HU e-ISSN 1787-2413

Miskolc Mathematical Notes

A Publication of the University of Miskolc

VOLUME 25 (2024), NUMBER 2

MISKOLC UNIVERSITY PRESS

Miskolc Mathematical Notes

VOLUME 25 (2024), NUMBER 2

CONTENTS

Mohammad Alnegga, Abdelbaki Choucha, Djamel Ouchenane, and Salah Boulaaras
General decay and blow up of solutions for a Kirchhoff-type equation with variable-exponents 537

Ilhame Amirali, Muhammet Enes Durmaz, and Gabil M. Amiraliyev
A numerical approach for solving Volterra delay integro-differential equation 553

Seon Hye An, Rekha Srivastava, and Nak Eun Cho
New approaches for subordination and superordination of multivalent functions associated with a family of linear operators 563

Najib Askouraye and Myelkebir Aitalioubrahim
On two and three and four point boundary value problems for differential inclusions with $\phi-$Laplacian 579

Souad Ayadi and Ozgur Ege
Existence of Solutions for a Class of BVPs on the Half-line 593

Umut Bas, Hüseyin Budak, and Hasan Kara
New extensions version of Hermite–Hadamard type inequalities by means of conformable fractional integrals 605

Bouharket Benaissa and Huseyin Budak
General $((k,p),\psi)$-Hilfer fractional integrals 619

Gülistan Butakın and Erhan Pişkin
Existence and blow up of solutions of a viscoelastic $m(x)$-biharmonic equation with logarithmic source term 631

Tingsong Du and Lei Xu
Hermite–Hadamard- and Pachpatte-type integral inequalities for generalized subadditive functions in the fractal sense 647

Miskolc Mathematical Notes

VOLUME 25 (2024), NUMBER 2

CONTENTS

Banh Duc Dung
Rings characterized by the extending property for finitely generated submodules 663

Ranran Fan, Min Zeng, and Yongxin Yuan
The solutions to some dual matrix equations 679

Sena Halıcı and Ayşegül Çetinkaya
k-Srivastava hypergeometric functions and their integral representations . 693

Emília Halušková
On discrete properties of continuous monotone functions 707

Murad Khan Hassani, Yasin Yazlik, and Nouressadat Touafek
Dynamics of positive solutions of a three dimensional higher order system of difference equations 723

Nguyen H. Hoc and Ravindra K. Bisht
Fixed point theorems of asymptotically regular maps 747

Eszter K. Horváth and Andreja Tepavčević
On the numbers of weak congruences of some finite lattices 759

Nazlı Irkıl
On the local existence of solutions to p-Laplacian equation with logarithmic nonlinearity and nonlinear damping term 771

Aisha Jabeen
On n-derivations of generalized matrix algebras 789

Waseem Ahmad Khan, Divesh Srivastava, and Kottakkaran Sooppy Nisar
A new class of generalized polynomials associated with Milne-Thomson-based poly-Bernoulli polynomials 805

Miskolc Mathematical Notes

VOLUME 25 (2024), NUMBER 2

CONTENTS

Gizem Köprülü Karakaş and Bayram Şahin
Biharmonic curves along Riemannian submersions 817

Péter Kórus, Juan Eduardo Nápoles Valdés, José Manuel Rodríguez, and José María Sigarreta Almira
Petrović-type inequality via fractional calculus 833

Ömer Küsmüş
Unit Group of Integral Group Ring $\mathbb{Z}(G \times C_3)$ 845

Kartikeswar Mahalik and Chandal Nahak
Well-posedness characterizations for system of mixed hemivariational inequalities 857

Farid Mihoubi and Brahim Nouiri
An inverse time-dependent source problem for a time-fractional diffusion equation with nonlocal boundary conditions 873

Mădălina Moga and Radu Truşcă
On some fixed point theorems for Ćirić operators 891

Babak Mohammadi, Mohammad Mursaleen, and Vahid Parvaneh
Solvability of integral equations via the technique of measures of noncompactness 907

Karima Mokni and Mohamed Ch-Chaoui
Dynamic analysis and Chaos control in a discrete predator-prey model with Holling type IV and nonlinear harvesting 921

Dijana Mosić, Daochang Zhang, and Jianping Hu
On operators whose core–EP inverse is n-potent 943

Miskolc Mathematical Notes

VOLUME 25 (2024), NUMBER 2

CONTENTS

Celil Nebiyev and Nurhan Sökmez
Some Properties of r-Supplemented Modules 957

Celil Nebiyev and Hasan Hüseyin Ökten
On Finitely g-Supplemented Modules 965

Zdeněk Opluštil
On non-oscillation for two dimensional systems of non-linear ordinary differential equations 971

Erhan Set, Ali Karaoğlan, İmdat İşcan, and Neslihan Kılıç
On (M,P)-functions with some features and new inequalities 985

Sergey Smirnov
Multiplicity of positive solutions for a third-order boundary value problem with nonlocal conditions of integral type 999

Swati Jain, Teerathram Raghuwanshi, Manoj Kumar Pandey, and Anil Goyal
On generalized M-projective curvature tensor of para-Sasakian manifold . 1009

Abdellah Taqbibt, M'Hamed El Omari, and Said Melliani
Solving Caputo fractional order differential equations via fixed point results of α-Geraghty contraction type mappings 1025

Zhengyan Wang, Yan Zhang, and Peibiao Zhao
Optimal cash holding model based on minimization of ruin probability . . 1043

Zhen Zhang and Shance Wang
The Auslander-Reiten correspondence about ∞-cotilting objects in extriangulated categories 1061

Hou Yu Zhao and Shan Shan Guo
Nondecreasing bounded continuous solutions of a q-difference equation . 1075

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 537–552

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4432

GENERAL DECAY AND BLOW UP OF SOLUTIONS FOR A KIRCHHOFF-TYPE EQUATION WITH VARIABLE-EXPONENTS

MOHAMMAD ALNEGGA, ABDELBAKI CHOUCHA, DJAMEL OUCHENANE, AND SALAH BOULAARAS

Received 02 October, 2022

Abstract. A nonlinear Kirchhoff-type equation with logarithmic nonlinearity and variable-exponents is studied. Firstly, the global existence is shown. Next, by using an integral inequality due to Komornik the general decay result is obtained. Finally, the blow-up of the solutions is proved.

2010 *Mathematics Subject Classification:* 35B40; 35L70; 93D20

Keywords: Kirchhoff equation, general decay, blow-up, variable exponents, logarithmic nonlinearity, nonlinear equations

1. Introduction

In this paper, we are interested in the following problem:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u(t) + \beta_1 |u_t(t)|^{m(x)-2} u_t(t) = u|u|^{p(x)-2}\ln|u|^k. \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad in \ \ \Omega \\ u(x,t) = 0, \quad in \ \ \partial\Omega \times (0,\infty), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $\beta_1, k > 0$, $M(s)$ is a positive C^1-function like $M(s) = 1 + s^\gamma$, $\gamma > 0$. $p(.)$ and $m(.)$ are the variable exponents given as measurable functions on $\overline{\Omega}$ such that:

$$\begin{aligned} &2 \le p^- \le p(x) \le p^+ \le p^*, \\ &2 \le m^- \le m(x) \le m^+ \le m^*, \end{aligned} \tag{1.2}$$

where

$$\begin{aligned} p^- &= ess \inf_{x\in\Omega} p(x), & p^+ &= ess \sup_{x\in\Omega} p(x), \\ m^- &= ess \inf_{x\in\Omega} m(x), & m^+ &= ess \sup_{x\in\Omega} m(x), \end{aligned} \tag{1.3}$$

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

and

$$p^*, m^* = \frac{2(n-1)}{n-2}, \quad \text{if } n \geq 3. \tag{1.4}$$

Of course, the problem we are going to study is of the Kirchhoff-type equation developed by Kirchhoff in the year 1876 [9], which is concerned with the study of describing small vibration amplitude of elastic stings.

The variable-exponents problems are among the most important and fertile research fields in recent years, and many researchers have touched on this matter. These problems appear in many physical phenomena and in many different branches of sciences and their mathematical modeling, such as image processing, electrical fluids, elasticity theory, and linear viscosity. For more in-depth study of this topic, we refer the reader to [1–4, 15]. Also, many results were reached regarding global existence, asymptotic behavior and blow-up of the solutions.

As for logarithmic nonlinearity, it appears in many fields of physical and applied sciences, including quantum mechanics, nuclear physics, and optics. The same goes for inflationary cosmology and symmetry theories. This is what led many mathematicians and researchers to address these applications and work on this type of problems.

In this context, we mention previous works that have a direct relationship with our present work. In case $\gamma = 0$, the authors in [5] determined the existence and uniqueness of a local solution in time by the Faedo-Galerkin method and they proved the blow-up of a solution under suitable conditions (see [6, 8–11]). Recently, in the present of a delay term also in the case $\gamma = 0$ in [14, 16] the authors established a global existence result under suitable conditions on the initial data only without imposing the Sobolev Logarithmic Inequality. After that, by the Komornik's lemma they proved the stability results, and they presented a numerical study that supports their results.

There are other works that study the variable exponent, including the followings. In [12] the authors considered a nonlinear Kirchhoff-type equation with distributed delay and variable-exponents in the presence of the source term ($k = 0$). Under suitable hypotheses they proved the blow-up of solutions, and by using an integral inequality due to Komornik they obtained the general decay result but in the case $b = 0$ (in the absence of the source term). Also, in [12–14], the authors considered a nonlinear $p(x)$-Laplacian equation with time delay and variable exponents, and they proved the blow-up of the solutions. Then, by applying an integral inequality due to Komornik, they obtained the decay result. Starting from all these works and supplementing them, we will try to study our problem (1.1), as we consider the coupling with the logarithmic nonlinearity and the variable exponent, with a thorough study it makes our problem different from what was previously studied.

The remainder of our work is organized as follows: in Section 2, we lay down the hypotheses, concepts and lemmas we need. In Section 3, the global existence is

shown and in the Section 4, we obtain the general decay result. Next, in Section 5, we prove the blow-up of the solutions. Finally, we put a general conclusion.

2. Preliminaries

In this section,we introduce some notations and materials that will be used throughout this work. Firstly, let $p\colon \Omega \to [1,\infty)$ be a measurable function. We define the variable-exponent Lebesgue space with a variable exponent $p(.)$ by

$$L^{p(.)}(\Omega) = \left\{ u : \Omega \to \mathbb{R};\ \text{ measurable in } \ \Omega : \ \int_\Omega |u|^{p(.)} dx < \infty \right\},$$

with a Luxemburg-type norm

$$\|u\|_{p(.)} = \inf \left\{ \sigma > 0 : \ \int_\Omega |\frac{u}{\sigma}|^{p(x)} dx \leq 1 \right\}.$$

Equipped with this norm, $L^{p(.)}(\Omega)$ is a Banach space (see [4]). Next, we define the variable-exponent Sobolev space $W^{1,p(.)}(\Omega)$ as follows

$$W^{1,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega); \nabla u \ \text{ exists and } \ |\nabla u| \in L^{p(.)}(\Omega) \right\},$$

with

$$\|u\|_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

$W^{1,p(.)}(\Omega)$ is a Banach space, and the closure of $C_0^\infty(\Omega)$ is defined by $W_0^{1,p(.)}(\Omega)$.

For $u \in W_0^{1,p(.)}(\Omega)$, we give the equivalent norm

$$\|u\|_{1,p(.)} = \|\nabla u\|_{p(.)}.$$

$W_0^{-1,p'(.)}(\Omega)$ denotes the dual of $W_0^{1,p(.)}(\Omega)$, where $\frac{1}{p(.)} + \frac{1}{p'(.)} = 1$. Also, we suppose that

$$|p(x) - p(y)| \leq -\frac{A}{\log|x-y|} \ \text{ and } \ |m(x) - m(y)| \leq -\frac{B}{\log|x-y|} \ \text{ for all } \ x, y \in \Omega, \tag{2.1}$$

$A, B > 0$ and $0 < \zeta < 1$ with $|x-y| < \zeta$. (Log-Hölder inequality).

Lemma 1 ([2]). *Suppose that $p(.)$ verifies 1.4 and let Ω be a bounded domain of $\mathbb{R}^n$. Then, $\exists c_* = c(\Omega, p^+, p^-) > 0$,*

$$\|u\|_{p(.)} \leq c_* \|\nabla u\|_{p(.)}, \ \ \forall u \in W_0^{1,p(.)}(\Omega).$$

Lemma 2 ([2]). *If $p : \overline{\Omega} \to [1,\infty)$ is continuous,*

$$2 \leq p^- \leq p(x) \leq p^+ \leq \frac{2n}{n-2}, \ \ n \geq 3, \tag{2.2}$$

is held, then the embedding $H_0^1(\Omega) \to L^{p(.)}(\Omega)$ is continuous.

Lemma 3 (Unit ball property, [3]). *Let $p \geq 1$ be a measurable function on Ω. Then,*

$$\|g\|_{p(.)} \leq 1, \quad \textit{if and only if} \quad \rho_{p(.)}(g) \leq 1,$$

where

$$\rho_{p(.)}(g) = \int_\Omega |g(x)|^{p(x)} dx.$$

Lemma 4 ([2]). *If $p \geq 1$ is a measurable function on Ω, then*

$$\min\left\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right\} \leq \rho_{p(.)}(u) \leq \max\left\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right\},$$

for all $u \in L^{p(.)}(\Omega)$ and for a.e. $x \in \Omega$.

Theorem 1. *Assume that (1.2) and (2.1) hold. Then, for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ there exists a unique solution u of problem (1.1) on $(0,T)$ such that*

$$\begin{aligned} &u \in C((0,T), H_0^1(\Omega)) \cap C^1((0,T), L^2(\Omega)), \\ &u_t \in L^{m(.)}(\Omega \times (0,T)). \end{aligned} \tag{2.3}$$

Firstly, we define the energy functional.

Lemma 5. *The functional of energy E, given by*

$$\begin{aligned} E(t) = &\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} \\ &+ k\int_\Omega \frac{|u|^{p(x)}}{p^2(x)} dx - \int_\Omega \frac{|u|^{p(x)} \ln |u|^k}{p(x)} dx, \end{aligned} \tag{2.4}$$

satisfies

$$E'(t) \leq -\beta_1 \int_\Omega |u_t(t)|^{m(x)} dx \leq 0. \tag{2.5}$$

Proof. By the inner product of (1.1) with u_t, we get

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + k\int_\Omega \frac{|u|^{p(x)}}{p^2(x)} dx - \int_\Omega \frac{|u|^{p(x)} \ln |u|^k}{p(x)} dx\right\} \\ &= -\beta_1 \int_\Omega |u_t(t)|^{m(x)} dx. \end{aligned} \tag{2.6}$$

Hence, we find (2.4) and (2.5), also E is a decreasing function. This completes the proof. □

3. Global existence

Now, we show that the solution of (1.1) is uniformly bounded and global in time. For this purpose, we set

$$I(t) = \|\nabla u\|_2^2 - \int_\Omega |u|^{p(x)} \ln |u|^k dx, \tag{3.1}$$

$$J(t) = \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + k\int_\Omega \frac{|u|^{p(x)}}{p^2(x)}dx - \int_\Omega \frac{|u|^{p(x)}\ln|u|^k}{p(x)}dx. \tag{3.2}$$

Hence,

$$E(t) = J(t) + \frac{1}{2}\|u_t\|_2^2. \tag{3.3}$$

Lemma 6. *Suppose that the initial data $u_0, u_1 \in H^1(\Omega) \times L^2(\Omega)$ satisfying $I(0) > 0$ and*

$$\xi := C_*(p^+,k)\left(\frac{2p^-}{p^- - 2}E(0)\right)^{\frac{p^+ + k - 2}{2}} < 1. \tag{3.4}$$

Then $I(t) > 0$, for any $t \in [0,T]$.

Proof. Of course, for the first case, if $\int_\Omega |u|^{p(x)} \ln |u|^k dx < 0$, the result is clear. Therefore, we can go directly to the second case and impose: $\int_\Omega |u|^{p(x)} \ln |u|^k dx > 0$. Since $I(0) > 0$ we deduce by continuity that there exists $T^* \le T$ such that $I(t) \ge 0$ for all $t \in [0,T^*]$. This implies that $\forall t \in [0,T^*]$,

$$\begin{aligned} J(t) &\ge \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + \frac{k}{(p^+)^2}\int_\Omega |u|^{p(x)}dx \\ &\quad - \frac{1}{p^-}\int_\Omega |u|^{p(x)}\ln|u|^k dx \\ &\ge \frac{p^- - 2}{2p^-}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + \frac{k}{(p^+)^2}\int_\Omega |u|^{p(x)}dx + \frac{1}{p^-}I(t) \\ &\ge \frac{p^- - 2}{2p^-}\|\nabla u(t)\|_2^2. \end{aligned}$$

Hence,

$$\|\nabla u(t)\|_2^2 \le \frac{2p^-}{p^- - 2}J(t) \le \frac{2p^-}{p^- - 2}E(t) \le \frac{2p^-}{p^- - 2}E(0). \tag{3.5}$$

On the other hand, using the facts that $\ln|u| < |u|$ and $|u| > 1$, we get

$$\int_\Omega |u|^{p(x)}\ln|u|^k dx < \int_\Omega |u|^{p^+ + k}dx. \tag{3.6}$$

Then, the embedding $H_0^1(\Omega)) \hookrightarrow L^{p^+ + k}(\Omega)$ yields

$$\int_\Omega |u|^{p^+ + k}dx \le C_*(p^+,k)\|\nabla u(t)\|_2^{p^+ + k} = C_*(p^+,k)\|\nabla u(t)\|_2^2\|\nabla u(t)\|_2^{p^+ + k - 2}. \tag{3.7}$$

Using (3.5), we find

$$\int_\Omega |u|^{p(x)} \ln |u|^k dx < C_*(p^+,k)\left(\frac{2p^- E(0)}{p^- - 2}\right)^{\frac{p^+ + k - 2}{2}} \|\nabla u(t)\|_2^2 < \xi \|\nabla u(t)\|_2^2, \tag{3.8}$$

where $\xi = C_*(p^+,k)\left(\frac{2p^- E(0)}{p^- - 2}\right)^{\frac{p^+ + k - 2}{2}}$ and $C_*(p^+,k)$ is the embedding constant.

According (3.1) and (3.5), we get

$$I(t) > (1-\xi)\|\nabla u(t)\|_2^2 > 0, \quad \forall t \in [0,T^*]. \tag{3.9}$$

By repeating this procedure, T^* can be extended to T. This completes the proof. □

Remark 1. Under the conditions of Lemma 6, we have $J(t) \geq 0$ and consequently $E(t) \geq 0, \ \forall t \in [0,T]$. Hence, by (3.2) and (3.5) we find

$$\begin{aligned} \|u_t(t)\|_2^2 &\leq 2E(0), \\ \|\nabla u(t)\|_{2(\gamma+1)}^2 &\leq 2(\gamma+1)E(0), \\ \int_\Omega |u|^{p(x)} dx &\leq \frac{(p^+)^2}{k} E(0). \end{aligned} \tag{3.10}$$

4. General decay

In this section, we state and prove the general decay of system (1.1). The following lemma will be used in the upcoming results.

Lemma 7 (Komornik, [10]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and suppose that $\exists \sigma, \omega > 0$ such that*

$$\int_s^\infty E^{1+\sigma}(t)dt \leq \frac{1}{\omega} E^\sigma(0) E(s) = cE(s), \ \ \forall s > 0. \tag{4.1}$$

Then, we have for $\forall t \geq 0$

$$\begin{cases} E(t) \leq cE(0)/(1+t)^{\frac{1}{\sigma}}, & \text{if } \sigma > 0, \\ E(t) \leq cE(0)e^{-\omega t}, & \text{if } \sigma = 0. \end{cases} \tag{4.2}$$

In the next step, we give the third result of our work.

Theorem 2. *Assume that (1.2) and (2.1) hold. Then $\exists c, \lambda > 0$ such that the solution of (1.1) satisfies*

$$\begin{cases} E(t) \leq cE(0)/(1+t)^{\frac{2}{m^+ - 2}}, & \text{if } m^+ > 2, \\ E(t) \leq cE^{-\lambda t}, & \text{if } m(x) = 2. \end{cases} \tag{4.3}$$

Proof. Multiplying (1.1) by $uE^q(t)$, for $q>0$ to be specified later, and integrating the result over $\Omega\times(s,T), s<T$, gives

$$\int_s^T E^q(t)\int_\Omega \left\{uu_{tt}-M\left(\|\nabla u\|_2^2\right)u\Delta u(t)+\beta_1 uu_t|u_t(t)|^{m(x)-2}-|u|^{p(x)}\ln|u|^k\right\}dxdt=0, \tag{4.4}$$

which implies that

$$\int_s^T E^q(t)\int_\Omega \left\{\frac{d}{dt}(uu_t)-|u_t|^2+M\left(\|\nabla u\|_2^2\right)|\nabla u|^2+\beta_1 uu_t|u_t(t)|^{m(x)-2}|u|^{p(x)}\ln|u|^k\right\}dxdt$$
$$=0. \tag{4.5}$$

By (2.4) and the relation

$$\frac{d}{dt}\left(E^q(t)\int_\Omega uu_t dx\right)=qE^{q-1}(t)E'(t)\int_\Omega uu_t dx+E^q(t)\frac{d}{dt}\left(\int_\Omega uu_t dx\right),$$

we get

$$\begin{aligned}
&2\int_s^T E^{q+1}(t)dt\\
&=\underbrace{\int_s^T \frac{d}{dt}\left(E^q(t)\int_\Omega uu_t dx\right)dt}_{I_1}-\underbrace{q\int_s^T\left(E^{q-1}(t)E'(t)\int_\Omega uu_t dx\right)dt}_{I_2}\\
&+\underbrace{2\int_s^T\left(E^q(t)\int_\Omega|\nabla u|^2dx\right)dt}_{I_3}+\underbrace{2k\int_s^T\left(E^q(t)\int_\Omega\frac{|u|^{p(x)}}{p^2(x)}dx\right)dt}_{I_4}\\
&+\underbrace{\frac{\gamma+2}{\gamma+1}\int_s^T\left(E^q(t)\int_\Omega\|\nabla u\|_2^{2\gamma}|\nabla u|^2dx\right)dt}_{I_5}\\
&+\underbrace{\beta_1\int_s^T\left(E^q(t)\int_\Omega uu_t|u_t(t)|^{m(x)-2}dx\right)dt}_{I_6}\\
&-\underbrace{\int_s^T\left(E^q(t)\int_\Omega|u|^{p(x)}\ln|u|^k dx\right)dt}_{I_7}-\underbrace{2\int_s^T\left(E^q(t)\int_\Omega\frac{|u|^{p(x)}\ln|u|^k}{p(x)}dx\right)dt}_{I_8}.
\end{aligned} \tag{4.6}$$

At this point, we estimate $I_i, i=1,\dots,8$ of the RHS in (4.6), we have

$$\begin{aligned}
I_1&=E^q(T)\int_\Omega uu_t(x,T)dx-E^q(s)\int_\Omega uu_t(x,s)dx\\
&\le\frac{1}{2}E^q(T)\left\{\int_\Omega u^2(x,T)dx+\int_\Omega u_t^2(x,T)dx\right\}+\frac{1}{2}E^q(s)\left\{\int_\Omega u^2(x,s)dx+\int_\Omega u_t^2(x,s)dx\right\}
\end{aligned}$$

$$\leq \frac{1}{2}E^q(T)\Big\{c_*\|\nabla u(T)\|_2^2+2E(T)\Big\}+E^q(s)\Big\{c_*\|\nabla u(s)\|_2^2+2E(s)\Big\}$$
$$\leq c_1\Big(E^{q+1}(T)+E^{q+1}(s)\Big). \tag{4.7}$$

Since E is decreasing, we get

$$I_1 \leq cE^{q+1}(s) \leq E^q(0)E(s) \leq cE(s). \tag{4.8}$$

Similarly, we find

$$I_2 \leq -q\int_s^T E^{q-1}(t)E'(t)\Big(c_*E(T)+E(T)\Big)dt$$
$$\leq -c\int_s^T E^q(t)E'(t)dt \leq cE^{q+1}(s) \leq cE(s). \tag{4.9}$$

Next, we get

$$I_3 = 2\int_s^T \Big(E^q(t)\|\nabla u\|^2\Big)dt \leq 2\int_s^T E^q(t)E(t)dt \leq cE^{q+1}(s) \leq cE(s). \tag{4.10}$$

We estimate the next term as follows,

$$I_4 = 2k\int_s^T E^q(t)\int_\Omega \frac{|u|^{p(x)}}{p^2(x)}dxdt \leq c\int_s^T E^q(t)E(t)dt \leq cE^{q+1}(s) \leq cE(s). \tag{4.11}$$

For the fifth term, we have

$$I_5 = 2(\gamma+2)\int_s^T \Big(E^q(t)\frac{\|\nabla u\|_2^{2(\gamma+1)}}{2(\gamma+1)}\Big)dt \leq 2(\gamma+2)\int_s^T E^q(t)E(t)dt \leq cE^{q+1}(s)$$
$$\leq cE(s), \tag{4.12}$$

and using Young's inequality, we find

$$I_6 = \beta_1\int_s^T \Big(E^q(t)\int_\Omega uu_t|u_t(t)|^{m(x)-2}dx\Big)dt$$
$$\leq \varepsilon\int_s^T \Big(E^q(t)\int_\Omega |u(t)|^{m(x)}dx\Big)dt + c\int_s^T \Big(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\Big)dt$$
$$\leq \varepsilon\int_s^T E^q(t)\Big[\int_{\Omega_+}|u(t)|^{m^+}dx+\int_{\Omega_-}|u(t)|^{m^-}dx\Big]dt$$
$$+c\int_s^T \Big(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\Big)dt.$$

By using the embeddings $H_0^1(\Omega)\hookrightarrow L^{m^-}(\Omega)$ and $H_0^1(\Omega)\hookrightarrow L^{m^+}(\Omega)$, we get

$$I_6 \leq \varepsilon\int_s^T E^q(t)\Big[c\|\nabla u(t)\|_2^{m^+}+c\|\nabla u(t)\|_2^{m^-}\Big]dt$$

$$
\begin{aligned}
&+c\int_s^T\left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt\\
&\le \varepsilon\int_s^T E^q(t)\left[cE^{\frac{m^+-2}{2}}(0)E(t)+cE^{\frac{m^--2}{2}}(0)E(t)\right]dt\\
&+c\int_s^T\left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt\\
&\le c\varepsilon\int_s^T E^{q+1}(t)dt+c\int_s^T\left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt. \qquad (4.13)
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
I_7&=\int_s^T\left(E^q(t)\int_\Omega |u|^{p(x)}\ln|u|^k dx\right)dt\\
&\le \xi\int_s^T E^q(t)\|\nabla u(t)\|_2^2 dt\\
&\le c\int_s^T E^{q+1}(t)dt\le cE^{q+1}(s)\le cE(s). \qquad (4.14)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
I_8&=\int_s^T\left(E^q(t)\int_\Omega \frac{|u|^{p(x)}\ln|u|^k}{p(x)}dx\right)dt\\
&\le \frac{1}{p^-}\int_s^T\left(E^q(t)\int_\Omega |u|^{p(x)}\ln|u|^k dx\right)dt\\
&\le cI_7\le cE(s). \qquad (4.15)
\end{aligned}
$$

By combining (4.6)-(4.15), we find

$$
\int_s^T E^{q+1}(t)dt\le c\varepsilon\int_s^T E^{q+1}(t)dt+c\int_s^T\left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt+cE(s). \qquad (4.16)
$$

Now, we choose ε small enough such that

$$
\int_s^T E^{q+1}(t)dt\le cE(s)+c\int_s^T\left(E^q(t)\int_\Omega c_\varepsilon(x)|u_t(t)|^{m(x)}dx\right)dt. \qquad (4.17)
$$

When ε is fixed, then $c_\varepsilon(x)\le M$ since $m(x)$ is bounded. Then,

$$
\begin{aligned}
\int_s^T E^{q+1}(t)dt&\le cE(s)+cM\int_s^T\left(E^q(t)\int_\Omega |u_t(t)|^{m(x)}dx\right)dt\\
&\le cE(s)-\frac{cM}{\beta_1}\int_s^T E^q(t)E'(t)dt\\
&\le cE(s)+\frac{cM}{\beta_1(q+1)}\left[E^{q+1}(s)-E^{q+1}(T)\right]\le cE(s). \qquad (4.18)
\end{aligned}
$$

Taking $T \to \infty$, we find

$$\int_s^\infty E^{q+1}(t)dt \leq cE(s). \tag{4.19}$$

Finally, Komornik's Lemma 7 (with $\sigma = q = \frac{m^+-2}{2}$) implies our result. The proof is complete. □

5. Blow-up

Now, to prove the blow-up, we assume that $E(0) < 0$ and $\int_\Omega |u|^{p(x)} \ln |u|^k dx > 0$. Let

$$\begin{aligned}\mathbb{H}(t) = -E(t) = &\int_\Omega \frac{|u|^{p(x)} \ln |u|^k}{p(x)} dx - k \int_\Omega \frac{|u|^{p(x)}}{p^2(x)} dx \\ &- \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)}.\end{aligned} \tag{5.1}$$

From (2.5), we have

$$E(t) \leq E(0) < 0. \tag{5.2}$$

Then

$$\mathbb{H}'(t) = -E'(t) \geq \beta_1 \int_\Omega |u_t(t)|^{m(x)} dx \geq 0, \tag{5.3}$$

and

$$0 < \mathbb{H}(0) \leq \mathbb{H}(t) \leq \int_\Omega \frac{|u|^{p(x)} \ln |u|^k}{p(x)} dx \leq \frac{k}{p^-} \int_\Omega |u|^{p(x)} \ln |u|, \tag{5.4}$$

where

$$\rho(u) = \rho_{p(.)}(u) = \int_\Omega |u|^{p(x)} dx.$$

Lemma 8 ([13]). *Assume that (1.2) is held. Then $\exists c = c(\Omega) > 1$, such that*

$$\rho^{s/p^-}(u) \leq c\Big(\|\nabla u\|_2^2 + \rho(u)\Big). \tag{5.5}$$

Then we have

$$\| u\|_{p^-}^s \leq c\Big(\|\nabla u\|_2^2 + \|u\|_{p^-}^{p^-}\Big), \tag{5.6}$$

$$\rho^{s/p^-}(u) \leq c\Big(|\mathbb{H}(t)| + \|u_t\|_2^2 + \rho(u)\Big), \tag{5.7}$$

$$\| u\|_{p^-}^s \leq c\Big(|\mathbb{H}(t)| + \|u_t\|_2^2 + \| u\|_{p^-}^{p^-}\Big), \tag{5.8}$$

$\forall u \in H_0^1(\Omega)$ and $2 \leq s \leq p^-$. Let (u) be a solution of (1.1), then

$$\rho(u) \geq c\|u\|_{p^-}^{p^-} \tag{5.9}$$

and

$$\int_\Omega |u|^{m(x)}dx \le c\left(\rho^{m^-/p^-}(u)+\rho^{m^+/p^-}(u)\right). \tag{5.10}$$

Theorem 3. *Suppose that (1.2), (2.1) and $E(0)<0$. Then, the solution of (1.1) blows up in finite time.*

Proof. We set

$$\mathcal{K}(t)=\mathbb{H}^{1-\alpha}(t)+\varepsilon\int_\Omega uu_t dx, \tag{5.11}$$

where $\varepsilon>0$ to be assigned later and

$$0<\alpha<\min\left\{\frac{p^--2}{2p^-},\frac{p^--m^+}{p^-(m^+-1)}\right\}. \tag{5.12}$$

By multiplying (1.1) by u and with a derivative of (5.11), we get

$$\begin{aligned}\mathcal{K}'(t)&=(1-\alpha)\mathbb{H}^{-\alpha}(t)\mathbb{H}'(t)+\varepsilon\|u_t\|_2^2-\varepsilon\|\nabla u\|_2^{2(\gamma+1)}\\&\quad-\varepsilon\|\nabla u\|_2^2+\varepsilon k\int_\Omega |u|^{p(x)}\ln|u|dx-\varepsilon\beta_1\int_\Omega uu_t|u_t|^{m(x)-2}dx.\end{aligned} \tag{5.13}$$

Using Young's inequality, we find for $\delta_1>0$

$$\varepsilon\beta_1\int_\Omega uu_t|u_t|^{m(x)-2}dx\le\varepsilon\beta_1\left\{\frac{1}{m^-}\int_\Omega\delta_1^{m(x)}|u|^{m(x)}dx+\frac{m^+-1}{m^-}\int_\Omega\delta_1^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}dx\right\}. \tag{5.14}$$

By substituting (5.14) into (5.13), we obtain

$$\begin{aligned}\mathcal{K}'(t)&\ge(1-\alpha)\mathbb{H}^{-\alpha}(t)\mathbb{H}'(t)+\varepsilon\|u_t\|_2^2-\varepsilon\|\nabla u\|_2^{2(\gamma+1)}\\&\quad-\varepsilon\|\nabla u\|_2^2+\varepsilon k\int_\Omega|u|^{p(x)}\ln|u|dx\\&\quad-\varepsilon\beta_1\left\{\frac{1}{m^-}\int_\Omega\delta_1^{m(x)}|u|^{m(x)}dx+\frac{m^+-1}{m^-}\int_\Omega\delta_1^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}dx\right\}.\end{aligned} \tag{5.15}$$

Therefore, by setting δ_1 so that

$$\delta_1^{-\frac{m(x)}{m(x)-1}}=\beta_1\kappa\mathbb{H}^{-\alpha}(t). \tag{5.16}$$

Replacing in (5.15) and by (5.3), we find

$$\begin{aligned}\mathcal{K}'(t)&\ge\left[(1-\alpha)-\varepsilon\kappa\widehat{m}\right]\mathbb{H}^{-\alpha}(t)\mathbb{H}'(t)+\varepsilon\|u_t\|_2^2\\&\quad-\varepsilon\|\nabla u\|_2^2-\varepsilon\|\nabla u\|_2^{2(\gamma+1)}+\varepsilon k\int_\Omega|u|^{p(x)}\ln|u|dx\\&\quad-\varepsilon\frac{\beta_1}{m^-}\int_\Omega(\beta_1\kappa)^{1-m(x)}\mathbb{H}^{\alpha(m(x)-1)}(t)|u|^{m(x)}dx,\end{aligned} \tag{5.17}$$

where $\widehat{m} = \frac{m^+-1}{m^-}$. Using (5.4) and (5.10), we have

$$\begin{aligned}\frac{\beta_1}{m^-}\int_\Omega (\beta_1\kappa)^{1-m(x)}\mathbb{H}^{\alpha(m(x)-1)}(t)|u|^{m(x)}dx &\le \frac{\beta_1}{m^-}\int_\Omega (\beta_1\kappa)^{1-m^-}\mathbb{H}^{\alpha(m^+-1)}(t)|u|^{m(x)}dx\\ &= \frac{C_1}{\kappa^{m^--1}}\mathbb{H}^{\alpha(m^+-1)}(t)\int_\Omega |u|^{m(x)}dx\\ &\le \frac{C_2}{\kappa^{m^--1}}\left[\rho^{\frac{m^-}{p^-}+\alpha(m^+-1)}(u)+\rho^{\frac{m^+}{p^-}+\alpha(m^+-1)}(u)\right].\end{aligned} \tag{5.18}$$

By (5.12), we find

$$s = m^- + \alpha p^-(m^+-1) \le p^-, \quad \text{and} \quad s = m^+ + \alpha p^-(m^+-1) \le p^-.$$

Further, Lemma 8 gives

$$C_1\mathbb{H}^{\alpha(m^+-1)}(t)\int_\Omega |u|^{m(x)}dx \le \frac{C_3}{\kappa^{m^--1}}\Big(\|\nabla u\|_2^2+\rho(u)\Big). \tag{5.19}$$

Combining (5.17) and (5.19), we obtain

$$\begin{aligned}\mathcal{K}'(t) \ge &\left[(1-\alpha)-\varepsilon\kappa\widehat{m}\right]\mathbb{H}^{-\alpha}(t)\mathbb{H}'(t)+\varepsilon\|u_t\|_2^2-\varepsilon C_4\|\nabla u\|_2^2\\ &-\varepsilon\|\nabla u\|_2^{2(\gamma+1)}-\varepsilon\frac{C_3}{\kappa^{m^--1}}\rho(u)+\varepsilon k\int_\Omega |u|^{p(x)}\ln|u|dx,\end{aligned} \tag{5.20}$$

where $C_4 = \frac{C_3}{\kappa^{m^--1}}+1$.

For $0<a<1$, we have from (5.1)

$$\begin{aligned}\frac{\varepsilon}{p^-}\int_\Omega |u|^{p(x)}\ln|u|^k dx &\ge \varepsilon\int_\Omega \frac{|u|^{p(x)}}{p(x)}\ln|u|^k dx\\ &= \varepsilon(1-a)\mathbb{H}(t)+\frac{\varepsilon(1-a)}{2}\|u_t\|_2^2+\varepsilon(1-a)k\int_\Omega \frac{|u|^{p(x)}}{p^2(x)}dx\\ &\quad+\frac{\varepsilon(1-a)}{2}\|\nabla u\|_2^2+\frac{\varepsilon(1-a)}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}\\ &\quad+a\varepsilon\int_\Omega \frac{|u|^{p(x)}}{p(x)}\ln|u|^k dx,\\ &\ge \varepsilon(1-a)\mathbb{H}(t)+\frac{\varepsilon(1-a)}{2}\|u_t\|_2^2+\frac{\varepsilon(1-a)k}{(p^+)^2}\int_\Omega |u|^{p(x)}dx\\ &\quad+\frac{\varepsilon(1-a)}{2}\|\nabla u\|_2^2+\frac{\varepsilon(1-a)}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}\\ &\quad+\frac{a\varepsilon}{p^+}\int_\Omega |u|^{p(x)}\ln|u|^k dx.\end{aligned} \tag{5.21}$$

We conclude that

$$\varepsilon\int_\Omega |u|^{p(x)}\ln|u|^k dx \geq \varepsilon(1-a)p^-\mathbb{H}(t)+\frac{\varepsilon(1-a)p^-}{2}\|u_t\|_2^2+\frac{\varepsilon(1-a)kp^-}{(p^+)^2}\int_\Omega |u|^{p(x)}dx$$
$$+\frac{\varepsilon(1-a)p^-}{2}\|\nabla u\|_2^2+\frac{\varepsilon(1-a)p^-}{2(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)}$$
$$+\frac{a\varepsilon p^-}{p^+}\int_\Omega |u|^{p(x)}\ln|u|^k dx, \tag{5.22}$$

substituting in (5.20), gives

$$\mathcal{K}'(t)\geq\left[(1-\alpha)-\varepsilon\kappa\widehat{m}\right]\mathbb{H}^{-\alpha}(t)\mathbb{H}'(t)+\varepsilon\left[\frac{p^-(1-a)}{2}+1\right]\|u_t\|_2^2$$
$$+\varepsilon\left[\frac{p^-(1-a)}{2}-1-\frac{C_3}{\kappa^{m^--1}}\right]\|\nabla u\|_2^2+\varepsilon p^-(1-a)\mathbb{H}(t)$$
$$+\varepsilon\left[\frac{p^-(1-a)}{2(\gamma+1)}-1\right]\|\nabla u\|_2^{2(\gamma+1)}-\varepsilon\left(\frac{(1-a)kp^-}{(p^+)^2}-\frac{C_3}{\kappa^{m^--1}}\right)\rho(u)$$
$$+\frac{a\varepsilon p^-}{p^+}\int_\Omega |u|^{p(x)}\ln|u|^k dx. \tag{5.23}$$

In this stage, we select $a>0$ so small that

$$\frac{p^-(1-a)}{2}-1>0 \quad\text{and}\quad \frac{p^-(1-a)}{2(\gamma+1)}-1>0,$$

then we choose κ so large that

$$\left(\frac{p^-(1-a)}{2}-1\right)-\frac{C_3}{\kappa^{m^--1}}>0 \quad\text{and}\quad \frac{(1-a)kp^-}{(p^+)^2}-\frac{C_3}{\kappa^{m^--1}}>0,$$

when κ, a are fixed. We can choose ε so small that

$$(1-\alpha)-\varepsilon\kappa\widehat{m}>0.$$

Thus, for some $\mu_1>0$, estimate (5.23) becomes

$$\mathcal{K}'(t)\geq\mu_1\left\{\mathbb{H}(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}+\rho(u)+\int_\Omega |u|^{p(x)}\ln|u|^k dx\right\}, \tag{5.24}$$

and

$$\mathcal{K}(t)\geq\mathcal{K}(0)>0,\quad t>0. \tag{5.25}$$

Therefore, by using Hölder's and Young's inequalities, we have

$$\|u\|_2=(\int_\Omega u^2dx)^{\frac{1}{2}}\leq[(\int_\Omega(|u|^2)^{p^-/2}dx)^{\frac{2}{p^-}}.(\int_\Omega 1dx)^{1-\frac{2}{p^-}}]^{\frac{1}{2}}\leq c\|u\|_{p-}, \tag{5.26}$$

and

$$|\int_\Omega uu_tdx|\leq\|u_t\|_2.\|u\|_2\leq c\|u_t\|_2.\|u\|_{p^-}.$$

Therefore

$$\begin{aligned}|\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}} &\geq c\|u_t\|_2^{\frac{1}{1-\alpha}}.\|u\|_{p^-}^{\frac{1}{1-\alpha}}\\ &\leq c[\|u_t\|_2^{\frac{\theta}{1-\alpha}}+\|u\|_{p^-}^{\frac{\mu}{1-\alpha}}],\end{aligned} \tag{5.27}$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$. We take $\theta=2(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha}=\frac{2}{1-2\alpha}\leq p^-.$$

Subsequently, for $s=\frac{2}{(1-2\alpha)}$, we obtain

$$|\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}}\leq c[\|u_t\|_2^2+\|u\|_{p^-}^s].$$

Therefore, Lemma 8 gives

$$|\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}}\geq c\Big\{\mathbb{H}(t)+\|u_t\|_2^2+\rho(u)\Big\}. \tag{5.28}$$

Subsequently,

$$\begin{aligned}\mathcal{K}^{\frac{1}{1-\alpha}}(t)&=\Big\{\mathbb{H}^{1-\alpha}+\varepsilon\int_\Omega uu_t dx\Big\}^{\frac{1}{1-\alpha}}\\ &\leq c\Big\{\mathbb{H}(t)+|\int_\Omega uu_t dx|^{\frac{1}{1-\alpha}}\Big\}\\ &\leq c\Big\{\mathbb{H}(t)+\|u_t\|_2^2+\rho(u)\Big\}\\ &\leq c\Big\{\mathbb{H}(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\|\nabla u\|_2^{2(\gamma+1)}+\rho(u)+\int_\Omega |u|^{p(x)}\ln|u|^k dx\Big\}.\end{aligned} \tag{5.29}$$

According (5.24) and (5.29), we have

$$\mathcal{K}'(t)\geq\lambda\mathcal{K}^{\frac{1}{1-\alpha}}(t), \tag{5.30}$$

where $\lambda(\mu_1,c)>0$, and by integration of (5.30), we find

$$\mathcal{K}^{\frac{\alpha}{1-\alpha}}(t)\geq\frac{1}{\mathcal{K}^{\frac{-\alpha}{1-\alpha}}(0)-\lambda\frac{\alpha}{(1-\alpha)}t}.$$

Hence, the solution blows up in a finite time T^*, such that

$$T^*=\frac{1-\alpha}{\lambda\alpha\mathcal{K}^{\alpha/(1-\alpha)}(0)}.$$

Then the proof is completed. □

6. Conclusion

The combination of the logarithmic term and the variable exponent in a nonlinear Kirchhoff-type equations makes our problem different from what was previously studied, and for this was the purpose of this work, as we showed the global existence. Also, we have reached the result of the general decay by using the integral inequality due to [10]. Next, we reached the result of blowing up of the solutions. This work is a solution to the open problem of our proposal in [5, 7].

What we want to perform in the future from the expected research works is an attempt to employ the same way with the same problem, but by adding another component of damping, which contributes to an actual solution to the required problem.

Acknowledgements

The authors would like to thanks the referees and handling editor for relevant remarks and comments which improved the final version of the paper.

References

[1] S. Boulaaras, A. Draifia, and K. Zennir, " General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping and logarithmic nonlinearity." *Math. Meth. Appl. Sci.*, vol. 42, pp. 4795–4814, 2019.

[2] S. Antontsev, " Wave equation with $p(x,t)-$Laplacian and damping term: blow-up of solutions." *Comptes Rendus Mecanique*, vol. 339, pp. 751–755, 2011.

[3] S. Antontsev, " Wave equation with $p(x,t)-$Laplacian and damping term: Existence and blow-up." *Differential Equation Appl.*, vol. 3, pp. 503–525, 2011.

[4] Y. Chen, S. Levine, and M. Rao, " Variable Exponent, Linear Growth Functionals in Image Restoration." *SIAM Journal on Applied Mathematics*, vol. 3, pp. 503–525, 2011.

[5] A. Choucha and S. Boulaaras, "Asymptotic behavior for a viscoelastic Kirchhoff equation with distributed delay and Balakrishnan–Taylor damping." *Boundary Value Problems*, vol. 2021, 2021, 77, doi: 10.1186/s13661-021-01555-0.

[6] A. Choucha, S. M. Boulaaras, and D. Ouchenane, "General decay rate for a viscoelastic wave equation with distributed delay and Balakrishnan-Taylor damping." *Open Mathematics*, vol. 19, no. 1, pp. 1120–1133, 2022, doi: 10.1515/math-2021-0108.

[7] A. Choucha, D. Ouchenane, and S. Boulaaras, " Well posedness and stability result for a thermoelastic laminated Timoshenko beam with distributed delay term." *Math. Meth. Appl. Sci.*, vol. 43, no. 17, pp. 9983–10 004, 2020, doi: 10.1002/mma.6673.

[8] L. Dienning, P. Hasto, P. Hrjulehto, and M. Ruzicka, *Lebesgue and Sobolev Space with Variable Exponents*. Springer-Verlag, 2011.

[9] S. Kumar, S. K. Dhiman, D. Baleanu, M. S. Osman, and A. Wazwaz, "Lie Symmetries, Closed-Form Solutions, and Various Dynamical Profiles of Solitons for the Variable Coefficient (2+1)-Dimensional KP Equations." *Symmetry*, vol. 14, no. 3, 2022, doi: 10.3390/sym14030597.

[10] S. Kumar, D. Kumar, and A. Wazwaz, "Group invariant solutions of (3+1)-dimensional generalized B-type Kadomstsev Petviashvili equation using optimal system of Lie subalgebra." *Phys. Scr.*, vol. 94, 2019, 065204, doi: 10.1088/1402-4896/aafc13.

[11] S. Kumar, D. Kumar, and A. Wazwaz, "Lie symmetries, optimal system, group-invariant solutions and dynamical behaviors of solitary wave solutions for a (3+1)-dimensional KdV-type equation." *Eur. Phys. J. Plus*, vol. 136, 2021, 531, doi: 10.1140/epjp/s13360-021-01528-3.

[12] S. Kumar and N. Mann, "Abundant closed-form solutions of the (3+1)-dimensional Vakhnenko-Parkes equation describing the dynamics of various solitary waves in ocean engineering." *Journal of Ocean Engineering and Science*, 2022, doi: 10.1016/j.joes.2022.04.007.

[13] W. Liu, B. Zhu, G. Li, and D. Wang, "General decay for a viscoelastic Kirchhoof equation with Balakrishnan-Taylor damping, dynamic boundary conditions and a time-varying delay term." *Evol. Equ. Control Theory*, vol. 6, pp. 239–260, 2017.

[14] S. Mesaoudi and M. Kafini, " On the decay and global nonexistence of solutions to a damped wave equation with variable-exponent nonlinearity and delay." *Ann. Pol. Math.*, vol. 122, no. 1, 2019.

[15] A. S. Nicaise and C. Pignotti, " Stabilization of the wave equation with boundary or internal distributed delay." *Diff. Int. Eqs.*, vol. 21, no. 9-10, pp. 935–958, 2008.

[16] M. Osman, H. Almusawa, K. Tariq, S. Anwar, S. Kumar, M. Younis, and W. Ma, " On global behavior for complex soliton solutions of the perturbed nonlinear Schrödinger equation in nonlinear optical fibers." *Journal of Ocean Engineering and Science*, vol. 7, no. 5, pp. 431–443, 2022, doi: 10.1016/j.joes.2021.09.018.

Authors' addresses

Mohammad Alnegga
Department of Mathematics, College of Science, Qassim University, 51452 Buraydah, Saudi Arabia
E-mail address: mnkaaa@qu.edu.sa

Abdelbaki Choucha
Department of Material Sciences, Faculty of Sciences, Amar Teleji Laghouat University, 03000 37G Ghardaia Road, Algeria
E-mail address: abdel.choucha@lagh-univ.dz

Djamel Ouchenane
Department of Mathematics, Faculty of Sciences, Amar Teleji Laghouat University, 03000 37G Ghardaia Road, Algeria
E-mail address: d.ouchenane@lagh-univ.dz

Salah Boulaaras
(**Corresponding author**) Department of Mathematics, College of Science, Qassim University, 51452 Buraydah, Saudi Arabia
E-mail address: s.boularas@qu.edu.sa

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 553–562

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4562

A NUMERICAL APPROACH FOR SOLVING VOLTERRA DELAY INTEGRO-DIFFERENTIAL EQUATION

ILHAME AMIRALI, MUHAMMET ENES DURMAZ, AND GABIL M. AMIRALIYEV

Received 30 January, 2023

Abstract. In this paper, we propose a finite difference technique for approximating solutions of first-order Volterra delay integro-differential equation. The presented numerical method acquires a second-order convergence in discrete maximum norm. The derived results are numerically validated in test problems to support the theoretical analysis.

2010 *Mathematics Subject Classification:* 45D05; 45J05; 65L20; 65L70; 65R20

Keywords: Volterra delay integro-differential equation, finite difference, uniform convergence, error estimate

1. Introduction

After Volterra's establishment, the Volterra integro-differential Equations (VIDEs) appeared [13]. It then arise in a variety of physical applications such as diffusion process, heat transfer, glassforming process, neutron diffusion, nanohydrodynamics [24]. Detailed information about VIDEs in literature can be found in biology, engineering and physics applications books [23].

In this work the following Volterra delay integro-differential equation (VDIDE) is being analyzed:

$$u'(t)+a(t)u(t)+b(t)u(t-r)+\int_{t-r}^{t}K(t,s)u(s)ds=f(t),\quad t\in I, \tag{1.1}$$

$$u(t)=\varphi(t),\quad -r\le t\le 0. \tag{1.2}$$

$I=(0,T]=\cup_{p=1}^{m}I_p,\quad I_p=\{t: r_{p-1}<t\le r_p\}, 1\le p\le m$ and $r_s=sr$, for $0\le s\le m, \bar{I}=[0,T]$, $I_0=[-r,0]$. $a(t)\ge 0, f(t), b(t)(t\in\bar{I}), \varphi(t)(t\in I_0)$ and $K(t,s)((t,s)\in \bar{I}\times\bar{I})$ given functions, r is a constant delay. Moreover, we will assume that $a,b,f\in C(\bar{I}), \varphi\in C^2(I_0)$ and $\frac{\partial^2 K}{\partial s^2}\in C(\bar{I}^2)(s=0,1,2)$.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

Integral equations (IEs) play an essential role in applied mathematics, since they arise from various biological, physical and engineering problems. They can be used to model various processes occurring in viscoelasticity, elasticity, diffraction problems, hydrodynamics, stochastic processes, epidemic studies [18], scattering in quantum mechanics, dynamical systems, scattering of time-harmonic acoustic waves [20], etc. Many initial value problems (IVPs) and boundary value problems involving ordinary differential equations and partial differential equations can also be rewritten as IEs. Many efforts have been made over time to study the solvability and characteristics of these equations [12], as well as approaches for approximating their solutions [22].

There has been a rise in interest in integro-differential equations (IDEs) in recent years, with various studies focusing on developing more advanced and efficient approaches [21]. The IDEs be an important branch of modern mathematics. Engineering, electrostatics, mechanics, elasticity theory, potential and mathematical physics are just a few of the fields where it commonly occurs.

Researchers have gradually developed an interest in the field of Volterra/Fredholm IEs during the last fifty years and significant efforts have been made to numerically solve them. The numerical solution to IEs can be found using [7]. There were also numerous notable research in the field of IDEs. [11], for example, investigated the numerical solution of non-linear IDEs using the meshless approach.

The Fredholm integro-differential equations (FIDEs) are solved using a different kind of analytical and computational approaches in the literature [8].

Many physical processes of interest are described by delay differential equation (DDE) in biology [9], medicine, chemistry, physics, engineering and economics, among other fields [14]. The initial-boundary value problem for a linear pseudo-parabolic equation (PPE) with delay is addressed in [4]. The stability bounds for the investigated problem are obtained using the method of energy estimates.The one dimensional initial-boundary problem for a PPE with time delay in the second spatial derivative is considered in [1]. In this work constructed a higher order difference approach to solve the problem numerically and obtained an error estimate for its solution. [6] discusses a singularly perturbed IVP for a quasilinear second-order DDE. An exponentially fitted difference scheme constructed in an equidistant mesh with first-order uniform convergence in the discrete maximum norm. The singularly perturbed IVP for a linear first-order VIDE with delay is addressed in [15]. The authors develop and analyze a numerical method with uniform convergence in the small parameter. The differential part of the problem is discretized using implicit difference rules, and the integral part is discretized using composite numerical quadrature rules. Established layer-adapted mesh error estimations for the approximate solution.

In recent years, the numerical solution of VDIDEs has gotten a lot of attention [2]. In [3] the researchers not only examined the stability inequalities of VDIDEs with a given order of derivatives, but also considered the high-order form of the VDIDEs.

For approximating solutions of second-order Fredholm-Volterra IEs, the author of [16] recommends a straightforward numerical approach. The method is based on Picard iteration and uses an appropriate quadrature formula. The author shows the existence and uniqueness of the solution under certain conditions and provides error estimates for the approximation. In terms of Taylor polynomials around any point, [17] presents a Taylor approach for finding the approximate solution to high-order linear Volterra-Fredholm IEs under mixed conditions. [5] give an overview of approximating techniques for VDIDEs and other similar problems.

Singularly perturbed Volterra/Fredholm IDEs appear in many scientific applications. [19] reviewed the literature on Volterra integral and IDEs that are singularly perturbed. The IVP for a quasilinear VDIDE with small parameter is studied in [25]. The method of integral identities is used to design and analyze a fitted difference scheme using exponential basis functions and interpolating quadrature rules with the weight and remainder term in integral form. The authors shown that the method displays first-order uniform convergence in perturbation parameter. [10] provide a comprehensive discussion of several strategies for the Volterra/Fredholm integral or IDEs.

The current article is divided into four sections and deals with a numerical approximation of the first-order VDIDE. In the Introduction, works in the area of IDEs were compared and newer studies were presented. Further, the authors construct the absolutely stable finite difference scheme in an equidistant mesh. The composite middle rectangle rule was used for the integral term of (1.1). The approximate problem's stability bounds are addressed in Section 3. In the discrete maximum norm, the authors shown that the method has a second-order accuracy. The paper concludes with numerical examples.

2. The Construction of Difference Scheme

We introduce the uniform mesh ω_{N_0} on

$$\bar{I} : \omega_{N_0} = \{t_i = ih, i = 1,2,...,N_0; h = T/N_0 = r/N\}$$

(for simplicity we suppose that T/r is integer; i.e., $T = mr$), which contains N mesh points at each subinterval $I_p (1 \leq p \leq m)$:

$$\omega_{N,p} = \{t_i : (p-1)N + 1 \leq i \leq pN\}, \quad 1 \leq p \leq m,$$

and consequently

$$\omega_{N_0} = \bigcup_{p=1}^{m} \omega_{N,p}.$$

To simplify the notation, we get $g_i = g(t_i)$ for any function $g(t)$ and y_i represent an approximation of $u(t)$ at t_i. Also, for any mesh function g_i defined on ω_{N_0} we use

$$g_{\bar{t},i} = \frac{g_i - g_{i-1}}{h}, \quad g_i^{(0.5)} = \frac{1}{2}(g_i + g_{i-1}), \quad g_{i-\frac{1}{2}} = g\left(t_{i-\frac{1}{2}}\right) = g\left(t_i - \frac{h}{2}\right)$$

and discrete maximum norms

$$\|g\|_{\infty,p} \equiv \|g\|_{\infty,\omega_{N,p}} = \max_{\bar{\omega}_{N,p}} |g(t)|, \quad \|g\|_{\infty,N_0} \equiv \|g\|_{\infty,\omega_{N_0}}.$$

Setting $t = t_{i-\frac{1}{2}}$ in (1.1) we have

$$u'\left(t_{i-\frac{1}{2}}\right) + a\left(t_{i-\frac{1}{2}}\right) u\left(t_{i-\frac{1}{2}}\right) + b\left(t_{i-\frac{1}{2}}\right) u\left(t_{i-\frac{1}{2}} - r\right) + \int_{t_{i-\frac{1}{2}}-r}^{t_{i-\frac{1}{2}}} K\left(t_{i-\frac{1}{2}}, s\right) u(s)ds = f\left(t_{i-\frac{1}{2}}\right), \qquad (1 \le i \le N_0). \tag{2.1}$$

Next, we use the relations

$$u'\left(t_{i-\frac{1}{2}}\right) = u_{\bar{t},i} + R_i^{(1)}, \qquad R_i^{(1)} = -\frac{h^2}{24} u'''(\xi_i^{(1)}), \quad \xi_i^{(1)} \in (t_{i-1}, t_i), \tag{2.2}$$

$$u\left(t_{i-\frac{1}{2}}\right) = u_i^{(0,5)} + R_i^{(2)}, \qquad R_i^{(2)} = -\frac{h^2}{8} u''(\xi_i^{(2)}), \quad \xi_i^{(2)} \in (t_{i-1}, t_i), \tag{2.3}$$

$$u\left(t_{i-\frac{1}{2}} - r\right) = u_{i-N}^{(0,5)} + R_i^{(3)}, \qquad R_i^{(3)} = -\frac{h^2}{8} u''(\xi_i^{(3)}), \quad \xi_i^{(3)} \in (t_{i-1-N}, t_{i-N}). \tag{2.4}$$

For the integral term of (2.1), after applying composite rectangle rule, we have

$$\int_{t_{i-\frac{1}{2}}-r}^{t_{i-\frac{1}{2}}} K\left(t_{i-\frac{1}{2}}, s\right) u(s)ds = h \sum_{j=i-N+1}^{i-1} K\left(t_{i-\frac{1}{2}}, t_i\right) u_i + R_i^{(4)},$$

where

$$R_i^{(4)} = \frac{h^2 r}{24} \frac{d^2}{ds^2} \left(K\left(t_{i-\frac{1}{2}}, (\xi_i^{(4)}\right) u\left(\xi_i^{(4)}\right)\right), \quad t_{i-\frac{1}{2}} - r < \xi_i < t_{i-\frac{1}{2}}. \tag{2.5}$$

Consequently we get the exact relation for $u(t_i)$

$$u_{\bar{t},i} + a_{i-\frac{1}{2}} u_i^{(0.5)} + b_{i-\frac{1}{2}} u_{i-N}^{(0.5)} + h \sum_{j=i-N+1}^{i-1} K_{i-\frac{1}{2},j} u_j = f_{i-\frac{1}{2}} - R_i, \tag{2.6}$$

with remainder term

$$R_i = R_i^{(1)} + a_{i-\frac{1}{2}} R_i^{(2)} + b_{i-\frac{1}{2}} R_i^{(3)} + R_i^{(4)}, \quad 1 \le i \le N_0,$$

where $R_i^{(k)} (k = 1,2,3,4)$ are defined by (2.2), (2.3), (2.4) and (2.5), respectively. Based on (2.6), we propose the following difference scheme for approximating (1.1)-(1.2):

$$y_{\bar{t},i} + a_{i-\frac{1}{2}} y_i^{(0.5)} + b_{i-\frac{1}{2}} y_{i-N}^{(0.5)} + h \sum_{j=i-N+1}^{i-1} K_{i-\frac{1}{2},j} y_j = f_{i-\frac{1}{2}}, \quad 1 \le i \le N_0, \tag{2.7}$$

$$y_i = \varphi_i, \quad -N \le i \le 0. \tag{2.8}$$

3. Stability Bound and Error Estimate

Lemma 1. *Consider the following difference problem:*

$$v_{\bar{t},i} + A_i v_i^{(0.5)} = F_i, \quad 1 \le i \le N, \tag{3.1}$$

$$v_0 = \mu \tag{3.2}$$

with $t_i = x_0 + ih, \quad h = (X - x_0)/N, A_i \ge 0$. *Let* $\mid F_i \mid \le \mathcal{F}_i$ *and* $\mathcal{F}_i$ *a nondecreasing function. Then the solution of* (3.1)-(3.2) *satisfies*

$$|v_i| \le |\mu| + (X - x_0)\, \mathcal{F}_i, \quad 1 \le i \le N. \tag{3.3}$$

Proof. The difference equation can be written as

$$v_i = \frac{2 - hA_i}{2 + hA_i} v_{i-1} + \frac{2h}{2 + hA_i} F_i.$$

From here, it follows that

$$|v_i| \le v_{i-1} + h|F_i|.$$

Therefore,

$$|v_i| \le |v_0| + h \sum_{j=1}^{i} |F_j| \le |v_0| + ih\mathcal{F}_i,$$

which implies the validity of (3.3). □

Let $z_i = y_i - u_i$. Then the error of approximate solution z_i by (2.7) and (2.8) will be

$$z_{\bar{t},i} + a_{i-\frac{1}{2}} z_i^{(0.5)} + b_{i-\frac{1}{2}} z_{i-N}^{(0.5)} + h \sum_{j=i-N+1}^{i-1} K_{i-\frac{1}{2},j} z_j = R_i, \quad 1 \le i \le N_0, \tag{3.4}$$

$$z_i = 0, \quad -N \le i \le 0. \tag{3.5}$$

Lemma 2. *The error function* z_i *will satisfy*

$$\|z\|_{\infty,N_0} \le \frac{\left[1 + r\left(\|b\|_{\infty,\bar{I}} + T\bar{K}\right) e^{r^2 T\bar{K}}\right]^m - 1}{\|b\|_{\infty,\bar{I}} + T\bar{K}} \|R\|_{\infty,N_0}, \tag{3.6}$$

where $\bar{K} = \max\limits_{\bar{I}\times\bar{I}} |K(t,s)|$.

Proof. For $t_i \in \omega_{N,p}$, we have

$$|R_i - b_{i-\frac{1}{2}} z_{i-N}^{(0.5)} - h \sum_{j=i-N+1}^{i-1} K_{i-\frac{1}{2},j} z_j| \le \|R\|_{\infty,N,p} + \|b\|_{\infty,\bar{I}} \|z\|_{\infty,p-1} + h\bar{K} \sum_{j=i-N+2}^{i} |z_{j-1}|$$

$$\le \|R\|_{\infty,p} + \|b\|_{\infty,\bar{I}} \|z\|_{\infty,p-1} + T\bar{K} \|z\|_{\infty,p-1} + h\bar{K} \sum_{j=(p-1)N+1}^{i} |z_{j-1}|$$

$$\leq \|R\|_{\infty,\omega_{N_0}} + (\|b\|_\infty + T\bar{K})\,\|z\|_{\infty,p-1} + h\bar{K}\sum_{j=(p-1)N+1}^{i}|z_{j-1}|.$$

Then after applying Lemma 1 to (3.4) on $\omega_{N,p}$ we get

$$|z_i| \leq |z_{(p-1)N}| + r\|R\|_{\infty,N_0} + r\left(\|b\|_{\infty,\bar{I}+T\bar{K}}\right)\|z\|_{\infty,p-1} + rh\bar{K}\sum_{j=(p-1)N+1}^{i}|z_{j-1}|,$$

so

$$|z_i| \leq r\|R\|_{\infty,N_0} + \left(1 + r\|b\|_{\infty,\bar{I}+T\bar{K}}\right)\|z\|_{\infty,p-1} + rh\bar{K}\sum_{j=(p-1)N+1}^{i}|z_{j-1}|.$$

From here, by using the difference analogue of Gronwall's inequality, we arrive at

$$|z_i| \leq re^{\bar{K}r^2t_i}\|R\|_{\infty,N_0} + \left[1 + r\left(\|b\|_{\infty,\bar{I}} + T\bar{K}\right)\right]e^{\bar{K}r^2t_i}\|z\|_{\infty,p-1}.$$

Therefore we have the first-order difference inequality of the form

$$\|z\|_{\infty,p} \leq A\|z\|_{\infty,p-1} + B \tag{3.7}$$

with

$$\begin{aligned} A &= 1 + r\left(\|b\|_{\infty,\bar{I}} + T\bar{K}\right)e^{r^2T\bar{K}}, \\ B &= re^{r^2T\bar{K}}\|R\|_{\infty,N_0}. \end{aligned} \tag{3.8}$$

Resolving (3.7), by taking into consideration (3.8) we arrive at

$$\|z\|_{\infty,p} \leq \frac{A^p - 1}{A - 1}B, \quad 1 \leq p \leq m,$$

which proves (3.6). □

Lemma 3. *Let be fulfilled the assumptions above and $u \in C^3[0,T]$. Then for the truncation error R_i, the following estimate holds:*

$$\|R\|_{\infty,\omega_{N_0}} \leq Ch^2.$$

Proof. From explicit expressions of $R_i^{(k)}$ $(k = 1,2,3,4)$, it is not hard to observe that $\|R^{(k)}\|_{\infty,\omega_{N_0}} \leq Ch^2$. This, along with (2.2), (2.3), (2.4) and (2.5) give the desired result. □

Theorem 1. *Let u the solution of* (1.1)-(1.2) *and y the solution of* (2.7)-(2.8). *Under the assuming conditions the solution of difference problem* (2.7)-(2.8) *second-order uniformly convergent to the solution of* (1.1)-(1.2)*:*

$$\|y - u\|_{\infty,\bar{\omega}_{N_0}} \leq Ch^2.$$

Proof. It is evidently from Lemma 2 and Lemma 3. □

4. Numerical Results

In this section, we present a numerical results which illustrate the presented method.

Example 1. Consider the test problem:

$$u'(t)+\frac{2}{3}u(t)-\frac{1}{3}\int_{t-1}^{t}u(s)ds=-\frac{1}{3}+\frac{1}{3}t-\frac{1}{2}(1-e^{-\frac{2t}{3}}),\quad t\in(0,2],$$

$$u(t)=1,\quad -1\le t\le 0.$$

The exact solution is

$$u(t)=\begin{cases} e^{-\frac{2}{3}t}, & 0\le t\le 1,\\ \frac{e^{\frac{1}{3}(t-1)}-e^{1-t}}{2}-1+e^{-\frac{2}{3}t}+e^{-\frac{2}{3}(1-t)}, & 1\le t\le 2.\end{cases}$$

We define the exact errors

$$e^N=\|y-u\|_{\infty,\bar{\omega}_{N_0}},$$

where y is the approximate solution of u for the various values of N. Experimental rates of convergence are computed by

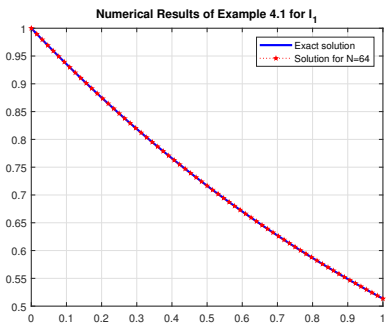
$$p^N=\frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$

From Table 1 we observe that the experimental rate of convergence is monotonically increasing towards 2, so in agreement with the theoretical analysis from Theorem 1.

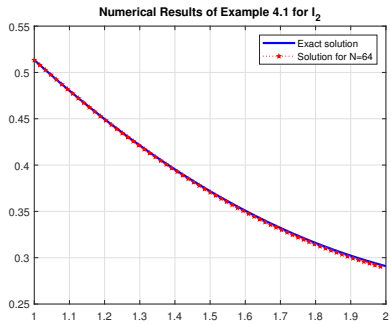
TABLE 1. The numerical results for the test problem.

$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
0.031121	0.008514	0.002250	0.000572	0.000144
1.87	1.92	1.98	1.99	

The graph of the exact and approximation solution of Example 1 in the interval I_1 is given in the graphs below:



The graph of the exact and approximation solution of Example 1 in the interval I_2 is given in the graphs below:



Example 2. Consider the IVP

$$u'(t)+tu(t)+u(t-1)+\int_{t-1}^{t} e^{ts}ds=\sin\frac{\pi t}{4}, \quad t\in(0,2],$$

$$u(t)=\sqrt{1-t}, \quad -1\le t\le 0,$$

whose exact solution is not available. Using the double mesh method, we evaluate errors in the computed solution and convergence rates. We calculate another approximate solution y^{2N} on a mesh that is obtained by uniformly bisecting the original mesh $\bar{\omega}_{N_0}$. We calculate the errors for various values of N by

$$E^N=\left\|y^N-y^{2N}\right\|_{\infty,\bar{\omega}_{N_0}}.$$

Furthermore, the experimental convergence rates are calculated in the same way as in Example 2.

TABLE 2. The numerical results for the test problem.

$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
0.048089	0.013620	0.003675	0.000958	0.000243
1.82	1.89	1.94	1.98	

5. CONCLUSION

We have considered the numerical approximation of first-order VDID problem. In discrete maximum norm, the method has a second-order accuracy. The obtained theoretical results have been tested on particular problems. The method given here can be used to solve more complex linear and nonlinear VDIDEs.

REFERENCES

[1] I. Amirali, "Analysis of higher order difference method for a pseudo-parabolic equation with delay," *Miskolc Math. Notes*, vol. 20, no. 2, pp. 755–766, 2019, doi: 10.18514/MMN.2019.2895.

[2] I. Amirali, "Stability properties for the delay integro-differential equation," *Gazi Univ. J. Sci.*, vol. 36, no. 2, pp. 862–868, 2023, doi: 10.35378/gujs.988728.

[3] I. Amirali and H. Acar, "A novel approach for the stability inequalities for high-order Volterra delay integro-differential equation," *J. Appl. Math. Comput.*, 2022, doi: 10.1007/s12190-022-01761-8.

[4] I. Amirali, S. Cati, and G. M. Amiraliyev, "Stability inequalities for the delay pseudo-parabolic equations," *Int. J. Appl. Math.*, vol. 32, no. 2, pp. 289–294, 2019, doi: 10.12732/ijam.v32i2.10.

[5] G. M. Amiraliyev and O. Yapman, "On the Volterra delay integro-differential equation with layer behavior and its numerical solution," *Miskolc Math. Notes*, vol. 20, no. 1, pp. 75–87, 2019, doi: 10.18514/MMN.2019.2424.

[6] I. G. Amiraliyeva, F. Erdogan, and G. M. Amiraliyev, "A uniform numerical method for dealing with a singularly perturbed delay initial-value problem," *Appl. Math. Lett.*, vol. 23, pp. 1221–1225, 2010, doi: 10.1016/j.aml.2010.06.002.

[7] P. Assari and M. Dehghan, "A meshless local discrete Galerkin (mldg) scheme for numerically solving two-dimensional nonlinear Volterra integral equations," *Appl. Math. Comput.*, vol. 350, pp. 249–265, 2019, doi: 10.1016/j.amc.2019.01.013.

[8] R. E. Bellman and K. L. Cooke, *Differential difference equations*. New York: Academic Press, 1963.

[9] G. A. Bocharov and F. A. Rihan, "Numerical modelling in biosciences with delay differential equations," *J. Comput. Appl. Math.*, vol. 125, pp. 183–199, 2000, doi: 10.1016/S0377-0427(00)00468-4.

[10] H. Brunner and P. J. van der Houwen, *The numerical solution of Volterra equations*, in: CWI Monographs 3, North-Holland, Amsterdam, 1986.

[11] M. Dehghan and R. Salehi, "The numerical solution of the non-linear integro-differential equations based on the meshless method," *J. Comput. Appl. Math.*, vol. 2012, no. 236, p. 9, 2012, doi: 10.1016/j.cam.2011.11.022.

[12] G. Izzo, Z. Jackiewicz, E. Messina, and A. Veccio, "General linear methods for Volterra integral equations," *J. Comput. Appl. Math.*, vol. 234, no. 9, pp. 2768–2782, 2010, doi: 10.1016/j.cam.2010.01.023.

[13] A. J. Jerri, *Introduction to Integral Equations with Applications*. New York: Wiley, 1999.

[14] M. Kudu, I. Amirali, and G. M. Amiraliyev, "Analysis of difference approximations to delay pseudo-parabolic equations," *Springer International Publishing Switzerland, Differential and Difference Equations with Applications, Springer Proceedings in Mathematics and Statistics*, vol. 164, pp. 159–168, 2016, doi: 10.1007/978-3-319-32857-7_16.

[15] M. Kudu, I. Amirali, and G. M. Amiraliyev, "A finite-difference method for a singularly perturbed delay integro-differential equation," *J. Comput. Appl. Math.*, vol. 308, pp. 379–390, 2016, doi: 10.1016/j.cam.2016.06.018.

[16] S. Micula, "An iterative numerical method for Fredholm-Volterra integral equations of the second kind," *Appl. Math. Comput.*, vol. 270, pp. 935–942, 2015, doi: 10.1016/j.amc.2015.08.110.

[17] F. Mirzaee and A. A. Hoseini, "Numerical solution of nonlinear Volterra-Fredholm integral equations using hybrid of block-pulse functions and Taylor series," *Alexandria Eng. J.*, vol. 52, pp. 551–555, 2013, doi: 10.1016/j.aej.2013.02.004.

[18] I. M. Olaru, "Generalization of an integral equation related to some epidemic models," *Carpathian J. Math.*, vol. 26, no. 1, pp. 92–96, 2010.

[19] A. Panda, J. Mohapatra, and I. Amirali, "A second order post-processing technique for singularly perturbed Volterra integro-differential equation," *Mediterr. J. Math.*, vol. 18, no. 231, 2021, doi: 10.1007/s00009-021-01873-8.

[20] Y. Qin and L. Liu, "Integral equation method for acoustic scattering by an inhomogeneous penetrable obstacle in a stratified medium," *App. Anal.*, vol. 93, no. 11, pp. 2402–2412, 2014, doi: 10.1080/00036811.2014.924111.

[21] M. T. Rashed, "Numerical solution of functional differential, integral and integro-differential equations," *Appl. Math. Comput.*, vol. 156, pp. 485–492, 2004, doi: 10.1016/j.amc.2003.08.021.

[22] H.-J. Reinhardt, *Analysis of approximation methods for differential and integral equations*. New York: Springer, 2012. doi: 10.1007/978-1-4612-1080-1.

[23] A. A. Samarskii, *The theory of difference schemes.* New York: Marcel Dekker, Inc.,, 2001.

[24] A.-M. Wazwaz, *Linear and nonlinear integral equations, methods and applications.* Berlin, Heidelberg: Springer, 2011. doi: 10.1007/978-3-642-21449-3.

[25] O. Yapman, G. M. Amiraliyev, and I. Amirali, "Convergence analysis of fitted numerical method for a singularly perturbed nonlinear Volterra integro-differential equation with delay," *J. Comput. Appl. Math.*, vol. 355, no. 301–309, 2019, doi: 10.1016/j.cam.2019.01.026.

Authors' addresses

Ilhame Amirali
Department of Mathematics, Faculty of Art and Science, Düzce University, Düzce 81620, Turkey
E-mail address: ailhame@gmail.com

Muhammet Enes Durmaz
(**Corresponding author**) Department of Information Technology, Kırklareli University, Kırklareli, 39100, Turkey
E-mail address: menesdurmaz025@gmail.com

Gabil M. Amiraliyev
Department of Mathematics, Faculty of Art and Science, Erzincan Binali Yıldırım University, Erzincan 24100, Turkey
E-mail address: gabilamirali@yahoo.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 563–577

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4328

NEW APPROACHES FOR SUBORDINATION AND SUPERORDINATION OF MULTIVALENT FUNCTIONS ASSOCIATED WITH A FAMILY OF LINEAR OPERATORS

SEON HYE AN, REKHA SRIVASTAVA, AND NAK EUN CHO

Received 14 July, 2022

Abstract. The aim of this paper is to study some subordination and superordination implication properties for multivalent functions in the open unit disk associated with a family of linear operators. Moreover, we apply the results and techniques presented here to a class of integral operators.

2010 *Mathematics Subject Classification:* 30C45; 30C80

Keywords: subordination, superordination, analytic function, Hadamard product, Carlson-Shaffer operator, Liu-Srivastava operator, integral operator

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\triangle)$ be the class of analytic functions defined over the unit disk

$$\triangle = \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \dots\}$, let

$$\mathcal{H}[c, n] = \{f \in \mathcal{H} : f(z) = c + c_n z^n + c_{n+1} z^{n+1} + \dots\}.$$

Let f and F belong to the class $\mathcal{H}$. Then we say that the function f is subordinate to F, or F is superordinate to f, denoted by $f \prec F$ or $f(z) \prec F(z)$ $(z \in \triangle)$, if there exists a Schwarz function w in $\triangle$, with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \triangle$, satisfying

$$f(z) = F(w(z)) \quad (z \in \triangle).$$

If the function F is univalent in $\triangle$, then the following relation holds (cf. [6]):

$$f \prec F \iff f(0) = F(0) \text{ and } f(\triangle) \subset F(\triangle).$$

The third author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2019R1I1A3A0105086).

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

Definition 1 ([6, page 16])**.** Let $\phi\colon \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\triangle$. If $\mathfrak{p}$ is analytic in $\triangle$ and satisfies the differential subordination

$$\phi(\mathfrak{p}(z), z\mathfrak{p}'(z)) \prec h(z) \quad (z \in \triangle), \tag{1.1}$$

then $\mathfrak{p}$ is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $\mathfrak{p} \prec q$ for all $\mathfrak{p}$ satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant.

Definition 2 ([7, page 815-817], Definition 1)**.** Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\triangle$. If $\mathfrak{p}$ and $\varphi(\mathfrak{p}(z), z\mathfrak{p}'(z))$ are univalent in $\triangle$ and satisfy the differential superordination

$$h(z) \prec \varphi(\mathfrak{p}(z), z\mathfrak{p}'(z)) \quad (z \in \triangle), \tag{1.2}$$

then $\mathfrak{p}$ is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec \mathfrak{p}$ for all $\mathfrak{p}$ satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Definition 3 ([6, page 21], Definition 2.2b)**.** Denote by Q the class of functions f that are analytic and injective on $\overline{\triangle} \backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\triangle : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\triangle \backslash E(f)$.

We also denote $\mathcal{M}_\beta^*$ by the class of univalent functions $q \in \mathcal{H}$ with $q(0) = 1$ satisfying the following condition:

$$\Re\left[(1-\beta)\frac{zq'(z)}{q(z)} + \beta\left(1 + \frac{zq''(z)}{q'(z)}\right)\right] > 0 \quad (\beta \in \mathbb{R};\ z \in \triangle).$$

Then we also note that $\mathcal{M}_1^*$ is the class of convex (not necessarily normalized) functions in $\triangle$ (cf. [8, 10]).

We denote by $\mathcal{A}_p$ the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} c_{k+p} z^{k+p} \quad (p \in \mathbb{N})$$

which are analytic and p-valent in $\triangle$. Now we define the function $\phi_p(a,b;z)$ by

$$\phi_p(a,b;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} z^{k+p} \quad (b \neq 0, -1, -2, \dots),$$

where $(\nu)_k$ is the Pochhammer symbol(or the shifted factorial) defined by

$$(\nu)_k = \begin{cases} 1 & \text{if } k = 0 \\ \nu(\nu+1)\dots(\nu+k-1) & \text{if } k \in \mathbb{N}. \end{cases}$$

Let $f \in \mathcal{A}_p$. Denote by $\mathcal{L}_p(a,b)\colon \mathcal{A}_p \to \mathcal{A}_p$ the operator defined by

$$\mathcal{L}_p(a,b)f(z) = \phi_p(a,b;z) * f(z) \quad (z \in \triangle), \tag{1.3}$$

where the symbol $(*)$ represents the Hadamard product (or convolution). We view that

$$\mathcal{L}_p(p+1,p)f(z) = zf'(z)/p \text{ and } \mathcal{L}_p(n+p,1)f(z) = D^{n+p-1}f(z),$$

where n is any real number with $n > -p$, and the operator D^n for $p = 1$ is the well-known Ruscheweyh derivative [14] (also, see [3]) for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Further, from the definition of the operator $\mathcal{L}_p(a,b)$ it is easily checked that

$$z(\mathcal{L}_p(a,b)f(z))' = a\mathcal{L}_p(a+1,b)f(z) - (a-p)\mathcal{L}_p(a,b)f(z). \tag{1.4}$$

The operator $\mathcal{L}_p(a,b)$ was introduced and studied by Saitoh [15], which is an extension of the recognized Carlson-Shaffer operator $\mathcal{L}_1(a,b)$. And also the operator $\mathcal{L}_p(a,b)$ has been extensively used by many researchers studying geometric function theory (see, for examples, [2, 18–20]). Moreover, we remark that Liu and Srivastava [5] considered a general class of linear convolution operators, popularly known as the Liu-Srivastava operator, which is much more general than the operator $\mathcal{L}_p(a,b)$ which we have studied in this paper.

By using the concepts of subordination, Miller *et al.* [9] investigated some interesting subordination-preserving results involving certain integral operators for analytic functions in $\triangle$. Also Owa and Srivastava [12] obtained the subordination properties for a family of integral operators. Moreover, Miller and Mocanu [7] introduced the concepts of differential superordinations, which is regarded as the dual problem of differential subordinations (see also [1]). We also note that several researchers studied many interesting results involving various operators in connection with differential subordinations and their dual problems (for details, see [16, 17, 21–25]). In this paper, motivated by the works stated above, we investigate the subordination and superordination implication properties of multivalent functions for the operator $\mathcal{L}_p(a,b)$ defined by (1.3). We also consider interesting applications to the integral operator.

The following lemmas will be required to derive our present investigation.

Lemma 1 ([6, page 24], Lemma 2.2d)**.** *Let $p \in Q$ with $\mathfrak{p}(0) = a$ and let*

$$q(z) = a + a_n z^n + \dots$$

be analytic in $\triangle$ with

$$q(z) \not\equiv a \text{ and } n \in \mathbb{N}.$$

If q is not subordinate to $\mathfrak{p}$, then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \text{ and } \zeta_0 \in \partial\triangle \backslash E(f),$$

for which

$$q(\mathbb{U}_{r_0}) \subset \mathfrak{p}(\triangle),\ q(z_0) = \mathfrak{p}(\zeta_0) \text{ and } z_0 q'(z_0) = m\zeta_0 \mathfrak{p}'(\zeta_0) \quad (m \geq n).$$

A function $L(z,t)$ defined on $\triangle \times [0,\infty)$ is the subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\triangle$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \triangle$ and

$$L(z,s) \prec L(z,t) \quad (z \in \triangle;\ 0 \le s < t).$$

Lemma 2 ([13, page 159], Theorem 6.2)**.** *The function*

$$L(z,t) = a_1(t)z + \dots$$

with

$$a_1(t) \neq 0 \text{ and } \lim_{t\to\infty} |a_1(t)| = \infty.$$

Suppose that $L(\cdot,t)$ ia analytic in $\triangle$ for all $t \ge 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \triangle$. If $L(z,t)$ satisfies the following two conditions

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad (z \in \triangle;\ 0 \le t < \infty)$$

and

$$|L(z,t)| \le K_0|a_1(t)| \quad (|z| < r_0 < 1;\ 0 \le t < \infty)$$

for some positive constants K_0 and r_0, then $L(z,t)$ is a subordination chain.

Lemma 3 ([7, page 822], Theorem 7)**.** *Let $q \in \mathcal{H}[a,1]$ and let $\varphi\colon \mathbb{C}^2 \to \mathbb{C}$. Also set*

$$\varphi(q(z), zq'(z)) \equiv h(z) \quad (z \in \triangle).$$

If

$$L(z,t) = \varphi(q(z), tzq'(z))$$

is a subordination chain and $\mathfrak{p} \in \mathcal{H}[a,1] \cap Q$, then

$$h(z) \prec \varphi(\mathfrak{p}(z), z\mathfrak{p}'(z)) \quad (z \in \triangle).$$

implies that

$$q(z) \prec \mathfrak{p}(z)\ (z \in \triangle).$$

Furthermore, if

$$\varphi(q(z), zq'(z)) = h(z)$$

has a univalent solution $q \in Q$, then q is the best subordinant.

2. Main Results

Firstly, we start by deriving the subordination implication stated below entailing the operator $\mathcal{L}_p(a,c)$ defined by (1.3).

Theorem 1. *Let $f,g \in \mathcal{A}_p$. Suppose also that*

$$\frac{\mathcal{L}_p(a,b)g(z)}{z^p} \in \mathcal{M}_\beta^* \quad (0 \le \beta \le 1;\ a > 0;\ z \in \triangle). \tag{2.1}$$

Then the following subordination relation:

$$\begin{aligned}\left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}\right]^{\beta} \\ \prec \left[\frac{\mathcal{L}_p(a,b)g(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p}\right]^{\beta} \quad (z \in \triangle)\end{aligned} \tag{2.2}$$

implies that

$$\frac{\mathcal{L}_p(a,b)f(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)g(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the function $\frac{\mathcal{L}_p(a,b)g(z)}{z^p}$ is the best dominant.

Proof. Let us define two functions F and G by

$$F(z) := \frac{\mathcal{L}_p(a,b)f(z)}{z^p} \text{ and } G(z) := \frac{\mathcal{L}_p(a,b)g(z)}{z^p} \quad (f,g \in \mathcal{A}_p;\ z \in \triangle). \tag{2.3}$$

By utilizing the equation (1.4) to (2.3), we get

$$\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p} = G(z) + \frac{1}{a}zG'(z). \tag{2.4}$$

Hence, combining (2.3) and (2.4), we have

$$\left[\frac{\mathcal{L}_p(a,b)g(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p}\right]^{\beta} = G(z)\left[1+\frac{1}{a}\frac{zG'(z)}{G(z)}\right]^{\beta}. \tag{2.5}$$

Thus, from (2.5), we need to prove the following subordination implication:

$$\begin{aligned}F(z)\left[1+\frac{1}{a}\frac{zF'(z)}{F(z)}\right]^{\beta} \prec G(z)\left[1+\frac{1}{a}\frac{zG'(z)}{G(z)}\right]^{\beta} \quad (z \in \triangle) \\ \Longrightarrow F(z) \prec G(z) \quad (z \in \triangle).\end{aligned} \tag{2.6}$$

Since $G \in \mathcal{M}^*(\beta)$, without loss of generality, we can assume that G satisfies the conditions of Theorem 1 on the closed disk $\overline{\triangle}$ and

$$G'(\zeta) \neq 0 \quad (\zeta \in \partial\triangle).$$

If not, then we replace F and G by

$$F_r(z) = F(rz) \text{ and } G_r(z) = G(rz),$$

respectively, where $0<r<1$ and then G_r is univalent on $\overline{\triangle}$. Since

$$F_r(z)\left[1+\frac{1}{a}\frac{zF_r'(z)}{F_r(z)}\right]^{\beta} \prec G_r(z)\left[1+\frac{1}{a}\frac{zG_r'(z)}{G_r(z)}\right]^{\beta} \quad (z\in\triangle),$$

where

$$F_r(z)=F(rz) \quad (0<r<1;\ z\in\triangle),$$

we would then prove that

$$F_r(z)\prec G_r(z) \quad (0<r<1;\ z\in\triangle),$$

and by letting $r\to 1^-$, we obtain

$$F(z)\prec G(z) \quad (z\in\triangle).$$

If we suppose that the implication (2.6) is not true, that is,

$$F(z)\not\prec G(z) \quad (z\in\triangle),$$

then, from Lemma 1, there exist points

$$z_0\in\triangle \text{ and } \zeta_0\in\partial\triangle$$

such that

$$F(z_0)=G(\zeta_0) \text{ and } z_0F'(z_0)=m\zeta_0G'(\zeta_0) \quad (m\geq 1). \tag{2.7}$$

To prove the implication (2.6), we define the function

$$L\colon \triangle\times[0,\infty)\longrightarrow\mathbb{C}$$

by

$$L(z,t)=G(z)\left[1+\frac{1+t}{a}\frac{zG'(z)}{G(z)}\right]^{\beta}=a_1(t)z+\dots,$$

and we will show that $L(z,t)$ is a subordination chain. At first, we note that $L(z,t)$ is analytic in $|z|<r<1$, for sufficient small $r>0$ and for all $t\geq 0$. We also have that $L(z,t)$ is continuously differentiable on $[0,\infty)$ for each $|z|<r<1$. A simple calculation shows that

$$a_1(t)=\frac{\partial L(0,t)}{\partial z}=G'(0)\left[1+\frac{(1+t)\beta}{a}\right].$$

Hence we obtain

$$a_1(t)\neq 0 \ (t\geq 0)$$

and also we can see that

$$\lim_{t\to\infty}|a_1(t)|=\infty.$$

While, by a direct computation, we have

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\}=\frac{a}{\beta}+\frac{1+t}{\beta}\Re\left[(1-\beta)\frac{zG'(z)}{G(z)}+\beta\left(1+\frac{zG''(z)}{G'(z)}\right)\right]. \tag{2.8}$$

By using the assumption of Theorem 1 to (2.8), we obtain

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\}>0 \quad (z\in\triangle;\ 0\le t<\infty),$$

which completes the proof of the first condition of Lemma 3. Moreover, we have

$$\begin{aligned}\left|\frac{L(z,t)}{a_1(t)}\right|^{1/\beta} &= \left|\frac{G(z)}{G'(0)}\right|^{1/\beta}\frac{\left|1+\frac{(1+t)}{a}\frac{zG'(z)}{G(z)}\right|}{\left|1+\frac{(1+t)\beta}{a}\right|^{1/\beta}}\\
&\le \frac{1}{\beta}\left|\frac{G(z)}{G'(0)}\right|^{1/\beta}\left[\left|\frac{zG'(z)}{G(z)}\right|+\frac{\left|\beta-\frac{zG'(z)}{G(z)}\right|}{\left|1+\frac{(1+t)\beta}{a}\right|}\right]\frac{1}{\left|1+\frac{(1+t)\beta}{a}\right|^{1/\beta-1}}\\
&\le \frac{1}{\beta G'(0)}\left|\frac{G(z)}{G'(0)}\right|^{1/\beta-1}\left[\left|zG'(z)\right|+\frac{\beta|G(z)|+|zG'(z)|}{\left|1+\frac{(1+t)\beta}{a}\right|}\right]\frac{1}{\left|1+\frac{(1+t)\beta}{a}\right|^{1/\beta-1}}\end{aligned} \tag{2.9}$$

Since G is univalent in $\triangle$, the function G can be written by

$$G(z)=G(0)+G'(0)K(z) \quad (z\in\triangle),$$

where K is a normalized univalent function in $\triangle$. Hence we get the following sharp growth and distortion results [13]:

$$\frac{r}{(1+r)^2}\le|K(z)|\le\frac{r}{(1-r)^2} \quad (|z|=r<1) \tag{2.10}$$

and

$$\frac{1-r}{(1+r)^3}\le|K'(z)|\le\frac{1+r}{(1-r)^3} \quad (|z|=r<1) \tag{2.11}$$

Hence, by using the equations (2.10) and (2.11) to (2.9), we can obtain easily an upper bound for the right-hand side of (2.9). Thus the function $L(z,t)$ satisfies the second condition of Lemma 3 and so $L(z,t)$ is a subordination chain. In particular, we note from the definition of subordination chain that

$$L(z,0)\prec L(z,t) \quad (z\in\triangle;\ t\ge 0).$$

Now, by utilizing the definition of $L(z,t)$ and the relation (2.7), we obtain

$$\begin{aligned}L(\zeta_0,t) &= G(\zeta_0)\left[1+\frac{1+t}{a}\frac{\zeta_0G'(\zeta_0)}{G(\zeta_0)}\right]^{\beta}\\
&= F(z_0)\left[1+\frac{1}{a}\frac{z_0F'(z_0)}{F(z_0)}\right]^{\beta}\\
&= [\mathcal{L}_p(a,b)f(z_0)]^{1-\beta}[\mathcal{L}_p(a+1,b)f(z_0)]^{\beta}\in L(\triangle,0),\end{aligned}$$

by virtue of the condition (2.2). This contradicts the above observation that

$$L(\zeta_0,t) \notin L(\triangle,0).$$

Therefore, the subordination condition (2.2) must imply the subordination given by (2.6). Taking $F=G$, we know that the function G is the best dominant. This evidently completes the proof of Theorem 1. □

Next, we give another subordination property by using the equation (1.4) in Theorem 2 below.

Theorem 2. *Let $f,g \in \mathcal{A}_p$ and suppose that*

$$\frac{\mathcal{L}_p(a,b)g(z)}{z^p} \in \mathcal{M}_1^* \quad (a>0;\ z\in\triangle). \tag{2.12}$$

Then the following subordination relation:

$$\beta\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}+(1-\beta)\frac{\mathcal{L}_p(a,b)f(z)}{z^p}$$
$$\prec \beta\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p}+(1-\beta)\frac{\mathcal{L}_p(a,b)g(z)}{z^p} \quad (0\le\beta\le 1;\ z\in\triangle)$$

implies that

$$\frac{\mathcal{L}_p(a,b)f(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)g(z)}{z^p} \quad (z\in\triangle).$$

Moreover, the function $\frac{\mathcal{L}_p(a,b)g(z)}{z^p}$ is the best dominant.

Proof. Let us define the functions F and G as (2.3) and by using the equation (1.4) to (2.3), we have (2.4). Hence, combining (2.3) and (2.4), we obtain

$$(1-\beta)\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p}+\beta\frac{\mathcal{L}_p(a,b)g(z)}{z^p}=G(z)\left(1+\frac{1-\beta}{a}\frac{zG'(z)}{G(z)}\right) \tag{2.13}$$

Thus, from (2.13), we need to prove the following subordination implication:

$$\begin{aligned} F(z)\left(1+\frac{1-\beta}{a}\frac{zF'(z)}{F(z)}\right) \prec G(z)\left(1+\frac{1-\beta}{a}\frac{zG'(z)}{G(z)}\right) \quad (z\in\triangle) \\ \Longrightarrow F(z)\prec G(z) \quad (z\in\triangle). \end{aligned} \tag{2.14}$$

Without loss of generality as in the proof of Theorem 1, we can suppose that G satisfies the conditions of Theorem 1 on the closed disk $\overline{\triangle}$ and

$$G'(\zeta)\neq 0 \quad (\zeta\in\partial\triangle).$$

To derive the implication (2.14), we consider the function

$$L\colon \triangle\times[0,\infty)\longrightarrow\mathbb{C}$$

by

$$L(z,t)=G(z)\left(1+\frac{(1-\beta)(1+t)}{a}\frac{zG'(z)}{G(z)}\right)=a_1(t)z+\dots,$$

and we want to prove that $L(z,t)$ is a subordination chain. But, by using a similar method given in the proof of Theorem 1 we can prove the remaining part of Theorem 2 and so we omit the detailed proof. □

We next consider dual problems of Theorem 1, in the point of view that the subordinations can be replaced by superordinations.

Theorem 3. *Let $f,g \in \mathcal{A}_p$. Suppose that the condition (2.1) is satisfied, the function*

$$\left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}\right]^{\beta}$$

is univalent and $\mathcal{L}_p(a,b)f(z)/z^p \in \mathcal{H}[1,1]\cap Q$. Then the following subordination relation:

$$\begin{aligned}\left[\frac{\mathcal{L}_p(a,b)g(z)}{z^p}\right]^{1-\beta}&\left[\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p}\right]^{\beta}\\ &\prec \left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}\right]^{\beta} \quad (z\in\triangle)\end{aligned} \tag{2.15}$$

implies that

$$\frac{\mathcal{L}_p(a,b)g(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)f(z)}{z^p} \quad (z\in\triangle).$$

Moreover, the function $\frac{\mathcal{L}_p(a,b)g(z)}{z^p}$ is the best subordinant.

Proof. Let us define the functions F and G by (2.3), respectively. By using (2.3), we have

$$\begin{aligned}\left[\frac{\mathcal{L}_p(a,b)g(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p}\right]^{\beta} &= G(z)\left[1+\frac{1}{a}\frac{zG'(z)}{G(z)}\right]^{\beta}\\ &=: \varphi(G(z),zG'(z)).\end{aligned} \tag{2.16}$$

Here, we note that the function G is univalent in $\triangle$ by the condition (2.1).

Next, we show that the subordination condition (2.15) implies that

$$F(z) \prec G(z) \quad (z\in\triangle). \tag{2.17}$$

Now considering the function $L(z,t)$ defined by

$$L(z,t) := G(z)\left[1+\frac{t}{a}\frac{zG'(z)}{G(z)}\right]^{\beta} \quad (z\in\triangle;\ 0\le t<\infty).$$

we can prove easily that $L(z,t)$ is a subordination chain as done in the proof of Theorem 1. Therefore according to Lemma 2, we conclude that the condition (2.15) must imply the superordination given by (2.17). Furthermore, since the equation (2.16) has the univalent solution G, we see that from Lemma 3, it is the best subordinant of the given differential superordination. Therefore the proof of Theorem 3 is completed. □

The proof of Theorem 4 below is similar to that of Theorem 3, and so we omit the details.

Theorem 4. *Let $f,g \in \mathcal{A}_p$. Suppose that the condition (2.12) is satisfied, the function*

$$\beta\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}+(1-\beta)\frac{\mathcal{L}_p(a,b)f(z)}{z^p}$$

is univalent in $\triangle$ and $\mathcal{L}_p(a,b)f(z)/z^p \in \mathcal{H}[1,1]\cap Q$. Then the following subordination relation:

$$\begin{aligned}\beta\frac{\mathcal{L}_p(a+1,b)g(z)}{z^p}&+(1-\beta)\frac{\mathcal{L}_p(a,b)g(z)}{z^p}\\ &\prec \beta\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}+(1-\beta)\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\quad (0\le\beta\le 1;\ z\in\triangle)\end{aligned}$$

implies that

$$\frac{\mathcal{L}_p(a,b)g(z)}{z^p}\prec\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\quad (z\in\triangle).$$

Moreover, the function $\frac{\mathcal{L}_p(a,b)g(z)}{z^p}$ is the best subordinant.

Combining Theorem 1 and Theorem 3, and Theorem 2 and Theorem 4, respectively, we get the following sandwich-type theorems.

Theorem 5. *Let $f,g_k \in \mathcal{A}_p$ $(k=1,2)$. Suppose that*

$$\frac{\mathcal{L}_p(a,b)g_k(z)}{z^p}\in\mathcal{M}_\beta^*\quad (\beta\ge 0;\ a>0;\ z\in\triangle),$$

and the function

$$\left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}\right]^{\beta}$$

is univalent and $\mathcal{L}_p(a,b)f(z)/z^p \in \mathcal{H}[1,1]\cap Q$. Then the following subordination relation:

$$\begin{aligned}&\left[\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)g_1(z)}{z^p}\right]^{\beta}\\ &\prec\left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p}\right]^{\beta}\\ &\prec\left[\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a+1,b)g_2(z)}{z^p}\right]^{\beta}\quad (z\in\triangle)\end{aligned}$$

implies that

$$\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)f(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)g_2(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the functions $\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p}$ *and* $\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p}$ *is the best subordinant and the best dominant.*

Theorem 6. *Let* $f, g_k \in \mathcal{A}_p$ $(k = 1,2)$. *Suppose that*

$$\frac{\mathcal{L}_p(a,b)g_k(z)}{z^p} \in \mathcal{M}_1^* \quad (a > 0;\ z \in \triangle),$$

and the function

$$\beta\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)f(z)}{z^p}$$

is univalent in $\triangle$ *and* $\mathcal{L}_p(a,b)f(z)/z^p \in \mathcal{H}[1,1] \cap Q$. *Then the following subordination relation*:

$$\begin{aligned}
&\beta\frac{\mathcal{L}_p(a+1,b)g_1(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p} \prec \beta\frac{\mathcal{L}_p(a+1,b)f(z)}{z^p} \\
&+ (1-\beta)\frac{\mathcal{L}_p(a,b)f(z)}{z^p} \prec \beta\frac{\mathcal{L}_p(a+1,b)g_2(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p} \\
&(0 \le \beta \le 1; \quad z \in \triangle)
\end{aligned}$$

implies that

$$\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)f(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)g_2(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the functions $\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p}$ *and* $\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p}$ *is the best subordinant and the best dominant.*

If we take

$$a = p,\ c = p \text{ and } \beta = 1$$

in Theorem 5 or Theorem 6, then we have the following result.

Corollary 1. *Let* $f, g_k \in \mathcal{A}_p$ $(k = 1,2)$. *Suppose also that*

$$\frac{g_k(z)}{z^p} \in \mathcal{M}_1^* \quad (z \in \triangle;\ k = 1,2)$$

the function $f'(z)/pz^{p-1}$ *is univalent in* $\triangle$ *and* $f(z)/z^p \in \mathcal{H}[1,1] \cap Q$. *Then we have the following implication*:

$$\frac{g_1'(z)}{pz^{p-1}} \prec \frac{f'(z)}{pz^{p-1}} \prec \frac{g_2'(z)}{pz^{p-1}} \quad (z \in \triangle) \Longrightarrow \frac{g_1(z)}{z^p} \prec \frac{f(z)}{z^p} \prec \frac{g_2(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the functions $\frac{g_1(z)}{z^p}$ *and* $\frac{g_2(z)}{z^p}$ *is the best subordinant and the best dominant.*

Next, we study the extended Libera integral operator F_ν $(\nu > -p)$ defined by (cf.[3,4,11])

$$F_\nu(f)(z) := \frac{\nu+p}{z^\nu}\int_0^z t^{\nu-1} f(t)\,dt \quad (f \in \mathcal{A}_p;\ \Re\{\nu\} > -p) \tag{2.18}$$

Now, we get the sandwich-type result below involving the integral operator defined by (2.18).

Theorem 7. *Let $f, g_k \in \mathcal{A}$ $(k = 1,2)$. Suppose also that*

$$\frac{\mathcal{L}_p(a,b)F_\nu(g_k)(z)}{z^p} \in \mathcal{M}_\beta^* \quad (\nu > -p;\ \beta \geq 0;\ z \in \triangle;\ k = 1,2),$$

the function

$$\left[\frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{\beta}$$

is univalent in $\triangle$ and $\mathcal{L}_p(a,b)F_\nu(f)(z)/z^p \in \mathcal{H}[1,1] \cap Q$. Then the following subordination relation:

$$\begin{aligned}
&\left[\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p}\right]^{\beta}\\
&\prec \left[\frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a,b)f(z)}{z^p}\right]^{\beta}\\
&\prec \left[\frac{\mathcal{L}_p(a,b)F_\nu(g_2)(z)}{z^p}\right]^{1-\beta}\left[\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p}\right]^{\beta} \quad (z \in \triangle)
\end{aligned}$$

implies that

$$\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)F_\nu(g_2)(z)}{z^p} \quad (z \in \triangle)$$

Moreover, the functions $\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p}$ and $\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p}$ are the best subordinant and the best dominant.

Proof. Let us define the functions F and G_k $(k = 1,2)$ by

$$F(z) := \frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p} \text{ and } G_k(z) := \frac{\mathcal{L}_p(a,b)F_\nu(g_k)(z)}{z^p} \quad (f, g \in \mathcal{A}_p;\ z \in \triangle). \tag{2.19}$$

By means of the definition of the integral operator F_ν defined by (2.19), we have

$$z(\mathcal{L}_p(a,b)F_\nu(f)(z))' = (\nu+p)\mathcal{L}_p(a+1,c)f(z) - \nu\mathcal{L}_p(a,b)F_\nu(f)(z) \tag{2.20}$$

Hence, by using (2.19), (2.20) and the same method as in the proof of Theorem 5, we can prove Theorem 7 and so we omit the details involved. □

Finally, we have the sandwich-type Theorem 8 below by using a similar method as in the proof of Theorem 6.

Theorem 8. *Let $f, g_k \in \mathcal{A}$ $(k = 1,2)$. Suppose that*

$$\frac{\mathcal{L}_p(a,b)F_\nu(g_k)(z)}{z^p} \in \mathcal{M}_1^* \quad (\nu > -p;\ \beta \geq 0;\ z \in \triangle;\ k = 1,2),$$

the function

$$\beta\frac{\mathcal{L}_p(a,b)f(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p}$$

is univalent in $\triangle$ and $\mathcal{L}_p(a,b)f(z)/z^p \in \mathcal{H}[1,1] \cap Q$. Then the following subordination relation:

$$\begin{aligned}
&\beta\frac{\mathcal{L}_p(a,b)g_1(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p} \\
&\prec \beta\frac{\mathcal{L}_p(a,b)f(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p} \\
&\prec \beta\frac{\mathcal{L}_p(a,b)g_2(z)}{z^p} + (1-\beta)\frac{\mathcal{L}_p(a,b)F_\nu(g_2)(z)}{z^p} \quad (0 \leq \beta \leq 1;\ z \in \triangle)
\end{aligned}$$

implies that

$$\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)F_\nu(f)(z)}{z^p} \prec \frac{\mathcal{L}_p(a,b)F_\nu(g_2)(z)}{z^p} \quad (z \in \triangle).$$

Moreover, the functions $\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p}$ and $\frac{\mathcal{L}_p(a,b)F_\nu(g_1)(z)}{z^p}$ are the best subordinant and the best dominant.

If we take

$$a = p,\ c = p \text{ and } \beta = 1$$

Theorem 7 or Theorem 8, then we have the following result.

Corollary 2. *Let $f, g_k \in \mathcal{A}$ $(k = 1,2)$. Suppose also that*

$$\frac{F_\nu(g_k)(z)}{z^p} \in \mathcal{M}_1^* \quad (\nu > -p;\ z \in \triangle;\ k = 1,2),$$

the function $f(z)/z^p$ is univalent in $\triangle$ and $F_\nu(f)(z)/z^p \in \mathcal{H}[1,1] \cap Q$. Then we have the following implication:

$$\frac{g_1(z)}{z^p} \prec \frac{f(z)}{z^p} \prec \frac{g_2(z)}{z^p} \quad (z \in \triangle) \Longrightarrow \frac{F_\nu(g_1)(z)}{z^p} \prec \frac{F_\nu(f)(z)}{z^p} \prec \frac{F_\nu(g_2)(z)}{z^p} \quad (z \in \triangle)$$

Moreover, the functions $\frac{F_\nu(g_1)(z)}{z^p}$ and $\frac{F_\nu(g_2)(z)}{z^p}$ are the best subordinant and the best dominant.

REFERENCES

[1] T. Bulboacă, "A class of superordination-preserving integral operators," *Indag. Math., New Ser.*, vol. 13, no. 3, pp. 301–311, 2002, doi: 10.1016/S0019-3577(02)80013-1.

[2] B. C. Carlson and D. B. Shaffer, "Starlike and prestarlike hypergeometric functions." *SIAM J. Math. Anal.*, vol. 159, pp. 737–745, 1984, doi: 10.1137/051505.

[3] R. M. Goel and N. S. Sohi, "A new criterion for p-valent functions." *Proc. Amer. Math. Soc.*, vol. 78, pp. 353–357, 1980, doi: 10.2307/2042324.

[4] R. J. Libera, "Some classes of regular univalent functions." *Proc. Amer. Math. Soc.*, vol. 16, pp. 755–758, 1965, doi: 10.2307/2033917.

[5] J.-L. Liu and H. M. Srivastava, "A class of multivalently analytic functions associated with the Dziok-Srivastava operator." *Integral Transforms Spec. Funct.*, vol. 20, pp. 401–417, 2009, doi: 10.1080/10652460802576138.

[6] S. S. Miller and P. T. Mocanu, *Differential subordinations: theory and applications.* New York: Marcel Dekker, Inc., 2000.

[7] S. S. Miller and P. T. Mocanu, "Subordinants of differential superordinations." *Complex Var. Theory Appl.*, vol. 48, pp. 815–826, 2003, doi: 10.1080/02781070310001599322.

[8] S. S. Miller, P. T. Mocanu, and M. O. Reade, "Bazilevič functions and generalized convexity." *Rev. Roumaine Math. Pures Appl.*, vol. 19, pp. 213–224, 1974.

[9] S. S. Miller, P. T. Mocanu, and M. O. Reade, "Subordination-preserving integral operators." *Trans. Amer. Math. Soc.*, vol. 283, pp. 605–615, 1984, doi: 10.1007/BFb0066538.

[10] P. T. Mocanu, "Une propri*été* de convexit*é* g*é*n*é*ralis*é*e dans la th*é*orie de la repr*é*sentation conforme." *Mathematica(Cluj)*, vol. 11(34), pp. 127–133, 1969.

[11] S. Owa and H. M. Srivastava, "Some applications of the generalized Libera integral operator." *Proc. Japan Acad. Ser. A Math. Sci.*, vol. 62), pp. 125–128, 1986, doi: 10.3792/pjaa.62.125.

[12] S. Owa and H. M. Srivastava, "Some subordination theorems involving a certain family of integral operators." *Integral Transforms Spec. Funct.*, vol. 15), pp. 445–454, 2004, doi: 10.1080/10652460410001727563.

[13] C. Pommerenke, *Univalent Functions.* Vanderhoeck and Ruprecht: Göttingen, 1975.

[14] S. Ruscheweyh, "New criteria for univalent functions." *Proc. Amer. Math. Soc.*, vol. 49), pp. 109–115, 1975, doi: 10.2307/2039801.

[15] H. Saitoh, "A linear operator and its applications of first order differential subordinations." *Math. Japon.*, vol. 44), pp. 31–38, 1996.

[16] H. M. Srivastava, S. Hussain, A. Raziq, and M. Raza, "The Fekete-Szegö functional for a subclass of analytic functions associated with quasi-subordination." *Carpathian J. Math.*, vol. 34, pp. 103–113, 2018.

[17] H. M. Srivastava, A. O. Mostafa, M. K. Aouf, and H. M. Zayed, "Basic and fractional q-calculus and associated Fekete-Szegö problem for p-valently q-starlike functions and p-valently q-convex functions of complex order." *Miskolc Math. Notes*, vol. 20, pp. 489–509, 2019, doi: 10.18514/MMN.2019.2405.

[18] H. M. Srivastava and S. Owa, "Some characterizations and distortions theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions." *Nagoya Math. J.*, vol. 106), pp. 1–28, 1987.

[19] H. M. Srivastava and S. Owa, *Current Topics in Analytic Function Theory.* Singapore, New Jersey, London, and Hong Kong: World Scientific, 1992. doi: 10.1142/1628.

[20] H. M. Srivastava and J. Patel, "Some subclasses of multivalent functions involving a certain linear operator." *J. Math. Anal. Appl.*, vol. 310), pp. 209–228, 2005, doi: 10.1016/j.jmaa.2005.02.003.

[21] H. M. Srivastava, A. Prajapati, and P. Gochhayat, "Third-order differential subordination and differential superordination results for analytic functions involving the Srivastava-Attiya operator." *Appl. Math. Inform. Sci.*, vol. 12, pp. 469–481, 2018, doi: 10.18576/amis/120301.

[22] H. M. Srivastava, D. Răducanu, and P. Zaprawa, "A certain subclass of analytic functions defined by means of differential subordination." *Filomat*, vol. 30), pp. 3743–3757, 2016, doi: 10.2298/FIL1614743S.

[23] H. Tang, H. M. Srivastava, and G.-T. Deng, "Some families of analytic functions in the upper half-plane and their associated differential subordination and differential superordination properties and problems." *Appl. Math. Inform. Sci.*, vol. 11, pp. 1247–1257, 2017, doi: 10.18576/amis/110502.

[24] H. Tang, H. M. Srivastava, G.-T. Deng, and S.-H. Li, "Second-order differential superordination for analytic functions in the upper half-plane." *J. Nonlinear Sci. Appl.*, vol. 10, pp. 5271–5280, 2017, doi: 10.22436/jnsa.010.10.13.

[25] Q.-H. Xu, H.-G. Xiao, and H. M. Srivastava, "Some applications of differential subordination and the Dziok-Srivastava convolution operator." *Appl. Math. Comput.*, vol. 230, pp. 496–508, 2014, doi: 10.1016/j.amc.2013.12.065.

Authors' addresses

Seon Hye An

Pukyong National University, Department of Applied Mathematics, 45 Yongso-ro, 48513, Busan, Korea

E-mail address: ansonhye@gmail.com

Rekha Srivastava

University of Victoria, Department of Mathematics and Statistics, Victoria, British Columbia V8W 3R4, Canada

E-mail address: rekhas@math.uvic.ca

Nak Eun Cho

(**Corresponding author**) Pukyong National University, Department of Applied Mathematics, 45 Yongso-ro, 48513, Busan, Korea

E-mail address: necho@pknu.ac.kr

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 579–592

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4425

ON TWO AND THREE AND FOUR POINT BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL INCLUSIONS WITH $\phi-$LAPLACIAN

NAJIB ASKOURAYE AND MYELKEBIR AITALIOUBRAHIM

Received 26 September, 2022

Abstract. We show the existence of solutions satisfying two point, three point or four point boundary conditions, for the differential inclusion

$$(\phi(x'(t)))' \in F(t,x(t)),$$

where $F(\cdot,\cdot)$ is a compact lower semi-continuous multi-valued map and ϕ is a homeomorphism function.

2010 *Mathematics Subject Classification:* 34A60; 34B10; 34B15

Keywords: differential inclusion, boundary value problem, existence theorem

1. Introduction

In this paper, we consider the following second-order non linear differential inclusion problems:

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t,x(t)) \text{ a.e. on } [0,T]; \\ x(0) = x'(\eta), \end{cases} \tag{1.1}$$

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t,x(t)) \text{ a.e. on } [0,T]; \\ x(0) = 0,\ x(T) = x(\eta) \end{cases} \tag{1.2}$$

and

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t,x(t)) \text{ a.e. on } [0,T]; \\ x(0) = x'(\eta),\ x(\tau) = x(T) \end{cases} \tag{1.3}$$

where $F : [0,T] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact multi-valued map, measurable and lower semi-continuous with respect to the second argument, $\phi :]-a,a[\to \mathbb{R}$ is a homeomorphism, $T > 0$ and $(\eta,\tau) \in]0,T[^2$.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

This kind of problems has been studied by several authors in the last few years. Sun, Yang and Ge [14], have proved the existence of a positive concave solution of the following two classes of three point p-Laplacian boundary value problems

$$\begin{aligned}
&\big(\phi_p(u'(t))\big)' + q(t)f(t,u(t),u'(t)) = 0, \quad t \in]0,1[,\\
&\big(\phi_p(u'(t))\big)' + q(t)f(t,u(t)) = 0, \quad t \in]0,1[,\\
&u(0) = 0,\ u(T) = u(\eta),
\end{aligned}$$

where f is continuous, $q(.)$ is a nonnegative continuous function defined on $]0,1[$, $\phi_p(s) = |s|^{p-2}s$, with $p > 1$, is a p-Laplacian operator, $\phi_p^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$. We recall here that the ϕ-Laplacian operator generalizes the p-Laplacian operator.

Ayyadi, El Khattabi and Frigon, see [11], have applied the method of upper and lower solutions, combined with the topological degree, to prove the existence of solutions for the problem $\big(\phi(x'(t))\big)' = f(t,x(t),x'(t))$, satisfying periodic, Dirichlet or Neumann boundary conditions. The right-hand side is a Carathéodory function satisfying a growth condition of Wintner-Nagumo type.

Chinní, Di Bella, Jebelean and Precup, see [8], proved the existence of solutions, satisfying four point value problem, for the differential equation

$$-\big(\phi(u'(t))\big)' = f(t,u(t),u'(t)),\ u(0) = \alpha u(\xi),\ u(T) = \beta u(\eta),$$

where $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\alpha, \beta \in [0,1[$, $0 < \xi < \eta < T$ and ϕ is an increasing homeomorphism such that $\phi(0) = 0$. Note that, if we take $\alpha = 0$ and $\beta = 1$ in the last problem, we obtain the same boundary conditions of problem (1.2).

The problems (1.2) and (1.3) are solved by Benchohra and Ntouyas in the case where $T = 1$, $\phi = id$ (id denotes identity mapping) and F is a multi-valued map. Their work is based on a fixed point theorem for contraction multi-valued maps due to Covitz and Nadler. For more details, see [5].

In [2,3], Aitalioubrahim and Sajid solved the problems (1.1), (1.2) and (1.3) in the case where $T = 1$, $\phi = id$ and F is a compact convex L^1-Carathéodory multifunction or is a closed multifunction, measurable in t and Lipschitz continuous in x.

Recently, Aitalioubrahim and Tebbaa [4] have applied the method of upper and lower solutions, combined with the topological degree, to prove the existence of solutions for the problem $\big(\phi(x'(t))\big)' \in F(t,x(t))$, satisfying periodic, Dirichlet boundary conditions. The set valued-map $F(\cdot,\cdot)$ is compact and lower semi-continuous. The function ϕ is a homeomorphism. For a review of results on this kind of boundary value problems for differential inclusions, we refer the reader to the references [1,6,10].

In this paper, we establish the existence results for the problems (1.1) , (1.2) and (1.3). Our approach is based on the topological degree. We study the case where the right hand side is lower semicontinuous. The main results of this work extend, to the multi-valued case, some existence results in [6,11,14] and in the literature related to this kind of problems.

2. Preliminaries and statement of the main results

In this section, we introduce notations, definitions, and preliminary facts from multi-valued analysis which are used throughout this paper. Let E be a Banach space equipped with the norm $\|.\|$. By $\mathcal{C}([0,T],E)$ we denote the Banach space of all continuous functions from $[0,T]$ into E equipped with the norm

$$\|x\|_\infty := \sup\left\{\|x(t)\|; t \in [0,T]\right\}.$$

$\mathcal{C}^1([0,T],E)$ denotes the Banach space of continuously differentiable functions on $[0,T]$ equipped with the norm

$$\|x\|_1 := \|x\|_\infty + \|x'\|_\infty.$$

The set $L^1([0,T],\mathbb{R})$ refers to the Banach space of Lebesgue integrable functions from $[0,T]$ into $\mathbb{R}$ equipped with the norm

$$\|x\|_{L^1} = \int_0^T |x(s)|ds.$$

We say that a subset A of $[0,T]\times\mathbb{R}$ is $\mathcal{L}\otimes\mathcal{B}$ -measurable if A belongs to the σ-algebra generated by all sets of the form $I\times D$ where I is Lebesgue measurable in $[0,T]$ and D is Borel measurable in $\mathbb{R}$. A multifunction is said to be measurable if its graph is measurable.

Definition 1. A subset B of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u,v \in B$ and $I \subset [0,T]$ measurable, the function $u\chi_I + v\chi_{[0,T]\setminus I} \in B$, where χ denotes the characteristic function.

Definition 2. Let E be a separable Banach space, X a nonempty closed subset of E and $G : X \to 2^E$ a multi-valued map with nonempty closed values. We say that G is lower semi-continuous if the set $\{x \in X : G(x)\cap C \neq \varnothing\}$ is open for any open set C in E.

Definition 3. Let $F : [0,T]\times\mathbb{R} \to 2^{\mathbb{R}}$ be a multi-valued map with nonempty compact values. Assign to F the multi-valued operator,

$$\mathcal{F} : \mathcal{C}([0,T],\mathbb{R}) \to 2^{L^1([0,T],\mathbb{R})},$$

defined by

$$\mathcal{F}(x) = \left\{ y \in L^1([0,T],\mathbb{R}) : y(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,T] \right\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F. We say that F is of the lower semi-continuous type if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

For multi-valued operator which is lower semi-continuous and has nonempty closed and decomposable values, we have the following well-known result.

Lemma 1 ([7]). *Let E be a separable metric space and let $\Gamma : E \to 2^{L^1([0,T],\mathbb{R})}$ be a multi-valued operator which is lower semi-continuous and has nonempty closed and decomposable values. Then Γ has a continuous selection, i.e. there exists a continuous function $g : E \to L^1([0,T],\mathbb{R})$ such that $g(y) \in \Gamma(y)$ for every $y \in E$.*

Now, we give the definition of compact and completely continuous function.

Definition 4. Let X and Y be topological spaces.

(1) A function $f : X \to Y$ is said compact if $f(X)$ is relatively compact.
(2) If X is a metric space, $f : X \to Y$ is called completely continuous if for any bounded subset $B \subset X$, $f(B)$ is relatively compact.

The notion of the topological degree is widely used in the sequel. Here we give its definition and some related results. For more details on topological degree, we refer the reader to [9, 13].

Definition 5. Let E be a real Banach space and id be the identity on E. A degree is an application, which associates to any open bounded set $U \subset E$ and any continuous compact function $f : \overline{U} \to E$, an integer $\deg(id - f, U)$ with the following properties:

(i) (Existence): If $0 \notin (id - f)(\partial U)$, where ∂U is the boundary of U, and $\deg(id - f, U) \neq 0$, then there exists $x \in U$ such that $x = f(x)$.
(ii) (Normalisation): If $0 \notin \partial U$, then $\deg(id, U) = 1$ if and only if $0 \in U$.
(iii) (Additivity): If $0 \notin (id - f)(\overline{U} \setminus U_1 \cup U_2)$, where U_1 and U_2 are disjoint open subsets of U and $\overline{U} \setminus U_1 \cup U_2$ is the relative complement of $U_1 \cup U_2$ in $\overline{U}$, then
$$\deg(id - f, U) = \deg(id - f, U_1) + \deg(id - f, U_2).$$
(iv) (Homotopy): If $H : [0,1] \times \overline{U} \to E$ is a continuous compact function such that $0 \notin (id - H(\lambda, \cdot))(\partial U)$, for every $\lambda \in [0,1]$, then
$$\deg(id - H(\lambda, \cdot), U) = \deg(id - H(0, \cdot), U), \quad \forall \lambda \in [0,1].$$
(v) (Excision): If $V \subset U$ is open and $0 \notin (id - f)(\overline{U} \setminus V)$, then
$$\deg(id - f, U) = \deg(id - f, V).$$

In this work, we will assume the following assumptions.

(H1) $F : [0,T] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a set-valued map with nonempty compact values satisfying
  (i) $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}-$ measurable,
  (ii) $x \mapsto F(t,x)$ is lower semi-continuous for almost all $t \in [0,T]$.
(H2) There exists a function $m \in L^1([0,T],\mathbb{R}^+)$ such that for almost all $t \in [0,T]$ and all $x \in \mathbb{R}$
$$\|F(t,x)\| := \sup\Big\{|y| : y \in F(t,x)\Big\} \leq m(t).$$
(H3) $\phi :]-a,a[\to \mathbb{R}$, $0 < a \leq +\infty$, is an increasing homeomorphism such that $\phi(0) = 0$.

We denote by $W_a^{2,1}([0,T],\mathbb{R})$ the class of all functions $x \in C^1([0,T],\mathbb{R})$ such that $\|x'\|_\infty < a$, $\phi(x')$ is absolutely continuous and $(\phi(x'))' \in L^1([0,T],\mathbb{R})$.

Definition 6. A function $x : [0,T] \to \mathbb{R}$ is said to be solution of (1.1) (resp. (1.2) and (1.3)) if $x \in W_a^{2,1}([0,T],\mathbb{R})$ and x satisfies the conditions of (1.1) (resp. (1.2) and (1.3)).

In the sequel, we will use the following important Lemmas.

Lemma 2 ([12])**.** *If assumptions (H1)-(H2) are satisfied, then F is of the lower semi-continuous type.*

Lemma 3. *Assume that (H3) holds. Let $\rho \in \{\tau,\eta\}$ and*

$$K : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R}),\ Kv(t) = \int_0^t v(s)\mathrm{d}s, \quad \forall t \in [0,T]$$

be a linear operator. For any $c \in \mathbb{R}$ and $v \in C([0,T],\mathbb{R})$, we set

$$\omega(c,v;t) = \phi^{-1}(c + Kv(t)), \quad \forall t \in [0,T] \text{ and } E_\rho(c,v) = \int_\rho^T \omega(c,v;t)\mathrm{d}t.$$

For each $v \in C([0,T],\mathbb{R})$, there exists an unique $Q(v) \in \mathbb{R}$, such that

$$E_\rho(Q(v),v) = 0.$$

Moreover, the map $Q : C([0,T],\mathbb{R}) \to \mathbb{R}$ is continuous and satisfies the following

$$|Q(v)| \le T\|v\|_\infty, \quad \forall v \in C([0,T],\mathbb{R}). \tag{2.1}$$

Proof. We use the same proof of Lemma 2.1 in [8]. From $|Kv(t)| \le T\|v\|_\infty$ and (H_3), we get

$$\omega(-T\|v\|_\infty, v;t) \le 0 \le \omega(T\|v\|_\infty, v;t), \quad \forall t \in [0,T],$$

which yields

$$E_\rho(-T\|v\|_\infty, v) \le 0 \le E_\rho(T\|v\|_\infty, v). \tag{2.2}$$

Since $\omega(\cdot,v;\cdot) : \mathbb{R} \times [0,T] \to \mathbb{R}$ is continuous, using Lebesgue's dominated convergence theorem, it is easy to see that, for any $0 \le b_1 < b_2 \le T$, the mapping

$$c \mapsto \int_{b_1}^{b_2} \omega(c,v;t)\mathrm{d}t$$

is continuous too. This implies that $E_\rho(\cdot,v)$ is continuous on $\mathbb{R}$. Then, the existence of some $Q(v)$ with $E_\rho(Q(v),v) = 0$ and satisfying (2.1). For the uniqueness of $Q(v)$, by the monotonicity of ϕ^{-1}, for any $c_1 < c_2$, one has

$$\omega(c_1,v;t) < \omega(c_2,v;t), \quad \forall t \in [0,T].$$

So $E_\rho(\cdot,v)$ is strictly increasing on $\mathbb{R}$. To see that Q is continuous, let $\{v_n\}_n$ be a sequence, in $C([0,T],\mathbb{R})$, convergent to some $v \in C([0,T],\mathbb{R})$. We want to show that $Q(v_n) \to Q(v)$, as $n \to \infty$. We have

$$\int_\rho^T \phi^{-1}(Q(v_n)+Kv_n(t))\mathrm{d}t = 0.$$

Then, there exists $t_0 \in [\rho,T]$ such that

$$\phi^{-1}(Q(v_n)+Kv_n(t_0)) = 0.$$

We deduce from the last equality that $\{Q(v_n)\}_n$ converges to some $q \in \mathbb{R}$. It remains to show that $q = Q(v)$. Since

$$Q(v_n)+Kv_n \to q+Kv \text{ in } C([0,T],\mathbb{R}), \text{ as } n \to \infty,$$

we infer that $\left\{\phi^{-1}(Q(v_n)+Kv_n)\right\}_n$ is bounded in $C([0,T],\mathbb{R})$ and

$$\omega(Q(v_n),v_n;t) = \phi^{-1}(Q(v_n)+Kv_n(t)) \to \phi^{-1}(q+Kv(t)) = \omega(q,v;t),$$

for all $t \in [0,T]$. Then, using Lebesgue's dominated convergence theorem, we deduce the following

$$0 = E_\rho(Q(v_n),v_n) \to E_\rho(q,v) \text{ as } k \to \infty.$$

Then $E_\rho(q,v) = 0$. However, by the uniqueness part, this means that $q = Q(v)$ and the proof is complete. □

Now, we are ready to give the main results of this paper.

Theorem 1. *If assumptions (H1), (H2) and (H3) are satisfied, then the problem (1.1) has at least one solution x on $[0,T]$.*

Theorem 2. *If assumptions (H1), (H2) and (H3) are satisfied, then the problem (1.2) has at least one solution x on $[0,T]$.*

Theorem 3. *If assumptions (H1), (H2) and (H3) are satisfied, then the problem (1.3) has at least one solution x on $[0,T]$.*

3. Proof of Theorem 1

By Lemma 2, F is of the lower semi-continuous type. Then, by Lemma 1, there exists a continuous function $g : \mathcal{C}([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $g(y) \in \mathcal{F}(y)$ for all $y \in \mathcal{C}([0,T],\mathbb{R})$, where $\mathcal{F}$ is the Niemytzki operator associated with F. Consider the problem

$$\begin{cases} (\phi(y'(t)))' = g(y)(t) \text{ a.e. on } [0,T]; \\ y(0) = y'(\eta). \end{cases} \tag{3.1}$$

Remark that, any solution of the problem (3.1) is a solution of the problem (1.1). Now, for all $\lambda \in [0,1]$, consider the following modified problem

$$\begin{cases} (\phi(y'(t)))' = \lambda g(y)(t) \text{ a.e. on } [0,T]; \\ y(0) = y'(\eta). \end{cases} \tag{3.2}$$

Let us consider the operators

$$N_g : \mathcal{C}([0,T],\mathbb{R}) \to \mathcal{C}([0,T],\mathbb{R})$$

and

$$\mathcal{H} : [0,1] \times \mathcal{C}([0,T],\mathbb{R}) \to \mathcal{C}([0,T],\mathbb{R}),$$

defined, for all $t \in [0,T]$, by

$$N_g(u)(t) = \int_0^t g(u)(s)ds$$

and

$$\mathcal{H}(\lambda,u)(t) = \phi^{-1}\big(\lambda N_g(u)(\eta)\big) + \int_0^t \phi^{-1}\big(\lambda N_g(u)(s)\big)ds.$$

From the assumptions (H1), (H2) and (H3), the function $\mathcal{H}$ is continuous and completely continuous. Also, observe that, the fixed points of $\mathcal{H}(\lambda,.)$ are solutions of (3.2). Indeed, if $u = \mathcal{H}(\lambda,u)$, then, for all $t \in [0,T]$, we have

$$u(t) = \phi^{-1}\big(\lambda N_g(u)(\eta)\big) + \int_0^t \phi^{-1}\big(\lambda N_g(u)(s)\big)ds.$$

Thus, in particular for $t = 0$, one has $u(0) = \phi^{-1}\big(\lambda N_g(u)(\eta)\big)$. Also by derivation, we get

$$u'(t) = \phi^{-1}\big(\lambda N_g(u)(t)\big) \text{ for almost all } t \in [0,T],$$

then $u'(\eta) = \phi^{-1}\big(\lambda N_g(u)(\eta)\big) = u(0)$. In addition, we have

$$\phi(u'(t)) = \lambda N_g(u)(t) \text{ for almost all } t \in [0,T],$$

which implies

$$(\phi(u'(t))' = \lambda g(u)(t) \text{ for almost all } t \in [0,T].$$

Hence u is a solution of (3.2).

Proposition 1. *Assume that (H1), (H2) and (H3) hold. Then, there exists $R > 0$ such that $deg(id - \mathcal{H}(\lambda,\cdot),\mathcal{U}) = 1$ for every $\lambda \in [0,1]$, where $\mathcal{U} = \{u \in \mathcal{C}([0,T],\mathbb{R}) : \|u\|_\infty < R\}$. In particular, the problem (3.2) has at least one solution for every $\lambda \in [0,1]$.*

Proof. Fix $R > a + Ta$. First note that $\|u\|_\infty < R$ for any fixed point u of $\mathcal{H}$. Indeed, let u be a fixed point of $\mathcal{H}$. Then

$$u(t) = \phi^{-1}\big(\lambda N_g(u)(\eta)\big) + \int_0^t \phi^{-1}\big(\lambda N_g(u)(s)\big)ds, \text{ for all } t \in [0,T].$$

which implies that $\|u\|_\infty \leq a + Ta < R$. Now, set

$$\mathcal{U} = \{u \in \mathcal{C}([0,T],\mathbb{R}) : \|u\|_\infty < R\}.$$

Since

$$u \neq \mathcal{H}(\lambda, u), \quad \forall (\lambda, u) \in [0,1] \times \partial\mathcal{U},$$

by the homotopy property of the topological degree, we get

$$\deg(id - \mathcal{H}(\lambda,\cdot), \mathcal{U}) = \deg(id - \mathcal{H}(0,\cdot), \mathcal{U}), \quad \forall \lambda \in [0,1],$$

which gives

$$\deg(id - \mathcal{H}(\lambda,\cdot), \mathcal{U}) = \deg(id, \mathcal{U}) = 1, \quad \forall \lambda \in [0,1].$$

Therefore, for every $\lambda \in [0,1]$, $\mathcal{H}(\lambda,\cdot)$ has a fixed point, and hence (3.2) has a solution. □

Now, Proposition 1 assures the existence of a solution $u \in W_a^{2,1}\big([0,T],\mathbb{R}\big)$ of (3.2) for $\lambda = 1$. Hence the problem (1.1) has at least one solution x on $[0,T]$.

4. Proof of Theorem 2

In order to establish the existence of a solution to (1.2), we consider the following family of problems defined, for $\lambda \in [0,1]$, by

$$\begin{cases} (\phi(u'(t)))' = \lambda g(u)(t) \text{ for a.e. } t \in [0,T], \\ u(0) = 0,\ u(T) = u(\eta), \end{cases} \tag{4.1}$$

where $g : \mathcal{C}([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ is defined in the previous section. We are going to transform the problem (4.1) to a fixed point problem. We consider the operator

$$\mathcal{D} : [0,1] \times \mathcal{C}([0,T],\mathbb{R}) \to \mathcal{C}([0,T],\mathbb{R})$$

defined by

$$\mathcal{D}(\lambda,u)(t) = \int_0^t \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds, \tag{4.2}$$

with $Q_{\lambda,u}$ is uniquely determined by Lemma 3 and satisfies the following

$$\int_\eta^T \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds = 0. \tag{4.3}$$

Note that, by taking $\lambda = 0$ in (4.3), we get

$$Q_{0,u} = 0, \quad \forall u \in \mathcal{C}([0,T],\mathbb{R}). \tag{4.4}$$

It follows from the assumptions (H1), (H2) and (H3) that $\mathcal{D}$ is continuous and completely continuous. On the other hand, if $u = \mathcal{D}(\lambda,u)$, we have

$$u(t) = \int_0^t \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds, \quad \text{for all} \quad t \in [0,T].$$

Hence $u(0)=0$. For $t=T$ and $t=\eta$, we have

$$u(T)=\int_0^T \phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(s)\big)ds \quad \text{and} \quad u(\eta)=\int_0^\eta \phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(s)\big)ds.$$

So

$$u(T)-u(\eta)=\int_0^T \phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(s)\big)ds-\int_0^\eta \phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(s)\big)ds.$$

Then, by (4.3), one has

$$u(T)-u(\eta)=\int_\eta^T \phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(s)\big)ds=0.$$

Hence $u(T)=u(\eta)$. Also, we get

$$u'(t)=\phi^{-1}\big(Q_{\lambda,u}+\lambda N_g(u)(t)\big), \quad \text{for almost all} \quad t\in[0,T].$$

So

$$(\phi(u'(t)))'=\lambda g(u)(t), \quad \text{for almost all} \quad t\in[0,T].$$

It follows that the fixed points of $\mathcal{D}(\lambda,.)$ are solutions of (4.1).

Proposition 2. *Assume that (H1), (H2) and (H3) hold. Then, there exists $R>0$ such that $deg(id-\mathcal{D}(\lambda,\cdot),\mathcal{U})=1$ for every $\lambda\in[0,1]$, where $\mathcal{U}=\{u\in\mathcal{C}([0,T],\mathbb{R}):\|u\|_\infty<R\}$. In particular, (4.1) has at least one solution for every $\lambda\in[0,1]$.*

Proof. Fix $R>aT$. It is clair that $\|u\|_\infty<R$ for any fixed point u of $\mathcal{D}$. Set

$$\mathcal{U}=\{u\in\mathcal{C}([0,T],\mathbb{R}):\|u\|_\infty<R\}.$$

Then, one has

$$u\neq\mathcal{D}(\lambda,u), \quad \forall(\lambda,u)\in[0,1]\times\partial\mathcal{U}.$$

Thus, by the homotopy property of the topological degree,

$$\deg(id-\mathcal{D}(\lambda,\cdot),\mathcal{U})=\deg(id-\mathcal{D}(0,\cdot),\mathcal{U}), \quad \forall\lambda\in[0,1].$$

Using (4.4), we get

$$\deg(id-\mathcal{D}(\lambda,\cdot),\mathcal{U})=\deg(id,\mathcal{U})=1, \quad \forall\lambda\in[0,1].$$

We conclude that, for every $\lambda\in[0,1]$, $D(\lambda,\cdot)$ has a fixed point, and hence (4.1) has a solution. □

Therefore, we showed the existence of a solution $u\in W_a^{2,1}\big([0,T],\mathbb{R}\big)$ of (4.1) for $\lambda=1$. Hence the problem (1.2) has at least one solution x on $[0,T]$.

5. Proof of Theorem 3

We consider the following family of problems defined, for $\lambda \in [0,1]$, by

$$\begin{cases} (\phi(u'(t)))' = \lambda g(u)(t) \text{ for a.e. } t \in [0,T], \\ u(0) = u'(\eta),\ u(\tau) = u(T). \end{cases} \tag{5.1}$$

Let us transform the problem (5.1) to a fixed point problem. We consider the operator

$$\mathcal{P} : [0,1] \times \mathcal{C}([0,T],\mathbb{R}) \to \mathcal{C}([0,T],\mathbb{R})$$

defined by

$$\mathcal{P}(\lambda,u)(t) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big) + \int_0^t \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds, \tag{5.2}$$

with $Q_{\lambda,u}$ is uniquely determined by Lemma 3 and satisfies the following

$$\int_\tau^T \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds = 0. \tag{5.3}$$

Taking $\lambda = 0$ in (5.3), we get

$$Q_{0,u} = 0, \quad \forall u \in \mathcal{C}([0,T],\mathbb{R}). \tag{5.4}$$

It follows from the assumptions (H1), (H2) and (H3) that $\mathcal{P}$ is continuous and completely continuous. On the other hand, if $u = \mathcal{P}(\lambda,u)$, we have

$$u(t) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big) + \int_0^t \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds, \quad \text{for all} \quad t \in [0,T].$$

Then for $t = 0$, we have $u(0) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big)$, and for $t = T$ and $t = \tau$, we get

$$u(T) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big) + \int_0^T \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds$$

and

$$u(\tau) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big) + \int_0^\tau \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds.$$

So

$$u(T) - u(\tau) = \int_0^T \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds - \int_0^\tau \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds.$$

Then, by (5.3),

$$u(T) - u(\tau) = \int_\tau^T \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(s)\big)ds = 0.$$

Hence $u(T) = u(\tau)$. On the other hand, we have

$$u'(t) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(t)\big), \quad \text{for almost all} \quad t \in [0,T].$$

So

$$(\phi(u'(t)))' = \lambda g(u)(t), \quad \text{for almost all} \quad t \in [0,T].$$

In addition, we have $u'(\eta) = \phi^{-1}\big(Q_{\lambda,u} + \lambda N_g(u)(\eta)\big) = u(0)$. It follows that the fixed points of $\mathcal{P}(\lambda,.)$ are solutions of (5.1).

Proposition 3. *Assume that (H1), (H2) and (H3) hold. Then, there exists $R > 0$ such that $deg(id - \mathcal{P}(\lambda,\cdot), \mathcal{U}) = 1$ for every $\lambda \in [0,1]$, where $\mathcal{U} = \{u \in \mathcal{C}([0,T],\mathbb{R}) : \|u\|_\infty < R\}$. In particular, (5.1) has at least one solution for every $\lambda \in [0,1]$.*

Proof. Fix $R > a + aT$. Set

$$\mathcal{U} = \{u \in \mathcal{C}([0,T],\mathbb{R}) : \|u\|_\infty < R\}.$$

We have $\|u\|_\infty < R$ for any fixed point u of $\mathcal{P}$. Then

$$u \neq \mathcal{P}(\lambda,u), \quad \forall(\lambda,u) \in [0,1] \times \partial\mathcal{U}.$$

By the homotopy property of the topological degree,

$$\deg(id - \mathcal{P}(\lambda,\cdot), \mathcal{U}) = \deg(id - \mathcal{P}(0,\cdot), \mathcal{U}), \quad \forall\lambda \in [0,1].$$

Using (5.4), we get

$$\deg(id - \mathcal{P}(\lambda,\cdot), \mathcal{U}) = \deg(id, \mathcal{U}) = 1, \quad \forall\lambda \in [0,1].$$

We conclude that, for every $\lambda \in [0,1]$, $P(\lambda,\cdot)$ has a fixed point, and hence (5.1) has a solution. □

Therefore, we showed the existence of a solution $u \in W_a^{2,1}\big([0,T],\mathbb{R}\big)$ of (5.1) for $\lambda = 1$. Hence the problem (1.3) has at least one solution x on $[0,T]$.

6. Illustrative example

Consider the ϕ-Laplacian differential inclusions with two, three and four boundary conditions

$$\begin{cases} \big(x'(t)^3\big)' \in F(t,x(t)) \text{ a.e. on } [0,4\pi]; \\ x(0) = x'(\frac{\pi}{2}), \end{cases} \tag{6.1}$$

$$\begin{cases} \big(x'(t)^3\big)' \in F(t,x(t)) \text{ a.e. on } [0,4\pi]; \\ x(0) = 0,\ x(4\pi) = x(2\pi) \end{cases} \tag{6.2}$$

and

$$\begin{cases} \big(x'(t)^3\big)' \in F(t,x(t)) \text{ a.e. on } [0,4\pi]; \\ x(0) = x'(\frac{\pi}{2}),\ x(2\pi) = x(4\pi), \end{cases} \tag{6.3}$$

where $F : [0,4\pi] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multi-valued map defined by

$$F(t,x) = \Big\{v \in \mathbb{R} : f_1(t,x) \leq v \leq f_2(t,x)\Big\}, \quad \forall t \in [0,4\pi],$$

and $f_1, f_2 : [0,4\pi] \times \mathbb{R} \to \mathbb{R}$ are single-valued functions such that, for each $t \in [0,4\pi]$,

$$f_1(t,x) = \frac{-3(1+3\sin(t))\cos^2(t)e^{(\ln(3+\sin(t))-\ln(3))^2}}{(3+\sin(t))^4 e^{x^2}} - 1,$$

$$f_2(t,x) = \frac{-3(1+3\sin(t))\cos^2(t)e^{(\ln(3+\sin(t))-\ln(3))^2}}{(3+\sin(t))^4 e^{x^2}} + 1.$$

It is clear that F has nonempty compact values and is measurable. For each $t \in [0,4\pi]$, $f_1(t,.)$ and $f_2(t,.)$ are continuous on $\mathbb{R}$. Then, for each $t \in [0,4\pi]$, $f_1(t,.)$ is upper semi-continuous on $\mathbb{R}$ and $f_2(t,.)$ is lower semi-continuous on $\mathbb{R}$. Hence, for each $t \in [0,4\pi]$, $F(t,.)$ is lower semi-continuous on $\mathbb{R}$. Moreover, for almost all $t \in [0,4\pi]$ and all $x \in \mathbb{R}$

$$\begin{aligned}
\|F(t,x)\| &= \sup\left\{|y| : y \in [f_1(t,x), f_2(t,x)]\right\} \\
&\leq max\{|f_1(t,x)|, |f_2(t,x)|\} \\
&\leq \left|\frac{-3(1+3\sin(t))\cos^2(t)e^{(\ln(3+\sin(t))-\ln(3))^2}}{(3+\sin(t))^4 e^{x^2}}\right| + 1 \\
&\leq \frac{3}{4}e^{(\ln(3+\sin(t))-\ln(3))^2} + 1.
\end{aligned}$$

Set

$$m(t) = \frac{3}{4}e^{(\ln(3+\sin(t))-\ln(3))^2} + 1, \quad \forall t \in [0,4\pi].$$

Then, for almost all $t \in [0,4\pi]$ and all $x \in \mathbb{R}$

$$\|F(t,x)\| \leq m(t),$$

with $m \in L^1([0,4\pi],\mathbb{R}^+)$. In this example $\phi(x) = x^3$. It is clear that ϕ is an increasing homeomorphism such that $\phi(0) = 0$. We conclude that all assumptions of Theorems 1, 2 and 3 are verified. Thus Problem (6.1) (resp. (6.2) and (6.3)) has at least one solution on $[0,4\pi]$. Set

$$u(t) = \ln\left(\frac{3+\sin(t)}{3}\right), \quad \forall t \in [0,4\pi].$$

u is a solution of Problem (6.1). Indeed, for almost all $t \in [0,4\pi]$,

$$u'(t) = \frac{\cos(t)}{3+\sin(t)} \text{ and } u''(t) = \frac{-1-3\sin(t)}{(3+\sin(t))^2}.$$

Then, for almost all $t \in [0,4\pi]$,

$$\begin{aligned}
\left(\phi(u'(t))\right)' &= \left(u'(t)^3\right)' \\
&= 3u''(t)u'(t)^2
\end{aligned}$$

$$
\begin{aligned}
&= 3 \times \frac{-1-3\sin(t)}{(3+\sin(t))^2} \times \frac{\cos^2(t)}{(3+\sin(t))^2} \\
&= \frac{-3(1+3\sin(t))\cos^2(t)}{(3+\sin(t))^4} \\
&= \frac{-3(1+3\sin(t))\cos^2(t)e^{(\ln(3+\sin(t))-\ln(3))^2}}{(3+\sin(t))^4 e^{(\ln(3+\sin(t))-\ln(3))^2}} \\
&= \frac{-3(1+3\sin(t))\cos^2(t)e^{(\ln(3+\sin(t))-\ln(3))^2}}{(3+\sin(t))^4 e^{u^2(t)}}.
\end{aligned}
$$

Hence, for almost all $t \in [0, 4\pi]$

$$\big(\phi(u'(t))\big)' \in [f_1(t,u(t)), f_2(t,u(t))] = F(t,u(t)).$$

In addition $u(0) = u'(\frac{\pi}{2}) = 0$. On the other hand, we have $u(0) = 0$ and $u(4\pi) = u(2\pi) = 0$. Then u is a solution of Problem (6.2). Also, we have $u(0) = u'(\frac{\pi}{2}) = 0$ and $u(2\pi) = u(4\pi)$. Then u is a solution of Problem (6.3).

ACKNOWLEDGEMENTS

The authors would like to thank the referee for his careful and thorough reading of the paper.

REFERENCES

[1] M. Aitalioubrahim and N. Ayyadi, "Existence of solutions for lower semi-continuous differential inclusions with the method of upper and lower solutions." *Nonlinear stud.*, vol. 25, no. 2, pp. 333–343, 2018.

[2] M. Aitalioubrahim and S. Sajid, "Higher-order boundary value problems for Carathéodory differential inclusions." *Miskolc Math. Notes*, vol. 9, no. 1, pp. 7–15, 2008, doi: 10.18514/MMN.2008.180.

[3] M. Aitalioubrahim and S. Sajid, "On boundary-value problems for higher-order differential inclusions." *Electron. J. Differential Equations*, vol. 2008, no. 62, pp. 1–11, 2008.

[4] M. Aitalioubrahim and A. Tebbaa, "Boundary value problems for differential inclusions with ϕ-Laplacian." *J. Dyn. Control Syst.*, vol. 28, no. 3, pp. 505–516, 2022, doi: 10.1007/s10883-021-09544-3.

[5] M. Benchohra and S. K. Ntouyas, "On three and four point boundary value problems for second order differential inclusions." *Miskolc Math. Notes*, vol. 2, no. 2, pp. 93–101, 2001, doi: 10.18514/MMN.2001.40.

[6] C. Bereanu and J. Mawhin, "Existence and multiplicity results for some nonlinear problems with singular $\phi-$Laplacian." *J. Differential Equations*, vol. 243, no. 2, pp. 536–557, 2007.

[7] A. Bressan and G. Colombo, "Extensions and selections of maps with decomposable values." *Studia Math.*, vol. 90, no. 1, pp. 69–86, 1988, doi: 10.4064/sm-90-1-69-86.

[8] A. Chinní, B. D. Bella, P. Jebelean, and R. Precup, "A four point boundary value problem with singular ϕ-Laplacian." *J. Fixed Point Theory Appl.*, vol. 21, no. 2, pp. 1–16, 2019, doi: 10.1007/s11784-019-0703-1.

[9] Y. J. Cho and Y.-Q. Chen, *Topological degree theory and applications.* New York: Chapman and Hall/CRC, 2006. doi: 10.1201/9781420011487.

[10] S. Djebali and A. Ouahab, "Existence results for ϕ-Laplacian Dirichlet BVP of differential inclusions with application to control theory." *Discuss. Math. Differ. Incl. Control Optim.*, vol. 30, no. 1, pp. 23–49, 2010.

[11] M. Frigon, N. El Khattabi, and N. Ayyadi, "Multiple solutions of boundary value problems with $\phi-$Laplacian operators and under a Wintner-Nagumo growth condition." *Bound. Value Probl.*, vol. 236, pp. 1–21, 2013, doi: 10.1186/1687-2770-2013-236.

[12] M. Frigon and A. Granas, "Théorèmes d'existence pour des inclusions différentielles sans convexité." *C. R. Acad. Sci.*, vol. Ser. I, no. 310, pp. 819–822, 1990.

[13] A. Granas and J. Dugundji, *Fixed Point Theory.* New York: Springer, 2003. doi: 10.1007/978-0-387-21593-8.

[14] B. Sun, A. Yang, and W. Ge, "Successive iteration and positive solutions for some second-order three point p-Laplacian boundary value problems." *Mathematical and Computer Modelling*, vol. 50, no. 3-4, pp. 344–350, 2009, doi: 10.1016/j.mcm.2008.08.024.

Authors' addresses

Najib Askouraye
University Sultan Moulay Slimane, Faculty Polydisciplinary, BP 145 Khouribga, Morocco
E-mail address: najib.ask22@gmail.com

Myelkebir Aitalioubrahim
(**Corresponding author**) University Sultan Moulay Slimane, Faculty Polydisciplinary, BP 145 Khouribga, Morocco
E-mail address: aitalifr@hotmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 593–604

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4545

EXISTENCE OF SOLUTIONS FOR A CLASS OF BVPS ON THE HALF-LINE

SOUAD AYADI AND OZGUR EGE

Received 24 January, 2023

Abstract. In this paper, we establish sufficient conditions to guarantee some existence results of nontrivial solutions for a class of nonlinear boundary value problem posed on an unbounded interval of $\mathbb{R}$. According to the behavior of the nonlinear term, an effective operator is considered. Our main tool is especially Schauder's fixed point theorem and the compactness criterion of the Sobolev space $H^1(0,\infty)$ into $C_{l,p}([0,+\infty))$.

2010 *Mathematics Subject Classification:* 47H10; 46E35; 35J25; 46C15

Keywords: fixed point theorem, compact embedding, BVP, Hilbert space

1. Introduction and preliminaries

The existence of solutions of boundary value problems has been studied widely in engineering and other related disciplines. To obtain existence results on the half line is not an easy task and many authors have worked on this subject [2,4]. Fixed point theory is one of the used tools to overcome the difficulty of proving the existence of solutions on unbounded domains of $\mathbb{R}$. Many researchers have concentred there work in this sense. For new trends on boundary value problems and theory of fixed point theorems, we refer the reader to [8–11]. In this area, there is an important activity among a wide number of investigators. Using available fixed points theorems in Banach space such as Leray-Schauder fixed point theorem and fixed point theorems in cone, authors have proven some existence results in [1,12,14]. In this work, we are concerned by some class of nonlinear boundary value problem, where we have used the compactness criterion of the Sobolev space $H^1(0,\infty)$ into $C_{l,p}([0,+\infty))$ to prove some existence results when different conditions on the nonlinear term are assumed.

Let us consider the boundary value problem

$$\begin{cases} -u''(x)+u(x)=f(x,u(x)), & x\in(0,+\infty), \\ u(0)=u(+\infty)=0, \end{cases} \tag{1.1}$$

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

posed on the half-line, where $f:[0,+\infty]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. According to the behavior of the nonlinear term, the study of Problem (1.1) is investigated. A suitable Hilbert space is chosen and an appropriate compactness criterion which enables us to deal with Schauder's fixed point theorem is used. Firstly, we recall some tools which will used in the sequel.

Theorem 1 (Schauder's fixed point theorem [13]). *Let E be a Banach space and $\mathcal{D}\subset E$ a bounded, closed and convex subset of E. Let $T:\mathcal{D}\to\mathcal{D}$ be a completely continuous operator. Then T has a fixed point in $\mathcal{D}$.*

Let us consider the space $H^1(0,+\infty)=\left\{u\in L^2(0,+\infty),\ u'\in L^2(0,+\infty)\right\}$ which is equipped with the following scalar product

$$\begin{aligned}(.,.):H^1(0,+\infty)\times H^1(0,+\infty)&\to\mathbb{R}\\ (u,v)=\textstyle\int_0^{+\infty}u'(x)v'(x)dx+\int_0^{+\infty}u(x)v(x)dx&\end{aligned}\tag{1.2}$$

and the natural norm

$$\|u\|=\left(\int_0^{+\infty}u^2(x)dx+\int_0^{+\infty}u'^2(x)dx\right)^{\frac{1}{2}}.$$

Note that if $u\in H^1(0,+\infty)$, then we get $u(+\infty)=0$ (see [3], Corollary 8.9). Let

$$C_{l,p}([0,+\infty))=\left\{u\in C([0,+\infty),\mathbb{R}):\lim_{x\to+\infty}p(x)u(x)\text{ exists}\right\}$$

endowed with the norm

$$\|u\|_{\infty,p}=\sup_{x\in[0,+\infty)}p(x)|u(x)|,$$

where $p:[0,+\infty)\to(0,+\infty)$ is continuously differentiable and bounded.

We also need the following results.

Lemma 1 ([6]). *$H^1(0,+\infty)$ embeds continuously in $C_{l,p}([0,+\infty))$, and*

$$\|u\|_{\infty,p}\leq M\|u\|,$$

with $M=max(\|p\|_{L^2},\|p'\|_{L^2})<\infty$.

Lemma 2 ([6]). *The embedding $H^1(0,+\infty)\hookrightarrow C_{l,p}([0,+\infty))$ is compact.*

2. Existence of Weak Solutions

A weak solution u of Problem (1.1) satisfies

$$\int_0^{+\infty}u'(x)v'(x)dx+\int_0^{+\infty}u(x)v(x)dx=\int_0^{+\infty}f(x,u(x))v(x)dx,\ \ \forall v\in H_0^1(0,+\infty),\tag{2.1}$$

that is,

$$(u,v)=\int_0^{+\infty}f(x,u(x))v(x)dx,\ \ \forall v\in H_0^1(0,+\infty).\tag{2.2}$$

Assume that there exist a constant $\sigma > 0, \sigma \neq 1$ and functions $a(.)$, $b(.)$ such that

$$(H_1) \quad |f(x,s)| \leq a(x) + b(x)|s|^{\sigma}, \ \forall x \in (0,+\infty), \ \forall s \in \mathbb{R},$$

$$(H_2) \quad \frac{a}{p} \in L^1(0,+\infty), \text{ and } \frac{b}{p^{\sigma+1}} \in L^1(0,+\infty),$$

$$(H_3) \quad \begin{cases} b^*(\frac{1}{\sigma b^*})^{\frac{\sigma}{\sigma-1}} - (\frac{1}{\sigma b^*})^{\frac{1}{\sigma-1}} + a^* \leq 0 \text{ when } \sigma > 1, \\ b^*(\frac{1}{\sigma b^*})^{\frac{\sigma}{\sigma-1}} - (\frac{1}{\sigma b^*})^{\frac{1}{\sigma-1}} + a^* \geq 0 \text{ when } 0 < \sigma < 1, \\ \text{where } \ a^* := M\|\frac{a}{p}\|_{L^1} \text{ and } b^* := M^{\sigma+1}\|\frac{b}{p^{\sigma+1}}\|_{L^1}. \end{cases} \tag{2.3}$$

For a fixed $u \in H_0^1(0,+\infty)$ and f satisfying (H_1), one can see that

$$\hat{A}_u : \ v \mapsto (u,v), \quad \hat{T}_u : \ v \mapsto \int_0^{+\infty} f(x,u(x))v(x)dx$$

are continuous linear functionals on $H_0^1(0,+\infty)$. Since $H_0^1(0,+\infty)$ is a Hilbert space, by Riesz Representation Theorem, there exist Au and Tu in $H_0^1(0,+\infty)$ such that

$$\hat{A}_u(v) = (Au,v) = (u,v) \text{ and } \hat{T}_u(v) = (Tu,v),$$

for all $v \in H_0^1(0,+\infty)$. Hence, from (2.2), we infer that

$$(u,v) = (Tu,v), \text{ for all } \ v \in H_0^1(0,+\infty),$$

that is, $(u - Tu, v) = 0$, for all $v \in H_0^1(0,+\infty)$. From the last equation, we can say that $u - Tu = 0$, for all $v \in H_0^1(0,+\infty)$, which means that, $Tu = u$ for all $v \in H_0^1(0,+\infty)$. For more details, see [5].

