_{(r_1+r_2)}}{2C_4(22^*_{\alpha,s} - (r_1 + r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{p\gamma_p - 2}{22^*_{\alpha,s}-2}} \\ &+ C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}\left[\frac{2 - (r_1 + r_2)\gamma_{(r_1+r_2)}}{2C_4(22^*_{\alpha,s} - (r_1 + r_2)\gamma_{(r_1+r_2)})}\right]^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}-2}{22^*_{\alpha,s}-2}}. \end{aligned} \tag{4.17}$$

Hence, we can choose $\tilde{\beta}_* > 0$, such that for any $\beta \in (0, \tilde{\beta}_*)$, there exist $\tilde{\mu}_{1,*} = \tilde{\mu}_{1,*}(\beta) > 0$ and $\tilde{\mu}_{2,*} = \tilde{\mu}_{2,*}(\beta) > 0$, for every $\mu_1 \in (0, \tilde{\mu}_{1,*})$, $\mu_2 \in (0, \tilde{\mu}_{2,*})$, such that (4.17) can not happen. Therefore, $\mathcal{P}^0_{\mu_1,\mu_2} = \emptyset$.

Case 2: As in $p < r_1 + r_2 = \bar{p}$, we get $p\gamma_p < (r_1 + r_2)\gamma_{(r_1+r_2)} = 2$. Similarly as in *Case 1*, from (4.1)–(4.3) and (4.15), we have $\hbar(p\gamma_p) = 0$, i.e.

$$\begin{aligned} (2 - p\gamma_p)t^2 &= \beta(r_1 + r_2)\gamma_{(r_1+r_2)}\left[(r_1 + r_2)\gamma_{(r_1+r_2)} - p\gamma_p\right] \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx \\ &\quad + 2(22^*_{\alpha,s} - p\gamma_p) \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx \\ &\leq 2C_3\beta(2 - p\gamma_p)\eta^2 + 2C_4(22^*_{\alpha,s} - p\gamma_p)\eta^{22^*_{\alpha,s}}. \end{aligned} \tag{4.18}$$

From $\hbar(22^*_{\alpha,s}) = 0$ we get

$$(22^*_{\alpha,s} - 2)\eta^2 \leq \gamma_p(\mu_1 C_1 + \mu_2 C_2)(22^*_{\alpha,s} - p\gamma_p)\eta^{p\gamma_p} + 2C_3\beta(22^*_{\alpha,s} - 2)\eta^2. \tag{4.19}$$

Combining with (4.18), we first suppose $1 - 2C_3\beta > 0$ and then

$$\left[\frac{(1 - 2C_3\beta)(2 - p\gamma_p)}{2C_4(22^*_{\alpha,s} - p\gamma_p)}\right]^{\frac{1}{22^*_{\alpha,s}-2}} \leq \left[\frac{\gamma_p(\mu_1 C_1 + \mu_2 C_2)(22^*_{\alpha,s} - p\gamma_p)}{(22^*_{\alpha,s} - 2)(1 - 2C_3\beta)}\right]^{\frac{1}{2-p\gamma_p}},$$

that is

$$\left(\frac{2-p\gamma_p}{2C_4}\right)^{2-p\gamma_p}\left(\frac{22^*_{\alpha,s}-2}{\gamma_p}\right)^{22^*_{\alpha,s}-2}\left(\frac{1}{22^*_{\alpha,s}-p\gamma_p}\right)^{22^*_{\alpha,s}-p\gamma_p}$$
$$\le(\mu_1C_1+\mu_2C_2)^{22^*_{\alpha,s}-2}\left(\frac{1}{1-2C_3\beta}\right)^{22^*_{\alpha,s}-p\gamma_p}.$$

Similar argument as in Case 1, choose appropriate $\tilde{\beta}_*, \tilde{\mu}_{1,*}=\tilde{\mu}_{1,*}(\beta), \tilde{\mu}_{2,*}=\tilde{\mu}_{2,*}(\beta)$, such that the last inequality may not happen. Therefore $\mathcal{P}^0_{\mu_1,\mu_2}=\emptyset$.

Case 3: If $p<\bar{p}<r_1+r_2$, then $p\gamma_p<2<(r_1+r_2)\gamma_{(r_1+r_2)}$. Also by (3.2), (4.1)–(4.3) and (4.15), since $\hbar(p\gamma_p)=0$ we have

$$(2-p\gamma_p)\eta^2\le C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}\left[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p\right]\eta^{(r_1+r_2)\gamma_{(r_1+r_2)}}$$
$$+2C_4(22^*_{\alpha,s}-p\gamma_p)\eta^{22^*_{\alpha,s}}. \tag{4.20}$$

By the definition of t_2 and t_* in Lemma 4.1, we need

$$\eta\ge t_2>t_*:=\left[\frac{(\mu_1C_1+\mu_2C_2)d((r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p)}{2-p\gamma_p}\right]^{\frac{1}{2-p\gamma_p}}.$$

Besides, from $\hbar((r_1+r_2)\gamma_{(r_1+r_2)})=0$ we have

$$((r_1+r_2)\gamma_{(r_1+r_2)}-2)\eta^2\le\gamma_p(\mu_1C_1+\mu_2C_2)\left[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p\right]\eta^{p\gamma_p}. \tag{4.21}$$

i.e.

$$\eta\le\left[\frac{\gamma_p(\mu_1C_1+\mu_2C_2)((r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p)}{((r_1+r_2)\gamma_{(r_1+r_2)}-2)}\right]^{\frac{1}{2-p\gamma_p}}.$$

This is a contradiction with $d>\frac{2(r_1+r_2)\gamma_{(r_1+r_2)}}{p((r_1+r_2)\gamma_{(r_1+r_2)}-2)}$ in Lemma 4.1. Hence, we can fix $\tilde{\beta}_*=\beta_*$, $\tilde{\mu}_{1,*}:=\tilde{\mu}_{1,*}(\beta)=\mu_{1,*}$ and $\tilde{\mu}_{2,*}:=\tilde{\mu}_{2,*}(\beta)=\mu_{2,*}$, to make sure $t_2>t_*$ and $\mathcal{P}^0_{\mu_1,\mu_2}=\emptyset$, where $\beta_*, \mu_{1,*}$ and $\mu_{2,*}$ are from Lemma 4.1.

Following, we prove $\mathcal{P}_{\mu_1,\mu_2}$ is a C^1-submanifold in H with codimension 3. For any $(u,v)\in\mathcal{P}_{\mu_1,\mu_2}$, we have $P_{\mu_1,\mu_2}(u,v)=0$, $G(u)=0$ and $F(v)=0$, where

$$G(u):=\int_{\mathbb{R}^N}u^2-a^2dx,\quad\text{and}\quad F(v):=\int_{\mathbb{R}^N}v^2-b^2dx.$$

Then we need to prove

$$d(P_{\mu_1,\mu_2}(u,v),G(u),F(v)):H\mapsto\mathbb{R}^3\quad\text{is surjective.}$$

If not, there exist $\nu_1,\nu_2\in\mathbb{R}$, for every $(\varphi,0)$ and $(0,\psi)$ in H such that

$$2s\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi dx=sp\gamma_p\int_{\mathbb{R}^N}\mu_1|u|^{p-2}u\varphi dx+s\beta(r_1+r_2)\gamma_{(r_1+r_2)}r_1\int_{\mathbb{R}^N}|u|^{r_1-2}u\varphi dx$$
$$+2s2^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha*|v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u\varphi dx+2\nu_1\int_{\mathbb{R}^N}u\varphi dx;$$

and

$$2s\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}v(-\Delta)^{\frac{s}{2}}\psi dx=sp\gamma_p\int_{\mathbb{R}^N}\mu_2|v|^{p-2}v\psi dx+s\beta(r_1+r_2)\gamma_{(r_1+r_2)}r_2\int_{\mathbb{R}^N}|v|^{r_2-2}v\psi dx$$
$$+2s2^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v\psi dx+2\nu_2\int_{\mathbb{R}^N}v\psi dx.$$

From which (u,v) is a weak solution of the system in $\mathbb{R}^N$

$$\begin{cases} 2s(-\Delta)^s u = 2\nu_1 u + sp\gamma_p\mu_1|u|^{p-2}u + s\beta(r_1+r_2)\gamma_{(r_1+r_2)}r_1|u|^{r_1-2}u|v|^{r_2} \\ \qquad\qquad + 2s2^*_{\alpha,s}(I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, \\ 2s(-\Delta)^s v = 2\nu_2 v + sp\gamma_p\mu_2|v|^{p-2}v + s\beta(r_1+r_2)\gamma_{(r_1+r_2)}r_2|v|^{r_2-2}|u|^{r_1} \\ \qquad\qquad + 2s2^*_{\alpha,s}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v, \\ \int_{\mathbb{R}^N}|u|^2dx = a^2, \quad \int_{\mathbb{R}^N}|v|^2dx = b^2. \end{cases}$$

The related Pohožaev identity of the above system is

$$2([u]^2+[v]^2) = p\gamma_p^2(\mu_1|u|_p^p + \mu_2|v|_p^p) + \beta(r_1+r_2)^2\gamma^2_{(r_1+r_2)}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx$$
$$+ 42^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx,$$

thus $P^0_{\mu_1,\mu_2}(u,v)=0$, which contradicts with $\mathcal{P}^0_{\mu_1,\mu_2}=\emptyset$. We complete this lemma. □

From Lemmas 4.1 and 4.2, we can have the geometry of Ψ_{μ_1,μ_2}.

Lemma 4.3. *For every $(u,v)\in S$, the function $\Psi_{\mu_1,\mu_2}(t)$ has exactly two critical points $s_{u,v} < t_{u,v} \in \mathbb{R}$ and two zeros $c_{u,v} < d_{u,v} \in \mathbb{R}$ with $s_{u,v} < c_{u,v} < t_{u,v} < d_{u,v}$. Moreover,*

(i) $s_{u,v} * (u,v) \in \mathcal{P}^+_{\mu_1,\mu_2}$ and $t_{u,v} * (u,v) \in \mathcal{P}^-_{\mu_1,\mu_2}$, and if $t*(u,v)\in\mathcal{P}_{\mu_1,\mu_2}$, then either $t = s_{u,v}$ or $t = t_{u,v}$.

(ii) $([t*u]^2+[t*v]^2)^{\frac12} \le R_2$ (R_2 is from Lemma 4.1) for every $t \le c_{u,v}$ and

$$J(s_{u,v}*(u,v)) = \min\{J(t*(u,v)) : t\in\mathbb{R} \quad \text{and} \quad ([t*u]^2+[t*v]^2)^{\frac12} \le R_2\}.$$

(iii) We get $J(t_{u,v}*(u,v)) = \max\{J(t*(u,v)) : t\in\mathbb{R}\} > 0$ and $\Psi_{\mu_1,\mu_2}(t)$ is strictly decreasing and concave on $(t_{u,v},\infty)$. In particular, if $t_{u,v}<0$, then $P_{\mu_1,\mu_2}(u,v)<0$.

(iv) The maps $(u,v)\mapsto s_{u,v}$, and $(u,v)\mapsto t_{u,v}$ for any $(u,v)\in S$ are of class C^1.

Proof. Let $(u,v)\in S$, then $t*(u,v)\in\mathcal{P}_{\mu_1,\mu_2}$ if and only if $\Psi'_{\mu_1,\mu_2}(t)=0$. By (4.4)-(4.5),

$$\Psi_{\mu_1,\mu_2}(t) = J(t*(u,v)) \ge k(e^{st}([u]^2+[v]^2)^{\frac12}),$$

thus from Lemma 4.1, $\Psi_{\mu_1,\mu_2}(t)$ is positive on

$$\left(s^{-1}\ln\frac{R_2}{([u]^2+[v]^2)^{\frac12}}, s^{-1}\ln\frac{R_3}{([u]^2+[v]^2)^{\frac12}}\right).$$

Since $p\gamma_p < 2$, we see $\Psi_{\mu_1,\mu_2}(-\infty)=0^-$ and $\Psi_{\mu_1,\mu_2}(+\infty)=-\infty$. Then $\Psi_{\mu_1,\mu_2}(t)$ has at least two critical points. Therefore, $\Psi_{\mu_1,\mu_2}(t)$ has a local minimum point $s_{u,v}$ at a negative level in $\big(-\infty, s^{-1}\ln\frac{R_2}{([u]^2+[v]^2)^{\frac12}}\big)$, and has a global maximum point $t_{u,v}$ at a positive level in $\big(s^{-1}\ln\frac{R_2}{([u]^2+[v]^2)^{\frac12}}, s^{-1}\ln\frac{R_3}{([u]^2+[v]^2)^{\frac12}}\big)$. We claim $\Psi_{\mu_1,\mu_2}(t)$ has exactly two critical points. Let $\Psi'_{\mu_1,\mu_2}(t)=0$, namely

$$\Psi'_{\mu_1,\mu_2}(t) = se^{2st}([u]^2+[v]^2) - s\gamma_p e^{sp\gamma_p t}(\mu_1|u|_p^p+\mu_2|v|_p^p) - 2se^{22^*_{\alpha,s}st}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx$$
$$- s\beta(r_1+r_2)\gamma_{(r_1+r_2)}e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx. \tag{4.22}$$

Case 1: $2 < p < r_1 + r_2 < \bar{p}$. From (4.22) we have

$$\Psi'_{\mu_1,\mu_2}(t) = e^{sp\gamma_p t}\big[se^{(2-p\gamma_p)st}([u]^2+[v]^2) - s\beta(r_1+r_2)\gamma_{(r_1+r_2)}e^{s[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]t}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx$$
$$-2se^{(22^*_{\alpha,s}-p\gamma_p)st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx - s\gamma_p(\mu_1|u|_p^p+\mu_2|v|_p^p)\big].$$

Denote

$$g_1(t) := se^{(2-p\gamma_p)st}([u]^2+[v]^2) - 2se^{(22^*_{\alpha,s}-p\gamma_p)st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx$$
$$- s\beta(r_1+r_2)\gamma_{(r_1+r_2)}e^{s[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]t}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx,$$

then

$$g_1'(t) = e^{[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]st}\Bigg[(2-p\gamma_p)s^2e^{(2-(r_1+r_2)\gamma_{(r_1+r_2)})st}([u]^2+[v]^2)$$
$$-2(22^*_{\alpha,s}-p\gamma_p)s^2e^{(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx$$
$$-[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]s^2(r_1+r_2)\gamma_{(r_1+r_2)}\beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx\Bigg].$$

Now define

$$f_1(t) := (2-p\gamma_p)s^2e^{[2-(r_1+r_2)\gamma_{(r_1+r_2)}]st}([u]^2+[v]^2)$$
$$-2(22^*_{\alpha,s}-p\gamma_p)s^2e^{(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx,$$

thus

$$f_1'(t) = e^{(2-(r_1+r_2)\gamma_{(r_1+r_2)})st}\Bigg[(2-p\gamma_p)(2-(r_1+r_2)\gamma_{(r_1+r_2)})s^3([u]^2+[v]^2)$$
$$-2(22^*_{\alpha,s}-p\gamma_p)(22^*_{\alpha,s}-(r_1+r_2)\gamma_{(r_1+r_2)})s^3e^{(22^*_{\alpha,s}-2)st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx\Bigg].$$

We see $f_1(t)$ has only one critical point $\bar{t}$, which is also a maximum point. Therefore if

$$f_1(\bar{t}) \le [(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]s^2(r_1+r_2)\gamma_{(r_1+r_2)}\beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx,$$

we have $g_1'(t) < 0$ and $g_1(t)$ is strictly decreasing in $\mathbb{R}\setminus\{\bar{t}\}$. Since $g_1(-\infty) = 0^-$ and $g_1(+\infty) = -\infty$, we get

$$g_1(t) < 0 < s\gamma_p(\mu_1|u|_p^p+\mu_2|v|_p^p),$$

and hence $\Psi'_{\mu_1,\mu_2}(t) < 0$, which means $\Psi_{\mu_1,\mu_2}(t)$ has no critical points. On the other hand, if

$$f_1(\bar{t}) > [(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]s^2(r_1+r_2)\gamma_{(r_1+r_2)}\beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx,$$

then by $f_1(-\infty) = 0^+$, $f_1(+\infty) = -\infty$, there exist two constants $\bar{t}_1 < \bar{t} < \bar{t}_2$, such that

$$f_1(\bar{t}_1) = f_1(\bar{t}_2) = [(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]s^2(r_1+r_2)\gamma_{(r_1+r_2)}\beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx.$$

Therefore, we find from the definitions of $g_1(t)$ and $\Psi_{\mu_1,\mu_2}(t)$ that

$$g_1(t) = s\gamma_p(\mu_1|u|_p^p+\mu_2|v|_p^p),$$

has at most two critical points, which implies $\Psi_{\mu_1,\mu_2}(t)$ has at most two critical points.

Case 2: $2 < p < (r_1 + r_2) = \bar{p}$. In this case, (4.22) becomes

$$\begin{aligned}\Psi'_{\mu_1,\mu_2}(t) = s\Big([u]^2 + [v]^2 - 2\beta \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}dx\Big)e^{2st} - s\gamma_p e^{sp\gamma_p t}(\mu_1|u|_p^p + \mu_2|v|_p^p)\\ - 2se^{22^*_{\alpha,s}st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx.\end{aligned} \tag{4.23}$$

If

$$[u]^2 + [v]^2 - 2\beta \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}dx \le 0,$$

we see $\Psi'_{\mu_1,\mu_2}(t) < 0$ and $\Psi_{\mu_1,\mu_2}(t)$ has no critical points. Now we suppose

$$[u]^2 + [v]^2 - 2\beta \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}dx > 0.$$

Then similarly as in *Case 1*, we conclude $\Psi_{\mu_1,\mu_2}(t)$ has at most two critical points.

Case 3: $2 < p < \bar{p} < r_1 + r_2$. From the definition of $g_1(t)$, we have

$$\begin{aligned}g_1'(t) &= e^{(2-p\gamma_p)st}\Big[(2-p\gamma_p)s^2([u]^2+[v]^2) - 2s^2(22^*_{\alpha,s} - p\gamma_p)e^{(22^*_{\alpha,s}-2)st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx\\ &\quad - s^2\beta(r_1+r_2)\gamma_{(r_1+r_2)}[(r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p]e^{((r_1+r_2)\gamma_{(r_1+r_2)}-2)st}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx\Big]\\ &=: e^{(2-p\gamma_p)st}[(2-p\gamma_p)s^2([u]^2+[v]^2) - Q(t)],\end{aligned}$$

where

$$\begin{aligned}Q(t) &= 2s^2(22^*_{\alpha,s} - p\gamma_p)e^{(22^*_{\alpha,s}-2)st}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx\\ &\quad + s^2\beta(r_1+r_2)\gamma_{(r_1+r_2)}[(r_1+r_2)\gamma_{(r_1+r_2)} - p\gamma_p]e^{((r_1+r_2)\gamma_{(r_1+r_2)}-2)st}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx.\end{aligned}$$

Moreover by

$$p\gamma_p < 2 < \min\{(r_1+r_2)\gamma_{(r_1+r_2)}, 22^*_{\alpha,s}\},$$

we see $Q(t)$ is strictly increasing in $\mathbb{R}$. Therefore $g_1(t)$ has a unique critical point $\hat{t}$, which is also a maximum point and $g_1(t)$ is strictly increasing in $(-\infty, \hat{t})$, strictly decreasing in $(\hat{t}, +\infty)$. On one hand, if

$$g_1(\hat{t}) \le s\gamma_p(\mu_1|u|_p^p + \mu_2|v|_p^p),$$

we have $\Psi_{\mu_1,\mu_2}(t)$ has no critical points. Besides, if

$$g_1(\hat{t}) > s\gamma_p(\mu_1|u|_p^p + \mu_2|v|_p^p),$$

then $\Psi'_{\mu_1,\mu_2}(t) = 0$ has at most two solutions, that is $\Psi_{\mu_1,\mu_2}(t)$ has at most two critical points. Hence, $\Psi_{\mu_1,\mu_2}(t)$ has exactly two critical points $s_{u,v} < t_{u,v}$.

By the definitions of $P_{\mu_1,\mu_2}(u,v)$ in (2.6) and $\Psi_{\mu_1,\mu_2}(t)$ in (2.7), we find $\Psi'_{\mu_1,\mu_2}(t) = P_{\mu_1,\mu_2}(t * u, t * v)$. Therefore we know $P_{\mu_1,\mu_2}(t * u, t * v) = 0$ if and only if t is a critical point of $\Psi_{\mu_1,\mu_2}(t)$. From above we find $\Psi_{\mu_1,\mu_2}(t)$ has exactly two critical points $s_{u,v}, t_{u,v}$, then we have $P_{\mu_1,\mu_2}(t * u, t * v) = 0$ if and only if $t = s_{u,v}$ or $t = t_{u,v}$. Moreover from Lemma 2.5 the definition of $\mathcal{P}_{\mu_1,\mu_2}$ here, by $(t * u, t * v) \in S$ we see that $(t * u, t * v) \in \mathcal{P}_{\mu_1,\mu_2}$ if and only if $t = s_{u,v}$ or $t = t_{u,v}$. Noticing $\Psi''_{\mu_1,\mu_2}(s_{u,v}) \ge 0$, $\Psi''_{\mu_1,\mu_2}(t_{u,v}) \le 0$ and $\mathcal{P}^0_{\mu_1,\mu_2} = \emptyset$, we obtain $s_{u,v} * (u,v) \in$

$\mathcal{P}^+_{\mu_1,\mu_2}$ and $t_{u,v} * (u,v) \in \mathcal{P}^-_{\mu_1,\mu_2}$. By the monotonicity and the behavior of $\Psi_{\mu_1,\mu_2}(t)$, we see $\Psi_{\mu_1,\mu_2}(t)$ has exactly two zeros $c_{u,v} < d_{u,v} \in \mathbb{R}$ with $s_{u,v} < c_{u,v} < t_{u,v} < d_{u,v}$, and $\Psi_{\mu_1,\mu_2}(t)$ has exactly two inflection points. Moreover, $\Psi_{\mu_1,\mu_2}(t)$ is concave on $[t_{u,v}, \infty)$, and if $t_{u,v} < 0$, then $P_{\mu_1,\mu_2}(u,v) = \Psi'_{\mu_1,\mu_2}(0) < 0$. Finally, we apply implicit function theorem on the C^1 function $\Phi(t,u,v) = \Psi'_{\mu_1,\mu_2}(t)$, then $\Phi(s_{u,v},u,v) = \Psi'_{\mu_1,\mu_2}(s_{u,v}) = 0$, $\partial_t\Phi(s_{u,v},u,v) = \Psi''_{\mu_1,\mu_2}(s_{u,v}) > 0$. Therefore we know $(u,v) \to s_{u,v}$ is of class C^1. Similarly, $(u,v) \to t_{u,v}$ is also of class C^1. □

For $r > 0$, define

$$B_r(a,b) := \{(u,v) \in S : ([u]^2 + [v]^2)^{\frac{1}{2}} < r\}, \quad \text{and} \quad \hat{m}(a,b) := \inf_{(u,v)\in B_{R_2}(a,b)} J(u,v).$$

From Lemma 4.3, we can deduce the following conclusion directly.

Corollary 4.4. *The set* $\mathcal{P}^+_{\mu_1,\mu_2} \subset B_{R_2}(a,b)$, *and*

$$\sup_{(u,v)\in\mathcal{P}^+_{\mu_1,\mu_2}} J(u,v) \le 0 \le \inf_{(u,v)\in\mathcal{P}^-_{\mu_1,\mu_2}} J(u,v).$$

Lemma 4.5. *We have* $\hat{m}(a,b) \in (-\infty, 0)$, *moreover*

$$\hat{m}(a,b) = m(a,b) = m^+(a,b) \quad \text{and} \quad \hat{m}(a,b) < \inf_{\overline{B_{R_2}(a,b)\setminus B_{R_2-\delta}(a,b)}} J(u,v),$$

for $\delta > 0$ *small enough.*

Proof. For any $(u,v) \in B_{R_2}(a,b)$, by (4.4) and (4.5), we get

$$J(u,v) \ge k(([u]^2 + [v]^2)^{\frac{1}{2}}) \ge \min_{t\in[0,R_2]} k(t) > -\infty.$$

Hence $\hat{m}(a,b) > -\infty$. Moreover, for any $(u,v) \in S$, when $t \ll -1$, we have $([t*u]^2 + [t*v]^2)^{\frac{1}{2}} < R_2$ and $J(t*(u,v)) < 0$. Hence $\hat{m}(a,b) < 0$. From Corollary 4.4, $\mathcal{P}^+_{\mu_1,\mu_2} \subset B_{R_2}(a,b)$, then $\hat{m}(a,b) \le m^+(a,b)$. On the other hand, for any $(u,v) \in B_{R_2}(a,b)$, from Lemma 4.3 we get

$$m^+(a,b) \le J(s_{u,v} * (u,v)) \le J(u,v).$$

Thus $m^+(a,b) = \hat{m}(a,b)$. Since $J(u,v) > 0$ on $\mathcal{P}^-_{\mu_1,\mu_2}$, we know $m(a,b) = m^+(a,b)$. Finally, by the continuity of $k(t)$ and $k(R_2) = 0$, we see from $-\infty < \hat{m}(a,b) < 0$ that there is $\delta > 0$ satisfying $k(t) \ge \frac{\hat{m}(a,b)}{2}$ if $t \in [R_2 - \delta, R_2]$. Thus

$$J(u,v) \ge k(([u]^2 + [v]^2)^{\frac{1}{2}}) \ge \frac{\hat{m}(a,b)}{2} \ge \hat{m}(a,b),$$

for any $(u,v) \in S$ with $R_2 - \delta \le ([u]^2 + [v]^2)^{\frac{1}{2}} \le R_2$. This completes the proof. □

Similarly from *Case 1* in Lemma 3.4, we obtain the monotonicity for this problem (1.1)–(1.2).

Lemma 4.6. *There exists* $\hat{\beta}_* > 0$, *for* $\beta \in (0, \hat{\beta}_*)$, *there are* $\hat{\mu}_{1,*} := \hat{\mu}_{1,*}(\beta), \hat{\mu}_{2,*} := \hat{\mu}_{2,*}(\beta) > 0$, *for any* $\mu_1 \in (0, \hat{\mu}_{1,*})$ *and* $\mu_2 \in (0, \hat{\mu}_{2,*})$, *the level satisfies* $m(a,b) \le m(a_1,b_1)$ *for any* $0 < a_1 \le a$, $0 < b_1 \le b$.

Proof. We also divide this proof into 3 cases.

Case 1: $2 < p < r_1 + r_2 < \bar{p}$. From Lemmas 4.1 and 4.3, we have $m(a,b) = \inf_{B_{t_0}(a,b)} J(u,v)$ and

$$t_0 = \left[\frac{(2 - p\gamma_p)(2 - (r_1 + r_2)\gamma_{(r_1+r_2)})}{2C_4(22^*_{\alpha,s} - p\gamma_p)(22^*_{\alpha,s} - (r_1 + r_2)\gamma_{(r_1+r_2)})} \right]^{\frac{1}{22^*_{\alpha,s}-2}},$$

which is independent with a, b. Besides, from (3.2), (4.1) and (4.2), we get C_1, C_2 and C_3 are increasing when a, b are increasing. Hence, we can choose $\hat{\beta}_* = \min\{\beta_*, \tilde{\beta}_*\}$, $\hat{\mu}_{1,*} = \min\{\mu_{1,*}, \tilde{\mu}_{1,*}\}$ and $\hat{\mu}_{2,*} = \min\{\mu_{1,*}, \tilde{\mu}_{1,*}\}$, such that there is $(u,v) \in B_{t_0}(a_1, b_1)$ with $J(u,v) \le m(a_1, b_1) + \frac{\varepsilon}{2}$, for ε is arbitrarily small. Using the same argument as Case 1 in Lemma 3.4, we get this result.

Case 2: $2 < p < r_1 + r_2 = \bar{p}$. Similarly, we have $m(a,b) = \inf_{B_{t_1}(a,b)} J(u,v)$ with

$$t_1 = \left[\frac{(1 - 2C_3\beta)(2 - p\gamma_p)}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{1}{22^*_{\alpha,s}-2}} \le \left[\frac{(2 - p\gamma_p)}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{1}{22^*_{\alpha,s}-2}} := t_{1,*},$$

which is independent with a, b. If there exists $\breve{\beta}_* > 0$. Then for any $\beta \in (0, \breve{\beta}_*)$, there are $\breve{\mu}_{1,*} = \breve{\mu}_{1,*}(\beta), \breve{\mu}_{2,*} = \breve{\mu}_{2,*}(\beta) > 0$, for any $\mu_1 \in (0, \breve{\mu}_{1,*})$ and $\mu_2 \in (0, \breve{\mu}_{2,*})$, such that $k(t_{1,*}) \ge 0$, that is

$$\frac{1}{2} - \frac{2 - p\gamma_p}{22^*_{\alpha,s}(22^*_{\alpha,s} - p\gamma_p)} \ge C_3\beta + \frac{\mu_1 C_1 + \mu_2 C_2}{p} \left[\frac{(2 - p\gamma_p)}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{p\gamma_p - 2}{22^*_{\alpha,s}-2}}. \tag{4.24}$$

Then from Lemmas 4.1 and 4.3, we can have $m(a,b) = \inf_{B_{t_{1,*}}(a,b)} J(u,v)$. Hence, there exists $\hat{\beta}_* = \min\{\beta_*, \tilde{\beta}_*, \breve{\beta}_*\}$, for $\beta \in (0, \hat{\beta}_*)$, there are $\hat{\mu}_{1,*}(\beta) = \min\{\mu_{1,*}, \tilde{\mu}_{1,*}, \breve{\mu}_{1,*}\}, \hat{\mu}_{2,*}(\beta) = \min\{\mu_{2,*}, \tilde{\mu}_{2,*}, \breve{\mu}_{2,*}\}$, for any $\mu_1 \in (0, \hat{\mu}_{1,*})$ and $\mu_2 \in (0, \hat{\mu}_{2,*})$, there is $(u,v) \in B_{t_{1,*}}(a_1, b_1)$ with $J(u,v) \le m(a_1, b_1) + \frac{\varepsilon}{2}$. The remainder of the proof is similar to Lemma 3.4, and so we omit the details here.

Case 3: $2 < p < \bar{p} < (r_1 + r_2) < 2^*_s$. First we have $m(a,b) = \inf_{B_{t_2}(a,b)} J(u,v)$ and

$$2 - p\gamma_p = C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}[(r_1 + r_2)\gamma_{(r_1+r_2)} - p\gamma_p] t_2^{(r_1+r_2)\gamma_{(r_1+r_2)}-2} + 2C_4(22^*_{\alpha,s} - p\gamma_p) t_2^{22^*_{\alpha,s}-2}.$$

If we choose

$$t_{2,*} = \left[\frac{2 - p\gamma_p}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{1}{22^*_{\alpha,s}-2}},$$

which is independent with a, b and satisfies

$$2 - p\gamma_p < C_3\beta(r_1 + r_2)\gamma_{(r_1+r_2)}[(r_1 + r_2)\gamma_{(r_1+r_2)} - p\gamma_p] t_{2,*}^{(r_1+r_2)\gamma_{(r_1+r_2)}-2} + 2C_4(22^*_{\alpha,s} - p\gamma_p) t_{2,*}^{22^*_{\alpha,s}-2},$$

then $t_2 \le t_{2,*}$. Furthermore, if $k(t_{2,*}) \ge 0$, that is

$$\frac{1}{2} - \frac{2 - p\gamma_p}{22^*_{\alpha,s}(22^*_{\alpha,s} - p\gamma_p)} \ge \frac{\mu_1 C_1 + \mu_2 C_2}{p} \left[\frac{(2 - p\gamma_p)}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{p\gamma_p - 2}{22^*_{\alpha,s}-2}} + C_3\beta \left[\frac{(2 - p\gamma_p)}{2C_4(22^*_{\alpha,s} - p\gamma_p)} \right]^{\frac{p\gamma_p - 2}{(r_1+r_2)\gamma_{(r_1+r_2)}-2}}.$$

Hence $m(a,b) = \inf_{B_{t_{2,*}}(a,b)} J(u,v)$. Like the same argument as before, choosing appropriate $\hat{\beta}_*, \hat{\mu}_{1,*}, \hat{\mu}_{2,*}$, using the same techniques in Lemma 3.4, we finish this problem. □

Lemma 4.7. *We have*

$$m(a,b) < \min\{m(a,0), m(0,b)\}.$$

Proof. From Theorem 3.1, we get $m(a,0)$ can be achieved by $\hat{u} \in S_a$. We choose a proper test function $\hat{v} \in S_b$ such that $(\hat{u}, t * \hat{v}) \in S$. By (3.3) and (4.5), we obtain $h(t) > k(t)$ for $t \in (0, +\infty)$. Hence, from Lemma 4.1, we have $R_0 < R_2$. By Theorem 3.1, we get

$$m(a,0) = \inf_{\mathcal{P}_{\mu_1,a}} I_{\mu_1,a}(u) = \inf_{B_{R_0}} I_{\mu_1,a}(u).$$

Therefore $[\hat{u}] \leq R_0 < R_2$. Thus, for $t \ll -1$, we have $(\hat{u}, t * \hat{v}) \in B_{R_2}(a,b)$ and

$$\begin{aligned} m(a,b) = & \inf_{(u,v)\in B_{R_2}(a,b)} J(u,v) \leq J(\hat{u}, t*\hat{v}) \\ = & \frac{1}{2}[\hat{u}]^2 - \frac{\mu_1}{p}\int_{\mathbb{R}^N} |\hat{u}|^p dx + \left(\frac{e^{2st}}{2}[\hat{v}]^2 - \frac{\mu_2 e^{p\gamma_p st}}{p}\int_{\mathbb{R}^N} |\hat{v}|^p dx \right. \\ & \left. - \beta \int_{\mathbb{R}^N} |\hat{u}|^{r_1} |t*\hat{v}|^{r_2} dx - \frac{1}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\hat{u}|^{2^*_{\alpha,s}}) |t*\hat{v}|^{2^*_{\alpha,s}} dx \right) \\ < & \frac{1}{2}[\hat{u}]^2 - \frac{\mu_1}{p}\int_{\mathbb{R}^N} |\hat{u}|^p dx = m(a,0). \end{aligned}$$

Analogously, we have $m(a,b) < m(0,b)$. Hence, the proof is completed. □

To obtain the compact result, we prove the boundedness first.

Lemma 4.8. *Let $2 < p < \bar{p}$, $p < r_1 + r_2 < 2^*_s$ and $\mu_1, \mu_2, a, b > 0$. Let $\{(u_n, v_n)\} \subset S_r$ be a Palais–Smale sequence, such that*

$$J(u_n, v_n) \to c; \quad J'(u_n, v_n)|_S \to 0 \quad \textit{and} \quad P_{\mu_1,\mu_2}(u_n, v_n) \to 0,$$

where $S_r = S \cap H_r$ and H_r is the space of radially symmetric functions in H. Then $\{(u_n, v_n)\}$ is bounded in H.

Proof. We divide this proof into two cases. *Case 1*: $2 < p < r_1 + r_2 < \bar{p}$. This implies $p\gamma_p < (r_1 + r_2)\gamma_{(r_1+r_2)} < 2$. Since (3.2), (4.1)–(4.3),

$$\begin{aligned} c + o_n(1) = & J(u_n, v_n) - \frac{1}{22^*_{\alpha,s}} P_{\mu_1,\mu_2}(u_n, v_n) \\ = & \frac{N+2s-\alpha}{2(2N-\alpha)}([u_n]^2 + [v_n]^2) - \left(\frac{1}{p} - \frac{\gamma_p}{22^*_{\alpha,s}}\right)(\mu_1 |u_n|_p^p + \mu_2 |v_n|_p^p) \\ & - \beta \left[1 - \frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{22^*_{\alpha,s}}\right] \int_{\mathbb{R}^N} |u_n|^{r_1} |v_n|^{r_2} dx \\ \geq & \frac{N+2s-\alpha}{2(2N-\alpha)}([u_n]^2 + [v_n]^2) - \left(\frac{1}{p} - \frac{\gamma_p}{22^*_{\alpha,s}}\right)(\mu_1 C_1 + \mu_2 C_2)([u_n]^2 + [v_n]^2)^{\frac{p\gamma_p}{2}} \\ & - \beta \left[1 - \frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{22^*_{\alpha,s}}\right] C_3([u_n]^2 + [v_n]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}}. \end{aligned}$$

Then, $\{(u_n, v_n)\}$ is bounded in H.

Case 2: $2 < p < \bar{p} \le r_1 + r_2 < 2_s^*$. From (3.2), (4.1)–(4.3) and $\alpha < N$, we can obtain

$$\begin{aligned}
c + o_n(1) &= J(u_n, v_n) - \frac{1}{(r_1+r_2)\gamma_{(r_1+r_2)}} P_{\mu_1,\mu_2}(u_n, v_n) \\
&= \left[\frac{1}{2} - \frac{1}{(r_1+r_2)\gamma_{(r_1+r_2)}}\right]([u_n]^2 + [v_n]^2) - \left[\frac{1}{p} - \frac{\gamma_p}{(r_1+r_2)\gamma_{(r_1+r_2)}}\right](\mu_1|u_n|_p^p + \mu_2|v_n|_p^p) \\
&\quad - \left[\frac{1}{2_{\alpha,s}^*} - \frac{2}{(r_1+r_2)\gamma_{(r_1+r_2)}}\right]\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2_{\alpha,s}^*})|v_n|^{2_{\alpha,s}^*}dx \\
&\ge \left[\frac{1}{2} - \frac{1}{(r_1+r_2)\gamma_{(r_1+r_2)}}\right]([u_n]^2 + [v_n]^2) \\
&\quad - \left[\frac{1}{p} - \frac{\gamma_p}{(r_1+r_2)\gamma_{(r_1+r_2)}}\right](\mu_1 C_1 + \mu_2 C_2)([u_n]^2 + [v_n]^2)^{\frac{p\gamma_p}{2}}.
\end{aligned}$$

From this, we have $\{(u_n, v_n)\}$ is bounded in H. □

In what follows, we discuss the convergence of a special Palais–Smale sequence, satisfying suitable additional conditions.

Proposition 4.9. *Let $\{(u_n, v_n)\} \subset S_r$ such that as $n \to \infty$,*

$$\begin{aligned}
J'(u_n, v_n) - \lambda_{1,n}u_n - \lambda_{2,n}v_n \to 0, &\quad \text{for some } \lambda_{1,n}, \lambda_{2,n} \in \mathbb{R}; \\
J(u_n, v_n) \to m(a,b), &\quad P_{\mu_1,\mu_2}(u_n, v_n) \to 0; \\
u_n^-, v_n^- \to 0, &\quad \text{a.e. in } \mathbb{R}^N,
\end{aligned} \tag{4.25}$$

with

$$m(a,b) \neq 0, \quad \text{and} \quad m(a,b) < \frac{N+2s-\alpha}{2N-\alpha}\left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

Then there exist $(u,v) \in H_r$ with $u, v > 0$ and $\lambda_1, \lambda_2 < 0$, such that up to a subsequence, $(u_n, v_n) \to (u,v)$ in H and $(\lambda_{1,n}, \lambda_{2,n}) \to (\lambda_1, \lambda_2)$ in $\mathbb{R}^2$.

Proof. From Lemma 4.8, we get $\{(u_n, v_n)\}$ is bounded in H_r. Moreover, by (4.25) we get

$$\lambda_{1,n} = \frac{1}{a^2}J'(u_n, v_n)(u_n, 0) + o_n(1), \quad \text{and} \quad \lambda_{2,n} = \frac{1}{b^2}J'(u_n, v_n)(0, v_n) + o_n(1),$$

thus $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ are bounded in $\mathbb{R}$. Therefore, there exist $(u,v) \in H_r$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ such that up to a subsequence,

$$\begin{aligned}
(u_n, v_n) &\rightharpoonup (u,v), && \text{in } H_r, \\
(u_n, v_n) &\to (u,v), && \text{in } L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N), \quad \text{for } 2 < q < 2_s^*, \\
(u_n, v_n) &\to (u,v), && \text{a.e. in } \mathbb{R}^N, \\
(\lambda_{1,n}, \lambda_{2,n}) &\to (\lambda_1, \lambda_2), && \text{in } \mathbb{R}^2.
\end{aligned}$$

Since

$$|v_n|^{2_{\alpha,s}^*} \rightharpoonup |v|^{2_{\alpha,s}^*}, \quad \text{in } L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N),$$

and the map $T : L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N) \mapsto L^{\frac{2N}{\alpha}}(\mathbb{R}^N)$ defined by $T(w) = I_\alpha * w$ is well defined , linear and continuous, we have

$$I_\alpha * |v_n|^{2_{\alpha,s}^*} \rightharpoonup I_\alpha * |v|^{2_{\alpha,s}^*}, \quad \text{in } L^{\frac{2N}{\alpha}}(\mathbb{R}^N).$$

Besides, by

$$|u_n|^{2^*_{\alpha,s}-2}u_n \rightharpoonup |u|^{2^*_{\alpha,s}-2}u, \quad \text{in } L^{\frac{2N}{N+2s-\alpha}}(\mathbb{R}^N),$$

we get

$$(I_\alpha * |v_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}-2}u_n \rightharpoonup (I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, \quad \text{in } L^{\frac{2N}{N+2s}}(\mathbb{R}^N).$$

Hence for any $\phi, \psi \in C_0^\infty(\mathbb{R}^N)$, we get

$$\int_{\mathbb{R}^N}(I_\alpha * |v_n|^{2^*_{\alpha,s}})|u_n|^{2^*_{\alpha,s}-2}u_n\phi dx \to \int_{\mathbb{R}^N}(I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u\phi dx,$$

and

$$\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}-2}v_n\psi dx \to \int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v\psi dx.$$

Therefore from (4.25), (u, v) satisfies

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1|u|^{p-2}u + \beta r_1|u|^{r_1-2}u|v|^{r_2} + (I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2|v|^{p-2}v + \beta r_2|u|^{r_1}|v|^{r_2-2}v + (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v, & \text{in } \mathbb{R}^N, \\ u \geq 0, \ v \geq 0, \end{cases} \tag{4.26}$$

and $P_{\mu_1,\mu_2}(u, v) = 0$.

Next, we will show $u \not\equiv 0$ and $v \not\equiv 0$. If not, we assume $u \equiv 0$. We claim $v \not\equiv 0$. Otherwise, from $P_{\mu_1,\mu_2}(u_n, v_n) \to 0$ and $u_n, v_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2^*_s)$, we get

$$[u_n]^2 + [v_n]^2 = 2\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}dx + o_n(1).$$

Since $\{(u_n, v_n)\}$ is bounded in H, we may assume $[u_n]^2 + [v_n]^2 \to l \in \mathbb{R}$. Then from (2.2), we have

$$l = 0 \quad \text{or } l \geq 2\left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

On one hand, if $l = 0$, we have $(u_n, v_n) \to (0, 0)$ in $D^s(\mathbb{R}^N) \times D^s(\mathbb{R}^N)$. Consequently $J(u_n, v_n) \to 0$ which gives a contradiction with $m(a, b) \neq 0$. On the other hand, if $l \geq 2(\frac{S^*}{2})^{\frac{2N-\alpha}{N+2s-\alpha}}$, from $P_{\mu_1,\mu_2}(u_n, v_n) \to 0$, we obtain

$$m(a, b) = J(u_n, v_n) + o_n(1) = J(u_n, v_n) - \frac{1}{2}P_{\mu_1,\mu_2}(u_n, v_n) + o_n(1) \geq \frac{N+2s-\alpha}{2N-\alpha}\left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}},$$

which can not happen since $m(a, b) < \frac{N+2s-\alpha}{2N-\alpha}(\frac{S^*}{2})^{\frac{2N-\alpha}{N+2s-\alpha}}$. Therefore $v \not\equiv 0$. From (4.26) and $u \equiv 0$, we have v satisfies

$$\begin{cases} (-\Delta)^s v = \lambda_2 v + \mu_2|v|^{p-2}v, & \text{in } \mathbb{R}^N, \\ v \geq 0. \end{cases}$$

Then we obtain from $|v|_2 \leq b$ and Lemma 3.4,

$$\begin{aligned} m(a, b) &= J(u_n, v_n) - \frac{1}{2}P_{\mu_1,\mu_2}(u_n, v_n) + o_n(1) \\ &= \left(\frac{\gamma_p}{2} - \frac{1}{p}\right)\mu_2|v|_p^p + \left(1 - \frac{1}{2^*_{\alpha,s}}\right)\int_{\mathbb{R}^N}(I_\alpha * |u_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}}dx + o_n(1) \\ &\geq \left(\frac{\gamma_p}{2} - \frac{1}{p}\right)\mu_2|v|_p^p \geq m(0, |v|_2) \geq m(0, b). \end{aligned}$$

From Lemma 4.7, which contradicts with $m(a,b) < m(0,b)$. Similar to [45, Lemma 3.7] and [32, Section 3], by the strong maximum principle [37, Proposition 2.17], we have $u > 0$. Analogously $v > 0$.

We claim $(u_n, v_n) \to (u, v)$ in $D^s(\mathbb{R}^N) \times D^s(\mathbb{R}^N)$. Indeed, if we let $(\hat{u}_n, \hat{v}_n) := (u_n - u, v_n - v)$, by [12, Lemma 2.2],

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} dx - \int_{\mathbb{R}^N} (I_\alpha * |\hat{u}_n|^{2^*_{\alpha,s}})|\hat{v}_n|^{2^*_{\alpha,s}} dx + o_n(1) = \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx,$$

and [45, Lemma 2.4],

$$\int_{\mathbb{R}^N} |u_n|^{r_1}|v_n|^{r_2} - |\hat{u}_n|^{r_1}|\hat{v}_n|^{r_2} - |u|^{r_1}|v|^{r_2} dx = o_n(1).$$

Therefore by the Brézis–Lieb Lemma [9], we have

$$P_{\mu_1,\mu_2}(\hat{u}_n, \hat{v}_n) = P_{\mu_1,\mu_2}(u_n, v_n) - P_{\mu_1,\mu_2}(u, v) + o_n(1) = o_n(1).$$

We deduce by the strong embedding in $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*_s)$ that,

$$\lim_{n\to\infty} ([\hat{u}_n]^2 + [\hat{v}_n]^2) = \lim_{n\to\infty} 2 \int_{\mathbb{R}^N} (I_\mu * |\hat{u}_n|^{2^*_{\alpha,s}})|\hat{v}_n|^{2^*_{\alpha,s}}.$$

Same argument as before, from (2.2) we can have

$$([\hat{u}_n]^2 + [\hat{v}_n]^2) \to 0,$$

or

$$([\hat{u}_n]^2 + [\hat{v}_n]^2) \geq 2 \left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}.$$

If the latter happens, we obtain from $|u|_2 \leq a$, $|v|_2 \leq b$ and Lemma 4.6,

$$\begin{aligned} m(a,b) + o_n(1) = J(u,v) + J(\hat{u}_n, \hat{v}_n) &= J(u,v) + J(\hat{u}_n, \hat{v}_n) - \frac{1}{22^*_{\alpha,s}} P_{\mu_1,\mu_2}(\hat{u}_n, \hat{v}_n) \\ &\geq m(|u|_2, |v|_2) + \frac{N+2s-\alpha}{2N-\alpha} \left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}} \\ &\geq m(a,b) + \frac{N+2s-\alpha}{2N-\alpha} \left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}, \end{aligned}$$

this can not happen. Therefore we have $([\hat{u}_n]^2 + [\hat{v}_n]^2) \to 0$, and we finish this claim.