In order to use Schauder's fixed point theorem to prove existence of weak solutions for (1.1) in $H_0^1(0,+\infty)$, let

$$\mathcal{D} = \overline{B}(0,R) = \left\{ u \in H_0^1(0,+\infty); \ \|u\| \leq R \right\}$$

with a positive number R which will be defined later and consider the operator

$$T : \mathcal{D} \to H_0^1(0,+\infty)$$

defined by

$$\forall v \in H_0^1(0,+\infty), \ (Tu,v) = \int_0^{+\infty} f(x,u(x))v(x)dx.$$

Hence, the fixed points of operator T are weak solutions for Problem (1.1).

Theorem 2. *Under assumptions $(H_1),(H_2)$ and (H_3) Problem (1.1) has at least one solution in $H_0^1(0,+\infty)$.*

Proof. By using assumption (H_3), we can show that according to $\sigma > 1$ and $0 < \sigma < 1$, there exists a suitable $\eta > 0$ which satisfies $a^* + b^*\eta^{\sigma} \leq \eta$. We take $R = \eta$.

Claim-1: T maps $\mathcal{D}$ into itself. Indeed, for any $u \in H_0^1(0,+\infty)$ with $\|u\| \leq R$, we have:

$$\begin{aligned}
\|Tu\| &= \sup_{\|v\|\leq 1} |(Tu,v)| = \sup_{\|v\|\leq 1} \left| \int_0^{+\infty} f(x,u(x))v(x)dx \right| \\
&\leq \sup_{\|v\|\leq 1} \int_0^{+\infty} |f(x,u(x))||v(x)|dx \\
&\leq \sup_{\|v\|\leq 1} \|v\|_{\infty,p} \int_0^{+\infty} |f(x,u(x))| \frac{1}{p(x)} dx \\
&\leq \sup_{\|v\|\leq 1} \|v\|_{\infty,p} \int_0^{+\infty} (a(x)+b(x)|u(x)|^{\sigma}) \frac{1}{p(x)} dx \\
&= \sup_{\|v\|\leq 1} \|v\|_{\infty,p} \left[\int_0^{+\infty} a(x)\frac{1}{p(x)}dx + \int_0^{+\infty} b(x)|u(x)|^{\sigma}\frac{1}{p(x)}dx \right] \\
&\leq \sup_{\|v\|\leq 1} \|v\|_{\infty,p} \left[\int_0^{+\infty} \frac{a(x)}{p(x)}dx + \|u\|_{\infty,p}^{\sigma} \int_0^{+\infty} \frac{b(x)}{(p(x))^{\sigma+1}}dx \right] \\
&\leq \sup_{\|v\|\leq 1} M\|v\| \left[\int_0^{+\infty} \frac{a(x)}{p(x)}dx + M^{\sigma}\|u\|^{\sigma} \int_0^{+\infty} \frac{b(x)}{(p(x))^{\sigma+1}}dx \right] \\
&\leq M \left[\int_0^{+\infty} \frac{a(x)}{p(x)}dx + M^{\sigma}R^{\sigma} \int_0^{+\infty} \frac{b(x)}{(p(x))^{\sigma+1}}dx \right]
\end{aligned}$$

$$\begin{aligned}
\|Tu\| &\leq M \int_0^{+\infty} \frac{a(x)}{p(x)}dx + M^{\sigma+1}R^{\sigma} \int_0^{+\infty} \frac{b(x)}{(p(x))^{\sigma+1}}dx \\
&\leq M\|\frac{a}{p}\|_{L^1} + M^{\sigma+1}R^{\sigma}\|\frac{b}{p^{\sigma+1}}\|_{L^1} \\
&\leq a^* + b^*R^{\sigma} \\
&\leq R,
\end{aligned}$$

hence, we have $\|Tu\| \leq R$. That is, $T(\mathcal{D}) \subset \mathcal{D}$.

Claim-2: T is compact.

To prove the continuity of T, we consider a sequence $(u_n) \subset \mathcal{D}$ such that $u_n \to u_0$ in $H_0^1(0,+\infty)$ and we show that $Tu_n \to Tu_0$ in $H_0^1(0,+\infty)$. Indeed, if $u_n \to u_0$ in $H_0^1(0,+\infty)$, as $n \to +\infty$, by Lemma 1, we have $u_n \to u_0$ in $C_{l,p}([0,+\infty))$, as $n \to +\infty$ and one can deduce that

$$p(x)|u_n(x) - u_0(x)| \to 0 \text{ as } n \to +\infty$$

for all $x \in [0,+\infty)$, then we have $u_n(x) \to u_0(x)$ as $n \to +\infty$ for all $x \in [0,+\infty)$. By the continuity of f with respect to its second variable, we obtain

$$f(x,u_n(x)) \to f(x,u_0(x)) \text{ as } n \to +\infty.$$

That is,

$$|f(x,u_n(x)) - f(x,u_0(x))| \to 0 \text{ as } n \to +\infty$$

and as $p(x)$ does not vanish on $[0,+\infty)$, we find

$$\frac{1}{p(x)}\,|f(x,u_n(x)) - f(x,u_0(x))| \to 0 \text{ as } n \to +\infty.$$

Moreover,

$$\frac{M}{p(x)}\,|f(x,u_n(x)) - f(x,u_0(x))| \le \frac{M}{p(x)}\left(2a(x) + b(x)\left(|u_n(x)|^\sigma + |u_0(x)|^\sigma\right)\right),$$

and one can see that

$$\begin{aligned}
&\int_0^{+\infty} \frac{M}{p(x)}\left(2a(x) + b(x)\left(|u_n(x)|^\sigma + |u_0(x)|^\sigma\right)\right)\\
&= M\int_0^{+\infty} \frac{2a(x)}{p(x)}dx + M\int_0^{+M\infty} \frac{b(x)}{p(x)}|u_n(x)|^\sigma dx + M\int_0^{+\infty} \frac{b(x)}{p(x)}|u_0(x)|^\sigma dx.
\end{aligned}$$

Hence, we get

$$\int_0^{+\infty} \frac{M}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))|dx \le 2a^* + 2b^*R^\sigma < \infty.$$

Then, we have

$$\begin{aligned}
\|Tu_n - Tu_0\| &= \sup_{\|v\|\le 1} |\,(Tu_n - Tu_0, v)\,|\\
&= \sup_{\|v\|\le 1} \left|\int_0^{+\infty} \left(f(x,u_n(x)) - f(x,u_0(x))\right) v(x)dx\right|\\
&\le \sup_{\|v\|\le 1} \int_0^{+\infty} |f(x,u_n(x)) - f(x,u_0(x))||v(x)|dx\\
&\le \sup_{\|v\|\le 1} \|v\|_{\infty,p} \int_0^{+\infty} \frac{1}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))|dx\\
&\le \sup_{\|v\|\le 1} M\|v\| \int_0^{+\infty} \frac{1}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))|dx\\
&\le \int_0^{+\infty} \frac{M}{p(x)}|f(x,u_n(x)) - f(x,u_0(x))|dx.
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we get

$$\|Tu_n - Tu_0\| \to 0 \text{ as } n \to +\infty, \text{ i.e., } Tu_n \to Tu_0 \text{ as } n \to +\infty,$$

which means that T is continuous.

Now, let $(u_n) \subset \mathcal{D}$ be a bounded sequence in $H_0^1(0,+\infty)$ and we show that $(T(u_n))$ is relatively compact. Since (u_n) is bounded in the reflexive space $H_0^1(0,+\infty)$, there exist a subsequence (u_{n_k}) and u_0 such that

$$u_{n_k} \rightharpoonup u_0 \text{ in } H_0^1(0,+\infty) \text{ as } n \to +\infty \text{ with } \|u_0\| \leq R$$

(see [7]). In view of the compact embedding of $H^1(0,+\infty)$ in $C_{l,p}([0,+\infty))$, we get $u_{n_k} \to u_0$ in $C_{l,p}([0,+\infty))$ as $n \to +\infty$ so $u_{n_k}(x) \to u_0(x)$ as $n \to +\infty$ for all $x \in [0,+\infty)$. Moreover, since f is Carathéodory, we obtain

$$f(x,u_{n_k}(x)) \to f(x,u_0(x)) \text{ as } n \to +\infty.$$

Following the same reasoning as above, we show that

$$T(u_{n_k}) \to Tu_0 \text{ as } n \to +\infty.$$

Then T is compact and the existence of solutions for Problem (1.1) follows from Schauder's fixed point theorem. □

Example 1. Let us consider the problem

$$\begin{cases} -u''(x)+u(x) = \frac{1}{3}\exp(-2x)+\exp(\frac{-7}{2}x)\dfrac{u^2(x)}{|u(x)|+1}, \ x \in (0,+\infty), \\ u(0)=u(+\infty)=0. \end{cases} \tag{2.4}$$

This problem has at least one solution. Indeed, if we put

$$f(x,u) = \frac{1}{3}\exp(-2x)+\exp(\frac{-7}{2}x)\frac{u^2}{|u|+1},$$

then we find

$$|f(x,u)| \leq \frac{1}{3}\exp(-2x)+\exp(\frac{-7}{2}x)|u|^2.$$

Furthermore, let $p:\ [0,+\infty) \to (0,+\infty)$ be continuously differentiable and bounded mapping defined by $p(x)=\exp(-x)$. It is easily seen that

$$\frac{a}{p} \in L^1(0,+\infty), \ and \ \frac{b}{p^{\sigma+1}} \in L^1(0,+\infty),$$

where

$$a(x) = \frac{1}{3}\exp(-2x), \ \ b(x) = \exp(\frac{-7}{2}x) \text{ and } \sigma = 2, \ a^* = \frac{\sqrt{2}}{6}, \ b^* = \frac{\sqrt{2}}{2}.$$

Then assumptions $(H_1),(H_2)$ and (H_3) of Theorem 2 are satisfied.

Now, we prove another existence result of solution for Problem (1.1). Suppose that the nonlinear term f satisfies

$$|f(x,u)| \leq p^2(x)H(x,p(x)|u|), \text{ for all } x \in (0,+\infty) \text{ and } u \in \mathbb{R},$$

with $p \in L^1(0,+\infty) \cap L^2(0,+\infty)$ and $H:(0,+\infty)\times\mathbb{R}^+ \to \mathbb{R}^+$ satisfies

(A_1) H is continuous, nondecreasing with respect to its second variable,

(A_2) There exists $R_0 > 0$ such that $H(x, MR_0) \leq \dfrac{R_0}{M\|p\|_{L^1}}$.

Theorem 3. *Under assumptions* (A_1) *and* (A_2)*, Problem (1.1) has at least one solution in* $H_0^1(0,+\infty)$.

Proof. Let $\mathcal{D} = \overline{B}(0,R_0) = \left\{u \in H_0^1(0,+\infty);\ \|u\| \leq R_0\right\}$ and the operator

$$T:\ \mathcal{D} \to \left(H_0^1(0,+\infty)\right)^*,\ \ u \mapsto Tu$$

with $Tu : H_0^1(0,+\infty) \to \mathbb{R}$ defined by $v \mapsto (Tu, v) = \int_0^{+\infty} f(x,u(x))v(x)dx$.

It can be easily seen that the functional Tu is linear and continuous. To apply Schauder's fixed point theorem, firstly, one can see that T is continuous. Indeed, if a sequence $(u_n) \subset \mathcal{D}$ converges in norm to u_0 as $n \to +\infty$ in $H_0^1(0,+\infty)$, by Lemma 1, we have $u_n \to u_0$ as $n \to +\infty$ in $C_{l,p}([0,+\infty))$, which gives

$$p(x)|u_n(x) - u_0(x)| \to 0 \text{ as } n \to +\infty$$

for all $x \in [0,+\infty)$, implying that $u_n(x) \to u_0(x)$ as $n \to +\infty$ for all $x \in [0,+\infty)$. Moreover, because f is Carathéodory, we get

$$\frac{1}{p(x)}\left|f(x,u_n(x)) - f(x,u_0(x))\right| \to 0 \text{ as } n \to +\infty.$$

By the continuity of the embedding of $H_0^1(0,+\infty)$ in $C_{l,p}([0,+\infty))$, we have

$$p(x)|u_n(x| \leq \|u_n\|_{\infty,\,p} \leq M\|u_n\| \leq MR_0$$

and

$$p(x)|u_0(x)| \leq \|u_0\|_{\infty,p} \leq M\|u_0\| \leq MR_0.$$

Then, $[H(x,p(x)|u_n(x)|) + H(x,p(x)|u(x)|)] \leq \dfrac{2R_0}{M\|p\|_{L^1}}$. Now, using assumptions on f, we find

$$\int_0^{+\infty} \frac{1}{p(x)}\left|f(x,u_n(x)) - f(x,u_0(x))\right| dx \leq \int_0^{+\infty} p(x)H(x,p(x)|u_n(x)|) + \int_0^{+\infty} p(x)H(x,p(x)|u_0(x)|)dx$$

with

$$\int_0^{+\infty} p(x)\left[H(x,p(x)|u_n(x)|) + H(x,p(x)|u(x)|)\right] \leq \frac{2R_0}{M} < \infty.$$

Hence, we can use the Lebesgue dominated convergence theorem to deduce that $Tu_n \to Tu_0$ as $n \to +\infty$, which means that T is continuous. On the other hand, if $(u_n) \subset \mathcal{D}$ is a bounded sequence in $H_0^1(0,+\infty)$, there exist a subsequence (u_{n_k}) and an element u_0 in $H_0^1(0,+\infty)$ such that $u_{n_k} \rightharpoonup u_0$ as $n \to +\infty$ in $H_0^1(0,+\infty)$, and by Lemma 2, we get $u_{n_k} \to u_0$ in $C_{l,p}([0,\ +\infty))$. Then using assumptions on the function H and the Lebesgue dominated convergence theorem, we can show that (Tu_n) has a

converging subsequence (Tu_{n_k}) which converges to Tu_0 as $n \to +\infty$. Consequently, T is compact.

Now, we claim that $T(\mathcal{D}) \subset \mathcal{D}$. Indeed, let $u \in H_0^1(0,+\infty)$ such that $\|u\| \leq R_0$, then we have

$$p(x)|u(x)| \leq MR_0$$

and because H is nondecreasing with respect to its second variable, we get

$$H(x, p(x)|u(x)|) \leq H(x, MR_0).$$

Then using (A_1) and (A_2), we obtain

$$\begin{aligned}
\|Tu\| &= \sup_{\|v\|\leq 1} |(Tu,v)| = \sup_{\|v\|\leq 1} \left|\int_0^{+\infty} f(x,u(x))v(x)dx\right| \\
&\leq \sup_{\|v\|\leq 1} \int_0^{+\infty} |f(x,u(x))||v(x)|dx \\
&\leq \sup_{\|v\|\leq 1} \|v\|_{\infty,p} \int_0^{+\infty} |f(x,u(x))| \frac{1}{p(x)} dx \\
&\leq \sup_{\|v\|\leq 1} M\|v\| \int_0^{+\infty} |f(x,u(x))| \frac{1}{p(x)} dx \\
&\leq M \int_0^{+\infty} \frac{1}{p(x)} p^2(x) H(x, p(x)|u(x)|)\, dx \\
&\leq M \int_0^{+\infty} p(x) H(x, MR_0)\, dx \\
&\leq M \frac{R_0}{M\|p\|_{L^1}} \int_0^{+\infty} p(x)dx \\
&\leq R_0.
\end{aligned}$$

Hence, Schauder's fixed point theorem ensures the existence of a fixed point of the operator T which is a solution of Problem (1.1). □

This result can be confirmed by the following example.

Example 2. Let us consider the problem

$$\begin{cases} -u''(x) + u(x) = \exp(-3x) \dfrac{u(x)}{\exp(-x)|u(x)| + 1} - \exp(-2x), \quad x \in (0,+\infty), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{2.5}$$

We can show that Problem (2.5) has a nontrivial solution. Let us consider functions p and H defined by

$$p: [0,+\infty) \to (0,+\infty), \; x \mapsto p(x) = \exp(-x)$$

and

$$H:\ (0,+\infty)\times\mathbb{R}^+\to\mathbb{R}^+,\ (x,s)\mapsto H(x,s)=\frac{s}{s+1}+1.$$

It can be easily seen that H is continuous, nondecreasing with respect to its second variable and that $p\in L^1(0,+\infty)\cap L^2(0,+\infty)$ is continuously differentiable and bounded. Moreover, if we put

$$f(x,u)=\exp(-3x)\frac{u}{\exp(-x)|u|+1}-\exp(-2x),$$

we have

$$|f(x,u)|\le\exp(-2x)\left(\frac{\exp(-x)|u|}{\exp(-x)|u|+1}+1\right),$$

that is,

$$|f(x,u)|\ \le\ \exp(-2x)H\left(x,\ \exp(-x)|u|\right).$$

Moreover, for any $R_0\ge 1$, we have $H\left(x,\ \dfrac{R_0}{\sqrt{2}}\right)\le\sqrt{2}R_0$. Then assumptions (A_1) and (A_2) in Theorem 3 are satisfied.

There is an another situation where we can prove the existence of solutions of (1.1). Assume that f satisfies:

(k_1) There exist a nondecreasing function $\Phi\in C(\mathbb{R}^+,\mathbb{R}^+)$ and a function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$|f(x,u)|\le\psi(x)\Phi(p(x)|u|),\quad\forall(x,u)\in(0,+\infty)\times\mathbb{R},\tag{2.6}$$

(k_2) $\dfrac{\psi}{p}\in L^1(0,+\infty)$ and $\exists r>0$ such that $\Phi(Mr)\le\dfrac{r}{M\|\frac{\psi}{p}\|_{L^1}}$.

Theorem 4. *Under assumptions (k_1) and (k_2), Problem (1.1) has at least one solution.*

Proof. Let $\mathcal{D}=\overline{B}(0,r)=\left\{u\in H_0^1(0,+\infty);\ \|u\|\le r\right\}$ and

$$T:\ \mathcal{D}\to\left(H_0^1(0,+\infty)\right)^*,\quad u\mapsto\ Tu:H_0^1(0,+\infty)\to\mathbb{R}$$

defined by $(Tu,v)=\int_0^{+\infty}f(x,u(x))v(x)dx$. One can easily check that Tu is linear and continuous. Moreover, when assumptions (k_1) and (k_2) are satisfied, we can prove that the operator T is compact. Indeed, if $(u_n)\subset\mathcal{D}$ is a sequence in $H_0^1(0,+\infty)$ which converges in $H_0^1(0,+\infty)$ to u_0, then we can show that

$$\frac{1}{p(x)}\left|f(x,u_n(x))-f(x,u_0(x))\right|\to 0\text{ as }\ n\to+\infty.$$

By the continuous embedding of $H_0^1(0,+\infty)$ in $C_{l,p}([0,+\infty))$, we find

$$\begin{aligned}p(x)|u_n(x)|&\le\|u_n\|_{\infty,\,p}\le M\|u_n\|\le Mr\\ p(x)|u_0(x)|&\le\|u\|_{\infty,\,p}\le M\|u_0\|\le Mr.\end{aligned}\tag{2.7}$$

Moreover, by using $(k_1),(k_2)$ and the Lebesgue dominated convergence theorem, we can deduce that T is continuous. The compactness of T follows in a same manner as in theorems above. Now, it remains to show that T maps $\mathcal{D}$ into itself. Let $u \in \mathcal{D}$, using $(k_1),(k_2)$, we get

$$\begin{aligned}
\|Tu\| &= \sup_{\|v\|\leq 1} |(Tu,v)| = \sup_{\|v\|\leq 1} |\int_0^{+\infty} f(x,u(x))v(x)dx| \\
&\leq \sup_{\|v\|\leq 1} \int_0^{+\infty} |f(x,u(x))||v(x)|dx \\
&\leq \sup_{\|v\|\leq 1} \|v\|_{\infty,p} \int_0^{+\infty} |f(x,u(x))| \frac{1}{p(x)} dx \\
&\leq \sup_{\|v\|\leq 1} M\|v\| \int_0^{+\infty} |f(x,u(x))| \frac{1}{p(x)} dx \\
&\leq M \int_0^{+\infty} \frac{1}{p(x)} \psi(x).\Phi(p(x)|u(x)|)dx \\
&\leq M \int_0^{+\infty} \frac{1}{p(x)} \psi(x).\Phi(Mr)dx \\
&\leq M \frac{r}{M\|\frac{\psi}{p}\|_{L^1}} \int_0^{+\infty} \frac{\psi(x)}{p(x)} dx \\
&\leq M \frac{r}{M\|\frac{\psi}{p}\|_{L^1}} \int_0^{+\infty} \frac{|\psi(x)|}{p(x)} dx \\
&\leq r.
\end{aligned}$$

Hence, Schauder's fixed point theorem gives the required result. □

Example 3. In the following, we show the existence of solution for the problem

$$\begin{cases} -u''(x) + u(x) = \dfrac{\exp(\frac{-5}{2}x)u(x)}{\exp(-x)|u(x)|+2} - \exp(\frac{-3}{2}x), \quad x \in (0,+\infty), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{2.8}$$

If we put $f(x,u) = \dfrac{\exp(\frac{-5}{2}x)u}{\exp(-x)|u|+2} - \exp(\frac{-3}{2}x)$, then

$$\begin{aligned}
|f(x,u)| &\leq \left| \frac{\exp(\frac{-5}{2}x)u}{\exp(-x)|u|+2} \right| + |\exp(\frac{-3}{2}x)| \leq \frac{\exp(\frac{-5}{2}x)|u|}{\exp(-x)|u|+1} + \exp(\frac{-3}{2}x) \\
&\leq \exp(\frac{-3}{2}x) \left(\frac{\exp(-x|u|}{\exp(-x)|u|+1} + 1 \right) \leq \psi(x)\Phi(x, \exp(-x)|u|),
\end{aligned}$$

with

$$\psi(x) = \exp(\frac{-3}{2}x),\ \Phi(x,s) = \frac{s}{s+1} + 1,\ p(x) = \exp(-x), \forall x \in (0,+\infty), \forall s \in \mathbb{R}^+.$$

In addition, $\frac{\psi}{p} \in L^1(0,+\infty)$ with $\|\frac{\psi}{p}\|_{L^1} = 2$ and for any $r \geq \frac{1+\sqrt{5}}{\sqrt{2}}$, we have

$$\Phi\left(\frac{r}{\sqrt{2}}\right) \leq \frac{r}{\sqrt{2}}.$$

Then assumptions (k_1), (k_2) in Theorem 4 are satisfied.

Remark 1. In all the above theorems, to obtain the solutions of problems is not trivial. So we must for example suppose that there exists $x_0 \in (0,+\infty)$ such that $f(x_0,0) \neq 0$.

ACKNOWLEDGEMENTS

The authors express their gratitude to the anonymous referees for their helpful suggestions and corrections.

REFERENCES

[1] C. Bai and J. Fang, "On positive solutions of boundary value problems for second-order functional differential equations on infinite intervals," *J. Math. Anal. Appl.*, vol. 282, no. 2, pp. 711–731, 2003, doi: 10.1016/S0022-247X(03)00246-4.

[2] J. Baxley, "Existence and uniqueness for nonlinear boundary value problems on infinite intervals," *J. Math. Anal. Appl.*, vol. 147, no. 1, pp. 127–133, 1990, doi: 10.1016/0022-247X(90)90388-V.

[3] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. New York: Springer, 2011.

[4] S. Chen and Y. Zhang, "Singular boundary value problems on a half-line," *J. Math. Anal. Appl.*, vol. 195, pp. 449–468, 1995.

[5] P. Drábek and J. Milota, *Methods of Nonlinear Analysis Applications to Differential Equations*. Berlin: Birkhäuser Basel, 2007.

[6] O. Frites, T. Moussaoui, and D. O'Regan, "Existence of solutions via variational methods for a problem with nonlinear boundary conditions on the half-line," *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.*, vol. 22, pp. 395–407, 2015.

[7] E. V. Groesen, "Variational methods for nonlinear operator equation," *In: Nonlinear Analysis: Proceeding of the Lectures of a Colloquium Nonlinear Analysis, Mathematisch Centrum, Amsterdam*, pp. 100–191, 1976.

[8] E. Guariglia and S. Silvestrov, "Fractional-wavelet analysis of positive definite distributions and wavelets on D'(C)," *In: Engineering Mathematics II, Silvestrov, Rancic (Eds.), Springer*, pp. 337–353, 2016.

[9] E. Guariglia and K. Tamilvanan, "On the stability of radical septic functional equations," *Mathematics*, vol. 8, no. 12, p. 2229, 2020, doi: 10.3390/math8122229.

[10] G. S. Guseinov, "Boundary value problems for nonlinear impulsive Hamiltonian systems," *J. Comput. Appl. Math.*, vol. 259, no. B, pp. 780–789, 2014, doi: 10.1016/j.cam.2013.06.034.

[11] M. Ragusa, "Elliptic boundary value problem in vanishing mean oscillation hypothesis," *Comment. Math. Univ. Carolin.*, vol. 404, no. 4, pp. 651–663, 1999.

[12] B. Yan, "Multiple unbounded solutions of boundary value problems for second-order differential equations on the half-line," *Nonlinear Anal.*, vol. 51, no. 6, pp. 1031–1044, 2002, doi: 10.1016/S0362-546X(01)00877-X.

[13] E. Zeidler, *Nonlinear Functional Analysis and its Applications: I: Fixed-Point Theorems*. Springer, 1985.

[14] M. Zima, "On positive solution of boundary value problems on the half-line," *J. Math. Anal. Appl.*, vol. 259, no. 1, pp. 127–136, 2001, doi: 10.1006/jmaa.2000.7399.

Authors' addresses

Souad Ayadi

Science Department of Matter, Faculty of Science, Djilali Bounaama University, Khemis Miliana, Algeria

E-mail address: souad.ayadi@univ-dbkm.dz

Ozgur Ege

(**Corresponding author**) Ege University, Faculty of Science, Department of Mathematics, Bornova, 35100, Izmir, Turkey

E-mail address: ozgur.ege@ege.edu.tr

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 605–618

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4506

NEW EXTENSIONS VERSION OF HERMITE–HADAMARD TYPE INEQUALITIES BY MEANS OF CONFORMABLE FRACTIONAL INTEGRALS

UMUT BAS, HÜSEYIN BUDAK, AND HASAN KARA

Received 29 November, 2022

Abstract. In the current investigation, we acquire the upper and lower bounds for inequalities of midpoint-type and trapezoid-type involving conformable fractional integral operators with the help of the mappings whose second derivatives are bounded. We support the established inequalities with examples. Moreover, we use graphs to demonstrate the correctness of the given examples. What's more, we prove the Hermite-Hadamard inequality, which includes conformable fractional integrals, with the aid of condition $f'(a+b-t)-f'(t)\geq 0$, $t\in\left[a,\frac{a+b}{2}\right]$ rather than the convexity of function.

2010 *Mathematics Subject Classification:* 26D07; 26D10; 26D15; 26A33

Keywords: Hermite–Hadamard inequality, conformable fractional integrals, bounded functions

1. Introduction

The theory of inequalities in literature is widely used in engineering, physics, convexity theory, optimization theory, time scale, and quantum analysis. Many mathematicians have obtained new inequalities in these subjects. The new inequalities acquired and the new bounds of the mean value of the mappings find use in many other fields of mathematics. In this respect, inequalities are closely related to other branches of mathematics. The most famous of these inequalities are Hermite-Hadamard type, Trapezoid type, Midpoint type, Simpson type, Newton type, and Bullen type inequalities.

The famous Hermite-Hadamard inequality, which was presented separately by Charles Hermite and Jacques Hadamard in the literature, has attracted the attention of many researchers. Studies on this inequality have been the subject of much scientific research. In particular, studies on the Trapezoid inequality and the Midpoint inequality, that is, the right and left sides of the Hermite–Hadamard inequality, constitute the majority of the studies on this subject. Dragomir and Agarwal first presented

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

trapezoid inequalities via convex mappings in [6], whereas Kirmacı first, acquired midpoint inequalities with the help of the convex mappings in [13]. Moreover, Dragomir et al. also proved new trapezoid and midpoint type extensions by utilizing the bounds of the twice differentiable instead of the condition of convexity in [8] and [7], respectively. For research with extensions on the Hermite-Hadamard inequality, midpoint type inequality, and trapezoid type inequality, refer to the following references [2, 4, 17].

Fractional calculus has taken its place in the literature to clarify the problems that classical analysis cannot answer. In particular, the function "Is it possible to take fractional derivatives and fractional integrals of a function?" answers the question. Many studies have been carried out that have contributed to the literature by using the convexity of the function in some studies and by using the bounds of the second derivative in some studies. In addition, new inequalities are obtained by using the condition $f'(a+b-x) \geq f'(x), x \in \left[a, \frac{a+b}{2}\right]$ instead of the convexity condition. There are two basic approaches to fractional calculations. One of these basic approaches is the Riemann-Liouville approach and the other is the Grünwald-Letnikov approach. Among these two approaches, research based on the Riemann-Liouville approach has been popular in recent years. In examples of these studies, Chen acquired extensions of the Hermite–Hadamard type inequality via convex mappings including Riemann-Liouville fractional integrals in [5]. Budak et al. presented the left and right-hand sides of fractional Hermite–Hadamard with the help of bounds in [3]. Sarikaya et al. gave Hermite–Hadamard type based on Riemann-Liouville fractional integrals in [15]. The conformable approach has emerged to eliminate the shortcomings of the Grünwald-Letnikov and Riemann-Liouville approaches. The conformable approach, based on the basic definition of the derivative, was developed by Khalil in [11]. Abdelhakim et al. presented that the conformable approach in [11] cannot yield good results when compared to the Caputo definition via specific mappings in [1]. This flaw in the conformable fractional approach has been fixed with some extensions in [9, 18]. Inspired by all these studies, Jarad et al. defined conformable fractional integrals. These definitions accelerated the research in fractional calculus.

One of the advantages of the conformable fractional integral is that it can be used to define a new class of functions that have the properties of both smooth and non-smooth functions. These functions are called conformable smooth functions, and they have been shown to have several interesting properties, such as the ability to capture sharp changes in a function without introducing discontinuities. Conformable fractional integrals have been used in several applications, such as signal processing, image processing, and control theory. They have been shown to be useful in the analysis of fractional-order systems, such as fractional-order differential equations, and in the design of fractional-order controllers.

This study consists of four sections including the introduction. In Section 2, we will give the necessary definitions and theorems to construct the main results.

In Section 3, we will present some Midpoint type, Trapezoid type, and Hermite-Hadamard type inequalities by means of conformable fractional integrals. In some of these obtained inequalities, we will use the bounds of the second derivative of the function instead of the convexity of the function. We will use the condition $f'(a+b-t)-f'(t)\geq 0$, $t\in\left[a,\frac{a+b}{2}\right]$ instead of the convexity when obtaining some inequalities. In Section 4, suggestions are made to researchers for future studies.

2. Preliminaries

This section presents the necessary definitions and theorems to form our principal outcomes.

Dragomir et al. obtained new bounds for classical midpoint type and trapezoid type inequalities.

Theorem 1 ([7])**.** *Assume that $f\colon [a,b]\to\mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m\leq f''\leq M$. Then, the following inequalities hold:*

$$m\frac{(b-a)^2}{24}\leq\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\leq M\frac{(b-a)^2}{24} \tag{2.1}$$

and

$$m\frac{(b-a)^2}{24}\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\leq M\frac{(b-a)^2}{24}. \tag{2.2}$$

Let's give the definitions of Riemann-Liouville fractional integrals in the literature.

Definition 1. The Euler Gamma function and Euler Beta function are defined

$$\Gamma(x):=\int_0^\infty t^{x-1}e^{-t}dt,$$

$$\mathcal{B}(x,y):=\int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

Here, $0<x,y<\infty$ and $x,y\in\mathbb{R}$.

Let's give the definitions of Riemann-Liouville fractional integrals in the literature.

Definition 2 ([12, 14])**.** Let $f\in L_1[a,b]$. The Riemann-Liouville fractional integrals of order $\beta>0$ are described by

$$J_{a^+}^{\beta}f(x)=\frac{1}{\Gamma(\beta)}\int_a^x(x-t)^{\beta-1}f(t)dt,\ \ x>a$$

and

$$J_{b^-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} f(t)dt, \quad x < b.$$

Here, Γ is the Euler Gamma function and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

Sarikaya first presented the inequality of Hermite–Hadamard-type involving Riemann-Liouville integrals as follows in [15].

Theorem 2. *Let consider $f\colon [a,b] \to \mathbb{R}$ be a positive mapping with $f \in L_1[a,b]$. If f is a convex mapping on $[a,b]$, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)}{2(b-a)^{\beta}} \left[J_{a^+}^{\beta} f(b) + J_{b^-}^{\beta} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2.3}$$

with $\beta > 0$.

In [5], Chen acquired extensions of the trapezoid type inequality and midpoint type inequality via convex mappings utilizing Riemann-Liouville fractional integrals.

Theorem 3 ([5])**.** *Consider $f\colon [a,b] \longrightarrow \mathbb{R}$ is a positive, twice differentiable function and $f \in L_1[a,b]$. If f'' is bounded $[a,b]$, then we derive*

$$\begin{aligned} m\frac{(b-a)^2(\beta^2-\beta+2)}{8(\beta+1)(\beta+2)} &\leq \frac{\Gamma(\beta+1)}{2(b-a)^{\beta}} \left[J_{a^+}^{\beta} f(b) + J_{b^-}^{\beta} f(a)\right] - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{(b-a)^2(\beta^2-\beta+2)}{8(\beta+1)(\beta+2)} M \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} m\frac{(b-a)^2\beta}{2(\beta+1)(\beta+2)} &\leq \frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)}{2(b-a)^{\beta}} \left[J_{a^+}^{\beta} f(b) + J_{b^-}^{\beta} f(a)\right] \\ &\leq \frac{(b-a)^2\beta}{2(\beta+1)(\beta+2)} M. \end{aligned} \tag{2.5}$$

for $\beta > 0$. Here, $m = \inf_{t\in[a,b]} f''(t)$, $M = \sup_{t\in[a,b]} f''(t)$.

Chen also established the inequality obtained in Theorem 2 by utilizing the condition $f'(a+b-x) \geq f'(x)$ instead of the convexity property of f in [5].

In 2017, Jarad et al. [10] defined the following fractional conformable integral operators. They also presented certain characteristics and relationships between these operators and several other fractional operators in the literature. The conformable fractional integral operators are described as follows.

Definition 3 ([10])**.** For $f \in L^1[a,b]$, the conformable fractional integral operators of order $\beta \in C$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are respectively given by

$$ {}^{\beta}_{a^+}\Upsilon^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt, $$

$$ {}^{\beta}\Upsilon^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left(\frac{(b-x)^{\alpha} - (b-t)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}} dt. $$

Set et al. achieved a new Hermite–Hadamard inequality with the help of the conformable fractional integral operators in [16].

Theorem 4 ([16])**.** *Note that f is a convex function on $[a,b]$. Then the following inequality is satisfied.*

$$ f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}} \left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a) \right] \leq \frac{f(a)+f(b)}{2}. \tag{2.6} $$

Here, $\beta > 0$, $\alpha \in (0,1]$ and Γ is the Euler Gamma function.

3. Principal Outcomes

In this section, some new midpoint-type, trapezoid-type, and Hermite-Hadamard-type inequalities are given.

Theorem 5. *Let $f\colon [a,b] \to \mathbb{R}$ be a twice differentiable function such that there exists real constants m and M so that $m \leq f'' \leq M$ and $\beta > 0$, $\alpha \in (0,1]$. The following inequality is*

$$ \begin{aligned} & m\frac{\beta(b-a)^2}{2}\left[\frac{1}{4\beta} + \mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right) - \mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right] \\ & \leq \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}} \left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a) \right] - f\left(\frac{a+b}{2}\right) \\ & \leq M\frac{\beta(b-a)^2}{2}\left[\frac{1}{4\beta} + \mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right) - \mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right] \end{aligned} \tag{3.1} $$

valid. Here, $\mathcal{B}$ is Euler Beta function.

Proof. With the help of Definition 3, we obtain

$$ \begin{aligned} & \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}} \left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a) \right] \\ & = \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}} \left[\frac{1}{\Gamma(\beta)} \int_a^b \left(\frac{(b-a)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt \right. \end{aligned} $$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\beta)}\int_a^b\left(\frac{(b-a)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}dt\Bigg]\\
&=\frac{\beta\alpha^\beta}{2(b-a)^{\alpha\beta}}\int_a^b\Bigg[\left(\frac{(b-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\\
&+\left(\frac{(b-a)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\Bigg]f(t)\,dt\\
&=\frac{\beta\alpha^\beta}{2(b-a)^{\alpha\beta}}\Bigg[\int_a^{\frac{a+b}{2}}\left(\frac{(b-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\\
&+\left(\frac{(b-a)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\\
&+\int_{\frac{a+b}{2}}^b\left(\frac{(b-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\\
&+\left(\frac{(b-a)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\Bigg]f(t)\,dt.
\end{aligned}
$$

With the aid of the change of variables, we get

$$
\begin{aligned}
&\frac{\Gamma(\beta+1)\alpha^\beta}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^\alpha f(b)+{}^{\beta}\Upsilon^\alpha_{b^-}f(a)\right] \qquad (3.2)\\
&=\frac{\beta\alpha^\beta}{2(b-a)^{\alpha\beta}}\Bigg[\int_a^{\frac{a+b}{2}}\Bigg(\left(\frac{(b-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\\
&+\left(\frac{(b-a)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\Bigg)f(t)dt\\
&+\int_a^{\frac{a+b}{2}}\Bigg(\left(\frac{(b-a)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\\
&+\left(\frac{(b-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\Bigg)f(a+b-t)dt\Bigg]\\
&=\frac{\beta\alpha^\beta}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\Bigg[\left(\frac{(b-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\\
&+\left(\frac{(b-a)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}\Bigg](f(t)+f(a+b-t))\,dt
\end{aligned}
$$

$$= \frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}} \int_{a}^{\frac{a+b}{2}} U_{\alpha}^{\beta}(t)\left[f(t)+f(a+b-t)\right]dt,$$

where

$$U_{\alpha}^{\beta}(t) = \left(\frac{(b-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}} + \left(\frac{(b-a)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(b-t)^{1-\alpha}}.$$

From equality (3.2), we can write

$$\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^{+}}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{b^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right) \tag{3.3}$$
$$= \frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_{a}^{\frac{a+b}{2}} U_{\alpha}^{\beta}(t)\left[f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right)\right]dt.$$

If we take advantage of the facts that

$$f(t)-f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^{t} f'(s)ds$$

and

$$f(a+b-t)-f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^{a+b-t} f'(s)ds,$$

then we obtain

$$\begin{aligned} f(t)+f(a+b-t)-2f(\frac{a+b}{2}) &= \int_{\frac{a+b}{2}}^{t} f'(s)ds+\int_{\frac{a+b}{2}}^{a+b-t} f'(s)ds \\ &= \int_{\frac{a+b}{2}}^{a+b-t} f'(s)ds-\int_{\frac{a+b}{2}}^{a+b-t} f'(a+b-s)ds \\ &= \int_{\frac{a+b}{2}}^{a+b-t}\left[f'(s)-f'(a+b-s)\right]ds. \end{aligned} \tag{3.4}$$

We also obtain

$$f'(s)-f'(a+b-s) = \int_{a+b-s}^{s} f''(y)dy. \tag{3.5}$$

By using the condition $m \le f''(y) \le M$ for all $y \in [a,b]$ and with the aid of equality (3.5), we get

$$\int_{a+b-s}^{s} m dy \le \int_{a+b-s}^{s} f''(y)dy \le \int_{a+b-s}^{s} M dy$$

which gives

$$m(2s-a-b) \le f'(s)-f'(a+b-s) \le M(2s-a-b).$$

From equality (3.4), we establish

$$m\int_{\frac{a+b}{2}}^{a+b-t}(2s-a-b) \le \int_{\frac{a+b}{2}}^{a+b-t}\left[f'(s)-f'(a+b-s)\right]ds \le M\int_{\frac{a+b}{2}}^{a+b-t}(2s-a-b).$$

This implies that

$$m\left(\frac{a+b}{2}-t\right)^2 \leq f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right) \leq M\left(\frac{a+b}{2}-t\right)^2. \tag{3.6}$$

Multiplying inequality (3.6) by $\frac{\beta\alpha^{\beta}U_{\alpha}^{\beta}(t)}{2(b-a)^{\alpha\beta}}$ and then integrating with respect to t on the interval $\left[a,\frac{a+b}{2}\right]$, we have

$$\begin{aligned}
&m\frac{\beta(b-a)^2}{2}\left[\frac{1}{4\beta}+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right]\\
&\leq \frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}} U_{\alpha}^{\beta}(t)\left[f(t)+f(a+b-t)-2f(\frac{a+b}{2})\right]dt\\
&\leq M\frac{\beta(b-a)^2}{2}\left[\frac{1}{4\beta}+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right].
\end{aligned}$$

By using equality (3.3), we establish the required result. □

Remark 1. If we take $\alpha=1$ in Theorem 5, then inequalities (3.1) reduce to inequalities (2.4).

Remark 2. If we choose $\alpha=1$ and $\beta=1$ in Theorem 5, then inequalities (3.1) turns into inequalities (2.1).

Example 1. If the function $f\colon [a,b]=[0,1]\to\mathbb{R}$ is described as $f(t)=t^4+t^2$ such that $2\leq f''(t)\leq 14$ for $t\in[0,1]$. Under these assumptions, the mid-term of inequality (3.1) becomes to

$$\begin{aligned}
&\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\\
&=\frac{\Gamma(\beta+1)\alpha^{\beta}}{2}\left[{}^{\beta}_{0^+}\Upsilon^{\alpha}f(1)+{}^{\beta}\Upsilon^{\alpha}_{1^-}f(0)\right]-f\left(\frac{1}{2}\right)\\
&=\frac{\Gamma(\beta+1)\alpha^{\beta}}{2}\left[\frac{1}{\Gamma(\beta)}\int_0^1\left(\frac{1^{\alpha}-t^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(t^4+t^2)}{t^{1-\alpha}}dt\right.\\
&\quad\left.+\frac{1}{\Gamma(\beta)}\int_0^1\left(\frac{1^{\alpha}-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(t^4+t^2)}{(1-t)^{1-\alpha}}dt\right]-\left(\frac{1}{2^4}+\frac{1}{2^2}\right)\\
&=\frac{\beta\alpha}{2}\left[\int_0^1(1^{\alpha}-t^{\alpha})^{\beta-1}\left(t^{\alpha+3}+t^{\alpha+2}\right)dt+\right.\\
&\quad\left.+\int_0^1\left(1^{\alpha}-(1-t)^{\alpha}\right)^{\beta-1}\left(t^4+t^2\right)(1-t)^{\alpha-1}dt\right]-\frac{5}{16}\\
&=\frac{\beta}{2}\left[2\mathcal{B}\left(\frac{4}{\alpha}+1,\beta\right)-4\mathcal{B}\left(\frac{3}{\alpha}+1,\beta\right)+8\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right.
\end{aligned}$$

$$-6\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)+\frac{2}{\beta}\Bigg]-\frac{5}{16}.$$

Consequently, inequalities (3.1) can be written as follows

$$\begin{aligned}
&\beta\left[\frac{1}{4\beta}+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right] \\
&\leq\frac{\beta}{2}\Bigg[2\mathcal{B}\left(\frac{4}{\alpha}+1,\beta\right)-4\mathcal{B}\left(\frac{3}{\alpha}+1,\beta\right) \\
&\quad+8\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-6\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)+\frac{2}{\beta}\Bigg]-\frac{5}{16} \\
&\leq 7\beta\left[\frac{1}{4\beta}+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)\right].
\end{aligned} \tag{3.7}$$

To illustrate the correctness of (3.7), one can refer to Figure 1.

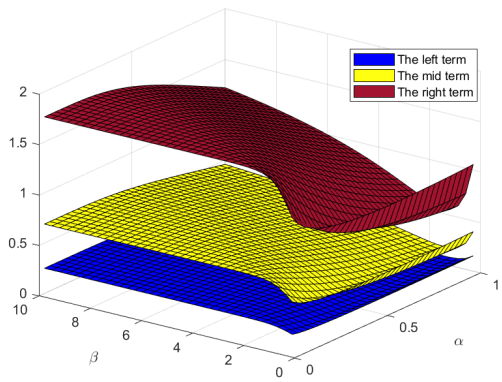


FIGURE 1. Graph for the result of Example 1 computed and plotted in MATLAB.

Theorem 6. *Let $f\colon [a,b]\to\mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m\leq f''\leq M$. Then, we establish*

$$\begin{aligned}
&m\frac{\beta(b-a)^2}{2}\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \\
&\leq\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f(a)\right] \\
&\leq M\frac{\beta(b-a)^2}{2}\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right]
\end{aligned} \tag{3.8}$$

where $\mathcal{B}$ is Euler Beta function.

Proof. By equality (3.2), we have

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^{+}}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{b^{-}}f(a)\right] \tag{3.9}$$
$$=\frac{f(a)+f(b)}{2}-\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_{a}^{\frac{a+b}{2}}U^{\beta}_{\alpha}(t)\left[f(t)+f(a+b-t)\right]dt$$
$$=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_{a}^{\frac{a+b}{2}}U^{\beta}_{\alpha}(t)\left[f(a)+f(b)-(f(t)+f(a+b-t))\right]dt.$$

With the aid of the equalities

$$f(a)-f(t)=-\int_{a}^{t}f'(s)ds$$

and

$$f(b)-f(a+b-t)=\int_{a+b-t}^{b}f'(s)ds,$$

we acquire

$$\begin{aligned} f(a)+f(b)-(f(t)+f(a+b-t)) &= \int_{a+b-t}^{b}f'(s)ds-\int_{a}^{t}f'(s)ds \\ &= \int_{a}^{t}f'(a+b-s)ds-\int_{a}^{t}f'(s)ds \\ &= \int_{a}^{t}\left[f'(a+b-s)-f'(s)\right]ds. \end{aligned} \tag{3.10}$$

We also acquire

$$f'(a+b-s)-f'(s)=\int_{s}^{a+b-s}f''(y)dy. \tag{3.11}$$

By equality (3.11) and the condition $m\leq f''\leq M$, we have

$$m(a+b-2s)\leq f'(s)-f'(a+b-s)\leq M(a+b-2s). \tag{3.12}$$

By utilizing equality (3.10) and inequality (3.12), we get

$$\int_{a}^{t}m(a+b-2s)ds\leq\int_{a}^{t}\left[f'(s)-f'(a+b-s)\right]ds\leq\int_{a}^{t}M(a+b-2s)ds$$

i.e.

$$m(b-t)(t-a)\leq f(a)+f(b)-(f(t)+f(a+b-t))\leq M(b-t)(t-a). \tag{3.13}$$

Multiplying inequality (3.13) by $\frac{\beta\alpha^{\beta}U^{\beta}_{\alpha}(t)}{2(b-a)^{\alpha\beta}}$ and then integrating with respect to t on the interval $\left[a,\frac{a+b}{2}\right]$, we derive

$$m\frac{\beta(b-a)^{2}}{2}\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \tag{3.14}$$

$$\leq \frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a)\right]$$
$$\leq M\frac{\beta(b-a)^2}{2}\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right) - \mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].$$

So, the proof is accomplished. □

Remark 3. If we take $\alpha = 1$ in Theorem 6, then inequalities (3.8) reduce to inequalities (2.5).

Remark 4. If we allow $\alpha = \beta = 1$ in Theorem 5, then inequalities (3.8) and inequalities (2.2) identical.

Example 2. Let us consider the function $f\colon [a,b] = [-2,1] \to \mathbb{R}$ is described as $f(t) = t^3$ such that $-12 \leq f''(t) \leq 6$ for $t \in [-2,1]$. By these assumptions, the midterm of (3.8) becomes as follows

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a)\right]$$
$$= \frac{f(-2)+f(1)}{2} - \frac{\Gamma(\beta+1)\alpha^{\beta}}{2\cdot 3^{\alpha\beta}}\left[{}^{\beta}_{-2^+}\Upsilon^{\alpha} f(1) + {}^{\beta}\Upsilon^{\alpha}_{1^-} f(-2)\right]$$
$$+ \frac{1}{\Gamma(\beta)}\int_{-2}^{1}\left(\frac{3^{\alpha}-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{t^3}{(1-t)^{1-\alpha}}dt\Bigg]$$
$$= -\frac{7}{2} - \frac{\beta}{2}\left[\left(27\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right) - 27\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right) - \frac{7}{\beta}\right)\right].$$

Finally, inequality (3.8) would be as follows

$$-54\beta\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right) - \mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \tag{3.15}$$
$$\leq -\frac{7}{2} - \frac{\beta}{2}\left[\left(27\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right) - 27\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right) - \frac{7}{\beta}\right)\right]$$
$$\leq 27\beta\left[\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right) - \mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].$$

To demonstrate the accuracy of (2), one can refer to Figure 2.

Theorem 7. *Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function and $f \in L_1[a,b]$. If $f'(a+b-x) \geq f'(x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$, then the following Hemite-Hadamard inequality for conformable fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{b^-} f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{3.16}$$

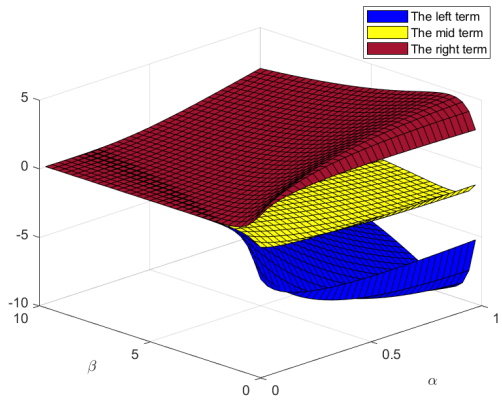


FIGURE 2. Graph for the result of Example 2 computed and plotted in MATLAB.

Proof. Using equalities (3.3) and (3.4), we can write

$$\begin{aligned}
&\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f(a)\right]-f\left(\frac{a+b}{2}\right)\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}U^{\beta}_{\alpha}(t)\left[f(t)+f(a+b-t)-2f(\frac{a+b}{2})\right]\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}U^{\beta}_{\alpha}(t)\left[\int_{\frac{a+b}{2}}^{a+b-t}\left[f'(s)-f'(a+b-s)\right]ds\right]dt\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}U^{\beta}_{\alpha}(t)\left[\int_{t}^{\frac{a+b}{2}}\left[f'(a+b-u)-f'(u)\right]ds\right]dt\\
&\geq 0
\end{aligned}$$

which presents the first inequality in (3.16).

Likewise, by equalities (3.9) and (3.10), we establish

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f(a)\right]\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}U^{\beta}_{\alpha}(t)\left[f(a)+f(b)-(f(t)+f(a+b-t))\right]dt\\
&=\frac{\beta\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}U^{\beta}_{\alpha}(t)\left[\int_{a}^{t}\left[f'(a+b-s)-f'(s)\right]ds\right]dt\\
&\geq 0.
\end{aligned}$$

Thus, the proof of Theorem 7 is completed. □

4. Conclusion

In this article, we derived some new midpoint-type and trapezoid-type inequalities for conformable fractional integrals. In deriving these inequalities, we took advantage of the bounds of the second derivative of the function. We also used a different technique for the conformable fractional Hermite Hadamard inequality instead of the convexity of the function. As can be seen from the paper, interested readers can use the condition $m \leq f''(t) \leq M$ for all $t \in [a,b]$ instead of convexity, and they can try to find better bounds using the condition $f'(a+b-t) - f'(t) \geq 0$, $t \in \left[a, \frac{a+b}{2}\right]$. More precisely, researchers can examine the other kinds of fractional integrals of the obtained these inequalities.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

References

[1] A. A. Abdelhakim, "The flaw in the conformable calculus: it is conformable because it is not fractional," *Fractional Calculus and Applied Analysis*, vol. 22, no. 2, pp. 242–254, 2019, doi: 10.1515/fca-2019-0016.

[2] M. U. Awan, M. A. Noor, T.-S. Du, and K. I. Noor, "New refinements of fractional Hermite-Hadamard inequality," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, vol. 113, pp. 21–29, 2019, doi: 10.1007/s13398-017-0448-x.

[3] H. Budak, H. Kara, M. Z. Sarikaya, and M. E. Kiris, "New extensions of the Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals," *Miskolc Mathematical Notes*, vol. 21, no. 2, pp. 665–678, 2020, doi: 10.18514/MMN.2020.3073.

[4] F. X. Chen, "On the generalization of some Hermite-Hadamard inequalities for functions with convex absolute values of the second derivatives via fractional integrals." *Ukrainian Mathematical Journal*, vol. 70, no. 12, pp. 1953–1966, 2019, doi: 10.1007/s10092-017-0213-8.

[5] F. Chen, "Extensions of the Hermite-Hadamard inequality for convex functions via fractional integrals," *J. Math. Inequal*, vol. 10, no. 1, pp. 75–81, 2016, doi: 10.7153/jmi-10-07.

[6] S. S. Dragomir and R. P. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998, doi: 10.1016/S0893-9659(98)00086-X.

[7] S. S. Dragomir, P. Cerone, and A. Sofo, "Some remarks on the midpoint rule in numerical integration," *Studia Univ. Babes-Bolyai, Math.*, vol. XLV, no. 1, pp. 63–74, 2000.

[8] S. S. Dragomir, P. Cerone, and A. Sofo, "Some remarks on the trapezoid rule in numerical integration," *Indian J. Pure Appl. Math.*, vol. 31, no. 5, pp. 475–494, 2000.

[9] A. A. Hyder and A. H. Soliman, "A new generalized theta-conformable calculus and its applications in mathematical physics," *Physica Scripta*, vol. 96, no. 1, p. 015208, 2020, doi: 10.1088/1402-4896/abc6d9.

[10] F. Jarad, E. Ugurlu, T. Abdeljawad, and D. Baleanu, "On a new class of fractional operators," *Advances in Difference Equations*, vol. 2017, pp. 1–16, 2017, doi: 10.1186/s13662-017-1306-z.

[11] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014, doi: 10.1016/j.cam.2014.01.002.

[12] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*. Elsevier, 2006, vol. 204.

[13] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 137–146, 2004, doi: 10.1016/S0096-3003(02)00657-4.

[14] S. G. Samko, A. Kilbas, and O. I. Marichev, *Fractional integrals and derivatives*. Switzerland: Gordon and Breach Science Publishers, 1993, no. 1, doi: 10.1515/fca-2016-0032.

[15] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities," *Mathematical and Computer Modelling*, vol. 57, no. 9-10, pp. 2403–2407, 2013, doi: 10.1016/j.mcm.2011.12.048.

[16] E. Set, J. Choi, and A. Gözpinar, "Hermite-Hadamard type inequalities involving nonlocal conformable fractional integrals," *Malaysian Journal of Mathematical Sciences*, vol. 15, no. 1, pp. 33–43, 2021.

[17] X.-X. You, M. A. Ali, H. Budak, P. Agarwal, and Y.-M. Chu, "Extensions of Hermite-Hadamard inequalities for harmonically convex functions via generalized fractional integrals," *Journal of Inequalities and Applications*, vol. 2021, no. 1, pp. 1–22, 2021, doi: 10.1186/s13660-021-02638-3.

[18] D. Zhao and M. Luo, "General conformable fractional derivative and its physical interpretation," *Calcolo*, vol. 54, pp. 903–917, 2017, doi: 10.1007/s10092-017-0213-8.

Authors' addresses

Umut Bas
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey
E-mail address: umutbas9661@gmail.com

Hüseyin Budak
Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey
E-mail address: hsyn.budak@gmail.com

Hasan Kara
(**Corresponding author**) Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-Turkey
E-mail address: hasan64kara@gmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 619–630

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4594

GENERAL $((k,p),\psi)$-HILFER FRACTIONAL INTEGRALS

BOUHARKET BENAISSA AND HUSEYIN BUDAK

Received 15 February, 2023

Abstract. The main motivation of this study is to establish a general version of the Riemann-Liouville fractional integrals with two exponential parameters k and p called $((k,p),\psi)$-Hilfer fractional integrals which is determined over the k-gamma function. We first prove that these operators are well-defined, continuous and have semi-group property. Then, particularly, we present the harmonic, geometric and arithmetic $(k,p),\psi$-Hilfer fractional integrals. Moreover, some special cases relating to general $((k,p),\psi)$-Riemann-Liouville fraction integrals are given.

2010 *Mathematics Subject Classification:* 26D15; 26A33; 26D10

Keywords: $((k,p),\psi)$-Hilfer fractional, k-gamma function, Riemann-Liouville operator, Hadamard operator, Katugampola operator

1. Introduction

The fractional calculus theory has been used as a mathematical tool in a variety of pure and practical fields. This approach has been used in different scientific fields. In applied mathematics, various fractional operators have been used to show a set of integral inequalities and their generalizations. One among the vital applications of fractional integrals is the k-Riemann-Liouville fractional integral operator which is an important tool and a source of many research works in field science such as the theory of inequalities, differential equations, integral inequalities. see for example [2], [9], [10], [11], [12]. The right and left-sided k-Riemann-Liouville fractional integrals of order $\alpha > 0$, for a continuous function f on $[a,b]$ are defined as

$$\mathrm{J}^{\alpha}_{a^+,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad a < x \leq b, \tag{1.1}$$

$$\mathrm{J}^{\alpha}_{b^-,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad a \leq x < b, \tag{1.2}$$

where the k-gamma function[3] verified

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \tag{1.3}$$

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

$$\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha)$$

for all $\alpha, k > 0$.

Some basic equations satisfied by the k-gamma function are given in [3].

Property 1. *For all $\alpha, k > 0$ and $n \in \mathbb{N}$, the fundamental formulae satisfied by k-gamma function are*

$$\Gamma_k(\alpha+nk) = k^n \left(\frac{\alpha}{k}\right)\left(\frac{\alpha}{k}+1\right)\ldots\left(\frac{\alpha}{k}+(n-1)\right)\Gamma_k(\alpha), \tag{1.4}$$

$$\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha), \tag{1.5}$$

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma(\frac{\alpha}{k}). \tag{1.6}$$

Remark 1. The above definition and properties reduce to the gamma function and its properties when $k \to 1$.

The k-beta function satisfies the following identities.

$$\beta_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\,dt. \tag{1.7}$$

$$\beta_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad x>0, \quad y>0. \tag{1.8}$$

In 2017, Kuldeep [6] defined the two-parameter gamma function called (k,p) gamma function which is a generalization of k-gamma.

Definition 1. Given $\alpha \in \mathbb{C}/k\mathbb{Z}^-; k,p \in \mathbb{R}^+ - \{0\}$ and $Re(\alpha) > 0$, then the integral representation of (k,p)-Gamma Function is given by

$$\Gamma_{(k,p)}(\alpha) = \int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{p}}\,dt. \tag{1.9}$$

We present certain base formulas related to the (k,p)-gamma function are mentioned in [5], [6].

Property 2. *For all $\alpha, k, p > 0$ and $n \in \mathbb{N}$, the fundamental formulae satisfied by (k,p)-gamma function are ,*

$$\Gamma_{(k,p)}(\alpha+nk) = p^n\left(\frac{\alpha}{k}\right)\left(\frac{\alpha}{k}+1\right)\cdots\left(\frac{\alpha}{k}+(n-1)\right)\Gamma_{(k,p)}(\alpha), \tag{1.10}$$

$$\Gamma_{(k,p)}(\alpha+k) = \frac{p\alpha}{k}\Gamma_{(k,p)}(\alpha), \tag{1.11}$$

$$\Gamma_{(k,p)}(\alpha) = \left(\frac{p}{k}\right)^{\frac{\alpha}{k}}\Gamma_k(\alpha) = \left(\frac{p^{\frac{\alpha}{k}}}{k}\right)\Gamma\left(\frac{\alpha}{k}\right). \tag{1.12}$$

We deduce that

$$\Gamma_{(k,p)}(1) = \left(\frac{p^{\frac{1}{k}}}{k}\right)\Gamma\left(\frac{1}{k}\right), \quad \Gamma_{(k,p)}(k) = \left(\frac{p}{k}\right), \tag{1.13}$$

$$\Gamma_{(k,p)}(p) = \left(\frac{p^{\frac{p}{k}}}{k}\right)\Gamma\left(\frac{p}{k}\right).$$

Remark 2. The above definition and properties reduce to the k-gamma function and its properties when $p = k$.

By using the formula (1.12), we get

$$\beta_k(x,y) = \frac{\Gamma_{(k,p)}(x)\Gamma_{(k,p)}(y)}{\Gamma_{(k,p)}(x+y)} = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad x > 0, \quad y > 0. \tag{1.14}$$

2. $((k,p),\psi)$-HILFER FRACTIONAL INTEGRALS

In this section, we present the general $((k,p),\psi)$-Hilfer fractional integrals of order α with two exponential parameters k and p which generalize the k-Riemann-Liouville fractional integrals.

Let $\varphi : (0,+\infty)\times(0,+\infty) \longrightarrow (0,+\infty)$ be a map satisfying the condition $\varphi(k,k) = k$. For example

(1) the arithmetic mean $\varphi_1(k,p) = \frac{k+p}{2}$,

(2) the geometric mean $\varphi_2(k,p) = \sqrt{kp}$,

(3) $\varphi_3(k,p) = \frac{k^2}{p}$, called the H-case.

Now we will give definitions of generalized fractional integrals.

Definition 2. (See [1], [7], [13]) Let $\psi(x)$ be an increasing positive monotone function on $[a,b]$ such that $\psi'(x)$ is continuous on (a,b). The space $X^r_\psi(a,b)$ $(1 \le r < +\infty)$ is defined as the set of those real-valued Lebesgue measurable functions f on $[a,b]$ for which

$$\|f\|_{X^r_\psi} = \left(\int_a^b |f(x)|^r \psi'(x)dx\right)^{\frac{1}{r}} < \infty. \tag{2.1}$$

In particular, when $\psi(x) = x$ the space $X^r_\psi(a,b)$ coincides with the $L_r[a,b]$.

Definition 3. Let $[a,b] \subseteq [0,+\infty]$, $f \in X^1_\psi(a,b)$ and $k,p > 0$ and for an strictly increasing function ψ, where $\psi \in \mathcal{C}^1[a,b]$ such that $\psi'(t) \neq 0$, for all $t \in [a,b]$. The left and right-sided general (k,p)-Hilfer fractional integrals of a function f with respect to the function ψ on $[a,b]$ are defined, respectively, as follows

$${}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1} f(t)dt, \quad a < x \le b, \tag{2.2}$$

$$ {}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{\frac{\alpha}{\varphi(k,p)}-1} f(t)dt, \quad a \le x < b. \tag{2.3} $$

Here, $\Gamma_{(k,p)}$ is the (k,p)-Gamma Function defined by as in (1.9).

Remark 3. Set $p = k$ in the above definition, we obtain the (k,ψ)-Hilfer fractional integrals [8].

In the following theorem, we show that the $((k,p),\psi)$-Hilfer fractional integrals are well-defined.

Theorem 1. *The fractional integrals (2.2), (2.3) are defined for functions $f \in L_1[a,b]$, existing almost everywhere and*

$$ {}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x),\ {}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x) \in X^1_\psi(a,b). \tag{2.4} $$

Moreover

$$ \left\| \mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x) \right\|_{X^1_\psi(a,b)} \le C \left\| f(t) \right\|_{X^1_\psi(a,b)}, \tag{2.5} $$

where

$$ C = \max\left(\frac{(\psi(b)-\psi(a))^{\frac{\alpha_1}{\varphi(k,p)}}}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}, \frac{(\psi(b)-\psi(a))^{\frac{\alpha_2}{\varphi(k,p)}}}{\Gamma_{(k,p)}\left(k\left(\frac{\alpha_2}{\varphi(k,p)}+1\right)\right)} \right), \quad 0 < \alpha_2 < \varphi(k,p) < \alpha_1. $$

Proof. Let $f(x) \in L_1[a,b]$.

- Let $\frac{\alpha}{\varphi(k,p)} = 1$, is clearly.
- Let $\frac{\alpha}{\varphi(k,p)} > 1$. Let $\Omega = [a,b] \times [a,b]$, we pose for all $(x,t) \in \Omega$, posing

$$ G_1(x,t) = \begin{cases} (\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1}, & a \le t \le x, \\ 0, & x \le t \le b, \end{cases} $$

and

$$ G_2(x,t) = \begin{cases} 0, & a \le t \le x, \\ (\psi(t)-\psi(x))^{\frac{\alpha}{\varphi(k,p)}-1}, & x \le t \le b. \end{cases} $$

Since ψ is continuous, strictly increasing function, for $i = 1,2$ we have

$$ \int_a^b G_i(x,t)\psi'(x)dx \le \int_a^b \psi'(x)(\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}-1}dx = (\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}}. $$

By applying Fubini's Theorem, we get

$$ \begin{aligned} \int_a^b \left(\int_a^b G_i(x,t)\,|f(t)|\,\psi'(t)dt \right) \psi'(x)dx &= \int_a^b \left(\int_a^b G_i(x,t)\psi'(x)dx \right) |f(t)|\,\psi'(t)dt \\ &\le (\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}} \int_a^b |f(t)|\,\psi'(t)dtdt \\ &= (\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}} \left\| f(t) \right\|_{X^1_\psi(a,b)} < \infty. \end{aligned} $$

Therefore

$$\int_a^b G_i(x,t)\,|f(t)|\,\psi'(t)dt \in X_\psi^1(a,b).$$

This gives us (2.4) for $i=1,2$.

- Let $0<\frac{\alpha}{\varphi(k,p)}<1$, put $\eta=1-\frac{\alpha}{\varphi(k,p)}$ hence $0<\eta<1$. By using Fubini's Theorem, we get

$$\begin{aligned}
&\int_a^b \left|{}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)\right|\psi'(x)dx\\
&\quad=\frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\int_a^b\left|\int_a^x(\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1}f(t)\psi'(t)dt\right|\psi'(x)dx\\
&\quad\le\frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\int_a^b\int_a^x\frac{|f(t)|}{(\psi(x)-\psi(t))^{\eta}}\psi'(t)\psi'(x)dtdx\\
&\quad=\frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\int_a^b|f(t)|\left(\int_t^b\frac{\psi'(x)}{(\psi(x)-\psi(t))^{\eta}}dx\right)\psi'(t)dt\\
&\quad=\frac{1}{(1-\eta)\,k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\int_a^b|f(t)|\,(\psi(b)-\psi(t))^{1-\eta}\psi'(t)dt\\
&\quad\le\frac{(\psi(b)-\psi(a))^{1-\eta}}{(1-\eta)\,k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\int_a^b|f(t)|\psi'(t)dt.
\end{aligned}$$

Hence, we obtain

$$\int_a^b\left|{}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)\right|\psi'(x)dx\le\frac{(\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}}}{\Gamma_{(k,p)}\left(k(\frac{\alpha}{\varphi(k,p)}+1)\right)}\|f(t)\|_{X_\psi^1(a,b)}<+\infty.$$

By the similar process, one can obtain

$$\int_a^b\left|{}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)\right|\psi'(x)dx\le\frac{(\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}}}{\Gamma_{(k,p)}\left(k(\frac{\alpha}{\varphi(k,p)}+1)\right)}\|f(t)\|_{X_\psi^1(a,b)}<+\infty.$$

This gives us (2.4).

□

Theorem 2. *Let* $\frac{\alpha}{\varphi(k,p)}>1$ *and* $f\in L_1[a,b]$, *then the fractional integrals* (2.2), (2.3) *are*

$$ {}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x),\ {}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)\in\mathcal{C}[a,b]. \tag{2.6}$$

Proof. Let $x,y\in[a,b]$, $y\leq x$ and $y\to x$. Since ψ is strictly increasing function and $\psi\in\mathcal{C}^1[a,b]$, we get

$$\begin{aligned}
&k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)\left|{}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)-{}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(y)\right|\\
&\quad=\left|\left(\int_x^b\left[(\psi(t)-\psi(x))^{\frac{\alpha}{\varphi(k,p)}-1}-(\psi(t)-\psi(y))^{\frac{\alpha}{\varphi(k,p)}-1}\right]f(t)\psi'(t)dt\right)\right.\\
&\qquad\left.-\int_y^x(\psi(t)-\psi(y))^{\frac{\alpha}{\varphi(k,p)}-1}f(t)\psi'(t)dt\right|\\
&\quad\leq\int_x^b\left|(\psi(t)-\psi(x))^{\frac{\alpha}{\varphi(k,p)}-1}-(\psi(t)-\psi(y))^{\frac{\alpha}{\varphi(k,p)}-1}\right||f(t)|\psi'(t)dt\\
&\qquad+(\psi(x)-\psi(y))^{\frac{\alpha}{\varphi(k,p)}-1}\|f(t)\|_{X^1_\psi(a,b)}.
\end{aligned}$$

Thus

$$\left|{}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)-{}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(y)\right|\to 0\quad\text{as } y\to x.$$

By similar process, one can prove

$$\left|{}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)-{}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(y)\right|\to 0\quad\text{as } y\to x,$$

which completes the proof. □

Now, we give the commutativity and the semigroup properties of the $((k,p),\psi)$-Hilfer fractional integrals.

Theorem 3. *Let $\alpha,\beta>0$. Then we have the following equalities for $((k,p),\psi)$-Hilfer fractional integrals.*

$$ {}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}\left({}_{a^+}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)}f(x)\right)={}_{a^+}\mathrm{I}^{\alpha+\beta,\psi}_{\varphi(k,p)}f(x)={}_{a^+}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)}\left({}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)\right). \tag{2.7}$$

$$ {}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}\left({}_{b^-}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)}f(x)\right)={}_{b^-}\mathrm{I}^{\alpha+\beta,\psi}_{\varphi(k,p)}f(x)={}_{b^-}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)}\left({}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)\right). \tag{2.8}$$

Equalities (2.7) and (2.8) are satisfied in any point for $f(t)\in\mathcal{C}([a,b])$ and in almost every point for $f(t)\in L_1[a,b]$.

Proof. Using Fubini's Theorem, we get

$$\begin{aligned}
&\left[k^2\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)\Gamma_k\left(\frac{k\beta}{\varphi(k,p)}\right)\right]{}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}\left({}_{a^+}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)}f(x)\right)\\
&\quad=\int_a^x(\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1}\left(\int_a^t(\psi(t)-\psi(s))^{\frac{\beta}{\varphi(k,p)}-1}f(s)\psi'(s)ds\right)\psi'(t)dt\\
&\quad=\int_a^x f(s)\left(\int_s^x(\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1}(\psi(t)-\psi(s))^{\frac{\beta}{\varphi(k,p)}-1}\psi'(t)dt\right)\psi'(s)ds.
\end{aligned}\tag{2.9}$$

If we use the change of variable $y=\frac{\psi(t)-\psi(s)}{\psi(x)-\psi(s)}$ in the inner integral in (2.9), we obtain

$$\begin{aligned}&\int_s^x(\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1}(\psi(t)-\psi(s))^{\frac{\beta}{\varphi(k,p)}-1}dt\\&=(\psi(x)-\psi(s))^{\frac{\alpha+\beta}{\varphi(k,p)}-1}\int_0^1(1-y)^{\frac{\alpha}{\varphi(k,p)}-1}(y)^{\frac{\beta}{\varphi(k,p)}-1}dy\\&=k(\psi(x)-\psi(s))^{\frac{\alpha+\beta}{\varphi(k,p)}-1}\beta_k\left(\frac{k\alpha}{\varphi(k,p)},\frac{k\beta}{\varphi(k,p)}\right).\end{aligned}\tag{2.10}$$

By using k-beta propriety (1.14) and (2.9) in (2.10), we deduce that

$${}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}\left({}_{a^+}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)}f(x)\right)=\frac{1}{k\Gamma_{(k,p)}\left(\frac{k(\alpha+\beta)}{\varphi(k,p)}\right)}\int_a^x(\psi(x)-\psi(s))^{\frac{\alpha+\beta}{\varphi(k,p)}-1}f(s)\psi'(s)ds.$$

This gives the equality (2.7). By similar way, the equality (2.8) can be proved easily. □

3. Special cases of general $((k,p),\psi)$-Hilfer fractional integrals

In this section, we present three interesting cases of the general integrals $(k,p),\psi)$-Hilfer depending on the choice of the function ψ.

3.1. (k,p)-*Riemann-Liouville fractional operators*

By setting $\psi(t)=t$, general fractional operators $(k,p),\psi)$-Hilfer reduce to fractional operators (k,p)-Riemann-Liouville as defined below.

Definition 4. Let $[a,b]\subseteq[0,+\infty]$, where $a<b$, $f\in X_t^1(a,b)$, for all $t\in[a,b]$ and $k,p>0$. The left and right-sided (k,p)- Riemann-Liouville fractional integrals of a function f are defined, respectively, as follows

$${}_{a^+}\mathrm{I}^{\alpha}_{\varphi(k,p)}f(x)=\frac{1}{k\Gamma_{(k,p)}(\frac{k\alpha}{\varphi(k,p)})}\int_a^x(x-t)^{\frac{\alpha}{\varphi(k,p)}-1}f(t)dt,\quad a<x\le b,\tag{3.1}$$

$${}_{b^-}\mathrm{I}^{\alpha}_{\varphi(k,p)}f(x)=\frac{1}{k\Gamma_{(k,p)}(\frac{k\alpha}{\varphi(k,p)})}\int_x^b(t-x)^{\frac{\alpha}{\varphi(k,p)}-1}f(t)dt,\quad a\le x<b,\tag{3.2}$$

where $\Gamma_{(k,p)}$ is the (k,p)-Gamma Function defined by (1.9).

We present some specially cases to (k,p)-Riemann-Liouville fractional operators

3.1.1. H-(k,p)-*Riemann-Liouville fractional integrals*

By choosing $\varphi(k,p)=\frac{k^2}{p}$ in Definition 4, we have:

Definition 5. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$ and $k,p>0$. The right and the left-sided harmonic (k,p)-Riemann-Liouville fractional integrals of order $\alpha>0$ are defined as

$$ {}_{a^+}\mathrm{H}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_a^x (x-t)^{\frac{p\alpha}{k^2}-1} f(t)dt, \quad a<x\leq b. \tag{3.3} $$

$$ {}_{b^-}\mathrm{H}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_x^b (t-x)^{\frac{p\alpha}{k^2}-1} f(t)dt, \quad a\leq x<b. \tag{3.4} $$

3.1.2. *Geometric (k,p)-Riemann-Liouville fractional integrals*

By taking $\varphi(k,p)=\sqrt{kp}$ in Definition 4, we have:

Definition 6. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$ and $k,p>0$. The right and the left-sided geometric (k,p)-Riemann-Liouville fractional integrals of order $\alpha>0$ are defined as

$$ {}_{a^+}\mathrm{G}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_a^x (x-t)^{\frac{\alpha}{\sqrt{kp}}-1} f(t)dt, \quad a<x\leq b. \tag{3.5} $$

$$ {}_{b^-}\mathrm{G}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_x^b (t-x)^{\frac{\alpha}{\sqrt{kp}}-1} f(t)dt, \quad a\leq x<b. \tag{3.6} $$

3.1.3. *Arithmetic (k,p)-Riemann-Liouville fractional integrals*

By putting $\varphi(k,p)=\frac{k+p}{2}$ in Definition 4, we have:

Definition 7. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$ and $k,p>0$. The right and the left-sided arithmetic (k,p)-Riemann-Liouville fractional integrals of order $\alpha>0$ are defined as

$$ {}_{a^+}\mathrm{A}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_a^x (x-t)^{\frac{2\alpha}{k+p}-1} f(t)dt, \quad a<x\leq b. \tag{3.7} $$

$$ {}_{b^-}\mathrm{A}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_x^b (t-x)^{\frac{2\alpha}{k+p}-1} f(t)dt, \quad a\leq x<b. \tag{3.8} $$

Remark 4. Choosing $p=k$ in Definitions 5, 6 and 7 give us the k-Riemann-Liouville fraction operators given in [9].

3.2. *(k,p)-Hadamard fractional operators*

Taking $\psi(t)=\ln(t)$ on the definition of general fractional operators $((k,p),\psi)$-Hilfer, we get (k,p)-Hadamard fractional operators defined out below.

Definition 8. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$, for all $t \in [a,b]$ and $k,p>0$. The left and right-sided (k,p)- Hadamard fractional integrals are defined,

respectively, as follows

$$ {}_{a^+}\mathrm{I}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{\varphi(k,p)}-1}\frac{f(t)}{t}dt, \quad a<x\le b, \tag{3.9}$$

$$ {}_{b^-}\mathrm{I}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\int_x^b \left(\ln\frac{t}{x}\right)^{\frac{\alpha}{\varphi(k,p)}-1}\frac{f(t)}{t}dt, \quad a\le x<b, \tag{3.10}$$

where $\Gamma_{(k,p)}$ is the (k,p)-Gamma Function defined by (1.9).

Now, we give some particular cases to (k,p)-Hadamard fractional operators.

3.2.1. *H-(k,p)-Hadamard fractional integrals*

By choosing $\varphi(k,p)=\frac{k^2}{p}$ in Definition 8, we get:

Definition 9. Let $[a,b]\subseteq[0,+\infty]$, where $a<b$, $f\in X_t^1(a,b)$ and $k,p>0$. The right and the left-sided harmonic (k,p) -Hadamard fractional integrals of order $\alpha>0$ are defined as

$$ {}_{a^+}\mathrm{H}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)}\int_a^x \left(\ln\frac{x}{t}\right)^{\frac{p\alpha}{k^2}-1}\frac{f(t)}{t}dt, \quad a<x\le b. \tag{3.11}$$

$$ {}_{b^-}\mathrm{H}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)}\int_x^b \left(\ln\frac{t}{x}\right)^{\frac{p\alpha}{k^2}-1}\frac{f(t)}{t}dt, \quad a\le x<b. \tag{3.12}$$

3.2.2. *Geometric (k,p)-Hadamard fractional integrals*

By taking $\varphi(k,p)=\sqrt{kp}$ in Definition 8, we get:

Definition 10. Let $[a,b]\subseteq[0,+\infty]$, where $a<b$, $f\in X_t^1(a,b)$ and $k,p>0$. The right and the left-sided geometric (k,p) -Hadamard fractional integrals of order $\alpha>0$ are defined as

$$ {}_{a^+}\mathrm{G}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)}\int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{\sqrt{kp}}-1}\frac{f(t)}{t}dt, \quad a<x\le b. \tag{3.13}$$

$$ {}_{b^-}\mathrm{G}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)}\int_x^b \left(\ln\frac{t}{x}\right)^{\frac{\alpha}{\sqrt{kp}}-1}\frac{f(t)}{t}dt, \quad a\le x<b. \tag{3.14}$$

3.2.3. *Arithmetic (k,p)-Hadamard fractional integrals*

By putting $\varphi(k,p)=\frac{k+p}{2}$ in Definition 8, we get:

Definition 11. Let $[a,b]\subseteq[0,+\infty]$, where $a<b$, $f\in X_t^1(a,b)$ and $k,p>0$. The right and the left-sided arithmetic (k,p) -Hadamard fractional integrals of order $\alpha>0$ are defined as

$$ {}_{a^+}\mathrm{A}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)}\int_a^x \left(\ln\frac{x}{t}\right)^{\frac{2\alpha}{k+p}-1}\frac{f(t)}{t}dt, \quad a<x\le b. \tag{3.15}$$

$$ {}_{b^-}\mathrm{A}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_x^b \left(\ln\frac{t}{x}\right)^{\frac{2\alpha}{k+p}-1} \frac{f(t)}{t}dt, \quad a \le x < b. \tag{3.16} $$

Remark 5. Choosing $p = k$ in Definitions 9, 10 and 11 give us the k-Hadamard fractional operators defined in [4].

3.3. (k,p)-*Katugompola fractional operators*

Putting $\psi(t) = \frac{t^{\rho+1}}{\rho+1}$ where $\rho > -1$, then the general $((k,p),\psi)$-Hilfer fractional operators reduce to the (k,p)-Katugompola operators define as follows.

Definition 12. Let $[a,b] \subseteq [0,+\infty]$, where $a < b$, $f \in X_t^1(a,b)$, for all $t \in [a,b]$, $\rho > -1$ and $k,p > 0$. The left and right-sided (k,p)- Katugompola fractional integrals of a function f are defined, respectively, as follows

$$ {}_{a^+}\mathrm{I}^{\alpha,\rho}_{\varphi(k,p)}f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\varphi(k,p)}}}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^x (x^{\rho+1}-t^{\rho+1})^{\frac{\alpha}{\varphi(k,p)}-1} t^{\rho} f(t)dt, \quad a < x \le b, \tag{3.17} $$

$$ {}_{b^-}\mathrm{I}^{\alpha,\rho}_{\varphi(k,p)}f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\varphi(k,p)}}}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_x^b (t^{\rho+1}-x^{\rho+1})^{\frac{\alpha}{\varphi(k,p)}-1} t^{\rho} f(t)dt, \quad a \le x < b, \tag{3.18} $$

where $\Gamma_{(k,p)}$ is the (k,p)-Gamma Function defined by (1.9).

The specially cases to (k,p)-Katugompola fractional operators are as follows.

3.3.1. H-(k,p)-*Katugompola fractional integrals*

By choosing $\varphi(k,p) = \frac{k^2}{p}$ in Definition 12, we have:

Definition 13. Let $[a,b] \subseteq [0,+\infty]$, where $a < b$, $f \in X_t^1(a,b)$, $\rho > -1$ and $k,p > 0$. The right and the left-sided harmonic (k,p) -Katugompola fractional integrals of order $\alpha > 0$ are defined as

$$ {}_{a^+}\mathrm{H}^{\alpha,\rho}_{\varphi(k,p)}f(x) = \frac{(\rho+1)^{\frac{p\alpha}{k^2}-1}}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_a^x (x^{\rho+1}-t^{\rho+1})^{\frac{p\alpha}{k^2}-1} t^{\rho} f(t)dt, \quad a < x \le b. \tag{3.19} $$

$$ {}_{b^-}\mathrm{H}^{\alpha,\rho}_{\varphi(k,p)}f(x) = \frac{(\rho+1)^{\frac{p\alpha}{k^2}-1}}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_x^b (t^{\rho+1}-x^{\rho+1})^{\frac{p\alpha}{k^2}-1} t^{\rho} f(t)dt, \quad a \le x < b. \tag{3.20} $$

3.3.2. *Geometric* (k,p)-*Katugompola fractional integrals*

By taking $\varphi(k,p) = \sqrt{kp}$ in Definition 12, we have:

Definition 14. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$, $\rho > -1$ and $k,p > 0$. The right and the left-sided geometric (k,p)-Katugompola fractional integrals of order $\alpha > 0$ are defined as

$$ {}_{a^+}\mathrm{G}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\sqrt{kp}}}}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_a^x (x^{\rho+1}-t^{\rho+1})^{\frac{\alpha}{\sqrt{kp}}-1} t^{\rho} f(t)dt, \quad a<x\le b. \tag{3.21}$$

$$ {}_{b^-}\mathrm{G}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\sqrt{kp}}}}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_x^b (t^{\rho+1}-x^{\rho+1})^{\frac{\alpha}{\sqrt{kp}}-1} t^{\rho} f(t)dt, \quad a\le x<b. \tag{3.22}$$

3.3.3. *Arithmetic (k,p)-Katugompola fractional integrals*

By putting $\varphi(k,p) = \frac{k+p}{2}$ in Definition 12, we have:

Definition 15. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$, $\rho > -1$ and $k,p > 0$. The right and the left-sided arithmetic (k,p)-Katugompola fractional integrals of order $\alpha > 0$ are defined as

$$ {}_{a^+}\mathrm{A}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{1-\frac{2\alpha}{k+p}}}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_a^x (x^{\rho+1}-t^{\rho+1})^{\frac{2\alpha}{k+p}-1} t^{\rho} f(t)dt, \quad a<x\le b. \tag{3.23}$$

$$ {}_{b^-}\mathrm{A}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{1-\frac{2\alpha}{k+p}}}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_x^b (t^{\rho+1}-x^{\rho+1})^{\frac{2\alpha}{k+p}-1} t^{\rho} f(t)dt, \quad a\le x<b. \tag{3.24}$$

Remark 6. Putting $p=k$ in Definitions 13, 14 and 15 give us the k-Katugompola fractional operators.

REFERENCES

[1] A. Akkurt, M. E. Yildirim, and H. Yildirim, "On some integral inequalities for (k,h)- riemann-liouville fractional integral," *New Trends in Mathematical Sciences*, vol. 4, no. 2, pp. 138–146, 2016, doi: 10.20852/ntmsci.2016217824.

[2] B. Benaissa, "Generalizing hardy type inequalities via k-riemann-liouville fractional integral operators involving two orders," *Honam Mathematical Journal*, vol. 44, no. 2, pp. 271–280, 2022, doi: 10.5831/HMJ.2022.44.2.271.

[3] R. Dıaz and E. Pariguan, "On hypergeometric functions and pochhammer k-symbol," *Divulgaciones Matemáticas*, vol. 15, no. 2, pp. 179–192, 2007.

[4] G. Farid and G. Habibullah, "An extension of hadamard fractional integral," *Int. J. Math. Anal*, vol. 9, no. 10, pp. 471–482, 2015, doi: 10.12988/ijma.2015.5118.

[5] K. S. Gehlot and K. S. Nisar, "Extension of two parameter gamma, beta functions and its properties," *Applications and Applied Mathematics: An International Journal (AAM)*, vol. 15, no. 3, p. 4, 2020.

[6] K. S. Gehlot, "Two parameter gamma function and its properties," *arXiv preprint arXiv:1701.01052*, 2017.
[7] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*. elsevier, 2006, vol. 204.
[8] K. D. Kucche and A. D. Mali, "On the nonlinear (k,ψ)-hilfer fractional differential equations," *Chaos, Solitons & Fractals*, vol. 152, p. 111335, 2021, doi: 10.1016/j.chaos.2021.111335.
[9] S. Mubeen and G. Habibullah, "k-fractional integrals and application," *Int. J. Contemp. Math. Sci*, vol. 7, no. 2, pp. 89–94, 2012.
[10] L. G. Romero, L. L. Luque, G. A. Dorrego, and R. A. Cerutti, "On the k-riemann-liouville fractional derivative," *Int. J. Contemp. Math. Sci*, vol. 8, no. 1, pp. 41–51, 2013.
[11] M. Z. Sarikaya and A. Karaca, "On the k -riemann-liouville fractional integral and applications," *Int. J. Stat. Math.*, vol. 1, no. 2, pp. 33–43, 2014.
[12] M. Sarikaya, C. C. Bilişik, and T. Tunç, "On hardy type inequalities via k-fractional integrals," *TWMS Journal of Applied and Engineering Mathematics*, vol. 10, no. 2, pp. 443–451, 2020.
[13] H. Yildirim and S. K. Yildirim, "On generalized fractional integral inequalities of ostrowski type," *Acta et Commentationes Universitatis Tartuensis de Mathematica*, vol. 25, no. 1, pp. 143–151, 2021, doi: 10.12697/ACUTM.2021.25.09.

Authors' addresses

Bouharket Benaissa
Faculty of Material Sciences, Laboratory of Informatics and Mathematics, University of Tiaret, Tiaret, Algeria
E-mail address: bouharket.benaissa@univ-tiaret.dz

Huseyin Budak
(**Corresponding author**) Department of Mathematics, Faculty of Science and Arts, Düzce University, 81620, Düzce, Türkiye
E-mail address: hsyn.budak@gmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 631–646

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4843

EXISTENCE AND BLOW UP OF SOLUTIONS OF A VISCOELASTIC $m(x)$-BIHARMONIC EQUATION WITH LOGARITHMIC SOURCE TERM

GÜLISTAN BUTAKIN AND ERHAN PIŞKIN

Received 03 November, 2023

Abstract. In this paper, we are concerned with a logarithmic nonlinear viscoelastic $m(x)$-biharmonic equation. Firstly, we proved the local existence of solutions by using the Faedo-Galerkin method. Later, we proved the blow up of solutions by using the concavity method.

2010 *Mathematics Subject Classification:* 35A01; 35B44; 35K55

Keywords: blow up, existence, m(x)-biharmonic equation, logarithmic term

1. Introduction

In this paper, we discuss a viscoelastic $m(x)$-biharmonic equation with logarithmic source term

$$\begin{cases} u_t + \Delta^2 u + \Delta^2_{m(x)} u - \int\limits_0^t g(t-z)\,\Delta^2 u dz = |u|^{p-2} u \ln|u|, (x,t) \in \Omega \times (0,T), \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, \ \ x \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \ \ x \in \Omega. \end{cases} \tag{1.1}$$

Here Ω denotes a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, with smooth boundary $\partial\Omega$. The variable exponents $m(.)$ are provided as measurable functions defined within Ω. The operator $\Delta^2_{m(x)}$ is the so-called $m(x)$-biharmonic operator and is defined by

$$\Delta^2_{m(x)} u = \Delta\left(|\Delta u|^{m(x)-2}\Delta u\right).$$

Also, $m(x)$ is given continuous and measurable function on $\overline{\Omega}$ such that

$$m_- = ess \inf_{x\in\Omega} m(x), \quad m_+ = ess \sup_{x\in\Omega} m(x).$$

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

We consider the following hypotheses:

$A_1)$ $2 < m_- \leq m(x) \leq m_+ < p < \frac{Nm(x)}{N-m(x)}$.

$A_2)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function with the following assumptions:

$$g(z) \geq 0,\ g'(z) \leq 0,\ \ 1 - \int_0^t g(z)\,dz = l > 0.$$

$A_3)$ $E(0) > 0$, is the energy functional defined by (4.2).

Up till now, there are so many results about the parabolic-type differential equations. Our main objective in the present paper is to consider the equation both with viscoelastic term $(\int_0^t g(t-z)\Delta^2 u dz)$, variable exponent term $(\Delta^2_{m(x)} u)$ and the logarithmic source term $(|u|^{p-2} u \ln|u|)$ which make the problem different from those considered in the literature. The problem we deal with is a very general problem:

- The equation with variable exponents arises in many branches in sciences such as image processing, electrorheological fluids and nonlinear elasticity theory (see [5, 8, 23]).
- The equation with logarithmic source term arises in many branches in quantum field theory, optics, inflation cosmology, nuclear physics and geophysics (see [2, 4, 10]).
- The fourth-order equation has its origin in the canonical model introduced by Petrovsky [18, 19]. This type equations arises in many branches in sciences such as acoustics, geophysics, ocean acoustics and optics [9].

Qu et al. [21] studied the fourth-order parabolic equation

$$u_t + \Delta^2 u = |u|^{p(x)}.$$

They established the asymptotic behavior of solutions. Later, Liu [15] demonstrated the local existence and blow-up of solutions for the same equation.

Han [11] investigated the fourth-order parabolic equation

$$u_t + \Delta^2 u - \nabla f(\nabla u) = h(x,t,u).$$

The author exhibited the global existence and finite-time blow-up of solutions.

Abita examined in [22] the pseudo-parabolic equation within the context of a linear memory term and a logarithmic nonlinear source term

$$u_t - \Delta u_t + \int_0^t g(t-s)\Delta u(x,s)\,ds - \Delta u = |u|^{p(x)-2} u \ln(|u|). \tag{1.2}$$

The author demonstrated that a solution to equation (1.2) blows up in finite time T, and gave the upper bound for the blow-up time.

Chuong et al. [7] investigated the following Cahn-Hilliard equation

$$u_t + \Delta^2 u - \Delta_{p(x)} u = |u|^{q(x)-2} u.$$

The authors established the existence and nonexistence of global solutions.

Chuong et al. [6] studied the pseudo-parabolic equations involving $p(x)-$ Laplacian and logarithmic nonlinearity

$$u_t - \Delta u_t - \Delta_{p(x)} u = |u|^{q(x)-2} u \ln(|u|).$$

They obtained some threshold results on existence or nonexistence of global weak solutions when the initial energy is subcritical.

Wang and Liu [24] studied the $p(x)$-biharmonic parabolic equation with the logarithmic nonlinearity

$$u_t + \Delta\left(|\Delta u|^{p(x)-2} \Delta u\right) = |u|^{q-2} u \ln|u|.$$

They proved the existence of the local weak solutions and the existence of the global weak solutions.

Narayanan and Soundararajan [16] studied the viscoelastic $p(x)-$ Laplacian equation with logarithmic nonlinearity

$$u_t - \Delta u - \beta\nabla\left(|\nabla u|^{p(x)-2} \nabla u\right) + \int_0^t g(t-z)\,\Delta u(x,z)\,dz = |u|^{q-2} u \ln|u|.$$

They proved the existence and finite time blow up of solutions of the problem.

This work consists of four sections. Firstly, in Section 2, we give some theories needed about Lebesgue and Sobolev space with variable exponents. Then, Section 3 is about the existence of weak solutions by using the Faedo Galerkin approximation method. Moreover, in Section 4, we obtain the analysis of blow-up phenomena and the determination of an upper bound for the blow-up time.

2. Preliminaries

Throughout this study, we represent the $L^p(\Omega)$ norm as $\|.\|_p$. Additionally, we provide essential background on Lebesgue spaces and Sobolev spaces with variable exponents (for detailed, see [1, 8, 20]).

Let $p\colon \Omega \to [1,\infty]$ be a measurable function. We introduce the Lebesgue space with variable exponent $p(.)$

$$L^{p(.)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable in } \Omega,\ \ \rho_{p(.)}(\lambda u) \ < \infty,\ \text{ for some } \lambda > 0\right\},$$

where

$$\rho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx.$$

The norm, known as Luxemburg's norm, is defined as follows:

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 : \int_{\Omega} \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

$L^{p(.)}(\Omega)$ is a Banach space.

Next we proceed to define the variable-exponent Sobolev space $W^{m,p(.)}(\Omega)$ as

$$W^{m,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) \text{ such that } D^{\alpha}u \text{ exists and } D^{\alpha}u \in L^{p(.)}(\Omega),\ |\alpha| \leq m\right\}.$$

Lemma 1 ([8]). *Assume that*

$$1 \leq m_{-} := ess \inf_{x\in\Omega} m(x) \leq m(x) \leq m_{+} := ess \sup_{x\in\Omega} m(x) < \infty,$$

and we get

$$\min\left\{\|u\|_{m(.)}^{m_{-}}, \|u\|_{m(.)}^{m_{+}}\right\} \leq \rho_{m(.)}(u) \leq \max\left\{\|u\|_{m(.)}^{m_{-}}, \|u\|_{m(.)}^{m_{+}}\right\}, \tag{2.1}$$

for any $u \in L^{m(.)}$.

Definition 1 ([8]). For any given points x and y belonging to the bounded domain Ω, there is a constant $M > 0$ such that the subsequent inequality is satisfied:

$$|p(x) - p(y)| \leq \frac{M}{\ln|x-y|}, \text{ for } x, y \in \Omega, \text{ with } |x-y| < \delta,\ 0 < \delta < 1,$$

then we say that $p(x)$ is log-Holder continuous.

Lemma 2 ([8]). *Suppose that* $q\colon \Omega \longrightarrow [1,\infty)$ *is a measurable function that fulfills inequality (2.1) and the continuity and compactness of the embedding* $L^{q(.)} \hookrightarrow L^{p(.)}$ *can be established. In this context, the variable exponents* $p(x)$ *and* $q(x)$ *adhere to the condition* $p(x) \leq q(x)$ *almost everywhere within the domain* Ω.

Lemma 3 ([8]). *Assume that* $q\colon \Omega \to [1,\infty)$ *is a measurable function that fulfills inequality (2.1) and there exists a continuous and compact Sobolev embedding from* $W_0^{1,m(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, *where the exponents* $m(x) \in C(\overline{\Omega})$, $q\colon \Omega \to [1,\infty)$,

$$ess \inf_{x\in\Omega} (m^{*}(x) - q(x)) > 0,$$

and

$$m^{*}(x) = \begin{cases} \dfrac{Nm(x)}{N - m(x)}, & \text{if } m(x) < N, \\ \infty, & \text{if } m(x) \geq N. \end{cases}$$

Lemma 4 ([12]). *Suppose that* v *be a positive number. We get*

$$\alpha^{q} \ln\alpha \leq (ev)^{-1} \alpha^{q+v},$$

holds for all $\alpha \in [1,\infty)$.

Lemma 5 ([13, 17], Concavity method)**.** *Suppose that* $\beta > 0$*, let* $\psi(t) \geq 0$ *be weakly twice-differentiable on* $(0,\infty)$ *such that* $\psi(0) > 0$, $\psi'(0) > 0$ *and*

$$\psi''(t)\psi(t) - (1+\beta)\left(\psi'(t)\right)^2 \geq 0,$$

for all $t \in (0,\infty)$*. Then there exists a* $T > 0$ *such that*

$$\lim_{t\to T^-} \psi(t) = \infty,$$

and

$$T \leq \frac{\psi(0)}{\beta\psi'(0)}.$$

Here we define a energy functional, which will be used in further calculations:

$$E(t) = \frac{1}{2}\left(1 - \int_0^t g(z)\,dz\right)\|\Delta u\|_2^2 + \frac{1}{2}\int_0^t g(t-z)\,\|\Delta u(x,z) - \Delta u(x,t)\|_2^2\,dz$$
$$- \int_\Omega \frac{|\Delta u|^{m(x)}}{m(x)}dx + \frac{1}{p^2}\|u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln|u|\,dx. \tag{2.2}$$

3. Weak solutions

In this section, we discuss the existence of weak solutions for the problem (1.1), by using the Faedo Galerkin approximation method.

Lemma 6. *A function* $u(x,t)$ *is considered a weak solution for the problem (1.1) if it satisfies the following conditions:*

$$u(x,t) \in L^2\left(0.T; W_0^{2,m(x)}(\Omega) \cap L^p(\Omega)\right) \cap L^2\left(0,T; H_0^2(\Omega)\right) \cap C\left(0,T; L^2(\Omega)\right),$$
$$u_t(x,t) \in L^2\left(0,T; L^2(\Omega)\right) \text{ and } u(x,0) = u_0(x),$$

which also satisfies:

$$\int_0^T\int_\Omega u_t\theta dxdt + \int_0^T\int_\Omega \Delta u\Delta\theta dxdt + \int_0^T\int_\Omega |\Delta u|^{m(x)-2}\Delta u\Delta\theta dxdt$$
$$- \int_0^T\int_0^t g(t-z)\int_\Omega \Delta u(x,z)\Delta\theta dxdzdt = \int_0^T\int_\Omega |u|^{p-2}u\ln|u|\,\theta dxdt, \tag{3.1}$$

$\forall\theta \in C^\infty(0,T; C_0^\infty(\Omega))$*, where the maximal interval of existence is defined as* $[0,T]$.

Theorem 1. *Assuming that* $u_0 \in W_0^{2,m(x)}(\Omega) \cap L^p(\Omega) \setminus \{0\}$*, the problem (1.1) admits a weak solution* $u(x,t)$*, which is provided as described in Lemma 6.*

Proof. We deal with an orthonormal basis of $L^2(\Omega)$, which is orthogonal in $H_0^2(\Omega)$ given by $\{\theta_i\}_{i=1}^{\infty}$. $\{\theta_i\}_{i=1}^{\infty}$ is a sequence of eigenfunctions of $-\Delta$ corresponding to the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$. Now, we proceed to seek finite-dimensional approximations of equation (1.1) denoted as $\{u_n\}$ where

$$u_n(x,t) = \sum_{i=1}^{n} a_{ni}(t)\,\theta_i(x), \tag{3.2}$$

so that

$$\int_\Omega u_n'\theta_i dx + \int_\Omega \Delta u_n \Delta\theta_i dx + \int_\Omega |\Delta u_n|^{m(x)-2}\Delta u_n \Delta\theta_i dx - \int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\,\Delta\theta_i dx dz$$
$$= \int_\Omega |u_n|^{p-2} u_n \ln|u_n|\,\theta_i dx, \tag{3.3}$$

and

$$u_n(x,0) = \sum_{i=1}^{n} a_{ni}(0)\,\theta_i(x) \longrightarrow u_0(x) \text{ in } W_0^{2,m(x)}(\Omega)\cap L^p(\Omega)\diagdown\{0\}. \tag{3.4}$$

Now, we must obtain the coefficients $\{a_{ni}\}_{i=1}^{n}$. For $k = 1,2,...n$,

$$a_{nk}'(t) = -\int_\Omega \Delta u_n \Delta\theta_k dx - \int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\,\Delta\theta_k dx dz$$
$$+ \int_\Omega |\Delta u_n|^{m(x)-2}\Delta u_n \Delta\theta_k dx + \int_\Omega |u_n|^{p-2} u_n \ln|u_n|\,\theta_k dx. \tag{3.5}$$

Here we take into account the maximum interval of existence as $[0,T]$. When we multiply equation (3.3) by $a_{ni}'(t)$ and sum for $i = 1,2,...n$, we obtain

$$\int_\Omega |u_n'|^2 dx + \int_\Omega \Delta u_n \Delta\theta_n' dx - \int_\Omega |\Delta u_n|^{m(x)-2}\Delta u_n \Delta\theta_n' dx$$
$$+ \int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\,\Delta\theta_n' dx dz = \int_\Omega |u_n|^{p-2} u_n \ln|u_n|\,\theta_n' dx.$$

Consequently,

$$\int_\Omega |u_n'|^2 dx - \frac{1}{2}\int_0^t g'(t-z)\,\|\Delta u_n(x,z) - \Delta u_n(x,t)\|_2^2 dz + \frac{1}{2} g(t)\,\|\Delta u_n\|_2^2$$
$$= -\frac{d}{dt} E(u_n(t)). \tag{3.6}$$

Considering E as the functional defined by equation (2.2), we proceed with the implications of hypothesis (A_2) to obtain

$$E'(u_n(t)) = -\int_\Omega |u_n'|^2 dx - \frac{1}{2} g(t) \|\Delta u_n\|_2^2 + \frac{1}{2}\int_0^t g'(t-z) \|\Delta u_n(x,z) - \Delta u_n(x,t)\|_2^2 dz$$
$$\leq 0. \tag{3.7}$$

This leads to the implication that $E(u_n)$ is non-increasing. Consequently, we establish $E(u_n(t)) \leq E(u_n(0))\ \forall t \in [0,T)$ and for every $n \in \mathbb{N}$. Multiplying (3.3) by $a_{ni}(t)$ and then taking the sum over i, we have

$$\int_\Omega u_n' u_n dx + \int_\Omega |\Delta u_n|^2 dx - \int_\Omega |\Delta u_n|^{m(x)} dx - \int_0^t g(t-z) \int_\Omega \Delta u_n(x,z) \Delta u_n(x,t) dxdz$$
$$= \int_\Omega |u_n|^p \ln |u_n| dx.$$

We get

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |u_n|^2 dx + \frac{d}{dt}\int_0^t \int_\Omega |\Delta u_n|^2 dxdz - \frac{d}{dt}\int_0^t \int_\Omega |\Delta u_n|^{m(x)} dxdz$$
$$= \int_0^t g(t-z) \int_\Omega \Delta u_n(x,z) \Delta u_n(x,t) dxdz + \int_\Omega |u_n|^p \ln |u_n| dx. \tag{3.8}$$

We call

$$H_n(t) = \frac{1}{2}\int_\Omega |u_n|^2 dx + \int_0^t \int_\Omega |\Delta u_n|^2 dxdz - \int_0^t \int_\Omega |\Delta u_n|^{m(x)} dxdz. \tag{3.9}$$

So,

$$\frac{d}{dt} H_n(t) = \int_0^t g(t-z) \int_\Omega \Delta u_n(x,z) \Delta u_n(x,t) dxdz + \int_\Omega |u_n|^p \ln |u_n| dx. \tag{3.10}$$

To proceed, we utilize Young's inequality. Consequently, we obtain

$$\int_0^t g(t-z) \int_\Omega \Delta u_n(x,z) \Delta u_n(x,t) dxdz$$
$$= \int_0^t g(t-z) \int_\Omega \Delta u_n(x,t) (\Delta u_n(x,z) - \Delta u_n(x,t)) dxdz + \int_0^t g(t-z) \|\Delta u_n\|_2^2 dz$$

$$\leq \frac{1}{2}\int_0^t g(t-z)\,\|\Delta u_n(x,z)-\Delta u_n(x,t)\|_2^2\,dz$$
$$+\frac{1}{2}\int_0^t g(t-z)\,\|\Delta u_n\|_2^2\,dz+\int_0^t g(t-z)\,\|\Delta u_n\|_2^2\,dz.$$

This results in

$$\int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\,\Delta u_n(x,t)\,dxdz$$
$$\leq \frac{1}{2}\int_0^t g(t-z)\,\|\Delta u_n(x,z)-\Delta u_n(x,t)\|_2^2\,dz+\frac{3}{2}\int_0^t g(t-z)\,\|\Delta u_n\|_2^2\,dz.$$

Therefore, based on hypothesis (A_2) and equation (3.7), we get

$$\frac{d}{dt}H_n(t)\leq E(u_n(t))-\frac{1}{2}\left(1-\int_0^t g(z)\,dz\right)\|\Delta u_n\|_2^2$$
$$+\int_\Omega \frac{|\Delta u_n|^{m(x)}}{m(x)}dx+\frac{1}{p}\int_\Omega |u_n|^p\ln|u_n|\,dx$$
$$-\frac{1}{p^2}\|u_n\|_p^p+\frac{3}{2}\int_0^t g(t-z)\,\|\Delta u_n\|_2^2\,dz+\int_\Omega |u_n|^p\ln|u_n|\,dx, \tag{3.11}$$

and

$$\frac{d}{dt}H_n(t)\leq E(u_n(t))+2\int_0^t g(z)\,dz\,\|\Delta u_n\|_2^2-\frac{1}{2}\|\Delta u_n\|_2^2+\int_\Omega \frac{|\Delta u_n|^{m(x)}}{m(x)}dx$$
$$-\frac{1}{p^2}\|u_n\|_p^p+\left(1+\frac{1}{p}\right)\int_\Omega |u_n|^p\ln|u_n|\,dx,$$

$$\frac{d}{dt}H_n(t)\leq E(u_n(0))+2(1-l)\,\|\Delta u_n\|_2^2+\left(1+\frac{1}{p}\right)\int_\Omega |u_n|^p\ln|u_n|\,dx. \tag{3.12}$$

Now, utilizing Lemma 6, we arrive at

$$\int_\Omega |u_n|^p\ln|u_n|\,dx\leq \int_{\{x\in\Omega:|u_n|\geq 1\}} |u_n|^p\ln|u_n|\,dx\leq (ev)^{-1}.\int_{\{x\in\Omega:|u_n|\geq 1\}} |u_n|^{p+v}\,dx$$
$$\leq (ev)^{-1}.\,\|u_n\|_{p+v}^{p+v}, \tag{3.13}$$

here $0<v<p^*-p$, $p^*=\frac{Np}{N-p}$. By using the Gagliardo-Nirenberg interpolation inequality as presented in [3], we obtain

$$\|u_n\|_{p+v}^{p+v}\leq C\|\Delta u_n\|_m^{\beta(p+v)}\|u_n\|_2^{(1-\beta)(p+v)}, \tag{3.14}$$

where $\beta=\frac{Nm(p+v-2)}{(p+v)(Nm-2N+2m)}$. By employing Young's inequality with ε we get

$$\|u_n\|_{p+v}^{p+v}\leq\varepsilon\|\Delta u_n\|_m^m+C(\varepsilon)\|u_n\|_2^{\frac{m_-(1-\beta)(p+v)}{p_--\beta(p+v)}}. \tag{3.15}$$

Consequently, equation (3.12) implies

$$\begin{aligned}\frac{d}{dt}H_n(t)\leq&\left(1+\frac{1}{p}\right)(ev)^{-1}\left[\varepsilon\|\Delta u_n\|_m^m+C(\varepsilon)\|u_n\|_2^{\frac{m_-(1-\beta)(p+v)}{p_--\beta(p+v)}}\right]\\&+E(u_n(0))+2(1-l)\|\Delta u_n\|_2^2.\end{aligned} \tag{3.16}$$

We obtain

$$\begin{aligned}\frac{d}{dt}H_n(t)\leq&\left(1+\frac{1}{p}\right)(ev)^{-1}\left[\varepsilon\|\Delta u_n\|_m^m+C(\varepsilon)\|u_n\|_2^{\frac{m_-(1-\beta)(p+v)}{p_--\beta(p+v)}}\right]\\&+E(u_n(0))+2(1-l)\|\Delta u_n\|_2^2.\end{aligned}$$

Set $\psi=\frac{m_-(1-\beta)(p+v)}{2(m-\beta)(p+v)}>1$ and suppose $\min\left\{\|\Delta u_n\|_{m(x)}^{m_-},\|\Delta u_n\|_{m(x)}^{m_+}\right\}=\|\Delta u_n\|_{m(x)}^{m_-}$, then by (2.1) we obtain

$$\begin{aligned}\frac{d}{dt}H_n(t)\leq&\left(1+\frac{1}{p}\right)(ev)^{-1}\left[C(\varepsilon)\|u_n\|_2^{2\psi}+\varepsilon\int_\Omega|\Delta u_n|^{m(x)}dx\right]\\&+E(u_n(0))+2(1-l)\|\Delta u_n\|_2^2.\end{aligned}$$

By integrating this inequality from 0 to t, we obtain

$$\begin{aligned}H_n(t)\leq&H_n(0)+E(u_n(0))t+\left(1+\frac{1}{p}\right)(ev)^{-1}2C(\varepsilon)\int_0^t\frac{1}{2}\|u_n\|_2^{2\psi}dz\\&+\left(1+\frac{1}{p}\right)(ev)^{-1}\varepsilon\int_0^t\int_\Omega|\Delta u_n|^{m(x)}dxdz+2(1-l)\int_0^t\|\Delta u_n\|_2^2dz.\end{aligned} \tag{3.17}$$

We define the constant $H_n(0)+E(u_n(0))t=c_1$, which is dependent on t, within the interval $t\in[0,T]$. Now, we select $l\geq\frac{3}{4}$ and $\varepsilon=\frac{evp}{2(p+1)}$, in order to derive

$$H_n(t)\leq c_1(t)+\frac{1}{2}\int_0^t\|\Delta u_n\|_2^2dz-\frac{1}{2}\int_0^t\int_\Omega|\Delta u_n|^{m(x)}dxdz+c_2\int_0^t\frac{1}{2}\|u_n\|_2^{2\psi}dz$$

$$\leq c_1(t) + \frac{1}{2}H_n(t) + c_2 \int_0^t H_n^{\psi} dz,$$

here $c_2 = 2C(\varepsilon)\left(1+\frac{1}{p}\right)(ev)^{-1}$. Consequently, we get

$$H_n(t) \leq c_1(t) + c_2 \int_0^t H_n^{\psi} dz. \tag{3.18}$$

Applying the Gronwall-Bellman-Bihari type inequality, we will obtain

$$H_n(t) = \frac{1}{2}\int_\Omega |u_n|^2 dx + \int_0^t \int_\Omega |\Delta u_n|^2 dxdz - \int_0^t \int_\Omega |\Delta u_n|^{m(x)} dxdz \leq C_T, \tag{3.19}$$

where, C_T depends on T. Our assumption $\min\left\{\|\Delta u_n\|_{m(x)}^{m_-}, \|\Delta u_n\|_{m(x)}^{m_+}\right\} = \|\Delta u_n\|_{m(x)}^{m_-}$, combined with (A_1) gives $\|\Delta u_n\|_{m(x)}^2 \leq \|\Delta u_n\|_{m(x)}^{m_-}$. Now, referring to equation (2.1) we obtain

$$\int_0^T \|\Delta u_n\|_{m(x)}^2 dz \leq \int_0^T \|\Delta u_n\|_{m(x)}^{m_-} dz \leq \int_0^T \int_\Omega |\Delta u_n|^{m(x)} dxdz \leq C_T. \tag{3.20}$$

Given the existence of equation (3.20), we can apply the Sobolev embedding to deduce

$$\int_0^T \|u_n\|_m^2 dx \leq C \int_0^T \|\Delta u_n\|_{m(x)}^2 dz \leq C_T. \tag{3.21}$$

The continuity of E yields a constant C and considering that $u_n(x,0) \longrightarrow u_0(x)$ in $W_0^{2,m(x)}(\Omega) \cap L^p(\Omega)$, we get

$$E(u_n(x,0)) \leq C, \text{ for any } n. \tag{3.22}$$

Additionally, we have as derived from equation (3.6)

$$\int_\Omega |u_{n_t}|^2 dx \leq -\frac{d}{dt}E(u_n),$$

integrating from 0 to t yields

$$\int_0^t \left\|u_n'(z)\right\|_2^2 dz + E(u_n) \leq E(u_n(x,0)) \leq C. \tag{3.23}$$

Together with the standard compactness results, the estimates (3.19), (3.20), (3.21) and (3.23), will lead to

$$u_n \longrightarrow u \text{ weakly in } L^2\left(0,T;W_0^{2,m(x)}(\Omega) \cap L^p(\Omega)\right) \cap L^2\left(0,T;H_0^2(\Omega)\right),$$

$$u_n' \longrightarrow u' \text{ weakly in } L^2\left(0,T;L^2(\Omega)\right),$$
$$|\Delta u_n|^{m(x)-2}\Delta u_n \longrightarrow \xi \text{ weakly in } L^2\left(0,T;L^{m'(x)}(\Omega)\right). \tag{3.24}$$

Now, utilizing the Aubin-Lions lemma, we obtain

$$u_n \longrightarrow u \text{ in } C\left(0,T;L^2(\Omega)\right). \tag{3.25}$$

The operator $|k|^{m(x)-2}k$ combined with the Minty-Browder condition leads to $\xi = |\Delta u|^{m(x)-2}\Delta u$. This concludes the proof. □

4. Upper bound for blow-up time

We obtain an upper bound for the blow-up time of solutions to equation (1.1).

Theorem 2. *Assume the presence of a weak solution $u(x,t)$ as defined in (1.1) for the problem indicated in equation (1.1). Also $(A_1)-(A_3)$ satisfied. We can conclude that the solution u blows up in finite time T^*,which is determined by*

$$T^* \leq \frac{(1+\gamma)\left(1+\frac{1}{\phi}\right)\|u_0\|_2^2}{2\mu\gamma J(0)}, \tag{4.1}$$

where ϕ,γ and μ are positive constants.

Proof. We multiply equation (1.1) by u_t and integrate over Ω, to obtain

$$\int_\Omega u_t^2 dx = -\int_\Omega \Delta u \Delta u_t dx + \int_\Omega |\Delta u|^{m(x)-2}\Delta u \Delta u_t dx + \int_\Omega |u|^{p-2} u \ln|u| u_t dx$$
$$+\int_\Omega \int_0^t g(t-z)\Delta^2 u(x,z) u_t(x,t) dz dx,$$

consequently

$$\int_\Omega u_t^2 dx - \frac{1}{2}\int_0^t g'(t-z)\|\Delta u(x,z)-\Delta u(x,t)\|_2^2 dz + \frac{1}{2}g(t)\|\Delta u\|_2^2$$
$$= -\frac{1}{2}\frac{d}{dt}\|\Delta u\|_2^2 + \frac{d}{dt}\int_\Omega \frac{|\Delta u|^{m(x)}}{m(x)} dx$$
$$-\frac{1}{2}\frac{d}{dt}\int_0^t g(t-z)\|\Delta u(x,z)-\Delta u(x,t)\|_2^2 dz + \frac{d}{dt}\int_0^t g(z) dz \|\Delta u\|_2^2$$
$$-\frac{1}{p^2}\frac{d}{dt}\|u\|_p^p + \frac{1}{p}\frac{d}{dt}\int_\Omega |u|^p \ln|u| dx.$$

Now, we set a functional

$$J(t) = -\frac{1}{2}\int_0^t g(t-z)\,\|\Delta u(x,z) - \Delta u(x,t)\|_2^2\,dz$$
$$-\frac{1}{2}\left(1-\int_0^t g(z)\,dz\right)\|\Delta u\|_2^2 + \int_\Omega \frac{|\Delta u|^{m(x)}}{m(x)}dx + \frac{1}{p}\int_\Omega |u|^p \ln|u|\,dx - \frac{1}{p^2}\|u\|_p^p, \tag{4.2}$$

where by utilizing hypothesis (A_2), we derive

$$\frac{dJ(t)}{dt} = \int_\Omega u_t^2 dx + \frac{1}{2}g(t)\,\|\Delta u\|_2^2 - \frac{1}{2}\int_0^t g'(t-z)\,\|\Delta u(x,z) - \Delta u(x,t)\|_2^2\,dz \geq 0. \tag{4.3}$$

We define an auxiliary functional,

$$K(t) = \int_0^t \int_\Omega u^2(x,z)\,dxdz + A. \tag{4.4}$$

Subsequently

$$K'(t) = 2\int_\Omega \int_0^t u(x,z)\,u_t(x,z)\,dzdx + \int_\Omega u_0^2(x)\,dx, \tag{4.5}$$

and

$$K''(t) = 2\int_\Omega uu_t dx$$
$$= -2\int_\Omega u(x,t)\,\Delta\left(|\Delta u|^{m(x)-2}\Delta u\right)dx + 2\int_\Omega u(x,t)\int_0^t g(t-z)\,\Delta^2 u(x,z)\,dzdx$$
$$-2\int_\Omega u(x,t)\,\Delta^2 u dx + 2\int_\Omega u(x,t)\,|u|^{p-2}u\ln|u|\,dx.$$

So that

$$K''(t) = -2\|\Delta u\|_2^2 - 2\int_\Omega |\Delta u|^{m(x)}\,dx + 2\int_\Omega |u|^p \ln|u|\,dx + 2\int_0^t g(t-z)\,dz\,\|\Delta u\|_2^2.$$
$$+2\int_0^t g(t-z)\int_\Omega \Delta u(x,t)\,(\Delta u(x,z) - \Delta u(x,t))\,dxdz.$$

Next, select a constant μ such that $m_+ < \mu < p$ and $\mu(\mu-2) > \frac{1-l}{l}$. This yields

$$\begin{aligned}
K''(t) = {} & 2\mu J(t) + \mu \int_0^t g(t-z) \|\Delta u(x,z) - \Delta u(x,t)\|_2^2 dz \\
& + 2\mu \int_\Omega \frac{|\Delta u|^{m(x)}}{m(x)} dx + \mu \left(1 - \int_0^t g(z)\, dz\right) \|\Delta u\|_2^2 \\
& - \frac{2\mu}{p} \int_\Omega |u|^p \ln|u|\, dx - 2\int_\Omega |\Delta u|^{m(x)} dx + \frac{2\mu}{p^2} \|u\|_p^p \\
& - 2\left(1 - \int_0^t g(t-z)\, dz\right) \|\Delta u\|_2^2 + 2\int_\Omega |u|^p \ln|u|\, dx \\
& + 2\int_0^t g(t-z) \int_\Omega \Delta u(x,t) (\Delta u(x,z) - \Delta u(x,t))\, dx dz.
\end{aligned}$$

Such that

$$\begin{aligned}
K''(t) \geq {} & 2\mu J(t) + \mu \int_\Omega g(t-z) \|\Delta u(x,z) - \Delta u(x,t)\|_2^2 dx dz \\
& + \frac{2\mu}{p^2} \|u\|_p^p + (\mu-2)\left(1 - \int_0^t g(t-z)\, dz\right) \|\Delta u\|_2^2 \\
& + \left(\frac{2\mu}{p} - 2\right) \int_\Omega |u|^p \ln|u|\, dx + \left(\frac{2\mu}{m_+} - 2\right) \int_\Omega |\Delta u|^{m(x)} dx \\
& + \int_0^t g(t-z) \int_\Omega \Delta u(x,t) (\Delta u(x,z) - \Delta u(x,t))\, dx dz.
\end{aligned}$$

Utilizing (A_2) and assuming $|u| \geq 1$, we obtain

$$\begin{aligned}
K''(t) \geq {} & 2\mu J(t) + \left[(\mu-2)\, l + \frac{(1-l)}{\mu}\right] \|\Delta u\|_2^2 + \left(\frac{2\mu}{m_+} - 2\right) \int_\Omega |\Delta u|^{m(x)} dx + \frac{2\mu}{p^2} \|u\|_p^p \\
& + 2\left(1 - \frac{\mu}{p}\right) \int_\Omega |u|^p \ln|u|\, dx \\
\geq {} & 2\mu J(t). \qquad (4.6)
\end{aligned}$$

From equation (4.3), we deduce

$$J(t) \geq J(0) + \int_0^t \int_\Omega u_t^2(x,z)\,dxdz. \tag{4.7}$$

Now, considering $\phi > 0$

$$\left(K'(t)\right)^2 \leq 4(1+\phi)\left[\int_\Omega \int_0^t u(x,z)\,u_t(x,z)\,dzdx\right]^2 + \left(1+\frac{1}{\phi}\right)\left[\int_\Omega u_0^2(x)\,dx\right]^2.$$

By utilizing Hölder's inequality, we obtain

$$\begin{aligned} K'^2 \leq{}& 4(1+\phi)\left(\int_0^t \int_\Omega u^2(x,z)\,dxdz\right)\left(\int_0^t \int_\Omega u_t^2(x,z)\,dxdz\right) \\ &+ \left(1+\frac{1}{\phi}\right)\left[\int_\Omega u_0^2(x)\,dx\right]^2. \end{aligned}$$

As ϕ is arbitrary and positive, we select $\gamma = \phi = \sqrt{\frac{\mu}{2}} - 1 > 0$, we get

$$K''(t)K(t) - (1+\gamma)\left(K'(t)\right)^2 \geq -(1+\gamma)\left(1+\frac{1}{\phi}\right)\left[\int_\Omega u_0^2(x)\,dx\right]^2 + 2\mu A J(0). \tag{4.8}$$

Since we possess equation (4.7) and $J(0) > 0$, in equation (4.8) we select $A > 0$ to be sufficiently large such that

$$K''(t)K(t) - (1+\gamma)\left(K'(t)\right)^2 \geq 0. \tag{4.9}$$

From this we infer the finite time blow up of the solution u at a time T^*, using Levine's [14] concavity method as detailed. By selecting A as

$$A = \frac{(1+\gamma)\left(1+\frac{1}{\phi}\right)\left[\int_\Omega u_0^2(x)\,dx\right]^2}{2\mu J(0)},$$

we obtain an upper bound for T^* as provided below:

$$0 < T^* \leq \frac{(1+\gamma)\left(1+\frac{1}{\phi}\right)\|u_0\|_2^2}{2\mu\gamma J(0)}. \tag{4.10}$$

□

5. Conclusion

As far as we know, there have not been any existence and blow up results in the literature known for the variable exponent biharmonic equation with logarithmic source term. Our work extends the works for some parabolic type equations with variable exponents treated in the literature to the variable exponent biharmonic equation with logarithmic source term.

References

[1] S. Antontsev and S. Shmarev, *Evolution PDEs with nonstandard growth conditions.* Springer, 2015, vol. 4, doi: 10.2991/978-94-6239-112-3.

[2] I. Bialynicki-Birula and J. Mycielski, "Wave equations with logarithmic nonlinearities." *Bulletin de l'Academie Polonaise des Sciences. Serie des Sciences, Mathematiques, Astronomiques et Physiques*, vol. 23, no. 4, pp. 461–466, 1975, doi: 10.1109/PROC.1973.9296.

[3] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations.* New York: Springer, 2011, vol. 2, no. 3, doi: 10.1007/978-0-387-70914-7.

[4] T. Cazenave and A. Haraux, "Équations d'évolution avec non linéarité logarithmique." *Annales de la Faculté des sciences de Toulouse: Mathématiques*, vol. 2, no. 1, pp. 21–51, 1980.

[5] Y. Chen, S. Levine, and M. Rao, "Variable exponent, linear growth functionals in image restoration." *SIAM journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006, doi: 10.1137/050624522.

[6] Q. V. Chuong, L. C. Nhan, and L. X. Truong, "Existence and non-existence of global solutions of pseudo-parabolic equations involving p (x)-Laplacian and logarithmic nonlinearity." *Journal of Elliptic and Parabolic Equations*, vol. 8, no. 1, pp. 483–512, 2022, doi: 10.58997/ejde.2022.46.

[7] Q. V. Chuong, L. C. Nhan, and L. X. Truong, "Existence and nonexistence of global solutions to the Cahn-Hilliard equation with variable exponent sources." *Electronic Journal of Differential Equations*, vol. 2022, no. 01-87, pp. 46–22, 2022, doi: 10.58997/ejde.2022.46.

[8] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents.* Springer, 2011. doi: 10.1007/978-3-642-18363-8.

[9] J. Ferreira, N. Irkıl, E. Pişkin, C. Raposo, and M. Shahrouzi, "Blow up of solutions for a Petrovsky type equation with logarithmic nonlinearity." *Bulletin of the Korean Mathematical Society*, vol. 59, no. 6, pp. 1495–1510, 2022, doi: 10.4134/BKMS.b210853.

[10] P. Górka, "Logarithmic Klein-Gordon Equation." *Acta Physica Polonica B*, vol. 40, no. 1, 2009.

[11] Y. Han, "A class of fourth-order parabolic equation with arbitrary initial energy." *Nonlinear Analysis: Real World Applications*, vol. 43, pp. 451–466, 2018, doi: 10.1016/j.nonrwa.2018.03.009.

[12] Y. Han, C. Cao, and P. Sun, "A p-Laplace equation with logarithmic nonlinearity at high initial energy level." *Acta Applicandae Mathematicae*, vol. 164, no. 1, pp. 155–164, 2019, doi: 10.1007/s10440-018-00230-4.

[13] V. Kalantarov and L. Olga, "The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types." *Journal of Soviet Mathematics*, vol. 10, pp. 53–70, 1978, doi: 10.1007/BF01109723.

[14] H. A. Levine, "Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + \mathfrak{F}(u)$." *Archive for Rational Mechanics and Analysis*, vol. 51, no. 5, pp. 371–386, 1973, doi: 10.1007/BF00263041.

[15] Y. Liu, W. Zhou, and B. Liang, "Existence and blow-up of solutions to a parabolic equation with nonstandard growth conditions." *Bulletin of the Australian Mathematical Society*, vol. 99, no. 2, pp. 242–249, 2019, doi: 10.1017/S0004972718001181.

[16] L. Narayanan and G. Soundararajan, "Nonexistence of global solutions of a viscoelastic p(x)-Laplacian equation with logarithmic nonlinearity." *AIP Conference proceedings*, vol. 2451, no. 1, 2022, doi: 10.1063/5.0095323.

[17] L. E. Payne, *Improperly posed problems in partial differential equations.* SIAM, 1975. doi: 10.1137/1.9781611970463.ch1.

[18] I. G. Petrowsky, "Uber das Cauchysche Problem fur Systeme von partiellen Differentialgleichungen." *Matematicheskii Sbornik*, vol. 2, no. 5, pp. 815–870, 1937.

[19] I. G. Petrowsky, "Sur l'analyticité des solutions des systèmes d'équations différentielles." *Matematicheskii Sbornik*, vol. 5, no. 1, pp. 3–70, 1939.

[20] E. Pişkin and B. Okutmuştur, *An introduction to Sobolev spaces.* Bentham Science Publishers, 2021. doi: 10.2174/97816810891331210101.

[21] C. Qu, W. Zhou, and B. Liang, "Asymptotic behavior for a fourth-order parabolic equation modeling thin film growth." *Applied Mathematics Letters*, vol. 78, pp. 141–146, 2018, doi: 10.1016/j.aml.2017.11.015.

[22] A. Rahmoune, "Bounds for blow-up solutions of a semilinear pseudo-parabolic equation with a memory term and logarithmic nonlinearity in variable space." *Mathematica Scandinavica*, vol. 128, no. 3, 2022, doi: 10.7146/math.scand.a-133418.

[23] M. Ruzicka, *Electrorheological fluids: modeling and mathematical theory.* Springer, 2007. doi: 10.1007/BFb0104029.

[24] J. Wang and C. Liu, "p-biharmonic parabolic equations with logarithmic nonlinearity." *Electronic Journal of Differential Equations*, vol. 2019, no. 08, pp. 1–18, 2019.

Authors' addresses

Gülistan Butakın
Dicle University, Institute of Natural and Applied Sciences, 21280 Diyarbakır, Turkey
E-mail address: gulistanbutakin@gmail.com

Erhan Pişkin
(**Corresponding author**) Dicle University, Institute of Natural and Applied Sciences, Diyarbakır, Turkey
E-mail address: episkin@dicle.edu.tr

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 647–661

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4511

HERMITE–HADAMARD- AND PACHPATTE-TYPE INTEGRAL INEQUALITIES FOR GENERALIZED SUBADDITIVE FUNCTIONS IN THE FRACTAL SENSE

TINGSONG DU AND LEI XU

Received 02 December, 2022

Abstract. In this paper, we establish the generalized Hermite–Hadamard- and Pachpatte-type integral inequalities for local fractional integrals via the generalized subadditive functions. In particular, we put forward a refined version of the generalized Hermite–Hadamard-type inequalities in the framework of fractal space.

2010 *Mathematics Subject Classification:* 26D07; 26D10; 26D15; 26A33

Keywords: Hermite–Hadamard inequality, Pachpatte inequalities, generalized subadditive functions

1. Introduction

A function $f\colon [0,\infty) \subset \mathbb{R} \to \mathbb{R}$ is said to be subadditive if the following inequality

$$f(x+y) \leq f(x) + f(y)$$

holds true for each $x, y \in [0,\infty)$. If equality holds, then the function f is called additive, and if the inequality is reversed, then the function f is named superadditive.

Subadditivity usually appears in thermodynamic properties of nonideal solutions and mixtures, such as excess molar volume, heat of mixing, or excess enthalpy. In addition, inequalities and subadditive functions can be occurred in electrical networks, quantum relative entropy, purification, ergodic theory and dynamical systems, equilibrium and repulsive perturbation. Here, we mention some findings of the published articles [6, 7, 12, 13, 16] and the bibliographies quoted therein.

A function $f\colon I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds true for each $x, y \in I$ and $t \in [0,1]$.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

The concept of convexity is well known in the literature. The classical Hermite–Hadamard's integral inequality is closely related to convexity, which can be noted as below:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2},$$

in which $f\colon I \to \mathbb{R}$ is a convex function as well as $a,b \in I$ with $a \neq b$.

The inequality above has been studied extensively by many scholars on Euclidean space. Quite a number of generalizations, refinements, extensions and variants have been existed in the literature, see for instance the published articles [3, 4, 10, 11] and the bibliographies quoted therein.

In [19], Tseng et al. acquired the following modified Hermite–Hadamard-type inequalities for convex functions.

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ &\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

In [14], Pachpatte developed the estimates regarding the products of two nonnegative convex functions.

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}f(x)g(x)\mathrm{d}x + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b)$$

and

$$\frac{1}{b-a}f(x)g(x)\mathrm{d}x \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b),$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

2. Preliminaries

Let us retrospect the theoretical knowledge in connection with local fractional operators over $\mathbb{R}^\alpha$ $(0 < \alpha \leq 1)$ advanced by Yang in [21]. Here, we want to emphasize that α denotes the fractal dimension of Cantor set, not an exponential symbol.

Let Υ_1^α, Υ_2^α and Υ_3^α belong to the set $\mathbb{R}^\alpha$ $(0 < \alpha \leq 1)$, then

(1) $\Upsilon_1^\alpha + \Upsilon_2^\alpha$ and $\Upsilon_1^\alpha\Upsilon_2^\alpha$ belong to the set $\mathbb{R}^\alpha$;
(2) $\Upsilon_1^\alpha + \Upsilon_2^\alpha = \Upsilon_2^\alpha + \Upsilon_1^\alpha = (\Upsilon_1 + \Upsilon_2)^\alpha = (\Upsilon_2 + \Upsilon_1)^\alpha$;
(3) $\Upsilon_1^\alpha + (\Upsilon_2^\alpha + \Upsilon_3^\alpha) = (\Upsilon_1^\alpha + \Upsilon_2^\alpha) + \Upsilon_3^\alpha$;
(4) $\Upsilon_1^\alpha\Upsilon_2^\alpha = \Upsilon_2^\alpha\Upsilon_1^\alpha = (\Upsilon_1\Upsilon_2)^\alpha = (\Upsilon_2\Upsilon_1)^\alpha$;
(5) $\Upsilon_1^\alpha(\Upsilon_2^\alpha\Upsilon_3^\alpha) = (\Upsilon_1^\alpha\Upsilon_2^\alpha)\Upsilon_3^\alpha$;
(6) $\Upsilon_1^\alpha(\Upsilon_2^\alpha + \Upsilon_3^\alpha) = \Upsilon_1^\alpha\Upsilon_2^\alpha + \Upsilon_1^\alpha\Upsilon_3^\alpha$;

(7) $\Upsilon_1^\alpha + 0^\alpha = 0^\alpha + \Upsilon_1^\alpha = \Upsilon_1^\alpha$ and $\Upsilon_1^\alpha 1^\alpha = 1^\alpha \Upsilon_1^\alpha = \Upsilon_1^\alpha$;
(8) $\Upsilon_1^\alpha \geq \Upsilon_2^\alpha$ when and only when $\Upsilon_1 \geq \Upsilon_2$, $\Upsilon_1, \Upsilon_2 \in \mathbb{R}$;
(9) $(\Upsilon_1^\alpha)^t = (\Upsilon_1^t)^\alpha, t > 0$ and $\Upsilon_1 > 0$;
(10) $\Upsilon_1^\alpha - \Upsilon_2^\alpha = (\Upsilon_1 - \Upsilon_2)^\alpha$;
(11) For any $\Upsilon_1^\alpha \in \mathbb{R}^\alpha$, its inverse element $(-\Upsilon_1)^\alpha$ may be noted as $-\Upsilon_1^\alpha$; for any $\Upsilon_2^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, its inverse element $(1/\Upsilon_2)^\alpha$ may be noted as $1^\alpha/\Upsilon_2^\alpha$, but not $1/\Upsilon_2^\alpha$.

Definition 1 ([21]). The non-differentiable function $f\colon \mathbb{R} \to \mathbb{R}^\alpha, x \to f(x)$ is named as local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\sigma > 0$, such that the following inequality

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

is valid for $|x - x_0| < \sigma$. If the function $f(x)$ is local fractional continuous defined over the interval (a,b), then one denotes it by $f(x) \in C_\alpha(a,b)$.

Definition 2 ([21]). Suppose that $f(x) \in C_\alpha[a,b]$, and if $\triangle = \{\xi_0, \xi_1, \cdots, \xi_N\}$, $(N \in \mathbb{N})$ is a partition with regard to the interval $[a,b]$ which satisfies that $a = \xi_0 < \xi_1 < \cdots < \xi_N = b$, then the local fractional integral of f defined over the interval $[a,b]$ of order α is defined by

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(\xi)(\mathrm{d}\xi)^\alpha := \frac{1}{\Gamma(\alpha+1)} \lim_{\triangle\xi \to 0} \sum_{j=0}^{N-1} f(\xi_j)(\Delta\xi_j)^\alpha,$$

where $\Delta\xi := \max\{\Delta\xi_1, \Delta\xi_2, \cdots, \Delta\xi_{N-1}\}$, $\Delta\xi_j := \xi_{j+1} - \xi_j$, $j = 0, \cdots, N-1$. For all $x \in [a,b]$, if there exists ${}_aI_x^{(\alpha)} f(x)$, then it is denoted by $f(x) \in I_x^\alpha[a,b]$.

Certain properties widely utilized in the fractal space regarding the integral ${}_aI_b^{(\alpha)} f(x)$ are established as below.

Assume that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b]$, then one achieves that

$${}_aI_b^{(\alpha)} f(x) = g(b) - g(a).$$

In accordance with this, the following two formulas are deduced:

$${}_aI_b^{(\alpha)}(1^\alpha) = \frac{1^\alpha}{\Gamma(1+\alpha)}(b^\alpha - a^\alpha)$$

and

$${}_aI_b^{(\alpha)}(x^{\tau\alpha}) = \frac{\Gamma(1+\tau\alpha)}{\Gamma(1+(\tau+1)\alpha)}\left(b^{(\tau+1)\alpha} - a^{(\tau+1)\alpha}\right),\ \tau > 0.$$

In [21], Yang presented the concept with regard to the generalized convexity.

A function $f\colon I \subset \mathbb{R} \to \mathbb{R}^\alpha$ is said to be generalized convex if the following inequality

$$f(tx + (1-t)y) \leq t^\alpha f(x) + (1-t)^\alpha f(y)$$

holds true for each $x, y \in I$ and $t \in [0,1]$. Here, we present the concept of generalized subadditive functions on fractal sets as follows.

Definition 3. The function $f\colon [0,\infty) \subset \mathbb{R} \to \mathbb{R}^{\alpha}$ is said to be generalized subadditive if the following inequality

$$f(x+y) \leq f(x) + f(y)$$

holds true for each $x, y \in [0,\infty)$. If equality holds, then f is called generalized additive, and if the inequality is reversed, then f is named generalized superadditive.

In 2019, Du et al. used the local fractional calculus to introduce the following generalized starshaped functions.

Definition 4 ([5])**.** The functions $f\colon [0,\infty) \subset \mathbb{R} \to \mathbb{R}^{\alpha}$ is referred to as generalized starshaped, if $f(tx) \leq t^{\alpha} f(x)$ holds true for each $x \in [0,\infty)$ and $t \in [0,1]$.

Remark 1. If the generalized subadditive function $f\colon [0,\infty) \to \mathbb{R}^{\alpha}$ is generalized starshaped, then the function f is also generalized convex. Moreover, if the function f is generalized subadditive and generalized convex over the interval $[0,\infty)$, as well as $f(0) = 0^{\alpha}$, then the function f is generalized additive.

In [9], Mo et al. gave the following generalized Hermite–Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_b^{(\alpha)} f(x) \leq \frac{f(a)+f(b)}{2^{\alpha}},$$

in which $f(x) \in C_{\alpha}[a,b]$ is a generalized convex function over the interval $[a,b]$ with $a < b$.

For recent outcomes in accordance with inequalities in the fractal sense, one may consults the published articles [1, 2, 8, 15, 17, 18, 20] and the bibliographies quoted therein.

In this paper we present certain local fractional Hermite–Hadamard-type integral inequalities via generalized subadditive functions, and the estimates of the products of two generalized subadditive functions are also considered.

3. Main results

Firstly, we construct the following generalized Hermite–Hadamard-type inequality under the generalized subadditivity in the settings of fractal sets.

Theorem 1. *Assume that $f\colon I = [0,\infty) \to \mathbb{R}^{\alpha}$ is a generalized subadditive function. If $a < b$ $(a, b \in I^{\circ})$ and $f \in C_{\alpha}[0,\infty)$, then the following Hermite–Hadamard-type inequalities in the settings of fractal sets hold*

$$\left(\frac{1}{2}\right)^{\alpha} f(a+b) \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_b^{(\alpha)} f(x) \leq \frac{\Gamma(1+\alpha)}{a^{\alpha}} {}_0I_a^{(\alpha)} f(x) + \frac{\Gamma(1+\alpha)}{b^{\alpha}} {}_0I_b^{(\alpha)} f(x).$$

Proof. Assume that $x = ta + (1-t)b \in [a,b]$, or $x = (1-t)a + tb \in [a,b]$ for any $t \in [0,1]$. On account of the generalized subadditivity of the function f, we can figure out that

$$f(ta + (1-t)b) \leq f(ta) + f((1-t)b) \tag{3.1}$$

and

$$f((1-t)a + tb) \leq f((1-t)a) + f(tb). \tag{3.2}$$

Combining the inequalities (3.1) and (3.2), and in terms of the generalized subadditivity of the function f again, we obtain that

$$\begin{aligned} f(a+b) &\leq f(ta + (1-t)b) + f((1-t)a + tb) \\ &\leq f(ta) + f((1-t)b) + f((1-t)a) + f(tb). \end{aligned} \tag{3.3}$$

Integrating both sides of the inequalities (3.3) in the fractal sense, we deduce that

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_0^1 f(a+b)(\mathrm{d}t)^\alpha \\ &\quad \leq \frac{1}{\Gamma(1+\alpha)} \int_0^1 f(ta + (1-t)b)(\mathrm{d}t)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_0^1 f((1-t)a + tb)(\mathrm{d}t)^\alpha \\ &\quad \leq \frac{2^\alpha}{\Gamma(1+\alpha)} \int_0^1 f(ta)(\mathrm{d}t)^\alpha + \frac{2^\alpha}{\Gamma(1+\alpha)} \int_0^1 f(tb)(\mathrm{d}t)^\alpha. \end{aligned} \tag{3.4}$$

Taking advantage of the appropriate substitutions, we attain that

$$\begin{aligned} &\left(\frac{1}{2}\right)^\alpha \frac{1^\alpha}{\Gamma(1+\alpha)} f(a+b) \\ &\quad \leq \left(\frac{1}{b-a}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} \int_a^b f(x)(\mathrm{d}x)^\alpha \\ &\quad \leq \left(\frac{1}{a}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} \int_0^a f(x)(\mathrm{d}x)^\alpha + \left(\frac{1}{b}\right)^\alpha \frac{1}{\Gamma(1+\alpha)} \int_0^b f(x)(\mathrm{d}x)^\alpha. \end{aligned}$$

As a consequence, we acquire that

$$\left(\frac{1}{2}\right)^\alpha f(a+b) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^{(\alpha)} f(x) \leq \frac{\Gamma(1+\alpha)}{a^\alpha} {}_0I_a^{(\alpha)} f(x) + \frac{\Gamma(1+\alpha)}{b^\alpha} {}_0I_b^{(\alpha)} f(x).$$

This fulfills the proof. □

Corollary 1. *If one attempts to take $f(tx) \leq t^\alpha f(x)$ in Theorem 1, then one acquires the following Hermite–Hadamard-type inequality for the generalized subadditive functions*:

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{2}\right)^\alpha f(a+b) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^{(\alpha)} f(x) \leq \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} \big(f(a) + f(b)\big).$$

Proof. Taking advantage of $f(tx) \le t^{\alpha} f(x)$ and the inequalities (3.4), we acquire that

$$f\left(\frac{a+b}{2}\right) \le \left(\frac{1}{2}\right)^{\alpha} f(a+b),$$

and

$$\begin{aligned}
&\left(\frac{1}{b-a}\right)^{\alpha} \frac{1}{\Gamma(1+\alpha)} \int_0^1 f(x)(\mathrm{d}x)^{\alpha} \\
&\le \frac{1}{\Gamma(1+\alpha)} \int_0^1 f(ta)(\mathrm{d}t)^{\alpha} + \frac{1}{\Gamma(1+\alpha)} \int_0^1 f(tb)(\mathrm{d}t)^{\alpha} \\
&\le \frac{1}{\Gamma(1+\alpha)} \int_0^1 t^{\alpha}(\mathrm{d}t)^{\alpha} [f(a)+f(b)] = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} [f(a)+f(b)].
\end{aligned}$$

As a consequence, we derive that

$$f\left(\frac{a+b}{2}\right) \le \left(\frac{1}{2}\right)^{\alpha} f(a+b) \le \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_b^{(\alpha)} f(x) \le \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} \big(f(a)+f(b)\big).$$

□

Next, we propose an another form of the generalized Hermite–Hadamard-type inequality for the generalized subadditive functions in the frame of fractal space.

Theorem 2. *Suppose that the hypotheses mentioned in Theorem 1 are met. Then we have the following inequalities for the generalized subadditivity*:

$$\begin{aligned}
&\left(\frac{1}{4}\right)^{\alpha} \left[f\left(\frac{3a+b}{2}\right) + f\left(\frac{a+3b}{2}\right)\right] \le \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_b^{(\alpha)} f(x) \\
&\le \left(\frac{1}{2}\right)^{\alpha} \left[\frac{\Gamma(1+\alpha)}{a^{\alpha}} {}_0I_a^{(\alpha)} f(x) + \frac{\Gamma(1+\alpha)}{b^{\alpha}} {}_0I_b^{(\alpha)} f(x)\right] + \frac{2^{\alpha}\Gamma(1+\alpha)}{(a+b)^{\alpha}} {}_0I_{\frac{a+b}{2}}^{(\alpha)} f(x).
\end{aligned} \tag{3.5}$$

Proof. Taking advantage of Theorem 1 over the intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$, we deduce that

$$\begin{aligned}
\left(\frac{1}{4}\right)^{\alpha} f\left(\frac{3a+b}{2}\right) &\le \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_{\frac{a+b}{2}}^{(\alpha)} f(x) \\
&\le \left(\frac{1}{2}\right)^{\alpha} \frac{\Gamma(1+\alpha)}{a^{\alpha}} {}_0I_a^{(\alpha)} f(x) + \frac{\Gamma(1+\alpha)}{(a+b)^{\alpha}} {}_0I_{\frac{a+b}{2}}^{(\alpha)} f(x)
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\left(\frac{1}{4}\right)^{\alpha} f\left(\frac{a+3b}{2}\right) &\le \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_{\frac{a+b}{2}}I_b^{(\alpha)} f(x) \\
&\le \left(\frac{1}{2}\right)^{\alpha} \frac{\Gamma(1+\alpha)}{b^{\alpha}} {}_0I_b^{(\alpha)} f(x) + \frac{\Gamma(1+\alpha)}{(a+b)^{\alpha}} {}_0I_{\frac{a+b}{2}}^{(\alpha)} f(x).
\end{aligned} \tag{3.7}$$

Adding the inequalities (3.6) and (3.7), we derive that

$$\left(\frac{1}{4}\right)^{\alpha}\left[f\left(\frac{3a+b}{2}\right)+f\left(\frac{a+3b}{2}\right)\right]\leq\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}I_{b}^{(\alpha)}f(x)$$
$$\leq\left(\frac{1}{2}\right)^{\alpha}\left[\frac{\Gamma(1+\alpha)}{a^{\alpha}}{}_{0}I_{a}^{(\alpha)}f(x)+\frac{\Gamma(1+\alpha)}{b^{\alpha}}{}_{0}I_{b}^{(\alpha)}f(x)\right]+\frac{2^{\alpha}\Gamma(1+\alpha)}{(a+b)^{\alpha}}{}_{0}I_{\frac{a+b}{2}}^{(\alpha)}f(x).$$

This closures the proof. □

On the basis of Theorem 2, if the function f is generalized starshaped, then we can acquire a refined version of the generalized Hermite–Hadamard-type inequalities.

Theorem 3. *Suppose that the hypotheses mentioned in Theorem 2 are met. If we attempt to take $f(tx)\leq t^{\alpha}f(x)$, then we have the following inequalities for the generalized subadditivity*:

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\leq\left(\frac{1}{2}\right)^{\alpha}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\\&\leq\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}I_{b}^{(\alpha)}f(x)\\&\leq\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{f(a)+f(b)}{2^{\alpha}}+f\left(\frac{a+b}{2}\right)\right]\\&\leq\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left[f(a)+f(b)\right].\end{aligned}\tag{3.8}$$

Proof. Taking advantage of the generalized subadditivity of the function f, we acquire that

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\leq\left(\frac{1}{2}\right)^{\alpha}f(a+b)\\&=\left(\frac{1}{2}\right)^{\alpha}f\left(\frac{3a+b}{4}+\frac{a+3b}{4}\right)\\&\leq\left(\frac{1}{2}\right)^{\alpha}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right].\end{aligned}\tag{3.9}$$

Using $f(tx)\leq t^{\alpha}f(x)$, it yields that

$$\left(\frac{1}{2}\right)^{\alpha}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\leq\left(\frac{1}{4}\right)^{\alpha}\left[f\left(\frac{3a+b}{2}\right)+f\left(\frac{a+3b}{2}\right)\right].\tag{3.10}$$

In accordance with Theorem 2, we know that

$$\left(\frac{1}{4}\right)^{\alpha}\left[f\left(\frac{3a+b}{2}\right)+f\left(\frac{a+3b}{2}\right)\right]\leq\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}I_{b}^{(\alpha)}f(x)$$

$$\leq \left(\frac{1}{2}\right)^{\alpha}\left[\frac{\Gamma(1+\alpha)}{a^{\alpha}}{}_0I_a^{(\alpha)}f(x)+\frac{\Gamma(1+\alpha)}{b^{\alpha}}{}_0I_b^{(\alpha)}f(x)\right]+\frac{2^{\alpha}\Gamma(1+\alpha)}{(a+b)^{\alpha}}{}_0I_{\frac{a+b}{2}}^{(\alpha)}f(x). \tag{3.11}$$

As a consequence, in view of the inequalities (3.6) and (3.7), and in terms of $f(tx)\leq t^{\alpha}f(x)$ again, we deduce that

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{a^{\alpha}}{}_0I_a^{(\alpha)}f(x)+\frac{\Gamma(1+\alpha)}{(a+b)^{\alpha}}{}_0I_{\frac{a+b}{2}}^{(\alpha)}f(x)\\
&=\frac{\Gamma(1+\alpha)}{2^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(at)(\mathrm{d}t)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f\left(\frac{a+b}{2}t\right)(\mathrm{d}t)^{\alpha}\right]\\
&\leq\frac{\Gamma(1+\alpha)}{2^{\alpha}}\left[f(a)+f\left(\frac{a+b}{2}\right)\right]\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(\mathrm{d}t)^{\alpha}\\
&=\left(\frac{1}{2}\right)^{\alpha}\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}\left[f(a)+f\left(\frac{a+b}{2}\right)\right]
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{b^{\alpha}}{}_0I_b^{(\alpha)}f(x)+\frac{\Gamma(1+\alpha)}{(a+b)^{\alpha}}{}_0I_{\frac{a+b}{2}}^{(\alpha)}f(x)\\
&=\frac{\Gamma(1+\alpha)}{2^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(bt)(\mathrm{d}t)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f\left(\frac{a+b}{2}t\right)(\mathrm{d}t)^{\alpha}\right]\\
&\leq\left(\frac{1}{2}\right)^{\alpha}\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}\left[f\left(\frac{a+b}{2}\right)+f(b)\right].
\end{aligned} \tag{3.13}$$

Adding the inequalities (3.12) and (3.13), we derive that

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{\alpha}\left[\frac{\Gamma(1+\alpha)}{a^{\alpha}}{}_0I_a^{(\alpha)}f(x)+\frac{\Gamma(1+\alpha)}{b^{\alpha}}{}_0I_b^{(\alpha)}f(x)\right]+\frac{2^{\alpha}\Gamma(1+\alpha)}{(a+b)^{\alpha}}{}_0I_{\frac{a+b}{2}}^{(\alpha)}f(x)\\
&\leq\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{f(a)+f(b)}{2^{\alpha}}+f\left(\frac{a+b}{2}\right)\right].
\end{aligned} \tag{3.14}$$

By virtue of the generalized subadditivity as well as $f(tx)\leq t^{\alpha}f(x)$, we acquire that

$$\begin{aligned}
\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{f(a)+f(b)}{2^{\alpha}}+f\left(\frac{a+b}{2}\right)\right]&\leq\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{f(a)+f(b)}{2^{\alpha}}+\frac{f(a)+f(b)}{2^{\alpha}}\right]\\
&=\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}\left[f(a)+f(b)\right].
\end{aligned} \tag{3.15}$$

Combining the inequalities (3.9), (3.10), (3.11), (3.14) and (3.15), we obtain the desired outcomes (3.8). This fulfills the proof. □

Corollary 2. *If one attempts to take $\alpha = 1$ in Theorem 3, then one acquires the following Hermite–Hadamard-type inequality for convex functions:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ &\le \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right] \le \frac{f(a)+f(b)}{2}, \end{aligned}$$

which is proposed by Tseng et al. in [19].

Finally, we present the Pachpatte-type integral inequalities in the settings of fractal sets.