Following, we claim $\lambda_1, \lambda_2 < 0$. If not, we may assume $\lambda_1 \geq 0$. From $u \geq 0$ we have

$$(-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2} u + \beta r_1 |u|^{r_1-2} u |v|^{r_2} + (I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2} u \geq 0.$$

From [29, Lemma 2.3] and $2s < N \leq 4s$, we have $u \equiv 0$, which is a contradiction. Hence, we obtain $\lambda_1 < 0$, and analogously $\lambda_2 < 0$. Then we deduce from taking $(u_n - u, v_n - v)$ into

(4.26) and the first formula of (4.25),

$$
\begin{aligned}
&[u_n - u]^2 + [v_n - v]^2 + o_n(1) \\
&= \int_{\mathbb{R}^N} (\lambda_{1,n} u_n - \lambda_1 u)(u_n - u) + (\lambda_{2,n} v_n - \lambda_2 v)(v_n - v) dx \\
&\quad + \mu_1 \int_{\mathbb{R}^N} (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u) dx \\
&\quad + \mu_2 \int_{\mathbb{R}^N} (|v_n|^{p-2} v_n - |v|^{p-2} v)(v_n - v) dx \\
&\quad + \beta r_1 \int_{\mathbb{R}^N} (|u_n|^{r_1-2} u_n |v_n|^{r_2} - |u|^{r_1-2} u |v|^{r_2})(u_n - u) dx \\
&\quad + \beta r_2 \int_{\mathbb{R}^N} (|u_n|^{r_1} |v_n|^{r_2-2} v_n - |u|^{r_1} |v|^{r_2-2} v)(v_n - v) dx \\
&\quad + \int_{\mathbb{R}^N} [(I_\alpha * |v_n|^{2^*_{\alpha,s}}) |u_n|^{2^*_{\alpha,s}-2} u_n - (I_\alpha * |v|^{2^*_{\alpha,s}}) |u|^{2^*_{\alpha,s}-2} u](u_n - u) dx \\
&\quad + \int_{\mathbb{R}^N} [(I_\alpha * |u_n|^{2^*_{\alpha,s}}) |v_n|^{2^*_{\alpha,s}-2} v_n - (I_\alpha * |u|^{2^*_{\alpha,s}}) |v|^{2^*_{\alpha,s}-2} v](v_n - v) dx.
\end{aligned}
$$

Since $(u_n, v_n) \to (u, v)$ in $D^s(\mathbb{R}^N) \times D^s(\mathbb{R}^N)$ and the embedding $D^s(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N)$ is continuous, we have

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \int_{\mathbb{R}^N} (\lambda_{1,n} u_n - \lambda_1 u)(u_n - u) + (\lambda_{2,n} v_n - \lambda_2 v)(v_n - v) dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \lambda_1 (u_n - u)^2 + \lambda_2 (v_n - v)^2 dx,
\end{aligned}
$$

by $\lambda_1, \lambda_2 < 0$, then $(u_n, v_n) \to (u, v)$ in H and we complete this proof. □

Proof of Theorem 1.1. Taking $\beta^* = \hat{\beta}_*$, there exist $\mu_1^*(\beta) = \min\{\hat{\mu}_{1,*}, \hat{\mu}_1\}$ and $\mu_2^*(\beta) = \min\{\hat{\mu}_{2,*}, \hat{\mu}_2\}$, for any $\mu_1 \in (0, \mu_1^*)$ and $\mu_2 \in (0, \mu_2^*)$, such that Lemmas 4.1, 4.2 and 4.7 are satisfied. Then, from Proposition 4.9, to finish this proof, it is sufficient to prove the existence of a sequence which satisfies Proposition 4.9. Let $\{(u_n, v_n)\}$ be a minimizing sequence for $m(a, b) = \inf_{B_{R_2}(a,b)} J(u, v)$, and assume that $\{(u_n, v_n)\} \subset S_r$ is radially decreasing, symmetry and non-negative for every $n \in \mathbb{N}$ (Firstly, due to $|(-\Delta)^{\frac{s}{2}} |u|| \le |(-\Delta)^{\frac{s}{2}} u|$, we can have (u_n, v_n) is non-negative. Secondly, we replace $|u_n|$ with $|u_n|^*$ and $|v_n|$ with $|v_n|^*$, where $|\cdot|^*$ is the Schwarz symmetrization rearrangement, then we can obtain another function in $B_{R_2}(a, b)$ with $J(|u_n|^*, |v_n|^*) \le J(u_n, v_n)$). Moreover by Lemma 4.3, $s_{u_n,v_n} * (u_n, v_n) \in \mathcal{P}^+_{\mu_1,\mu_2}$ such that

$$
([u_n]^2 + [v_n]^2)^{\frac{1}{2}} < R_2,
$$

and

$$
\begin{aligned}
J(s_{u_n,v_n} * (u_n, v_n)) &= \min\{J(t * (u_n, v_n)) : t \in \mathbb{R} \text{ and } ([t * u_n]^2 + [t * v_n]^2)^{\frac{1}{2}} < R_2\} \\
&\le J(u_n, v_n).
\end{aligned}
$$

Thus, we get another minimizing sequence $\{\tilde{u}_n := s_{u_n,v_n} * u_n, \tilde{v}_n := s_{u_n,v_n} * v_n\}$ with $\{(\tilde{u}_n, \tilde{v}_n)\} \subset S_r$. By Lemma 4.5, we have $([\tilde{u}_n]^2 + [\tilde{v}_n]^2)^{\frac{1}{2}} \le R_2 - \delta$. Then, from Ekeland's Variational Principle [17], we know there exists a radially Palais–Smale sequence $\{(w_n, z_n)\}$ for $J|_S$ satisfying $\|(w_n, z_n) - (\tilde{u}_n, \tilde{v}_n)\|_H \to 0$ as $n \to \infty$. Following, we claim $P_{\mu_1,\mu_2}(w_n, z_n) = P(\tilde{u}_n, \tilde{v}_n) + o_n(1) = o_n(1)$. Firstly, by the Brézis–Lieb Lemma and Sobolev's embedding Theorem, we have

$$
[w_n]^2 = [w_n - \tilde{u}_n]^2 + [\tilde{u}_n]^2 + o_n(1) = [\tilde{u}_n]^2 + o_n(1),
$$

and

$$\int_{\mathbb{R}^N} |w_n|^p dx = \int_{\mathbb{R}^N} |w_n - \tilde{u}_n|^p dx + \int_{\mathbb{R}^N} |\tilde{u}_n|^p dx + o_n(1) = \int_{\mathbb{R}^N} |\tilde{u}_n|^p dx + o_n(1).$$

Moreover, by the Hölder inequality and Lemma 2.1, we get

$$\int_{\mathbb{R}^N} |w_n|^{r_1} |z_n|^{r_2} dx = \int_{\mathbb{R}^N} |\tilde{u}_n|^{r_1} |\tilde{v}_n|^{r_2} dx + o_n(1),$$

and

$$\int_{\mathbb{R}^N} (I_\alpha * |w_n|^{2^*_{\alpha,s}}) |z_n|^{2^*_{\alpha,s}} dx = \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_n|^{2^*_{\alpha,s}}) |\tilde{v}_n|^{2^*_{\alpha,s}} dx + o_n(1).$$

The same relationship happen to z_n and $\tilde{v}_n$. Therefore, we obtain

$$P_{\mu_1,\mu_2}(w_n, z_n) = P_{\mu_1,\mu_2}(\tilde{u}_n, \tilde{v}_n) + o_n(1) = o_n(1), \quad \text{and} \quad w_n^-, z_n^- \to 0, \quad \text{a.e. in } \mathbb{R}^N.$$

Thus from Proposition 4.9, we obtain there is $(u, v) \in H_r$ and $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ with $\lambda_1, \lambda_2 < 0$, such that $(w_n, z_n) \to (u, v)$ in H and $(\lambda_{1,n}, \lambda_{2,n}) \to (\lambda_1, \lambda_2)$ in $\mathbb{R}^2$. Hence, $(u, v) \in \mathcal{P}_{\mu_1,\mu_2}$ is a solution for (1.1)–(1.2), which is a normalized ground state with $J(u, v) = m(a, b)$. Moreover, for any ground state solution (u, v), from $m(a, b) < 0$ and Lemma 4.2, we have

$$J(u, v) = m(a, b) = \inf_{B_{R_2}(a,b)} J(u, v), \quad \text{and} \quad ([u]^2 + [v]^2)^{\frac{1}{2}} < R_2,$$

i.e. (u, v) is a local minimizer for $J(u, v)$ on $B_{R_2}(a, b)$. □

5 The case: $\bar{p} < p < r_1 + r_2 < 2^*_s$

Firstly, we show the boundedness result for this case.

Lemma 5.1. *Let $\bar{p} < p < r_1 + r_2 < 2^*_s$ and $\mu_1, \mu_2, a, b > 0$. Let $\{(u_n, v_n)\} \subset S_r$ be a Palais–Smale sequence such that*

$$J(u_n, v_n) \to c; \quad J'(u_n, v_n)|_S \to 0, \quad \text{and} \quad P_{\mu_1,\mu_2}(u_n, v_n) \to 0.$$

Then $\{(u_n, v_n)\}$ is bounded in H.

Proof. In this case, we have $2 < p\gamma_p < (r_1 + r_2)\gamma_{(r_1+r_2)}$. Then

$$\begin{aligned} c + o_n(1) &= J_{\mu_1,\mu_2}(u_n, v_n) - \frac{1}{2} P_{\mu_1,\mu_2}(u_n, v_n) \\ &= \left(\frac{\gamma_p}{2} - \frac{1}{p}\right)(\mu_1 |u_n|_p^p + \mu_2 |v_n|_p^p) + \beta \left[\frac{(r_1 + r_2)\gamma_{(r_1+r_2)}}{2} - 1\right] \int_{\mathbb{R}^N} |u_n|^{r_1} |v_n|^{r_2} dx \\ &\quad + \left(1 - \frac{1}{2^*_{\alpha,s}}\right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_{\alpha,s}}) |v_n|^{2^*_{\alpha,s}} dx, \end{aligned}$$

by each coefficient is positive, we get $\{(u_n, v_n)\}$ is bounded in H. The proof is completed. □

Recalling the decomposition of $\mathcal{P}_{\mu_1,\mu_2} = \mathcal{P}^+_{\mu_1,\mu_2} \cup \mathcal{P}^0_{\mu_1,\mu_2} \cup \mathcal{P}^-_{\mu_1,\mu_2}$, we have

Lemma 5.2. *$\mathcal{P}^0_{\mu_1,\mu_2} = \emptyset$ and $\mathcal{P}_{\mu_1,\mu_2}$ is a C^1-submanifold in H with codimension 3.*

Proof. If there is $(u,v) \in \mathcal{P}^0_{\mu_1,\mu_2}$, then

$$[u]^2+[v]^2 = \gamma_p(\mu_1|u|_p^p+\mu_2|v|_p^p)+\beta(r_1+r_2)\gamma_{(r_1+r_2)}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx+2\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx,$$

and

$$2([u]^2+[v]^2) = p\gamma_p^2(\mu_1|u|_p^p+\mu_2|v|_p^p)+\beta(r_1+r_2)^2\gamma^2_{(r_1+r_2)}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx + 4\cdot 2^*_{\alpha,s}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx.$$

From above, we obtain

$$(2-p\gamma_p)([u]^2+[v]^2) = \beta(r_1+r_2)\gamma_{(r_1+r_2)}[(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p]\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx + 2(22^*_{\alpha,s}-p\gamma_p)\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx.$$

Since $2-p\gamma_p<0$, $(r_1+r_2)\gamma_{(r_1+r_2)}-p\gamma_p>0$ and $22^*_{\alpha,s}-p\gamma_p>0$, we have $(u,v)=(0,0)$, which contradicts with $(u,v)\in S$. The remainder parts of this proof is similar with Lemma 4.2, and we omit the details here. □

Following, we show the geometry for this mass supercritical case.

Lemma 5.3. *For every $(u,v)\in S$, the function $\Psi_{\mu_1,\mu_2}(t)$ has exactly one critical point $t_{u,v}\in\mathbb{R}$ such that $t_{u,v}*(u,v)\in\mathcal{P}_{\mu_1,\mu_2}$. Moreover:*

(i) $\mathcal{P}_{\mu_1,\mu_2}=\mathcal{P}^-_{\mu_1,\mu_2}$;

(ii) $\Psi_{\mu_1,\mu_2}(t)$ is strictly decreasing and concave on $(t_{u,v},+\infty)$, and $\Psi_{\mu_1,\mu_2}(t_{u,v})=\max_{t\in\mathbb{R}}\Psi_{\mu_1,\mu_2}(t)>0$;

(iii) The map $(u,v)\mapsto t_{u,v}$ is of class C^1;

(iv) If $\mathcal{P}_{\mu_1,\mu_2}(u,v)<0$, then $t_{u,v}<0$.

Proof. From the definition of $\Psi_{\mu_1,\mu_2}(t)$, we have

$$\begin{aligned}\Psi'_{\mu_1,\mu_2}(t) &= e^{2st}\Big[s([u]^2+[v]^2)-s\gamma_p e^{(p\gamma_p-2)st}(\mu_1|u|_p^p+\mu_2|v|_p^p)-2se^{(22^*_{\alpha,s}-2)st}\int_{\mathbb{R}^N}(I_\alpha*|u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx \\ &\quad - s\beta(r_1+r_2)\gamma_{(r_1+r_2)}e^{[(r_1+r_2)\gamma_{(r_1+r_2)}-2]st}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx\Big],\end{aligned}$$

which implies $\Psi_{\mu_1,\mu_2}(t)$ has exactly one critical point $t_{u,v}$. Since

$$\Psi_{\mu_1,\mu_2}(-\infty)=0^+, \quad \text{and} \quad \Psi_{\mu_1,\mu_2}(+\infty)=-\infty,$$

we get $t_{u,v}$ is a strict maximum point at a positive level and $t_{u,v}*(u,v)\in\mathcal{P}_{\mu_1,\mu_2}$. From $\Psi''_{\mu_1,\mu_2}(t_{u,v})\le 0$ and $\mathcal{P}^0_{\mu_1,\mu_2}=\emptyset$, we have $\Psi''_{\mu_1,\mu_2}(t_{u,v})<0$, this implies $t_{u,v}*(u,v)\in\mathcal{P}^-_{\mu_1,\mu_2}$ and $\mathcal{P}_{\mu_1,\mu_2}=\mathcal{P}^-_{\mu_1,\mu_2}$. To see (iii), we use the implicit function theorem as in Lemma 4.3. Finally, since $\Psi'_{\mu_1,\mu_2}(t)<0$ if and only if $t>t_{u,v}$, we get $P_{\mu_1,\mu_2}(u,v)=\Psi'_{\mu_1,\mu_2}(0)<0$ if and only if $t_{u,v}<0$. □

Remark 5.4. From Lemma 5.3, we see $m(a,b)=m^-(a,b)$.

Lemma 5.5. $m(a,b) = \inf_{\mathcal{P}_{\mu_1,\mu_2}} J(u,v) > 0.$

Proof. For $(u,v) \in \mathcal{P}_{\mu_1,\mu_2}$, then from (3.2), (4.1)-(4.3), we get

$$([u]^2+[v]^2) \le \gamma_p(C_1\mu_1 + C_2\mu_2)([u]^2+[v]^2)^{\frac{p\gamma_p}{2}} + 2C_4([u]^2+[v]^2)^{2^*_{\alpha,s}}$$
$$+ C_3\beta(r_1+r_2)\gamma_{(r_1+r_2)}([u]^2+[v]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}}.$$

Since $p\gamma_p > 2$, we obtain $\inf_{\mathcal{P}_{\mu_1,\mu_2}}([u]^2+[v]^2) > 0$ and so

$$\begin{aligned}\inf_{\mathcal{P}_{\mu_1,\mu_2}} J(u,v) &= \inf_{\mathcal{P}_{\mu_1,\mu_2}}\left[J(u,v) - \frac{1}{2}P_{\mu_1,\mu_2}(u,v)\right]\\ &= \inf_{\mathcal{P}_{\mu_1,\mu_2}}\left[\left(\frac{\gamma_p}{2}-\frac{1}{p}\right)(\mu_1|u|_p^p + \mu_2|v|_p^p) + \beta\left(\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}-1\right)\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx\right.\\ &\quad\left. + \left(1-\frac{1}{2^*_{\alpha,s}}\right)\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx\right] > 0.\end{aligned}$$

Therefore we have $m(a,b) > 0$. □

Lemma 5.6. *For any $\delta > 0$ sufficiently small, we have $0 < \sup_{\overline{B_\delta}} J(u,v) < m(a,b)$ and*

$$u \in \overline{B_\delta} \Rightarrow J(u,v),\ P_{\mu_1,\mu_2}(u,v) > 0,$$

where $B_\delta := \{(u,v) \in S : ([u]^2+[v]^2)^{\frac{1}{2}} < \delta\}$.

Proof. Since (3.2), (4.1)–(4.3), we get

$$J(u,v) \ge \frac{1}{2}([u]^2+[v]^2) - \frac{(\mu_1C_1+\mu_2C_2)}{p}([u]^2+[v]^2)^{\frac{p\gamma_p}{2}} - \beta C_3([u]^2+[v]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}}$$
$$- \frac{C_4}{2^*_{\alpha,s}}([u]^2+[v]^2)^{2^*_{\alpha,s}},$$

and

$$P_{\mu_1,\mu_2}(u,v) \ge s([u]^2+[v]^2) - s\gamma_p(\mu_1C_1+\mu_2C_2)([u]^2+[v]^2)^{\frac{p\gamma_p}{2}} - 2sC_4([u]^2+[v]^2)^{2^*_{\alpha,s}}$$
$$- s\beta(r_1+r_2)\gamma_{(r_1+r_2)}C_3([u]^2+[v]^2)^{\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2}}.$$

Thus for $\delta > 0$ small enough, we have $J(u,v) > 0$ and $P_{\mu_1,\mu_2}(u,v) > 0$. Moreover, by Lemma 5.5, we can choose δ with smaller quantity, such that

$$J(u,v) \le ([u]^2+[v]^2) < m(a,b).$$ □

To use the Proposition 4.9, we need some properties about $m(a,b)$. Firstly, we get the monotonicity of $m(a,b)$. The proof is similar with *Case 2* in Lemma 3.4 and we omit this process here.

Lemma 5.7. $m(a,b) \le m(a_1,b_1)$ *for any* $0 < a_1 \le a$, $0 < b_1 \le b$.

Lemma 5.8. *For $a,b > 0$ fixed, we have* $\lim_{\beta\to+\infty} m(a,b) = 0^+$.

Proof. This lemma is equivalent to prove, for any $\varepsilon > 0$, there exists $\bar{\beta} > 0$ such that

$$m(a,b) < \varepsilon \quad \text{for any } \beta \geq \bar{\beta}.$$

Firstly, from Lemma 5.6, for any $\beta > 0$, we have $m(a,b) > 0$. If we choose $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $|\varphi|_2 \leq \min\{a,b\}$, by Lemmas 5.3 and 5.7 we obtain,

$$m(a,b) \leq m(|\varphi|_2, |\varphi|_2) \leq \max_{t\in\mathbb{R}} J(t*(\varphi,\varphi)) = \max_{t\in\mathbb{R}} \left[E(t) - \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t} \int_{\mathbb{R}^N} |\varphi|^{(r_1+r_2)} dx\right],$$

where

$$E(t) := e^{2ts}[\varphi]^2 - \frac{e^{p\gamma_p st}}{p}(\mu_1+\mu_2)|\varphi|_p^p - \frac{e^{22^*_{\alpha,s}st}}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\varphi|^{2^*_{\alpha,s}})|\varphi|^{2^*_{\alpha,s}} dx.$$

From $p\gamma_p - 2 > 0$ and $22^*_{\alpha,s} - 2 > 0$ we see

$$\begin{aligned} E(t) &= e^{2ts}\left([\varphi]^2 - \frac{e^{(p\gamma_p-2)st}}{p}(\mu_1+\mu_2)|\varphi|_p^p - \frac{e^{(22^*_{\alpha,s}-2)st}}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\varphi|^{2^*_{\alpha,s}})|\varphi|^{2^*_{\alpha,s}} dx\right) \\ &= e^{2ts}([\varphi]^2 + o(1)) \to 0^+, \quad \text{as } t \to -\infty. \end{aligned}$$

And there exists $\tilde{t} > 0$ such that $E(t) < \frac{\varepsilon}{4}$ for $t < -\tilde{t}$. Moreover, there exists $\bar{\beta} > 0$, such that for any $\beta \geq \bar{\beta}$,

$$\begin{aligned} &\max_{t\geq -\tilde{t}} \left[E(t) - \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t} |\varphi|_{r_1+r_2}^{r_1+r_2}\right] \\ &\leq \max_{t\geq -\tilde{t}} \left[e^{2ts}[\varphi]^2 - \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t} |\varphi|_{(r_1+r_2)}^{(r_1+r_2)} - \frac{e^{22^*_{\alpha,s}st}}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\varphi|^{2^*_{\alpha,s}})|\varphi|^{2^*_{\alpha,s}} dx\right] \\ &\leq \max_{t\in\mathbb{R}} \left[e^{2ts}[\varphi]^2 - \frac{e^{22^*_{\alpha,s}st}}{2^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_\alpha * |\varphi|^{2^*_{\alpha,s}})|\varphi|^{2^*_{\alpha,s}} dx\right] - \beta e^{-s(r_1+r_2)\gamma_{(r_1+r_2)}\tilde{t}} |\varphi|_{r_1+r_2}^{r_1+r_2} \\ &\leq \left(1 - \frac{1}{2^*_{\alpha,s}}\right)[\varphi]^{\frac{22^*_{\alpha,s}}{2^*_{\alpha,s}-1}} \left(\int_{\mathbb{R}^N} (I_\alpha * |\varphi|^{2^*_{\alpha,s}})|\varphi|^{2^*_{\alpha,s}} dx\right)^{\frac{-1}{2^*_{\alpha,s}-1}} - \beta e^{-s(r_1+r_2)\gamma_{(r_1+r_2)}\tilde{t}} |\varphi|_{r_1+r_2}^{r_1+r_2}. \end{aligned}$$

Hence, we have $\max_{t\in\mathbb{R}} \left[E(t) - \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t} |\varphi|_{r_1+r_2}^{r_1+r_2}\right] < \varepsilon$ for $\beta \geq \bar{\beta}$, and $m(a,b) < \varepsilon$. □

Thus by the above lemma, we have the following conclusion:

Lemma 5.9. *There exists $\hat{\beta}_1 > 0$, we get $m(a,b) < \frac{N+2s-\alpha}{2N-\alpha}\left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}$ for any $\beta > \hat{\beta}_1$.*

Lemma 5.10. *There exists $\hat{\beta}_2 > 0$ such that for any $\beta > \hat{\beta}_2$, the level satisfies*

$$m(a,b) < \min\{m(a,0), m(0,b)\}.$$

Proof. From Theorem 3.3, $m(a,0) > 0$ can be achieved by $u^* \in S_a$. Similarly, $m(0,b) > 0$ can be achieved by $v^* \in S_b$. Since

$$I_{\mu_1,a}(t*u^*) \to 0, \quad \text{and} \quad I_{\mu_2,b}(t*v^*) \to 0, \quad \text{as } t \to -\infty,$$

there is $t^* \ll -1$ which is independent of β, such that

$$\max_{t<t^*} J(u^*,v^*) < \max_{t<t^*} I_{\mu_1,a}(t*u^*) + \max_{t<t^*} I_{\mu_2,b}(t*v^*) < \min\{m(a,0), m(0,b)\}.$$

On the other hand, for $t > t^*$, firstly we have

$$\int_{\mathbb{R}^N} |t * u^*|^{r_1} |t * v^*|^{r_2} dx = e^{st(r_1+r_2)\gamma_{(r_1+r_2)}} \int_{\mathbb{R}^N} |u^*|^{r_1} |v^*|^{r_2} dx \geq C e^{st^*(r_1+r_2)\gamma_{(r_1+r_2)}},$$

for some $C > 0$. Then by Theorem 3.3, we get

$$\begin{aligned}\max_{t\geq t^*} J(t*(u^*,v^*)) &\leq \max_{t\geq t^*} I_{\mu_1,a}(t*u^*) + \max_{t\geq t^*} I_{\mu_2,b}(t*v^*) - C\beta e^{st^*(r_1+r_2)\gamma_{(r_1+r_2)}} \\ &\leq m(a,0) + m(0,b) - C\beta e^{st^*(r_1+r_2)\gamma_{(r_1+r_2)}}.\end{aligned}$$

Hence, there is $\hat{\beta}_2 > 0$, for any $\beta > \hat{\beta}_2$ such that $m(a,b) < \min\{m(a,0), m(0,b)\}$. □

To prove Theorem 1.2, we give the following minimax theorem to establish the existence of Palais–Smale sequence. At first, we show some definitions.

Definition 5.11 ([21, Definition 3.1])**.** Let Θ be a closed subset of a metric space $X \subset H$. We say that a class $\mathcal{F}$ of compact subsets of X is a homotopy-stable family with closed boundary Θ provided

(i) every set in $\mathcal{F}$ contains Θ;

(ii) for any set $Y \in \mathcal{F}$ and any $\eta \in C([0,1]\times X, X)$ satisfying $\eta(t,x) = x$ for all $(t,x) \in (\{0\}\times X) \cup ([0,1]\times\Theta)$, we have that $\eta(\{1\}\times Y) \in \mathcal{F}$.

Definition 5.12. [33] Let M be a C^∞ m-dimensional manifold and $\widetilde{TM} = TM \setminus \{0\}$, where TM is a tangent bundle. A function $F : TM \to [0,\infty)$ is called a Finsler structure on M if F has the following properties:

(i) $F(tY) = tF(Y)$, $\forall t \in \mathbb{R}^+$;

(ii) F is C^∞ on $\widetilde{TM}$;

(iii) for every non-zero $Y \in T_xM$, the induced quadratic form g_Y is an inner product in T_xM, where

$$g_Y(U,V) := \frac{1}{2}\frac{\partial^2}{\partial_s\partial_t}(F^2(Y+sU+tV))|_{s=t=0},$$

and T_xM is the tangent space at the point x. A Finsler manifold is a C^∞-manifold M with its Finsler structure F.

Remark 5.13. From [14], we know Riemannian manifolds are special cases of Finsler manifolds. Denote $X := \mathbb{R} \times S_r$. Since $\mathbb{R}$ is a Banach space and $S_r \subset H^s(\mathbb{R}^N, R) \times H^s(\mathbb{R}^N, R)$ is a Banach manifold, similar to [24] (see (7.2) there), [26, Lemma 4.8] and [32, Theorem 6.12], we know X is a para-compact space with satisfying the requirement of locally limited refinement for each open coverage. Moreover, by [41, Section 3], we can assign X a Finsler structure and we know X is a Finsler manifold.

Proposition 5.14 ([21, Theorem 3.2])**.** *Let φ be a C^1-functional on a complete connected C^1-Finsler manifold X (without boundary) and consider a homotopy stable family $\mathcal{F}$ of compact subsets of X with a closed boundary B.*

$$c = c(\varphi, \mathcal{F}) = \inf_{Y\in\mathcal{F}} \max_{u\in Y} \varphi(u),$$

and suppose that $\sup \varphi(\Theta) < c$. *Then for any sequence of sets* $\{Y_n\}$ *in* $\mathcal{F}$ *such that* $\lim_{n\to\infty} \sup_{Y_n} \varphi = c$, *there exists a sequence* $\{u_n\}$ *in* X *such that*

$$\lim_{n\to\infty} \varphi(u_n) = c; \quad \lim_{n\to\infty} \|d\varphi(u_n)\| = 0; \quad \textit{and} \quad \lim_{n\to\infty} \operatorname{dist}(u_n, Y_n) = 0.$$

Furthermore, if $d\varphi$ *is uniformly continuous, then* u_n *can be chosen to be in* Y_n *for each* n.

Proof of Theorem 1.2. Using the strategy from [25], for $\delta > 0$ be defined by Lemma 5.6, let the function $\tilde{J} : \mathbb{R} \times H \mapsto \mathbb{R}$ as

$$\begin{aligned}\tilde{J}(t,(u,v)) := J(t*(u,v)) = \frac{e^{2st}}{2}([u]^2+[v]^2) - \frac{e^{sp\gamma_p t}}{p}(\mu_1|u|_p^p + \mu_2|v|_p^p) \\ - \beta e^{s(r_1+r_2)\gamma_{(r_1+r_2)}t}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}dx \\ - \frac{e^{22^*_{\alpha,s}st}}{2^*_{\alpha,s}}\int_{\mathbb{R}^N}(I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}}dx,\end{aligned}$$

then $\tilde{J} \in C^1$ and a Palais–Smale sequence for $\tilde{J}|_{\mathbb{R}\times S_r}$ is a Palais–Smale sequence for $\tilde{J}|_{\mathbb{R}\times S}$. Setting $J^c := \{(u,v) \in S, J(u,v) \le c\}$, we introduce the minimax class

$$\Gamma := \{\gamma = (\alpha,\beta) \in C([0,1], \mathbb{R}\times S_r) : \gamma(0) \in (0,\overline{B}_\delta), \gamma(1) \in (0,J^0)\},$$

with the minimax level

$$\sigma(a,b) := \inf_{\gamma\in\Gamma} \max_{(t,(u,v))\in\gamma([0,1])} \tilde{J}(t,(u,v)).$$

Let $(u,v) \in S_r$. From $[t*u]^2 + [t*v]^2 \to 0^+$ as $t \to -\infty$ and $J(t*(u,v)) \to -\infty$ as $t \to +\infty$, there is $t_0 \ll -1$ and $t_1 \gg 1$ such that

$$\gamma_{(u,v)} : \tau \in [0,1] \mapsto (0, ((1-\tau)t_0 + \tau t_1) * (u,v)) \in \mathbb{R}\times S_r, \tag{5.1}$$

which is a path in Γ and $\sigma(a,b)$ is a real number. For any $\gamma = (\alpha,\beta) \in \Gamma$, we study the function

$$\Pi_\gamma : \tau \in [0,1] \mapsto P_{\mu_1,\mu_2}(\alpha(\tau) * \beta(\tau)) \in \mathbb{R}.$$

From Lemma 5.6, we find $\Pi_\gamma(0) = P_{\mu_1,\mu_2}(\beta(0)) > 0$. Besides, from Lemma 5.3, $\Psi_{\mu_1,\mu_2}(t) > 0$ for any $t \in (-\infty, t_{u,v})$. If $(u,v) = \beta(1)$, we have $\Psi_{\mu_1,\mu_2}(0) = J(\beta(1)) \le 0$. Hence, we obtain $t_{\beta(1)} < 0$ and $\Pi_\gamma(1) = P_{\mu_1,\mu_2}(0 * \beta(1)) < 0$. Since the map $\tau \mapsto \alpha(\tau) * \beta(\tau)$ is continuous from $[0,1]$ to H, there exists $\tau_\gamma \in (0,1)$ such that $\Pi_\gamma(\tau_\gamma) = 0$, which implies $\alpha(\tau_\gamma) * \beta(\tau_\gamma) \in \mathcal{P}_{\mu_1,\mu_2} \cap S_r$ and

$$\max_{\gamma([0,1])} \tilde{J} \ge \tilde{J}(\gamma(\tau_\gamma)) = J(\alpha(\tau_\gamma) * \beta(\tau_\gamma)) \ge \inf_{\mathcal{P}_{\mu_1,\mu_2}\cap S_r} J(u,v) = m_r(a,b).$$

Therefore, $\sigma(a,b) \ge m_r(a,b)$. On the other hand, for any $(u,v) \in \mathcal{P}_{\mu_1,\mu_2} \cap S_r$, from (5.1), $\gamma_{(u,v)}$ is a path in Γ and by Lemma 5.3,

$$J(u,v) = \max_{\gamma_{(u,v)}([0,1])} \tilde{J} \ge \sigma(a,b),$$

then $m_r(a,b) \ge \sigma(a,b)$. Combining this with (5.6), we get

$$\sigma(a,b) = m_r(a,b) > \sup_{(\overline{B}_\delta\cup J^0)\cap S_r} J(u,v) = \sup_{((0,\overline{B}_\delta)\cup(0,J^0))\cap(\mathbb{R}\times S_r)} \tilde{J}.$$

From Definition 5.11, the set $\{\gamma([0,1]) : \gamma \in \Gamma\}$ is a homotopy stable family of compact subsets of $\mathbb{R} \times S_r$ with closed boundary $(0, \overline{B}_\delta) \cup (0, J^0)$. By Proposition 5.14, similar to [24,32], taking any minimizing sequence $\{\gamma_n = (\alpha_n, \beta_n)\} \subset \Gamma_n$ for $\sigma(a,b)$ with $\alpha_n \equiv 0$, and $\beta_n(\tau) \geq 0$ a.e. in $\mathbb{R}$, for every $\tau \in [0,1]$, there exists a Palais–Smale sequence $\{t_n, w_n\} \subset \mathbb{R} \times S_r$ for $\tilde{J}|_{\mathbb{R}\times S_r}$ at the level $\sigma(a,b)$, where $w_n = (u_n, v_n)$, such that,

$$\partial t \tilde{J}(t_n, w_n) \to 0, \quad \|\partial w \tilde{J}(t_n, w_n)\|_{(T_{w_n} S_r)*} \to 0, \quad \text{as } n \to \infty, \tag{5.2}$$

with an additional property

$$|t_n| + \text{dist}_H(w_n, \beta_n([0,1])) \to 0, \quad \text{as } n \to \infty. \tag{5.3}$$

From (5.3), we have t_n is bounded in both side. Besides, from the first formula of (5.2), we have $P_{\mu_1,\mu_2}(t_n * (u_n, v_n)) \to 0$, and from the second formula of (5.2) with the boundedness of t_n, for any $\varphi \in T_{w_n} S_r$,

$$dJ(t_n * w_n)[t_n * \varphi] = o_n(1)\|\varphi\| = o_n(1)\|t_n * \varphi\|, \quad \text{as } n \to \infty.$$

Following, we define $\hat{w}_n := t_n * w_n$ with $\hat{w}_n = (\hat{u}_n, \hat{v}_n)$. Therefore, $\{(\hat{u}_n, \hat{v}_n)\}$ is a Palais–Smale sequence for $J(u,v)|_{S_r}$ at the level $\sigma(a,b)$ with an additional condition $P_{\mu_1,\mu_2}(\hat{u}_n, \hat{v}_n) \to 0$. From Lemma 5.8, there exists $\hat{\beta}_1 > 0$, $m_r(a,b) \in \left(0, \frac{N+2s-\alpha}{2N-\alpha}\left(\frac{S^*}{2}\right)^{\frac{2N-\alpha}{N+2s-\alpha}}\right)$ for any $\beta > \hat{\beta}_1$. Besides, from Lemma 5.10, there exists $\hat{\beta}_2 > 0$ such that $m_r(a,b) < \min\{m(a,0), m(0,b)\}$. Following, we may require $\beta_0 = \max\{\hat{\beta}_1, \hat{\beta}_2\}$. Then for $\beta \in (\beta_0, +\infty)$, by Proposition 4.9, we know there is $(u,v) \in H$ with $u, v > 0$ a.e. in $\mathbb{R}$, such that $(\hat{u}_n, \hat{v}_n) \to (u,v)$ in H and $J(u,v) = m_r(a,b)$. Hence, we need to show

$$\inf_{\mathcal{P}_{\mu_1,\mu_2} \cap S_r} J(u,v) = \inf_{\mathcal{P}_{\mu_1,\mu_2}} J(u,v) = m(a,b).$$

If this does not happen, there is $(\bar{u}, \bar{v}) \in \mathcal{P}_{\mu_1,\mu_2} \setminus S_r$ such that $J(\bar{u}, \bar{v}) < m_r(a,b)$. Denote $(\tilde{u}, \tilde{v}) := (|\bar{u}|^*, |\bar{v}|^*)$ as the symmetric decreasing rearrangement of $(\bar{u}, \bar{v})$ such that

$$[\tilde{u}]^2 + [\tilde{v}]^2 \leq [\hat{u}]^2 + [\hat{v}]^2, \quad J(\tilde{u}, \tilde{v}) \leq J(\bar{u}, \bar{v}), \quad \text{and} \quad P_{\mu_1,\mu_2}(\tilde{u}, \tilde{v}) \leq P_{\mu_1,\mu_2}(\bar{u}, \bar{v}) = 0.$$

If $P_{\mu_1,\mu_2}(\tilde{u}, \tilde{v}) = 0$, which $(\tilde{u}, \tilde{v}) \in \mathcal{P}_{\mu_1,\mu_2} \cap S_r$, there is a contradiction. On the other hand, if $P_{\mu_1,\mu_2}(\tilde{u}, \tilde{v}) < 0$, from Lemma 5.3, we get $t_{\tilde{u},\tilde{v}} < 0$. Therefore, by $t_{\tilde{u},\tilde{v}} * (\tilde{u}, \tilde{v}) \in \mathcal{P}_{\mu_1,\mu_2}$, we have

$$\begin{aligned}
J(\bar{u}, \bar{v}) &\leq J(t_{\tilde{u},\tilde{v}} * (\tilde{u}, \tilde{v})) - \frac{1}{2} P_{\mu_1,\mu_2}(t_{\tilde{u},\tilde{v}} * (\tilde{u}, \tilde{v})) \\
&= \left(\frac{\gamma_p}{2} - \frac{1}{p}\right) e^{p\gamma_p s t_{\tilde{u},\tilde{v}}} (\mu_1 |\tilde{u}|_p^p + \mu_2 |\tilde{v}|_p^p) \\
&\quad + \beta \left[\frac{(r_1+r_2)\gamma_{(r_1+r_2)}}{2} - 1\right] e^{(r_1+r_2)\gamma_{(r_1+r_2)} s t_{\tilde{u},\tilde{v}}} \int_{\mathbb{R}^N} |\tilde{u}|^{r_1} |\tilde{v}|^{r_2} dx \\
&\quad + \left(1 - \frac{1}{2^*_{\alpha,s}}\right) e^{22^*_{\alpha,s} s t_{\tilde{u},\tilde{v}}} \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}|^{2^*_{\alpha,s}}) |\tilde{v}|^{2^*_{\alpha,s}} dx \\
&< J(\bar{u}, \bar{v}),
\end{aligned}$$

which is a contradiction. Thus, we have $m_r(a,b) = m(a,b)$ and (u,v) is a ground state solution.

□

6 The case: $p = r_1 + r_2 = 2_s^*$

Lemma 6.1. *Assume $s \in (0,1)$, $2s < N \le 4s$, $p = r_1 + r_2 = 2_s^*$ and $a, b, \mu_1, \mu_2, \beta > 0$. Then the following system*

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2}u + \beta r_1 |u|^{r_1-2}u|v|^{r_2} + (I_\alpha * |v|^{2^*_{\alpha,s}})|u|^{2^*_{\alpha,s}-2}u, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{p-2}v + \beta r_2 |v|^{r_2-2}v|u|^{r_1} + (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}-2}v, \\ \int_{\mathbb{R}^N} u^2 dx = a^2, \quad \int_{\mathbb{R}^N} v^2 dx = b^2, \quad u, v \in H^s(\mathbb{R}^N), \end{cases} \tag{6.1}$$

has no positive solution.

Proof. Assume by contradiction that there is a positive solution (u, v) of (6.1) for some $\lambda_1, \lambda_2 \in \mathbb{R}$. On one hand, from Proposition 4.9 and [29, Lemma 2.3], we see that $\lambda_1, \lambda_2 < 0$ for $2s < N \le 4s$. On the other hand, by Proposition 2.4 we know (u, v) satisfies the Pohožaev identity such that

$$[u]^2 + [v]^2 = (\mu_1 |u|_{2_s^*}^{2_s^*} + \mu_2 |v|_{2_s^*}^{2_s^*}) + \beta 2_s^* \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx + 2\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx. \tag{6.2}$$

Moreover since (u, v) is a weak solution to (1.1)-(1.2), it satisfies

$$\begin{aligned} [u]^2 + [v]^2 = &\int_{\mathbb{R}^N} (\lambda_1 |u|^2 + \lambda_2 |v|^2) dx + (\mu_1 |u|_{2_s^*}^{2_s^*} + \mu_2 |v|_{2_s^*}^{2_s^*}) + \beta 2_s^* \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} dx \\ &+ 2\int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_{\alpha,s}})|v|^{2^*_{\alpha,s}} dx. \end{aligned} \tag{6.3}$$

Combining (6.2)–(6.3), we show

$$\int_{\mathbb{R}^N} \lambda_1 |u|^2 + \lambda_2 |v|^2 dx = \lambda_1 a^2 + \lambda_2 b^2 = 0.$$

From which we obtain $\lambda_1 = \lambda_2 = 0$. This is clearly a contradiction with $\lambda_1, \lambda_2 < 0$. The proof is complete. □

Proof of Theorem 1.3. Theorem 1.3 follows from Lemma 6.1, then we finish the proof. □

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Declarations

We would like to thank you for following the above instructions. This will definitely speed up the publication process of your paper.

Data Availability

Date sharing is not applicable to this article as no new data were created and analyzed in this study.

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Center manifolds for random dynamical systems with generalized trichotomies

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Abstract. In this paper, we consider small perturbations of linear random dynamical systems evolving on a Banach space and admitting a general form of trichotomy. We prove the existence of invariant center manifolds in both continuous and discrete-time. Furthermore, we provide several illustrative examples.

Keywords: invariant manifolds, random dynamical systems, trichotomies.

2020 Mathematics Subject Classification: 37L55, 37D10, 37H99.

1 Introduction

The theory of center manifolds plays a crucial role in stability and bifurcation theory, as it often enables the reduction of the dimension of the state space (see [19,29,31–33]). The origins of this theory date back to the 1960s, with the works of Pliss [49] and Kelley [34,35]. Subsequently, various results on this subject were developed by several authors. In the context of autonomous differential equations, we recommend the surveys by Vanderbauwhede [54] (see also Vanderbauwhede and Gils [56]) for the finite-dimensional case and by Vanderbauwhede and Iooss [55] in the infinite-dimensional case. For the nonautonomous case we recommend the survey by Aulbach and Wanner [3]. We also recommend [23,24] and [22,25,26,44,53] for, respectively, finite and infinite dimension.

The concept of trichotomy is an essential tool for obtaining center manifolds. The (uniform) exponential trichotomies were introduced, independently, by Sacker and Sell [51], Aulbach [2] and Elaydi and Hájek [28]. This notion was motivated by the idea of (uniform) exponential dichotomy that started in the thirties with Perron [47,48].

Several generalizations of exponential trichotomies have since emerged. Fenner and Pinto [42] introduced the (h,k)-trichotomies that use non exponential growth rates and Barreira and Valls [4,5] introduced nonuniform exponential trichotomies that take into account the initial time. Later, Barreira and Valls [6,7] introduced the ρ-nonuniform exponential trichotomies that are nonuniform and non exponential, but do not include the (h,k)-trichotomies.

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In [12, 15], a general type of trichotomies was introduced, for linear differential equations and linear difference equations, respectively. This new framework contains as special cases the notions of trichotomies mentioned above and also contains additional new cases (the case of dichotomies was done in [13, 14]).

Invariant manifold theory has also been extended to dynamical systems with randomness. In this work, we focus on random dynamical systems (RDS), which can be generated, for instance, by random or stochastic differential equations. In this context, various studies have addressed center, stable, unstable, and inertial invariant manifolds, both locally and globally, across a range of spaces that goes from finite to infinite dimension, including Hilbert spaces and separable Banach spaces. Arnold's monograph [1] provides a detailed exposition on the Multiplicative Ergodic Theorem and invariant manifold theory for finite-dimensional RDS. Smooth systems are discussed in [41]. For results on infinite-dimensional RDS, we refer to [8–11, 18, 27, 37, 40, 43, 45, 46, 50, 52] and the references therein.

Center manifolds for RDS have also garnered attention, either in finite or infinite dimensions. In the finite-dimensional context, Wanner [57] discusses invariant manifolds, including center manifolds, in terms of linearization in $\mathbb{R}^n$. Boxler [17] proved the existence of center manifolds for discrete random maps (random diffeomorphisms). Existence, smooth conjugacy theorems, and Takens-type theorems based on Lyapunov exponents were established by Li and Lu in [38] and by Guo and Shen in [30], in the presence of zero Lyapunov exponents. On the other hand, infinite-dimensional RDS hold significant interest not only due to their inherent mathematical richness but also for their applications in understanding stochastic and partial differential equations. Under the assumption of an exponential trichotomy, Chen, Roberts, and Duan [20] established the existence and smoothness of center manifolds for a class of stochastic evolution equations with linear multiplicative noise. In [21], Chen, Roberts and Duan established the existence of center manifolds for both discrete and continuous-time infinite-dimensional RDS, assuming an exponential trichotomy, by employing the Lyapunov–Perron method. Moreover, they provided examples illustrating the application of these results to stochastic evolution equations through their conversion into infinite-dimensional RDS. In a similar vein, Kuehn and Neamţu [36] addressed the issue of center manifolds for rough partial differential equations, which also translates into center manifolds within the RDS framework. Li, Zeng and Huan [39] established the existence and smoothness of center-unstable invariant manifolds and center-stable foliations for a class of stochastic PDE with non-dense domain, by converting them into infinite-dimensional RDS.

Exponential trichotomies have played an important role in invariant manifold theory for infinite-dimensional dynamical systems and non-autonomous systems, whether in deterministic or random scenarios, as discussed. In this work, we extend the results on the existence of center manifolds for infinite-dimensional RDS by assuming a generalized trichotomy. This type of general assumption was considered in [16] for dichotomies, and in this work, it is extended to include a central direction. This generalization allows various types of non exponential behaviours along the three subspaces of the invariant splitting. In our context, each subspace is governed by a very general type of rate for controlling the growth of the evolution operator, described in terms of a cocycle. In specific cases, these subspaces correspond to the traditional central, stable, and unstable subspaces. However, our assumptions are sufficiently general to accommodate behaviours beyond exponential-type, such as those observed in (non)uniformly (pseudo-)hyperbolic settings.

This paper is organized as follows. Section 2 introduces the setup and provides preliminaries on RDS and generalized random trichotomies, as well as a description of auxiliary spaces of functions which are essential for handling nonlinear RDS components and deriving the center manifold as the graph of a suitable regular function. Section 3 presents the main result for continuous-time RDS (Theorem 3.1), while Section 4 focuses on the discrete-time counterpart (Theorem 4.1). In Section 5, continuous-time examples are discussed, including tempered exponential trichotomies and a general framework called ψ-trichotomies, which extend beyond exponential bounds. Corresponding discrete-time examples are provided in Section 6.

2 Generalized trichotomies for RDS

2.1 Random Dynamical Systems

Consider *time* $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{R}$, and set $\mathbb{T}^- = \mathbb{T} \cap]-\infty, 0]$ and $\mathbb{T}^+ = \mathbb{T} \cap [0, +\infty[$. A *measure-preserving dynamical system* is a quadruplet $\Sigma \equiv (\Omega, \mathcal{F}, \mathbb{P}, \theta)$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a measure space and

$\theta\colon \mathbb{T} \times \Omega \to \Omega$ is measurable;

$\theta^t(\cdot) = \theta(t, \cdot)\colon \Omega \to \Omega$ preserves $\mathbb{P}$ for all $t \in \mathbb{T}$;

$\theta^0 = \mathrm{Id}_\Omega$;

$\theta^{t+s} = \theta^t \circ \theta^s$ for all $t, s \in \mathbb{T}$.

A (Bochner) measurable *random dynamical system*, henceforth abbreviated as RDS, on a Banach space X over a measure-preserving dynamical system Σ with time $\mathbb{T}$ is a map

$$\Phi : \mathbb{T} \times \Omega \times X \to X$$

such that

i) $\Phi(\cdot, \cdot, x)$ is (Bochner) measurable for all $x \in X$;

ii) $\Phi^t_\omega(\cdot) = \Phi(t, \omega, \cdot)\colon X \to X$ satisfies

a) $\Phi^0_\omega = \mathrm{Id}_X$ for all $\omega \in \Omega$;

b) $\Phi^{t+s}_\omega = \Phi^t_{\theta^s\omega} \circ \Phi^s_\omega$, for all $\omega \in \Omega$ and all $s, t \in \mathbb{T}$.

When Φ^t_ω is a bounded linear operator for all $(t, \omega) \in \mathbb{T} \times \Omega$, the RDS Φ is called *linear*.

We may restrict the *driving system* Σ to a θ^t-invariant subset $\Omega' \subset \Omega$ with $\mathbb{P}$-full measure, obtaining a (Bochner) measurable RDS $\Phi|_{\mathbb{T}\times\Omega'\times X}$ over $\Sigma' \equiv (\Omega', \mathcal{F}', \mathbb{P}|_{\mathcal{F}'}, \theta|_{\Omega'})$, where $\mathcal{F}' = \{B \cap \Omega'\colon B \in \mathcal{F}\}$. In view of this, without any loss of generality, throughout this work, requiring a property to hold for all $\omega \in \Omega'$, for a θ^t-invariant subset $\Omega' \subset \Omega$ with $\mathbb{P}$-full measure, can be replaced by simply requiring it for all $\omega \in \Omega$ by restricting, if necessary, the RDS Φ to $\Phi|_{\mathbb{T}\times\Omega'\times X}$ over Σ'.