Theorem 4. *Suppose that $f,g\colon I=[0,\infty)\to\mathbb{R}_+^\alpha$ are both nonnegative generalized subadditive functions. If $a<b$ $(a,b\in I^\circ)$ and $f\in C_\alpha[0,\infty)$, then the following inequalities in the frame of fractal space hold*

$$\begin{aligned} &\left(\frac{2}{b-a}\right)^\alpha {}_aI_b^{(\alpha)} f(x)g(x) \\ &\le \left(\frac{2}{a}\right)^\alpha {}_0I_a^{(\alpha)} f(x)g(x) + \left(\frac{2}{b}\right)^\alpha {}_0I_b^{(\alpha)} f(x)g(x) \\ &\quad + \frac{2^\alpha}{\Gamma(1+\alpha)}\int_0^1 f(ta)g((1-t)b)(\mathrm{d}t)^\alpha + \frac{2^\alpha}{\Gamma(1+\alpha)}\int_0^1 f((1-t)a)g(tb)(\mathrm{d}t)^\alpha \\ &\le \left(\frac{1}{a}\right)^\alpha {}_0I_a^{(\alpha)}\left[f^2(x)+g^2(x)\right] + \left(\frac{1}{b}\right)^\alpha {}_0I_b^{(\alpha)}\left[f^2(x)+g^2(x)\right] \\ &\quad + \frac{2^\alpha}{\Gamma(1+\alpha)}\int_0^1 f(ta)f((1-t)b)(\mathrm{d}t)^\alpha + \frac{2^\alpha}{\Gamma(1+\alpha)}\int_0^1 g(ta)g((1-t)b)(\mathrm{d}t)^\alpha. \end{aligned}$$

Proof. Taking advantage of the generalized subadditivity of the functions f and g, we can figure out that

$$f(ta+(1-t)b) \le f(ta)+f((1-t)b) \tag{3.16}$$

and

$$g(ta+(1-t)b) \le g(ta)+g((1-t)b). \tag{3.17}$$

Multiplying the inequalities (3.16) and (3.17), as well as noticing that all these terms are nonnegative, we obtain that

$$\begin{aligned} &f(ta+(1-t)a)g(ta+(1-t)a) \\ &\quad \le f(ta)g(ta)+f(ta)g((1-t)b)+f((1-t)b)g(ta)+f((1-t)b)g((1-t)b) \\ &\quad \le \left(\frac{1}{2}\right)^\alpha\left[\big(f(ta)+f((1-t)b)\big)^2+\big(g(ta)+g((1-t)b)\big)^2\right]. \end{aligned} \tag{3.18}$$

Integrating the resulting inequalities (3.18) regarding the variate t over the interval $[0,1]$ in the fractal sense, it yields that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)(\mathrm{d}t)^\alpha \\
&\le \frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta)g(ta)(\mathrm{d}t)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta)g((1-t)b)(\mathrm{d}t)^\alpha \\
&\quad + \frac{1}{\Gamma(1+\alpha)}\int_0^1 f((1-t)b)g(ta)(\mathrm{d}t)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_0^1 f((1-t)b)g((1-t)b)(\mathrm{d}t)^\alpha \\
&\le \left(\frac{1}{2}\right)^\alpha \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left[f^2(ta)+g^2(ta)\right](\mathrm{d}t)^\alpha \\
&\quad + \left(\frac{1}{2}\right)^\alpha \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left[f^2((1-t)b)+g^2((1-t)b)\right](\mathrm{d}t)^\alpha \\
&\quad + \frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta)f((1-t)b)(\mathrm{d}t)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_0^1 g(ta)g((1-t)b)(\mathrm{d}t)^\alpha.
\end{aligned} \tag{3.19}$$

As a consequence, by making appropriate substitutions for the inequalities (3.19), we deduce the desired results. This fulfills the proof. □

Corollary 3. *If one attempts to take $f(tx) \le t^\alpha f(x)$ in Theorem 4, then one acquires the following inequalities for the generalized subadditive functions*:

$$\begin{aligned}
\left(\frac{1}{b-a}\right)^\alpha {}_aI_b^{(\alpha)} f(x)g(x) &\le \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}R(a,b) + \left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]S(a,b) \\
&\le \left(\frac{1}{2}\right)^\alpha \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[\left(f(a)-f(b)\right)^2 + \left(g(a)-g(b)\right)^2\right] \\
&\quad + \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[f(a)f(b)+g(a)g(b)\right],
\end{aligned}$$

where

$$R(a,b) = f(a)g(a) + f(b)g(b)$$

and

$$S(a,b) = f(a)g(b) + f(b)g(a).$$

Proof. Taking advantage of $f(tx) \le t^\alpha f(x)$ and the inequalities (3.19), it yields that

$$\begin{aligned}
&\left(\frac{1}{b-a}\right)^\alpha {}_aI_b^{(\alpha)} f(x)g(x) \\
&\le f(a)g(a)\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2\alpha}(\mathrm{d}t)^\alpha + f(a)g(b)\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^\alpha(1-t)^\alpha(\mathrm{d}t)^\alpha
\end{aligned}$$

$$
\begin{aligned}
&+ f(b)g(a)\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha} + f(b)g(b)\frac{1}{\Gamma(1+\alpha)}\int_0^1 (1-t)^{2\alpha}(\mathrm{d}t)^{\alpha} \\
\leq &\left(\frac{1}{2}\right)^{\alpha}\left[\left(f^2(a)+g^2(a)\right)\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2\alpha}(\mathrm{d}t)^{\alpha}\right. \\
&\left.+\left(f^2(b)+g^2(b)\right)\frac{1}{\Gamma(1+\alpha)}\int_0^1 (1-t)^{2\alpha}(\mathrm{d}t)^{\alpha}\right] \\
&+\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha}\left[f(a)f(b)+g(a)g(b)\right].
\end{aligned}
$$

Calculating the following integrals, we derive that

$$
\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2\alpha}(\mathrm{d}t)^{\alpha} = \frac{1}{\Gamma(1+\alpha)}\int_0^1 (1-t)^{2\alpha}(\mathrm{d}t)^{\alpha} = \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)},
$$

and

$$
\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha} = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}.
$$

This terminates the proof. □

Theorem 5. *Suppose that $f,g\colon I=[0,\infty)\to\mathbb{R}_+^{\alpha}$ are both nonnegative generalized subadditive functions. If $a<b$ $(a,b\in I^{\circ})$ and $f\in C_{\alpha}[0,\infty)$, then the following inequalities in the fractal sense hold*

$$
\begin{aligned}
&\left(\frac{1}{2}\right)^{\alpha}\frac{1^{\alpha}}{\Gamma(1+\alpha)}f(a+b)g(a+b) \\
&\leq \left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x) \\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1\left[f(ta)g((1-t)a)+f(tb)g((1-t)b)\right](\mathrm{d}t)^{\alpha} \\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1\left[f(ta)g(tb)+f(tb)g(ta)\right](\mathrm{d}t)^{\alpha}
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
&\left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x) \\
&\leq \left(\frac{1}{a}\right)^{\alpha}{}_0I_a^{(\alpha)}f(x)g(x)+\left(\frac{1}{b}\right)^{\alpha}{}_0I_b^{(\alpha)}f(x)g(x) \\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta)g((1-t)b)(\mathrm{d}t)^{\alpha} \\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(tb)g((1-t)a)(\mathrm{d}t)^{\alpha}.
\end{aligned}
\tag{3.21}
$$

Proof. Taking advantage of the generalized subadditivity of the functions f and g, we can figure out that

$$f(a+b) \le f(ta+(1-t)b)+f((1-t)a+tb),$$

and

$$g(a+b) \le g(ta+(1-t)b)+g((1-t)a+tb).$$

Multiplying the above inequalities, as well as noticing that all these terms are non-negative, we attain that

$$\begin{aligned}
&f(a+b)g(a+b)\\
&\le f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\\
&\quad+[f(ta)+f((1-t)b)]\,[g((1-t)a)+g(tb)]\\
&\quad+[f((1-t)a)+f(tb)]\,[g(ta)+g((1-t)b)]\\
&=f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\\
&\quad+f(ta)g((1-t)a)+f(ta)g(tb)+f((1-t)b)g((1-t)a)+f((1-t)b)g(tb)\\
&\quad+f((1-t)a)g(ta)+f((1-t)a)g((1-t)b)+f(tb)g(ta)+f(tb)g((1-t)b).
\end{aligned} \tag{3.22}$$

Integrating the resulting inequality (3.22) regarding the variate t over the interval $[0,1]$ in the fractal sense, it yields that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 f(a+b)g(a+b)(\mathrm{d}t)^\alpha\\
&\le \frac{1}{\Gamma(1+\alpha)}\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)(\mathrm{d}t)^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 f((1-t)a+tb)g((1-t)a+tb)(\mathrm{d}t)^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left[f(ta)g((1-t)a)+f(tb)g((1-t)b)\right](\mathrm{d}t)^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left[f((1-t)a)g(ta)+f((1-t)b)g(tb)\right](\mathrm{d}t)^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left[f(ta)g(tb)+f(tb)g(ta)\right](\mathrm{d}t)^\alpha\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left[f((1-t)a)g((1-t)b)+f((1-t)b)g((1-t)a)\right](\mathrm{d}t)^\alpha.
\end{aligned} \tag{3.23}$$

By virtue of appropriate substitutions for the inequality (3.23), we can deduce inequality (3.20).

On the other hand, taking advantage of the subadditivity of functions f and g again, it follows that

$$\begin{aligned}
&f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\\
&\le f(ta)g(ta)+f(ta)g((1-t)b)+f((1-t)b)g(ta)+f((1-t)b)g((1-t)b)\\
&\quad+f((1-t)a)g((1-t)a)+f((1-t)a)g(tb)+f(tb)g((1-t)a)+f(tb)g(tb).
\end{aligned} \tag{3.24}$$

Similarly, integrating the inequalities (3.24) with regard to the variate t over the interval $[0,1]$ in the settings of fractal sets, as well as making appropriate substitutions, we can acquire the desired inequality (3.21). This fulfills the proof. □

Corollary 4. *If one attempts to take $f(tx)\le t^{\alpha}f(x)$ in Theorem 5, then one acquires the following inequalities for the generalized subadditive functions*:

$$\begin{aligned}
&\frac{2^{\alpha}}{\Gamma(1+\alpha)}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\le\left(\frac{1}{2}\right)^{\alpha}\frac{1^{\alpha}}{\Gamma(1+\alpha)}f(a+b)g(a+b)\\
&\le\left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x)+\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]R(a,b)+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}S(a,b)
\end{aligned}$$

and

$$\left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x)\le\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}R(a,b)+\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]S(a,b),$$

where $R(a,b)$ and $S(a,b)$ are same defined as Corollary 3.

Proof. Taking advantage of the inequality (3.23) and $f(tx)\le t^{\alpha}f(x)$, we can figure out that

$$\begin{aligned}
&\left(\frac{1}{2}\right)^{\alpha}\frac{1^{\alpha}}{\Gamma(1+\alpha)}f(a+b)g(a+b)\\
&\le\left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x)\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1-t)^{\alpha}(\mathrm{d}t)^{\alpha}\left[f(a)g(a)+f(b)g(b)\right]\\
&\quad+\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{2\alpha}(\mathrm{d}t)^{\alpha}\left[f(a)g(b)+f(b)g(a)\right]\\
&=\left(\frac{1}{b-a}\right)^{\alpha}{}_aI_b^{(\alpha)}f(x)g(x)\\
&\quad+\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]R(a,b)+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}S(a,b).
\end{aligned}$$

As a consequence, we can acquire the first desired outcome. Similarly, we can deduce another inequality by taking advantage of the same procedure. □

4. Conclusions

A class of generalized subadditive functions is first presented in the current article. With the aid of the proposed functions and generalized starshaped functions, several Hermite–Hadamard- and Pachpatte-type fractal integral inequalities with different bounds are derived as well, respectively. We believe that the present article can provide ideas for interested researchers to develop the fractal inequalities similarly using the fractal-fractional integral operators, see for example the published article [22].

Acknowledgments

The authors would like to thank the reviewer for his/her valuable comments and suggestions.

References

[1] O. Almutairi and A. Kiliçman, "Generalized Fejér–Hermite–Hadamard type via generalized $(h-m)$-convexity on fractal sets and applications." *Chaos, Solitons and Fractals*, vol. 147, 2021, doi: 10.1016/j.chaos.2021.110938.

[2] G. Anastassiou, A. Kashuri, and R. Liko, "Local fractional integrals involving generalized strongly m-convex mappings." *Arab. J. Math.*, vol. 8, pp. 95–107, 2019, doi: 10.1007/s40065-018-0214-8.

[3] H. Budak, T. Tunç, and M. Z. Sarikaya, "Fractional Hermite–Hadamard–type inequalities for interval-valued functions." *Proc. Amer. Math. Soc.*, vol. 148, no. 2, pp. 705–718, 2020, doi: 10.1090/proc/14741.

[4] M. R. Delavar and M. D. L. Sen, "A mapping associated to h-convex version of the Hermite–Hadamard inequality with applications." *J. Math. Inequal.*, vol. 14, no. 2, pp. 329–335, 2020, doi: 10.7153/jmi-2020-14-22.

[5] T. S. Du, H. Wang, M. A. Khan, and Y. Zhang, "Certain integral inequalities considering generalized m-convexity on fractal sets and their applications." *Fractals*, vol. 27, no. 7, 2019, doi: 10.1142/S0218348X19501172.

[6] H. Kadakal, "Hermite–Hadamard type inequalities for subadditive functions." *AIMS Mathematics*, vol. 5, no. 2, pp. 930–939, 2020, doi: 10.3934/math.2020064.

[7] J. Matkowski and T. Swiatkowski, "On subadditive functions." *Proc. Amer. Math. Soc.*, vol. 119, no. 1, pp. 187–197, 1993, doi: 10.1090/s0002-9939-1993-1176072-2.

[8] B. Meftah, M. Benssaad, W. Kaidouchi, and S. Ghomrani, "Conformable fractional Hermite–Hadamard type inequalities for product of two harmonic s-convex functions." *Proc. Amer. Math. Soc.*, vol. 149, no. 4, pp. 1495–1506, 2021, doi: 10.1090/proc/15396.

[9] H. X. Mo, X. Sui, and D. Y. Yu, "Generalized convex functions on fractal sets and two related inequalities." *Abstr. Appl. Anal.*, vol. 2014, 2014, doi: 10.1155/2014/636751.

[10] P. O. Mohammed, "Hermite–Hadamard inequalities for Riemann–Liouville fractional integrals of a convex function with respect to a monotone function." *Math. Meth. Appl. Sci.*, vol. 44, no. 3, pp. 2314–2324, 2021, doi: 10.1002/mma.5784.

[11] M. A. Noor, K. I. Noor, and M. U. Awan, "Simpson-type inequalities for geometrically relative convex functions." *Ukrainian Math. J.*, vol. 70, no. 7, pp. 1145–1154, 2018, doi: 10.1007/s11253-018-1558-0.

[12] L. P. Østerdal, "Subadditive functions and their (pseudo-) inverses." *J. Math. Anal. Appl.*, vol. 317, no. 2, pp. 724–731, 2006, doi: 10.1016/j.jmaa.2005.05.039.

[13] S. Özcan, "Integral inequalities of Hermite–Hadamard type for exponentially subadditive functions." *AIMS Mathematics*, vol. 5, no. 4, pp. 3002–3009, 2020, doi: 10.3934/math.2020194.

[14] B. G. Pachpatte, "On some inequalities for convex functions." *RGMIA Res. Rep. Coll.*, 2003.

[15] Z. Robles, J. E. Sanabria, and R. V. Sánchez C., "On a generalization of strongly η-convex functions via fractal sets." *Fractals*, vol. 29, no. 7, 2021, doi: 10.1142/S0218348X21502030.

[16] M. Z. Sarikaya and M. A. Ali, "Hermite–Hadamard type inequalities and related inequalities for subadditive functions." *Miskolc Math. Notes*, vol. 22, no. 2, pp. 929–937, 2021, doi: 10.18514/MMN.2021.3154.

[17] M. Z. Sarikaya and H. Budak, "Generalized Ostrowski type inequalities for local fractional integrals." *Proc. Amer. Math. Soc.*, vol. 145, no. 4, pp. 1527–1538, 2017, doi: 10.1090/proc/13488.

[18] W. B. Sun, "On generalization of some inequalities for generalized harmonically convex functions via local fractional integrals." *Quaestiones Math.*, vol. 42, no. 9, pp. 1159–1183, 2019, doi: 10.2989/16073606.2018.1509242.

[19] K. L. Tseng, S. R. Hwang, and S. S. Dragomir, "Fejér-type inequalities (I)." *J. Inequal. Appl.*, vol. 2010, 2010, doi: 10.1155/2010/531976.

[20] L. Xu, S. H. Yu, and T. S. Du, "Properties and integral inequalities arising from the generalized n-polynomial convexity in the frame of fractal space." *Fractals*, vol. 30, no. 4, 2022, doi: 10.1142/S0218348X22500840.

[21] X. J. Yang, *Advanced local fractional calculus and its applications.* New York: World Science Publisher, 2012.

[22] S. H. Yu, Y. X. Zhou, and T. S. Du, "Certain midpoint-type integral inequalities involving twice differentiable generalized convex mappings and applications in fractal domain." *Chaos, Solitons and Fractals*, vol. 164, 2022, doi: 10.1016/j.chaos.2022.112661.

Authors' addresses

Tingsong Du

(**Corresponding author**) Three Gorges Mathematical Research Center, China Three Gorges University, Yichang 443002, P. R. China

Current address: Department of Mathematics, College of Science, China Three Gorges University, Yichang 443002, P. R. China

E-mail address: tingsongdu@ctgu.edu.cn

Lei Xu

Three Gorges Mathematical Research Center, China Three Gorges University, Yichang 443002, P. R. China

E-mail address: leixu9903@163.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 663–677

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4568

RINGS CHARACTERIZED BY THE EXTENDING PROPERTY FOR FINITELY GENERATED SUBMODULES

BANH DUC DUNG

Received 03 February, 2023

Abstract. A module M is called ef-extending if every closed submodule which contains essentially a finitely generated submodule is a direct summand of M. In this paper, we prove some properties of rings via ef-extending modules and essentially finite injective modules. It is shown that a module M is an ef-extending module and whenever $M = H \oplus K$ with H essentially finite, then H is essentially finite K-injective if and only if for essentially finite submodules N_1, N_2 of M with $N_1 \cap N_2 = 0$, there exist submodules M_1, M_2 of M such that N_i is essential in M_i $(i = 1, 2)$ and $M_1 \oplus M_2$ is a direct summand of M. A ring R is right co-Harada if and only if R is right (or left) perfect with ACC on right annihilators and $R \oplus R$ is ef-extending as a right R-module, iff R is right (or left) perfect and $R_R^{(\mathbb{N})}$ is an ef-extending module. Some properties of ef-extending modules over excellent extension rings are considered.

2010 *Mathematics Subject Classification:* 16D50; 16D70; 16D80

Keywords: ef-extending module, extending (or CS) module, co-Harada ring, excellent extension ring

1. Introduction

The class of injective modules and their generalizations have been researched and developed (in [4, 13–15]). The structure of rings through the properties of the classes of these modules has been considered. One of the important generalizations of the injective module class is the extending (or C1, CS) module class. It can be said that this class of modules has been strongly researched and developed over the past decades. Many authors have given many important properties and characteristics of this class of modules. In addition, QF-rings and co-Harada rings have been studied via extending modules ([3, 5, 6, 16]). In 1991, Thuyet and Wisbauer ([17]) introduced the concept of ef-extending modules. This is a generalization of the class of extending modules. Then the authors also gave many properties of this class of modules. Continuing this work, the authors Quynh, Thuyet, Chien ([2, 16]) have obtained more properties of the class of ef-extending modules.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

In this paper, we give some other properties of ef-extending modules and the relatively injective of modules.

Firstly, we introduce the essentially finite N-injective modules. A module M is called *essentially finite N-injective* if any homomorphism from every essentially finite submodule of N to M can be extended to a homomorphism from N to M. We study the property of modules that whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2, then M_2 is essentially finite M_1-injective. It is shown that an essentially finite ef-extending module M satisfies C3 if and only if, whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2, then M_2 is essentially finite M_1-injective (Proposition 4). We also prove that a module M is an ef-extending module and whenever $M = H \oplus K$ with H essentially finite, then H is essentially finite K-injective if and only if for essentially finite submodules N_1, N_2 of M with $N_1 \cap N_2 = 0$, there exist submodules M_1, M_2 of M such that N_i is essential in M_i $(i = 1,2)$ and $M_1 \oplus M_2$ is a direct summand of M (Proposition 5).

Secondly, we characterize the structure of co-Harada rings and show that a ring R is right co-Harada if and only if R is right (or left) perfect with ACC on right annihilators and $R \oplus R$ is ef-extending as a right R-module, iff R is right (or left) perfect and $R_R^{(\mathbb{N})}$ is an ef-extending module (Theorem 1). The structure of rings in which the direct sum of any two ef-extending right R-modules is ef-extending are considered. It is shown that if R has property the direct sum of any two ef-extending right R-modules is ef-extending and $E(R_R) = \bigoplus_{i \in I} E_i$ where E_i is indecomposable for all $i \in I$, then R is a right Artinian ring whose uniform right R-modules have length at most two (Theorem 8).

Finally, we study ef-extendingmodules over excellent extension rings and show that if M is a right S-module and M_R is an ef-extending module, then M_S is an ef-extending module (Theorem 2). On the other hand, if M is a right R-module and $(M \otimes_R S)_S$ is an ef-extending module then M_R is an ef-extending module (Theorem 3).

Throughout this paper, R will denote an associative ring with unit and Mod-R the category of unital right R-modules. We write M_R (resp., ${}_RM$) to denote that M is a right (resp., left) R-module. Unless otherwise mentioned, by a module we will mean a right R-module. We denote the Jacobson radical of a ring R by $J(R)$ and the injective hull of M by $E(M)$. If A is a submodule of M, we denote by $A \leq M$. A submodule K of M is *essential* in M if $K \cap L \neq 0$ for every non-zero submodule L of M. In this case, M is called an *essential extension* of K and we write $K \leq^e M$. A submodule C of M is *closed* in M if C has no proper essential extension in M. A module M is called uniform if $M \neq 0$ and every non-zero submodule of M is essential in M. A submodule N of M is called *small* in M, denote by $N \ll M$ whenever for any submodule L of M, $N + L = M$ implies $L = M$. A module M is called a *small module* if M is small in $E(M)$. If M is not a small module, we say that M is *non-small*. We denote the radical

of a right R-module by $Rad(M)$. We refer to [1, 4, 10] for any undefined notion used in the text.

2. Direct sum of EF-extending modules are EF-extending

A module M is called *essentially finitely generated* (*essentially finite* for short) if M contains a finitely generated submodule that it is essential in M ([4, page 60]). A submodule N of M is called an *essentially finite submodule* if N is an essential finite right R-module.

Lemma 1. *Every direct summand of an essentially finite module is essentially finite.*

Proof. Let $M = N \oplus N'$ be an essentially finite module. Call $H = m_1R + m_2R + \cdots + m_kR$ an essential submodule of M. Write $m_i = n_i + n'_i$ with $n_i \in N$ and $n'_i \in N'$ for all $i = 1, 2, \ldots, k$. This shows that $K = n_1R + n_2R + \cdots + n_kR$ is an essential submodule of N. In fact, for every nonzero element y of N, there is x in R such that yx is nonzero in H, and so $yx = m_1r_1 + m_2r_2 + \cdots + m_kr_k$ for some r_i in R. It follows that $yx = n_1r_1 + n_2r_2 + \cdots + n_kr_k$ is nonzero in K. We deduce that K is essential in N. □

Let M and N be right R-modules. We define M to be *essentially finite N-injective* if any homomorphism from every essentially finite submodule of N to M can be extended to a homomorphism from N to M. We say to a module M is *essentially finite quasi-injective* if M is essentially finite M-injective.

Proposition 1. *Let M and N be right R modules.*

(1) *Any direct product $\prod_I M_i$ is essentially finite N-injective if and only if every M_i is essentially finite N-injective.*
(2) *If M is essentially finite N-injective, then M is essentially finite K-injective and essentially finite N/A-injective for all submodules A and K of N with A finitely generated.*

Proof. (1) Let $M = \prod_I M_i$. For each $i \in I$, let $\pi_i \colon M \to M_i$ be the canonical projection and $\iota_i \colon M_i \to M$ be the inclusion map.

Assume that M is essentially finite N-injective. Let C be an essentially finite submodule of N and $\psi \colon C \to M_i$ be a homomorphism. Let $\iota \colon C \to N$ be the inclusion map.

$$\begin{tikzcd} & M_i & \\ 0 \arrow[r] & C \arrow[u, "\psi"] \arrow[r, "\iota"] & N \end{tikzcd}$$

Since M is essentially finite N-injective, there is a homomorphism $\alpha\colon N \to M$ such that $\alpha\iota = \iota_i\psi$. Call $\gamma = \pi_i\alpha$. It follows that $\gamma\iota = \psi$. Thus, M_i is essentially finite N-injective.

Conversely, assume that M_i is essentially finite N-injective for all $i \in I$. Let C be an essentially finite submodule of N, $\iota\colon C \to N$ be the inclusion map and $\psi\colon C \to M$ be a homomorphism. For each $i \in I$, since M_i is essentially finite N-injective, there exists a homomorphism $h_i\colon M \to M_i$ such that $h_i\iota = \pi_i\psi$. Consider the homomorphism $h\colon M \to \prod_{i\in I} M_i$ defined by $h(m) = (h_i(m))_{i\in I}$ for all $m \in M$. Then, $h\iota = \psi$. We deduce that M is essentially finite N-injective.

(2) Let $f\colon U \to M$ be a homomorphism from an essentially finite submodule U of K to M. Since M is essentially finite N-injective, f is extended to a homomorphism $g\colon N \to M$. It follows that $g|_K\colon K \to M$ is an extension of f. Thus, M is essentially finite K-injective.

Next, we show that M is essentially finite N/A-injective with each finitely generated submodule A of N. Let $f\colon X/A \to M$ be a homomorphism from an essentially finite submodule X/A of N/A to M. Call $A_1 = (a_1+A)R + (a_2+A)R + \cdots + (a_k+A)R$ an essential submodule of X/A. One can check that $a_1R + a_2R + \cdots + a_kR + A$ is essential in X and so X is an essentially finite submodule of N. Let $\pi\colon N \to N/A$ be the natural projection. Since M is essentially finite N-injective, there is a homomorphism $g\colon N \to M$ such that g is an extension of $f \circ \pi|_X$. Then, $g(A) = (f \circ \pi|_X)(A) = 0$ so that there exists a homomorphism $h\colon N/A \to M$ such that $h \circ \pi = g$. For each $x \in X$, we have

$$h(x+A) = (h \circ \pi)(x) = g(x) = (f \circ \pi|_X)(x) = f(x).$$

It is shown that h is an extension of f. We deduce that M is essentially finite N/A-injective. □

Corollary 1. *Let M be an essentially finite quasi-injective right R-module. Then, every essentially finite submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.*

Proof. Let A be an essentially finite submodule of M that is isomorphic to a direct summand B of M. Since M is essentially finite M-injective, B is essentially finite M-injective by Proposition 1(1) and so A is essentially finite M-injective.

$$\begin{tikzcd} & A & \\ 0 \arrow[r] & A \arrow[u, "1_A"] \arrow[r, "\iota"] & M \end{tikzcd}$$

It follows that there is a homomorphism $f\colon M \to A$ such that $g \circ \iota = 1_A$ with $\iota\colon A \to M$ the inclusion map. Therefore, ι splits and so A is a direct summand of M. □

A nonzero module M does not contain an infinite direct sum of nonzero submodules if and only if there exists a positive integer n such that M contains an essential submodule of the form $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ for some uniform submodules U_i of M. Furthermore, the integer n is an invariant of the module M called the *uniform dimension* or *Goldie dimension* and denoted by $\dim(M)$. If $M = 0$ we write $\dim(M) = 0$.

Proposition 2. *Let M be a right R-module.*

(1) *If M is a right R-module with finite Goldie dimension, then M is quasi-injective if and only if M is essentially finite quasi-injective.*
(2) *If $M = M_1 \oplus M_2$ is essentially finite quasi-injective then M_1 is essentially finite M_2-injective.*

Proof. (1) Assume that M is a right R-module with finite Goldie dimension. Let $f : A \to M$ be a homomorphism from a submodule A of M to M. It is well-known that every submodule of a right R-module with finite Goldie dimension has finite Goldie dimension. Then, A is an essentially finite submodule of M. We have that M is quasi-injective and obtain that f is extended to an endomorphism of M. We deduce that M is quasi-injective.

(2) Assume that $M = M_1 \oplus M_2$ is essentially finite quasi-injective. Then, M_1 is essentially finite M-injective by Proposition 1(1). Again by Proposition 1(2) we have that M_1 is essentially finite M_2-injective. □

Corollary 2. *Let M be a right R-module. If M is a Noetherian module, then M is quasi-injective if and only if M is essentially finite quasi-injective.*

Proof. Since every Noetherian right R-module has finite Goldie dimension, M is quasi-injective if M is essentially finite quasi-injective by Proposition 2. □

Recall that a right R-module M is called *Zelmanowitz regular* if, every finitely generated submodule of M is projective and is a direct summand of M ([18]). Note that R_R is a Zelmanowitz regular module if and only if R is a von Neumann regular ring.

Lemma 2. *If M is a Zelmanowitz regular module, then M is essentially finite quasi-injective.*

Proof. Let H be an essentially finite submodule of M. Then, there is a finitely generated submodule I of M such that I is essential in H. We have that M is Zelmanowitz regular and obtain that I is a direct summand of M and so, it is a direct summand of H. This implies that $H = I$ is a direct summand of M. We deduce that every homomorphism from H to M can be extended to an edomorphism of M. Thus, M is essentially finite quasi-injective. □

From Lemma 2, we have the following example of an essentially finite quasi-injective module which is not quasi-injective.

Example 1. Let D be a division ring and ${}_DV$ be a left D-vector space of infinite dimension. Take $R = \mathrm{End}_D(V)$. It is well-known that R is von Neumann regular but not right self-injective. By Lemma 2, R_R is essentially finite quasi-injective.

Lemma 3. *Let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2. Then, the following conditions are equivalent:*

(1) *M_2 is essentially finite M_1-injective.*
(2) *For each essentially finite submodule N of M with $N \cap M_2 = 0$, there exists a submodule M' of M containing N such that $M = M' \oplus M_2$.*

Proof. $(1) \Rightarrow (2)$. For $i = 1, 2$, let $\pi_i \colon M \longrightarrow M_i$ denote the canonical projection. Consider the following diagram with N an essentially finite submodule of M:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "\beta"] & M_1 \arrow[dl, dotted, "\phi"] \\ & M_2 & \end{tikzcd}$$

where $\alpha = \pi_1|_N$, $\beta = \pi_2|_N$. It is easy to see that α is a monomorphism, and so $N \cong \alpha(N) \leq M_1$. By (1), there exists a homomorphism $\phi \colon M_1 \longrightarrow M_2$ such that $\phi\alpha = \beta$. Let $M' = \{x + \phi(x) | x \in M_1\}$. One can check that $M = M' \oplus M_2$ and N is contained in M'.

$(2) \Rightarrow (1)$. Let K be an essentially finite submodule of M_1, and $f \colon K \longrightarrow M_2$ a homomorphism. Put $L = \{y - f(y) \mid y \in K\}$. Since K is essentially finite, L is also an essentially finite submodule of M with $L \cap M_2 = 0$. By (2), $M = L' \oplus M_2$ for some submodule L' of M containing L. Let $\pi \colon M \longrightarrow M_2$ denote the canonical projection (for the direct sum $M = L' \oplus M_2$). Let $\bar{f} = \pi|_{M_1} \colon M_1 \longrightarrow M_2$ and, for any $y \in K$, we have $\bar{f}(y) = \bar{f}(y - f(y) + f(y)) = f(y)$. It means that $\bar{f}$ is an extension of f and so M_2 is essentially finite M_1-injective. $\square$

Corollary 3. *Let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2. If M_2 is essentially finite M_1-injective, then for every essentially finite closed submodule N of M such that $N \cap M_1$ is essential in M_1 and $N \cap M_2 = 0, M = N \oplus M_2$.*

Proof. Let K be an essentially finite closed submodule of M such that $K \cap M_1$ is essential in M_1 and $K \cap M_2 = 0$. We have $K \cap M_1 = K \cap ((K \cap M_1) \oplus M_2)$. It follows that $K \cap M_1$ is essential in K. By Lemma 3, there exists a submodule M' of M such that $M = M' \oplus M_2$ and $K \leq M'$. Next, we show that K is essential in M'. In fact, let $x = m_1 + m_2$ be a nonzero element of M with $m_i \in M_i$. If m_1 is nonzero, there exists r in R such that $m_1 r \in K \cap M_1$ is nonzero. It follows that $xr = m_1 r + m_2 r \in K \oplus M_2$ is nonzero. From this, we deduce that there exists $y \in R$ such that $xy \in K \oplus M_2$ is nonzero. It is shown that $K \oplus M_2$ is essential in M. On the other hand, we have that $K = M' \cap (K \oplus M_2)$ and obtain that K is essential in M'. But, K is closed in M, and so $K = M'$. $\square$

Recall that a module M is *extending* (resp., *ef-extending*) if every closed (resp., essentially finite closed) submodule of M is a direct summand of M (see [4, 17]).

Example 2. Let D be a division ring and V a left D-vector space of infinite dimension. Take $S = \mathrm{End}_D(V)$ and $R = \begin{pmatrix} S & S \\ S & S \end{pmatrix}$. It is well-known that S is a von Neumann regular ring. One can check that R_R is ef-extending but not extending.

It is well-known that a module M is extending if every submodule of M is essential in a direct summand of M. We have a similar situation for ef-extending modules.

Proposition 3. *The following conditions are equivalent for a right R-module M:*

(1) *M is ef-extending.*
(2) *Every essentially finite submodule of M is essential in a direct summand of M.*

Proof. $(1) \Rightarrow (2)$. Let N be an essentially finite submodule of M. Then, N contains a finitely generated submodule which is essential in N. Call H a maximal essential extension of N in M. One can check that H is an essentially finite closed submodule of M. By (1), H is a direct summand of M.

$(2) \Rightarrow (1)$. Let C be an essentially finite closed submodule of M. By (2), there is a direct summand K of M such that C is essential in K. We have that C is closed and obtain that $C = K$ is a direct summand of M. □

Corollary 4. *Let M be a right R-module with finite Goldie dimension or finitely cogenerated. Then, M is an extending module if and only if M is an ef-extending module.*

Corollary 5. *Let M be a finitely generated ef-extending right R-module. Assume that*

(1) *every local direct summand of M is a direct summand, or*
(2) *$End(M)$ does not contain an infinite set of orthogonal idempotents.*

Then M is an extending module.

Proof. Assume that every local direct summand of M is a direct summand, or $End(M)$ does not contain an infinite set of orthogonal idempotents. Then, M is a direct sum of uniform submodules by [17, Corollary 2.3]. Moreover, we have that M is finitely generated and obtain that M has finite Goldie dimension. We deduce that M is extending by Corollary 4. □

Recall that a module M satisfies $C3$ if, $M_1 \oplus M_2$ is a direct summand of M for any two direct summands M_1 and M_2 of M with $M_1 \cap M_2 = 0$.

Lemma 4 ([5, Lemma 6])**.** *The following statements are equivalent for a right R-module M.*

(1) *M satisfies $C3$.*

(2) *For all direct summands P and Q of M with $P \cap Q = 0$, there exists a submodule P' of M such that $M = P \oplus P'$ and $Q \leq P'$.*

Proposition 4. *An essentially finite ef-extending module M satisfies C3 if and only if whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1, M_2, then M_2 is essentially finite M_1-injective.*

Proof. ($\Rightarrow$). Assume that M is ef-extending satisfying C3. Let N be an essentially finite submodule N of M with $N \cap M_2 = 0$. Since M is ef-extending, there exists a direct summand N' of M such that N is essential in N'. Clearly $N' \cap M_2 = 0$. By Lemma 4, $M = M' \oplus M_2$ for some submodule M' such that $N' \leq M'$, and so $N \leq M'$. Thus M_2 is essentially finite M_1-injective by Lemma 3.

($\Leftarrow$) Assume that M_2 is essentially finite M_1-injective whenever $M = M_1 \oplus M_2$. By Lemma 3 and Lemma 4, M satisfies C3. □

Corollary 6. *If $M = M_1 \oplus M_2$ is an essentially finite ef-extending right R-module satisfying C3, then M_i is essentially finite M_j-injective for all $i, j \in \{1,2\}$, $i \neq j$.*

Proposition 5. *The following conditions are equivalent for a right R-module M:*

(1) *For essentially finite submodules N_1, N_2 of M with $N_1 \cap N_2 = 0$, there exist submodules M_1 and M_2 of M such that N_i is essential in M_i ($i = 1,2$) and $M_1 \oplus M_2$ is a direct summand of M.*

(2) *M is an ef-extending module and whenever $M = H \oplus K$ with H essentially finite, then H is essentially finite K-injective.*

Proof. (1) $\Rightarrow$ (2) Let N be an essentially finite submodule of M. Call N' a complement of N in M. Then, $N \cap N' = 0$. Take K a finitely generated submodule of N' and so $N \cap K = 0$. By (1), there are submodules M_1 and M_2 of M such that N is essential in M_1, K is essential in M_2 and $M_1 \oplus M_2$ is a direct summand of M. It follows that M is an ef-extending module. Next, we show that if M has a decomposition $M = H \oplus K$ with H essentially finite, then H is essentially finite K-injective. In fact, let L be an essentially finite submodule of M with $L \cap H = 0$. By (1), there are submodules P and Q of M such that L is essential in P, H is essential in Q and $P \oplus Q$ is a direct summand of M. We have that H is closed and obtain that $H = Q$. Write $M = P \oplus Q \oplus W$. Then, $M = H \oplus (P \oplus W)$ and $L \leq P \oplus W$. By Lemma 3, H is essentially finite K-injective.

(2) $\Rightarrow$ (1) Let N_1 and N_2 be essentially finite submodules of M with $N_1 \cap N_2 = 0$. Call L a finitely generated submodule of M such that L is essential in N_1. Let C be a complement of L in M, and so $N_1 \cap C = 0$. By Zorn's lemma, take D a maximal essential extension of N_1 in M. One can check that $D \cap C = 0$, D is closed and L is essential in D. By (2), $M = D \oplus D'$ for some submodule D' of M. Moreover, from the essentiality of N_1 in D, it immediately infers that $N_2 \cap D = 0$. Again, by (2) and Lemma 3, we have a decomposition $M = D \oplus D$" with D" a direct summand of M containing N_2. Since M is an ef-extending module, D" is also an ef-extending module. By Proposition 3, N_2 is essential in a direct summand W of D". Write D" $= W \oplus W'$. Then, $M = D \oplus W \oplus W'$ with N_1 essential in D and N_2 essential in W. □

Note that the direct sum of two direct summands which are ef-extending is not necessarily ef-extending.

Example 3. Let p be a prime number. Then $\mathbb{Z}$-modules $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/p^3\mathbb{Z}$ are ef-extending. But $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not ef-extending because $(1+p\mathbb{Z}, p+p^3\mathbb{Z})\mathbb{Z}$ is a closed submodule of M (which contains a finitely generated, essential submodule) and it is not direct summand of M.

Proposition 6. *Let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2. Assume that the following conditions hold:*

(1) *every closed submodule K of M with $K \cap M_1$ essential in K is a direct summand of M;*
(2) *every essentially finite closed submodule L of M with $L \cap M_1 = 0$ is a direct summand of M.*

Then M is ef-extending.

Proof. Let N be an essentially finite closed submodule of M. If $N \cap M_1 = 0$, we are done. Otherwise, there exists a closed submodule H of N such that $N \cap M_1$ is essential in H. Then, H is closed in M and $H \cap M_1 = N \cap M_1$ is essential in H. By (1), H is a direct summand of M. Take $M = H \oplus H'$, and so $N = H \oplus (N \cap H')$. It follows that $N \cap H'$ is a closed submodule of N, and so it is closed in M. By Lemma 1, $N \cap H'$ is an essentially finite closed submodule of M. We have $(N \cap H') \cap M_1 = 0$ and obtain, from (2), that $N \cap H'$ is a direct summand of M. We write $H' = (N \cap H') \oplus L$. Thus, $M = H \oplus H' = H \oplus (N \cap H') \oplus L = N \oplus L$. □

Corollary 7. *Assume that $M = M_1 \oplus M_2$ with M_1 is extending and M_2 is ef-extending. If one of the following conditions holds, then M is ef-extending.*

(1) *M_2 is M_1-injective and M_1 is essentially finite M_2-injective.*
(2) *M_2 is essentially finite M_1- injective and every closed submodule K of M such that $K \cap M_2 = 0$, is a direct summand of M.*

Proof. Assume that M_2 is M_1-injective and M_1 is essentially finite M_2-injective.

Firstly, we show that the condition (1) in Proposition 6 is satisfied. Let K be a closed submodule of M with $K \cap M_1$ an essential submodule of K. Then, $K \cap M_2 = 0$. Sine M_2 is M_1-injective, there exists a decomposition $M = L \oplus M_2$ such that L contains K. It follows that K is closed in L and $L \cong M_1$ an extending module. We deduce that K is a direct summand L, and so it is a direct summand of M.

Next, we show that the condition (1) in Proposition 6 is satisfied. Let N be an essentially finite closed submodule of M with $N \cap M_1 = 0$. Since M_1 is essentially finite M_2-injective, there is a submodule H of M such that $M = H \oplus M_1$ and $N \leq H$. Then, N is an essentially finite closed submodule of H and $H \cong M_2$ is an ef-extending module. We deduce that N is a direct summand of H. Thus, N is a direct summand of M. From Proposition 6, we deduce that M is an ef-extending module.

Case (2) has a similar proof. □

Lemma 5. *Let A and B be uniform modules with local endomorphism rings such that $M = A \oplus B$ is ef-extending. Let C be a submodule of A and $f\colon C \to B$ a homomorphism. Then the following hold.*

(1) *If f cannot be extended to a homomorphism from A to B, then f is a monomorphism and B is embedded in A.*
(2) *If any monomorphism $B \to A$ is an isomorphism, then B is A-injective.*
(3) *If B is not embedded in A, then B is A-injective.*

Proof. (1). Suppose f cannot be extended to A. Let

$$U = \{x - f(x) | x \in C\} \leq A \oplus B.$$

Then $U \cong C$ is a uniform submodule of M and clearly $U \cap B = 0$. Hence there is a direct summand U^* of M such that U is essential in U^*. By the Krull-Schmidt-Azumaya Theorem ([1, Corollary 12.7]), we have $M = A \oplus U^*$ or $M = U^* \oplus B$. Suppose that $M = B \oplus U^*$. Let $\pi\colon B \oplus U^* \to B$ be the projection. For every $c \in C$, we have $c = f(c) + c - f(c)$ and so $(\pi|_A)(c) = \pi(f(c) + c - f(c)) = f(c)$. It is shown that $\pi|_A : A \to B$ is an extension of $f\colon C \to B$, a contradiction to our assumption. Thus $M = A \oplus U^*$ which implies that $f(x) \neq 0$ for all $x \neq 0$, i.e., f is a monomorphism. We have $U^* \cap B = 0$ and obtain that B is embedded in A.

(2). As in the proof of (1), given any homomorphism $f\colon C \to B$ with C a submodule of A. Suppose that $M = A \oplus U^*$. Let $\psi\colon A \oplus U^* \to A$ be the projection. Then, clearly $\psi|_B$ is a monomorphism (because U is essential in U^*, hence an isomorphism by the hypothesis). It follows easily that $M = B \oplus U^*$, so that, as in (1) f can be extended to a homomorphism from A to B. It follows that B is A-injective.

(3). It is clear by (1). □

A module M is called a *non-cosmall* module if M is a homomorphic image of a projective module P whose kernel is not essential in P. A ring R is called right *co-Harada* (or co-H) if it satisfies ACC on right annihilators and every non-cosmall right R-module contains a non-zero projective direct summand ([6]). We have a characterization of right co-Harada via ef-extending.

Theorem 1. *Let R be a ring. Then, the following statements are equivalent.*

(1) *R is right co-Harada.*
(2) *R is right (or left) perfect with ACC on right annihilators and $R \oplus R$ is ef-extending as a right R-module.*
(3) *R is right (or left) perfect and $R_R^{(\mathbb{N})}$ is an ef-extending module.*

Proof. $(1) \Rightarrow (2), (3)$ are clear.

$(2) \Rightarrow (1)$. Let R be a right perfect ring. We need only show that $E(R_R)$ is Σ-injective (i.e., $E(R_R)^{(I)}$ is injective for every index set I). Since R has ACC on right annihilators, it suffices to show that $E(R_R)$ is projective. Let e be any primitive idempotent of R. Suppose that eR is non-small. Then for any primitive idempotent f of R,

assume that $h\colon eR \to fR$ is a monomorphism, if $eR \cong A = h(eR) < fR$. Since fR has only maximal, small submodule in fR. Thus $A \ll fR$ and so $A \ll E(A)$. Moreover, there exists $E(A) \cong E(eR)$. It implies that $eR \ll E(eR)$. This is a contradiction. It means that $h(eR) = fR$ and so h is an isomorphism. Note that eR and fR are uniform modules. Since $(R \oplus R)_R$ is ef-extending, $eR \oplus fR$ is ef-extending, too. By Lemma 5, eR is fR- injective. It is shown that eR is injective as a right R-module. Now suppose that eR is small. Since R is right perfect, there is a projective cover $\varphi\colon P \to E(eR)$, where $P = \oplus_I P_i$ each P_i, is indecomposable projective. If P_j is small for some $j \in I$, then $\varphi(P_j)$ is small in $E(eR)$ by [10, Lemma 4.2]. Let $P = P_j \oplus Q$, then $E(eR) = \varphi(Q)$. But $Ker(\varphi)$ is small in P, so it follows that $Q = P$, a contradiction. Thus each P_i is non-small, hence injective. It is easy to see that, because eR is projective, eR can be embedded into a finite direct sum of the P_i. Therefore $E(eR)$ is projective. We write $R_R = e_1R \oplus \cdots \oplus e_kR \oplus f_1R \oplus \cdots \oplus f_nR$, where $\{e_i\}_{i=1}^k \cup \{f_j\}_{j=1}^n$ is a complete set of orthogonal primitive idempotents of R, in which each e_iR is a non-small module $(i = 1,\ldots,k)$, each f_jR is a small module $(j = 1,\ldots,n)$. Therefore $E(R_R)$ is projective. Thus R is semiprimary and $Soc(R_R)$ is essential in R_R by [6, Theorem 6]. This implies that $R \oplus R$ is extending as a right R-module by Corollary 4. Thus R is right co-Harada by [3, Theorem II].

Now, we assume that R is left perfect. It follows that $Soc(R_R)$ is essential in R_R, and so R_R is finitely cogenerated. It follows that $(R \oplus R)_R$ is extending by Corollary 4. From [15, Theorem 3.7], it infers that R is right co-Harada.

(3) $\Rightarrow$ (2). By (3), R^n is ef-extending for each $n \in \mathbb{N}$. We will prove that R has ACC on right annihilators. Now we claim that $E(R_R)^{(\mathbb{N})}$ is injective. Since R is right perfect, $R_R = e_1R \oplus \cdots \oplus e_kR \oplus f_1R \oplus \cdots \oplus f_nR$, where $\{e_i\}_{i=1}^k \cup \{f_j\}_{j=1}^n$ is a complete set of orthogonal primitive idempotents of R such that:

- e_iR is non-small for all $i = 1,2,\ldots,k$.
- f_jR is small for all $j = 1,2,\ldots,n$.

By proving the same (2) $\Rightarrow$ (1), we have e_iR is injective for all $i = 1,2,\ldots,k$ and obtain that $E(R_R)$ is projective. For any $j = 1,2,\ldots,n$, since R is right ef-extending, f_jR is uniform and so $E(f_jR)$ is indecomposable. Moreover $E(f_jR)$ is projective, there is $i \in \{1,2,\ldots,k\}$ such that $E(f_jR) \cong e_iR$. From this, it is easy to see that

$$E(R_R) = (\oplus_{I_1} e_1R) \oplus \cdots \oplus (\oplus_{I_k} e_kR),$$

where I_i are finite sets. Then, we have

$$E(R_R)^{(\mathbb{N})} \cong ((\oplus_{I_1} e_1R) \oplus \cdots \oplus (\oplus_{I_k} e_kR))^{(\mathbb{N})} \cong (e_1R \oplus e_2R \oplus \cdots \oplus e_kR)^{(\mathbb{N})}.$$

Note that $(e_1R \oplus e_2R \oplus \cdots \oplus e_kR)^{(\mathbb{N})}$ is isomorphic to a direct summand of $R_R^{(\mathbb{N})}$ and each e_iR is injective and indecomposable. We have that $R_R^{(\mathbb{N})}$ is ef-extending and obtain that $(e_1R \oplus e_2R \oplus \cdots \oplus e_kR)^{(\mathbb{N})}$ is ef-extending. Thus, by [4, Lemma 8.10], $E(R_R)^{(\mathbb{N})} \cong (e_1R \oplus e_2R \oplus \cdots \oplus e_kR)^{(\mathbb{N})}$ is self-injective, and so it is injective. This implies that R has ACC on right annihilators.

It is similar if R is left perfect. Note that $E(R_R)$ is projective by Corollary 4 and [15, Theorem 3.3]. □

Corollary 8 ([6, Theorem 3])**.** *Let R be a ring. Then the following statements are equivalent.*

(1) *R is right co-Harada.*
(2) *R is right perfect and $R_R^{(\mathbb{N})}$ is an extending module.*

Proposition 7. *Let R be a right nonsingular ring. Then the following statements are equivalent.*

(1) *R is right co-Harada.*
(2) *R is right (or left) perfect and $R \oplus R$ is ef-extending as a right R-module.*

Proof. By Lemma 6, Theorem 7 in [3]. □

Lemma 6. *The following statements are equivalent:*

(1) *The direct sum of any two uniform modules is ef-extending.*
(2) *Any uniform self-injective module has length at most 2.*
(3) *Any direct sum of uniform modules is extending.*

Proof. $(1) \Rightarrow (2)$. Consider any uniform injective module U. Suppose $x \in Rad(U)$ and T is a maximal nonzero submodule of xR. Then, U and xR/T have local endomorphism rings and $U \oplus xR/T$ is ef-extending by assumption. Hence the map $f\colon xR \to xR/T$ can be extended to $\bar{f}\colon U \to xR/T$ by Lemma 5. However $xR \leq Rad(U) \leq Ker\bar{f}$, which yields a contradiction. We conclude that $Rad(U)$ is semisimple and hence simple.

Assume that K_1 and K_2 are two distinct maximal submodules of U. Then any monomorphism $f\colon K_i \to K_j$ is onto for $i, j \in \{1,2\}$, since f extends to a monomorphism of U which has to be an automorphism. So the endomorphism rings of K_1 and K_2 are local. Now $K_i \oplus K_j$ is extending for $i, j \in \{1,2\}$ and hence K_1 is both K_2-injective and K_1-injective by Lemma 5. We have $K_1 + K_2 = U$ and obtain that K_1 is U-injective and hence it is a direct summand in U, a contradiction.

$(2) \Rightarrow (3)$. By [4, 13.1].

$(3) \Rightarrow (1)$. Obvious. □

Proposition 8. *Assume that R has property the direct sum of any two ef-extending right R-modules is ef-extending and $E(R_R) = \bigoplus_{i \in I} E_i$ where E_i is indecomposable for all $i \in I$. Then R is a right Artinian ring whose uniform right R-modules have length at most two.*

Proof. Assume that $E(R_R) = \bigoplus_{i \in I} E_i$ where E_i is indecomposable for all $i \in I$. Then, E_i is a uniform module for every $i \in I$. Let $\{V_i\}_{i \in I_1}$ be any nonempty family of injective hulls of simple modules. Call $V = \oplus_i V_i$. Then the module

$$M = E(R_R) \oplus V$$

is extending by Lemma 6, hence quasi-injective by [4, Lemma 8.10]. Therefore V is a quasi-injective module which is $E(R_R)$-injective. Thus, V is injective. This implies that R is right Noetherian.

By the above argument, every injective module is a direct sum of uniform modules. Since by Lemma 5, each uniform module has length at most two, every injective is a direct sum of injective hulls of simple modules. This proves that R is right Artinian. Hence the proof is complete. □

Corollary 9. *Assume that R has finite Goldie dimension and the direct sum of any two ef-extending right R-modules is ef-extending. Then, R is a right Artinian ring whose uniform right R-modules have length at most two.*

3. EF-EXTENDING MODULES AND EXCELLENT EXTENSIONS

When R is a subring of the ring S, and R and S have the same identity 1, the ring S is a *right excellent extension* of R if

(1) S_R and ${}_RS$ are free modules with a basis $\{1 = a_1, a_2, ..., a_n\}$ such that $a_iR = Ra_i$ for $i = 1, ..., n$.
(2) For any submodule A_S of a module M_S, if A_R is a direct summand of M_R, then A_S is a direct summand of M_S.

Excellent extensions were introduced by Passman ([11]). Let R be an associative ring with identity and let G be a finite multiplicative group. Then the crossed product $R * G$ is an associative ring constructed from R and G analogous. If $|G|^{-1} \in R$, then $R * G$ is a right excellent extension of R (in [7] and [12]).

Through this section, the ring S is a right excellent extension of R.

Lemma 7 ([8, Proposition 1.6], [7, Proposition 1.1])**.** *Let A_S be a submodule of an S-module M. Then*

(1) *A_S is closed in M_S if and only if A_R is closed in M_R.*
(2) *A_S is essential in M_S if and only if A_R is essential in M_R.*

Lemma 8 ([9, Lemma 2.4])**.** *Let A_R be a submodule of M_R. Then A_R is a closed submodule of M_R if and only if $(A \otimes_R S)_S$ is a closed submodule of $(M \otimes_R S)_S$.*

Theorem 2. *Let M be a right S-module. If M_R is an ef-extending module then M_S is an ef-extending module.*

Proof. Let M_R be an ef-extending module. Let A_S be a closed submodule of M_S which contains essentially a finitely generated submodule T_S. By Lemma 7, A_R is a closed submodule of M_R and T_R is essetial in A_R. Take $T = t_1S + t_2S + \cdots + t_kS$. One can check that T_R is generated by $\{t_ia_j | 1 \leq i \leq k, 1 \leq j \leq n\}$. We have that M_R is an ef-extending module and obtain that A_R is a direct summand of M_R. Since S is a right excellent extension of R, A_S is a direct summand of M_S. We deduce that M_S is an ef-extending module. □

Theorem 3. *Let M be a right R-module. If $(M \otimes_R S)_S$ is an ef-extending module then M_R is an ef-extending module.*

Proof. Let $(M \otimes_R S)_S$ be an ef-extending module. Let A_R be a closed submodule of M_R which contains essentially a finitely generated submodule T_R. By Lemma 8, $(A \otimes_R S)_S$ is a closed submodule of $(M \otimes_R S)_S$. We have that $(T \otimes_R S)_S$ is a finitely generated submodule of $(A \otimes_R S)_S$. On the other hand, we have

$$(T \otimes_R S)_R = (T \otimes a_1) \oplus (T \otimes a_2) \oplus \cdots \oplus (T \otimes a_n),$$
$$(A \otimes_R S)_R = (A \otimes a_1) \oplus (A \otimes a_2) \oplus \cdots \oplus (A \otimes a_n).$$

Note that $T \otimes a_i \cong T \otimes_R R \cong T$ and $A \otimes a_i \cong A \otimes_R R \cong A$ for all $i = 1, 2, \ldots, n$. One can check that $(T \otimes a_i)_R$ is essential in $(A \otimes a_i)_R$ for all $i = 1, 2, \ldots, n$. It follows that $(T \otimes_R S)_R$ is essential in $(A \otimes_R S)_R$, and so $(T \otimes_R S)_S$ is essential in $(A \otimes_R S)_S$ by Lemma 7. Since $(M \otimes_R S)_S$ is an ef-extending module, $(A \otimes_R S)_S$ is a direct summand of $(M \otimes_R S)_S$. Then, $(A \otimes_R S)_R$ is a direct summand of $(M \otimes_R S)_R$. We have that $(A \otimes 1)_R$ is a direct summand of $(M \otimes_R S)_R$ and $(A \otimes 1)_R \leq (M \otimes 1)_R$ and obtain that $(A \otimes 1)_R$ is a direct summand of $(M \otimes 1)_R$. We deduce that A is a direct summand of M. Thus, M_R is an ef-extending module. □

An R-module M is called *finitely Σ-ef-extending* if every finite direct sum of copies of M is ef-extending. The ring R is called *right finitely Σ-ef-extending* if R_R is finitely Σ-ef-extending ([17]).

Theorem 4. *Let S be a right excellent extension of R. Then R is right finitely Σ-ef-extending if and only if S is right finitely Σ-ef-extending.*

Proof. Suppose R_R is finitely Σ-ef-extending. Then for any $k > 0$, $(S^k)_R \cong (R^{nk})_R$ and so $(S^k)_S$ is ef-extending by Theorem 2. For the converse, suppose S_S is finitely Σ-ef-extending. Then for any $k > 0$, $(R^k \otimes_R S)_S \cong (S^k)_S$ is ef-extending. By Theorem 3, $(R^k)_R$ is ef-extending. □

It can easily be checked that the theorem still holds (with the same proof) if "finitely Σ-ef-extending" is replaced by "coutably Σ-ef-extending" or "Σ-ef-extending".

Corollary 10. *Let S be a right excellent extension of R.*

(1) *Every right R-module is ef-extending if and only if every right S-module is ef-extending.*
(2) *Every singular right R-module is ef-extending if and only if every singular right S-module is ef-extending.*
(3) *Every nonsingular right R-module is ef-extending if and only if every nonsingular right S-module is ef-extending.*

Proof. It follows immediately by Theorem 2 and 3. □

ACKNOWLEDGEMENTS

The authors are grateful to the referee for her/his valuable comments.

REFERENCES

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules.* New York: Springer Science+Business Media, LLC, 2012. doi: 10.1007/978-1-4684-9913-1.

[2] N. Chien and L. V. Thuyet, "On ef-extending modules," *Southeast Asian Bull. Math.*, vol. 26, no. 6, pp. 909–916, 2003.

[3] P. Dan, "Right perfect rings with the extending property on finitely generated free modules," *Osaka J. Math.*, vol. 26, no. 2, pp. 265–273, 1989.

[4] N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer, *Extending Modules*. New York: Pitman Research Notes in Mathematics Series, 1994. doi: 10.1201/9780203756331.

[5] A. Harmanci and P. F. Smith, "Finite direct sums of CS-Modules," *Houston J. of Algebra*, vol. 19, no. 4, pp. 523–532, 1995.

[6] D. V. Huynh and P.Dan, "Some characterizations of right co-h-rings," *Math. J. Okayama Univ.*, vol. 34, pp. 165–174, 1992.

[7] M. M. Parmenter and P. N. Stewart, "Excellent extensions," *Comm. Algebra*, vol. 16, no. 4, p. 703–713, 1988, doi: 10.1080/00927878808823597.

[8] M. M. Parmenter and Y. Zhou, "Relative injectivity of modules and excellent extensions," *Quaestiones Mathematicae*, vol. 22, no. 1, p. 101–107, 1999, doi: 10.1080/16073606.1999.9632062.

[9] M. M. Parmenter and Y. Zhou, "Finitely σ-cs property of excellent extensions of rings," *Algebra Colloquium*, vol. 10, no. 1, p. 17–21, 2003, doi: 10.1007/s100110300003.

[10] S. H. Mohammed and B. J. Müller, *Continuous and Discrete Modules*. London Math.Soc. LN: Cambridge University Press, 1990. doi: 10.1017/CBO9780511600692.

[11] D. Passman, *The algebraic structure of group rings*. New York: Wiley Interscience, 1977.

[12] D. Passman, "It's essentially Maschke's theorem," *Rocky Mountain J. Math*, vol. 13, no. 1, pp. 37–54, 1983, doi: 10.1216/RMJ-1983-13-1-37.

[13] T. C. Quynh, A. Abyzov, N. T. T. Ha, and T. Yildirim, "Modules close to the automorphism invariant and coinvariant," *J. Algebra Appl.* , vol. 14, no. 12, p. 19502359, 2019, doi: 10.1142/S0219498819502359.

[14] T. C. Quynh, A. Abyzov, and D. T. Trang, "Rings all of whose finitely generated ideals are automorphism-invariant," *J. Algebra Appl.* , vol. 21, no. 8, p. 2250159, 2022, doi: 10.1142/S0219498822501596.

[15] L. V. Thuyet, P. Dan, and B. D. Dung, "On a class of semiperfect rings," *J. Algebra Appl.* , vol. 12, no. 6, p. 1350009, 2013, doi: 10.1142/S0219498813500096.

[16] L. V. Thuyet and T. C. Quynh, "On ef-extending modules and rings with chain conditions," *Trends Math., Birkhauser/Springer Basel AG, Base* , pp. 327–334, 2010, doi: 10.1007/978-3-0346-0286-0_21.

[17] L. V. Thuyet and R. Wisbauer, "Extending property for finitely generated submodules," *Vietnam J. Math.* , vol. 25, no. 1, pp. 65 – 73, 1997.

[18] J. Zelmanowitz, "Regular modules," *Trans. Am. Math. Soc.*, vol. 163, pp. 341–355, 1972, doi: 10.2307/1995726.

Author's address

Banh Duc Dung

HCMC University of Technology and Education, Faculty of Applied Sciences, 1 Vo Van Ngan St., Ho Chi Minh City, Vietnam

E-mail address: dungbd@hcmute.edu.vn

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 679–691

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4434

THE SOLUTIONS TO SOME DUAL MATRIX EQUATIONS

RANRAN FAN, MIN ZENG, AND YONGXIN YUAN

Received 03 October, 2022

Abstract. The solvability conditions for the dual matrix equation $AXB = D$, a pair of dual matrix equations $AX = B, XC = D$ and the dual matrix equation $AX + YB = D$ are established by using the generalized inverses and the singular value decompositions of some real matrices, and the expressions of the general solutions to these dual matrix equations are presented. Finally, a numerical experiment is given to validate the accuracy of our result.

2010 *Mathematics Subject Classification:* 15A24; 15B99

Keywords: dual matrix equation, generalized inverse, singular value decomposition

1. Introduction

Throughout this paper, $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices, and the symbols $A^\top, A^-, A^\dagger$ and $\|A\|$ denote the transpose, the g-inverse, the Moore-Penrose inverse and the Frobenius norm of a real matrix A, respectively. I_n represents the identity matrix of order n. For a matrix $A \in \mathbb{R}^{m\times n}$, E_A and F_A stand for the orthogonal projectors: $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$.

In real and complex matrix spaces, many scholars have considered the following linear matrix equations:

$$AXB = D, \tag{1.1}$$

$$AX = B, \qquad XC = D, \tag{1.2}$$

and

$$AX + YB = D. \tag{1.3}$$

For Eq. (1.1), Penrose [15] obtained the solvability condition and the general solution of Eq. (1.1) by using the generalized inverse. Tian and Wang [18] analyzed the relations between least-squares and least-rank solutions of Eq. (1.1). Yuan and Dai [22] considered the reflexive solutions of Eq. (1.1) and the associated optimal approximation problem by applying the generalized singular value decomposition. For Eq. (1.2), Mitra [11] provided the solvability conditions and the general solution of

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

Eq. (1.2) by using the g-inverses of matrices. Dajić and Koliha [7] considered the common Hermitian and positive solutions to Eq. (1.2) for Hilbert space operators. For Eq. (1.3), Baksalary and Kala [1] and Chu [2] have been derived its general solution by using the g-inverse and the singular value decomposition (SVD), respectively. For inconsistent equations, l_p and Chebyshev solutions of Eq. (1.3) were considered in [24] and [25].

The dual number originally was introduced by Clifford [3], and then was further developed to represent the dual angle in spatial geometry [17]. For any dual number a can be writhen as $a = a_1 + \varepsilon a_2$, where $a_1, a_2 \in \mathbb{R}$ are the real part and the dual part of the dual number a, respectively, and ε denotes the dual unit with the operational rules:$\varepsilon \neq 0,\ \varepsilon 0 = 0\varepsilon = 0,\ \varepsilon 1 = 1\varepsilon = \varepsilon,\ \varepsilon^2 = 0$. A matrix whose elements are dual numbers is called a dual matrix, that is, for an $m \times n$ dual matrix A can be written as $A = A_1 + \varepsilon A_2$, where the matrices A_1 and $A_2 \in \mathbb{R}^{m\times n}$. Today, dual matrices are used in a various areas of engineering like the kinematic analysis and synthesis of spatial mechanisms [4, 16] and the robotics [8]. The growing applicability of dual matrices in science and engineering has sparked a renewal of interest in the linear algebra and numerical algorithms [12, 20].

The solution to some linear dual equations is also a task often required in the kinematic analysis of many spatial mechanisms [13, 14]. For example, Li et al.[9] used dual-quaternions to solve homogeneous transformation equation $AX = ZB$ for the robot-to-world and hand-eye calibration problem. Wang et al.[21] converted the multi-coordinate calibration problem of a dual-robot system into solving the $AXB = YCZ$ problem. Condurache and Burlacu [5] provided a new approach for solving $AX = XB$ sensor calibration problem by applying orthogonal dual tensor methods. Udwadia [19] explored the solution of the linear dual equation $Ax = \beta$ which is commonly encountered in areas of kinematics and robotics, where A is a p-by-q dual matrix and β is a p-by-1 dual vector.

We observe that the dual matrix equations of (1.1), (1.2) and (1.3) are seldom considered in the literature. In this paper, we will discuss the solvability conditions and general solutions of these dual matrix equations, which can be stated as the following problems:

Problem 1. *Given dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{p\times q}$ and $D_i \in \mathbb{R}^{m\times q}$. Find a dual matrix $X = X_1 + \varepsilon X_2$, where $X_i \in \mathbb{R}^{n\times p}$ $(i = 1,2)$ such that Eq. (1.1) is satisfied.*

Problem 2. *Given dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2, C = C_1 + \varepsilon C_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{m\times p}, C_i \in \mathbb{R}^{p\times q}$ and $D_i \in \mathbb{R}^{n\times q}$. Find a dual matrix $X = X_1 + \varepsilon X_2$, where $X_i \in \mathbb{R}^{n\times p}$ $(i = 1,2)$ such that Eq. (1.2) is satisfied.*

Problem 3. *Given dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{p\times q}$ and $D_i \in \mathbb{R}^{m\times q}$. Find dual matrices $X = X_1 + \varepsilon X_2$ and $Y = Y_1 + \varepsilon Y_2$, where $X_i \in \mathbb{R}^{n\times q}$ and $Y_i \in \mathbb{R}^{m\times p}$ $(i = 1,2)$ such that Eq. (1.3) is satisfied.*

In the following sections, each dual matrix equation will firstly be decomposed into two real matrix equations. Then, by making use of the generalized inverses and the SVDs of matrices, the solvability conditions and explicit solutions of Problems 1, 2 and 3 are derived. A numerical example is given to verify its correctness.

2. The solution of Problem 1

To begin with, we introduce some lemmas.

Lemma 1 ([23, page 907]). *Suppose that $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{p\times q}$ and $D \in \mathbb{R}^{m\times q}$. Then Eq. (1.1) has a solution $X \in \mathbb{R}^{n\times p}$ if and only if $AA^{\dagger}DB^{\dagger}B = D$. In this case, the general solution is $X = A^{\dagger}DB^{\dagger} + F_A V_1 + V_2 E_B$, where V_1 and V_2 are arbitrary matrices.*

Lemma 2 ([6, page 30]). *Let $A \in \mathbb{R}^{m\times p}, B \in \mathbb{R}^{m\times q}$ and $C = (I_m - AA^{-})B$. Then the g-inverse of the matrix $[A,B]$ is*

$$[A,B]^{-} = \begin{bmatrix} A^{-} - A^{-}BC^{-}(I_m - AA^{-}) \\ C^{-}(I_m - AA^{-}) \end{bmatrix}.$$

Lemma 3 ([1, page 145]). *Given $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, $C \in R^{m\times r}$, $D \in \mathbb{R}^{s\times q}$ and $L \in \mathbb{R}^{m\times q}$. Then the matrix equation $AXB + CYD = L$ is consistent if and only if*

$$E_G E_A L = 0, \quad E_A L F_D = 0, \quad E_C L F_B = 0, \quad L F_B F_H = 0,$$

where $G = E_A C$ and $H = DF_B$. In this case, the general solution is

$$X = A^{-}(L - CYD)B^{-} + Z - A^{-}AZBB^{-},$$
$$Y = G^{-}E_A L D^{-} + (F_G C^{-} + F_C G^{-} E_A) L F_B H^{-} + W - C^{-}CF_G W H H^{-} - G^{-}GWDD^{-},$$

where W and Z are arbitrary matrices.

Theorem 1. *Let dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{p\times q}$ and $D_i \in \mathbb{R}^{m\times q}$ $(i = 1,2)$, and let $H = E_{A_1} A_2 F_{A_1}, K = E_{B_1} B_2 F_{B_1}$. Then Eq. (1.1) is solvable if and only if*

$$A_1 A_1^{\dagger} D_1 B_1^{\dagger} B_1 = D_1, \qquad (D_2 - A_1 A_1^{\dagger} D_1 B_1^{\dagger} B_2) F_{B_1} F_K = 0, \tag{2.1}$$

$$E_{A_1} D_2 F_{B_1} = 0, \qquad E_H E_{A_1} (D_2 - A_2 A_1^{\dagger} D_1 B_1^{\dagger} B_1) = 0. \tag{2.2}$$

In this case, the solution set S_1 of Problem 1 can be expressed as

$$S_1 = \left\{ X = X_1 + \varepsilon X_2 \middle| X_1, X_2 \in \mathbb{R}^{n\times p} \right\},$$

where

$$\begin{aligned} X_1 =& A_1^{\dagger} D_1 B_1^{\dagger} + H^{\dagger}(D_2 - A_2 A_1^{\dagger} D_1) B_1^{\dagger} + A_1^{\dagger}(D_2 - D_1 B_1^{\dagger} B_2) K^{\dagger} \\ &+ F_{A_1} W_{22} - H^{\dagger} H W_{22} B_1 B_1^{\dagger} + W_1 E_{B_1} - A_1^{\dagger} A_1 W_1 K K^{\dagger}, \\ X_2 =& A_1^{\dagger}(I_m - A_2 H^{\dagger})(D_2 - A_2 A_1^{\dagger} D_1) B_1^{\dagger} - A_1^{\dagger} D_1 B_1^{\dagger} B_2 B_1^{\dagger} \end{aligned} \tag{2.3}$$

$$-A_1^\dagger(D_2-D_1B_1^\dagger B_2)K^\dagger B_2B_1^\dagger - A_1^\dagger A_1W_{21}B_1B_1^\dagger + W_{21}$$
$$-A_1^\dagger A_1W_1E_KE_{B_1}B_2B_1^\dagger - A_1^\dagger A_2F_{A_1}F_HW_{22}B_1B_1^\dagger, \tag{2.4}$$

and W_1, W_{21}, W_{22} *are arbitrary matrices.*

Proof. Clearly, Eq. (1.1) can be equivalently written as

$$A_1X_1B_1 = D_1, \tag{2.5}$$
$$A_2X_1B_1 + A_1X_1B_2 + A_1X_2B_1 = D_2. \tag{2.6}$$

By Lemma 1, Eq. (2.5) is solvable if and only if the first condition of (2.1) is satisfied, and the general solution is

$$X_1 = A_1^\dagger D_1B_1^\dagger + F_{A_1}V_1 + V_2E_{B_1}, \tag{2.7}$$

where V_1 and $V_2 \in \mathbb{R}^{n\times p}$ are arbitrary matrices. Inserting (2.7) into Eq. (2.6), we have

$$[A_1, A_2F_{A_1}]\begin{bmatrix} X_2 \\ V_1 \end{bmatrix} B_1 + A_1V_2E_{B_1}B_2 = D_2 - A_2A_1^\dagger D_1B_1^\dagger B_1 - A_1A_1^\dagger D_1B_1^\dagger B_2. \tag{2.8}$$

By Lemma 2, we know that $\begin{bmatrix} A_1^\dagger - A_1^\dagger A_2F_{A_1}H^\dagger E_{A_1} \\ H^\dagger E_{A_1} \end{bmatrix} = \begin{bmatrix} A_1^\dagger - A_1^\dagger A_2H^\dagger \\ H^\dagger \end{bmatrix}$ is one of the g-inverse of $[A_1, A_2F_{A_1}]$. Thus

$$E_{[A_1,A_2F_{A_1}]} = I_m - [A_1, A_2F_{A_1}][A_1, A_2F_{A_1}]^- = E_HE_{A_1}.$$

By Lemma 3, Eq. (2.8) is solvable if and only if the second condition of (2.1) and conditions (2.2) are satisfied. In which case, the general solution is

$$V_2 = A_1^\dagger(D_2 - D_1B_1^\dagger B_2)K^\dagger + W_1 - A_1^\dagger A_1W_1KK^\dagger, \tag{2.9}$$
$$\begin{aligned}\begin{bmatrix} X_2 \\ V_1 \end{bmatrix} =& [A_1, A_2F_{A_1}]^-(D_2 - A_2A_1^\dagger D_1 - A_1^\dagger A_1D_1B_1^\dagger B_2 - A_1V_2E_{B_1}B_2)B_1^\dagger \\ &+ \begin{bmatrix} W_{21} \\ W_{22} \end{bmatrix} - [A_1, A_2F_{A_1}]^-[A_1, A_2F_{A_1}]\begin{bmatrix} W_{21} \\ W_{22} \end{bmatrix} B_1B_1^\dagger,\end{aligned} \tag{2.10}$$

where W_1, W_{21} and $W_{22} \in \mathbb{R}^{n\times p}$ are arbitrary matrices. From (2.9) and (2.10), we can get the relation of (2.4) and

$$V_1 = H^\dagger(D_2 - A_2A_1^\dagger D_1)B_1^\dagger + W_{22} - H^\dagger HW_{22}B_1B_1^\dagger. \tag{2.11}$$

Substituting (2.9) and (2.11) into (2.7), we can get (2.3). □

Clearly, the dual matrix equation $AXA = A$ is a special case of the dual matrix equation (1.1). According to Theorem 1, we can get the following result about $\{1\}$-dual generalized inverse of the dual matrix A, which was discussed by Udwadia in [19].

Corollary 1. *Given the dual matrix $A = A_1 + \varepsilon A_2$, where $A_1, A_2 \in \mathbb{R}^{m\times n}$. Then the dual matrix equation $AXA = A$ is solvable if and only if $H = E_{A_1}A_2F_{A_1} = 0$. In this case, the general solution is $X = X_1 + \varepsilon X_2$, where*

$$X_1 = A_1^\dagger + W_1E_{A_1} + F_{A_1}W_{22},$$
$$X_2 = -A_1^\dagger A_2A_1^\dagger - A_1^\dagger A_1W_1E_{A_1}A_2A_1^\dagger + W_{21} - A_1^\dagger A_1W_{21}A_1A_1^\dagger - A_1^\dagger A_2F_{A_1}W_{22}A_1A_1^\dagger,$$

and W_1, W_{21}, W_{22} are arbitrary matrices.

3. The solution of Problem 2

Before solving Problem 2, we need the following Lemma.

Lemma 4 ([10, page 215])**.** *Suppose that $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{m\times p}, C \in \mathbb{R}^{p\times q}$ and $D \in \mathbb{R}^{n\times q}$. Then Eq. (1.2) has a solution $X \in \mathbb{R}^{n\times p}$ if and only if $E_AB = 0, DF_C = 0$ and $AD = BC$. In which case, the general solution is $X = A^\dagger B + F_ADC^\dagger + F_AWE_C$, where $W \in \mathbb{R}^{n\times p}$ is an arbitrary matrix.*

Theorem 2. *Assume that dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2, C = C_1 + \varepsilon C_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{m\times p}, C_i \in \mathbb{R}^{p\times q}$ and $D_i \in \mathbb{R}^{n\times q}$ $(i = 1,2)$. Let the SVDs of the matrices A_1 and C_1 be*

$$A_1 = P\begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix}Q^\top, \qquad C_1 = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}V^\top, \tag{3.1}$$

where $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_r)$, $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_s)$, $r = \mathrm{rank}(A_1)$, $s = \mathrm{rank}(C_1)$, and $P = [P_1, P_2] \in \mathbb{R}^{m\times m}$, $Q = [Q_1, Q_2] \in \mathbb{R}^{n\times n}$, $U = [U_1, U_2] \in \mathbb{R}^{p\times p}$, $V = [V_1, V_2] \in \mathbb{R}^{q\times q}$ are orthogonal matrices with $P_2 \in \mathbb{R}^{m\times(m-r)}$, $Q_2 \in \mathbb{R}^{n\times(n-r)}$, $U_2 \in \mathbb{R}^{p\times(p-s)}$ and $V_2 \in \mathbb{R}^{q\times(q-s)}$. If write

$$G_1 = B_2 - A_2A_1^\dagger B_1 - A_2F_{A_1}D_1C_1^\dagger, \qquad G_2 = D_2 - A_1^\dagger B_1C_2 - F_{A_1}D_1C_1^\dagger C_2,$$
$$M_1 = U_2^\top C_2V_2, \quad N_1 = P_2^\top A_2Q_2, \quad H_1 = Q_2F_{N_1}, \quad H_2 = E_{M_1}U_2^\top,$$
$$S = Q_2N_1^\dagger P_2^\top(B_2 - A_2A_1^\dagger B_1)E_{C_1} + Q_2F_{N_1}Q_2^\top G_2V_2M_1^\dagger U_2^\top.$$

Then Eq. (1.2) is solvable if and only if

$$E_{A_1}B_1 = 0, \qquad D_1F_{C_1} = 0, \qquad A_1D_1 = B_1C_1, \tag{3.2}$$
$$A_1G_2 = G_1C_1, \qquad A_1^\dagger A_1G_2F_{C_1} = 0, \qquad E_{A_1}G_1C_1C_1^\dagger = 0, \tag{3.3}$$
$$E_{N_1}P_2^\top G_1U_2 = 0, \qquad Q_2^\top G_2V_2F_{M_1} = 0, \qquad N_1Q_2^\top G_2V_2 = P_2^\top G_1U_2M_1. \tag{3.4}$$

The solution set S_2 of Problem 2 can be expressed as

$$S_2 = \left\{X = X_1 + \varepsilon X_2 \middle| X_1, X_2 \in \mathbb{R}^{n\times p}\right\},$$

where

$$X_1 = A_1^\dagger B_1 + F_{A_1}D_1C_1^\dagger + S + H_1J_1H_2, \tag{3.5}$$

$$\begin{aligned} X_2 =& A_1^\dagger(G_1 - A_2S) + F_{A_1}G_2C_1^\dagger - SC_2C_1^\dagger - A_1^\dagger A_2H_1J_1H_2 \\ & - H_1J_1H_2C_2C_1^\dagger + F_{A_1}R_1E_{C_1}, \end{aligned} \tag{3.6}$$

and J_1, R_1 are arbitrary matrices.

Proof. Eq. (1.2) can be equivalently written as

$$A_1X_1 = B_1, \qquad X_1C_1 = D_1, \tag{3.7}$$

$$A_2X_1 + A_1X_2 = B_2, \qquad X_1C_2 + X_2C_1 = D_2. \tag{3.8}$$

By using Lemma 4, we know Eq. (3.7) is solvable if and only if the conditions (3.2) are satisfied, and the general solution is

$$X_1 = A_1^\dagger B_1 + F_{A_1}D_1C_1^\dagger + F_{A_1}W_1E_{C_1}, \tag{3.9}$$

where $W_1 \in \mathbb{R}^{n\times p}$ is an arbitrary matrix. Substituting (3.9) into Eq. (3.8), we can get

$$A_1X_2 = G_1 - A_2F_{A_1}W_1E_{C_1},\ X_2C_1 = G_2 - F_{A_1}W_1E_{C_1}C_2. \tag{3.10}$$

Using Lemma 4 again, Eq. (3.10) with respect to X_2 is solvable if and only if the first condition of (3.3) is satisfied and

$$F_{A_1}W_1E_{C_1}C_2F_{C_1} = G_2F_{C_1}, \tag{3.11}$$

$$E_{A_1}A_2F_{A_1}W_1E_{C_1} = E_{A_1}G_1. \tag{3.12}$$

In this case, the general solution is

$$X_2 = A_1^\dagger(G_1 - A_2F_{A_1}W_1E_{C_1}) + F_{A_1}(G_2 - F_{A_1}W_1E_{C_1}C_2)C_1^\dagger + F_{A_1}R_1E_{C_1}, \tag{3.13}$$

where $R_1 \in \mathbb{R}^{n\times p}$ is an arbitrary matrix. Inserting (3.1) to Eq. (3.11) and Eq. (3.12), we can obtain

$$\begin{aligned} & Q\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^\top W_1 U \begin{bmatrix} 0 & 0 \\ 0 & I_{p-s} \end{bmatrix} U^\top C_2 V \begin{bmatrix} 0 & 0 \\ 0 & I_{q-s} \end{bmatrix} V^\top \\ & = G_2 V \begin{bmatrix} 0 & 0 \\ 0 & I_{q-s} \end{bmatrix} V^\top, \end{aligned} \tag{3.14}$$

$$\begin{aligned} & P\begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top A_2 Q \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^\top W_1 U \begin{bmatrix} 0 & 0 \\ 0 & I_{p-s} \end{bmatrix} U^\top \\ & = P\begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top G_1. \end{aligned} \tag{3.15}$$

If let

$$Q^\top W_1 U = \begin{bmatrix} W_{11} & W_{12} \\ W_{13} & W_{14} \end{bmatrix}, \quad U^\top C_2 V = \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & C_{24} \end{bmatrix}, \quad Q^\top G_2 V = \begin{bmatrix} G_{21} & G_{22} \\ G_{23} & G_{24} \end{bmatrix},$$

$$P^\top A_2 Q = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix}, \quad P^\top G_1 U = \begin{bmatrix} G_{11} & G_{12} \\ G_{13} & G_{14} \end{bmatrix}.$$

Then Eq. (3.14) and Eq. (3.15) can be equivalently written as

$$G_{22} = 0, \qquad G_{13} = 0, \tag{3.16}$$

$$W_{14}M_1 = Q^\top G_2 V_2, \qquad N_1 W_{14} = P_2^\top G_1 U_2. \tag{3.17}$$

We note that

$$G_{22} = 0 \quad \Leftrightarrow \quad Q_1^\top G_2 V_2 = 0 \quad \Leftrightarrow \quad Q_1 Q_1^\top G_2 V_2 V_2^\top = 0 \quad \Leftrightarrow \quad A_1^\dagger A_1 G_2 F_{C_1} = 0,$$

which is the second condition of (3.3). Similarly, the condition $G_{13} = 0$ is equivalent to the third condition of (3.3). Using Lemma 4, Eq. (3.17) with respect to W_{14} is solvable if and only if conditions (3.4) hold, and the general solution is

$$W_{14} = N_1^\dagger p_2^\top G_1 U_2 + F_{N_1} Q_2^\top G_2 V_2 M_1^\dagger + F_{N_1} J_1 E_{M_1},$$

where $J_1 \in \mathbb{R}^{(n-r)\times(p-s)}$ is an arbitrary matrix. Thus, we have

$$F_{A_1} W_1 E_{C_1} = Q_2 W_{14} U_2^\top = S + H_1 J_1 H_2, \tag{3.18}$$

Substituting (3.18) into (3.9) and (3.13), we can obtain (3.5) and (3.6). □

4. The solution of Problem 3

Theorem 3. *Assume that dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m\times n}, B_i \in \mathbb{R}^{p\times q}$ and $D_i \in \mathbb{R}^{m\times q} (i = 1,2)$. Let the SVDs of the matrices A_1 and B_1 be*

$$A_1 = P \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} Q^\top, \qquad B_1 = M \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} N^\top, \tag{4.1}$$

where $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_r)$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_t)$, $r = \mathrm{rank}(A_1)$, $t = \mathrm{rank}(B_1)$, and $P = [P_1, P_2] \in \mathbb{R}^{m\times m}$, $Q = [Q_1, Q_2] \in \mathbb{R}^{n\times n}$, $M = [M_1, M_2] \in \mathbb{R}^{p\times p}$, $N = [N_1, N_2] \in \mathbb{R}^{q\times q}$ are orthogonal matrices with $P_2 \in \mathbb{R}^{m\times(m-r)}$, $Q_2 \in \mathbb{R}^{n\times(n-r)}$, $M_2 \in \mathbb{R}^{p\times(p-t)}$ and $N_2 \in \mathbb{R}^{q\times(q-t)}$. If write

$$J = D_2 - A_2 A_1^\dagger D_1 - E_{A_1} D_1 B_1^\dagger B_2, \quad K_1 = P_2^\top A_2 Q_2, \quad K_2 = M_2^\top B_2 N_2. \tag{4.2}$$

Then Eq. (1.3) is solvable if and only if

$$E_{A_1} D_1 F_{B_1} = 0, \qquad E_{K_1} P_2^\top J N_2 F_{K_2} = 0. \tag{4.3}$$

The solution set S_3 of Problem 3 can be expressed as

$$S_3 = \{X = X_1 + \varepsilon X_2, Y = Y_1 + \varepsilon Y_2 | X_1, X_2 \in \mathbb{R}^{n\times q}, Y_1, Y_2 \in \mathbb{R}^{m\times p}\},$$

where

$$\begin{aligned} X_1 =& A_1^\dagger D_1 + Q_2 K_1^\dagger P_2^\top J F_{B_1} - A_1^\dagger P_1 V_{11} M_1^\top B_1 + Q_2 V_{23} N_1^\top \\ &+ Q_2 (F_{K_1} W_4 - K_1^\dagger W_3 K_2) N_2^\top, \end{aligned} \tag{4.4}$$

$$\begin{aligned} Y_1 =& E_{A_1} D_1 B_1^\dagger + P_2 E_{K_1} P_2^\top J N_2 K_2^\dagger M_2^\top + P_1 (V_{11} M_1^\top + V_{12} M_2^\top) \\ &+ P_2 (W_3 - E_{K_1} W_3 K_2 K_2^\dagger) M_2^\top, \end{aligned} \tag{4.5}$$

$$\begin{aligned}X_2 =& A_1^\dagger J - A_1^\dagger A_2 Q_2 (K_1^\dagger P_2^\top J N_2 - K_1^\dagger W_3 K_2 + F_{K_1} W_4) N_2^\top \\ & - A_1^\dagger P_1 (V_{11} M_1^\top + V_{12} M_2^\top) B_2 - A_1^\dagger R_1 B_1 + F_{A_1} R_2 \\ & + A_1^\dagger A_2 (A_1^\dagger P_1 V_{11} M_1^\top B_1 - Q_2 V_{23} N_1^\top), \end{aligned} \tag{4.6}$$

$$\begin{aligned}Y_2 =& E_{A_1} J B_1^\dagger + E_{A_1} A_2 (A_1^\dagger P_1 V_{11} M_1^\top B_1 - Q_2 V_{23} N_1^\top) B_1^\dagger \\ & - P_2 E_{K_1} P_2^\top J N_2 K_2^\dagger M_2^\top B_2 B_1^\dagger - P_2 (W_3 - E_{K_1} W_3 K_2 K_2^\dagger) M_2^\top B_2 B_1^\dagger \\ & + R_1 - E_{A_1} R_1 B_1 B_1^\dagger, \end{aligned} \tag{4.7}$$

and V_{11}, V_{12}, V_{23}, W_3, W_4, R_1, R_2 *are arbitrary matrices.*

Proof. Obviously, Eq. (1.3) can be equivalently written as

$$A_1 X_1 + Y_1 B_1 = D_1, \tag{4.8}$$

$$A_2 X_1 + A_1 X_2 + Y_2 B_1 + Y_1 B_2 = D_2. \tag{4.9}$$

From Lemma 3, Eq. (4.8) is solvable if and only if the first condition of (4.3) is satisfied, and the general solution is

$$Y_1 = E_{A_1} D_1 B_1^\dagger + V_1 - E_{A_1} V_1 B_1 B_1^\dagger, \tag{4.10}$$

$$X_1 = A_1^\dagger D_1 - A_1^\dagger V_1 B_1 + F_{A_1} V_2, \tag{4.11}$$

where V_1 and V_2 are arbitrary matrices. Plugging (4.10) and (4.11) into Eq. (4.9), we have

$$A_1 X_2 + Y_2 B_1 = J - A_2 F_{A_1} V_2 + A_2 A_1^\dagger V_1 B_1 - V_1 B_2 + E_{A_1} V_1 B_1 B_1^\dagger B_2. \tag{4.12}$$

Using Lemma 3 again, Eq. (4.12) with respects to X_2 and Y_2 is solvable if and only if

$$E_{A_1} A_2 F_{A_1} V_2 F_{B_1} + E_{A_1} V_1 E_{B_1} B_2 F_{B_1} = E_{A_1} J F_{B_1}. \tag{4.13}$$

In this case, the general solution is

$$\begin{aligned}Y_2 =& E_{A_1} J B_1^\dagger + E_{A_1} A_2 A_1^\dagger V_1 B_1 B_1^\dagger - E_{A_1} V_1 E_{B_1} B_2 B_1^\dagger \\ & - E_{A_1} A_2 F_{A_1} V_2 B_1^\dagger + R_1 - E_{A_1} R_1 B_1 B_1^\dagger, \end{aligned} \tag{4.14}$$

$$X_2 = A_1^\dagger (J - A_2 F_{A_1} V_2 + A_2 A_1^\dagger V_1 B_1 - V_1 B_2 - Y_2 B_1) + F_{A_1} R_2, \tag{4.15}$$

where R_1 and R_2 are arbitrary matrices. Inserting (4.1) to Eq. (4.13), we can get

$$\begin{aligned}& P \begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top A_2 Q \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^\top V_2 N \begin{bmatrix} 0 & 0 \\ 0 & I_{q-t} \end{bmatrix} N^\top \\ & + P \begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top V_1 M \begin{bmatrix} 0 & 0 \\ 0 & I_{p-t} \end{bmatrix} M^\top B_2 N \begin{bmatrix} 0 & 0 \\ 0 & I_{q-t} \end{bmatrix} N^\top \\ & = P \begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top J N \begin{bmatrix} 0 & 0 \\ 0 & I_{q-t} \end{bmatrix} N^\top. \end{aligned} \tag{4.16}$$

If let

$$P^\top A_2 Q = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix}, \quad Q^\top V_2 N = \begin{bmatrix} V_{21} & V_{22} \\ V_{23} & V_{24} \end{bmatrix}, \quad P^\top V_1 M = \begin{bmatrix} V_{11} & V_{12} \\ V_{13} & V_{14} \end{bmatrix},$$
$$M^\top B_2 N = \begin{bmatrix} B_{21} & B_{22} \\ B_{23} & B_{24} \end{bmatrix}, \quad P^\top J N = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}.$$

Then Eq. (4.16) can be equivalently written as

$$K_1 V_{24} + V_{14} K_2 = P_2^\top J N_2. \tag{4.17}$$

Using Lemma 3, Eq. (4.17) is solvable if and only if the second condition of (4.3) holds, and the general solution is

$$V_{14} = E_{K_1} P_2^\top J N_2 K_2^\dagger + W_3 - E_{K_1} W_3 K_2 K_2^\dagger, \tag{4.18}$$
$$V_{24} = K_1^\dagger P_2^\top J N_2 - K_1^\dagger W_3 K_2 + F_{K_1} W_4, \tag{4.19}$$

where W_3 and W_4 are arbitrary matrices. Thus,

$$V_1 = P_1 V_{11} M_1^\top + P_2 V_{13} M_1^\top + P_1 V_{12} M_2^\top + P_2 V_{14} M_2^\top, \tag{4.20}$$
$$V_2 = Q_1 V_{21} N_1^\top + Q_2 V_{23} N_1^\top + Q_1 V_{22} N_2^\top + Q_2 V_{24} N_2^\top, \tag{4.21}$$

where V_{11}, V_{12}, V_{13}, V_{21}, V_{22} and V_{23} are arbitrary matrices. Substituting (4.18)–(4.21) into (4.10),(4.11),(4.14),(4.15), and noticing

$$\begin{aligned}
&E_{A_1} P_1 V_{11} M_1^\top = 0, && P_1 V_{11} M_1^\top E_{B_1} = 0, && A_1^\dagger P_2 V_{13} M_1^\top = 0, && P_2 V_{13} M_1^\top E_{B_1} = 0,\\
&E_{A_1} P_1 V_{12} M_2^\top = 0, && P_1 V_{12} M_2^\top B_1 = 0, && A_1^\dagger P_2 V_{14} M_2^\top = 0, && P_2 V_{14} M_2^\top B_1 = 0,\\
&F_{A_1} Q_1 V_{21} N_1^\top = 0, && Q_1 V_{21} N_1^\top F_{B_1} = 0, && A_1 Q_2 V_{23} N_1^\top = 0, && Q_2 V_{23} N_1^\top F_{B_1} = 0,\\
&F_{A_1} Q_1 V_{22} N_2^\top = 0, && Q_1 V_{22} N_2^\top B_1^\dagger = 0, && A_1 Q_2 V_{24} N_2^\top = 0, && Q_2 V_{24} N_2^\top B_1^\dagger = 0,
\end{aligned}$$

we can obtain (4.4)–(4.7). □

Based on Theorem 3, we can formulate the following algorithm to solve Problem 3.

Algorithm 1.

(1) *Input matrices A_i, B_i and D_i $(i = 1, 2)$.*
(2) *Compute the SVDs of the matrices A_1 and B_1 by (4.1).*
(3) *Calculate J, K_1 and K_2 by (4.2).*
(4) *If the conditions (4.3) are satisfied, go to (5); otherwise, the Eq. (1.3) has no solution, and stop.*
(5) *Randomly choose the matrices V_{11}, V_{12}, V_{23}, W_3, W_4, R_1 and R_2.*
(6) *Compute dual matrix $X = X_1 + \varepsilon X_2$ by (4.4)–(4.7).*

Example 1. Let $m=7,\ n=6,\ p=5,\ q=8$ and the matrices $A_i,\ B_i$ and $D_i\ (i=1,2)$ be given by

$$A_1=\begin{bmatrix} 0.9295 & 0.8406 & 0.3948 & 0.8918 & 1.1059 & 0.6344 \\ 1.6982 & 1.7141 & 0.7129 & 1.7326 & 1.6930 & 1.3104 \\ 1.0853 & 1.1207 & 0.5291 & 0.9446 & 0.9248 & 0.7579 \\ 1.3420 & 1.2970 & 0.7564 & 0.8606 & 1.1432 & 0.8302 \\ 1.1715 & 1.2305 & 0.4552 & 1.3524 & 1.1679 & 0.7921 \\ 0.7213 & 0.8033 & 0.3125 & 0.7792 & 0.5855 & 0.4736 \\ 1.0188 & 1.1001 & 0.3871 & 1.2182 & 0.9771 & 0.6932 \end{bmatrix},$$

$$A_2=\begin{bmatrix} 0.4243 & 0.7691 & 0.9493 & 0.8620 & 0.4070 & 0.1117 \\ 0.2703 & 0.3968 & 0.3276 & 0.9899 & 0.7487 & 0.1363 \\ 0.1971 & 0.8085 & 0.6713 & 0.5144 & 0.8256 & 0.6787 \\ 0.8217 & 0.7551 & 0.4386 & 0.8843 & 0.7900 & 0.4952 \\ 0.4299 & 0.3774 & 0.8335 & 0.5880 & 0.3185 & 0.1897 \\ 0.8878 & 0.2160 & 0.7689 & 0.1548 & 0.5341 & 0.4950 \\ 0.3912 & 0.7904 & 0.1673 & 0.1999 & 0.0900 & 0.1476 \end{bmatrix},$$

$$B_1=\begin{bmatrix} 0.6079 & 1.1159 & 0.5956 & 0.6270 & 1.2325 & 1.0054 & 0.4038 & 0.9286 \\ 1.1726 & 1.7527 & 1.2573 & 1.0184 & 1.7030 & 1.8189 & 1.2731 & 1.1029 \\ 1.1426 & 1.8468 & 1.1734 & 1.0666 & 1.8938 & 1.8022 & 1.0471 & 1.3055 \\ 1.2331 & 1.8517 & 1.3963 & 1.0324 & 1.8061 & 1.9805 & 1.4456 & 1.2211 \\ 0.7972 & 1.0332 & 0.6332 & 0.7645 & 0.8880 & 0.9636 & 0.6552 & 0.3169 \end{bmatrix},$$

$$B_2=\begin{bmatrix} 0.8449 & 0.6147 & 0.1231 & 0.6352 & 0.5358 & 0.8739 & 0.4170 & 0.1420 \\ 0.2094 & 0.3624 & 0.2055 & 0.2819 & 0.4452 & 0.2703 & 0.2060 & 0.1665 \\ 0.5523 & 0.0495 & 0.1465 & 0.5386 & 0.1239 & 0.2085 & 0.9479 & 0.6210 \\ 0.6299 & 0.4896 & 0.1891 & 0.6952 & 0.4904 & 0.5650 & 0.0821 & 0.5737 \\ 0.0320 & 0.1925 & 0.0427 & 0.4991 & 0.8530 & 0.6403 & 0.1057 & 0.0521 \end{bmatrix},$$

$$D_1=\begin{bmatrix} 6.1406 & 8.0527 & 7.0589 & 5.8597 & 9.0681 & 8.7133 & 6.2116 & 7.1038 \\ 5.9687 & 6.8809 & 7.6271 & 6.0367 & 8.8409 & 8.1612 & 6.6011 & 7.8182 \\ 5.1132 & 6.4170 & 5.9975 & 5.0679 & 7.5120 & 7.2419 & 5.5252 & 6.0488 \\ 3.5206 & 3.7180 & 4.3838 & 3.6134 & 5.0063 & 4.7191 & 3.9893 & 4.7616 \\ 5.1204 & 6.1849 & 6.2281 & 5.2350 & 7.5842 & 7.1172 & 5.4687 & 6.3310 \\ 3.2391 & 4.1829 & 3.8441 & 3.4011 & 5.1168 & 4.6853 & 3.3821 & 4.2400 \\ 3.3819 & 3.8221 & 4.3239 & 3.6439 & 5.1230 & 4.6124 & 3.6780 & 4.5859 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 8.0148 & 8.6320 & 7.9558 & 8.7138 & 10.4261 & 10.2622 & 7.6799 & 8.7502 \\ 8.0950 & 8.6298 & 9.7167 & 8.8261 & 9.6116 & 9.5072 & 7.0522 & 8.1786 \\ 6.9905 & 7.4758 & 7.4314 & 7.6470 & 9.1041 & 8.3395 & 5.7783 & 6.9511 \\ 7.2252 & 8.3537 & 9.3238 & 7.8794 & 9.5693 & 9.1666 & 7.2035 & 7.9500 \\ 6.2462 & 6.5371 & 6.6738 & 6.9087 & 7.8271 & 7.9660 & 5.7225 & 6.5788 \\ 6.5421 & 7.5126 & 6.5417 & 6.4456 & 8.0466 & 8.4030 & 6.0659 & 6.3854 \\ 6.2975 & 7.3790 & 6.7705 & 6.7915 & 8.1127 & 8.4531 & 5.7726 & 5.9501 \end{bmatrix}.$$

It is easy to verify that the solvability conditions (4.3) are satisfied:

$$\|E_{A_1} D_1 F_{B_1}\| = 2.1298e-15, \quad \|E_{K_1} P_2^\top J N_2 F_{K_2}\| = 1.9240e-15.$$

According to Algorithm 1, and taking the matrices V_{11}, V_{12}, V_{23}, W_3, W_4, R_1 and R_2 as zero matrices, we can obtain a pair of solution $X = X_1 + \varepsilon X_2$ and $Y = Y_1 + \varepsilon Y_2$ of Problem 3 as follows:

$$X_1 = \begin{bmatrix} 0.8679 & 1.5792 & 1.0708 & 1.1229 & 1.3338 & 1.4621 & 1.0616 & 1.2846 \\ -1.1069 & -2.0598 & -1.2459 & -0.5771 & -1.3804 & -1.5612 & -1.0286 & -0.8204 \\ 0.4778 & 0.1432 & -0.1177 & 0.4213 & 0.3441 & 0.4644 & 0.4361 & 0.5040 \\ 1.3293 & 1.9433 & 1.5302 & 1.7165 & 2.4117 & 1.9648 & 1.1896 & 1.7743 \\ 6.3263 & 8.7531 & 6.6704 & 5.4077 & 8.9556 & 8.9486 & 5.6859 & 6.2824 \\ -4.8304 & -7.3331 & -4.0270 & -5.1423 & -7.4170 & -7.3118 & -3.6731 & -4.9208 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 3.9555 & 5.5750 & 4.1899 & 3.9725 & 6.0625 & 5.9144 & 3.9358 & 3.9953 \\ 5.4489 & 8.3834 & 5.1434 & 5.5083 & 8.5723 & 8.3644 & 4.1819 & 4.6228 \\ 3.6170 & 5.4246 & 3.9892 & 3.6878 & 5.9108 & 5.5483 & 3.6692 & 3.7562 \\ 6.1274 & 8.2455 & 4.9003 & 5.4792 & 8.5528 & 9.4694 & 5.3390 & 5.3433 \\ 2.9899 & 2.4317 & 3.4133 & 2.4154 & 3.5625 & 4.2695 & 5.2274 & 4.4004 \\ -23.5531 & -33.3170 & -21.8236 & -21.3747 & -36.1257 & -37.4695 & -24.4449 & -23.8245 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0.2731 & 0.1583 & 0.2593 & 0.1392 & 0.1734 \\ -0.0619 & -0.1359 & -0.1246 & -0.1256 & -0.1735 \\ 0.1409 & 0.3755 & 0.3286 & 0.3476 & 0.4810 \\ -0.1599 & -0.1836 & -0.2151 & -0.1660 & -0.2172 \\ -0.2442 & 0.0414 & -0.0900 & 0.0411 & 0.0510 \\ 0.4746 & 0.0933 & 0.3429 & 0.0674 & 0.0352 \\ -0.1406 & -0.1897 & -0.2349 & -0.1644 & -0.1789 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 0.5453 & -0.0005 & 0.4083 & -0.2108 & 0.3890 \\ -0.4241 & 0.0642 & -0.2844 & 0.2672 & -0.3632 \\ 1.1377 & -0.2030 & 0.7443 & -0.7519 & 0.9533 \\ -0.5297 & 0.0924 & -0.3426 & 0.3190 & -0.3464 \\ -0.3304 & -0.3287 & -0.4576 & -0.1886 & -0.6673 \\ 0.1814 & -0.0026 & 0.1125 & 0.0463 & -0.2845 \\ -0.0533 & 0.3677 & 0.2065 & 0.3118 & 0.6601 \end{bmatrix}.$$

The absolute errors are estimated by

$$\|A_1 X_1 + Y_1 B_1 - D_1\| = 4.0468e-14,$$

$$\|A_2X_1 + A_1X_2 + Y_2B_1 + Y_1B_2 - D_2\| = 5.6005e - 14,$$

which implies that X and Y is a pair of solution of Problem 3.

5. Acknowledgments

The authors are grateful to an anonymous referee for useful comments and suggestions which helped to improve the presentation of this paper.

References

[1] J. K. Baksalary and R. Kala, "The matrix equation $AXB + CYD = E$," *Linear Algebra and its Applications*, vol. 30, pp. 141–147, 1980, doi: 10.1016/0024-3795(80)90189-5.

[2] K.-W. E. Chu, "Singular value and generalized singular value decompositions and the solution of linear matrix equations," *Linear Algebra and its Applications*, vol. 88/89, pp. 83–98, 1987, doi: 10.1016/0024-3795(87)90104-2.

[3] M. A. Clifford, "Preliminary sketch of biquaternions," *Proceedings of the London Mathematical Society*, vol. s1-4, pp. 381–395, 1873, doi: 10.1112/plms/s1-4.1.381.

[4] A. Cohen and M. Shoham, "Application of hyper-dual numbers to rigid bodies equations of motion," *Mechanism and Machine Theory*, vol. 111, pp. 76–84, 2017.

[5] D. Condurache and A. Burlacu, "Orthogonal dual tensor method for solving the $AX = XB$ sensor calibration problem," *Mechanism and Machine Theory*, vol. 104, pp. 382–404, 2016, doi: 10.1016/j.mechmachtheory.2016.06.002.

[6] D. S. Cvetković-Ilić, J. N. Radenković, and Q. Wang, "Algebraic conditions for the solvability to some systems of matrix equations," *Linear and Multilinear Algebra*, pp. 1–31, 2019, doi: 10.1080/03081087.2019.1633993.

[7] A. Dajić and J. J. Koliha, "Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators," *Journal of Mathematical Analysis and Applications*, vol. 333, pp. 567–576, 2007, doi: 10.1016/j.jmaa.2006.11.016.

[8] Y.-L. Gu and J. Y. S. Luh, "Dual-number transformation and its applications to robotics," *IEEE Journal of Robotics and Automation*, vol. 3, pp. 615–623, 1987, doi: 10.1109/JRA.1987.1087138.

[9] A. Li, L. Wang, and D. Wu, "Simultaneous robot-world and hand–eye calibration using dual–quaternions and kronecker product," *International Journal of the Physical Sciences*, vol. 5, pp. 1530–1536, 2010.

[10] S. K. Mitra, "Common solutions to a pair of linear matrix equations $A_1XB_1 = C1, A_2XB_2 = C_2$," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 74, pp. 213–216, 1973, doi: 10.1017/S030500410004799X.

[11] S. K. Mitra, "The matrix equations $AX = C, XB = D$," *Linear Algebra and its Applications*, vol. 59, pp. 171–181, 1984, doi: 10.1016/0024-3795(84)90166-6.

[12] E. Pennestrí and R. Stefanelli, "Linear algebra and numerical algorithms using dual numbers," *Multibody System Dynamics*, vol. 18, pp. 323–344, 2007, doi: 10.1007/s11044-007-9088-9.

[13] E. Pennestrí and P. P. Valentini, "Linear dual algebra algorithms and their application to kinematics," *Computational Methods in Applied Sciences*, vol. 12, pp. 207–229, 2009.

[14] E. Pennestri, P. P. Valentini, G. Figliolini, and J. Angeles, "Dual Cayley-Klein parametes and mobius transform: theory and applications," *Mechanism and Machine Theory*, vol. 106, pp. 50–67, 2016.

[15] R. Penrose, "A generalized inverse for matrices," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 51, pp. 406–413, 1955, doi: 10.1017/S0305004100030401.

[16] A. Perez and J. M. Mccarthy, "Bennett's linkage and the cylindroid," *Mechanism and Machine Theory*, vol. 37, pp. 1245–1260, 2002, doi: 10.1016/s0094-114x(02)00055-1.

[17] E. Study, *Geometrie der Dynamen*. Germany: Leipzig, 1903.
[18] Y. Tian and H. Wang, "Relations between least-squares and least-rank solutions of the matrix equation $AXB = C$," *Applied Mathematics and Computation*, vol. 219, pp. 10 293–10 301, 2013, doi: 10.1016/j.amc.2013.03.137.
[19] F. E. Udwadia, "Dual generalized inverses and their use in solving systems of linear dual equations," *Mechanism and Machine Theory*, vol. 156, 104158, 2021, doi: 10.1016/j.mechmachtheory.2020.104158.
[20] F. E. Udwadia, E. Pennestri, and D. Falco, "Do all dual matrices have dual Moore-Penrose generalized inverses," *Mechanism and Machine Theory*, vol. 151, 103878, 2020, doi: 10.1016/j.mechmachtheory.2020.103878.
[21] G. Wang, W. Li, C. Jiang, D. Zhu, H. Xie, X. Liu, and H. Ding, "Simultaneous calibration of multicoordinates for a dual-robot system by solving the $AXB = YCZ$ problem," *IEEE Transactions on Robotics*, vol. 37, 2021, doi: 10.1109/TRO.2020.3043688.
[22] Y. Yuan and H. Dai, "Generalized reflexive solutions of the matrix equation $AXB = D$ and an associated optimal approximation problem," *Computers and Mathematics with Applications*, vol. 56, pp. 1643–1649, 2008, doi: 10.1016/j.camwa.2008.03.015.
[23] Y. Yuan and K. Zuo, "The Re-nonnegative definite and Re-positive definite solutions to the matrix equation $AXB = D$," *Applied Mathematics and Computation*, vol. 256, pp. 905–912, 2015, doi: 10.1016/j.amc.2015.01.098.
[24] K. Ziętak, "The l_p–solution of the linear matrix equation $AX + YB = C$," *Computing*, vol. 32, pp. 153–162, 1984.
[25] K. Ziętak, "The Chebyshev solution of the linear matrix equation $AX + YB = C$," *Numerische Mathematik*, vol. 46, pp. 455–478, 1985, doi: 10.1007/BF01389497.

Authors' addresses

Ranran Fan
School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, PR China
E-mail address: 1140340634@qq.com

Min Zeng
School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, PR China
E-mail address: 1812750442@qq.com

Yongxin Yuan
(**Corresponding author**) School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, PR China
E-mail address: yxyuan@hbnu.edu.cn

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 693–706

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4351

k-SRIVASTAVA HYPERGEOMETRIC FUNCTIONS AND THEIR INTEGRAL REPRESENTATIONS

SENA HALICI AND AYŞEGÜL ÇETİNKAYA

Received 28 July, 2022

Abstract. In this study, we introduce the k-Srivastava hypergeometric functions by means of the Pochhammer k-symbol. Also, we obtain the relations between k-Srivastava hypergeometric and classical Srivastava hypergeometric functions. Then, we show that with the help of these relations, integral representations of k-Srivastava hypergeometric functions can be easily proved without the need for lengthy proofs.

2010 *Mathematics Subject Classification:* 33B15; 33C05; 33C60; 33C70

Keywords: Srivastava hypergeometric functions, Pochhammer k-symbol, integral representation

1. Introduction

Various generalizations of special functions have been frequently encountered in recent years [1, 3–5, 7–11, 14]. One of them is the k-generalization of special functions, and some of the studies on this subject are as follows:

In 2007, for $k \in \mathbb{R}^+$, Diaz and Pariguan [4] introduced, respectively, the k-gamma function $\Gamma_k(x)$, k-beta function $B_k(x,y)$ and Pochhammer k-symbol $(\alpha)_{n,k}$ as follows:

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}}\, dt, \quad \mathrm{Re}(x) > 0, \tag{1.1}$$

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\, dt, \quad \mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0, \tag{1.2}$$

$$(\alpha)_{n,k} = \begin{cases} \alpha(\alpha+k)(\alpha+2k)\cdots(\alpha+(n-1)k), & n = 1,2,\ldots \\ 1, & n = 0. \end{cases} \tag{1.3}$$

These functions and Pochhammer k-symbol have the following properties [4, 9, 11]:

$$\Gamma_k(x+k) = x\Gamma_k(x),$$
$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \tag{1.4}$$

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

$$(\alpha)_{n,k} = \frac{\Gamma_k(\alpha+nk)}{\Gamma_k(\alpha)}, \tag{1.5}$$

$$\frac{(\beta)_{n,k}}{(\gamma)_{n,k}} = \frac{B_k(\beta+kn,\gamma-\beta)}{B_k(\beta,\gamma-\beta)}, \tag{1.6}$$

$$(\alpha)_{m+n,k} = (\alpha)_{m,k}(\alpha+mk)_{n,k}, \tag{1.7}$$

$$\sum_{n=0}^{\infty}(\alpha)_{n,k}\frac{x^n}{n!} = (1-kx)^{-\frac{\alpha}{k}}, \quad |x| < \frac{1}{k}.$$

Clearly, taking $k=1$ in (1.1), (1.2) and (1.3), the classical gamma function $\Gamma(x)$, classical beta function $B(x,y)$ and classical Pochhammer symbol $(\alpha)_n$ are obtained respectively. That is,

$$\Gamma_1(x) = \Gamma(x), \quad B_1(x,y) = B(x,y), \quad (\alpha)_{n,1} = (\alpha)_n.$$

Note that, the relations between the definitions of k-generalizations given by (1.1), (1.2), (1.3) and their corresponding classical definitions (that is, the $k=1$ case), are as follows (see [4]):

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right), \tag{1.8}$$

$$B_k(x,y) = \frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right),$$

$$(\alpha)_{n,k} = k^n\left(\frac{\alpha}{k}\right)_n. \tag{1.9}$$

Moreover, Diaz and Pariguan [4] also introduced k-hypergeometric function defined by

$${}_2F_{1,k}(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}}\frac{x^n}{n!}, \quad |x| < \frac{1}{k} \tag{1.10}$$

$$(k \in \mathbb{R}^+ \text{and } \gamma \neq 0,-k,-2k,\ldots).$$

For $k=1$, k-hypergeometric function reduces to classical hypergeometric function

$${}_2F_{1,1}(\alpha,\beta;\gamma;x) = {}_2F_1(\alpha,\beta;\gamma;x)$$

and the relation between them is

$${}_2F_{1,k}(\alpha,\beta;\gamma;x) = {}_2F_1\left(\frac{\alpha}{k},\frac{\beta}{k};\frac{\gamma}{k};kx\right). \tag{1.11}$$

In 2012, Mubeen and Habibullah [9] presented an integral representation of k-hypergeometric function as

$${}_2F_{1,k}(\alpha,\beta;\gamma;x) = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-kxt)^{-\frac{\alpha}{k}}\,dt,$$

$$(\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0).$$

In 2015, Mubeen et al. [10] defined the k-generalization of Appell hypergeometric function F_1 as

$$F_{1,k}(\alpha,\beta,\beta';\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<\frac{1}{k},|y|<\frac{1}{k} \tag{1.12}$$
$$(k\in\mathbb{R}^+\text{and }\gamma\neq 0,-k,-2k,\ldots)$$

and obtained an integral representation of this function as

$$\begin{aligned}F_{1,k}(\alpha,\beta,\beta';\gamma;x,y)&=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}\\&\quad\times\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\gamma-\alpha}{k}-1}(1-kxt)^{-\frac{\beta}{k}}(1-kyt)^{-\frac{\beta'}{k}}\,dt,\\&\quad(\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0).\end{aligned}$$

In 2017, Kıymaz et al. [7] introduced k-generalizations of Appell hypergeometric functions F_2,F_3 and F_4 as

$$F_{2,k}(\alpha,\beta,\beta';\gamma,\gamma';x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m,k}(\gamma')_{n,k}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|+|y|<\frac{1}{k}$$
$$F_{3,k}(\alpha,\alpha',\beta,\beta';\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m,k}(\alpha')_{n,k}(\beta)_{m,k}(\beta')_{n,k}}{(\gamma)_{m+n,k}}\frac{x^m}{m!}\frac{y^n}{n!},\quad |x|<\frac{1}{k},\ |y|<\frac{1}{k}$$
$$F_{4,k}(\alpha,\beta;\gamma,\gamma';x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n,k}(\beta)_{m+n,k}}{(\gamma)_{m,k}(\gamma')_{n,k}}\frac{x^m}{m!}\frac{y^n}{n!},\quad \sqrt{|x|}+\sqrt{|y|}<\frac{1}{\sqrt{k}}$$

where $k\in\mathbb{R}^+$ and $\gamma,\gamma'\neq 0,-k,-2k,\ldots$. They also presented following relations between k-Appell hypergeometric and classical Appell hypergeometric functions (that is, the $k=1$ case), in the forms

$$F_{1,k}(\alpha,\beta,\beta';\gamma;x,y)=F_1\left(\frac{\alpha}{k},\frac{\beta}{k},\frac{\beta'}{k};\frac{\gamma}{k};kx,ky\right), \tag{1.13}$$

$$F_{2,k}(\alpha,\beta,\beta';\gamma,\gamma';x,y)=F_2\left(\frac{\alpha}{k},\frac{\beta}{k},\frac{\beta'}{k};\frac{\gamma}{k},\frac{\gamma'}{k};kx,ky\right), \tag{1.14}$$

$$F_{3,k}(\alpha,\alpha',\beta,\beta';\gamma;x,y)=F_3\left(\frac{\alpha}{k},\frac{\alpha'}{k},\frac{\beta}{k},\frac{\beta'}{k};\frac{\gamma}{k};kx,ky\right), \tag{1.15}$$

$$F_{4,k}(\alpha,\beta;\gamma,\gamma';x,y)=F_4\left(\frac{\alpha}{k},\frac{\beta}{k};\frac{\gamma}{k},\frac{\gamma'}{k};kx,ky\right). \tag{1.16}$$

In the same article, they gave integral representations and double Mellin transforms to show that many properties of k-Appell hypergeometric functions can be easily deduced using relations (1.13)-(1.16) and the well-known properties of classical Appell hypergeometric functions.

In 2020, Gürel Yılmaz et.al. [5] acquired some transformation formulas, reduction formulas, linear and bilinear generating relations for the k-Appell hypergeometric functions.

The purpose of this study is to firstly introduce the k-Srivastava hypergeometric functions using the Pochhammer k-symbol. Then, it is to obtain the relations between k-Srivastava hypergeometric and classical Srivastava hypergeometric functions corresponding to $k=1$ case. Finally, it is to point out that many properties of k-Srivastava hypergeometric functions, such as their integral representations, can be easily obtained using these relations and the well-known properties of classical Srivastava hypergeometric functions.

2. k-SRIVASTAVA HYPERGEOMETRIC FUNCTIONS

In this section, k-generalizations of the classical Srivastava hypergeometric functions H_A, H_B and H_C are defined using the Pochhammer k-symbol. The definitions of these k-generalizations called k-Srivastava hypergeometric functions $H_{A,k}$, $H_{B,k}$ and $H_{C,k}$ are as follows:

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n+p,k}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{2.1}$$
$$\big(r<1,s<1,t<(1-r)(1-s)\big);$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n,k}(\gamma_3)_{p,k}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{2.2}$$
$$\big(r+s+t+2\sqrt{rst}<1\big);$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma)_{m+n+p,k}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{2.3}$$
$$\big(r<1,s<1,t<1,r+s+t-2\sqrt{(1-r)(1-s)(1-t)}<2\big)$$

where $k\in\mathbb{R}^+$; $\gamma,\gamma_1,\gamma_2,\gamma_3\neq 0,-k,-2k,\ldots$ and $r:=k|x|, s:=k|y|, t:=k|z|$.

Throughout this paper, we assume that k is any positive real number.

Remark 1. For $k=1$, the k-Srivastava hypergeometric functions $H_{A,k}$, $H_{B,k}$ and $H_{C,k}$ in (2.1)-(2.3) are reduced to the classical Srivastava hypergeometric functions H_A, H_B and H_C. That is,

$$H_{A,1}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=H_A(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z), \tag{2.4}$$
$$H_{B,1}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z)=H_B(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z), \tag{2.5}$$

$$H_{C,1}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = H_C(\alpha,\beta_1,\beta_2;\gamma;x,y,z). \tag{2.6}$$

Remark 2. Using properties (1.4)-(1.7), the k-Srivastava hypergeometric functions can be also expressed as follows:

$$\begin{aligned}
H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}}{(\gamma_1)_{m,k}} \\
&\quad \times \frac{B_k(\beta_2+kn+kp,\gamma_2-\beta_2)}{B_k(\beta_2,\gamma_2-\beta_2)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \\
H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha+\beta_1)_{2m+n+p,k}(\beta_2)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n,k}(\gamma_3)_{p,k}} \\
&\quad \times \frac{B_k(\alpha+km+kp,\beta_1+km+kn)}{B_k(\alpha,\beta_1)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \\
H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) &= \sum_{m,n,p=0}^{\infty} \frac{(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma)_{n,k}} \\
&\quad \times \frac{B_k(\alpha+km+kp,\gamma-\alpha+kn)}{B_k(\alpha,\gamma-\alpha+kn)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}.
\end{aligned}$$

Here $B_k(x,y)$ is the k-beta function defined in (1.2).

Remark 3. Considering (1.7) and (1.12), the functions $H_{A,k}$ and $H_{C,k}$ which have the three-fold series representations in (2.1) and (2.3), can also be represented by the single-fold series as follows, respectively.

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) = \sum_{m=0}^{\infty} \frac{(\alpha)_{m,k}(\beta_1)_{m,k}}{(\gamma_1)_{m,k}} F_{1,k}(\beta_2,\beta_1+km,\alpha+km;\gamma_2;y,z)\frac{x^m}{m!},$$

and

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = \sum_{n=0}^{\infty} \frac{(\beta_1)_{n,k}(\beta_2)_{n,k}}{(\gamma)_{n,k}} F_{1,k}(\alpha,\beta_1+kn,\beta_2+kn;\gamma+kn;x,z)\frac{y^n}{n!}.$$

Here $F_{1,k}$ is the k-Appell hypergeometric function defined in (1.12).

Theorem 1. *The relations between the k-Srivastava hypergeometric functions given in* (2.1)-(2.3) *and the classical Srivastava hypergeometric functions given in* (2.4)-(2.6) *are,*

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) = H_A\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k};kx,ky,kz\right), \tag{2.7}$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = H_B\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k},\frac{\gamma_3}{k};kx,ky,kz\right), \tag{2.8}$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = H_C\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma}{k};kx,ky,kz\right). \tag{2.9}$$

Proof. By using relation (1.9) in the definition of function $H_{A,k}$ in (2.1) and considering the definition of function H_A in (2.4), we obtain

$$\begin{aligned}
H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z) &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n+p,k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{k^{m+p}(\frac{\alpha}{k})_{m+p}k^{m+n}(\frac{\beta_1}{k})_{m+n}k^{n+p}(\frac{\beta_2}{k})_{n+p}}{k^m(\frac{\gamma_1}{k})_m k^{n+p}(\frac{\gamma_2}{k})_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{(\frac{\alpha}{k})_{m+p}(\frac{\beta_1}{k})_{m+n}(\frac{\beta_2}{k})_{n+p}}{(\frac{\gamma_1}{k})_m(\frac{\gamma_2}{k})_{n+p}} \frac{(kx)^m}{m!}\frac{(ky)^n}{n!}\frac{(kz)^p}{p!} \\
&= H_A\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k};kx,ky,kz\right)
\end{aligned}$$

which completes the proof of (2.7). Similarly, we can prove relation (2.8) by using (1.9), (2.2) and (2.5) as follows:

$$\begin{aligned}
H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n,k}(\gamma_3)_{p,k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{k^{m+p}(\frac{\alpha}{k})_{m+p}k^{m+n}(\frac{\beta_1}{k})_{m+n}k^{n+p}(\frac{\beta_2}{k})_{n+p}}{k^m(\frac{\gamma_1}{k})_m k^n(\frac{\gamma_2}{k})_n k^p(\frac{\gamma_3}{k})_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{(\frac{\alpha}{k})_{m+p}(\frac{\beta_1}{k})_{m+n}(\frac{\beta_2}{k})_{n+p}}{(\frac{\gamma_1}{k})_m(\frac{\gamma_2}{k})_n(\frac{\gamma_3}{k})_p} \frac{(kx)^m}{m!}\frac{(ky)^n}{n!}\frac{(kz)^p}{p!} \\
&= H_B\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k},\frac{\gamma_3}{k};kx,ky,kz\right).
\end{aligned}$$

Finally, by using (1.9), (2.3) and (2.6), we have (2.9) as follows:

$$\begin{aligned}
H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) &= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p,k}(\beta_1)_{m+n,k}(\beta_2)_{n+p,k}}{(\gamma)_{m+n+p,k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{k^{m+p}(\frac{\alpha}{k})_{m+p}k^{m+n}(\frac{\beta_1}{k})_{m+n}k^{n+p}(\frac{\beta_2}{k})_{n+p}}{k^{m+n+p}(\frac{\gamma}{k})_{m+n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \\
&= \sum_{m,n,p=0}^{\infty} \frac{(\frac{\alpha}{k})_{m+p}(\frac{\beta_1}{k})_{m+n}(\frac{\beta_2}{k})_{n+p}}{(\frac{\gamma}{k})_{m+n+p}} \frac{(kx)^m}{m!}\frac{(ky)^n}{n!}\frac{(kz)^p}{p!} \\
&= H_C\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma}{k};kx,ky,kz\right).
\end{aligned}$$

□

Remark 4. The convergence regions of the series (2.1)-(2.3) can easily seen from the relations of (2.7)-(2.9) and the convergence regions of the classic Srivastava hypergeometric series.

3. INTEGRAL REPRESENTATIONS

In this section, we show that the integral representations of k-Srivastava hypergeometric functions can be easily proved using relations (2.7)-(2.9) and the well-known integral representations of classical Srivastava hypergeometric functions given in [2] (see also [6, 12, 13]).

Theorem 2. *The function $H_{A,k}$ has the following integral representations.*

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\frac{\Gamma_k(\gamma_1)\Gamma_k(\gamma_2)}{k^2\Gamma_k(\beta_1)\Gamma_k(\beta_2)\Gamma_k(\gamma_1-\beta_1)\Gamma_k(\gamma_2-\beta_2)}$$
$$\times\int_0^1\int_0^1\xi^{\frac{\beta_1}{k}-1}\eta^{\frac{\beta_2}{k}-1}(1-\xi)^{\frac{\gamma_1-\beta_1}{k}-1}(1-\eta)^{\frac{\gamma_2-\beta_2}{k}-1}$$
$$\times(1-ky\eta)^{\frac{-\beta_1}{k}}(1-kx\xi-kz\eta)^{\frac{-\alpha}{k}}\left(1-\frac{k^2xy\xi\eta}{(1-ky\eta)(1-kx\xi-kz\eta)}\right)^{\frac{-\alpha}{k}}d\xi d\eta, \quad (3.1)$$
$$\big(\mathrm{Re}(\gamma_1)>\mathrm{Re}(\beta_1)>0,\mathrm{Re}(\gamma_2)>\mathrm{Re}(\beta_2)>0\big);$$

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\frac{\Gamma_k(\gamma_2)}{k\Gamma_k(\beta_2)\Gamma_k(\gamma_2-\beta_2)}\int_0^1\xi^{\frac{\beta_2}{k}-1}(1-\xi)^{\frac{\gamma_2-\beta_2}{k}-1}$$
$$\times(1-ky\xi)^{\frac{-\beta_1}{k}}(1-kz\xi)^{\frac{-\alpha}{k}}{}_2F_{1,k}\left(\alpha,\beta_1;\gamma_1;\frac{x}{(1-ky\xi)(1-kz\xi)}\right)d\xi, \quad (3.2)$$
$$\big(\mathrm{Re}(\gamma_2)>\mathrm{Re}(\beta_2)>0\big);$$

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\frac{\Gamma_k(\gamma_2)(1+\lambda)^{\frac{\beta_2}{k}}}{k\Gamma_k(\beta_2)\Gamma_k(\gamma_2-\beta_2)}\int_0^1\xi^{\frac{\beta_2}{k}-1}(1-\xi)^{\frac{\gamma_2-\beta_2}{k}-1}$$
$$\times(1+\lambda\xi)^{\frac{\alpha+\beta_1-\gamma_2}{k}}[1+\lambda\xi-(1+\lambda)ky\xi]^{\frac{-\beta_1}{k}}[1+\lambda\xi-(1+\lambda)kz\xi]^{\frac{-\alpha}{k}}$$
$$\times{}_2F_{1,k}\left(\alpha,\beta_1;\gamma_1;\frac{x(1+\lambda\xi)^2}{[1+\lambda\xi-(1+\lambda)ky\xi][1+\lambda\xi-(1+\lambda)kz\xi]}\right)d\xi, \quad (3.3)$$
$$\big(\mathrm{Re}(\gamma_2)>\mathrm{Re}(\beta_2)>0;\lambda>-1\big);$$

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\frac{\Gamma_k(\gamma_2)}{k\Gamma_k(\beta_2)\Gamma_k(\gamma_2-\beta_2)}\frac{(b-c)^{\frac{\beta_2}{k}}(a-c)^{\frac{\gamma_2-\beta_2}{k}}}{(b-a)^{\frac{\gamma_2-\alpha-\beta_1}{k}-1}}$$
$$\times\int_a^b(b-\xi)^{\frac{\gamma_2-\beta_2}{k}-1}(\xi-a)^{\frac{\beta_2}{k}-1}(\xi-c)^{\frac{\alpha+\beta_1-\gamma_2}{k}}[(b-a)(\xi-c)-(b-c)(\xi-a)ky]^{\frac{-\beta_1}{k}}$$
$$\times[(b-a)(\xi-c)-(b-c)(\xi-a)kz]^{\frac{-\alpha}{k}}{}_2F_{1,k}(\alpha,\beta_1;\gamma_1;x\sigma)d\xi, \quad (3.4)$$
$$\big(\mathrm{Re}(\gamma_2)>\mathrm{Re}(\beta_2)>0;c<a<b\big),$$
$$\sigma:=\frac{(b-a)^2(\xi-c)^2}{[(b-a)(\xi-c)-(b-c)(\xi-a)ky][(b-a)(\xi-c)-(b-c)(\xi-a)kz]};$$

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\frac{\Gamma_k(\gamma_2)}{k\Gamma_k(\beta_2)\Gamma_k(\gamma_2-\beta_2)}\int_0^\infty \xi^{\frac{\beta_2}{k}-1}(1+\xi)^{\frac{\alpha+\beta_1-\gamma_2}{k}}$$
$$\times(1+\xi-ky\xi)^{\frac{-\beta_1}{k}}(1+\xi-kz\xi)^{\frac{-\alpha}{k}}$$
$$\times {}_2F_{1,k}\left(\alpha,\beta_1;\gamma_1;\frac{x(1+\xi)^2}{(1+\xi-ky\xi)(1+\xi-kz\xi)}\right)d\xi, \tag{3.5}$$
$$\big(\mathrm{Re}(\gamma_2)>\mathrm{Re}(\beta_2)>0\big);$$

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\frac{2\Gamma_k(\gamma_2)}{k\Gamma_k(\beta_2)\Gamma_k(\gamma_2-\beta_2)}\int_0^{\frac{\pi}{2}}(\sin^2\xi)^{\frac{\beta_2}{k}-\frac{1}{2}}(\cos^2\xi)^{\frac{\gamma_2-\beta_2}{k}-\frac{1}{2}}$$
$$\times(1-ky\sin^2\xi)^{\frac{-\beta_1}{k}}(1-kz\sin^2\xi)^{\frac{-\alpha}{k}}$$
$$\times {}_2F_{1,k}\left(\alpha,\beta_1;\gamma_1;\frac{x}{(1-ky\sin^2\xi)(1-kz\sin^2\xi)}\right)d\xi, \tag{3.6}$$
$$\big(\mathrm{Re}(\gamma_2)>\mathrm{Re}(\beta_2)>0\big).$$

Here ${}_2F_{1,k}$ is the k-hypergeometric function defined in (1.10).

Proof. In [12] (see also [2, 6]), the well-known integral representation of H_A is given by

$$H_A(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma_1-\beta_1)\Gamma(\gamma_2-\beta_2)}$$
$$\times\int_0^1\int_0^1\xi^{\beta_1-1}\eta^{\beta_2-1}(1-\xi)^{\gamma_1-\beta_1-1}(1-\eta)^{\gamma_2-\beta_2-1}$$
$$\times(1-y\eta)^{-\beta_1}(1-x\xi-z\eta)^{-\alpha}\left(1-\frac{xy\xi\eta}{(1-y\eta)(1-x\xi-z\eta)}\right)^{-\alpha}d\xi d\eta.$$

Using this integral representation in (2.7) and making use of (1.8), we have

$$H_{A,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=H_A\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k};kx,ky,kz\right)$$
$$=\frac{\Gamma(\frac{\gamma_1}{k})\Gamma(\frac{\gamma_2}{k})}{\Gamma(\frac{\beta_1}{k})\Gamma(\frac{\beta_2}{k})\Gamma(\frac{\gamma_1-\beta_1}{k})\Gamma(\frac{\gamma_2-\beta_2}{k})}$$
$$\times\int_0^1\int_0^1\xi^{\frac{\beta_1}{k}-1}\eta^{\frac{\beta_2}{k}-1}(1-\xi)^{\frac{\gamma_1-\beta_1}{k}-1}(1-\eta)^{\frac{\gamma_2-\beta_2}{k}-1}$$
$$\times(1-ky\eta)^{\frac{-\beta_1}{k}}(1-kx\xi-kz\eta)^{\frac{-\alpha}{k}}\left(1-\frac{k^2xy\xi\eta}{(1-ky\eta)(1-kx\xi-kz\eta)}\right)^{\frac{-\alpha}{k}}d\xi d\eta$$
$$=\frac{k^{1-\frac{\gamma_1}{k}}\Gamma_k(\gamma_1)k^{1-\frac{\gamma_2}{k}}\Gamma_k(\gamma_2)}{k^{1-\frac{\beta_1}{k}}\Gamma_k(\beta_1)k^{1-\frac{\beta_2}{k}}\Gamma_k(\beta_2)k^{1-\frac{\gamma_1-\beta_1}{k}}\Gamma_k(\gamma_1-\beta_1)k^{1-\frac{\gamma_2-\beta_2}{k}}\Gamma_k(\gamma_2-\beta_2)}$$

$$\times \int_0^1 \int_0^1 \xi^{\frac{\beta_1}{k}-1} \eta^{\frac{\beta_2}{k}-1} (1-\xi)^{\frac{\gamma_1-\beta_1}{k}-1} (1-\eta)^{\frac{\gamma_2-\beta_2}{k}-1}$$
$$\times (1-ky\eta)^{\frac{-\beta_1}{k}} (1-kx\xi-kz\eta)^{\frac{-\alpha}{k}} \left(1-\frac{k^2xy\xi\eta}{(1-ky\eta)(1-kx\xi-kz\eta)}\right)^{\frac{-\alpha}{k}} d\xi d\eta$$

which completes the proof of (3.1).

The other integral representations (3.2)-(3.6) of $H_{A,k}$ are proved similarly by considering (1.8) and (1.11) after the well-known integral representations of H_A given in [2] are used in the right-hand side of (2.7). □

Theorem 3. *The function $H_{B,k}$ has the following integral representations.*

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \frac{\Gamma_k(\alpha+\beta_1)}{k\Gamma_k(\alpha)\Gamma_k(\beta_1)} \int_0^1 \xi^{\frac{\alpha}{k}-1}(1-\xi)^{\frac{\beta_1}{k}-1}$$
$$\times X_{4,k}(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\xi(1-\xi),y(1-\xi),z\xi)\,d\xi, \tag{3.7}$$
$$\big(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1)\}>0\big);$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \frac{\Gamma_k(\alpha+\beta_1)}{k\Gamma_k(\alpha)\Gamma_k(\beta_1)} \frac{(b-c)^{\frac{\alpha}{k}}(a-c)^{\frac{\beta_1}{k}}}{(b-a)^{\frac{\alpha+\beta_1}{k}-1}}$$
$$\times \int_a^b (b-\xi)^{\frac{\beta_1}{k}-1}(\xi-a)^{\frac{\alpha}{k}-1}(\xi-c)^{\frac{-\alpha-\beta_1}{k}}$$
$$\times X_{4,k}(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\sigma_1,y\sigma_2,z\sigma_3)\,d\xi, \tag{3.8}$$
$$\big(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1)\}>0; c<a<b\big),$$
$$\sigma_1 := \frac{(a-c)(b-c)(\xi-a)(b-\xi)}{(b-a)^2(\xi-c)^2}, \sigma_2 := \frac{(a-c)(b-\xi)}{(b-a)(\xi-c)}, \sigma_3 := \frac{(b-c)(\xi-a)}{(b-a)(\xi-c)};$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \frac{2\Gamma_k(\alpha+\beta_1)}{k\Gamma_k(\alpha)\Gamma_k(\beta_1)} \int_0^{\frac{\pi}{2}} (\sin^2\xi)^{\frac{\alpha}{k}-\frac{1}{2}}(\cos^2\xi)^{\frac{\beta_1}{k}-\frac{1}{2}}$$
$$\times X_{4,k}(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\sigma_1,y\sigma_2,z\sigma_3)\,d\xi, \tag{3.9}$$
$$\big(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1)\}>0\big),$$
$$\sigma_1 := \sin^2\xi\cos^2\xi, \sigma_2 := \cos^2\xi, \sigma_3 := \sin^2\xi;$$

$$H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \frac{2\Gamma_k(\alpha+\beta_1)(1+\lambda)^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)\Gamma_k(\beta_1)}$$
$$\times \int_0^{\frac{\pi}{2}} \frac{(\sin^2\xi)^{\frac{\alpha}{k}-\frac{1}{2}}(\cos^2\xi)^{\frac{\beta_1}{k}-\frac{1}{2}}}{(1+\lambda\sin^2\xi)^{\frac{\alpha+\beta_1}{k}}}$$
$$\times X_{4,k}(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\sigma_1,y\sigma_2,z\sigma_3)\,d\xi, \tag{3.10}$$
$$\big(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1)\}>0;\lambda>-1\big),$$

$$\sigma_1 := \frac{(1+\lambda)\sin^2\xi\cos^2\xi}{(1+\lambda\sin^2\xi)^2}, \sigma_2 := \frac{\cos^2\xi}{1+\lambda\sin^2\xi}, \sigma_3 := \frac{(1+\lambda)\sin^2\xi}{1+\lambda\sin^2\xi};$$

$$\begin{aligned}
&H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \frac{2\Gamma_k(\alpha+\beta_1)\lambda^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)\Gamma_k(\beta_1)} \\
&\times\int_0^{\frac{\pi}{2}} \frac{(\sin^2\xi)^{\frac{\alpha}{k}-\frac{1}{2}}(\cos^2\xi)^{\frac{\beta_1}{k}-\frac{1}{2}}}{(\cos^2\xi+\lambda\sin^2\xi)^{\frac{\alpha+\beta_1}{k}}} \\
&\times X_{4,k}(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\sigma_1,y\sigma_2,z\sigma_3)\,d\xi, \qquad (3.11)\\
&\left(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1)\}>0;\lambda>0\right), \\
&\sigma_1 := \frac{\lambda\sin^2\xi\cos^2\xi}{(\cos^2\xi+\lambda\sin^2\xi)^2}, \sigma_2 := \frac{\cos^2\xi}{\cos^2\xi+\lambda\sin^2\xi}, \sigma_3 := \frac{\lambda\sin^2\xi}{\cos^2\xi+\lambda\sin^2\xi}.
\end{aligned}$$

Here, the k-generalization of the classical Exton hypergeometric function X_4 (see [15]) is defined as

$$X_{4,k}(\alpha,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{2m+n+p,k}(\beta_1)_{n+p,k}}{(\gamma_1)_{m,k}(\gamma_2)_{n,k}(\gamma_3)_{p,k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$

which has the following properties:

$$X_{4,1}(\alpha,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z) = X_4(\alpha,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z),$$

and

$$X_{4,k}(\alpha,\beta_1;\gamma_1,\gamma_2,\gamma_3;x,y,z) = X_4\left(\frac{\alpha}{k},\frac{\beta_1}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k},\frac{\gamma_3}{k};kx,ky,kz\right). \qquad (3.12)$$

Proof. In [2], the well-known integral representation of H_B is given by

$$\begin{aligned}
H_B(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) &= \frac{\Gamma(\alpha+\beta_1)}{\Gamma(\alpha)\Gamma(\beta_1)}\int_0^1 \xi^{\alpha-1}(1-\xi)^{\beta_1-1} \\
&\times X_4(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\xi(1-\xi),y(1-\xi),z\xi)\,d\xi.
\end{aligned}$$

Using this integral representation in relation (2.8) and considering (1.8) and (3.12), we obtain

$$\begin{aligned}
H_{B,k}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z) &= H_B\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k},\frac{\gamma_3}{k};kx,ky,kz\right) \\
&= \frac{\Gamma(\frac{\alpha+\beta_1}{k})}{\Gamma(\frac{\alpha}{k})\Gamma(\frac{\beta_1}{k})}\int_0^1 \xi^{\frac{\alpha}{k}-1}(1-\xi)^{\frac{\beta_1}{k}-1} \\
&\times X_4\left(\frac{\alpha+\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma_1}{k},\frac{\gamma_2}{k},\frac{\gamma_3}{k};kx\xi(1-\xi),ky(1-\xi),kz\xi\right)d\xi
\end{aligned}$$

$$=\frac{k^{1-\frac{\alpha+\beta_1}{k}}\Gamma_k(\alpha+\beta_1)}{k^{1-\frac{\alpha}{k}}\Gamma_k(\alpha)k^{1-\frac{\beta_1}{k}}\Gamma_k(\beta_1)}\int_0^1 \xi^{\frac{\alpha}{k}-1}(1-\xi)^{\frac{\beta_1}{k}-1}$$
$$\times X_{4,k}\left(\alpha+\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x\xi(1-\xi),y(1-\xi),z\xi\right)d\xi,$$

which completes the proof of (3.7).

The other integral representations (3.8)-(3.11) of $H_{B,k}$ are proved similarly by considering (1.8) and (3.12) after the well-known integral representations of H_B given in [2] are used in the right-hand side of (2.8). □

Theorem 4. *The function $H_{C,k}$ has the following integral representations.*

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=\frac{\Gamma_k(\gamma)}{k^2\Gamma_k(\alpha)\Gamma_k(\beta_1)\Gamma_k(\gamma-\alpha-\beta_1)}$$
$$\times\int_0^1\int_0^1 \xi^{\frac{\alpha}{k}-1}\eta^{\frac{\beta_1}{k}-1}(1-\xi)^{\frac{\gamma-\alpha}{k}-1}(1-\eta)^{\frac{\gamma-\alpha-\beta_1}{k}-1}(1-kx\xi)^{\frac{\beta_2-\beta_1}{k}}$$
$$\times\left(1-kx\xi-ky\eta-kz\xi+ky\xi\eta+k^2xz\xi^2\right)^{\frac{-\beta_2}{k}}d\xi d\eta, \tag{3.13}$$
$$\left(\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta_1),\mathrm{Re}(\gamma-\alpha-\beta_1)\}>0\right);$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}\int_0^1 \xi^{\frac{\alpha}{k}-1}(1-\xi)^{\frac{\gamma-\alpha}{k}-1}(1-kx\xi)^{\frac{-\beta_1}{k}}$$
$$\times(1-kz\xi)^{\frac{-\beta_2}{k}}{}_2F_{1,k}\left(\beta_1,\beta_2;\gamma-\alpha;\frac{y(1-\xi)}{(1-kx\xi)(1-kz\xi)}\right)d\xi, \tag{3.14}$$
$$\left(\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0\right);$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=\frac{\Gamma_k(\gamma)(1+\lambda)^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}\int_0^1 \xi^{\frac{\alpha}{k}-1}(1-\xi)^{\frac{\gamma-\alpha}{k}-1}(1+\lambda\xi)^{\frac{\beta_1+\beta_2-\gamma}{k}}$$
$$\times[1+\lambda\xi-(1+\lambda)kx\xi]^{\frac{-\beta_1}{k}}[1+\lambda\xi-(1+\lambda)kz\xi]^{\frac{-\beta_2}{k}}$$
$$\times{}_2F_{1,k}\left(\beta_1,\beta_2;\gamma-\alpha;\frac{y(1+\lambda\xi)(1-\xi)}{[1+\lambda\xi-(1+\lambda)kx\xi][1+\lambda\xi-(1+\lambda)kz\xi]}\right)d\xi, \tag{3.15}$$
$$\left(\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0;\lambda>-1\right);$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z)=\frac{\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}\frac{(b-c)^{\frac{\alpha}{k}}(a-c)^{\frac{\gamma-\alpha}{k}}}{(b-a)^{\frac{\gamma-\beta_1-\beta_2}{k}-1}}\int_a^b (b-\xi)^{\frac{\gamma-\alpha}{k}-1}$$
$$\times(\xi-a)^{\frac{\alpha}{k}-1}(\xi-c)^{\frac{\beta_1+\beta_2-\gamma}{k}}[(b-a)(\xi-c)-(b-c)(\xi-a)kx]^{\frac{-\beta_1}{k}}$$
$$\times[(b-a)(\xi-c)-(b-c)(\xi-a)kz]^{\frac{-\beta_2}{k}}{}_2F_{1,k}(\beta_1,\beta_2;\gamma-\alpha;y\sigma)d\xi, \tag{3.16}$$
$$\left(\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0;c<a<b\right),$$

$$\sigma := \frac{(b-a)(a-c)(\xi-c)(b-\xi)}{[(b-a)(\xi-c)-(b-c)(\xi-a)kx][(b-a)(\xi-c)-(b-c)(\xi-a)kz]};$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}\int_0^\infty \xi^{\frac{\alpha}{k}-1}(1+\xi)^{\frac{\beta_1+\beta_2-\gamma}{k}}$$
$$\times(1+\xi-kx\xi)^{\frac{-\beta_1}{k}}(1+\xi-kz\xi)^{\frac{-\beta_2}{k}}{}_2F_{1,k}(\beta_1,\beta_2;\gamma-\alpha;y\sigma))\,d\xi, \tag{3.17}$$
$$\big(\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0\big),$$
$$\sigma := \frac{1+\xi}{(1+\xi-kx\xi)(1+\xi-kz\xi)};$$

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = \frac{2\Gamma_k(\gamma)}{k\Gamma_k(\alpha)\Gamma_k(\gamma-\alpha)}\int_0^{\frac{\pi}{2}} (\sin^2\xi)^{\frac{\alpha}{k}-\frac{1}{2}}(\cos^2\xi)^{\frac{\gamma-\alpha}{k}-\frac{1}{2}}$$
$$\times(1-kx\sin^2\xi)^{\frac{-\beta_1}{k}}(1-kz\sin^2\xi)^{\frac{-\beta_2}{k}}{}_2F_{1,k}(\beta_1,\beta_2;\gamma-\alpha;y\sigma)\,d\xi, \tag{3.18}$$
$$\big(\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0\big),$$
$$\sigma := \frac{\cos^2\xi}{(1-kx\sin^2\xi)(1-kz\sin^2\xi)}.$$

Here, ${}_2F_{1,k}$ is the k-hypergeometric function defined in (1.10).

Proof. In [13] (see also [2, 6]), the well-known integral representation of H_C is given by

$$H_C(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\gamma-\alpha-\beta_1)}$$
$$\times\int_0^1\int_0^1 \xi^{\alpha-1}\eta^{\beta_1-1}(1-\xi)^{\gamma-\alpha-1}(1-\eta)^{\gamma-\alpha-\beta_1-1}(1-x\xi)^{\beta_2-\beta_1}$$
$$\times\left(1-x\xi-y\eta-z\xi+y\xi\eta+xz\xi^2\right)^{-\beta_2}d\xi d\eta.$$

Using this integral representation in (2.9) and applying (1.8), we obtain

$$H_{C,k}(\alpha,\beta_1,\beta_2;\gamma;x,y,z) = H_C\left(\frac{\alpha}{k},\frac{\beta_1}{k},\frac{\beta_2}{k};\frac{\gamma}{k};kx,ky,kz\right)$$
$$=\frac{\Gamma(\frac{\gamma}{k})}{\Gamma(\frac{\alpha}{k})\Gamma(\frac{\beta_1}{k})\Gamma(\frac{\gamma-\alpha-\beta_1}{k})}$$
$$\times\int_0^1\int_0^1 \xi^{\frac{\alpha}{k}-1}\eta^{\frac{\beta_1}{k}-1}(1-\xi)^{\frac{\gamma-\alpha}{k}-1}(1-\eta)^{\frac{\gamma-\alpha-\beta_1}{k}-1}(1-kx\xi)^{\frac{\beta_2-\beta_1}{k}}$$
$$\times\left(1-kx\xi-ky\eta-kz\xi+ky\xi\eta+k^2xz\xi^2\right)^{\frac{-\beta_2}{k}}d\xi d\eta$$
$$=\frac{k^{1-\frac{\gamma}{k}}\Gamma_k(\gamma)}{k^{1-\frac{\alpha}{k}}\Gamma_k(\alpha)k^{1-\frac{\beta_1}{k}}\Gamma_k(\beta_1)k^{1-\frac{\gamma-\alpha-\beta_1}{k}}\Gamma_k(\gamma-\alpha-\beta_1)}$$

$$\times \int_0^1 \int_0^1 \xi^{\frac{\alpha}{k}-1} \eta^{\frac{\beta_1}{k}-1} (1-\xi)^{\frac{\gamma-\alpha}{k}-1} (1-\eta)^{\frac{\gamma-\alpha-\beta_1}{k}-1} (1-kx\xi)^{\frac{\beta_2-\beta_1}{k}}$$
$$\times \left(1-kx\xi-ky\eta-kz\xi+ky\xi\eta+k^2xz\xi^2\right)^{\frac{-\beta_2}{k}} d\xi d\eta$$

which completes the proof of (3.13).

The other integral representations (3.14)-(3.18) of $H_{C,k}$ are similarly proved by considering (1.8) and (1.11) after the well-known integral representations of H_C given in [2] are used in the right-hand side of (2.9). □

Remark 5. For $k = 1$, the integral representations obtained in this study are reduced to the well-known integral representations of the classical Srivastava hypergeometric functions given in [2] (see also [6, 12, 13]).

ACKNOWLEDGEMENTS

A part of this study was presented at the *6th International Conference on Computational Mathematics and Engineering Sciences, Ordu, Turkey, May 20-22, 2022*. Also, the authors would like to thank the referees for their valuable comments and suggestions for improving this article.

REFERENCES

[1] P. Agarwal, "Some inequalities involving Hadamard-type k-fractional integral operators," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 11, pp. 3882–3891, 2017, doi: 10.1002/mma.4270.

[2] J. Choi, A. Hasanov, H. M. Srivastava, and M. Turaev, "Integral representations for Srivastava's triple hypergeometric functions," *Taiwanese J. Math.*, vol. 15, no. 6, pp. 2751–2762, 2011, doi: 10.11650/twjm/1500406495.

[3] J. Choi and R. K. Parmar, "Generalized Srivastava's triple hypergeometric functions and their associated properties," *J. Nonlinear Sci. Appl*, vol. 10, pp. 817–827, 2017, doi: 10.22436/jnsa.010.02.41.

[4] R. Díaz and E. Pariguan, "On hypergeometric functions and Pochhammer k-symbol," *Divulg. Mat.*, vol. 15, no. 2, pp. 179–192, 2007.

[5] Ö. Gürel Yılmaz, R. Aktaş, and F. Taşdelen, "On some formulas for the k-analogue of Appell functions and generating relations via k-fractional derivative," *Fractal Fract.*, vol. 4, no. 4, p. 48, 2020, doi: 10.3390/fractalfract4040048.

[6] A. Hasanov, H. M. Srivastava, and M. Turaev, "Decomposition formulas for some triple hypergeometric functions," *J. Math. Anal. Appl.*, vol. 324, no. 2, pp. 955–969, 2006, doi: 10.1016/j.jmaa.2006.01.006.

[7] İ. O. Kıymaz, A. Çetinkaya, and P. Agarwal, "A study on the k-generalizations of some known functions and fractional operators," *J. Inequal. Spec. Funct.*, vol. 8, no. 4, pp. 31–41, 2017.

[8] M.-J. Luo, G. V. Milovanovic, and P. Agarwal, "Some results on the extended beta and extended hypergeometric functions," *Applied Mathematics and Computation*, vol. 248, pp. 631–651, 2014, doi: 10.1016/j.amc.2014.09.110.

[9] S. Mubeen and G. M. Habibullah, "An integral representation of some k-hypergeometric functions," *Int. Math. Forum*, vol. 7, no. 4, pp. 203–207, 2012.

[10] S. Mubeen, S. Iqbal, and G. Rahman, "Contiguous function relations and an integral representation for Appell k-series $F_{1,k}$," *Inter. J. Math. Research*, vol. 4, no. 2, pp. 53–63, 2015, doi: 10.18488/journal.24/2015.4.2/24.2.53.63.

[11] S. Mubeen and A. Rehman, "A note on k-gamma function and Pochhammer k-symbol," *J. Inf. Math. Sci.*, vol. 6, no. 2, pp. 93–107, 2014.

[12] H. M. Srivastava, "Hypergeometric functions of three variables," *Gaṇita*, vol. 15, no. 2, pp. 97–108, 1964.

[13] H. M. Srivastava, "Some integrals representing triple hypergeometric functions," *Rend. Circ. Mat. Palermo*, vol. 16, pp. 99–115, 1967, doi: 10.1007/BF02844089.

[14] H. M. Srivastava, P. Agarwal, and S. Jain, "Generating functions for the generalized Gauss hypergeometric functions," *Applied Mathematics and Computation*, vol. 247, pp. 348–352, 2014, doi: 10.1016/j.amc.2014.08.105.

[15] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian hypergeometric series.* New York, Chichester, Brisbane and Toronto: Halsted Press (Ellis Horwood Limited), John Wiley and Sons, 1985.

Authors' addresses

Sena Halıcı

Kırşehir Ahi Evran University, Faculty of Arts and Science, Department of Mathematics, 40100, Kırşehir, Turkey

E-mail address: halici.sena@ogr.ahievran.edu.tr

Ayşegül Çetinkaya

(**Corresponding author**) Kırşehir Ahi Evran University, Faculty of Arts and Science, Department of Mathematics, 40100, Kırşehir, Turkey

E-mail address: acetinkaya@ahievran.edu.tr

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 707–721

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4459

ON DISCRETE PROPERTIES OF CONTINUOUS MONOTONE FUNCTIONS

EMÍLIA HALUŠKOVÁ

Received 19 October, 2022

Abstract. We deal with strictly monotone continuous functions $h\colon I \to I$, where I is an interval of real numbers. The monounary algebra (I,h) contains at most 4 non-isomorphic components. We derive that there are 10 possibilities only how these components can be combined in (I,h). Two more options are cancelled in case that $I = \langle a,b\rangle$ for some real numbers $a,b,a<b$.

2010 *Mathematics Subject Classification:* 08A60; 26A15; 26A18; 37C20; 37C25; 97I20

Keywords: strictly monotone function, continuous function, interval, fixed point, monounary algebra, component

1. Introduction

The notion of a function belongs to the fundamental notions in mathematics. If range of a function is a subset of its domain, then this function determines an algebraic structure which is called a monounary algebra, cf. e.g. [10]. Therefore these algebras naturally occur in many areas of mathematics and their properties are applied in many ways. A lot of results outside universal algebra can be formulated in notions of monounary algebras. For example, Sharkovski Theorem from the theory of dynamical systems is about monounary algebras which contain cycles of all natural lengths, cf. [12, 13]. The famous Łojasiewicz Theorem describes component partitions of all monounary algebras with a bijective operation such that relevant functional equations of iterative roots have a solution, cf. [1, 9].

Monounary algebras can be visualised in a natural way, as directed graphs with exactly one out-edge at each vertex. Many tasks in universal algebra can be simplified to monounary algebras. To investigate congruences means to go to unary operations,

This publication was supported the grant VEGA 2/0104/24 and by the Operational Programme Integrated Infrastructure (OPII) for the project 313011BWH2: "InoCHF – Research and development in the field of innovative technologies in the management of patients with CHF", co-financed by the European Regional Development Fund.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

cf. [6]. To find a homomorphism is possible via a monounary algebras on enlarged sets, cf. [11].

Monounary algebras are elaborated from the point of view of universal algebra in great details, for overview see [2, 7]. This offers a tool for solving many questions e.g. in numerical analysis and functional equations, cf. [2, 3, 8]. Notice that methods to get the solution are constructive and they provide the ability to obtain algorithms which can be used by computer algebra systems.

Functions of $\mathbb{R}$ into $\mathbb{R}$ are called *real functions*. A function of I into I, where I is an interval of real numbers, is said to be an *interval function*. Sharkovski Theorem mentioned above is on continuous interval functions.

We deal with strictly monotone continuous interval functions. Strictly increasing (strictly decreasing) functions or sequences will be called increasing (decreasing) in short.

Monounary algebras with continuous increasing real functions were studied by O. Kopeček in [8]. We develop methods from this paper for monounary algebras with continuous increasing interval functions. We obtain a classification of all such functions according to types of components which occur in corresponding algebras. Then we apply this classification to classify continuous decreasing interval functions according to types of components which occur in corresponding algebras.

2. Preliminaries

The set of all positive integers is denoted by $\mathbb{N}$. If A is a set, then $\|A\|$ is the cardinality of this set.

Let A be a non-empty set and h be a function from A into A. We define by induction $h^n(x) = h(h^{n-1}(x))$ for every $x \in A$ and $n \in \mathbb{N} \setminus \{1\}$.

The pair (A,h) is called a *monounary algebra*. To (A,h) there corresponds an oriented graph with the vertex set A and the edge set $\{(a,h(a)) \mid a \in A\}$. We use the symbol $\cong$ for isomorphism between algebras. For monounary terminology see, e.g., [4, 5, 7, 10].

An algebra (A,h) is said to be *connected* if for every $x,y \in A$ there exist $m,n \in \mathbb{N}$ such that $h^m(x) = h^n(y)$.

Let (A,h) be a monounary algebra. An element $a \in A$ is called *cyclic* if there exists $k \in \mathbb{N}$ such that $h^k(a) = a$. If $h^k(a) = a$ for some $k \in \mathbb{N}$ and $h^l(a) \neq a$ for each $l < k, l \in \mathbb{N}$, then we say that a belongs to a k-element cycle. The algebra (A,h) is a *cycle* if it is connected and every element of A is cyclic. Then $\|A\| = i$ for some $i \in \mathbb{N}$ and (A,h) is said to be an i-element cycle.

Let $B \subseteq A$. We denote $h(B) = \{h(b), b \in B\}$ and $h|B$ the function that is the restriction of h onto B. If $h(B) \subseteq B$ and $(B,h|B)$ is a maximal connected subalgebra of (A,h), then we say that $(B,h|B)$ is a *component of* (A,h).

Suppose that h is injective and x is an element of range of h. Then we denote by $h^{-1}(x)$ the unique element y of A such that $h(y) = x$. Next, by induction, if $h^{-n}(x)$

is an element of range of h, then $h^{-(n+1)}(x)$ is the unique element $z \in A$ such that $h^{n+1}(z) = x$.

The next lemma is trivial.

Lemma 1. *Let $h\colon A \to A$ and $c \in A$. The following assertions are equivalent:*

(1) $h^2(c) = c$,

(2) *c belongs to a 1-element cycle in the algebra (A,h) or c belongs to a 2-element cycle in (A,h).*

We denote by N the monounary algebra that is defined on the set of all natural numbers with the successor function; analogously, Z is defined on the set of all integers. Further, let C_1, C_2 be fixed 1- or 2-element cycles, respectively.

Lemma 2. *Let $h\colon A \to A$ and $n \in \mathbb{N}$. Then*

(1) *An element $a \in A$ is cyclic in the algebra (A,h) if and only if it is cyclic in the algebra (A,h^n).*

(2) *h is injective if and only if h^n is injective.*

(3) *h is surjective if and only if h^n is surjective.*

(4) *$(A,h) \cong Z$ if and only if the algebra (A,h^n) consists of n components and each of them is isomorphic to Z.*

(5) *$(A,h) \cong N$ if and only if the algebra (A,h^n) consists of n components and each of them is isomorphic to N.*

Proof. Let $h^m(a) = a$. That means that the element a is cyclic in algebras (A,h) and (A,h^n).

Statements (2), (3) are obvious.

If $(A,h) \cong Z$, then it is easy to see that the algebra (A,h^n) consists of n components and each of them is isomorphic to Z.

Let (A,h^n) consist of n components and each of them is isomorphic to Z. In view of (2), (3) the operation h is bijective on A. Let B be a component of (A,h). Then B is a cycle or $B \cong Z$ since h is bijective. Further, B does not contain a cycle according to (1). Therefore $B \cong Z$. If (A,h) is not connected then (A,h^n) consists of more than n components.

If $(A,h) \cong N$, then the algebra (A,h^n) obviously consists of n components isomorphic to N.

Let (A,h^n) consist of n components and each of them is isomorphic to N. In view of (2) the operation h is injective on A and it is not surjective on A according to (3). That means that there exists a component B of the algebra (A,h) such that $(B,h) \cong N$. The algebra (B,h^n) consists of n components and thus $B = A$. $\square$

If $a \in A \setminus h(A)$ then the element a is called a *source* of (A,h). If $(A,h) \cong N$, then there exists exactly one source of (A,h).