2.2 Generalized trichotomies

For every $i \in \{c, s, u\}$, consider a map $P^i\colon \Omega \times X \to X$, and set $P^i_\omega(\cdot) = P(\omega, \cdot)\colon X \to X$. Let $\mathcal{P} = (P^c, P^s, P^u)$. A (Bochner) measurable linear RDS Φ over Σ admits a (Bochner) measurable *$\mathcal{P}$-invariant splitting* if *i*) $P^i(\cdot, x)$ is (Bochner) measurable, for all $x \in X$ and every $i \in \{c, s, u\}$;

ii) P^i_ω is a bounded linear projection, for all $\omega \in \Omega$ and every $i \in \{c, s, u\}$;

iii) $P^c_\omega + P^s_\omega + P^u_\omega = \mathrm{Id}$, for all $\omega \in \Omega$;

iv) $P^c_\omega P^s_\omega = 0$, for all $\omega \in \Omega$;

v) $P^i_{\theta^t\omega}\Phi^t_\omega = \Phi^t_\omega P^i_\omega$, for all $(t, \omega) \in \mathbb{T} \times \Omega$ and every $i \in \{c, s, u\}$;

Notice that for all $\omega \in \Omega$ and $i, j \in \{c, s, u\}$, with $i \neq j$, we have $P^i_\omega P^j_\omega = 0$. To shorten the writing during future computations, for $t \in \mathbb{T}$, $\omega \in \Omega$, and $i \in \{c, s, u\}$ we will adopt the notation

$$\Phi^{i,t}_\omega = \Phi^t_\omega P^i_\omega.$$

We define the linear subspaces $E^i_\omega = P^i_\omega(X)$ for each $i \in \{c, s, u\}$. As usual, we identify $E^c_\omega \times E^s_\omega \times E^u_\omega$ and $E^c_\omega \oplus E^s_\omega \oplus E^u_\omega$. Given the maps

$$\begin{aligned} \alpha^c &: \mathbb{T} \times \Omega \to (0, +\infty), \\ \alpha^s &: \mathbb{T}^+ \times \Omega \to (0, +\infty), \\ \alpha^u &: \mathbb{T}^- \times \Omega \to (0, +\infty), \end{aligned}$$

we define $\alpha = (\alpha^c, \alpha^s, \alpha^u)$. Denote $\alpha^i(t, \omega)$ by $\alpha^i_{t,\omega}$. We say that a (Bochner) measurable linear RDS Φ over Σ exhibits a *generalized trichotomy with bounds* α (or simply an α*-trichotomy*) if it admits a (Bochner) measurable $\mathcal{P}$-invariant splitting satisfying

(T1) $\|\Phi^{c,t}_\omega\| \leqslant \alpha^c_{t,\omega}$ for all $(t, \omega) \in \mathbb{T} \times \Omega$,

(T2) $\|\Phi^{s,t}_\omega\| \leqslant \alpha^s_{t,\omega}$ for all $(t, \omega) \in \mathbb{T}^+ \times \Omega$,

(T3) $\|\Phi^{u,t}_\omega\| \leqslant \alpha^u_{t,\omega}$ for all $(t, \omega) \in \mathbb{T}^- \times \Omega$,

where the operators in (T1)-(T3) are considered as operators from X into X. In what follows, we always consider the operators defined in the whole Banach space X.

In Section 5 and Section 6, we present several examples of generalized trichotomies with both exponential and non-exponential bounds α.

In the remainder of this article, Φ will always denote a measurable (when $\mathbb{T} = \mathbb{Z}$) or Bochner measurable (when $\mathbb{T} = \mathbb{R}$) linear RDS on a Banach space X over a measure-preserving dynamical system $\Sigma \equiv (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ exhibiting a trichotomy with bounds $\alpha = (\alpha^c, \alpha^s, \alpha^u)$.

2.3 Auxiliary spaces

Let $\mathscr{F}$ denote the space of maps $f\colon \Omega \times X \to X$ such that $f(\cdot, x)$ is measurable for every $x \in X$, and for which, setting $f_\omega(\cdot) = f(\omega, \cdot)$, for every $\omega \in \Omega$ we have

$$f_\omega(0) = 0 \tag{2.1}$$

and

$$\mathrm{Lip}(f_\omega) = \sup\left\{ \frac{\|f_\omega(x) - f_\omega(y)\|}{\|x - y\|} : x, y \in X,\ x \neq y \right\} < +\infty. \tag{2.2}$$

Conditions (2.2) and (2.1) ensure that for all $\omega \in \Omega$ and $x, y \in X$

$$\|f_\omega(x) - f_\omega(y)\| \leqslant \mathrm{Lip}(f_\omega)\|x - y\|, \tag{2.3}$$

and

$$\|f_\omega(x)\| \leqslant \mathrm{Lip}(f_\omega)\|x\|. \tag{2.4}$$

Let $\mathscr{F}^{(B)}$ represent the collection of functions $f \in \mathscr{F}$ for which $f(\cdot, x)$ is Bochner measurable for each $x \in X$. Additionally, define $\mathscr{F}_\alpha^{(B)}$ as the subset of $\mathscr{F}^{(B)}$ consisting of functions f such that, for every $\omega \in \Omega$, the maps

$$[a,b] \ni r \mapsto \alpha^c_{t-r,\theta^r\omega} \,\mathrm{Lip}(f_{\theta^r\omega})\alpha^c_{r,\omega},$$
$$[c,0] \ni r \mapsto \alpha^s_{-r,\theta^r\omega} \,\mathrm{Lip}(f_{\theta^r\omega})\alpha^c_{r,\omega},$$
$$[0,d] \ni r \mapsto \alpha^u_{-r,\theta^r\omega} \,\mathrm{Lip}(f_{\theta^r\omega})\alpha^c_{r,\omega}$$

are measurable for every $a < b$, $c < 0$, $d > 0$ and $t \in \mathbb{R}$.

We define the set

$$\mathcal{C} = \{(t, \omega, \xi) \in \mathbb{T} \times \Omega \times X \colon \xi \in E^c_\omega\}.$$

For a given $M > 0$, let $\mathfrak{C}_M$ (resp. $\mathfrak{C}_M^{(B)}$) denote the space of all functions $h\colon \mathcal{C} \to X$ such that, for each $(t, \omega) \in \mathbb{T} \times \Omega$, the map $h_{t,\omega}(\cdot) = h(t, \omega, \cdot)$ satisfies

$$h(\cdot, \cdot, P^c_\omega x) \text{ is measurable (resp. Bochner measurable) for all } x \in X; \tag{2.5}$$
$$h_{t,\omega}(0) = 0 \text{ for all } (t, \omega) \in \mathbb{T} \times \Omega; \tag{2.6}$$
$$h_{0,\omega} = \mathrm{Id}_{E^c_\omega} \text{ for all } \omega \in \Omega; \tag{2.7}$$
$$h_{t,\omega}(E^c_\omega) \subseteq E^c_{\theta^t\omega} \text{ for all } (t, \omega) \in \mathbb{T} \times \Omega; \tag{2.8}$$
$$\|h_{t,\omega}(\xi) - h_{t,\omega}(\xi')\| \leqslant M\alpha^c_{t,\omega}\|\xi - \xi'\| \text{ for all } (t, \omega, \xi), (t, \omega, \xi') \in \mathcal{C}. \tag{2.9}$$

From (2.9) and (2.6), it follows that

$$\|h_{t,\omega}(\xi)\| \leqslant M\alpha^c_{t,\omega}\|\xi\| \text{ for all } (t, \omega, \xi) \in \mathcal{C}. \tag{2.10}$$

Defining

$$d_1(h, g) = \sup\left\{\frac{\|h_{t,\omega}(\xi) - g_{t,\omega}(\xi)\|}{\alpha^c_{t,\omega}\|\xi\|} : (t, \omega, \xi) \in \mathcal{C},\ \xi \neq 0\right\} \tag{2.11}$$

we have that $(\mathfrak{C}_M, d_1)$ and $(\mathfrak{C}_M^{(B)}, d_1)$ are complete metric spaces.

We now consider the set

$$\mathcal{D} = \{(\omega, \xi) \in \Omega \times X \colon \xi \in E^c_\omega\}.$$

For a given $N > 0$, let $\mathfrak{D}_N$ (resp. $\mathfrak{D}_N^{(B)}$) denote the space of all functions $\varphi\colon \mathcal{D} \to X$ such that, for each $\omega \in \Omega$, the map $\varphi_\omega(\cdot) = \varphi(\omega, \cdot)$ satisfies

$$\varphi(\cdot, P^c_\omega x) \text{ is measurable (resp. Bochner measurable) for all } x \in X; \tag{2.12}$$
$$\varphi_\omega(0) = 0 \text{ for all } \omega \in \Omega; \tag{2.13}$$
$$\varphi_\omega(E^c_\omega) \subseteq E^s_\omega \oplus E^u_\omega \text{ for all } \omega \in \Omega; \tag{2.14}$$
$$\|\varphi_\omega(\xi) - \varphi_\omega(\xi')\| \leqslant N\|\xi - \xi'\| \text{ for all } (\omega, \xi), (\omega, \xi') \in \mathcal{D}. \tag{2.15}$$

By (2.15) and (2.13), taking $\xi' = 0$, we get

$$\|\varphi_\omega(\xi)\| \leqslant N\|\xi\| \text{ for all } (\omega, \xi) \in \mathcal{D}. \tag{2.16}$$

For future use, we set the notation $\varphi^s_\omega = P^s_\omega \varphi_\omega$ and $\varphi^u_\omega = P^u_\omega \varphi_\omega$. Given $\varphi \in \mathfrak{D}_N$ and $\omega \in \Omega$, we denote the *graph* of φ_ω by

$$\Gamma_{\varphi,\omega} = \{(\xi, \varphi_\omega(\xi)) : \xi \in E^c_\omega\} \subseteq X.$$

Defining now

$$d_2(\varphi, \psi) = \sup \left\{ \frac{\|\varphi_\omega(\xi) - \psi_\omega(\xi)\|}{\|\xi\|} : (\omega, \xi) \in \mathcal{D}, \xi \neq 0 \right\} \tag{2.17}$$

it follows that $(\mathfrak{D}_N, d_2)$ and $(\mathfrak{D}^{(B)}_N, d_2)$ are complete metric spaces.

To conclude this section, let $\mathfrak{U}_{M,N} = \mathfrak{C}_M \times \mathfrak{D}_N$ and $\mathfrak{U}^{(B)}_{M,N} = \mathfrak{C}^{(B)}_M \times \mathfrak{D}^{(B)}_N$. Setting

$$d((h, \varphi), (g, \psi)) = d_1(h, g) + d_2(\varphi, \psi),$$

we also have that $(\mathfrak{U}_{M,N}, d)$ and $(\mathfrak{U}^{(B)}_{M,N}, d)$ are complete metric spaces.

3 Invariant manifolds in continuous-time RDS

Throughout this section, we focus on the continuous-time case by considering $\mathbb{T} = \mathbb{R}$. Given a Bochner measurable linear RDS Φ and a map $f \in \mathscr{F}^{(B)}_\alpha$, we define

$$\sigma = \sup_{(t,\omega)\in\mathbb{R}\times\Omega} \frac{1}{\alpha^c_{t,\omega}} \left| \int_0^t \alpha^c_{t-r,\theta^r\omega} \operatorname{Lip}(f_{\theta^r\omega}) \alpha^c_{r,\omega} \, dr \right| \tag{3.1}$$

and

$$\tau = \sup_{\omega\in\Omega} \int_{-\infty}^0 \alpha^s_{-r,\theta^r\omega} \operatorname{Lip}(f_{\theta^r\omega}) \alpha^c_{r,\omega} \, dr + \int_0^{+\infty} \alpha^u_{-r,\theta^r\omega} \operatorname{Lip}(f_{\theta^r\omega}) \alpha^c_{r,\omega} \, dr. \tag{3.2}$$

If for every $(\omega, x) \in \Omega \times X$ there is a unique solution $\Psi(\cdot, \omega, x)$ of the equation

$$u(t) = \Phi^t_\omega x + \int_0^t \Phi^{t-r}_{\theta^r\omega} f_{\theta^r\omega}(u(r)) \, dr \tag{3.3}$$

then $\Psi \colon \mathbb{R} \times \Omega \times X \to X$ is a Bochner measurable RDS on X over Σ. In particular, $\Psi(\cdot, \cdot, x)$ is Bochner measurable for all $x \in X$, and

$$\Psi^t_\omega x = \Phi^t_\omega x + \int_0^t \Phi^{t-r}_{\theta^r\omega} f_{\theta^r\omega}(\Psi^r_\omega x) \, dr. \tag{3.4}$$

Theorem 3.1. *Let Φ be a Bochner measurable linear RDS exhibiting an α-trichotomy, and let $f \in \mathscr{F}^{(B)}_\alpha$. Suppose that Ψ is a Bochner measurable RDS such that $\Psi(\cdot, \omega, x)$ is the unique solution of (3.3) for all $(\omega, x) \in \Omega \times X$. If*

$$\lim_{t\to-\infty} \alpha^s_{-t,\theta^t\omega} \alpha^c_{t,\omega} = \lim_{t\to+\infty} \alpha^u_{-t,\theta^t\omega} \alpha^c_{t,\omega} = 0 \tag{3.5}$$

for all $\omega \in \Omega$, and

$$\sigma + \tau < 1/2, \tag{3.6}$$

then there are $N \in]0,1[$ *and a unique* $\varphi \in \mathfrak{D}_N^{(B)}$ *such that*

$$\Psi_\omega^t(\Gamma_{\varphi,\omega}) \subseteq \Gamma_{\varphi,\theta^t\omega} \tag{3.7}$$

for all $(t,\omega) \in \mathbb{R} \times \Omega$. *Moreover, for all* $(t,\omega,\xi), (t,\omega,\xi') \in \mathcal{C}$ *we have*

$$\|\Psi_\omega^t(\xi, \varphi_\omega(\xi)) - \Psi_\omega^t(\xi', \varphi_\omega(\xi'))\| \leqslant (N/\tau)\alpha_{t,\omega}^c \|\xi - \xi'\|. \tag{3.8}$$

The remaining part of this section is devoted to proving Theorem 3.1.

From [15, Lemma 5.1], we may find constants $M \in]1,2[$ and $N \in]0,1[$ such that

$$\sigma = \frac{M-1}{M(1+N)} \quad \text{and} \quad \tau = \frac{N}{M(1+N)}. \tag{3.9}$$

Lemma 3.2. *Consider* $(h,\varphi) \in \mathfrak{U}_{M,N}^{(B)}$.

a) For every $x \in X$ *the maps*

$$\begin{aligned}
&(t,r,\omega) \mapsto \Phi_{\theta^r\omega}^{c,t-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x))),\\
&(r,\omega) \mapsto \Phi_{\theta^r\omega}^{s,-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x))),\\
&(r,\omega) \mapsto \Phi_{\theta^r\omega}^{u,-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x)))
\end{aligned}$$

are Bochner measurable on $\mathbb{R} \times \mathbb{R} \times \Omega$, $\mathbb{R}^- \times \Omega$ *and* $\mathbb{R}^+ \times \Omega$, *respectively.*

b) For every $(t,\omega,x) \in \mathbb{R} \times \Omega \times X$ *the map*

$$r \mapsto \Phi_{\theta^r\omega}^{c,t-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x)))$$

is Bochner integrable in every closed interval with bounds 0 *and* t.

c) For every $(\omega,x) \in \Omega \times X$ *and* $t > 0$, *the maps*

$$\begin{aligned}
&r \mapsto \Phi_{\theta^r\omega}^{s,-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x))),\\
&r \mapsto \Phi_{\theta^r\omega}^{u,-r} f_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x), \varphi_{\theta^r\omega}(h_{r,\omega}(P_\omega^c x)))
\end{aligned}$$

are Bochner integrable in $[-t,0]$ *and* $[0,t]$, *respectively.*

The proof follows similarly as [16, Lemma 3.6]. Given $\omega \in \Omega$ and $x_\omega = (x_\omega^c, x_\omega^s, x_\omega^u) \in E_\omega^c \times E_\omega^s \times E_\omega^u$, it follows from (3.4) that the trajectory $x_{\theta^t\omega} = \Psi_\omega^t x_\omega = \left(x_{\theta^t\omega}^c, x_{\theta^t\omega}^s, x_{\theta^t\omega}^u\right)$ satisfies, for all $i \in \{c,s,u\}$ and all $t \in \mathbb{R}$,

$$x_{\theta^t\omega}^i = \Phi_\omega^{i,t} x_\omega + \int_0^t \Phi_{\theta^s\omega}^{i,t-s} f_{\theta^s\omega}(x_{\theta^s\omega}^c, x_{\theta^s\omega}^s, x_{\theta^s\omega}^u)\, ds. \tag{3.10}$$

Taking into account the invariance required in (3.7), for any given $x_\omega \in \Gamma_{\varphi,\omega}$ and $t \in \mathbb{R}$ we must have $x_{\theta^t\omega} \in \Gamma_{\varphi,\theta^t\omega}$. Thus, in this situation, the equations given by (3.10) can be written as

$$\begin{aligned}
x_{\theta^t\omega}^c &= \Phi_\omega^{c,t} x_\omega + \int_0^t \Phi_{\theta^s\omega}^{c,t-s} f_{\theta^s\omega}(x_{\theta^s\omega}^c, \varphi_{\theta^s\omega}(x_{\theta^s\omega}^c))\, ds,\\
\varphi_{\theta^t\omega}^s(x_{\theta^t\omega}) &= \Phi_\omega^t \varphi_\omega^s(x_\omega^c) + \int_0^t \Phi_{\theta^s\omega}^{s,t-s} f_{\theta^s\omega}(x_{\theta^s\omega}^c, \varphi_{\theta^s\omega}(x_{\theta^s\omega}^c))\, ds,\\
\varphi_{\theta^t\omega}^u(x_{\theta^t\omega}) &= \Phi_\omega^t \varphi_\omega^u(x_\omega^c) + \int_0^t \Phi_{\theta^s\omega}^{u,t-s} f_{\theta^s\omega}(x_{\theta^s\omega}^c, \varphi_{\theta^s\omega}(x_{\theta^s\omega}^c))\, ds.
\end{aligned}$$

Lemma 3.3. *Consider* $(h,\varphi) \in \mathfrak{U}^{(B)}_{M,N}$ *such that, for all* $(t,\omega,\xi) \in \mathcal{C}$,

$$h_{t,\omega}(x) = \Phi^t_\omega \xi + \int_0^t \Phi^{c,t-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\;dr. \tag{3.11}$$

The following properties a) and b) are equivalent:

a) For each $j \in \{s,u\}$ *and all* $(t,\omega,\xi) \in \mathcal{C}$,

$$\varphi^j_{\theta^t\omega}(h_{t,\omega}(\xi)) = \Phi^t_\omega \varphi^j_\omega(\xi) + \int_0^t \Phi^{j,t-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\;dr \tag{3.12}$$

b) For all $(\omega,\xi) \in \mathcal{D}$

$$\varphi^s_\omega(\xi) = \int_{-\infty}^0 \Phi^{s,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\;dr \tag{3.13}$$

and

$$\varphi^u_\omega(\xi) = -\int_0^{+\infty} \Phi^{u,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\;dr. \tag{3.14}$$

Proof. From (2.4), (2.16) and (2.10) we have

$$\begin{aligned}\|f_{\theta^r\omega}(h_{r,\omega}(\xi),\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\| &\leqslant \operatorname{Lip}(f_{\theta^r\omega})(\|h_{r,\omega}(\xi)\| + \|\varphi_{\theta^r\omega}(h_{r,\omega}(\xi))\|)\\ &\leqslant M(1+N)\operatorname{Lip}(f_{\theta^r\omega})\alpha^c_{r,\omega}\|\xi\|\end{aligned}$$

for every $(\omega,\xi) \in \mathcal{D}$. Thus, by (T2),

$$\int_{-\infty}^0 \left\|\Phi^{s,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\right\|\;dr \leqslant M(1+N)\tau\|\xi\|,$$

and by (T3) we obtain

$$\int_0^{+\infty} \left\|\Phi^{u,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\right\|\;dr \leqslant M(1+N)\tau\|\xi\|.$$

Hence the integrals are convergent.

Suppose that (3.12) holds for $j = s$ and all $(t,\omega,\xi) \in \mathcal{C}$. By applying $\Phi^{-t}_{\theta^t\omega}$ to both sides, it is equivalent to

$$\varphi^s_\omega(\xi) = \Phi^{s,-t}_{\theta^t\omega}\varphi^s_{\theta^t\omega}(h_{t,\omega}(\xi)) - \int_0^t \Phi^{s,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\;dr. \tag{3.15}$$

Using (T2), (2.16) and (2.10), for $t \leqslant 0$ we have

$$\left\|\Phi^{s,-t}_{\theta^t\omega}\varphi^s_{\theta^t\omega}(h_{t,\omega}(\xi))\right\| \leqslant MN\alpha^s_{-t,\theta^t\omega}\,\alpha^c_{t,\omega}\|\xi\|,$$

which converges to zero as $t \to -\infty$ by (3.5). Thus, by taking $t \to -\infty$ in equation (3.15) we obtain (3.13). Similarly, equation (3.12) with $j = u$ can be written as

$$\varphi^u_\omega(\xi) = \Phi^{u,-t}_{\theta^t\omega}\varphi_{\theta^t\omega}(h_{t,\omega}(\xi)) - \int_0^t \Phi^{u,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi)\,,\varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\;dr. \tag{3.16}$$

Using (T3), (2.16) and (2.10), for $t \geqslant 0$ we have

$$\left\|\Phi^{u,-t}_{\theta^t\omega}\varphi_{\theta^t\omega}(h_{t,\omega}(\xi))\right\| \leqslant MN\alpha^u_{-t,\theta^t\omega}\alpha^c_{t,\omega}\,\|\xi\|\,,$$

which, by (3.5), converges to zero as $t \to +\infty$. Thus, we obtain (3.14) by taking $t \to +\infty$ in equation (3.16).

For the converse, assume now that (3.13) and (3.14) hold for all $(\omega, \xi) \in \mathcal{D}$. For all $t \in \mathbb{R}$, we have

$$\begin{aligned}\Phi^t_\omega \varphi^s_\omega(\xi) = &\int_t^0 \Phi^{s,t-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr \\ &+ \int_{-\infty}^0 \Phi^{s,-r}_{\theta^{t+r}\omega} f_{\theta^{t+r}\omega}(h_{t+r,\omega}(\xi), \varphi_{\theta^{t+r}\omega}(h_{t+r,\omega}(\xi)))\, dr\end{aligned}$$

and

$$\begin{aligned}\Phi^t_\omega \varphi^u_\omega(\xi) = &-\int_0^t \Phi^{u,t-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr \\ &- \int_{-\infty}^0 \Phi^{u,-r}_{\theta^{t+r}\omega} f_{\theta^{t+r}\omega}(h_{t+r,\omega}(\xi), \varphi_{\theta^{t+r}\omega}(h_{t+r,\omega}(\xi)))\, dr.\end{aligned}$$

Since $h_{t+s,\omega}(\xi) = h_{s,\theta^t\omega}(h_{t,\omega}(\xi))$ due to the uniqueness of the solution of (3.3), we get the identity (3.12) for $j = s$ and $j = u$. □

Consider the operator C, which assigns each pair $(h, \varphi) \in \mathfrak{U}^{(B)}_{M,N}$ to the map $C(h, \varphi) \colon \mathcal{C} \to X$ given by

$$[C(h,\varphi)]\,(t,\omega,\xi) = \Phi^t_\omega \xi + \int_0^t \Phi^{c,t-r}_{\theta^r\omega} \varphi(r,\omega)\, dr.$$

Lemma 3.4. $C(\mathfrak{U}^{(B)}_{M,N}) \subseteq \mathfrak{C}^{(B)}_M$.