Let T be a set of connected monounary algebras. The algebra (A,h) will be called the *T-algebra*, if the following two conditions are satisfied:

(1) every component of (A,h) is isomorphic to some algebra from the set T;
(2) if $B \in T$, then there exists a component B' of (A,h) such that $B' \cong B$.

Lemma 3. *Let $A \subseteq \mathbb{R}$, $h\colon A \to A$ be increasing and $a \in A$.*

(1) *If $a < h(a)$, then $h^k(a) < h^{k+1}(a)$ for every $k \in \mathbb{N}$.*
(2) *If $h(a) < a$, then $h^{k+1}(a) < h^k(a)$ for every $k \in \mathbb{N}$.*

Proof. Let $h(a) < a$. Then $h^2(a) = h(h(a)) < h(a)$ since h is increasing. Suppose that $h^{k+1}(a) < h^k(a)$ for $k \in \mathbb{N}$. We use that h is increasing to obtain

$$h^{k+2}(a) = h(h^{k+1}(a)) < h(h^k(a)) = h^{k+1}(a).$$

□

Lemma 4. *If $A \subseteq \mathbb{R}$ and $h\colon A \to A$ is decreasing, then the function h^2 is increasing.*

Proof. It follows from definitions. □

Lemma 5. *Let h be decreasing and $a \in A$ be such that $h(a) = a$. If $b \in A \setminus \{a\}$, then $h(b) \neq b$.*

Proof. If $a < b$, then the assumption that h is decreasing yields that

$$h(b) < h(a) = a < b.$$

If $b < a$, then $b < h(b)$. □

Theorem 1. *Let $A \subseteq \mathbb{R}$ and $h\colon A \to A$.*

(1) *If h is an increasing function, then there exists $T \subseteq \{C_1, N, Z\}$ such that (A,h) is a T-algebra.*
(2) *If h is an decreasing function, then there exists $T \subseteq \{C_1, C_2, N, Z\}$ such that (A,h) is a T-algebra and (A,h) contains at most one 1-element cycle.*

Proof. Suppose h is increasing. Let $a \in A$ and $h(a) \neq a$. The sequence $\{h^n(a)\}_{n\in\mathbb{N}}$ consists of infinitely many elements according to Lemma 3. It means the component of (A,h) which contains the element a is isomorphic to N or Z since h is injective.

Now let h be decreasing. Then the function h^2 is increasing by Lemma 4 and components of (A,h^2) are algebras isomorphic to C_1, N or Z. That means components of (A,h) are algebras isomorphic to C_1, C_2, N or Z according to Lemmas 2 and 1. In view of Lemma 5 the function h has at most one fixed point. □

We denote

$$\mathrm{Fix}_h = \{a \in A\colon h(a) = a\}.$$

Let us remark that if $\mathrm{Fix}_h \neq \varnothing$, then the algebra $(\mathrm{Fix}_h, h|\mathrm{Fix}_h)$ is the greatest $\{C_1\}$-subalgebra of (A,h).

Let $a \in \mathbb{R}$. The interval $\langle a,a\rangle = \{a\}$ will be called *trivial*.

In the next two lemmas we suppose that $I \subseteq \mathbb{R}$ is an interval and $h\colon I \to I$.

Lemma 6. *Let h be continuous and $a,b \in \mathrm{Fix}_h, a < b$. If $c \in (a,b) \setminus \mathrm{Fix}_h$, then there exist $a',b' \in \mathrm{Fix}_h$ such that*

$$a \le a' < c < b' \le b \text{ and } (a',b') \cap \mathrm{Fix}_h = \varnothing.$$

Proof. Let $(a,b) \cap \mathrm{Fix}_h \neq \varnothing$. Consider $A = \langle a,c) \cap \mathrm{Fix}_h$ and $B = (c,b\rangle \cap \mathrm{Fix}_h$. The set A has a supremum $a' \in \langle a,c\rangle$. To see that $a' \in A$ suppose that $a' \notin A$. Then there is a sequence $\{a_i\}_{i=1}^{\infty}$ such that $a_i \in A$ and $\lim_{i\to\infty} a_i = a'$. Since h is continuous we obtain $a_i = h(a_i) \to h(a')$. Therefore $a' = h(a')$. Conclude $a' \in A$ according to $c \notin \mathrm{Fix}_h$, a contradiction.

Analogously the set B has a minimum $b', b' > c$. □

Lemma 7. *Let h be increasing, $m \in \mathbb{N}$ and $a \in I$.*

If $a < h(a)$, then $a < h^m(a)$ and $\langle a, h^m(a)\rangle \cap \mathrm{Fix}_h = \varnothing$.
If $a > h(a)$, then $a > h^m(a)$ and $\langle h^m(a), a\rangle \cap \mathrm{Fix}_h = \varnothing$.

Proof. Assume that $h(a) > a$. Then $h^m(a) > a$ according to Lemma 3. Consider $d \in (a, h(a))$. Then $h(a) < h(d)$ since h is increasing. Therefore $d < h(d)$, so $d \notin \mathrm{Fix}_h$. We obtained $\langle a, h(a)\rangle \cap \mathrm{Fix}_h = \varnothing$. That means the statement $\langle a, h^m(a)\rangle \cap \mathrm{Fix}_h = \varnothing$ is valid because the function h^m is increasing and $\mathrm{Fix}_h = \mathrm{Fix}_{h^m}$.

The second implication can be proved analogously. □

3. Continuous increasing interval functions

In this section we give a classification of algebras (I,h), where $I \subseteq \mathbb{R}$ is an interval and h is a continuous increasing function in the common sense. The algebras under consideration are T-algebras for some $T \subseteq \{C_1, N, Z\}$ (see Theorem 1). We will show that $T = \{N, Z\}$ is not possible. Moreover, $C_1 \in T$ in the case that $I = \langle a,b\rangle$ for some $a,b \in \mathbb{R}$.

Let $I \subseteq \mathbb{R}$ be an interval. The closure of the interval I will be denoted by $\overline{I}$.

We suppose that h is a function from I into I and it is a continuous increasing function in this section.

Lemma 8. *Let function $\overline{h}$ be defined on $\overline{I}$ by*

$$\overline{h}(x) = \begin{cases} h(x) & \text{if } x \in I, \\ \lim_{a\to x+} h(a) & \text{if } x \notin I \text{ and } x \le a \text{ for } a \in I, \\ \lim_{a\to x-} h(a) & \text{if } x \notin I \text{ and } x \ge a \text{ for } a \in I. \end{cases}$$

Then $\overline{h}\colon \overline{I} \to \overline{I}$ and it is continuous increasing. If $J \subset I$ is an interval such that $h(J) \subseteq J$, then $\overline{h|J} = \overline{h}|\overline{J}$.

Proof. Suppose $x \in \overline{I} \setminus I$. Then $x \le a$ for each $a \in I$ or $x \ge a$ for each $a \in I$. If $x \le a$ for $a \in I$, then $x \le h(a)$ since $h(I) \subseteq I$. Therefore $\lim_{a\to x+} h(a)$ exists because h is continuous. Analogously if $x \ge a$ for each $a \in I$, then $\lim_{a\to x-} h(a)$ exists. So, the function $\overline{h}$ is correctly defined. Obviously, it is continuous and increasing. The equality $\overline{h|J} = \overline{h}|\overline{J}$ is valid according to the definition. □

3.1. *On the set of fixed points*

Lemma 9. *Let $x \in I$ and $x < h(x)$. Then*

(1) *the sequence $\{h^n(x)\}_{n\in\mathbb{N}}$ is increasing and*
  (a) $\langle x, lim_{n\to\infty} h^n(x)) \cap \mathrm{Fix}_h = \varnothing$,
  (b) $lim_{n\to\infty} h^n(x) \in \mathrm{Fix}_{\overline{h}} \cup \{\infty\}$;

(2) *if $M = \{n \in \mathbb{N} \colon h^{-n}(x)$ is defined$\} \neq \varnothing$, then the sequence $\{h^{-n}(x)\}_{n\in M}$ is decreasing and if $M = \mathbb{N}$, then*
  (a) $(lim_{n\to\infty} h^{-n}(x), x\rangle \cap \mathrm{Fix}_h = \varnothing$,
  (b) $lim_{n\to\infty} h^{-n}(x) \in \mathrm{Fix}_{\overline{h}} \cup \{-\infty\}$.

Proof. The sequence $\{h^n(x)\}_{n\in\mathbb{N}}$ is increasing according to Lemma 3. Therefore it is convergent and its limit is ∞ or it is from $\overline{I}$ since $h^n(x) \in I$ for each $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} h^n(x) = a, a \in \overline{I}$. In view of $\overline{h}$ is continuous we obtain

$$\overline{h}(a) = \overline{h}(\lim_{n\to\infty} h^n(x)) = \lim_{n\to\infty} \overline{h}(h^n(x)) = \lim_{n\to\infty} h(h^n(x)) = a.$$

Put $y_n = h^{-n}(x)$ for each $n \in M$. Let $m \in \mathbb{N}$ be such that $y_{m+1} \geq y_m$. Then $y_{m+1} > y_m$. The assumption that h is increasing implies

$$x = h^{m+1}(y_{m+1}) > h^{m+1}(y_m) = h(x),$$

a contradiction. Thus $y_{n+1} < y_n$ for each $n \in M$. If $M = \mathbb{N}$, then $\{y_n\}_{n\in\mathbb{N}}$ is convergent and its limit is $-\infty$ or it is from $\overline{I}$ since $y_n \in I$ for each $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} y_n = b, b \in \overline{I}$. In view of $\overline{h}$ is continuous we obtain

$$\overline{h}(b) = \overline{h}(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} \overline{h}(y_n) = \lim_{n\to\infty} h(y_n) = \lim_{n\to\infty} y_{n-1} = b.$$

Statements (a) for $h^n(x)$ and $h^{-n}(x)$ follow from Lemma 7. □

Analogously we can prove

Lemma 10. *Let $x \in I$ and $x > h(x)$. Then*

(1) *the sequence $\{h^n(x)\}_{n\in\mathbb{N}}$ is decreasing and*
  (a) $(lim_{n\to\infty} h^n(x), x\rangle \cap \mathrm{Fix}_h = \varnothing$,
  (b) $lim_{n\to\infty} h^n(x) \in \mathrm{Fix}_{\overline{h}} \cup \{-\infty\}$;

(2) *if $M = \{n \in \mathbb{N} \colon h^{-n}(x)$ is defined$\} \neq \varnothing$, then the sequence $\{h^{-n}(x)\}_{n\in M}$ is increasing and if $M = \mathbb{N}$, then*
  (a) $\langle x, lim_{n\to\infty} h^{-n}(x)) \cap \mathrm{Fix}_h = \varnothing$,
  (b) $lim_{n\to\infty} h^{-n}(x) \in \mathrm{Fix}_{\overline{h}} \cup \{\infty\}$.

Lemma 11. *If $\overline{I} = \langle a, b\rangle$, $a, b \in \mathbb{R}$, then $\mathrm{Fix}_{\overline{h}} \neq \varnothing$.*

Proof. Consider the function $g \colon \overline{I} \to \overline{I}$ defined by $g(x) = \overline{h}(x) - x$. Then g is continuous and $g(a) \geq 0, g(b) \leq 0$. Thus Intermediate Value Theorem gives that there exists $c \in \langle a, b\rangle$ such that $g(c) = 0$. We have $c \in \mathrm{Fix}_{\overline{h}}$. □

Lemma 12. *Let $a \in I, (a-\varepsilon, a+\varepsilon) \subseteq I$ and $\varepsilon \in \mathbb{R}, \varepsilon > 0$.*

(1) *If* $(a-\varepsilon,a)\cap \mathrm{Fix}_h=\varnothing$ *and* $(a,a+\varepsilon)\subseteq \mathrm{Fix}_h$, *then* $a\in \mathrm{Fix}_h$.
(2) *If* $(a,a+\varepsilon)\cap \mathrm{Fix}_h=\varnothing$ *and* $(a-\varepsilon,a)\subseteq \mathrm{Fix}_h$, *then* $a\in \mathrm{Fix}_h$.

Proof. Let assumptions of (1) be satisfied. Then continuity of h implies

$$h(a)=\lim_{x\to a+} h(x)=\lim_{x\to a+} x=a.$$

□

Corollary 1. *Let* $\mathrm{Fix}_{\overline{h}}\neq\varnothing$. *Then the following conditions are equivalent:*
(1) $\mathrm{Fix}_{\overline{h}}$ *is an interval,*
(2) $\mathrm{Fix}_{\overline{h}}\in\{(-\infty,a\rangle,\langle a,b\rangle,\langle b,\infty),\mathbb{R}\}$ *for some* $a,b\in\mathbb{R}, a\leq b$.
(3) $\langle c,d\rangle\cap\mathrm{Fix}_{\overline{h}}\neq\{c,d\}$ *for each* $c,d\in\overline{I}$ *such that* $c<d$.

Proof. The equivalence of (1) and (2) follows from the definition of $\overline{h}$ and Lemma 12. It is trivial that (2) implies (3).

Assume $\mathrm{Fix}_{\overline{h}}$ is not an interval. Then there exist $a,b\in\mathrm{Fix}_{\overline{h}}, a<b$ such that $\langle a,b\rangle\not\subseteq \mathrm{Fix}_{\overline{h}}$. Let $c\in\langle a,b\rangle\setminus\mathrm{Fix}_{\overline{h}}$. Consider elements $a',b'\in\overline{I}$ such that a' is a supremum of $\{u\in\mathrm{Fix}_{\overline{h}}\colon a\leq u<c\}$ and b' is an infimum of $\{v\in\mathrm{Fix}_{\overline{h}}\colon c<v\leq b\}$. If $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of elements of $\mathrm{Fix}_{\overline{h}}$ such that $a\leq a_n<c$ and $a_n\to a'$, then

$$\overline{h}(a')=h(\lim_{n\to\infty} a_n)=\lim_{n\to\infty}\overline{h}(a_n)=\lim_{n\to\infty} a_n=a'.$$

Therefore $a'\in\mathrm{Fix}_{\overline{h}}$. Analogously $b'\in\mathrm{Fix}_{\overline{h}}$. We obtained

$$\langle a',b'\rangle\cap\mathrm{Fix}_{\overline{h}}=\{a',b'\}.$$

□

3.2. $\{N\}$-*algebras*

Lemma 13. *Let* h *be not surjective and* $\mathrm{Fix}_{\overline{h}}=\varnothing$. *Then* (I,h) *is an* $\{N\}$-*algebra.*

Proof. In view of the Lemma 11 we have $I=\mathbb{R},\overline{I}=\langle a,\infty)$ or $\overline{I}=(-\infty,a\rangle$ for some $a\in\mathbb{R}$. Further, $\lim_{n\to\infty}h^n(x)\in\{-\infty,\infty\}$ for each $x\in I$ in view of Lemmas 9 and 10.

Let $x\in I$ and $\lim_{n\to\infty}h^n(x)=\infty$. Then the sequence $\{h^n(x)\}_{n=1}^{\infty}$ is increasing. If $h^{-k}(x)$ is defined for every $k\in\mathbb{N}$ then the sequence $\{h^{-k}(x)\}_{k=1}^{\infty}$ is decreasing. In view of Lemma 9 $\lim_{n\to\infty}h^n(x)=-\infty$. Therefore $I=\mathbb{R}$ and h is surjective, a contradiction. We conclude that $h^{-k}(x)$ is not defined for some $k\in\mathbb{N}$ and the component of x is isomorphic to N. □

Lemma 14. *Let* $\overline{I}=\langle a,b\rangle$, $a,b\in\mathbb{R}$. *If* $\mathrm{Fix}_h=\varnothing$ *and* $\|\mathrm{Fix}_{\overline{h}}\|=1$, *then* (I,h) *is an* $\{N\}$-*algebra.*

Proof. We have $\mathrm{Fix}_{\overline{h}}\subset\{a,b\}$ by Lemma 11. Let $\mathrm{Fix}_{\overline{h}}=\{a\}$. Take $x\in I$. Then $h(x)<x$ since $\mathrm{Fix}_h=\varnothing$. The sequence $\{h^n(x)\}_{n=1}^{\infty}$ is decreasing and its limit is equal to a according to Lemma 10. If $h^{-k}(x)$ is defined for every $k\in\mathbb{N}$, then in view of Lemma 10 the sequence $\{h^{-k}(x)\}_{k=1}^{\infty}$ is increasing and it has a limit from the set

$\{a,\infty\}$, what is not possible. Therefore $h^{-k}(x)$ is not defined for some $k \in \mathbb{N}$ and the component of x in (I,h) is isomorphic to N.

Analogously if $\mathrm{Fix}_{\overline{h}} = \{b\}$. □

Lemma 15. *Let $I = (a,\infty)$ or $I = (-\infty,a)$, $a \in \mathbb{R}$ and h be not surjective. If $\mathrm{Fix}_h = \varnothing$ and $\|\mathrm{Fix}_{\overline{h}}\| = 1$, then (I,h) is an $\{N\}$-algebra.*

Proof. We have $\mathrm{Fix}_{\overline{h}} = \{a\}$. Let $I = (a,\infty)$. Then h is bounded from above since h is not surjective. Take $x \in I$. Then $h(x) < x$ and the sequence $\{h^n(x)\}_{n=0}^{\infty}$ is decreasing with the limit equal to a according to Lemma 10. If $h^{-k}(x)$ is defined for every $k \in \mathbb{N}$, then the sequence $\{h^{-k}(x)\}_{k=1}^{\infty}$ is increasing. In view of Lemma 10 it has a limit equal to ∞, what is not possible since h is bounded. Therefore $h^{-k}(x)$ is not defined for some $k \in \mathbb{N}$ and the component of x in (I,h) is isomorphic to N.

For $I = (-\infty,a)$ the function h is bounded from below and the sequence $\{h^n(x)\}_{n=0}^{\infty}$ is increasing. □

3.3. *Main result*

Lemma 16. *Let $a \in \mathrm{Fix}_h$.*

(1) *If $\mathrm{Fix}_h \cap (-\infty,a) = \varnothing$, then $I \cap (-\infty,a)$ is a subalgebra of (I,h).*
(2) *If $\mathrm{Fix}_h \cap (a,\infty) = \varnothing$, then $I \cap (a,\infty)$ is a subalgebra of (I,h).*

Proof. Let $x \in I$ and $x < a$. If $h(x) < x$, then $h(x) < a$. If $h(x) > x$, then $h(x) < a$ in view of Lemma 9(1a).

Analogously we can see the second part of the assertion. □

Lemma 17. *Let h be not surjective, sets $\mathrm{Fix}_h, \mathrm{Fix}_{\overline{h}}$ be intervals and h is bounded on the set $I \setminus \mathrm{Fix}_h$. Then (I,h) is $\{N,C_1\}$-algebra.*

Proof. We have $\mathrm{Fix}_h \neq \varnothing$. The set $\overline{I} \setminus \mathrm{Fix}_{\overline{h}}$ is an interval or there exist intervals I_1, I_2 such that $\overline{I} \setminus \mathrm{Fix}_{\overline{h}} = I_1 \cup I_2$.

Let $\overline{I} \setminus \mathrm{Fix}_{\overline{h}}$ be an interval. Denote $A = I \setminus \mathrm{Fix}_h$. Then $\overline{h}|\overline{A} = \overline{h|A}$ according to Lemma 8. We have that A is a subalgebra of (I,h) according to Lemma 16. Further,

$$\mathrm{Fix}_{h|A} = \varnothing, \|\mathrm{Fix}_{\overline{h|A}}\| = 1$$

and h is not surjective on A. Therefore the algebra $(A,h|A)$ satisfies assumptions of Lemmas 14 or 15 and so it is an $\{N\}$-algebra.

Let $\overline{I} \setminus \mathrm{Fix}_{\overline{h}}$ be not an interval. Then $\mathrm{Fix}_{\overline{h}} = \mathrm{Fix}_h = \langle a,b \rangle$ for some $a,b \in \mathbb{R}, a \leq b$. Denote $J_1 = I_1 \cap I, J_2 = I_2 \cap I$. Intervals J_1, J_2 are non-empty and they create subalgebras of (I,h) according to Lemma 16. Further,

$$J_1 \cup \mathrm{Fix}_h \cup J_2 = I, \mathrm{Fix}_{h|J_1} = \mathrm{Fix}_{h|J_2} = \varnothing \text{ and } \|\mathrm{Fix}_{\overline{h|J_1}}\| = \|\mathrm{Fix}_{\overline{h|J_2}}\| = 1.$$

Moreover, $h(J_1) \neq J_1, h(J_2) \neq J_2$ since h is bounded. Thus algebras $(J_1,h|J_1)$, $(J_2,h|J_2)$ satisfy assumptions of Lemmas 14 or 15 and therefore they are $\{N\}$-algebras. □

Lemma 18. *Let $a,b \in \bar{I}$ be such that $a < b$. If $\langle a,b\rangle \cap \mathrm{Fix}_{\bar{h}} = \{a,b\}$, then (a,b) forms a subalgebra of (I,h) with all components isomorphic to Z.*

Proof. Let x be such that $a < x < b$. Then $x \in I$ and

$$a = \bar{h}(a) < h(x) = \bar{h}(x) < \bar{h}(b) = b$$

according to $\bar{h}$ is increasing. Therefore (a,b) forms a subalgebra of (I,h). The function h is bijective on (a,b) since it is continuous. Thus every component of $((a,b),h|(a,b))$ is isomorphic to Z or to C_1 according to h is increasing. Further,

$$(a,b) \cap \mathrm{Fix}_h = (a,b) \cap \mathrm{Fix}_{\bar{h}} = \varnothing$$

according to the assumption. □

Theorem 2. *Let I be an interval and $h\colon I \to I$ be continuous increasing function.*

(1) *Let h be surjective. Then the algebra (I,h) is one of the following 3 types:*
   (a) *$\{C_1\}$-algebra, if $\mathrm{Fix}_h = I$.*
   (b) *$\{Z\}$-algebra, if $\mathrm{Fix}_h = \varnothing$.*
   (c) *$\{C_1, Z\}$-algebra, if $\mathrm{Fix}_h \notin \{I, \varnothing\}$.*

(2) *Let h be not surjective. Then the algebra (I,h) is one of the following 3 types:*
   (a) *$\{N\}$-algebra, if $\mathrm{Fix}_h = \varnothing$.*
   (b) *$\{C_1, N\}$-algebra, if $\mathrm{Fix}_{\bar{h}}$ is an interval and h is bounded (above and below) on $I \setminus \mathrm{Fix}_h$.*
   (c) *$\{C_1, N, Z\}$-algebra, otherwise.*

Proof. The statement (1) is obvious.

Let h be not surjective. Suppose that $\mathrm{Fix}_h = \varnothing$. If $\mathrm{Fix}_{\bar{h}} = \varnothing$, then (I,h) is $\{N\}$-algebra according to Lemma 13. Suppose that $\mathrm{Fix}_{\bar{h}} \neq \varnothing$. We have $\bar{I} = \langle a,b\rangle$ or $I \in \{(a,\infty),(-\infty,a)\}$ for some $a,b \in \mathbb{R}, a < b$. If $I \in \{(a,\infty),(-\infty,a)\}$, then $\mathrm{Fix}_{\bar{h}} = \{a\}$ and (I,h) is $\{N\}$-algebra according to Lemma 15. If $\bar{I} = \langle a,b\rangle$, then $\mathrm{Fix}_{\bar{h}} = \{a\}$ or $\mathrm{Fix}_{\bar{h}} = \{b\}$ since h is continuous and not surjective. Therefore (I,h) is $\{N\}$-algebra according to Lemma 14.

The assertion (b) is proved in Lemma 17.

Let suppositions of (a),(b) be not valid. Then $\mathrm{Fix}_h \neq \varnothing$. Therefore the algebra (I,h) contains an 1-element cycle. Further, $h(I) \neq I$ since h is not surjective. Thus every $u \in I$ such that $u \notin h(I)$ is a source of (I,h). That means (I,h) contains components isomorphic to N according to Theorem 1.

Non-surjectivity of h means that h is bounded from above or h is bounded from below. Assume that $\mathrm{Fix}_{\bar{h}}$ is not an interval. Then there are $a,b \in \mathrm{Fix}_{\bar{h}}$ such that there exists $c \in (a,b) \setminus \mathrm{Fix}_{\bar{h}}$. In view of Lemma 6 there are $a',b' \in \mathrm{Fix}_{\bar{h}}$ such that

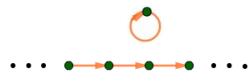
$$a \le a' < c < b' \le b \text{ and } (a',b') \cap \mathrm{Fix}_{\bar{h}} = \varnothing.$$

Thus by Lemma 18 the interval (a',b') determines a subalgebra of (I,h) and every component of $((a',b'),h|(a',b'))$ is isomorphic to Z.

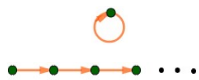
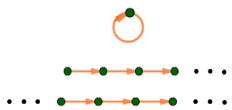
Now let $\mathrm{Fix}_{\overline{h}}$ be an interval. Then $\mathrm{Fix}_h \neq \varnothing$ by (a). That means Fix_h is an interval. Therefore h is not bounded from above on $I \setminus \mathrm{Fix}_h$ or h is not bounded from below on $I \setminus \mathrm{Fix}_h$ according to (b). Suppose that h is not bounded from above on $I \setminus \mathrm{Fix}_h$. Then $\overline{I} \in \{\mathbb{R}, \langle a', \infty)\}$ for some $a' \in \mathbb{R}$. Further, $\mathrm{Fix}_{\overline{h}} \in \{(-\infty, b\rangle, \langle a, b\rangle\}$ for some $a, b \in \overline{I}$ according to Corollary 1. Thus $(b, \infty) \cap \mathrm{Fix}_h = \varnothing$. We obtained the interval (b, ∞) is a subalgebra of (I, h) and every component of the algebra $((b, \infty), h|(b, \infty))$ is isomorphic to Z according to Lemma 18. Analogously we proceed if h is not bounded from below on $I \setminus \mathrm{Fix}_h$. □

The following tables illustrate Theorem 2. They show all types of components that occur in the corresponding algebras.

(1) h surjective

$\mathrm{Fix}_h = \varnothing$	$\mathrm{Fix}_h = I$	otherwise
		

(2) h not surjective

$\mathrm{Fix}_h = \varnothing$	h bounded on $I \setminus \mathrm{Fix}_h$, $\mathrm{Fix}_{\overline{h}}$ interval	otherwise
		

Corollary 2. *Let I be an interval and $h\colon I \to I$ be continuous increasing function. Then the following assertions are equivalent:*

(i) *(I, h) is an $\{C_1, N, Z\}$-algebra.*

(ii) *$h(I) \neq I$, $\mathrm{Fix}_h \neq \varnothing$ and at least one of the following conditions is satisfied*

  (a) *$\mathrm{Fix}_{\overline{h}}$ is not an interval,*

  (b) *h is not bounded on $I \setminus \mathrm{Fix}_h$.*

Corollary 3. *Let I be an interval and $h\colon I \to I$ be continuous increasing function. Then the algebra (I, h) is not an $\{N, Z\}$-algebra.*

If $I = \langle a, b\rangle$, $a, b \in \mathbb{R}$, then (A, h) is an T-algebra such that

$$T \in \{\{C_1\}, \{C_1, Z\}, \{C_1, N\}, \{C_1, N, Z\}\}.$$

Proof. It follows from Theorem 2 and Lemma 11. □

Example 1. Let $h(x) = 2x$ for $x \in \mathbb{R}$. The algebra $(\mathbb{R}, h)$ is $\{C_1, Z\}$-algebra. It contains exactly one 1-element cycle.

Example 2. Let $h(x) = e^x$ for $x \in \mathbb{R}$. The algebra $(\mathbb{R}, h)$ is $\{N\}$-algebra.

Example 3. Let $h(x) = e^x - 1$ for $x \in \mathbb{R}$. The algebra $(\mathbb{R}, h)$ is $\{C_1, N, Z\}$-algebra with exactly one 1-element cycle. The interval $(-\infty, 0)$ forms the maximal $\{N\}$-subalgebra of $(\mathbb{R}, h)$. If $a \leq -1$, then intervals $\langle a, 0), (a, 0)$ form $\{N\}$-subalgebras of $(\mathbb{R}, h)$. The interval $(0, \infty)$ forms the maximal $\{Z\}$-subalgebra of $(\mathbb{R}, h)$ and there is no interval $I' \subset (0, \infty)$ such that I' forms a subalgebra of $(\mathbb{R}, h)$.

Example 4. Let

$$h(x) = \begin{cases} e^x - 1 & \text{if } x \in \langle -1, 0\rangle, \\ x & \text{if } x \in (0, 1\rangle. \end{cases}$$

The algebra $(\langle -1, 1\rangle, h)$ is $\{C_1, N\}$-algebra. The interval $\langle -1, 0)$ forms the maximal $\{N\}$-subalgebra of $(\langle -1, 1\rangle, h)$. The interval $\langle 0, 1\rangle$ the a maximal $\{C_1\}$-subalgebra of $(\langle -1, 1\rangle, h)$.

4. Continuous decreasing interval functions

Let $I \subseteq \mathbb{R}$ be an interval. In this section we will suppose that h is a function from I into I and it is continuous and decreasing.

The algebra (I, h) is T-algebra with at most one 1-element cycle for some $T \subseteq \{C_1, C_2, N, Z\}$ by Theorem 1. We will show in this section that there are six possibilities for T only and that T is determined by the set Fix_{h^2}.

Lemma 19. *There exists $o \in I$ such that* $\mathrm{Fix}_h = \{o\}$.

Proof. In view of Lemma 5 we have $\|\mathrm{Fix}_h\| \leq 1$. Let $a \notin \mathrm{Fix}_h$. Without loss of generality let $h(a) < a$. Then $h^2(a) > h(a)$. Put $g(x) = h(x) - x$. We have $g(a) < 0 < g(h(a))$. The function g is continuous and thus there is

$$o \in (h(a), a) \text{ such that } g(o) = 0$$

according to Intermediate Value Theorem. We obtained $\mathrm{Fix}_h = \{o\}$. □

Corollary 4. *The algebra (I, h) contains exactly one component isomorphic to C_1.*

Lemma 20. *Let $a \in I \setminus \mathrm{Fix}_h$. Then the following properties are equivalent:*

(1) *$\{a, h(a)\}$ creates 2-element cycle of (I, h).*
(2) *there exists $c \in \mathbb{R}$ such that the line $y = -x + c$ contains points $[a, h(a)]$, $[h(a), h^2(a)]$.*
(3) *the point $[a, h(a)]$ is symmetric with the point $[h(a), h^2(a)]$ according to the line $x = y$.*

Proof. Let (1) be satisfied. Then $h^2(a) = a$. The property (3) is valid since points $[a, a], [h(a), a], [h(a), h(a)], [a, h(a)]$ are vertices of a square.

Let (3) be fulfilled. Therefore $h^2(a) = a$. The statement (2) follows from the fact that the line determined by points $[a, h(a)], [h(a), h^2(a)]$ is perpendicular to $x = y$.

If (2) is valid, then

$$h(a) = -a + c \text{ and } h^2(a) = -h(a) + c = -(-a + c) + c = a.$$

□

Lemma 21. *Let h be not surjective and $a \in \mathbb{R}$.*

(1) *If $\overline{h(I)} = (-\infty, a\rangle$, then*
  (a) $h(I) = (-\infty, a)$,
  (b) *$I = \mathbb{R}$ or there exists $b \in \mathbb{R}$, $b > a$ such that $I = (-\infty, b)$,*
  (c) $h^2(I) = (h(a), a)$.

(2) *If $\overline{h(I)} = \langle a, \infty)$, then*
  (a) $h(I) = (a, \infty)$,
  (b) *$I = \mathbb{R}$ or there exists $b \in \mathbb{R}$,$b < a$ such that $I = (b, \infty)$,*
  (c) $h^2(I) = (a, h(a))$.

Proof. Let $\overline{h(I)} = (-\infty, a\rangle$. We have $(-\infty, a) \subset I$ and $a \in I$ since h is not surjective. The equality $h(I) = (-\infty, a\rangle$ is not possible since if $c < a$, then $c \in I$ and $h(c) > h(a)$. Thus $h(I) = (-\infty, a)$. Assume that $I = (-\infty, b\rangle$ for some $b > a, b \in \mathbb{R}$. Then the number $h(b)$ is the minimum of the the function h on I since h is decreasing. Therefore the interval $h(I)$ is closed on the left side, a contradiction. We obtained $I = \mathbb{R}$ or $I = (-\infty, b)$. Further, $\lim_{x \to -\infty} h(x) = a$. Therefore

$$h^2(I) = h((-\infty, a)) = (h(a), \lim_{x \to -\infty} h(x)) = (h(a), a).$$

Similarly we argue if $h(I) = (a, \infty)$. □

Lemma 22. *Let h be not surjective, $a, b \in \mathbb{R}$ are such that $a < b$ and $\overline{h(I)} = \langle a, b\rangle$. Then $a \in I$ or $b \in I$ and*

(1) *if $a, b \in I$, then $h^2(I) \subseteq \langle h(b), h(a)\rangle$.*
(2) *if $b \notin I$, then $h^2(I) \subseteq (a, h(a)\rangle$.*
(3) *if $a \notin I$, then $h^2(I) \subseteq \langle h(b), b)$.*

Proof. There are $a \in I$ or $b \in I$ since h is not surjective. Assume that $a, b \in I$. Then

$$h^2(I) \subseteq h(\overline{h(I)}) = h(\langle a, b\rangle) = \langle h(b), h(a)\rangle.$$

Let $b \notin I$. Then $b \in \bar{I}$ since h is continuous and $h(I) \subset I$. Consider a sequence $\{b_n\}_{n \in \mathbb{N}}$ such that $b_n \in I$ and $\lim_{n \to \infty} b_n = b$. We obtain $b_n < b$ and $\lim_{n \to \infty} h(b_n) = a$ since h is decreasing. Therefore

$$h^2(I) \subseteq h(\overline{h(I)} \setminus \{b\}) = h(\langle a, b)) = (\lim_{n \to \infty} h(b_n), h(a)\rangle = (a, h(a)\rangle.$$

If $a \notin I$, then $h^2(I) \subseteq \langle h(b), b)$ by a similar way. □

Lemma 23. *Let h be not surjective. Then h^2 is bounded (above and below).*

Proof. The interval I is not trivial since h is not surjective. Suppose that $a, b \in \mathbb{R}$, $a < b$.

If $\overline{h(I)} = \langle a,b\rangle$, then h^2 is bounded according to Lemma 22. If $\overline{h(I)} = (-\infty,a\rangle$ or $\overline{h(I)} = \langle a,\infty)$, then h^2 is bounded in view of Lemma 21. □

The function h^2 is increasing according to Lemma 4. Thus the function $\overline{h^2}$ is defined by Lemma 8.

Lemma 24. *Let h be not surjective. Then* $\mathrm{Fix}_{h^2} = \mathrm{Fix}_{\overline{h^2}}$.

Proof. Suppose that $a,b \in \mathbb{R}$, $a < b$. To see that $\mathrm{Fix}_{\overline{h^2}} \subset I$ let $\overline{h(I)} = (-\infty,a\rangle$. Then $h(I) = (-\infty,a)$. We have $a \in I$ and $h^2(I) = (h(a),a)$ according to Lemma 21. Therefore $\mathrm{Fix}_{\overline{h^2}} \subseteq \langle h(a),a\rangle \subset I$. Analogously we can proceed if $\overline{h(I)} = \langle a,\infty)$. If $\overline{h(I)} = \langle a,b\rangle$, then use Lemma 22.

Let $x \in \mathrm{Fix}_{\overline{h^2}}$. Then $x \in I$ and $h^2(x) = \overline{h^2}(x) = x$. That means $x \in \mathrm{Fix}_{h^2}$. This yields that $\mathrm{Fix}_{h^2} = \mathrm{Fix}_{\overline{h^2}}$.

□

Theorem 3. *Let I be an interval and $h\colon I \to I$ be continuous decreasing function.*

(1) *Let h be surjective. Then the algebra (I,h) is one of the following 3 types:*
   (a) *$\{C_1,C_2\}$-algebra, if* $\mathrm{Fix}_{h^2} = I$,
   (b) *$\{C_1,C_2,Z\}$-algebra, if* $\mathrm{Fix}_{h^2} \neq I$ *and* $\|\mathrm{Fix}_{h^2}\| > 1$,
   (c) *$\{C_1,Z\}$-algebra, if* $\|\mathrm{Fix}_{h^2}\| = 1$.

(2) *Let h be not surjective. Then the algebra (I,h) is one of the following 3 types:*
   (a) *$\{C_1,N\}$-algebra, if* $\|\mathrm{Fix}_{h^2}\| = 1$,
   (b) *$\{C_1,C_2,N\}$-algebra, if* Fix_{h^2} *is a non-trivial interval,*
   (c) *$\{C_1,C_2,N,Z\}$-algebra, if* Fix_{h^2} *is not an interval.*

Proof. Suppose h is surjective. Then there is $T \subseteq \{C_1,C_2,Z\}$ such that (I,h) is an T-algebra according to Theorem 1. We have $C_1 \in T$ according to Lemma 19. If $\mathrm{Fix}_{h^2} \neq I$, then $Z \in T$ according to Lemma 1. If $\|\mathrm{Fix}_{h^2}\| > 1$, then $C_2 \in T$ according to Lemmas 1 and 19.

Suppose h is not surjective. The function h^2 is increasing according to Lemma 4 and it is not surjective according to Lemma 2(3). Moreover, it is bounded on both sides by Lemma 23. We will apply Theorem 2 to the algebra (I,h^2). The equality $\mathrm{Fix}_{h^2} = \mathrm{Fix}_{\overline{h^2}}$ is valid according to Lemma 24.

Let $\|\mathrm{Fix}_{h^2}\| = 1$. Then Fix_{h^2} is a trivial interval. The algebra (I,h^2) is an $\{C_1,N\}$-algebra according to Theorem 2(2b). Therefore the algebra (I,h) is an $\{C_1,N\}$-algebra according to Lemma 2(4).

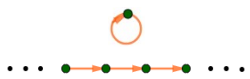
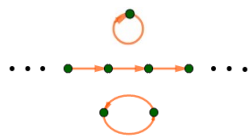
Let Fix_{h^2} be a non-trivial interval. Then the algebra (I,h^2) is an $\{C_1,N\}$-algebra according to Theorem 2(2b). That means the algebra (I,h) is an $\{C_1,C_2,N\}$-algebra according to Lemma 2, Lemma 1 and Lemma 19.

Now suppose Fix_{h^2} is not an interval. Then the set Fix_{h^2} contains at least 3 points according to Lemma 1 and Lemma 19. The algebra (I,h^2) is an $\{C_1,N,Z\}$-algebra

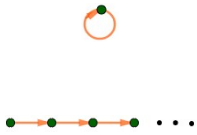
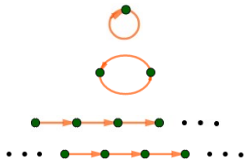
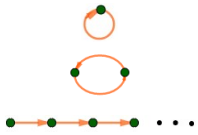
according to Theorem 2(2c). In view of Lemma 2, Lemma 1 and Lemma 19 we obtain that (I,h) is an $\{C_1,C_2,N,Z\}$-algebra. □

The following tables illustrate Theorem 3. They show all types of components that occur in the corresponding algebras.

(1) $h\colon I \to I$ surjective

$\|\mathrm{Fix}_{h^2}\| = 1$	$\mathrm{Fix}_{h^2} = I$	otherwise
		

(2) $h\colon I \to I$ not surjective

$\|\mathrm{Fix}_{h^2}\| = 1$	Fix_{h^2} not interval	otherwise
		

Corollary 5. *The algebra (I,h) is not an $\{C_1,N,Z\}$-algebra.*

Example 5. Let

$$h(x) = \begin{cases} x^2 & \text{if } x \in \langle -1,0\rangle, \\ -x^2 & \text{if } x \in (0,1\rangle. \end{cases}$$

Then

$$h^2(x) = \begin{cases} -x^4 & \text{if } x \in \langle -1,0\rangle, \\ x^4 & \text{if } x \in (0,1\rangle. \end{cases}$$

The algebra $(\langle -1,1\rangle,h)$ is $\{C_1,C_2,Z\}$-algebra. Algebras $(\langle -1,1\rangle,h^2)$, $((-1,1),h)$ are $\{C_1,Z\}$-algebras.

References

[1] L. Bartłomiejczyk, "Iterative roots with big graph," *Real Analysis Exchange*, vol. 26, no. 1, pp. 391–399, 2001.

[2] J. Chvalina, O. Kopeček, and M. Novotný, *Homomorphic transformations - why and possible ways to how*. Masaryk University Brno, 2012.

[3] J. Chvalina and Z. Svoboda, "Sandwich semigroups of solutions of certain functional equations and hyperstructures determined by sandwiches of functions," *Aplimat-J.of Applied Math.*, vol. II, no. 1, pp. 35–42, 2009.

[4] Z. Farkasová and D. Jakubíková-Studenovská, "On radical classes of monounary algebras," *Miskolc Math. Notes*, vol. 14, no. 3, pp. 871–886, 2013.

[5] E. Halušková, "Monounary algebras with easy direct limits," *Miskolc Math. Notes*, vol. 19, no. 1, pp. 291–302, 2018, doi: 10.18514/MMN.2018.1744.

[6] D. Jakubíková-Studenovská and L. Janičková, "Congruence lattices of connected monounary algebras," *Algebra Univers.*, vol. 81, no. 4/54, p. 12, 2020, doi: 10.1007/s00012-020-00686-2.

[7] D. Jakubíková-Studenovská and J. Pócs, *Monounary algebras*. P. J. Šafárik University in Košice, 2009.

[8] O. Kopeček, "Equation $f(p(x)) = q(f(x))$ for given real functions p,q," *Czech. Math. J.*, vol. 62, no. 137, pp. 1011–1032, 2012, doi: 10.1007/s10587-012-0061-2.

[9] S. Łojasiewicz, "Solution générale de l'équation fonctionelle $f(f(\ldots f(x))) = g(x)$.(french)," *Ann. Soc. Polon. Math.*, vol. 24, pp. 88–91, 1951.

[10] R. McKenzie, G. McNulty, and W. Taylor, *Algebras, lattices, varieties. Volume I*. Wadsworth, 1987.

[11] M. Novotný, "Homomorphisms of algebras," *Czechoslovak Math. J.*, vol. 52, pp. 345—-364, 2002, doi: 10.1023/A:1021782813143.

[12] A. N. Sharkovskii, "Co-existence of cycles of a continuous map of the line into itself," *Ukr. Mat. Zh.*, vol. 16, no. 1, pp. 61–71, 1964.

[13] A. N. Sharkovskii, "On cycles and structure of a continuous map," *Ukr. Mat. Zh.*, vol. 17, no. 3, pp. 104–111, 1965.

Author's address

Emília Halušková
Slovak Academy of Sciences, Mathematical Institute, Grešákova 6, 040 01 Košice, Slovakia
E-mail address: ehaluska@saske.sk

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 723–746

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4722

DYNAMICS OF POSITIVE SOLUTIONS OF A THREE DIMENSIONAL HIGHER ORDER SYSTEM OF DIFFERENCE EQUATIONS

MURAD KHAN HASSANI, YASIN YAZLIK, AND NOURESSADAT TOUAFEK

Received 04 May, 2023

Abstract. In this paper, we study the global behavior of positive solutions of the following system of difference equations

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = A + \frac{z_n}{z_{n-k}}, \quad z_{n+1} = A + \frac{x_n}{x_{n-k}}, \quad n \in \mathbb{N}_0,$$

where the parameter $A > 0$, the initial values $x_{-i}, y_{-i}, z_{-}i$, $i \in \{0, 1, 2, \ldots, k\}$, are arbitrary positive real number and $k \in \mathbb{N}$. Moreover, we provide semi-cycle analysis of positive solutions to the above system of difference equations. Finally, we also give some numerical examples which support our analytical results.

2010 *Mathematics Subject Classification:* 39A10; 39A23; 40A05

Keywords: system of difference equations, semi-cycle analysis, bounded solutions and periodic solutions, global asymptotic stability

1. Introduction and preliminaries

Let $\mathbb{N}$ be set of all natural numbers, $\mathbb{N}_0$ be set of non-negative integers, $\mathbb{Z}$ be set of all integers, $\mathbb{R}$ be set of all real numbers and for $k \in \mathbb{Z}$ the notation $\mathbb{N}_k$ represents the set of $\{n \in \mathbb{Z} : n \geq k\}$.

Discrete dynamical systems and difference equations have attracted attention of researchers in the last years, particularly these equations and systems, which arise in mathematical models describing the real world phenomenon, are used in natural sciences, engineering, operations research, social sciences and linguistics, etc. Recently, there has been a lot of study concerning the qualitative behaviour of nonlinear difference equation and system of difference equations [1–24]. Even though difference equations with higher-order and their systems have sometimes very simple in their form, they are actually difficult to understand exact the behaviour of their solutions. Hence, studying the qualitative behaviour of such difference equations and system of difference equations is worth further consideration. Later on, making a historical

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

flash back for the system of difference equation we consider in this paper, we should mention that in paper [1], Abu-Saris and Devault investigated the global asymptotic stability results for the equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \; n \in \mathbb{N}_0, \tag{1.1}$$

where $A \in (0,\infty)$, $k \in \mathbb{N}_2$ and x_{-i}, for $i \in \{0,1,\ldots,k\}$, are positive real numbers, while the next difference equation

$$y_{n+1} = \alpha + \frac{y_{n-k}}{y_n}, \; n \in \mathbb{N}_0, \tag{1.2}$$

where $\alpha \in (1,\infty)$, $k \in \mathbb{N}_1$ and x_{-i}, for $i \in \{0,1,\ldots,k\}$, are positive real numbers, was studied in [8]. Eqs. (1.1) and (1.2) were good prototypes for inspiring the investigation of their several extensions. A natural extension of (1.2) is the next system of difference equations

$$x_{n+1} = A + \frac{y_{n-k}}{y_n}, \; y_{n+1} = A + \frac{x_{n-k}}{x_n}, \; n \in \mathbb{N}_0, \tag{1.3}$$

with $A > 0$, and the initial conditions x_{-i}, y_{-i}, for $i \in \{0,1,\ldots,k\}$, are arbitrary positive real numbers. Zhang et al. [21] overcomed the asymptotic behavior of positive solutions to the system (1.3) in the cases $0 < A < 1$, $A = 1$ and $A > 1$. Additionally, in [9], Gümüş dealt with the global asymptotic stability of positive equilibrium, the rate of convergence of positive solutions and semi-cycle analysis of system (1.3). The other extensions of Eq. (1.2) can be found in references [2, 13, 15, 20, 22, 24]. Similarly, a natural extension of (1.1) is the following two-dimensional system of difference equation

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \; y_{n+1} = A + \frac{x_n}{x_{n-k}}, \; n \in \mathbb{N}_0, \tag{1.4}$$

with $A > 0$, and the initial conditions x_{-i}, y_{-i}, for $i \in \{0,1,\ldots,k\}$, are arbitrary positive real numbers. Abualrub and Aloqeili performed semi-cycle analysis of positive solutions of system (1.3) and also studied the dynamical behavior of the solutions of system (1.3) in the cases $0 < A < 1$, $A = 1$ and $A > 1$. Some related to extensions of Eq. (1.1) have extensively studied. For more details see [4, 16, 20, 23] and the reference therein

Inspired with the previous valuable theoretical results, we consider the following three-dimensional symmetrical system of difference equation with high order

$$x_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad y_{n+1} = A + \frac{z_n}{z_{n-k}}, \quad z_{n+1} = A + \frac{x_n}{x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where the parameter $A > 0$, the initial conditions x_{-i}, y_{-i}, z_{-i}, $i \in \{0,1,2,\ldots,k\}$, are arbitrary positive number and $k \in \mathbb{N}$, which is natural extensions of Eq. (1.1) and system (1.4). The main purpose of this paper is to study global asymptotic stability of positive equilibrium, boundedness character of positive solutions, the

rate of convergence of positive solutions and semi-cycle analysis of positive solutions to system (1.5). In addition, the theoretical results are verified by numerical examples. It is clearly seen that system (1.5) has a unique positive equilibrium $(\bar{x},\bar{y},\bar{z}) = (A+1, A+1, A+1)$.

2. Semi-cycle Analysis of system (1.5)

In this section we investigate the behavior of positive solution to system (1.5) by means of semi-cycle analysis method. It is easy to see that system (1.5) has a unique positive equilibrium point $(\bar{x},\bar{y},\bar{z}) = (A+1, A+1, A+1)$.

Theorem 1. *Assume that $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ is a solution to system (1.5). Then either this solution is non-oscillatory solution or it oscillates about the equilibrium $(\bar{x},\bar{y},\bar{z}) = (A+1, A+1, A+1)$, with semi-cycles having $k+1$ terms.*

Proof. Let $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5) and for any integer $n_0 \in \mathbb{N}_0$, $(x_{n_0}, y_{n_0}, z_{n_0})$ be the last term of a semi-cycle which has at least k terms. Then, from this assumption, we can write either

$$\begin{cases} \dots, x_{n_0-k+1}, \dots, x_{n_0-1}, x_{n_0} < 1+A \le x_{n_0+1}, \\ \dots, y_{n_0-k+1}, \dots, y_{n_0-1}, y_{n_0} < 1+A \le y_{n_0+1}, \\ \dots, z_{n_0-k+1}, \dots, z_{n_0-1}, z_{n_0} < 1+A \le z_{n_0+1}, \end{cases} \tag{2.1}$$

or

$$\begin{cases} \dots, x_{n_0-k+1}, \dots, x_{n_0-1}, x_{n_0} \ge 1+A > x_{n_0+1}, \\ \dots, y_{n_0-k+1}, \dots, y_{n_0-1}, y_{n_0} \ge 1+A > y_{n_0+1}, \\ \dots, z_{n_0-k+1}, \dots, z_{n_0-1}, z_{n_0} \ge 1+A > z_{n_0+1}. \end{cases} \tag{2.2}$$

Now, in here we will only present the first case since the other case can be done in a similar way. From (2.1) and (1.5), we have

$$\begin{cases} x_{n_0+2}, x_{n_0+3}, \dots, x_{n_0+k+1} > A+1, \\ y_{n_0+2}, y_{n_0+3}, \dots, y_{n_0+k+1} > A+1, \\ z_{n_0+2}, z_{n_0+3}, \dots, z_{n_0+k+1} > A+1, \end{cases} \tag{2.3}$$

from which it implies that the semi-cycle beginning with $(x_{n_0+1}, y_{n_0+1}, z_{n_0+1})$ has at least $k+1$ terms. Now, we may assume that the semi-cycle beginning with $(x_{n_0+1}, y_{n_0+1}, z_{n_0+1})$ include exactly $k+1$ terms. Then, the next semi-cycle will start with $(x_{n_0+k+2}, y_{n_0+k+2}, z_{n_0+k+2})$ such that

$$\begin{cases} x_{n_0+1}, x_{n_0+2}, \dots, x_{n_0+k+1} \ge 1+A > x_{n_0+k+2}, \\ y_{n_0+1}, y_{n_0+2}, \dots, y_{n_0+k+1} \ge 1+A > y_{n_0+k+2}, \\ z_{n_0+1}, z_{n_0+2}, \dots, z_{n_0+k+1} \ge 1+A > z_{n_0+k+2}, \end{cases} \tag{2.4}$$

from which along with (1.5), for $i = 1,2,\ldots,k$, it follows that

$$\begin{cases} x_{n_0+k+2+i} = A + \frac{y_{n_0+k+1+i}}{y_{n_0+1+i}} < A+1, \\ y_{n_0+k+2+i} = A + \frac{z_{n_0+k+1+i}}{z_{n_0+1+i}} < A+1, \\ z_{n_0+k+2+i} = A + \frac{x_{n_0+k+1+i}}{x_{n_0+1+i}} < A+1, \end{cases} \tag{2.5}$$

which is desired. □

Theorem 2. *System (1.5) has no nontrivial periodic solutions of period k (not necessarily prime period k).*

Proof. Assume that system (1.5) has k-periodic solutions in the following form

$$\ldots,(\alpha_1,\beta_1,\gamma_1),(\alpha_2,\beta_2,\gamma_2),\ldots,(\alpha_k,\beta_k,\gamma_k),\ldots. \tag{2.6}$$

Then, from (1.5), it is not hard to see that for all $n \geq 0$, $(x_{n-k},y_{n-k},z_{n-k}) = (x_n,y_n,z_n)$, and so

$$x_{n+1} = A + \frac{y_n}{y_{n-k}} = A+1,\ y_{n+1} = A + \frac{z_n}{z_{n-k}} = A+1,\ z_{n+1} = A + \frac{x_n}{x_{n-k}} = A+1, \tag{2.7}$$

for all $n \geq 0$, which means that the solution to system (1.5) is the equilibrium solution $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$, which is desired. □

Theorem 3. *Any increasing solution to system (1.5) is non-oscillatory positive.*

Proof. Assume that $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$is an increasing non-oscillatory solution to system (1.5). Then, from (1.5), we can write either

$$\begin{cases} A+1 \leq x_1, \\ A+1 \leq y_1, \\ A+1 \leq z_1, \end{cases} \tag{2.8}$$

or

$$\begin{cases} x_1 < A+1, \\ y_1 < A+1, \\ z_1 < A+1. \end{cases} \tag{2.9}$$

From (1.5), it is easy to see that the inequalities in (2.8) hold. On the other hand, if the inequalities in (2.9) are satisfied, then we may assume that the semi-cycle involving (x_1,y_1,z_1) ends with (x_i,y_i,z_i) such that $i \in \{1,2,\ldots,k+1\}$. If $i = k+2$, then

$$\begin{cases} x_{k+2} = A + \frac{y_{k+1}}{y_1} < A+1, \\ y_{k+2} = A + \frac{z_{k+1}}{z_1} < A+1, \\ z_{k+2} = A + \frac{x_{k+1}}{x_1} < A+1, \end{cases} \tag{2.10}$$

which means that $y_{k+1} < y_1$, $z_{k+1} < z_1$ and $x_{k+1} < x_1$, where $k+1 > 1$, which contradicts that the solution to system (1.5) is increasing. Hence, the proof is completed. □

Theorem 4. *System* (1.5) *has no non-oscillatory negative solutions (has no infinite negative semi-cycle)*

Proof. Let $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5). We may suppose that system (1.5) has an infinite negative semi-cycle beginning with (x_N, y_N, z_N) for $N \geq -k$. Then, it is clearly to see that $x_n < A+1$, $y_n < A+1$ and $z_n < A+1$, for every $n \geq N$. From this and system (1.5), we get $y_n < y_{n-k}$, $z_n < z_{n-k}$ and $x_n < x_{n-k}$, for $n \geq \max\{1, N-1\}$, from which along with system (1.5) it follows that for every $n \geq \max\{1, N\}$

$$\begin{cases} A < \cdots < x_{n+k} < x_n < x_{n-k} < A+1, \\ A < \cdots < y_{n+k} < y_n < y_{n-k} < A+1, \\ A < \cdots < z_{n+k} < z_n < z_{n-k} < A+1. \end{cases} \tag{2.11}$$

From the inequalities in (2.11), we get that there exists α_i, β_i and γ_i for all $i \in \{0, 1, \ldots, k-1\}$ such that

$$\lim_{n\to\infty} x_{nk+i} = \alpha_i, \quad \lim_{n\to\infty} y_{nk+i} = \beta_i, \quad \lim_{n\to\infty} z_{nk+i} = \gamma_i. \tag{2.12}$$

Therefore,

$$(\alpha_0, \beta_0, \gamma_0), (\alpha_1, \beta_1, \gamma_1), \ldots, (\alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}) \tag{2.13}$$

is a periodic solution (not necessarily prime period) of system (1.5) with period k, which contradicts with Theorem 2 in the case when the solution is not trivial solution. So, the solution to system (1.5) converge to the equilibrium, which contradicts that the solution to system (1.5) is diverging from the equilibrium. Thus, the proof is completed. □

Theorem 5. *System* (1.5) *has no decreasing non-oscillatory solutions.*

Proof. Let $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5) which is a decreasing non-oscillatory solutions. Then, the solution to system (1.5) is either of the form

$$\begin{cases} \cdots \leq x_3 \leq x_2 \leq x_1 \leq A+1, \\ \cdots \leq y_3 \leq y_2 \leq y_1 \leq A+1, \\ \cdots \leq z_3 \leq z_2 \leq z_1 \leq A+1, \end{cases} \tag{2.14}$$

or there exists a positive integer $n_0 \geq k+1$ such that

$$\begin{cases} \cdots \leq x_{n_0+2} \leq x_{n_0+1} \leq A+1 \leq x_{n_0} \leq x_{n_0-1}, \\ \cdots \leq y_{n_0+2} \leq y_{n_0+1} \leq A+1 \leq y_{n_0} \leq y_{n_0-1}, \\ \cdots \leq z_{n_0+2} \leq z_{n_0+1} \leq A+1 \leq z_{n_0} \leq z_{n_0-1}. \end{cases} \tag{2.15}$$

In the second case, it is clearly seen that the positive semi-cycle ends with $(x_{n_0}, y_{n_0}, z_{n_0})$ and furthermore it contains at most $2k+2$ terms. From (2.14) and (2.15), the solution to system (1.5) has an infinite negative semi-cycle, which contradicts with Theorem 4. Thus, the proof is completed. □

Theorem 6. *Let k be even and $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5). Then the following statements hold.*

(a) *Every semi-cycle has length at most $3k+1$.*
(b) *The extreme term in a semi-cycle occurs in the first $k+3$ term of the semi-cycle.*
(c) *Every solution oscillates about $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$.*

Proof. Let $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5). In here we will just prove the case of a negative semi-cycle since the proof the other case can be done in a similar way. We may assume that system (1.5) has a negative semi-cycle beginning with (x_μ,y_μ,z_μ), for $\mu \geq -k$, and furthermore it involves $2k+1$ terms. Then, from definition of negative semi-cycle we can write the following inequalities

$$\begin{cases} x_\mu, x_{\mu+1}, \dots, x_{\mu+2k} < A+1, \\ y_\mu, y_{\mu+1}, \dots, y_{\mu+2k} < A+1, \\ z_\mu, z_{\mu+1}, \dots, z_{\mu+2k} < A+1. \end{cases} \tag{2.16}$$

By considering system (1.5) and the inequalities in (2.16), we have

$$\begin{cases} x_{\mu+k+1+i} = A + \frac{y_{\mu+k+i}}{y_{\mu+i}} > A + \frac{y_{\mu+k+i}}{A+1} > y_{\mu+k+i}, \\ y_{\mu+k+i} = A + \frac{z_{\mu+k+i-1}}{z_{\mu+i-1}} > A + \frac{z_{\mu+k+i-1}}{A+1} > z_{\mu+k+i-1}, \\ z_{\mu+k+i-1} = A + \frac{x_{\mu+k+i-2}}{x_{\mu+i-2}} > A + \frac{x_{\mu+k+i-2}}{A+1} > x_{\mu+k+i-2}, \end{cases} \tag{2.17}$$

for $i \in \{2,3,\dots,2k-1\}$, which means that

$$\begin{cases} x_{\mu+k} < z_{\mu+k+1} < y_{\mu+k+2} < x_{\mu+k+3} < z_{\mu+k+4} < y_{\mu+k+5} < x_{\mu+k+6} < \cdots < x_{\mu+3k}, \\ y_{\mu+k} < x_{\mu+k+1} < z_{\mu+k+2} < y_{\mu+k+3} < x_{\mu+k+4} < z_{\mu+k+5} < y_{\mu+k+6} < \cdots < y_{\mu+3k}, \\ z_{\mu+k} < y_{\mu+k+1} < x_{\mu+k+2} < z_{\mu+k+3} < y_{\mu+k+4} < x_{\mu+k+5} < z_{\mu+k+6} < \cdots < z_{\mu+3k}. \end{cases} \tag{2.18}$$

From (2.18), we have

$$x_{\mu+k} < x_{\mu+3k}, \quad y_{\mu+k} < y_{\mu+3k}, \quad z_{\mu+k} < z_{\mu+3k}, \tag{2.19}$$

from which along with system (1.5), it follows that

$$\begin{cases} x_{\mu+3k+1} = A + \frac{y_{\mu+3k}}{y_{\mu+2k}} > A+1, \\ y_{\mu+3k+1} = A + \frac{z_{\mu+3k}}{z_{\mu+2k}} > A+1, \\ z_{\mu+3k+1} = A + \frac{x_{\mu+3k}}{x_{\mu+2k}} > A+1. \end{cases} \tag{2.20}$$

Thus, from (2.20), we conclude that a negative semi-cycle has at most $3k+1$ terms. Furthermore, the first $k+3$ terms of a semi-cycle are formed as the extreme term in this semi-cycle. Finally, every solution to system (1.5) oscillates about the equilibrium $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$. Therefore the proof is completed. □

3. THE ASYMPTOTIC BEHAVIOR OF SYSTEM (1.5)

In this section, we study the asymptotic behavior of system (1.5) according to values of A.

Theorem 7. *Assume that $A = 1$. Then every positive solutions of system (1.5) is bounded and persists.*

Proof. Assume that $A = 1$ and $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ is a positive solutions of system (1.5). It is easily seen that for every $n \in \mathbb{N}_0$, $x_n, y_n, z_n > A = 1$. Thus, we can write

$$x_i, y_i, z_i \in \left[K, \frac{K}{K-1}\right], \ i = 1, 2, \ldots, k+1, \tag{3.1}$$

where $K = min\{\alpha, \frac{\beta}{\beta-1}\} > 1$, $\alpha = \min\limits_{1 \le i \le k+1}\{x_i, y_i, z_i\}$, $\beta = \max\limits_{1 \le i \le k+1}\{x_i, y_i, z_i\}$. From system (1.5), we have

$$K = 1 + \frac{K}{\frac{K}{K-1}} \le x_{k+2} = 1 + \frac{y_{k+1}}{y_1} \le 1 + \frac{\frac{K}{K-1}}{K} = \frac{K}{K-1}, \tag{3.2}$$

$$K = 1 + \frac{K}{\frac{K}{K-1}} \le y_{k+2} = 1 + \frac{z_{k+1}}{z_1} \le 1 + \frac{\frac{K}{K-1}}{K} = \frac{K}{K-1}, \tag{3.3}$$

$$K = 1 + \frac{K}{\frac{K}{K-1}} \le z_{k+2} = 1 + \frac{x_{k+1}}{x_1} \le 1 + \frac{\frac{K}{K-1}}{K} = \frac{K}{K-1}. \tag{3.4}$$

By using the method of induction, it follows that

$$x_i, y_i, z_i \in \left[K, \frac{K}{K-1}\right], \text{ for all } i = 1, 2, \ldots, \tag{3.5}$$

which is desired. □

Theorem 8. *Assume that $A = 1$ and $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ is a positive solutions of system (1.5). Then the following result holds true.*

$$\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \inf y_n = \lim_{n\to\infty} \inf z_n$$

$$\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} \sup y_n = \lim_{n\to\infty} \sup z_n$$

Proof. Assume that

$$\begin{cases} l_1 = \lim_{n\to\infty} \inf x_n, \ u_1 = \lim_{n\to\infty} \sup x_n, \\ l_2 = \lim_{n\to\infty} \inf y_n, \ u_2 = \lim_{n\to\infty} \sup y_n, \\ l_3 = \lim_{n\to\infty} \inf z_n, \ u_3 = \lim_{n\to\infty} \sup z_n. \end{cases} \tag{3.6}$$

From (3.6) and system (1.5), we get the following inequalities

$$1 < l_1 \le u_1, \ 1 < l_2 \le u_2, \ 1 < l_3 \le u_3 \tag{3.7}$$

and

$$\begin{cases} 1+\frac{l_2}{u_2} \leq l_1,\ u_1 \leq 1+\frac{u_2}{l_2}, \\ 1+\frac{l_3}{u_3} \leq l_2,\ u_2 \leq 1+\frac{u_3}{l_3}, \\ 1+\frac{l_1}{u_1} \leq l_3,\ u_3 \leq 1+\frac{u_1}{l_1}, \end{cases} \tag{3.8}$$

from which it follows that

$$\begin{cases} u_1 l_2 \leq l_2+u_2 \leq l_1 u_2, \\ u_2 l_3 \leq l_3+u_3 \leq l_2 u_3, \\ u_3 l_1 \leq u_1+l_1 \leq u_1 l_3. \end{cases} \tag{3.9}$$

Also, from the inequalities in (3.9), without loss of generality assume that we continue only with the following equality cases

$$l_2 u_1 = l_1 u_2, \tag{3.10}$$

$$l_2 u_1 = l_2 + u_2, \tag{3.11}$$

$$l_3 u_2 = l_2 u_3, \tag{3.12}$$

$$l_2 u_3 = l_3 + u_3, \tag{3.13}$$

$$l_1 u_3 = u_1 l_3, \tag{3.14}$$

$$l_3 u_1 = l_1 + u_1. \tag{3.15}$$

From (3.10), (3.12) and (3.14), it follows that

$$\frac{l_1}{l_2} = \frac{u_1}{u_2}, \tag{3.16}$$

$$\frac{l_2}{l_3} = \frac{u_2}{u_3}, \tag{3.17}$$

$$\frac{l_1}{l_3} = \frac{u_1}{u_3}. \tag{3.18}$$

Now, upon dividing both side of (3.13) by l_2 and both side of (3.15) by l_3, we obtain

$$u_3 = \frac{l_3}{l_2} + \frac{u_3}{l_2}, \tag{3.19}$$

$$u_1 = \frac{l_1}{l_3} + \frac{u_1}{l_3}, \tag{3.20}$$

from which along with (3.17), (3.12) and after some basic calculations, it follows that

$$l_2 = \frac{u_2}{u_2 - 1}, \tag{3.21}$$

$$l_3 = \frac{u_3}{u_1 - 1}. \tag{3.22}$$

Replacing l_2 by $\frac{u_2}{u_2-1}$ in (3.11) and l_3 by $\frac{u_3}{u_1-1}$ in (3.13), we get

$$u_1 \frac{u_2}{u_2 - 1} = \frac{u_2}{u_2 - 1} + u_2, \tag{3.23}$$

$$u_2\frac{u_3}{u_3-1}=\frac{u_3}{u_3-1}+u_3, \tag{3.24}$$

which can be written as follows

$$\frac{u_2}{u_2-1}=\frac{u_1}{u_2-1}, \tag{3.25}$$

$$\frac{u_3}{u_3-1}=\frac{u_2}{u_3-1}. \tag{3.26}$$

From this and the equalities (3.16) and (3.17), consequently

$$u_2=u_1=u_3, \tag{3.27}$$

$$l_2=l_1=l_3 \tag{3.28}$$

and this complete the proof. □

Theorem 9. *Assume that $A=1$. Then, system (1.5) has no positive nontrivial solution prime period-two.*

Proof. Let $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ be a positive solution of system (1.5). Assume that $a,b,c,d,e,f>1$ and for all $n\in\mathbb{N}_0$, $x_{2n-k}=a$, $y_{2n-k}=b$, $z_{2n-k}=c$, $x_{2n+1-k}=d$, $y_{2n+1-k}=e$ and $z_{2n+1-k}=f$. Then, it is clearly seen that the positive solutions of system (1.5) has the following form

$$\ldots,(a,b,c),(d,e,f),(a,b,c),(d,e,f),\ldots. \tag{3.29}$$

If it was $a=d$, $b=e$ and $c=f$, then we would have that the solution to system (1.5) is not two-periodic. So, it must be $a\neq d$, $b\neq e$ and $c\neq f$. By considering Theorem (8), we can write the following equalities

$$\min\{a,d\}=\lim_{n\to\infty}\inf x_n=\lim_{n\to\infty}\inf y_n=\min\{b,e\}=\lim_{n\to\infty}\inf z_n=\min\{c,f\} \tag{3.30}$$

and

$$\max\{a,d\}=\lim_{n\to\infty}\sup x_n=\lim_{n\to\infty}\sup y_n=\max\{b,e\}=\lim_{n\to\infty}\sup z_n=\max\{c,f\}. \tag{3.31}$$

There are eight cases to be considered.

a) If $a<d$, $b<e$ and $c<f$, then $a=b=c$ and $d=e=f$ and the solution to system (1.5) can be written in the following form

$$\ldots,(a,a,a),(d,d,d),(a,a,a),(d,d,d),\ldots. \tag{3.32}$$

b) If $a<d$, $b<e$ and $c>f$, then $a=b=f$ and $d=c=e$ and the solution to system (1.5) is expressed in the following form

$$\ldots,(a,a,d),(d,d,a),(a,a,d),(d,d,a),\ldots. \tag{3.33}$$

c) If $a<d$, $b>e$ and $c>f$ then $a=e=f$ and $d=b=c$ and the solution to system (1.5) is written as in the form

$$\ldots,(a,d,d),(d,a,a),(a,d,d),(d,a,a),\ldots. \tag{3.34}$$

d) If $d > a$, $b > e$, $f > c$ then $a = e = c$ and $d = b = f$ and the solution to system (1.5) can be written in the following form

$$\ldots,(a,d,a),(d,a,d),(a,d,a),(d,a,d),\ldots. \tag{3.35}$$

e) If $a > d$, $e > b$, $f > c$ then $d = b = c$ and $a = e = f$ and we obtain the case (c).
f) If $a > d$, $e > b$, $c > f$ then $d = b = f$ and $a = e = c$ and we get the case (d).
g) If $a > d$, $b > e$, $f > c$ then $d = e = c$ and $a = b = f$ and we have the case (b).
h) If $a > d$, $b > e$, $c > f$ then $d = e = f$ and $a = b = c$ and we reach the case (a).

In here, we will only prove in the case when k is odd, since the proof of other case can be done in a similar way. Then, from our above assumptions and (3.32), we can write $x_{2n-k} = a$, $y_{2n-k} = a$, $z_{2n-k} = a$, $x_{2n+1-k} = d$ $y_{2n+1-k} = d$, $z_{2n+1-k} = d$ for the case (a), from which it follows that $x_1 = 1 + \frac{y_0}{y_{-k}}$, so $a = 1 + \frac{d}{a}$ that also means $a^2 = a + d$. Further, $x_2 = 1 + \frac{y_1}{y_{-k+1}}$, so $d = 1 + \frac{a}{d}$ which implies that $d^2 = a + d$. Then $d^2 = a^2$, but $d \neq a$, so $d = -a$. Therefore, we conclude that in this case system (1.5) has no positive prime period-two solutions. The other cases can be obtained similar to the case (a). So we will omit proofs of other cases.

□

Theorem 10. *Assume that $A > 1$. Then every positive solutions of system (1.5) is bounded and persists.*

Proof. Let $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ be a positive solutions of system (1.5). Since $A > 1$ and the initial conditions x_{-i}, y_{-i}, z_{-i}, for all $i \in \{0,1,2,\ldots,k\}$, are arbitrary positive numbers, then we can write the following inequalities

$$x_n > A,\ y_n > A,\ z_n > A, \quad \text{for all } n \geq 1. \tag{3.36}$$

By considering system (1.5) and (3.36), we get for all $n \geq k+2$

$$x_n = A + \frac{y_{n-1}}{y_{n-k-1}} < A + \frac{1}{A} y_{n-1}, \tag{3.37}$$

$$y_n = A + \frac{z_{n-1}}{z_{n-k-1}} < A + \frac{1}{A} z_{n-1}, \tag{3.38}$$

$$z_n = A + \frac{x_{n-1}}{x_{n-k-1}} < A + \frac{1}{A} x_{n-1}. \tag{3.39}$$

Now assume that $\{(u_n, v_n, w_n)\}$ is solutions of the following difference equations systems

$$u_n = A + \frac{1}{A} v_{n-1},\ v_n = A + \frac{1}{A} w_{n-1},\ w_n = A + \frac{1}{A} u_{n-1},\ n \geq k+2, \tag{3.40}$$

such that $u_i = x_i,\ v_i = y_i,\ w_i = z_i$, where $i \in \{1,2,\ldots,k+1\}$. Now we will prove the following inequalities using the induction method for $n \geq k+2$

$$x_n < u_n, \quad y_n < v_n, \quad z_n < w_n. \tag{3.41}$$

It is easy to see that, for $n = k+2$, from system (1.5) and (3.41), we have

$$x_{k+2} = A + \frac{y_{k+1}}{y_1} < A + \frac{y_{k+1}}{A} = A + \frac{1}{A}v_{k+1} = u_{k+2}. \tag{3.42}$$

Assume that the inequalities in (3.41) hold for $m > k+2$. Then, from this assumption, system (1.5) and (3.40), we obtain

$$\begin{cases} x_{m+1} < A + \frac{1}{A}y_m < A + \frac{1}{A}v_m = u_{m+1}, \\ y_{m+1} < A + \frac{1}{A}z_m < A + \frac{1}{A}w_m = v_{m+1}, \\ z_{m+1} < A + \frac{1}{A}x_m < A + \frac{1}{A}u_m = w_{m+1}, \end{cases} \tag{3.43}$$

which ends up the induction. Hence, considering (3.40), the following equations are obtained for $m \geq k$

$$\begin{cases} u_{m+3} &= \frac{1}{A^3}u_m + \frac{1}{A} + A + 1, \\ v_{m+3} &= \frac{1}{A^3}w_m + \frac{1}{A} + A + 1, \\ w_{m+3} &= \frac{1}{A^3}v_m + \frac{1}{A} + A + 1. \end{cases} \tag{3.44}$$

From (3.44), we have that the sequences $(u_{k+3l+i})_{l\geq 0}$, $(v_{k+3l+i})_{l\geq 0}$ and $(w_{k+3l+i})_{l\geq 0}$, for $i \in \{0,1,2\}$, provide the following recurrence relation

$$r_{n+1} = \frac{1}{A^3}r_n + \frac{1}{A} + A + 1,\ n \geq k, \tag{3.45}$$

from which it follows that for all $l \geq 0$,

$$\begin{cases} u_{k+3l+i} &= \left(x_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}, \\ v_{k+3l+i} &= \left(y_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}, \\ w_{k+3l+i} &= \left(z_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}. \end{cases} \tag{3.46}$$

By considering (3.36), (3.41) and (3.46), we can write the boundaries of the solutions to system (1.5) for all $l \geq 0$ and $i \in \{0,1,2\}$ as follows

$$\begin{cases} A < x_{k+3l+i} \leq \left(x_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}, \\ A < y_{k+3l+i} \leq \left(y_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}, \\ A < z_{k+3l+i} \leq \left(z_{k+i} + \frac{A^2}{1-A}\right)\left(\frac{1}{A}\right)^{3l} + \frac{A^2}{A-1}, \end{cases} \tag{3.47}$$

which is desired.

□

Theorem 11. *Assume that $A \geq 1$ and $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ is a solution to system (1.5). Then, system (1.5) has no non-oscillatory positive solutions (has no infinite positive semi-cycle).*

Proof. Let $\{(x_n, y_n, z_n)\}_{n=-k}^{\infty}$ be a solution to system (1.5). Conversely, we suppose that system (1.5) has a non-oscillatory positive solutions and also this solution has an infinite positive semi-cycle beginning with (x_μ, y_μ, z_μ) for $\mu \geq -k$. By considering definition of positive semi-cycle and system (1.5), we have

$$\begin{cases} x_{n+1} = A + \frac{y_n}{y_{n-k}} \geq A+1, \\ y_{n+1} = A + \frac{z_n}{z_{n-k}} \geq A+1, \\ z_{n+1} = A + \frac{x_n}{x_{n-k}} \geq A+1, \end{cases} \tag{3.48}$$

for $n \geq \mu - 1$. From Theorem (10), there exist three real numbers K, L and M such that for every $n \geq k+2$, $x_n \leq K$, $y_n \leq L$ and $z_n \leq M$. From this and (3.48), we can write

$$\begin{cases} A+1 \leq x_{n-k} \leq x_n \leq x_{n+k} \leq \cdots \leq K, \\ A+1 \leq y_{n-k} \leq y_n \leq y_{n+k} \leq \cdots \leq L, \\ A+1 \leq z_{n-k} \leq z_n \leq z_{n+k} \leq \cdots \leq M, \end{cases} \tag{3.49}$$

for all $n \geq max\{\mu - 1, 2k+1\}$. From the inequalities in (3.49), we have that there exists β_i, γ_i and δ_i for all $i \in \{0, 1, \ldots, k-1\}$ such that

$$\lim_{n\to\infty} x_{nk+i} = \beta_i, \quad \lim_{n\to\infty} y_{nk+i} = \gamma_i, \quad \lim_{n\to\infty} z_{nk+i} = \delta_i. \tag{3.50}$$

Hence,

$$(\beta_0, \gamma_0, \delta_0), (\beta_1, \gamma_1, \delta_1), \ldots, (\beta_{k-1}, \gamma_{k-1}, \delta_{k-1}) \tag{3.51}$$

is a periodic solution of (not necessarily prime period) system (1.5) with period k, which contradicts with Theorem 2 in the case when the solution is not trivial solution. So, the solution to system (1.5) converge to the equilibrium, which contradicts that the solution to system (1.5) is diverging from the equilibrium. We complete the proof. □

Lemma 1. *[3] Assume that $A > 1$ and $0 < \varepsilon < \frac{A-1}{(A+1)(k+1)}$, for $k \in \mathbb{N}$. Then $\frac{2}{(1-(k+1))\varepsilon(A+1)} < 1$.*

The following theorem gives us the local stability of the unique equilibrium point of the system (1.5) when $A > 1$.

Theorem 12. *If $A > 1$ then the unique positive equilibrium $(\bar{x}, \bar{y}, \bar{z}) = (A+1, A+1, A+1)$ of system (1.5) is locally asymptotically stable.*

Proof. For analysing the local stability of the unique equilibrium point of the system (1.5), firstly, the system (1.5) is transformed into the following equivalent system as

$$X_{n+1} = F(X_n), \; n \in \mathbb{N}_0, \tag{3.52}$$

where $X_n = \left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k+1)}, y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(k+1)}, z_n^{(1)}, z_n^{(2)}, \ldots, z_n^{(k+1)}\right)^T$, where

$$\begin{cases} x_n^{(1)} = x_n, x_n^{(2)} = x_{n-1}, \ldots, x_n^{(k+1)} = x_{n-k}, \\ y_n^{(1)} = y_n, y_n^{(2)} = y_{n-1}, \ldots, y_n^{(k+1)} = y_{n-k}, \\ z_n^{(1)} = z_n, z_n^{(2)} = z_{n-1}, \ldots, z_n^{(k+1)} = z_{n-k}, \end{cases} \tag{3.53}$$

and $F : [0,\infty)^{3k+3} \to [0,\infty)^{3k+3}$ such that for all

$$X = \left(x^{(1)}, x^{(2)}, \ldots, x^{(k+1)}, y^{(1)}, y^{(2)}, \ldots, y^{(k+1)}, z^{(1)}, z^{(2)}, \ldots, z^{(k+1)}\right) \in [0,\infty)^{3k+3},$$

$$F(X) = \left(f_1(X), x^{(2)}, \ldots, x^{(k+1)}, f_2(X), y^{(2)}, \ldots, y^{(k+1)}, f_3(X), z^{(2)}, \ldots, z^{(k+1)}\right),$$

where

$$f_1(X) = A + \frac{y^{(1)}}{y^{(k+1)}},\ f_2(X) = A + \frac{z^{(1)}}{z^{(k+1)}},\ f_3(X) = A + \frac{x^{(1)}}{x^{(k+1)}}.$$

Then, we have

$$\begin{cases} \frac{\partial f_1}{\partial y^{(1)}}(X) = \frac{1}{y^{(k+1)}}, & \frac{\partial f_1}{\partial y^{(k+1)}}(X) = -\frac{y^{(1)}}{(y^{(k+1)})^2}, \\ \frac{\partial f_2}{\partial z^{(1)}}(X) = \frac{1}{z^{(k+1)}}, & \frac{\partial f_2}{\partial z^{(k+1)}}(X) = -\frac{z^{(1)}}{(z^{(k+1)})^2}, \\ \frac{\partial f_3}{\partial x^{(1)}}(X) = \frac{1}{x^{(k+1)}}, & \frac{\partial f_3}{\partial x^{(k+1)}}(X) = -\frac{x^{(1)}}{(x^{(k+1)})^2}. \end{cases} \tag{3.54}$$

$\mathcal{J}_{\mathcal{F}}$ is the Jacobian matrix of F at the equilibrium point $(\bar{x}, \bar{y}, \bar{z}) = (A+1, A+1, A+1)$, which is given by

$$\mathcal{J}_{\mathcal{F}} = \begin{pmatrix} 0 & 0\cdots0 & 0 & \frac{1}{A+1} & 0\cdots0 & \frac{-1}{A+1} & 0 & 0\cdots0 & 0 \\ 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & \frac{1}{A+1} & 0\cdots0 & \frac{-1}{A+1} \\ 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 \\ \frac{1}{A+1} & 0\cdots0 & \frac{-1}{A+1} & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 \end{pmatrix} \tag{3.55}$$

Assume that $\lambda_1, \lambda_2, \ldots, \lambda_{3k+3}$ are the eigenvalues of matrix $\mathcal{J}_{\mathcal{F}}$ and that $\mathcal{D} = diag(d_1, d_2, \ldots, d_{3k+3})$ is a diagonal matrix, where

$$d_1 = d_{k+2} = d_{2k+3} = 1, \quad d_m = d_{k+1+m} = d_{2k+2+m} = 1 - m\varepsilon,$$

for $m \in \{2,3,\dots,k+1\}$ and $0 < \varepsilon < \frac{A-1}{(k+1)(A+1)}$. From this and taking into account the fact that $1 - m\varepsilon > 0$, for all $m \in \{2,3,\dots,k+1\}$, we conclude that the matrix $\mathcal{D}$ is invertible. The matrix DBD^{-1} is given by

$$\begin{pmatrix}
0 & 0\cdots 0 & 0 & \frac{1}{A+1}\frac{d_1}{d_{k+2}} & 0\cdots 0 & \frac{-1}{A+1}\frac{d_1}{d_{2k+2}} & 0 & 0\cdots 0 & 0 \\
\frac{d_2}{d_1} & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\
0 & 0\cdots \frac{d_{k+1}}{d_k} & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\
0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & \frac{1}{A+1}\frac{d_{k+2}}{d_{2k+3}} & 0\cdots 0 & \frac{-1}{A+1}\frac{d_{k+2}}{d_{3k+3}} \\
0 & 0\cdots 0 & 0 & \frac{d_{k+3}}{d_{k+2}} & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\
0 & 0\cdots 0 & 0 & 0 & 0\cdots \frac{d_{2k+2}}{d_{2k+1}} & 0 & 0 & 0\cdots 0 & 0 \\
\frac{1}{A+1}\frac{d_{2k+3}}{d_1} & 0\cdots 0 & \frac{-1}{A+1}\frac{d_{2k+3}}{d_{k+1}} & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 \\
0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & \frac{d_{2k+4}}{d_{2k+3}} & 0\cdots 0 & 0 \\
0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots \frac{d_{3k+3}}{d_{3k+2}} & 0
\end{pmatrix} \tag{3.56}$$

To calculate the infinity norm of $\mathcal{D}\mathcal{J}_{\mathcal{F}}\mathcal{D}^{-1}$, we will show that inequality $\max\limits_{1\le i\le 3k+3} \sum_{j=1}^{3k+3} |a_{ij}| < 1$ holds. For this, by considering the fact that $d_1 > d_2 > \cdots > d_{k+1}$, $d_{k+2} > d_{k+3} > \cdots > d_{2k+2}$ and $d_{2k+3} > d_{2k+4} > \cdots > d_{3k+3}$, we can write the following inequalities for every $j \in \{1,2,\dots,3k+2\}$

$$\frac{d_{j+1}}{d_j} < 1 \tag{3.57}$$

from which along with $A > 1$ and lemma (1), it yields

$$\begin{aligned}
\frac{1}{A+1}\frac{d_1}{d_{k+2}} + \frac{1}{A+1}\frac{d_1}{d_{2k+2}} &= \frac{1}{A+1} + \frac{1}{(1-(k+1)\varepsilon)(A+1)} \\
&< \frac{1}{(1-(k+1)\varepsilon)(A+1)} + \frac{1}{(1-(k+1)\varepsilon)(A+1)} \\
&< \frac{2}{(1-(k+1)\varepsilon)(A+1)} \\
&< 1,
\end{aligned} \tag{3.58}$$

$$\begin{aligned}
\frac{1}{A+1}\frac{d_{k+2}}{d_{2k+3}} + \frac{1}{A+1}\frac{d_{k+2}}{d_{3k+3}} &= \frac{1}{A+1} + \frac{1}{(1-(k+1)\varepsilon)(A+1)} \\
&< \frac{1}{(1-(k+1)\varepsilon)(A+1)} + \frac{1}{(1-(k+1)\varepsilon)(A+1)} \\
&< \frac{2}{(1-(k+1)\varepsilon)(A+1)} \\
&< 1
\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}\frac{1}{A+1}\frac{d_{2k+3}}{d_1}+\frac{1}{A+1}\frac{d_{2k+3}}{d_{k+1}} &= \frac{1}{A+1}+\frac{1}{(1-(k+1)\varepsilon)(A+1)}\\ &< \frac{1}{(1-(k+1)\varepsilon)(A+1)}+\frac{1}{(1-(k+1)\varepsilon)(A+1)}\\ &< \frac{2}{(1-(k+1)\varepsilon)(A+1)}\\ &< 1. \end{aligned} \tag{3.60}$$

Since $\mathcal{J}_{\mathcal{F}}$ and $\mathcal{D}\mathcal{J}_{\mathcal{F}}\mathcal{D}^{-1}$ similar matrices, they have the same eigenvalues. Consequently, it can easily seen that the following inequality holds

$$\begin{aligned}\rho(\mathcal{J}_{\mathcal{F}}) &= \max|\lambda_i|\\ &\leq ||\mathcal{D}\mathcal{J}_{\mathcal{F}}\mathcal{D}^{-1}||_\infty\\ &= \max\{d_2d_1^{-1},\dots,d_{k+1}d_k^{-1},\dots,d_{k+3}d_{k+2}^{-1},\dots,d_{3k+3}d_{3k+2}^{-1} < 1. \end{aligned} \tag{3.61}$$

This means that all eigenvalues of $\mathcal{J}_{\mathcal{F}}$ lie inside the unit disk. Therefore, the unique equilibrium point $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$ of system (1.5) is locally asymptotically stable. □

The following theorem gives us the global attractor to the unique positive equilibrium point of system (1.5) when $A > 1$.

Theorem 13. *Assume that $A > 1$. Then the unique equilibrium point $(A+1,A+1,A+1)$ of system (1.5) is globally asymptotically stable.*

Proof. Evidently, since the unique positive equilibrium $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$ of system (1.5) is locally asymptotically stable when $A > 1$, we only prove that the unique positive equilibrium $(\bar{x},\bar{y},\bar{z}) = (A+1,A+1,A+1)$ of system (1.5) is a global attractor. For this, firstly, assume that

$$\begin{cases} l_1 = \lim_{n\to\infty}\inf x_n,\ u_1 = \lim_{n\to\infty}\sup x_n,\\ l_2 = \lim_{n\to\infty}\inf y_n,\ u_2 = \lim_{n\to\infty}\sup y_n,\\ l_3 = \ lim_{n\to\infty}\inf z_n,\ u_3 = \ lim_{n\to\infty}\sup z_n. \end{cases} \tag{3.62}$$

From (3.62) and system (1.5), it can be easily seen that the inequalities

$$\begin{cases} 1 < l_1 \leq u_1,\\ 1 < l_2 \leq u_2,\\ 1 < l_3 \leq u_3, \end{cases} \tag{3.63}$$

and

$$\begin{cases} A+\frac{l_2}{u_2} \leq l_1,\ u_1 \leq A+\frac{u_2}{l_2},\\ A+\frac{l_3}{u_3} \leq l_2,\ u_2 \leq A+\frac{u_3}{l_3},\\ A+\frac{l_1}{u_1} \leq l_3,\ u_3 \leq A+\frac{u_1}{l_1}, \end{cases} \tag{3.64}$$

hold. From (3.63), we can write the following inequalities

$$\begin{cases} Au_2 + l_2 \leq l_1 u_2, \\ l_2 u_1 \leq Al_2 + u_2, \\ Au_3 + l_3 \leq l_2 u_3, \\ l_3 u_2 \leq Al_3 + u_3, \\ Au_1 + l_1 \leq l_3 u_1, \\ l_1 u_3 \leq Al_1 + u_1. \end{cases} \tag{3.65}$$

Now, by multiplying both sides of the first inequality in (3.65) by u_3 and both sides of the sixth one in (3.65) by u_2, then we can write

$$Au_2u_3 + l_2u_3 \leq l_1u_2u_3, \; u_2u_3l_1 \leq Al_1u_2 + u_1u_2, \tag{3.66}$$

from which it follows that

$$Au_2u_3 + l_2u_3 \leq Al_1u_2 + u_1u_2. \tag{3.67}$$

Similarly, upon multiplying both sides of the second inequality in (3.65) by u_3 and both sides of the third one in (3.65) by u_1, then we have

$$l_2u_1u_3 \leq Al_2u_3 + u_2u_3, \; Au_1u_3 + l_3u_1 \leq l_2u_1u_3, \tag{3.68}$$

from which it follows that

$$Au_1u_3 + l_3u_1 \leq Al_2u_3 + u_2u_3 \tag{3.69}$$

Likewise, upon multiplying both sides of the fourth inequality (3.65) by u_1 and both sides of the fifth one in (3.65) by u_2, we reach the following inequality

$$l_3u_1u_2 \leq Al_3u_1 + u_1u_3, \; Au_1u_2 + l_1u_2 \leq l_3u_1u_2, \tag{3.70}$$

from which it follows that

$$Au_1u_2 + l_1u_2 \leq Al_3u_1 + u_1u_3. \tag{3.71}$$

From (3.67), (3.69) and (3.71), we have

$$\begin{aligned} &Au_2u_3 + l_2u_3 + Au_1u_3 + l_3u_1 + Au_1u_2 + l_1u_2 \\ &\quad \leq Al_1u_2 + u_1u_2 + Al_2u_3 + u_2u_3 + Al_3u_1 + u_1u_3 \end{aligned} \tag{3.72}$$

which implies that

$$Au_3(u_2 - l_2) + Au_1(u_3 - l_3) + Au_2(u_1 - l_1) - u_3(u_2 - l_2) - u_1(u_3 - l_3) - u_2(u_1 - l_1) \leq 0 \tag{3.73}$$

and, consequently

$$(A-1)u_3(u_2 - l_2) + (A-1)u_1(u_3 - l_3) + (A-1)u_2(u_1 - l_1) \leq 0. \tag{3.74}$$

Since $A > 1$, from (3.74), we can reach the following results

$$u_1 = l_1, \; u_2 = l_2, \; u_3 = l_3, \tag{3.75}$$

from which along with (3.62), it follows that

$$\lim_{n\to\infty} x_n = A+1,\ \lim_{n\to\infty} y_n = A+1,\ \lim_{n\to\infty} z_n = A+1, \tag{3.76}$$

which is desired. □

4. Rate of convergence

In this section, we investigate the rate of convergence of a solutions which converges to the equilibrium point $(\bar{x},\bar{y},\bar{z}) = (A+1, A+1, A+1)$ of the system (1.5) in the region of parameters described by $A > 1$. The following result gives the rate of convergence of solutions of the system of difference equations

$$U_{n+1} = (A+B_n)U_n \tag{4.1}$$

where U_n is a $(3k+3)$-dimensional vector, $A \in C^{(3k+3)\times(3k+3)}$ is a constant matrix and $B : \mathbb{Z}^+ \to C^{(3k+3)\times(3k+3)}$ is a matrix function providing

$$\|B_n\| \to 0, \quad \text{as} \quad n \to \infty, \tag{4.2}$$

where $\|.\|$ denotes any matrix norm.

Theorem 14. *(Perron's Theorem, see, [14]) Assume that condition (4.2) holds. If U_n is a solution of (4.1), then either $U_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|U_{n+1}\|}{\|U_n\|}$$

or

$$\rho = \lim_{n\to\infty} (\|U_n\|)^{\frac{1}{n}}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 15. *Assume that $\{(x_n, y_n, z_n)\}_{n\geq -k}$ is a solution of system (1.5) such that the equalities in (3.76) are provided. Then, the error vector*

$$e_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ \vdots \\ e_{n-k}^1 \\ e_n^2 \\ e_{n-1}^2 \\ \vdots \\ e_{n-k}^2 \\ e_n^3 \\ e_{n-1}^3 \\ \vdots \\ e_{n-k}^3 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ x_{n-1} - \bar{x} \\ \vdots \\ x_{n-k} - \bar{x} \\ y_n - \bar{y} \\ y_{n-1} - \bar{y} \\ \vdots \\ y_{n-k} - \bar{y} \\ z_n - \bar{z} \\ z_{n-1} - \bar{z} \\ \vdots \\ z_{n-k} - \bar{x} \end{pmatrix}$$

of every solution of system (1.5) *satisfies both of the asymptotic relations for some* $i \in \{1,2,\dots,k\}$,

$$\rho = \lim_{n\to\infty}(||U_n||)^{\frac{1}{n}} = |\lambda_i J_F(\bar{x},\bar{y},\bar{z})|, \quad \rho = \lim_{n\to\infty}\frac{||U_{n+1}||}{||U_n||} = |\lambda_i J_F(\bar{x},\bar{y},\bar{z})|, \tag{4.3}$$

where ρ *is equal to the modulus of one of the eigenvalues of* $\mathcal{J}_{\mathcal{F}}$ *about* $(\bar{x},\bar{y},\bar{z})$.

Proof. Let $\{(x_n,y_n,z_n)\}_{n=-k}^{\infty}$ be any solution to system (1.5) that satisfies the following conditions

$$\lim_{n\to\infty} x_n = \bar{x} = A+1, \quad \lim_{n\to\infty} y_n = \bar{y} = A+1, \quad \lim_{n\to\infty} z_n = \bar{z} = A+1. \tag{4.4}$$

To obtain the error terms, one gets

$$x_{n+1} - \bar{x} = A + \frac{y_n}{y_{n-k}} - A - 1 = \frac{y_n - y_{n-k}}{y_{n-k}} = \frac{y_n - \bar{y}}{y_{n-k}} - \frac{y_{n-k} - \bar{y}}{y_{n-k}}, \tag{4.5}$$

$$y_{n+1} - \bar{y} = A + \frac{z_n}{z_{n-k}} - A - 1 = \frac{z_n - z_{n-k}}{z_{n-k}} = \frac{z_n - \bar{z}}{z_{n-k}} - \frac{z_{n-k} - \bar{z}}{z_{n-k}}, \tag{4.6}$$

$$z_{n+1} - \bar{z} = A + \frac{x_n}{x_{n-k}} - A - 1 = \frac{x_n - x_{n-k}}{x_{n-k}} = \frac{x_n - \bar{x}}{x_{n-k}} - \frac{x_{n-k} - \bar{x}}{x_{n-k}}, \tag{4.7}$$

for $i \in \{1,2,\dots,k\}$. Now, let

$$e_n^1 = x_n - \bar{x}, \quad e_n^2 = y_n - \bar{y}, \quad e_n^3 = z_n - \bar{z}, \tag{4.8}$$

and then the equations in (4.5)-(4.7) become

$$e_{n+1}^1 = \frac{e_n^2}{y_{n-k}} - \frac{e_{n-k}^2}{y_{n-k}}, \tag{4.9}$$

$$e_{n+1}^2 = \frac{e_n^3}{z_{n-k}} - \frac{e_{n-k}^3}{z_{n-k}}, \tag{4.10}$$

$$e_{n+1}^3 = \frac{e_n^1}{x_{n-k}} - \frac{e_{n-k}^1}{x_{n-k}}. \tag{4.11}$$

Now, we let $\mathcal{A}_{1i} = \mathcal{C}_{1i} = \mathcal{A}_{2i} = \mathcal{B}_{2i} = \mathcal{B}_{3i} = \mathcal{C}_{3i} = 0$, for $i \in \{0,1,\dots,k\}$, $\mathcal{B}_{10} = \frac{1}{y_{n-k}}$, $\mathcal{B}_{1j} = 0$, for $j \in \{1,2,\dots,k-1\}$, $\mathcal{B}_{1k} = -\frac{1}{y_{n-k}}$, $\mathcal{C}_{20} = \frac{1}{z_{n-k}}$, $\mathcal{C}_{2j} = 0$, for $j \in \{1,2,\dots,k-1\}$, $\mathcal{C}_{2k} = -\frac{1}{z_{n-k}}$, $\mathcal{A}_{30} = \frac{1}{x_{n-k}}$, $\mathcal{A}_{3j} = 0$, for $j \in \{1,2,\dots,k-1\}$ and $\mathcal{A}_{3k} = -\frac{1}{x_{n-k}}$. Then, the equations in (4.9)-(4.11) can be written the next form of

$$e_{n+1}^1 = \sum_{i=0}^{k} \mathcal{A}_{1i} e_{n-i}^1 + \sum_{i=0}^{k} \mathcal{B}_{1i} e_{n-i}^2 + \sum_{i=0}^{k} \mathcal{C}_{1i} e_{n-i}^3, \tag{4.12}$$

$$e_{n+1}^2 = \sum_{i=0}^{k} \mathcal{A}_{2i} e_{n-i}^1 + \sum_{i=0}^{k} \mathcal{B}_{2i} e_{n-i}^2 + \sum_{i=0}^{k} \mathcal{C}_{2i} e_{n-i}^3, \tag{4.13}$$

$$e^3_{n+1} = \sum_{i=0}^{k} \mathcal{A}_{3i} e^1_{n-i} + \sum_{i=0}^{k} \mathcal{B}_{3i} e^2_{n-i} + \sum_{i=0}^{k} \mathcal{C}_{3i} e^3_{n-i}. \tag{4.14}$$

Then we get

$$\lim_{n\to\infty} \mathcal{A}_{1i} = \lim_{n\to\infty} \mathcal{C}_{1i} = \lim_{n\to\infty} \mathcal{A}_{2i} = \lim_{n\to\infty} \mathcal{B}_{2i} = \lim_{n\to\infty} \mathcal{B}_{3i} = \lim_{n\to\infty} \mathcal{C}_{3i} = 0, \tag{4.15}$$

for $i \in \{0, 1, \dots, k\}$,

$$\lim_{n\to\infty} \mathcal{B}_{10} = \frac{1}{\bar{y}}, \quad \lim_{n\to\infty} \mathcal{C}_{20} = \frac{1}{\bar{z}}, \quad \lim_{n\to\infty} \mathcal{A}_{30} = \frac{1}{\bar{x}}, \tag{4.16}$$

$$\lim_{n\to\infty} \mathcal{B}_{1j} = \lim_{n\to\infty} \mathcal{C}_{2j} = \lim_{n\to\infty} \mathcal{A}_{3j} = 0, \tag{4.17}$$

for $j \in \{1, 2, \dots, k-1\}$, and

$$\lim_{n\to\infty} \mathcal{B}_{1k} = -\frac{1}{\bar{y}}, \quad \lim_{n\to\infty} \mathcal{C}_{2k} = -\frac{1}{\bar{z}}, \quad \lim_{n\to\infty} \mathcal{A}_{3k} = -\frac{1}{\bar{x}}, \tag{4.18}$$

that is,

$$\begin{cases} \mathcal{B}_{10} = \frac{1}{\bar{y}} + a_n, & \mathcal{B}_{1k} = -\frac{1}{\bar{y}} + b_n, \\ \mathcal{C}_{20} = \frac{1}{\bar{z}} + \alpha_n, & \mathcal{C}_{2k} = -\frac{1}{\bar{z}} + \beta_n, \\ \mathcal{A}_{30} = \frac{1}{\bar{x}} + \gamma_n, & \mathcal{A}_{3k} = -\frac{1}{\bar{x}} + \delta_n, \end{cases} \tag{4.19}$$

where $a_n \to 0,\ b_n \to 0,\ \alpha_n \to 0,\ \beta_n \to 0,\ \gamma_n \to 0,\ \delta_n \to 0$, for $n \to \infty$. Then, we obtain the system

$$\mathcal{E}_{n+1} = (A + B_n)\mathcal{E}_n, \tag{4.20}$$

where $\mathcal{E}_n = (e^1_n, e^1_{n-1}, \dots, e^1_{n-k}, e^2_n, e^2_{n-1}, \dots, e^2_{n-k}, e^3_n, e^3_{n-1}, \dots, e^3_{n-k})^T$ and, the constant matrix A is of the form

$$A = \begin{pmatrix} 0 & 0\cdots0 & 0 & a_n & 0\cdots0 & b_n & 0 & 0\cdots0 & 0 \\ 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & \alpha_n\cdots0 & \beta_n \\ 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 \\ \gamma_n & 0\cdots0 & \delta_n & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 \\ & & & & & & & & \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 \end{pmatrix} \tag{4.21}$$

and

$$B_n = \begin{pmatrix} 0 & 0\cdots0 & 0 & \frac{1}{A+1} & 0\cdots0 & \frac{-1}{A+1} & 0 & 0\cdots0 & 0 \\ 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ \\ 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & \frac{1}{A+1} & 0\cdots0 & \frac{-1}{A+1} \\ 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 & 0 & 0\cdots0 & 0 \\ \frac{1}{A+1} & 0\cdots0 & \frac{-1}{A+1} & 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 1 & 0\cdots0 & 0 \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 1\cdots0 & 0 \\ \\ 0 & 0\cdots0 & 0 & 0 & 0\cdots0 & 0 & 0 & 0\cdots1 & 0 \end{pmatrix} \tag{4.22}$$

where $\|B_n\| \to 0$ as $n \to \infty$. The matrix A is equal to the $\mathcal{J}_{\mathcal{F}}$. So, by applying Theorem 14 to system (1.5), the result holds. □

5. Numerical example

To support our theoretical results, we will present some interesting numerical examples. These examples present different types of qualitative behavior of solutions to system (1.5).

Example 1. Consider the system (1.5) with the initial values $x_{-2} = 2.1$, $x_{-1} = 0.67$, $x_0 = 10.21$, $y_{-2} = 6.45$, $y_{-1} = 9.38$, $y_0 = 4.45$, $z_{-2} = 0.47$, $z_{-1} = 4.41$ and $z_0 = 0.67$. Further, we take the parameters $k = 2$ and $A = 1.5$, i.e.,

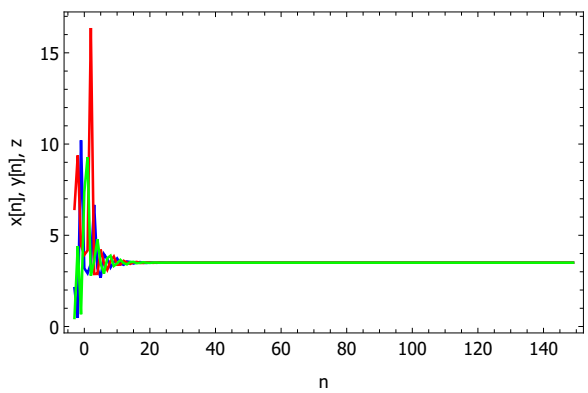


FIGURE 1. The plot of system (1.5) when $A > 1$ and $k = 2$.

This figure shows the global attractivity of the equilibrium point

$$(\bar{x}, \bar{y}, \bar{z}) = (2.5, 2.5, 2.5)$$

of system (1.5).

Example 2. Consider the system (1.5) with the initial values $x_{-3} = 3.7$, $x_{-2} = 0.1$, $x_{-1} = 5.47$, $x_0 = 1.17$, $y_{-3} = 0.45$, $y_{-2} = 1.1$, $y_{-1} = 4.38$, $y_0 = 0.87$, $z_{-3} = 1.8$, $z_{-2} = 0.67$, $z_{-1} = 0.41$ and $z_0 = 11.28$. Further, we take the parameters $k = 3$ and $A = 1$, i.e.,

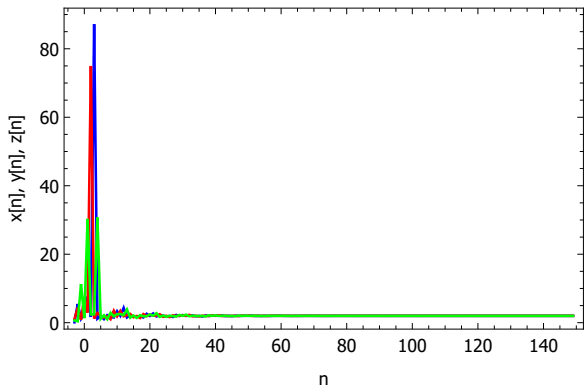


FIGURE 2. The plot of system (1.5) when $A = 1$ and $k = 3$.

Example 3. Consider the system (1.5) with the initial values $x_{-4} = 2.41$, $x_{-3} = 3.39$, $x_{-2} = 0.27$, $x_{-1} = 1.91$, $x_0 = 0.19$, $y_{-4} = 7.82$, $y_{-3} = 2.71$, $y_{-2} = 4.33$, $y_{-1} = 0.38$, $y_0 = 0.01$, $z_{-4} = 0.82$, $z_{-3} = 2.73$, $z_{-2} = 0.08$, $z_{-1} = 0.31$ and $z_0 = 5.28$. Further, we take the parameters $k = 4$ and $A = 0.19$, i.e.,

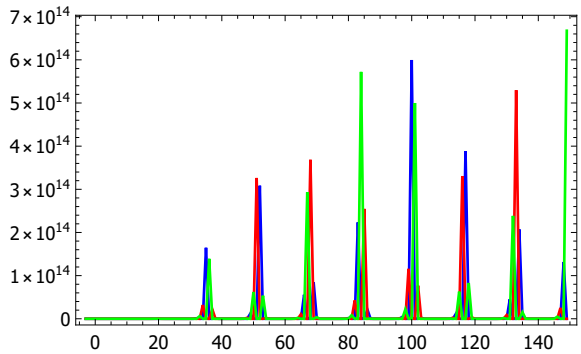


FIGURE 3. The plot of system (1.5) when $A < 1$ and $k = 4$.

This figure shows that the equilibrium point $(\bar{x},\bar{y},\bar{z}) = (1.19, 1.19, 1.19)$ of system (1.5) is not global asymptotic stable. Further, system (1.5) has unbounded solutions.

6. Conclusion

This study represents a contribution to the analysis of three-dimensional concrete nonlinear system of difference equations. This paper mainly discusses the dynamic properties of a class of higher-order system of difference equations by utilizing semi-cycle analysis, stability theory and rate of convergence. The main results are as follows.

i) From semi-cycle analysis of system (1.5), it is obtained that system (1.5) has no non-oscillatory negative solutions, no decreasing non-oscillatory solutions, no nontrivial periodic solutions of period k. It is also obtained that the solution of system (1.5) is either non-oscillatory solution or it oscillates about the equilibrium point of system (1.5), with semi-cycles having $k+1$ terms.
ii) When $A \geq 1$, the positive solution of system (1.5) is bounded and persists. Further system (1.5) has no positive nontrivial solution prime period-two in this case $A = 1$.
iii) When $A \geq 1$, system (1.5) has no non-oscillatory positive solutions.
iv) When $A > 1$, the unique equilibrium point of system (1.5) is globally asymptotically stable.

Acknowledgements

This study is a part of the first author's Ph.D. Thesis.

References

[1] R. M. Abu-Saris and R. DeVault, "Global stability of yn+ 1= a+ ynyn- k," *Applied Mathematics Letters*, vol. 16, no. 2, pp. 173–178, 2003, doi: 10.1016/S0893-9659(03)80028-9.
[2] S. Abualrub and M. Aloqeili, "Dynamics of the system of difference equations $x_{n+1} = a + \frac{y_{n-k}}{y_n}$, $y_{n+1} = b + \frac{x_{n-k}}{x_n}$," *Qualitative Theory of Dynamical Systems*, vol. 19, no. 69, pp. 1–19, 2020, doi: 10.1007/s12346-020-00408-y.
[3] S. Abualrub and M. Aloqeili, "Dynamics of positive solutions of a system of difference equations $x_{n+1} = a + \frac{y_n}{y_{n-k}}$, $y_{n+1} = a + \frac{x_n}{x_{n-k}}$," *Journal of Computational and Applied Mathematics*, vol. 392, no. 113489, pp. 1–15, 2021, doi: 10.1016/j.cam.2021.113489.
[4] M. Aloqeili, "On the difference equation $x_{n+1} = \alpha + \frac{x_n^p}{x_{n-1}^p}$," *Journal of Applied Mathematics and Computing*, vol. 25, no. 1–2, pp. 375–382, 2007.
[5] M. N. Bešo E, Kalabušić S and P. E, "Boundedness of solutions and stability of certain second-order difference equation with quadratic term," *Advances in Difference Equations*, vol. 2020, no. 1, pp. 1–22, 2020, doi: 10.1186/s13662-019-2490-9.
[6] E. Camouzis and G. Papaschinopoulos, "Global asymptotic behaviour of positive solutions on the system of rational difference equations $x_{n+1} = 1 + \frac{x_n}{y_{n-m}}$, $y_{n+1} = 1 + \frac{y_n}{x_{n-m}}$,," *Applied Mathematics Letters*, vol. 2004, no. 17, pp. 733–737, 2004, doi: 10.1016/S0893-9659(04)90113-9.

[7] I. Dekkar, N. Touafek, and Y. Yazlik, "Global stability of a third-order nonlinear system of difference equations with period-two coefficients," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, vol. 111, pp. 325–347, 2017, doi: 10.1007/s13398-016-0297-z.

[8] H. El-Owaidy, A. Ahmed, and M. Mousa, "On asymptotic behaviour of the difference equation $x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}$," *Applied mathematics and computation*, vol. 147, no. 1, pp. 163–167, 2004.

[9] M. Gümüş, "The global asymptotic stability of a system of difference equations," *Journal of Difference Equations and Applications*, vol. 24, no. 6, pp. 976–991, 2018, doi: 10.1080/10236198.2018.1443445.

[10] Y. Halim, A. Allam, and Z. Bengueraichi, "Dynamical behavior of a p-dimensional system of nonlinear difference equations," *Mathematica Slovaca*, vol. 71, no. 4, pp. 903–924, 2021, doi: 10.1515/ms-2021-0030.

[11] H. Hamioud, N. Touafek, I. Dekkar, and Y. Yazlik, "On a three dimensional nonautonomous system of difference equations," *Journal of Applied Mathematics and Computing*, vol. 68, pp. 3901–3936, 2022, doi: 10.1007/s12190-021-01693-9.

[12] A. Q. Khan *et al.*, "Global dynamics of a nonsymmetric system of difference equations," *Mathematical Problems in Engineering*, vol. 2022, 2022, doi: 10.1155/2022/4435613.

[13] A. Khelifa and Y. Halim, "Global behavior of p-dimensional difference equations system," *Electronic Research Archive*, vol. 29, no. 5, pp. 3121–3139, 2021, doi: 10.3934/era.2021029.

[14] İ. Okumuş and Y. Soykan, "Dynamical behavior of a system of three-dimensional nonlinear difference equations," *Advances in Difference Equations*, vol. 2018, pp. 1–15, 2018, doi: 10.1186/s13662-018-1667-y.

[15] S. Oudina, M. A. Kerker, and S. Abdelouahab, "Dynamics of a system of higher order difference equations with a period-two coefficient," *International Journal of Nonlinear Analysis and Applications*, vol. 13, no. 2, pp. 2043–2058, 2022, doi: 10.22075/IJNAA.2022.26716.3398.

[16] G. Papaschinopoulos and C. Schinas, "On a system of two nonlinear difference equations," *Journal of Mathematical Analysis and Applications*, vol. 219, no. 2, pp. 415–426, 1998.

[17] E. Taşdemir, "Global dynamics of a higher order difference equation with a quadratic term," *Journal of Applied Mathematics and Computing*, pp. 1–15, 2021, doi: 10.1007/s12190-021-01497-x.

[18] E. Taşdemir, "On the global asymptotic stability of a system of difference equations with quadratic terms," *Journal of Applied Mathematics and Computing*, vol. 66, pp. 423–437, 2021, doi: 10.1007/s12190-020-01442-4.

[19] D. T. Tollu and İ. Yalçinkaya, "Global behavior of a three-dimensional system of difference equations of order three," *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, vol. 68, no. 1, pp. 1–16, 2019, doi: 10.31801/cfsuasmas.443530.

[20] N. Touafek, "On a general system of difference equations defined by homogeneous functions," *Mathematica Slovaca*, vol. 71, no. 3, pp. 697–720, 2021, doi: 10.1515/ms-2021-0014.

[21] D. Zhang, W. Ji, L. Wang, and X. Li, "On the symmetrical system of rational difference equation $x_{n+1} = a + \frac{y_{n-k}}{y_n}$, $y_{n+1} = a + \frac{x_{n-k}}{x_n}$," *Applied Mathematics*, vol. 4, pp. 834–837, 2013.

[22] Q. Zhang, L. Yang, and J. Liu, "On the recursive system $x_{n+1} = a + \frac{x_{n-m}}{y_n}, y_{n+1} = b + \frac{y_{n-m}}{x_n}$," *Acta Mathematica Universitatis Comenianae*, vol. 82, no. 2, pp. 201–208, 2017.

[23] Q. Zhang, W. Zhang, Y. Shao, and J. Liu, "On the system of high order rational difference equations," *International Scholarly Research Notices*, vol. 2014, 2014, doi: 10.1155/2014/760502.

[24] Y. Zhang, X. Yang, D. J. Evans, and C. Zhu, "On the nonlinear difference equation system $x_{n+1} = a + \frac{x_{n-m}}{y_n}, y_{n+1} = a + \frac{x_{n-m}}{y_n}$," *Computers & Mathematics with Applications*, vol. 53, no. 10, pp. 1561–1566, 2007, doi: 10.1016/j.camwa.2006.04.030.

Authors' addresses

Murad Khan Hassani
Nevsehir Hacı Bektas Veli University, Faculty of Science and Art, Department of Mathematics, 50300, Nevşehir, Turkey
E-mail address: muradkhanhassani@gmail.com

Yasin Yazlik
Nevsehir Hacı Bektas Veli University, Faculty of Science and Art, Department of Mathematics, 50300, Nevşehir, Turkey
E-mail address: yyazlik@nevsehir.edu.tr

Nouressadat Touafek
(**Corresponding author**) Mohamed Seddik Ben Yahia University, LMAM Laboratory, Department of Mathematics, 28212, Jijel, Algeria
E-mail address: ntouafek@gmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 747–758

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4012

FIXED POINT THEOREMS OF ASYMPTOTICALLY REGULAR MAPS

NGUYEN H. HOC AND RAVINDRA K. BISHT

Received 30 January, 2023

Abstract. We prove some fixed point theorems for asymptotically regular self-mappings, not necessarily orbitally continuous or k-continuous, satisfying weaker Proinov contraction. Our results extend several recent results in the literature and provide more answers to an open question raised by Rhoades. Some nontrivial examples are also given to illustrate our results.

2010 *Mathematics Subject Classification:* 47H09; 54E50; 47H10

Keywords: fixed point, φ-contraction, Proinov contraction, weakly orbital continuity, orbital lower semi-continuity

1. Introduction and preliminaries

In 2006, Proinov [25] proved an interesting fixed point theorem which includes a wide class of well-known fixed point theorems in the existing literature (see, [11, 14–16, 18, 24]). Before stating the result by Proinov, we recall here some related definitions and notations. We also refer the reader to [1] for various related terminologies in fixed point iterative methods in metric spaces.

Definition 1. [13] Let (X,d) be a metric space and $T : X \to X$ be a mapping. A Picard iterate of T is a sequence $\{x_n\}$ in X defined by $x_n = T^n x$ for some $x \in X$.

Definition 2. [13, 25] Let T be a self-mapping on a metric space (X,d). A fixed point z of T is said to be contractive if all of the Picard iterates of T converge to this fixed point.

Note that if z is a contractive fixed point of $T : X \to X$, then z is the unique fixed point of T [13].

Definition 3. [6] A self-mapping T on a metric space (X,d) is called asymptotically regular at $x \in X$, if $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$. If T is asymptotically regular at all $x \in X$, then T is said to be asymptotically regular.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

For some fixed point results concerning asymptotically regular mappings, we refer the reader to, e.g., [8, 9, 12, 17, 25] and references therein.

Let Φ [25] denote the class of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying: for any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon < t < \delta$ implies $\varphi(t) \leq \varepsilon$. Proinov's fixed point theorem is stated as follows.

Theorem 1. *[25] Let (X,d) be a complete metric space. Let T be a continuous and asymptotically regular self-mapping on X such that:*

(i) *There exists $\varphi \in \Phi$ such that $d(Tx,Ty) \leq \varphi(D(x,y))$ for all $x,y \in X$, where $D(x,y) = d(x,y) + \gamma[d(x,Tx) + d(y,Ty)]$, $\gamma \geq 0$;*
(ii) *$d(Tx,Ty) < D(x,y)$ for all $x,y \in X$ with $x \neq y$.*

Then T has a contractive fixed point.

Moreover, if $D(x,y) = d(x,y) + d(x,Tx) + d(y,Ty)$ and φ is continuous and satisfies $\varphi(t) < t$ for all $t > 0$, then the continuity of T can be dropped.

The first conclusion of Theorem 1 still holds true if the condition (i) is replaced by another condition. More precisely, as shown in [25], the first part of Theorem 1 is equivalent to the following.

Theorem 2. *[25] Let (X,d) be a complete metric space. Let T be a continuous and asymptotically regular self-mapping on X such that:*

(i) *For any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon < D(x,y) < \delta$ implies $d(Tx,Ty) \leq \varepsilon$, where $D(x,y) = d(x,y) + \gamma[d(x,Tx) + d(y,Ty)]$, $\gamma \geq 0$;*
(ii) *$d(Tx,Ty) < D(x,y)$ for all $x,y \in X$ with $x \neq y$.*

Then T has a contractive fixed point.

We see that, except the cases $0 \leq \gamma < 1$, the above result demands the continuity of the considered mapping T.

Recently, the study of contractive conditions for mappings which are not continuous at fixed points is an active research direction. The main idea is to give new notions which are weaker than continuity of the mapping such that the existence of fixed point for mappings satisfying a contractive type condition is ensured. Some results in this direction can be found in ([2–5, 18–20, 22, 23] and references therein). These results also provide answers to an open question raised by Rhoades in [26].

Let (X,d) be a metric space and $T : X \to X$ be a mapping. The orbit of T at a point $x \in X$ is the set $O(T,x) = \{x, Tx, T^2x, \cdots, T^nx, \cdots\}$. We now recall some weaker notions of continuity.

Definition 4. [7] Let (X,d) be a metric space. A mapping $T : X \to X$ is called orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(T,x)$ for some $x \in X$, $x_n \to z$ implies $Tx_n \to Tz$ as $n \to \infty$.

Definition 5. [18] Let (X,d) be a metric space. A mapping $T : X \to X$ is called k-continuous, $k = 1,2,\cdots$, at a point $z \in X$ if $T^kx_n \to Tz$ whenever $\{x_n\}$ is a sequence in X such that $T^{k-1}x_n \to z$ as $n \to \infty$.

Definition 6. [19] Let (X,d) be a metric space. A mapping $T : X \to X$ is called weakly orbitally continuous if the set $\{y \in X : \lim_{i\to\infty} T^{m_i} y = u \Longrightarrow \lim_{i\to\infty} TT^{m_i} y = Tu\}$ is non-empty whenever the set $\{x \in X : \lim_{i\to\infty} T^{m_i} x = u\}$ is non-empty.

Remark 1. The following observations are evident (see, e.g., [19, 20]).

(i) Continuity of T^k and k-continuity of T are independent conditions when $k > 1$.

(ii) 1- continuity of T is equivalent to the continuity of T and

$$\text{continuity} \implies 2-\text{continuity} \implies 3-\text{continuity} \implies \ldots,$$

but not conversely.

(iii) Continuity of T implies orbital continuity of T, but not conversely.

(iv) Orbital continuity of T implies weak orbital continuity of T, but the converse need not be true.

(v) k-continuous mapping of T is orbitally continuous but the converse need not be true.

In [2, 4], replacing the continuity of the mapping T in Theorem 1 by k-continuity of T and orbital continuity of T, respectively, the authors proved the following improvement of Theorem 1.

Theorem 3. *[2, 4] Let (X,d) be a complete metric space. Let T be an asymptotically regular self-mapping on X satisfying (i) and (ii) of Theorem 1. Suppose T is k-continuous for some $k \in \mathbb{N}$ or T is orbitally continuous. Then T has a contractive fixed point.*

Hicks and Rhoades [10] introduced the following notion of orbital lower semi-continuity.

Definition 7. [10] Let (X,d) be a metric space and $T : X \to X$. A mapping $G : X \to \mathbb{R}$ is said to be T-orbitally lower semi-continuous at a point $z \in X$ if $\{x_n\}$ is a sequence in $O(T,x)$ for some $x \in X$, $\lim_{n\to\infty} x_n = z$ implies $G(z) \le \liminf_{n\to\infty} G(x_n)$.

In [17], Nguyen showed that the T-orbital lower semi-continuity of $x \to d(x,Tx)$ is weaker than both orbital continuity and k-continuity of T. The following example illustrates that fact.

Example 1. Let $X = \{0\} \cup \{1/n : n = 1,2,\ldots\} \cup \{1 + 1/n : n = 1,2,\ldots\}$ with the metric $d(x,y) = |x-y|$ for all $x,y \in X$. We define a mapping $T : X \to X$ as follows:

$$Tx = \begin{cases} 0 & \text{if} \quad x = 0 \\ 2 & \text{if} \quad x = 1 \\ 1 + \frac{1}{n+1} & \text{if} \quad x = \frac{1}{n}, \qquad n = 2,\ldots \\ \frac{1}{n} & \text{if} \quad x = 1 + \frac{1}{n}, \quad n = 1,2,\ldots \end{cases}$$

We can check that T is neither orbitally continuous nor k-continuous for any $k \geq 1$. Indeed, one has

$$O(T,1) = \left\{1, 1+\frac{1}{2}, \frac{1}{2}, 1+\frac{1}{3}, \frac{1}{3}, \ldots, 1+\frac{1}{n}, \frac{1}{n}, \ldots\right\}.$$

Let $\{x_n\}$ be a sequence in $O(T,1)$ with $x_n = \frac{1}{n}$ for $n \geq 1$. Then, $x_n \to 0$ as $n \to \infty$. However, $Tx_n = 1 + \frac{1}{n+1} \to 1 \neq T0$. Hence, T is not orbitally continuous. Let $\{u_n\}$ be in X with $u_n = \frac{1}{n}$ for $n \geq 1$. For $i = 1, 2, \cdots$, we have $T^{2i-1}u_n = 1 + \frac{1}{n+i}$, $T^{2i}u_n = \frac{1}{n+i}$ and $T^{2i+1}u_n = 1 + \frac{1}{n+i+1}$. Since $\lim_{n\to\infty} T^{2i-1}u_n = 1$ and $\lim_{n\to\infty} T^{2i}u_n = 0 \neq 2 = T1$, T is not $2i$-continuous. Since $\lim_{n\to\infty} T^{2i+1}u_n = 1 \neq T0$, T is not $(2i+1)$-continuous. Therefore, T is not k-continuous for any $k \geq 1$.

Moreover,

$$d(x,Tx) = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x = 1 \\ 1 - \frac{1}{n(n+1)} & \text{if} \quad x = \frac{1}{n}, \qquad n = 1,2,\ldots \\ 1 & \text{if} \quad x = 1 + \frac{1}{n}, \quad n = 1,2,\ldots \end{cases}$$

and one can check that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous in X.

The following examples show that the weak orbital continuity of a mapping T on a metric space (X,d) and the T-orbital lower semi-continuity of the mapping $x \to d(x,Tx)$ are independent.

Example 2. Let (X,d) as in Example 1. We define a mapping $T : X \to X$ as follows.

$$Tx = \begin{cases} \frac{1}{2} & \text{if} \quad x = 0 \\ 1 + \frac{1}{n+1} & \text{if} \quad x = \frac{1}{n}, \qquad n = 1,2,\cdots \\ \frac{1}{n} & \text{if} \quad x = 1 + \frac{1}{n}, \quad n = 1,2,\cdots \end{cases}$$

Let $y \in X$. If $\{x_n\} \subset O(T,y)$ converges, then it converges either to 0 or 1. If $\{x_n\}$ converges to 0, then $x_n = \frac{1}{n}$ for large n. In this case, $\lim_{n\to\infty} Tx_n = 1 \neq \frac{1}{2} = T1$. If $\{x_n\}$ converges to 1, then $x_n = 1 + \frac{1}{n}$ for n large. In this case, $\lim_{n\to\infty} Tx_n = 0 \neq \frac{3}{2} = T1$. Thus, T is not weakly orbitally continuous

Moreover,

$$d(x,Tx) = \begin{cases} \frac{1}{2} & \text{if} \quad x = 0 \\ 1 - \frac{1}{n(n+1)} & \text{if} \quad x = \frac{1}{n}, \qquad n = 1,2,\ldots \\ 1 & \text{if} \quad x = 1 + \frac{1}{n}, \quad n = 1,2,\ldots \end{cases}$$

and we can check that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous in X.

Example 3. Let $X = [0,1]$ with the usual metric. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} 1 & \text{if} \quad x = 0, \\ \frac{x}{2} & \text{if} \quad 0 < x < 1, \\ 1 & \text{if} \quad x = 1. \end{cases}$$

It is easy to see that T is weakly orbitally continuous on X. We have

$$d(x,Tx) = \begin{cases} 1 & \text{if} \quad x=0, \\ \frac{x}{2} & \text{if} \quad 0<x<1, \\ 0 & \text{if} \quad x=1. \end{cases}$$

Let $\{x_n\}$ in $O(T,x)$ with $x \in X$ be such that $\{x_n\}$ converges to 0. Then $x \in (0,1)$ and in this case we may assume that $x_n = \frac{x}{2^n}$ for all n. We have

$$\liminf_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{x}{2^{n+1}} = 0 < 1 = f(0).$$

Thus, mapping $x \to f(x) = d(x,Tx)$ is not T-orbitally lower semi-continuous at $z=0$.

In this paper, we show that contractive condition given by Proinov still guarantees the existence of fixed points under weaker forms of continuity. We not only relax the continuity requirement in the results proved by Proinov and others, but also relax the condition (i) in Theorems 1 and 2. Our results also provide more answers to an open question posed in [26]. Some examples are also given to support our results.

2. Fixed point results

Our first result is the following theorem:

Theorem 4. *Let (X,d) be a complete metric space and $T : X \to X$ be an asymptotically regular mapping such that:*

(i) *There exists $\varphi \in \Phi$ such that $d(T^{i+1}x, T^{j+1}x) \le \varphi(D(T^i x, T^j x))$ for all $x \in X$, $i,j \in \mathbb{N} \cup \{0\}$, where $\gamma \ge 0$ and $D(x,y) = d(x,y) + \gamma[d(x,Tx) + d(y,Ty)]$;*

(ii) *$d(Tx,Ty) < D(x,y)$ for all $x,y \in X$ with $x \ne y$.*

Then T has a contractive fixed point $z \in X$ provided that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous or T is weakly orbitally continuous on X.

Proof. Let $x \in X$ and define the sequence $\{x_n\}$ as follows:

$$x_n = T^n x, \quad n = 0,1,2\ldots$$

Since T is asymptotically regular, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0. \tag{2.1}$$

We will show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ and $\varepsilon < \delta < 2\varepsilon$. By (2.1), there exists $N > 0$ such that

$$d(T^n x, T^{n+1}x) = d(x_n, x_{n+1}) < \frac{\delta - \varepsilon}{1+2\gamma}, \quad \text{for all } n \ge N. \tag{2.2}$$

We will prove that

$$d(T^n x, T^m x) < \frac{\delta + 2\gamma\varepsilon}{1+2\gamma}, \quad \text{for all } m \ge n \ge N. \tag{2.3}$$

Fix $n \geq N$. We prove (2.3) by induction on m. It is obvious that (2.2) holds for $m = n$. Since

$$\frac{\delta - \varepsilon}{1+2\gamma} < \frac{\delta + 2\gamma\varepsilon}{1+2\gamma},$$

(2.3) also holds for $m = n+1$. Assume now that (2.3) holds for an integer $m \geq n+1$, we shall show that it holds for $m+1$.

It is enough to show that

$$d(T^{n+1}x, T^{m+1}x) \leq \varepsilon. \tag{2.4}$$

Indeed, if (2.4) holds, then by the triangle inequality and (2.2),

$$\begin{aligned} d(T^n x, T^{m+1}x) &\leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{m+1}x) \\ &< \frac{\delta - \varepsilon}{1+2\gamma} + \varepsilon = \frac{\delta + 2\gamma\varepsilon}{1+2\gamma}, \end{aligned}$$

i.e., (2.3) holds for $m+1$.

We consider two cases.

Case 1. $D(T^n x, T^m x) \leq \varepsilon$. By (ii), it follows that

$$d(T^{n+1}x, T^{m+1}x) \leq D(T^n x, T^m x) \leq \varepsilon,$$

i.e., (2.4) holds.

Case 2. $D(T^n x, T^m x) > \varepsilon$. Using (2.2) and (2.3), we have

$$\begin{aligned} D(T^n x, T^m x) &= d(T^n x, T^m x) + \gamma\left[d(T^n x, T^{n+1}x) + d(T^m x, T^{m+1}x)\right] \\ &< \frac{\delta + 2\gamma\varepsilon}{1+2\gamma} + 2\gamma.\frac{\delta - \varepsilon}{1+2\gamma} = \delta. \end{aligned}$$

Hence, $\varepsilon < D(T^n x, T^m x) < \delta$. By the property of φ, it holds $\varphi(D(T^n x, T^m x)) \leq \varepsilon$. Thus, by (i), one has

$$d(T^{n+1}x, T^{m+1}x) \leq \varphi(D(T^n x, T^m x)) \leq \varepsilon,$$

i.e., (2.4) holds.

We have proved (2.3). Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x = z.$$

Assume that $x \to d(x, Tx)$ is T-orbitally lower semi-continuous. Since $\{x_n\} \subset O(T, x)$ and $x_n \to z$ satisfying $d(x_n, Tx_n) = d(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$, by the T-orbital lower semi-continuity of $x \to d(x, Tx)$, we have $d(z, Tz) \leq \liminf_{n\to\infty} d(x_n, Tx_n) = 0$. Thus, z is a fixed point of T.

Suppose now that T is weakly orbitally continuous. Since $T^n x_0 \to z$ for each x_0, by virtue of weak orbital continuity of T we get, $T^n y_0 \to z$ and $T^{n+1}y_0 \to Tz$ for some $y_0 \in X$. This implies that $z = Tz$ since $T^{n+1}y_0 \to z$. Therefore, z is a fixed point of T. Uniqueness of the fixed point z follows from (ii). □

Remark 2. Theorem 4 improves Theorem 1 in the following ways. Firstly, the continuity condition on the considered mapping T is replaced by a weaker condition: the weakly orbital continuity of T or T-orbitally lower semi-continuity of $x \to d(x,Tx)$. Secondly, in Theorem 1, it is required that the inequality $d(Tx,Ty) \leq \varphi(D(x,y))$ holds for all $x,y \in X$ whereas, in Theorem 4, we only ask the inequality holds for x,y in the same orbit of T. Later, we will provide examples (see, Example1 and Example 3) which can apply to our result but cannot apply to the result by Proinov.

Equivalent version of Theorem 4 is the following result:

Theorem 5. *Let (X,d) be a complete metric space and $T : X \to X$ be an asymptotically regular mapping such that:*

(i) *For any $\varepsilon > 0$, there exists $\delta > \varepsilon$ such that $\varepsilon < D(T^i x, T^j x) < \delta$ implies $d(T^{i+1}x, T^{j+1}x) \leq \varepsilon$ for all $x \in X$, $i,j \in \mathbb{N} \cup \{0\}$;*
(ii) *$d(Tx,Ty) < D(x,y)$ for all $x,y \in X$ with $x \neq y$.*

Then T has a contractive fixed point $z \in X$ provided that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous or T is weakly orbitally continuous on X.

Inspired by [27], we have the following result:

Proposition 1. *Let $T : X \to X$ be a mapping. Consider the following statements:*

(a) *T has a contractive fixed point;*
(b) *$d(Tx,Ty) < D(x,y)$ for all $x,y \in X$, $x \neq y$;*
(c) *For any $\varepsilon > 0$, there exists $\delta > \varepsilon$ such that $\varepsilon < D(T^i x, T^j x) < \delta$ implies $d(T^{i+1}x, T^{j+1}x) \leq \varepsilon$ for all $x \in X, i,j \in \mathbb{N} \cup \{0\}$.*

If (a) and (b) hold, then (c) holds.

Proof. Let $z \in X$ be a contractive fixed point of T. Then, $\{T^n x\}$ converges to z for each $x \in X$. Assume that (a) and (b) hold but (c) does not hold. Then, there exist $x \in X$, $\varepsilon > 0$ and sequences $\{u(n)\}$ and $\{w(n)\}$ in $\mathbb{N} \cup \{0\}$ such that $u(n) < w(n)$ and

$$\varepsilon < D\left(x_{u(n)}, x_{w(n)}\right) < \varepsilon + \frac{1}{n} \quad \text{and} \quad \varepsilon < d\left(x_{u(n)+1}, x_{w(n)+1}\right) \tag{2.5}$$

for any $n \in \mathbb{N}$, where $x_k = T^k x$ for each $k \in \mathbb{N} \cup \{0\}$.

Without loss of generality, we may assume that either $\lim_{n\to\infty} u(n) = \infty$ holds or there exists $p \in \mathbb{N} \cup \{0\}$ such that $u(n) = p$ for any $n \in \mathbb{N}$. Similarly, we can also assume that either $\lim_{n\to\infty} w(n) = \infty$ or there exists $q \in \mathbb{N} \cup \{0\}$ such that $w(n) = q$ for any $n \in \mathbb{N}$. From (b) and (2.5), we have

$$\varepsilon < d\left(x_{u(n)+1}, x_{w(n)+1}\right) < D\left(x_{u(n)}, x_{w(n)}\right) < \varepsilon + \frac{1}{n}.$$

Thus,

$$0 < \varepsilon = \lim_{n\to\infty} d\left(x_{u(n)+1}, x_{w(n)+1}\right) = \lim_{n\to\infty} D\left(x_{u(n)}, x_{w(n)}\right). \tag{2.6}$$

If $\lim_{n\to\infty} u(n) = \infty$, then $\lim_{n\to\infty} w(n) = \infty$. By (2.6) and the fact that $\{T^n x\}$ converges to z, we have $\varepsilon = d(z,z) = 0$ which is a contradiction. We have two possible cases:

Case 1: $u(n) = p$ and $w(n) = q$ for all n. In this case, using (2.6) we have

$$d(Tx_p, Tx_q) = d(x_{p+1}, x_{q+1}) = \varepsilon = D(x_p, x_q)$$

which contradicts (b).

Case 2: $u(n) = p$ for all n and $\lim\limits_{n\to\infty} w(n) = \infty$. Using (2.6), we obtain

$$d(Tx_p, Tz) = d(x_{p+1}, z) = D(x_p, z) = \varepsilon$$

which contradicts (b). Therefore, (c) holds. □

We now provide some examples to illustrate our results.

Example 4. Consider the set ℓ^∞ of all bounded real sequences with the metric

$$d(x,y) = \sup\{|x_n - y_n| : n = 1,2,3,\cdots\} \quad \text{for all } x = (x_n), y = (y_n) \in \ell^\infty.$$

For each $n \in \mathbb{N}$, denote by $e(n)$ the sequence with all elements equal to 0 except the nth element equal to 1, i.e.,

$$e(n) = (0,\cdots,0,1,0,\cdots).$$

Let $a > 1$ and κ be a positive integer. Consider the set

$$X = \left\{a^{1-m/n} e(n) : m,n = 1,2,,3,\cdots\right\} \cup \{\mathbf{0}\},$$

where $\mathbf{0}$ is the zero sequence in ℓ^∞. We define a mapping $T : X \to X$ by

$$T(\mathbf{0}) = \mathbf{0}, \quad \text{and} \quad T\left(a^{1-m/n} e(n)\right) = a^{1-(m+\kappa)/n} e(n) \text{ for all } m,n = 1,2,3,\cdots.$$

Then,

(a) (X,d) is a complete metric space.
(b) Condition (i) of Theorem 1 does not hold and Theorems 1, 2 and 3 are not applicable to T.
(c) Conditions (a) and (b) of Proposition 1 hold and therefore condition (i) of Theorem 5 holds.
(d) Theorems 5 and 4 are applicable to T.

Proof. (a) It is evident that (ℓ^∞, d) is a complete metric space. One can easily check that X is a closed subset of ℓ^∞ and thus (X,d) is a complete metric space.

(b) Assume to the contrary that condition (i) of Theorem 1 holds. That is, there exist $\varphi \in \Phi$ and $\gamma > 0$ such that $d(Tx,Ty) \le \varphi(D(x,y))$ for all $x,y \in X$. Let $x = \mathbf{0}$ and $y = a^{1-(n-\kappa-1)/n}$ with $n > \kappa + 1$. We have

$$D_n := D(x,y) = a^{(\kappa+1)/n} + \gamma(a^{(\kappa+1)/n} - a^{1/n}) \quad \text{and} \quad d(Tx,Ty) = a^{1/n}.$$

Let $\varepsilon = 1$. Since $D_n > 1$ for all $n > \kappa + 1$ and $\lim_{n\to\infty} D_n = 1$, there exists $\delta > 1 = \varepsilon$ such that

$$\varepsilon < D_n = D(x,y) < \delta$$

for all n sufficiently large. By the property of Φ, one has

$$a^{1/n} = d(Tx,Ty) \le \varphi(D(x,y)) \le \varepsilon = 1.$$

This is a contradiction. Thus, condition (i) of Theorem 1 does not hold. Therefore, we cannot apply Theorems 1, 2 and 3 to the mapping T.

(c) It is evident that T has a unique fixed point $z = \mathbf{0}$. Moreover, for $x \in X$ and $i \in \mathbb{N} \cup \{0\}$, we have

$$T^i x = \begin{cases} \mathbf{0} & \text{if} \quad x = \mathbf{0} \\ a^{1-\frac{m+i\kappa}{n}} e(n) & \text{if} \quad x = a^{1-\frac{m}{n}} e(n). \end{cases}$$

It is obvious that $\{T^i x\}$ converges to $\mathbf{0}$ for each $x \in X$. Hence, condition (a) of Proposition 1 holds. One can also check, for $\gamma = 2$, that $d(Tx,Ty) < D(x,y)$ for all $x,y \in X$ with $x \neq y$, i.e., condition (b) of Proposition 1 holds. By Proposition 1, condition (i) of Theorem 5 holds.

(d) It is easy to check that T is asymptotically regular and continuous on X. As in (c), conditions (i) and (ii) of Theorem 5 hold. Thus, all the conditions of Theorem 5 hold and we can apply this theorem to T. □

Example 5. Let (X,d) and T be as in Example 3 and let $\gamma = 2$. One can easily check that

(a) T is asymptotically regular on X,
(b) $d(T^{i+1}x, T^{j+1}x) \le 1/2 D(T^i x, T^j x)$ for all $x \in X$ and $i, j \in \mathbb{N} \cup \{0\}$,
(c) $d(Tx,Ty) < D(x,y)$ for all $x,y \in X$, $x \neq y$.

Since (X,d) is complete and T is weakly orbitally continuous on X, we can apply Theorem 4 to T. In fact, T has a unique fixed point $z = 1$. Since T is neither $k-$continuous nor orbitally continuous, hence we cannot apply Theorems 1 and 3 to this example. Note that T is not continuous at the fixed point $z = 1$.

The following corollaries are easy consequences of Theorem 4.

Corollary 1. *Let (X,d) be a complete metric space and $T : X \to X$ be an asymptotically regular mapping such that:*

(i) *There exists $\varphi \in \Phi$ such that $d(T^{i+1}x, T^{j+1}x) \le \varphi(m(T^i x, T^j x))$ for all $x \in X$, $i, j \in \mathbb{N} \cup \{0\}$, where*

$$m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\};$$

(ii) *$d(Tx,Ty) < m(x,y)$ for all $x,y \in X$ with $x \neq y$.*

Then T has a contractive fixed point $z \in X$ provided that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous or T is weakly orbitally continuous on X.

Corollary 2. *Let (X,d) be a complete metric space and $T : X \to X$ be an asymptotically regular mapping such that:*

(i) *For any $\varepsilon > 0$, there exists $\delta > \varepsilon$ such that $\varepsilon < m(T^i x, T^j x) < \delta$ implies $d(T^{i+1}x, T^{j+1}x) \leq \varepsilon$ for all $x \in X$, $i, j \in \mathbb{N} \cup \{0\}$;*

(ii) *$d(Tx,Ty) < m(x,y)$ for all $x,y \in X$ with $x \neq y$.*

Then T has a contractive fixed point $z \in X$ provided that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous or T is weakly orbitally continuous on X.

Existence of a fixed point of T is still guaranteed if condition (ii) in Theorem 4 holds for all $x,y \in O(T,x_0)$ and $x \neq y$ for some $x_0 \in X$. In this case, we only require T to be asymptotically regular at x_0 and condition (i) of Theorem 4 can also be relaxed. However, in this case, T may have more than one fixed point. More precisely, we have the following result whose proof follows from the proof of Theorem 4.

Theorem 6. *Let (X,d) be a complete metric space and $x_0 \in X$. Assume that $T : X \to X$ is asymptotically regular at $x_0 \in X$ such that:*

(i) *There exists $\varphi \in \Phi$ such that $d(T^{i+1}x, T^{j+1}x) \leq \varphi(D(T^i x, T^j x))$ for all $x \in O(T,x_0)$, $i, j \in \mathbb{N} \cup \{0\}$;*

(ii) *$d(Tx,Ty) < D(x,y)$ for all $x,y \in O(T,x_0)$, with $x \neq y$.*

Then T has a fixed point $z \in X$ and $T^n x$ converges to z for each $x \in O(T,x_0)$.

Example 6. Let (X,d) and T be as in Example 1. Then, T is asymptotically regular at $x_0 = 0$. It is obvious that all conditions of Theorem 6 are satisfied and T has a fixed point $z = 0$.

Remark 3. Our results generalize and subsume various results given in [2–6, 11, 13, 15, 16, 19–25]. In addition to it, we provide more answers to the open problem regarding the existence of contractive mapping which has fixed point but is discontinuous at the fixed point [26].

Acknowledgment

The authors express their gratitude to the esteemed reviewer for providing valuable suggestions on the manuscript.

REFERENCES

[1] V. Berinde, A. Petrusel, and I. A. Rus, " Remarks on the terminology of the mappings in fixed point iterative methods in metric spaces," *Fixed Point Theory*, vol. 24, no. 2, pp. 525–540, 2023, doi: 10.24193/fpt-ro.2023.2.05.

[2] R. K. Bisht, "A remark on convergence theory for iterative processes of Proinov contraction," *Commun. Korean Math. Soc.*, vol. 34, no. 4, pp. 1157–1162, 2019, doi: 10.4134/CKMS.c180382.

[3] R. K. Bisht and R. P. Pant, "A remark on discontinuity at fixed point," *J. Math. Anal. Appl.*, vol. 445, no. 2, pp. 1239–1242, 2017, doi: 10.1016/j.jmaa.2016.02.053.

[4] R. K. Bisht, R. P. Pant, and V. Rakočević, "Proinov contractions and discontinuity at fixed point," *Miskolc Math. Notes*, vol. 20, no. 1, pp. 131–137, 2019, doi: 10.18514/MMN.2019.2277.

[5] R. K. Bisht and V. Rakočević, "Generalized Meir-Keeler type contractions and discontinuity at fixed point," *Fixed Point Theory*, vol. 19, no. 1, pp. 57–64, 2018, doi: 10.24193/fpt-ro.2018.1.06.

[6] F. E. Browder and W. V. Petryshyn, "The solution by iteration of nonlinear functional equations in Banach spaces," *Bull. Am. Math. Soc.*, vol. 72, pp. 571–575, 1966, doi: 10.1090/S0002-9904-1966-11544-6.

[7] L. B. Ćirić, "On contraction type mappings," *Math. Balk.*, vol. 1, pp. 52–57, 1971.

[8] P. Debnath and M. de La Sen, "Contractive inequalities for some asymptotically regular set-valued mappings and their fixed points," *Symmetry*, vol. 12, no. 3, 2020, doi: 10.3390/sym12030411.

[9] P. Debnath and M. de La Sen, "On some new contractive conditions for asymptotically regular set-valued mappings," *International Journal of Analysis and Application*, vol. 18, no. 4, pp. 586–593, 2020.

[10] T. L. Hicks and B. E. Rhoades, "A Banach type fixed point theorem," *Math. Japon.*, vol. 24, pp. 327–330, 1979.

[11] J. Jachymski, "Equivalent conditions and the Meir-Keeler type theorems," *J. Math. Anal. Appl.*, vol. 194, no. 1, pp. 293–303, 1995, doi: 10.1006/jmaa.1995.1299.

[12] A. Jeribi, N. Kaddachi, and Z. Laouar, "Fixed point theorems for weakly asymptotically regular mappings in Banach spaces with an application," *Numer. Funct. Anal. Optim.*, vol. 43, no. 1, pp. 68–87, 2022, doi: 10.1080/01630563.2021.2001820.

[13] S. Leader, "Fixed points for operators on metric spaces with conditional uniform equivalence of orbits," *J. Math. Anal. Appl.*, vol. 61, pp. 466–474, 1977, doi: 10.1016/0022-247X(77)90131-7.

[14] T.-C. Lim, "On characterizations of Meir-Keeler contractive maps," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 46, no. 1, pp. 113–120, 2001, doi: 10.1016/S0362-546X(99)00448-4.

[15] J. Matkowski, "Integrable solutions of functional equations," *Diss. Math.*, vol. 127, 1975.

[16] A. Meir and E. Keeler, "A theorem on contraction mappings," *J. Math. Anal. Appl.*, vol. 28, pp. 326–329, 1969, doi: 10.1016/0022-247X(69)90031-6.

[17] L. V. Nguyen, "On fixed points of asymptotically regular mappings," *Rend. Circ. Mat. Palermo (2)*, vol. 70, no. 2, pp. 709–719, 2021, doi: 10.1007/s12215-020-00527-0.

[18] A. Pant and R. P. Pant, "Fixed points and continuity of contractive maps," *Filomat*, vol. 31, no. 11, pp. 3501–3506, 2017, doi: 10.2298/FIL1711501P.

[19] A. Pant, R. P. Pant, and M. C. Joshi, "Caristi type and Meir-Keeler type fixed point theorems," *Filomat*, vol. 33, no. 12, pp. 3711–3721, 2019, doi: 10.2298/FIL1912711P.

[20] A. Pant, R. P. Pant, V. Rakočević, and R. K. Bisht, "Generalized Meir-Keeler type contractions and discontinuity at fixed point. II," *Math. Slovaca*, vol. 69, no. 6, pp. 1501–1507, 2019, doi: 10.1515/ms-2017-0325.

[21] A. Pant, R. P. Pant, and W. Sintunavarat, "Analytical Meir-Keeler type contraction mappings and equivalent characterizations," *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, vol. 115, no. 1, p. 15, 2021, id/No 37, doi: 10.1007/s13398-020-00939-8.

[22] R. P. Pant, "Discontinuity and fixed points," *J. Math. Anal. Appl.*, vol. 240, no. 1, pp. 284–289, 1999, doi: 10.1006/jmaa.1999.6560.

[23] R. P. Pant, N. Y. Özgür, and N. Taş, "On discontinuity problem at fixed point," *Bull. Malays. Math. Sci. Soc. (2)*, vol. 43, no. 1, pp. 499–517, 2020, doi: 10.1007/s40840-018-0698-6.

[24] S. park and J. S. Bae, "Extension of a fixed point theorem of Meir-Keeler," *Ark. Mat.*, vol. 19, no. 0, pp. 223–228, 1991.

[25] P. D. Proinov, "Fixed point theorems in metric spaces," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 64, no. 3, pp. 546–557, 2006, doi: 10.1016/j.na.2005.04.044.

[26] B. E. Rhoades, "Contractive definitions and continuity," Fixed point theory and its applications, Proc. Conf., Berkeley/Calif. 1986, Contemp. Math. 72, 233-245 (1988)., 1988.

[27] T. Suzuki, "The weakest contractive conditions for Edelstein's mappings to have a fixed point in complete metric spaces," *J. Fixed Point Theory Appl.*, vol. 19, no. 4, pp. 2361–2368, 2017, doi: 10.1007/s11784-017-0430-4.

Authors' addresses

Nguyen H. Hoc
Faculty of Natural Sciences, Hong Duc University, Thanh Hoa, Vietnam
E-mail address: nguyenhuuhoc@hdu.edu.vn

Ravindra K. Bisht
(**Corresponding author**) Department of Mathematics, National Defence Academy, Khadakwasla-411023, Pune, India
E-mail address: ravindra.bisht@yahoo.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 759–769

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4385

ON THE NUMBERS OF WEAK CONGRUENCES OF SOME FINITE LATTICES

ESZTER K. HORVÁTH AND ANDREJA TEPAVČEVIĆ

This paper is dedicated to Gábor Czédli on his 70th birthday.

Received 23 August, 2022

Abstract. We determine the two greatest numbers of weak congruences of lattices. The number of weak congruences of some special lattices are deduced via combinatorial considerations.

2010 *Mathematics Subject Classification:* 06B10; 06A07; 08A30

Keywords: lattice, subuniverse, congruence, weak congruence

1. Introduction, preliminaries

A weak congruence on the algebra A is a compatible weak equivalence on A, i.e., symmetric and transitive relation on A being a subuniverse of A^2. The collection $\mathrm{Cw}A$ of weak congruences on an algebra A is an algebraic lattice under inclusion. The congruence lattice $\mathrm{Con}A$, the subuniverse lattice $\mathrm{Sub}A$ are sublattices of $\mathrm{Cw}A$ as well as $\mathrm{Con}B$, for every subalgebra B of A. See [7] and the book of B. Šešelja and A. Tepavčević [8] for more details. The lattice of weak congruences of a lattice was studied in [9]. Weak congruence lattices of groups have been studied recently with the aim of characterizing various types of groups by weak congruence lattices [4, 5].

This research of the first author was supported by the National Research, Development and Innovation Fund of Hungary under funding scheme K138892.

This research of the first author was supported by project TKP2021-NVA-09. Project no TKP2021-NVA-09 has been implemented with the support provided by the Ministry of Culture and Innovation of Hungary from the National Research, Development and Innovation Fund, financed under the TKP2021-NVA funding scheme." .

This research of the second author was supported by the Science Fund of the Republic of Serbia, # Grant no 6565, Advanced Techniques of Mathematical Aggregation and Approximative Equations Solving in Digital Operational Research- AT-MATADOR and by the Ministry of Science, Technological Development and Innovation, through Mathematical Institute of the Serbian Academy of Sciences and Arts in Belgrade and Faculty of Science, University of Novi Sad.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

The greatest numbers of subuniverses of lattices were determined in [3] and [1], the greatest numbers of congruences of lattices were determined in [2] and [6] .

Let P and Q be posets with disjoint underlying sets. Then the *ordinal sum* $P+_{ord}Q$ is the poset on $P\cup Q$ with $s\le t$ if, either $s,t\in P$ and $s\le t$; or $s,t\in Q$ and $s\le t$; or $s\in P$ and $t\in Q$. In other words, every element of P is less than every element of Q, and the relations in P and Q stay the same; to draw the Hasse diagram of $P+_{ord}Q$, we place the Hasse diagram of Q above that of P and then connect any minimal element of Q with any maximal element of P. If K with 1 and L with 0 are finite posets, then their *glued sum* $K+_{\mathrm{glu}}L$ is the ordinal sum of the posets $K\setminus\{1_K\}$, the singleton poset, and $L\setminus\{0_L\}$, in this order. Note that $+_{\mathrm{glu}}$ and $+_{ord}$ are associative but not commutative operations.

2. The greatest number of weak congruences of finite lattices

Theorem 1. *If L is a finite lattice of size $n=|L|$, then L has at most $\frac{3^n+1}{2}$ weak congruences. Furthermore, $|\mathrm{Cw}\,L|=\frac{3^n+1}{2}$ if and only if L is a chain.*

Proof. First, we prove that if L is a chain, then $|\mathrm{Cw}\,L|=\frac{3^n+1}{2}$. By [3], an n-element lattice can have at most 2^n subuniverses. Furthermore, $|\mathrm{Sub}\,L|=2^n$ if and only if it is a chain. By [2], an n-element lattice can have at most 2^{n-1} congruences; furthermore, $|\mathrm{Con}\,L|=2^{n-1}$ if and only if it is a chain. Now

$$\begin{aligned}|\mathrm{Cw}\,L| &= 1+\sum_{\substack{L^*\in\mathrm{Sub}\,L\\ L^*\neq\varnothing}}|\mathrm{Con}\,L^*| = 1+\sum_{i=1}^{n}\binom{n}{i}2^{i-1}\\ &= 1+\frac{\sum_{i=1}^{n}\binom{n}{i}2^i}{2} = 1+\frac{-1+\sum_{i=0}^{n}\binom{n}{i}2^i}{2}\\ &= 1+\frac{-1+(1+2)^n}{2} = \frac{3^n+1}{2}.\end{aligned}$$

We have to show that all the n-element lattices have fewer weak congruences than $\frac{3^n+1}{2}$. We denote the elements of L by $a_1\prec\cdots\prec a_n$. If L' is not a chain, then it has at least two incomparable elements, say $p||q$. Of course $p\vee q\in L'$ and $p\wedge q\in L'$. We denote the remaining elements of L' by $b_1,\dots,b_{n-4}$ arbitrarily. Now

$$\left|\mathrm{Cw}\,L'\right| = 1+\sum_{\substack{L^*\in\mathrm{Sub}\,L'\\ L^*\neq\varnothing}}|\mathrm{Con}\,L^*|$$

By [3], the sum $|\mathrm{Cw}\,L'|$ has less summands than the sum $|\mathrm{Cw}\,L|$. We make an injection from the summands of $|\mathrm{Cw}\,L'|$ to the summands of $|\mathrm{Cw}\,L|$ in such a way that the image of each summand is not greater than the summand itself. For this, we define a bijective map $\varphi\colon L'\to L$, $(p\wedge q)\mapsto a_1$, $p\mapsto a_2$, $q\mapsto a_3$, $(p\vee q)\mapsto a_4$, and if $x\notin\{p,q,p\wedge q,p\vee q\}$, then $\varphi(x)\in L\setminus\{a_1,a_2,a_3,a_4\}$ arbitrarily. This φ induces an injective map from the set of sublattices of L' to the set of sublattices of L in a natural

way (notice that all the subsets of L are subuniverses), and also an injective map from the summands of $|\mathrm{Cw}\,L'|$ to the summands of $|\mathrm{Cw}\,L|$ in such a way that the image of each summand of $|\mathrm{Cw}\,L'|$ is not greater than the summand itself.

□

3. Lantern: the n-element lattice M_{n-2}

We use the notation M_1 for the 3-element chain and M_2 for the 4-element Boolean lattice. For $n \geq 3$, M_{n-2} consists of $n-2$ atoms, which are also coatoms, and of 0 and 1. So, the lattice M_{n-2} has $n-2$ atoms and n elements. We call the lattice M_{n-2} a *lantern.*

Theorem 2. *For $n \geq 3$, the lantern M_{n-2} has $2^{n-1}+n^2+2n-5$ weak congruences.*

Proof. In the considered lantern M_{n-2} there are n congruences of n one-element sublattices, they provide n weak congruences.

Moreover, there are $2n-3$ two-element sublattices, and each has two congruences, so here we have $4n-6$ weak congruences.

Further, there are $n-2$ three element sublattices (3-element chains) and each has 4 congruences, providing $4n-8$ weak congruences.

Each sublattice with 4 elements contains 1 and 0 and two more middle elements. There are $\frac{(n-2)(n-3)}{2}$ sublattices of 4 elements and each has 4 congruences, so there are $2(n-2)(n-3)$ weak congruences here.

Each sublattice of 5 and more elements contains 1 and 0 and three or more middle elements. Those sublattices have only 2 trivial congruences. There are

$$\sum_{k=3}^{n-2} \binom{n-2}{k}$$

such sublattices. Since

$$\sum_{k=0}^{n-2} \binom{n-2}{k} = 2^{n-2},$$

the number of congruences on sublattices with five and more elements is: $2 \cdot (2^{n-2} - 1 - (n-2) - \binom{n-2}{2})$ which is equal to $2^{n-1} - n^2 + 3n - 4$.