Proof. Fix a pair $(h, \varphi) \in \mathfrak{U}^{(B)}_{M,N}$. It is straightforward to check that $C(h, \varphi)$ satisfies conditions (2.5) to (2.8). Define

$$\gamma_{\theta^r\omega}(\xi, \xi') = \|f_{\theta^r\omega}(h_{r,\omega}(\xi), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi))) - f_{\theta^r\omega}(h_{r,\omega}(\xi'), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi')))\|.$$

From (2.3), (2.15) and (2.9) we have

$$\gamma_{\theta^r\omega}(\xi, \xi') \leqslant \mathrm{Lip}(f_{\theta^r\omega}) M(1+N) \|\xi - \xi'\| \alpha^c_{r,\omega}. \tag{3.17}$$

Following the previous notation, $C(h, \varphi)_{t,\omega}(\xi)$ stands for $[C(h, \varphi)](t, \omega, \xi)$. By (T1), (3.1), (3.17) and (3.9), we have

$$\begin{aligned}\|C(h,\varphi)_{t,\omega}(\xi) - C(h,\varphi)_{t,\omega}(\xi')\| &\leqslant \|\Phi^{c,t}_\omega\| \|\xi - \xi'\| + \int_0^t \|\Phi^{c,t-r}_{\theta^r\omega}\| \gamma_{\theta^r\omega}(\xi,\xi')\, dr \\ &\leqslant (1 + \sigma M(1+N))\, \alpha^c_{t,\omega} \|\xi - \xi'\| \\ &= M \alpha^c_{t,\omega} \|\xi - \xi'\|.\end{aligned}$$

Hence $C(h, \varphi)$ also satisfies (2.9). □

Consider now the operator D, which assigns each pair $(h, \varphi) \in \mathfrak{U}^{(B)}_{M,N}$ the map

$$D(h, \varphi) \colon \mathcal{D} \to X$$

given by

$$[D(h,\varphi)]\,(\omega,\xi) = [D^s(h,\varphi)]\,(\omega,\xi) + [D^u(h,\varphi)]\,(\omega,\xi)$$

where

$$[D^s(h,\varphi)](\omega,\xi) = \int_{-\infty}^{0} \Phi^{s,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr$$

and

$$[D^u(h,\varphi)](\omega,\xi) = -\int_{0}^{+\infty} \Phi^{u,-r}_{\theta^r\omega} f_{\theta^r\omega}(h_{r,\omega}(\xi), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi)))\, dr.$$

Lemma 3.5. $D(\mathfrak{U}^{(B)}_{M,N}) \subseteq \mathfrak{D}^{(B)}_N$.

Proof. Fix $(h,\varphi) \in \mathfrak{U}^{(B)}_{M,N}$. It is immediate to check that $[D(h,\varphi)](\omega,\xi)$ satisfies conditions (2.12) to (2.14). Again, $D(h,\varphi)_\omega(\xi)$ stands for $[D(h,\varphi)](\omega,\xi)$. From (T2), (T3), (3.17), (3.2) and (3.9) we have

$$\begin{aligned}\|D(h,\varphi)_\omega(\xi) - D(h,\varphi)_\omega(\xi')\| &\leqslant \int_{-\infty}^{0} \|\Phi^{s,-r}_{\theta^r\omega}\|\, \gamma_{\theta^r\omega}(\xi,\xi')\, dr + \int_{0}^{+\infty} \|\Phi^{u,-r}_{\theta^r\omega}\|\, \gamma_{\theta^r\omega}(\xi,\xi')\, dr \\ &\leqslant \tau M(1+N)\,\|\xi-\xi'\| \\ &= N\,\|\xi-\xi'\|.\end{aligned}$$

Hence (2.15) also holds for $D(h,\varphi)$. □

Consider now $U\colon \mathfrak{U}^{(B)}_{M,N} \to \mathfrak{U}^{(B)}_{M,N}$ given by

$$U(h,\varphi) = (C(h,\varphi), D(h,\varphi)).$$

Lemma 3.6. *The operator U is a contraction in $(\mathfrak{U}^{(B)}_{M,N}, d)$.*

Proof. Consider $(h,\varphi), (g,\psi) \in \mathfrak{U}^{(B)}_{M,N}$. Define

$$\hat{\gamma}_{\theta^r\omega}(\xi) = \|f_{\theta^r\omega}(h_{r,\omega}(\xi), \varphi_{\theta^r\omega}(h_{r,\omega}(\xi))) - f_{\theta^r\omega}(g_{r,\omega}(\xi), \psi_{\theta^r\omega}(g_{r,\omega}(\xi)))\|.$$

By (2.3), (2.15), (2.11), (2.17) and (2.10), for all $(r,\omega) \in \mathbb{R}^+_0 \times \Omega$ and all $\xi \in E_\omega$,

$$\hat{\gamma}_{\theta^r\omega}(\xi) \leqslant \mathrm{Lip}(f_{\theta^r\omega})\alpha^c_{r,\omega}((1+N)d_1(h,g) + Md_2(\varphi,\psi))\,\|\xi\|. \tag{3.18}$$

Hence, in one hand, from (T1), (3.18) and (3.1), we have

$$\begin{aligned}\|C(h,\varphi)_{t,\omega}(\xi) - C(g,\psi)_{t,\omega}(\xi)\| &\leqslant \int_0^t \left\|\Phi^{c,t-r}_{\theta^r\omega}\right\| \hat{\gamma}_{\theta^r\omega}(\xi)\, dr \\ &\leqslant \sigma\alpha^c_{t,\omega}((1+N)d_1(h,g) + Md_2(\varphi,\psi))\,\|\xi\|,\end{aligned}$$

which implies

$$d_1(C(h,\varphi), C(g,\psi)) \leqslant \sigma((1+N)d_1(h,g) + Md_2(\varphi,\psi)).$$

On the other hand, from (T2), (T3), (3.18) and (3.2) we get

$$\begin{aligned}\|D(h,\varphi)_\omega(\xi) - D(g,\psi)_\omega(\xi)\| &\leqslant \int_{-\infty}^{0} \|\Phi^{s,-r}_{\theta^r\omega}\|\, \hat{\gamma}_{\theta^r\omega}(\xi)\, dr + \int_0^{+\infty} \|\Phi^{u,-r}_{\theta^r\omega}\|\, \hat{\gamma}_{\theta^r\omega}(\xi)\, dr \\ &\leqslant \tau((1+N)d_1(h,g) + Md_2(\varphi,\psi))\,\|\xi\|,\end{aligned}$$

which implies

$$d_2(D(h,\varphi), D(g,\psi)) \leqslant \tau((1+N)d_1(h,g) + Md_2(\varphi,\psi)).$$

In overall we get

$$\begin{aligned} d(U(h,\varphi),U(g,\psi)) &\leqslant (\sigma+\tau)((1+N)d_1(h,g)+Md_2(\varphi,\psi)) \\ &\leqslant \frac{1}{2}\max\{1+N,M\}\, d((h,\varphi),(g,\psi)) \end{aligned}$$

and because $N<1$ and $M<2$, U is a contraction. □

Proof of Theorem 3.1. Since U is a contraction, by the Banach Fixed Point Theorem, U has a unique fixed point (h,φ), that satisfies (3.11), (3.13) and (3.14). By Lemma 3.3, the pair (h,φ) also satisfies conditions (3.12). Therefore, for given initial condition $x_\omega = (\xi, \varphi^s_\omega(\xi), \varphi^u_\omega(\xi)) \in E^c_\omega \times E^s_\omega \times E^u_\omega$, the trajectory $x_{\theta^t\omega} = (h_{t,\omega}(\xi), \varphi_{\theta^t\omega}(h_{t,\omega}(\xi)))$ is the solution of (3.3). The graphs $\Gamma_{\varphi,\omega}$ are the required invariant manifolds of Ψ. To obtain (3.8), it follows from (2.15), (2.9) and (3.9) that, for each $(t,\omega,\xi),(t,\omega,\xi') \in \mathcal{C}$

$$\begin{aligned} &\|\Psi^t_\omega(\xi,\varphi^s_\omega(\xi),\varphi^u_\omega(\xi)) - \Psi^t_\omega(\xi',\varphi^s_\omega(\xi'),\varphi^u_\omega(\xi'))\| \\ &\quad = \|(h_{t,\omega}(\xi),\varphi_{\theta^t\omega}(h_{t,\omega}(\xi))) - (h_{t,\omega}(\xi'),\varphi_{\theta^t\omega}(h_{t,\omega}(\xi')))\| \\ &\quad \leqslant M(1+N)\alpha^c_{t,\omega}\|\xi-\bar{\xi}\| \\ &\quad \leqslant \frac{N}{\tau}\alpha^c_{t,\omega}\|\xi-\bar{\xi}\|. \end{aligned}$$

□

4 Invariant manifolds in discrete-time RDS

Throughout this section we consider $\mathbb{T}=\mathbb{Z}$. Given a measurable linear RDS Φ and a map $f\in\mathscr{F}$, we define

$$\begin{aligned} \sigma^-_\omega &= \sup_{n\in\mathbb{N}} \frac{1}{\alpha^c_{-n,\omega}} \sum_{k=-n}^{-1} \alpha^c_{-n-k-1,\theta^{k+1}\omega}\operatorname{Lip}(f_{\theta^k\omega})\alpha^c_{k,\omega}, \\ \sigma^+_\omega &= \sup_{n\in\mathbb{N}} \frac{1}{\alpha^c_{n,\omega}} \sum_{k=0}^{n-1} \alpha^c_{n-k-1,\theta^{k+1}\omega}\operatorname{Lip}(f_{\theta^k\omega})\alpha^c_{k,\omega} \end{aligned}$$

and

$$\sigma = \sup_{\omega\in\Omega}\max\{\sigma^-_\omega,\sigma^+_\omega\}.$$

Moreover, writing

$$\begin{aligned} \tau^-_\omega &= \sum_{k=-\infty}^{-1} \alpha^s_{-k-1,\theta^{k+1}\omega}\operatorname{Lip}(f_{\theta^k\omega})\alpha^c_{k,\omega}, \\ \tau^+_\omega &= \sum_{k=0}^{+\infty} \alpha^u_{-k-1,\theta^{k+1}\omega}\operatorname{Lip}(f_{\theta^k\omega})\alpha^c_{k,\omega}, \end{aligned}$$

we also define

$$\tau = \sup_{\omega\in\Omega}(\tau^-_\omega+\tau^+_\omega).$$

Consider the measurable RDS $\Psi\colon \mathbb{Z}\times\Omega\times X\to X$ given by

$$\Psi^n_\omega(x) = \begin{cases} \Phi^n_\omega x + \sum_{k=0}^{n-1} \Phi^{n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(\Psi^k_\omega(x)) & \text{if } n\geqslant 1, \\ x & \text{if } n=0, \\ \Phi^n_\omega x - \sum_{k=n}^{-1} \Phi^{n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(\Psi^k_\omega(x)) & \text{if } n\leqslant -1 \end{cases} \tag{4.1}$$

which encapsulates the solutions of the random nonlinear difference equation

$$x_{n+1} = \Phi^1_{\theta^n\omega} x_n + f_{\theta^n\omega}(x_n).$$

Theorem 4.1. *Let Φ be a measurable linear RDS exhibiting an α-trichotomy and let $f \in \mathscr{F}$. If*

$$\lim_{n\to-\infty} \alpha^s_{-n,\theta^n\omega}\alpha^c_{n,\omega} = \lim_{n\to+\infty} \alpha^u_{-n,\theta^n\omega}\alpha^c_{n,\omega} = 0$$

for all $\omega \in \Omega$, and

$$\sigma + \tau < 1/2,$$

then there are $N \in]0,1[$ and a unique $\varphi \in \mathfrak{D}_N$ such that for the RDS Ψ given by (4.1) we have

$$\Psi^n_\omega(\Gamma_{\varphi,\omega}) \subseteq \Gamma_{\varphi,\theta^n\omega} \tag{4.2}$$

for all $(n,\omega) \in \mathbb{Z} \times \Omega$. Moreover, for every $(n,\omega,\xi), (n,\omega,\xi') \in \mathcal{C}$ we have

$$\|\Psi^n_\omega(\xi, \varphi_\omega(\xi)) - \Psi^n_\omega(\xi', \varphi_\omega(\xi'))\| \leqslant (N/\tau)\alpha^c_{n,\omega} \|\xi - \xi'\|.$$

The proof of Theorem 4.1 is analogous to the proof of Theorem 3.1. Therefore, in the remainder of this section, we provide a guide to the necessary adaptations. Fix M and N as in (3.9). Given $\omega \in \Omega$ and

$$x_\omega = (x^c_\omega, x^s_\omega, x^u_\omega) \in E^c_\omega \times E^s_\omega \times E^u_\omega,$$

the trajectory

$$x_{\theta^n\omega} = \Psi^n_\omega x_\omega = (x^c_{\theta^n\omega}, x^s_{\theta^n\omega}, x^u_{\theta^n\omega}) \in E^c_\omega \times E^s_\omega \times E^u_\omega$$

satisfies the following equations for each $i \in \{c, s, u\}$:

$$x^i_{\theta^n\omega} = \begin{cases} \Phi^{i,n}_\omega x^i_\omega + \sum_{k=0}^{n-1} \Phi^{i,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, x^s_{\theta^k\omega}, x^u_{\theta^k\omega}) & \text{if } n \geqslant 1, \\ \Phi^{i,n}_\omega x^i_\omega - \sum_{k=n}^{-1} \Phi^{i,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, x^s_{\theta^k\omega}, x^u_{\theta^k\omega}) & \text{if } n \leqslant -1. \end{cases} \tag{4.3}$$

In view of the invariance required in (4.2), if $x_\omega \in \Gamma_{\varphi,\omega}$ then $x_{\theta^n\omega}$ must be in $\Gamma_{\varphi,\theta^n\omega}$ for every $n \in \mathbb{Z}$, and thus, in this situation, the equations from (4.3) can be written as

$$x^c_{\theta^n\omega} = \begin{cases} \Phi^{c,n}_\omega x^c_\omega + \sum_{k=0}^{n-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, \varphi_{\theta^k\omega}(x^c_{\theta^k\omega})) & \text{if } n \geqslant 1, \\ \Phi^{c,n}_\omega x^c_\omega - \sum_{k=n}^{-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, \varphi_{\theta^k\omega}(x^c_{\theta^k\omega})) & \text{if } n \leqslant -1 \end{cases} \tag{4.4}$$

and, for $j \in \{s, u\}$,

$$\varphi^j_{\theta^n\omega}(x_{\theta^n\omega}) = \begin{cases} \Phi^{j,n}_\omega \varphi_\omega(x_\omega) + \sum_{k=0}^{n-1} \Phi^{j,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, \varphi_{\theta^k\omega}(x^c_{\theta^k\omega})) & \text{if } n \geqslant 1, \\ \Phi^{j,n}_\omega \varphi_\omega(x_\omega) - \sum_{k=n}^{-1} \Phi^{j,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(x^c_{\theta^k\omega}, \varphi_{\theta^k\omega}(x^c_{\theta^k\omega})) & \text{if } n \leqslant -1. \end{cases} \tag{4.5}$$

Let us prove prove that equations (4.4) and (4.5) have solutions. First, we rewrite them, by a discrete version of Lemma 3.3.

Lemma 4.2. *Consider* $(h,\varphi) \in \mathfrak{U}_{M,N}$ *such that, for all* $(n,\omega) \in \mathbb{Z} \times \Omega$ *and all* $\xi \in E^c_\omega$

$$h_{n,\omega}(\xi) = \begin{cases} \Phi^{c,n}_\omega \xi + \sum_{k=0}^{n-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \geqslant 1, \\ \Phi^{c,n}_\omega \xi - \sum_{k=n}^{-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \leqslant -1. \end{cases} \tag{4.6}$$

Then the following conditions a) and b) are equivalent:

a) For each $j \in \{u,s\}$ *and all* $(n,\omega,\xi) \in \mathcal{C}$

$$\varphi^j_{\theta^n\omega}(h_{n,\omega}(\xi)) = \begin{cases} \Phi^{j,n}_\omega \varphi_\omega(\xi) + \sum_{k=0}^{n-1} \Phi^{j,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \geqslant 1, \\ \Phi^{j,n}_\omega \varphi_\omega(\xi) - \sum_{k=n}^{-1} \Phi^{j,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \leqslant -1. \end{cases} \tag{4.7}$$

b) For all $(\omega,\xi) \in \mathcal{D}$

$$\varphi^s_\omega(\xi) = \sum_{k=-\infty}^{-1} \Phi^{s,-(k+1)}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) \tag{4.8}$$

and

$$\varphi^u_\omega(\xi) = -\sum_{k=0}^{+\infty} \Phi^{u,-(k+1)}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))). \tag{4.9}$$

Consider here the operator C, which assigns each pair $(h,\varphi) \in \mathfrak{U}^{(B)}_{M,N}$ to the map

$$C(h,\varphi) \colon \mathcal{C} \to X$$

given by

$$[C(h,\varphi)]\,(n,\omega,\xi) = \begin{cases} \Phi^{c,n}_\omega \xi + \sum_{k=0}^{n-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \geqslant 1, \\ \Phi^{c,n}_\omega \xi - \sum_{k=n}^{-1} \Phi^{c,n-k-1}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))) & \text{if } n \leqslant -1, \end{cases}$$

and D be the operator that assigns to each pair $(h,\varphi) \in \mathfrak{U}_{M,N}$ the map $D(h,\varphi) \colon \mathcal{D} \to X$ defined by

$$[D(h,\varphi)]\,(\omega,\xi) = [D^s(h,\varphi)]\,(\omega,\xi) + [D^u(h,\varphi)]\,(\omega,\xi),$$

where

$$[D^s(h,\varphi)]\,(\omega,\xi) = \sum_{k=-\infty}^{-1} \Phi^{s,-(k+1)}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi)))$$

and

$$[D^u(h,\varphi)]\,(\omega,\xi) = -\sum_{k=0}^{+\infty} \Phi^{u,-(k+1)}_{\theta^{k+1}\omega} f_{\theta^k\omega}(h_{k,\omega}(\xi), \varphi_{\theta^k\omega}(h_{k,\omega}(\xi))).$$

To finalize, define $U\colon \mathfrak{U}_{M,N} \to \mathfrak{U}_{M,N}$ by

$$U(h,\varphi) = (C(h,\varphi), D(h,\varphi)) .$$

The operator U is a contraction in $(\mathfrak{U}_{M,N}, d)$. By the Banach Fixed Point Theorem, U as a unique fixed point (h,φ), which satisfies conditions (4.6), (4.8) and (4.9). By Lemma 4.2 the pair (h,φ) also satisfy the conditions in (4.7). Hence, by (4.4) and (4.5), we get that $(h_{n,\omega}(\xi), \varphi_{\theta^n\omega}(h_{n,\omega}(\xi)))$ is the orbit by Ψ of the initial condition

$$(\xi, \varphi^s_\omega(\xi), \varphi^u_\omega(\xi)) \in E^c_\omega \times E^s_\omega \times E^u_\omega.$$

The graphs $\Gamma_{\varphi,\omega}$ are the required invariant manifolds of Ψ. Furthermore, for all $\omega \in \Omega$, all $n \in \mathbb{Z}$ and all $\xi, \xi' \in E^c_\omega$ it follows from (2.15), (2.9) and (3.9) that

$$\|\Psi^n_\omega(\xi, \varphi_\omega(\xi)) - \Psi^n_\omega(\xi', \varphi_\omega(\xi'))\| \leqslant \frac{N}{\tau} \alpha^c_{n,\omega} \|\xi - \bar{\xi}\|,$$

which finishes the proof of Theorem 4.1.

5 Continuous-time examples

For this section assume $\mathbb{T} = \mathbb{R}$. Throughout this entire section we consider a constant $\delta \in]0, 1/6[$ and a random variable $G\colon \Omega \to]0, +\infty[$ satisfying

$$\int_{-\infty}^{+\infty} G(\theta^r\omega)\, dr \leqslant 1 \quad \text{for all } \omega \in \Omega.$$

In all the following examples we may consider different growth rates along the *central directions* E^c_ω, depending if we are looking to the *future* ($t \to +\infty$) or to the *past* ($t \to -\infty$).

5.1 Tempered exponential trichotomies

Let

$$\lambda^{\bar{c}}, \lambda^{\underline{c}}, \lambda^s, \lambda^u \colon \Omega \to \mathbb{R}$$

be θ-invariant random variables, i.e. satisfying $\lambda^\ell(\theta^t\omega) = \lambda^\ell(\omega)$ for all $\omega \in \Omega$, $t \in \mathbb{R}$ and $\ell \in \{\bar{c}, \underline{c}, s, u\}$. A Bochner measurable linear RDS Φ exhibits an *exponential trichotomy* if it exhibits a generalized trichotomy with bounds

$$\alpha^c_{t,\omega} = \begin{cases} K(\omega)\, \mathrm{e}^{\lambda^{\bar{c}}(\omega)t}, & t \geqslant 0, \\ K(\omega)\, \mathrm{e}^{\lambda^{\underline{c}}(\omega)t}, & t \leqslant 0, \end{cases}$$
$$\alpha^s_{t,\omega} = K(\omega)\, \mathrm{e}^{\lambda^s(\omega)t}, \quad t \geqslant 0,$$
$$\alpha^u_{t,\omega} = K(\omega)\, \mathrm{e}^{\lambda^u(\omega)t}, \quad t \leqslant 0$$

for some random variable $K : \Omega \to [1, +\infty[$. If the random variable K is *tempered*, *i.e.*, if

$$\Lambda_{K,\gamma,\omega} := \sup_{t \in \mathbb{T}} \left[\mathrm{e}^{-\gamma|t|} K(\theta^t w) \right] < +\infty \tag{5.1}$$

for all $\gamma > 0$ and all $\omega \in \Omega$, we say that Φ exhibits an *tempered exponential trichotomy*.

Corollary 5.1. *Let Φ be a Bochner measurable linear RDS exhibiting a tempered exponential trichotomy such that*

$$\lambda^{\underline{c}}(\omega) > \lambda^{s}(\omega) \quad \text{and} \quad \lambda^{\overline{c}}(\omega) < \lambda^{u}(\omega)$$

for all $\omega \in \Omega$, and let $f \in \mathscr{F}_{\alpha}^{(B)}$. Assume that Ψ is a Bochner measurable RDS such that (3.3) has unique solution $\Psi(\cdot,\omega,x)$ for every $(\omega,x) \in \Omega \times X$. Consider a θ-invariant random variable $\gamma(\omega) > 0$ satisfying

$$a(\omega) := \lambda^{\underline{c}}(\omega) - \lambda^{s}(\omega) - \gamma(\omega) > 0 \quad \text{and} \quad b(\omega) := \lambda^{u}(\omega) - \lambda^{\overline{c}}(\omega) - \gamma(\omega) > 0$$

for all $\omega \in \Omega$. If

$$\operatorname{Lip}(f_\omega) \leqslant \frac{\delta}{K(\omega)} \min\left\{G(\omega), \frac{a(\omega)}{\Lambda_{K,\gamma(\omega),\omega}}, \frac{b(\omega)}{\Lambda_{K,\gamma(\omega),\omega}}\right\}$$

for all $\omega \in \Omega$, then the same conclusions of Theorem 3.1 hold.

Proof. Since K is a tempered random variable, we have

$$\lim_{t\to-\infty} \alpha^{s}_{-t,\theta^t\omega}\alpha^{c}_{t,\omega} = \lim_{t\to-\infty} K(\omega)K(\theta^t\omega)\, \mathrm{e}^{(\lambda^{\underline{c}}(\omega)-\lambda^{s}(\omega))t} \leqslant \lim_{t\to-\infty} K(\omega)\Lambda_{K,a(\omega),\omega}\, \mathrm{e}^{\gamma(\omega)t} = 0$$

and

$$\lim_{t\to+\infty} \alpha^{u}_{-t,\theta^t\omega}\alpha^{c}_{t,\omega} = \lim_{t\to+\infty} K(\omega)K(\theta^t\omega)\, \mathrm{e}^{(\lambda^{\overline{c}}(\omega)-\lambda^{u}(\omega))t} \leqslant \lim_{t\to+\infty} K(\omega)\Lambda_{K,b(\omega),\omega}\, \mathrm{e}^{\gamma(\omega)t} = 0$$

for all $\omega \in \Omega$. Therefore condition (3.5) holds. Let us check (3.6). Indeed, for every $t \geqslant 0$ and every $\omega \in \Omega$ we have

$$\begin{aligned}\frac{1}{\alpha^{c}_{t,\omega}} \int_0^t \alpha^{c}_{t-r,\theta^r\omega} \operatorname{Lip}(f_{\theta^r\omega})\alpha^{c}_{r,\omega}\, dr &= \int_0^t K(\theta^r\omega) \operatorname{Lip}(f_{\theta^r\omega})\, dr \\ &\leqslant \delta \int_{-\infty}^{+\infty} G(\theta^r\omega)\, dr \\ &\leqslant \delta,\end{aligned}$$

and, similarly, for every $t \leqslant 0$ and every $\omega \in \Omega$ we have

$$\frac{1}{\alpha^{c}_{t,\omega}} \int_t^0 \alpha^{c}_{t-r,\theta^r\omega} \operatorname{Lip}(f_{\theta^r\omega})\alpha^{c}_{r,\omega}\, dr \leqslant \delta.$$

Thus, $\sigma \leqslant \delta$. Moreover, since $K(\omega) \leqslant \mathrm{e}^{\gamma(\omega)|r|}\, \Lambda_{K,\gamma(\omega),\theta^r\omega}$ for every $\omega \in \Omega$ and $r \in \mathbb{R}$, we have

$$\begin{aligned}\int_{-\infty}^0 \alpha^{s}_{-r,\theta^r\omega} \operatorname{Lip}(f_{\theta^r\omega})\alpha^{c}_{r,\omega}\, dr &= \int_{-\infty}^0 K(\omega)K(\theta^r\omega)\, \mathrm{e}^{(\lambda^{\underline{c}}(\omega)-\lambda^{s}(\omega))r} \operatorname{Lip}(f_{\theta^r\omega})\, dr \\ &\leqslant \delta \int_{-\infty}^0 a(\omega)\, \mathrm{e}^{a(\omega)r}\, dr \\ &\leqslant \delta.\end{aligned}$$

and

$$\begin{aligned}\int_0^{+\infty} \alpha^{u}_{-r,\theta^r\omega} \operatorname{Lip}(f_{\theta^r\omega})\alpha^{c}_{r,\omega}\, dr &= \int_0^{+\infty} K(\omega)K(\theta^r\omega)\, \mathrm{e}^{(\lambda^{\overline{c}}(\omega)-\lambda^{u}(\omega))r} \operatorname{Lip}(f_{\theta^r\omega})\, dr \\ &\leqslant \delta \int_0^{+\infty} b(\omega)\, \mathrm{e}^{-b(\omega)r}\, dr \\ &\leqslant \delta.\end{aligned}$$

Henceforth, $\sigma + \tau \leqslant 3\delta < 1/2$. □

5.2 ψ-trichotomies

Consider measurable functions

$$\psi^{\bar{c}}, \psi^{\underline{c}}, \psi^{s}, \psi^{u} \colon \mathbb{R} \times \Omega \to]0, +\infty[$$

such that for $\ell \in \{\bar{c}, \underline{c}, s, u\}$ we have

$$\psi^{\ell}(t+s, \omega) = \psi^{\ell}(t, \theta^{s}\omega)\psi^{\ell}(s, \omega) \tag{5.2}$$

for all $t, s \in \mathbb{R}$ and all $\omega \in \Omega$. A *ψ-trichotomy* is a generalized trichotomy with bounds

$$\begin{aligned} \alpha^{c}_{t,\omega} &= \begin{cases} K(\omega)\psi^{\bar{c}}(t,\omega), & t \geqslant 0, \\ K(\omega)\psi^{\underline{c}}(t,\omega), & t \leqslant 0, \end{cases} \\ \alpha^{s}_{t,\omega} &= K(\omega)\psi^{s}(t,\omega), \quad t \geqslant 0, \\ \alpha^{u}_{t,\omega} &= K(\omega)\psi^{u}(t,\omega), \quad t \leqslant 0 \end{aligned}$$

for a random variable $K \colon \Omega \to [1, +\infty[$.

For all $\ell \in \{\underline{c}, \bar{c}, u, s\}$ set

$$d_{\psi^{\ell}}(\omega) = \lim_{h \to 0} \frac{\psi^{\ell}(h, \omega) - 1}{h}. \tag{5.3}$$

Since $\psi^{\ell}(0, \omega) = 1$, from (5.2) we have

$$\frac{d}{dt}\psi^{\ell}(t, \omega) = d_{\psi^{\ell}}(\theta^{t}\omega)\psi^{\ell}(t, \omega)$$

whenever limits (5.3) exist. Moreover, in this situation we also have

$$\frac{d}{dt}\psi^{\ell}(-t, \theta^{t}\omega) = \frac{d}{dt}\frac{1}{\psi^{\ell}(t, \omega)} = -d_{\psi^{\ell}}(\theta^{t}\omega)\psi^{\ell}(-t, \theta^{t}\omega).$$

From now on we also assume that for all $\omega \in \Omega$ the following limit exists:

$$d_{K}(\omega) = \lim_{h \to 0} \frac{K(\theta^{h}\omega) - K(\omega)}{h}. \tag{5.4}$$

We notice that for all $t \in \mathbb{R}$, $\frac{d}{dt}K(\theta^{t}\omega) = d_{K}(\theta^{t}\omega)$.

Corollary 5.2. *Let Φ be a Bochner measurable linear RDS exhibiting a ψ-trichotomy such that the limits in* (5.3) *and* (5.4) *exist and satisfy*

$$d_{\psi^{\bar{c}}}(\omega) - d_{\psi^{u}}(\omega) < \frac{d_{K}(\omega)}{K(\omega)} < d_{\psi^{\underline{c}}}(\omega) - d_{\psi^{s}}(\omega)$$

for all $\omega \in \Omega$. Let $f \in \mathscr{F}^{(B)}_{\alpha}$ be such that

$$\operatorname{Lip}(f_{\omega}) \leqslant \frac{\delta}{K(\omega)} \min\left\{G(\omega), \frac{a(\omega)}{K(\omega)}, \frac{b(\omega)}{K(\omega)}\right\}$$

for all $\omega \in \Omega$, where

$$a(\omega) = \frac{d_{K}(\omega)}{K(\omega)} - d_{\psi^{\bar{c}}}(\omega) + d_{\psi^{u}}(\omega) \quad \text{and} \quad b(\omega) = -\frac{d_{K}(\omega)}{K(\omega)} + d_{\psi^{\underline{c}}}(\omega) - d_{\psi^{s}}(\omega).$$

Assume that Ψ is a Bochner measurable RDS such that (3.3) *has unique solution $\Psi(\cdot,\omega,x)$ for every $\omega \in \Omega$ and every $x \in X$. If, for all $\omega \in \Omega$,*

$$\lim_{t\to-\infty} K(\theta^t\omega)\psi^s(-t,\theta^t\omega)\psi^{\underline{c}}(t,\omega) = \lim_{t\to+\infty} K(\theta^t\omega)\psi^u(-t,\theta^t\omega)\psi^{\overline{c}}(t,\omega) = 0 \tag{5.5}$$

then the same conclusions of Theorem 3.1 hold.

Proof. Conditions in (5.5) are equivalent to those in (3.5), and, as in the proof of Corollary 5.1 we have $\sigma \leqslant \delta$. Moreover, since

$$\begin{aligned}\frac{d}{dt}\left(\frac{\psi^u(-t,\theta^t\omega)\psi^{\overline{c}}(t,\omega)}{K(\theta^t\omega)}\right) &= \frac{\left(-d_{\psi^u}(\theta^t\omega)+d_{\psi^{\overline{c}}}(t,\omega)\right)K(\theta^t\omega)-d_K(\theta^t\omega)}{[K(\theta^t\omega)]^2}\,\psi^u(-t,\theta^t\omega)\,\psi^{\overline{c}}(t,\omega)\\ &= -\frac{a(\theta^t\omega)}{K(\theta^t\omega)}\,\psi^u(-t,\theta^t\omega)\,\psi^{\overline{c}}(t,\omega),\end{aligned}$$

we have

$$\begin{aligned}\int_0^{+\infty}\alpha^u_{-r,\theta^r\omega}\operatorname{Lip}(f_{\theta^r\omega})\alpha^{\overline{c}}_{r,\omega}\,dr &= K(\omega)\int_0^{+\infty}K(\theta^r\omega)\psi^u(-r,\theta^r\omega)\operatorname{Lip}(f_{\theta^r\omega})\psi^{\overline{c}}(r,\omega)\,dr\\ &\leqslant \delta K(\omega)\int_0^{+\infty}\frac{a(\theta^r\omega)}{K(\theta^r\omega)}\,\psi^u(-r,\theta^r\omega)\,\psi^{\overline{c}}(r,\omega)\,dr\\ &= \delta-\delta K(\omega)\lim_{r\to+\infty}\frac{\psi^u(-r,\theta^r\omega)\psi^{\overline{c}}(r,\omega)}{K(\theta^r\omega)}\\ &= \delta.\end{aligned}$$

Similarly, since

$$\begin{aligned}\frac{d}{dt}\left(\frac{\psi^s(-t,\theta^t\omega)\psi^{\underline{c}}(t,\omega)}{K(\theta^t\omega)}\right) &= \frac{\left(-d_{\psi^s}(\theta^t\omega)+d_{\psi^{\underline{c}}}(t,\omega)\right)K(\theta^t\omega)-d_K(\theta^t\omega)}{[K(\theta^t\omega)]^2}\,\psi^s(-t,\theta^t\omega)\,\psi^{\underline{c}}(t,\omega)\\ &= \frac{b(\theta^t\omega)}{K(\theta^t\omega)}\,\psi^s(-t,\theta^t\omega)\,\psi^{\underline{c}}(t,\omega),\end{aligned}$$

we have

$$\begin{aligned}\int_{-\infty}^{0}\alpha^s_{-r,\theta^r\omega}\operatorname{Lip}(f_{\theta^r\omega})\alpha^{\underline{c}}_{r,\omega}\,dr &= K(\omega)\int_{-\infty}^{0}K(\theta^r\omega)\psi^s(-r,\theta^r\omega)\operatorname{Lip}(f_{\theta^r\omega})\psi^{\underline{c}}(r,\omega)\,dr\\ &\leqslant \delta K(\omega)\int_{-\infty}^{0}\frac{b(\theta^r\omega)}{K(\theta^r\omega)}\,\psi^s(-r,\theta^r\omega)\,\psi^{\underline{c}}(r,\omega)\,dr\\ &= \delta-\delta K(\omega)\lim_{r\to-\infty}\frac{\psi^s(-r,\theta^r\omega)\psi^{\underline{c}}(r,\omega)}{K(\theta^r\omega)}\\ &= \delta.\end{aligned}$$

Thus $\sigma+\tau\leqslant 3\delta<1/2$. □

In the following we provide a particular example of a ψ-trichotomy in $\mathbb{R}^4$.

Example 5.3. Let $\psi^{\overline{c}},\psi^{\underline{c}},\psi^s,\psi^u\colon \mathbb{R}\times\Omega\to]0,+\infty[$ be measurable functions satisfying (5.2) and let $K\colon\Omega\to[1,+\infty[$ be a random variable. In $X=\mathbb{R}^4$, equipped with the maximum norm, consider the projections

$$\begin{aligned}P^{\overline{c}}_\omega(x_1,x_2,x_3,x_4) &= (0,0,x_3+(K(\omega)-1)x_4,0)\\ P^{\underline{c}}_\omega(x_1,x_2,x_3,x_4) &= ((1-K(\omega))x_2,x_2,0,0)\\ P^{s}_\omega(x_1,x_2,x_3,x_4) &= (x_1+(K(\omega)-1)x_2,0,0,0)\\ P^{u}_\omega(x_1,x_2,x_3,x_4) &= (0,0,(1-K(\omega))x_4,x_4)\end{aligned}$$

For all $\omega', \omega \in \Omega$,

$$\begin{aligned} P^{\bar{c}}_{\omega'} P^u_\omega &= (0, 0, (K(\omega') - K(\omega))x_4, 0), \\ P^s_{\omega'} P^{\underline{c}}_\omega &= ((K(\omega') - K(\omega))x_2, 0, 0, 0) \end{aligned}$$

and for all the remaining $i, j \in \{\bar{c}, \underline{c}, s, u\}$, with $i \neq j$,

$$P^i_{\omega'} P^j_\omega = 0.$$

Notice that for all $\omega, \omega' \in \Omega$

$$P^s_{\omega'} P^s_\omega = P^s_\omega, \quad P^u_{\omega'} P^u_\omega = P^u_{\omega'}, \quad P^{\bar{c}}_{\omega'} P^{\bar{c}}_\omega = P^{\bar{c}}_\omega \quad \text{and} \quad P^{\underline{c}}_{\omega'} P^{\underline{c}}_\omega = P^{\underline{c}}_{\omega'}.$$

Moreover,

$$\|P^{\bar{c}}_\omega\| = \|P^s_\omega\| = K(\omega)$$

and

$$\|P^{\underline{c}}_\omega\| = \|P^u_\omega\| = \max\{K(\omega) - 1, 1\} \leqslant K(\omega).$$

We define $\Phi \colon \mathbb{R} \times \Omega \times \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\Phi^t_\omega = \psi^{\bar{c}}(t, \omega)\, P^{\bar{c}}_\omega + \frac{K(\omega)}{K(\theta^t \omega)} \psi^{\underline{c}}(t, \omega)\, P^{\underline{c}}_{\theta^t \omega} + \psi^s(t, \omega) P^s_\omega + \frac{K(\omega)}{K(\theta^t \omega)} \psi^u(t, \omega)\, P^u_{\theta^t \omega}.$$

Let $P^c = P^{\bar{c}} + P^{\underline{c}}$ and $\mathcal{P} = (P^c, P^s, P^u)$. We have that Φ is a measurable linear RDS over Σ that admits a measurable $\mathcal{P}$-invariant splitting, and

$$\begin{aligned} \|\Phi^{c,t}_\omega\| &= \max\left\{\psi^{\bar{c}}(t, \omega)\|P^{\bar{c}}_\omega\|, \frac{1}{K(\theta^t \omega)} \psi^{\underline{c}}(t, \omega)\|P^{\underline{c}}_{\theta^t \omega}\|\right\} \leqslant K(\omega) \max\left\{\psi^{\bar{c}}(t, \omega), \psi^{\underline{c}}(t, \omega)\right\}, \\ \|\Phi^{s,t}_\omega\| &= \psi^s(t, \omega)\|P^s_\omega\| = K(\omega)\psi^s(t, \omega), \\ \|\Phi^{u,t}_\omega\| &= \frac{K(\omega)}{K(\theta^t \omega)} \psi^u(t, \omega)\|P^u_{\theta^t \omega}\| \leqslant K(\omega)\psi^u(t, \omega). \end{aligned}$$

Hence the linear RDS Φ exhibits a generalized trichotomy with bounds

$$\begin{aligned} \alpha^c_{t,\omega} &= K(\omega) \max\left\{\psi^{\bar{c}}(t, \omega), \psi^{\underline{c}}(t, \omega)\right\}, \\ \alpha^s_{t,\omega} &= K(\omega)\psi^s(t, \omega), \\ \alpha^u_{t,\omega} &= K(\omega)\psi^u(t, \omega). \end{aligned}$$

If we assume $\psi^{\bar{c}}(t, \omega) \geqslant \psi^{\underline{c}}(t, \omega)$ for all $t \geqslant 0$ then

$$\alpha^c_{t,\omega} = \begin{cases} K(\omega)\psi^{\bar{c}}(t, \omega) & \text{if } t \geqslant 0, \\ K(\omega)\psi^{\underline{c}}(t, \omega) & \text{if } t \leqslant 0, \end{cases}$$

and Φ exhibits a ψ-trichotomy.

Next, based on the previous example, we provide an example of a ψ-trichotomy on an infinite dimensional Banach space.

Example 5.4. Let $K_n \colon \Omega \to [1, +\infty[$ be a sequence of random variables such that

$$K(\omega) := \sup_{n \in \mathbb{N}} K_n(\omega) < +\infty \text{ for all } \omega \in \Omega.$$

In ℓ_∞, the space of bounded sequences equipped with the supremum norm, consider, for all $n \in \mathbb{N}$ and for all $\omega \in \Omega$, the projections

$$P^{\bar{c}}_{n,\omega}, P^{\underline{c}}_{n,\omega}, P^{s}_{n,\omega}, P^{u}_{n,\omega} \colon \ell_\infty \to \ell_\infty$$

defined by

$$\begin{aligned}
P^{\bar{c}}_{n,\omega}(x_1, x_2, x_3, x_4, x_5, \ldots) &= (\underbrace{0, \ldots, 0}_{4n-4 \text{ zeros}}, 0, 0, x_{4n-1} + L_n(\omega) x_{4n}, 0, 0, 0, \ldots),\\
P^{\underline{c}}_{n,\omega}(x_1, x_2, x_3, x_4, x_5, \ldots) &= (\underbrace{0, \ldots, 0}_{4n-4 \text{ zeros}}, -L_n(\omega) x_{4n-2}, x_{4n-2}, 0, 0, 0, \ldots),\\
P^{s}_{n,\omega}(x_1, x_2, x_3, x_4, x_5, \ldots) &= (\underbrace{0, \ldots, 0}_{4n-4 \text{ zeros}}, x_{4n-3} + L_n(\omega) x_{4n-2}, 0, 0, 0, \ldots),\\
P^{u}_{n,\omega}(x_1, x_2, x_3, x_4, x_5, \ldots) &= (\underbrace{0, \ldots, 0}_{4n-4 \text{ zeros}}, 0, 0, -L_n(\omega) x_{4n}, x_{4n}, 0, 0, 0, \ldots),
\end{aligned}$$

where $L_n(\omega) = K_n(\omega) - 1$. It follows that for all $\omega', \omega \in \Omega$, for all $i, j \in \{\bar{c}, \underline{c}, s, u\}$ and for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
P^{i}_{n,\omega} P^{j}_{m,\omega'} &= 0 \quad \text{with} \quad n \neq m,\\
P^{i}_{n,\omega'} P^{j}_{n,\omega} &= 0 \quad \text{with} \quad i \neq j, \ (i,j) \neq (\bar{c}, u) \quad \text{and} \quad (i,j) \neq (s, \underline{c}),\\
P^{\bar{c}}_{n,\omega'} P^{u}_{n,\omega} &= (\underbrace{0, \ldots, 0}_{4n-4 \text{ zeros}}, 0, 0, (K_n(\omega') - K_n(\omega)) x_{4n}, 0, 0, 0, \ldots),\\
P^{s}_{n,\omega'} P^{\underline{c}}_{n,\omega} &= (\underbrace{0, \ldots, 0}_{4n-4 \text{ zeros}}, (K_n(\omega') - K_n(\omega)) x_{4n-2}, 0, 0, 0, \ldots).
\end{aligned}$$

Moreover, for all $\omega, \omega' \in \Omega$ and for all $n \in \mathbb{N}$,

$$P^{i}_{n,\omega'} P^{i}_{n,\omega} = P^{i}_{n,\omega} \quad \text{and} \quad P^{j}_{n,\omega'} P^{j}_{n,\omega} = P^{j}_{n,\omega'} \text{ with } i \in \{\bar{c}, s\} \text{ and } j \in \{\underline{c}, u\}.$$

Let $\psi^{\bar{c}}_n, \psi^{\underline{c}}_n, \psi^{s}_n, \psi^{u}_n \colon \mathbb{R} \times \Omega \to]0, +\infty[$ be sequences of measurable functions satisfying (5.2) and such that

$$\psi^i(t, \omega) := \sup_{n \in \mathbb{N}} \psi^i_n(t, \omega) < +\infty$$

for all $\omega \in \Omega$, for all $t \in \mathbb{R}$ and for all $i \in \{\bar{c}, \underline{c}, s, u\}$. If $\Phi \colon \mathbb{R} \times \Omega \times \ell_\infty \to \ell_\infty$ is given by

$$\Phi^t_\omega = \sum_{n=1}^{+\infty} \left[\psi^{\bar{c}}_n(t, \omega)\, P^{\bar{c}}_{n,\omega} + \frac{K_n(\omega)}{K_n(\theta^t \omega)} \psi^{\underline{c}}_n(t, \omega)\, P^{\underline{c}}_{n,\theta^t \omega} + \psi^{s}_n(t, \omega) P^{s}_{n,\omega} + \frac{K_n(\omega)}{K_n(\theta^t \omega)} \psi^{u}_n(t, \omega)\, P^{u}_{n,\theta^t \omega} \right],$$

then Φ is a measurable linear RDS over Σ that admits a measurable (P^c, P^s, P^u)-invariant splitting, where, for all $\omega \in \Omega$,

$$P^i_\omega = \sum_{n=1}^{+\infty} P^i_{n,\omega}, \quad i \in \{\bar{c}, \underline{c}, s, u\}, \quad \text{and} \quad P^c_\omega = P^{\bar{c}}_\omega + P^{\underline{c}}_\omega.$$

Moreover, since

$$\|P^{\bar{c}}_{n,\omega}\| = \|P^{s}_{n,\omega}\| = K_n(\omega)$$

and

$$\|P^{\underline{c}}_{n,\omega}\| = \|P^{u}_{n,\omega}\| = \max\{K_n(\omega)-1,1\},$$

it follows that

$$\left\|P^{\bar{c}}_{\omega}\right\| = \sup_{n\in\mathbb{N}} \left\|P^{s}_{n,\omega}\right\| = \sup_{n\in\mathbb{N}} K_n(\omega) = K(\omega)$$

and

$$\left\|P^{\underline{c}}_{\omega}\right\| = \left\|P^{u}_{n,\omega}\right\| = \sup_{n\in\mathbb{N}} (\max\{K_n(\omega)-1,1\}) = \max\{K(\omega)-1,1\} \leqslant K(\omega).$$

Hence

$$\begin{aligned}
\|\Phi^{c,t}_{\omega}\| &= \sup_{n\in\mathbb{N}} \left[\max\left\{\psi^{\bar{c}}_n(t,\omega)\|P^{\bar{c}}_{n,\omega}\|, \frac{1}{K_n(\theta^t\omega)}\psi^{\underline{c}}_n(t,\omega)\|P^{\underline{c}}_{n,\theta^t\omega}\|\right\}\right]\\
&\leqslant \sup_{n\in\mathbb{N}} \left[K_n(\omega)\max\left\{\psi^{\bar{c}}_n(t,\omega),\psi^{\underline{c}}_n(t,\omega)\right\}\right]\\
&\leqslant K(\omega)\max\left\{\psi^{\bar{c}}(t,\omega),\psi^{\underline{c}}(t,\omega)\right\},\\
\|\Phi^{s,t}_{\omega}\| &= \sup_{n\in\mathbb{N}} \left[\psi^{s}_n(t,\omega)\|P^{s}_{n,\omega}\|\right] = \sup_{n\in\mathbb{N}} \left[\psi^{s}_n(t,\omega)K_n(\omega)\right] \leqslant K(\omega)\psi^{s}(t,\omega),\\
\|\Phi^{u,t}_{\omega}\| &= \sup_{n\in\mathbb{N}} \left[\frac{K_n(\omega)}{K_n(\theta^t\omega)}\psi^{u}_n(t,\omega)\|P^{u}_{n,\theta^t\omega}\|\right] \leqslant K(\omega)\psi^{u}(t,\omega).
\end{aligned}$$

This implies that the linear RDS Φ admits a generalized trichotomy with bounds

$$\begin{aligned}
\alpha^{c}_{t,\omega} &= K(\omega)\max\left\{\psi^{\bar{c}}(t,\omega),\psi^{\underline{c}}(t,\omega)\right\},\\
\alpha^{s}_{t,\omega} &= K(\omega)\psi^{s}(t,\omega),\\
\alpha^{u}_{t,\omega} &= K(\omega)\psi^{u}(t,\omega).
\end{aligned}$$

If, in addition, $\psi^{\bar{c}}, \psi^{\underline{c}}, \psi^{s}, \psi^{u}$ satisfy (5.2) and $\psi^{\bar{c}}(t,\omega) \geqslant \psi^{\underline{c}}(t,\omega)$, for all $t \geqslant 0$ and for all $\omega \in \Omega$, then

$$\alpha^{c}_{t,\omega} = \begin{cases} K(\omega)\psi^{\bar{c}}(t,\omega) & \text{if } t \geqslant 0,\\ K(\omega)\psi^{\underline{c}}(t,\omega) & \text{if } t \leqslant 0,\end{cases}$$

and Φ exhibits a ψ-trichotomy.

In the next sections we consider particular ψ-trichotomies.