Summing all together, there are $1+n+4n-6+4n-8+2n^2-10n+12+2^{n-1}-n^2+3n-4 = 2^{n-1}+n^2+2n-5$ weak congruences of the lantern M_{n-2}. □

4. The number of weak congruences of ordinal sum of lattices

Lemma 1. *Given finite lattices L_1 and L_2, let $L = L_1 +_{ord} L_2$. Then*

$$|\mathrm{Sub}\,L| = |\mathrm{Sub}\,L_1||\mathrm{Sub}\,L_2|.$$

Proof. It is easy to see that any subuniverse of L can be obtained by the union of a subuniverse chosen from L_1 with a subuniverse chosen from L_2. □

Lemma 2. *Given finite lattices L_1 and L_2, let $L = L_1 +_{glu} L_2$. Then*

$$|\mathrm{Con}\, L| = |\mathrm{Con}\, L_1||\mathrm{Con}\, L_2|.$$

Proof. It is well-known that the congruence classes of lattices are convex. It is easy to see that any congruence of L can be obtained by chosing a congruence of L_1 and choosing a congruence of L_2. □

Lemma 3. *Given finite lattices L_1 and L_2, Let $L = L_1 +_{ord} L_2$. Then*

$$|\mathrm{Cw}\, L| = 2 \cdot (|\mathrm{Cw}\, L_1| - 1)(|\mathrm{Cw}\, L_2| - 1) + |\mathrm{Cw}\, L_1| + |\mathrm{Cw}\, L_2| - 1.$$

Proof. Any weak congruence of L_i is a weak congruence of L. Now $|\mathrm{Cw}\, L_1| + |\mathrm{Cw}\, L_2|$ count the empty set twice, so we have to subtract 1. All the remaining subuniverses are ordinal sums of nonempty subuniverses of form $B_1 \subseteq L_1$ and $B_2 \subseteq L_2$. Ordinal sum is nothing else than glued sum of them with a 2-element chain in the middle:

$$L_1 +_{ord} L_2 = L_1 +_{glu} C_2 +_{glu} L_2.$$

Now we can note that any weak congruence on L which have some elements from both L_1 and L_2 we can get in two ways. First, as the union of a nonempty weak congruence on L_1 and a nonempty weak congruence on L_2. Second, the transitive closure of the relation which is the union of the full relation on the two element chain with the union of a nonempty weak congruence on L_1 and a nonempty weak congruence on L_2. Therefore, we get $2 \cdot (|\mathrm{Cw}\, L_1| - 1)(|\mathrm{Cw}\, L_2| - 1)$ such weak congruences. □

5. Lantern on chain

Theorem 3. *If $L \simeq C_1 +_{ord} M_{k-2} +_{ord} C_2$, where C_1 and C_2 are chains or the empty set, and $|C_1| + |C_2| = l$, then*

$$|\mathrm{Cw}\, L| = \frac{(2^k + 2k^2 + 4k - 11) \cdot 3^l + 1}{2}$$

Proof. We use Theorem 1, Theorem 2 and Lemma 3. Let $|C_1| = l_1$ and $|C_1| = l_2$ and $L^* \simeq C_1 +_{ord} M_{k-2}$. First

$$\begin{aligned} |\mathrm{Cw}\, L^*| &= 2 \cdot \left(\frac{3^{l_1} + 1}{2} - 1\right)(2^{k-1} + k^2 + 2k - 5 - 1) + \frac{3^{l_1} + 1}{2} \\ &\quad + 2^{k-1} + k^2 + 2k - 5 - 1 \\ &= \frac{(2^k + 2k^2 + 4k - 11) \cdot 3^{l_1} + 1}{2}. \end{aligned}$$

Second

$$
\begin{aligned}
|\operatorname{Cw} L| &= 2\cdot\left(\frac{3^{l_2}+1}{2}-1\right)\left(\frac{(2^k+2k^2+4k-11)\cdot 3^{l_1}+1}{2}-1\right)\\
&\quad+\frac{3^{l_2}+1}{2}+\frac{(2^k+2k^2+4k-11)\cdot 3^{l_1}+1}{2}-1=\\
&=\frac{(2^k+2k^2+4k-11)\cdot 3^{l}+1}{2}.
\end{aligned}
$$

□

6. The second greatest number of weak congruences of finite lattices

Theorem 4. *If $|L|=n\geq 4$ and L has less than $\frac{3^n+1}{2}$ weak congruences, then the second greatest value in weak congruences is $\frac{53\cdot 3^{n-4}+1}{2}$. Furthermore, L has $\frac{53\cdot 3^{n-4}+1}{2}$ weak congruences if and only if $L\simeq C_1+_{glu}B_4+_{glu}C_2$, where C_1 and C_2 are chains or the empty set and B_4 is the four element Boolean lattice.*

Proof. By Theorem 3, we obtain the result $\frac{53\cdot 3^{n-4}+1}{2}$.

We prove that all the other n-element lattices have less weak congruences. To show this, first we calculate the above number in a different way. By [3], L has $13\cdot n^{n-4}$ sublattices. By [2], this form of an n-element lattice L has 2^{n-2} congruences. We denote the non-comparable elements of B_4 by a and b. Now

$$
\begin{aligned}
|\operatorname{Cw} L| &= 1+\sum_{\substack{L^*\in\operatorname{Sub}L\\ L^*\neq\varnothing}}|\operatorname{Con}L^*|\\
&= 1+\sum_{\substack{L^*\in\operatorname{Sub}L\\ B_4\subseteq L^*}}|\operatorname{Con}L^*|+\sum_{\substack{L^*\in\operatorname{Sub}L\\ b\notin L^*\\ a\in L^*}}|\operatorname{Con}L^*|+\sum_{\substack{L^*\in\operatorname{Sub}L\\ \{a\}\cap L^*=\varnothing}}|\operatorname{Con}L^*|=(**)
\end{aligned}
$$

Now

$$
1+\sum_{\substack{L^*\in\operatorname{Sub}L\\ \{a\}\cap L^*=\varnothing}}|\operatorname{Con}L^*|=|\operatorname{Cw}C_{n-1}|
$$

so

$$
\begin{aligned}
(**) &= \sum_{i=0}^{n-4}\binom{n-4}{i}2^{i+4-2}+\sum_{i=0}^{n-2}\binom{n-2}{i}2^{i+1-1}+\frac{3^{n-1}+1}{2}\\
&= 4(1+2)^{n-4}+(1+2)^{n-2}+\frac{3^{n-1}+1}{2}=\frac{53\cdot 3^{n-4}+1}{2}.
\end{aligned}
$$

Consider an arbitrary n-element lattice L' that is neither a chain, nor of form $C_1 +_{glu} B_4 +_{glu} C_2$,. Clearly

$$\left|\mathrm{Cw}\, L'\right| = 1 + \sum_{\substack{L^* \in \mathrm{Sub}\, L' \\ L^* \neq \emptyset}} \left|\mathrm{Con}\, L^*\right|.$$

This sum contains not more summands than that of L by [3].

We show that $|\mathrm{Cw}\, L'| \leq |\mathrm{Cw}\, L|$. If L' is neither a chain, nor of the form $C_1 +_{glu} B_4 +_{glu} C_2$, then it has antichains, let $a||b$ one of them. We make an injection from the summands of $|\mathrm{Cw}\, L'|$ to the summands of $|\mathrm{Cw}\, L|$ in such a way that the image of each summand is not greater than the summand itself. For this, we define a bijective map $\varphi\colon L' \to L$. Denote the elements of B_4 in L by $\{p, q, p \wedge q, p \vee q\}$. Let $a\varphi = p$, $b\varphi = q$, $(a \wedge b)\varphi = p \wedge q$, $(a \vee b)\varphi = p \vee q$; otherwise we define φ arbitrarily but bijectively. The image of any sublattice of L' is a sublattice of L because if the considered sublattice contains both a and b, then the image of it is a sublattice of form $C_1 +_{glu} B_4 +_{glu} C_2$. If the considered sublattice contains at most one of a and b, then its image is a chain. Now clearly by [2], the image of each summand is not greater than the summand itself because the image of a sublattice is a chain or of form $C_1 +_{glu} B_4 +_{glu} C_2$, but the latter case happens only when the sublattice is not a chain. □

7. Chandelier

Let $N_{m_1,m_2,\ldots,m_n}$ be a lattice of width n, containing n chains with m_1, m_2,...,m_n elements. They have intersection $\{0,1\}$, any other element of it belongs exactly to one chain. The index i in m_i denote the i-th chain. We call the lattice $N_{m_1,m_2,\ldots,m_n}$ a *chandelier.*

Theorem 5. *The chandelier $N_{m,k}$ has*

$$\frac{3^m-1}{2}\frac{3^k-1}{2} + 3\cdot(2^m-1)(2^k-1) + \frac{3^{m+2}+3^{k+2}}{2} - 4$$

weak congruences.

Proof. This lattice has $m+k+2$ elements. By Theorem 1 the number of weak congruences on the chain with $m+2$ elements is $\frac{3^{m+2}+1}{2}$, and the number of weak congruences on the chain with $k+2$ elements is $\frac{3^{k+2}+1}{2}$. Here, we counted twice the weak congruences on the sublattice $\{0,1\}$, and there are 5 of them.

Further, we calculate the sublattices having non-comparable elements and congruences on them.

We note that 0 and 1 must belong to any of such sublattices L_1 and at least one element different from 0 and 1 must belong to L_1 from each of the two chains. When we fix such a sublattice with non-comparable elements, we can note that there are

two types of congruences on such a sublattice, depending on whether 0 and 1 are one-element congruence classes or not.

First, suppose that in a fixed sublattice L_1, $\{0\}$ and $\{1\}$ are one-element classes in a congruence and we calculate the number of all such congruences. We can chose independently the congruence classes on each of the chains and if this chain has e.g. m_1 and k_1 elements in a sublattice L_1, then there are $2^{m_1-1} \cdot 2^{k_1-1}$ different congruences. Moreover, on each sublattice L_1 we have the congruence L_1^2. When we calculate all congruences on all combinations of such sublattices, there are

$$\sum_{l=1}^{m}\sum_{s=1}^{k}\binom{m}{l}\binom{k}{s}(2^{l-1}2^{s-1}+1)$$

weak congruences of this type.

Second, if $\{0\}$ or $\{1\}$ are not one-element classes, then due to the properties of congruences, if 0 is in a congruence relation with one of the elements of one of the chains, then the whole chain is a congruence block, and 1 is in the relation with all elements of the other chain. Hence, for each of the sublattices there are two possibilities (depending on the chain which is in the congruence relation with 0). Hence, there are

$$2\cdot\sum_{l=1}^{m}\sum_{s=1}^{k}\binom{m}{l}\binom{k}{s}$$

such weak congruences.

Altogether there are

$$\sum_{l=1}^{m}\sum_{s=1}^{k}\binom{m}{l}\binom{k}{s}2^{l-1}2^{s-1}+3\cdot\sum_{l=1}^{m}\sum_{s=1}^{k}\binom{m}{l}\binom{k}{s}+\frac{3^{m+2}+3^{k+2}}{2}-4$$

weak congruences. This is equal to

$$\frac{3^m-1}{2}\frac{3^k-1}{2}+3\cdot(2^m-1)(2^k-1)+\frac{3^{m+2}+3^{k+2}}{2}-4.$$

□

Let $N_{m_1,m_2,\dots,m_n}$ be a chandelier of width n, containing n chains with m_1, m_2,...,m_n elements. Let $w^{(k)}(m_{l_1},\dots,m_{l_k})$ be the number of special weak congruences on $N_{m_1,m_2,\dots,m_n}$, which are congruences of sublattices of $N_{m_1,m_2,\dots,m_n}$ of width k where $\{m_{l_1},\dots m_{l_k}\}$ is a fixed subset of the set $\{m_1,m_2,\dots,m_n\}$ containing k different elements. If $n=1$, then $w^{(n)}(m)$ is the number of weak congruences on the chain with n-elements (not counting the empty set).

It is easy to see that

$$|\mathrm{Cw}\,N_{m,k}| = w^{(1)}(m)+w^{(1)}(k)+w^{(2)}(m,k)-3.$$

Further, $|\mathrm{Cw}\,N_{m,k,l}| = w^{(1)}(m)+w^{(1)}(k)+w^{(1)}(l)+w^{(2)}(m,k)+w^{(2)}(m,l)+w^{(2)}(k,l)+w^{(3)}(m,k,l)-7$.

Lemma 4. *Let $k \geq 3$. Then,*

$$w^{(k)}(m_1,\dots m_n) = \prod_{i=1}^{k} \frac{3^{m_i}-1}{2} + (2^{m_1}-1)\cdot(2^{m_2}-1)\cdot\ldots\cdot(2^{m_k}-1)$$

Proof. $w^{(k)}(m_1,\dots m_k)$ is the set of congruences on sublattices of $N_{m_1,m_2,\dots,m_n}$ of width k. This means that 0 and 1 belong to each of these sublattices as well as at least one element from each of k chains. We obtain different weak congruences as union of weak congruences on chains. Besides the squares of all subalgebras are also weak congruences. 0 and 1 are one-element congruence classes unless the weak congruence is the square of the sublattice.

The number of sublattices of width k is $(2^{m_1}-1)\cdot(2^{m_2}-1)\cdot\ldots\cdot(2^{m_k}-1)$ and the number of weak congruences (which are not square) is $\prod_{i=1}^{k}\frac{3^{m_i}-1}{2}$ □

We use the notation $w^{(i)}A$ for $w^{(i)}(m_1,\dots m_i)$, if $A=\{m_1,\dots m_i\}$.

The proof of the following theorem is straightforward.

Theorem 6. *The number of weak congruences of a chandelier of width n is*

$$|\mathrm{Cw}\,N_{m_1,m_2,\dots,m_n}| = \sum_{i=1}^{n} \sum_{A\in\mathcal{P}^i(\{m_1,m_2,\dots,m_n\})} w^{(i)}A - 4n + 5,$$

where $\mathcal{P}^i(\{m_1,m_2,\dots,m_n\})$ is the set of all subsets of $\{m_1,m_2,\dots,m_n\}$ with i elements.

8. Appendix: N_5, M_3 and M_4 on chains

In this chapter, we determine the number of weak congruences of some special lattices, which play significant role in lattice theory. These results can be obtained by using Lemma 3, but we prove them without it.

Theorem 7. *If $|L|\geq 5$ and $L \simeq C_1 +_{glu} N_5 +_{glu} C_2$, where C_1 and C_2 are chains or the empty set, then L has $\frac{125\cdot 3^{n-5}+1}{2}$ weak congruences.*

Proof. By [6], the n-element lattice of form $C_1 +_{glu} N_5 +_{glu} C_2$ has $5\cdot 2^{n-5}$ congruences. We denote by a and b the atoms of N_5 and by c and b the coatoms of N_5; of course $a \prec c$.

$$\begin{aligned}
|\mathrm{Cw}\,L| &= 1 + \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ L^*\neq\varnothing}} |\mathrm{Con}\,L^*| = \\
&= \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ N_5\subseteq L^*}} |\mathrm{Con}\,L^*| + \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ a,b\in L^*\\ c\notin L^*}} |\mathrm{Con}\,L^*| + \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ c,b\in L^*\\ a\notin L^*}} |\mathrm{Con}\,L^*| + \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ b\in L^*\\ a,c\notin L^*}} |\mathrm{Con}\,L^*| \\
&\quad + |\mathrm{Cw}\,C_{n-1}| = \\
&= \sum_{i=0}^{n-5}\binom{n-5}{i} 5\cdot 2^{i+5-5} + 2\sum_{i=0}^{n-5}\binom{n-5}{i} 2^{i+4-2} + \sum_{i=0}^{n-3}\binom{n-3}{i} 2^{i+1-1}
\end{aligned}$$

$$+\frac{3^{n-1}+1}{2}=$$
$$=5\cdot(1+2)^{n-5}+8\cdot(1+2)^{n-5}+9\cdot(1+2)^{n-5}+\frac{3^{n-1}+1}{2}$$
$$=\frac{125\cdot 3^{n-5}+1}{2}.$$

□

Remark 1. We conjecture that $\frac{125\cdot 3^{n-5}+1}{2}$ is the third greatest number of weak congruences of finite lattices, and the corresponding lattice is $L\simeq C_1+_{glu}N_5+_{glu}C_2$, where C_1 and C_2 are chains or the empty set.

Theorem 8. *If $|L|\geq 5$ and $L\simeq C_1+_{glu}M_3+_{glu}C_2$, where C_1 and C_2 are chains or the empty set, then L has $\frac{91\cdot 3^{n-5}+1}{2}$ weak congruences.*

Proof. It is easy to see that the n-element lattice of form $C_1+_{glu}M_3+_{glu}C_2$ has 2^{n-4} congruences. We denote by a, b and c the atoms of M_3, which are also coatoms of M_3. We denote the sublattice of this lattice by deleting one coatom (denoted by a) of M_3 by $\hat{L}$. Now

$$|\mathrm{Cw}\,L|=1+\sum_{\substack{L^*\in\mathrm{Sub}\,L\\ L^*\neq\varnothing}}|\mathrm{Con}\,L^*|$$
$$=\sum_{\substack{L^*\in\mathrm{Sub}\,L\\ M_3\subseteq L^*}}|\mathrm{Con}\,L^*|+\sum_{\substack{L^*\in\mathrm{Sub}\,L\\ a\in L^*\\ b,c\notin L^*}}|\mathrm{Con}\,L^*|+\sum_{\substack{L^*\in\mathrm{Sub}\,L\\ a,b\in L^*\\ c\notin L^*}}|\mathrm{Con}\,L^*|+\sum_{\substack{L^*\in\mathrm{Sub}\,L\\ a,c\in L^*\\ b\notin L^*}}|\mathrm{Con}\,L^*|+$$
$$+|\mathrm{Cw}\,\hat{L}|$$
$$=\sum_{i=0}^{n-5}\binom{n-5}{i}2^{i+5-4}+\sum_{i=0}^{n-3}\binom{n-3}{i}2^{i+1-1}+2\sum_{i=0}^{n-5}\binom{n-5}{i}2^{i+4-2}$$
$$+\frac{53\cdot 3^{n-5}+1}{2}$$
$$=2\cdot(1+2)^{n-5}+(1+2)^{n-3}+8\cdot(1+2)^{n-5}+\frac{53\cdot 3^{n-5}+1}{2}$$
$$=\frac{91\cdot 3^{n-5}+1}{2}$$

□

Theorem 9. *If $|L|\geq 5$ and $L\simeq C_1+_{glu}M_4+_{glu}C_2$, where C_1 and C_2 are chains or the empty set, then L has $\frac{149\cdot 3^{n-6}+1}{2}$ weak congruences.*

Proof. It is easy to see that the n-element lattice of form $C_1+_{glu}M_4+_{glu}C_2$ has 2^{n-5} congruences. We denote by a, b, c and d the atoms of M_4, which are also

coatoms of M_4. We denote the sublattice of this lattice by deleting one coatom (denoted by a) of M_4 by $\hat{L}$. Now

$$\begin{aligned}
|\mathrm{Cw}\,L| &= 1+\sum_{\substack{L^*\in \mathrm{Sub}\,L\\ L^*\neq\varnothing}}|\mathrm{Con}\,L^*| = \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ M_4\subseteq L^*}}|\mathrm{Con}\,L^*| + \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ a\in L^*\\ b,c,d\notin L^*}}|\mathrm{Con}\,L^*| \\
&\quad + \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ a,b\in L^*\\ c,d\notin L^*}}|\mathrm{Con}\,L^*| + \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ a,c\in L^*\\ b,d\notin L^*}}|\mathrm{Con}\,L^*| + \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ a,d\in L^*\\ b,c\notin L^*}}|\mathrm{Con}\,L^*| \\
&\quad + \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ a,b,c\in L^*\\ d\notin L^*}}|\mathrm{Con}\,L^*| + \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ a,b,d\in L^*\\ c\notin L^*}}|\mathrm{Con}\,L^*| + \sum_{\substack{L^*\in \mathrm{Sub}\,L\\ a,c,d\in L^*\\ b\notin L^*}}|\mathrm{Con}\,L^*| + |\mathrm{Cw}\,\hat{L}| \\
&= \sum_{i=0}^{n-6}\binom{n-6}{i}2^{i+6-5} + \sum_{i=0}^{n-4}\binom{n-4}{i}2^{i+1-1} + 3\sum_{i=0}^{n-6}\binom{n-6}{i}2^{i+4-2} \\
&\quad + 3\sum_{i=0}^{n-6}\binom{n-6}{i}2^{i+5-4} + \frac{91\cdot 3^{n-6}+1}{2} \\
&= 2\cdot(1+2)^{n-6} + (1+2)^{n-4} + 12\cdot(1+2)^{n-6} + 6\cdot(1+2)^{n-6} \\
&\quad + \frac{91\cdot 3^{n-6}+1}{2} \\
&= \frac{149\cdot 3^{n-6}+1}{2.}
\end{aligned}$$

□

REFERENCES

[1] D. Ahmed and E. K. Horváth, "Yet two additional large numbers of subuniverses of finite lattices." *Discussiones Mathematicae: General Algebra & Applications*, vol. 39, no. 2, pp. 251–261, 2019, doi: 10.7151/dmgaa.1309.

[2] G. Czédli, "A note on finite lattices with many congruences," *Acta Univ. M. Belii Ser. Math.*, pp. 22–28, 2018.

[3] G. Czédli and E. K. Horváth, "A note on lattices with many sublattices," *Miskolc Mathematical Notes*, vol. 20, no. 2, p. 839–848, 2019, doi: 10.18514/MMN.2019.2821.

[4] J. Jovanović, B. Šešelja, and A. Tepavčević, "Lattice characterization of finite nilpotent groups," *Algebra Universalis*, vol. 82, no. 3, 2021, doi: 10.1007/s00012-021-00716-7.

[5] J. Jovanović, B. Šešelja, and A. Tepavčević, "Lattices with normal elements," *Algebra Universalis*, vol. 83, no. 2, 2022, doi: 10.1007/s00012-021-00759-w.

[6] C. Mureşan and J. Kulin, "On the largest numbers of congruences of finite lattices," *Order*, vol. 37, no. 3, pp. 445–460, 2020, doi: 10.1007/s11083-019-09514-2.

[7] G. Vojvodić and B. Šešelja, "On the lattice of weak congruence relations," *Algebra universalis*, vol. 25, no. 1, pp. 121–130, 1988, doi: 10.1007/BF01229965.

[8] B. Šešelja and A. Tepavčević, *Weak Congruences in Universal Algebra*. Novi Sad: Institute of Mathematics, 2001.

[9] B. Šešelja and G. Vojvodić, "Weak congruences of a lattice," *Review of Research Faculty of Science, Mathematics Series*, vol. 18, pp. 205–209, 1988.

Authors' addresses

Eszter K. Horváth
(**Corresponding author**) University of Szeged, Bolyai Institute, Aradi vértanúk tere 1, 6720 Szeged, Hungary
E-mail address: horeszt@math.u-szeged.hu

Andreja Tepavčević
Mathematical Institute SANU, Kneza Mihaila 36, Belgrade, and University of Novi Sad, Faculty of Sciences, Department of Mathematics and Informatics, Trg Dositeja Obradovica 4, Novi Sad, Serbia.
E-mail address: andreja@dmi.uns.ac.rs

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 771–788

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4360

ON THE LOCAL EXISTENCE OF SOLUTIONS TO p-LAPLACIAN EQUATION WITH LOGARITHMIC NONLINEARITY AND NONLINEAR DAMPING TERM

NAZLI IRKIL

Received 06 August, 2022

Abstract. This paper is concerned with the interaction between logarithmic source term and p-Laplacian term for nonlinear damped semilinear wave equation. We established the local existence and uniquenes under appropriate conditions.

2010 *Mathematics Subject Classification:* 35A01; 35B44; 35L05

Keywords: existence, p-Laplacian equation, logarithmic nonlinearity

1. Introduction

In this work, we investigate the following p-Laplacian hyperbolic type equation with logarithmic nonlinearity and nonlinear damping

$$\begin{cases} u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u_t|^{k-2} u_t = |u|^{q-2} u \ln|u|, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \end{cases} \tag{1.1}$$

where $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $u_1 \in H_0^1(\Omega)$ are given initial data and $\Omega \subset R^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. The parameter $k \geq 2$ and the exponents p, q satisfy

$$2 < p < q < p\left(1 + \frac{2}{n}\right). \tag{1.2}$$

The logarithmic nonlinearity occured naturally in quantum mechanics, inflation cosmolog, supersymmetric field theories, and a lot of different areas of physics such as, optics, geophysics and nuclear physics It was a classical field equation whose popularity increased especially when it was shown in [1, 3]. The qualitative behavior of solutions for problems with logarithmic nonlinearity in the absence of the

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

p-Laplacian term

$$u_{tt} - \Delta u + f(u_t) = |u|^{q-2} u \ln |u|$$

have attracted the attention of several mathematicians. Some of the based work in this subject are [2, 5–7, 9, 10, 12, 16, 18, 19]. In [14], Nhan and Truong investigated

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) - \Delta u_t = |u|^{p-2} u \ln |u|, \tag{1.3}$$

and they established the global existence, blow up and decay of the solutions for $p > 2$. The problem (1.3) was studied by Cao and Liu[4] and they proved global boundedness and blowing-up at ∞ for $1 < p < 2$. Ding and Zhou [8] studied the problem (1.3) replaced $|u|^{p-2} u \ln |u|$ with $|u|^{q-2} u \ln |u|$. They established global existence, blow up in finite time and blow up at infinite time. He et al. [11] studied the decay of solutions the same problem. Our aim in this study will be existence of solution hyperbolic type equation with logarithmic source term and p-Laplacian term.

2. Preliminaries

In order to state the main results to problem 1.1 more clearly, we start to our work by introducing some notations, lemmas and definitions which will be used in this paper. Throughout this paper, we denote

$$\|u\|_m = \|u\|_{L^m(\Omega)}, \qquad \|u\|_{1,m} = \|u\|_{W_0^{1,m}(\Omega)} = (\|u\|_m^m + \|\nabla u\|_m^m)^{\frac{1}{m}}$$

for $1 < m < \infty$. We consider $W_0^{-1,m'}(\Omega)$ to denote the dual space of $W_0^{1,m'}(\Omega)$ where m' is Hölder conjugate exponent for $m > 1$.

We define energy function as follows

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \ln |u| \, dx + \frac{1}{q^2}\|u\|_q^q.$$

Let us define some useful funcionals as follows

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \ln |u| \, dx + \frac{1}{q^2}\|u\|_q^q, \tag{2.1}$$

and

$$I(u) = \|\nabla u\|_p^p - \int_\Omega |u|^q \ln |u| \, dx. \tag{2.2}$$

By the Gagliardo-Nirenberg multiplicative embedding inequality that $J(u)$ and $I(u)$ are continuous. Then, by (2.1) and (2.2), it tells us that

$$J(u) = \frac{1}{q} I(u) + \left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q \tag{2.3}$$

and

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(u).$$

We can define the mountain-pass level

$$d = \inf_{u\in\aleph} J(t), \tag{2.4}$$

where $\aleph$ is the Nehari manifold, which is defined by

$$\aleph = \left\{u \in W_0^{1,p}(\Omega)\setminus\{0\} : I(u) = 0\right\}.$$

As in [17], we put the potential well depth

$$0 < d = \inf_u \left\{\sup_{\lambda\geq 0} J(\lambda u) : u \in W_0^{1,p}(\Omega), \|u\|_p^p \neq 0\right\}.$$

Now, we introduce the potential well U and its corresponding set K

$$U = \left\{u \in W_0^{1,p}(\Omega) : I(u) > 0,\ J(u) < d\right\} \cup \{0\},$$
$$K = \left\{u \in W_0^{1,p}(\Omega) : I(u) < 0,\ J(u) < d\right\}.$$

Lemma 1. *For any $u \in W_0^{1,p}(\Omega)$, we get*

$$\|u\|_s \leq C_p \|\nabla u\|_p, \text{ for } u \in H_0^1(\Omega)$$

for all $1 \leq p \leq \frac{pn}{n-p}$ if $n > p$; $1 \leq p < \infty$ if $n \leq p$, where C_p is the best embedding constant.

Lemma 2 ([13, Chapter II, Lemma 1.1])**.**

(i) For any function $u \in W_0^{1,p}(\Omega)$, we have

$$\|u\|_q \leq B_{q,p}\|\nabla u\|_p,$$

for all $q \in [1,\infty]$ if $n \leq p$, and $1 \leq q \leq \frac{np}{n-p}$ if $n > p$. The best constant $B_{q,p}$ depends only on Ω, n, p and q. We will denote the constant $B_{p,p}$ by B_p.

(ii) For any $u \in W_0^{1,p}(\Omega)$, $p \geq 1$ and $r \geq 1$, we have

$$\|u\|_q \leq C\|\nabla u\|_p^{\mu}\|u\|_r^{1-\mu},$$

where C is a positive constant

$$\mu = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{r}\right)^{-1},$$

and

- *for $p \geq n = 1$, $r \leq q \leq \infty$*
- *for $p \geq n = 1$, $r \leq q \leq \infty$*
- *for $n > 1$ and $p < n$, $q \in \left[r, \frac{np}{n-p}\right]$ if $r \leq \frac{np}{n-p}$ and $q \in \left[r, \frac{np}{n-p}\right]$ if $r \leq \frac{np}{n-p}$*

- *for* $p = n > 1$, $r \leq q < \infty$
- *for* $p > n > 1$, $r \leq q \leq \infty$.

Lemma 3. *$E(t)$ is a nonincreasing function, for $t \geq 0$*

$$E'(t) = -\|u_t\|_k^k \leq 0.$$

Proof. Multiplying the equation (1.1) by u_t and integrating on Ω, we have

$$\int_\Omega u_{tt} u_t \, dx - \int_\Omega \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) u_t \, dx + \int_\Omega |u_t|^{k-1} u_t \, dx = \int_\Omega u^{q-2} u \ln |u| \, u_t \, dx,$$

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \ln |u| \, dx + \frac{1}{q^2}\|u\|_q^q\right) = -\|u_t\|_k^k,$$

$$E'(t) = -\|u_t\|_k^k.$$

□

Lemma 4. *Suppose that $\lambda > 0$, $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and $\|u\|_q \neq 0$. Then we get*

(i) $\lim_{\lambda \to 0^+} J(\lambda u) = 0$, $\lim_{\lambda \to \infty} J(\lambda u) = -\infty$;

(ii) there exists a unique λ^* such that

$$\frac{d}{d\lambda} J(\lambda u) \,|_{\lambda = \lambda^*} = 0;$$

(iii) $J(\lambda u)$ is strictly decreasig on $\lambda^* < \lambda < \infty$, strictly increasing on $0 \leq \lambda \leq \lambda^*$, and takes maximum at $\lambda = \lambda^*$;

(iv) For any $\lambda \geq 0$, we get

$$I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u) = \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < \infty. \end{cases} \tag{2.5}$$

Proof.

(i) It is obvious that by the definition of $J(u)$,

$$\begin{aligned} J(\lambda u) &= \frac{1}{p}\|\lambda \nabla u\|_p^p + \frac{1}{q^2}\|\lambda u\|_q^q - \frac{1}{q}\int_\Omega (\lambda u)^q \ln |\lambda u| \, dx \\ &= \frac{\lambda^p}{p}\|\nabla u\|_p^p + \frac{\lambda^q}{q^2}\|u\|_q^q - \frac{\lambda^q}{q}\ln |\lambda| \, \|u\|_q^q - \frac{\lambda^q}{q}\int_\Omega \ln |u| \, |u|^q \, dx. \end{aligned}$$

By virtue of $\|u\|_p^p \neq 0$, we obtain $\lim_{\lambda \to 0} g(\lambda) = 0$, $\lim_{\lambda \to \infty} g(\lambda) = -\infty$.

(ii) Now, differentiating $J(\lambda u)$ with respect to λ, we obtain

$$\begin{aligned}\frac{d}{d\lambda}J(\lambda u) &= \lambda^{p-1}\|\nabla u\|_p^p - \lambda^{q-1}\ln|\lambda|\,\|u\|_q^q - \lambda^{q-1}\int_\Omega |u|^q \ln|u|\,dx \\ &= \lambda\left(\lambda^{p-2}\|\nabla u\|_p^p - \lambda^{q-2}\ln|\lambda|\,\|u\|_q^q - \lambda^{q-2}\int_\Omega |u|^q\ln|u|\,dx\right) \\ &= \lambda\varphi(\lambda)\end{aligned}$$

where

$$\varphi(\lambda) = \lambda^{p-2}\|\nabla u\|_p^p - \lambda^{q-2}\ln|\lambda|\,\|u\|_q^q - \lambda^{q-2}\int_\Omega |u|^q\ln|u|\,dx$$

We observe from $2<p<q$ that

$$\begin{aligned}\varphi(\lambda) &= \lambda^{p-2}\|\nabla u\|_p^p - \lambda^{q-2}\ln|\lambda|\,\|u\|_q^q - \lambda^{q-2}\int_\Omega |u|^q\ln|u|\,dx \\ &= \lambda^{q-2}\left(\lambda^{p-q}\|\nabla u\|_p^p - \ln|\lambda|\,\|u\|_q^q - \int_\Omega |u|^q\ln|u|\,dx\right) \\ &= \lambda^{q-2}\left(x\lambda^{p-q} - y\ln|\lambda| - z\right)\end{aligned}$$

where $x=\|\nabla u\|_p^p \geq 0$, $y=\|u\|_q^q \geq 0$ and $z=\int_\Omega |u|^q\ln|u|\,dx$. Also we obtain

$$\begin{aligned}\varphi'(\lambda) &= (q-2)\lambda^{q-3}\left(x\lambda^{p-q} - y\ln|\lambda| - z\right) + \lambda^{q-3}\left(x(p-q)\lambda^{p-q} - y\right) \\ &= \lambda^{q-3}\left[(p-2)x\lambda^{p-q} - y((q-2)\ln|\lambda|+1) - (q-2)z\right].\end{aligned}$$

Let

$$g(\lambda) = (p-2)x\lambda^{p-q} - y((q-2)\ln|\lambda|+1) - (q-2)z$$

which together with $2<p<q$ satisfies that

$$\lim_{\lambda\to 0} g(\lambda) = \infty, \qquad \lim_{\lambda\to\infty} g(\lambda) = -\infty$$

and

$$g'(\lambda) = \frac{(p-q)(p-1)\lambda^{p-q} - (q-1)z}{\lambda} < 0.$$

Now, we deduce that there exist a unique λ_0 such that $g(\lambda)\,|_{\lambda=\lambda_0} = 0$, which satisfies

$$\begin{cases}\varphi'(\lambda) > 0, & \text{for } 0<\lambda<\lambda_0, \\ \varphi'(\lambda) = 0, & \text{for } \lambda=\lambda_0, \\ \varphi'(\lambda) < 0, & \text{for } \lambda>\lambda_0.\end{cases}$$

Therefore, we conclude that there exists a unique $\lambda_1 > \lambda_0$ such that $\varphi(\lambda)\mid_{\lambda=\lambda_1} = 0$ and $\varphi(\lambda)$ is monotone decreasing $\lambda > \lambda_1$. Hence, there exists $\lambda^* > \lambda_1$ such that $\left(\|\nabla u\|^2 + \varphi(\lambda)\right) = 0$,which means $\frac{d}{d\lambda}J(\lambda u)\mid_{\lambda=\lambda^*}$.

(iii) From (ii), we can see clearly

$$\frac{d}{d\lambda}J(\lambda u) > 0 \qquad \text{for } 0 \le \lambda \le \lambda^*,$$

$$\frac{d}{d\lambda}J(\lambda u) < 0 \qquad \text{for } \lambda^* < \lambda < \infty,$$

which gives (iii).

(iv) Thus, by definition of $I(u)$ we have the desired results such that

$$I(\lambda u) = \lambda^p \|\nabla u\|_p^p - \lambda^q \ln|\lambda| \, \|u\|_q^q - \lambda^q \int_\Omega |u|^q \ln|u| \, dx = \lambda \frac{d}{d\lambda} J(\lambda u) \tag{2.6}$$

We obtain (2.5) from the proof of the (ii) and (2.6).

□

Lemma 5.

(i) d is positive and there exists a positive function $u \in \aleph$ such that $J(u) = d$

(ii) The depth of potential well d is defined as

$$d = \left(\frac{q-p}{pq}\right)\left(\frac{e\alpha}{C}\right)^{\frac{p}{q+\alpha-p}}$$

Proof.

(i) By (2.3), our aim is to show that there is a positive function $u \in \aleph$ such that $J(u) = d$. Let $\{u_m\}_{m=1}^{\infty} \subset \aleph$ be a minimum sequence of $J(u)$, i.e.

$$\lim_{m\to\infty} J(u_m) = d.$$

Hence, we have $\{|u_m|\}_{m=1}^{\infty} \subset \aleph$ is a minimum sequence of $J(u)$ from $|u_m| \subset u_m \in \aleph$ and $J(|u_m|) = J(|u_m|)$ .Morever, we can assume that $u_m > 0$ a.e. for all $m \in \mathbb{N}$.

Otherwise, we have already observed that, $J(u)$ is coercive on $\aleph$ which satisfies that $\{u_m\}_{m=1}^{\infty} \subset \aleph$ is bounded in $u \in W_0^{1,p}(\Omega)$. Let $\alpha > 0$ is a sufficiently small such that $q + \alpha < \frac{np}{n-p}$, so the embedding $W_0^{1,p} \hookrightarrow L^{q+\alpha}$ is compact, and there is a function u and subsequence $\{u_m\}_{m=1}^{\infty}$, still denoted by $\{u_m\}_{m=1}^{\infty}$, such that

$$u_m \to u, \text{weakly in } W_0^{1,p}(\Omega),$$
$$u_m \to u, \text{strongly in } L^{q+\alpha}(\Omega),$$
$$u_m \to u, \text{a.e. in } \Omega.$$

Thus, we get $u \geq 0$ a.e. in Ω. By Lebesgue dominated convergence theorem, we see that

$$\int_{\Omega} |u|^q \ln |u| \, dx = \lim_{m\to\infty} \int_{\Omega} |u_m|^q \ln |u_m| \, dx, \tag{2.7}$$

$$\int_{\Omega} |u|^q \, dx = \lim_{m\to\infty} \int_{\Omega} |u_m|^q \, dx. \tag{2.8}$$

The weak lower semicontinuity of $\|.\|_{W_0^{1,p}}$ implies

$$\|\nabla u\|_p \leq \lim_{m\to\infty} \inf \|\nabla u_m\|_p . \tag{2.9}$$

Combining definition of the $J(u)$ and $I(u)$, (2.7) - (2.9), we conclude that

$$J(u) \leq \lim_{m\to\infty} \inf J(u_m) = d, \tag{2.10}$$

$$I(u) \leq \lim_{m\to\infty} \inf I(u_m) = 0. \tag{2.11}$$

Thanks to $u_m \in \aleph$ one has $u_m \in W_0^{1,p}(\Omega)$ and $I(u_m) = 0$. Therefore, by using the fact

$$\ln x \leq \frac{1}{e\alpha} x^{\alpha} \text{ for } x \geq 1 \tag{2.12}$$

and the Sobolev embedding inequality, we have

$$\begin{aligned}
\|\nabla u_m\|_p^p &= \int_{\Omega} |u_m|^q \ln |u_m| \, dx \\
&= \int_{\{x\in\Omega:|u_m(x)|\geq 1\}} |u_m|^q \ln |u_m| \, dx + \int_{\{x\in\Omega:|u_m(x)|<1\}} |u_m|^q \ln |u_m| \, dx \\
&\leq \int_{\{x\in\Omega:|u_m(x)|\geq 1\}} |u_m|^q \ln |u_m| \, dx \\
&\leq \frac{1}{e\alpha} \int_{\{x\in\Omega:|u_m(x)|\geq 1\}} |u_m|^{q+\alpha} \, dx \leq C \|\nabla u_m\|_{p+\alpha}^{p+\alpha},
\end{aligned}$$

for some positive constant C, which implies

$$\int_{\Omega} |u_m|^q \ln |u_m| \, dx = \|\nabla u_m\|_p^p \geq C. \tag{2.13}$$

From (2.13) and (2.7), we reproduce

$$\int_{\Omega} |u|^q \ln |u| \, dx \geq C.$$

Therefore, we obtain $u \in W_0^{1,p}(\Omega)$. By (2.11), we easily have $I(u) \leq 0$. Now, we show that $I(u) = 0$. Indeed, if it false, we get $I(u) < 0$, then by Lemma 2.4, there exists a λ^* such that $0 < \lambda^* < 1$ and $I(\lambda^* u) = 0$. Thus, we conclude that

$$\begin{aligned}
d &\leq J(\lambda^* u) \\
&= \frac{1}{q} I(\lambda^* u) + \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla(\lambda^* u)\|_p^p + \frac{1}{q^2} \|\lambda^* u\|_q^q \\
&= \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla(\lambda^* u)\|_p^p + \frac{1}{q^2} \|\lambda^* u\|_q^q \\
&\leq (\lambda^*)^p \left(\left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p + \frac{1}{q^2} \|u\|_q^q\right) \\
&\leq (\lambda^*)^p \liminf_{m\to\infty} \left(\left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u_m\|_p^p + \frac{1}{q^2} \|u_m\|_q^q\right) \\
&\leq (\lambda^*)^p \liminf_{m\to\infty} J(u_m) = (\lambda^*)^p d < d.
\end{aligned}$$

This is impossible, so we derive $I(u) = 0$ and $u_m \in \aleph$. From (2.10) and (2.4), we obtain $J(u) = d$, and the proof of (i) is complete.

(ii) By $I(u) = 0$ and the definition of $I(u)$, we obtain

$$\|\nabla u\|_p^p = \int_\Omega |u|^q \ln|u| \, dx. \tag{2.14}$$

Then, by using tha fact (2.12) and Sobolev embedding theorem, (2.14) becomes

$$\|\nabla u\|_p^p < \frac{1}{e\alpha} \|u\|_{q+\alpha}^{q+\alpha} \leq \frac{C}{e\alpha} \|\nabla u\|_p^{q+\alpha}$$

where $C > 0$, which means that

$$\left(\frac{e\alpha}{C}\right)^{\frac{1}{q+\alpha-p}} \leq \|\nabla u\|_p. \tag{2.15}$$

From the (i) we know that, $u \in \aleph$. By $I(u) = 0$, (2.3) and (2.15), we note that

$$\begin{aligned}
J(u) &= \frac{1}{q} I(u) + \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p + \frac{1}{q^2} \|u\|_q^q \geq \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p \\
&\geq \left(\frac{q-p}{pq}\right) \left(\frac{e\alpha}{C}\right)^{\frac{p}{q+\alpha-p}}
\end{aligned}$$

where $q > p$, which implies that

$$d = \left(\frac{q-p}{pq}\right) \left(\frac{e\alpha}{C}\right)^{\frac{p}{q+\alpha-p}}.$$

This completes the proof.

□

3. Local existence of solution for $E(0) < d$

In this part, we established the global existence of the problem (1.1). Firstly, we start the definition of the weak solution to the problem (1.1).

Definition 1. *A function $u(t)$ is called a weak solution to problem (1.1) on $\Omega \times [0,T)$, if*

$$u \in L^{\infty}\left(0,T;W_0^{1,p}(\Omega)\right)$$

and

$$u_t \in L^{\infty}\left(0,T;L^k(\Omega)\right)$$

satisfy for $t \in [0,T)$ and $llw \in W_0^{1,p}(\Omega)$

$$\begin{cases} \int\limits_{\Omega} u_{tt}(x,t)\,w(x)\,dx + \int\limits_{\Omega} |u_t(x,t)|^{k-2} u_t(x,t)\,w(x)\,dx \\ + \int\limits_{\Omega} |\nabla u(x,t)|^{p-2} \nabla u(x,t)\,\nabla w(x)\,dx \\ = \int\limits_{\Omega} \ln|u(x,t)|\,u^{q-2}(x,t)\,w(x)\,dx, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \end{cases}$$

Theorem 1.

Rational case: *Let $(u_0,u_1) \in W_0^{1,p}(\Omega) \times L^k(\Omega)$ and $2 < p < q < p\left(1+\frac{2}{n}\right)$ for every $T > 0$. Then problem (1.1) has a unique weak solution*

$$u \in C\left([0,T);W_0^{1,p}(\Omega)(\Omega)\right), \qquad u_t \in C\left([0,T);L^k(\Omega)\right).$$

Irrational case: *Moreover, u satisfies the following energy inequality*

$$E(t) + \int_0^t \|u_t(s)\|_k^k\,ds \le E(0) \qquad \text{for } 0 \le t \le T.$$

Proof. To consider the well-posedness of problem (1.1), we employ the standard Faedo–Galerkin method. The proof will consist of three steps.

Step 1: Approximate Problem: Let $\{w_j\}_{j=1}^{\infty}$ be the orthogonal basis of $W_0^{1,p}(\Omega)$ space. We take the finite dimensional space

$$V_m = \operatorname{span}\{w_1, w_2, \ldots, w_m\}.$$

Let the projections of the initial data on the finite dimensional subspace V_m be given by

$$u_m(0) = u_{m0}(x) = \sum_{j=1}^{m} a_{jm} w_j(x) \to u_0 \qquad \text{in } W_0^{1,p}(\Omega),$$

$$u_{mt}(0) = u_{m1}(x) = \sum_{j=1}^{m} b_{jm} w_j(x) \to u_1 \qquad \text{in } L^k(\Omega), \tag{3.1}$$

for $j=1,2,\ldots,m$.

We construct the approximate solutions $u_m(x,t)$ for problem (1.1) in the form

$$u_m(x,t)=\sum_{j=1}^{m} h_{jm}(t)\,w_j(x) \tag{3.2}$$

which satisfy the approximate problem in V_m

$$\begin{aligned}&(u_{mtt},w_s)+\left(|\nabla u_m|^{p-2}\nabla u_m,\nabla w_s\right)\\&=\left(|u_m|^{q-2}u_m\log|u_m|,w_s\right)ds-\left(|u_{mt}|^{k-2}u_{mt},w_s\right)\end{aligned} \tag{3.3}$$

for conditions

$$\begin{cases}u_0^m(x)=\sum_{j=1}^{m}a_jw_j(x)\to u_0 & \text{in } W_0^{1,p}(\Omega),\\ u_1^m(x)=\sum_{j=1}^{m}b_jw_j(x)\to u_1 & \text{in } L^k(\Omega),\end{cases}$$

$s=1,2,\ldots m$, where $w\in V_m$ as $m\to\infty$.

This leads to a system of ordinary differantial equations for unknown functions $h_j^m(t)$. Based on standard existence theory for ordinary differantial equation, one can obtain functions

$$h_j:\ [0,t_m)\to R,\qquad j=1,2,\ldots,m,$$

which satisfy (3.3) in a maximal interval $[0,t_m)$, $0<t_m\le T$ and therefore $u_m\in C\left([0,t_m);W_0^{1,p}(\Omega)\right)$, $u_{mt}\in C\left([0,t_m);H^1(\Omega)\right)$.

Step 2: A priori estimates: Our purpose is to show that $t_m=T$ and that the local solution is uniformly bounded independent of m and t. Now, taking the derivative of (3.3) with respect to t, multiplying the obtained equation by $h'_{mj}(t)$ and summing for j=1,2,...,m, we obtain

$$(u_{mtt},w)+\left(|u_{mt}|^{k-2}u_{mt},w\right)+\left(|\nabla u_m|^{p-2}\nabla u_m,\nabla w\right)=\left(|u_m|^{q-2}\ln|u_m|,w\right)$$

for $\forall w\in H_0^1(\Omega)$. Let us replace w by u_{mt} in and integrate by parts we obtain

$$\frac{d}{dt}E_m(t)=-\int_0^t\|u_{mt}(s)\|_k^k\,ds$$

where

$$E_m(t)=\frac{1}{2}\|u_{mt}\|^2+\frac{1}{p}\|\nabla u_m\|_p^p-\frac{1}{q}\int_\Omega|u_m|^q\ln|u_m|\,dx+\frac{1}{q^2}\|u_m\|_q^q.$$

Then Integrating (1.2) with respect to t from 0 to t, we have

$$E_m(t)+\int_0^t\|u_{mt}(s)\|_k^k\,ds=E_m(0). \tag{3.4}$$

Otherwise, for $\alpha > 0$, we obtain

$$\int_{\Omega} |u_m|^q \ln |u_m| \, dx \leq \frac{1}{e\alpha} \|u_m\|_{q+\alpha}^{q+\alpha},$$

where α is taken such that $0 < \alpha < p\left(1+\frac{2}{n}\right) - q$. Then by using Lemma 2.2 and Young's inequality

$$ab \leq \delta a^k + C(\delta) b^l$$

with $k = \frac{p}{q+\alpha}$ and $l = \frac{p(1-\mu)(q+\alpha)}{p-\mu(q+\alpha)}$ for $\delta \in (0,1)$, we have

$$\begin{aligned} \int_{\Omega} |u_m|^q \ln |u_m| \, dx &\leq B \|\nabla u_m\|_p^{\mu(q+\alpha)} \|u_m\|_2^{(1-\mu)(q+\alpha)} \\ &\leq \delta \|\nabla u_m\|_p^p + C(\delta) \|u_m\|_2^{\frac{p(1-\mu)(q+\alpha)}{p-\mu(q+\alpha)}}, \end{aligned} \tag{3.5}$$

where

$$\mu = \left(\frac{1}{2} - \frac{1}{q+\alpha}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{2}\right)^{-1}.$$

Here, we take $\alpha > 0$ such that $p - \mu(q+\alpha)$ and $0 < \alpha < p\left(1+\frac{2}{n}\right) - q$ hold. Let

$$h = \frac{p(1-\mu)(q+\alpha)}{p-\mu(q+\alpha)} = \frac{p(n+q+\alpha) - n(q+\alpha)}{p(2+n) - n(q+\alpha)}$$

then $h > 1$ because $2 < p < q < p\left(1+\frac{2}{n}\right)$. Morever, by the combination of (3.2), (3.4) and (3.5), we obtain

$$E_m(t) \leq C_1 + C_2 \int_0^t E_m^h(s) ds, \tag{3.6}$$

where C_1, C_2 are positive constants independent of m. By using of the Gronwall inequality, we have a positive consatant

$$T < \frac{C_1^{1-h}}{C_2(h-1)}$$

such that

$$E_m(t) \leq C_T \tag{3.7}$$

for any $t \in [0,T]$.

Subsequently, there exists the solution of (3.3) on $[0,T]$, for any m.

On the other hand, multiplying (3.3) by $h_{mj}'(t)$ and summing for s, we derive

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) = E_m(0) \tag{3.8}$$

for $\forall t \in [0,T]$.

By the continuity of J and (3.1), we consider

$$E_m(0) \leq C \tag{3.9}$$

where C is the positive constant for any m.

Therefore, it follows from the definion of $E(t)$, (3.5), (3.7)-(3.9) and using Hölder's inequality, we have

$$\begin{aligned}
C \geq E_m(t) &= \frac{1}{2}\|u_{mt}\|^2 + \frac{1}{p}\|\nabla u_m\|_p^p - \frac{1}{q}\int_\Omega |u_m|^q \ln|u_m|\,dx + \frac{1}{q^2}\|u_m\|_q^q \\
&\geq \frac{1}{p}\|\nabla u_m\|_p^p - \frac{1}{q}\int_\Omega |u_m|^q \ln|u_m|\,dx + \frac{1}{q^2}\|u_m\|_q^q \\
&\geq \frac{1}{p}\|\nabla u_m\|_p^p - \frac{\delta}{q}\|\nabla u_m\|_p^p - \frac{C(\delta)}{q}\|u_m\|_2^{2h} + \frac{1}{q^2}\|u_m\|_q^q \\
&\geq \left(\frac{1}{p} - \frac{\delta}{q}\right)\|\nabla u_m\|_p^p - \frac{C(\delta)}{q}p^{ph}E_m^{ph}(t) + \frac{1}{q^2}\|u_m\|_q^q \\
&\geq \left(\frac{1}{p} - \frac{\delta}{q}\right)\|\nabla u_m\|_p^p + \frac{1}{q^2}\|u_m\|_q^q - C_3.
\end{aligned} \tag{3.10}$$

Combining (3.10) and (3.8), we have

$$\begin{aligned}
\|u_m\|_{L^\infty(0,T;W^{1,p}(\Omega))} &\leq C, \\
\|u_{mt}\|_{L^\infty(0,T;H^1(\Omega))} &\leq C.
\end{aligned} \tag{3.11}$$

It follows from (3.4) and (3.7) that

$$\left\| |\nabla u_m|^{p-2}\nabla u_m \right\|_{L^\infty(0,T;W^{-1,p'}(\Omega))} \leq C. \tag{3.12}$$

Step 3: Passage to the limit: Combining **(3.11)-(3.12),** there are functions u and χ and a subsequence of $\{u_m\}_{m=1}^\infty$ which we still denoted by $\{u_m\}_{m=1}^\infty$ such that

$$\begin{cases}
u_m \to u, \ \text{ weakly}^* \text{in } L^\infty\left(0,T;W_0^{1,p}(\Omega)\right), \\
u_{mt} \to u_t, \text{weakly in } L^\infty\left(0,T;L^k(\Omega)\right), \\
|\nabla u_m|^{p-2}\nabla u_m \to \chi \text{ weakly}^* \text{in } L^\infty\left(0,T;W_0^{-1,p'}(\Omega)\right).
\end{cases}$$

By Aubin–Lions–Simon Lemma we obtain

$$\begin{aligned}
&u_m \to u \text{ strongly in } C([0,T];W_0^{1,p}(\Omega)), \\
&u_m \to u, \text{ a.e. } (x,t) \in \Omega \times (0,T),
\end{aligned}$$

which implies that

$$|u_m|^{q-2}u_m \ln|u_m| \to |u|^{q-2}u\ln|u|, \text{a.e.} \qquad (x,t) \in \Omega \times (0,T). \tag{3.13}$$

On the other side, since $2<p<q<p\left(1+\frac{2}{n}\right)<\frac{np}{n-p}$, we can choose $\alpha>0$ such that $(q-1+\mu)q'<\frac{np}{n-p}$. So, by direct calculation and Sobolev inequality, we note that

$$\begin{aligned}\int_{\Omega}|\psi_m(x,t)|^{q'}dx &= \int_{\{x\in\Omega:|u_m(x,t)|\le 1\}}|\psi_m(x,t)|^{q'}dx+\int_{\{x\in\Omega:|u_m(x,t)|>1\}}|\psi_m(x,t)|^{q'}dx\\ &\le (e(q-1))^{-q'}|\Omega|+(e\alpha)^{-q'}\int_{\{x\in\Omega:|u_m(x,t)|>1\}}|\psi_m(x,t)|^{(q-1+\alpha)q'}dx\\ &\le C_4+C_5\|\nabla u_m(t)\|_p^{(q-1+\alpha)q'}\le C \qquad (3.14)\end{aligned}$$

where $\psi_m(x,t)=|u_m|^{q-2}u_m\ln|u_m|$. And we have used

$$\left|x^{p-1}\log x\right|\le (e(p-1))^{-1}\qquad \text{for } 0<x<1,$$

while $x^{-\alpha}\log x\le\frac{1}{e\alpha}$ for $x\ge 1$, $\alpha>0$, where $\psi_m(x,t)=|u_m|^{q-2}u_m\ln|u_m|$. And we have used $\left|x^{p-1}\log x\right|\le(e(p-1))^{-1}$ for $0<x<1$, while $x^{-\alpha}\log x\le\frac{1}{e\alpha}$ for $x\ge 1$, $\alpha>0$.

Hence, from (3.13), (3.14) and Lions Lemma [15], we get

$$|u_m|^{q-2}u_m\ln|u_m|\to|u|^{q-2}u\ln|u| \text{ weakly}^*\text{in } L^{\infty}\left(0,T;L^{q'}(\Omega)\right).$$

Now, taking the limit in (3.1) as $m\to\infty$, it follows that u satisfies the initial conditions $u(x,0)=u_0$ in $W_0^{1,p}(\Omega)$ and $u_t(x,0)=u_1$ in $H^1(\Omega)$. Additionally, passing to the limit in (3.3), it follows that $t\in[0,T]$

$$\begin{aligned}&(u_t,w_s)+\int_0^t\int_{\Omega}|\nabla u|^{p-2}\nabla u\nabla w_s ds+\int_0^t\int_{\Omega}|u_{mt}|^{k-2}u_{mt}w_s ds\\ &=\int_0^t\left(|u|^{q-2}u\log|u|,w_s\right)ds+(u_1,w_s)\end{aligned}$$

for all $w\in W_0^{1,p}(\Omega)$.

Step 3: Uniqueness: : Firstly, we consider linear problem

$$\begin{cases} v_{tt}+|v_t|^{k-2}v_t-\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) & (x,t)\in\Omega\times(0,T),\\ =f(u_1)-f(u_2), & \\ v(x,0)=v_0(x),\quad v_t(x,0)=v_1(x), & x\in\Omega,\\ v=\frac{\partial v}{\partial n}=0, & x\in\partial\Omega\times R^+.\end{cases}\qquad (3.15)$$

where $f(s)=|s|^{q-2}s\ln|s|$. Suppose there are two solutions u_1 and u_2 to problem (1.1). Then, $v=u_1-u_2$ solves the problem (3.15).

Multiplying both sides of the first equation for above problem (3.15) by v_t and integrating the obtained result over $\Omega \times (0,T)$, then we obtain

$$\int_0^t \int_\Omega v_{tt} v_t \, dx ds + \int_0^t \int_\Omega |v_t|^k \, dx ds + \int_0^t \int_\Omega |\nabla v|^{p-2} \nabla v \nabla v_t dx ds$$
$$= \int_0^t \int_\Omega \left(|u_1|^{q-2} u_1 \ln |u_1| - |u_2|^{q-2} u_2 \ln |u_2| \right) v_t \, dx ds. \tag{3.16}$$

Making use of mean value theorem, we get

$$|f(u_1) - f(u_2)| \times \left| f'(\vartheta u_1 + (1-\vartheta) u_2)(u_1 - u_2) \right|$$
$$\leq [1 + (q-1) \ln |(u_1 + \vartheta u_2)|] \, |(u_1 + \vartheta u_2)|^{q-2} |u_1 - u_2| \tag{3.17}$$

where $0 < \vartheta < 1$. Inserting (3.17) into (3.16), we denote

$$\int_0^t \int_\Omega v_{tt} v_t \, dx ds + \int_0^t \int_\Omega |v_t|^k \, dx ds + \int_0^t \int_\Omega |\nabla v|^{p-2} \nabla v \nabla v_t \, dx ds$$
$$\leq \int_0^t \int_\Omega \left([1 + (q-1) \ln |(u_1 + \vartheta u_2)|] \, |(u_1 + \vartheta u_2)|^{q-2} \right) v v_t \, dx ds$$
$$\leq \int_0^t \int_\Omega |(u_1 + \vartheta u_2)|^{q-2} v v_t \, dx ds$$
$$+ (q-1) \int_0^t \int_\Omega \ln |(u_1 + \vartheta u_2)| \, |(u_1 + \vartheta u_2)|^{q-2} v v_t \, dx ds. \tag{3.18}$$

Morever, from the Lebesgue and Sobolev inequality and Hölder inequality, we obtain

$$\int_0^t \int_\Omega |(u_1 + \vartheta u_2)|^{q-2} v v_t \, dx ds \leq \int_0^t \|u_1 + \vartheta u_2\|_{n(q-2)}^{q-2} \|v\|_{\frac{2n}{n-2}} \|v_t\|_2 \, ds$$
$$\leq C_5^{q-2} C_6 \int_0^t \|\nabla u_1 + \vartheta \nabla u_2\|_2^{q-2} \|\nabla v\|_2 \|v_t\|_2 \, ds$$
$$\leq C_7 \int_0^t \|\nabla v\|_2 \|v_t\|_2 \, ds,$$

$$\leq C_7 \int_0^t \|\nabla v\|_p \|v_t\|_2 \, ds, \tag{3.19}$$

where C_5, C_6, C_7 are the best constants satisfying Sobolev inequality. We used the condition $n(q-2) < p\left(1+\frac{2}{n}\right)$.

Now, our purpose is to estimate the second term of the (3.18). Furthermore, taking $\alpha > 0$ such that $(q-2+\alpha)n < p\left(1+\frac{2}{n}\right)$, and by using the calculation similar to (3.14), it follows that

$$\begin{aligned}
&\int_0^t \int_\Omega \left| \ln|(u_1+\vartheta u_2)| \, |(u_1+\vartheta u_2)|^{q-2} \right|^n v v_t \, dx \, ds \\
&\leq (e(q-1))^{-n} |\Omega| + (e\alpha)^{-n} C_8^{(q-1+\alpha)n} \|(\nabla u_1 + \vartheta \nabla u_2)\|^{(q-1+\alpha)n} \\
&\leq (e(q-1))^{-n} |\Omega| + (e\alpha)^{-n} C_8^{(q-1+\alpha)n} \|(\nabla u_1 + \vartheta \nabla u_2)\|_p^{(q-1+\alpha)n}
\end{aligned} \tag{3.20}$$

where C_8 is the optimal constant satisfying

$$\|(u_1+\vartheta u_2)\|_{(q-1+\alpha)n} \leq \|(\nabla u_1 + \vartheta \nabla u_2)\|^{(q-1+\alpha)n}.$$

Inserting (3.20) into (3.18), we obtain

$$\begin{aligned}
&(q-1) \int_0^t \int_\Omega \ln|(u_1+\vartheta u_2)| \, |(u_1+\vartheta u_2)|^{q-2} v v_t \, dx \, ds \\
&\leq (q-1) \int_0^t \left(\int_\Omega \ln|(u_1+\vartheta u_2)| \, |(u_1+\vartheta u_2)|^{q-2} \right)^{\frac{1}{n}} \|v\|_{\frac{2n}{n-2}} \|v_t\|_2 \, ds \\
&\leq C_9 \int_0^t \|\nabla v\|_2 \|v_t\|_2 \, ds \leq C_{10} \int_0^t \|\nabla v\|_p \|v_t\|_2 \, ds.
\end{aligned} \tag{3.21}$$

Inserting (3.19) and (3.21) into (3.18) and using $v(x,0) = 0, v_t(x,0) = 0$, we have

$$\|v_t\|^2 + \|\nabla v\|_p^p \leq C \int_0^t \|\nabla v\|_p \|v_t\|_2 \, ds \leq \int_0^t \left(\|v_t\|^2 + \left(\|\nabla v\|_p^p \right)^{\frac{2}{p}} \right).$$

Using the algebraic inequality

$$z^\nu \leq z+1 \leq \left(1+\frac{1}{\alpha}\right)(z+\alpha), \qquad \forall z \geq 0,\ 0 < \nu \leq 1,\ \alpha \geq 0,$$

we obtain

$$\left(\|\nabla v\|_p^p \right)^{\frac{2}{p}} \leq 1 + \|\nabla v\|_p^p$$

where $p > 2$. The uniqueness is derived from the Gronwall's inequality.

Step 3: Energy inequality : We will show that the solutions u satisfy (3.4). First, we prove that

$$\int_\Omega |u|^q \ln|u|\, dx = \lim_{m\to\infty} \int_\Omega |u_m|^q \ln|u_m|\, dx, \tag{3.22}$$

$$\int_\Omega |u|^q\, dx = \lim_{m\to\infty} \int_\Omega |u_m|^q\, dx. \tag{3.23}$$

Additionally, for each fixed $t > 0$, by similar calculation to (3.18) and Hölder inequality, we obtain

$$\begin{aligned}
&\left| \int_\Omega |u_m|^q \ln|u_m|\, dx - \int_\Omega |u|^q \ln|u|\, dx \right| \\
&\leq \int_\Omega \left| q\,|\sigma_{1m}|^{q-1} \ln|\sigma_m| + |\sigma_{1m}|^{q-1} \right| |u-u_m|\, dx \\
&\leq q \int_\Omega \left(\left| |\sigma_{1m}|^{q-1} \ln|\sigma_{1m}| \right|^{q'} dx \right)^{\frac{1}{q'}} \|u-u_m\|_q + \|\sigma_{1m}\|_q^{q-1} \|u-u_m\|_q \\
&\leq C \|u-u_m\|_q \to 0
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_\Omega |u_m|^q\, dx - \int_\Omega |u|^q\, dx \right| &\leq \int_\Omega \left| |u_m|^q - |u|^q \right| dx \\
&\leq q \int_\Omega |\sigma_{2m}|^{q-1} |u-u_m|\, dx \\
&\leq q \|\sigma_{2m}\|_q^{q-1} \|u-u_m\|_q \leq C \|u-u_m\|_q \to 0,
\end{aligned}$$

as $m \to \infty$, where $\sigma_i = u + \vartheta_i u_m$, $0 < \vartheta_i < 1$ $(i = 1,2)$. Morever, (3.22) and (3.23) hold.

On the other hand, from initial and boundary condition of the (3.3), it follows that $E(u_{0m}, u_{1m}) \to E(u_0, u_1) = E(0)$ as $m \to \infty$. Therefore, making use of Fatou Lemma and (3.4), we note that

$$\begin{aligned}
&\frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p + \int_0^t \|u_t(s)\|_k^k\, ds \\
&\leq \frac{1}{2} \liminf_{m\to\infty} \|u_{mt}\|^2 + \frac{1}{p} \liminf_{m\to\infty} \|\nabla u_m\|_p^p + \int_0^t \liminf_{m\to\infty} \|u_{mt}(s)\|_k^k\, ds
\end{aligned}$$

$$\leq \lim_{m\to\infty} \inf \left[\frac{1}{2} \|u_{mt}\|^2 + \frac{1}{p} \|\nabla u_m\|_p^p + \int_0^t \|u_t(s)\|_k^k ds \right]$$

$$= \lim_{m\to\infty} \inf \left[E_m(0) + \frac{1}{q} \int_\Omega |u_m|^q \ln |u_m| \, dx - \frac{1}{q^2} \|u_m\|_q^q \right]$$

$$= E(0) + \frac{1}{q} \int_\Omega |u|^q \ln |u| \, dx - \frac{1}{q^2} \|u\|_q^q .$$

So that the proof is completed. □

REFERENCES

[1] I. Bialynicki-Birula and J. Mycielski, "Wave equations with logarithmic nonlinearities," *Bulletin de l'Academie Polonaise des Sciences. Serie des Sciences, Mathematiques, Astronomiques et Physiques*, vol. 23, no. 4, pp. 461–466, 1975.

[2] S. M. Boulaaras, A. Choucha, A. Zara, M. Abdalla, and B.-B. Cheri, "Global existence and decay estimates of energy of solutions for a new class of p-Laplacian heat equations with logarithmic nonlinearity," *J. Funct. Spaces*, vol. 2021, p. 11, 2021, id/No 5558818, doi: 10.1155/2021/5558818.

[3] H. Buljan, A. Šiber, M. Soljačić, T. Schwartz, M. Segev, and D. N. Christodoulides, "Incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media," *Physical Review E*, vol. 68, no. 3, p. 036607, 2003, doi: 10.1103/PhysRevE.68.036607.

[4] Y. Cao and C. Liu, "Initial boundary value problem for a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity," *Electron. J. Differ. Equ.*, vol. 2018, p. 19, 2018, id/No 116. [Online]. Available: ejde.math.txstate.edu/Volumes/2018/116/abstr.html#latest

[5] T. Cazenave and A. Haraux, "Equations d'évolution avec non linearite logarithmique," vol. 2," doi: 10.5802/afst.543, 1980, pp. 21–51.

[6] T. G. Chakuvinga and F. S. Topal, "Existence of positive solutions for the nonlinear fractional boundary value problems with p-Laplacian," *Filomat*, vol. 35, no. 9, pp. 2927–2949, 2021, doi: 10.2298/FIL2109927C.

[7] H. Chen and S. Tian, "Initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity," *J. Differ. Equations*, vol. 258, no. 12, pp. 4424–4442, 2015, doi: 10.1016/j.jde.2015.01.038.

[8] H. Ding and J. Zhou, "Corrigendum to: "Global existence and blow-up for a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity"," *J. Math. Anal. Appl.*, vol. 489, no. 1, p. 2, 2020, id/No 124164, doi: 10.1016/j.jmaa.2020.124164.

[9] P. Górka, "Logarithmic Klein-Gordon equation," *Acta Phys. Pol. B*, vol. 40, no. 1, pp. 59–66, 2009. [Online]. Available: www.actaphys.uj.edu.pl/fulltext?series=Reg&vol=40&page=59

[10] X. Han, "Global existence of weak solutions for a logarithmic wave equation arising from Q-ball dynamics," *Bull. Korean Math. Soc.*, vol. 50, no. 1, pp. 275–283, 2013, doi: 10.4134/BKMS.2013.50.1.275. [Online]. Available: mathnet.or.kr/mathnet/kms_content.php?no=411018

[11] Y. He, H. Gao, and H. Wang, "Blow-up and decay for a class of pseudo-parabolic p-Laplacian equation with logarithmic nonlinearity," *Comput. Math. Appl.*, vol. 75, no. 2, pp. 459–469, 2018, doi: 10.1016/j.camwa.2017.09.027.

[12] N. Irkıl, "On the p-Laplaciantype equation with logarithmic nonlinearity: existence, decay and blow up," *Filomat*, vol. 37, no. 16, pp. 5485–5507, 2023, doi: 10.2298/FIL2316485I.
[13] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, "Linear and quasilinear equations of parabolic type.(russian)," Moskva: Izdat. 'Nauka'. 736 pp. (1967)., 1967.
[14] C. N. Le and X. T. Le, "Global solution and blow-up for a class of p-Laplacian evolution equations with logarithmic nonlinearity," *Acta Appl. Math.*, vol. 151, no. 1, pp. 149–169, 2017, doi: 10.1007/s10440-017-0106-5.
[15] J.-L. Lions, "Quelques méthodes de résolution de problemes aux limites non linéaires," 1969.
[16] G. Liu, "The existence, general decay and blow-up for a plate equation with nonlinear damping and a logarithmic source term," *Electron. Res. Arch.*, vol. 28, no. 1, pp. 263–289, 2020, doi: 10.3934/era.2020016.
[17] L. E. Payne and D. H. Sattinger, "Saddle points and instability of nonlinear hyperbolic equations," *Isr. J. Math.*, vol. 22, pp. 273–303, 1976, doi: 10.1007/BF02761595.
[18] E. Pişkin, S. Boulaaras, and N. Irkil, "Qualitative analysis of solutions for the p-Laplacian hyperbolic equation with logarithmic nonlinearity," *Math. Methods Appl. Sci.*, vol. 44, no. 6, pp. 4654–4672, 2021, doi: 10.1002/mma.7058.
[19] M. A. Ragusa and A. Razani, "Existence of a periodic solution for a coupled system of differential equations," in *AIP Conference Proceedings*, vol. 2425, no. 1, doi: 10.1063/5.0081381, 2022, p. 370004.

Author's address

Nazlı Irkıl
Mardin Anatolian High School, Department of Mathematics, 47100 Mardin, Turkey
E-mail address: nazliirkil@gmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 789–804

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.3838

ON n-DERIVATIONS OF GENERALIZED MATRIX ALGEBRAS

AISHA JABEEN

Received 14 May, 2021

Abstract. In this article, we study n-derivations on generalized matrix algebras under certain restrictions and find that every n-derivation is an extremal n-derivation on generalized matrix algebras.

2010 *Mathematics Subject Classification:* 16W25; 15A78; 47L35

Keywords: generalized matrix algebra, n-derivation, derivation

1. Historical Development

As far as we know numerous algebraists studied n-derivations on variety of rings and algebras which can be seen in [1–4, 6–15, 18] and bibliographic content existing therein. In the year 1993, Brešar et al. [6] proved that 'every biderivation over a noncommutative prime ring can be described as inner biderivation'. Also, in [8] Brešar investigated biderivations on semiprime rings. The readers are encouraged to read the survey paper [10, Section 3] where applications of biderivations to other fields are also described. Benkovič in [4] defined the concept of an extremal biderivation and proved that 'under certain conditions a biderivation of a triangular algebra is a sum of an extremal and an inner biderivation.' Ghosseiri [13] showed that 'every biderivation of upper triangular matrix rings is decomposed into the sum of three biderivations D, ψ and Δ, where $D(E_{11}, E_{11}) = 0$, ψ is an extremal biderivation and Δ is a special kind of biderivation.' Moreover, they proved that ' every biderivation of upper triangular matrices over a noncommutative prime ring is inner which extended some results due to Benkovič [4].' Du and Wang [12] proved that 'under certain conditions a biderivation of a generalized matrix algebra is a sum of an extremal and an inner biderivation.' Also they considered the question 'when a biderivation of a generalized matrix algebra is an inner biderivation?' and showed that 'every biderivation of a full matrix algebra over a unital algebra is inner.' Apart from associative algebras or rings, many authors studied biderivation and related maps on various types of Lie algebras for example see [7, 11] and references therein.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

In [18], Wang et al. explored 'n-derivations ($n \geq 3$) on a certain class of triangular algebras.' Also, they put on their major findings on upper triangular matrix algebras and nest algebras. In the light of above literature, we study the n-derivations on generalized matrix algebras and prove that every n-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ is an extremal n-derivation ζ such that $\zeta(x_1,x_2,\ldots,x_n)=[x_1,[x_2,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$, where $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$ under certain restrictions.

2. Basic Definitions & Preliminaries

Let $\mathcal{A}$ be an algebra over a commutative ring R with unity. For any $x,y\in\mathcal{A}$, $[x,y]=xy-yx$ denotes the commutator and $\mathfrak{Z}(\mathcal{A})$ denote the center of $\mathcal{A}$. An R-linear map $d\colon \mathcal{A}\to\mathcal{A}$ is said to be a derivation if $d(xy)=d(x)y+xd(y)$ for all $x,y\in\mathcal{A}$. If derivation d takes form $d(x)=[x,a]$ for some fixed $a\in\mathcal{A}$, then d is called an inner derivation on $\mathcal{A}$.

An n-linear map $\phi\colon \mathcal{A}\times\mathcal{A}\times\cdots\times\mathcal{A}\to\mathcal{A}$ is said to be an n-derivation, if it is a derivation in each component. In particular, a 2-derivation is a biderivation. A permuting n-derivation $\zeta\colon \mathcal{A}\times\mathcal{A}\times\cdots\times\mathcal{A}\to\mathcal{A}$ is said to be an extremal n-derivation if it is of the form $\zeta(x_1,x_2,\ldots,x_n)=[x_1,[x_2,\ldots,[x_n,a]\ldots]]$ for all $x_1,x_2,\ldots,x_n\in\mathcal{A}$, where $a\in\mathcal{A}$ and $a\notin\mathfrak{Z}(\mathcal{A})$ such that $[[\mathcal{A},\mathcal{A}],a]=0$. An extremal 2-derivation is said to be an extremal biderivation. If $\mathcal{A}$ is a noncommutative algebra, then the map $\phi(x,y)=\lambda[x,y]$ for all $x,y\in\mathcal{A}$, where $\lambda\in\mathfrak{Z}(\mathcal{A})$ is called an inner biderivation.

Let A and B be two unital algebras with unity 1_{A} and 1_{B}, respectively. A *Morita context* consists of two unital R-algebras A and B, two bimodules (A,B)-bimodule M and (B,A)-bimodule N, and two bimodule homomorphisms called the bilinear pairings $\xi_{\mathrm{MN}}\colon \mathrm{M}\underset{\mathrm{B}}{\otimes}\mathrm{N}\longrightarrow\mathrm{A}$ and $\Omega_{\mathrm{NM}}\colon \mathrm{N}\underset{\mathrm{A}}{\otimes}\mathrm{M}\longrightarrow\mathrm{B}$ satisfying the following commutative diagrams:

$$\begin{tikzcd}
\mathrm{M}\underset{\mathrm{B}}{\otimes}\mathrm{N}\underset{\mathrm{A}}{\otimes}\mathrm{M} \arrow[r, "\xi_{\mathrm{MN}}\otimes I_{\mathrm{M}}"] \arrow[d, "I_{\mathrm{M}}\otimes\Omega_{\mathrm{NM}}"] & \mathrm{A}\underset{\mathrm{A}}{\otimes}\mathrm{M} \arrow[d, "\cong"] \\
\mathrm{M}\underset{\mathrm{B}}{\otimes}\mathrm{B} \arrow[r, "\cong"] & \mathrm{M}
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathrm{N}\underset{\mathrm{A}}{\otimes}\mathrm{M}\underset{\mathrm{B}}{\otimes}\mathrm{N} \arrow[r, "\Omega_{\mathrm{NM}}\otimes I_{\mathrm{N}}"] \arrow[d, "I_{\mathrm{N}}\otimes\xi_{\mathrm{MN}}"] & \mathrm{B}\underset{\mathrm{B}}{\otimes}\mathrm{N} \arrow[d, "\cong"] \\
\mathrm{N}\underset{\mathrm{A}}{\otimes}\mathrm{A} \arrow[r, "\cong"] & \mathrm{N}\,.
\end{tikzcd}$$

If $(\mathrm{A},\mathrm{B},\mathrm{M},\mathrm{N},\,\xi_{\mathrm{MN}},\Omega_{\mathrm{NM}})$ is a Morita context (refer [17] basic properties of Morita context), then the set

$$\begin{bmatrix} \mathrm{A} & \mathrm{M} \\ \mathrm{N} & \mathrm{B} \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \,\middle|\, a\in\mathrm{A}, m\in\mathrm{M}, n\in\mathrm{N}, b\in\mathrm{B} \right\}$$

forms an R-algebra under matrix addition and matrix-like multiplication, where at least one of the two bimodules M and N is distinct from zero. This kind of R-algebra

introduced by Morita [17] is generally called as *generalized matrix algebra* of order 2 and represented by

$$\mathfrak{G} = \mathfrak{G}(\mathrm{A},\mathrm{M},\mathrm{N},\mathrm{B}) = \begin{bmatrix} \mathrm{A} & \mathrm{M} \\ \mathrm{N} & \mathrm{B} \end{bmatrix}.$$

All associative algebras with nontrivial idempotents are isomorphic to generalized matrix algebras. The familiar examples of generalized matrix algebras are full matrix algebras and triangular algebras [12]. Also, if $\mathrm{N} = 0$, then $\mathfrak{G}$ is called a triangular algebra.

The center of $\mathfrak{G}$ is $\mathfrak{Z}(\mathfrak{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \, am = mb, na = bn \text{ for all } m \in \mathrm{M}, n \in \mathrm{N} \right\}$. Also, note that the center $\mathfrak{Z}(\mathfrak{G})$ consists of all diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a \in \mathfrak{Z}(\mathrm{A}), b \in \mathfrak{Z}(\mathrm{B})$ and $am = mb, na = bn$ for all $m \in \mathrm{M}, n \in \mathrm{N}$. However, if we assume that M is faithful as a left A-module and also as a right B-module, then the conditions $a \in \mathfrak{Z}(\mathrm{A})$ and $b \in \mathfrak{Z}(\mathrm{B})$ become redundant and can be deleted.

Define two natural projections $\pi_{\mathrm{A}} \colon \mathfrak{G} \to \mathrm{A}$ and $\pi_{\mathrm{B}} \colon \mathfrak{G} \to \mathrm{B}$ by $\pi_{\mathrm{A}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = a$ and $\pi_{\mathrm{B}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = b$. Moreover, $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathrm{A})$ and $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathrm{B})$ and there exists a unique algebraic isomorphism $\eta \colon \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) \to \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$ such that $am = m\eta(a)$ and $na = \eta(a)n$ for all $a \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})), m \in \mathrm{M}$ and $n \in \mathrm{N}$.

Let 1_{A} (resp. 1_{B}) be the unit of the algebra A (resp. B) and let I be the unity of generalized matrix algebra $\mathfrak{G}$, $e = \begin{bmatrix} 1_{\mathrm{A}} & 0 \\ 0 & 0 \end{bmatrix}$, $f = I - e = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathrm{B}} \end{bmatrix}$ and $\mathfrak{G}_{11} = e\mathfrak{G}e$, $\mathfrak{G}_{12} = e\mathfrak{G}f$, $\mathfrak{G}_{21} = f\mathfrak{G}e$, $\mathfrak{G}_{22} = f\mathfrak{G}f$. Thus $\mathfrak{G} = e\mathfrak{G}e + e\mathfrak{G}f + f\mathfrak{G}e + f\mathfrak{G}f = \mathfrak{G}_{11} + \mathfrak{G}_{12} + \mathfrak{G}_{21} + \mathfrak{G}_{22}$ where $\mathfrak{G}_{11}$ is subalgebra of $\mathfrak{G}$ isomorphic to A, $\mathfrak{G}_{22}$ is subalgebra of $\mathfrak{G}$ isomorphic to B, $\mathfrak{G}_{12}$ is $(\mathfrak{G}_{11}, \mathfrak{G}_{22})$-bimodule isomorphic to M and $\mathfrak{G}_{21}$ is $(\mathfrak{G}_{22}, \mathfrak{G}_{11})$-bimodule isomorphic to N. Also, $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ and $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$ are isomorphic to $e\mathfrak{Z}(\mathfrak{G})e$ and $f\mathfrak{Z}(\mathfrak{G})f$ respectively. Then there is an algebra isomorphisms $\eta \colon e\mathfrak{Z}(\mathfrak{G})e \to f\mathfrak{Z}(\mathfrak{G})f$ such that $am = m\eta(a)$ and $na = \eta(a)n$ for all $m \in e\mathfrak{G}f$ and $n \in f\mathfrak{G}e$. Also, through the rest of paper, it is assume that M is faithful as a left A-module and also as a right B-module.

An (A,B)-bimodule homomorphism $\mathfrak{f} \colon \mathrm{M} \to \mathrm{M}$ is of the standard form if there exist $a_0 \in \mathfrak{Z}(\mathrm{A})$, $b_0 \in \mathfrak{Z}(\mathrm{B})$ such that $\mathfrak{f}(m) = a_0 m + m b_0$ for all $m \in \mathrm{M}$. Similarly, (B,A)-bimodule homomorphism $\mathfrak{g} \colon \mathrm{N} \to \mathrm{N}$ is of the standard form if there exist $a \in \mathfrak{Z}(\mathrm{A}), b \in \mathfrak{Z}(\mathrm{B})$ such that $\mathfrak{g}(n) = na + bn$ for all $n \in \mathrm{N}$. We say that a pair of bimodule homomorphisms $\mathfrak{f} \colon \mathrm{M} \to \mathrm{M}$ and $\mathfrak{g} \colon \mathrm{N} \to \mathrm{N}$ is special if $\mathfrak{f}(m)n + m\mathfrak{g}(n) = 0 = n\mathfrak{f}(m) + \mathfrak{g}(n)m$ for all $m \in \mathrm{M}$ and $n \in \mathrm{N}$.

A special pair of bimodule homomorphisms $\mathfrak{f} \colon \mathrm{M} \to \mathrm{M}$ and $\mathfrak{g} \colon \mathrm{N} \to \mathrm{N}$ is of the standard form if there exist $a_0 \in \mathfrak{Z}(\mathrm{A}), b_0 \in \mathfrak{Z}(\mathrm{B})$ such that $\mathfrak{f}(m) = a_0 m + m b_0$ and $\mathfrak{g}(n) = -na_0 - b_0 n$ for all $m \in \mathrm{M}, n \in \mathrm{N}$.

Now we should mention some important results which are subsequently used in this article.

Lemma 1 ([9, Corollary 2.4])**.** *Let* $\phi\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ *be a biderivation. Then*

$$\phi(x,y)[u,v]=[x,y]\phi(u,v) \qquad \text{for all } x,y,u,v\in\mathcal{A}.$$

Lemma 2 ([12, Proposition 3.3])**.** *Let* $\mathfrak{G}$ *be a generalized matrix algebra such that*

(1) *for each* $n\in\mathrm{N}$ *the condition* $\mathrm{M}n=0=n\mathrm{M}$ *implies* $n=0$,

(2) *every* (A,B)*-bimodule homomorphism of* M *has the standard form.*

Then each special pair of bimodule homomorphisms $\mathfrak{f}\colon \mathrm{M}\to\mathrm{M}$ *and* $\mathfrak{g}\colon \mathrm{N}\to\mathrm{N}$ *has the standard form.*

Lemma 3 ([12, Proposition 3.4])**.** *Suppose that every derivation of a generalized matrix algebra* $\mathfrak{G}$ *is inner. Then every special pair of bimodule homomorphisms* $\mathfrak{f}\colon \mathrm{M}\to\mathrm{M}$ *and* $\mathfrak{g}\colon \mathrm{N}\to\mathrm{N}$ *has the standard form.*

3. Key Content

In this part, we study significant results of the article and we initiate with the following basic facts:

Lemma 4. *Let* $\mathfrak{G}$ *be a generalized matrix algebra and* $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ *be a 3-derivation. If* $[x,y]=0$*, then* $\phi(x,y,z)\in\mathrm{M}+\mathrm{N}\ \forall\, x,y,z\in\mathfrak{G}$*. Moreover, suppose that*

(1) *for each* $n\in\mathrm{N}$ *the condition* $\mathrm{M}n=0=n\mathrm{M}$ *implies* $n=0$,

(2) *for each* $m\in\mathrm{M}$ *the condition* $m\mathrm{N}=0=\mathrm{N}m$ *implies* $m=0$,

then $\phi(x,y,z)=0$ *for all* $x,y,z\in\mathfrak{G}$.

Proof. Define a map $\phi_z\colon \mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ for some fix $z\in\mathfrak{G}$ by

$$\phi_z(x,y)=\phi(x,y,z) \qquad \text{for all } x,y\in\mathfrak{G}.$$

Thence ϕ_z is a biderivation on $\mathfrak{G}$ and with [9, Corollory 2.4], we find that

$$\begin{aligned}\phi_z(x,y)[u,v]&=[x,y]\phi_z(u,v)\\ \phi(x,y,z)[u,v]&=[x,y]\phi(u,v,z) \qquad \text{for all } x,y,z,u,v\in\mathfrak{G}.\end{aligned} \tag{3.1}$$

Again, we define a map $\psi_y\colon \mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ for any fixed $y\in\mathfrak{G}$ by

$$\psi_y(x,z)=\phi(x,y,z) \qquad \text{for all } x,y,z\in\mathfrak{G}.$$

It follows that

$$\phi(x,y,z)[u,v]=[x,z]\phi(u,y,v) \qquad \text{for all } x,y,z,u,v\in\mathfrak{G}. \tag{3.2}$$

On comparison of (3.1) and (3.2), we find that

$$[x,y]\phi(u,v,z)=[x,z]\phi(u,y,v) \qquad \text{for all } x,y,z,u,v\in\mathfrak{G}. \tag{3.3}$$

Further (3.2) can be rewritten as

$$\phi(x,v,y)[u,z]=[x,y]\phi(u,v,z) \qquad \text{for all } x,y,z,u,v\in\mathfrak{G}. \tag{3.4}$$

From (3.1) and (3.4), we find that

$$\phi(x,y,z)[u,v] = \phi(x,v,y)[u,z] \qquad \text{for all } x,y,z,u,v \in \mathfrak{G}. \tag{3.5}$$

In view of (3.1), we obtain

$$\begin{aligned} \phi(x,y,z)[e,m] &= [x,y]\phi(e,m,z) = 0 \\ [e,m]\phi(x,y,z) &= \phi(e,m,z)[x,y] = 0. \end{aligned}$$

Implying to $\phi(x,y,z)\mathrm{M} = 0 = \mathrm{M}\phi(x,y,z)$ for all $x,y,z \in \mathfrak{G}$ and hence we have

$$\begin{aligned} e\phi(u,v,z)e\mathrm{M} &= 0 = \mathrm{M}f\phi(u,v,z)f, \\ f\phi(u,v,z)e\mathrm{M} &= 0 = \mathrm{M}f\phi(u,v,z)e \ \forall\ u,v,z \in \mathfrak{G}. \end{aligned} \tag{3.6}$$

Since M is a faithful left A-module and a faithful right B-module, we find $e\phi(u,v,z)e = 0 = f\phi(u,v,z)f$. Hence $\phi(u,v,z) = e\phi(u,v,z)f + f\phi(u,v,z)$, that is, $\phi(u,v,z) \in \mathrm{M} + \mathrm{N}$ for all $u,v,z \in \mathfrak{G}$.

Now, suppose that the conditions (1) and (2) are true. It follows from (3.6), $f\phi(u,v,z)e = 0$. Similarly, we can show that $e\phi(u,v,z)f = 0$. Therefore, $\phi(u,v,z) = 0$ for all $u,v,z \in \mathfrak{G}$. □

Now it is easy to verify following lemma:

Lemma 5. *Let $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \to \mathfrak{G}$ be a 3-derivation. Then*

(1) $\phi(x,y,1) = \phi(1,x,y) = \phi(x,1,y) = 0$ *for all* $x,y \in \mathfrak{G}$,
(2) $\phi(x,y,0) = \phi(0,x,y) = \phi(x,0,y) = 0$ *for all* $x,y \in \mathfrak{G}$,
(3) $\phi(e,e,e) = -\phi(e,e,f) = \phi(e,f,f) = -\phi(f,f,f)$
$= \phi(f,e,f) = -\phi(e,f,e) = -\phi(f,e,e) = \phi(f,f,e)$.

Proposition 1. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring* R *and $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \to \mathfrak{G}$ be a 3-derivation on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{B})$,
(2) *If* $\alpha a = 0, \alpha \in \mathfrak{Z}(\mathfrak{G}), 0 \neq a \in \mathfrak{G}$, *then* $\alpha = 0$.
(3) *If* $\mathrm{MN} = 0 = \mathrm{NM}$, *then at least one of the algebras* A *and* B *is noncommutative.*
(4) *Every special pair of bimodule homomorphisms has the standard form.*

Then every 3-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \to \mathfrak{G}$ is an extremal 3-derivation ζ such that $\zeta(x_1,x_2,x_3) = [x_1,[x_2,[x_3,\phi(e,e,e)]]]$, where $\phi(e,e,e) \notin \mathfrak{Z}(\mathfrak{G})$.

Proof. Let ϕ be a 3-derivation with $\phi(e,e,e) \neq 0$ and $\phi(e,e,e) \notin \mathfrak{Z}(\mathfrak{G})$. Then with Lemma 4, we see that $\phi(x,y,z) \notin \mathfrak{Z}(\mathfrak{G})$. From (3.1) it follows that

$$\begin{aligned} \phi(e,e,e)[u,v] &= [e,e]\phi(u,v,e) = 0 \qquad \text{for all } u,v \in \mathfrak{G}, \\ [x,y]\phi(e,e,e) &= \phi(x,y,e)[e,e] = 0 \qquad \text{for all } x,y \in \mathfrak{G}, \end{aligned}$$

this leads to $[\phi(e,e,e),[\mathfrak{G},\mathfrak{G}]] = 0$, then the map defined by

$$\zeta(x,y,z) = [x,[y,[z,\phi(e,e,e)]]]$$

is an extremal 3-derivation of $\mathfrak{G}$. We note that

$$\begin{aligned}\zeta(e,e,e) &= [e,[e,[e,\phi(e,e,e)]]]\\ &= [e,[e,[e,e\phi(e,e,e)f - f\phi(e,e,e)e]]]\\ &= e\phi(e,e,e)f + f\phi(e,e,e)e = \phi(e,e,e).\end{aligned}$$

Let $\psi = \phi - \zeta$. Then ψ is a 3-derivation of $\mathfrak{G}$ satisfying $\psi(e,e,e) = 0$. Now we have to show that every 3-derivations $\psi = 0$ with $\psi(e,e,e) = 0$. We will prove this argument via following sequence of claims:

Claim 1. *For any $x \in \mathrm{A} \cup \mathrm{B}$, $m \in \mathrm{M}$ and $n \in \mathrm{N}$, we have*

(i) $\psi(x,y,m) = \psi(y,x,m) = \psi(x,m,y) = \psi(y,m,x) = \psi(m,x,y) = \psi(m,y,x) = 0$ *for all* $y \in \mathrm{A} \cup \mathrm{M} \cup \mathrm{B}$,

(ii) $\psi(x,y,n) = \psi(y,x,n) = \psi(x,n,y) = \psi(y,n,x) = \psi(n,x,y) = \psi(n,y,x) = 0$ *for all* $y \in \mathrm{A} \cup \mathrm{N} \cup \mathrm{B}$.

(iii) $\psi(x,y,z) = 0$ *for all* $y,z \in \mathrm{A} \cup \mathrm{B}$.

Since $\psi(e,e,e) = 0$, we find that

$$\begin{aligned}\psi(a_1,a_2,a_3) &= \psi(ea_1e,a_2,a_3)\\ &= e\psi(a_1,a_2,a_3)e + \psi(e,a_2,a_3)a_1 + a_1\psi(e,a_2,a_3)\\ &= e\psi(a_1,a_2,a_3)e + f\psi(e,ea_2e,a_3)a_1 + a_1\psi(e,ea_2e,a_3)f\\ &= e\psi(a_1,a_2,a_3)e + f\psi(e,e,ea_3e)a_2a_1 + a_1a_2\psi(e,e,ea_3e)f\\ &= e\psi(a_1,a_2,a_3)e + f\psi(e,e,e)a_3a_2a_1 + a_1a_2a_3\psi(e,e,e)f\\ &= e\psi(a_1,a_2,a_3)e \in \mathrm{A}. \qquad (3.7)\end{aligned}$$

for all $a_1,a_2,a_3 \in \mathrm{A}$. Since $\psi(f,f,f) = -\psi(e,e,e) = 0$ and by similar calculation, we have $\psi(b_1,b_2,b_3) \in \mathrm{B}$ for all $b_1,b_2,b_3 \in \mathrm{B}$. Also note that $\psi(e,e,f) = -\psi(e,e,e) = 0$. Then in view of Lemma 4, we have

$$\begin{aligned}\psi(a_1,a_2,b) &= e\psi(a_1,a_2,b)f + f\psi(a_1,a_2,b)e\\ &= e\psi(ea_1e,a_2,b)f + f\psi(ea_1e,a_2,b)e\\ &= a_1\psi(e,ea_2e,b)f + f\psi(e,ea_2e,b)a_1\\ &= a_1a_2\psi(e,e,fbf)f + f\psi(e,e,fbf)a_2a_1\\ &= a_1a_2\psi(e,e,f)b + b\psi(e,e,f)a_2a_1 = 0. \qquad (3.8)\end{aligned}$$

for all $a_1,a_2 \in \mathrm{A}$ and $b \in \mathrm{B}$. In view of (3.7), for any $x \in \mathrm{A}, y \in \mathrm{A}, m \in \mathrm{M}$, we have

$$\begin{aligned}\psi(x,y,m) &= \psi(x,y,emf)\\ &= em\psi(x,y,f) + \psi(x,y,em)f\\ &= em\psi(x,y,f)e + e\psi(x,y,m)f + f\psi(x,y,e)mf\\ &= e\psi(x,y,m)f \in \mathrm{M}.\end{aligned}$$

Similarly, we can have $\psi(x,y,m) \in \mathrm{M}$ for all $x \in \mathrm{A} \cup \mathrm{B}, y \in \mathrm{A} \cup \mathrm{M} \cup \mathrm{B}$ and $m \in \mathrm{M}$. With similar reasons, we can obtain $\psi(x,y,n) \in \mathrm{N}$ for all $x \in \mathrm{A} \cup \mathrm{B}, y \in \mathrm{A} \cup \mathrm{N} \cup \mathrm{B}, n \in \mathrm{N}$ and also rest of the cases.

For fix $y \in \mathfrak{G}$, define maps $\mathfrak{f}\colon \mathrm{M} \to \mathrm{M}$ and $\mathfrak{g}\colon \mathrm{N} \to \mathrm{N}$ by $\mathfrak{f}(m) = e\psi(e,y,m)f$ and $\mathfrak{g}(n) = f\psi(e,y,n)e$ for all $m \in \mathrm{M},\ n \in \mathrm{N}$ respectively. Then $\mathfrak{f}$ and $\mathfrak{g}$ are bimodule homomorphisms. Namely for all $a \in \mathrm{A}, b \in \mathrm{B}, m \in \mathrm{M}$ we get

$$\begin{aligned}\mathfrak{f}(amb) &= e\psi(e,y,amb)f \\ &= e\psi(e,y,a)mb + a\psi(e,y,m)b + am\psi(e,y,b)f \\ &= a\psi(e,y,m)b = a\mathfrak{f}(m)b.\end{aligned}$$

Similarly we obtain that $\mathfrak{g}(bna) = b\mathfrak{g}(n)a$ for all $n \in \mathrm{N}$. Moreover, we find that

$$\begin{aligned}\mathfrak{f}(m)n + m\mathfrak{g}(n) &= e\psi(e,y,m)fn + mf\psi(e,y,n)e = e\psi(e,y,mn)e = 0, \\ \mathfrak{g}(n)m + n\mathfrak{f}(m) &= f\psi(e,y,n)em + ne\psi(e,y,m)f = f\psi(e,y,nm)f = 0.\end{aligned}$$

By assumption (4) the bimodule homomorphisms $\mathfrak{f}$, $\mathfrak{g}$ have the standard form, then

$$\mathfrak{f}(m) = a_0m + mb_0 \text{ and } \mathfrak{g}(n) = -na_0 - b_0n \text{ for } a_0 \in \mathfrak{Z}(\mathrm{A}), b_0 \in \mathfrak{Z}(\mathrm{B}).$$

With assumption (1), we see that $a_0 \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ and $b_0 \in \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$. We may write

$$\begin{aligned}\mathfrak{f}(m) &= (a_0 + \eta^{-1}(b_0))m = \alpha_y m \qquad \text{for all } m \in \mathrm{M}, \\ \mathfrak{g}(n) &= -n(a_0 + \eta^{-1}(b_0)) = -n\alpha_y \qquad \text{for all } n \in \mathrm{N},\end{aligned}$$

where $\alpha_y = a_0 + \eta^{-1}(b_0) \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ (depending on y). Suppose first that $\mathrm{MN} \neq 0$ or $\mathrm{NM} \neq 0$. Then by (3.5) we have

$$\begin{aligned}\psi(e,y,m)[f,n] &= \psi(e,n,y)[f,m] \\ \Longrightarrow\ e\psi(e,y,m)n &= -e\psi(e,n,y)m \\ \alpha_y mne &= -e\psi(e,n,y)me \\ \alpha_y \mathrm{MN} &= 0 \text{ for all } y \in \mathfrak{G}.\end{aligned}$$

Further with (3.4), we obtain that

$$\begin{aligned}[f,n]\psi(e,y,m) &= [f,m]\psi(e,n,y) \\ n\psi(e,y,m)f &= -m\psi(e,n,y)f \\ fn\alpha_y m &= -fm\psi(e,n,y)f \\ \eta(\alpha_y)\mathrm{NM} &= 0 \qquad \text{for all } y \in \mathfrak{G}.\end{aligned}$$

The assumption (2) imply $\alpha_y = 0$ or $\eta(\alpha_y) = 0$ and hence $\alpha_y = 0$ for all $y \in \mathfrak{G}$.

Suppose next that $\mathrm{MN} = 0 = \mathrm{NM}$. By assumption (3), one of A and B is noncommutative. Without loss of generality, assume B is a noncommutative algebra and let $b_1, b_2 \in \mathrm{B}$ be fixed elements with $[b_1,b_2] \neq 0$. With $e\psi(e,y,m)f = \alpha_y m$, we obtain

$$\psi(e,y,m)[b_1,b_2] = \psi(e,b_2,y)[b_1,m]$$

$$\begin{aligned}\alpha_y m[b_1,b_2] &= -e\psi(e,b_2,y)mb_1\\ m\eta(\alpha_y)[b_1,b_2] &= -e\psi(e,b_2,y)mb_1\\ \mathrm{M}\eta(\alpha_y)[b_1,b_2] &= 0 \qquad \text{for all } y\in\mathfrak{G}.\end{aligned}$$

The faithfulness of M as a right B-module imply to $\eta(\alpha_y)[b_1,b_2]=0$ and from the assumption (2), we get $\eta(\alpha_y)=0$ and hence $\alpha_y=0$ for all $y\in\mathfrak{G}$. It follows that $e\psi(e,y,m)f=0$ for all $y\in\mathfrak{G}$ and $m\in\mathrm{M}$. For any $a\in\mathrm{A}$ and $y\in\mathrm{A}\cup\mathrm{M}\cup\mathrm{B}$, we have

$$\begin{aligned}\psi(a,y,m) &= e\psi(ae,y,m)f\\ &= a\psi(e,y,m)f+e\psi(a,y,m)ef=0.\end{aligned}$$

Likewise, we have $\psi(b,y,m)=0$ for all $b\in\mathrm{B}$. Therefore, $\psi(x,y,m)=0$ for all $x\in\mathrm{A}\cup\mathrm{B}, y\in\mathrm{A}\cup\mathrm{M}\cup\mathrm{B}$ and $m\in\mathrm{M}$. Similarly, we can prove the other relations of part (ii) and part (iii) also.

In view of (3.1) and (ii), for any $m\in\mathrm{M}$

$$\begin{aligned}&\psi(a_1,a_2,a_3)[e,m]=[a_1,a_2]\psi(e,m,a_3)\\ \Longrightarrow\ &\psi(a_1,a_2,a_3)m=[a_1,a_2]\psi(e,m,a_3)\\ \Longrightarrow\ &\psi(a_1,a_2,a_3)\mathrm{M}=0.\end{aligned}$$

By faithfulness of M as a left A-module implies $e\psi(a_1,a_2,a_3)e=0$ and hence $\psi(a_1,a_2,a_3)=0$ for all $a_1,a_2,a_3\in\mathrm{A}$. Taking into account (3.8), similarly we can have the other cases of part (i).

Claim 2. *For any $m\in\mathrm{M}$ and $n\in\mathrm{N}$, we have*

$$\begin{aligned}\psi(m_1,m_2,n)=\psi(m_1,n,m_2)=\psi(n,m_1,m_2)=0 &\qquad \textit{for all } m_1,m_2\in\mathrm{M},\\ \psi(n_1,n_2,m)=\psi(n_1,m,n_2)=\psi(m,n_1,n_2)=0 &\qquad \textit{for all } n_1,n_2\in\mathrm{N}.\end{aligned}$$

In view of Lemma 4, for any $m_1,m_2\in\mathrm{M}$ and $n\in\mathrm{N}$, we have

$$\begin{aligned}\psi(m_1,m_2,n) &= e\psi(m_1,m_2,ne)f+f\psi(m_1,m_2,ne)e\\ &= fn\psi(m_1,m_2,e)e+f\psi(m_1,m_2,n)e\\ &= f\psi(m_1,em_2,n)e=0.\end{aligned}$$

In similar manner, we can find the other relations.

Claim 3. *For any $x\in\mathrm{A}\cup\mathrm{B}, m\in\mathrm{M}$ and $n\in\mathrm{N}$, we have*

$$\begin{aligned}&\psi(x,n,m)=\psi(n,x,m)=\psi(x,m,n)=\psi(n,m,x)=\psi(m,x,n)=\psi(m,n,x)=0,\\ &\psi(x,m,n)=\psi(m,x,n)=\psi(x,n,m)=\psi(m,n,x)=\psi(n,x,m)=\psi(n,m,x)=0.\end{aligned}$$

For any $m\in\mathrm{M},\ n\in\mathrm{N}$, we find that

$$\begin{aligned}\psi(x,n,m) &= \psi(x,n,em)\\ &= e\psi(x,n,m)+\psi(x,n,e)m\\ &= e\psi(x,fn,m)=0.\end{aligned}$$

Similarly, we can prove rest of the cases.

Claim 4. $\psi(m,m_1,m_2)=0\ \forall\ m,m_1,m_2\in \mathrm{M}$, *and* $\psi(n,n_1,n_2)=0\ \forall\ n,n_1,n_2\in \mathrm{N}$.

In view of Lemma 4, it is easy to see that

$$\begin{aligned}\psi(m,m_1,m_2)&=e\psi(m,em_1,m_2)f+f\psi(m,em_1,m_2)e\\&=e\psi(m,m_1,m_2)f\in \mathrm{M}\qquad \text{for all } m,m_1,m_2\in \mathrm{M}.\end{aligned}$$

For fix $m',m''\in \mathrm{M}$, the map $\mathfrak{l}\colon \mathrm{M}\to \mathrm{M}$ defined by $\mathfrak{l}(m)=\psi(m,m',m'')$ for all $m\in \mathrm{M}$ is a bimodule homomorphism.

$$\begin{aligned}\mathfrak{l}(amb)&=\psi(amb,m',m'')\\&=\psi(am,m',m'')b+am\psi(b,m',m'')\\&=a\psi(m,m',m'')b+\psi(a,m',m'')mb+am\psi(b,m',m'')\\&=a\psi(m,m',m'')b=a\mathfrak{l}(m)b.\end{aligned}$$

Now we have to show that $\mathfrak{l}(m)n=0=n\mathfrak{l}(m)$ for all $m\in \mathrm{M},\ n\in \mathrm{N}$.

$$\begin{aligned}\psi(mn,m',m'')&=m\psi(n,m',m'')e+\psi(m,m',m'')n\\0&=\psi(m,m',m'')n=\mathfrak{l}(m)n.\end{aligned}$$

Similarly, we can obtain that $n\mathfrak{l}(m)=0$ for all $m\in \mathrm{M}$ and $n\in \mathrm{N}$.

Likewise, we have $\psi(n,n_1,n_2)\in \mathrm{N}$. Fix $n',n''\in \mathrm{N}$, the map $\mathfrak{h}\colon \mathrm{N}\to \mathrm{N}$ defined by $\mathfrak{h}(n)=\psi(n,n',n'')$ is a bimodule homomorphism and $\mathfrak{h}(bna)=b\mathfrak{h}(n)a$ for all $a\in \mathrm{A},b\in \mathrm{B},n\in \mathrm{N}$. Also, we can see that $\mathfrak{h}(n)m=0=m\mathfrak{h}(n)$ for all $m\in \mathrm{M},n\in \mathrm{N}$. Particularly, we see $\mathfrak{k}$ and $\mathfrak{h}$ are a special pair of bimodule homomorphisms. By assumptions (1) and (4), we get $\mathfrak{k}(m)=\gamma_{m',m''}m$ and $\mathfrak{h}(n)=-n\gamma_{m',m''}$, where $\gamma_{m',m''}\in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$. Suppose first that $\mathrm{MN}\neq 0$ or $\mathrm{NM}\neq 0$. Then

$$\gamma_{m',m''}mn=\mathfrak{k}(m)n=0=n\mathfrak{k}(m)=n\gamma_{m',m''}m.$$

That is, $\gamma_{m',m''}\mathrm{MN}=0=\eta(\gamma_{m',m''})\mathrm{NM}$. The assumption (2) implies that $\gamma_{m',m''}=0$ or $\eta(\gamma_{m',m''})=0$. So $\gamma_{m',m''}=0$. Hence $\mathfrak{k}(m)=0=\mathfrak{h}(n)$ for all $m\in \mathrm{M},n\in \mathrm{N}$.

Suppose next that $\mathrm{MN}=0=\mathrm{NM}$. The assumption (3) implies that one of A and B is noncommutative. Without loss of generality, we assume that B is a noncommutative algebra and let $b_1,b_2\in \mathrm{B}$ be fixed elements with $[b_1,b_2]\neq 0$. By (3.1) and $\psi(m,m',m'')=\gamma_{m',m''}m$, we obtain that

$$\begin{aligned}\psi(m,m',m'')[b_1,b_2]&=[m,m']\psi(b_1,b_2,m'')\\\gamma_{m',m''}m[b_1,b_2]&=0\\\mathrm{M}\eta(\gamma_{m',m''})[b_1,b_2]&=0.\end{aligned}$$

By faithfulness of the right B-module M, $\eta(\gamma_{m',m''})[b_1,b_2]=0$ and by assumption (2), we get $\eta(\gamma_{m',m''})=0$ and hence $\gamma_{m',m''}=0$. Therefore, $\mathfrak{k}(m)=0=\mathfrak{h}(n)$ for all $m\in \mathrm{M},n\in \mathrm{N}$. This proves our claim.

Thence, we see that $\psi(x,y,z) = 0$ for all $x,y,z \in \mathfrak{G}$. Since ψ is linear in each argument, we obtain that $\psi = 0$. This completes the proof. $\square$

Proposition 2. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring* R *and $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \to \mathfrak{G}$ be a 3-derivation on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{B})$,
(2) *For each $m \in \mathrm{M}$, the condition $m\mathrm{N} = 0 = \mathrm{N}m$ implies $m = 0$.*
(3) *For each $n \in \mathrm{N}$, the condition $\mathrm{M}n = 0 = n\mathrm{M}$ implies $n = 0$.*
(4) *Every special pair of bimodule homomorphisms has the standard form.*

Then every 3-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \to \mathfrak{G}$ is an extremal 3-derivation ζ such that $\zeta(x_1,x_2,x_3) = [x_1,[x_2,[x_3,\phi(e,e,e)]]]$, where $\phi(e,e,e) \notin \mathfrak{Z}(\mathfrak{G})$.

Proof. The whole proof is similar to proof of Proposition 1 except some modifications in Claim 1 & Claim 4 according to the assumptions of present proposition. Now we attempt to rewrite these proofs as follows:

Let ϕ be a 3-derivation with $\phi(e,e,e) \neq 0$ and $\phi(e,e,e) \notin \mathfrak{Z}(\mathfrak{G})$. It is easy to see $[\phi(e,e,e),[\mathfrak{G},\mathfrak{G}]] = 0$ and the map $\zeta(x,y,z) = [x,[y,[z,\phi(e,e,e)]]]$ is an extremal 3-derivation of $\mathfrak{G}$. Also note that $\zeta(e,e,e) = \phi(e,e,e)$. Let $\psi = \phi - \zeta$. Then ψ is a 3-derivation of $\mathfrak{G}$ satisfying $\psi(e,e,e) = 0$.

Now we have to show that every 3-derivations $\psi = 0$ with $\psi(e,e,e) = 0$. We will verify this argument via upcoming sequence of claims:

Claim 5. *For any $x \in \mathrm{A}\cup\mathrm{B}$, $m \in \mathrm{M}$ and $n \in \mathrm{N}$, we have*

(i) $\psi(x,y,m) = \psi(y,x,m) = \psi(x,m,y) = \psi(y,m,x) = \psi(m,x,y) = \psi(m,y,x) = 0$ *for all $y \in \mathrm{A}\cup\mathrm{M}\cup\mathrm{B}$,*
(ii) $\psi(x,y,n) = \psi(y,x,n) = \psi(x,n,y) = \psi(y,n,x) = \psi(n,x,y) = \psi(n,y,x) = 0$ *for all $y \in \mathrm{A}\cup\mathrm{N}\cup\mathrm{B}$.*
(iii) $\psi(x,y,z) = 0$ *for all $y,z \in \mathrm{A}\cup\mathrm{B}$.*

It is easy to verify that $\psi(x,y,m) \in \mathrm{M}$ for all $y \in \mathrm{A}\cup\mathrm{M}\cup\mathrm{B}, x \in \mathrm{A}\cup\mathrm{B}$ and $\psi(x,y,n) \in \mathrm{N}$ for all $y \in \mathrm{A}\cup\mathrm{N}\cup\mathrm{B}, x \in \mathrm{A}\cup\mathrm{B}$ and rest of the cases follow similarly. For fix $y \in \mathfrak{G}$, define maps $\mathfrak{f}\colon \mathrm{M} \to \mathrm{M}$ and $\mathfrak{g}\colon \mathrm{N} \to \mathrm{N}$ by $\mathfrak{f}(m) = e\psi(e,y,m)f$ and $\mathfrak{g}(n) = f\psi(e,y,n)e$ for all $m \in \mathrm{M}, n \in \mathrm{N}$ respectively. Then $\mathfrak{f}$ and $\mathfrak{g}$ are bimodule homomorphisms. For all $a \in \mathrm{A}, b \in \mathrm{B}, m \in \mathrm{M}, n \in \mathrm{N}$, we have $\mathfrak{f}(amb) = a\mathfrak{f}(m)b$ and $\mathfrak{g}(bna) = b\mathfrak{g}(n)a$. Moreover, $\mathfrak{f}(m)n + m\mathfrak{g}(n) = 0 = \mathfrak{g}(n)m + n\mathfrak{f}(m)$ for all $m \in \mathrm{M}, n \in \mathrm{N}$. By assumption (4) the bimodule homomorphism $\mathfrak{f}$ and $\mathfrak{g}$ have the standard form, then

$$\mathfrak{f}(m) = a_0m + mb_0 \text{ and } \mathfrak{g}(n) = -na_0 - b_0n \qquad \textit{for } a_0 \in \mathfrak{Z}(\mathrm{A}), b_0 \in \mathfrak{Z}(\mathrm{B}).$$

Now we use the assumption (1) to see that $a_0 \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ and $b_0 \in \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$. We may write

$$\mathfrak{f}(m) = (a_0 + \eta^{-1}(b_0))m = \alpha_y m \qquad \text{for all } m \in \mathrm{M},$$
$$\mathfrak{g}(n) = -n(a_0 + \eta^{-1}(b_0)) = -n\alpha_y \qquad \text{for all } n \in \mathrm{N},$$

where $\alpha_y = a_0 + \eta^{-1}(b_0) \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ (depending on y). By (3.5) we have

$$\begin{aligned} \psi(e,y,m)[f,n] &= \psi(e,n,y)[f,m] \\ e\psi(e,y,m)ne &= -e\psi(e,n,y)me \\ e\psi(e,y,m)f\mathrm{N} &= 0 \qquad \text{for all } y \in \mathfrak{G}. \end{aligned}$$

Further with (3.4), we obtain that

$$\begin{aligned} [f,n]\psi(e,y,m) &= [f,m]\psi(e,n,y) \\ fn\psi(e,y,m)f &= -fm\psi(e,n,y)f \\ \mathrm{N}e\psi(x,y,m)f &= 0 \qquad \text{for all } y \in \mathfrak{G}. \end{aligned}$$

The above two expression with assumption (2) imply $\mathfrak{f}(m) = e\psi(e,y,m)f = 0$ for all $y \in \mathfrak{G}$. Now for any $a \in \mathrm{A}$ and $y \in \mathrm{A} \cup \mathrm{M} \cup \mathrm{B}$, we may write

$$\begin{aligned} \psi(a,y,m) &= e\psi(ae,y,m)f \\ &= a\psi(e,y,m)f + e\psi(a,y,m)ef = 0. \end{aligned}$$

Likewise, we have $\psi(b,y,m) = 0$ for all $b \in \mathrm{B}$. Therefore, $\psi(x,y,m) = 0$ for all $x \in \mathrm{A} \cup \mathrm{B}, y \in \mathrm{A} \cup \mathrm{M} \cup \mathrm{B}$ and $m \in \mathrm{M}$. Similarly, we can prove the other relations of part (ii) and part (iii) also.

Claim 6. *For any $m \in \mathrm{M}$ and $n \in \mathrm{N}$, we have*

$$\begin{aligned} \psi(m_1,m_2,n) = \psi(m_1,n,m_2) = \psi(n,m_1,m_2) = 0 \qquad & \textit{for all } m_1,m_2 \in \mathrm{M}, \\ \psi(n_1,n_2,m) = \psi(n_1,m,n_2) = \psi(m,n_1,n_2) = 0 \qquad & \textit{for all } n_1,n_2 \in \mathrm{N}. \end{aligned}$$

Claim 7. *For any $x \in \mathrm{A} \cup \mathrm{B}, m \in \mathrm{M}$ and $n \in \mathrm{N}$, we have*

$$\begin{aligned} &\psi(x,n,m) = \psi(n,x,m) = \psi(x,m,n) = \psi(n,m,x) = \psi(m,x,n) = \psi(m,n,x) = 0, \\ &\psi(x,m,n) = \psi(m,x,n) = \psi(x,n,m) = \psi(m,n,x) = \psi(n,x,m) = \psi(n,m,x) = 0. \end{aligned}$$

Claim 8. $\psi(m,m_1,m_2) = 0 \; \forall \; m,m_1,m_2 \in \mathrm{M}$, *and* $\psi(n,n_1,n_2) = 0 \; \forall \; n,n_1,n_2 \in \mathrm{N}$.

In view of Lemma 4 and assumptions (2), (3), we can have $\psi(m,m_1,m_2) = 0$ for all $m,m_1,m_2 \in \mathrm{M}$, and $\psi(n,n_1,n_2) = 0$ for all $n,n_1,n_2 \in \mathrm{N}$.

Thence, we see that $\psi(x,y,z) = 0$ for all $x,y,z \in \mathfrak{G}$. Since ψ is linear in each argument, we obtain that $\psi = 0$. Therefore, ϕ is an extremal 3-derivation ζ. □

At this moment, we are equipped to demonstrate a significant result of this article for $n \geq 3$ as below:

Theorem 1. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring R and $\phi\colon \mathfrak{G} \times \mathfrak{G} \times \cdots \times \mathfrak{G} \to \mathfrak{G}$ be a n-derivation (for $n \geq 3$) on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{B})$,
(2) *If* $\alpha a = 0, \alpha \in \mathfrak{Z}(\mathfrak{G}), 0 \neq a \in \mathfrak{G}$, *then* $\alpha = 0$.

(3) *If* $\mathrm{MN} = 0 = \mathrm{NM}$, *then at least one of the algebras* A *and* B *is noncommutative.*
(4) *Every special pair of bimodule homomorphisms has the standard form.*

Then every n-derivation $\phi\colon \mathfrak{G} \times \mathfrak{G} \times \cdots \times \mathfrak{G} \to \mathfrak{G}$ *is an extremal n-derivation* ζ *such that* $\zeta(x_1, x_2, \ldots, x_n) = [x_1, [x_2, \ldots, [x_n, \phi(e, e, \ldots, e)] \ldots]]$, *where* $\phi(e, e, \ldots, e) \notin \mathfrak{Z}(\mathfrak{G})$.

Proof. For $n = 3$ result follows from the Proposition 1. For $n \geq 4$ we apply induction method. Now fix $x_4, \ldots, x_n \in \mathfrak{G}$. Set

$$\phi_{x_4,\ldots,x_n}(x_1, x_2, x_3) = \phi(x_1, x_2, x_3, x_4, \ldots, x_n) \qquad \text{for all } x_1, x_2, x_3 \in \mathfrak{G}.$$

Then $\phi_{x_4,\ldots,x_n}(x_1, x_2, x_3)$ is a 3-derivation. By Proposition 1, it follows that

$$\phi_{x_4,\ldots,x_n}(x_1, x_2, x_3) = [x_1, [x_2, [x_3, \phi(e, e, e)]]] \qquad \text{for all } x_1, x_2, x_3 \in \mathfrak{G},$$

where $\phi(e, e, e) \notin \mathfrak{Z}(\mathfrak{G})$ (depending on $x_4, \ldots, x_n$) with the property $[\phi(e, e, e), [\mathfrak{G}, \mathfrak{G}]] = 0$. Particularly, we have that $\phi_{x_4,\ldots,x_n}(e, e, e) = y$ and so $\phi(e, e, e, x_4, \ldots, x_n) = y$ for all $y \notin \mathfrak{Z}(\mathfrak{G})$. Hence

$$\phi(x_1, x_2, \ldots, x_n) = [x_1, [x_2, [x_3, \phi(e, e, e, x_4, \ldots, x_n)]]] \text{ for all } x_1, x_2, \ldots, x_n \in \mathfrak{G}. \tag{3.9}$$

Clearly, $\phi(e, x_2, x_3, \ldots, x_n)$ is a $(n-1)$-derivation on $\mathfrak{G}$. By induction, we get

$$\phi(e, x_2, x_3, \ldots, x_n) = [x_2, [x_3, \ldots, [x_n, \phi(e, e, \ldots, e)] \ldots]]$$

for all $x_2, \ldots, x_n \in \mathfrak{G}$, where $\phi(e, e, \ldots, e) \notin \mathfrak{Z}(\mathfrak{G})$ and $[\phi(e, e, \ldots, e), [\mathfrak{G}, \mathfrak{G}]] = 0$. Particularly,

$$\phi(e, e, e, x_4, \ldots, x_n) = [x_4, [x_5, \ldots, [x_n, \phi(e, e, \ldots, e)] \ldots]]$$

for all $x_4, \ldots, x_n \in \mathfrak{G}$, where we used that $\phi(e, e, \ldots, e) \notin \mathfrak{Z}(\mathfrak{G})$. From (3.9) we have

$$\phi(x_1, x_2, x_3, \ldots, x_n) = [x_1, [x_2, \ldots, [x_n, \phi(e, e, \ldots, e)] \ldots]]$$

for all $x_1, x_2, \ldots, x_n \in \mathfrak{G}$. Hence we obtain the expected result. □

In view of [12, Proposition 3.4], we come with the following consequence:

Corollary 1. *Let* $\mathfrak{G}$ *be a generalized matrix algebra over a commutative ring* R *and* $\phi\colon \mathfrak{G} \times \mathfrak{G} \times \cdots \times \mathfrak{G} \to \mathfrak{G}$ *be a n-derivation (for* $n \geq 3$*) on* $\mathfrak{G}$. *Suppose that* ϕ *satisfies*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{B})$,
(2) *If* $\alpha a = 0, \alpha \in \mathfrak{Z}(\mathfrak{G}), 0 \neq a \in \mathfrak{G}$, *then* $\alpha = 0$.
(3) *If* $\mathrm{MN} = 0 = \mathrm{NM}$, *then at least one of the algebras* A *and* B *is noncommutative.*
(4) *Every derivation* $\mathfrak{G}$ *is inner.*

Then every n-derivation $\phi\colon \mathfrak{G} \times \mathfrak{G} \times \cdots \times \mathfrak{G} \to \mathfrak{G}$ *is an extremal n-derivation* ζ *such that* $\zeta(x_1, x_2, \ldots, x_n) = [x_1, [x_2, \ldots, [x_n, \phi(e, e, \ldots, e)] \ldots]]$, *where* $\phi(e, e, \ldots, e) \notin \mathfrak{Z}(\mathfrak{G})$.

Forthwith, we present another significant result of this article as follows:

Theorem 2. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring* R *and $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ be a n-derivation (for $n\geq 3$) on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(\mathrm{B})$,
(2) *For each $m\in\mathrm{M}$, the condition $m\mathrm{N}=0=\mathrm{N}m$ implies $m=0$.*
(3) *For each $n\in\mathrm{N}$, the condition $\mathrm{M}n=0=n\mathrm{M}$ implies $n=0$.*
(4) *Every special pair of bimodule homomorphisms has the standard form.*

Then every n-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ is an extremal n-derivation ζ such that $\zeta(x_1,x_2,\ldots,x_n)=[x_1,[x_2,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$, where $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$.

Proof. In view of Proposition 2, proof is similar to the proof of Theorem 1. □

On account of [12, Proposition 3.3, Proposition 3.4], we come with the following results respectively.

Corollary 2. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring* R *and $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ be a n-derivation (for $n\geq 3$) on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(\mathrm{B})$,
(2) *For each $m\in\mathrm{M}$, the condition $m\mathrm{N}=0=\mathrm{N}m$ implies $m=0$.*
(3) *For each $n\in\mathrm{N}$, the condition $\mathrm{M}n=0=n\mathrm{M}$ implies $n=0$.*
(4) *Every (A,B)-bimodule homomorphism of* M *is of the standard form.*

Then every n-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ is an extremal n-derivation ζ such that $\zeta(x_1,x_2,\ldots,x_n)=[x_1,[x_2,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$, where $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$.

Corollary 3. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring* R *and $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ be a n-derivation (for $n\geq 3$) on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(\mathrm{B})$,
(2) *For each $m\in\mathrm{M}$, the condition $m\mathrm{N}=0=\mathrm{N}m$ implies $m=0$.*
(3) *For each $n\in\mathrm{N}$, the condition $\mathrm{M}n=0=n\mathrm{M}$ implies $n=0$.*
(4) *Every derivation $\mathfrak{G}$ is inner.*

Then every n-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ is an extremal n-derivation ζ such that $\zeta(x_1,x_2,\ldots,x_n)=[x_1,[x_2,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$, where $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$.

4. Applications

On application of our significant results to some classical examples of generalized matrix algebras, we prevail the following consequences:

Corollary 4. *Let $\mathfrak{M}_s(\mathrm{R})$ be the algebra of all $s\times s$ matrices over a commutative ring* R*, where $s\geq 2$ is an integer. Then every n-derivation (for $n\geq 3$) is an extremal n-derivation on $\mathfrak{M}_s(\mathrm{R})$.*

Corollary 5 ([18, Theorem 2])**.** *Let $\mathfrak{T} = Tri(\mathrm{A},\mathrm{M},\mathrm{B})$ be a triangular algebra. If the following conditions hold:*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{B})$,
(2) *either* A *or* B *does not contain nonzero central ideals,*
(3) *each derivation of* A *is inner,*

then every n-derivation $(n \geq 3)$ $\phi\colon \mathfrak{T}\times\mathfrak{T}\times\cdots\times\mathfrak{T} \to \mathfrak{T}$ *is an extremal n-derivation.*

5. For Future Discussions

In this part, we make an effort to collect a few specific queries related to the literature of the article. But before that, we should bring up some basic notions of related subject matter. In view of [5, Propostion 2.1, 2.2], we can write the structure of automorphisms on generalized matrix algebras respectively as follows:

Lemma 6. *Let $\mathfrak{G} = \mathfrak{G}(\mathrm{A},\mathrm{M},\mathrm{N},\mathrm{B})$ be a generalized matrix algebra and $(\gamma,\delta,\mu,\nu,$ $m_0,n_0)$ be a 6-tuple such that $\gamma\colon \mathrm{A}\to\mathrm{A}$ & $\delta\colon \mathrm{B}\to\mathrm{B}$ are algebraic automorphisms, $\mu\colon \mathrm{M}\to\mathrm{M}$ is a $\gamma-\delta-$bimodule automorphism, $\nu\colon \mathrm{N}\to\mathrm{N}$ is a $\delta-\gamma-$bimodule automorphism and $m_0\in\mathrm{M}$ & $n_0\in\mathrm{N}$ are fixed elements such that following conditions are satisfied:*

(i) $[m_0,\mathrm{N}] = 0$ *and* $(\mathrm{N},m_0) = 0$,
(ii) $[\mathrm{M},n_0] = 0$ *and* $(n_0,\mathrm{M}) = 0$,
(iii) $[\mu(m),\nu(n)] = \gamma([m,n])$ *and* $(\nu(n),\mu(m)) = \delta((n,m))$.

Then the map $\alpha_1\colon \mathfrak{G}\to\mathfrak{G}$ defined by

$$\alpha_1\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \gamma(a) & \gamma(a)m_0 - m_0\delta(b) + \mu(m) \\ n_0\gamma(a) - \delta(b)n_0 + \nu(n) & \delta(b) \end{bmatrix}$$

is an algebraic automorphism.

Lemma 7. *Let $\mathfrak{G} = \mathfrak{G}(\mathrm{A},\mathrm{M},\mathrm{N},\mathrm{B})$ be a generalized matrix algebra and $(\rho,\sigma,\mu,\nu,$ $m_*,n_*)$ be a 6-tuple such that $\rho\colon \mathrm{A}\to\mathrm{B}$ & $\sigma\colon \mathrm{B}\to\mathrm{A}$ are algebraic automorphisms, $\mu\colon (\mathrm{M},+)\to(\mathrm{N},+)$ & $\nu\colon (\mathrm{N},+)\to(\mathrm{M},+)$ are group automorphisms such that $\mu(amb) = \rho(a)\mu(m)\sigma(b)$ & $\nu(bna) = \sigma(b)\nu(n)\rho(a)$ for all $a\in\mathrm{A}, b\in\mathrm{B}, m\in\mathrm{M}, n\in\mathrm{N}$ and $m_*\in\mathrm{M}$ & $n_*\in\mathrm{N}$ are fixed elements such that following conditions are satisfied:*

(i) $[m_*,\mathrm{N}] = 0$ *and* $(\mathrm{N},m_*) = 0$,
(ii) $[\mathrm{M},n_*] = 0$ *and* $(n_*,\mathrm{M}) = 0$,
(iii) $(\mu(m),\nu(n)) = \rho([m,n])$ *and* $[\nu(n),\mu(m)] = \sigma((n,m))$.

Then the map $\alpha_2\colon \mathfrak{G}\to\mathfrak{G}$ defined by

$$\alpha_2\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \sigma(a) & m_*\rho(a) - \sigma(b)m_* + \nu(n) \\ \rho(a)n_* - n_*\sigma(b) + \mu(m) & \rho(b) \end{bmatrix}$$

is an algebraic automorphism.

Let α be an automorphism on R-algebra $\mathcal{A}$. An R-linear map $d\colon \mathcal{A} \to \mathcal{A}$ is said to be an α-derivation if $d(xy) = d(x)y + \alpha(x)d(y)\ \forall\, x, y \in \mathcal{A}$. An R-linear map $g\colon \mathcal{A} \to \mathcal{A}$ is said to be a generalized α-derivation associated with an α-derivation d if $g(xy) = g(x)y + \alpha(x)d(y)$ for all $x, y \in \mathcal{A}$. An n-linear map $\Phi\colon \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A} \to \mathcal{A}$ is said to be a generalized $\alpha - n-$derivation, if it is a generalized α-derivation in each component. In particular, a generalized $\alpha - 2-$derivation is a generalized α-biderivation. Also, if $\alpha = I_{\mathcal{A}}$, then a generalized $I_{\mathcal{A}} - n-$derivation is a generalized $n-$derivation. Now in view of [9, 16], it is reasonable to raise the following questions as:

Question 1. *What is the most general form of generalized n-derivations on triangular algebras and which constraints are needed to apply on triangular algebras?*

Question 2. *What is the most general form of generalized α-biderivations on generalized matrix algebras and which constraints are needed to apply on generalized matrix algebras?*

In general, one can also explore the following query:

Question 3. *What is the most general form of generalized $\alpha - n-$derivations on generalized matrix algebras and which constraints are needed to apply on generalized matrix algebras?*

6. Acknowledgments

The author would like to express her sincere thanks to the referees for carefully reading the manuscript and their useful suggestions. This research is supported by Dr. D. S. Kothari Postdoctoral Fellowship under University Grants Commission (Grant No. F.4-2/2006 (BSR)/MA/18-19/0014), awarded to the author.

References

[1] S. Ali and M. S. Khan, "On $\star$-bimultipliers, generalized $\star$-biderivations and related mappings," *Kyungpook Math. J.*, vol. 51, no. 3, pp. 301–309, 2011, doi: 10.5666/KMJ.2011.51.3.301.

[2] M. Ashraf, N. ur Rehman, S. Ali, and M. R. Mozumder, "On generalized (σ,τ)-biderivations in rings," *Asian-Eur. J. Math.*, vol. 51, no. 3, pp. 389–402, 2011, doi: 10.1142/S1793557111000319.

[3] M. Ashraf and N. ur Rehman, "On symmetric (σ,σ)-biderivations," *Aligarh Bull. Math.*, vol. 17, pp. 9–16, 1997.

[4] D. Benkovič, "Biderivations of triangular algebras," *Linear Algebra Appl.*, vol. 431, pp. 1587–1602, 2009, doi: 10.1016/j.laa.2009.05.029.

[5] C. Boboc, S. S. Dascalescu, and L. Wyk, "Isomorphisms between Morita context rings," *Linear Multilinear Algebra*, vol. 60, pp. 545–563, 2012, doi: 10.1080/03081087.2011.611946.

[6] M. Brešar, W. S. I. Martindale, and C. R. Miers, "Centralizing maps in prime rings with involution," *J. Algebra*, vol. 161, no. 2, pp. 342–357, 1993, doi: 10.1006/jabr.1993.1223. [Online]. Available: hdl.handle.net/1828/2661

[7] M. Brešar and K. Zhao, "Biderivations and commuting linear maps on Lie algebras," *J. Lie Theory*, vol. 28, no. 3, pp. 885–900, 2018. [Online]. Available: www.heldermann.de/JLT/JLT28/JLT283/jlt28042.htm#jlt283

[8] M. Brešar, "On certain pairs of functions of semiprime rings," *Proc. Amer. Math. Soc.*, vol. 120, no. 3, pp. 709–713, 1994, doi: 10.2307/2160460.

[9] M. Brešar, "On generalized biderivations and related maps," *J. Algebra*, vol. 172, pp. 764–786, 1995, doi: 10.1006/jabr.1995.1069.

[10] M. Brešar, "Commuting maps: a survey," *Taiwanese J. Math.*, vol. 8, pp. 361–397, 2004, doi: 10.11650/twjm/1500407660.

[11] X. Cheng, M. Wang, J. Sun, and H. Zhang, "Biderivations and linear commuting maps on the Lie algebra $\mathfrak{gca}$," *Linear Multilinear Algebra*, vol. 65, no. 12, p. 2483–2493, 2017, doi: 10.1080/03081087.2016.1277688.

[12] Y. Du and Y. Wang, "Biderivations of generalized matrix algebras," *Linear Algebra Appl.*, vol. 438, pp. 4483–4499, 2013, doi: 10.1016/j.laa.2013.02.017.

[13] N. M. Ghosseiri, "On biderivations of upper triangular matrix rings," *Linear Algebra Appl.*, vol. 438, no. 1, pp. 250–260, 2013, doi: 10.1016/j.laa.2012.07.039.

[14] N. M. Ghosseiri, "On derivations and biderivations of trivial extensions and triangular matrix rings," *Bull. Iranian Math. Soc.*, vol. 43, no. 6, pp. 1629–1644, 2017.

[15] A. Jabeen, "On n-Lie derivations of triangular algebras," *Oper. Matrices*, vol. 16, no. 3, pp. 611–622, 2022, doi: 10.7153/oam-2022-16-45.

[16] A. Jabeen, M. Ashraf, and M. Ahmad, "σ-derivations on generalized matrix algebras," *An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat.*, vol. 28, no. 2, pp. 115–135, 2020, doi: 10.2478/auom-2020-0022.

[17] K. Morita, "Duality for modules and its applications to the theory of rings with minimum condition," *Rep. Tokyo Kyoiku Diagaku Sect. A*, vol. 6, pp. 83–142, 1958.

[18] Y. Wang, Y. Wang, and Y. Du, "n-derivations of triangular algebras," *Linear Algebra Appl.*, vol. 439, no. 2, pp. 463–471, 2013, doi: 10.1016/j.laa.2013.03.032.

Author's address

Aisha Jabeen
Department of Applied Sciences & Humanities, Jamia Millia Islamia, New Delhi 110025, India
E-mail address: ajabeen329@gmail.com, g.ajabeen@jmi.ac.in

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 805–816

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.2820

A NEW CLASS OF GENERALIZED POLYNOMIALS ASSOCIATED WITH MILNE-THOMSON-BASED POLY-BERNOULLI POLYNOMIALS

WASEEM AHMAD KHAN, DIVESH SRIVASTAVA,
AND KOTTAKKARAN SOOPPY NISAR

Received 19 January, 2019

Abstract. Motivated by their importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis, and other field of applied mathematics, a variety of polynomials and numbers with their variants and extensions have recently been introduced and investigated. In this sequel, we modify the known generating functions of polynomials, due to both Milne-Thomson and Dere and Simsek, to introduce a new class of generalized polynomials and present some of their involved properties. As obvious special cases of the newly introduced polynomials, we also called power sum-Laguerre-Hermite polynomials and generalized Laguerre and poly-Bernoulli polynomials and present some of their involved identities and formulas. The results presented here, being very general, are pointed out to be specialized to yield a number of known and new identities involving relatively simple and familiar polynomials.

2010 *Mathematics Subject Classification:* 05A10; 11B68; 33C45; 33C99

Keywords: Hermite polynomials, Laguerre polynomials, Milne-Thomson polynomials, generalized poly-Bernoulli polynomials, Milne-Thomson-based poly-Bernoulli polynomials

1. Introduction

Throughout this presentation, we use the following standard notions $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,\ldots\} = \mathrm{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1,-2,\ldots\}$. Also as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The two variable Laguerre polynomials (2-VLP) $L_n(x;y)$ are defined by the following generating function [1,5]:

$$\frac{1}{1-yt}\exp\left(\frac{-xt}{1-yt}\right) = \sum_{n=0}^{\infty} L_n(x;y)\frac{t^n}{n!} \qquad (|\,yt\,| < 1).$$

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

Also, equivalently, the polynomials $L_n(x,y)$ are given by [4]

$$e^{yt}C_0(xt) = \sum_{n=0}^{\infty} L_n(x;y)\frac{t^n}{n!}, \tag{1.1}$$

where $C_0(x)$ denotes the 0th order Tricomi function. The nth order Tricomi functions $C_n(x)$ are defined by [4]

$$\exp\left(t-\frac{x}{t}\right) = \sum_{n=0}^{\infty} C_n(x)t^n \qquad (t\in\mathbb{C}\setminus\{0\},\ x\in\mathbb{C}).$$

We have

$$C_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r x^r}{r!(n+r)!}. \tag{1.2}$$

The Tricomi functions $C_n(x)$ are associated with the Bessel function of the first kind $J_n(x)$ [3,5]

$$C_n(x) = x^{-\frac{n}{2}}J_n(2\sqrt{x}).$$

From (1.1) and (1.2), we note that

$$L_n(x,y) = n!\sum_{k=0}^{n}\frac{(-1)^k y^{n-k}x^k}{(k!)^2(n-k)!} = y^n L_n(x,y), \tag{1.3}$$

where $L_n(x)$ are the ordinary Laguerre polynomials [25]. We have

$$L_n(x,0) = \frac{(-1)^n x^n}{n!}, \qquad L_n(0,y) = y^n \qquad L_n(x,1) = L_n(x).$$

For $s\in\mathbb{Z}$, the polylogarithm function is defined by a power series in z as

$$\mathrm{Li}_s(z) = \sum_{j=1}^{\infty}\frac{j^n}{j^s} = z+\frac{z^2}{2^s}+\frac{z^3}{3^s}, \qquad (|\,z\,|<1).$$

It is notice that

$$\mathrm{Li}_1(z) = \sum_{j=1}^{\infty}\frac{z^j}{j} = -\log(1-z).$$

In (2015), Jolany *et al.* [7] introduced the generalized poly-Bernoulli polynomials are defined by means of the following generating function:

$$\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{xt} = \sum_{n=0}^{\infty}B_n^{(k)}(x;a,b)\frac{t^n}{n!}; \qquad |\,t\,|<\frac{2\pi}{|\,\ln a+\ln b\,|}.$$

In the case when $x=0$, $B_n^{(k)}(a,b) = B_n^{(k)}(0;a,b)$ are called the generalized poly-Bernoulli numbers [8].

Milne-Thomson [27] defined polynomials $\Phi_n^{(\alpha)}(x)$ of degree n and order α by the following generating function:

$$f(t,\alpha)e^{xt+g(t)} = \sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x)\frac{t^n}{n!},$$

where $f(t,\alpha)$ is a function of t and $\alpha \in \mathbb{Z}$ and $g(t)$ is a function of t. Then, by choosing some explicit functions of $f(t,\alpha)$ and $g(t)$, he [27] presented several interesting properties for certain polynomials such as Bernoulli polynomials and Hermite polynomials.

Dere and Simsek [6] made a slight modification of the Milne-Thomson polynomials $\Phi_n^{(\alpha)}(x)$ to give polynomials $\Phi_n^{(\alpha)}(x,\nu)$ of degree n and order α by means of the following generating function:

$$G_1(t,x;\alpha,\nu) = f(t,\alpha)e^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x;\nu)\frac{t^n}{n!}, \tag{1.4}$$

where $f(t,\alpha)$ and $h(t,\nu)$ are functions of t and $\alpha \in \mathbb{Z}$ and t and $\nu \in \mathbb{N}_0$, respectively, which are analytic in a neighborhood of $t = 0$.

Observe that $\Phi_n^{(\alpha)}(\nu) = \Phi_n^{(\alpha)}(0;\nu)$, (see for details [26,27]). In particular, choosing $f(t,\alpha) = \left(\frac{\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{t}})}{1-e^{-t}}\right)^{\alpha}$ in (1.4), we obtain the following following polynomials given by the generating function:

$$G_2(t,x;k,\nu) = \frac{\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{t}})}{1-e^{-t}}e^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} B_n^{(k)}(x;\nu)\frac{t^n}{n!}, \tag{1.5}$$

Note that the polynomials $B_n^{(k)}(x;\nu)$ are related to both Bernoulli polynomials and Hermite polynomials. For example, if $h(t,0) = 0$ in (1.5), we get

$$B_n^{(k)}(x;0) = B_n^{(k)}(x),$$

where $B_n^{(k)}(x)$ are called the poly-Bernoulli polynomials [2].

The two-variable Hermite-Kampé de Fériet polynomials $H_n(x,y)$ [2,3] are generated by

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.6}$$

Note that

$$H_n(x,y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} n!\frac{y^r x^{n-2r}}{r!(n-2r)!} \qquad \text{(see [9, 10, 14, 16])},$$

It is clear that

$$H_n(2x,-1) = H_n(x), \qquad H_n(x,0) = x^n.$$

Dere and Simsek [6] generalized the polynomials $H_n(x,y)$ in (1.6) to define two variable Hermite polynomials $H_n^{(l)}(x,y)$ by the following generating function:

$$e^{xt+yt^l} = \sum_{n=0}^{\infty} H_n^{(l)}(x,y)\frac{t^n}{n!}; \qquad (l \in \mathbb{N}\setminus\{0\}).$$

Taking $h(t,y)=yt^2$ in (1.5), we get the generalized Hermite poly-Bernoulli polynomials of two variables ${}_HB^{(k)}(x,y)$ introduced by Pathan and Khan [21]:

$$\frac{\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{t}})}{1-e^{-t}}e^{xt+yt^2)}=\sum_{n=0}^{\infty}{}_HB_n^{(k)}(x,y)\frac{t^n}{n!}. \tag{1.7}$$

In this paper, we introduce generalized Laguerre-based poly-Bernoulli polynomials by using Milne-Thomson polynomials and investigate some of their properties such as explicit summation formulas, addition formulas, implicit formula and symmetry identities.

2. Milne-Thomson based poly-Bernoulli polynomials

In this section, we introduce generalized Laguerre-based poly-Bernoulli polynomials by using Milne-Thomson polynomials. By replacing x by y and ν by z in (1.4) and $f(t,\alpha)=\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}C_0(xt)$. First, we start with the following definition as.

Definition 1. Let $a,b\in\mathbb{R}^+$ with $a\neq b$. The generalized poly-Bernoulli polynomials $B_n^{(k)}(x,y,z;a;b,e)$ are defined by the following generating function:

$$\begin{aligned}&\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{yt+h(t,z)}C_0(xt)=\sum_{n=0}^{\infty}B_n^{(k)}(x,y,z;a,b,e)\frac{t^n}{n!},\\&\left(x,y,z\in\mathbb{R};\ |\,t\,|<\frac{2\pi}{|\,\ln a+\ln b\,|}\right).\end{aligned} \tag{2.1}$$

In particular, taking $h(t,z)=zt^2$ in (2.1), we get the following.

Definition 2. Let $a,b>0,\ a\neq b$. The generalized Laguerre-based poly-Bernoulli polynomials ${}_LB_n^{(k)}(x,y,z;a,b)$ are defined by

$$\begin{aligned}&\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}\exp(yt+zt^2)C_0(xt)=\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y,z;a,b)\frac{t^n}{n!},\\&\left(x,y,z\in\mathbb{R};\ |\,t\,|<\frac{2\pi}{|\,\ln a+\ln b\,|}\right).\end{aligned} \tag{2.2}$$

Remark 1. On setting $z=0$ in (2.2) reduces to the generalized Laguerre-based poly-Bernoulli polynomials in two variables defined as

$$\frac{\mathrm{Li_k}(1+(ab)^{-t})}{b^t-a^{-t}}e^{yt}C_0(xt)=\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y;a,b)\frac{t^n}{n!},\qquad\left(|\,t\,|<|\,\frac{2\pi}{(\ln a+\ln b)\,|}\right).$$

Remark 2. Taking $k=1,\ a=1,\ b=e$ and $z=0$ in (2.2) reduces to the known Laguerre Bernoulli polynomials defined as

$$\left(\frac{t}{e^t-1}\right)e^{yt}C_0(xt)=\sum_{n=0}^{\infty}{}_LB_n(x,y)\frac{t^n}{n!};\qquad(|\,t\,|<|\,2\pi).$$

Remark 3. Letting $x=0$ in (2.2) reduces to the known generalized Hermite poly-Bernoulli polynomials defined by [19]

$$\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{yt+zt^2}=\sum_{n=0}^{\infty}{}_HB_n^{(k)}(y,z;a,b)\frac{t^n}{n!},\qquad\left(\mid t\mid<\mid\frac{2\pi}{(\ln a+\ln b)}\mid\right).$$

By using generalized Laguerre-based poly-Bernoulli polynomials ${}_LB_n^{(k)}(x,y,z;a,b)$ defined by (2.2), we have the following properties which are stated as theorems below.

Theorem 1. *For $n\geq 0$, we have*

$$ {}_LB_n^{(k)}(x,y,z;a,b)=\sum_{m=0}^{n}\binom{n}{m}{}_HB_m^{(k)}(0,z;a,b)L_{n-m}(x,y). \tag{2.3}$$

Proof. Using (1.3), (1.7) and (2.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_LH^{B_n^{(k)}}(x,y,z;a,b)\frac{t^n}{n!}&=\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}\exp(yt+zt^2)C_0(xt)\\&=\sum_{m=0}^{\infty}{}_HB_m^{(k)}(0,z;a,b)\frac{t^m}{m!}\sum_{n=0}^{\infty}L_n(x,y)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}{}_HB_m^{(k)}(0,z;a,b)L_{n-m}(x,y)\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients of t^n in the above equation, we obtain the result (2.3). □

Theorem 2. *For $n\geq 0$, we have*

$$ {}_LB_n^{(k)}(x,y+u,z+v;a,b)=\sum_{m=0}^{n}\binom{n}{m}{}_LB_{n-m}^{(k)}(x,y,z;a,b)H_m(u,v). \tag{2.4}$$

$$ {}_LB_n^{(k)}(x,y+u,z;a,b)=\sum_{m=0}^{n}\binom{n}{m}{}_LB_{n-m}^{(k)}(x,y;a,b)H_m(u,z). \tag{2.5}$$

Proof. On replacing y by $y+u$ and z by $z+v$ in (2.2), we get

$$\begin{aligned}\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y+u,z+v;a,b)\frac{t^n}{n!}&=\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}\exp(yt+zt^2)C_0(xt)\exp\left(ut+vt^2\right)\\&=\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y,z;a,b)\frac{t^n}{n!}\sum_{m=0}^{\infty}H_m(u,v)\frac{t^m}{m!},\end{aligned}$$

$$\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y+u,z+v;a,b)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\binom{n}{m}{}_LB_{n-m}^{(k)}(x,y,z;a,b)H_m(u,v)\right]\frac{t^n}{n!}.$$

Now equating the coefficients of the like powers of t in the above equation, we get the result (2.4).

Again, from (2.2), we have

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y+u,z;a,b)\frac{t^n}{n!} = \frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}\exp(yt+zt^2)C_0(xt)\exp(ut)$$
$$= \frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{yt}C_0(xt)\exp(ut+zt^2)$$
$$= \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y;a,b)\frac{t^n}{n!}\sum_{m=0}^{\infty} H_m(u,z)\frac{t^m}{m!}.$$

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y+u,z;a,b)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\binom{n}{m}{}_LB_{n-m}^{(k)}(x,y;a,b)H_m(u,z)\right]\frac{t^n}{n!}. \tag{2.6}$$

On comparing the coefficients of equal powers of t^n in (2.6), we acquire the desired result (2.5). □

Theorem 3. *Let $p,q>0$ and $n\geq 0$, we have*

$${}_LB_n^{(k)}(x,py,qz;a,b) = \sum_{m=0}^{n}\binom{n}{m}{}_LB_{n-m}^{(k)}(x,(p-1)y,(q-1)z;a,b)H_m(y,z). \tag{2.7}$$

Proof. Rewriting the generating function (2.2), we have

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,py,qz;a,b)\frac{t^n}{n!} = \frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{yt+zt^2}e^{(p-1)yt+(q-1)zt^2}C_0(xt)$$
$$= \left(\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,(p-1)y,(q-1)z;a,b)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} H_m(y,z)\frac{t^m}{m!}\right) \tag{2.8}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}{}_LB_{n-m}^{(k)}(x,(p-1)y,(q-1)z;a,b)H_m(y,z)\right)\frac{t^n}{n!}.$$

On comparing the coefficients of $\frac{t^n}{n!}$ in (2.8), we obtain at the desired result (2.7). □

3. Summation formulae for generalized Laguerre-based poly-Bernoulli polynomials.

In this section, we derive some implicit summation formulae for generalized Laguerre-based poly-Bernoulli polynomials ${}_LB_n^{(k)}(x,y,z;a,b)$. We start with the following theorem as.

Theorem 4. *The following relation holds true:*

$${}_LB_{l+p}^{(k)}(x,v,z;a,b) = \sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(v-y)^{m+n}{}_LB_{l+p-m-n}^{(k)}(x,y,z;a,b). \tag{3.1}$$

Proof. Replace t by $t+u$ in (2.2) and rewrite the generating function (2.2), we have

$$\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-(\mathrm{t+u})})}{b^t-a^{-(t+u)}}\exp(z(t+u)^2)C_0(x(t+u))$$
$$=\exp(-y(t+u))\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,y,z;a,b)\frac{t^lu^p}{l!p!}. \quad (3.2)$$

Upon setting y by v in (3.2), it is not difficult to show that

$$\exp((v-y)(t+u))\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,y,z;a,b)\frac{t^lu^p}{l!p!}=\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,v,z;a,b)\frac{t^lu^p}{l!p!}. \quad (3.3)$$

Expanding the exponential part from the above equation (3.3) with the help of result [19], we get

$$\begin{aligned}&\sum_{N=0}^{\infty}\frac{[(v-y)(t+u)]^N}{N!}\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,y,z;a,b)\frac{t^lu^p}{l!p!}\\&=\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,v,z;a,b)\frac{t^lu^p}{l!p!}.\\&\sum_{m,n=0}^{\infty}\frac{(v-y)^{m+n}t^mu^n}{m!n!}\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,y,z;a,b)\frac{t^lu^p}{l!p!}\\&=\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,v,z;a,b)\frac{t^lu^p}{l!p!}.\end{aligned} \quad (3.4)$$

On replacing l with $l-m$ and p with $p-n$ in L.H.S. of (3.4), yields

$$\begin{aligned}&\sum_{l,p=0}^{\infty}\sum_{m,n=0}^{l,p}\frac{(v-y)^{m+n}}{m!n!}{}_LB_{l+p-m-n}^{(k)}(x,y,z;a,b)\frac{t^lu^p}{(l-m)!(p-n)!}\\&=\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(v,x,z;a,b)\frac{t^lu^p}{l!p!}.\end{aligned} \quad (3.5)$$

On comparing the coefficients of equal powers of t and u form (3.5), we obtain the result (3.1). □

Remark 4. For $l=0$ in (3.1) reduces to

$${}_LB_p^{(k)}(x,v,z;a,b)=\sum_{n=0}^{p}\binom{p}{n}(v-y)^n{}_LB_{p-n}^{(k)}(x,y,z;a,b).$$

Remark 5. On taking $v=0$ in (3.1), we get

$${}_LB_{l+p}^{(k)}(x,z;a,b)=\sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(-y)^{m+n}{}_LB_{l+p-m-n}^{(k)}(x,y,z;a,b).$$

Theorem 5. *The following relation holds true:*

$$ {}_LB_n^{(k)}(x,y+1,z;a,b)=\sum_{m=0}^{n-2j}\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n-2j}{m}{}_LB_{n-2j-m}^{(k)}(x,y;a,b)\frac{n!}{j!(n-2j)!}z^j. \tag{3.6} $$

Proof. On replacing y by $y+1$, (2.2) converts as

$$ \sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y+1,z;a,b)\frac{t^n}{n!}=\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{(y+1)t+zt^2}C_0(xt). $$

$$ \begin{aligned} \sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y+1,z;a,b)\frac{t^n}{n!}&=\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{yt}C_0(xt)e^{t+zt^2}\\ &=\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y;a,b)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{j=0}^{\infty}z^j\frac{t^{2j}}{j!}. \end{aligned} $$

$$ \sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y+1,z;a,b)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}{}_LB_{n-m}^{(k)}(x,y;a,b)\frac{t^n}{n!}\sum_{j=0}^{\infty}z^j\frac{t^{2j}}{j!}. $$

$$ L.H.S=\sum_{n=0}^{\infty}\left[\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{m=0}^{n-2j}\binom{n-2j}{m}z^j{}_LB_{n-2j-m}^{(k)}(y;a,b)\frac{n!}{j!(n-2j)!}\right]\frac{t^n}{n!}. $$

On comparing of the coefficients of $\frac{t^n}{n!}$, we obtain at the desired result (3.6). □

Theorem 6. *The following relation holds true:*

$$ {}_LB_n^{(k)}(x,y+1,z;a,b)\frac{t^n}{n!}=\sum_{m=0}^{n}\binom{n}{m}{}_LB_m^{(k)}(x,y,z;a,b). \tag{3.7} $$

Proof. We start with the following relation

$$ \begin{aligned} &\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y+1,z;a,b)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y,z;a,b)\frac{t^n}{n!}\\ &=\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{yt+zt^2}C_0(xt)(e^t-1)\\ &=\sum_{m=0}^{\infty}{}_LB_m^{(k)}(x,y,z;a,b)\frac{t^m}{m!}\sum_{n=0}^{\infty}\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y,z;a,b)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}{}_LB_m^{(k)}(x,y,z;a,b)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y,z;a,b)\frac{t^n}{n!}. \end{aligned} $$

On comparing the coefficients of equal powers of t^n produce the result (3.7). □

4. Symmetry identities for the generalized Laguerre-based poly-Bernoulli polynomials

Recently, Khan *et al.* [11–13, 15, 17, 18] and Pathan and Khan [19–24] have established some interesting symmetry identities for various polynomials. Here, we present certain symmetry identities for the generalized Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x,y,z;a,b)$ in the following form.