5.2.1 Integral exponential trichotomy

Let

$$\lambda^{\bar{c}}, \lambda^{\underline{c}}, \lambda^{s}, \lambda^{u} \colon \Omega \to \mathbb{R}$$

be random variables such that for all $\omega \in \Omega$ and $\ell \in \{\bar{c}, \underline{c}, s, u\}$, the map $r \mapsto \lambda^{\ell}(\theta^r\omega)$ is integrable in every interval $[0,t]$ An *integral exponential trichotomy* is a ψ-trichotomy with

$$\psi^{\ell}(t,\omega) = e^{\int_0^t \lambda^{\ell}(\theta^r\omega)dr}$$

for all $\ell \in \{\bar{c}, \underline{c}, s, u\}$, i.e., is a generalized trichotomy with bounds

$$\alpha^c_{t,\omega} = \begin{cases} K(\omega)\, e^{\int_0^t \lambda^{\bar{c}}(\theta^r\omega)dr}, & t \geqslant 0, \\ K(\omega)\, e^{\int_0^t \lambda^{\underline{c}}(\theta^r\omega)dr}, & t \leqslant 0, \end{cases}$$
$$\alpha^s_{t,\omega} = K(\omega)\, e^{\int_0^t \lambda^{s}(\theta^r\omega)dr}, \quad t \geqslant 0,$$
$$\alpha^u_{t,\omega} = K(\omega)\, e^{\int_0^t \lambda^{u}(\theta^r\omega)dr}, \quad t \leqslant 0.$$

Notice that if

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \lambda^\ell(\theta^r\omega)\, dr = \lambda^\ell(\omega) \tag{5.6}$$

then $d_{\psi^\ell}(\omega) = \lambda^\ell(\omega)$ for all $\ell \in \{\bar{c}, \underline{c}, s, u\}$. From Corollary 5.2 we get the following.

Corollary 5.5. *Let Φ be a Bochner measurable linear RDS exhibiting an integral exponential trichotomy such that (5.6) holds and the limit (5.4) exists and satisfyies*

$$\lambda^{\bar{c}}(\omega) - \lambda^u(\omega) < \frac{d_K(\omega)}{K(\omega)} < \lambda^{\underline{c}}(\omega) - \lambda^s(\omega)$$

for all $\omega \in \Omega$. Let $f \in \mathscr{F}_\alpha^{(B)}$ be such that

$$\mathrm{Lip}(f_\omega) \leqslant \frac{\delta}{K(\omega)} \min\left\{ G(\omega), \frac{a(\omega)}{K(\omega)}, \frac{b(\omega)}{K(\omega)} \right\}$$

for all $\omega \in \Omega$, where

$$a(\omega) = \frac{d_K(\omega)}{K(\omega)} - \lambda^{\bar{c}}(\omega) + \lambda^u(\omega) \quad \text{and} \quad b(\omega) = -\frac{d_K(\omega)}{K(\omega)} + \lambda^{\underline{c}}(\omega) - \lambda^s(\omega).$$

Assume that Ψ is a Bochner measurable RDS such that (3.3) has a unique solution $\Psi(\cdot, \omega, x)$ for every $\omega \in \Omega$ and every $x \in X$. If for all $\omega \in \Omega$,

$$\lim_{t\to+\infty} K(\theta^t\omega)\, e^{\int_0^t \lambda^{\bar{c}}(\theta^r\omega) - \lambda^u(\theta^r\omega)\, dr} = \lim_{t\to-\infty} K(\theta^t\omega)\, e^{\int_0^t \lambda^{\underline{c}}(\theta^r\omega) - \lambda^s(\theta^r\omega)\, dr} = 0,$$

then the same conclusions of Theorem 3.1 hold.

5.2.2 Non exponential trichotomies

We provide now a particular type of ψ-trichotomies that can be easily handled to construct trichotomies beyhond the exponential bounds. Let

$$\lambda^{\bar{c}}, \lambda^{\underline{c}}, \lambda^s, \lambda^u \colon \Omega \to \mathbb{R}$$

be random variables such that for all $\ell \in \{\bar{c}, \underline{c}, s, u\}$ the following limit exists for all ω:

$$d_{\lambda^\ell}(\omega) := \lim_{h\to 0} \frac{\lambda^\ell(\theta^h\omega) - \lambda^\ell(\omega)}{h}. \tag{5.7}$$

Consider a ψ-trichotomy with

$$\psi^\ell(t,\omega) = \frac{\lambda^\ell(\omega)}{\lambda^\ell(\theta^t\omega)}$$

for all $\ell \in \{\bar{c}, \underline{c}, s, u\}$, i.e., is a generalized trichotomy with bounds

$$\begin{aligned} \alpha^c_{t,\omega} &= \begin{cases} K(\omega)\dfrac{\lambda^{\bar{c}}(\omega)}{\lambda^{\bar{c}}(\theta^t\omega)}, & t \geqslant 0, \\ K(\omega)\dfrac{\lambda^{\underline{c}}(\omega)}{\lambda^{\underline{c}}(\theta^t\omega)}, & t \leqslant 0, \end{cases} \\ \alpha^s_{t,\omega} &= K(\omega)\frac{\lambda^s(\omega)}{\lambda^s(\theta^t\omega)}, \quad t \geqslant 0 \\ \alpha^u_{t,\omega} &= K(\omega)\frac{\lambda^u(\omega)}{\lambda^u(\theta^t\omega)}, \quad t \leqslant 0. \end{aligned} \tag{5.8}$$

Notice that

$$d_{\psi^\ell}(\omega) = -\frac{d_{\lambda^\ell}(\omega)}{\lambda^\ell(\omega)}.$$

for all $\ell \in \{\bar{c}, \underline{c}, s, u\}$. From Corollary 5.2 we get the following.

Corollary 5.6. *Let Φ be a Bochner measurable linear RDS exhibiting an α-trichotomy, with bounds* (5.8) *and such that* (5.7) *and* (5.4) *exist and satisfy*

$$\frac{d_{\lambda^u}(\omega)}{\lambda^u(\omega)} - \frac{d_{\lambda^{\bar{c}}}(\omega)}{\lambda^{\bar{c}}(\omega)} < \frac{d_K(\omega)}{K(\omega)} < \frac{d_{\lambda^s}(\omega)}{\lambda^s(\omega)} - \frac{d_{\lambda^{\underline{c}}}(\omega)}{\lambda^{\underline{c}}(\omega)}.$$

Let $f \in \mathscr{F}^{(B)}_\alpha$ be such that for all $\omega \in \Omega$ we have

$$\operatorname{Lip}(f_\omega) \leqslant \frac{\delta}{K(\omega)} \min\left\{G(\omega), \frac{a(\omega)}{K(\omega)}, \frac{b(\omega)}{K(\omega)}\right\},$$

where

$$a(\omega) = \frac{d_K(\omega)}{K(\omega)} + \frac{d_{\lambda^{\bar{c}}}(\omega)}{\lambda^{\bar{c}}(\omega)} - \frac{d_{\lambda^u}(\omega)}{\lambda^u(\omega)} \quad \text{and} \quad b(\omega) = -\frac{d_K(\omega)}{K(\omega)} - \frac{d_{\lambda^{\underline{c}}}(\omega)}{\lambda^{\underline{c}}(\omega)} + \frac{d_{\lambda^s}(\omega)}{\lambda^s(\omega)}.$$

Assume that Ψ is a Bochner measurable RDS such that (3.3) *has a unique solution $\Psi(\cdot, \omega, x)$ for every $\omega \in \Omega$ and every $x \in X$. If for all $\omega \in \Omega$,*

$$\lim_{t\to+\infty} K(\theta^t\omega)\frac{\lambda^s(\theta^t\omega)}{\lambda^{\underline{c}}(\theta^t\omega)} = \lim_{t\to+\infty} K(\theta^t\omega)\frac{\lambda^u(\theta^t\omega)}{\lambda^{\bar{c}}(\theta^t\omega)} = 0$$

then the same conclusions of Theorem 3.1 hold.

Example 5.7 (Non exponential trichotomy)**.** Consider for the driving system the horizontal flow in $\mathbb{R}^2$ given by $\theta^t(x,y) = (x+t, y)$, which preserves the Lebesgue measure. Let $C, \xi^{\bar{c}}, \xi^{\underline{c}}, \xi^s, \xi^u$ and ε be some real constants with $C \geqslant 1$ and $\varepsilon \geqslant 0$, and set:

$$\begin{aligned} \lambda^\ell(x,y) &= (1+x^2)^{-(1+y^2)\xi_\ell}, \quad \ell \in \{\bar{c}, \underline{c}, s, u\}, \\ K(x,y) &= C(1+x^2)^{(1+y^2)\varepsilon}. \end{aligned}$$

In this case we obtain a polynomial type trichotomy. Let us assume $\lambda_{\overline{c}} \geqslant \lambda_{\underline{c}}$. Thus we have a trichotomy with

$$\alpha^c_{t,(x,y)} = \begin{cases} C\left(\dfrac{1+(x+t)^2}{1+x^2}\right)^{(1+y^2)\xi^{\overline{c}}} (1+x^2)^{(1+y^2)\varepsilon}, & t \geqslant 0, \\ C\left(\dfrac{1+(x+t)^2}{1+x^2}\right)^{(1+y^2)\xi^{\underline{c}}} (1+x^2)^{(1+y^2)\varepsilon}, & t \leqslant 0, \end{cases}$$

$$\alpha^s_{t,(x,y)} = C\left(\frac{1+(x+t)^2}{1+x^2}\right)^{(1+y^2)\xi^s} (1+x^2)^{(1+y^2)\varepsilon}, \quad t \geqslant 0,$$

$$\alpha^u_{t,(x,y)} = C\left(\frac{1+(x+t)^2}{1+x^2}\right)^{(1+y^2)\xi^u} (1+x^2)^{(1+y^2)\varepsilon}, \quad t \leqslant 0.$$

Notice that $d_{\lambda^\ell}(x,y) = \frac{\partial}{\partial x}\lambda^\ell(x,y)$.

6 Discrete-time examples

In this section we assume $\mathbb{T} = \mathbb{Z}$ and provide some corollaries to Theorem 4.1. Let X be a Banach space and let $\Sigma \equiv (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a measure-preserving dynamical system. Throughout this subsection we consider a real number $\delta \in]0, 1/6[$ and a random variable $G \colon \Omega \to]0, +\infty[$ such that for all $\omega \in \Omega$ we have

$$\sum_{k=-\infty}^{+\infty} G(\theta^k \omega) \leqslant 1.$$

6.1 Tempered exponential trichotomies

Consider θ-invariant random variables

$$\lambda^{\overline{c}}, \lambda^{\underline{c}}, \lambda^s, \lambda^u \colon \Omega \to \mathbb{R}.$$

We say that a measurable linear RDS Φ on X over Σ exhibits an *exponential trichotomy* if it admits a generalized trichotomy with bounds

$$\alpha^c_{n,\omega} = \begin{cases} K(\omega)\, \mathrm{e}^{\lambda^{\overline{c}}(\omega) n}, & n \geqslant 0, \\ K(\omega)\, \mathrm{e}^{\lambda^{\underline{c}}(\omega) n}, & n \leqslant 0, \end{cases}$$

$$\alpha^s_{n,\omega} = K(\omega)\, \mathrm{e}^{\lambda^s(\omega) n}, \quad n \geqslant 0,$$

$$\alpha^u_{n,\omega} = K(\omega)\, \mathrm{e}^{\lambda^u(\omega) n}, \quad n \leqslant 0$$

for some random variable $K : \Omega \to [1, +\infty[$. If the random variable K is *tempered* we say that Φ exhibits an *tempered exponential trichotomy*. Notice that in the discrete-time case the condition (5.1) is equivalent to

$$\lim_{n \to \pm\infty} \frac{1}{|n|} \log K(\theta^n \omega) = 0 \quad \text{for all } \omega \in \Omega.$$

Corollary 6.1. *Let Φ be a measurable linear RDS exhibiting a tempered exponential trichotomy such that, for all $\omega \in \Omega$, satisfies*

$$\lambda^{\underline{c}}(\omega) > \lambda^s(\omega) \quad \textit{and} \quad \lambda^{\overline{c}}(\omega) < \lambda^u(\omega)$$

and let $f \in \mathscr{F}$. Consider a θ-invariant random variable $\gamma(\omega) > 0$ satisfying for all $\omega \in \Omega$

$$a(\omega) := \lambda^{\underline{c}}(\omega) - \lambda^{s}(\omega) - \gamma(\omega) > 0 \quad \text{and} \quad b(\omega) := \lambda^{u}(\omega) - \lambda^{\overline{c}}(\omega) - \gamma(\omega) > 0.$$

If

$$\operatorname{Lip}(f_\omega) \leqslant \frac{\delta}{K(\theta\omega)} \min\left\{ \mathrm{e}^{\min\{\lambda^{\underline{c}}(\omega),\lambda^{\overline{c}}(\omega)\}}\, G(\omega), \mathrm{e}^{\lambda^{u}(\omega)}\, \frac{\mathrm{e}^{a(\omega)} - 1}{\Lambda_{K,\gamma(\omega),\omega}}, \mathrm{e}^{\lambda^{s}(\omega)}\, \frac{1 - \mathrm{e}^{-b(\omega)}}{\Lambda_{K,\gamma(\omega),\omega}} \right\}$$

for all $\omega \in \Omega$ then the same conclusions of Theorem 3.1 hold.

6.2 ψ-trichotomies

Consider measurable functions

$$\psi^{\overline{c}}, \psi^{\underline{c}}, \psi^{s}, \psi^{u} \colon \mathbb{Z} \times \Omega \to]0, +\infty[$$

such that for $\ell \in \{\overline{c}, \underline{c}, s, u\}$ we have

$$\psi^{\ell}(t+s,\omega) = \psi^{\ell}(t,\theta^{s}\omega)\psi^{\ell}(s,\omega)$$

for all $t, s \in \mathbb{Z}$ and all $\omega \in \Omega$. A *ψ-trichotomy* is a generalized trichotomy with bounds

$$\begin{aligned} \alpha^{c}_{n,\omega} &= \begin{cases} K(\omega)\psi^{\overline{c}}(n,\omega), & t \geqslant 0, \\ K(\omega)\psi^{\underline{c}}(n,\omega), & t \leqslant 0, \end{cases} \\ \alpha^{s}_{t,\omega} &= K(\omega)\psi^{s}(n,\omega), \quad t \geqslant 0, \\ \alpha^{u}_{t,\omega} &= K(\omega)\psi^{u}(n,\omega), \quad t \leqslant 0 \end{aligned}$$

where $K \colon \Omega \to [1, +\infty[$ is a random variable. We notice that, as in the continuous-time case, we may consider different growth rates along the central directions E^{c}_{ω}, depending if we are looking to the *future* ($n \to +\infty$) or to the *past* ($n \to -\infty$).

Corollary 6.2. *Let Φ be a measurable linear RDS exhibiting a ψ-trichotomy such that*

$$\frac{\psi^{\overline{c}}(1,\omega)}{\psi^{u}(1,\omega)} < \frac{K(\theta\omega)}{K(\omega)} < \frac{\psi^{\underline{c}}(1,\omega)}{\psi^{s}(1,\omega)}. \tag{6.1}$$

Let $f \in \mathscr{F}$ be such that

$$\operatorname{Lip}(f_\omega) \leqslant \frac{\delta}{K(\theta\omega)} \min\left\{ \psi^{\overline{c}}(1,\omega)G(\omega), \psi^{\underline{c}}(1,\omega)G(\omega), a(\omega), b(\omega) \right\},$$

where

$$a(\omega) = \frac{\psi^{u}(1,\omega)}{K(\omega)} - \frac{\psi^{\overline{c}}(1,\omega)}{K(\theta\omega)} \quad \text{and} \quad b(\omega) = \frac{\psi^{\underline{c}}(1,\omega)}{K(\theta\omega)} - \frac{\psi^{s}(1,\omega)}{K(\omega)}.$$

If

$$\lim_{n\to-\infty} K(\theta^{n}\omega)\psi^{s}(-n,\theta^{n}\omega)\psi^{\underline{c}}(n,\omega) = \lim_{n\to+\infty} K(\theta^{n}\omega)\psi^{u}(-n,\theta^{n}\omega)\psi^{\overline{c}}(n,\omega) = 0$$

for all $\omega \in \Omega$, then the same conclusion of Theorem 4.1 holds.

Proof. We will check that we are in conditions to apply Theorem 4.1. Notice that from (6.1) we conclude $a(\omega), b(\omega) > 0$. We have

$$\begin{aligned}
\sigma_\omega^- &= \sup_{n\in\mathbb{N}} \frac{1}{\psi^{\underline{c}}(-n,\omega)} \sum_{k=-n}^{-1} K(\theta^{k+1}\omega)\psi^{\underline{c}}(-n-k-1,\theta^{k+1}\omega)\operatorname{Lip}(f_{\theta^k\omega})\psi^{\overline{c}}(k,\omega) \\
&\leqslant \delta \sum_{k=-\infty}^{+\infty} G(\theta^k\omega) \leqslant \delta,
\end{aligned}$$

and, similarly, $\sigma_\omega^+ \leqslant \delta$. Thus $\sigma \leqslant \delta$. Moreover,

$$\begin{aligned}
\tau_\omega^+ &= \sum_{k=0}^{+\infty} K(\theta^{k+1}\omega)\psi^u(-k-1,\theta^{k+1}\omega)\operatorname{Lip}(f_{\theta^k\omega})K(\omega)\psi^{\overline{c}}(k,\omega) \\
&\leqslant \delta K(\omega) \sum_{k=0}^{+\infty} \left[\frac{\psi^u(-k,\theta^k\omega)\psi^{\overline{c}}(k,\omega))}{K(\theta^k\omega)} - \frac{\psi^u(-(k+1),\theta^{k+1}\omega)\psi^{\overline{c}}(k+1,\omega))}{K(\theta^{k+1}\omega)} \right] \\
&\leqslant \delta K(\omega) \left(\frac{1}{K(\omega)} - \lim_{k\to+\infty} \frac{\psi^u(-k,\theta^k\omega)\psi^{\overline{c}}(k,\omega))}{K(\theta^k\omega)} \right) \\
&= \delta.
\end{aligned}$$

Similarly we get $\tau_\omega^- \leqslant \delta$. Therefore $\sigma + \tau \leqslant 3\delta < 1/2$. □

In the following we consider particular ψ-trichotomies.

6.2.1 Summable exponential trichotomies

We start by considering the integral (or summable) exponential trichotomies, which are a generalization of the exponential trichotomies and can be seen as the discrete counterpart of those in Section 5.2.1. Let

$$\lambda^{\overline{c}}, \lambda^{\underline{c}}, \lambda^s, \lambda^u \colon \Omega \to \mathbb{R}$$

be random variables and set For all $\ell \in \{\overline{c}, \underline{c}, s, u\}$ we set

$$S^\ell(n,\omega) = \begin{cases} \lambda^\ell(\omega) + \cdots + \lambda^\ell(\theta^{n-1}\omega), & n \geqslant 1, \\ 0, & n = 0, \\ -\lambda^\ell(\theta^n\omega) - \cdots - \lambda^\ell(\theta^{-1}\omega), & n \leqslant -1. \end{cases}$$

A *summable exponential trichotomy* is a ψ-trichotomy with

$$\psi^\ell(t,\omega) = \mathrm{e}^{S^\ell(n,\omega)}$$

for all $\ell \in \{\overline{c}, \underline{c}, s, u\}$, i.e., is a generalized trichotomy with bounds

$$\begin{aligned}
\alpha_{n,\omega}^c &= \begin{cases} K(\omega)\,\mathrm{e}^{S^{\overline{c}}(n,\omega)}, & n \geqslant 0, \\ K(\omega)\,\mathrm{e}^{S^{\underline{c}}(n,\omega)}, & n \leqslant 0, \end{cases} \\
\alpha_{n,\omega}^s &= K(\omega)\,\mathrm{e}^{S^s(n,\omega)}, \quad n \geqslant 0, \\
\alpha_{n,\omega}^u &= K(\omega)\,\mathrm{e}^{S^u(n,\omega)}, \quad n \leqslant 0
\end{aligned}$$

for some tempered random variable $K : \Omega \to [1, +\infty[$.

Corollary 6.3. *Let Φ be a measurable linear RDS exhibiting a summable exponential trichotomy such that*

$$\frac{e^{\lambda^{\bar{c}}(\omega)}}{e^{\lambda^{u}(\omega)}} < \frac{K(\theta\omega)}{K(\omega)} < \frac{e^{\lambda^{\underline{c}}(\omega)}}{e^{\lambda^{s}(\omega)}}.$$

Let $f \in \mathscr{F}$ be such that

$$\operatorname{Lip}(f_\omega) \leqslant \frac{\delta}{K(\theta\omega)} \min\left\{ e^{\lambda^{\bar{c}}(\omega)} G(\omega), e^{\lambda^{\underline{c}}(\omega)} G(\omega), a(\omega), b(\omega) \right\},$$

where

$$a(\omega) = \frac{e^{\lambda^{u}(\omega)}}{K(\omega)} - \frac{e^{\lambda^{\bar{c}}(\omega)}}{K(\theta\omega)} \quad \text{and} \quad b(\omega) = \frac{e^{\lambda^{\underline{c}}(\omega)}}{K(\theta\omega)} - \frac{e^{\lambda^{s}(\omega)}}{K(\omega)}.$$

If

$$\lim_{n\to-\infty} K(\theta^n\omega)\, e^{S^s(-n,\theta^n\omega)+S^{\underline{c}}(n,\omega)} = \lim_{n\to+\infty} K(\theta^n\omega)\, e^{S^u(-n,\theta^n\omega)+S^{\bar{c}}(n,\omega)} = 0$$

for all $\omega \in \Omega$, then the same conclusion of Theorem 4.1 holds.

6.2.2 Non exponential trichotomies

We provide a particular type of ψ-trichotomies that can be easily handled to construct trichotomies beyhond the exponential bounds in the discrete-time scenario. Consider a ψ-trichotomy with

$$\psi^\ell(n,\omega) = \frac{\lambda^\ell(\omega)}{\lambda^\ell(\theta^n\omega)}$$

for all $\ell \in \{\bar{c}, \underline{c}, s, u\}$, *i.e.*, is a generalized trichotomy with bounds

$$\begin{aligned} \alpha^c_{n,\omega} &= \begin{cases} K(\omega)\dfrac{\lambda^{\bar{c}}(\omega)}{\lambda^{\bar{c}}(\theta^n\omega)}, & n \geqslant 0, \\ K(\omega)\dfrac{\lambda^{\underline{c}}(\omega)}{\lambda^{\underline{c}}(\theta^n\omega)}, & n \leqslant 0, \end{cases} \\ \alpha^s_{n,\omega} &= K(\omega)\frac{\lambda^s(\omega)}{\lambda^s(\theta^n\omega)}, \quad n \geqslant 0, \\ \alpha^u_{n,\omega} &= K(\omega)\frac{\lambda^u(\omega)}{\lambda^u(\theta^n\omega)}, \quad n \leqslant 0. \end{aligned} \tag{6.2}$$

For future use let us define

$$a(\omega) = \frac{\lambda^u(\omega)}{\lambda^u(\theta\omega)K(\omega)} - \frac{\lambda^{\bar{c}}(\omega)}{\lambda^{\bar{c}}(\theta\omega)K(\theta\omega)} \quad \text{and} \quad b(\omega) = \frac{\lambda^{\underline{c}}(\omega)}{\lambda^{\underline{c}}(\theta\omega)K(\theta\omega)} - \frac{\lambda^s(\omega)}{\lambda^s(\theta\omega)K(\omega)}.$$

Corollary 6.4. *Let Φ be a measurable linear RDS exhibiting an α-trichotomy with bounds (6.2) and such that*

$$\frac{\lambda^{\bar{c}}(\omega)\lambda^u(\theta\omega)}{\lambda^{\bar{c}}(\theta\omega)\lambda^u(\omega)} < \frac{K(\theta\omega)}{K(\omega)} < \frac{\lambda^{\underline{c}}(\omega)\lambda^s(\theta\omega)}{\lambda^{\underline{c}}(\theta\omega)\lambda^s(\omega)}.$$

Let $f \in \mathscr{F}$ be such that

$$\operatorname{Lip}(f_\omega) \leqslant \frac{\delta}{K(\theta\omega)} \min\left\{ \frac{\lambda^{\bar{c}}(\omega)}{\lambda^{\bar{c}}(\theta\omega)} G(\omega), \frac{\lambda^{\underline{c}}(\omega)}{\lambda^{\underline{c}}(\theta\omega)} G(\omega), a(\omega), b(\omega) \right\}.$$

If

$$\lim_{n\to-\infty} \frac{K(\theta^n\omega)\lambda^s(\theta^n\omega)}{\lambda^{\underline{c}}(\theta^n\omega)} = \lim_{n\to+\infty} \frac{K(\theta^n\omega)\lambda^u(\theta^n\omega)}{\lambda^{\bar{c}}(\theta^n\omega)} = 0$$

for all $\omega \in \Omega$, then the same conclusion of Theorem 4.1 holds.

We may consider Example 5.7 with $\mathbb{T} = \mathbb{Z}$ to get an apllication of this result in a non exponential trichotomy situation.

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