Theorem 7. *The following symmetry identities hold true:*

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m{}_LB_{n-m}^{(k)}(bx_1,by,b^2z;A,B){}_LB_m^{(k)}(ax_2,ay,a^2z;A,B)$$
$$=\sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^m{}_LB_{n-m}^{(k)}(ax_1,ay,a^2z;A,B){}_LB_m^{(k)}(bx_2,by,b^2z;A,B). \quad (4.1)$$

Proof. Consider the following function

$$A(t)=\left[\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))^2}{(B^{at}-A^{-at})(B^{bt}-A^{-bt})}\right]e^{abyt+a^2b^2zt^2}C_0(abx_1t)C_0(abx_2t).$$

We see that the function $A(t)$ is symmetric in a and b and we can expand $A(t)$ into series in two ways to obtain

$$A(t)=\left(\sum_{n=0}^{\infty}{}_LB_n^{(k)}(bx_1,by,b^2z;A,B)\frac{(at)^n}{n!}\right)\left(\sum_{m=0}^{\infty}{}_LB_m^{(k)}(ax_2,ay,a^2z;A,B)\frac{(bt)^m}{m!}\right).$$

$$A(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m{}_LB_{n-m}^{(k)}(bx_1,by,b^2z;A,B)\right.$$
$$\left.\times{}_LB_m^{(k)}(ax_2,ay,a^2z;A,B)\right)\frac{t^n}{n!}. \quad (4.2)$$

Similarly, we have

$$A(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^m{}_LB_{n-m}^{(k)}(ax_1,ay,a^2z;A,B)\right.$$
$$\left.\times{}_LB_m^{(k)}(bx_2,by,b^2z;A,B)\right)\frac{t^n}{n!}. \quad (4.3)$$

On comparing the coefficients of t^n in (4.2) and (4.3), we obtain the result (4.1). □

Remark 6. If $b=1$, Theorem 7 reduces to

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}{}_LB_{n-m}^{(k)}(x_1,y,z;A,B){}_LB_m^{(k)}(ax_2,ay,a^2z;A,B)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^m{}_LB_{n-m}^{(k)}(ax_1,ay,a^2z;A,B){}_LB_m^{(k)}(x_2,y,z;A,B).$$

Theorem 8. *The following symmetry identities hold true:*

$$\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} b^m a^{n-m}{}_L B_n^{(k)}\left(bx_1, by+\frac{b}{a}i+j, b^2z; A, B\right){}_L B_m^{(k)}(ax_2, av; A, B)\\
&=\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1} a^m b^{n-m}{}_L B_n^{(k)}\left(ax_1, ay+\frac{a}{b}i+j, a^2z; A, B\right)\\
&\qquad\qquad \times {}_L B_m^{(k)}(bx_2, bv; A, B).
\end{aligned}\tag{4.4}$$

Proof. Let

$$\begin{aligned}
B(t) &= \left[\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))^2}{(B^{at}-A^{-at})(B^{bt}-A^{-bt})}\right]\frac{(e^{abt}-1)^2 e^{ab(y+v)t+a^2b^2zt^2}}{(e^{at}-1)(e^{bt}-1)}C_0(abx_1t)C_0(abx_2t)\\
&= \frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))}{(B^{at}-A^{-at})}e^{abyt+a^2b^2zt^2}C_0(abx_1t)\left(\frac{(e^{abt}-1)}{(e^{bt}-1)}\right)\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))}{(B^{bt}-A^{-bt})}\\
&\quad\times e^{abvt}C_0(abx_2t)\left(\frac{(e^{abt}-1)}{(e^{at}-1)}\right)\\
&= \frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))}{(B^{at}-A^{-at})}e^{abyt+a^2b^2zt^2}C_0(abx_1t)\\
&\quad\times\sum_{i=0}^{a-1}e^{bti}\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))}{(B^{bt}-A^{-bt})}e^{abvt}C_0(abx_2t)\sum_{j=0}^{b-1}e^{atj}\\
&= \frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))}{(B^{at}-A^{-at})}e^{a^2b^2zt^2}C_0(abx_1t)\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}e^{(by+\frac{b}{a}i+j)at}\\
&\quad\times\sum_{m=0}^{\infty}{}_L B_m^{(k)}(ax_2, av; A, B)\frac{(bt)^m}{m!}\\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_L B_n^{(k)}\left(bx_1, by+\frac{b}{a}i+j, b^2z; A, B\right)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}{}_L B_m^{(k)}(ax_2, av; A, B)\frac{(bt)^m}{m!}.
\end{aligned}$$

$$\begin{aligned}
B(t) = \sum_{n=0}^{\infty}\Bigg(&\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} b^m a^{n-m}{}_L B_n^{(k)}\left(bx_1, by+\frac{b}{a}i+j, b^2z; A, B\right)\\
&\times {}_L B_m^{(k)}(ax_2, av; A, B)\Bigg)\frac{t^n}{n!}.
\end{aligned}\tag{4.5}$$

On the other hand,

$$\begin{aligned}
B(t) = \sum_{n=0}^{\infty}\Bigg(&\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1} a^m b^{n-m}{}_L B_n^{(k)}\left(ax_1, ay+\frac{a}{b}i+j, a^2z; A, B\right)\\
&\times {}_L B_m^{(k)}(bx_2, bv; A, B)\Bigg)\frac{t^n}{n!}.
\end{aligned}\tag{4.6}$$

Comparing the coefficients of t^n in (4.5) and (4.6), we get the result (4.4). □

REFERENCES

[1] A. A. Atash, "On the two-variable generalized Laguerre polynomials." *Open J. Math. Sci.*, vol. 3, no. 1, pp. 152–158, 2019, doi: 10.30538/oms2019.0058.

[2] A. Bayad and Y. Hamahata, "Polylogarithms and poly-Bernoulli polynomials." *Kyushu J. Math.*, vol. 65, no. 11, pp. 15–24, 2011, doi: 10.2206/kyushujm.65.15.

[3] G. Dattoli, "Generalized polynomials, operational identities and their applications." *JCAM*, vol. 118, pp. 111–123, 2000, doi: 10.1016/S0377-0427(00)00283-1.

[4] G. Dattoli and A. Torre, *Theory and applications of generalized Bessel function.* Aracne: Aracne,, 1996.

[5] G. Dattoli, A. Torre, and A. M. Mancho, "The generalized Laguerre polynomials, the associated Bessel functions and applications to propagation problems." *Radiat Phys. Chem.*, vol. 59, no. 3, pp. 229–237, 2000, doi: 10.1016/S0969-806X(00)00273-5.

[6] R. Dere and Y. Simsek, "Bernoulli type polynomials on umbral algebra." *Russ. J. Math. Phy.*, vol. 23, no. 1, pp. 1–6, 2015, doi: 10.48550/arXiv.1110.1484.

[7] H. Jolany and R. B. Corcino, "Explicit formula for generalization of poly-Bernoulli numbers and polynomials with a,b,c parameters." *J. Class. Anal.*, vol. 6, no. 2, pp. 119–135, 2015, doi: 10.48550/arXiv.1109.1387.

[8] M. Kaneko, "Poly-Bernoulli numbers." *J. Théor Nombres Bordeaux*, vol. 9, pp. 221–228, 1997.

[9] W. A. Khan, "Some properties of the generalized Apostol type Hermite-based polynomials." *Kyung. Math. J.*, vol. 55, pp. 597–614, 2015, doi: 10.5666/KMJ.2015.55.3.597.

[10] W. A. Khan, "A note on Hermite-based poly-Euler and multi poly-Euler polynomials," *Palest. J. Math.*, vol. 6, pp. 204–214, 2017.

[11] W. A. Khan, "A new class of higher-order hypergeometric Bernoulli polynomials associated with Hermite polynomials." *BSPM*, vol. 40, pp. 1–14, 2022, doi: 10.5269/bspm.51845.

[12] W. A. Khan, "A note on q-analogue of degenerate Catalan numbers associated p-adic integral on $\mathbb{F}_p$." *Symmetry*, vol. 14, no. 6, pp. 1–10, 2022, doi: 10.3390/sym14061119.

[13] W. A. Khan, "A note on q-analogues of degenerate Catalan-Daehee numbers and polynomials." *Journal of Mathematics*, 2022, doi: 10.1155/2022/9486880.

[14] W. A. Khan, "On generalized Lagrange-based Apostol type and related polynomials." *Kragujevac J. Math.*, vol. 46, no. 6, pp. 865–882, 2022, doi: 10.46793/KgJMat2206.865K.

[15] W. A. Khan, R. Ali, K. A. H. Alzobydi, and N. Ahmad, "A new family of degenerate poly-Genocchi polynomials with its certain properties." *JFS*, vol. 4, pp. 1–18, 2021, doi: 10.1155/2021/6660517.

[16] W. A. Khan, K. S. Nisar, M. Acikgoz, U. Duran, and H. A. Abusufian, "On unified Gould-Hopper based Apostol-type polynomials." *J. Math. Computer Sci.*, vol. 24, pp. 287–298, 2022, doi: 10.22436/jmcs.024.04.01.

[17] W. A. Khan and S. K. Sharma, "A new class of Hermite-based higher-order central Fubini polynomials." *IJACM*, vol. 87, pp. 1–14, 2020, doi: 10.1007/s40819-020-00840-3.

[18] W. A. Khan and D. Srivastava, "On the generalized Apostol-type Frobenius-Genocchi polynomials." *Filomat*, vol. 33, no. 7, pp. 1967–1977, 2019, doi: 10.2298/FIL1907967K.

[19] M. A. Pathan and W. A. Khan, "Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials." *Mediterr. J. Math.*, vol. 12, pp. 675–695, 2015, doi: 10.1007/s00009-014-0423-0.

[20] M. A. Pathan and W. A. Khan, "A new class of generalized polynomials associated with Hermite and Euler polynomials." *Mediterr. J. Math.*, vol. 13, pp. 913–328, 2016, doi: 10.1007/s00009-015-0551-1.

[21] M. A. Pathan and W. A. Khan, "A new class of generalized polynomials associated with Hermite and poly-Bernoulli polynomials." *Miskolc Math. J.*, vol. 22, no. 1, pp. 317–330, 2021, doi: 10.18514/MMN.2021.1684.

[22] M. A. Pathan and W. A. Khan, "On three families of extended Laguerre-based Apostol type polynomials." *Proyecciones J. Math.*, vol. 40, no. 2, pp. 291–312, 2021, doi: 10.22199/issn.0717-6279-2021-02-0019.

[23] M. A. Pathan and W. A. Khan, "On a class of generalized Humbert-Hermite polynomials via generalized fibonacci polynomials." *TJM*, vol. 46, pp. 929–945, 2022, doi: 10.55730/1300-0098.3133.

[24] M. A. Pathan and W. A. Khan, "On λ-Changhee-Hermite polynomials." *Analysis*, vol. 42, no. 2, pp. 1–13, 2022, doi: 10.1515/anly-2021-1012.

[25] E. D. Rainville, *Special functions.* New York: Chelsea Publishing Company, 1971.

[26] Y. Simsek, "Formulas for Poisson-Charlier, Hermite, Milne-Thomson and other type polynomials by their generating functions and p-adic integral approach." *Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat. RACSAM*, vol. 113, pp. 931–948, 2019, doi: 10.1007/s13398-018-0528-6.

[27] L. M. Thomson, "Two classes of generalized polynomials." *Proc. London Math. Soc.*, vol. 35, no. 1, pp. 514–522, 1933, doi: 10.1112/plms/s2-35.1.514.

Authors' addresses

Waseem Ahmad Khan

Prince Mohammad Bin Fahd University, Department of Electrical Engineering, P.O Box 1664, Al Khobar, Saudi Arabia

E-mail address: wkhan1@pmu.edu.sa

Divesh Srivastava

Allenhouse Business School, Allenhouse Institute of Management (Mathematics), Rooma, Kanpur, India

E-mail address: mba.divesh@allenhouse.ac.in

Kottakkaran Sooppy Nisar

(**Corresponding author**) Prince Sattam Bin Abdulaziz University, Department of Mathematics, College of Science and Humanities in Alkharj, Alkharj,, 11942, Saudi Arabia

Current address: Prince Sattam Bin Abdulaziz University, Department of Mathematics, College of Science and Humanities in Alkharj, Alkharj,, 11942, Saudi Arabia

E-mail address: n.sooppy@psau.edu.sa

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 817–831

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4375

BIHARMONIC CURVES ALONG RIEMANNIAN SUBMERSIONS

GIZEM KÖPRÜLÜ KARAKAŞ AND BAYRAM ŞAHIN

Received 17 August, 2022

Abstract. The purpose of this paper is to study biharmonic curves along Riemannian submersions. We first consider a Riemannian submersion from a Riemannian manifold onto Riemannian manifold and investigate under what conditions a biharmonic curve on the total manifold is transformed to a biharmonic curve on the base manifold. We obtain several results with certain restrictions on curvatures. We then consider a Riemannian submersion from a Kaehler manifold onto a Riemannian manifold. Necessary and sufficient conditions were obtained for a curve that is biharmonic in the total manifold of Riemannian submersion to be biharmonic on the base manifold along the Riemannian submersion. In addition, considering the special cases of the curvatures of the curve, the biharmonicity of the curve on the base manifold is discussed.

2010 *Mathematics Subject Classification:* 53C15; 53C43

Keywords: Riemannian manifold, real space form, complex space form, Riemannian submersion, biharmonic curves

1. Introduction

Riemannian submersions between Riemannian manifolds were studied by O'Neill [14] and Gray [9]. Riemannian submersions have been used as an effective tool to obtain new manifolds with certain curvatures and to compare their geometry when given two manifolds. In addition, Riemannian submersions are very useful tools for the applications of Kaluza-Klein theory [8, 10] and robotic theory [2].

Harmonic maps $F\colon (M,g) \to (N,g_N)$ between Riemannian manifolds are the critical points of the energy $E(F) = \frac{1}{2}\int_M |dF|^2 \upsilon_g$, and they are therefore the solutions of the corresponding Euler-Lagrange equation. This equation is giving by the vanishing of the tension field $\tau(F) = \mathrm{trace}\nabla dF$. On the other hand, Jiang [11] studied first and second variation formulas of the bienergy functional $E_2(F)$ whose critical points are called as biharmonic maps.

Biharmonic maps have been a very active research area in recent years and many interesting results have been obtained [1, 3–7, 12, 14–16, 18–21], for update results about biharmonic theory, see:[17]. However, as far as we know, there are no studies

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

on whether a given biharmonic curve is biharmonic in the base manifold along a Riemann submersion.

In this paper, we study curves along Riemannian submersions between Riemannian manifolds We first considered the curve as horizontal curve and check the biharmonicity of a curve on the base manifold along a Riemannian submersion. We also study curves along Riemannian submersions from complex space form onto Riemannian manifolds. The paper is organized as follows. In Section 2, we present the basic information needed for this paper. In Section 3, we investigate necessary and sufficient conditions for the curves along Riemannian submersions from Riemannian manifolds to be biharmonic. We show that, if horizontal vector field $\mathcal{A}$ is parallel, then this curve is biharmonic. Then, we investigate necessary and sufficient conditions for the Frenet curves along Riemannian submersions from Riemannian manifolds to be biharmonic. In Section 4, we investigate necessary and sufficient conditions for the curves along Riemannian submersions from complex space forms to be biharmonic. We also investigate necessary and sufficient conditions for the Frenet curves along Riemannian submersions from complex space forms to be biharmonic.

2. Preliminaries

In this section, we recall some basic notions from [3, 8] and [22] which will be needed throughout the paper.

Let F be a Riemannian submersion between Riemannian manifolds (M, g_M) and (N, g_N). The geometry of Riemannian submersion is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ defined for vector fields E, F on M by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{2.1}$$

where ∇ is the Levi-Civita connection of g_M, $\mathcal{H}$ and $\mathcal{V}$ are projections to horizontal and vertical subbundles, respectively. It is easy to see that a Riemannian submersion $F\colon M \to N$ has totally geodesic fibres if and only if $\mathcal{T}$ vanishes identically. For any $E \in \Gamma(TM)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew -symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. $\mathcal{A}_E$ and $\mathcal{T}_E$ are anti-symmetric with respect to g. It is easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor field $\mathcal{T}$ is symmetric, $\mathcal{T}_V W = \mathcal{T}_W V$, $\forall V, W \in \Gamma(\ker F_*)$ and $\mathcal{A}$ is anti-symmetric, $A_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]$, $\forall X, Y \in \Gamma((\ker F_*)^{\perp})$. On the other hand, from (2.1), we have

$$\begin{aligned} \nabla_X V &= \mathcal{A}_X V + \mathcal{V}\nabla_X V \\ \nabla_X Y &= \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \end{aligned} \tag{2.2}$$

for $X, Y \in \Gamma((\ker F_*)^{\perp})$ and $V \in \Gamma(\ker F_*)$. If X is basic, then $\mathcal{H}\nabla_V X = \mathcal{A}_X V$.

Let $\alpha\colon I \to M$ be a curve parametrized by arc length in an n-dimensional Riemannian manifold (M, g). If there exists orthonormal vector fields $E_1 = \alpha' = T, E_2, \ldots, E_r$ along α such that

$$\nabla_T E_1 = \kappa_1 E_2,$$

$$\nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3,$$
$$\cdots$$
$$\nabla_T E_r = -\kappa_{r-1} E_{r-1}. \tag{2.3}$$

then α is called a Frenet curve of osculating order r, where $\kappa_1, \ldots, \kappa_{r-1}$ are positive functions on I and $1 \leq r \leq n$. A Frenet curve of osculating order 1 is a geodesic. A Frenet curve of osculating order 2 is called a circle if κ_1 is a nonzero positive constant. A Frenet curve of osculating order $r \geq 3$ is called a helix of order r if $\kappa_1, \ldots, \kappa_{r-1}$ are nonzero positive constants. A helix of order 3 is shortly called a helix.

Let $F\colon (M,g) \to (N,h)$ be a map between two Riemannian manifolds of dimensions m and n respectively. The second fundamental form of a map is defined by

$$(\nabla F_*)(X,Y) = \overset{N}{\nabla^F}_X F_* Y - F_*(\overset{M}{\nabla}_X Y) \tag{2.4}$$

for any vector fields X, Y on M, where $\overset{M}{\nabla}$ is the Levi-Civita cennection of M and $\overset{N}{\nabla^F}$ is the pull-back of the connection $\overset{N}{\nabla}$ of N to the induced vector bundle $F^{-1}(TN)$. It is well known that ∇F_* is symmetric. It is known that, F is a harmonic map if and only if the tension field $\tau(F) = \mathrm{trace}(\nabla F_*) = 0$, which is called the harmonic equation or the Euler-Lagrange equation.

A map $F\colon (M,g_M) \to (N,g_N)$ between Riemannian manifolds is a biharmonic map if the bitension field of F

$$\tau_2(F) = -\Delta_F \tau(F) + \mathrm{trace} R^N(\tau(F), F_*)F_*$$

vanishes, where R^N denotes the curvature tensor field of N. The operator Δ_F is the rough Laplacian acting on $\Gamma(F^*TM)$ defined by $\Delta_F := -\sum_{i=1}^n (\overset{N}{\nabla^F}_{e_i} \overset{N}{\nabla^F}_{e_i} - \overset{N}{\nabla^F}_{\overset{N}{\nabla}_{e_i} e_i})$, where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field of N.

3. Biharmonic curves along Riemannian submersions from Riemannian manifolds

In this section, we study biharmonic curves along Riemannian submersions from Riemannian manifolds. Then, we will investigate necessary and sufficient conditions for the curves along Riemannian submersions from Riemannian manifolds to be biharmonic. We first recall the biharmonic equation for curves from [3]. Let $\alpha\colon I \to M$ be a curve defined on an open interval I and parametrized by arc-length. Then the bitension field is given by

$$\tau_2(\alpha) = \nabla_T^3 T - R(T, \nabla_T T)T \tag{3.1}$$

where $T = \alpha'$ and $\nabla_T^3 T = \nabla_T \nabla_T \nabla_T T$. Then, using Frenet equations, the bitension field of α becomes

$$\tau_2(\alpha) = -3\kappa_1 \kappa_1' E_1 + (\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2 + c\kappa_1)E_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2')E_3$$

$$+\kappa_1\kappa_2\kappa_3 E_4.$$

Theorem 1. *Let F be a Riemannian submersion from a space form $(M(c), g_M)$ with constant sectional curvature c to a Riemannian manifold (N, g_N). Let $\alpha\colon I \to (M(c), g_M)$ be a biharmonic horizontal curve. Then $F\circ\alpha\colon \gamma\colon I \to (N, g_N)$ is a biharmonic curve if and only if the following condition is satisfied*

$$-2\kappa_1' F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} = 0, \tag{3.2}$$

where E_{1h} and E_{1v} denote the horizontal part and the vertical part of E_1.

Proof. Let $F\colon (M(c), g_M) \to (N, g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha\colon I \to (M(c), g_M)$ be a biharmonic horizontal curve. Then, we have

$$\alpha' = T = E_{1h}, \qquad \gamma' = F_*T = \tilde{T}, \tag{3.3}$$

where E_{1h} is horizontal part of $T = E_1$. Note that $\gamma' = \tilde{T}$ is the unit tangent vector field along the curve. Since F is Riemannian submersion, then $(\nabla F_*)(X,Y) = 0$, where $X, Y \in \Gamma(\mathcal{H})$, using (2.3) and (3.3) we get

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T} = \kappa_1 F_* E_{2h}.$$

and

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} = \kappa_1' F_* E_{2h} + \kappa_1((\nabla F_*)(E_{1h}, E_{2h}) + F_*\overset{M}{\nabla}_{E_{1h}}E_{2h}). \tag{3.4}$$

From (2.2), (2.3) and Frenet formulas, we have

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{2h} = -\kappa_1 E_{1h} + \kappa_2 E_{3h} - \mathcal{A}_{E_{1h}}E_{2v}. \tag{3.5}$$

Using (3.5) in (3.4), we derive

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} = \kappa_1' F_* E_{2h} - \kappa_1^2 F_* E_{1h} + \kappa_1\kappa_2 F_* E_{3h} - \kappa_1 F_*\mathcal{A}_{E_{1h}}E_{2v}. \tag{3.6}$$

Taking the covariant dervivative of (3.6) and using the second fundamental form of the Riemannian submersion, we get

$$\begin{aligned}\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} = {}& \kappa_1'' F_* E_{2h} + \kappa_1' F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{2h} - 2\kappa_1\kappa_1' F_* E_{1h} - \kappa_1^2 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{1h}\\ &+ \kappa_1'\kappa_2 F_* E_{3h} + \kappa_1\kappa_2' F_* E_{3h} + \kappa_1\kappa_2 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{3h}\\ &- \kappa_1' F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.\end{aligned} \tag{3.7}$$

Since

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{3h} = -\kappa_2 E_{2h} + \kappa_3 E_{4h} - \mathcal{A}_{E_{1h}}E_{3v} \tag{3.8}$$

due (3.5), (3.8) and Frenet formulas, using (3.7), we arrive at

$$\begin{aligned}\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} &= -3\kappa_1\kappa_1^{'}F_*E_{1h} + (\tilde{\kappa}_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2)F_*E_{2h} + (2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})F_*E_{3h} \\ &\quad + \kappa_1\kappa_2\kappa_3F_*E_{4h} - 2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v} \\ &\quad - \kappa_1F_*\mathcal{H}\overset{1}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}. \end{aligned} \tag{3.9}$$

It is easy to see that

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = R^N(F_*E_{1h}, \kappa_1F_*E_{2h})F_*E_{1h},$$

Taking the vertical and horizontal parts of E_2, we find

$$R^M(T, \overset{M}{\nabla}_T T)T = R^M(E_{1h}, \kappa_1E_{2v})E_{1h} + R^M(E_{1h}, \kappa_1E_{2h})E_{1h}.$$

Hence, we obtain

$$F_*(R^M(T, \overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h}, \kappa_1E_{2v})E_{1h}) + F_*(R^M(E_{1h}, \kappa_1E_{2h})E_{1h}).$$

Since F is Riemannian submersion, we have

$$\begin{aligned}&F_*(R^M(T, \overset{M}{\nabla}_T T)T) \\ &= F_*(R^M(E_{1h}, \kappa_1E_{2v})E_{1h}) + R^N(F_*E_{1h}, \kappa_1F_*E_{2h})F_*E_{1h}.\end{aligned}$$

On the other hand, since M is a space form, we obtain

$$R^N(F_*E_{1h}, \kappa_1F_*E_{2h})F_*E_{1h} = -c\kappa_1F_*E_{2h}. \tag{3.10}$$

Putting (3.9) and (3.10) in (3.1), we have

$$\begin{aligned}\tau_2(\gamma) &= \overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} - R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} \\ &= -3\kappa_1\kappa_1^{'}F_*E_{1h} + (\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2 + c\kappa_1)F_*E_{2h} \\ &\quad + (2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})F_*E_{3h} + \kappa_1\kappa_2\kappa_3F_*E_{4h} - 2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} \\ &\quad - \kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.\end{aligned}$$

Since $\tau_2(\alpha) = 0$, we can write $F_*\tau_2(\alpha) = 0$. Then, using this equation in $\tau_2(\gamma)$, we get

$$\tau_2(\gamma) = -2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.$$

Thus $F\circ\alpha\colon \gamma\colon I \to (N, g_N)$ is a biharmonic curve if and only if (3.2) is satisfied. □

In particular cases, we have the following results.

Theorem 2. *Let $F\colon (M(c),g_M)\to(N,g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I\to(M(c),g_M)$ be a biharmonic horizontal curve and $\kappa_1=$ constant $\neq 0$. Then $F\circ\alpha\colon\gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

$$\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v}+F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}=0.$$

Proof. The assertion follows from Theorem 1. □

Theorem 3. *Let $F\colon (M(c),g_M)\to(N,g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I\to(M(c),g_M)$ be a biharmonic horizontal curve and horizontal vector field $\mathcal{A}$ be a parallel. Then $F\circ\alpha\colon\gamma\colon I\to(N,g_N)$ is a biharmonic curve.*

Proof. Since, horizontal vector field $\mathcal{A}$ is a parallel, we have $\mathcal{A}=0$. Then the assertion follows from Theorem 1 . □

If the curve is the Frenet curve, the following theorem regarding the biharmonicity of the curve on the base manifold is obtained.

Theorem 4. *Let $F\colon (M(c),g_M)\to(N,g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I\to(M(c),g_M)$ be a horizontal Frenet curve. Then Frenet curve $F\circ\alpha\colon\gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

$$\tilde{\kappa_1}=\text{constant}\neq 0,\qquad \tilde{\kappa_2}=\text{constant},\qquad \tilde{\kappa_1}^2+\tilde{\kappa_2}^2=c,\qquad \tilde{\kappa_2}\tilde{\kappa_3}=0.$$

where $\tilde{\kappa_1},\dots,\tilde{\kappa_{r-1}}$ are positive functions of γ on I.

Proof. Let $F\colon (M(c),g_M)\to(N,g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c),g_M)$ to a Riemannian manifold (N,g_N). Since $\alpha\colon I\to(M(c),g_M)$ is a horizontal Frenet curve, we have

$$\alpha'=T=E_{1h},\qquad \gamma'=F_*T=\tilde{T},$$

where E_{1h} is horizontal part of $T=E_1$. Then for an orthonormal frame $E_1=\alpha'=T,E_2,\dots,E_r$ along α, we have Frenet formulas of γ as follows

$$\begin{aligned}\overset{N}{\nabla}_{\tilde{T}}\tilde{T}&=\tilde{\kappa_1}F_*E_{2h}\\ \overset{N}{\nabla}_{\tilde{T}}F_*E_{2h}&=-\tilde{\kappa_1}F_*E_{1h}+\tilde{\kappa_2}F_*E_{3h}\\ &\cdots\\ \overset{N}{\nabla}_{\tilde{T}}F_*E_{rh}&=-\tilde{\kappa}_{r-1}F_*E_{(r-1)h}.\end{aligned}$$

Thus, by direct computations, we have

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T}=\overset{N}{\nabla}_{F_*E_{1h}}F_*E_{1h}=\tilde{\kappa_1}F_*E_{2h},$$

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} = -\tilde{\kappa_1}^2 F_* E_{1h} + \tilde{\kappa_1}' F_* E_{2h} + \tilde{\kappa_1}\tilde{\kappa_2} F_* E_{3h},$$

and

$$\begin{aligned}\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} &= -3\tilde{\kappa_1}\tilde{\kappa_1}' F_* E_{1h} + (\tilde{\kappa_1}'' - \tilde{\kappa_1}^3 - \tilde{\kappa_1}\tilde{\kappa_2}^2) F_* E_{2h} \\ &\quad + (2\tilde{\kappa_1}'\tilde{\kappa_2} + \tilde{\kappa_1}\tilde{\kappa_2}') F_* E_{3h} + \tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3} F_* E_{4h}. \qquad (3.11)\end{aligned}$$

Then, using the Frenet formulas, we obtain

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = R^N(F_* E_{1h}, \tilde{\kappa_1} F_* E_{2h}) F_* E_{1h} = \tilde{\kappa_1} R^N(F_* E_{1h}, F_* E_{2h}) F_* E_{1h}. \qquad (3.12)$$

Now, taking the vertical and horizontal parts of E_2, we find

$$R^M(T, \overset{M}{\nabla}_T T)T = R^M(E_{1h}, \kappa_1 E_2) E_{1h} = R^M(E_{1h}, \kappa_1 E_{2v}) E_{1h} + R^M(E_{1h}, \kappa_1 E_{2h}) E_{1h}.$$

Thus, we obtain

$$F_*(R^M(T, \overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h}, \kappa_1 E_{2v}) E_{1h}) + F_*(R^M(E_{1h}, \kappa_1 E_{2h}) E_{1h}).$$

Since F is a Riemannian submersion, we have

$$F_*(R^M(T, \overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h}, \kappa_1 E_{2v}) E_{1h}) + R^N(F_* E_{1h}, \kappa_1 F_* E_{2h}) F_* E_{1h}.$$

Using, Riemannian curvature tensor of M, we get

$$\begin{aligned}R^N(F_* E_{1h}, F_* E_{2h}) F_* E_{1h} &= \frac{1}{\kappa_1} F_*(R^M(T, \overset{M}{\nabla}_T T)T) - \frac{1}{\kappa_1} F_*(R^M(E_{1h}, \kappa_1 E_{2v}) E_{1h}). \\ &= -c F_* E_{2h}. \qquad (3.13)\end{aligned}$$

Then, using (3.13) into (3.12), we have

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = -c\tilde{\kappa_1} F_* E_{2h}. \qquad (3.14)$$

Thus putting (3.11) and (3.14) in (3.1), we have

$$\begin{aligned}\tau_2(\gamma) &= -3\tilde{\kappa_1}\tilde{\kappa_1}' F_* E_{1h} + (\tilde{\kappa_1}'' - \tilde{\kappa_1}^3 - \tilde{\kappa_1}\tilde{\kappa_2}^2 + c\tilde{\kappa_1}) F_* E_{2h} + (2\tilde{\kappa_1}'\tilde{\kappa_2} + \tilde{\kappa_1}\tilde{\kappa_2}') F_* E_{3h} \\ &\quad + \tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3} F_* E_{4h}.\end{aligned}$$

Thus, Frenet curve $F \circ \alpha$: γ: $I \to (N, g_N)$ is a biharmonic curve if and only if

$$\begin{aligned}&-3\tilde{\kappa_1}\tilde{\kappa_1}' F_* E_{1h} + (\tilde{\kappa_1}'' - \tilde{\kappa_1}^3 - \tilde{\kappa_1}\tilde{\kappa_2}^2 + c\tilde{\kappa_1}) F_* E_{2h} + (2\tilde{\kappa_1}'\tilde{\kappa_2} + \tilde{\kappa_1}\tilde{\kappa_2}') F_* E_{3h} \\ &+ \tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3} F_* E_{4h} = 0\end{aligned}$$

which gives the assertion. □

From Theorem 4, we have the following result.

Corollary 1. *Let $F\colon (M(c),g_M) \to (N,g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I \to (M(c),g_M)$ be a horizontal Frenet curve. Then Frenet curve $F\circ\alpha\colon \gamma\colon I \to (N,g_N)$ is a biharmonic curve such that $\tilde{\kappa}_2 = 0$ if and only if*

- *γ is a circle with $\tilde{\kappa}_1 = \sqrt{c}$ such that $n \geq 2$,*
  or
- *γ is a helix with $\tilde{\kappa}_1{}^2 + \tilde{\kappa}_2{}^2 = c$ such that $n \geq 3$.*

4. Biharmonic curves along Riemannian submersions from complex space forms

A $2n-$dimensional Riemannian manifold (M,g,J) is called an almost Hermitian manifold if there exist a tensor field J of type $(1,1)$ on M such that $J^2 = -I$ and $g(JX,JY) = g(X,Y)$, $\forall X,Y \in \chi(M)$ where I denotes the identity transformation of T_pM. Consider an almost Hermitian manifold (M,g,J) and denote by ∇ the Levi-Civita connection on M with respect to g. Then M is called a Kaehler manifold if J is parallel with respect to ∇, i.e., $(\nabla_X J)Y = 0$, for $X,Y \in \chi(M)$. Let $M^m(4c)$ be a complex space form of holomorphic sectional curvature $4c$. Let us denote by J the complex structure and by g the Riemannian metric on $M^m(4c)$. Then its curvature operator is given by

$$\begin{aligned} R^{M^m(4c)}(X,Y)Z =& c\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY \\ &+ 2g(X,JY)JZ\} \end{aligned}$$

for $X,Y,Z \in \chi(M)$.

In this section, we study biharmonic curves along Riemannian submersions from complex space forms. More precisely, we will investigate necessary and sufficient conditions for the curves along Riemannian submersions from complex space forms to be biharmonic. We first recall the criteria for a curve on a complex manifold to be biharmonic from [3]. Let (M,g) be a complex space form and $\alpha\colon I \to M$ be a curve defined on an open interval I and parametrized by arc-length. Then, using Frenet equations for an orthonormal frame $E_1 = \alpha' = T, E_2, \ldots, E_r$ along α, the bitension field of α becomes

$$\begin{aligned} \tau_2(\alpha) =& -3\kappa_1\kappa_1' E_1 + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + c\kappa_1)E_2 + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 + \kappa_1\kappa_2\kappa_3 E_4 \\ &- 3c\kappa_1\tau_{12}JE_1. \end{aligned}$$

Following S. Maeda and Y. Ohnita [13], we define the complex torsions of the curve α by $\tau_{ij} = g(E_i, JE_j)$, $1 \leq i < j \leq r$.

The following theorem gives the appropriate condition for a biharmonic curve given on the total manifold to be biharmonic on the base manifold along a Riemannian submersion from a complex space form to a Riemannian manifold.

Theorem 5. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) and $\alpha\colon I\to(M(4c),g_M)$ a biharmonic horizontal curve. Then $F\circ\alpha\colon\gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

$$-2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v}-\kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v}-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}-3c\kappa_1\tau_{12mix}F_*JE_{1h}=0,$$

where $\tau_{12mix}=g_M(E_{1h},JE_{2v})$.

Proof. Let $F\colon(M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I\to(M(4c),g_M)$ be a biharmonic horizontal curve. Then, we have the following equation.

$$\begin{aligned}\tau_2(\alpha)=&-3\kappa_1\kappa_1^{'}E_1+(\kappa_1^{''}-\kappa_1^3-\kappa_1\kappa_2^2+c\kappa_1)E_2+(2\kappa_1^{'}\kappa_2+\kappa_1\kappa_2^{'})E_3+\kappa_1\kappa_2\kappa_3E_4\\&-3c\kappa_1\tau_{12}JE_1.\end{aligned}$$

Since α is horizontal curve, we have

$$\alpha^{'}=T=E_{1h},\qquad\gamma^{'}=F_*T=\tilde{T},$$

where E_{1h} is horizontal part of $T=E_1$. Note that $\gamma^{'}=\tilde{T}$ is the unit tangent vector field along the curve. Using (2.4) we obtain

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T}=\overset{N}{\nabla}_{F_*E_{1h}}F_*E_{1h}=(\nabla F_*)(E_{1h},E_{1h})+F_*\overset{M}{\nabla}_{E_{1h}}E_{1h}.$$

Since F is Riemannian submersion, we get

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T}=\kappa_1F_*E_{2h}. \tag{4.1}$$

From this we derive

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T}=\kappa_1^{'}F_*E_{2h}+\kappa_1((\nabla F_*)(E_{1h},E_{2h})+F_*\overset{M}{\nabla}_{E_{1h}}E_{2h}). \tag{4.2}$$

By using (2.2), (2.3) and Frenet formulas, we have

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{2h}=-\kappa_1E_{1h}+\kappa_2E_{3h}-\mathcal{A}_{E_{1h}}E_{2v}. \tag{4.3}$$

Inserting (4.3) in (4.2), we have

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T}=\kappa_1^{'}F_*E_{2h}-\kappa_1^2F_*E_{1h}+\kappa_1\kappa_2F_*E_{3h}-\kappa_1F_*\mathcal{A}_{E_{1h}}E_{2v}.$$

From (2.4) we obtain,

$$\begin{aligned}\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T}=&\kappa_1^{''}F_*E_{2h}+\kappa_1^{'}F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{2h}-2\kappa_1\kappa_1^{'}F_*E_{1h}-\kappa_1^2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{1h}\\&+\kappa_1^{'}\kappa_2F_*E_{3h}+\kappa_1\kappa_2^{'}F_*E_{3h}+\kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{3h}-\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v}\\&-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.\end{aligned} \tag{4.4}$$

Similar to expression (4.3), we have

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{3h} = -\kappa_2 E_{2h} + \kappa_3 E_{4h} - \mathcal{A}_{E_{1h}}E_{3v} \tag{4.5}$$

Putting (4.1), (4.3) and (4.5) in (4.4), we have

$$\begin{aligned}\overset{N}{\nabla}^3_{\tilde{T}}\tilde{T} &= -3\kappa_1\kappa_1^{'}F_*E_{1h} + (\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2)F_*E_{2h} + (2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})F_*E_{3h}\\ &\quad + \kappa_1\kappa_2\kappa_3F_*E_{4h} - 2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v}\\ &\quad - \kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.\end{aligned} \tag{4.6}$$

On the other hand, using (4.1) and (2.4), we obtain

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = R^N(F_*E_{1h}, \kappa_1F_*E_{2h})F_*E_{1h}.$$

and

$$R^M(T, \overset{M}{\nabla}_T T)T = R^M(T, \kappa_1E_2)T, \tag{4.7}$$

respectively. Now, taking the vertical and horizontal parts of E_2 in (4.7), we find

$$R^M(T, \overset{M}{\nabla}_T T)T = R^M(E_{1h}, \kappa_1E_{2v})E_{1h} + R^M(E_{1h}, \kappa_1E_{2h})E_{1h}.$$

Since F is a Riemannian submersion, we derive

$$F_*(R^M(T, \overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h}, \kappa_1E_{2v})E_{1h}) + R^N(F_*E_{1h}, \kappa_1F_*E_{2h})F_*E_{1h}.$$

Using curvature operator, we get

$$\begin{aligned}R^N(F_*E_{1h}, \kappa_1F_*E_{2h})F_*E_{1h} &= -c\kappa_1F_*E_{2h} - 3c\kappa_1\tau_{12}F_*JE_{1h}\\ &\quad - 3c\kappa_1g_M(E_{2v}, JE_{1h})F_*JE_{1h}.\end{aligned} \tag{4.8}$$

Thus putting (4.6) and (4.8) in (2.4), we have

$$\begin{aligned}\tau_2(\gamma) &= -3\kappa_1\kappa_1^{'}F_*E_{1h} + (\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2 + c\kappa_1)F_*E_{2h} + +(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})F_*E_{3h}\\ &\quad + \kappa_1\kappa_2\kappa_3F_*E_{4h} - 2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}\\ &\quad + 3c\kappa_1\tau_{12}F_*JE_{1h} + 3c\kappa_1g_M(E_{2v}, JE_{1h})F_*JE_{1h}.\end{aligned}$$

Since $\tau_2(\alpha) = 0$, we can write $F_*\tau_2(\alpha) = 0$. Then, using this equation in $\tau_2(\gamma)$, we have

$$\tau_2(\gamma) = -2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - 3c\kappa_1\tau_{12mix}F_*JE_{1h}.$$

Thus, proof is complete. □

In particular, if κ_1 =constant$\neq 0$, then we have the following result.

Theorem 6. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I\to(M(4c),g_M)$ be a biharmonic horizontal curve and κ_1 =constant$\neq 0$. Then $F\circ\alpha\colon\gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

$$\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v}+F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}+3c\tau_{12mix}F_*JE_{1h}=0$$

Proof. Since κ_1 =constant$\neq 0$, we have $\kappa_1^{'}=0$. Then the assertion follows from Theorem 5. □

Theorem 7. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) . Let $\alpha\colon I\to(M(4c),g_M)$ be a biharmonic horizontal curve such that κ_1 =constant$\neq 0$ and horizontal tensor field $\mathcal{A}$ is parallel. Then $F\circ\alpha\colon\gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if either $c=0$ or $\tau_{12mix}=0$, here $\tau_{12mix}=g_M(E_{1h},JE_{2v})$.*

Proof. Since κ_1 =constant$\neq 0$, we have $\kappa_1^{'}=0$. The parallelity of $\mathcal{A}$ implies that $\mathcal{A}=0$. The assertion follows from Theorem 5. □

Theorem 8. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be an invariant Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) . Let $\alpha\colon I\to(M(4c),g_M)$ be a biharmonic horizontal curve. Then $F\circ\alpha\colon\gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

$$-2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v}-\kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v}-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}=0.$$

Proof. Since F is an invariant Riemannian submersion, we have $J(\ker F_*)\subseteq\ker F_*$. The asserion follows from Theorem 5. □

Theorem 9. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be an invariant Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) . Let $\alpha\colon I\to(M(4c),g_M)$ be a biharmonic horizontal curve such that κ_1 =constant$\neq 0$ and horizontal tensor field $\mathcal{A}$ is parallel. Then $F\circ\alpha\colon\gamma\colon I\to(N,g_N)$ is a biharmonic curve.*

Proof. The parallelity of $\mathcal{A}$ implies that $\mathcal{A}=0$. The assertion follows from Theorem 8. □

The following theorem gives the appropriate condition for a horizontal Frenet curve given on the total manifold to be biharmonic on the base manifold along a Riemannian submersion from a complex space form to a Riemannian manifold.

Theorem 10. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) and $\alpha\colon I\to(M(4c),g_M)$ a horizontal Frenet curve. Then Frenet curve $F\circ\alpha\colon\gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

$$\tilde{\kappa}_1=\text{constant}\neq 0,$$

$$\tilde{\kappa_1}^2 + \tilde{\kappa_2}^2 = c\{1 + 3(\tau_{12} - \tau_{12mix})g_N(F_*JE_{1h}, F_*E_{2h})\},$$
$$\tilde{\kappa_2}' = -c\{3(\tau_{12} - \tau_{12mix})g_N(F_*JE_{1h}, F_*E_{3h})\},$$
$$\tilde{\kappa_2}\tilde{\kappa_3} = -c\{3(\tau_{12} - \tau_{12mix})g_N(F_*JE_{1h}, F_*E_{4h})\}.$$

where $\tilde{\kappa_1}, \ldots, \tilde{\kappa_{r-1}}$ *are positive functions of* γ *on* I.

Proof. Let $F\colon (M(4c), g_M) \to (N, g_N)$ be a Riemannian submersion from a complex space form $(M(4c), g_M)$ to a Riemannian manifold (N, g_N). Since $\alpha\colon I \to (M(4c), g_M)$ is a horizontal Frenet curve, we have

$$\alpha' = T = E_{1h}, \qquad \gamma' = F_*T = \tilde{T},$$

where E_{1h} is horizontal part of $T = E_1$. Note that $\gamma' = \tilde{T}$ is the unit tangent vector field along the curve. Then we have Frenet formulas of γ as follows

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T} = \tilde{\kappa_1}F_*E_{2h}$$
$$\overset{N}{\nabla}_{\tilde{T}}F_*E_{2h} = -\tilde{\kappa_1}F_*E_{1h} + \tilde{\kappa_2}F_*E_{3h}$$
$$\cdots$$
$$\overset{N}{\nabla}_{\tilde{T}}F_*E_{rh} = -\tilde{\kappa}_{r-1}F_*E_{(r-1)h}.$$

Thus we have

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T} = \overset{N}{\nabla}_{F_*E_{1h}}F_*E_{1h} = \tilde{\kappa_1}F_*E_{2h}.$$

Then, using Frenet formulas of γ, we get

$$\overset{N}{\nabla}^2_{\tilde{T}}\tilde{T} = -\tilde{\kappa_1}^2F_*E_{1h} + \tilde{\kappa_1}'F_*E_{2h} + \tilde{\kappa_1}\tilde{\kappa_2}F_*E_{3h}.$$

Thus $\overset{N}{\nabla}^3_{\tilde{T}}\tilde{T}$ becomes

$$\begin{aligned}\overset{N}{\nabla}^3_{\tilde{T}}\tilde{T} &= -3\tilde{\kappa_1}\tilde{\kappa_1}'F_*E_{1h} + (\tilde{\kappa_1}'' - \tilde{\kappa_1}^3 - \tilde{\kappa_1}\tilde{\kappa_2}^2)F_*E_{2h} \\ &\quad + (2\tilde{\kappa_1}'\tilde{\kappa_2} + \tilde{\kappa_1}\tilde{\kappa_2}')F_*E_{3h} + \tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}F_*E_{4h}. \qquad (4.9)\end{aligned}$$

Then, using the Frenet formulas, we obtain

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = R^N(F_*E_{1h}, \tilde{\kappa_1}F_*E_{2h})F_*E_{1h} = \tilde{\kappa_1}R^N(F_*E_{1h}, F_*E_{2h})F_*E_{1h}. \qquad (4.10)$$

Now, taking the vertical and horizontal parts of E_2, we find

$$\begin{aligned}R^M(T, \overset{M}{\nabla}_T T)T &= R^M(E_{1h}, \kappa_1E_2)E_{1h} \\ &= R^M(E_{1h}, \kappa_1E_{2v})E_{1h} + R^M(E_{1h}, \kappa_1E_{2h})E_{1h}. \qquad (4.11)\end{aligned}$$

From (4.11), we obtain

$$F_*(R^M(T,\overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h},\kappa_1 E_{2v})E_{1h}) + F_*(R^M(E_{1h},\kappa_1 E_{2h})E_{1h}).$$

Since F is a Riemannian submersion, we have

$$F_*(R^M(T,\overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h},\kappa_1 E_{2v})E_{1h}) + R^N(F_*E_{1h},\kappa_1 F_*E_{2h})F_*E_{1h}.$$

Using curvature operator, we get

$$\begin{aligned} R^N(F_*E_{1h},F_*E_{2h})F_*E_{1h} &= \frac{1}{\kappa_1}F_*(R^M(T,\overset{M}{\nabla}_T T)T) - \frac{1}{\kappa_1}F_*(R^M(E_{1h},\kappa_1 E_{2v})E_{1h}). \\ &= \frac{1}{\kappa_1}F_*(-c\kappa_1 E_2 - 3c\kappa_1\tau_{12}JE_{1h}) \\ &\quad - \frac{1}{\kappa_1}F_*c\{g_M(\kappa_1 E_{2v},E_{1h})E_{1h} - g_M(E_{1h},E_{1h})\kappa_1 E_{2v} \\ &\quad + g_M(JE_{1h},E_{1h})J\kappa_1 E_{2v} - g_M(J\kappa_1 E_{2v},E_{1h})JE_{1h} \\ &\quad + 2g_M(JE_{1h},\kappa_1 E_{2v})JE_{1h}\} \\ &= -cF_*E_{2h} - 3c\tau_{12}F_*JE_{1h} + 3cg_M(JE_{2v},E_{1h})F_*JE_{1h}. \end{aligned} \tag{4.12}$$

Then, using (4.12) into (4.10), we have

$$\begin{aligned} R^N(\tilde{T},\overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} &= \tilde{\kappa_1}(-cF_*E_{2h} - 3c\tau_{12}F_*JE_{1h} + 3cg_M(JE_{2v},E_{1h})F_*JE_{1h}) \\ &= -c\tilde{\kappa_1}F_*E_{2h} - 3c\tilde{\kappa_1}\tau_{12}F_*JE_{1h} + 3c\tilde{\kappa_1}\tau_{12mix}F_*JE_{1h}. \end{aligned} \tag{4.13}$$

Thus putting (4.9) and (4.13) in (3.1), we have

$$\begin{aligned} \tau_2(\gamma) &= \overset{N}{\nabla}^3_{\tilde{T}}\tilde{T} - R^N(\tilde{T},\overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} \\ &= -3\tilde{\kappa_1}\tilde{\kappa_1}'F_*E_{1h} + (\tilde{\kappa_1}'' - \tilde{\kappa_1}^3 - \tilde{\kappa_1}\tilde{\kappa_2}^2 + c\tilde{\kappa_1})F_*E_{2h} + (2\tilde{\kappa_1}'\tilde{\kappa_2} + \tilde{\kappa_1}\tilde{\kappa_2}')F_*E_{3h} \\ &\quad + \tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}F_*E_{4h} + 3c\tilde{\kappa_1}\tau_{12}F_*JE_{1h} - 3c\tilde{\kappa_1}\tau_{12mix}F_*JE_{1h}. \end{aligned}$$

Thus proof is complete. □

In particular cases, we have the following results.

Theorem 11. *Let $F\colon (M(4c),g_M) \to (N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I \to (M(4c),g_M)$ be a horizontal Frenet curve such that $\tau_{12} = \tau_{12mix}$. Then Frenet curve $F\circ\alpha\colon \gamma\colon I \to (N,g_N)$ is a biharmonic curve if and only if*

$$\tilde{\kappa_1} = \text{constant} \neq 0, \quad \tilde{\kappa_2} = \text{constant}, \quad \tilde{\kappa_1}^2 + \tilde{\kappa_2}^2 = 0, \quad \tilde{\kappa_2}\tilde{\kappa_3} = 0.$$

Proof. Since $\tau_{12} = \tau_{12mix}$, we have $E_{1h} \perp JE_{2h}$. The assertion follows from Theorem 10. □

Corollary 2. *Let $F:(M^m(4c),g_M)\to(N^n,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I\to(M(4c),g_M)$ be a horizontal Frenet curve such that $\tau_{12}=\tau_{12mix}$. Then Frenet curve $F\circ\alpha\colon\gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

- *γ is a circle with $\tilde{\kappa}_1=\sqrt{c}$ such that $n\geq 2$ or*
- *γ is a helix with $\tilde{\kappa_1}^2+\tilde{\kappa_2}^2=c$ such that $n\geq 3$.*

REFERENCES

[1] M. A. Akyol and Y. L. Ou, "Biharmonic Riemannian submersions," *Annali di Matematica Pura ed Applicata*, vol. 198, no. 2, pp. 559–570, 2019, doi: 10.1007/s10231-018-0789-x.

[2] C. Altafini, "Redundant robotic chains on Riemannian submersions," *IEEE Transactions on Robotics and Automation*, vol. 20, pp. 335–340, 2004.

[3] A. Balmus, "Biharmonic maps and submanifolds," *University of Cagliari, Geometry Balkan Press.*, 2007.

[4] A. Balmus, S. Montaldo, and C. Oniciuc, "Biharmonic hypersurfaces in 4-dimensional space forms," *Mathematische Nachrichten*, vol. 283, no. 12, pp. 1696–1705, 2010.

[5] A. Bennouar and S. Ouakkas, "Some constructions of biharmonic maps on the warped product manifolds," *Comment. Math. Univ. Carolin.*, vol. 58, no. 4, pp. 481–500, 2017.

[6] N. E. H. Djaa, A. Boulal, and A. Zagane, "Generalized warped product manifolds and biharmonic maps," *Acta Math. Univ. Comenian. (N.S.)*, vol. 81, no. 2, pp. 283–289, 2012, doi: 10.1307/mmj/1028999604.

[7] I. K. Erken and C. Murathan, "Biharmonic pseudo-Riemannian submersions from 3-manifolds," *Filomat*, vol. 32, no. 2, pp. 543–552, 2018, doi: 10.2298/FIL1802543E.

[8] M. Falcitelli, S. Ianus, and A. M. Pastore, *Riemannian submersions and related topics*. World Scientific, 2004. doi: 10.1142/5568.

[9] A. Gray, "Pseudo-Riemannian almost product manifolds and submersions," *Journal of Mathematics and Mechanics*, vol. 16, no. 7, pp. 715–737, 1967.

[10] P. A. Hogan, "Kaluza–Klein theory derived from a Riemannian submersion," *J. Math. Phys.*, vol. 25, pp. 2301–2305, 1984, doi: 10.1063/1.526407.

[11] G. Y. Jiang, "2-harmonic maps and their first and second variational formulas," *Chinese Ann. Math. Ser. A.*, vol. 7, pp. 389–402, 1986, doi: 10.1285/i15900932v28n1supplp209.

[12] K. Madani and S. Ouakkas, "Biharmonic maps on doubly warped product manifolds," *Kyungpook Math. J.*, vol. 60, no. 3, pp. 599–627, 2020, doi: 10.5666/KMJ.2020.60.3.599.

[13] S. Maeda and Y. Ohnita, "Helical geodesic immersions into complex space forms," *Geom. Dedicata*, vol. 30, pp. 93–114, 1983, doi: 10.1007/BF02424315.

[14] B. O'Neill, "The fundamental equations of a submersion," *Michigan Math. J.*, vol. 13, pp. 459–469, 1966, doi: 10.1307/mmj/1028999604.

[15] C. Oniciuc, "Biharmonic maps between Riemannian manifolds," *A1. I. Cuza Iasi. Mat.(N. S.*, vol. 48, no. 2, pp. 237–248, 2002.

[16] C. Oniciuc, "Biharmonic submanifolds in space forms," *Habilitation Thesis, Universitatea Alexandru Ioan Cuza*, p. 149, 2012, doi: 10.13140/2.1.4980.5605.

[17] Y.-L. Ou and B.-Y. Chen, *Biharmonic submanifolds and biharmonic maps in Riemannian geometry*. World Scientific, 2020. doi: 10.1142/11610.

[18] B. Sahin, "Biharmonic Riemannian maps," *Ann. Polon. Math.*, vol. 102, pp. 39–49, 2011, doi: 10.4064/ap102-1-4.

[19] B. Sahin, *Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications*. Cambridge: Elsevier, Academic Press, 2017.

[20] S. M. K. Torbaghan and M. M. Rezaii, "Warped products, biharmonic and semi-conformal maps," *Math. Rep. (Bucur.)*, vol. 21, no. 4, pp. 441–459, 2019.
[21] H. Urakawa, *Calculus of variation and harmonic maps*. American Mathematical Society, 2013.
[22] K. Yano and M. Kon, *Structures on manifolds*. World Scientific, 1984. doi: 10.1142/0067.

Authors' addresses

Gizem Köprülü Karakaş
Ege University, Department of Mathematics, 35100, Bornova-Izmir, Türkiye
E-mail address: koprulu1gizem@gmail.com

Bayram Şahin
(**Corresponding author**) Ege University, Department of Mathematics, 35100., Bornova-Izmir, Türkiye
E-mail address: bayram.sahin@ege.edu.tr

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 833–843

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4366

PETROVIĆ-TYPE INEQUALITY VIA FRACTIONAL CALCULUS

PÉTER KÓRUS, JUAN EDUARDO NÁPOLES VALDÉS,
JOSÉ MANUEL RODRÍGUEZ, AND JOSÉ MARÍA SIGARRETA ALMIRA

Received 11 August, 2022

Abstract. Inequalities play an important role in pure and applied mathematics. In particular, Petrović inequality is an important inequality which have several interesting generalizations. In this work we prove a new Petrović-type inequality for measurable functions defined on a space with finite measure, and we apply it to generalized Riemann–Liouville-type integral operators.

2010 *Mathematics Subject Classification:* 26A33; 26A51; 26D15

Keywords: Petrović's inequality, fractional derivatives and integrals, fractional integral inequalities, convex functions

1. Introduction

Integral inequalities are used in countless mathematical problems such as approximation theory, spectral analysis, statistical analysis, distribution theory, etc. Studies involving integral inequalities play an important role in several areas of science and engineering.

In recent years there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types of fractional derivatives, since integral inequalities and their applications play a vital role in the theory of differential equations and applied mathematics. Some of the inequalities studied are Gronwall, Chebyshev, Hermite–Hadamard-type, Ostrowski-type, Opial-type, Grüss-type, Hardy-type, Gagliardo–Nirenberg-type, reverse Minkowski and reverse Hölder inequalities (see, e.g., [3, 5, 9–11, 14–16, 20, 21]).

In 1905, J. Jensen was the first to define convex functions (see [7] and [19, p. 8]) and to draw attention to their importance. One of the most significant inequalities is the distinguished Petrović's inequality for convex functions (see Theorem 1). There are many generalizations of Petrović's inequality (see, e.g., [1, 6, 13, 17, 18] and the references therein).

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

In this work we obtain a new Petrović-type inequality for measurable functions defined on a space with finite measure, and we apply it to the generalized Riemann–Liouville-type integral operators defined in [2], which include the well-known Riemann–Liouville-type integral operators.

2. Preliminaries

One of the first operators that can be called fractional is the Riemann–Liouville fractional integrals and derivatives of order $\alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 0$, defined as follows (see [4]).

Definition 1. Let $a < b$ and $f \in L^1([a,b])$. The *left and right side Riemann–Liouville fractional integrals of order* α, with $\mathrm{Re}(\alpha) > 0$, are defined, respectively, by

$$ {}^{RL}J^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \tag{2.1} $$

and

$$ {}^{RL}J^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds, \tag{2.2} $$

with $t \in (a,b)$.

When $\alpha \in (0,1)$, their corresponding *Riemann–Liouville fractional derivatives* are given by

$$ \left({}^{RL}D^{\alpha}_{a^+} f\right)(t) = \frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{a^+} f(t)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^{\alpha}}\, ds, $$
$$ \left({}^{RL}D^{\alpha}_{b^-} f\right)(t) = -\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{b^-} f(t)\right) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^{\alpha}}\, ds. $$

Other definitions of fractional operators are the following ones.

Definition 2. Let $a < b$ and $f \in L^1([a,b]))$. The *left and right side Hadamard fractional integrals of order* α, with $\mathrm{Re}(\alpha) > 0$, are defined, respectively, by

$$ H^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, \tag{2.3} $$

and

$$ H^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, \tag{2.4} $$

with $t \in (a,b)$.

When $\alpha \in (0,1)$, *Hadamard fractional derivatives* are given by the following expressions:

$$\left({}^{H}D^{\alpha}_{a^+}f\right)(t) = t\,\frac{d}{dt}\left(H^{1-\alpha}_{a^+}f(t)\right) = \frac{1}{\Gamma(1-\alpha)}\,t\,\frac{d}{dt}\int_a^t \left(\log\frac{t}{s}\right)^{-\alpha}\frac{f(s)}{s}\,ds,$$

$$\left({}^{H}D^{\alpha}_{b^-}f\right)(t) = -t\,\frac{d}{dt}\left(H^{1-\alpha}_{b^-}f(t)\right) = \frac{-1}{\Gamma(1-\alpha)}\,t\,\frac{d}{dt}\int_t^b \left(\log\frac{s}{t}\right)^{-\alpha}\frac{f(s)}{s}\,ds,$$

with $t \in (a,b)$.

Definition 3. Let $0 < a < b$, $g\colon [a,b] \to \mathbb{R}$ an increasing positive function on $(a,b]$ with continuous derivative on (a,b), $f\colon [a,b] \to \mathbb{R}$ an integrable function, and $\alpha \in (0,1)$ fixed real number. The left and right side fractional integrals in [8] of order α of f with respect to g are defined, respectively, by

$$I^{\alpha}_{g,a^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{g'(s)f(s)}{\left(g(t)-g(s)\right)^{1-\alpha}}\,ds, \tag{2.5}$$

and

$$I^{\alpha}_{g,b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \frac{g'(s)f(s)}{\left(g(s)-g(t)\right)^{1-\alpha}}\,ds, \tag{2.6}$$

with $t \in (a,b)$.

There are other definitions of integral operators in the global case, but they are slight modifications of the previous ones.

3. General fractional integral of Riemann–Liouville type

Now, we give the definition of a general fractional integral introduced in [2].

Definition 4. Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g\colon [a,b] \to \mathbb{R}$ be a positive function on $(a,b]$ with continuous positive derivative on (a,b), and $G\colon [0, g(b)-g(a)] \times (0,\infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b)-g(a)] \times (0,\infty)$. Let us define the function $T\colon [a,b] \times [a,b] \times (0,\infty) \to \mathbb{R}$ by

$$T(t,s,\alpha) = \frac{G\left(|g(t)-g(s)|,\alpha\right)}{g'(s)}.$$

The *left and right integral operators*, denoted respectively by J^{α}_{T,a^+} and J^{α}_{T,b^-}, are defined for each measurable function f on $[a,b]$ as

$$J^{\alpha}_{T,a^+}f(t) = \int_a^t \frac{f(s)}{T(t,s,\alpha)}\,ds,$$

$$J^{\alpha}_{T,b^-}f(t) = \int_t^b \frac{f(s)}{T(t,s,\alpha)}\,ds,$$

with $t \in [a,b]$.

We say that $f \in L^1_T([a,b])$ if $J^{\alpha}_{T,a^+}|f|(t), J^{\alpha}_{T,b^-}|f|(t) < \infty$ for every $t \in [a,b]$.

Note that these operators generalize the integral operators in Definitions 1, 2 and 3:

(A) If we choose

$$g(t)=t, \quad G(x,\alpha)=\Gamma(\alpha)x^{1-\alpha}, \quad T(t,s,\alpha)=\Gamma(\alpha)\,|t-s|^{1-\alpha},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} become the left and right Riemann–Liouville fractional integrals ${}^{RL}J^{\alpha}_{a^+}$ and ${}^{RL}J^{\alpha}_{b^-}$ in (2.1) and (2.2), respectively. Its corresponding left and right Riemann–Liouville fractional derivatives are

$$\left({}^{RL}D^{\alpha}_{a^+}f\right)(t)=\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{a^+}f(t)\right), \quad \left({}^{RL}D^{\alpha}_{b^-}f\right)(t)=-\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{b^-}f(t)\right).$$

(B) If we choose

$$g(t)=\log t, \quad G(x,\alpha)=\Gamma(\alpha)x^{1-\alpha}, \quad T(t,s,\alpha)=\Gamma(\alpha)\,t\left|\log\frac{t}{s}\right|^{1-\alpha},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} become the left and right Hadamard fractional integrals $H^{\alpha}_{a^+}$ and $H^{\alpha}_{b^-}$ in (2.3) and (2.4), respectively. Its corresponding left and right Hadamard fractional derivatives are

$$\left({}^{H}D^{\alpha}_{a^+}f\right)(t)=t\,\frac{d}{dt}\left(H^{1-\alpha}_{a^+}f(t)\right), \quad \left({}^{H}D^{\alpha}_{b^-}f\right)(t)=-t\,\frac{d}{dt}\left(H^{1-\alpha}_{b^-}f(t)\right).$$

(C) If we choose a function g with the properties in Definition 4 and

$$G(x,\alpha)=\Gamma(\alpha)x^{1-\alpha}, \quad T(t,s,\alpha)=\Gamma(\alpha)\,\frac{|g(t)-g(s)|^{1-\alpha}}{g'(s)},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the left and right fractional integrals I^{α}_{g,a^+} and I^{α}_{g,b^-} in (2.5) and (2.6), respectively.

Definition 5. Let $a<b$ and $\alpha\in\mathbb{R}^+$. Let $g\colon[a,b]\to\mathbb{R}$ be a positive function on $(a,b]$ with continuous positive derivative on (a,b), and $G\colon[0,g(b)-g(a)]\times(0,\infty)\to\mathbb{R}$ a continuous function which is positive on $(0,g(b)-g(a)]\times(0,\infty)$. For each function $f\in L^1_T[a,b]$, its *left and right generalized derivative of order* α are defined, respectively, by

$$D^{\alpha}_{T,a^+}f(t)=\frac{1}{g'(t)}\,\frac{d}{dt}\left(J^{1-\alpha}_{T,a^+}f(t)\right),$$
$$D^{\alpha}_{T,b^-}f(t)=\frac{-1}{g'(t)}\,\frac{d}{dt}\left(J^{1-\alpha}_{T,b^-}f(t)\right).$$

for each $t\in(a,b)$.

Note that if we choose

$$g(t)=t, \quad G(x,\alpha)=\Gamma(\alpha)x^{1-\alpha}, \quad T(t,s,\alpha)=\Gamma(\alpha)\,|t-s|^{1-\alpha},$$

then $D^{\alpha}_{T,a^+}f(t)={}^{RL}D^{\alpha}_{a^+}f(t)$ and $D^{\alpha}_{T,b^-}f(t)={}^{RL}D^{\alpha}_{b^-}f(t)$. Also, we can obtain other fractional derivatives such as Hadamard type as particular cases of this generalized derivative.

4. Petrović-type inequality

The famous Petrović inequality is stated as follows, see [12]:

Theorem 1. *Let φ be a convex function on $[0,a]$, and $w_1,\dots,w_n \geq 0$. If $t_1,\dots,t_n \in [0,a]$ satisfy $\sum_{k=1}^n t_k w_k \in (0,a]$, and*

$$\sum_{k=1}^n t_k w_k \geq t_j, \qquad j=1,\dots,n,$$

then

$$\sum_{k=1}^n \varphi(t_k) w_k \leq \varphi\Big(\sum_{k=1}^n t_k w_k\Big) + \Big(\sum_{k=1}^n w_k - 1\Big)\varphi(0).$$

There are many generalizations of Petrović's inequality (see, e.g., [1, 6, 13, 17, 18] and the references therein).

Next, we present a continuous version of the above Petrović inequality.

Theorem 2. *Let μ be a finite measure on the space X, φ be a convex function on $[0,a]$, and $f\colon X \to [0,a]$ be a measurable function with $\int_X f\,d\mu \in (0,a]$. Then $\varphi\circ f$ is a μ-integrable function. If*

$$f(x) \leq \int_X f\,d\mu$$

for every $x \in X$, then

$$\int_X \varphi\circ f\,d\mu \leq \varphi\Big(\int_X f\,d\mu\Big) + \big(\mu(X)-1\big)\varphi(0).$$

Proof. Assume first that f is constant a.e. $f=c$. Thus, Theorem 1 with $n=1$, $t_1=c$ and $w_1=\mu(X)$ gives the conclusion.

Assume now that f is not constant a.e. We have

$$0 \leq f(x) \leq A = \int_X f\,d\mu$$

for every $x \in X$ and $0 < A = \int_X f\,d\mu$, since f is not constant a.e. Since φ is a convex function on $[0,a]$, $\varphi\circ f$ is a bounded function. Since μ is a finite measure, $\varphi\circ f$ is a μ-integrable function.

For each $n \geq 1$ and $1 \leq k < 2^n$, consider the sets

$$\begin{aligned} I_{n,k} &= \big((k-1)2^{-n}A,\, k2^{-n}A\big], \\ I_{n,2^n} &= \big((2^n-1)2^{-n}A,\, A\big), \\ I_{n,0} &= \{0\}, \qquad I_{n,2^n+1} = \{A\}. \end{aligned}$$

Note that $\{I_{n,k}\}_{k=0}^{2^n+1}$ is a partition of $[0,A]$ for each $n \geq 1$.

For each $n \geq 1$ and $0 \leq k \leq 2^n+1$, define the sets

$$S_{n,k} = f^{-1}\big(I_{n,k}\big)$$

and choose constants $a_{n,k} \in I_{n,k}$ satisfying

$$a_{n,k}\mu\big(S_{n,k}\big) = \int_{S_{n,k}} f\,d\mu.$$

Thus, $a_{n,0} = 0$ and $a_{n,2^n+1} = A$.

Since f is a measurable function satisfying $0 \le f \le A$, we have that $\{S_{n,k}\}_{k=0}^{2^n+1}$ are pairwise disjoint measurable sets and $X = \cup_{k=0}^{2^n+1} S_{n,k}$ for each n.

Recall that the characteristic function of a set E is defined as $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. If we define

$$f_n = \sum_{k=0}^{2^n+1} a_{n,k}\chi_{S_{n,k}},$$

then

$$\int_X f_n\,d\mu = \sum_{k=0}^{2^n+1} a_{n,k}\mu(S_{n,k}) = \sum_{k=0}^{2^n+1} \int_{S_{n,k}} f\,d\mu = \int_X f\,d\mu.$$

It is clear that

$$0 \le a_{n,k} \le \int_X f\,d\mu = \int_X f_n\,d\mu, \qquad 0 \le f_n \le \int_X f\,d\mu, \qquad |f_n - f| \le 2^{-n}A.$$

Hence, f_n uniformly converges to f and

$$0 \le f_n \le \int_X f\,d\mu = \int_X f_n\,d\mu.$$

Since $\{S_{n,k}\}_{k=0}^{2^n+1}$ are pairwise disjoint sets and $X = \cup_{k=0}^{2^n+1} S_{n,k}$, we have

$$\varphi \circ f_n = \sum_{k=0}^{2^n+1} \varphi(a_{n,k})\chi_{S_{n,k}},$$

$$\int_X \varphi \circ f_n\,d\mu = \sum_{k=0}^{2^n+1} \varphi(a_{n,k})\mu(S_{n,k}).$$

Since φ is a convex function on $[0,A]$, it is continuous on $(0,A)$ and it can be written on $[0,A]$ as

$$\varphi = \varphi_0 + \delta_0\chi_{\{0\}} + \delta_A\chi_{\{A\}},$$

where φ_0 is a continuous convex function on $[0,A]$ and $\delta_0, \delta_A \ge 0$. Hence

$$\begin{aligned}
\varphi \circ f_n &= \sum_{k=0}^{2^n+1} \varphi(a_{n,k})\chi_{S_{n,k}} \\
&= \sum_{k=0}^{2^n+1} \varphi_0(a_{n,k})\chi_{S_{n,k}} + \delta_0\chi_{f^{-1}(\{0\})} + \delta_A\chi_{f^{-1}(\{A\})} \\
&= \varphi_0 \circ f_n + \delta_0\chi_{f^{-1}(\{0\})} + \delta_A\chi_{f^{-1}(\{A\})}
\end{aligned}$$

and

$$\begin{aligned}\int_X \varphi\circ f_n\,d\mu &= \sum_{k=0}^{2^n+1}\varphi(a_{n,k})\mu(S_{n,k})\\ &= \sum_{k=0}^{2^n+1}\varphi_0(a_{n,k})\mu(S_{n,k})+\delta_0\mu(f^{-1}(\{0\}))+\delta_A\mu(f^{-1}(\{A\}))\\ &= \delta_0\mu(f^{-1}(\{0\}))+\delta_A\mu(f^{-1}(\{A\}))+\int_X \varphi_0\circ f_n\,d\mu.\end{aligned}$$

Since

$$0\le a_{n,j}\le \int_X f\,d\mu=\int_X f_n\,d\mu=\sum_{k=0}^{2^n+1}a_{n,k}\mu(S_{n,k})$$

for $0\le j\le 2^n+1$, Petrović's inequality gives

$$\sum_{k=0}^{2^n+1}\varphi(a_{n,k})\mu(S_{n,k})\le \varphi\Big(\sum_{k=0}^{2^n+1}a_{n,k}\mu(S_{n,k})\Big)+\big(\mu(X)-1\big)\varphi(0). \tag{4.1}$$

Then the right hand side of (4.1) is equal to

$$\varphi\Big(\int_X f\,d\mu\Big)+\big(\mu(X)-1\big)\varphi(0).$$

Since $0\le f_n\le A=\int_X f\,d\mu$ for every n, $\lim_{n\to\infty}f_n=f$ and φ_0 is a continuous function on $[0,A]$, we have $\lim_{n\to\infty}\varphi_0\circ f_n=\varphi_0\circ f$.

Since φ_0 is a continuous function on $[0,A]$, there exists a constant M with $|\varphi_0|\le M$ on $[0,A]$, hence $|\varphi_0\circ f_n|\le M$ for every n.

Since μ is a finite measure, $M\in L^1(X,\mu)$ and the dominated convergence theorem gives

$$\lim_{n\to\infty}\int_X \varphi_0\circ f_n\,d\mu=\int_X \varphi_0\circ f\,d\mu.$$

Therefore, the left hand side of (4.1) has limit

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=0}^{2^n+1}\varphi(a_{n,k})\mu(S_{n,k}) &= \lim_{n\to\infty}\int_X \varphi\circ f_n\,d\mu\\ &= \delta_0\mu(f^{-1}(\{0\}))+\delta_A\mu(f^{-1}(\{A\}))+\lim_{n\to\infty}\int_X \varphi_0\circ f_n\,d\mu\\ &= \delta_0\mu(f^{-1}(\{0\}))+\delta_A\mu(f^{-1}(\{A\}))+\int_X \varphi_0\circ f\,d\mu\\ &= \int_X \varphi\circ f\,d\mu.\end{aligned}$$

This fact finishes the proof. □

Jensen's inequality and Theorem 2 provide the following result.

Corollary 1. *Let μ be a finite measure on the space X, φ be a convex function on $[0,a]$, and $f\colon X \to [0,a]$ be a measurable function with $\int_X f\,d\mu \in (0,a]$. Then $\varphi \circ f$ is a μ-integrable function. If*

$$f(x) \le \int_X f\,d\mu$$

for every $x \in X$, then

$$\mu(X)\varphi\Big(\frac{1}{\mu(X)}\int_X f\,d\mu\Big) \le \int_X \varphi\circ f\,d\mu \le \varphi\Big(\int_X f\,d\mu\Big) + \big(\mu(X)-1\big)\varphi(0).$$

5. Application to fractional integrals

Corollary 1 has the following direct consequence for general fractional integrals of Riemann–Liouville type.

Proposition 1. *Let $a<b$, $\alpha, B>0$ be real constants and T be a function as in Definition 4. If $f\colon [a,b] \to [0,B]$ is a measurable function,*

$$\mathbb{T}(\alpha) = \int_a^b \frac{1}{T(b,s,\alpha)}\,ds = \int_0^{g(b)-g(a)} \frac{dx}{G(x,\alpha)} < \infty, \qquad \int_a^b \frac{f(s)}{T(b,s,\alpha)}\,ds \in (0,B],$$

φ is a convex function on $[0,B]$, and

$$f(x) \le \int_a^b \frac{f(s)}{T(b,s,\alpha)}\,ds$$

for every $x \in [a,b]$, then $\varphi(f(s))/T\big(b,s,\alpha\big) \in L^1([a,b])$ and

$$\begin{aligned}\mathbb{T}(\alpha)\varphi\Big(\frac{1}{\mathbb{T}(\alpha)}\int_a^b \frac{f(s)}{T\big(b,s,\alpha\big)}\,ds\Big) &\le \int_a^b \frac{\varphi(f(s))}{T\big(b,s,\alpha\big)}\,ds \\ &\le \varphi\Big(\int_a^b \frac{f(s)}{T\big(b,s,\alpha\big)}\,ds\Big) + \big(\mathbb{T}(\alpha)-1\big)\varphi(0).\end{aligned}$$

Proposition 1 for the classical Riemann–Liouville fractional integrals ${}^{RL}J^{\alpha}_{a^+}$, ${}^{RL}J^{\alpha}_{b^-}$ and for the Hadamard fractional integrals $H^{\alpha}_{a^+}$, $H^{\alpha}_{b^-}$ reads as follows.

Corollary 2. *Let $a<b$, $\alpha, B>0$ be real constants. If $f\colon [a,b] \to [0,B]$ is a measurable function,*

$${}^{RL}J^{\alpha}_{a^+} f(b) = \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s)\,ds \in (0,B],$$

φ is a convex function on $[0,B]$, and

$$f(x) \le \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s)\,ds$$

for every $x \in [a,b]$, *then* $(b-s)^{\alpha-1}\varphi(f(s)) \in L^1([a,b])$ *and*

$$\begin{aligned}
&\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\varphi\Big(\frac{\alpha}{(b-a)^\alpha}\int_a^b (b-s)^{\alpha-1} f(s)\,ds\Big)\\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\varphi(f(s))\,ds\\
&\quad\le \varphi\Big(\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s)\,ds\Big) + \Big(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} - 1\Big)\varphi(0).
\end{aligned}$$

Similarly for the right Riemann–Liouville fractional integral ${}^{RL}J^\alpha_{b^-} f(a)$.

Corollary 3. *Let* $a < b$, $\alpha, B > 0$ *be real constants. If* $f\colon [a,b] \to [0,B]$ *is a measurable function,*

$$H^\alpha_{a^+} f(b) = \frac{1}{\Gamma(\alpha)}\int_a^b \Big(\log\frac{b}{s}\Big)^{\alpha-1}\frac{f(s)}{s}\,ds \in (0,B],$$

φ *is a convex function on* $[0,B]$, *and*

$$f(x) \le \frac{1}{\Gamma(\alpha)}\int_a^b \Big(\log\frac{b}{s}\Big)^{\alpha-1}\frac{f(s)}{s}\,ds$$

for every $x \in [a,b]$, *then* $\big(\log\frac{b}{s}\big)^{\alpha-1}\frac{\varphi(f(s))}{s} \in L^1([a,b])$ *and*

$$\begin{aligned}
&\frac{\big(\log\frac{b}{a}\big)^\alpha}{\Gamma(\alpha+1)}\varphi\Big(\frac{\alpha}{\big(\log\frac{b}{a}\big)^\alpha}\int_a^b \Big(\log\frac{b}{s}\Big)^{\alpha-1}\frac{f(s)}{s}\,ds\Big)\\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_a^b \Big(\log\frac{b}{s}\Big)^{\alpha-1}\frac{\varphi(f(s))}{s}\,ds\\
&\quad\le \varphi\Big(\frac{1}{\Gamma(\alpha)}\int_a^b \Big(\log\frac{b}{s}\Big)^{\alpha-1}\frac{f(s)}{s}\,ds\Big) + \Big(\frac{\big(\log\frac{b}{a}\big)^\alpha}{\Gamma(\alpha+1)} - 1\Big)\varphi(0).
\end{aligned}$$

Similarly for the right Hadamard fractional integral $H^\alpha_{b^-} f(a)$.

ACKNOWLEDGEMENTS

The research of José Manuel Rodríguez and José María Sigarreta Almira is supported by a grant from Agencia Estatal de Investigación (PID2019-106433GB-I00 / AEI / 10.13039 / 501100011033), Spain. The research of José Manuel Rodríguez is supported by the Madrid Government (Comunidad de Madrid-Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M23), and in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

REFERENCES

[1] I. A. Baloch and Y.-M. Chu, "Petrović-type inequalities for harmonic h-convex functions," *J. Funct. Spaces*, vol. 2020, p. 3075390, 2020, doi: 10.1155/2020/3075390.

[2] P. Bosch, H. J. Carmenate, J. M. Rodríguez, and J. M. Sigarreta, "Generalized inequalities involving fractional operators of Riemann-Liouville type," *AIMS Math.*, vol. 7, no. 1, pp. 1470–1485, 2022, doi: 10.3934/math.2022087.

[3] Z. Dahmani, "On Minkowski and Hermite-Hadamard integral inequalities via fractional integral," *Ann. Funct. Anal.*, vol. 1, pp. 51–58, 2010, doi: 10.15352/afa/1399900993.

[4] R. Gorenflo and F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, 1st ed. Vienna: Springer, 1997.

[5] J. Han, P. O. Mohammed, and H. Zeng, "Generalized fractional integral inequalities of Hermite-Hadamard-type for a convex function," *Open Math.*, vol. 18, pp. 794–806, 2020, doi: 10.1515/math-2020-0038.

[6] W. Iqbal, K. M. Awan, A. U. Rehman, and G. Farid, "An extension of Petrović's inequality for h-convex (h-concave) functions in plane," *Open J. Math. Sci.*, vol. 2019, no. 3, pp. 398–403, 2019, doi: 10.30538/oms2019.0082.

[7] J. L. W. V. Jensen, "Om konvekse funktioner og uligheder mellem middelvaerdier [About convex functions and inequalities between mean values]," *Nyt Tidsskr. Math.*, vol. 16 B, pp. 49–69, 1905.

[8] A. A. Kilbas, O. I. Marichev, and S. G. Samko, *Fractional integrals and derivatives. Theory and applications*, 1st ed. Pennsylvania: Gordon & Breach, 1993.

[9] P. Kórus and J. E. Nápoles Valdés, "q-hermite-hadamard inequalities for functions with convex or h-convex q-derivative," *Math. Ineq. Appl.*, vol. 25, no. 2, pp. 601–610, 2022, doi: 10.7153/mia-2022-25-36.

[10] S. Mubeen, S. Habib, and M. N. Naeem, "The Minkowski inequality involving generalized k-fractional conformable integral," *J. Inequal. Appl.*, vol. 2019, p. 81, 2019, doi: 10.1186/s13660-019-2040-8.

[11] K. S. Nisar, F. Qi, G. Rahman, S. Mubeen, and M. Arshad, "Some inequalities involving the extended gamma function and the Kummer confluent hypergeometric k-function," *J. Inequal. Appl.*, vol. 2018, p. 135, 2018, doi: 10.1186/s13660-018-1717-8.

[12] M. Petrović, "Sur une fonctionnelle," *Publ. Math. Univ. Belgrade*, vol. 1, pp. 146–149, 1932.

[13] J. E. Pečarić, "On the Petrović inequality for convex functions," *Glasnik Matem.*, vol. 18, no. 38, pp. 77–85, 1983.

[14] G. Rahman, T. Abdeljawad, F. Jarad, A. Khan, and K. S. Nisar, "Certain inequalities via generalized proportional Hadamard fractional integral operators," *Adv. Difference Eq.*, vol. 2019, p. 454, 2019, doi: 10.1186/s13662-019-2381-0.

[15] G. Rahman, K. S. Nisar, B. Ghanbari, and T. Abdeljawad, "On generalized fractional integral inequalities for the monotone weighted Chebyshev functionals," *Adv. Difference Eq.*, vol. 2020, p. 368, 2020, doi: 10.1186/s13662-020-02830-7.

[16] S. Rashid, M. A. Noor, K. I. Noor, and Y.-M. Chu, "Ostrowski type inequalities in the sense of generalized $\mathcal{K}$-fractional integral operator for exponentially convex functions," *AIMS Math.*, vol. 5, no. 3, pp. 2629–2645, 2020, doi: 10.3934/math.2020171.

[17] A. U. Rehman, G. Farid, and V. N. Mishra, "Generalized convex function and associated Petrović's inequality," *Int. J. Anal. Appl.*, vol. 17, no. 1, pp. 122–131, 2019, doi: 10.28924/2291-8639-17-2019-122.

[18] A. U. Rehman, M. Mudessir, H. T. Fazal, and G. Farid, "Petrović's inequality on coordinates and related results," *Cogent Math.*, vol. 3, p. 1227298, 2016, doi: 10.1080/23311835.2016.1227298.

[19] A. W. Roberts and D. E. Varberg, *Convex functions*. New York: Academic Press, 1973.

[20] Y. Sawano and H. Wadade, "On the Gagliardo-Nirenberg type inequality in the critical Sobolev-Morrey space," *J. Fourier Anal. Appl.*, vol. 19, no. 1, pp. 20–47, 2013, doi: 10.1007/s00041-012-9223-8.

[21] E. Set, M. Tomar, and M. Z. Sarikaya, "On generalized Grüss type inequalities for k-fractional integrals," *Appl. Math. Comput.*, vol. 269, pp. 29–34, 2015, doi: 10.1016/j.amc.2015.07.026.

Authors' addresses

Péter Kórus
(**Corresponding author**) Institute of Applied Pedagogy, Juhász Gyula Faculty of Education, University of Szeged, Hattyas utca 10, H-6725 Szeged, Hungary
E-mail address: korus.peter@szte.hu

Juan Eduardo Nápoles Valdés
UNNE, FaCENA, Av. Libertad 5450, Corrientes 3400, Argentina
E-mail address: jnapoles@exa.unne.edu.ar

José Manuel Rodríguez
Universidad Carlos III de Madrid, Departamento de Matemáticas, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain
E-mail address: jomaro@math.uc3m.es

José María Sigarreta Almira
Universidad Autónoma de Guerrero, Centro Acapulco, CP 39610, Acapulco de Juárez, Guerrero, Mexico
E-mail address: josemariasigarretaalmira@hotmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 845–855

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4666

UNIT GROUP OF INTEGRAL GROUP RING $\mathbb{Z}(G \times C_3)$

ÖMER KÜSMÜŞ

Received 28 March, 2023

Abstract. Presenting an explicit descryption of unit group in the integral group ring of a given non-abelian group is a classical and open problem. Let S_3 be a symmetric group of order 6 and C_3 be a cyclic group of order 3. In this study, we firstly explore the commensurability in unit group of integral group ring $\mathbb{Z}(S_3 \times C_3)$ by showing the existence of a subgroup as $(F_{55} \rtimes F_3) \rtimes (S_3^* \times C_2)$ where F_ρ denotes a free group of rank ρ. Later, we introduce an explicit structure of the unit group in $\mathbb{Z}(S_3 \times C_3)$ in terms of semi-direct product of torsion-free normal complement of S_3 and the group of units in RS_3 where $R = \mathbb{Z}[\omega]$ is the complex integral domain since ω is the primitive 3rd root of unity. At the end, we give a general method that determines the structure of the unit group of $\mathbb{Z}(G \times C_3)$ for an arbitrary group G depends on torsion-free normal complement $V(G)$ of G in $U(\mathbb{Z}(G \times C_3))$ in an implicit form. As a consequence, a conjecture which is found in [21] is solved.

2010 *Mathematics Subject Classification:* 16S34; 16U60

Keywords: unit group, integral group ring, symmetric group, direct product

1. Introduction

We denote the integral group ring of a given finite group G over the ring of integers by $\mathbb{Z}G$. Its elements are all finite sums of the form $\sum_{g \in G} \alpha_g g$ where $\alpha_g \in \mathbb{Z}$. The ring epimorphism defined by $\varepsilon : \mathbb{Z}G \longrightarrow \mathbb{Z}$, $\varepsilon(g) = 1$, is called the *augmentation map*. Hence, naturally kernel of ε which is named by *augmentation ideal* and denoted by $\Delta_{\mathbb{Z}}(G)$ is obtained as a $\mathbb{Z}$-module based on the set $S = \{g - 1 : g \in G\}$.

The group of all units in $\mathbb{Z}G$ is generally shown by $U(\mathbb{Z}G)$ and the subgroup of units whose augmentations are 1 denoted by $U_1(\mathbb{Z}G)$. It is well known that $\pm U_1(\mathbb{Z}G) = U(\mathbb{Z}G)$ and the set $\pm G$ is said to be *trivial units* in $U(\mathbb{Z}G)$ [19].

Describing units of integral group rings is a classical hard problem for various type of groups. The subject matter has captured the interest of researchers in the fields of algebra, number theory, and algebraic topology throughout the years. As stated in [16], most descriptions of $U(\mathbb{Z}G)$ in the mathematical literature either give an explicit description of unit group, the general structure of $U(\mathbb{Z}G)$, or a subgroup of

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

finite index of the unit group $U(\mathbb{Z}G)$. Results were often attained by using techniques from representation theory and algebraic number theory [22].

In 1940, a crucial work on the unit problem was done by Graham Higman who displayed that if $U(\mathbb{Z}G) = \pm G$, then $U(\mathbb{Z}(G \times C_2)) = \pm(G \times C_2)$ [6, 7]. Using this, he showed that $U(\mathbb{Z}G) = \pm G$ if and only if G is an abelian group of exponent 2, 3, 4, or 6 or $G = E \times K_8$ where K_8 is the quaternion group of order 8, E is an elementary abelian 2–group and moreover, Higman gave a general structure theorem for $U(\mathbb{Z}A)$, where A is a finite abelian group [6, 7]. Some additional results have been provided for these groups: A_4 and S_4 by Allen-Hobby [1, 2], D_{2p} written by Passman-Smith [18], $G = C_p \rtimes C_q$, such that q is a prime dividing $p-1$ made by Galovitch-Reiner-Ullom [5], $|G| = p^3$ given in [20], and a version of $U(\mathbb{Z}S_3)$ can be found at [8]. In [10], Jespers and Parmenter submitted an explicit description of $U(\mathbb{Z}S_3)$ in terms of bicyclic units as follows:

Theorem 1. *[10] In $U_1(\mathbb{Z}S_3)$, S_3 has a torsion-free normal complement which is generated by bicyclic units as $U_1(\mathbb{Z}S_3) = V \rtimes S_3$ such that $V = \langle u_{b,a}, u_{ba,a}, u_{ba^2,a} \rangle$ where*

$$u_{b,a} = 1 + (1-b)a(1+b)$$
$$u_{ba,a} = 1 + (1-ba)a(1+ba)$$
$$u_{ba^2,a} = 1 + (1-ba^2)a(1+ba^2)$$

In 1993, Jespers and Parmenter described $U(\mathbb{Z}G)$ for all groups of order 16 [11]. In [9], Jespers introduced $U(\mathbb{Z}G)$ for the dihedral group of order 12 and for $G = D_8 \times C_2$. Kelebek and Bilgin gave the structure of $U(\mathbb{Z}(C_n \times K_4))$ where C_n is a cyclic group of order n and K_4 is a Klein-4 group [14]. Köklüce and Tüfekçi presented the symmetric unit groups whose rank is less than 4 of integral group rings of cyclic groups [15]. In [16, 21], a general algebraic framework was implicitly developed to study $U(\mathbb{Z}G^*)$, where $G^* = G \times C_p$, p is a prime integer and G is a finite group. It was stated that $U(\mathbb{Z}G^*)$ is an open problem, where $G^* = T \times C_2$ such that T is a binary dihedral group (or dicyclic group) of order 12 and C_2 is a cyclic group of order 2 was posed as a conjecture [21]. In [3], Bilgin, Küsmüş and Low studied this open problem and introduced some substantial proofs which explain the isomorphic form of unit group in integral group ring of direct product

$$T \times C_2 = \langle a, b, x : a^6 = x^2 = 1, a^3 = b^2, bab^{-1} = a^{-1}, ax = xa, bx = xb \rangle$$

in terms of semi-direct products of finitely generated free-groups. In [21], Low stated that succeeding a complete characterization of $U(\mathbb{Z}(G \times C_p))$ depends on clarifying the structure of $U(RC_p)$ where R is a complex integral domain for $p \geq 3$ which keeps its enigma. In this study, we focus on the case where $G = S_3$, $p = 3$ and we then study the open problem which related to characterization of unit group of integral group ring $\mathbb{Z}(G \times C_p)$ where G is a non-abelian group and $p = 3$ found in [21].

As a solution to this problem, we characterize the unit group in integral group ring $\mathbb{Z}(S_3\times C_3)$ of

$$S_3^* := S_3\times C_3 = \left\langle a,b,x : a^3=b^2=x^3=1, bab^{-1}=a^{-1}, ax=xa, bx=xb\right\rangle.$$

2. Decomposition of $U(\mathbb{Z}S_3^*)$

It is worth mentioning that the groups possessing a non-abelian torsion-free normal complement have been classified in [12]. Furthermore, various studies [9], [17], [21], [3] have described the free group structure of $U(\mathbb{Z}(G\times C_2))$ for $G=S_3,D_4,P$ and T, respectively. It is crucial to note that these characterizations heavily rely on the fact that $\omega^2=1$ implies $\mathbb{Z}[\omega]=\mathbb{Z}$ which provides an advantage according to the scheme presented in [3]. However, this is not the case for $\omega^n=1$ where $n\geq 3$, as we have $\mathbb{Z}\subsetneq\mathbb{Z}[\omega]$. In this section, we firstly concentrate on investigating a subgroup of $U(\mathbb{Z}S_3^*)$ that is commensurable [13]. To proceed further, we require some preliminary propositions.

Proposition 1. *Since* $S_3=\left\langle a,b : a^3=b^2=1, bab^{-1}=a^{-1}\right\rangle$,

$$U_1(\mathbb{F}_3S_3)\simeq (3_+^{1+2}\rtimes C_3)\rtimes K_4$$

where K_4 is Klein 4-group, C_3 is a cyclic group of order 3 and 3_+^{1+2} extraspecial group of order 27.

Proof. Let $N=\langle a\rangle$. We can define the projection $\varphi : S_3\longrightarrow S_3/N$ by $\varphi(g)=gN$. Linearity of φ over the ring $\mathbb{F}_3$ follows that $\overline{\varphi}:\mathbb{F}_3S_3\longrightarrow\mathbb{F}_3(S_3/N)$ is a projection at ring level by

$$\overline{\varphi}(\alpha_i a^i+\beta_i ba^i)=(\alpha_0+\alpha_1+\alpha_2)N+(\beta_0+\beta_1+\beta_2)bN \tag{2.1}$$

Let $\tilde{\varphi}$ be the restriction of $\overline{\varphi}$ on the unit level which is defined in the same as the rule in (2.1).

This indicates that $\tilde{\varphi}(U(\mathbb{F}_3S_3))=U(\mathbb{F}_3C_2)$. Furthermore, we have from [4] that $U(\mathbb{F}_{3^k}C_2)=C_{p^k-1}\times C_{p^k-1}$. Therefore $U(\mathbb{F}_3C_2)=K_4$ where K_4 is Klein 4-group and thus $Im\tilde{\varphi}=K_4$.

On the other hand,

$$Ker\tilde{\varphi}=\left\{\sum_{i=0}^{2}\alpha_i a^i+\beta_i ba^i : (\sum_{i=0}^{2}\alpha_i,\sum_{i=0}^{2}\beta_i)=(1,0), \forall\alpha_i,\beta_i\in\mathbb{F}_3\right\}$$

is composed of 81 elements. Since $\langle a\rangle\subset Ker\tilde{\varphi}$, consider the quotient $G^{'}=Ker\tilde{\varphi}/\langle a\rangle$. One can discern that $\left\langle -1+a+a^2\right\rangle$ is a cyclic group of order 3 and moreover it is central in $G^{'}$.

As a result, we conclude that $G^{'}/\left\langle -1+a+a^2\right\rangle$ is an elementary abelian 3-group and hence $G^{'}$ is an extraspecial 3-group which can generally be denoted as 3_+^{1+2}. This

manifests that $Ker\tilde{\varphi}$ is isomorphic to $3_+^{1+2} \rtimes C_3$. Therefore, it is possible to achieve an exact sequence as

$$Ker\tilde{\varphi} \xrightarrow{\iota} U_1(\mathbb{F}_3S_3) \xrightarrow{\tilde{\varphi}} Im\tilde{\varphi} = U_1(\mathbb{F}_3C_2) \simeq K_4$$

Reminding the identity map is the reverse direction of embedding i, we can phrase that the sequence splits as $U_1(\mathbb{F}_3S_3) = Ker\tilde{\varphi} \rtimes Im\tilde{\varphi}$ which substantiates the proof. □

Proposition 2. *Let $\sigma_3 : \mathbb{Z}G \to \mathbb{Z}_3G$ be a surjective ring homomorphism that reduces the coefficients modulo 3 and $V = \langle u_{b,a}, u_{ba,a}, u_{ba^2,a} \rangle$ be the torsion free normal complement of S_3. Then, $\sigma_3(U_1(\mathbb{Z}S_3)) = \sigma_3(V) \rtimes S_3$.*

Proof. Clearly, $\sigma_3(U_1(\mathbb{Z}S_3)) = \sigma_3(V \rtimes S_3) = \sigma_3(V) \cdot S_3$. In addition, $\sigma_3(V)$ is normalized by S_3 due to the following equalities which can be easily shown:

$$\begin{aligned}
a\sigma_3(u_{b,a})a^2 &= \sigma_3(u_{ba,a}) \\
a\sigma_3(u_{ba,a})a^2 &= \sigma_3(u_{ba^2,a}) \\
a\sigma_3(u_{ba^2,a})a^2 &= \sigma_3(u_{b,a}) \\
b\sigma_3(u_{b,a})b &= \sigma_3(u_{b,a})^{-1} \\
b\sigma_3(u_{ba,a})b &= \sigma_3(u_{ba^2,a})^{-1} \\
b\sigma_3(u_{ba^2,a})b &= \sigma_3(u_{ba,a})^{-1}
\end{aligned}$$

On the other hand, the fact that $\sigma_3(V) \cap S_3 = 1$ infers that $\sigma_3(U_1(\mathbb{Z}S_3))$ is indeed a semi-direct product as claimed. □

Proposition 3. $\sigma_3(V) \simeq C_3 \times C_3 \times C_3$.

Proof. Since $V = \langle u_{b,a}, u_{ba,a}, u_{ba^2,a} \rangle$, check that

$$\sigma_3(u_{b,a})^3 = \sigma_3(u_{ba,a})^3 = \sigma_3(u_{ba^2,a})^3 = 1.$$

We can conclude by the previous proposition that each of $\sigma_3(u_{b,a}), \sigma_3(u_{ba,a})$ and $\sigma_3(u_{ba^2,a})$ gives a normal subgroup. Moreover, as the bicyclic units in V and the powers of images under σ_3 of them are distinct, the product is direct. □

Now, we can introduce the following commutative diagram by taking inspration from [16]:

$$\begin{array}{ccccc}
K & \xrightarrow{\iota} & U(\mathbb{Z}S_3^*) & \xrightarrow{\pi} & U(\mathbb{Z}S_3) \\
{\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \sigma_3}\downarrow \\
M & \xrightarrow{\iota} & U(\mathbb{Z}[\omega]S_3) & \xrightarrow{\alpha} & U(\mathbb{Z}_3S_3)
\end{array}$$

Lemma 1. *K and M are isomorphic to each other.*

Proof. Using the definitions of π, α, f and σ_3 defined on $\mathbb{Z}S_3^*$, we have the following diagram:

$$\begin{array}{ccccc} K^* & \xrightarrow{\iota} & \mathbb{Z}S_3^* & \xrightarrow{\pi} & \mathbb{Z}S_3 \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \sigma_3}\downarrow \\ M^* & \xrightarrow{\iota} & \mathbb{Z}[\omega]S_3 & \xrightarrow{\alpha} & \mathbb{Z}_3S_3 \end{array}$$

Let $(x-1)c_1+(x^2-1)c_2 \in K^*$. Then, linearity of f follows that

$$f((x-1)c_1+(x^2-1)c_2) = (\omega-1)c_1+(\omega^2-1)c_2.$$

Due to the fact that ω is the primitive root of unity, rearranging the terms, we have

$$(\omega-1)c_1+(\omega^2-1)c_2 = (\omega-1)(c_1-c_2)-3c_2.$$

It requires that $f((x-1)c_1+(x^2-1)c_2)=0$ if and only if $(x-1)c_1+(x^2-1)c_2=0$. Thus, f is injective. Taking the pre-image of ω as x, it is easy to perceive that f is surjective. We deduce from [16] that $K = U(1+K^*)$ and $M = U(1+M^*)$ are isomorphic groups of units in multiplicative monoids K^* and M^* respectively. $\square$

Corollary 1.

$$\begin{array}{ccccc} K & \xrightarrow{\iota} & U(\mathbb{Z}S_3^*) & \xrightarrow{\pi} & U(\mathbb{Z}S_3) \\ {\scriptstyle \simeq}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \sigma_3}\downarrow \\ M & \xrightarrow{\iota} & U(\mathbb{Z}[\omega]S_3) & \xrightarrow{\sigma_3} & U(\mathbb{Z}_3S_3) \\ {\scriptstyle \iota}\uparrow & & {\scriptstyle \iota}\uparrow & & {\scriptstyle \iota}\uparrow \\ M^+ & \xrightarrow{\iota} & U_1(\mathbb{Z}S_3) & \xrightarrow{onto} & \sigma_3(V)\rtimes S_3 \\ {\scriptstyle \iota}\uparrow & & {\scriptstyle \iota}\uparrow & & {\scriptstyle \iota}\uparrow \\ M^+\cap V & \xrightarrow{\iota} & V & \xrightarrow{onto} & \sigma_3(V) \end{array}$$

Corollary 2. *M^+ is a free group of rank 55.*

Proof. Let $w(\sigma_3(u_{b,a}),\sigma_3(u_{ba,a}),\sigma_3(u_{ba^2,a}))$ be a word in $\sigma_3(V)$ and $g\in S_3$. As $\sigma_3(V)\cap S_3=\{1\}$, observe that if $w(\sigma_3(u_{b,a}),\sigma_3(u_{ba,a}),\sigma_3(u_{ba^2,a}))\cdot g=1$, then

$$w(\sigma_3(u_{b,a}),\sigma_3(u_{ba,a}),\sigma_3(u_{ba^2,a})) = g^{-1}\in\sigma_3(V)\cap S_3.$$

This implies that $g=1$. This means that M^+ consists of words of the form

$$u_{b,a}^{i_1}\cdot u_{ba,a}^{i_2}\cdot u_{ba^2,a}^{i_3}$$

where $0\le i_1,i_2,i_3\le 2$. This shows that $M^+=M^+\cap V$. Besides $\frac{V}{M^+\cap V}\simeq\sigma_3(V)$ implies that

$$[V:M^+] = |\sigma_3(V)| = 27$$

Since V is a free group of rank 3, we conclude by Schreier's index formula and Schreier's Theorem that M^+ is a free group of rank 55 via Schreier's index formula. $\square$

Theorem 2. *$U(\mathbb{Z}S_3^*)$ is commensurable which contains a subgroup of finite index as*

$$(F_{55} \rtimes F_3) \rtimes (S_3^* \times C_2)$$

Proof. As $U(\mathbb{Z}[\omega]S_3) = U_1(\mathbb{Z}[\omega]S_3) \rtimes U(\mathbb{Z}[\omega])$ and $U(\mathbb{Z}[\omega]) = \langle -1 \rangle \times \langle \omega \rangle \simeq C_6$, we can recognize that $U_1(\mathbb{Z}S_3) \hookrightarrow U_1(\mathbb{Z}[\omega]S_3) \times C_6$ and hence $M^+ \hookrightarrow M^+ \times C_6 \subsetneq M$. We know from the diagram in Corollary 1 and Lemma 1 that

$$U_1(\mathbb{Z}S_3^*) = K \rtimes (V \rtimes S_3) \simeq M \rtimes (V \rtimes S_3)$$

Since,

$$(M^+ \times C_6) \rtimes (V \rtimes S_3) \subsetneq M \rtimes (V \rtimes S_3)$$

we conclude that

$$(M^+ \times C_6) \rtimes (V \rtimes S_3) = (M^+ \rtimes V) \rtimes (S_3 \times C_3) \times C_2 = (F_{55} \rtimes F_3) \rtimes (S_3^* \times C_2)$$

as claimed. $\square$

Let $R = \mathbb{Z}[\omega]$ and $\{u_{ba^i,a}\}_{i=1}^3$ be the set of generators in torsion-free normal complement of S_3. Note that $\mathbb{Z}[\omega]S_3$ can be decomposed as $\mathbb{Z}S_3 \oplus \omega\mathbb{Z}S_3$. Defining a ring epimorphism $\psi : \mathbb{Z}S_3 \oplus \omega\mathbb{Z}S_3 \longrightarrow \mathbb{Z}S_3$ by $\psi(\alpha, \omega\beta) = \alpha$ follows that $Ker\psi = \omega\mathbb{Z}S_3$ and the following exact sequence

$$\omega\mathbb{Z}S_3 \xrightarrow{\iota} \mathbb{Z}S_3 \oplus \omega\mathbb{Z}S_3 \xrightarrow{\psi} \mathbb{Z}S_3$$

At unit group level, this sequence can be extended as

$$(1 + \omega\mathbb{Z}S_3) \cap U(\mathbb{Z}S_3) \xrightarrow{\iota} U(\mathbb{Z}[\omega]S_3) \xrightarrow{\psi} U(\mathbb{Z}S_3)$$

Let $u = 1 + \omega r \in (1 + \omega\mathbb{Z}S_3) \cap U(\mathbb{Z}S_3)$. So there exists $v = 1 + \omega s$ where $r, s \in \mathbb{Z}S_3$ such that $uv = 1$. Notice that u and v are inverses of each other if and only if

$$uv = (1 - rs) + \omega(r + s - rs) = 1$$

and so r and s are either trivial or nilpotent elements of index 2 in $\mathbb{Z}S_3$. It follows that either $1 - r = 1$ or $1 - r \in V$ due to the fact that every unipotent unit is lifted from a nilpotent element of index 2. Hence we can take $r = u_{ba^i,a} - 1$. Accordingly,

$$(1 + \omega\mathbb{Z}S_3) \cap U(\mathbb{Z}S_3) = F$$

and

$$U(\mathbb{Z}[\omega]S_3) = F \rtimes U(\mathbb{Z}S_3) \tag{2.2}$$

where

$$F = \left\langle 1 + \omega(u_{ba^i,a} - 1) : u_{ba^i,a} \in V, i = 0, 1, 2 \right\rangle$$

Theorem 3.

$$U(RS_3) \simeq \frac{F \rtimes V \rtimes S_3}{(3_+^{1+2} \rtimes C_3) \rtimes K_4}$$

where 3_+^{1+2} *denotes extraspecial group of order 27.*

Proof. In Corollary 1, consider the second row of the diagram. As the map i is an embedding, this row can be split into two parts: That is $U(RS_3)$ is isomorphic to the semidirect product of M and $U(\mathbb{Z}_3S_3)$. Additionally, equation (2.2) tells us that $U(RS_3)$ is isomorphic to the direct product of F and $U(\mathbb{Z}S_3)$. Combining these different forms of $U(RS_3)$, we can conclude that F and $U(\mathbb{Z}S_3)$ together form a group which is isomorphic to the semidirect product of M and $U(\mathbb{Z}_3S_3)$ that is

$$U(RS_3) = F \rtimes U(\mathbb{Z}S_3) \simeq M \rtimes U(\mathbb{Z}_3S_3)$$

Using this result, we can infer that M is isomorphic to the quotient group obtained by dividing the direct product of F and $U(\mathbb{Z}S_3)$ by the normal subgroup $U(\mathbb{Z}_3S_3)$. In other words,

$$M \simeq \frac{F \rtimes U(\mathbb{Z}S_3)}{U(\mathbb{Z}_3S_3)}$$

Furthermore, Proposition 1 tells us that $U(\mathbb{Z}_3S_3)$ is an extraspecial 3-group of order 27, denoted by 3_+^{1+2}. This establishes the proof. □

An alternative way to describe $U(RS_3)$ is through the use of cyclic groups instead of 3_+^{1+2} as follows.

Theorem 4.

$$U(RS_3) \simeq \frac{F \rtimes V \rtimes S_3}{[(C_3^3 \rtimes C_3) \rtimes C_2] \times C_2}$$

where $C_3^3 = C_3 \times C_3 \times C_3$.

Proof. To enhance clarity of the proof, it is adequate to take into account $k = 1$ in the main result of [4] concerning $U(\mathbb{Z}_3S_3)$. □

Lemma 2. *F is a free group on the generator set $S = \{v_{b,a}, v_{ba,a}, v_{ba^2,a}\}$ where $v_{ba^i,a} = 1 + \omega(u_{ba^i,a} - 1)$.*

Proof. Let $v_{ba^i,a}$ denote $1 + \omega(u_{ba^i,a} - 1)$. Then F can be stated as

$$F = \langle v_{b,a}, v_{ba,a}, v_{ba^2,a} \rangle$$

Assume that F is not a free group. Then any unit in F can be written in different ways as a product of finitely many elements of $S = \{v_{b,a}, v_{ba,a}, v_{ba^2,a}\}$ and their inverses disregarding trivial variations. Let any element $v_{ba^i,a}$ in F be phrased as

$$v_{ba^i,a} = v_{ba^{i_1},a} v_{ba^{i_2},a} \tag{2.3}$$

Then the equation (2.3) hold if and only if

$$v_{ba^i,a} = 1 + (u_{ba^{i_1},a} + u_{ba^{i_2},a} - u_{ba^{i_1},a} u_{ba^{i_2},a} - 1) + \omega(2(u_{ba^{i_1},a} + u_{ba^{i_2},a}) - 3)$$

and so

$$2u_{ba^{i_1},a}u_{ba^{i_2},a} = u_{ba^i,a}$$

which is a contradiction due to the fact that V is free. Besides, $T : V \longrightarrow F$ defined by

$$T(u_{ba^i,a}) = v_{ba^i,a}$$

is a bijective map. Therefore, F is a free group as well. □

We are now prepared to establish our principal structural theorem in this section, as follows.

Theorem 5. *$U(\mathbb{Z}S_3^*) = M \rtimes V \rtimes S_3$ where $M \simeq \frac{F \rtimes V \rtimes S_3}{(3_+^{1+2} \rtimes C_3) \rtimes K_4}$.*

Proof. Remind that we have the split extension

$$U(\mathbb{Z}S_3^*) = K \rtimes U(\mathbb{Z}S_3)$$

from the commutative diagram of maps in Corollary 1. We also know from Lemma 1 that $K \simeq M$. Besides this,

$$U(RS_3) = M \rtimes (3_+^{1+2} \rtimes C_3) \rtimes K_4 \tag{2.4}$$

because of Theorem 3. On the other hand, equation (2.2) and [10] follows that

$$U(RS_3) = F \rtimes V \rtimes S_3 \tag{2.5}$$

Let

$$\delta : M \rtimes (3_+^{1+2} \rtimes C_3) \rtimes K_4 \longrightarrow M$$

by $\delta(m.(a.b.c)) = m$ where $(a.b).c \in (3_+^{1+2} \rtimes C_3) \rtimes K_4$. Then, it is clear that

$$Ker\delta = U_1(\mathbb{F}_3S_3) = (3_+^{1+2} \rtimes C_3) \rtimes K_4$$

Thus,

$$M \simeq U(RS_3)/U_1(\mathbb{F}_3S_3) \tag{2.6}$$

Using equations (2.4), (2.5) and (2.6), we conclude that $U(\mathbb{Z}S_3^*) = M \rtimes V \rtimes S_3$ where

$$M \simeq \frac{F \rtimes V \rtimes S_3}{(3_+^{1+2} \rtimes C_3) \rtimes K_4}$$

as required. □

3. Conclusion and Discussion

In this study, we have examined the commensurability of $U(\mathbb{Z}S_3^*)$, demonstrating the existence of a subgroup in $U(\mathbb{Z}S_3^*)$, one of whose factors is free-by-free group, specifically $F_{55} \rtimes F_3$. We then proceeded to investigate the whole unit group of the integral group ring $\mathbb{Z}S_3^*$ by characterizing it through the description of the unit group of RS_3, where $R = \mathbb{Z}[\omega]$, with ω being the primitive 3rd root of unity, and its torsion-free subgroup generated by units which are lifted from square-zero elements.

Regarding the characterization of the entire unit group in $\mathbb{Z}S_3^*$, we have established that the torsion-free normal complement of S_3^* is derived from the unipotent unit generators of $\mathbb{Z}S_3^*$. In fact, by modifying the definitions of maps π, σ_3 and f in the commutative diagram established in Corollary 1 over an arbitrary finite group G rather than specifically on the symmetric group S_3, we can derive the aforementioned diagram in the following manner:

$$\begin{tikzcd} K \arrow[r, "\iota"] \arrow[d, "\simeq"'] & U(\mathbb{Z}G^*) \arrow[r, "\pi"] \arrow[d, "f"'] & U(\mathbb{Z}G) \arrow[d, "\sigma_3"'] \\ M \arrow[r, "\iota"] & U(\mathbb{Z}[\omega]G) \arrow[r, "\alpha"] & U(\mathbb{Z}_3G) \end{tikzcd} \tag{3.1}$$

where $G^* = G \times C_3$.

To achieve a comprehensive description of $U(\mathbb{Z}G^*)$ using (3.1), it is imperative to possess an explicit version of $U(\mathbb{Z}G)$ that either elucidates its general structure or identifies a subgroup of finite index. Moreover, as can be seen from the split extension of the second row of our diagram, since $U(\mathbb{Z}[\omega]G) = M \rtimes U(\mathbb{Z}_3G)$, another characterization of the unit group $U(\mathbb{Z}G^*)$ in terms of the unit group of the finite group algebra $\mathbb{Z}_3G$ can be given. The commutative diagram constructed on an arbitrary group G only yields an implicit version of $U(\mathbb{Z}G^*)$. Our diagram can extend the integral group ring $\mathbb{Z}G$, which has been specifically examined by relevant researchers to explore the unit group and provide an explicit description, to $U(\mathbb{Z}G^*)$. In particular, when working with a group G for which G has a torsion-free normal complement, denoted by $V(G)$ in $U(\mathbb{Z}G)$, one can establish split extension form of $U(\mathbb{Z}G^*)$ in the context of $V(G)$. We know that $U(\mathbb{Z}G^*) \simeq M \rtimes U(\mathbb{Z}G)$ and $U(RG) = F \rtimes U(\mathbb{Z}G)$. As another homomorphic image, we have $M \simeq U(RG)/U(\mathbb{Z}_3G)$. Therefore, we can implicitly characterize $U(\mathbb{Z}G^*)$ as

$$U(\mathbb{Z}G^*) \simeq \frac{F \rtimes V(G) \rtimes G}{U(\mathbb{Z}_3G)} \rtimes V(G) \rtimes G \tag{3.2}$$

The isomorphism established in (3.2) can be applied to provide a precise characterization for all groups G, for which the unit group in group algebra of G over $\mathbb{Z}_3$ is completely characterized and has a known torsion-free normal complement in $U(\mathbb{Z}G)$. From this perspective, this study may serve as a source of inspiration for future investigations.

ACKNOWLEDGEMENTS

The author would like to express gratitude to the anonymous reviewers and editors who generously contributed their valuable time and effort to provide insightful feedback and constructive suggestions for improving the quality of this article.

REFERENCES

[1] P. J. Allen and C. Hobby, "A characterization of units in $\mathbb{Z}[A_4]$," *J. Algebra*, vol. 66, pp. 534–543, 1980, doi: 10.1016/0021-8693(80)90102-7.

[2] P. J. Allen and C. Hobby, "A characterization of units in ZS_4," *Commun. Algebra*, vol. 16, no. 7, pp. 1479–1505, 1988, doi: 10.1080/00927878808823639.

[3] T. Bilgin, O. Kusmus, and R. M. Low, "A characterization of the unit group in $\mathbb{Z}[T \times C_2]$." *Bull. Korean Math. Soc.*, vol. 53, no. 4, pp. 1105–1112, 2016, doi: 10.4134/BKMS.b150526.

[4] L. Creedon and J. Gildea, "The structure of the unit group of the group algebra $\mathbb{F}_{3^k}D_6$." *Int. J. Pure Appl. Math.*, vol. 45, no. 2, pp. 315–320, 2008.

[5] S. Galovich, I. Reiner, and S. Ullom, "Class groups for integral representations of metacyclic groups," *Mathematika*, vol. 19, pp. 105–111, 1972, doi: 10.1112/S0025579300005015.

[6] G. Higman, "The units of group-rings," Ph.D. dissertation, University of Oxford, 1940.

[7] G. Higman, "The units of group-rings," *Proc. Lond. Math. Soc. (2)*, vol. 46, pp. 231–248, 1940, doi: 10.1112/plms/s2-46.1.231.

[8] I. Hughes and K. R. Pearson, "The group of units of the integral group ring ZS_3," *Can. Math. Bull.*, vol. 15, pp. 529–534, 1972, doi: 10.4153/CMB-1972-093-1.

[9] E. Jespers, "Bicyclic units in some integral group rings," *Can. Math. Bull.*, vol. 38, no. 1, pp. 80–86, 1995, doi: 10.4153/CMB-1995-010-4.

[10] E. Jespers and M. M. Parmenter, "Bicyclic units in $\mathbb{Z}S_3$," *Bull. Soc. Math. Belg., Sér. B*, vol. 44, no. 2, pp. 141–146, 1992.

[11] E. Jespers and M. M. Parmenter, "Units of group rings of groups of order 16," *Glasg. Math. J.*, vol. 35, no. 3, pp. 367–379, 1993, doi: 10.1017/S0017089500009952.

[12] E. Jespers, "Free normal complements and the unit group of integral group rings." *Proc. Am. Math. Soc.*, vol. 122, no. 1, pp. 59–66, 1994, doi: 10.2307/2160841.

[13] E. Jespers, A. Pita, Á. del Río, M. Ruiz, and P. Zalesskii, "Groups of units of integral group rings commensurable with direct products of free-by-free groups." *Adv. Math.*, vol. 212, no. 2, pp. 692–722, 2007, doi: 10.1016/j.aim.2006.11.005.

[14] I. G. Kelebek and T. Bilgin, "Characterization of $U_1(\mathbb{Z}[C_n \times K_4])$," *Eur. J. Pure Appl. Math.*, vol. 7, no. 4, pp. 462–471, 2014.

[15] B. Köklüce and N. Tüfekçi, "Characterization of $V(\mathbb{Z}C_n^+)$ of rank ≤ 4." *Miskolc Math. Notes*, vol. 15, no. 2, pp. 571–584, 2014, doi: 10.18514/MMN.2014.746.

[16] R. M. Low, "On the units of the integral group ring $\mathbb{Z}[G \times C_p]$." *J. Algebra Appl.*, vol. 7, no. 3, pp. 393–403, 2008, doi: 10.1142/S0219498808002898.

[17] M. M. Parmenter, "Free torsion-free normal complements in integral group rings," *Commun. Algebra*, vol. 21, no. 10, pp. 3611–3617, 1993, doi: 10.1080/00927879308824751.

[18] D. S. Passman and P. F. Smith, "Units in integral group rings," *J. Algebra*, vol. 69, pp. 213–239, 1981, doi: 10.1016/0021-8693(81)90139-3.

[19] C. Polcino Milies and S. K. Sehgal, *An introduction to group rings*, ser. Algebr. Appl. Dordrecht: Kluwer Academic Publishers, 2002, vol. 1.

[20] J. Ritter and S. K. Sehgal, "Units of group rings of solvable and Frobenius groups over large rings of cyclotomic integers," *J. Algebra*, vol. 158, no. 1, pp. 116–129, 1993, doi: 10.1006/jabr.1993.1126.

[21] L. R.M., "Units in integral group rings for direct products," Ph.D. dissertation, Western Michigan University, Kalamazoo, MI, 1998.

[22] S. K. Sehgal, *Units in integral group rings*, ser. Pitman Monogr. Surv. Pure Appl. Math. Harlow: Longman Scientific & Technical; New York, N.Y.: John Wiley & Sons, 1993, vol. 69.

Author's address

Ömer Küsmüş
Department of Math., Faculty of Science, Van Yuzuncu Yil University, 65080 Van, Turkey
E-mail address: omerkusmus@yyu.edu.tr

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 857–872

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4201

WELL-POSEDNESS CHARACTERIZATIONS FOR SYSTEM OF MIXED HEMIVARIATIONAL INEQUALITIES

KARTIKESWAR MAHALIK AND CHANDAL NAHAK

Received 11 April, 2022

Abstract. In this paper, we concern with the concept of well-posedness and well-posedness in generalized sense for a system of mixed hemivariational inequality (SMHVI) with perturbations. We establish several metric characterizations and equivalent conditions of well-posedness for SMHVI. Our main results improve and extend some announced work in the literature.

2010 *Mathematics Subject Classification:* 49J40; 47J20

Keywords: system of mixed hemivariational inequalities, approximating sequence, well-posedness, perturbation

1. Introduction

The theory of well-posedness in optimization problems and various comprehensive problems correlated to it such as variational inequalities, fixed point problems, equilibrium problems, saddle point problems, and inclusion problems has been paid deep impact attention by researchers due to their essential contributions to physics, mechanics, engineering, and economics. Tykhonov [20] first proposed the concept of well-posedness for the unconstrained global minimization problems, i.e., a minimization problem is well-posed, if it possesses a unique minimizer and the limit of every subsequence of the minimizing sequence is the unique minimizer. Indeed, an optimization problem may not have a unique solution. Hence, the notion of generalized well-posedness was introduced, which enhances the precise solution of the problem, under some mild conditions, by ensuring the convergence of every approximating solutions to the exact solution. Accordingly, the approximating approach captured the concept of well-posedness that plays a vital aspect in well-posedness theory. For more related research works concerning to the well-posedness results

The first author was supported by the Council of Scientific and Industrial Research (CSIR), India, under grant No. 09/081(1306)/2017-EMR-I. The second author was partially supported by the SERB, DST, Govt. of India, Grant No MTR/2022/000296.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

of optimization problems, we refer to [11, 14]. Later, the notion of well-posedness turned toward variational inequalities in different aspects. Lucchetti and Patrone [17] first studied and developed the concept of well-posedness for variational inequalities by Ekeland's technique. From then on, researchers are dedicated to investigate the well-posedness for variational inequalities and build their links with the well-posedness of inclusion and minimization problems. Lignola [13] and Lignola and Morgan [14] coined the notion of $L-$well-posedness for quasivariational inequalities and parametric well-posedness for variational inequalities, respectively. Ceng et al. [3] explored various necessary and sufficient conditions of the well-posedness for the mixed quasivariational-like inequalities. Meanwhile, Fang et al. [8, 9] demonstrated different forms of well-posedness results for mixed variational inequalities, which were further generalized by Ceng et al. [2]. In the works of [4, 8, 9], the authors discussed various necessary and sufficient metric characterizations of well-posedness. There, the authors also elaborated on the corresponding inclusion problems as well as the corresponding fixed point problems and established the equivalent relations between the well-posedness of mixed variational inequalities and their corresponding inclusion problems.

In the last decade, the study of well-posedness devoted special attention to hemivariational inequality problem (HVIP), which is a powerful and important generalization of variational inequalities. The HVIP was first investigated and explored by Panagiotopoulos [19] at the beginning of 1980s to formulate several classes of unilateral mechanical problems with nonsmooth and nonconvex energy functional. The HVIP is described through Clarke's generalized gradient for nonconvex and nondifferentiable functions. It is straightforward to see that if the involved functions are convex, then the HVIP weakens to the variational inequality (see [6]). Consequently, several extensive well-posedness results (for e.g. see, [2, 5, 10, 15, 22–24]) have been investigated and widely discussed for HVIP. Goeleven and Mentagui [10] first considered the well-posedness consequences for HVIP. Xiao et al. [23] discussed the well-posedness characterizations for a class of HVIP. Recently, Xiao et al. [24] also explored the well-posedness results for hemivariational inequalities by utilizing the notion of strongly relaxed monotonicity. By introducing $\alpha-\eta$ monotone mappings, Liu et al. [15] established $L-\alpha$ well-posedness for mixed quasivariational hemivariational inequality through Mosco convergence of sets. Xiao and Huang [22] proposed the well-posedness results for variational-hemivariational inequalities with perturbations. In 2013, Costea and Varga [7] considered the existence results for the system of hemivariational inequality and also derived the Nash generalized derivative points. Recently, Wang et al. [21] generalized the concept of well-posedness to a system of hemivariational inequalities in product Banach space. Besides, the authors in [21–23] also built the equivalence results between the well-posedness of the inequality problems and the well-posedness of its derived inclusion problems. Further, Ceng et al. [5]

studied the well-posedness results for system of time-dependent hemivariational inequalities. However, the research works based on the well-posedness of solutions for the system of hemivariational inequalities are quite limited due to the complexity of its structure. The aim of this work is to develop the theoretical framework on well-posedness for a new class of system of mixed hemivariational inequality with perturbations.

Motivated by the aforesaid research works on well-posedness, in this work, our aim is to introduce and generalize the concept of well-posedness for the SMHVI with perturbations, which includes the HVIP as a special case. Section 2 briefly revisits some preliminary definitions and notations, which will be used to achieve the main results. Section 3 formulates the SMHVI with perturbations and the concept of approximating sequence as well as strong (resp., weak) well-posedness. Section 4 explores the necessary theorems and the metric characterizations for the well-posedness of SMHVI.

2. Preliminaries

Let X be a Banach space with dual space X^*. We signify $\langle\cdot,\cdot\rangle_{X^*\times X}$ as the duality pairing between X^* and X. We denote $\to, \rightharpoonup$ and $\xrightarrow{w^*}$ as the strong, weak and weak* convergence, respectively, in the appointed space.

Definition 1 ([18, Definition 1.31])**.** A sequence $\{x_n\} \subset X$ is weakly convergent iff there exists $x \in X$ s.t.

$$\langle x^*, x_n\rangle_{X^*\times X} \to \langle x^*, x\rangle_{X^*\times X}, \qquad \forall\, x^* \in X^*.$$

Definition 2 ([18, Definition 1.42])**.** A sequence of functional $\{x_n^*\} \subset X^*$ is called weakly* convergent to $x^* \in X^*$ iff

$$\langle x_n^*, x\rangle_{X^*\times X} \to \langle x^*, x\rangle_{X^*\times X}, \qquad \forall\, x \in X.$$

Theorem 1 ([18, Proposition 1.37 and Proposition 1.45])**.** *If $\{x_n\} \subset X$ and $\{x_n^*\} \subset X^*$ with $x_n \to x$ (resp. $x_n \rightharpoonup x$) in X and $x_n^* \xrightarrow{w^*} x^*$ (resp. $x_n^* \to x^*$) in X^*, then*

$$\langle x_n^*, x_n\rangle_{X^*\times X} \to \langle x^*, x\rangle_{X^*\times X}.$$

A function $\phi\colon X \to \mathbb{R}\cup\{\infty\}$ is called lower semicontinuous (l.s.c.) if it fulfills

$$\phi(x) \le \liminf_{n\to\infty} \phi(x_n)$$

for all sequence $x_n \in X$ with $x_n \to x$.

Definition 3 ([21, Definition 2.13])**.** We say that a functional $J\colon X \to \mathbb{R}$ is locally Lipschitz, if for every $z \in X$, there exists a neighborhood V of z and a constant $K_z > 0$ s.t. $|J(x) - J(y)| \le K_z\|x-y\|_X$ for all $x, y \in V$.

Definition 4 ([21, Definition 2.16])**.** Let $J\colon X \to \mathbb{R}$ be a locally Lipschitz functional. We denote $J^\circ(x;y)$ and $\bar{\partial}J(x)$ as the Clarke's generalized directional derivative at the point $x \in X$ in the direction $y \in X$ and Clarke's generalized gradient of J at $x \in X$, respectively, which are defined as:

$$J^\circ(x;y) = \limsup_{w\to x, \varepsilon\to 0^+} \frac{J(w+\varepsilon y) - J(w)}{\varepsilon},$$

and $\bar{\partial}J(x) = \{\xi \in X^* : \langle \xi, y\rangle_{X^*\times X} \le J^\circ(x;y), \forall\, y \in X\}$, respectively.

Lemma 1 ([6, Proposition 2.1.1 and Proposition 2.1.2])**.** *Let $J\colon X \to \mathbb{R}$ be a locally Lipschitz functional of rank L. Then*

(i) the function $y \to J^\circ(x;y)$ is finite, homogenous, subadditive on X and satisfies $|J^\circ(x;y)| \le L\|y\|_X$;
(ii) the functional $(x,y) \to J^\circ(x,y)$ is upper semicontinuous;
(iii) $J^\circ(x;-y) = (-J)^\circ(x;y)$;
(iv) for each $x \in X, \bar{\partial}J(x)$ is a nonempty, convex, weak*-compact subset of X^* and $\|\xi\|_{X^*} \le L$ for all $\xi \in \bar{\partial}J(x)$;
(v) for each $y \in X$, one has $J^\circ(x;y) = \max\limits_{\xi\in\bar{\partial}J(x)} \langle \xi, y\rangle_{X^*\times X}$.

Definition 5 ([22, Definition 2.3])**.** The measure of non-compactness for the set $D \subset X$ is defined by

$$\mu(D) = \inf\left\{\gamma > 0 : D \subset \bigcup_{i=1}^{n} D_i,\ \operatorname{diam}|D_i| < \gamma,\ i = 1,2,\ldots,n\right\},$$

where $\operatorname{diam}|D_i|$ denotes the diameter of the set D_i.

If D is a bounded subset of X then $\mu(D) = 0$ iff $\overline{D}$ is compact. Also, if D_1 is a subset of D_2 then $\mu(D_1) \le \mu(D_2)$.

Definition 6 ([16, Definition 8.1.7])**.** Let C and D be two nonempty subsets of X. The Hausdroff distance $\mathscr{H}(\cdot,\cdot)$ between C and D is defined by

$$\mathscr{H}(C,D) = \max\{e(C,D), e(D,C)\},$$

where $e(C,D) = \sup\limits_{c\in C} d(c,D)$ with $d(c,D) = \inf\limits_{d\in D} \|c-d\|_X$.

Here, we want to emphasize that if $\{C_n\}$ be a sequence of nonempty subsets of X. Then we say that C_n converges to C with respect to $\mathscr{H}(\cdot,\cdot)$ iff $\mathscr{H}(C_n,C) \to 0$.

Definition 7 ([22, Definition 2.1 and Definition 2.2])**.** Let $G\colon X \to X^*$ be a single-valued operator, then G is

(i) hemicontinuous, iff for all $x,y \in X$, the function $\varepsilon \to \langle G(x+\varepsilon(y-x)), y-x\rangle_{X^*\times X}$ from $[0,1]$ into R is continuous at 0^+;
(ii) monotone, iff $\langle G(x)-G(y), x-y\rangle_{X^*\times X} \ge 0,\ \forall\, x,y \in X$.

3. Formulation of the problem and approximating sequence

Let $k \in \mathbb{N}$. For each $i \in \{1,\dots,k\}$, let $(X_i, \|\cdot\|_{X_i})$ be a real Banach space with dual space $(X_i^*, \|\cdot\|_{X_i^*})$. It is well-known that $X_1 \times \cdots \times X_i \times \cdots \times X_k$ is also a Banach space with the product norm $\|x\| = \sum_{i=1}^k \|x_i\|_{X_i}$, $\forall\, x = (x_1,\dots,x_i,\dots,x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$. For $i \in \{1,\dots,k\}$, let $A_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ be a mapping (nonlinear), $S_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ be a perturbation, $\phi_i\colon X_i \to \mathbb{R} \cup \{\infty\}$ be a proper, convex, and l.s.c. functional, $f_i \in X_i^*$, and $J_i\colon X_i \to \mathbb{R}$ be a locally Lipschitz functional. The effective domain of ϕ_i is specified as:

$$\mathrm{dom}(\phi_i) = \{x_i \in X_i\colon \phi_i(x_i) < \infty\} \neq \varnothing,$$

for $i \in \{1,\dots,k\}$.

In this work, we are going to establish the well-posedness of the following SMHVI, which consists in finding $x = (x_1,\dots,x_i,\dots,x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ s.t.

$$\text{(SMHVI)} \quad \begin{cases} \langle A_1(x) + S_1(x) - f_1, y_1 - x_1\rangle_{X_1^* \times X_1} + J_1^\circ(x_1; y_1 - x_1) \\ +\phi_1(y_1) - \phi_1(x_1) \geq 0, \qquad \forall\, y_1 \in X_1, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_i(x) + S_i(x) - f_i, y_i - x_i\rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i) \\ +\phi_i(y_i) - \phi_i(x_i) \geq 0, \qquad \forall\, y_i \in X_i, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_k(x) + S_k(x) - f_k, y_k - x_k\rangle_{X_k^* \times X_k} + J_k^\circ(x_k; y_k - x_k) \\ +\phi_k(y_k) - \phi_k(x_k) \geq 0, \qquad \forall\, y_k \in X_k, \end{cases}$$

where $J_i^\circ(x_i; y_i - x_i)$ is the Clarke's generalized directional derivative at $x_i \in X_i$ in the direction $y_i - x_i \in X_i$. We denote θ_i as the zero element of X_i^* for $i \in \{1,\dots,k\}$ and Q as the set of all solutions to SMHVI.

If $k = 1$, then the SMHVI weakens to the variational-hemivariational inequality, whose well-posedness was considered by Xiao and Huang [22]. Furthermore, if $k = 1$ with $S_1 = \theta_1$ and $\phi_1 = 0$, then the SMHVI is equivalent to the hemivariational inequality, whose well-posedness was given by Xiao et al. [23]. Moreover, letting $k = 1$ with $S_1 = \theta_1, J_1 = 0$, and $\phi_1 = 0$, SMHVI collapses to the classical variational inequality, i.e.,

$$\langle A_1(x_1) - f_1, y_1 - x_1\rangle \geq 0, \qquad \forall\, y_1 \in X_1.$$

Also, if $k = 1$ with $A_1 = \theta_1, S_1 = \theta_1, J_1 = 0$ and $f_1 = 0$ then SMHVI reduces to the global minimization problem, i.e.,

$$\min_{x_1 \in X_1} \phi(x_1).$$

To investigate the well-posedness of SMHVI, let us review some primitive definitions and results.

Definition 8. The sequence $x^n = (x_1^n, \dots, x_i^n, \dots, x_k^n) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ is an approximating sequence for SMHVI, if there exists a sequence $\{\rho_n\} \subset \mathbb{R}_+$ with $\rho_n \to 0$ as $n \to \infty$ s.t.

$$\begin{cases} \langle A_1(x^n) + S_1(x^n) - f_1, y_1 - x_1^n \rangle_{X_1^* \times X_1} + J_1^\circ(x_1^n; y_1 - x_1^n) + \phi_1(y_1) - \phi_1(x_1^n) \\ \geq -\rho_n \|y_1 - x_1^n\|_{X_1}, \qquad \forall\, y_1 \in X_1, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_i(x^n) + S_i(x^n) - f_i, y_i - x_i^n \rangle_{X_i^* \times X_i} + J_i^\circ(x_i^n; y_i - x_i^n) + \phi_i(y_i) - \phi_i(x_i^n) \\ \geq -\rho_n \|y_i - x_i^n\|_{X_i}, \qquad \forall\, y_i \in X_i, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_k(x^n) + S_k(x^n) - f_k, y_k - x_k^n \rangle_{X_k^* \times X_k} + J_k^\circ(x_k^n; y_k - x_k^n) + \phi_k(y_k) - \phi_k(x_k^n) \\ \geq -\rho_n \|y_k - x_k^n\|_{X_k}, \qquad \forall\, y_k \in X_k. \end{cases} \tag{3.1}$$

Definition 9. The SMHVI is said to be strongly (resp., weakly) well-posed iff it has a unique solution and every approximating sequence converges strongly (resp., weakly) to the unique solution.

Definition 10. The SMHVI is said to be strongly (resp., weakly) well-posed in generalized sense iff Q is nonempty and every approximating sequence has a subsequence which converges strongly (resp., weakly) to some point of Q.

Remark 1. Definition 9 and Definition 10 extend the definition of well-posedness studied in [8, 10, 21–24]. Also, it is worth noting that strong well-posedness (in the generalized sense) implies the weak well-posedness (in the generalized sense), however, the converse is not true in general.

For any $\rho > 0$, we define the sets $\Upsilon(\rho)$ and $\Psi(\rho)$ in $X_1 \times \cdots \times X_i \times \cdots \times X_k$ as:

$$\Upsilon(\rho) = \begin{cases} (x_1, \dots, x_i, \dots, x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k : \\ \langle A_1(x_1, \dots, x_i, \dots, x_k) + S_1(x_1, \dots, x_i, \dots, x_k) - f_1, y_1 - x_1 \rangle_{X_1^* \times X_1} \\ + J_1^\circ(x_1; y_1 - x_1) + \phi_1(y_1) - \phi_1(x_1) \geq -\rho \|y_1 - x_1\|_{X_1}, \qquad \forall\, y_1 \in X_1, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_i(x_1, \dots, x_i, \dots, x_k) + S_i(x_1, \dots, x_i, \dots, x_k) - f_i, y_i - x_i \rangle_{X_i^* \times X_i} \\ + J_i^\circ(x_i; y_i - x_i) + \phi_i(y_i) - \phi_i(x_i) \geq -\rho \|y_i - x_i\|_{X_i}, \qquad \forall\, y_i \in X_i, \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots \\ \langle A_k(x_1, \dots, x_i, \dots, x_k) + S_k(x_1, \dots, x_i, \dots, x_k) - f_k, y_k - x_k \rangle_{X_k^* \times X_k} \\ + J_k^\circ(x_k; y_k - x_k) + \phi_k(y_k) - \phi_k(x_k) \geq -\rho \|y_k - x_k\|_{X_k}, \qquad \forall\, y_k \in X_k, \end{cases}$$

and

$$\Psi(\rho)=\begin{cases}(x_1,\dots,x_i,\dots,x_k)\in X_1\times\cdots\times X_i\times\cdots\times X_k:\\ \langle A_1(y_1,\dots,x_i,\dots,x_k)+S_1(x_1,\dots,x_i,\dots,x_k)-f_1,y_1-x_1\rangle_{X_1^*\times X_1}\\ +J_1^\circ(x_1;y_1-x_1)+\phi_1(y_1)-\phi_1(x_1)\geq-\rho\|y_1-x_1\|_{X_1},\qquad\forall\, y_1\in X_1,\\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\\ \langle A_i(x_1,\dots,y_i,\dots,x_k)+S_i(x_1,\dots,x_i,\dots,x_k)-f_i,y_i-x_i\rangle_{X_i^*\times X_i}\\ +J_i^\circ(x_i;y_i-x_i)+\phi_i(y_i)-\phi_i(x_i)\geq-\rho\|y_i-x_i\|_{X_i},\qquad\forall\, y_i\in X_i,\\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\\ \langle A_k(x_1,\dots,x_i,\dots,y_k)+S_k(x_1,\dots,x_i,\dots,x_k)-f_k,y_k-x_k\rangle_{X_k^*\times X_k}\\ +J_k^\circ(x_k;y_k-x_k)+\phi_k(y_k)-\phi_k(x_k)\geq-\rho\|y_k-x_k\|_{X_k},\qquad\forall\, y_k\in X_k.\end{cases}$$

Motivated by Remark 2.4 of [1], we have the following results.

Theorem 2. *If SMHVI is strongly well-posed in generalized sense and admits a unique solution then SMHVI is strongly well-posed.*

Proof. Let $x^n=(x_1^n,\dots,x_i^n,.,.,x_k^n)\in X_1\times\cdots\times X_i\times\cdots\times X_k$ be an approximating sequence for SMHVI. Accordingly, there exists $\{\rho_n\}\subset\mathbb{R}_+$ with $\{\rho_n\}\to 0$ as $n\to\infty$ and $x^n\in\Upsilon(\rho_n)\ \forall\, n\in\mathbb{N}$. Since SMHVI is strongly well-posed in generalized sense and has a unique solution $x=(x_1,\dots,x_i,\dots,x_k)$, then $\{x^n\}$ has a subsequence $\{x^{n_l}\}$, which converges strongly to x. It suffices to prove that $\{x^n\}$ converges strongly to x. Arguing by contradiction, let $x^n\not\to x$ which leads to $\|x^n-x\|\not\to 0$ as $n\to\infty$. Thus there exists $\eta>0$ and a subsequence $\{x^{m_l}\}$ of $\{x^n\}$ s.t.

$$\|x^{m_l}-x\|\geq\eta. \tag{3.2}$$

Now, it is simple to see that $\{x^{m_l}\}$ is an approximating sequence for SMHVI. Since SMHVI is strongly well-posed in generalized sense and also has a unique solution x, therefore, there exists a subsequence $\{x^{m_{l_q}}\}$ of $\{x^{m_l}\}$ s.t. $x^{m_{l_q}}\to x$ as $q\to\infty$, i.e.,

$$\|x^{m_{l_q}}-x\|<\frac{\eta}{2}.$$

Letting $m_{l_q}=m_{l_r}$, we obtain

$$\|x^{m_{l_r}}-x\|<\frac{\eta}{2},$$

which contradicts (3.2). This confirms that $x^n\to x$. □

Let $A_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k\to X_i^*$. We say A_i is norm to weak*-sequentially continuous iff $x^n=(x_1^n,\dots,x_i^n,\dots,x_k^n)\to x=(x_1,\dots,x_i,\dots,x_k)$ in $X_1\times\cdots\times X_i\times\cdots\times X_k$ implies $A_i(x^n)\xrightarrow{w^*}A_i(x)$ in X_i^*. Further, we also need the following assumptions throughout this paper. For each $i\in\{1,\dots,k\}$,

(H_m) $A_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k\to X_i^*$ is monotone with respect to X_i,
(H_a) $A_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k\to X_i^*$ is norm to weak*-sequentially continuous,

(H_j) $J_i\colon X_i \to \mathbb{R}$ is locally Lipschitz functional,
(H_ϕ) $\phi_i\colon X_i \to \mathbb{R}\cup\{\infty\}$ is proper, convex and l.s.c. map with effective domain: $\mathrm{dom}(\phi_i) = \{x_i \in X_i : \phi_i(x_i) < \infty\} \neq \varnothing$,
(H_c) $S_i\colon X_i \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ is continuous on $X_1 \times \cdots \times X_i \times \cdots \times X_k$,
(H_h) $A_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ is hemicontinuous with respect to X_i.

We end this section by addressing the following equivalence results between the sets $\Upsilon(\rho)$ and $\Psi(\rho)$ for any $\rho > 0$.

Lemma 2. *For each $i \in \{1,\dots,k\}$, suppose that $A_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ satisfies the hypothesis (H_m) and (H_h). Further assume that for each $i \in \{1,\dots,k\}$, $\phi_i\colon X_i \to \mathbb{R}\cup\{\infty\}$ is a proper and convex functional, and $J_i\colon X_i \to \mathbb{R}$ satisfies the hypothesis (H_j). Then $\Upsilon(\rho) = \Psi(\rho)$ for any $\rho > 0$.*

Proof. Let $x = (x_1,\dots,x_i,\dots,x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ and $x \in \Upsilon(\rho)$. Thus, for each $i \in \{1,\dots,k\}$, we have

$$\begin{aligned}&\langle A_i(x_1,\dots,x_i,\dots,x_k) + S_i(x_1,\dots,x_i,\dots,x_k) - f_i, y_i - x_i\rangle_{X_i^*\times X_i} + J_i^\circ(x_i; y_i - x_i)\\ &+ \phi_i(y_i) - \phi_i(x_i) \geq -\rho\|y_i - x_i\|_{X_i}, \qquad \forall\, y_i \in X_i. \end{aligned} \tag{3.3}$$

Since A_i is monotone with respect to X_i (by H_m). We have

$$\langle A_i(x_1,\dots,y_i,\dots,x_k) - A_i(x_1,\dots,x_i,\dots,x_k), y_i - x_i\rangle_{X_i^*\times X_i} \geq 0, \qquad \forall\, y_i, x_i \in X_i,$$

i.e.,

$$\begin{aligned}&\langle A_i(x_1,\dots,y_i,\dots,x_k), y_i - x_i\rangle_{X_i^*\times X_i}\\ &\geq \langle A_i(x_1,\dots,x_i,\dots,x_k), y_i - x_i\rangle_{X_i^*\times X_i}, \qquad \forall\, x_i, y_i \in X_i. \end{aligned} \tag{3.4}$$

By combining (3.3) and (3.4), we get

$$\begin{aligned}&\langle A_i(x_1,\dots,y_i,\dots,x_k) + S_i(x_1,\dots,x_i,\dots,x_k) - f_i, y_i - x_i\rangle_{X_i^*\times X_i} + J_i^\circ(x_i; y_i - x_i)\\ &+ \phi_i(y_i) - \phi_i(x_i) \geq -\rho\|y_i - x_i\|_{X_i}, \qquad \forall\, y_i \in X_i.\end{aligned}$$

i.e., $x = (x_1,\dots,x_i,\dots,x_k) \in \Psi(\rho)$, so, $\Upsilon(\rho) \subset \Psi(\rho)$.

Conversely, let $x = (x_1,\dots,x_i,\dots,x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ and $x \in \Psi(\rho)$ for any $\rho > 0$. So, we yield that for each $i \in \{1,\dots,k\}$

$$\begin{aligned}&\langle A_i(x_1,\dots,y_i,\dots,x_k) + S_i(x_1,\dots,x_i,\dots,x_k) - f_i, y_i - x_i\rangle_{X_i^*\times X_i} + J_i^\circ(x_i; y_i - x_i)\\ &+ \phi_i(y_i) - \phi_i(x_i) \geq -\rho\|y_i - x_i\|_{X_i}, \qquad \forall\, y_i \in X_i.\end{aligned} \tag{3.5}$$

Now for any $w = (w_1,\dots,w_i,\dots,w_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ and $t \in (0,1)$, let

$$z_i = x_i + t(w_i - x_i) = tw_i + (1-t)x_i \in X_i, \qquad i \in \{1,\dots,k\}.$$

Inserting z_i in (3.5), we have that for each $i \in \{1,\dots,k\}$

$$\begin{aligned}&\langle A_i(x_1,\dots,x_i + t(w_i - x_i),\dots,x_k) + S_i(x_1,\dots,x_i,\dots,x_k) - f_i, t(w_i - x_i)\rangle_{X_i^*\times X_i}\\ &+ J_i^\circ(x_i; t(w_i - x_i)) + \phi_i(x_i + t(w_i - x_i)) - \phi_i(x_i) \geq -\rho t\|w_i - x_i\|_{X_i}.\end{aligned}$$

By the convexity of ϕ_i and Lemma 1, we estimate that for each $i \in \{1,\dots,k\}$

$$\begin{aligned}&\langle A_i(x_1,\dots,x_i+t(w_i-x_i),\dots,x_k)+S_i(x_1,\dots,x_i,\dots,x_k)-f_i,w_i-x_i\rangle_{X_i^*\times X_i}\\&+J_i^\circ(x_i;w_i-x_i)+\phi_i(w_i)-\phi_i(x_i)\geq-\rho\|w_i-x_i\|_{X_i}.\end{aligned}\tag{3.6}$$

From (H_h), A_i is hemicontinuous with respect to X_i for each $i \in \{1,\dots,k\}$. Taking the limit as $t \to 0$ in (3.6), we achieve

$$\begin{aligned}&\langle A_i(x_1,\dots,x_i,\dots,x_k)+S_i(x_1,\dots,x_i,\dots,x_k)-f_i,w_i-x_i\rangle_{X_i^*\times X_i}\\&+J_i^\circ(x_i;w_i-x_i)+\phi_i(w_i)-\phi_i(x_i)\geq-\rho\|w_i-x_i\|_{X_i},\qquad\forall\, w_i\in X_i.\end{aligned}$$

Hence, the arbitrariness of w implies that $x \in \Upsilon(\rho)$. Thus, $\Psi(\rho) \subset \Upsilon(\rho)\ \forall\, \rho > 0$. □

Remark 2. Lemma 2 improves and generalizes Lemma 3.1 in [22] and Lemma 3.1 in [23].

4. Main results

This section presents the metric characterizations and deduces some criteria, under which, SMHVI is strongly well-posed.

Proposition 1. *For each $i \in \{1,..,k\}$, let X_i be a real Banach space with dual space X_i^*. Further, assume that for each $i \in \{1,\dots,k\}$, $A_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k \to X_i^*$, $S_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k \to X_i^*$, $J_i\colon X_i \to \mathbb{R}$ and $\phi_i\colon X_i \to \mathbb{R}\cup\{\infty\}$ satisfy the hypothesis (H_a), (H_c), (H_j) and (H_ϕ), respectively. Then, $\Psi(\rho)$ is closed in $X_1\times\cdots\times X_i\times\cdots\times X_k$ for any $\rho > 0$.*

Proof. Let $x^n = (x_1^n,\dots,x_i^n,\dots,x_k^n) \in \Psi(\rho)$ be a sequence s.t. $(x_1^n,\dots,x_i^n,\dots,x_k^n) \to x = (x_1,\dots,x_i,\dots,x_k)$. Then for each $i \in \{1,\dots,k\}$,

$$\begin{aligned}&\langle A_i(x_1^n,\dots,y_i,\dots,x_k^n)+S_i(x_1^n,\dots,x_i^n,\dots,x_k^n)-f_i,y_i-x_i^n\rangle_{X_i^*\times X_i}+J_i^\circ(x_i^n;y_i-x_i^n)\\&+\phi_i(y_i)-\phi_i(x_i^n)\geq-\rho\|y_i-x_i^n\|_{X_i},\qquad\forall\, y_i\in X_i.\end{aligned}\tag{4.1}$$

Since A_i, S_i, J_i, and ϕ_i, satisfy $(H_a), (H_c), (H_j)$ and (H_ϕ), respectively. Thus, for each $i \in \{1,\dots,k\}$, by using (H_a) and Theorem 1, we obtain that

$$\langle A_i(x_1^n,\dots,y_i,\dots,x_k^n),y_i-x_i^n\rangle_{X_i^*\times X_i}\to\langle A_i(x_1,\dots,y_i,\dots,x_k),y_i-x_i\rangle_{X_i^*\times X_i}.\tag{4.2}$$

Also, taking the upper limit as $n \to \infty$ in (4.1), it follows from $(H_c), (H_j), (H_\phi)$ and (4.2) that for each $i \in \{1,2\dots,k\}$

$$\begin{aligned}&\langle A_i(x_1,\dots,y_i,\dots,x_k)+S_i(x_1,\dots,x_i,\dots,x_k)-f_i,y_i-x_i\rangle_{X_i^*\times X_i}+J_i^\circ(x_i;y_i-x_i)\\&+\phi_i(y_i)-\phi_i(x_i)\geq-\rho\|y_i-x_i\|_{X_i},\qquad\forall\, y_i\in X_i,\end{aligned}$$

which confirms that $x = (x_1,\dots,x_i,\dots,x_k) \in \Psi(\rho)$. Therefore, $\Psi(\rho)$ is closed. □

The following result can be obtained by applying Lemma 2 and Proposition 1.

Corollary 1. *For each $i \in \{1,..,k\}$, let X_i be real Banach space with dual space X_i^* and assume that $A_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ satisfies the hypothesis $(H_a),(H_h)$ and (H_m). Further, suppose that for each $i \in \{1,\ldots,k\}, S_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*, J_i\colon X_i \to \mathbb{R}$ and $\phi_i\colon X_i \to \mathbb{R} \cup \{\infty\}$ satisfy the hypothesis (H_c), (H_j) and (H_ϕ), respectively. Then $\Upsilon(\rho)$ is closed for any $\rho > 0$.*

We now present the main results concerning to the strongly well-posedness of SMHVI.

Theorem 3. *For each $i \in \{1,..,k\}$, let X_i be a real Banach space with dual space X_i^* and assume that $A_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*$ satisfies the hypothesis (H_a), (H_m) and (H_h). Further, suppose that for each $i \in \{1,\ldots,k\}, S_i\colon X_1 \times \cdots \times X_i \times \cdots \times X_k \to X_i^*, J_i\colon X_i \to \mathbb{R}$ and $\phi_i\colon X_i \to \mathbb{R} \cup \{\infty\}$ satisfy the hypothesis (H_c), (H_j) and (H_ϕ), respectively. Then SMHVI is strongly well-posedness iff*

$$\Upsilon(\rho) \neq \varnothing \quad \forall\, \rho > 0 \quad \textit{and} \quad diam(\Upsilon(\rho)) \to 0 \quad \textit{as}\ \rho \to 0.$$

Proof. Let SMHVI be strongly well-posed. Then SMHVI has a unique solution, say, $x = (x_1,\ldots,x_i,\ldots,x_k)$ and hence $x \in \Upsilon(\rho)\ \forall\, \rho > 0$, which relates to $Q \neq \varnothing$ and $\Upsilon(\rho) \neq \varnothing\ \forall\, \rho > 0$. Also, suppose that $diam(\Upsilon(\rho)) \not\to 0$ as $\rho \to 0$. Thus, there exists $\eta > 0$ and a sequence $\{\rho_n\} \subset \mathbb{R}_+$ with $\rho_n \to 0$ as $n \to \infty$ s.t. for each $n \in \mathbb{N}$, there exists $w^n = (w_1^n,\ldots,w_i^n,\ldots,w_k^n)$ and $z^n = (z_1^n,\ldots,z_i^n,\ldots,z_k^n)$ with $w^n, z^n \in \Upsilon(\rho_n)$ s.t.

$$\|w^n - z^n\| > \eta. \tag{4.3}$$

On the other hand, since $w^n, z^n \in \Upsilon(\rho_n)$, we have that the sequences $\{w^n\}$ and $\{z^n\}$ are both approximating sequences for SMHVI. Further, SMHVI is strongly well-posed, so the sequence $\{w^n\}$ and $\{z^n\}$ converges strongly to $x \in Q$, which contradicts (4.3).

In order to prove the sufficiency, let $x^n = (x_1^n,\ldots,x_i^n,\ldots,x_k^n)$ be an approximating sequence for SMHVI. Thus, there exists a sequence $\{\rho_n\} \subset \mathbb{R}_+$ with $\rho_n \to 0$ s.t. (3.1) holds, i.e., $x^n \in \Upsilon(\rho_n)$. Since $diam(\Upsilon(\rho_n)) \to 0$ as $\rho_n \to 0$, it is easy to draw that $\{x^n\}$ is a Cauchy sequence in $X_1 \times \cdots \times X_i \times \cdots \times X_k$. Since $X_1 \times \cdots \times X_i \times \cdots \times X_k$ is complete, and so $\{x^n\}$ converges strongly to some point $x = (x_1,\ldots,x_i,\ldots,x_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$. Using the hypothesis (H_a), (H_j), (H_c), (H_ϕ) and Lemma 1, it follows that for each $i \in \{1,\ldots,k\}$,

$$\begin{aligned}
&\langle A_i(x_1,\ldots,y_i,\ldots,x_k) + S_i(x_1,\ldots,x_i,\ldots,x_k) - f_i, y_i - x_i\rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i)\\
&+ \phi_i(y_i) - \phi_i(x_i),\\
&\geq \limsup_{n\to\infty} \Big[\langle A_i(x_1^n,\ldots,y_i,\ldots,x_k^n) + S_i(x_1^n,\ldots,x_i^n,\ldots,x_k^n) - f_i, y_i - x_i^n\rangle_{X_i^* \times X_i} +\\
&\qquad J_i^\circ(x_i^n; y_i - x_i^n) + \phi_i(y_i) - \phi_i(x_i^n)\Big],
\end{aligned}$$

$$\geq \limsup_{n\to\infty}\Bigg[\langle A_i(x_1^n,\ldots,x_i^n,\ldots,x_k^n)+S_i(x_1^n,\ldots,x_i^n,\ldots,x_k^n)-f_i,y_i-x_i^n\rangle_{X_i^*\times X_i}+$$
$$J_i^\circ(x_i^n;y_i-x_i^n)+\phi_i(y_i)-\phi_i(x_i^n)\Bigg],\quad [\text{by}\quad (H_m)]$$
$$\geq \limsup_{n\to\infty}\Bigg[-\rho_n\|y_i-x_i^n\|_{X_i}\Bigg],\quad [\text{by}\quad (3.1)],$$
$$=0,\qquad \forall\, y_i\in X_i.$$

Now, due to the convexity of $\phi_i,(H_h)$ and Lemma 1, we can derive (by invoking the analogous proof outline of Lemma 2) that

$$\langle A_i(x_1,\ldots,x_i,\ldots,x_k)+S_i(x_1,\ldots,x_i,\ldots,x_k)-f_i,y_i-x_i\rangle_{X_i^*\times X_i}+J_i^\circ(x_i;y_i-x_i)$$
$$+\phi_i(y_i)-\phi_i(x_i)\geq 0,\qquad \text{for each } i\in\{1,\ldots,k\},$$

which means that $x\in Q$. For the completeness of the proof, let us assume by contradiction that the SMHVI has two distinct solutions, say, $x,z\in Q$. Since $Q\subset\Upsilon(\rho)\ \forall\,\rho>0$, so as $x,z\in\Upsilon(\rho)\ \forall\,\rho>0$. Also, since $diam(\Upsilon(\rho))\to 0$ as $\rho\to 0$, it is straightforward to deduce that

$$\|x-z\|\leq\ diam(\Upsilon(\rho))\to 0,$$

which contradicts $x\neq z$, and this confirms the uniqueness of the solution. □

Theorem 4. *For each $i\in\{1,..,k\}$, let X_i be a real Banach space with dual space X_i^* and assume that $A_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k\to X_i^*$ satisfies the hypothesis (H_a), (H_h) and (H_m). Further, suppose that for each $i\in\{1,\ldots,k\}, S_i\colon X_1\times\cdots\times X_i\times\cdots\times X_k\to X_i^*, J_i\colon X_i\to\mathbb{R}$ and $\phi_i\colon X_i\to\mathbb{R}\cup\{\infty\}$ satisfy the hypothesis (H_c), (H_j) and (H_ϕ), respectively. Then SMHVI is strongly well-posedness in generalized sense iff*

$$\Upsilon(\rho)\neq\varnothing\quad\forall\,\rho>0\quad \textit{and}\quad \mu(\Upsilon(\rho))\to 0\quad \textit{as}\ \rho\to 0. \tag{4.4}$$

Proof. Assume that SMHVI is strongly well-posed in generalized sense. For any $\rho>0$, then $Q\neq\varnothing$ and $Q\subset\Upsilon(\rho)$. Moreover, we claim that the set Q is compact. For any sequence, say, $x^n\in X_1\times\cdots\times X_i\times\cdots\times X_k$ s.t. $x^n\in Q$ and so $x^n\in\Upsilon(\rho)$ for all $\rho>0$. For any $\rho>0$, it follows from $Q\subset\Upsilon(\rho)$ that $\{x^n\}$ is an approximating sequence for SMHVI. Since the SMHVI is strongly well-posed in generalized sense, therefore $\{x^n\}$ has a subsequence that converges to some point of the set Q, which indicates that the set Q is compact and hence $\mu(Q)=0$. Now, we claim that $\mu(\Upsilon(\rho))\to 0$. In fact, for any $\rho>0, Q\subset\Upsilon(\rho)$ implies that

$$\mathscr{H}(\Upsilon(\rho),Q)=max\{e(\Upsilon(\rho),Q),e(Q,\Upsilon(\rho))\}=e(\Upsilon(\rho),Q). \tag{4.5}$$

Also, from the compactness of Q and (4.5), we obtain

$$\mu(\Upsilon(\rho))\leq 2\mathscr{H}(\Upsilon(\rho),Q)+\mu(Q)\leq 2\mathscr{H}(\Upsilon(\rho),Q)=2e(\Upsilon(\rho),Q).$$

Hence, to prove $\mu(\Upsilon(\rho)) \to 0$ as $\rho \to 0$, it is enough to show that $e(\Upsilon(\rho), Q) \to 0$ as $\rho \to 0$. Arguing by contradiction, we assume that $e(\Upsilon(\rho), Q) \not\to 0$ as $\rho \to 0$. Thus, there exists a constant $\varepsilon > 0$ and a sequence $\{\rho_n\} \subset \mathbb{R}_+$ with $\rho_n \to 0$ as $n \to \infty$ and $w^n \in \Upsilon(\rho_n)$ satisfying

$$d(w^n, Q) > \varepsilon, \qquad \forall\, n \in \mathbb{N}. \tag{4.6}$$

Since $\{w^n\}$ is an approximating sequence for SMHVI and SMHVI is well-posed in generalized sense, therefore, there exists a subsequence $\{w^{n_l}\}$ of $\{w^n\}$, which converges to some point of Q. So we get

$$0 < \varepsilon < d(w^{n_l}, Q) \to 0,$$

which contradicts (4.6).

Conversely, assume that (4.4) holds ($\Upsilon(\rho) \neq \varnothing$). By Corollary 1, we have that $\Upsilon(\rho)$ is closed for all $\rho > 0$. Next, we claim that

$$Q = \bigcap_{\rho > 0} \Upsilon(\rho).$$

It is evident that $Q \subset \bigcap_{\rho>0} \Upsilon(\rho)$. Conversely, let $x = (x_1, \ldots, x_i, \ldots, x_k) \in \bigcap_{\rho>0} \Upsilon(\rho)$. So, there exists $\rho_n > 0$ s.t. $\rho_n \to 0$, which leads to $x = (x_1, \ldots, x_i, \ldots, x_k) \in \Upsilon(\rho_n)$. Thus for each $i \in \{1, \ldots, k\}$, we get

$$\begin{aligned} &\langle A_i(x) + S_i(x) - f_i, y_i - x_i \rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i) + \phi_i(y_i) - \phi_i(x_i) \\ &\geq -\rho_n \|y_i - x_i\|_{X_i}, \qquad \forall\, y_i \in X_i. \end{aligned} \tag{4.7}$$

Letting $n \to \infty$ in (4.7), we get

$$\langle A_i(x) + S_i(x) - f_i, y_i - x_i \rangle_{X_i^* \times X_i} + J_i^\circ(x_i; y_i - x_i) + \phi_i(y_i) - \phi_i(x_i) \geq 0, \qquad \forall\, y_i \in X_i.$$

For any $z = (z_1, \ldots, z_i, \ldots, z_k) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ and $t \in (0,1)$, let $y_i = x_i + t(z_i - x_i) \in X_i, i \in \{1, \ldots, k\}$. Now using the convexity of ϕ_i and Lemma 1, it is easy to draw that $x \in Q$ and hence $\bigcap_{\rho>0} \Upsilon(\rho) \subset Q$. This shows that $Q = \bigcap_{\rho>0} \Upsilon(\rho)$. Since $\mu(\Upsilon(\rho)) \to 0$ as $\rho \to 0$. By generalized Cantor theorem (see [12]), it follows that Q is nonempty and compact with $\Upsilon(\rho) \to Q$ as $\rho \to 0$ w.r.t. $\mathscr{H}$, i.e.,

$$\mathscr{H}(\Upsilon(\rho), Q) \to 0 \quad \text{as } \rho \to 0. \tag{4.8}$$

As,

$$e(\Upsilon(\rho), Q) = \mathscr{H}(\Upsilon(\rho), Q), \quad \text{for } \rho > 0.$$

Hence, from (4.8), we get

$$e(\Upsilon(\rho), Q) \to 0 \quad \text{as } \rho \to 0. \tag{4.9}$$

In order to complete the proof, let $x^n = (x_1^n, \dots, x_i^n, \dots, x_k^n) \in X_1 \times \cdots \times X_i \times \cdots \times X_k$ be an approximating sequence for SMHVI, thus there exists $\{\rho_n\} \subset \mathbb{R}_+$ with $\rho_n \to 0$ s.t. for each $i \in \{1, \dots, k\}$, we have

$$\begin{aligned}&\langle A_i(x^n) + S_i(x^n) - f_i, y_i - x_i^n\rangle_{X_i^* \times X_i} + J_i^\circ(x_i^n; y_i - x_i^n) + \phi_i(y_i) - \phi_i(x_i^n)\\ &\geq -\rho_n \|y_i - x_i^n\|_{X_i}, \qquad \forall\, y_i \in X_i,\end{aligned}$$

i.e., $x^n \in \Upsilon(\rho_n)$. It ensures from (4.9) that

$$d(x^n, Q) \leq e(\Upsilon(\rho_n), Q) \to 0.$$

As the solution set Q is compact, so there exists $w^n = (w_1^n, \dots, w_i^n, \dots, w_k^n) \in Q$ s.t.

$$\|x^n - w^n\| = d(x^n, Q) \to 0. \tag{4.10}$$

Also, from the compactness of Q, the sequence $\{w^n\}$ has a subsequence $\{w^{n_l}\}$ s.t. $w^{n_l} \to w^{'}$, (for some $w^{'} \in Q$). Hence, it follows from (4.10) that

$$\|x^{n_l} - w^{'}\| \leq \|x^{n_l} - w^{n_l}\| + \|w^{n_l} - w^{'}\| \to 0,$$

which estimates $x^{n_l} \to w^{'}$. This derives the strongly well-posedness of the SMHVI. □

The next theorem is arranged by setting $X_i = \mathbb{R}$ for each $i \in \{1, \dots, k\}$. Hence $X_1 \times \cdots \times X_i \times \cdots \times X_k = \mathbb{R}^k$. Here, we present some conditions by which the SMHVI is well-posed in generalized sense.

Theorem 5. *For each $i \in \{1, \dots, k\}$, let $A_i \colon \mathbb{R}^k \to \mathbb{R}$ be a continuous mapping satisfying the hypothesis (H_h) and (H_m). Further, assume that for each $i \in \{1, \dots, k\}, S_i \colon \mathbb{R}^k \to \mathbb{R}, J_i \colon \mathbb{R} \to \mathbb{R}$ and $\phi_i \colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ satisfy the hypothesis (H_c), (H_j) and (H_ϕ), respectively. If there exist $n_0 \in \mathbb{N}$ and $\rho > 0$ s.t. $\Upsilon(\rho)$ is nonempty, bounded and for every approximating sequence $\{x^n\}$ of SMHVI $x^n \in \Upsilon(\rho)\ \forall\, n > n_0$. Then SMHVI is strongly well-posedness in generalized sense.*

Proof. Let $x^n = (x_1^n, \dots, x_i^n, \dots, x_k^n) \in \mathbb{R}^k$ be an approximating sequence of the SMHVI. Then there exists $\{\rho_n\} \subset \mathbb{R}_+$ s.t. $\rho_n \to 0$ as $n \to \infty$. Thus for each $i \in \{1, \dots, k\}$,

$$\begin{aligned}&\langle A_i(x^n) + S_i(x^n) - f_i, y_i - x_i^n\rangle + J_i^\circ(x_i^n; y_i - x_i^n) + \phi_i(y_i) - \phi_i(x_i^n)\\ &\geq -\rho_n \|y_i - x_i^n\|_{\mathbb{R}}, \qquad \forall\, y_i \in \mathbb{R},\end{aligned} \tag{4.11}$$

On the other hand, let $\rho > 0$ be s.t. $\Upsilon(\rho)$ is nonempty and bounded. Accordingly, there exists $n_0 \in \mathbb{N}$ s.t. $x^n \in \Upsilon(\rho)\ \forall\, n > n_0$. This formulates that $\{x^n\}$ is bounded. Thus, there exists a subsequence $\{x^{n_l}\}$ of $\{x^n\}$ s.t. $x^{n_l} = (x_1^{n_l}, \dots, x_i^{n_l}, \dots, x_k^{n_l}) \to \hat{x} = (\hat{x_1}, \dots, \hat{x_i}, \dots, \hat{x_k})$. Further, due to the hypothesis $(H_c), (H_m), (H_j)$ and the lower semi-continuity of ϕ_i along with Lemma 1, we have the following estimates

$$\begin{aligned}&\langle A_i(\hat{x_1}, \dots, y_i, \dots, \hat{x_k}) + S_i(\hat{x_1}, \dots, \hat{x_i}, \dots, \hat{x_k}) - f_i, y_i - \hat{x_i}\rangle\\ &+ J_i^\circ(\hat{x_i}; y_i - \hat{x_i}) + \phi_i(y_i) - \phi_i(\hat{x_i}),\end{aligned}$$

$$
\begin{aligned}
&\geq \limsup_{l\to\infty}\Big[\langle A_i(x_1^{n_l},\ldots,y_i,\ldots,x_k^{n_l})+S_i(x_1^{n_l},\ldots,x_i^{n_l},\ldots,x_k^{n_l})-f_i,y_i-x_i^{n_l}\rangle \\
&\qquad\qquad +J_i^{\circ}(x_i^{n_l};y_i-x_i^{n_l})+\phi_i(y_i)-\phi_i(x_i^{n_l})\Big], \\
&\geq \limsup_{l\to\infty}\Big[\langle A_i(x_1^{n_l},\ldots,x_i^{n_l},\ldots,x_k^{n_l})+S_i(x_1^{n_l},\ldots,x_i^{n_l},\ldots,x_k^{n_l})-f_i,y_i-x_i^{n_l}\rangle \\
&\qquad\qquad +J_i^{\circ}(x_i^{n_l};y_i-x_i^{n_l})+\phi_i(y_i)-\phi_i(x_i^{n_l})\Big], \\
&\geq \limsup_{l\to\infty}[-\rho_n\|y_i-x_i^{n_l}\|_{\mathbb{R}}], \qquad [\text{by} \quad (4.11)], \\
&= 0.
\end{aligned}
$$

By following the arguments similar to Lemma 2, we can deduce (by hemicontinuity of A_i and the convexity of ϕ_i) that

$$
\begin{aligned}
&\langle A_i(\hat{x_1},\ldots,\hat{x_i},\ldots,\hat{x_k})+S_i(\hat{x_1},\ldots,\hat{x_i},\ldots,\hat{x_k})-f_i,y_i-\hat{x_i}\rangle \\
&+J_i^{\circ}(\hat{x_i};y_i-\hat{x_i})+\phi_i(y_i)-\phi_i(\hat{x_i})\geq 0,
\end{aligned}
$$

which means that $(\hat{x_1},\ldots,\hat{x_i},\ldots,\hat{x_k})\in Q$. Thus, the SMHVI is strongly well-posed in generalized sense. □

Remark 3. Proposition 1 generalizes Lemma 3.2 of [22] and Lemma 3.2 of [23]. Further, Theorem 3 and Theorem 4 generalize Theorem 3.1 of [22] and Theorem 3.2 of [22], respectively. Also, if $k=1$ with $S_1=\theta_1$ and $\phi_1=0$ then Theorem 3 and Theorem 4 reduce to Theorem 3.1 of [23] and Theorem 3.2 of [23], respectively.

ACKNOWLEDGEMENTS

The authors are grateful to the referees for their insightful comments and suggestions, which have greatly enhanced the clarity and coherence of the article.

REFERENCES

[1] J. Cen, C. Min, G. J. Tang, and V. T. Nguyen, "Generalized well-posedness results for a class of new mixed variational inequalities." *Optimization.*, vol. 72, no. 2, pp. 411–437, 2023, doi: 10.1080/02331934.2021.1970752.

[2] L. C. Ceng, H. Gupta, and C. F. Wen, "Well-posedness by perturbations of variational-hemivariational inequalities with perturbations." *Filomat.*, vol. 26, no. 5, pp. 881–895, 2012, doi: 10.2298/FIL1205881C.

[3] L. C. Ceng, N. Hadjisavvas, S. Schaible, and J. C. Yao, "Well-posedness for mixed quasivariational-like inequalities," *J. Optim. Theory Appl.*, vol. 139, pp. 109–125, 2008, doi: 10.1007/s10957-008-9428-9.

[4] L. C. Ceng and J. C. Yao, "Well-posedness of generalized mixed variational inequalities, inclusion problems and fixed-point problems." *Nonlinear Anal.*, vol. 69, no. 12, pp. 4585–4603, 2008, doi: 10.1016/j.na.2007.11.015.

[5] L. C. Ceng, Y. C. Liou, J. C. Yao, and Y. Yao, "Well-posedness for systems of time-dependent hemivariational inequalities in Banach spaces." *J. Nonlinear Sci. Appl.*, vol. 10, no. 8, pp. 4318–4336, 2017, doi: 10.22436/jnsa.010.08.26.

[6] F. H. Clarke, *Optimization and Nonsmooth Analysis.* Philadelphia: SIAM, 1987.

[7] N. Costea and C. Varga, "Systems of nonlinear hemivariational inequalities and applications." *Topol. Methods Nonlinear Anal.*, vol. 41, no. 1, pp. 39–65, 2013.

[8] Y. P. Fang, N. J. Huang, and J. C. Yao, "Well-posedness of mixed variational inequalities inclusion problems and fixed point problems." *J. Global Optim.*, vol. 41, no. 1, pp. 117–133, 2008, doi: 10.1007/s10898-007-9169-6.

[9] Y. P. Fang, N. J. Huang, and J. C. Yao, "Well-posedness by perturbations of mixed variational inequalities in Banach spaces." *European J. Oper. Res.*, vol. 201, no. 3, pp. 682–692, 2010, doi: 10.1016/j.ejor.2009.04.001.

[10] D. Goeleven and D. Mentagui, "Well-posed hemivariational inequalities." *Numer. Funct. Anal. Optim.*, vol. 16, no. 7-8, pp. 909–921, 1995, doi: 10.1080/01630569508816652.

[11] X. X. Huang and X. Q. Yang, "Generalized Levitin–Polyak well-posedness in constrained optimization." *SIAM J. Optim.*, vol. 17, no. 1, pp. 243–258, 2006, doi: 10.1137/040614943.

[12] K. Kuratowski, "Sur les espaces complets," *Fund. Math.*, vol. 15, pp. 301–309, 1930, doi: 10.4064/fm-15-1-301-309.

[13] M. B. Lignola, "Well-posedness and L-well-posedness for quasivariational inequalities," *J. Optim. Theory Appl.*, vol. 128, no. 1, pp. 119–138, 2006, doi: 10.1007/s10957-005-7561-2.

[14] M. B. Lignola and J. Morgan, "Well-posedness for optimization problems with constraints defined by variational inequalities having a unique solution," *J. Global Optim.*, vol. 16, no. 1, pp. 57–67, 2000, doi: 10.1023/A:1008370910807.

[15] Z. Liu, S. Zeng, and B. Zeng, "Well-posedness for mixed quasi-variational-hemivariational inequalities," *Topol. Methods Nonlinear Anal.*, vol. 47, no. 2, pp. 561–578, 2016, doi: 10.12775/TMNA.2016.016.

[16] R. Lucchetti, *Convexity and well-posed problems*, ser. CMS Books Math./Ouvrages Math. SMC. New York, NY: Springer, 2006, vol. 22, doi: 10.1007/0-387-31082-7.

[17] R. Lucchetti and F. Patrone, "A characterization of tyhonov well-posedness for minimum problems with applications to variational inequalities." *Numer. Funct. Anal. Optim.*, vol. 3, no. 4, pp. 461–476, 1981, doi: 10.1080/01630568108816100.

[18] S. Migórski, A. Ochal, and M. Sofonea, "Nonlinear inclusions and hemivariational inequalities. Models and analysis of contact problems," vol. 26, 2013, doi: 10.1007/978-1-4614-4232-5.

[19] P. D. Panagiotopoulos, "Nonconvex energy functions. Hemivariational inequalities and substationarity principles." *Acta Mech.*, vol. 48, no. 3–4, pp. 111–130, 1983, doi: 10.1007/BF01170410.

[20] A. N. Tykhonov, "On the stability of the functional optimization problem," *USSR J Comput Math Math Phys.*, vol. 6, no. 4, pp. 28–33, 1966, doi: 10.1016/0041-5553(66)90003-6.

[21] Y. M. Wang, Y. B. Xiao, X. Wang, and Y. J. Cho, "Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems." *J. Nonlinear Sci. Appl.*, vol. 9, no. 3, pp. 1178–1192, 2016, doi: 10.22436/jnsa.009.03.44.

[22] Y. B. Xiao and N. J. Huang, "Well-posedness for a class of variational hemivariational inequalities with perturbations." *J. Optim. Theory Appl.*, vol. 151, no. 1, pp. 33–51, 2011, doi: 10.1007/s10957-011-9872-9.

[23] Y. B. Xiao, N. J. Huang, and M. M. Wong, "Well-posedness of hemivariational inequalities and inclusion problems." *Taiwanese J. Math.*, vol. 15, no. 3, pp. 1261–1276, 2011, doi: 10.11650/twjm/1500406298.

[24] Y. B. Xiao, X. Yang, and N. J. Huang, "Some equivalence results for well-posedness of hemivariational inequalities." *J. Global Optim.*, vol. 61, no. 4, pp. 789–802, 2015, doi: 10.1007/s10898-014-0198-7.

Authors' addresses

Kartikeswar Mahalik
Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721302, India
E-mail address: kartikiit1234@gmail.com

Chandal Nahak
(**Corresponding author**) Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721302, India
E-mail address: cnahak@maths.iitkgp.ac.in

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 873–889

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4495

AN INVERSE TIME-DEPENDENT SOURCE PROBLEM FOR A TIME-FRACTIONAL DIFFUSION EQUATION WITH NONLOCAL BOUNDARY CONDITIONS

FARID MIHOUBI AND BRAHIM NOUIRI

Received 21 November, 2022

Abstract. In this paper, we investigate an inverse time-dependent source problem for a time-fractional diffusion equation with nonlocal boundary and integral over-determination conditions. The fractional derivative is described in the generalized Caputo sense. The nonlocal boundary conditions are regular but not strongly regular. The special thing about this problem is: the system of eigenfunctions is not complete, but the system of eigen-and associated functions forming a basis in $L^2(0,1)$. Under some natural regularity and consistency conditions on the input data the existence, uniqueness and continuously dependence upon the data of the solution are shown by using the generalized Fourier method, the estimates of Mittag-Leffler function and Banach's contraction mapping principle.

2010 *Mathematics Subject Classification:* 34B07; 42A16; 34K37; 35R30; 33E12

Keywords: bi-orthogonal system, generalized Fourier method, generalized Caputo fractional derivative, inverse source problem, Mittag-Leffler function

1. Introduction

Fractional calculus (FC) is the study of integrals and derivatives of arbitrary order, which has been considered by many researchers in recent years. The arbitrary order integrals and derivatives are used in the modelling of many physical, chemical, and biological phenomena, [2, 5, 8, 20, 24, 28]. The FC has applications in the reaction-diffusion equations [18, 27], and is used to explain the well-known phenomena of anomalous diffusion observed in experiments, [4, 18]. The time fractional diffusion equations are obtained by replacing the standard time derivative with fractional derivative in time variable to explain the sub-diffusion or super-diffusion, [19, 21].

In this paper, we are interested with the following time-fractional diffusion equation

$$ {}^{c}\mathcal{D}_t^{\alpha,\rho} u(x,t) = u_{xx} + F(x,t), \quad (x,t) \in \Omega_T, \tag{1.1} $$

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

with the initial condition

$$u(x,0)=\varphi(x), \quad 0<x<1, \tag{1.2}$$

and family of nonlocal boundary conditions

$$u(0,t)=u(1,t), \quad \beta u_x(0,t)=u_x(1,t), \quad 0<t\leq T, \quad \beta\in\mathbb{R}\backslash\{-1,1\}, \tag{1.3}$$

where $\Omega_T:=\{(x,t):0<x<1,\ 0<t\leq T\}$ for some fixed $T>0$, ${}^c\mathcal{D}_t^{\alpha,\rho}$ stands for left-sided generalized Caputo fractional derivative of order $0<\alpha\leq 1$, $\rho>0$ is a real constant, $F(x,t)$ is the source term and $\varphi(x)$ is the initial data. The fractional derivative in (1.1) is a generalization of left Caputo and Caputo-Hadamard fractional derivatives, which can be obtained by taking $\rho=1$ and $\rho\to 0^+$, respectively. For $\beta=0$, the boundary conditions (1.3) are well-known and called in literature as Samarskii-Ionkin conditions, [10]. More general boundary conditions of the type (1.3) have been considered in [7, 12, 22, 25, 29].

The determination of a function $u(x,t)$ which satisfies the initial boundary value problem (IBVP) (1.1)-(1.3) such that $u(\cdot,t)\in\mathcal{C}^2(0,1)$ and ${}^c\mathcal{D}_t^{\alpha,\rho}u(x,\cdot)\in\mathcal{C}(0,T)$, whenever the source term $F(x,t)$ and the initial data $\varphi(x)$ are given and continuous, is called strong or classical solution of the IBVP (1.1)-(1.3). This problem is usually known as the direct problem.

Letting the source term have the form $F(x,t)=r(t)f(x,t)$, the inverse source problem consists of determining $u(x,t)$ and $r(t)$, from the initial data $\varphi(x)$, the source term $f(x,t)$ and nonlocal boundary conditions (1.3). This problem is not uniquely solvable. To have the inverse source problem uniquely solvable, we impose the integral over-determination condition

$$\int_0^1 u(x,t)\,dx=g(t), \quad t\in[0,T], \tag{1.4}$$

where $g\in AC[0,T]$ (the space of absolutely continuous functions). The solvability of inverse source problems with such condition has been considered earlier [3, 11, 26].

In [26] the inverse source problem (1.1)-(1.4) is studied for $\alpha=\rho=1$. The case $0<\alpha<1$ and $\rho\neq 1$ is considered for the first time in this paper. We have analysed the inverse source problem (1.1)-(1.4). Our strategy is mainly based on Fourier's method for construction of the series solution using a bi-orthogonal system of functions obtained from the eigen-and associated functions of a spectral problem and its conjugate problem, see [16]. We provide existence, uniqueness, and stability results for solution of inverse source problem.

The rest of the paper is organized as follows: in Section 2, we provide some preliminaries and basic result needed for the forthcoming sections. In Section 3, we present associated spectral problem of this problem and its properties, whilst the bi-orthogonal system is constructed. Our main results concerning the existence, uniqueness and continuous dependence of the solution of the inverse problem is present in Section 4.

2. Preliminaries and some basic results

In this section, we recall some definitions, notations from fractional calculus, and some basic results for the convenience of the readers.

Definition 2.1 ([15, page 861])**.** Let $[a,b]$ be a finite interval and $f\colon [a,b]\to\mathbb{R}$ be an integrable function. The generalized left fractional integral (in the sense of Katugampola) is defined by

$$(I_a^{\alpha,\rho})f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}f(s)\,\frac{ds}{s^{1-\rho}},\quad 0<\alpha<1,\quad \rho>0,$$

where $\Gamma(\cdot)$ is the Euler Gamma function defined by $\Gamma(\alpha):=\int_0^{+\infty}t^{\alpha-1}e^{-t}\,dt$.

Definition 2.2 ([17, 1.1.5])**.** Let $[a,b]$ be a finite interval. Then, $AC[a,b]$ is the space of absolute continuous functions on $[a,b]$, defined by

$$AC[a,b]=\left\{f\colon [a,b]\to\mathbb{R}\text{ such that }f(x)=c+\int_a^x\varphi(t)\,dt,\,\varphi\in L^1(a,b)\right\}.$$

Definition 2.3 ([14])**.** Let $\rho>0$ and $f\in AC[a,b]$. The left generalized Caputo fractional derivative of f of order $0<\alpha<1$ is defined by

$${}_a^C\mathcal{D}^{\alpha,\rho}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{-\alpha}f'(s)\,ds.$$

If $\alpha=\rho=1$, then ${}_a^c\mathcal{D}^{\alpha,\rho}f(t)=f'(t)$.

Theorem 2.1 ([14])**.** *Let $f\in AC[a,b]$, $0<\alpha<1$ and $\rho>0$. Then, we have:*

$$I_a^{\alpha,\rho}\left({}_a^C\mathcal{D}^{\alpha,\rho}f(x)\right)=f(x)-f(a).$$

Definition 2.4 ([1])**.** Let $f\colon [0,+\infty[\,\to\mathbb{R}$ be a real valued function. The ρ-Laplace transform of f is defined by

$$\mathcal{L}_\rho\{f(t)\}(s)=\int_0^{+\infty}e^{-s\frac{t^\rho}{\rho}}f(t)\,\frac{dt}{t^{1-\rho}},\quad \rho>0,$$

for all values of s, the integral is valid.

Theorem 2.2 ([1])**.** *If the ρ-Laplace transform of $f\colon [0,+\infty[\,\to\mathbb{R}$ exists for $s>c_1$ and the ρ-Laplace transform of $g\colon [0,+\infty[\,\to\mathbb{R}$ for $s>c_2$. Then, for any constants a and b, the ρ-Laplace transform of $af+bg$ exists and*

$$\mathcal{L}_\rho\{af(t)+bg(t)\}(s)=a\mathcal{L}_\rho\{f(t)\}(s)+b\mathcal{L}_\rho\{g(t)\}(s),\quad \textit{for } s>\max\{c_1,c_2\}.$$

Definition 2.5 ([13])**.** Let f and g be two functions which are piecewise continuous at each interval $[0,T]$. We define the ρ-convolution of f and g by

$$(f*g)(t)=\int_0^t f\left[(t^\rho-s^\rho)^{1/\rho}\right]g(s)\,\frac{ds}{s^{1-\rho}}.$$

Theorem 2.3 ([13]). *Let f and g be two functions which are piecewise continuous at each interval $[0,T]$. Then,*

$$\mathcal{L}_\rho\{(f*g)(t)\} = \mathcal{L}_\rho\{f(t)\}\mathcal{L}_\rho\{g(t)\}.$$

Theorem 2.4 ([13]). *Let $\alpha > 0$ and $f \in AC[0,T]$. Then,*

$$\mathcal{L}_\rho\left\{\left({}_0^C\mathcal{D}^{\alpha,\rho} f\right)(t)\right\}(s) = s^\alpha \mathcal{L}_\rho\{f(t)\} - s^{\alpha-1} f(0).$$

Definition 2.6 ([6]). The Mittag-Leffler function of two parameters is defined as

$$E_{\xi,\eta}(x) := \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\xi k+\eta)}, \quad z \in \mathbb{C}, \quad \mathrm{Re}(\xi) > 0, \quad \mathrm{Re}(\eta) > 0.$$

For $\eta = 1$, the Mittag-Leffler function is reduced to classical one parameter Mittag-Leffler function, that is,

$$E_{\xi,1}(x) := E_\xi(x) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\xi k+1)}.$$

Let $e_\xi(t,\mu) := E_\xi\left(-\mu t^\xi\right)$ and $e_{\xi,\eta}(t,\mu) := t^{\eta-1}E_{\xi,\eta}\left(-\mu t^\xi\right)$, where μ is a positive real number. The Mittag-Leffler functions $e_\xi(t,\mu)$, $e_{\xi,\eta}(t,\mu)$ for $0 < \xi \le 1$, $0 < \xi \le \eta \le 1$, respectively, are completely monotone functions, i.e.

$$(-1)^n \frac{\partial^n}{\partial t^n}\left[e_\xi(t,\mu)\right] \ge 0 \text{ and } (-1)^n \frac{\partial^n}{\partial t^n}\left[e_{\xi,\eta}(t,\mu)\right] \ge 0, \quad n \in \mathbb{N}.$$

Using Theorem 1.6 in [23], we can have the following estimate

$$\left|\mu e_{\xi,\xi}(t,\mu)\right| \le \frac{N\mu t^\xi}{t\left(1+\mu^\xi\right)} \le \frac{N}{t} \le C, \quad t \in \left]\varepsilon, T\right], \tag{2.1}$$

where $\varepsilon > 0$, N and C are some constants.

Lemma 2.1 ([13]). *Let $\xi > 0$ and $\left|\frac{\lambda}{s^\xi}\right| < 1$. Then, we have:*

$$\mathcal{L}_\rho\left\{e_\xi\left(\frac{t^\rho}{\rho},\lambda\right)\right\} = \frac{s^{\alpha-1}}{s^\alpha+\lambda} \text{ and } \mathcal{L}_\rho\left\{e_{\xi,\xi}\left(\frac{t^\rho}{\rho},\lambda\right)\right\} = \frac{1}{s^\alpha+\lambda}.$$

Theorem 2.5 ([13]). *The Cauchy problem*

$$\begin{cases} {}_0^C\mathcal{D}^{\alpha,\rho} y(t) + \lambda y(t) = f(t), & t > 0, \quad 0 < \alpha < 1, \quad \rho > 0, \quad \lambda \in \mathbb{R}, \\ y(0) = y_0, & y_0 \in \mathbb{R}, \end{cases}$$

has the solution

$$y(t) = y_0 e_\alpha\left(\frac{t^\rho}{\rho},\lambda\right) + \int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho - s^\rho}{\rho},\lambda\right) f(s)\frac{ds}{s^{1-\rho}}.$$

3. The auxiliary spectral problem and bi-orthogonal systems

Applying the Fourier method for solving the direct problem (1.1)-(1.3) leads to the spectral problem

$$-X'' = \lambda X, \quad 0 < x < 1, \tag{3.1}$$

$$X(0) = X(1), \quad \beta X'(0) = X'(1), \quad \beta \in \mathbb{R} \backslash \{-1, 1\}. \tag{3.2}$$

According to [25, Theorem 3.109], the boundary conditions (3.2) are regular but not strongly regular. Obviously, the boundary-value problem (3.1)-(3.2) is not self-adjoint. But the problem

$$-Y'' = \lambda Y, \quad 0 < x < 1, \tag{3.3}$$

$$Y(0) = \beta Y(1), \quad Y'(0) = Y'(1), \quad \beta \in \mathbb{R} \backslash \{-1, 1\}, \tag{3.4}$$

will be a conjugated problem.

The two spectral problems (3.1)-(3.2) and (3.3)-(3.4) have the same double eigenvalues $\lambda_k = (2\pi k)^2$ (except for the first $\lambda_0 = 0$). Following the idea presented in [9], the system of eigen-and associated functions of problem (3.1)-(3.2) is given by:

$$X_0(x) = 2, \quad X_{2k-1}(x) = 4\cos(2\pi kx), \quad X_{2k}(x) = 4(1-b-ax)\sin(2\pi kx) \tag{3.5}$$

and the system of eigen-and associated functions of conjugate problem (3.3)-(3.4) is the following:

$$Y_0(x) = ax+b, \quad Y_{2k-1}(x) = (ax+b)\cos(2\pi kx), \quad Y_{2k}(x) = \sin(2\pi kx), \tag{3.6}$$

where $a = (1-\beta)/(1+\beta)$ and $b = \beta/(1+\beta)$.

Lemma 3.1. *The systems of functions* (3.5) *and* (3.6) *are bi-orthonormal in* $L^2(0,1)$.

Proof. It is easy to show that the systems (3.5) and (3.6) form a bi-orthogonal system on $[0,1]$, i.e. $\langle X_i, Y_i \rangle = 1$ and $\langle X_i, Y_j \rangle = 0, i \neq j$. □

Lemma 3.2. *The systems of functions* (3.5) *and* (3.6) *are complete in* $L^2(0,1)$.

Proof. Let $f \in L^2(0,1)$ be orthogonal with the system of functions (3.5). $f(x)$ can be presented by the series

$$f(x) = \sum_{n=1}^{+\infty} B_n \sin(2\pi nx), \tag{3.7}$$

which converges in $L^2(0,1)$. Since $f(x)$ is orthogonal with (3.5), we have

$$\begin{aligned} 0 &= \int_0^1 4(1-b-ax) f(x) \sin(2\pi kx)\, dx \\ &= \sum_{n=1}^{+\infty} B_n \int_0^1 4(1-b-ax)\sin(2\pi kx)\sin(2\pi nx)\, dx = B_k, \quad k \in \mathbb{N}^*. \end{aligned}$$

From (3.7), $f(x)=0$. Then, (3.5) is complete in $L^2(0,1)$. □

Theorem 3.1. *The system of functions* (3.5) *forms a Riesz basis in* $L^2(0,1)$.

Proof. From [25, page 211], the system (3.5) is a Riesz basis in $L^2(0,1)$ if there exist two constants $m,M>0$ such that for any $f\in L^2(0,1)$, the following inequality holds:

$$m\|f\|^2_{L^2(0,1)}\leq\sum_{i=0}^{+\infty}f_i^2\leq M\|f\|^2_{L^2(0,1)},$$

where

$$f_i=\langle f,Y_i\rangle=\int_0^1 f(x)Y_i(x)\,dx \text{ and } \bar{f}_i=\langle f,X_i\rangle=\int_0^1 f(x)X_i(x)\,dx.$$

For $i=0$, and using the Cauchy-Schwarz inequality we have

$$\begin{aligned} f_0^2=\langle f,Y_0\rangle^2=\left[\int_0^1 Y_0(x)f(x)\,dx\right]^2 &\leq\int_0^1 Y_0^2(x)\,dx\int_0^1 f^2(x)\,dx \\ &\leq\frac{1+\beta+\beta^2}{3(1+\beta)^2}\|f\|^2_{L^2(0,1)}. \end{aligned}\tag{3.8}$$

For $i=2k-1$, and using the Bessel inequality we obtain:

$$\begin{aligned} \sum_{k=1}^{+\infty}f_{2k-1}^2=\sum_{k=1}^{+\infty}\langle f,Y_{2k-1}\rangle^2 &\leq\|Y_{2k-1}\|^2_{L^2(0,1)}\|f\|^2_{L^2(0,1)} \\ &\leq\frac{7-11\beta+7\beta^2}{6(1+\beta)^2}\|f\|^2_{L^2(0,1)}. \end{aligned}\tag{3.9}$$

For $i=2k$, and using the Bessel inequality we obtain:

$$\sum_{k=1}^{+\infty}f_{2k}^2=\sum_{k=1}^{+\infty}\langle f,Y_{2k}\rangle^2\leq\|Y_{2k}\|^2_{L^2(0,1)}\|f\|^2_{L^2(0,1)}\leq\frac{1}{2}\|f\|^2_{L^2(0,1)}.\tag{3.10}$$

From (3.8)-(3.10), we have

$$\sum_{i=0}^{+\infty}f_i^2=f_0^2+\sum_{k=1}^{+\infty}f_{2k-1}^2+\sum_{k=1}^{+\infty}f_{2k}^2\leq M\|f\|^2_{L^2(0,1)},\tag{3.11}$$

where $M=\frac{4-\beta+4\beta^2}{2(1+\beta)^2}$.

On the other hand we have:

$$\bar{f}_0^2=\langle f,X_0\rangle^2=\left[\int_0^1 X_0(x)f(x)\,dx\right]^2\leq 4\|f\|^2_{L^2(0,1)}.\tag{3.12}$$

Using the Bessel inequality, we obtain:

$$\sum_{k=1}^{+\infty}\bar{f}_{2k-1}^2=\sum_{k=1}^{+\infty}\langle f,X_{2k-1}\rangle^2\leq 8\|f\|^2_{L^2(0,1)},\tag{3.13}$$

$$\sum_{k=1}^{+\infty} \bar{f}_{2k}^2 = \sum_{k=1}^{+\infty} \langle f, X_{2k} \rangle^2 \leq \frac{8\left(1+\beta+\beta^2\right)}{3\left(1+\beta\right)^2} \|f\|_{L^2(0,1)}^2 . \tag{3.14}$$

Then, from (3.12)-(3.14) we have:

$$\sum_{i=0}^{+\infty} \bar{f}_i^2 \leq \frac{44+80\beta+44\beta^2}{3\left(1+\beta\right)^2} \|f\|_{L^2(0,1)}^2 . \tag{3.15}$$

Using the Cauchy-Schwarz inequality and (3.15), we get

$$\begin{aligned} \|f\|_{L^2(0,1)}^2 = \langle f, f \rangle &= \sum_{i=0}^{+\infty} \bar{f}_i f_i \\ &\leq \left[\sum_{i=0}^{+\infty} \bar{f}_i^2\right]^{1/2} \left[\sum_{i=0}^{+\infty} f_i^2\right]^{1/2} \\ &\leq \left[\frac{44+80\beta+44\beta^2}{3\left(1+\beta\right)^2}\right]^{1/2} \|f\|_{L^2(0,1)} \left[\sum_{i=0}^{+\infty} f_i^2\right]^{1/2} . \end{aligned}$$

Consequently, we have:

$$m\|f\|_{L^2(0,1)}^2 \leq \sum_{i=0}^{+\infty} f_i^2, \quad m = \frac{3\left(1+\beta\right)^2}{44+80\beta+44\beta^2}. \tag{3.16}$$

From (3.11) and (3.16), the system (3.5) is a Riesz basis in $L^2(0,1)$. □

Corollary 3.1. *From Lemme 3.1 and Theorem 3.1, the systems* (3.5) *and* (3.6) *are equivalent bases in* $L^2(0,1)$.

4. Main results

4.1. *Existence and uniqueness of the solution*

In this subsection, we give the main result on existence and uniqueness of the solution of the inverse problem (1.1)-(1.4) is presented as follows.

Theorem 4.1. *Let the following assumptions be satisfied*

(A1) $\varphi \in \mathcal{C}^4(0,1)$, $\varphi(1) = \varphi(0)$, $\varphi'(1) = \beta\varphi'(0)$, $\varphi''(1) = \varphi''(0)$, $\varphi'''(1) = \beta \times \varphi'''(0)$;

(A2) $f(x,\cdot) \in \mathcal{C}[0,T]$ *and for* $t \in [0,T]$, $f(\cdot,t) \in \mathcal{C}^4[0,1]$;

$$f(0,t) = f(1,t), \qquad f_x(1,t) = \beta f_x(0,t), \qquad f_{xx}(0,t) = f_{xx}(1,t),$$

$$f_{xxx}(1,t) = \beta f_{xxx}(0,t), \qquad \int_0^1 f(x,t)\,dx \neq 0$$

and there exists a constant $M > 0$ *such that*

$$0 < \left|\int_0^1 f(x,t)\,dx\right|^{-1} \leq M;$$

(A3) *$g \in C^1(0,T)$, and g satisfies the consistency condition $\int_0^1 \varphi(x)\,dx = g(0)$.*

If the following condition

$$T < \left(\frac{\alpha|1+\beta|\rho^{\alpha}}{MC'|1-\beta|}\right)^{1/\rho\alpha}, \tag{4.1}$$

where C' is defined in (4.16), then the inverse problem (1.1)-(1.4) has a unique solution.

Proof. According to assumptions (A1)-(A3), there are positive constants, L_1, L_2, M_i, $i = 0,\dots,2$, such that

$$L_1 := \max_{0\le t\le T} e_\alpha\left(\frac{t^\rho}{\rho},\lambda_k\right), \quad L_2 := \max_{0\le s\le t\le T} E_{\alpha,\alpha}\left[-\lambda_k\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha}\right],$$
$$M_0 := \|r\|_{C(0,T)}. \quad M_1 := \max\left(\|f_0\|_{C(0,T)}, \left\|f_{2k-1}^{(4)}\right\|_{C(0,T)}, \left\|f_{2k}^{(4)}\right\|_{C(0,T)}\right),$$
$$M_2 := \max\left(|\varphi_0|, \left|\varphi_{2k-1}^{(4)}\right|, \left|\varphi_{2k}^{(4)}\right|\right).$$

The proof of this theorem takes place in three steps:

Step 1: Construction of solution: By applying the Fourier's method, the solution $u(x,t)$ of the direct problem (1.1)-(1.3), can be developed in uniformly convergent series form using the eigenfunctions (3.5) in $L^2(0,1)$ as follows

$$u(x,t) = 2u_0(t) + \sum_{k=1}^{+\infty} u_{2k-1}(t)X_{2k-1}(x) + \sum_{k=1}^{+\infty} u_{2k}(t)X_{2k}(x), \tag{4.2}$$

We define the coefficients $u_0(t)$, $u_{2k-1}(t)$ and $u_{2k}(t)$ for $k \in \mathbb{N}^*$ by multiplying (4.2) by the eigenfunctions of (3.6) and integrating over $[0,1]$ and using Lemma 3.1, we get

$$\begin{aligned} &u_0(t) = \langle u(x,t), Y_0(x)\rangle, \quad u_{2k-1}(t) = \langle u(x,t), Y_{2k-1}(x)\rangle, \\ &u_{2k}(t) = \langle u(x,t), Y_{2k}(x)\rangle, \end{aligned} \tag{4.3}$$

where $\langle\cdot,\cdot\rangle$ represents the inner product in $L^2(0,1)$.

The expansion coefficients of the functions $f(x,t)$ and $\varphi(x)$ into eigenfunctions (3.6) are given by

$$\begin{aligned} &f_0(t) = \langle f(x,t), Y_0(x)\rangle, \quad f_{2k-1}(t) = \langle f(x,t), Y_{2k-1}(x)\rangle, \\ &f_{2k}(t) = \langle f(x,t), Y_{2k}(x)\rangle, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} &\varphi_0 = \langle \varphi(x), Y_0(x)\rangle, \quad \varphi_{2k-1} = \langle \varphi(x), Y_{2k-1}(x)\rangle, \\ &\varphi_{2k} = \langle \varphi(x), Y_{2k}(x)\rangle. \end{aligned} \tag{4.5}$$

From (1.1), (4.3)-(4.5), Lemma 3.1, integration by parts twice and (1.3), we obtain

$$\begin{cases} {}^{c}\mathcal{D}_{t}^{\alpha,\rho} u_0(t) = r(t) f_0(t), \\ u_0(0) = \varphi_0, \end{cases} \tag{4.6}$$

$$\begin{cases} {}^{c}\mathcal{D}_{t}^{\alpha,\rho} u_{2k}(t) + \lambda_k u_{2k}(t) = r(t) f_{2k}(t), \\ u_{2k}(0) = \varphi_{2k}, \end{cases} \tag{4.7}$$

$$\begin{cases} {}^{c}\mathcal{D}_{t}^{\alpha,\rho} u_{2k-1}(t) + \lambda_k u_{2k-1}(t) = -4\pi a k u_{2k}(t) + r(t) f_{2k-1}(t), \\ u_{2k-1}(0) = \varphi_{2k-1}. \end{cases} \tag{4.8}$$

Applying $I_0^{\alpha,\rho}$ on (4.6) and using Theorem 2.1, we obtain

$$u_0(t) = \varphi_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} r(s) f_0(s) \frac{ds}{s^{1-\rho}}. \tag{4.9}$$

Applying Theorem 2.5 on (4.7) and (4.8) , we obtain:

$$u_{2k}(t) = \varphi_{2k} e_\alpha\left(\frac{t^\rho}{\rho}, \lambda_k\right) + \int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho - s^\rho}{\rho}, \lambda_k\right) r(s) f_{2k}(s) \frac{ds}{s^{1-\rho}}, \tag{4.10}$$

and

$$\begin{aligned} u_{2k-1}(t) = {} & \varphi_{2k-1} e_\alpha\left(\frac{t^\rho}{\rho}, \lambda_k\right) + \int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho - s^\rho}{\rho}, \lambda_k\right) r(s) f_{2k-1}(s) \frac{ds}{s^{1-\rho}} \\ & - 4\pi a k \int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho - s^\rho}{\rho}, \lambda_k\right) u_{2k}(s) \frac{ds}{s^{1-\rho}}. \end{aligned} \tag{4.11}$$

After substituting expressions $u_0(t)$, $u_{2k}(t)$, and $u_{2k-1}(t)$, respectively described by (4.9), (4.10), and (4.11), into (4.2), we have:

$$\begin{aligned} u(x,t) = {} & 2\varphi_0 + \frac{2}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} r(s) f_0(s) \frac{ds}{s^{1-\rho}} \\ & + \sum_{k=1}^{+\infty} \left\{ \varphi_{2k} e_\alpha\left(\frac{t^\rho}{\rho}, \lambda_k\right) + \int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho - s^\rho}{\rho}, \lambda_k\right) r(s) f_{2k}(s) \frac{ds}{s^{1-\rho}} \right\} X_{2k}(x) \\ & + \sum_{k=1}^{+\infty} \left\{ \varphi_{2k-1} e_\alpha\left(\frac{t^\rho}{\rho}, \lambda_k\right) + \int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho - s^\rho}{\rho}, \lambda_k\right) r(s) f_{2k-1}(s) \frac{ds}{s^{1-\rho}} \right. \\ & \left. -4\pi a k \int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho - s^\rho}{\rho}, \lambda_k\right) u_{2k}(s) \frac{ds}{s^{1-\rho}} \right\} X_{2k-1}(x), \end{aligned}$$

Taking the generalized Caputo fractional derivative ${}^{c}\mathcal{D}_{t}^{\alpha,\rho}$ of the overdetermination condition (1.4), and integrating the equation (1.1) on $[0,1]$ and

using (1.3), we obtain

$$r(t)=\frac{{}^{c}\mathcal{D}_{t}^{\alpha,\rho}g(t)+(1-\beta)u_{x}(0,t)}{\int_{0}^{1}f(x,t)\,dx}\text{ where }\int_{0}^{1}f(x,t)\,dx=2f_{0}(t)+\frac{2a}{\pi}\sum_{k=1}^{+\infty}\frac{f_{2k}(t)}{k}$$

and

$$u_{x}(0,t)=\sum_{k=1}^{+\infty}8\pi k(1-b)\left(\varphi_{2k}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_{k}\right)\right.$$
$$\left.+\int_{0}^{t}e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda_{k}\right)r(s)f_{2k}(s)\frac{ds}{s^{1-\rho}}\right).$$

Hence, we get following implicit representation of $r(t)$

$$r(t)=\eta(t)+\left[2f_{0}(t)+\frac{2a}{\pi}\sum_{k=1}^{+\infty}\frac{f_{2k}(t)}{k}\right]^{-1}\int_{0}^{t}K(t,s)r(s)\frac{ds}{s^{1-\rho}},\tag{4.12}$$

where

$$\eta(t)=\frac{{}^{c}\mathcal{D}_{t}^{\alpha,\rho}g(t)+a\sum_{k=1}^{+\infty}8\pi k\varphi_{2k}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_{k}\right)}{\int_{0}^{1}f(x,t)\,dx},\tag{4.13}$$

and

$$K(t,s)=a\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\sum_{k=1}^{+\infty}8\pi kf_{2k}(s)E_{\alpha,\alpha}\left[-\lambda_{k}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right].$$

Step 2: Existence of the solution: We consider the following map:

$$\mathcal{P}(r(t)):=\eta(t)+\left[2f_{0}(t)+\frac{2a}{\pi}\sum_{k=1}^{+\infty}\frac{f_{2k}(t)}{k}\right]^{-1}\int_{0}^{t}K(t,s)r(s)\frac{ds}{s^{1-\rho}}.$$

on the space $\mathcal{C}[0,T]$ with $\|\phi\|:=\max_{0\leq t\leq T}|\phi(t)|$. To show $\mathcal{P}$ is well defined. Since, under the assumptions (A1), (A2) and integration by parts four times, for $t,s\in[0,T]$, we obtain

$$\sum_{k=1}^{+\infty}8\pi k\varphi_{2k}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_{k}\right)\leq\sum_{k=1}^{+\infty}\frac{L_{1}\left|\varphi_{2k}^{(4)}\right|}{2\pi^{3}k^{3}},\tag{4.14}$$

$$\sum_{k=1}^{+\infty}8\pi kf_{2k}(s)E_{\alpha,\alpha}\left[-\lambda_{k}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right]\leq\sum_{k=1}^{+\infty}\frac{L_{2}\left|f_{2k}^{(4)}(s)\right|}{2\pi^{3}k^{3}},\tag{4.15}$$

where $\varphi_{2k}^{(4)}=\int_{0}^{1}\varphi^{(4)}(x)\sin(2\pi kx)\,dx$, $f_{2k}^{(4)}(t)=\int_{0}^{1}\frac{\partial^{4}f(x,t)}{\partial x^{4}}\sin(2\pi kx)\,dx$.

Using the Cauchy-Schwarz and Bessel inequalities, we obtain

$$\sum_{k=1}^{+\infty}\frac{L_2\left|f_{2k}^{(4)}(s)\right|}{2\pi^3k^3}\leq\left[\sum_{k=1}^{+\infty}\frac{L_2^2}{4\pi^6k^6}\right]^{1/2}\left[\sum_{k=1}^{+\infty}\left(f_{2k}^{(4)}(s)\right)^2\right]^{1/2}\leq c\left\|\frac{\partial^4 f(x,t)}{\partial x^4}\right\|_{L^2(0,1)},$$

where c is a constant independent of t and k. Thus, we have

$$\begin{aligned}&\sum_{k=1}^{+\infty}8\pi k f_{2k}(s)E_{\alpha,\alpha}\left[-\lambda_k\left(\frac{t^\rho-s^\rho}{\rho}\right)^\alpha\right]\leq C',\\&\qquad\qquad C'=c\max_{0\leq t\leq T}\left\|\frac{\partial^4 f(x,t)}{\partial x^4}\right\|_{L^2(0,1)}.\end{aligned}\tag{4.16}$$

By (4.14) and (4.15), the series functions

$$\sum_{k=1}^{+\infty}8\pi k f_{2k}(s)E_{\alpha,\alpha}\left[-\lambda_k\left(\frac{t^\rho-s^\rho}{\rho}\right)^\alpha\right]\text{ and }\sum_{k=1}^{+\infty}8\pi k\varphi_{2k}e_\alpha\left(\frac{t^\rho}{\rho},\lambda_k\right)$$

are uniformly convergent. Then, $\eta(t)$ and $K(t,s)$ are continuous functions on $[0,T]$ and $[0,T]\times[0,T]$, respectively. Hence, the operator $\mathcal{P}$ is well defined.

Let $r_1,r_2\in\mathcal{C}(0,T)$. From (4.16) and the change of variable $\tau=\frac{t^\rho-s^\rho}{\rho}$, we get

$$\|\mathcal{P}(r_1)-\mathcal{P}(r_2)\|\leq\frac{MC'|1-\beta|T^{\rho\alpha}}{\alpha|1+\beta|\rho^\alpha}\|r_1-r_2\|.\tag{4.17}$$

With the condition (4.1), $\frac{MC'|1-\beta|T^{\rho\alpha}}{\alpha|1+\beta|\rho^\alpha}<1$, then the mapping $\mathcal{P}$ is a contraction. Consequently, by Banach fixed point theorem, the mapping $\mathcal{P}$ has a unique fixed point $r\in\mathcal{C}[0,T]$.

To establish the regularity of the obtained solution, it remains to show

$$u(x,t),u_x(x,t),u_{xx}(x,t),{}^c\mathcal{D}_t^{\alpha,\rho}u(x,t)\in\mathcal{C}(\Omega_T).$$

Under assumptions (A1)-(A2) and integration by parts four times, we have

$$\begin{aligned}f_{2k}(t)&=\frac{f_{2k}^{(4)}(t)}{16\pi^4k^4},\quad f_{2k-1}(t)=\frac{1}{16\pi^4k^4}\left(f_{2k-1}^{(4)}(t)-\frac{2a}{\pi k}f_{2k}^{(4)}(t)\right),\\\varphi_{2k}&=\frac{\varphi_{2k}^{(4)}}{16\pi^4k^4},\quad\varphi_{2k-1}=\frac{1}{16\pi^4k^4}\left(\varphi_{2k-1}^{(4)}-\frac{2a}{\pi k}\varphi_{2k}^{(4)}\right).\end{aligned}\tag{4.18}$$

From (4.9)-(4.11), (4.18) and (2.1), we get

$$\begin{aligned}
|u_0(t)| &\leq M_2 + \frac{M_0 M_1 T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} := M_3, \quad t \in [0,T], \\
|u_{2k}(t)| &\leq \frac{L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha}{16\pi^4 k^4}, \quad t \in [0,T], \\
|u_{2k-1}(t)| &\leq \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)(1+|a|+|a|CT^\rho/\rho)}{16\pi^4 k^4}, \\
&\quad t \in [\varepsilon, T], \quad \varepsilon > 0.
\end{aligned} \tag{4.19}$$

By using (4.2) and (4.19), following relations hold for $x \in [0,1]$ and $t \in [\varepsilon, T]$ with $\varepsilon > 0$ such that

$$\begin{aligned}
|u(x,t)| &\leq 2M_1 + \sum_{k=1}^{+\infty} \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)(1+|a|+|a|CT^\rho/\rho)}{4\pi^4 k^4} \\
&\quad + \sum_{k=1}^{+\infty} \frac{(1+|b|+|a|)(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)}{4\pi^4 k^4}, \\
|u_x(x,t)| &\leq \sum_{k=1}^{+\infty} \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)(1+|a|+|a|CT^\rho/\rho)}{2\pi^3 k^3} \\
&\quad + \sum_{k=1}^{+\infty} \frac{(|a| + 2\pi k(1+|b|+|a|))(1+|b|+|a|)}{4\pi^4 k^4} \\
&\qquad \times (L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha) \\
|u_{xx}(x,t)| &\leq \sum_{k=1}^{+\infty} \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)(1+|a|+|a|CT^\rho/\rho)}{\pi k^2} \\
&\quad + \sum_{k=1}^{+\infty} \frac{(a + \pi k(1+|b|+|a|))(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)}{\pi^3 k^3}.
\end{aligned} \tag{4.20}$$

From (4.6)-(4.8), (4.19) and for $t \in [\varepsilon, T]$, we have

$$\begin{aligned}
\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u_0(t)\right| &\leq M_0 M_2, \\
\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u_{2k}(t)\right| &\leq \frac{M_0 M_2}{16\pi^4 k^4} + \frac{L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha}{4\pi^2 k^2}, \\
\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u_{2k-1}(t)\right| &\leq \frac{(1+|a|) M_0 M_2}{16\pi^4 k^4} + \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)}{4\pi^2 k^2} \\
&\quad \times (1+|a|+|a|CT^\rho/\rho) + \frac{|a|(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)}{4\pi^3 k^3}.
\end{aligned}$$

Consequently,

$$\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u(x,t)\right| \leq 2M_0 M_2 + \sum_{k=1}^{+\infty} \frac{(2+|b|+2|a|) M_0 M_2}{4\pi^4 k^4} \tag{4.21}$$

$$+\sum_{k=1}^{+\infty}\frac{|a|\left(L_1M_2+L_2M_0M_1T^{\rho\alpha}/\alpha\rho^{\alpha}\right)}{\pi^3k^3}$$
$$+\sum_{k=1}^{+\infty}\frac{\left(L_1M_2+L_2M_0M_1T^{\rho\alpha}/\alpha\rho^{\alpha}\right)\left(2+|b|+2|a|+|a|CT^{\rho}/\rho\right)}{\pi^2k^2}.$$

From (4.20), (4.21) and by Weierstrass M-test, the series corresponding to $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, ${}_0^c\mathcal{D}_t^{\alpha,\rho}u(x,t)$ are uniformly convergent on $[0,1]\times[\varepsilon,T]$ for $\varepsilon>0$. Hence, $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, ${}_0^c\mathcal{D}_t^{\alpha,\rho}u(x,t)$ are continuous functions on Ω_T.

Step 3: Uniqueness of the solution: Let $\{u(x,t),r_1(t)\}$ and $\{v(x,t),r_2(t)\}$ be two solution sets of the inverse problem (1.1)-(1.4). By using (4.2), we obtain

$$\begin{aligned}u(x,t)-v(x,t)&=2\left(u_0(t)-v_0(t)\right)\\&\quad+\sum_{k=1}^{+\infty}\left(u_{2k-1}(t)-v_{2k-1}(t)\right)X_{2k-1}(x)\\&\quad+\sum_{k=1}^{+\infty}\left(u_{2k}(t)-v_{2k}(t)\right)X_{2k}(x),\end{aligned}\tag{4.22}$$

Due to the estimate (4.17) and condition (4.1), we have $r_1=r_2$, and by substituting $r_1=r_2$ in (4.22) and (4.9)-(4.11), we obtain $u=v$.

□

4.2. *Continuous dependence of the solution on the data*

Let $\mathcal{H}$ be the set of triples $\{\varphi,f,g\}$ where the functions φ, f and g satisfy the assumptions of Theorem 4.1 and

$$\|\varphi\|_{C^4(0,1)}\le M_4,\quad \|f\|_{C^4(\Omega_T)}\le M_5,\quad \|g\|_{C^1(0,1)}\le M_6.$$

For $\phi\in\mathcal{H}$, we define the norm $\|\phi\|_{\mathcal{H}}:=\|\varphi\|_{C^4(0,1)}+\|f\|_{C^4(\Omega_T)}+\|g\|_{C^1(0,1)}$.

By using the Cauchy-Schwarz and Bessel inequalities, the series functions

$$\sum_{k=1}^{+\infty}\frac{\left|f_{2k}^{(4)}(s)\right|}{2\pi^3k^3}\le M_7,$$

is uniformly convergent, where $f_{2k}^{(4)}(s)$ are the coefficients of the sine Fourier expansion of the function $\frac{\partial^4 f(x,s)}{\partial x^4}$.

Theorem 4.2. *The solution $\{u(x,t),r(t)\}$ of the inverse problem (1.1)-(1.4) under the assumptions of Theorem 4.1, depends continuously upon the data for $T<\left(\frac{\alpha|1+\beta|\rho^{\alpha}}{MC'|1-\beta|}\right)^{1/\rho\alpha}$.*

Proof. Let $\{u(x,t), r(t)\}$ and $\{\tilde{u}(x,t), \tilde{r}(t)\}$ be two solution sets of the inverse problem (1.1)-(1.4), corresponding to the data $\phi = \{\varphi, f, g\}$ and $\phi = \{\tilde{\varphi}, \tilde{f}, \tilde{g}\}$, respectively.

For $g, \tilde{g} \in C^1(0,T)$, we have $\left\| {}_0^c\mathcal{D}_t^{\alpha,\rho} g - {}_0^c\mathcal{D}_t^{\alpha,\rho} \tilde{g} \right\|_{C(0,T)} \leq M_8 \|g - \tilde{g}\|_{C^1(0,T)}$, where $M_8 = \frac{T^{1-\rho\alpha}}{\rho^{1-\alpha}\Gamma(2-\alpha)}$. From (4.13), we have

$$\begin{aligned}
\eta(t) - \tilde{\eta}(t) = &\left(\int_0^1 f(x,t)\,dx \int_0^1 \tilde{f}(x,t)\,dx \right)^{-1} \\
&\times \left[\int_0^1 \tilde{f}(x,t)\,dx \left({}_0^c\mathcal{D}_t^{\alpha,\rho} g(t) - {}_0^c\mathcal{D}_t^{\alpha,\rho} \tilde{g}(t) \right) \right. \\
&+ a \sum_{k=1}^{+\infty} 8\pi k (\varphi_{2k} - \tilde{\varphi}_{2k}) E_\alpha \left(-\lambda_k \left(\frac{t^\rho}{\rho} \right)^\alpha \right) \\
&+ {}_0^c\mathcal{D}_t^{\alpha,\rho} \tilde{g}(t) \left(\int_0^1 \tilde{f}(x,t)\,dx - \int_0^1 f(x,t)\,dx \right) \\
&+ a \sum_{k=1}^{+\infty} 8\pi k \tilde{\varphi}_{2k} E_\alpha \left(-\lambda_k \left(\frac{t^\rho}{\rho} \right)^\alpha \right) \\
&\times \left. \left(\int_0^1 \tilde{f}(x,t)\,dx - \int_0^1 f(x,t)\,dx \right) \right].
\end{aligned}$$

From (4.18), we have $\varphi_{2k} - \tilde{\varphi}_{2k} = \int_0^1 (\varphi(x) - \tilde{\varphi}(x)) X_{2k}(x)\,dx = \frac{\varphi_{2k}^{(4)} - \tilde{\varphi}_{2k}^{(4)}}{16\pi^4 k^4}$.

We have the estimate

$$\|\eta - \tilde{\eta}\| \leq N_1 \|\varphi - \tilde{\varphi}\|_{C^4(0,1)} + N_2 \left\| f - \tilde{f} \right\|_{C(\Omega_T)} + N_3 \|g - \tilde{g}\|_{C^1(0,1)},$$

where $N_1 = M^2 |a| L_1 C^*$, $\quad N_2 = M^2 (|a| L_1 M_7 + M_6 M_8)$, $\quad N_3 = M^2 M_5 M_8$.

From (4.12), we have the estimate

$$\begin{aligned}
\|r - \tilde{r}\| \leq \|\eta - \tilde{\eta}\| &+ \frac{M M_0 |a| T^{\rho\alpha}}{\alpha\rho^\alpha} \left\| f^{(4)} - \tilde{f}^{(4)} \right\|_{C(\Omega_T)} + \frac{M |a| C' T^{\rho\alpha}}{\alpha\rho^\alpha} \|r - \tilde{r}\| \\
&+ \frac{M^2 M_0 C' T^{\rho\alpha}}{\alpha\rho^\alpha} \left\| f - \tilde{f} \right\|_{C(\Omega_T)}.
\end{aligned}$$

Due to the estimate of $\|\eta - \tilde{\eta}\|$, we have

$$\begin{aligned}
\left(1 - \frac{M |a| C' T^{\rho\alpha}}{\alpha\rho^\alpha} \right) \|r - \tilde{r}\| \leq &N_1 \|\varphi - \tilde{\varphi}\|_{C^4(0,1)} \\
&+ \left(N_2 + \frac{M M_0 |a| T^{\rho\alpha}}{\alpha\rho^\alpha} + \frac{M^2 M_0 C' T^{\rho\alpha}}{\alpha\rho^\alpha} \right) \left\| f - \tilde{f} \right\|_{C^4(\Omega_T)} \\
&+ N_3 \|g - \tilde{g}\|_{C^1(0,1)}.
\end{aligned}$$

Hence

$$\left(1-\frac{M|a|C'T^{\rho\alpha}}{\alpha\rho^{\alpha}}\right)\|r-\tilde{r}\| \leq N_4\left\|\phi-\tilde{\phi}\right\|_{\mathcal{H}},$$

where $N_4 := \max\left\{N_1, N_2+\frac{MM_0|a|T^{\rho\alpha}}{\alpha\rho^{\alpha}}+\frac{M^2M_0C'T^{\rho\alpha}}{\alpha\rho^{\alpha}}, N_3\right\}$.

For $T<\left(\frac{\alpha\rho^{\alpha}}{M|a|C'}\right)^{1/\rho\alpha}$, we have $\|r-\tilde{r}\| \leq \frac{N_4}{1-\frac{M|a|C'T^{\rho\alpha}}{\alpha\rho^{\alpha}}}\left\|\phi-\tilde{\phi}\right\|_{\mathcal{H}}$.

From (4.2), a similar estimate can be also obtained for the difference $u-\tilde{u}$:

$$\|u-\tilde{u}\|_{C(\bar{\Omega}_T)} \leq N_5\left\|\phi-\tilde{\phi}\right\|_{\mathcal{H}}.$$

□

ACKNOWLEDGEMENT

The authors would like to thank the referee for his/her careful reading and valuable comments, which greatly improved the paper.

REFERENCES

[1] T. Abdeljawad, "On conformable fractional calculus," *J. Comput. Appl. Math.*, vol. 279, pp. 57–66, 2015, doi: 10.1016/j.cam.2014.10.016.

[2] A. O. Akdemir, A. Karaoğblan, M. A. Ragusa, and E. Set, "Fractional integral inequalities via Atangana-Baleanu operators for convex and concave functions," *J. Funct. Spaces*, vol. 2021, no. Article ID 1055434, 10 pages, 2021, doi: 10.1155/2021/1055434.

[3] M. Ali and S. Aziz, "Some inverse problems for time-fractional diffusion equation with nonlocal Samarskii-Ionkin type condition," *Math. Meth. Appl. Sci.*, vol. 44, no. 10, pp. 8447–8462, 2021, doi: 10.1002/mma.6330.

[4] J. A. Ferreira, G. Pena, and G. Romanazzi, "Anomalous diffusion in porous media," *Appl. Math. Model.*, vol. 40, no. 3, pp. 1850–1862, 2016, doi: 10.1016/j.apm.2015.09.034.

[5] G. Gonzalez-Parra, A. J. Arenas, and B. M. C. Charpentier, "A fractional order epidemic model for the simulation of outbreak of influenza A(H1N1)," *Math. Meth. Appl. Sci.*, vol. 37, no. 15, pp. 2218–2226, 2014, doi: 10.1002/mma.2968.

[6] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*, ser. Springer Monogr. Math. Springer Berlin, Heidelberg, 2014. doi: 10.1007/978-3-662-43930-2.

[7] A. Hazanee and D. Lesnic, "Determination of a time-dependent heat source from nonlocal boundary conditions," *Eng. Anal. Bound. Elem.*, vol. 37, pp. 936–956, 2013, doi: 10.1016/j.enganabound.2013.03.003.

[8] R. Hilfer, *Applications of fractional calculus in physics*. World Scientific, Singapore, 2000. doi: 10.1142/3779.

[9] V. A. Il'in, "Necessary and sufficient conditions for a subsystem of the eigen- and associated functions of a Keldys bundle of ordinary differential operators to be a basis," *Sov. Math., Dokl.*, vol. 17, pp. 513–516, 1976.

[10] N. I. Ionkin, "The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition," *Differ. Uravn.*, vol. 13, no. 2, pp. 294–304, 1977.

[11] M. I. Ismailov, F. Kanca, and D. Lesnic, "Determination of a time-dependent heat source under nonlocal boundary and integral overdetermination conditions," *Appl. Math. Comp.*, vol. 218, no. 8, pp. 4138–4146, 2011, doi: 10.1016/j.amc.2011.09.044.

[12] N. I. Ivanchov, "On the determination of unknown source in the heat equation with nonlocal boundary conditions," *Ukr. Math. J.*, vol. 47, pp. 1647–1652, 1995, doi: 10.1007/BF01060166.

[13] F. Jarad and T. Abdeljawad, "A modified Laplace transform for certain generalized fractional operators," *Results in Nonlinear Analysis*, vol. 1, no. 2, pp. 88–98, 2018.

[14] F. Jarad, T. Abdeljawad, and D. Baleanu, "On the generalized fractional derivatives and their Caputo modification," *J. Nonlinear Sci. Appl.*, vol. 10, no. 5, pp. 2607–2619, 2017, doi: 10.22436/jnsa.010.05.27.

[15] U. N. Katugampola, "New approach to a generalized fractional integral," *Appl. Math. Comput.*, vol. 218, no. 3, pp. 860–865, 2011, doi: 10.1016/j.amc.2011.03.062.

[16] M. V. Keldysh, "On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators," *Russ. Math. Surv.*, vol. 26, no. 4, pp. 15–44, 1971, doi: 10.1070/RM1971v026n04ABEH003985.

[17] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations, Volume 204*. Elsevier Science B.V., Amsterdam, 2006.

[18] R. Klages, G. Radons, and M. I. Sokolov, *Anomalous transport: foundations and applications*. WILEY-VCH Verlag GmbH & Co. KGaA, 2008. doi: 10.1002/9783527622979.

[19] L. Ma and R. He, "Numerical and experimental study of general diffusuion equation within fractal pores," *Combust. Sci. Technol.*, vol. 188, no. 7, pp. 1073–1094, 2016, doi: 10.1080/00102202.2016.1151877.

[20] F. Mainardi, *Fractional calculus and waves in linear viscoelaticity*. Imperial College Press, 2010. doi: 10.1142/p614.

[21] R. Metzler and J. Klafter, "The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics," *J. Phys. A: Math. Gen.*, vol. 37, no. 31, pp. R161–R208, 2004, doi: 10.1088/0305-4470/37/31/R01.

[22] M. A. Naimark, *Linear differential operators: two volumes bound as one*. Dover Publications, Incorporated, 2012.

[23] I. Podlubny, *Fractional differential equations*. Academic Press, 1999.

[24] S. Rashid, A. O. Akdemir, M. A. Noor, and K. I. Noor, "Integral inequalities for exponentially harmonically convex functions via fractional integral operators," *Miskolc Math. Notes*, vol. 22, no. 2, pp. 875–888, 2021, doi: 10.18514/MMN.2021.3078.

[25] M. Ruzhansky, M. Sadybekov, and D. Suragan, *Spectral geometry of partial differential operators*. CRC Press, 2020. doi: 10.1201/9780429432965.

[26] M. Sadybekov, G. Oralsyn, and M. Ismailov, "Determination of a time-dependent heat source under not strengthened regular boundary and integral overdetermination conditions," *Filomat*, vol. 32, no. 3, pp. 809–814, 2018, doi: 10.2298/FIL1803809S.

[27] D. Sierociuk, T. Skovranek, and M. Macias, "Diffusion process modeling by using fractional-order models," *Appl. Math. Comp.*, vol. 257, pp. 2–11, 2015, doi: 10.1016/j.amc.2014.11.028.

[28] S. Valizadeh, A. Malek, and A. Borhanifara, "Compact ADI method for two-dimensional Riesz space fractional diffusion equation," *Filomat*, vol. 35, no. 5, pp. 1543–1554, 2021, doi: 10.2298/FIL2105543V.

[29] R. M. Young, *An introduction to nonharmonic Fourier series.*, revised edition ed. Orlando, FL: Academic Press, 2001.

Authors' addresses

Farid Mihoubi
Department of Mathematics, Faculty of Mathematics and Computer Science, University of M'sila, PO Box 166 Ichebilia, 28000 M'sila, Algeria
E-mail address: farid.mihoubi@univ-msila.dz

Brahim Nouiri
(**Corresponding author**) Laboratory of Pure and Applied Mathematics, Faculty of Mathematics and Computer Science, University of M'sila, PO Box 166 Ichebilia, 28000 M'sila, Algeria
E-mail address: brahim.nouiri@univ-msila.dz

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 891–906

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4502

ON SOME FIXED POINT THEOREMS FOR ĆIRIĆ OPERATORS

MĂDĂLINA MOGA AND RADU TRUŞCĂ

Received 28 November, 2022

Abstract. In this paper, we will present existence and localization fixed point theorems and stability results for the fixed point problem involving some very general classes of operators (single-valued and multi-valued), namely for Ćirić type operators. Then, an application to homotopy principles is given. Our results complement and extend the works in the literature.

2010 *Mathematics Subject Classification:* 47H10; 54H25

Keywords: metric space, fixed point, Ćirić type operator, data dependence, well-posedness, Ulam-Hyers stability

1. Introduction and Preliminaries

Let (X,d) be a metric space and $\mathrm{f}\colon X \to X$ be an operator. For each $x \in X$, we denote $O(x,\infty) = \{x, \mathrm{f}(x), \dots, \mathrm{f}^n(x), \dots\}$.

Let x_0 be a given point in X and $r > 0$. The set $B(x_0;r) := \{x \in X\colon \mathrm{d}(x_0,x) < r\}$ is the open ball of center x_0 and radius r, while $\tilde{B}(x_0;r) := \{x \in X\colon \mathrm{d}(x_0,x) \le r\}$ is the closed ball of center x_0 and radius r.

We will now recall some definitions and well-known results, which will be useful throughout the paper.

Definition 1 ([4, Cirić Definition page 268])**.** Let (X,d) be a metric space and $\mathrm{f}\colon X \to X$ be an operator. Then X is said to be f-orbitally complete if every Cauchy sequence contained in $O(x,\infty)$, for some $x \in X$, converges in X.

In the above context, a sequence of Picard iterates starting from $x_0 \in X$ is a sequence $x_n := \mathrm{f}^n(x_0)$, for $n \in \mathbb{N}^*$.

Definition 2 ([4, Cirić Definition 1])**.** Let (X,d) be a metric space. Then, $\mathrm{f}\colon Y \subseteq X \to X$ is a single-valued Ćirić type operator with constant q if there exists a number $q \in (0,1)$, such that for all $x,y \in Y$ we have

$$\mathrm{d}(\mathrm{f}(x),\mathrm{f}(y)) \le q \cdot \max\{\mathrm{d}(x,y), \mathrm{d}(x,\mathrm{f}(x)), \mathrm{d}(y,\mathrm{f}(y)), \mathrm{d}(x,\mathrm{f}(y)), \mathrm{d}(y,\mathrm{f}(x))\}.$$

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

Let (X,d) be a metric space. By $P(X)$ we denote the family of all nonempty subsets of X, and the family of all nonempty and closed subsets of X is denoted with $P_{cl}(X)$. Throughout the paper, we consider the following distances (see, e.g., [9,10]):

(1) The gap functional (generated by d) between a point $a \in X$ and a set $Y \in P(X)$ is
$$\mathrm{D}(a,Y) := \inf\{\mathrm{d}(a,y) \mid y \in Y\}.$$

(2) The Pompeiu-Hausdorff functional (generated by d) between two sets $A,B \in P(X)$ is
$$\mathrm{H}(A,B) := \max\left\{\sup_{a\in A}(\inf_{b\in B}\mathrm{d}(a,b)), \sup_{b\in B}(\inf_{a\in A}\mathrm{d}(a,b))\right\}.$$

If $\mathrm{F}\colon X \to P(X)$ is a multi-valued operator, then its fixed point set is denoted by $\mathrm{Fix}(F) := \{x \in X \mid x \in \mathrm{F}(x)\}$, while the graph of F is the set $\mathrm{Graph}(F) := \{(x,y) \in X \times X \mid y \in \mathrm{F}(x)\}$. The set of all strict fixed points of F is denoted by $\mathrm{SFix}(F)$, i.e., there exists $x^* \in X$ such that $\mathrm{F}(x^*) = \{x^*\}$.

In this paper, we will present several existence and localization fixed point theorems and stability results for the fixed point problem involving some very general classes of operators (single-valued and multi-valued), namely for Ćirić type operators. Then, some applications to homotopy principles are given. Our results complement and extend some works in the literature, see e.g. [1–4,6,8,11].

2. A STUDY OF THE FIXED POINT EQUATION WITH GENERALIZED ĆIRIĆ OPERATORS

In this section, the single-valued case is taken into consideration. We first recall Ćirić's Theorem which appeared in the well-known paper from 1974, see [4].

Theorem 1 ([4, Cirić Theorem 1])**.** *Let (X,d) be a metric space and $\mathrm{f}\colon X \to X$ be a Ćirić type operator with constant $q \in (0,1)$. Suppose that X is f-orbitally complete. Then:*

(i) f has a unique fixed point x^* in X and $\lim_{n\to\infty} f^n(x) = x^*$, i.e., f is a Picard operator;

(ii) $\mathrm{d}(f^n(x),x^*) \le \dfrac{q^n}{1-q}\mathrm{d}(x,\mathrm{f}(x))$, for every $x \in X$ and every $n \in \mathbb{N}^*$.

Our first main result, which generalizes the above theorem, is an existence, uniqueness and localization for the unique fixed point of a single-valued Ćirić type operator.

Theorem 2. *Let (X,d) be a complete metric space, $x_0 \in X$ and $r > 0$. We consider $\mathrm{f}\colon B(x_0;r) \to X$ a single-valued Ćirić type operator with constant $q \in \left(0,\frac{1}{2}\right)$. We also suppose that*
$$\mathrm{d}(x_0,\mathrm{f}(x_0)) < \frac{1-2q}{1-q}r.$$

Then f *has a unique fixed point* $x^* \in B(x_0;r)$, $f^n(x_0) \in B(x_0;r)$, *for all* $n \in \mathbb{N}$ *and the sequence* $(f^n(x_0))_{n\in\mathbb{N}}$ *of Picard iterates starting from* x_0 *converges to* x^* *as* $n \longrightarrow \infty$.

Proof. Let $0 < s < r$ such that

$$\mathrm{d}(x_0, \mathrm{f}(x_0)) \leq \frac{1-2q}{1-q}s < \frac{1-2q}{1-q}r.$$

The sequence $(x_n)_{n\in\mathbb{N}}$, with $x_n := \mathrm{f}^n(x_0)$, has the recurrent form $x_{n+1} = \mathrm{f}(x_n)$, for all $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\mathrm{d}(x_1,x_2) &= \mathrm{d}(\mathrm{f}(x_0),\mathrm{f}(x_1)) \\
&\leq q\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_2)\mathrm{d}(x_1,x_1)\} \\
&= q\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_2)\} \\
&\leq q\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)\} \\
&= q\left(\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)\right),
\end{aligned}$$

implying

$$\mathrm{d}(x_1,x_2) \leq \frac{q}{1-q}\mathrm{d}(x_0,x_1).$$

We denote $h := \dfrac{q}{1-q}$, thus $\dfrac{1-2q}{1-q} = 1-h$ with $h \in (0,1)$. Using the mathematical induction, we can prove the inequality

$$\mathrm{d}(x_{n-1},x_n) \leq h^{n-1}\mathrm{d}(x_0,x_1)$$

holds for all $n \in \mathbb{N}^*$. We also know that $\mathrm{d}(x_0,x_1) \leq (1-h)s$.

By taking a point $n \in \mathbb{N}^*$ arbitrarily, we obtain

$$\begin{aligned}
\mathrm{d}(x_0,x_n) &\leq \mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)+\cdots+\mathrm{d}(x_{n-1},x_n) \\
&\leq \mathrm{d}(x_0,x_1)+h\mathrm{d}(x_0,x_1)+\cdots+h^{n-1}\mathrm{d}(x_0,x_1) \\
&= \mathrm{d}(x_0,x_1)(1+h+\cdots+h^{n-1}) \\
&= \frac{1-h^n}{1-h}\mathrm{d}(x_0,x_1) \leq \frac{1}{1-h}\mathrm{d}(x_0,x_1) \leq s,
\end{aligned}$$

proving that all elements of the sequence are in the closed ball $\tilde{B}(x_0;s)$.

We will continue by proving that the sequence considered is Cauchy in X. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}^*$. We compute

$$\begin{aligned}
\mathrm{d}(x_m,x_{m+n}) &\leq \mathrm{d}(x_m,x_{m+1})+\cdots+\mathrm{d}(x_{m+n-1},x_{m+n}) \\
&\leq h^m\mathrm{d}(x_0,x_1)+\cdots+h^{m+n-1}\mathrm{d}(x_0,x_1) \\
&= h^m\mathrm{d}(x_0,x_1)\left(1+h+\cdots+h^{n-1}\right) \\
&= h^m\frac{1-h^n}{1-h}\mathrm{d}(x_0,x_1)
\end{aligned}$$

$$\leq \frac{h^m}{1-h}\mathrm{d}(x_0,x_1).$$

This relation leads us to the conclusion that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Moreover, due to the completeness of (X,d), we obtain it is also convergent to a point $x^* \in \tilde{B}(x_0;s)$. We will prove that x^* is a fixed point. We compute the following inequality

$$\begin{aligned}
\mathrm{d}(x^*,\mathrm{f}(x^*)) &\leq \mathrm{d}(x^*,x_{n+1}) + \mathrm{d}(x_{n+1},\mathrm{f}(x^*)) \\
&\leq \mathrm{d}(x^*,x_{n+1}) + q\max\{\mathrm{d}(x_n,x^*),\mathrm{d}(x_n,x_{n+1}),\mathrm{d}(x^*,\mathrm{f}(x^*)), \\
&\qquad\qquad \mathrm{d}(x^*,x_{n+1}),\mathrm{d}(x_n,\mathrm{f}(x^*))\} \\
&\leq \mathrm{d}(x^*,x_{n+1}) + q\max\{\mathrm{d}(x_n,x_{n+1}),\mathrm{d}(x^*,x_{n+1}),\mathrm{d}(x_n,x^*)+\mathrm{d}(x^*,\mathrm{f}(x^*))\} \\
&\leq \mathrm{d}(x^*,x_{n+1}) + q\left[\mathrm{d}(x_n,x_{n+1})+\mathrm{d}(x_{n+1},x^*)+\mathrm{d}(x^*,\mathrm{f}(x^*))\right],
\end{aligned}$$

which implies

$$\mathrm{d}(x^*,\mathrm{f}(x^*)) \leq \frac{1+q}{1-q}\mathrm{d}(x_{n+1},x^*) + \frac{q}{1-q}\left[\mathrm{d}(x_n,x_{n+1})\right], \text{ for all } n\in\mathbb{N}.$$

We only need to let $n \longrightarrow \infty$ in the above inequality, and we will obtain $\mathrm{d}(x^*,\mathrm{f}(x^*)) = 0$, proving that x^* is a fixed point for f.

For the uniqueness of the fixed point, we suppose by contradiction that there exists another fixed point y^*, with $x^* \neq y^*$. Then,

$$\begin{aligned}
\mathrm{d}(x^*,y^*) &= \mathrm{d}(\mathrm{f}(x^*),\mathrm{f}(y^*)) \\
&\leq q\max\{\mathrm{d}(x^*,y^*),\mathrm{d}(x^*,x^*),\mathrm{d}(y^*,y^*),\mathrm{d}(x^*,y^*),\mathrm{d}(y^*,x^*)\} = q\mathrm{d}(x^*,y^*),
\end{aligned}$$

which is a contradiction due to the fact that $q < \frac{1}{2}$. □

We now denote by $\mathcal{S}(Y,X)$ the family of all operators from Y to X, where X is a metric space and Y a closed subset of X, and by

$$\mathcal{S}_{\partial Y}(Y,X) := \left\{\mathrm{f}\in\mathcal{S}(Y,X) \text{ such that } \mathrm{f}_{|_{\partial Y}} : \partial Y \to X \text{ is fixed point free}\right\}.$$

We will introduce the concept of a family of single-valued Ćirić type operators with constant q.

Definition 3. Let (X,d) be a metric space and (J,ρ) be a metric space. We say that $\{\mathrm{f}_\lambda : \lambda\in J\} \subset \mathcal{S}(Y,X)$ is a family of single-valued Ćirić type operators with constant $q\in(0,1)$ if the following conditions are satisfied: there exist $p\in(0,1]$ and $M>0$ such that

(i) for all $x_1,x_2\in Y$ and $\lambda\in J$, we have

$$\begin{aligned}
\mathrm{d}(\mathrm{f}_\lambda(x_1),\mathrm{f}_\lambda(x_2)) \leq q\max\{&\mathrm{d}(x_1,x_2),\mathrm{d}(x_1,\mathrm{f}_\lambda(x_1)),\mathrm{d}(x_2,\mathrm{f}_\lambda(x_2)), \\
&\mathrm{d}(x_1,\mathrm{f}_\lambda(x_2)),\mathrm{d}(x_2,\mathrm{f}_\lambda(x_1))\};
\end{aligned}$$

(ii) for all $x\in Y$ and $\lambda,\mu\in J$, we have

$$\mathrm{d}(\mathrm{f}_\lambda(x),\mathrm{f}_\mu(x)) \leq M\left[\rho(\lambda,\mu)\right]^p.$$

The following homotopy result can now be proved.

Theorem 3. *Let* (X,d) *be a complete metric space and* Y *be a closed subset such that* $\operatorname{int} Y \neq \varnothing$. *Let* (J,ρ) *be a connected metric space and* $\{\mathrm{f}_\lambda \colon \lambda \in J\}$ *be a family of single-valued Ćirić type operators with constant* $q \in \left(0, \frac{1}{2}\right)$ *from* $\mathcal{S}_{\partial Y}(Y,X)$. *Then the following conclusions occur:*

(i) If there exists a point $\lambda_0^* \in J$, *such that the equation* $\mathrm{f}_{\lambda_0^*}(x) = x$ *has a solution, then the equation* $\mathrm{f}_\lambda(x) = x$ *has a unique solution for any* $\lambda \in J$;

(ii) If $\mathrm{f}_\lambda(x_\lambda) = x_\lambda$, *for any* $\lambda \in J$, *then the operator*

$$\mathrm{j} \colon J \to \operatorname{int} Y, \mathrm{j}(\lambda) = x_\lambda$$

is continuous.

Proof. We will begin the proof by considering two fixed points, x_λ a fixed point of f_λ and x_μ a fixed point of f_μ. Then,

$$\begin{aligned}\mathrm{d}(x_\lambda, x_\mu) &= \mathrm{d}(\mathrm{f}_\lambda(x_\lambda), \mathrm{f}_\mu(x_\mu)) \\ &\leq \mathrm{d}(\mathrm{f}_\lambda(x_\lambda), \mathrm{f}_\lambda(x_\mu)) + \mathrm{d}(\mathrm{f}_\lambda(x_\mu), \mathrm{f}_\mu(x_\mu)).\end{aligned}$$

Taking $\mathrm{d}(\mathrm{f}_\lambda(x_\lambda), \mathrm{f}_\lambda(x_\mu))$ separately, we compute

$$\begin{aligned}\mathrm{d}(\mathrm{f}_\lambda(x_\lambda), \mathrm{f}_\lambda(x_\mu)) &\leq q \max\{\mathrm{d}(x_\lambda, x_\mu), \mathrm{d}(x_\lambda, \mathrm{f}_\lambda(x_\lambda)), \mathrm{d}(x_\mu, \mathrm{f}_\lambda(x_\mu)), \mathrm{d}(x_\lambda, \mathrm{f}_\lambda(x_\mu)), \\ &\qquad \mathrm{d}(x_\mu, \mathrm{f}_\lambda(x_\lambda))\} \\ &= q \max\left\{\mathrm{d}(x_\lambda, x_\mu), \mathrm{d}(x_\mu, \mathrm{f}_\lambda(x_\mu)), \mathrm{d}(x_\lambda, \mathrm{f}_\lambda(x_\mu))\right\} \\ &\leq q \max\left\{\mathrm{d}(x_\lambda, x_\mu), \mathrm{d}(x_\lambda, x_\mu) + \mathrm{d}(x_\lambda, \mathrm{f}_\lambda(x_\mu)), \mathrm{d}(x_\lambda, \mathrm{f}_\lambda(x_\mu))\right\} \\ &= q\left[\mathrm{d}(x_\lambda, x_\mu) + \mathrm{d}(x_\lambda, \mathrm{f}_\lambda(x_\mu))\right],\end{aligned}$$

which implies

$$\mathrm{d}(\mathrm{f}_\lambda(x_\lambda), \mathrm{f}_\lambda(x_\mu)) \leq \frac{q}{1-q}\mathrm{d}(x_\lambda, x_\mu).$$

Using the latter inequality together with the first one, we obtain

$$\begin{aligned}\mathrm{d}(x_\lambda, x_\mu) &\leq \frac{q}{1-q}\mathrm{d}(x_\lambda, x_\mu) + \mathrm{d}(\mathrm{f}_\lambda(x_\mu), \mathrm{f}_\mu(x_\mu)) \\ &\leq \frac{q}{1-q}\mathrm{d}(x_\lambda, x_\mu) + M\left[\rho(\lambda, \mu)\right]^p\end{aligned}$$

entailing

$$\mathrm{d}(x_\lambda, x_\mu) \leq \frac{1-q}{1-2q} M\left[\rho(\lambda, \mu)\right]^p.$$

Let us consider the set

$$Q = \{\lambda \in J \mid \exists x_\lambda \in \operatorname{int} Y \text{ such that } x_\lambda = \mathrm{f}_\lambda(x_\lambda)\}.$$

In addition to J being a connected space, by proving that Q is both closed and open, will lead us to $Q = J$, proving (i). For the closedness of Q, let $(\lambda_n)_{n\in\mathbb{N}} \subset Q$ such that

$\lambda_n \to \lambda^*$, and we show that $\lambda^* \in Q$. We consider $x_{\lambda_m} = \mathrm{f}_{\lambda_m}(x_{\lambda_m})$ and $x_{\lambda_n} = \mathrm{f}_{\lambda_n}(x_{\lambda_n})$ and we know that

$$\mathrm{d}(x_{\lambda_m}, x_{\lambda_n}) \leq \frac{1-q}{1-2q} M \left[\rho(\lambda_m, \lambda_n)\right]^p. \tag{2.1}$$

We already know that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is Cauchy in J, which implies that for an arbitrary $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ with $m, n \in \mathbb{N}$, $m, n > n_\varepsilon$ such that

$$\rho(\lambda_m, \lambda_n) < \varepsilon. \tag{2.2}$$

We will denote $\dfrac{\varepsilon^p M(1-q)}{1-2q} =: \varepsilon' > 0$. Using this notation, together with relations (2.1) and (2.2), we get

$$\mathrm{d}(x_{\lambda_m}, x_{\lambda_n}) < \varepsilon',$$

which proves the sequence (x_{λ_n}) is Cauchy in X. Also, since (X, d) is a complete space, we obtain (x_{λ_n}) is a convergent sequence in Y. Let us now denote the limit of this sequence by x_{λ^*}, and we compute

$$\begin{aligned}
\mathrm{d}(x_{\lambda^*}, \mathrm{f}(x_{\lambda^*})) &\leq \mathrm{d}(x_{\lambda^*}, x_{\lambda_{n+1}}) + \mathrm{d}(x_{\lambda_{n+1}}, \mathrm{f}(x_{\lambda^*})) \\
&\leq \mathrm{d}(x_{\lambda^*}, x_{\lambda_{n+1}}) + q \max\big\{\mathrm{d}(x_{\lambda_n}, x_{\lambda^*}), \mathrm{d}(x_{\lambda_n}, x_{\lambda_{n+1}}), \mathrm{d}(x_{\lambda^*}, \mathrm{f}(x_{\lambda^*})), \\
&\qquad\qquad \mathrm{d}(x_{\lambda_n}, \mathrm{f}(x_{\lambda^*})), \mathrm{d}(x_{\lambda^*}, x_{\lambda_{n+1}})\big\} \\
&\leq \mathrm{d}(x_{\lambda^*}, x_{\lambda_{n+1}}) + q\left[\mathrm{d}(x_{\lambda_n}, x_{\lambda^*}) + \mathrm{d}(x_{\lambda_n}, x_{\lambda_{n+1}}) + \mathrm{d}(x_{\lambda^*}, \mathrm{f}(x_{\lambda^*}))\right. \\
&\qquad\qquad \left. + \mathrm{d}(x_{\lambda_n}, \mathrm{f}(x_{\lambda^*})) + \mathrm{d}(x_{\lambda^*}, x_{\lambda_{n+1}})\right].
\end{aligned}$$

The inequality obtained above implies that for all $n \in \mathbb{N}$,

$$\mathrm{d}(x_{\lambda^*}, \mathrm{f}(x_{\lambda^*})) \leq \frac{1}{1-2q}\left[(1+2q)\mathrm{d}(x_{\lambda_n}, x_{\lambda^*}) + q\left(\mathrm{d}(x_{\lambda_n}, x_{\lambda_{n+1}}) + \mathrm{d}(x^*_{\lambda}, x_{\lambda_{n+1}})\right)\right],$$

and by letting $n \longrightarrow \infty$, we get that x_{λ^*} is a fixed point for f. Since f is fixed point free on its boundary, then λ^* belongs to Q, proving it is closed.

In order to show that Q is open, we consider $\lambda_0 \in Q$. Then, there exists a point $x_{\lambda_0} \in \operatorname{int} Y$ such that $x_{\lambda_0} = \mathrm{f}_{\lambda_0}(x_{\lambda_0})$. Now, we will prove the existence of an $\varepsilon > 0$ and an open ball $B(\lambda_0; \varepsilon) \subset Q$. Due to $\operatorname{int} Y$ being an open set and $x_{\lambda_0} \in \operatorname{int} Y$, there exists an open ball $B(x_{\lambda_0}; r) \subseteq \operatorname{int} Y$. We consider arbitrary $\varepsilon > 0$ such that $\varepsilon^p < \dfrac{1-2q}{M(1-q)} r$ and an arbitrary $\lambda \in B(\lambda_0; \varepsilon)$, and we prove that $\lambda \in Q$. Let us begin by estimating the following distance

$$\begin{aligned}
\mathrm{d}(\mathrm{f}_\lambda(x_{\lambda_0}), x_{\lambda_0}) &= \mathrm{d}(\mathrm{f}_\lambda(x_{\lambda_0}), \mathrm{f}_{\lambda_0}(x_{\lambda_0})) \\
&\leq M(\rho(\lambda, \lambda_0))^p \leq M\varepsilon^p \\
&\leq \frac{1-2q}{1-q} r.
\end{aligned}$$

From the inequality above, we get that the operator

$$\mathrm{f}_\lambda \colon B(x_{\lambda_0}; r) \to X$$

is a Ćirić type operator, and using the local fixed point theorem for Ćirić type operators, we obtain that $\mathrm{Fix}(\mathrm{f}_\lambda) \neq \varnothing$, implying $\lambda \in Q$.

Based on what we proved so far, the operator j is single-valued. We consider $\lambda,\ \mu \in J$ and we have

$$\mathrm{d}(\mathrm{j}(\lambda), \mathrm{j}(\mu)) \leq \frac{1-q}{1-2q} M\left[\rho(\lambda,\mu)\right]^p .$$

Letting

$$\rho(\lambda,\mu) < \delta := \left[\frac{\varepsilon(1-2q)}{M(1-q)}\right]^{\frac{1}{p}},$$

we immediately obtain that $\mathrm{d}(\mathrm{j}(\lambda), \mathrm{j}(\mu)) < \varepsilon$, proving that j is a continuous operator. □

3. A STUDY OF THE FIXED POINT EQUATION WITH MULTI-VALUED GENERALIZED ĆIRIĆ OPERATORS

We first consider some notions related to our main results.

Definition 4. An operator $\mathrm{F}\colon X \to P_{cl}(X)$ is said to be a multi-valued generalized contraction if for every $x, y \in X$ there exist non-negative numbers p, q, r, which may depend on both x and y, such that $sup\{p+2q+2r \mid x, y \in X\} < 1$ and

$$\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq p \cdot \mathrm{d}(x,y) + q \cdot [\mathrm{D}(x, \mathrm{F}(x)) + \mathrm{D}(y, \mathrm{F}(y))] + r \cdot [\mathrm{D}(x, \mathrm{F}(y)) + \mathrm{D}(y, \mathrm{F}(x))] .$$

Definition 5 ([2, A. Amini-Harandi Definition 2.1])**.** Let (X, d) be a metric space. The set-valued map $\mathrm{F}\colon Y \subseteq X \to P_{b,cl}(X)$ is said to be a multi-valued Ćirić type operator with constant k (named a k-set-valued quasi-contraction in [2]) if

$$\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq k \max\{\mathrm{d}(x,y), \mathrm{D}(x, \mathrm{F}(x)), \mathrm{D}(y, \mathrm{F}(y)), \mathrm{D}(x, \mathrm{F}(y)), \mathrm{D}(y, \mathrm{F}(x))\},$$

for any $x, y \in X$, where $0 \leq k < 1$.

We have the following example of a multi-valued Ćirić type operator, which is not a multi-valued generalized contraction.

Example 1. Let

$$X_1 = \left\{\frac{m}{n} : m = 0,1,2,4,6,\ldots;\ n = 1,3,7,\ldots,2k+1,\ldots\right\},$$

$$X_2 = \left\{\frac{m}{n} : m = 1,2,4,6,8,\ldots;\ n = 2,5,8,\ldots,3k+2,\ldots\right\},$$

where $k \in \mathbb{N}$ and let $X = X_1 \cup X_2$. Let us define $\mathrm{F}\colon X \to X$ by

$$\mathrm{F}(x) = \begin{cases} \left\{\frac{2}{3}x, \frac{6}{7}x\right\}, & x \in X_1, \\ \frac{1}{5}x, & x \in X_2. \end{cases}$$

The mapping F is a multi-valued Ćirić type operator with $q = \dfrac{6}{7}$. If both x and y are in X_1 or in X_2, then

$$\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq \frac{6}{7}\mathrm{d}(x,y).$$

If we take $x \in X_1$ and $y \in X_2$, then we have that

$$x \geq \frac{7}{30}y \text{ implies } \mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) = \frac{6}{7}\left(x - \frac{7}{30}y\right) \leq \frac{6}{7}\left(x - \frac{1}{5}y\right) = \frac{6}{7}\mathrm{D}(x, \mathrm{F}(y)),$$

$$x < \frac{7}{30}y \text{ implies } \mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) = \frac{6}{7}\left(\frac{7}{30}y - x\right) \leq \frac{6}{7}(y - x) = \frac{6}{7}\mathrm{d}(x,y).$$

Therefore, we have that F satisfies the following condition:

$$\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq \frac{6}{7}\max\left\{\mathrm{d}(x,y), \mathrm{D}(x, \mathrm{F}(y)), \mathrm{D}(y, \mathrm{F}(x))\right\},$$

and hence, it is a multi-valued Ćirić type operator.

In the following step, we show that F is not a multi-valued generalized contraction on X. Let $x = 1$ and $y = \frac{1}{2}$. Then we have that

$$\begin{aligned} p \cdot \mathrm{d}(x,y) + q \cdot [\mathrm{D}(x, \mathrm{F}(x)) + \mathrm{D}(y, \mathrm{F}(y))] + r \cdot [\mathrm{D}(x, \mathrm{F}(y)) + \mathrm{D}(y, \mathrm{F}(x))] = \\ = \frac{1}{2}p + \frac{4}{10}q + \frac{88}{70}r < (p + 2q + 2r)\frac{88}{140} < \\ < \frac{88}{140} < \frac{53}{70} = \mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)), \end{aligned}$$

as $p + 2q + 2r < 1$. Thus, we can see that F is not a multi-valued generalized contraction.

If (X, d) is a metric space and $\mathrm{F}\colon X \to P(X)$ is a multi-valued operator, then a sequence $(x_n)_{n \in \mathbb{N}}$ from X is called a sequence of Picard type starting from $(x,y) \in \mathrm{Graph}(F)$ if $x_0 = x, x_1 = y$ and $x_n \in \mathrm{F}(x_{n-1}), n \in \mathbb{N}^*$.

The following lemma is useful for our following results.

Lemma 1 (Cauchy's Lemma)**.** *Let $(a_n), (b_n)$ be two sequences of positive numbers such that $\sum_{n \geq 0} a_n < \infty$ and $\lim_{n \to \infty} b_n = 0$. Then $\lim_{n \to \infty}\left(\sum_{k=0}^{n} a_{n-k} b_k\right) = 0$.*

Theorem 4 ([2, A. Amini-Harandi Theorem 2.2])**.** *Let* (X,d) *be a complete metric space. Let* $\mathrm{F}\colon X \to P_{b,cl}(X)$ *be a multi-valued Ćirić type operator with constant* $k < \frac{1}{2}$*. Then,* F *has a fixed point.*

Here we will give a constructive proof of this theorem, as well as some data dependence and stability results for the fixed point problem $x \in \mathrm{F}(x)$.

Theorem 5. *Let* (X,d) *be a complete metric space. Let* $\mathrm{F}\colon X \to P_{cl}(X)$ *be a multi-valued Ćirić type operator with constant* $k < \frac{1}{2}$*. Then:*

(i) $\mathrm{Fix}(F) \neq \varnothing$ *and for every* $(x,y) \in \mathrm{Graph}(F)$ *there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *of Picard type starting from* $x_0 := x$*,* $x_1 := y$ *which converge to a fixed point* x^* *of* F*;*

(ii) the fixed point equation $x \in \mathrm{F}(x)$ *has the data dependence property, i.e., for any* $x^* \in \mathrm{Fix}(F)$ *and any* $G\colon X \to P(X)$ *such that* $\mathrm{Fix}(G) \neq \varnothing$ *and the inequality* $\mathrm{H}(\mathrm{F}(x),\mathrm{G}(x)) \leq \eta$ *holds for all* $x \in X$ *and some* $\eta > 0$*, there is* $u^* \in \mathrm{Fix}(\mathrm{G})$ *such that*

$$\mathrm{d}(x^*,u^*) \leq \frac{(1+k)q}{1-k}\eta,$$

where $1 < q < \dfrac{1}{2k}$*;*

(iii) the fixed point equation is well-posed, i.e., for every sequence $(u_n)_{n\in\mathbb{N}} \subset X$ *such that*

$$\mathrm{D}(u_n,\mathrm{F}(u_n)) \longrightarrow 0,$$

as $n \longrightarrow \infty$*, we have that* $u_n \longrightarrow x^*$*, as* $n \longrightarrow \infty$*.*

(iv) if $q < \frac{1}{2}$*, then the fixed point equation has the Ostrowski stability property, i.e., for any sequence* $(u_n)_{n\in\mathbb{N}} \subset X$ *with* $\mathrm{D}(u_{n+1},\mathrm{F}(u_n)) \longrightarrow 0$ *as* $n \longrightarrow \infty$*, we have that* $u_n \longrightarrow x^*$*;*

Proof. In order to prove (i), let $x_0 \in X$ and we construct the sequence $(x_n)_{n\in\mathbb{N}}$ of Picard type starting from $x_0 := x$ having the general term $x_n \in \mathrm{F}(x_{n-1}), n \in \mathbb{N}^*$. We prove that this sequence is Cauchy.

Let $x_1 \in \mathrm{F}(x_0)$ and $1 < q < \frac{1}{2k}$. Then, there exists $x_2 \in \mathrm{F}(x_1)$ such that $\mathrm{d}(x_1,x_2) \leq q\mathrm{H}(\mathrm{F}(x_0),\mathrm{F}(x_1))$. Then, we have:

$$\begin{aligned}
\mathrm{d}(x_1,x_2) &\leq qk \cdot \max\{\mathrm{d}(x_0,x_1),\mathrm{D}(x_0,\mathrm{F}(x_0),\mathrm{D}(x_1,\mathrm{F}(x_1)),\\
&\qquad \mathrm{D}(x_0,\mathrm{F}(x_1)),\mathrm{D}(x_1,\mathrm{F}(x_0))\}\\
&\leq qk \cdot \max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{D}(x_0,\mathrm{F}(x_1))\}\\
&\leq qk \cdot \max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_2))\}\\
&\leq qk \cdot \max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)\}\\
&\leq qk(\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)).
\end{aligned}$$

Hence,

$$\mathrm{d}(x_1,x_2) \leq \frac{qk}{1-qk}\mathrm{d}(x_0,x_1).$$

We denote $\beta := \dfrac{qk}{1-qk} < 1$. Then $\mathrm{d}(x_1,x_2) \leq \beta \mathrm{d}(x_0,x_1)$. Using mathematical induction we get that:

$$\mathrm{d}(x_n,x_{n+1}) \leq \beta^n \mathrm{d}(x_0,x_1).$$

and

$$\begin{aligned}\mathrm{d}(x_m,x_{m+n}) &\leq \mathrm{d}(x_m,x_{m+1}) + \cdots + \mathrm{d}(x_{m+n+1},x_{m+n}) \\ &\leq \beta^m \mathrm{d}(x_0,x_1) + \cdots + \beta^{m+n+1}\mathrm{d}(x_0,x_1) = \beta^m \frac{1-\beta^n}{1-\beta}\mathrm{d}(x_0,x_1).\end{aligned}$$

It follows that

$$\mathrm{d}(x_m,x_{m+n}) \leq \frac{\beta^m}{1-\beta}\mathrm{d}(x_0,x_1).$$

Due to the fact that the series $\sum \beta^m$ is convergent, we get the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Since (X,d) is a complete metric space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to an element $x^* \in X$. We show first that $x^* \in \mathrm{Fix}(F)$. Indeed, we have

$$\begin{aligned}0 \leq \mathrm{D}(x^*,\mathrm{F}(x^*)) &\leq \mathrm{d}(x^*,x_{n+1}) + \mathrm{D}(x_{n+1},\mathrm{F}(x^*)) \\ &\leq \mathrm{d}(x^*,x_{n+1}) + \mathrm{H}(\mathrm{F}(x_n),\mathrm{F}(x^*)) \\ &\leq \mathrm{d}(x^*,x_{n+1}) + k\cdot\max\{\mathrm{d}(x_n,x^*),\mathrm{D}(x_n,\mathrm{F}(x_n),\mathrm{D}(x^*,\mathrm{F}(x^*)), \\ &\qquad \mathrm{D}(x_n,\mathrm{F}(x^*)),\mathrm{D}(x^*,\mathrm{F}(x_n))\} \\ &\leq \mathrm{d}(x^*,x_{n+1}) + k\cdot\max\{\mathrm{d}(x_n,x^*),\mathrm{d}(x_n,x_{n+1}),\mathrm{D}(x^*,\mathrm{F}(x^*)), \\ &\qquad \mathrm{D}(x_n,\mathrm{F}(x^*)),\mathrm{d}(x^*,x_{n+1})\}.\end{aligned}$$

In the above inequality if we let $n \longrightarrow \infty$, then we get that

$$0 \leq \mathrm{D}(x^*,\mathrm{F}(x^*)) \leq k\mathrm{D}(x^*,\mathrm{F}(x^*)).$$

Thus $\mathrm{D}(x^*,\mathrm{F}(x^*)) = 0$ and so $x^* \in \mathrm{Fix}(F)$.

For proving (ii), let $x^* \in \mathrm{Fix}(F)$ and $1 < q < \frac{1}{2k}$. Then, there exists $u^* \in G(u^*)$ such that

$$\begin{aligned}\mathrm{d}(x^*,u^*) &\leq q\mathrm{H}(\mathrm{F}(x^*),\mathrm{G}(u^*)) \\ &\leq q\mathrm{H}(\mathrm{F}(x^*),\mathrm{F}(u^*)) + q\eta \\ &\leq qk\cdot\max\{\mathrm{d}(x^*,u^*),\mathrm{D}(u^*,\mathrm{F}(u^*)),\mathrm{D}(x^*,\mathrm{F}(u^*)),\mathrm{D}(u^*,\mathrm{F}(x^*))\} + q\eta \\ &\leq qk\cdot\max\{\mathrm{d}(x^*,u^*),\mathrm{D}(u^*,\mathrm{G}(u^*)) + \eta,\mathrm{D}(x^*,\mathrm{G}(u^*)) + \eta\} + q\eta \\ &\leq qk\cdot\max\{\mathrm{d}(x^*,u^*),\eta,\mathrm{d}(x^*,u^*) + \eta\} + q\eta \\ &\leq qk(\mathrm{d}(x^*,u^*) + \eta) + q\eta,\end{aligned}$$

thus we obtain

$$\mathrm{d}(x^*,u^*) \leq \frac{(1+k)q}{1-kq}\eta.$$

Thus, the fixed point equation with a multi-valued Ćirić type operator has the data dependence property.

Concerning conclusion (iii), in order to prove that the fixed point equation for F is well-posed, we take the sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that $\mathrm{D}(u_n,\mathrm{F}(u_n)) \longrightarrow 0$, as $n \longrightarrow \infty$. Then, we have $\mathrm{d}(u_n,x^*) \leq \mathrm{D}(u_n,\mathrm{F}(u_n)) + \mathrm{H}(\mathrm{F}(u_n),\mathrm{F}(x^*))$. Furthermore, we can write:

$$\begin{aligned}
\mathrm{d}(u_n,x^*) &\leq \mathrm{D}(u_n,\mathrm{F}(u_n)) + k\cdot\max\{\mathrm{d}(u_n,x^*),\mathrm{D}(u_n,\mathrm{F}(u_n)),\mathrm{D}(x^*,\mathrm{F}(x^*)),\\
&\qquad \mathrm{D}(x^*,\mathrm{F}(u_n)),\mathrm{D}(u_n,\mathrm{F}(x^*))\}\\
&\leq \mathrm{D}(u_n,\mathrm{F}(u_n)) + k\cdot\max\{\mathrm{d}(u_n,x^*),\mathrm{d}(u_n,x^*)+\mathrm{D}(x^*,\mathrm{F}(u_n)),\\
&\qquad \mathrm{D}(x^*,\mathrm{F}(u_n)),\mathrm{D}(u_n,\mathrm{F}(x^*))\}\\
&\leq \mathrm{D}(u_n,\mathrm{F}(u_n)) + k(\mathrm{d}(u_n,x^*)+\mathrm{D}(x^*,\mathrm{F}(u_n)))\\
&\leq \mathrm{D}(u_n,\mathrm{F}(u_n)) + k(2\mathrm{d}(u_n,x^*)+\mathrm{D}(u_n,\mathrm{F}(u_n))),
\end{aligned}$$

implying

$$\mathrm{d}(u_n,x^*) \leq \frac{1+k}{1-2k}\mathrm{D}(u_n,\mathrm{F}(u_n)) \longrightarrow 0, n \longrightarrow \infty.$$

Regarding (iv), we will show that the operator $\mathrm{F}\colon X \to P(X)$ has the Ostrowski property. Let us take the sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that

$$\mathrm{d}(u_{n+1},x^*) \leq \mathrm{D}(u_{n+1},\mathrm{F}(u_n)) + \mathrm{D}(\mathrm{F}(u_n),x^*). \tag{3.1}$$

We take separately $\mathrm{D}(\mathrm{F}(u_n),x^*)$ from the above inequality and we have that

$$\begin{aligned}
\mathrm{D}(\mathrm{F}(u_n),x^*) = \mathrm{H}(\mathrm{F}(u_n),\mathrm{F}(x^*)) &\leq k\cdot\max\{\mathrm{d}(u_n,x^*),\mathrm{D}(u_n,\mathrm{F}(u_n)),\mathrm{D}(x^*,\mathrm{F}(x^*)),\\
&\qquad \mathrm{D}(x^*,\mathrm{F}(u_n)),\mathrm{D}(u_n,\mathrm{F}(x^*))\}\\
&\leq k(\mathrm{d}(u_n,x^*)+\mathrm{D}(x^*,\mathrm{F}(u_n))).
\end{aligned}$$

Thus $\mathrm{D}(\mathrm{F}(u_n),x^*) \leq \frac{k}{1-k}\mathrm{d}(u_n,x^*)$ and denote $\alpha := \frac{k}{1-k} < 1$. We replace this result in the relation (3.1) and it follows that

$$\begin{aligned}
\mathrm{d}(u_{n+1},x^*) &\leq \mathrm{D}(u_{n+1},\mathrm{F}(u_n)) + \alpha\mathrm{d}(u_n,x^*)\\
&\leq \mathrm{D}(u_{n+1},\mathrm{F}(u_n)) + \alpha\mathrm{D}(u_n,\mathrm{F}(u_{n-1})) + \alpha^2\mathrm{d}(u_{n-1},x^*)\\
&\leq \cdots \leq \mathrm{D}(u_{n+1},\mathrm{F}(u_n)) + \alpha\mathrm{D}(u_n,\mathrm{F}(u_{n-1})) + \alpha^2\mathrm{d}(u_{n-1},x^*) + \cdots\\
&\quad + \alpha^n\mathrm{D}(u_1,\mathrm{F}(u_0)) + \alpha^{n+1}\mathrm{d}(u_0,x^*)\\
&= \sum_{k=0}^{n}\alpha^{n-k}\mathrm{D}(u_{k+1},\mathrm{F}(u_k)) + \alpha^{n+1}\mathrm{d}(u_0,x^*)
\end{aligned}$$

Since $\alpha < 1$, using Cauchy's lemma (see 1), we get $\mathrm{d}(u_{n+1}, x^*) \longrightarrow 0$. □

We will now give a theorem that shows that, under an additional condition, the fixed point set and the strict fixed point set of a multi-valued Ćirić type operator coincide.

Theorem 6. *Let (X, d) be a complete metric space. Let $\mathrm{F}\colon X \to P_{cl}(X)$ be a multi-valued Ćirić type operator with constant $k < 1$. Suppose that $\mathrm{SFix}(\mathrm{F}) \neq \varnothing$. Then $\mathrm{Fix}(\mathrm{F}) = \mathrm{SFix}(\mathrm{F}) = \{x^*\}$.*

Proof. We will prove that F has a unique fixed point in X. Since $\mathrm{SFix}(\mathrm{F}) \neq \varnothing$ we know that there exists $x^* \in X$ such that $\mathrm{F}(x^*) = \{x^*\}$. We suppose that there exists $z \in \mathrm{Fix}(\mathrm{F})$ such that $z \neq x^*$. We have

$$\begin{aligned}\mathrm{d}(x^*, z) &\leq \mathrm{H}(\mathrm{F}(x^*), \mathrm{F}(z)) \\ &\leq k \max\{\mathrm{d}(z, x^*), \mathrm{D}(z, \mathrm{F}(z)), \mathrm{D}(x^*, \mathrm{F}(x^*)), \mathrm{D}(x^*, \mathrm{F}(z)), \mathrm{D}(z, \mathrm{F}(x^*))\} \\ &\leq k\mathrm{d}(z, x^*).\end{aligned}$$

This is a contradiction for $k < 1$. Therefore $\mathrm{SFix}(F) = \mathrm{Fix}(F) = \{x^*\}$. □

Now we will prove a local fixed point theorem.

Theorem 7. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. We consider the multi-valued operator $\mathrm{F}\colon \tilde{B}(x_0; r) \to P_{cl}(X)$ such that there exists $k \in \left(0, \frac{1}{2}\right)$ with*

$$\begin{aligned}\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq k \max\{&\mathrm{d}(x, y), \mathrm{D}(x, \mathrm{F}(x)), \mathrm{D}(y, \mathrm{F}(y)), \mathrm{D}(x, \mathrm{F}(y)), \\ &\mathrm{D}(y, \mathrm{F}(x))\}, \textit{ for all } x, y \in \tilde{B}(x_0; r).\end{aligned}$$

We also suppose that

$$\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} r.$$

Then, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of Picard iterates starting from x_0 which converges to a fixed point of F.

Proof. Since $\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} r$ we get there exists $x_1 \in \mathrm{F}(x_0)$ such that

$$\mathrm{d}(x_0, x_1) < \frac{1-2k}{1-k} r.$$

Moreover,

$$\begin{aligned}\mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1)) &\leq k \max\{\mathrm{d}(x_0, x_1), \mathrm{D}(x_0, \mathrm{F}(x_0)), \mathrm{D}(x_1, \mathrm{F}(x_1)), \mathrm{D}(x_0, \mathrm{F}(x_1)), \\ &\qquad \mathrm{D}(x_1, \mathrm{F}(x_0))\} \\ &= k \max\{\mathrm{d}(x_0, x_1), \mathrm{D}(x_0, \mathrm{F}(x_0)), \mathrm{D}(x_1, \mathrm{F}(x_1)), \mathrm{D}(x_0, \mathrm{F}(x_1))\} \\ &\leq k \max\{\mathrm{d}(x_0, x_1), \mathrm{D}(x_1, \mathrm{F}(x_1)), \mathrm{d}(x_0, x_1) + \mathrm{D}(x_1, \mathrm{F}(x_1)))\} \\ &\leq k \max\{\mathrm{d}(x_0, x_1), \mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1)), \mathrm{d}(x_0, x_1) + \mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1))\}\end{aligned}$$

$$= k \max\left(\mathrm{d}(x_0,x_1) + \mathrm{H}(\mathrm{F}(x_0),\mathrm{F}(x_1))\right),$$

and thus

$$\mathrm{H}(\mathrm{F}(x_0),\mathrm{F}(x_1)) \leq \frac{k}{1-k}\mathrm{d}(x_0,x_1) < \frac{k}{1-k}\frac{1-2k}{1-k}r.$$

We will now denote $h := \dfrac{k}{1-k}$, which immediately implies $\dfrac{1-2k}{1-k} = 1-h$, with $h \in (0,1)$. Hence,

$$\mathrm{H}(\mathrm{F}(x_0),\mathrm{F}(x_1)) < h(1-h)r.$$

Thus, there exists $x_2 \in \mathrm{F}(x_1)$ such that $\mathrm{d}(x_1,x_2) < h(1-h)r$. We assume

$$p(n): \text{ there exists } x_n \in \mathrm{F}(x_{n-1}) \text{ such that } \mathrm{d}(x_{n-1},x_n) < h^{n-1}(1-h)r,$$

and compute

$$\begin{aligned}
\mathrm{H}(\mathrm{F}(x_{n-1}),\mathrm{F}(x_n)) &\leq k \max\{\mathrm{d}(x_{n-1},x_n), \mathrm{D}(x_{n-1},\mathrm{F}(x_{n-1})), \mathrm{D}(x_n,\mathrm{F}(x_n)),\\
&\qquad \mathrm{D}(x_{n-1},\mathrm{F}(x_n)), \mathrm{D}(x_n,\mathrm{F}(x_{n-1}))\}\\
&\leq k \max\{\mathrm{d}(x_{n-1},x_n), \mathrm{D}(x_n,\mathrm{F}(x_n)), \mathrm{D}(x_{n-1},\mathrm{F}(x_n))\}\\
&\leq k \max\{\mathrm{d}(x_{n-1},x_n), \mathrm{D}(x_n,\mathrm{F}(x_n)), \mathrm{d}(x_{n-1},x_n) + \mathrm{D}(x_n,\mathrm{F}(x_n))\}\\
&\leq k(\mathrm{d}(x_{n-1},x_n) + \mathrm{H}(\mathrm{F}(x_{n-1}),\mathrm{F}(x_n))),
\end{aligned}$$

which implies

$$\mathrm{H}(\mathrm{F}(x_{n-1}),\mathrm{F}(x_n)) \leq h\mathrm{d}(x_{n-1},x_n) < h^n(1-h)r.$$

Using the latter inequality, we get the existence of a point $x_{n+1} \in \mathrm{F}(x_n)$ such that the relation $p(n+1)$ holds, and therefore we proved $p(n)$ by mathematical induction. Again, by means of mathematical induction, one can easily prove the assumption

$$t(n): \mathrm{d}(x_0,x_n) < (1-h^n)r,$$

which shows that all the elements of the sequence $(x_n)_{n\in\mathbb{N}}$ are in the closed ball $\tilde{B}(x_0;r)$. Due to the following inequality

$$\begin{aligned}
\mathrm{d}(x_m,x_{m+n}) &\leq \mathrm{d}(x_m,x_{m+1}) + \cdots + \mathrm{d}(x_{m+n-1},x_{m+n})\\
&\leq h^m(1-h)(1+\cdots+h^{n-1})r \leq h^m(1-h)\frac{1-h^n}{1-h}r \leq h^m r,
\end{aligned}$$

the sequence $(x_n)_{n\in\mathbb{N}} \subset B(x_0;s)$ is Cauchy, thus convergent to a point $x^* \in \tilde{B}(x_0;r)$.

We finish the proof with showing $x^* \in \mathrm{Fix}(F)$, for which we compute

$$\begin{aligned}
\mathrm{D}(x^*,\mathrm{F}(x^*)) &\leq \mathrm{d}(x^*,x_{n+1}) + \mathrm{H}(\mathrm{F}(x_n),\mathrm{F}(x^*))\\
&\leq \mathrm{d}(x^*,x_{n+1}) + k \max\{\mathrm{d}(x_n,x^*), \mathrm{D}(x_n,\mathrm{F}(x_n)), \mathrm{D}(x^*,\mathrm{F}(x^*)),\\
&\qquad \mathrm{D}(x_n,\mathrm{F}(x^*)), \mathrm{D}(x^*,\mathrm{F}(x_n))\}\\
&\leq \mathrm{d}(x^*,x_{n+1}) + k \max\{\mathrm{d}(x_n,x^*) + \mathrm{D}(x_n,\mathrm{F}(x_n)),\\
&\qquad \mathrm{d}(x_n,x^*) + \mathrm{D}(x^*,\mathrm{F}(x^*))\}
\end{aligned}$$

$$\leq \mathrm{d}(x^*, x_{n+1}) + k\mathrm{d}(x_n, x^*) + k\mathrm{d}(x_n, x_{n+1}) + k\mathrm{D}(x^*, \mathrm{F}(x^*)).$$

By considering $n \longrightarrow \infty$, we get the desired conclusion. □

By the above proof, we immediately get the following result.

Theorem 8. *Let (X, d) be a complete metric space, $x_0 \in X$ and $r > 0$. We consider the multi-valued operator $\mathrm{F}\colon B(x_0; r) \to P_{cl}(X)$ such that there exists $k \in \left(0, \frac{1}{2}\right)$ with*

$$\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq k \max\{\mathrm{d}(x,y), \mathrm{D}(x, \mathrm{F}(x)), \mathrm{D}(y, \mathrm{F}(y)), \mathrm{D}(x, \mathrm{F}(y)), \mathrm{D}(y, \mathrm{F}(x))\}, \text{ for all } x, y \in B(x_0; r).$$

We also suppose that

$$\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} r.$$

Then, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of Picard iterates starting from x_0 which converges to a fixed point of F.

Proof. Let $s \in (0, r)$ such that

$$\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} s < \frac{1-2k}{1-k} r.$$

For our conclusion, we follow the approach given in the above proof for the operator $\mathrm{F}\colon \tilde{B}(x_0, s) \to P(X)$. □

Remark 1. It is an open question to obtain, by the above approach, a local fixed point theorem and related stability results for a multi-valued Ćirić type operators with constant $k \in (0,1)$. For a different approach and a general existence result, see [7].

We now introduce the notion of a family of multi-valued Ćirić type operators with constant $k \in (0,1)$.

Definition 6. Let (X, d) be a metric space. Then, the family $(\mathrm{F}_t)_{t\in[0,1]}$ (where $\mathrm{F}_t\colon Y \subseteq X \to P(X)$, for each $t \in [0,1]$) is a family of multi-valued Ćirić type operators with constant k if $k \in (0,1)$ and the following conditions are satisfied:

(i)

$$\mathrm{H}(\mathrm{F}_t(x_1), \mathrm{F}_t(x_2)) \leq k \max\{\mathrm{d}(x_1, x_2), \mathrm{D}(x_1, \mathrm{F}_t(x_1)), \mathrm{D}(x_2, \mathrm{F}_t(x_2)), \mathrm{D}(x_1, \mathrm{F}_t(x_2)), \mathrm{D}(x_2, \mathrm{F}_t(x_1))\}, \text{ for all } x_1, x_2 \in Y, t \in [0,1].$$

(ii) $\mathrm{H}(\mathrm{F}_t(x), \mathrm{F}_s(x)) \leq |\phi(t) - \phi(s)|$, for all $t, s \in [0,1]$ and $x \in Y$,

where $\phi\colon [0,1] \to \mathbb{R}$ is strictly increasing and continuous.

Using the previous definitions, we can state, as an application of the multi-valued local fixed point theorem, a homotopy principle for multi-valued Ćirić type operators. The result generalizes a similar theorem given for multi-valued contraction, given by Frigon and Granas, see [5].

Theorem 9. *Let (X,d) be a complete metric space, $U \subset X$ be an open set and $\mathrm{F}\colon [0,1]\times \overline{U} \to P_{cl}(X)$ be a multi-valued operator with closed graph. We denote $\mathrm{F}_t := \mathrm{F}(t,\cdot)$, for $t \in [0,1]$. We suppose:*

(i) $(\mathrm{F}_t)_{t\in[0,1]}$ is a family of multi-valued Ćirić type operators with a constant $k \in \left(0,\frac{1}{2}\right)$;

(ii) $x \notin \mathrm{F}_t(x)$. for all $(t,x) \in [0,1]\times \partial U$.

Then F_0 has a fixed point if and only if F_1 has a fixed point.

Proof. Let $x^* \in U$ such that $x^* \in \mathrm{F}_0(x^*)$. We define the set

$$Q = \{(t,x) \in [0,1]\times U : x \in \mathrm{Fix}(\mathrm{F}_t)\}.$$

We observe that Q is nonempty, since $(0,x^*) \in Q$. Next, we consider the following partial order relation on Q

$$(t,x) \le (s,y) \text{ if and only if } t \le s \text{ and } \mathrm{d}(x,y) \le \frac{2\,(1-k)\,(\phi(s)-\phi(t))}{1-2k},$$

where ϕ is the function associated to the family $(\mathrm{F}_t)_{t\in[0,1]}$ of multi-valued Ćirić type operators with constant $k \in (0,1)$. We will use for Q the Kuratowski-Zorn Lemma (saying that if a partially ordered set Q has the property that every chain P in Q has an upper bound in Q, then the set Q contains at least one maximal element.)

We consider $P \subset Q$ a totally ordered subset (a chain in Q) and define

$$t^* = \sup\{t : (t,x) \in P\}.$$

We also consider a sequence $\{(t_n,x_n)\}$ in P such that

$$(t_n,x_n) \le (t_{n+1},x_{n+1}) \text{ and } t_n \longrightarrow t^*.$$

Then, taking into consideration the partial order relation on Q, we obtain that

$$\mathrm{d}(x_m,x_n) \le \frac{2(1-k)\,(\phi(t_m)-\phi(t_n))}{1-2k}, \text{ for all } m > n.$$

As a consequence, the sequence (x_n) is Cauchy, therefore it converges to an element $x^* \in \overline{U}$. Since F has closed graph, and it is fixed point free on the boundary of U, we get that $(t^*,x^*) \in Q$. Moreover, we have $(t,x) \le (t^*,x^*)$ for every $(t,x) \in P$, proving that (t^*,x^*) is an upper bound of P. Due to the Kuratowski-Zorn lemma, Q admits a maximal element $(t_0,x_0) \in Q$. Thus, x_0 is a fixed point of $\mathrm{F}_{t_0}(x_0)$.

We will show now, by contradiction, that $t_0 = 1$. We assume that $t_0 \neq 1$. Hence, there exist $t_1 \in (t_0,1]$ and $r > 0$ such that

$$0 < \frac{(1-k)(\phi(t_1)-\phi(t_0))}{1-2k} < r$$

and $B(x_0;r) \subset U$. We also have the following inequality

$$\mathrm{D}(x_0,\mathrm{F}_{t_1}(x_0)) \le \mathrm{D}(x_0,\mathrm{F}_{t_0}(x_0)) + \mathrm{H}(\mathrm{F}_{t_0}(x_0),\mathrm{F}_{t_1}(x_0)) \le |\phi(t_1)-\phi(t_0)|.$$

This implies

$$\mathrm{D}(x_0, \mathrm{F}_{t_1}(x_0)) < \frac{1-2k}{1-k} r.$$

Using the local fixed point theorem for multi-valued Ćirić type operators, we obtain that there exists a fixed point x_1 of F_{t_1} such that $\mathrm{d}(x_0, x_1) \leq r$. Hence, (t_1, x_1) belongs to Q and $(t_0, x_0) < (t_1, x_1)$, which contradicts the maximality of (t_0, x_0).

Conversely, if $\mathrm{F}(1, \cdot)$ has a fixed point, then taking $t := 1 - t$ in the previous approach, we get that $\mathrm{F}(0, \cdot)$ has a fixed point. The proof is complete. □

REFERENCES

[1] C. D. Alecsa and A. Petruşel, "Some variants of Ćirić's multi-valued contraction principle," *An. Univ. Vest Timiş. Ser. Mat.-Inform.*, vol. LVII, no. 1, pp. 23–42, 2019, doi: 10.2478/awutm-2019-0004.

[2] A. Amini-Harandi, "Fixed point theory for set-valued quasi-contraction maps in metric spaces," *Appl. Math. Lett.*, vol. 24, pp. 1791–1794, 2011, doi: 10.1016/j.aml.2011.04.033.

[3] L. B. Ćirić, "Fixed points for generalized multi-valued contractions," *Math. Vesnik*, vol. 24, pp. 265–272, 1972.

[4] L. B. Ćirić, "A generalization of Banach's contraction principle," *Proc. Amer. Math. Soc.*, vol. 45, pp. 267–273, 1974, doi: 10.2307/2040075.

[5] M. Frigon and A. Granas, "Resultats du type de Leray-Schauder pour des contractions multivoques," *Topol. Methods Nonlinear Anal*, vol. 4, no. 1, pp. 197–208, 1994.

[6] A. Granas and J. Dugundji, *Fixed Point Theory*. Berlin: Springer, 2003. doi: 10.1007/978-0-387-21593-8.

[7] R. H. Haghi, S. Rezapour, and N. Shahzad, "On fixed points of quasi-contraction type multifunctions," *Applied Math. Letters*, vol. 25, pp. 843–845, 2012, doi: 10.1016/j.aml.2011.10.029.

[8] M. Moga, "On some qualitative properties of Ćirić's fixed point theorem," *Stud. Univ. Babeş-Bolyai Math.*, vol. 67, no. 1, pp. 47–54, 2022, doi: 10.24193/subbmath.2022.1.04.

[9] G. Petruşel, "Generalized multivalued contractions which are quasi-bounded," *Demonstratio Math.*, vol. 40, pp. 639–648, 2007, doi: 10.1515/dema-2007-0314.

[10] I. A. Rus, A. Petruşel, and G. Petruşel, "Fixed point theorems for set-valued Y-contractions," *In: Fixed Point Theory and its Applications, Banach Center Publ.*, vol. 77, pp. 227–237, 2007.

[11] R. Truşcă, "Some local fixed point theorems and applications to open mapping principles and continuation results," *Arab. J. Math.*, vol. 10, no. 3, pp. 711–723, 2021, doi: 10.1007/s40065-021-00331-3.

Authors' addresses

Mădălina Moga

Babeş-Bolyai University Cluj-Napoca, Faculty of Mathematics and Computer Science, Kogălniceanu Street, No. 1, 400084 Cluj-Napoca, Romania

E-mail address: madalina.moga@math.ubbcluj.ro

Radu Truşcă

(**Corresponding author**) Babeş-Bolyai University Cluj-Napoca, Faculty of Mathematics and Computer Science, Kogălniceanu Street, No. 1, 400084 Cluj-Napoca, Romania

E-mail address: radu.trusca@math.ubbcluj.ro

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 907–919

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.3617

SOLVABILITY OF INTEGRAL EQUATIONS VIA THE TECHNIQUE OF MEASURES OF NONCOMPACTNESS

BABAK MOHAMMADI, MOHAMMAD MURSALEEN, AND VAHID PARVANEH

Received 26 December, 2020

Abstract. Motivated by the work of [Mohammadi et al., Mathematics, 2019, 7, 575.], we extend here Darbo's fixed point theorem in a Banach space using the combined technique of Wardowski-Mizoguchi-Takahashi contraction. The existence of solution for a system of integral equations is provided and an example to support the effectiveness of our results is also given. Our results generalize and extend some famous comparable obtained results.

2010 *Mathematics Subject Classification:* 47H09; 47H10

Keywords: measure of noncompactness, Darbo's fixed point theorem, Wardowski-Mizoguchi-Takahashi contraction, integral equations

1. Introduction and preliminaries

The first "measure of non-compactness (MNC, for short)" was defined by Kuratowski [12], that is, for any bounded set Ω in a metric space, it is the infimum of all the numbers $\varepsilon > 0$ such that Ω can be covered by a finite number of sets with diameters $< \varepsilon$. The Hausdorff measure is another MNC, which is defined as

$$\mu(\Omega) = \inf\{\varepsilon > 0 : \Omega \subset \cup_{i=1}^{n} B(x_i, r_i) : x_i \in \Omega, r_i < \varepsilon\,(i = 1, 2, \ldots, n), n \in \mathbb{N}\}.$$

In the study of infinite systems of differential equations (ISDEs), the MNC plays a signifying role. In recent times, the MNC has been effectively applied in sequence spaces and function spaces for variant classes of differential equations. One of the most applicable results in organizing the existence of solutions of differential equations, integral equations and integro-differential equations is Darbo's fixed point theorem [10]. The controllability problem of dynamical systems represented by implicit differential equations (see [3]) is another field in which Darbo's fixed-point theorem can be used. Schauder's fixed point principle [1] and Darbo's fixed point theorem [10] are two important results in this study.

For more details on applications of MNC, we refer the reader to [1, 2, 4–8, 11, 14, 15, 17].

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

In this paper, first we generalize the Darbo's fixed point theorem in a Banach space via Wardowski-Mizoguchi-Takahashi combined contraction and secondly we study the existence of solution for the following system of integral equations:

$$\begin{cases} \rho(\iota) = f\Big(\iota, h(\iota, \rho(\theta(\iota)), \sigma(\theta(\iota))), \displaystyle\int_0^{\zeta(\iota)} g\Big(\iota, \kappa, \rho(\theta(\kappa)), \sigma(\theta(\kappa))\Big) d\kappa\Big) \\ \sigma(\iota) = f\Big(\iota, h(\iota, \sigma(\theta(\iota)), \rho(\theta(\iota))), \displaystyle\int_0^{\zeta(\iota)} g\Big(\iota, \kappa, \sigma(\theta(\kappa)), \rho(\theta(\kappa))\Big) d\kappa\Big) \end{cases} \tag{1.1}$$

where $\iota \in [0,T]$.

We gather several concepts which are used throughout the text. We denote by $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+ = [0,+\infty)$. Let $(\mathfrak{B}, \|\cdot\|)$ denote a real Banach space. In addition, let $\overline{B}(\rho, r)$ be the closed ball with center ρ and radius r and $\overline{B}_r := \overline{B}(0,r)$. For a nonempty subset Q of $\mathfrak{B}$, let $\overline{Q}$ and $Conv(Q)$ denote the closure and the closed convex hull of Q, respectively. Let $\mathcal{NB}_{\mathfrak{B}}$ be the collection of all nonempty bounded subsets of $\mathfrak{B}$ and $\mathcal{RC}_{\mathfrak{B}}$ be the family of all relatively compact subsets of $\mathfrak{B}$.

Definition 1 ([8]). $\chi\colon \mathcal{NB}_{\mathfrak{B}} \longrightarrow \mathbb{R}_+$ is called a "measure of noncompactness" in $\mathfrak{B}$ if:

1° $ker\chi = \{Q \in \mathcal{NB}_{\mathfrak{B}} : \chi(Q) = 0\}$ is nonempty and $ker\chi \subset \mathcal{RC}_{\mathfrak{B}}$;
2° $Q \subset \Delta \Longrightarrow \chi(Q) \leq \chi(\Delta)$;
3° $\chi(\overline{Q}) = \chi(Q)$;
4° $\chi(Conv(Q)) = \chi(Q)$;
5° $\chi(\lambda Q + (1-\lambda)\Delta) \leq \lambda\chi(Q) + (1-\lambda)\chi(\Delta)$ for all $\lambda \in [0,1]$;
6° for all sequence $\{Q_n\}$ of closed sets in $\mathcal{NB}_{\mathfrak{B}}$ with the reservations $Q_{n+1} \subset Q_n$ for all $n = 1,2,3,\cdots$, and $\lim\limits_{n\to\infty} \chi(Q_n) = 0$, then $Q_\infty = \cap_{n=1}^{\infty} Q_n \neq \varnothing$.

2. Main results

Recall the Mizoguchi-Takahashi mapping.

Definition 2 ([13]). $Ł\colon (0,\infty) \longrightarrow [0,1)$ with the restriction $\limsup\limits_{\omega\longrightarrow t^+} Ł(\omega) < 1$, for any $t \geq 0$ is called a Mizoguchi-Takahashi (MT) mapping. We denote this class by $\mathcal{M}-\mathcal{T}$.

Let Π be the collection of all maps $\pi\colon [0,\infty) \longrightarrow [0,\infty)$ so that

(1) $\pi(\eta_n) \to 0$ if and only if $\eta_n \to 0$ for any sequence $\{\eta_n\}$ in $[0,\infty)$;
(2) π is a nondecreasing mapping.

In this paper, some enlargements of Darbo's fixed point theorem via Wardowski type Mizoguchi-Takahashi's contractions have been obtained. Moreover, we provide an application for a system of functional integral equations.

Let $\mathbb{A}$ be the family of all functions $\Gamma\colon (0,\infty) \to \mathbb{R}$ so that:

(Γ_1) Γ is continuous and increasing;

(Γ_2) $\lim_{n\to\infty} \iota_n = 1$ iff $\lim_{n\to\infty} \Gamma(\iota_n) = 0$ for all $\{\iota_n\} \subseteq (0,\infty)$.

Note that from (Γ_1) and (Γ_2), we have $\Gamma(1) = 0$. Some examples of elements in $\mathbb{A}$ are as follows:

(i) $\Gamma_1(\iota) = \ln(\iota)$,
(ii) $\Gamma_2(\iota) = \ln(\iota - \frac{1}{\iota} + 1)$,
(iii) $\Gamma_3(\iota) = -\frac{1}{\sqrt{\iota}} + 1$,
(iv) $\Gamma_5(\iota) = -\frac{1}{\iota} + \iota$.

Here and afterwards we take C as a nonempty, bounded, closed and convex subset of a Banach space $\mathfrak{B}$ and $\mathcal{G}\colon C \to C$ a continuous operator.

Theorem 1. *Let*

$$\Gamma(\pi(\chi(\mathcal{G}(Q)))) \leq \Gamma(Ł(\pi(\chi(Q)))) + \Gamma(\pi(\chi(Q)))$$

for all $Q \subseteq C$ with $\chi(Q) \neq 0$ and $\chi(\mathcal{G}(Q)) \neq 0$ where $\Gamma \in \mathbb{A}$, $Ł \in \mathcal{M} - \mathcal{T}$, $\pi \in \Pi$ and χ is an arbitrary MNC. Then $\mathcal{G}$ has at least one fixed point in C.

Proof. Let $\{C_n\}$ be defined by $C_0 = C$ and $C_{n+1} = \overline{Conv}(\mathcal{G}(C_n))$ for all $n \in \mathbb{N}$.

Assume that there exists an integer $N \in \mathbb{N}$ such that $\chi(C_N) = 0$. Then C_N is relatively compact and so according to the Schauder Theorem $\mathcal{G}$ admits a fixed point. So, let $\chi(C_n) > 0$ for each $n \in \mathbb{N}$.

Obviously, $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of nonempty, bounded, closed and convex sets such that

$$C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n \supseteq C_{n+1}.$$

On the other hand

$$\Gamma(\pi(\chi(C_{n+1}))) = \Gamma(\pi(\chi(\mathcal{G}(C_n)))) \leq \Gamma(Ł(\pi(\chi(C_n)))) + \Gamma(\pi(\chi(C_n))), \tag{2.1}$$

for each $n \in \mathbb{N}$.

According to the properties of measure of noncompactness χ and the properties of function π, $\{\pi(\chi(C_n)\}$ is a positive decreasing and bounded below sequence of real numbers.

Thus, $\{\pi(\chi(C_n)\}_{n\in\mathbb{N}}$ is a convergent sequence. Suppose that $\lim_{n\to\infty} \pi(\chi(C_n)) = r$.

We prove that $r = 0$. Suppose to the contrary that $r > 0$. Taking the upper limit in (2.1) when $n \to \infty$, we obtain that

$$\Gamma(r) \leq \Gamma(\limsup_{n\to\infty} Ł(\pi(\chi(C_n)))) + \Gamma(r) < \Gamma(r),$$

which is a contradiction. So, $\lim_{n\to\infty} \pi(\chi(C_n)) = r = 0$.

Therefore $\lim_{n\to\infty} \chi(C_n) = 0$. According to principle (6°) of Definition 1 we derive that the set $C_\infty = \bigcap_{n=1}^{\infty} C_n$ is a nonempty, closed and convex set and it is invariant under $\mathcal{G}$ and belongs to $Ker\chi$. Then, in the light of the Schauder Theorem $\mathcal{G}$ possesses a fixed point. $\square$

Taking $\Gamma(\iota) = \ln(\iota)$ in Theorem 1, we have

Corollary 1. *Let*

$$\pi(\chi(\mathcal{G}(\mathcal{Q}))) \leq Ł(\pi(\chi(\mathcal{Q})))\pi(\chi(\mathcal{Q})) \tag{2.2}$$

where $\mathcal{Q} \subseteq C$ with $\chi(\mathcal{Q}) \neq 0$ and $\chi(\mathcal{G}(\mathcal{Q})) \neq 0$, $Ł \in \mathcal{M} - \mathcal{T}$, $\pi \in \Pi$ and χ is an arbitrary MNC. Then $\mathcal{G}$ possesses at least one fixed point in C.

Taking $\Gamma(\iota) = -\frac{1}{\sqrt{\iota}} + 1$ and if π be the identity function in Theorem 1, we have:

Corollary 2. *Let*

$$\chi(\mathcal{G}(\mathcal{Q})) \leq \frac{Ł(\chi(\mathcal{Q}))\chi(\mathcal{Q})}{(\sqrt{Ł(\chi(\mathcal{Q}))} + \sqrt{\chi(\mathcal{Q})} - \sqrt{\chi(\mathcal{Q})Ł(\chi(\mathcal{Q}))})^2} \tag{2.3}$$

where $\mathcal{Q} \subseteq C$ with $\chi(\mathcal{Q}) \neq 0$ and $\chi(\mathcal{G}(\mathcal{Q})) \neq 0$, $Ł \in \mathcal{M} - \mathcal{T}$ and χ is an arbitrary MNC. Then $\mathcal{G}$ has at least one fixed point in C.

3. Coupled fixed point

Definition 3 ([9])**.** The ordered pair $(\rho, \sigma) \in \mathfrak{B}^2$ is called a coupled fixed point of a mapping $\mathcal{G} : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ if $\mathcal{G}(\rho, \sigma) = \rho$ and $\mathcal{G}(\sigma, \rho) = \sigma$.

Theorem 2 ([8])**.** *Let $\chi_1, \chi_2, \ldots, \chi_n$ be measures of noncompactness in Banach spaces $\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_n$, respectively, and let $\mathcal{G}\colon [0,\infty)^n \longrightarrow [0,\infty)$ be a convex mapping with the condition $\mathcal{G}(\rho_1, \ldots, \rho_n) = 0$ if and only if $\rho_i = 0$ for all $i = 1, 2, \ldots, n$. Then*

$$\tilde{\chi}(\mathcal{Q}) = \mathcal{G}(\chi_1(\mathcal{Q}_1), \chi_2(\mathcal{Q}_2), \ldots, \chi_n(\mathcal{Q}_n)),$$

is a measure of noncompactness in $\mathfrak{B}_1 \times \mathfrak{B}_2 \times \ldots \times \mathfrak{B}_n$, where $\mathcal{Q}_i$ is the natural projection of $\mathcal{Q}$ into $\mathfrak{B}_i$, for all $i = 1, 2, \ldots, n$.

Now, let Γ be a subadditive mapping. As in the previous section, let C be a nonempty, bounded, closed and convex subset of a Banach space $\mathfrak{B}$ and $\mathcal{G}\colon C \times C \to C$ be a continuous function. Then we have the following results.

Theorem 3. *Let*

$$\Gamma\Big(\pi\big(\chi(\mathcal{G}(\mathcal{Q}_1 \times \mathcal{Q}_2))\big)\Big) \leq \frac{1}{2}\Big[\Gamma\big(Ł(\pi(\chi(\mathcal{Q}_1) + \chi(\mathcal{Q}_2)))\big) + \Gamma\big(\pi(\chi(\mathcal{Q}_1) + \chi(\mathcal{Q}_2))\big)\Big] \tag{3.1}$$

for all subsets $\mathcal{Q}_1, \mathcal{Q}_2$ of C with $\chi(\mathcal{G}(\mathcal{Q}_1 \times \mathcal{Q}_2)) \neq 0$ and $\chi(\mathcal{Q}_1)$ or $\chi(\mathcal{Q}_2) \neq 0$, where $\Gamma \in \mathbb{A}$, $Ł \in \mathcal{M} - \mathcal{T}$, $\pi \in \Pi$ and χ is an arbitrary MNC. Then $\mathcal{G}$ has at least a coupled fixed point.

Proof. Define $\widetilde{\mathcal{G}}\colon C^2 \to C^2$ by

$$\widetilde{\mathcal{G}}(\rho, \sigma) = (\mathcal{G}(\rho, \sigma), \mathcal{G}(\sigma, \rho)).$$

Clearly, $\widetilde{\mathcal{G}}$ is continuous.

Let $Q \subset C^2$ be a nonempty subset. Evidently, $\widetilde{\chi}(Q) = \chi(Q_1) + \chi(Q_2)$ is a (MNC) [8], where Q_1 and Q_2 are the natural projections of Q into $\mathfrak{B}$. From (3.1) we have

$$\begin{aligned}
\Gamma\Big(\pi(\widetilde{\chi}(\widetilde{\mathcal{G}}(Q)))\Big) &= \Gamma\Big(\pi(\widetilde{\chi}(\mathcal{G}(Q_1 \times Q_2) \times \mathcal{G}(Q_2 \times Q_1)))\Big) \\
&= \Gamma\Big(\pi(\chi(\mathcal{G}(Q_1 \times Q_2)) + \chi(\mathcal{G}(Q_2 \times Q_1)))\Big) \\
&\leq \frac{1}{2}\Big[\Gamma\big(Ł(\pi(\chi(Q_1) + \chi(Q_2)))\big) + \Gamma\big(\pi(\chi(Q_1) + \chi(Q_2))\big)\Big] \\
&\quad + \frac{1}{2}\Big[\Gamma\big(Ł(\pi(\chi(Q_1) + \chi(Q_2)))\big) + \Gamma\big(\pi(\chi(Q_1) + \chi(Q_2))\big)\Big] \\
&\leq \Gamma\Big(Ł(\pi(\chi(Q_1) + \chi(Q_2)))\Big) + \Gamma\Big(\pi(\chi(Q_1) + \chi(Q_2))\Big) \\
&= \Gamma\Big(Ł(\pi(\widetilde{\chi}(Q)))\Big) + \Gamma\Big(\pi(\widetilde{\chi}(Q))\Big).
\end{aligned}$$

Now, according to Theorem 1 we derive that $\widetilde{\mathcal{G}}$ admits at least a fixed point which yields that $\mathcal{G}$ possesses at least a coupled fixed point. □

Theorem 4. *Suppose that*

$$\Gamma\Big(\pi\big(\chi(\mathcal{G}(Q_1 \times Q_2))\big)\Big) \leq \Gamma\big(Ł(\pi(\max\{\chi(Q_1), \chi(Q_2)\}))\big) + \Gamma\big(\pi(\max\{\chi(Q_1), \chi(Q_2)\})\big) \tag{3.2}$$

for all subsets Q_1, Q_2 of C with $\chi(\mathcal{G}(Q_1 \times Q_2)) \neq 0$ and $\chi(Q_1)$ or $\chi(Q_2) \neq 0$, where $\Gamma \in \mathbb{A}$ is a mapping, $Ł \in \mathcal{M} - \mathcal{T}$, $\pi \in \Pi$ is also a mapping and χ is an arbitrary MNC. Then $\mathcal{G}$ has at least a coupled fixed point.

Proof. Define the mapping $\widetilde{\mathcal{G}} \colon C^2 \to C^2$ by

$$\widetilde{\mathcal{G}}(\rho, \sigma) = (\mathcal{G}(\rho, \sigma), \mathcal{G}(\sigma, \rho)).$$

It is clear that $\widetilde{\mathcal{G}}$ is continuous. Clearly, $\widetilde{\chi}(Q) = \max\{\chi(Q_1), \chi(Q_2)\}$ is an MNC, where Q_1 and Q_2 denote the natural projections of Q into $\mathfrak{B}$. Let $Q \subset C^2$ be a nonempty subset. From (3.2) we obtain that

$$\begin{aligned}
\Gamma\Big(\pi(\widetilde{\chi}(\widetilde{\mathcal{G}}(Q)))\Big) &= \Gamma\Big(\pi(\widetilde{\chi}(\mathcal{G}(Q_1 \times Q_2) \times \mathcal{G}(Q_2 \times Q_1)))\Big) \\
&= \Gamma\Big(\pi(\max\{\chi(\mathcal{G}(Q_1 \times Q_2)), \chi(\mathcal{G}(Q_2 \times Q_1))\})\Big) \\
&= \max\Big\{\Gamma\Big(\pi(\chi(\mathcal{G}(Q_1 \times Q_2)))\Big), \Gamma\Big(\pi(\chi(\mathcal{G}(Q_2 \times Q_1)))\Big)\Big\} \\
&\leq \Gamma\Big(Ł(\pi(\max\{\chi(Q_1), \chi(Q_2)\}))\Big) + \Gamma\Big(\pi(\max\{\chi(Q_1), \chi(Q_2)\})\Big) \\
&= \Gamma\Big(Ł(\pi(\widetilde{\chi}(Q)))\Big) + \Gamma\Big(\pi(\widetilde{\chi}(Q))\Big).
\end{aligned}$$

Theorem 1 deduces that $\widetilde{\mathcal{G}}$ has at least a fixed point and so, $\mathcal{G}$ has at least a coupled fixed point. □

4. Application

Now, we study the existence of solutions for the following system of equations:

$$\begin{cases} \rho(\iota) = f\Big(\iota, h(\iota, \rho(\theta(\iota)), \sigma(\theta(\iota))), \displaystyle\int_0^{\zeta(\iota)} g\Big(\iota, \kappa, \rho(\theta(\kappa)), \sigma(\theta(\kappa))\Big) d\kappa\Big) \\ \sigma(\iota) = f\Big(\iota, h(\iota, \sigma(\theta(\iota)), \rho(\theta(\iota))), \displaystyle\int_0^{\zeta(\iota)} g\Big(\iota, \kappa, \sigma(\theta(\kappa)), \rho(\theta(\kappa))\Big) d\kappa\Big) \end{cases} \tag{4.1}$$

in the space $C([0,T])$ consisting of all bounded and continuous real functions ρ on $[0,T]$ with $\|\rho\| = \sup\{|\rho(\iota)| : 0 \le \iota \le T\}$.

The modulus of the continuity of $\rho \in C([0,T])$ is defined by

$$\omega(\rho, \varepsilon) = \sup\{|\rho(\iota) - \rho(\kappa)| : \iota, \kappa \in [0,T], |\iota - \kappa| \le \varepsilon\}.$$

Let

$$\omega(Q, \varepsilon) = \sup\{\omega(\rho, \varepsilon) : \rho \in Q\},$$

$$\omega_0(Q) = \lim_{\varepsilon \to 0} \omega(Q, \varepsilon).$$

It is well known that the above function is a measure of noncompactness in the space $Q = C([0,T])$.

Theorem 5. *Suppose that:*

(i) $\theta, \zeta : [0,T] \longrightarrow [0,T]$ *are continuous functions,*

(ii) *The functions* $f, h : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *are continuous and*

$$\Gamma(|f(\iota, \rho_1, \rho_2) - f(\iota, \sigma_1, \sigma_2)|) \le \Gamma(Ł(\max\{|\rho_1 - \sigma_1|, |\rho_2 - \sigma_2|\})) + \Gamma(\max\{|\rho_1 - \sigma_1|, |\rho_2 - \sigma_2|\})$$

and

$$|h(\iota_1, \rho_1, \rho_2) - h(\iota_2, \sigma_1, \sigma_2)| \le \max\{|\iota_1 - \iota_2|, |\rho_1 - \sigma_1|, |\rho_2 - \sigma_2|\},$$

for all $\iota_1, \iota_2 \in [0,T]$ *and* $\rho_1, \rho_2, \sigma_1, \sigma_2 \in \mathbb{R}$,

(iii) $M := \sup\{\big|f(\iota, 0, 0)\big| : \iota \in [0,T]\}$ *and* $N := \sup\{\big|h(\iota, 0, 0)\big| : \iota \in [0,T]\}$,

(iv) $g : [0,T] \times [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is bounded and continuous and*

$$G := \sup\Big\{\Big|\int_0^{\zeta(\iota)} g(\iota, \kappa, \rho(\theta(\kappa)), \sigma(\theta(\kappa))) d\kappa\Big| : \iota, \kappa \in [0,T], \rho, \sigma, \in C([0,T])\Big\},$$

(v) *There exists a positive solution* r_0 *to the inequality*

$$\Gamma^{-1}[\Gamma(Ł(\max\{r + N, G\})) + \Gamma(\max\{r + N, G\})] + M \le r.$$

Then the system of integral equations (4.1) *has at least one solution in the space* $(C([0,T]))^2$.

Proof. Let

$$\mathcal{G}\colon C([0,T])\times C([0,T])\longrightarrow C([0,T])$$

be defined by

$$\mathcal{G}(\rho,\sigma)(\iota)=f\Big(\iota,h(\iota,\rho(\theta(\iota)),\sigma(\theta(\iota))),\int_0^{\zeta(\iota)}g\Big(\iota,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa))\Big)d\kappa\Big). \quad (4.2)$$

We observe that the function $\mathcal{G}$ is continuous. Applying the assumptions (i) – (iv) we have

$$\begin{aligned}
&\Big|\mathcal{G}(\rho,\sigma)(\iota)\Big|\\
&\le\Big|f\big(\iota,h(\iota,\rho(\theta(\iota)),\sigma(\theta(\iota))),\int_0^{\zeta(\iota)}g(\iota,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\big)-f\big(\iota,0,0\big)\Big|+\Big|f\big(\iota,0,0\big)\Big|\\
&\le\Gamma^{-1}\Bigg[\Gamma\Big(Ł(\max\{\Big|h(\iota,\rho(\theta(\iota)),\sigma(\theta(\iota)))\Big|,\Big|\int_0^{\zeta(\iota)}g(\iota,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\Big|\})\Big)\\
&\quad+\Gamma\Big(\max\{\Big|h(\iota,\rho(\theta(\iota)),\sigma(\theta(\iota)))\Big|,\Big|\int_0^{\zeta(\iota)}g(\iota,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\Big|\}\Big)\Bigg]+\Big|f\big(\iota,0,0\big)\Big|\\
&\le\Gamma^{-1}\Bigg[\Gamma\Big(Ł(\max\{\max\{|\rho(\theta(\iota))|,|\sigma(\theta(\iota))|\}+|h(\iota,0.0)|,G\}\Big)\\
&\quad+\Gamma\Big(\max\{\max\{|\rho(\theta(\iota))|,|\sigma(\theta(\iota))|\}+|h(\iota,0.0)|,G\}\Big)\Bigg]+\Big|f\big(\iota,0,0\big)\Big|\\
&\le\Gamma^{-1}\Bigg[\Gamma\Big(Ł(\max\{\max\{||\rho||,||\sigma||\}+N,G\}\Big)+\Gamma\Big(\max\{\max\{||x||,||y||\}+N,G\}\Big)\Bigg]+M.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathcal{G}(\rho,\sigma)\|\le\Gamma^{-1}[&\Gamma(Ł(\max\{\max\{||\rho||,||\sigma||\}+N,G\}))\\
&+\Gamma(\max\{\max\{||\rho||,||\sigma||\}+N,G\})]+M. \quad (4.3)
\end{aligned}$$

Due to inequality (4.3) and using (v), the function $\mathcal{G}$ maps $(\bar{B}_{r_0})^2$ into $\bar{B}_{r_0}$.

For the proof of continuity of G on $\bar{B}_{r_0}$ we refer the reader to [16].

Let Q_1 and Q_2 are nonempty and bounded subsets of $\bar{B}_{r_0}$, and assume that $\varepsilon>0$ is an arbitrary constant. Let $\iota_1,\iota_2\in[0,T]$, with $|\iota_1-\iota_2|\le\varepsilon$ and $\zeta(\iota_1)\le\zeta(\iota_2)$ and $(\rho,\sigma)\in Q_1\times Q_2$. Then we have

$$\begin{aligned}
&\Big|\mathcal{G}(\rho,\sigma)(\iota_2)-\mathcal{G}(\rho,\sigma)(\iota_1)\Big| \quad (4.4)\\
&\le\Big|f\big(\iota_2,h(\iota_2,\rho(\theta(\iota_2)),\sigma(\theta(\iota_2))),\int_0^{\zeta(\iota_2)}g(\iota_2,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\big)\\
&\quad-f\big(\iota_2,h(\iota_1,\rho(\theta(\iota_1)),\sigma(\theta(\iota_1))),\int_0^{\zeta(\iota_2)}g(\iota_2,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\big)|
\end{aligned}$$

$$
\begin{aligned}
&+|f\big(\iota_2,h(\iota_1,\rho(\theta(\iota_1)),\sigma(\theta(\iota_1))),\int_0^{\zeta(\iota_2)} g(\iota_2,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\big)\\
&-f\big(\iota_1,h(\iota_1,\rho(\theta(\iota_1)),\sigma(\theta(\iota_1))),\int_0^{\zeta(\iota_2)} g(\iota_2,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\big)|\\
&+|f\big(\iota_1,h(\iota_1,\rho(\theta(\iota_1)),\sigma(\theta(\iota_1))),\int_0^{\zeta(\iota_2)} g(\iota_2,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\big)\\
&-f\big(\iota_1,h(\iota_1,\rho(\theta(\iota_1)),\sigma(\theta(\iota_1))),\int_0^{\zeta(\iota_2)} g(\iota_1,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\big)|\\
&+|f\big(\iota_1,h(\iota_1,\rho(\theta(\iota_1)),\sigma(\theta(\iota_1))),\int_0^{\zeta(\iota_2)} g(\iota_1,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\big)\\
&-f\big(\iota_1,h(\iota_1,\rho(\theta(\iota_1)),\sigma(\theta(\iota_1))),\int_0^{\zeta(\iota_1)} g(\iota_1,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\big)|\\
\le\ &\Gamma^{-1}\Big(\Gamma\Big(Ł\big(\{|h(\iota_2,\rho(\theta(\iota_2)),\sigma(\theta(\iota_2)))-h(\iota_1,\rho(\theta(\iota_1)),\sigma(\theta(\iota_1)))|\}\big)\Big)\\
&+\Gamma\Big(\{|h(\iota_2,\rho(\theta(\iota_2)),\sigma(\theta(\iota_2)))-h(\iota_1,\rho(\theta(\iota_1)),\sigma(\theta(\iota_1)))|\}\Big)\Big)\\
&+\omega_{r_0,\overline{G}}(f,\varepsilon)+\Gamma^{-1}\Big[\Gamma\Big(Ł\big(\Big|\int_{\zeta(\iota_1)}^{\zeta(\iota_2)} g(\iota_1,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\Big|\big)\Big)\\
&+\Gamma\Big(\Big|\int_{\zeta(\iota_1)}^{\zeta(\iota_2)} g(\iota_1,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))d\kappa\Big|\Big)\Big]\\
&+\Gamma^{-1}\Big[\Gamma\Big(Ł\big(\Big|\int_0^{\zeta(\iota_2)} \big|g(\iota_1,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))-g(\iota_2,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))\big|d\kappa\Big|\big)\Big)\\
&+\Gamma\Big(\Big|\int_0^{\zeta(\iota_2)} \big|g(\iota_1,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))-g(\iota_2,\kappa,\rho(\theta(\kappa)),\sigma(\theta(\kappa)))\big|d\kappa\Big|\Big)\Big]\\
\le\ &\Gamma^{-1}\Big(\Gamma\Big(Ł\big(\max\{|\iota_2-\iota_1|,|\rho(\theta(\iota_2))-\rho(\theta(\iota_1))|,|\sigma(\theta(\iota_2))-\sigma(\theta(\iota_1)))|\}\big)\Big)\\
&+\Gamma\Big(\max\{|\iota_2-\iota_1|,|\rho(\theta(\iota_2))-\rho(\theta(\iota_1))|,|\sigma(\theta(\iota_2))-\sigma(\theta(\iota_1))|\}\Big)\Big)\\
&+\omega_{r_0,\overline{G}}(f,\varepsilon)+\Gamma^{-1}\Big[\Gamma\Big(Ł\big(U_{r_0}\omega(\zeta,\varepsilon)\big)\Big)+\Gamma\Big(U_{r_0}\omega(\zeta,\varepsilon)\Big)\Big]\\
&+\Gamma^{-1}\Big[\Gamma\Big(Ł\big(\zeta_T\omega_{r_0}(g,\varepsilon)\big)\Big)+\Gamma\Big(\zeta_T\omega_{r_0}(g,\varepsilon)\Big)\Big]
\end{aligned}
$$

where

$$\omega(\zeta,\varepsilon)=\sup\{|\zeta(\iota_2)-\zeta(\iota_1)|:\iota_1,\iota_2\in[0,T],\ |\iota_2-\iota_1|\le\varepsilon\},$$

$$\zeta_T=\sup\{\zeta(\iota):\iota\in[0,T]\},$$

$$U_{r_0}=\sup\{|g(\iota,\kappa,\mu_1,\mu_2)|:\iota\in[0,T],\kappa\in[0,\zeta_T],\ \mu_1,\mu_2\in[-r_0,r_0]\},$$

$$\overline{G}=\zeta_T\sup\{|g(\iota,\kappa,\mu_1,\mu_2)|:\iota\in[0,T],\kappa\in[0,\zeta_T],\ \mu_1,\mu_2\in[-r_0,r_0]\},$$

$$H = \sup\{|h(\iota,\mu_1,\mu_2)| : \iota \in [0,T], \mu_1,\mu_2 \in [-r_0,r_0]\},$$

$$\begin{aligned}\omega_{r_0,\overline{G}}(f,\varepsilon) = \sup\{&|f(\iota_2,\mu,z) - f(\iota_1,\mu,z)| : \iota_1,\iota_2 \in [0,T],\\ &|\iota_2-\iota_1| \le \varepsilon,\ \mu \in [-H,H],\ \ z \in [-\overline{G},\overline{G}]\}\end{aligned}$$

and

$$\begin{aligned}\omega_{r_0}(g,\varepsilon) = \sup\{&|g(\iota_2,\kappa,\mu_1,\mu_2) - g(\iota_1,\kappa,\mu_1,\mu_2)| : \iota_1,\iota_2 \in [0,T],\\ &|\iota_2-\iota_1| \le \varepsilon,\ \mu_1,\mu_2 \in [-r_0,r_0], \kappa \in [0,\zeta_T]\}.\end{aligned}$$

Since (ρ,σ) was an arbitrary element of $Q_1 \times Q_2$ in (4.4), we have

$$\begin{aligned}\omega(\mathcal{G}(Q_1 \times Q_2),\varepsilon) \le \Gamma^{-1}\Big[&\Gamma\Big(Ł\big(\max\{\varepsilon,\omega(\rho,\varepsilon),\omega(\sigma,\varepsilon)\}\big)\Big)\\ &+\Gamma\Big(\max\{\varepsilon,\omega(\rho,\varepsilon),\omega(\sigma,\varepsilon)\}\Big)\Big] + \omega_{r_0,\overline{G}}(f,\varepsilon)\\ &+\Gamma^{-1}\Big[\Gamma\Big(Ł\big(U_{r_0}\omega(\zeta,\varepsilon)\big)\Big) + \Gamma\Big(U_{r_0}\omega(\zeta,\varepsilon)\Big)\Big]\\ &+\Gamma^{-1}\Big[\Gamma\Big(Ł\big(\zeta_T\omega_{r_0}(g,\varepsilon)\big)\Big) + \Gamma\Big(\zeta_T\omega_{r_0}(g,\varepsilon)\Big)\Big].\end{aligned}$$

Moreover, in the light of the uniform continuity of the functions f and g on

$$[0,T] \times [-H,H] \times [-\overline{G},\overline{G}]$$

and

$$[0,T] \times [0,\zeta_T] \times [-r_0,r_0] \times [-r_0,r_0]$$

respectively, $\omega_{r_0,\overline{G}}(f,\varepsilon) \longrightarrow 0$ and $\omega_{r_0}(g,\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$. Also, because of the uniform continuity of ζ on $[0,T]$, $\omega(\zeta,\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$.

Now, this remarks and the above inequality via taking the sup imply that

$$\begin{aligned}\omega(\mathcal{G}(Q_1 \times Q_2),\varepsilon) \le \Gamma^{-1}\Big[&\Gamma\Big(Ł\big(\max\{\omega(Q_1,\varepsilon),\omega(Q_2,\varepsilon)\}\big)\Big)\\ &+\Gamma\Big(\max\{\omega(Q_1,\varepsilon),\omega(Q_2,\varepsilon)\}\Big)\Big]\end{aligned}$$

which by tending $\varepsilon \to 0$ and using the continuity of Ł implies that

$$\begin{aligned}\omega_0(\mathcal{G}(Q_1 \times Q_2)) \le \Gamma^{-1}\Big[&\Gamma\Big(Ł\big(\max\{\omega_0(Q_1),\omega_0(Q_2)\}\big)\Big)\\ &+\Gamma\Big(\max\{\omega_0(Q_1),\omega_0(Q_2)\}\Big)\Big].\end{aligned}$$

Therefore,

$$\begin{aligned}\Gamma\Big(\omega_0(\mathcal{G}(Q_1 \times Q_2))\Big) \le \Gamma\Big(&Ł\big(\max\{\omega_0(Q_1),\omega_0(Q_2)\}\big)\Big)\\ &+\Gamma\Big(\max\{\omega_0(Q_1),\omega_0(Q_2)\}\Big).\end{aligned}$$

Thus, Theorem 4 yields that the operator $\mathcal{G}$ has a coupled fixed point. Thus, the system (4.1) has at least one solution in $(C([0,T]))^2$. □

5. Example

Example 1. Let:

$$\begin{cases} \rho(\iota) = \frac{1}{7}e^{-\iota^2} + \frac{1}{2}\dfrac{\frac{\iota}{3} + \frac{tanh(\rho(\iota)+\sigma(\iota))}{3}}{1+\frac{\iota}{3}+\frac{tanh(\rho(\iota)+\sigma(\iota))}{3}} + \dfrac{1}{2}\dfrac{\int_0^{\iota^2}\frac{e^{-(\iota+\kappa)}sech^2(\rho(\kappa)+\sigma^2(\kappa))}{e^{(2\rho^2(\kappa)+\sigma(\kappa))}}d\kappa}{1+\int_0^{\iota^2}\frac{e^{-(\iota+\kappa)}sech^2(\rho(\kappa)+\sigma^2(\kappa))}{e^{(2\rho^2(\kappa)+\sigma(\kappa))}}d\kappa} \\ \sigma(\iota) = \frac{1}{7}e^{-\iota^2} + \frac{1}{2}\dfrac{\frac{\iota}{3} + \frac{tanh(\rho(\iota)+\sigma(\iota))}{3}}{1+\frac{\iota}{3}+\frac{tanh(\rho(\iota)+\sigma(\iota))}{3}} + \dfrac{1}{2}\dfrac{\int_0^{\iota^2}\frac{e^{-(\iota+\kappa)}sech^2(\sigma(\kappa)+\rho^2(\kappa))}{e^{(2\sigma^2(\kappa)+\rho(\kappa))}}d\kappa}{1+\int_0^{\iota^2}\frac{e^{-(\iota+\kappa)}sech^2(\sigma(\kappa)+\rho^2(\kappa))}{e^{(2\sigma^2(\kappa)+\rho(\kappa))}}d\kappa}. \end{cases} \tag{5.1}$$

The above system is a case of (4.1) with

$$\theta(\iota) = \iota,\ \zeta(\iota) = \iota^2, \qquad \iota \in [0,1],$$

$$f(\iota,\rho,\sigma) = \frac{1}{7}e^{-\iota^2} + \frac{1}{2}\cdot\frac{\rho}{1+\rho} + \frac{1}{2}\cdot\frac{\sigma}{1+\sigma},$$

$$g(\iota,\kappa,\rho,\sigma) = \frac{e^{-(\iota+\kappa)}sech^2(\rho+\sigma^2)}{e^{(2\rho^2+\sigma)}},$$

and

$$h(\iota,\rho,\sigma) = \frac{\iota}{3} + \frac{\tanh(\rho+\sigma)}{3}.$$

Also, take $Ł(\iota) = \frac{2}{3}$. To prove the existence of a solution for this system, we should survey the conditions (i)-(v) of Theorem 5.

Condition (i) is clearly evident.

Now

$$\begin{aligned} |h(\iota_1,\rho,\sigma) - h(\iota_2,u,v)| &\le \frac{|\iota_1-\iota_2|}{3} + \frac{|tanh(\rho+\sigma) - tanh(u+v)|}{3} \\ &\le \frac{|\iota_1-\iota_2|}{3} + \frac{1}{3}(|\rho-u| + |\sigma-v|) \\ &\le \max\{|\iota_1-\iota_2|, |\rho-u|, |\sigma-v|\} \end{aligned}$$

and

$$\begin{aligned} \big|f(\iota,\rho,\sigma) - f(\iota,u,v)\big| &= \frac{1}{2}\Big|\frac{\rho}{1+\rho} + \frac{\sigma}{1+\sigma} - \frac{u}{1+u} - \frac{v}{1+v}\Big| \\ &\le \frac{1}{2}\Big(\frac{|\rho-u|}{(1+\rho)(1+u)} + \frac{|\sigma-v|}{(1+\sigma)(1+v)}\Big) \end{aligned}$$

$$\leq \frac{1}{2}\left(\frac{|\rho-u|}{1+\frac{1}{2}|\rho-u|}+\frac{|\sigma-v|}{1+\frac{1}{2}|\sigma-v|}\right)$$
$$\leq \frac{\max\{|\rho-u|,|\sigma-v|\}}{1+\frac{1}{2}\max\{|\rho-u|,|\sigma-v|\}}.$$

Therefore,

$$\frac{-1}{\left|f(\iota,\rho,\sigma)-f(\iota,u,v)\right|}+1 \leq \frac{-1}{2}+\frac{-1}{\max\{|\rho-u|,|\sigma-v|\}}+1.$$

Taking $\Gamma(\iota)=\frac{-1}{t}+1$, we have

$$\Gamma(Ł(\max\{|\rho-u|,|\sigma-v|\}))=\Gamma(\frac{2}{3})=\frac{-1}{2}.$$

Thus,

$$\Gamma(\left|f(\iota,\rho,\sigma)-f(\iota,u,v)\right|) \leq \Gamma(Ł(\max\{|\rho-u|,|\sigma-v|\}))+\Gamma(\max\{|\rho-u|,|\sigma-v|\}).$$

We can find that f satisfies condition (ii) of Theorem 5. Also,

$$M=\sup\{|f(\iota,0,0)|:\iota\in[0,1]\}=\sup\{\frac{1}{7}e^{-\iota^2}:\iota\in[0,1]\}=\frac{1}{7}$$

and

$$N=\sup\{|h(\iota,0,0)|:\iota\in[0,1]\}=\sup\{\frac{1}{2}e^{-\iota}:\iota\in[0,1]\}=\frac{1}{2}.$$

Moreover, g is continuous on $[0,1]\times[0,1]\times\mathbb{R}^2$.

On the other hand,

$$G=\sup\{\int_0^{\iota^2} g(\iota,\kappa,\rho,\sigma)d\kappa:\iota\in[0,1],\kappa\in[0,\iota^2]\}$$
$$\leq \sup\{|\int_0^{\iota^2} e^{-(\iota+\kappa)}d\kappa|:\iota\in[0,1],\kappa\in[0,\iota^2]\}$$
$$\leq \sup\{|e^{-\iota}(1-e^{-\iota^2})|:\iota\in[0,1]\}\leq 1.$$

Furthermore, for $r=1$,

$$\Gamma^{-1}[\Gamma(Ł(\max\{r+N,G\}))+\Gamma(\max\{r+N,G\})]+M$$
$$\leq \Gamma^{-1}[\Gamma(Ł(\max\{1+\frac{1}{2},1\}))+\Gamma(\max\{1+\frac{1}{2},1\})]+\frac{1}{7}$$
$$=\Gamma^{-1}[\Gamma(Ł(\frac{3}{2}\}))+\Gamma(\frac{3}{2})]+\frac{1}{7}=\Gamma^{-1}[\Gamma(\frac{2}{3})+\Gamma(\frac{3}{2})]+\frac{1}{7}$$
$$=\Gamma^{-1}[\frac{-1}{2}+\frac{1}{3}]+\frac{1}{7}=\Gamma^{-1}[\frac{-1}{6}]+\frac{1}{7}=\frac{6}{7}+\frac{1}{7}=1=r.$$

Consequently, all the reservations of Theorem 5 are fulfilled. Hence, the system (5.1) has at least one solution in $(C([0,1]))^2$.

REFERENCES

[1] R. P. Agarwal, M. Meehan, and D. O'Regan, *Fixed point theory and applications*. Cambridge University Press, 2001, vol. 141, doi: 10.1017/CBO9780511543005.

[2] J. Ahmad, A. Al-Rawashdeh, and A. Azam, "New fixed point theorems for generalized F-contractions in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2015, no. 1, p. 80, 2015, doi: 10.1186/s13663-015-0333-2.

[3] K. Balachandran and J. Dauer, "Controllability of nonlinear systems via fixed-point theorems," *Journal of optimization theory and applications*, vol. 53, no. 3, pp. 345–352, 1987, doi: 10.1007/BF00938943.

[4] S. Banaei, "Solvability of a system of integral equations of Volterra type in the Fréchet space $L^p_{loc}(\mathbb{R}_+)$ via measure of noncompactness," *Filomat*, vol. 32, no. 15, pp. 5255–5263, 2018, doi: 10.2298/FIL1815255B.

[5] S. Banaei, "An extension of Darbo's theorem and its application to existence of solution for a system of integral equations," *Cogent Mathematics & Statistics*, vol. 6, no. 1, p. 1614319, 2019, doi: 10.1080/25742558.2019.1614319.

[6] S. Banaei and M. B. Ghaemi, "A generalization of the Meir-Keeler condensing operators and its application to solvability of a system of nonlinear functional integral equations of volterra type," *Sahand Communications in Mathematical Analysis*, vol. 15, no. 1, pp. 19–35, 2019, doi: 10.22130/scma.2018.74869.322.

[7] S. Banaei, M. B. Ghaemi, and R. Saadati, "An extension of Darbo's theorem and its application to system of neutral differential equation with deviating argument," *Miskolc Mathematical Notes*, vol. 18, no. 1, pp. 83–94, 2017, doi: 10.18514/MMN.2017.2086.

[8] J. Banaś, M. Jleli, M. Mursaleen, B. Samet, C. Vetro *et al.*, *Advances in nonlinear analysis via the concept of measure of noncompactness*. Springer, 2017. doi: 10.1007/978-981-10-3722-1.

[9] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006, doi: 10.1016/j.na.2005.10.017.

[10] G. Darbo, "Punti uniti in trasformazioni a codominio non compatto," *Rendiconti del Seminario matematico della Università di Padova*, vol. 24, pp. 84–92, 1955.

[11] A. Das, B. Hazarika, and M. Mursaleen, "Application of measure of noncompactness for solvability of the infinite system of integral equations in two variables in ℓ_p, $1 < p < \infty$," *Revista De La Real Academia De Ciencias Exactas, Físicas Y Naturales. Serie A. Matemáticas*, vol. 113, no. 1, pp. 31–40, 2019, doi: 10.1007/s13398-017-0452-1.

[12] K. Kuratowski, "Sur les espaces complets," *Fundamenta Mathematicae*, vol. 1, no. 15, pp. 301–309, 1930.

[13] N. Mizoguchi and W. Takahashi, "Fixed point theorems for multivalued mappings on complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 141, no. 1, pp. 177–188, 1989, doi: 10.1016/0022-247X(89)90214-X.

[14] B. Mohammadi, V. Parvaneh, H. Aydi, and H. Işik, "Extended Mizoguchi-Takahashi type fixed point theorems and their application," *Mathematics*, vol. 7, no. 7, p. 575, 2019, doi: 10.3390/math7070575.

[15] M. Mursaleen and S. M. Rizvi, "Solvability of infinite systems of second order differential equations in c_0 and ℓ_1 by Meir–Keeler condensing operators," *Proceedings of The American Mathematical Society*, vol. 144, no. 10, pp. 4279–4289, 2016, doi: 10.1090/proc/13048.

[16] V. Parvaneh, M. Khorshidi, M. De La Sen, and H. Işik, "Measure of noncompactness and a generalized Darbo's fixed point theorem and its applications to a system of integral equations," *Advances in Difference Equations*, vol. 2020, no. 1, p. 243, 2020, doi: 10.1186/s13662-020-02703-z.

[17] A. Samadi, "Applications of measure of noncompactness to coupled fixed points and systems of integral equations," *Miskolc Mathematical Notes*, vol. 19, no. 1, pp. 537–553, 2018, doi: 10.18514/MMN.2018.2532.

Authors' addresses

Babak Mohammadi
Department of Mathematics, Marand Branch, Islamic Azad University, Marand, Iran
E-mail address: babakmohammadi28@yahoo.com

Mohammad Mursaleen
(**Corresponding author**) China Medical University Hospital, China Medical University (Taiwan), Department of Medical Research, Taichung, Taiwan; Aligarh Muslim University, Department of Mathematics, Aligarh-202 002, India
E-mail address: mursaleenm@gmail.com

Vahid Parvaneh
Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran
E-mail address: zam.dalahoo@gmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 921–942

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4219

DYNAMIC ANALYSIS AND CHAOS CONTROL IN A DISCRETE PREDATOR-PREY MODEL WITH HOLLING TYPE IV AND NONLINEAR HARVESTING

KARIMA MOKNI AND MOHAMED CH-CHAOUI

Received 23 April, 2022

Abstract. In this paper, we investigate the dynamical behavior of a two dimensional discretized prey predator system. The model is formulated in terms of difference equations and derived by using the higher-order implicit Runge Kutta method with a very small step size to attain a discrete time version of its continuous counterpart. The existence of fixed points as well as their local asymptotic stability are proved. Further, it is shown that the model experiences Neimark-Sacker bifurcation (NSB for short) and the periodic doubling bifurcation (PDB) in a small neighborhood of the coexistence fixed point under certain parametric conditions. This analysis utilizes bifurcation theory and the center manifold theorem. The chaos influenced by NSB is stabilized. Finally, we use numerical simulations and computer analysis to check our theories and show more complex behaviors.

2010 *Mathematics Subject Classification:* 65Q10; 34H10; 34G20; 39A28; 39A30; 39A33; 92D25

Keywords: discrete model, asymptotic stability, bifurcation analysis, chaos control

1. Introduction

When a predator and its prey interact, the population dynamics may be represented using a prey-predator model that is either discrete or continuous (in time) [10, 14]. Discrete time prey-predator models based on difference equations are best suited for predicting and describing the dynamics of populations with non-overlapping generations [6]. The discrete case is especially important because it can exhibit more interesting dynamic behaviors, including bifurcations and chaos. It may also offer a more efficient computational model for numerical simulations. Controlling chaos in nonlinear dynamical systems is a challenging area of research. Chaos occurs suddenly and might lead to unpredictable behavior.

In [8], a discrete time predator prey model with six parameters is investigated, and the hybrid control strategy is applied to control Neimark-Sacker bifurcation. The Neimark-Sacker bifurcation in a discrete time glycolytic oscillator model have been

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

studied in [11]. Prime period and periodic points of the discrete model are investigated. A further hybrid control strategy is applied to control chaos influenced by Neimark-Sacker bifurcation. In [1], nonlinear dynamics and chaotic behavior of a multi-strain tuberculosis model under the fractal fractional operator in Atangana-Baleanu sense are established. In [20], the local dynamics of a discrete nonlinear prey–predator model are obtained on asymptotic properties of fixed points and Neimark–Sacker bifurcations. Some results on the global stability are studied. In [17], a new approach related to the global stability are established for a special class of discrete time evolutionary models for both single species and multi-species dynamics, that are derived according to the evolutionary game theory. The reader interested to other works on the asymptotic stability, bifurcation theory and chaos control is referred to, among many others, to [3, 4, 7, 9, 11, 12, 14, 15, 15, 16, 19, 21].

Motivated by the previous cited works, we consider the following class of prey-predator system incorporating Holling functional response type IV and nonlinear harvesting in the prey [22]

$$\frac{dx}{dt} = x(1-x) - \underbrace{\frac{xy}{\frac{1}{\alpha}x^2 + x + \beta}}_{\text{Functional response Holling IV}} - \underbrace{\frac{ax}{x+b}}_{\text{Nonlinear harvesting}}, \tag{1.1}$$

$$\frac{dy}{dt} = cy\left(1 - \frac{dy}{x}\right), \tag{1.2}$$

here x and y model the prey and predator densities, respectively. All the model's parameters $a, b, c, d, \alpha, \beta$ are nonnegative. The coefficient α is the tolerance of the prey, β represent half saturation constant. The parameters b stands for the effort of harvesting, and c is intrinsic growth rate of the predator. Finally, the parameter d models the amount of prey required to support one predator at equilibrium. The qualitative behavior of (1.1)-(1.2) is investigated in [22]. In [2], Euler scheme is implemented to discretize the system (1.1)-(1.2). In this work, arguing as in [18], and considering that the variables and functions switch only at regular intervals, the following formula is built through the conversion form (1.1)-(1.2) by implementing the largest integer function $[t]$:

$$\frac{1}{x(t)}\frac{dx}{dt} = (1-x) - \frac{y[t]}{\frac{1}{\alpha}x^2[t] + x[t] + \beta} - \frac{a}{x[t]+b}, \tag{1.3}$$

$$\frac{1}{y(t)}\frac{dy}{dt} = c\left(1 - \frac{dy[t]}{x[t]})\right), \tag{1.4}$$

applying the piecewise constant arguments method for differential equations $[t]$, thus integrating the system (1.3)-(1.4) on the interval $[n, n+1]$ we have

$$\ln\frac{x_{n+1}}{x_n} = (1-x_n) - \frac{y_n}{\frac{1}{\alpha}x_n^2 + x_n + \beta} - \frac{a}{x_n+b}, \tag{1.5}$$

$$\ln\frac{y_{n+1}}{y_n} = c\left(1-\frac{dy_n}{x_n})\right). \tag{1.6}$$

For $n = 0,1,2...$, we obtain the following discrete time model.

$$x_{n+1} = x_n \exp\left((1-x_n) - \frac{y_n}{\frac{1}{\alpha}x_n^2 + x_n + \beta} - \frac{a}{x_n+b}\right), \tag{1.7a}$$

$$y_{n+1} = y_n \exp\left(c\left(1-\frac{dy_n}{x_n}\right)\right). \tag{1.7b}$$

The aim of this research is to find the system's fixed points (1.7a)-(1.7b) and analyze the asymptotic stability conditions of these fixed points. Furthermore, the interesting aspect of this study, is to prove rigorously, by using center manifold theory, that the system possesses NSB and PDB near the positive fixed point. Moreover, we control the chaos influenced by NSB. The paper is organized as follows: In Section 2, the existence and asymptotic stability of the fixed points are investigated. In Section 3, we prove that the discretized system admits NSB near the coexistence fixed point. The existence of PDB is also proved analytically by using center manifold theory in Section 4. A state feedback method is implemented to control chaos in Section 5. Finally, some numerical simulations are given in 6, followed by a conclusion in 7.

2. Existence of the fixed points and their stability

The fixed points for the system (1.7) are the solutions of the two isoclines

$$(1-x) - \frac{y}{\frac{1}{\alpha}x^2 + x + \beta} - \frac{a}{x+b} = 0, \quad 1 - \frac{dy}{x} = 0. \tag{2.1}$$

For $y = 0$, it follows that the system (1.7) has two boundary fixed points noted $(x_1, 0)$ and $(x_2, 0)$, where x_1 and x_2 are the roots of the quadratic equation $x^2 + (b-1)x + a - b = 0$, where $x_{1,2} = \frac{1-b\pm\sqrt{(1+b)^2-4a}}{2}$ *and* $(1+b)^2 > 4a$. Moreover, we assume that $0 < b < 1$ and $b < a < \frac{1}{4}(b+1)^2$, then both x_1 and x_2 are positive. Let $x_{1,2} = \bar{x}$ and we write the boundary fixed point $E_1(\bar{x}, 0)$.

For the coexistence fixed point $E_2(x^*, y^*)$ the components are given as follows $y^* = \frac{1}{d}x^*$, where x^* is a positive real root for the quadratic equation

$$\alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5 = 0.$$

where $\alpha_1 = d$, $\alpha_2 = d(\alpha + b - 1)$, $\alpha_3 = d(\alpha(\beta - 1) + a + b(\alpha_1)) + \alpha$, and $\alpha_4 = \alpha(ad + (b-1)\beta b - bd + b)$, $\alpha_5 = \alpha\beta d(a-b)$. Arguing as [22], there exist a unique coexistence fixed point. We rewrite (2.1) as $y = \frac{x}{d}$ and $y = (1 - x - \frac{a}{b+x})(\beta + \frac{x^2}{a} + x)$. The Jacobian matrix of the system (1.7a)-(1.7b) at any fixed point (x, y) is given by

$$J(x,y) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}, \tag{2.2}$$

where

$$
\begin{aligned}
j_{11} &= \exp\left(1-x-\frac{y}{(\frac{1}{\alpha}x^2+x+\beta)}-\frac{a}{x+b}\right)\\
&\quad\times\left[1-x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)\right],\\
j_{12} &= -\frac{x}{\frac{1}{\alpha}x^2+x+\beta}\exp\left(1-x-\frac{y}{(\frac{1}{\alpha}x^2+x+\beta)}-\frac{a}{x+b}\right),\\
j_{21} &= \frac{cdy^2}{x^2}\exp\left(c(1-\frac{dy}{x})\right),\\
j_{22} &= \exp\left(c(1-d\frac{y}{x})\right)\left(1-\frac{ycd}{x}\right).
\end{aligned}
$$

The fixed point $E_1(K,0)$ has two eigenvalues $\lambda_1 = 1-x\left(1-\frac{a}{(x+b)^2}\right)$ and $\lambda_2 = \exp(c)$ with $\lambda_2 > 1$. Thus both the boundary fixed point $E_1(\bar{x},0)$ of the system (1.7) are unstable.

For the fixed point $E_2(x^*, y^* = \frac{x^*}{d})$, we have

$$
J(E_2) = \begin{pmatrix} 1-x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right) & -\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta} \\ \frac{c}{d} & 1-c \end{pmatrix}. \tag{2.3}
$$

The characteristic equation associated to (2.3) is [6]

$$
\eta^2 - \mathrm{tr}J(E_2)\eta + \det J(E_2) = 0, \tag{2.4}
$$

where

$$
\mathrm{tr}J(E_2) = 2 - x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right) - c, \tag{2.5}
$$

and

$$
\begin{aligned}
\det J(E_2) &= \left(1-x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)\right)(1-c)\\
&\quad+\frac{cx^*}{\frac{d}{\alpha}x^{*2}+dx^*+d\beta}.
\end{aligned} \tag{2.6}
$$

The discriminant of (2.4) is

$$
\Delta = \mathrm{tr}\Big(J(E_2)\Big)^2 - 4\det\Big(J(E_2)\Big). \tag{2.7}
$$

Theorem 1. *We assume* $x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)>2$, *and we set*

$$\Lambda_1=\frac{2d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-2\right)}{d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-2\right)+\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}},$$

$$\Lambda_2=\frac{dx^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)}{d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-1\right)+\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}}.$$

The coexistent fixed point E_2 *is locally asymptotically stable if* $\Lambda_1<c<\Lambda_2$ *and unstable otherwise.* E_2 *is nonhyperbolic if either* $c=\Lambda_1$ *or* $c=\Lambda_2$.

Theorem 2. *We assume* $x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)>2$, *and we set*

$$\Lambda_1=\frac{2d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-2\right)}{d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-2\right)+\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}},$$

$$\Lambda_2=\frac{dx^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)}{d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-1\right)+\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}}.$$

The coexistent fixed point E_2 *is locally asymptotically stable if* $\Lambda_1<c<\Lambda_2$ *and unstable otherwise.* E_2 *is nonhyperbolic if either* $c=\Lambda_1$ *or* $c=\Lambda_2$.

3. Neimark Sacker bifurcation

If the discriminant Δ defined in (2.7) is negative and $c=\Lambda_2$ hold, then the eigenvalues of (2.7) are pair of conjugate complex numbers with modulus 1. Thus, this conditions can be written as

$$N_s=\left\{(\alpha,\beta,a,b,c,d)>0,\ \Delta<0,\ c=\Lambda_2\right\}. \tag{3.1}$$

If we vary c in the neighborhood of $c=\bar{c}$ keeping other parameters in (3.1) constant, then the coexistence fixed point E_2 undergoes Neimark-Sacker bifurcation. Taking a perturbation c^* where ($c^*\ll 1$) of the parameter c in the neighborhood of $c=\bar{c}$ in the system (1.7a)-(1.7b), we have

$$x_{n+1}=x_n\exp\left((1-x_n)-\frac{y_n}{\frac{1}{\alpha}x_n^2+x_n+\beta}-\frac{a}{x_n+b}\right)=f(x_n,y_n,c^*), \tag{3.2a}$$

$$y_{n+1} = y_n \exp\left((c+c^*)\left(1-\frac{dy_n}{x_n})\right)\right) = g(x_n, y_n, c^*). \tag{3.2b}$$

We transform the (3.2) into the origin ($v_n = x_n - x^*$ and $w_n = y_n - y^*$), and expanding the resulted system up to second order near the origin, one obtain

$$v_{n+1} = \gamma_1 v_n + \gamma_2 w_n + \gamma_{12} v_n w_n + \gamma_{11} v_n^2 + \gamma_{22} w_n^2 + O\left((\mid v_n \mid + \mid w_n \mid)^2\right), \tag{3.3a}$$

$$w_{n+1} = \delta_1 v_n + \delta_2 w_n + \delta_{12} v_n w_n + \delta_{11} v_n^2 + \delta_{22} w_n^2 + O\left((\mid v_n \mid + \mid w_n \mid)^2\right), \tag{3.3b}$$

where

$$\gamma_1 = f_x(x^*, y^*, 0) = 1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right),$$

$$\gamma_2 = f_y(x^*, y^*, 0) = -\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta},$$

$$\gamma_{12} = f_{xy}(x^*, y^*, 0) = \frac{-1+x^*}{\frac{x^{*2}}{\alpha}+x^*+\beta} + \frac{2x^{*2}+\alpha x^*}{\alpha(\frac{x^{*2}}{\alpha}+x^*+\beta)^2} - \frac{2x^{*2}y^*+\alpha x^* y^*}{\alpha(\frac{x^{*2}}{\alpha}+x^*+\beta)^3}$$
$$- \frac{ax^*}{(x+b)^2(\frac{x^{*2}}{\alpha}+x^*+\beta)},$$

$$\gamma_{11} = f_{xx}(x^*, y^*, 0) = -1 + \frac{2x^*y^*+\alpha y^*+y^*x^*}{\alpha(\frac{x^{*2}}{\alpha}+x^*+\beta)^2} - \frac{4x^{*2}y^*+\alpha^2 x^* y^* + 4\alpha x^{*2} y^*}{\alpha^2(\frac{x^{*2}}{\alpha}+x^*+\beta)^3}$$
$$+ \frac{2x^{*3}y^{*2}+2\alpha x^{*2}y^*}{\alpha^2(\frac{x^{*2}}{\alpha}+x^*+\beta)^4} + \frac{y^{*2}x^*}{2(\frac{x^{*2}}{\alpha}+x^*+\beta)^4} + \frac{2x^{*2}y^*a+\alpha a x^* y^*}{\alpha(x+b)^2(\frac{x^{*2}}{\alpha}+x^*+\beta)^2}$$
$$+ \frac{a}{(x+b)^2} - \frac{ax}{(x+b)^3} + \frac{a^2x^*}{2(x^*+b)^4} - \frac{2x^{*2}y^*+\alpha x^* y^*}{\alpha(\frac{x^{*2}}{\alpha}+x^*+\beta)^2} - \frac{ax^*}{(x^*+b)^3} + \frac{x^*}{2},$$

$$\gamma_{22} = f_{yy}(x^*, y^*, 0) = \frac{x^*}{2((\frac{x^{*2}}{\alpha}+x^*+\beta)^2)},$$

$$\delta_1 = \frac{c}{d},$$

$$\delta_2 = 1 - c,$$

$$\delta_{12} = g_{xy}(x^*, y^*, 0) = \frac{c}{x^*}(2 - \frac{c}{x^*}),$$

$$\delta_{11} = g_{xx}(x^*, y^*, 0) = \frac{cx^*}{2d}(-1 + \frac{c}{x^{*2}}),$$

$$\delta_{22} = g_{yy}(x^*, y^*, 0) = \frac{cd}{x^*}(-1 + \frac{c}{2}).$$

The roots of the characteristic equation associated with the linearized map (3.3) at $(v_n, w_n) = (0,0)$ are given by

$$\eta_{1,2}(c^*) = \frac{\mathrm{tr}J(c^*) \pm i\sqrt{4\det J(c^*) - (\mathrm{tr}(c^*))^2}}{2}, \quad |\eta_{1,2}(c^*)| = \sqrt{\det J(c^*)}.$$

When $c^* = 0$, we have

$$\det(J(0)) = 1 \text{ , and } \frac{d|\eta_{1,2}|}{dc^*}\,|_{c^*=0} \neq 0. \tag{3.4}$$

Additionally, we required that when $c^* = 0$, $\eta_{1,2}^m \neq 1$, $m = 1,2,3,4$. This is equivalent to $trJ(0) \neq -2,-1,1,2$.

Let $\zeta = Re(\eta_{1,2})$, and $\xi = Im(\eta_{1,2})$. The model (3.3) is written as

$$\begin{pmatrix} v_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \delta_1 & \delta_2 \end{pmatrix} \begin{pmatrix} v_n \\ w_n \end{pmatrix} + \begin{pmatrix} \gamma_{12} v_n w_n + \gamma_{11} v_n^2 + \gamma_{22} w_n^2 \\ \delta_{12} v_n w_n + \delta_{11} v_n^2 + \delta_{22} w_n^2 \end{pmatrix}. \tag{3.5}$$

Let us consider the invertible matric P associated to the eigenvalue $\eta_{1,2} = \zeta \pm i\xi$,

$$P = \begin{pmatrix} \gamma_2 & 0 \\ \zeta - \gamma_1 & -\xi \end{pmatrix}.$$

Using the following translation

$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = \begin{pmatrix} \gamma_2 & 0 \\ \zeta - \gamma_1 & -\xi \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}.$$

The system (3.5) can be written as

$$P \begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \delta_1 & \delta_2 \end{pmatrix} P \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} \widetilde{f}(X_n, Y_n) \\ \widetilde{g}(X_n, Y_n) \end{pmatrix},$$

where

$$\widetilde{f}(X_n, Y_n) = \Big(\gamma_{12}\gamma_2(\zeta - \gamma_1) + \gamma_{11}\gamma_2^2 + \gamma_{22}(\zeta - \gamma_1)^2\Big) X_n^2 - \Big(\gamma_{12}\gamma_2\xi + 2(\zeta - \gamma_1)\gamma_{22}\xi\Big) X_n Y_n + \gamma_{22}\xi^2 Y_n^2,$$

$$\widetilde{g}(X_n, Y_n) = \Big(\delta_{12}\gamma_2(\zeta - \gamma_1) + \delta_{11}\gamma_2^2 + \delta_{22}(\zeta - \gamma_1)^2\Big) X_n^2 - \Big(\delta_{12}\gamma_2\xi + 2(\zeta - \gamma_1)\delta_{22}\xi\Big) X_n Y_n + \delta_{22}\xi^2 Y_n^2.$$

Thus

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \zeta & -\xi \\ -\xi & \zeta \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + P^{-1} \begin{pmatrix} \widetilde{f}(X_n, Y_n) \\ \widetilde{g}(X_n, Y_n) \end{pmatrix},$$

where

$$P^{-1} = \begin{pmatrix} \frac{1}{\gamma_2} & 0 \\ \frac{\zeta - \gamma_1}{\xi\gamma_2} & -\frac{1}{\xi} \end{pmatrix}.$$

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \zeta & -\xi \\ -\xi & \zeta \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} F(X_n,Y_n) \\ G(X_n,Y_n) \end{pmatrix}, \tag{3.6}$$

with

$$\begin{aligned} F(X_n,Y_n) = {} & \frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big)X_n^2 \\ & -\frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\Big)X_nY_n+\frac{1}{\gamma_2}\gamma_{22}\xi^2Y_n^2, \end{aligned}$$

and

$$\begin{aligned} G(X_n,Y_n) = {} & \Bigg(\frac{\zeta-\gamma_1}{\xi\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big) \\ & -\frac{1}{\xi}\Big(\delta_{12}\gamma_2(\zeta-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\zeta-\gamma_1)^2\Big)\Bigg)X_n^2 \\ & -\Bigg(\frac{\zeta-\gamma_1}{\xi\gamma_2}\Big(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\Big)-\frac{1}{\xi}\Big(\delta_{12}\gamma_2\xi \\ & +2(\zeta-\gamma_1)\delta_{22}\xi\Big)\Bigg)X_nY_n+\Bigg(\frac{(\zeta-\gamma_1)\gamma_{22}\xi^2}{\xi\gamma_2}-\frac{\delta_{22}\xi^2}{\xi}\Bigg)Y_n^2. \end{aligned}$$

In order for (3.6) to undergo a Neimark Sacker bifurcation, it is required that the following quantity is non zero [13]

$$N = -\Re\left[\frac{(1-2\overline{\eta})\overline{\eta}^2}{1-\eta}\tau_{11}\tau_{20}\right]-\frac{1}{2}\mid\tau_{11}\mid^2-\mid\tau_{02}\mid^2+\Re(\overline{\eta}\tau_{21}), \tag{3.7}$$

where

$$\begin{aligned} \tau_{02} = {} & \frac{1}{4}\Bigg[\Bigg(\frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big)-\frac{1}{\gamma_2}\gamma_{22}\xi^2 \\ & +\Bigg(\frac{\zeta-\gamma_1}{\xi\alpha_2}\Big(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\Big)-\frac{1}{\xi}\Big(\delta_{12}\gamma_2\xi+2(\zeta-\gamma_1)\delta_{22}\xi\Big)\Bigg)\Bigg) \\ & +i\Bigg(\Bigg(\frac{\zeta-\gamma_1}{\xi\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big) \\ & -\frac{1}{\xi}\Big(\delta_{12}\gamma_2(\zeta-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\zeta-\gamma_1)^2\Big)\Bigg)-\Bigg(\frac{(\zeta-\gamma_1)\gamma_{22}\xi^2}{\xi\gamma_2}-\frac{\delta_{22}\xi^2}{\xi}\Bigg) \\ & -\frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\Big)\Bigg)\Bigg], \\ \tau_{11} = {} & \frac{1}{2}\Bigg[\Bigg(\frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big)+\frac{1}{\gamma_2}\gamma_{22}\xi^2\Bigg) \\ & +i\Bigg(\Bigg(\frac{\zeta-\gamma_1}{\xi\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\xi}\Big(\delta_{12}\gamma_2(\zeta-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\zeta-\gamma_1)^2\Big)\Big)\\
&+\Big(\frac{(\zeta-\gamma_1)\gamma_{22}\xi^2}{\xi\gamma_2}-\frac{\delta_{22}\xi^2}{\xi}\Big)\Big)\Big],\\
\tau_{20}=\frac{1}{4}\Big[&\Big(\frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big)\\
&-\frac{1}{\gamma_2}\gamma_{22}\xi^2-\Big(\frac{\zeta-\gamma_1}{\xi\gamma_2}\Big(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\Big)\\
&-\frac{1}{\xi}\Big(\delta_{12}\gamma_2\xi+2(\zeta-\gamma_1)\delta_{22}\xi\Big)\Big)\Big)\\
&+i\Big(\Big(\frac{\zeta-\gamma_1}{\xi\gamma_1 2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big)\\
&-\frac{1}{\xi}\Big(\delta_{12}\gamma_2(\zeta-\gamma_1)+\delta_{11}\gamma_2^2+\zeta_{22}(\zeta-\gamma_1)^2\Big)\Big)\\
&+\Big(\frac{\zeta-\gamma_1}{\xi\gamma_2}\Big(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\Big)\\
&-\frac{1}{\xi}\Big(\delta_{12}\gamma_2\xi+2(\zeta-\gamma_1)\delta_{22}\xi\Big)\Big)+\frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\Big)\Big)\Big],\\
\tau_{21}=0.&
\end{aligned}
$$

Based on the above analysis, we state the following result on NSB.

Theorem 3. *If the condition (3.4) holds and N defined in (3.7) is nonzero then the model (1.7a)-(1.7b) undergoes NSB about the coexistent fixed point $E_2(x^*,y^*)$ when c^* varies near the origin. and $(\alpha,\beta,a,b,c,d)\in N_b$. Moreover, if $N<0$ ($N>0$) then an attracting (respectively repelling) invariant closed curve bifurcates from the fixed point $E_2(x^*,y^*)$ for $c>\bar{c}$ (respectively, $c<\bar{c}$).*

4. Period Doubling Bifurcation

For the fixed point $E_2(x^*,y^*)$ associated to the system (1.7a)-(1.7b). We define the space P_d as

$$
P_d=\Big\{c=\Lambda_1,\quad \mathrm{tr}^2J(E_2)>4\det J(E_2),\ \ (\alpha,\beta,a,b,c,d)>0\Big\},\tag{4.1}
$$

one of the eigenvalues of $J(x^*,y^*)$ is -1 and the other is neither 1 nor -1. Therefore the system (1.7a)-(1.7b) undergoes PDB at the fixed point $E(x^*,y^*)$ if c varies in the small neighborhood of $c=\widehat{c}$ and $(\alpha,\beta,a,b,c,d)\in P_d$. Giving a perturbation c^*

(where $c^* \ll 1$) of the parameter c in the neighborhood of $c = \widehat{c}$ to the system (1.7a)-(1.7b) which is rewritten as follows

$$x_{n+1} = x_n \exp\left((1-x_n) - \frac{y}{\frac{1}{\alpha}x_n^2 + x_n + \beta} - \frac{a}{x_n + b}\right) = f(x_n, y_n, c^*), \tag{4.2a}$$

$$y_{n+1} = y_n \exp\left((c+c^*)\left(1 - \frac{dy_n}{x_n})\right)\right) = g(x_n, y_n, c^*). \tag{4.2b}$$

Let $v_n = x_n - x^*$, $w_n = y_n - y^*$, Then from (4.2) we set

$$v_{n+1} = (v_n + x^*)\exp\left((1-(v_n+x^*)) - \frac{(w_n + y^*)}{\frac{1}{\alpha}(v_n+x^*)^2 + (v_n+x^*) + \beta} - \frac{a}{(v_n+x^*)+b}\right) - x^*, \tag{4.3a}$$

$$w_{n+1} = (w_n + y^*)\exp\left((c+c^*)(1 - \frac{d(w_n+y^*)}{(v_n+x^*)})\right) - y^*. \tag{4.3b}$$

Expanding (4.3) in Taylor series about $(v_n, w_n, c^*) = (0,0,0)$, and considering the terms up to second order, we have

$$v_{n+1} = \gamma_1 v_n + \gamma_2 w_n + \gamma_{12} v_n w_n + \gamma_{11} v_n^2 + \gamma_{22} w_n^2, \tag{4.4}$$

$$\begin{aligned} w_{n+1} =& \delta_1 v_n + \delta_2 w_n + \delta_{12} v_n w_n + \delta_{11} v_n^2 + \delta_{22} w_n^2 + \delta_{13} c^* v_n + \delta_{13} c^* v_n \\ &+ \delta_{23} c^* w_n + \delta_{123} c^* v_n w_n + \delta_{113} c^* v_n^2 + \delta_{223} c^* w_n^2. \end{aligned} \tag{4.5}$$

where

$$\gamma_1 = f_x(x^*, y^*, 0) = 1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha} + 1)}{(\frac{1}{\alpha}x^{*2} + x^* + \beta)^2} + \frac{a}{(x^*+b)^2}\right)\right),$$

$$\gamma_2 = f_y(x^*, y^*, 0) = -\frac{x^*}{\frac{1}{\alpha}x^{*2} + x^* + \beta},$$

$$\begin{aligned} \gamma_{12} = f_{xy}(x^*, y^*, 0) =& \frac{-1 + x^*}{\frac{x^{*2}}{\alpha} + x^* + \beta} + \frac{2x^{*2} + \alpha x^*}{\alpha(\frac{x^{*2}}{\alpha} + x^* + \beta)^2} - \frac{2x^{*2}y^* + \alpha x^* y^*}{\alpha(\frac{x^{*2}}{\alpha} + x^* + \beta)^3} \\ &- \frac{ax^*}{(x+b)^2(\frac{x^{*2}}{\alpha} + x^* + \beta)} \end{aligned}$$

$$\begin{aligned} \gamma_{11} = f_{xx}(x^*, y^*, 0) =& -1 + \frac{2x^*y^* + \alpha y^* + y^* x^*}{\alpha(\frac{x^{*2}}{\alpha} + x^* + \beta)^2} - \frac{4x^{*2}y^* + \alpha^2 x^* y^* + 4\alpha x^{*2} y^*}{\alpha^2(\frac{x^{*2}}{\alpha} + x^* + \beta)^3} \\ &+ \frac{a}{(x+b)^2} + \frac{2x^{*3}y^{*2} + 2\alpha x^{*2} y^*}{\alpha^2(\frac{x^{*2}}{\alpha} + x^* + \beta)^4} + \frac{y^{*2}x^*}{2(\frac{x^{*2}}{\alpha} + x^* + \beta)^4} \\ &+ \frac{2x^{*2}y^* a + \alpha a x^* y^*}{\alpha(x+b)^2(\frac{x^{*2}}{\alpha} + x^* + \beta)^2} - \frac{ax}{(x+b)^3} + \frac{a^2 x^*}{2(x^*+b)^4} \end{aligned}$$

$$-\frac{2x^{*2}y^*+\alpha x^*y^*}{\alpha(\frac{x^{*2}}{\alpha}+x^*+\beta)^2}-\frac{ax^*}{(x^*+b)^3}+\frac{x^*}{2},$$

$$\gamma_{22}=f_{yy}(x^*,y^*,0)=\frac{x^*}{2(\frac{x^{*2}}{\alpha}+x^*+\beta)^2)},$$

$$\beta_1=g_x(x^*,y^*,0)=\frac{c}{d},$$

$$\beta_2=g_y(x^*,y^*,0)=1-c,$$

$$\beta_{12}=g_{xy}(x^*,y^*,0)=\frac{c}{x^*}(2-\frac{c}{x^*}),$$

$$\beta_{13}=g_{xc^*}(x^*,y^*,0)=\frac{1}{d},$$

$$\beta_{11}=g_{xx}(x^*,y^*,0)=\frac{cx^*}{2h}(-1+\frac{s}{x^{*2}}),$$

$$\beta_{22}=g_{yy}(x^*,y^*,0)=\frac{cd}{x^*}(-1+\frac{s}{2}),$$

$$\beta_{23}=g_{yc^*}(x^*,y^*,0)=-1,$$

$$\beta_{123}=g_{xyc^*}(x^*,y^*,0)=\frac{2}{x^*}(1-\frac{c}{x^*}),$$

$$\beta_{113}=g_{xxc^*}(x^*,y^*,0)=\frac{-x^*}{2d}+\frac{c}{dx^*},$$

$$\beta_{223}=g_{yyc^*}(x^*,y^*,0)=\frac{d}{x^*}(-1+\frac{c}{2})+\frac{cd}{2x^*}.$$

Now we define an invertible matrix $T=\begin{pmatrix}\gamma_2 & \gamma_2\\ -1-\gamma_1 & \eta_2-\gamma_1\end{pmatrix}$, and use the transformation $\begin{pmatrix}v_n\\ w_n\end{pmatrix}=T\begin{pmatrix}X_n\\ Y_n\end{pmatrix}$. Writing

$$v_n=\gamma_2(X_n+Y_n), w_n=-(1+\gamma_1)X_n+(\eta_2-\gamma_1)Y_n.$$

Then the system (4.4) becomes

$$T\begin{pmatrix}X_{n+1}\\ Y_{n+1}\end{pmatrix}=\begin{pmatrix}\gamma_1 & \gamma_2\\ \delta_1 & \delta_2\end{pmatrix}T\begin{pmatrix}X_n\\ Y_n\end{pmatrix}+T\begin{pmatrix}f_1(X_n,Y_n,c^*)\\ g_1(X_n,Y_n,c^*)\end{pmatrix},$$

where

$$\begin{aligned}f_1(X_n,Y_n,c^*)=&\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\Big)X_n^2\\ &+\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)-\gamma_{12}\gamma_2(1+\gamma_1)+2\gamma_{11}\gamma_2^2-\gamma_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)X_nY_n\\ &+\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\eta_2-\gamma_1)^2\Big)Y_n^2,\end{aligned}\tag{4.6}$$

and

$$
\begin{aligned}
g_1(X_n,Y_n,c^*) = {} & \Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big)X_n^2 \\
& + \Big(\delta_{113}\gamma_2^2+\delta_{223}(1+\gamma_1)^2-\delta_{123}\gamma_2(1+\gamma_1)\Big)X_n^2c^* \\
& + \Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\eta_2-\gamma_1)^2\Big)Y_n^2 \\
& + \Big(\delta_{223}\gamma_2^2+\delta_{223}(\eta_2-\gamma_1)^2+\delta_{123}\gamma_2(\eta_2-\gamma_1)\Big)Y_n^2c^* \\
& + \Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)-\delta_{12}\gamma_2(1+\gamma_1)+2\delta_{11}\gamma_2^2-2\delta_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)X_nY_n \\
& + \Big(2\delta_{113}\gamma_2^2+2\delta_{223}(1+\gamma_1)(\eta_2-\gamma_1)+\delta_{123}\gamma_2(\eta_2-\gamma_1) \\
& - \delta_{123}\gamma_2(1+\gamma_1)\Big)X_nY_nc^* + \Big(\delta_{13}\gamma_2-\delta_{23}(1+\gamma_1)\Big)X_nc^* \\
& + \Big(\delta_{13}\gamma_2+\delta_{23}(\eta_2-\gamma_1)\Big)Y_nc^*.
\end{aligned} \tag{4.7}
$$

Thus

$$
\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \eta_2 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} F_1(X_n,Y_n,c^*) \\ G_1(X_n,Y_n,c^*) \end{pmatrix}, \tag{4.8}
$$

where

$$
\begin{aligned}
F_1(X_n,Y_n,c^*) = {} & \frac{\eta_2-\gamma_1}{1+\eta_2}\Bigg(\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\Big)X_n^2 \\
& + \Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)-\gamma_{12}\gamma_2(1+\gamma_1)+2\gamma_{11}\gamma_2^2-2\gamma_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)X_nY_n \\
& + \Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\eta_2-\gamma_1)^2\Big)Y_n^2 \\
& - \frac{1}{\eta_2+1}\Bigg(\Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big)X_n^2 \\
& + \Big(\delta_{113}\gamma_2^2+\delta_{223}(1+\gamma_1)^2-\delta_{123}\gamma_2(1+\gamma_1)\Big)X_n^2c^* \\
& + \Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\eta_2-\gamma_1)^2\Big)Y_n^2 \\
& + \Big(\delta_{223}\gamma_2^2+\delta_{223}(\eta_2-\gamma_1)^2+\delta_{123}\gamma_2(\eta_2-\gamma_1)\Big)Y_n^2c^*
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)-\delta_{12}\gamma_2(1+\gamma_1)+2\delta_{11}\gamma_2^2-2\delta_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)X_nY_n\\
&+\Big(2\delta_{113}\gamma_2^2+2\delta_{223}(1+\gamma_1)(\eta_2-\gamma_1)+\delta_{123}\gamma_2(\eta_2-\gamma_1)\\
&-\delta_{123}\gamma_2(1+\gamma_1)\Big)X_nY_nc^*+\Big(\delta_{13}\gamma_2-\delta_{23}(1+\gamma_1)\Big)X_nc^*\\
&+\Big(\delta_{13}\gamma_2+\delta_{23}(\eta_2-\gamma_1)\Big)Y_nc^*\Big)\Big).
\end{aligned}
$$

and

$$
\begin{aligned}
G_1(X_n,Y_n,c^*)=&\frac{1+\gamma_1}{1+\eta_2}\Big(\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2\Big)X_n^2\\
&+\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)-\gamma_{12}\gamma_2(1+\gamma_1)+2\gamma_{11}\gamma_2^2\Big)X_nY_n\\
&+\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)+\gamma_{11}\gamma_2^2\Big)Y_n^2\Big)\\
&+\frac{1}{\eta_2+1}\Big(\Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big)X_n^2\\
&+\Big(\delta_{113}\gamma_2^2+\delta_{223}(1+\gamma_1)^2-\delta_{123}\gamma_2(1+\gamma_1)\Big)X_n^2c^*\\
&+\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\eta_2-\gamma_1)^2\Big)Y_n^2\\
&+\Big(\delta_{223}\gamma_2^2+\delta_{223}(\eta_2-\gamma_1)^2+\delta_{123}\gamma_2(\eta_2-\gamma_1)\Big)Y_n^2c^*\\
&+\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)-\delta_{12}\gamma_2(1+\gamma_1)+2\delta_{11}\gamma_2^2-2\delta_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)X_nY_n\\
&+\Big(2\delta_{113}\gamma_2^2+2\delta_{223}(1+\gamma_1)(\eta_2-\gamma_1)+\delta_{123}\gamma_2(\eta_2-\gamma_1)\\
&-\delta_{123}\gamma_2(1+\gamma_1)\Big)X_nY_nc^*+\Big(\delta_{13}\gamma_2-\delta_{23}(1+\gamma_1)\Big)X_nc^*\\
&+\Big(\delta_{13}\gamma_2+\delta_{23}(\eta_2-\gamma_1)\Big)Y_nc^*\Big)\Big).
\end{aligned}
$$

Hereafter we determine the center manifold $\mathcal{W}_c(0,0)$ of (4.8) about $(0,0)$ in a small neighborhood of c^*. By center manifold theorem, there exists a center manifold

$\mathcal{W}_c(0,0)$ that can be represented as follows:

$$\mathcal{W}_c(0,0)=\{(X_n,Y_n):Y_n=\phi(X_n,c^*)=a_1X_n^2+a_2X_nc^*+a_3c^{*2}+O((\mid X_n\mid+\mid c^*\mid)^2)\}, \tag{4.9}$$

where $O((\mid X_n\mid+\mid c^*\mid)^2)$ is a function with order at least three in their variables (X_t,s^*), Moreover, the center manifold must satisfy

$$\phi\Big(-X_n+F_1(X_n,\phi(X_n,c^*)),c^*),c^*\Big)-\eta_2\phi(X_n,c^*)-G_1(X_n,\phi(X_n,c^*),c^*)=0. \tag{4.10}$$

By equating (4.10), we obtain

$$\begin{aligned}
a_1&=\frac{1+\gamma_1}{1-\eta_2^2}\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\Big)\\
&\quad+\frac{1}{1-\eta_2^2}\Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big),\\
a_2&=\frac{-1}{1+\eta_2}\Big(\delta_{13}\gamma_2-\beta_{23}(1+\alpha_1)\Big),\quad a_3=0.
\end{aligned}$$

Therefore, we consider the map which is the map (4.8) restricted to the center manifold $\mathcal{W}_c(0,0)$

$$f=X_{n+1}=-X_n+c_1X_nc^*+c_2X_n^2+c_3X_n^2c^*+c_4X_n^3, \tag{4.11}$$

where

$$\begin{aligned}
c_1&=-\frac{1}{1+\eta_2}\Big(\delta_{13}\gamma_2-\delta_{23}(1+\gamma_1)\Big),\\
c_2&=\frac{\eta_2-\gamma_1}{1+\eta_2}\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\Big)\\
&\quad-\frac{1}{\eta_2+1}\Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big),\\
c_3&=\frac{-1}{1+\eta_2}\Big(\delta_{13}\gamma_2-\delta_{23}(1+\gamma_1)\Big)\Big[\frac{\eta_2-\gamma_1}{1+\eta_2}\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)-\gamma_{12}\gamma_2(1+\gamma_1)+2\gamma_{11}\gamma_2^2\\
&\quad-2\gamma_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)-\frac{1}{1+\eta_2}\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)-\delta_{12}\gamma_2(1+\gamma_1)+2\delta_{11}\gamma_2^2\\
&\quad-2\gamma_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)\Big]-\frac{1}{1+\eta_2}\Big(\delta_{113}\gamma_2^2+\delta_{223}(1+\gamma_1)^2-\delta_{123}\gamma_2(1+\gamma_1)\Big)\\
&\quad\times\Big(\frac{1+\gamma_1}{(1-\eta_2^2)(1+\eta_2)}\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\Big)\\
&\quad+\frac{1}{(1-\eta_2^2)(1+\eta_2)}\Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big)\Big)
\end{aligned}$$

$$\times\left(\delta_{13}\gamma_2+\delta_{23}(\eta_2-\gamma_1)\right),$$

$$c_4=\left(\frac{1+\gamma_1}{1-\eta_2^2}\left(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\right)+\frac{1}{1-\eta_2^2}\left(-\delta_{12}\gamma_2(1+\gamma_1)\right.\right.$$
$$\left.\left.+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\right)\right)\left[\frac{\eta_2-\gamma_1}{1+\eta_2}\left(\gamma_{12}\gamma_2(\eta_2-\gamma_1)-\gamma_{12}\gamma_2(1+\gamma_1)+2\gamma_{11}\gamma_2^2\right)\right.$$
$$-\frac{1}{1+\eta_2}\left(\delta_{12}\gamma_2(\eta_2-\gamma_1)-\delta_{12}\gamma_2(1+\gamma_1)+2\delta_{11}\gamma_2^2\right.$$
$$\left.\left.-2\delta_{22}(1+\gamma_1)(\eta_2-\gamma_1)\right)\right].$$

In order for the map (4.11) to undergo a period-doubling bifurcation, we require that the following discriminatory quantities are non-zero [13]:

$$\sigma_1=\left(\frac{\partial^2 f}{\partial X_n\partial c^*}+\frac{1}{2}\frac{\partial f}{\partial c^*}\frac{\partial^2 f}{\partial^2 X_n}\right)|_{(0,0)}\neq 0,$$

and

$$\sigma_2=\left(\frac{1}{6}\frac{\partial^3 f}{\partial X_n^3}+(\frac{1}{2}\frac{\partial^2 f}{\partial X_n^2})^2\right)|_{(0,0)}\neq 0.$$

After calculating we get

$$\sigma_1=c_1+c_3,\ \ \sigma_2=c_4+c_2^2.$$

From the above analysis we have the following theorem.

Theorem 4. *If $\sigma_2\neq 0$, and $\sigma_1\neq 0$, then the system (1.7a)-(1.7b) experiences a period-doubling bifurcation about the unique positive fixed point $E_2(x^*,y^*)$ when c^* varies in a small neighborhood of $O(0,0)$. Moreover, if $\sigma_2>0$ (resp $\sigma_2<0$), then the period 2 points that bifurcate from $E_2(x^*,y^*)$ are stable (unstable).*

5. Chaos Control

In this paper, we apply the state feedback method [3] to stabilize the chaotic orbits at an unstable fixed point of system. Towards this we introduce a feedback control force P_n such that

$$x_{n+1}=x_n\exp\left((1-x_n)-\frac{y_n}{\frac{1}{\alpha}x_n^2+x_n+\beta}-\frac{a}{x_n+b}\right),\tag{5.1}$$

$$y_{n+1}=y_n\exp\left(\widehat{c}\left(1-\frac{dy_n}{x_n}\right)\right)-\underbrace{\mu_1(x_n-x^*)-\mu_2(y_n-y^*)}_{\text{The control force } P_n}.\tag{5.2}$$

where μ_1, μ_2 are feedback gains and $\widehat{c}$ is the nominal value for c which belongs to some chaotic regions. The Jacobian matrix of (5.1)-(5.2) at E_2 is

$$J(E_2) = \begin{pmatrix} 1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right) & -\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta} \\ \frac{\widehat{c}}{d} - \mu_1 & 1-\widehat{c}-\mu_2 \end{pmatrix}. \tag{5.3}$$

The corresponding characteristic equation of (5.3) is

$$\begin{aligned} \lambda^2 - \left(2 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right) - \widehat{c} - \mu_2\right)\right)\lambda \\ + \left(1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right)\left(1-\widehat{c}-\mu_2\right) \\ + \frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}\left(\frac{\widehat{c}}{d} - \mu_1\right) = 0. \end{aligned} \tag{5.4}$$

Let λ_1 , λ_2 are the eigenvalues of the characteristic Eq.(5.4) then sum and the product of their roots are given by

$$\lambda_1 + \lambda_2 = 2 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right) - \widehat{c} - \mu_2\right), \tag{5.5}$$

$$\begin{aligned} \lambda_1\lambda_2 = \left(1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right)\left(1-\widehat{c}-\mu_2\right) \\ + \frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}\left(\frac{\widehat{c}}{d} - \mu_1\right). \end{aligned} \tag{5.6}$$

Lemma 1. *The system* (5.1)-(5.2) *is asymptotically stable if all the eigenvalues of the characteristic Eq.* (5.4) *have modulus less than 1.*

Proof. The marginal stability lines can be obtained from the conditions $\lambda_1 = \pm 1$, $\lambda_1\lambda_2 = 1$. For the conditions $\lambda_1\lambda_2 = 1$, Eq.(5.6) gives

$$\begin{aligned} L_1 : \frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}\mu_1 + \left(1 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^*+\alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right)\mu_2 \\ = \left[1 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^*+\alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right]\left(1-\widehat{c}\right) \\ + \frac{x^*\widehat{c}}{d(\frac{1}{\alpha}x^{*2}+x^*+\beta)}, \end{aligned} \tag{5.7}$$

The Eq. (5.7) expresses the first condition for marginal stability. For $\lambda_1 = 1$, the Eq. (5.5) yields

$$L_2 : \frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}\mu_1 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^*+\alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\mu_2\right.$$

$$= \widehat{c} + \frac{x^*\widehat{c}}{d(\frac{1}{\alpha}x^{*2}+x^*+\beta)} - \widehat{c}\left[1 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^*+\alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right],$$

similarly for $\lambda_1 = -1$, it gives

$$\begin{aligned} L_3 : & \frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}\mu_1 + \left(2 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^*+\alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right. \\ & = \left[1 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^*+\alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right]\left(2-\widehat{c}\right) + 2 - \widehat{c} \\ & \quad + \frac{x^*\widehat{c}}{d(\frac{1}{\alpha}x^{*2}+x^*+\beta)}. \end{aligned}$$

The lines L_1, L_2, L_3 give the conditions for the eigenvalues to have absolute value less than 1. The triangular region bounded by these lines accommodates stable eigenvalues. □

6. Numerical simulations

Let $(\alpha,\beta,a,b,d) = (0.19,1,0.0001,0.75,0.3)$ [2] and initial conditions $(x_0,y_0) = (0.5,0.5)$. The coexistent fixed point $E_2(0.4,1.35)$ is locally asymptotically stable if c takes higher values than 3, which means that all the orbits attract towards the positive fixed point $E_2(0.4,1.35)$. By decreasing the value of c from $c = 0.3$ to $c = 0.25$, the system (1.7a)-(1.7b) loses its stability. In particular, we observe the appearance of a closed invariant curve which indicate that the discrete-time model (1.7a)-(1.7b) experiences Neimark-Sacker bifurcation about $E_2 = (0.4,1.35)$. To see this, we compute the value of N defined in (3.7), that is $N = -0.0081364312 < 0$. Hence, the model (1.7a)-(1.7b) admits supercritical Neimark-Sacker bifurcation if $c < 0.3$ and meanwhile, stable curve appears, see Fig. 2.

Additional simulations are plotted to show the emergence of periodic doubling bifurcation for $(\alpha,\beta,a,b,d) = (0.19,1,0.0001,1,0.3)$ and initial conditions $(0.5,0.5)$. For these values E_2 undergo periodic doubling bifurcation, see Fig. 3.

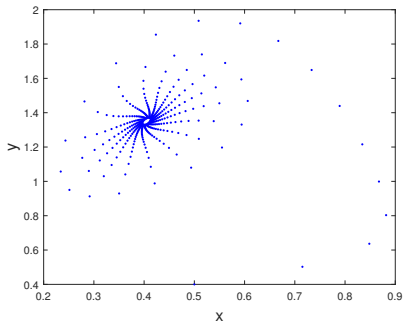


FIGURE 1. *Phase portrait of the system* (1.7) *for* $c = 0.3$.

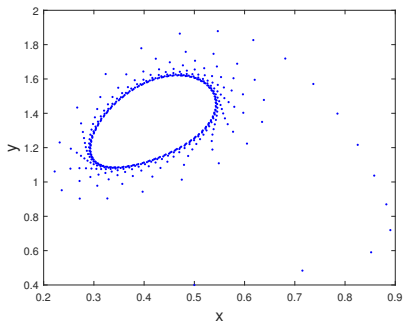


FIGURE 2. *Supercritical Neimark Sacker bifurcation of the system* (1.7) *for* $c = 0.25$.

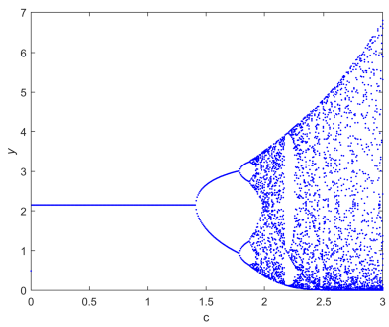


FIGURE 3. Period doubling bifurcation with respect to c.

Now, we use the state feedback method described in Section. 5 to control the chaos influenced by N-S bifurcation, the following set of parameter is taken : $(\alpha, \beta, a, b, d) = (0.19, 1, 0.0001, 0.75, 0.3)$ and $\widehat{c} = 0.25$. Using lemma 1, the domain of stability of the controlled system (5.1)-(5.2) is drawn in Fig.4.

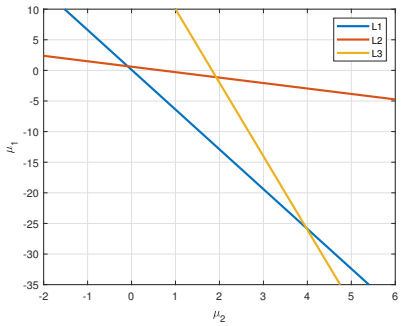


FIGURE 4. Stability region of the controlled system (5.1)-(5.2).

Now, in order to stabilize the chaos in the system (1.7), we consider the feedback controlling force $P_n = \mu_1(x_n - 0.4) + \mu_2(y_n - 1.35)$ with feedback gains $\mu_1 = -0.2$, $\mu_2 = 0.35$chosen from the triangular region from Fig.4. Accordingly to these values, the system (1.3)-(1.4) converges to the fixed point $(0.4, 1.35)$ as shown in Fig.5. The periodic solution is stabilized at time $t = 150$ and reproduce the efficiency of the implemented method.

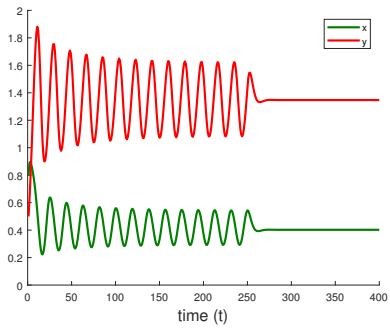


FIGURE 5. The chaos is controlled after time t=250.

7. CONCLUDING REMARKS

In this paper, we explored the dynamical proprieties of a discrete-time two-dimensional prey predator system. The model is formulated in terms of difference

equations, by a discretization of a differential prey predator system involving a nonlinear harvesting effect on the prey and Holling functional response type IV. The discretization is based on the higher-order implicit Runge Kutta method with a very small step size to attain a discrete time version of its continuous counterpart. The existence and local asymptotic stability of the fixed points are investigated. In order to support the complexity of (1.7a)-(1.7b) the presence of Neimark-Sacker bifurcation and periodic doubling bifurcation for the coexistence fixed point $E_2(x^*, y^*)$ is proved analytically by using bifurcation theory and center manifold theory. In addition, the paper provides the method of state feedback and parameter perturbation for the bifurcation control. Numerical simulations carried out to verify our theoretical analysis. In particular, we showed that the discrete model (1.7a)-(1.7b) loses its asymptotic stability when the intrinsic growth rate of the predator $c$varies in the neighborhood of 3.1 and 4.1, and when the fixed point $E_2 = (0.4, 1.35)$ loses stability (all orbits are not attracted to this fixed point), a feedback chaos control strategy is used to stabilize the chaos. Numerical simulations give evidence of the successful implementation of the method. Using Evolutionary Game theory to explain how phenotypic traits change over time will be a fascinating topic for our future work [17]. We want to do an asymptotic study of the global stability of the present discrete model. As a result, new methods for determining the global stability of fixed points, such as the method of Lyapunov functions, are required [5].

ACKNOWLEDGMENT

We would like to thank anonymous referees very much for their valuable comments and suggestions. Professor Saber Elaydi is thanked by K.M for his excellent guidance during his Ph.D.

REFERENCES

[1] Adnan, S. Ahmad, A. Ullah, M. B. Riaz, A. Ali, A. Akgül, and M. Partohaghighi, "Complex dynamics of multi strain tb model under nonlocal and nonsingular fractal fractional operator," *Results in Physics*, vol. 30, p. 104823, 2021, doi: 10.1016/j.rinp.2021.104823.

[2] M. Bilal Ajaz, U. Saeed, Q. Din, I. Ali, and M. Israr Siddiqui, "Bifurcation analysis and chaos control in discrete-time modified Leslie-Gower prey harvesting model," *Adv. Difference Equ.*, vol. 2020, p. 24, 2020, id/No 45, doi: 10.1186/s13662-020-2498-1.

[3] Q. Din, "Complexity and chaos control in a discrete-time prey-predator model," *Communications in Nonlinear Science and Numerical Simulation*, vol. 49, pp. 113–134, 2017, doi: 10.1016/j.cnsns.2017.01.025.

[4] Q. Din, "Controlling chaos in a discrete time prey-predator model with allee effects," *Int. J. Dynam. Control*, vol. 6, pp. 858–872, 2018, doi: 10.1007/s40435-017-0347-1.

[5] S. Elaydi, "Global dynamics of discrete dynamical systems and difference equations," in *Difference equations, discrete dynamical systems and applications, ICDEA 23, Timişoara, Romania, July 24–28, 2017. Proceedings of the 23rd international conference on difference equations and applications*. Cham: Springer, 2019, pp. 51–81, doi: 10.1007/978-3-030-20016-9_3.

[6] S. N. Elaydi, *An introduction to difference equations*, 3rd ed., ser. Undergraduate Texts Math. New York, NY: Springer, 2005.

[7] S. N. Elaydi, *Discrete chaos. With applications in science and engineering*, 2nd ed. Boca Raton, FL: Chapman & Hall/CRC, 2008.

[8] L. Fei, X. Chen, and B. Han, "Bifurcation analysis and hybrid control of a discrete-time predator-prey model," *J. Difference Equ. Appl.*, vol. 27, no. 1, pp. 102–117, 2021, doi: 10.1080/10236198.2021.1876038.

[9] Ö. A. Gümüs and M. Feckan, "Stability, Neimark-Sacker bifurcation and chaos control for a prey-predator system with harvesting effect on predator," *Miskolc Math. Notes*, vol. 22, no. 2, pp. 663–679, 2021, doi: 10.18514/MMN.2021.3450.

[10] Z. Jingwen, W. Ranchao, and C. Mengxin, "Bifurcation analysis in a predator–prey model with strong allee effect," *Zeitschrift für Naturforschung A*, vol. 76, no. 12, pp. 1091–1105, 2021, doi: 10.1515/zna-2021-0178.

[11] A. Q. Khan and T. Khalique, "Bifurcations and chaos control in a discrete-time biological model," *Int. J. Biomath.*, vol. 13, no. 4, p. 31, 2020, id/No 2050022, doi: 10.1142/S1793524520500229.

[12] A. Q. Khan, J. Ma, and D. Xiao, "Global dynamics and bifurcation analysis of a host-parasitoid model with strong Allee effect," *J. Biol. Dyn.*, vol. 11, no. 1, pp. 121–146, 2017, doi: 10.1080/17513758.2016.1254287.

[13] Y. A. Kuznetsov, *Elements of applied bifurcation theory*, 3rd ed., ser. Appl. Math. Sci. New York, NY: Springer, 2004, vol. 112.

[14] Z. Limin and Z. Lan, "Bifurcations and control in a discrete predator prey model with strong allee effect," *International Journal of Bifurcation and Chaos*, vol. 28, no. 5, 2018, doi: 10.1142/S0218127418500621.

[15] X. Liu, A. Shabir, U. Aman, S. Sayed, A. Ali, and Q. Haidong, "Bifurcations, stability analysis and complex dynamics of caputo fractal-fractional cancer model," *Chaos, Solitons & Fractals*, vol. 159, p. 112113, 2022, doi: 10.1016/j.chaos.2022.112113.

[16] P. Majumdar, S. Debnath, B. Mondal, S. Sarkar, and U. Ghosh, "Complex dynamics of a prey-predator interaction model with Holling type-II functional response incorporating the effect of fear on prey and non-linear predator harvesting," *Rend. Circ. Mat. Palermo (2)*, vol. 72, no. 2, pp. 1017–1048, 2023, doi: 10.1007/s12215-021-00701-y.

[17] K. Mokni, S. Elaydi, M. CH-Chaoui, and A. Eladdadi, "Discrete evolutionary population models: a new approach," *Journal of Biological Dynamics*, vol. 14, no. 1, pp. 454–478, 2020, doi: 10.1080/17513758.2020.1772997.

[18] S. Pal, N. Pal, and J. Chattopadhyay, "Hunting cooperation in a discrete-time predator-prey system," *Int. J. Bifurcation Chaos Appl. Sci. Eng.*, vol. 28, no. 7, p. 22, 2018, id/No 1850083, doi: 10.1142/S0218127418500839.

[19] S. H. Streipert, G. S. K. Wolkowicz, and M. Bohner, "Derivation and analysis of a discrete predator-prey model," *Bull. Math. Biol.*, vol. 84, no. 7, p. 34, 2022, id/No 67, doi: 10.1007/s11538-022-01016-4.

[20] J. Wang and M. Fečkan, "Dynamics of a discrete nonlinear prey-predator model," *Int. J. Bifurcation Chaos Appl. Sci. Eng.*, vol. 30, no. 4, p. 15, 2020, id/No 2050055, doi: 10.1142/S0218127420500558.

[21] L.-G. Yuan and Q.-G. Yang, "Bifurcation, invariant curve and hybrid control in a discrete-time predator-prey system," *Appl. Math. Modelling*, vol. 39, no. 8, pp. 2345–2362, 2015, doi: 10.1016/j.apm.2014.10.040.

[22] Z. Zhang, R. K. Upadhyay, and J. Datta, "Bifurcation analysis of a modified Leslie-Gower model with Holling type-IV functional response and nonlinear prey harvesting," *Adv. Difference Equ.*, vol. 2018, p. 21, 2018, id/No 127, doi: 10.1186/s13662-018-1581-3.

Authors' addresses

Karima Mokni
Faculte Polydisciplinaire, Sultan moulay Slimane University, MRI Laboratory, BP: 145 Khouribga principale, 25000, Morocco
E-mail address: moknirima@gmail.com

Mohamed Ch-Chaoui
(**Corresponding author**) Faculte Polydisciplinaire, Sultan moulay Slimane University, MRI Laboratory, BP: 145 Khouribga principale, 25000, Kingdom of Morocco.
E-mail address: m.chchaoui@usms.ma

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 943–955

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4228

ON OPERATORS WHOSE CORE–EP INVERSE IS n-POTENT

DIJANA MOSIĆ, DAOCHANG ZHANG, AND JIANPING HU

Received 05 May, 2022

Abstract. The main contribution of this paper is to establish a number of equivalent conditions for the core–EP inverse of an operator, to be n-potent. We prove that the core–EP inverse of an operator is n-potent if and only if the Drazin inverse of the same operator is n-potent. Thus, we present new characterizations for n-potency of the Drazin inverse. Consequently, we get many characterizations for the core–EP inverse (and Drazin inverse) to be an idempotent. We observe that the core–EP inverse of an operator is idempotent if and only it is the orthogonal projector. Furthermore, we show that the n-potency of an operator implies n-potency of its core–EP inverse and develop the condition for the converse to hold. Applying these results, we obtain necessary and sufficient conditions for the n-potency and idempotency of the core inverse. Notice that the core inverse of an operator is n-potent (or idempotent) if and only if the given operator is n-potent (idempotent).

2010 *Mathematics Subject Classification:* 47A05; 47A99; 47B40; 15A09

Keywords: EP operator, generalized Drazin inverse, core–EP inverse, partial order, Hilbert space

1. Introduction

Let X and Y be arbitrary Hilbert spaces, and let $\mathcal{B}(X,Y)$ be the set of all bounded linear operators from X to Y. Especially, $\mathcal{B}(X) = \mathcal{B}(X,X)$. Denote by A^*, $R(A)$ and $N(A)$ the adjoint, range and null space of $A \in \mathcal{B}(X,Y)$, respectively.

The first author was supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, grant number 451-03-47/2023-01/200124, and the bilateral project between Serbia and France (Generalized inverses on algebraic structures and applications), grant number 337-00-93/2023-05/13.

The second author was supported by the National Natural Science Foundation of China (NSFC) (No. 11901079), and China Postdoctoral Science Foundation (No. 2021M700751), and the Scientific and Technological Research Program Foundation of Jilin Province (No. JJKH20190690KJ; No. 20200401085GX; No. JJKH20220091KJ)..

The third author is supported by the National Natural Science Foundation of China (NSFC) (No. 61672149).

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

For $n \geq 2$, an operator $A \in \mathcal{B}(X)$ is called n-potent if $A^n = A$. In the case that $n = 2$, an operator A satisfying $A^2 = A$ is an idempotent (or projector). If an idempotent A satisfies $A = A^*$, we say that A is the orthogonal projector. Some interesting results about n-potent elements of rings can be found in [12].

It is well known that $B \in \mathcal{B}(Y,X)$ is the Moore–Penrose inverse of $A \in \mathcal{B}(X,Y)$ if

$$ABA = A, \qquad BAB = B, \qquad (AB)^* = AB, \qquad (BA)^* = BA.$$

The Moore–Penrose inverse of A is unique (if it exists) and denoted by $A^\dagger$ [13, 14]. Recall that $A^\dagger$ exists if and only if $R(A)$ is closed in Y. For $A \in \mathcal{B}(X,Y)$, we set $A\{1,3\} = \{B \in \mathcal{B}(Y,X) : ABA = A \text{ and } (AB)^* = AB\}$.

The Drazin inverse of $A \in \mathcal{B}(X)$ is an operator $B \in \mathcal{B}(X)$ for which

$$AB = BA, \quad BAB = B \quad \text{and} \quad A^{k+1}B = A^k,$$

where k is the index of A (denoted by $ind(A)$), i.e. the smallest non-negative integer k such that the previous three equations are satisfied. The Drazin inverse of A is unique (if it exists) and denoted by A^D [14]. In the case that $ind(A) = 1$, the Drazin inverse becomes the group inverse $A^\#$ of A. We use $\mathcal{B}(X)^D$ and $\mathcal{B}(X)^\#$ to denote the sets of all Drazin invertible and group invertible operators in $\mathcal{B}(X)$, respectively.

Firstly, the core–EP inverse was presented in [17] for a square matrix and then it was generalized in [15, 16] for Hilbert space operators. If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, there exists the unique core–EP inverse $B \in \mathcal{B}(X)$ of A (denoted by $A^{\textcircled{D}}$) satisfying [15]:

$$BAB = B \quad \text{and} \quad R(B) = R(B^*) = R(A^k).$$

When $ind(A) = 1$, the core–EP inverse of A reduces to the core inverse $A^{\textcircled{\#}}$ of A [1]. The core and core–EP inverses have attracted attentions of many authors [3, 4, 8, 10, 23]. In particular, various expressions of core-EP inverse were given in [5, 11, 20], iterative method for computing core-EP inverse was proved in [18]; limit representations for core-EP inverse in [21]; continuity of core-EP inverse was presented in [7]. Some generalizations of the core-EP inverse were considered for tensors in [19].

Inspired by some matrix equations occurring in physics and involving assumption that the matrices in them have an an idempotent Moore-Penrose inverse, the class of square matrices which have idempotent Moore-Penrose inverse was investigated in [2]. Beside of characterizations for an idempotent Moore-Penrose inverse, the authors studied the relation between idempotency of a given matrix and its Moore-Penrose inverse. These results were extended to elements of rings in [22].

The aim of this paper is to consider the n-potency of the core–EP inverse of an operator. Precisely, we prove many equivalent conditions for the core–EP inverse of a Drazin invertible operator A, to be n-potent. We observe that $A^{\textcircled{D}}$ is n-potent if and only if A^D is n-potent. As a consequence, we obtain a set of characterizations for $A^{\textcircled{D}}$ to be an idempotent. Remark that we present new characterizations for n-potency and idempotency of the Drazin inverse. Notice that $A^{\textcircled{D}}$ is an idempotent if and only $A^{\textcircled{D}}$ is the orthogonal projector. Also, we verify that the n-potency of A

implies n-potency of its core–EP inverse and consider a condition for the converse to be satisfied. Applying these results, we get necessary and sufficient conditions for the n-potency and idempotency of the core inverse. Remark that $A^{\textcircled{\#}}$ is n-potent (or idempotent) if and only if A is n-potent (idempotent).

This is the content of this paper. In Section 2, we present a number of characterizations for the n-potency and idempotency of the core–EP inverse as well as the relation between the n-potency of a given operator and its core–EP inverse. Section 3 contains equivalent conditions for the n-potency and idempotency of the core inverse.

2. Main results

To develop a list of equivalent conditions for the core–EP inverse to be n-potent, we firstly present one auxiliary result related to the expressions for the power of the core–EP inverse.

Lemma 1. *If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, then*

$$(A^{\textcircled{D}})^n = (A^D)^n A^k (A^k)^\dagger = (A^D)^n A^k (A^k)^{(1,3)},$$

for any $n \geq 1$ and $(A^k)^{(1,3)} \in (A^k)\{1,3\}$.

Proof. It is well known, by [6, Theorem 2.3], that $A^{\textcircled{D}} = A^D A^k (A^k)^\dagger$. Assume that $(A^{\textcircled{D}})^n = (A^D)^n A^k (A^k)^\dagger$, for $n \geq 1$. Then

$$(A^{\textcircled{D}})^{n+1} = (A^{\textcircled{D}})^n A^{\textcircled{D}} = (A^D)^n (A^k (A^k)^\dagger A^k) A^D (A^k)^\dagger = (A^D)^{n+1} A^k (A^k)^\dagger.$$

The rest is clear. □

In the following theorem, for a Drazin invertible operator A, notice that $A^{\textcircled{D}}$ is n-potent if and only if A^D is n-potent.

Theorem 1. *If $A \in \mathcal{B}(X)^D$, $k = ind(A)$ and $n \geq 2$, then the following statements are equivalent:*

(i) $A^{\textcircled{D}}$ *is n-potent;*
(ii) $(A^{\textcircled{D}})^{n-1} A^k = A^k$;
(iii) $A^{k+n-1} = A^k$;
(iv) $A^D A^k = A^{k+n-2}$;
(v) $A^k (A^k)^{(1,3)} = (A^{\textcircled{D}})^{n-1}$, *for* $(A^k)^{(1,3)} \in (A^k)\{1,3\}$;
(v') $A^k (A^k)^\dagger = (A^{\textcircled{D}})^{n-1}$;
(vi) $A^D (A^{\textcircled{D}})^{n-1} = A^{\textcircled{D}}$;
(vii) $A A^{\textcircled{D}} = (A^{\textcircled{D}})^{n-1}$;
(viii) $(A^{\textcircled{D}})^{n-1}$ *is orthogonal projector;*
(ix) $(A^{\textcircled{D}})^{n-1}$ *is an idempotent;*
(x) $A^D A = (A^D)^{n-1}$;
(xi) A^D *is n-potent;*
(xii) $(A^k)^* = (A^k)^* (A^{\textcircled{D}})^{n-1}$;

(xiii) $(A^{\mathbb{D}})^m = (A^{\mathbb{D}})^{m+n-1}$, *for some/any* $m \geq 1$*;*
(xiv) $(A^D)^m = (A^D)^{m+n-1}$, *for some/any* $m \geq 1$.

Proof. (i) $\Rightarrow$ (ii): By the hypothesis $(A^{\mathbb{D}})^n = A^{\mathbb{D}}$, we obtain

$$A^k = A^{\mathbb{D}}A^{k+1} = (A^{\mathbb{D}})^n A^{k+1} = (A^{\mathbb{D}})^{n-1}A^k.$$

(ii) $\Rightarrow$ (iii): Using $(A^{\mathbb{D}})^{n-1}A^k = A^k$, we get

$$A^{k+n-1} = A^k A^{n-1} = (A^{\mathbb{D}})^{n-1}A^{k+n-1} = (A^{\mathbb{D}})^{n-2}A^{k+n-2} = \cdots = A^{\mathbb{D}}A^{k+1} = A^k.$$

(iii) $\Rightarrow$ (iv): Multiplying the equality $A^{k+n-1} = A^k$ by A^D from the left hand side, notice that $A^D A^k = A^D A^{k+n-1} = A^{k+n-2}$.

(iv) $\Rightarrow$ (v): The hypothesis $A^D A^k = A^{k+n-2}$ and Lemma 1 imply

$$A^k(A^k)^{(1,3)} = (A^D)^{n-2}A^{k+n-2}(A^k)^{(1,3)} = (A^D)^{n-1}A^k(A^k)^{(1,3)} = (A^{\mathbb{D}})^{n-1},$$

for $(A^k)^{(1,3)} \in (A^k)\{1,3\}$.

(v) $\Rightarrow$ (vi): If $A^k(A^k)^{(1,3)} = (A^{\mathbb{D}})^{n-1}$, for $(A^k)^{(1,3)} \in (A^k)\{1,3\}$, then $A^D(A^{\mathbb{D}})^{n-1} = A^D A^k(A^k)^{(1,3)} = A^{\mathbb{D}}$.

(vi) $\Rightarrow$ (vii): Applying $A^D(A^{\mathbb{D}})^{n-1} = A^{\mathbb{D}}$ and Lemma 1, we have

$$AA^{\mathbb{D}} = AA^D(A^{\mathbb{D}})^{n-1} = AA^D(A^D)^{n-1}A^k(A^k)^{(1,3)} = (A^D)^{n-1}A^k(A^k)^{(1,3)} = (A^{\mathbb{D}})^{n-1},$$

for $(A^k)^{(1,3)} \in (A^k)\{1,3\}$.

(vii) $\Rightarrow$ (i): Multiplying $AA^{\mathbb{D}} = (A^{\mathbb{D}})^{n-1}$ by $A^{\mathbb{D}}$ from the left hand side, we see that $A^{\mathbb{D}} = (A^{\mathbb{D}})^n$.

(v) $\Rightarrow$ (viii) $\Rightarrow$ (ix): These implications are evident.

(ix) $\Rightarrow$ (vii): Because $(A^{\mathbb{D}})^{n-1}$ is orthogonal projector, then

$$(A^{\mathbb{D}})^n = A^{\mathbb{D}}(A^{\mathbb{D}})^{n-1} = A^{\mathbb{D}}(A^{\mathbb{D}})^{2n-2} = (A^{\mathbb{D}})^{2n-1}.$$

Therefore,

$$AA^{\mathbb{D}} = A^n(A^{\mathbb{D}})^n = A^n(A^{\mathbb{D}})^{2n-1} = AA^{\mathbb{D}}(A^{\mathbb{D}})^{n-1} = (A^{\mathbb{D}})^{n-1}.$$

(iii) $\Rightarrow$ (x): Since $A^{k+n-1} = A^k$, then

$$A^D A = (A^D)^{k+n-1}A^{k+n-1} = (A^D)^{k+n-1}A^k = (A^D)^{n-1}.$$

(x) $\Rightarrow$ (xi): The condition $A^D A = (A^D)^{n-1}$ yields $A^D = (A^D A)A^D = (A^D)^n$.

(xi) $\Rightarrow$ (iii): From $A^D = (A^D)^n$, we get

$$A^{k+n-1} = A^{k+n}A^D = A^{k+n}(A^D)^n = A^{k+1}A^D = A^k.$$

(v') $\Rightarrow$ (xii): Multiplying $A^k(A^k)^\dagger = (A^{\mathbb{D}})^{n-1}$ by $(A^k)^*$ from the left hand side, we see that $(A^k)^* = (A^k)^*(A^{\mathbb{D}})^{n-1}$.

(xii) $\Rightarrow$ (v'): The assumption $(A^k)^* = (A^k)^*(A^{\mathbb{D}})^{n-1}$ and Lemma 1 imply

$$\begin{aligned} A^k(A^k)^\dagger &= ((A^k)^\dagger)^*(A^k)^* = ((A^k)^\dagger)^*(A^k)^*(A^{\mathbb{D}})^{n-1} \\ &= (A^k(A^k)^\dagger A^k)(A^D)^{n-1}(A^k)^\dagger = (A^D)^{n-1}A^k(A^k)^\dagger = (A^{\mathbb{D}})^{n-1}. \end{aligned}$$

(i) $\Rightarrow$ (xiii): It is clear.

(xiii) $\Rightarrow$ (vii): Assume that $(A^{\mathbb{D}})^m = (A^{\mathbb{D}})^{m+n-1}$, for $m \geq 1$. Then

$$AA^{\mathbb{D}} = A^m(A^{\mathbb{D}})^m = A^m(A^{\mathbb{D}})^{m+n-1} = AA^{\mathbb{D}}(A^{\mathbb{D}})^{n-1} = (A^{\mathbb{D}})^{n-1}.$$

(x) $\Rightarrow$ (xiv): This implication is obvious.

(xiv) $\Rightarrow$ (x): Applying $(A^D)^m = (A^D)^{m+n-1}$, for $m \geq 1$, we have

$$A^D = A^{m-1}(A^D)^m = A^{m-1}(A^D)^{m+n-1} = (A^D)^n.$$

□

By Theorem 1, we can obtain more characterizations for $A^{\mathbb{D}}$ to be n-potent operator. Recall that $B \in \mathcal{B}(Y,X)\backslash\{0\}$ is an outer inverse of $A \in \mathcal{B}(X,Y)$ if $BAB = B$ is satisfied.

Corollary 1. *If $A \in \mathcal{B}(X)^D$, $k = ind(A)$ and $n \geq 2$, then the following statements are equivalent:*

(i) *$A^{\mathbb{D}}$ is n-potent;*
(ii) *$(A^{\mathbb{D}})^n$ is an outer inverse of A;*
(iii) *$A^D(A^{\mathbb{D}})^{n-1}$ is an outer inverse of A;*
(iv) *$(A^D)^n$ is an outer inverse of A;*
(v) *A^DA is an outer inverse of A^{n-1}.*

Proof. (i) $\Rightarrow$ (ii): Since $(A^{\mathbb{D}})^n = A^{\mathbb{D}}$, we conclude that

$$(A^{\mathbb{D}})^n = A^{\mathbb{D}} = A^{\mathbb{D}}AA^{\mathbb{D}} = (A^{\mathbb{D}})^nA(A^{\mathbb{D}})^n,$$

i.e. $(A^{\mathbb{D}})^n$ is an outer inverse of A.

(ii) $\Rightarrow$ (i): Notice that $(A^{\mathbb{D}})^n = (A^{\mathbb{D}})^nA(A^{\mathbb{D}})^n = (A^{\mathbb{D}})^{2n-1}$. By Theorem 1(xiii), for $m = n$, we deduce that $A^{\mathbb{D}}$ is n-potent.

(i) $\Rightarrow$ (iii): Using Theorem 1(vi), $A^D(A^{\mathbb{D}})^{n-1} = A^{\mathbb{D}}$ and so $A^D(A^{\mathbb{D}})^{n-1}$ is an outer inverse of A.

(iii) $\Rightarrow$ (i): Firstly, by Lemma 1, we can check that $AA^D(A^{\mathbb{D}})^m = (A^{\mathbb{D}})^m$, for $m \geq 1$. Now, from $A^D(A^{\mathbb{D}})^{n-1} = A^D(A^{\mathbb{D}})^{n-1}AA^D(A^{\mathbb{D}})^{n-1} = A^D(A^{\mathbb{D}})^{2n-2}$, we get $AA^D(A^{\mathbb{D}})^{n-1} = AA^D(A^{\mathbb{D}})^{2n-2}$, that is, $(A^{\mathbb{D}})^{n-1} = (A^{\mathbb{D}})^{2n-2}$. The rest is clear by Theorem 1(xiii).

(i) $\Rightarrow$ (iv): Applying Theorem 1(x), we see that $(A^D)^n = A^D$ is an outer inverse of A.

(iv) $\Rightarrow$ (i): Because $(A^D)^n = (A^D)^nA(A^D)^n = (A^D)^{2n-1}$, by Theorem 1(xiii), we conclude that $A^{\mathbb{D}}$ is n-potent.

(i) $\Rightarrow$ (v): According to Theorem 1(ix), $A^DA = (A^D)^{n-1} = (A^{n-1})^D$ is an outer inverse of A^{n-1}.

(v) $\Rightarrow$ (i): Suppose that A^DA is an outer inverse of A^{n-1}. Then $A^DA = A^DAA^{n-1}A^DA$ $= A^DA^n$ gives

$$(A^D)^{n-1} = (A^DA)(A^D)^{n-1} = A^DA^n(A^D)^{n-1} = (A^D)^nA^n = A^DA.$$

By Theorem 1(ix), $A^{\mathbb{D}}$ is n-potent. $\square$

In the case that $n = 2$ in Theorem 1 and Corollary 1, we present necessary and sufficient conditions for $A^{\mathbb{D}}$ to be an idempotent. Remark that $A^{\mathbb{D}}$ is an idempotent if and only if $A^{\mathbb{D}}$ is the orthogonal projector.

Corollary 2. *If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, then the following statements are equivalent:*

(i) $A^{\mathbb{D}}$ *is an idempotent;*
(ii) $A^{\mathbb{D}}A^k = A^k$;
(iii) $A^{k+1} = A^k$;
(iv) $A^D A^k = A^k$;
(v) $A^k(A^k)^{(1,3)} = A^{\mathbb{D}}$, *for* $(A^k)^{(1,3)} \in (A^k)\{1,3\}$;
(v') $A^k(A^k)^{\dagger} = A^{\mathbb{D}}$;
(vi) $A^D A^{\mathbb{D}} = A^{\mathbb{D}}$;
(vii) $AA^{\mathbb{D}} = A^{\mathbb{D}}$;
(viii) $A^{\mathbb{D}}$ *is orthogonal projector;*
(ix) $A^D A = A^D$;
(x) A^D *is an idempotent;*
(xi) $(A^k)^* = (A^k)^* A^{\mathbb{D}}$;
(xii) $(A^{\mathbb{D}})^m = (A^{\mathbb{D}})^{m+1}$, *for some/any* $m \geq 1$;
(xiii) $(A^D)^m = (A^D)^{m+1}$, *for some/any* $m \geq 1$;
(xiv) $(A^{\mathbb{D}})^2$ *is an outer inverse of* A;
(xv) $A^D A^{\mathbb{D}}$ *is an outer inverse of* A;
(xvi) $(A^D)^2$ *is an outer inverse of* A;
(xvii) $A^D A$ *is an outer inverse of* A.

Now, we consider the relation between n-potency of A and $A^{\mathbb{D}}$. We firstly show that if A is n-potent, then $A^{\mathbb{D}}$ is n-potent.

Lemma 2. *Let $n \geq 2$. If $A \in \mathcal{B}(X)^D$ is n-potent, then $A^{\mathbb{D}}$ is n-potent.*

Proof. Using $A^n = A$, we obtain $A^{\mathbb{D}} = A(A^{\mathbb{D}})^2 = A^n(A^{\mathbb{D}})^{n+1} = A(A^{\mathbb{D}})^{n+1} = (A^{\mathbb{D}})^n$. $\square$

In the following example, we remark that the converse of Lemma 2 does not hold in general.

Example 1. Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

on $X = \mathbb{C}^2$. Because

$$A^{\mathbb{D}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A^n = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad n \geq 2,$$

we deduce that $(A^{\mathbb{D}})^n = A^{\mathbb{D}}$ and $A^n \neq A$, for $n \geq 2$. Hence, $A^{\mathbb{D}}$ is n-potent and A is not n-potent, for $n \geq 2$.

We establish an additional condition under which n-potency of $A^{\mathbb{D}}$ implies n-potency of A. Also, we give more characterizations of n-potency of A which involve the core–EP inverse of A.

Theorem 2. *If $A \in \mathcal{B}(X)^D$ and $n \geq 2$, then the following statements are equivalent:*

(i) *A is n-potent;*
(ii) *$A^{\mathbb{D}}$ is n-potent and $(I - A^{n-1})A(I - AA^{\mathbb{D}}) = 0$;*
(iii) *$A^{\mathbb{D}}A^n = A^{\mathbb{D}}A$ and $A - AA^{\mathbb{D}}A$ is n-potent;*
(iv) *$A^{\mathbb{D}}A^n = A^{\mathbb{D}}A$ and $(I - AA^{\mathbb{D}})A(I - A^{n-1}) = 0$;*
(v) *$A^2A^{\mathbb{D}}$ is n-potent and $(I - A^{n-1})A(I - AA^{\mathbb{D}}) = 0$.*

Proof. (i) $\Rightarrow$ (ii): By Lemma 2, A is n-potent implies that $A^{\mathbb{D}}$ is n-potent. The rest is evident by $A^n = A$.

(ii) $\Rightarrow$ (i): According to [15, Corollary 2.2], for $k = ind(A)$, the operators A and $A^{\mathbb{D}}$ can be represented with respect to the orthogonal sum $X = R(A^k) \oplus N((A^k)^*)$ as:

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \qquad \text{and} \qquad A^{\mathbb{D}} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.1}$$

where $A_1 \in \mathcal{B}(R(A^k))$ is invertible and $A_3 \in \mathcal{B}[N((A^k)^*)]$ is nilpotent. We observe that

$$A^n = \begin{bmatrix} A_1^n & U \\ 0 & A_3^n \end{bmatrix} \qquad \text{and} \qquad I - AA^{\mathbb{D}} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix},$$

where $U = \sum_{j=0}^{n-1} A_1^{n-1-j} A_2 A_3^j$. Since $A^{\mathbb{D}}$ is n-potent, from

$$\begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = A^{\mathbb{D}} = (A^{\mathbb{D}})^n = \begin{bmatrix} A_1^{-n} & 0 \\ 0 & 0 \end{bmatrix},$$

notice that $A_1^{n-1} = I$. Further, $(I - A^{n-1})A(I - AA^{\mathbb{D}}) = 0$ is equivalent to

$$\begin{bmatrix} 0 & A_2 \\ 0 & A_3 \end{bmatrix} = A(I - AA^{\mathbb{D}}) = A^n(I - AA^{\mathbb{D}}) = \begin{bmatrix} 0 & U \\ 0 & A_3^n \end{bmatrix},$$

which gives $A_2 = U$ and $A_3 = A_3^n$. Thus, $A = A^n$.

(i) $\Leftrightarrow$ (iii): Using the same representations of A and $A^{\mathbb{D}}$ as in (2.1), we have that A is n-potent if and only if $A_1^{n-1} = I$, $A_2 = U$ and $A_3 = A_3^n$. We can see that

$$\begin{bmatrix} A_1^{n-1} & A_1^{-1}U \\ 0 & 0 \end{bmatrix} = A^{\mathbb{D}}A^n = A^{\mathbb{D}}A = \begin{bmatrix} I & A_1^{-1}A_2 \\ 0 & 0 \end{bmatrix}$$

is equivalent to $A_1^{n-1} = I$ and $A_2 = U$. Since

$$A - AA^{\mathbb{D}}A = (I - AA^{\mathbb{D}})A = \begin{bmatrix} 0 & 0 \\ 0 & A_3 \end{bmatrix},$$

then $A - AA^{\mathbb{D}}A$ is n-potent if and only if A_3 is n-potent.

(i) $\Leftrightarrow$ (iv): This equivalence can be proved as (i) $\Leftrightarrow$ (iii) when we note that $(I - AA^{\mathbb{D}})A(I - A^{n-1}) = 0$ is equivalent to

$$\begin{bmatrix} 0 & 0 \\ 0 & A_3 \end{bmatrix} = (I - AA^{\mathbb{D}})A = (I - AA^{\mathbb{D}})A^n = \begin{bmatrix} 0 & 0 \\ 0 & A_3^n \end{bmatrix},$$

that is $A_3 = A_3^n$.

(i) $\Leftrightarrow$ (v): Applying (2.1), $A^2A^{\mathbb{D}} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ is n-potent if and only if $A_1^n = A_1$, i.e. $A_1^{n-1} = I$. The rest follows as in the part (ii) $\Rightarrow$ (i) of this proof. $\square$

For $n = 2$ in Theorem 2, we get the next result.

Corollary 3. *If $A \in \mathcal{B}(X)^D$, then the following statements are equivalent:*

(i) *A is an idempotent;*
(ii) *$A^{\mathbb{D}}$ is an idempotent and $(I - A)A(I - AA^{\mathbb{D}}) = 0$;*
(iii) *$A^{\mathbb{D}}A^2 = A^{\mathbb{D}}A$ and $A - AA^{\mathbb{D}}A$ is an idempotent;*
(iv) *$A^{\mathbb{D}}A^2 = A^{\mathbb{D}}A$ and $(I - AA^{\mathbb{D}})A(I - A) = 0$;*
(v) *$A^2A^{\mathbb{D}}$ is an idempotent and $(I - A)A(I - AA^{\mathbb{D}}) = 0$.*

We also study equivalent conditions under which $A^{\mathbb{D}} = A^{k+1}$.

Theorem 3. *If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, then the following statements are equivalent:*

(i) *$A^{\mathbb{D}} = A^{k+1}$;*
(ii) *$A^k(A^k)^\dagger = A^{k+2}$;*
(ii') *$A^k(A^k)^{(1,3)} = A^{k+2}$, for $(A^k)^{(1,3)} \in (A^k)\{1,3\}$;*
(iii) *A^{k+2} is orthogonal projector;*
(iv) *$A^k = A^{2k+2}$ and $(A^{k+2})^* = A^{k+2}$;*
(v) *$A^DA^k = A^{2k+1}$ and $(A^{k+2})^* = A^{k+2}$.*

Proof. (i) $\Rightarrow$ (ii): The equalities $A^{\mathbb{D}} = A^{k+1}$ and $A^{\mathbb{D}} = A^DA^k(A^k)^\dagger$ yield

$$A^{k+2} = AA^{k+1} = AA^{\mathbb{D}} = (AA^DA^k)(A^k)^\dagger = A^k(A^k)^\dagger.$$

(ii) $\Rightarrow$ (iii): It is clear.

(iii) $\Rightarrow$ (iv): Because A^{k+2} is orthogonal projector, $(A^{k+2})^* = A^{k+2}$ and $A^{k+2} = A^{2k+4}$ which gives $A^k = (A^D)^2 A^{k+2} = (A^D)^2 A^{2k+4} = A^{2k+2}$.

(iv) $\Rightarrow$ (v): Multiplying $A^k = A^{2k+2}$ by A^D from the left hand side, we obtain $A^D A^k = A^{2k+1}$.

(v) $\Rightarrow$ (i): The assumptions $A^D A^k = A^{2k+1}$ and $(A^{k+2})^* = A^{k+2}$ imply

$$\begin{aligned} A^{\mathbb{D}} &= (A^D A^k)(A^k)^\dagger = A^{2k+1}(A^k)^\dagger = A^D A^{k+2} A^k (A^k)^\dagger = A^D (A^{k+2})^* A^k (A^k)^\dagger \\ &= A^D (A^{k+2})^* = A^D A^{k+2} = A^{k+1}. \end{aligned}$$

□

The equality $A^{\mathbb{D}} = (A^k)^\dagger$ is studied in the next result.

Theorem 4. *If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, then the following statements are equivalent:*

(i) $A^{\mathbb{D}} = (A^k)^\dagger$;
(ii) $A^D A^k = (A^k)^\dagger A^k$;
(iii) $A^k = (A^k)^\dagger A^{k+1}$;
(iv) $(A^k)^* = A^*(A^k)^\dagger A^k$;
(v) $A^D A^k (A^k)^* = (A^k)^*$;
(vi) $A^k (A^k A^D)^* = A^k$.

Proof. (i) $\Rightarrow$ (ii): Because $(A^k)^\dagger = A^{\mathbb{D}} = A^D A^k (A^k)^\dagger$, we have

$$(A^k)^\dagger A^k = A^D A^k (A^k)^\dagger A^k = A^D A^k.$$

(ii) $\Rightarrow$ (iii): The hypothesis $A^D A^k = (A^k)^\dagger A^k$ gives $A^k = (A^D A^k)A = (A^k)^\dagger A^{k+1}$.

(iii) $\Rightarrow$ (i): From $A^k = (A^k)^\dagger A^{k+1}$, we get

$$A^{\mathbb{D}} = A^D A^k (A^k)^\dagger = A^k A^D (A^k)^\dagger = (A^k)^\dagger (A^{k+1} A^D)(A^k)^\dagger = (A^k)^\dagger A^k (A^k)^\dagger = (A^k)^\dagger.$$

(iii) $\Leftrightarrow$ (iv) and (v) $\Leftrightarrow$ (vi): These equivalences follow by properties of the adjoint operator.

(ii) $\Rightarrow$ (v): The condition $A^D A^k = (A^k)^\dagger A^k$ yield $A^D A^k (A^k)^* = (A^k)^\dagger A^k (A^k)^* = (A^k)^*$.

(v) $\Rightarrow$ (ii): Multiplying $A^D A^k (A^k)^* = (A^k)^*$ by $(A^\dagger)^*$ from the right hand side, we obtain $A^D A^k = (A^k)^\dagger A^k$. □

We can consider the equality $A^{\mathbb{D}} = (A^k)^*$ too.

Theorem 5. *If $A \in \mathcal{B}(X)^D$ and $k = ind(A)$, then the following statements are equivalent:*

(i) $A^{\mathbb{D}} = (A^k)^*$;
(ii) $A^D A^k = (A^k)^* A^k$;
(iii) $A^D [(A^k)^\dagger]^* = (A^k)^\dagger A^k$.

Proof. (i) $\Rightarrow$ (ii)–(iii): Since $(A^k)^* = A^{\textcircled{D}} = A^D A^k (A^k)^\dagger$, we deduce that $(A^k)^* A^k = A^D A^k$ and

$$(A^k)^\dagger A^k = (A^k)^*[(A^k)^\dagger]^* = A^D A^k (A^k)^\dagger [(A^k)^\dagger]^* = A^D [(A^k)^\dagger]^*.$$

(ii) $\Rightarrow$ (i): Multiplying $A^D A^k = (A^k)^* A^k$ by $(A^k)^\dagger$ from the right hand side, we get $A^{\textcircled{D}} = (A^k)^*$.

(iii) $\Rightarrow$ (i): The assumption $A^D[(A^k)^\dagger]^* = (A^k)^\dagger A^k$ implies

$$A^{\textcircled{D}} = A^D A^k (A^k)^\dagger = A^D [(A^k)^\dagger]^* (A^k)^* = (A^k)^\dagger A^k (A^k)^* = (A^k)^*.$$

□

3. Application to the core inverse

Applying the results of Section 2, we obtain many characterizations for n-potency of the core inverse.

When $ind(A) = 1$ in Lemma 1, we have the next representations for the power of the core inverse.

Lemma 3. *If $A \in \mathcal{B}(X)^\#$, then*

$$(A^{\textcircled{\#}})^n = (A^\#)^n A A^\dagger = (A^\#)^n A A^{(1,3)},$$

for any $n \geq 1$ and $A^{(1,3)} \in A\{1,3\}$.

Taking $ind(A) = 1$ in Theorem 1 and Corollary 1, we characterize n-potency of the core inverse. Note that $A^{\textcircled{\#}}$ is n-potent if and only if A is n-potent.

Corollary 4. *If $A \in \mathcal{B}(X)^\#$ and $n \geq 2$, then the following statements are equivalent:*

(i) *$A^{\textcircled{\#}}$ is n-potent;*
(ii) *$(A^{\textcircled{\#}})^{n-1} A = A$;*
(iii) *A is n-potent;*
(iv) *$A^\# A = A^{n-1}$;*
(v) *$A A^{(1,3)} = (A^{\textcircled{\#}})^{n-1}$, for $A^{(1,3)} \in A\{1,3\}$;*
(v') *$A A^\dagger = (A^{\textcircled{\#}})^{n-1}$;*
(vi) *$A^\# (A^{\textcircled{\#}})^{n-1} = A^{\textcircled{\#}}$;*
(vii) *$A A^{\textcircled{\#}} = (A^{\textcircled{\#}})^{n-1}$;*
(viii) *$(A^{\textcircled{\#}})^{n-1}$ is orthogonal projector;*
(ix) *$(A^{\textcircled{\#}})^{n-1}$ is an idempotent;*
(x) *$A^\# A = (A^\#)^{n-1}$;*
(xi) *$A^\#$ is n-potent;*
(xii) *$A^* = A^* (A^{\textcircled{\#}})^{n-1}$;*
(xiii) *$(A^{\textcircled{\#}})^m = (A^{\textcircled{\#}})^{m+n-1}$, for some/any $m \geq 1$;*
(xiv) *$(A^\#)^m = (A^\#)^{m+n-1}$, for some/any $m \geq 1$;*
(xv) *$(A^{\textcircled{\#}})^n$ is an outer inverse of A;*
(xvi) *$A^\# (A^{\textcircled{\#}})^{n-1}$ is an outer inverse of A;*

(xvi) $(A^{\#})^n$ *is an outer inverse of* A;
(xvii) $A^{\#}A$ *is an outer inverse of* A^{n-1};
(xviii) $A^{n-1} = A^{\textcircled{\#}}A$;
(xix) $A^2A^{\textcircled{\#}}$ *is n-potent and* $(I - A^{n-1})A(I - AA^{\textcircled{\#}}) = 0$.

Choosing $n = 2$ in Corollary 4, we get equivalent conditions for $A^{\textcircled{\#}}$ to be an idempotent. We can observe that $A^{\textcircled{\#}}$ is an idempotent if and only if A is an idempotent.

Corollary 5. *If* $A \in \mathcal{B}(X)^{\#}$, *then the following statements are equivalent:*

(i) $A^{\textcircled{\#}}$ *is an idempotent;*
(ii) $A^{\textcircled{\#}}A = A$;
(iii) A *is an idempotent;*
(iv) $A^{\#}A = A$;
(v) $AA^{(1,3)} = A^{\textcircled{\#}}$, *for* $A^{(1,3)} \in A\{1,3\}$;
(v') $AA^{\dagger} = A^{\textcircled{\#}}$;
(vi) $A^{\#}A^{\textcircled{\#}} = A^{\textcircled{\#}}$;
(vii) $AA^{\textcircled{\#}} = A^{\textcircled{\#}}$;
(viii) $A^{\textcircled{\#}}$ *is orthogonal projector;*
(ix) $A^{\#}A = A^{\#}$;
(x) $A^{\#}$ *is an idempotent;*
(xi) $A^* = A^*A^{\textcircled{\#}}$;
(xii) $(A^{\textcircled{\#}})^m = (A^{\textcircled{\#}})^{m+1}$, *for some/any* $m \geq 1$;
(xiii) $(A^{\#})^m = (A^{\#})^{m+1}$, *for some/any* $m \geq 1$;
(xiv) $(A^{\textcircled{\#}})^2$ *is an outer inverse of* A;
(xv) $A^{\#}A^{\textcircled{\#}}$ *is an outer inverse of* A;
(xvi) $(A^{\#})^2$ *is an outer inverse of* A;
(xvii) $A^{\#}A$ *is an outer inverse of* A;
(xviii) $A^2A^{\textcircled{\#}}$ *is an idempotent and* $(I - A)A(I - AA^{\textcircled{\#}}) = 0$;
(xix) $A^{\textcircled{\#}}A^2 = A^{\textcircled{\#}}A$.

Theorem 3 implies several characterizations for the equality $A^{\textcircled{\#}} = A^2$ to be satisfied.

Corollary 6. *If* $A \in \mathcal{B}(X)^{\#}$, *then the following statements are equivalent:*

(i) $A^{\textcircled{\#}} = A^2$;
(ii) $AA^{\dagger} = A^3$;
(ii') $AA^{(1,3)} = A^3$, *for* $A^{(1,3)} \in A\{1,3\}$;
(iii) A^3 *is orthogonal projector;*
(iv) $A = A^4$ *and* $(A^3)^* = A^3$;
(v) $A^{\#}A = A^3$ *and* $(A^3)^* = A^3$.

Recall that $A \in \mathcal{B}(X)^{\#}$ is an EP operator if $A^{\#} = A^{\dagger}$ [9, 14]. By Theorem 7, we obtain some well-known characterizations of an EP operator.

Corollary 7. *If* $A \in \mathcal{B}(X)^{\#}$, *then the following statements are equivalent:*

(i) $A^{\textcircled{\#}} = A^{\dagger}$;
(ii) $A^{\#}A = A^{\dagger}A$;
(iii) $A = A^{\dagger}A^{2}$;
(iv) $A^{*} = A^{*}A^{\dagger}A$;
(v) $A^{\#}AA^{*} = A^{*}$;
(vi) $A(AA^{\#})^{*} = A$;
(vii) *A is EP.*

Theorem 5 gives necessary and equivalent conditions for $A^{\textcircled{\#}} = A^{*}$ to hold.

Corollary 8. *If $A \in \mathcal{B}(X)^{\#}$, then the following statements are equivalent:*

(i) $A^{\textcircled{\#}} = A^{*}$;
(ii) $A^{\#}A = A^{*}A$;
(iii) $A^{\#}(A^{\dagger})^{*} = A^{\dagger}A$.

In addition, if any of statements (i)–(iii) *holds, then A is EP.*

REFERENCES

[1] O. Baksalary and G. Trenkler, "Core inverse of matrices." *Linear Multilinear Algebra*, vol. 58, no. 6, pp. 681–697, 2010, doi: 10.1080/03081080902778222.

[2] O. Baksalary and G. Trenkler, "On matrices whose Moore-Penorse inverse is idempotent." *Linear Multilinear Algebra*, vol. 70, pp. 2014–2026, 2022, doi: 10.1080/03081087.2020.1781038.

[3] R. Behera, G. Maharana, and J. Sahoo, " Further results on weighted core-EP inverse of matrices." *Results Math.*, vol. 75, 2020, 174, doi: 10.1007/s00025-020-01296-z.

[4] G. Dolinar, B. Kuzma, J. Marovt, and B.Ungor, "Properties of core-EP order in rings with involution." *Front. Math. China*, vol. 14, no. 4, pp. 715–736, 2019, doi: 10.1007/s11464-019-0782-8.

[5] D. Ferreyra, F. Levis, and N. Thome, "Revisiting the core EP inverse and its extension to rectangular matrices." *Quaest. Math.*, vol. 41, no. 2, pp. 265–281, 2018, doi: 10.2989/16073606.2017.1377779.

[6] Y. Gao and J. Chen, "Pseudo core inverses in rings with involution." *Commun. Algebra*, vol. 46, no. 1, pp. 38–50, 2018, doi: 10.1080/00927872.2016.1260729.

[7] Y. Gao, J. Chen, and P. Patricio, "Continuity of the core-EP inverse and its applications." *Linear Multilinear Algebra*, vol. 69, no. 3, pp. 557–571, 2019, doi: 10.1080/03081087.2019.1608899.

[8] Y. Ke, L. Wang, and J. Chen, "The core inverse of a product and 2×2 matrices." *Bull. Malays. Math. Sci. Soc.*, vol. 42, pp. 51–66, 2019, doi: 10.1007/s40840-017-0464-1.

[9] J. Koiha and P. Patricio, "Elements of rings with equal spectral idempotents." *J. Aust. Math. Soc.*, vol. 72, pp. 137–152, 2002, doi: 10.1017/S1446788700003657.

[10] I. Kyrchei, "Weighted quaternion core-EP, DMP, MPD, and CMP inverses and their determinantal representations." *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 114, 2020, 198, doi: 10.1007/s13398-020-00930-3.

[11] H. Ma and P. Stanimirović, "Characterizations, approximation and perturbations of the core-EP inverse," *Appl. Math. Comput.*, vol. 359, no. 6, pp. 404–417, 2019, doi: 10.1016/j.amc.2019.04.071.

[12] D. Mosić, "Characterizations of k-potent elements in rings." *Annali di Matematica*, vol. 194, pp. 1157–1168, 2015, doi: 10.1007/s10231-014-0415-5.

[13] D. Mosić, "On the Moore–Penrose inverse in rings with involution." *Miskolc Mathematical Notes*, vol. 18, no. 1, pp. 347–351, 2017, doi: 10.18514/MMN.2017.1913.

[14] D. Mosić, *Generalized inverses.* Niš: Faculty of Sciences and Mathematics, University of Niš, 2018.

[15] D. Mosić, "Weighted core–EP inverse of an operator between Hilbert spaces." *Linear Multilinear Algebra*, vol. 67, no. 2, pp. 278–298, 2019, doi: 10.1080/03081087.2017.1418824.
[16] D. Mosić and D. Djordjević, "The gDMP inverse of Hilbert space operators." *Journal of Spectral Theory*, vol. 8, no. 2, pp. 555–573, 2018, doi: 10.4171/JST/207.
[17] K. Prasad and K. Mohana, "Core–EP inverse." *Linear Multilinear Algebra*, vol. 62, no. 6, pp. 792–802, 2014, doi: 10.1080/03081087.2013.791690.
[18] K. Prasad, M. Raj, and M. Vinay, "Iterative method to find core-EP inverse." *Bull. Kerala Math. Assoc., Special Issue*, vol. 16, no. 1, pp. 139–152, 2018.
[19] J. Sahoo, R. Behera, P. Stanimirović, V. Katsikis, and H. Ma, "Core and core-EP inverses of tensors." *Comput. Appl. Math.*, vol. 39, 2020, 9, doi: 10.1007/s40314-019-0983-5.
[20] H. Wang, "Core-EP decomposition and its applications." *Linear Algebra Appl.*, vol. 508, pp. 289–300, 2016, doi: 10.1016/j.laa.2016.08.008.
[21] M. Zhou, J. Chen, T. Li, and D. Wang, " Three limit representations of the core-EP inverse." *Filomat*, vol. 32, pp. 5887–5894, 2018, doi: 10.2298/FIL1817887Z.
[22] H. Zhu, J. Chen, and Y. Zhou, "On elements whose Moore-Penrose inverse is idempotent in a $*$-ring." *Turk. J. Math.*, vol. 45, pp. 878–889, 2021, doi: 10.3906/mat-2101-93.
[23] H. Zou, J. Chen, and P. Patricio, " Reverse order law for the core inverse in rings." *Mediterr. J. Math.*, vol. 15, 2018, 145, doi: 10.1007/s00009-018-1189-6.

Authors' addresses

Dijana Mosić
Faculty of Sciences and Mathematics, University of Niš, P.O. Box 224, 18000 Niš, Serbia
E-mail address: dijana@pmf.ni.ac.rs

Daochang Zhang
(**Corresponding author**) College of Sciences, Northeast Electric Power University, Jilin, P.R. China
E-mail address: daochangzhang@126.com

Jianping Hu
College of Sciences, Northeast Electric Power University, Jilin, P.R. China
E-mail address: neduhjp307@163.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 957–963

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4273

SOME PROPERTIES OF r-SUPPLEMENTED MODULES

CELIL NEBIYEV AND NURHAN SÖKMEZ

Received 03 June, 2022

Abstract. In this work, r-supplemented modules are defined and some properties of these modules are investigated. It is proved that every factor module and every homomorphic image of an r-supplemented module are r-supplemented. Let M be an $R-$module and $M = M_1 + M_2 + ... + M_n$. If M_i is r-supplemented for each $i = 1, 2, ..., n$, then M is also r-supplemented. Let M be an r-supplemented module. Then every finitely $M-$generated $R-$module is r-supplemented.

2010 *Mathematics Subject Classification:* 16D10; 16D70

Keywords: small submodules, radical, r-small submodules, supplemented modules

1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* submodule of M and denoted by $N \ll M$. Let M be an R -module. M is called a *hollow* module if every proper submodule of M is small in M. M is said to be *local* if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented* module if every submodule of M has a supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement $V^{'}$ with $V^{'} \leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. Let M be an R-module and $U, V \leq M$. If $M = U + V$

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

and $U \cap V \leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. Let M be an $R-$module and $K \leq M$. If $K \ll RadM$, then K is called an *r-small* submodule of M and denoted by $K \ll_r M$. Let M be an $R-$module. It is defined the relation $'\beta^*{}'$ on the set of submodules of an $R-$module M by $X\beta^*Y$ if and only if $Y+K=M$ for every $K \leq M$ such that $X+K=M$ and $X+T=M$ for every $T \leq M$ such that $Y+T=M$. Let M be an $R-$module and $K \leq V \leq M$. We say V *lies above* K in M if $V/K \ll M/K$.

More informations about (amply) supplemented modules are in [2, 3, 8, 9]. More results about Rad-supplemented modules are in [7]. The definition of r-small submodules and some properties of them are in [4, 5]. The definition of β^* relation and some properties of this relation are in [1].

Lemma 1. *Let M be an R -module. The following assertions hold.*

(i) *If $K \ll_r M$, then $K \ll M$.*
(ii) *If $L \ll_r M$ and $K \leq L$, then $K \ll_r M$.*
(iii) *If $K \ll_r L \leq M$, then $K \ll_r M$.*
(iv) *If $K_i \ll_r L_i \leq M$ for $i = 1,2,\ldots,n$, then $K_1+K_2+\ldots+K_n \ll_r L_1+L_2+\ldots+L_n$.*
(v) *If $K_i \ll_r M$ for $i = 1,2,\ldots,n$, then $K_1+K_2+\ldots+K_n \ll_r M$.*
(vi) *If $K \ll_r M$, then $(K+L)/L \ll_r M/L$ for every $L \leq M$.*
(vii) *If $K \ll M$ and $RadM$ is a supplement submodule in M, then $K \ll_r M$.*
(viii) *Let N be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll_r M$, then $f(K) \ll_r f(M)$.*
(ix) *$Rad(RadM) = \sum_{K \ll_r M} K$.*

Proof. See [4, 5]. □

2. R-Supplemented Modules

Definition 1. Let M be an $R-$module and $U, V \leq M$. If $M = U+V$ and $U \cap V \ll_r V$, then V is called an r-supplement of U in M. If every submodule of M has an r-supplement in M, then M is called an r-supplemented module. (See also [6])

Clearly we can see that every r-supplemented module is supplemented. But the converse is not true in general (See Example 1).

Lemma 2. *Let V be a Rad-supplement of U in M. Then V is an r-supplement of U in M if and only if $RadV$ is a supplement of U in $U+RadV$.*

Proof.

($\Longrightarrow$) Let V be an r-supplement of U in M. Then $U \cap V \ll RadV$ and $U \cap RadV = U \cap V \cap RadV \ll RadV$. Hence $RadV$ is a supplement of U in $U+RadV$.

($\Longleftarrow$) Let $RadV$ be a supplement of U in $U+RadV$. Since V is a Rad-supplement of U in M, $M=U+V$ and $U\cap V\leq RadV$. Then $U\cap V=U\cap V\cap RadV=U\cap RadV$. Since $RadV$ is a supplement of U in $U+RadV$, $U\cap RadV\ll RadV$. Hence $U\cap V=U\cap RadV\ll RadV$ and $U\cap V\ll_r V$. Therefore, V is an r-supplement of U in M.

□

Corollary 1. *Let V be an r-supplement of U in M. Then $Rad\,(RadV)=RadV\cap Rad\,(U+RadV)$.*

Proof. Since V is an r-supplement of U in M, by Lemma 2, $RadV$ is a supplement of U in $U+RadV$. Then by [8, 41.1 (5)], $Rad\,(RadV)=RadV\cap Rad\,(U+RadV)$, as desired. □

Lemma 3. *Let V be an r-supplement of U in M and $U\beta^*X$ in $U+RadV$ with $X\leq U+RadV$. Then V is an r-supplement of X in M.*

Proof. Since V is an r-supplement of U in M, by Lemma 2, $RadV$ is a supplement of U in $U+RadV$. Since $U\beta^*X$ in $U+RadV$, by [1, Theorem 2.6 (ii)], $RadV$ is a supplement of X in $U+RadV$. Here $X+RadV=U+RadV$ and $X\cap RadV\ll RadV$. Hence $RadV$ is a supplement of X in $X+RadV$. Let $N=U+RadV$ and $U+K=M$ with $K\leq M$. Since $U+K=M$, $N=N\cap M=N\cap(U+K)=U+N\cap K$ and since $U\beta^*X$ in N, $X+N\cap K=N$. Here $X+K=X+N\cap K+K=N+K=U+N\cap K+K=U+K=M$. Interchanging the roles of U and X, we can see that $U+T=M$ for every $T\leq M$ with $X+T=M$. Hence $U\beta^*X$ in M. Since V is an r-supplement of U in M, V is a supplement of U in M and since $U\beta^*X$ in M, by [1, Theorem 2.6 (ii)], V is a supplement of X in M. Then V is a Rad-supplement of X in M and since $RadV$ is a supplement of X in $X+RadV$, by Lemma 2, V is an r-supplement of X in M. □

Corollary 2. *Let V be an r-supplement of U in M and U lies above X in $U+RadV$. Then V is an r-supplement of X in M.*

Proof. Clear from Lemma 3. □

Lemma 4. *Let V be a supplement of U in M. If $RadV$ is a supplement submodule in V, then V is an r-supplement of U in M.*

Proof. Since V is a supplement of U in M, $M=U+V$ and $U\cap V\ll V$. Since $RadV$ is a supplement submodule in V, by Lemma 1, $U\cap V\ll_r V$. Hence V is an r-supplement of U in M, as desired. □

Corollary 3. *Let V be a supplement of U in M. If $RadV$ is a is direct summand of V, then V is an r-supplement of U in M.*

Proof. Since $RadV$ is a direct summand of V, $RadV$ is a supplement submodule in V. Then by Lemma 4, V is an r-supplement of U in M. □

Corollary 4. *Let V be a supplement of U in M. If $RadV$ is a supplement submodule in M, then V is an r-supplement of U in M.*

Proof. Let $RadV$ be a supplement of X in M. Then $M = X + RadV$ and $X \cap RadV \ll RadV$. Since $M = X + RadV$, by Modular law, $V = V \cap M = V \cap (X + RadV) = V \cap X + RadV$ and since $V \cap X \cap RadV = X \cap RadV \ll RadV$, $RadV$ is a supplement of $V \cap X$ in V. Then by Lemma 4, V is an r-supplement of U in M. □

We can also prove this Corollary 4 as follows:

Proof. Since V is a supplement of U in M, $M = U + V$ and $U \cap V \ll V$. Then $U \cap V \leq RadV$ and $U \cap V \ll M$. Since $RadV$ is a supplement submodule in M, by [8, 41.1 (5)], $U \cap V = U \cap V \cap RadV \ll RadV$. Hence $U \cap V \ll_r V$ and V is an r-supplement of U in M. □

Corollary 5. *Let V be a supplement of U in M. If $RadV$ is a is direct summand of M, then V is an r-supplement of U in M.*

Proof. Clear from Corollary 4. □

Lemma 5. *Let M be a supplemented module. If $RadV$ is a supplement submodule in V for every supplement submodule V in M, then M is r-supplemented.*

Proof. Let $U \leq M$. Since M is supplemented, U has a supplement V in M. By hypothesis, $RadV$ is a supplement submodule in V. Then by Lemma 4, V is an r-supplement of U in M. Hence M is r-supplemented, as desired. □

Corollary 6. *Let M be a supplemented module. If $RadV$ is a direct summand of V for every supplement submodule V in M, then M is r-supplemented.*

Proof. Clear from Lemma 5. □

Proposition 1. *Let M be an r-supplemented module. Then $RadM$ is a supplement submodule in M.*

Proof. Since M is r-supplemented, $RadM$ has an r-supplement V in M. Here $M = RadM + V$ and $V \cap RadM \ll_r V$. Since $V \cap RadM \ll_r V$, $V \cap RadM \ll RadV \leq RadM$. Hence $RadM$ is a supplement of V in M. □

Remark 1. The converse of Proposition 1 is not true in general. Consider the $\mathbb{Z}-$module ${}_{\mathbb{Z}}\mathbb{Q}$. Since $Rad_{\mathbb{Z}}\mathbb{Q} =_{\mathbb{Z}} \mathbb{Q}$, $Rad_{\mathbb{Z}}\mathbb{Q}$ is a supplement submodule in ${}_{\mathbb{Z}}\mathbb{Q}$. But ${}_{\mathbb{Z}}\mathbb{Q}$ is not r-supplemented.

Proposition 2. *Let M be an r-supplemented module. Then $M/Rad\,(RadM)$ is semisimple.*

Proof. Let $\frac{K}{Rad(RadM)}$ be any submodule of $\frac{M}{Rad(RadM)}$. Since M is r-supplemented, K has an r-supplement V in M. Then $M = K + V$ and $K \cap V \ll_r V$. Since $M = K + V$, $\frac{M}{Rad(RadM)} = \frac{K}{Rad(RadM)} + \frac{V + Rad(RadM)}{Rad(RadM)}$. Since $K \cap V \ll_r V$, by Lemma 1, $K \cap V \leq$

$Rad\,(RadM)$. Then $\frac{K}{Rad(RadM)} \cap \frac{V+Rad(RadM)}{Rad(RadM)} = \frac{K\cap V+Rad(RadM)}{Rad(RadM)} = 0$ and $\frac{M}{Rad(RadM)} = \frac{K}{Rad(RadM)} \oplus \frac{V+Rad(RadM)}{Rad(RadM)}$. Hence every submodule of $\frac{M}{Rad(RadM)}$ is a direct summand of $\frac{M}{Rad(RadM)}$ and $\frac{M}{Rad(RadM)}$ is semisimple. □

Lemma 6. *Let M be an $R-$module, $U \leq M$ and $M_1 \leq M$. If X is an r-supplement of $U+M_1$ in M and Y is an r-supplement of $(U+X)\cap M_1$ in M_1, then $X+Y$ is an r-supplement of U in M.*

Proof. Since X is an r-supplement of $U+M_1$ in M, $M = U+M_1+X$ and $X \cap (U+M_1) \ll_r X$. Since Y is an r-supplement of $(U+X)\cap M_1$ in M_1, $M_1 = (U+X)\cap M_1 + Y$ and $(U+X)\cap Y = (U+X)\cap M_1 \cap Y \ll_r Y$. Then $M = U+M_1+X = U+X+(U+X)\cap M_1+Y = U+X+Y$ and $U\cap(X+Y) \leq (U+X)\cap Y+(U+Y)\cap X \leq (U+M_1)\cap X+(U+X)\cap Y \ll_r X+Y$. Hence $X+Y$ is a r-supplement of U in M. □

Lemma 7. *Let M be an $R-$module, $U \leq M$ and $M_1 \leq M$. If M_1 is r-supplemented and $U+M_1$ has an r-supplement in M, then U has an r-supplement in M.*

Proof. Clear from Lemma 6. □

Corollary 7. *Let M be an $R-$module, $U \leq M$ and $M_i \leq M$ for $i = 1,2,...,n$. If M_i is r-supplemented for every $i = 1,2,...,n$ and $U+M_1+M_2+...+M_n$ has an r-supplement in M, then U has an r-supplement in M.*

Proof. Clear from Lemma 7. □

Lemma 8. *Let $M = M_1+M_2$. If M_1 and M_2 are r-supplemented, then M is also r-supplemented.*

Proof. Let $U \leq M$. Then 0 is an r-supplement of $U+M_1+M_2$ in M. Since M_2 is r-supplemented, by Lemma 7, $U+M_1$ has an r-supplement in M. Since M_1 is r-supplemented, by Lemma 7 again, U has an r-supplement in M. Hence M is r-supplemented. □

Corollary 8. *Let $M = M_1+M_2+...+M_n$. If M_i is r-supplemented for each $i = 1,2,...,n$, then M is also r-supplemented.*

Proof. Clear from Lemma 8. □

Lemma 9. *Let V be an r-supplement of U in M and $K \leq U$. Then $\frac{V+K}{K}$ is an r-supplement of $\frac{U}{K}$ in $\frac{M}{K}$.*

Proof. Since V is an r-supplement of U in M, $M = U+V$ and $U\cap V \ll_r V$. Since $M = U+V$ and $K \leq U$, $\frac{M}{K} = \frac{U+V}{K} = \frac{U}{K}+\frac{V+K}{K}$. Since $U\cap V \ll_r V$, by Lemma 1, $\frac{U}{K}\cap\frac{V+K}{K} = \frac{U\cap V+K}{K} \ll_r \frac{V+K}{K}$. Hence $\frac{V+K}{K}$ is an r-supplement of $\frac{U}{K}$ in $\frac{M}{K}$. □

Lemma 10. *Every factor module of an r-supplemented module is r-supplemented.*

Proof. Let M be an r-supplemented $R-$module and $\frac{M}{K}$ be any factor module of M. Let $\frac{U}{K} \leq \frac{M}{K}$. Since M is r-supplemented, U has an r-supplement V in M. Since $K \leq U$, by Lemma 9, $\frac{V+K}{K}$ is an r-supplement of $\frac{U}{K}$ in $\frac{M}{K}$. Hence $\frac{M}{K}$ is r-supplemented. □

Corollary 9. *Every homomorphic image of an r-supplemented module is r-supplemented.*

Proof. Clear from Lemma 10. □

Corollary 10. *Every direct summand of an r-supplemented module is r-supplemented.*

Proof. Clear from Lemma 10. □

Lemma 11. *Let M be an r-supplemented module. Then every finitely $M-$ generated $R-$module is r-supplemented.*

Proof. Let N be a finitely $M-$generated $R-$module. Then there exist a finite index set Λ and an $R-$module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is r-supplemented, by Corollary 8, $M^{(\Lambda)}$ is r-supplemented. Then by Corollary 9, N is r-supplemented. □

Proposition 3. *Let R be a ring. Then ${}_RR$ is r-supplemented if and only if every finitely generated $R-$module is r-supplemented.*

Proof. Clear from Lemma 11. □

Lemma 12. *Let M be a local module with $RadM \neq 0$. Then M is not r-supplemented. But M is supplemented.*

Proof. Since M is local, M is the only supplement of $RadM$ in M. Since $M \cap RadM = RadM$ is not small in $RadM$, M is not an r-supplement of $RadM$ in M. Hence M is not r-supplemented. But since M is local, M is supplemented. □

Lemma 13. *Let M be a hollow module with $RadM = M$. Then M is r-supplemented.*

Proof. Clear from definitions. □

Example 1. Consider the $\mathbb{Z}-$module $\mathbb{Z}_8$. Since $\mathbb{Z}_8$ is local and $Rad\mathbb{Z}_8 = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\} \neq 0$, by Lemma 12, $\mathbb{Z}_8$ is supplemented, but not r-supplemented.

Example 2. Consider the $\mathbb{Z}-$module $\mathbb{Z}_{p^\infty}$. Since $\mathbb{Z}_{p^\infty}$ is hollow and $Rad\mathbb{Z}_{p^\infty} = \mathbb{Z}_{p^\infty}$, by Lemma 13, $\mathbb{Z}_{p^\infty}$ is r-supplemented.

Example 3. Consider the $\mathbb{Z}-$module $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_5$. Since $\mathbb{Z}_5$ is simple, $\mathbb{Z}_5$ is r-supplemented. Since $\mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_5$ are r-supplemented, by Lemma 8, $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_5$ is r-supplemented. Here $RadM = Rad\mathbb{Z}_{p^\infty} \oplus Rad\mathbb{Z}_5 = \mathbb{Z}_{p^\infty} \oplus 0 = \mathbb{Z}_{p^\infty} \neq M$.

REFERENCES

[1] G. F. Birkenmeier, F. Takil Mutlu, C. Nebiyev, N. Sokmez, and A. Tercan, "Goldie*-supplemented modules." *Glasg. Math. J.*, vol. 52A, pp. 41–52, 2010, doi: 10.1017/S0017089510000212.

[2] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting modules. Supplements and projectivity in module theory.*, ser. Front. Math. Basel: Birkhäuser, 2006.

[3] C. Nebiyev and A. Pancar, "On supplement submodules." *Ukr. Math. J.*, vol. 65, no. 7, pp. 1071–1078, 2013, doi: 10.1007/s11253-013-0842-2.

[4] C. Nebiyev and H. H. Ökten, "r-small submodules," in *3rd International E-Conference on Mathematical Advances and Applications (ICOMAA-2020)*, 2020.

[5] C. Nebiyev and H. H. Ökten, "Some properties of r-small submodules," *Erzincan University Journal of Science and Technology*, vol. 15, no. 3, pp. 996–1001, 2022, doi: 10.18185/erzifbed.876557.

[6] C. Nebiyev and N. Sökmez, "r-supplemented modules," in *4th International E-Conference on Mathematical Advances and Applications (ICOMAA-2021)*, 2021.

[7] Y. Wang and N. Ding, "Generalized supplemented modules." *Taiwanese J. Math.*, vol. 10, no. 6, pp. 1589–1601, 2006, doi: 10.11650/twjm/1500404577.

[8] R. Wisbauer, *Foundations of module and ring theory. A handbook for study and research.*, ser. Algebra Log. Appl. Philadelphia etc.: Gordon and Breach Science Publishers, 1991, vol. 3.

[9] H. Zöschinger, "Komplementierte Moduln über Dedekindringen," *J. Algebra*, vol. 29, pp. 42–56, 1974, doi: 10.1016/0021-8693(74)90109-4. [Online]. Available: epub.ub.uni-muenchen.de/4100/

Authors' addresses

Celil Nebiyev

(**Corresponding author**) Ondokuz Mayıs University, Department of Mathematics, Kurupelit, Atakum, Samsun, TURKEY

E-mail address: cnebiyev@omu.edu.tr

Nurhan Sökmez

E-mail address: nurhanozkan82@hotmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 965–969

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4409

ON FINITELY G-SUPPLEMENTED MODULES

CELIL NEBIYEV AND HASAN HÜSEYIN ÖKTEN

Received 17 September, 2022

Abstract. In this work, some properties of finitely g-supplemented modules are investigated. Let M be a finitely g-supplemented $R-$module and N be a finitely generated or small submodule of M. Then M/N is finitely g-supplemented. Let $f : M \longrightarrow N$ be an $R-$module epimomorphism with small kernel. If M is finitely g-supplemented, then N is also finitely g-supplemented. Let M be a finitely g-supplemented module, $Rad_g M \leq U \leq M$ and U be finitely generated. Then $U/Rad_g M$ is a direct summand of $M/Rad_g M$.

2010 *Mathematics Subject Classification:* 16D10; 16D70

Keywords: small submodules, essential submodules, g-small submodules, f-supplemented modules

1. Introduction

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an $R-$module. We denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*)submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an $R-$module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$, we denote this by $K \ll_g M$ (in [6], it is called an *e-small* submodule of M and denoted by $K \ll_e M$). Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *finitely supplemented* (briefly, *f-supplemented*) if every finitely generated submodule of M has a supplement in M. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T =$

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

V, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. The intersection of essential maximal submodules of an R-module M is called a *generalized radical* (briefly, *g-radical*) of M and denoted by Rad_gM (in [6], it is denoted by Rad_eM). If M have no essential maximal submodules, then we denote $Rad_gM = M$. An $R-$module M is said to be *noetherian* if every submodule of M is finitely generated. Let M be an $R-$module and $K \leq V \leq M$. We say V *lies above* K in M if $V/K \ll M/K$.

More details about supplemented modules are in [1, 5]. More informations about g-small submodules and g-supplemented modules are in [2, 3].

Lemma 1. *Let M be an R -module and $K, N \leq M$. Consider the following conditions.*

(1) *If $K \leq N$ and N is generalized small submodule of M, then K is a generalized small submodule of M.*
(2) *If K is contained in N and a generalized small submodule of N, then K is a generalized small submodule in submodules of M which contain N.*
(3) *If $K \ll_g L$ and $N \ll_g T$ with $L, T \leq M$, then $K + N \ll_g L + T$.*
(4) $Rad_gM = \sum_{L \ll_g M} L$.
(5) *Let T be an $R-$module and $f : M \to T$ be an R -module homomorphism. If $K \ll_g M$, then $f(K) \ll_g T$. Here $f(Rad_gM) \leq Rad_gT$.*

Proof. See [3, Lemma 1 and Lemma 3]. □

2. Finitely g-supplemented modules

Definition 1. Let M be an $R-$module. If every finitely generated submodule of M has a g-supplement in M, then M is called a finitely g-supplemented (or briefly fg-supplemented) module. (See also [4])

Clearly we can see that every f-supplemented module is fg-supplemented.

Proposition 1. *Every g-supplemented module is fg-supplemented.*

Proof. Clear from definitions. □

Proposition 2. *Let M be a fg-supplemented $R-$module. If M is noetherian, then M is g-supplemented.*

Proof. Let $U \leq M$. Since M is noetherian, U is finitely generated and since M is fg-supplemented, U has a g-supplement in M. Hence M is g-supplemented. □

Lemma 2. *Let M be a fg-supplemented $R-$module and N be a finitely generated submodule of M. Then M/N is fg-supplemented.*

Proof. Let U/N be a finitely generated submodule of M/N. Since U/N finitely generated, there exists a finitely generated submodule K of M such that $U = K + N$. Since K and N are finitely generated, $U = K + N$ is also finitely generated. By hypothesis, U has a g-supplement V in M. Then by [2, Lemma 4], $(V+N)/N$ is a g-supplement of U/N in M/N. Hence M/N is fg-supplemented. □

Corollary 1. *Let M be a fg-supplemented $R-$module and N be a cyclic submodule of M. Then M/N is fg-supplemented.*

Proof. Clear from Lemma 2. □

Corollary 2. *Let $f : M \longrightarrow N$ be an $R-$module epimomorphism and Kef be finitely generated. If M is fg-supplemented, then N is also fg-supplemented.*

Proof. Since M is fg-supplemented and Kef is finitely generated, by Lemma 2, M/Kef is fg-supplemented. Then by $M/Kef \cong N$, N is also fg-supplemented. □

Corollary 3. *Let $f : M \longrightarrow N$ be an $R-$module epimomorphism with cyclic kernel. If M is fg-supplemented, then N is also fg-supplemented.*

Proof. Clear from Corollary 2. □

Lemma 3. *Let M be an fg-supplemented module, $Rad_gM \leq U \leq M$ and U be finitely generated. Then U/Rad_gM is a direct summand of M/Rad_gM.*

Proof. Since M is fg-supplemented and U is a finitely generated submodule of M, U has a g-supplement V in M. Here $M = U + V$ and $U \cap V \ll_g V$. By Lemma 1, $U \cap V \leq Rad_gM$. Then $\frac{M}{Rad_gM} = \frac{U+V}{Rad_gM} = \frac{U}{Rad_gM} + \frac{V+Rad_gM}{Rad_gM}$ and $\frac{U}{Rad_gM} \cap \frac{V+Rad_gM}{Rad_gM} = \frac{U\cap V+Rad_gM}{Rad_gM} = \frac{Rad_gM}{Rad_gM} = 0$. Hence $\frac{M}{Rad_gM} = \frac{U}{Rad_gM} \oplus \frac{V+Rad_gM}{Rad_gM}$ and U/Rad_gM is a direct summand of M/Rad_gM. □

Corollary 4. *Let M be a fg-supplemented module and Rad_gM be finitely generated. Then every finitely generated submodule of M/Rad_gM is a direct summand of M/Rad_gM.*

Proof. Let U/Rad_gM be a finitely generated submodule of M/Rad_gM. Then there exists a finitely generated submodule K of M such that $U = K + Rad_gM$. Since K and Rad_gM are finitely generated, $U = K + Rad_gM$ is also finitely generated. Then by Lemma 3, U/Rad_gM is a direct summand of M/Rad_gM. □

Lemma 4. *Let M be a fg-supplemented $R-$module and $N \ll M$. Then M/N is fg-supplemented.*

Proof. Let U/N be a finitely generated submodule of M/N. Then there exists a finitely generated submodule K of M such that $U = K+N$. Since M is fg-supplemented, K has a g-supplement V in M. Here $M = K+V$ and $K \cap V \ll_g V$. Since $K \leq U$, $M = K+V = U+V$. Let $M = U+T$ with $T \trianglelefteq V$. Then $M = U+T = K+N+T$ and

since $N \ll M$, $K+T=M$. Since V is a g-supplement of K in M and $T \trianglelefteq V$, by definition, $T=V$. Hence V is a g-supplement of U in M. By [2, Lemma 4], $(V+N)/N$ is a g-supplement of U/N in M/N. Hence M/N is fg-supplemented. □

Corollary 5. *Let $f: M \longrightarrow N$ be an $R-$module epimomorphism with small kernel. If M is fg-supplemented, then N is also fg-supplemented.*

Proof. Since M is fg-supplemented and $Kef \ll M$, by Lemma 4, M/Kef is fg-supplemented. Then by $M/Kef \cong N$, N is also fg-supplemented. □

Lemma 5. *Let M be a fg-supplemented $R-$module and $Rad_gM \ll M$. Then every finitely generated submodule of M/Rad_gM is a direct summand of M/Rad_gM.*

Proof. Let U/Rad_gM be a finitely generated submodule of M/Rad_gM. Then there exists a finitely generated submodule K of M such that $U=K+Rad_gM$. Since M is fg-supplemented, K has a g-supplement V in M. Here $M=K+V$ and $K \cap V \ll_g V$. Since $K \leq U$, $M=K+V=U+V$. Let $M=U+T$ with $T \trianglelefteq V$. Then $M=U+T=K+Rad_gM+T$ and since $Rad_gM \ll M$, $K+T=M$. Since V is a g-supplement of K in M and $T \trianglelefteq V$, by definition, $T=V$. Hence V is a g-supplement of U in M. Here $M=U+V$ and $U \cap V \ll_g V$. By Lemma 1, $U \cap V \leq Rad_gM$. Then $\frac{M}{Rad_gM} = \frac{U+V}{Rad_gM} = \frac{U}{Rad_gM} + \frac{V+Rad_gM}{Rad_gM}$ and $\frac{U}{Rad_gM} \cap \frac{V+Rad_gM}{Rad_gM} = \frac{U \cap V+Rad_gM}{Rad_gM} = \frac{Rad_gM}{Rad_gM} = 0$. Hence $\frac{M}{Rad_gM} = \frac{U}{Rad_gM} \oplus \frac{V+Rad_gM}{Rad_gM}$ and U/Rad_gM is a direct summand of M/Rad_gM. □

Corollary 6. *Let M be a fg-supplemented $R-$module and $Rad_gM \ll M$. Then every finitely generated submodule of $M/RadM$ is a direct summand of $M/RadM$.*

Proof. Since $Rad_gM \ll M$, $RadM = Rad_gM$. Then by Lemma 5, every finitely generated submodule of $M/RadM$ is a direct summand of $M/RadM$. □

REFERENCES

[1] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting modules. Supplements and projectivity in module theory.*, ser. Front. Math. Basel: Birkhäuser, 2006.

[2] B. Koşar, C. Nebiyev, and N. Sökmez, "g-supplemented modules," *Ukr. Math. J.*, vol. 67, no. 6, pp. 975–980, 2015, doi: 10.1007/s11253-015-1127-8.

[3] B. Koşar, C. Nebiyev, and A. Pekin, "A generalization of g-supplemented modules," *Miskolc Math. Notes*, vol. 20, no. 1, pp. 345–352, 2019, doi: 10.18514/MMN.2019.2586.

[4] C. Nebiyev and H. H. Ökten, "Finitely g-supplemented modules," in *3rd International E-Conference on Mathematical Advances and Applications ICOMAA-2020*, 2020.

[5] R. Wisbauer, *Foundations of module and ring theory. A handbook for study and research.*, revised and updated Engl. ed. ed., ser. Algebra Log. Appl. Philadelphia etc.: Gordon and Breach Science Publishers, 1991, vol. 3.

[6] D. X. Zhou and X. R. Zhang, "Small-essential submodules and Morita duality." *Southeast Asian Bull. Math.*, vol. 35, no. 6, pp. 1051–1062, 2011.

Authors' addresses

Celil Nebiyev
Ondokuz Mayıs University, Department of Mathematics, Kurupelit-Atakum, Samsun, TURKEY
E-mail address: cnebiyev@omu.edu.tr

Hasan Hüseyin Ökten
(**Corresponding author**) Amasya University, Technical Sciences Vocational School, Amasya, Turkey
E-mail address: hokten@gmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 971–983

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4420

ON NON-OSCILLATION FOR TWO DIMENSIONAL SYSTEMS OF NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

ZDENĚK OPLUŠTIL

Received 23 September, 2022

Abstract. The paper studies the non-oscillatory properties of two-dimensional systems of non-linear differential equations

$$u' = g(t)|v|^{\frac{1}{\alpha}}\operatorname{sgn} v, \quad v' = -p(t)|u|^{\alpha}\operatorname{sgn} u,$$

where the functions $g\colon [0,+\infty[\to [0,+\infty[$, $p\colon [0,+\infty[\to \mathbb{R}$ are locally integrable and $\alpha > 0$. We are especially interested in the case of $\int^{+\infty} g(s)\,\mathrm{d}s < +\infty$.

In the paper, new non-oscillation criteria are established. Among others, they generalize well-known results for linear systems as well as second order linear and also half-linear differential equations. The criteria presented complement the results of Hartman-Wintner's type for the system in question.

2010 *Mathematics Subject Classification:* 34C10

Keywords: two dimensional system of non-linear differential equations, oscillatory properties

1. Introduction

On the half-line $\mathbb{R}_+ = [0,+\infty[$, we consider the two-dimensional system of non-linear ordinary differential equations

$$\begin{aligned} u' &= g(t)|v|^{\frac{1}{\alpha}}\operatorname{sgn} v, \\ v' &= -p(t)|u|^{\alpha}\operatorname{sgn} u, \end{aligned} \tag{1.1}$$

where $\alpha > 0$ and $p, g\colon \mathbb{R}_+ \to \mathbb{R}$ are locally Lebesgue integrable functions.

By a solution to system (1.1) on the interval $J \subseteq [0,+\infty[$ we understand a vector function (u,v), where functions $u,v\colon J \to \mathbb{R}$ are absolutely continuous on every compact interval contained in J and satisfy equalities (1.1) almost everywhere in J.

It was proved in [9] that all non-extendable solutions to system (1.1) are defined on the whole interval $[0,+\infty[$. Consequently, speaking about a solution to system (1.1), we assume, without loss of generality, that it is defined on $[0,+\infty[$.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

Definition 1. A solution (u,v) of system (1.1) is called *non-trivial*, if

$$|u(t)| + |v(t)| \neq 0 \quad \text{for } t \geq 0.$$

We say that a non-trivial solution (u,v) of system (1.1) is *non-oscillatory* if at least one of its components does not have any sequence of zeroes tending to infinity, and *oscillatory* otherwise.

In [9, Theorem 1.1], it is shown that a certain analogue of Sturm's theorem holds for system (1.1) if the function g is non-negative. Especially if system (1.1) has a non-oscillatory solution, then any other of its non-trivial solutions is also non-oscillatory. Therefore, it is natural to assume

$$g(t) \geq 0 \quad \text{for a. e. } t \geq 0 \tag{1.2}$$

throughout the paper.

On the other hand, if $g(t) \equiv 0$ on some neighborhood of $+\infty$, then all non-trivial solutions to system (1.1) are non-oscillatory. Consequently, we also suppose that the inequality

$$\operatorname{meas}\{\tau \geq t\colon g(\tau) > 0\} > 0 \quad \text{for } t \geq 0 \tag{1.3}$$

holds.

Definition 2. We say that system (1.1) is *non-oscillatory* if all its non-trivial solutions are non-oscillatory.

The oscillation and non-oscillation theory for ordinary differential equations is widely studied in the literature. The criteria presented below are close to those established in [1–4, 6–8, 10, 12]. Namely, many of them (see, e.g., the survey given in [1]) are known for the so-called "half-linear" equation

$$\big(r(t)|u'|^{q-1}\operatorname{sgn} u'\big)' + p(t)|u|^{q-1}\operatorname{sgn} u = 0, \tag{1.4}$$

where $q > 1$, $p,r\colon [0,+\infty[\to \mathbb{R}$ are continuous and r is positive. We can see that (1.4) is a particular case of system (1.1). Indeed, if the function u, with the properties $u \in C^1$ and $r|u'|^{q-1}\operatorname{sgn} u' \in C^1$, is a solution to equation (1.4), then the vector function $(u, r|u'|^{q-1}\operatorname{sgn} u')$ is a solution to system (1.1) with $g(t) := r^{\frac{1}{1-q}}(t)$ for $t \geq 0$ and $\alpha := q-1$. In the case of $\int_0^{+\infty} g(s)\,\mathrm{d}s = +\infty$, some of the above-mentioned results are generalized in [11].

Throughout the paper, we assume that the function g is integrable on $[0,+\infty[$, i.e.

$$\int_0^{+\infty} g(s)\,\mathrm{d}s < +\infty. \tag{1.5}$$

In this case, the interesting results dealing with the oscillation of the system (1.1) are presented in [2]. Below formulated criteria complement these ones in certain sense.

On the other hand, as far as we know, not many non-oscillation criteria are known under the assumption (1.5). In particular, for the half-linear equation (1.4), one can

find some non-oscillation criteria, e.g., in [1,5]. But there are some "sign" restrictions on the coefficient p.

We introduce the following notations. Let

$$f(t) := \int_t^{+\infty} g(s)\,\mathrm{d}s \quad \text{for } t \geq 0.$$

In view of assumptions (1.2), (1.3) and (1.5), we have

$$\lim_{t\to+\infty} f(t) = 0$$

and

$$f(t) > 0 \quad \text{for } t \geq 0.$$

Further, for any $\lambda > \alpha$, we put

$$c_\alpha(t;\lambda) := (\lambda-\alpha) f^{\lambda-\alpha}(t) \int_0^t \frac{g(s)}{f^{\lambda-\alpha+1}(s)} \left(\int_0^s f^\lambda(\xi) p(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s \quad \text{for } t \geq 0.$$

It is known that some analogy of the Hartmann–Wintner theorem (see [2, Corollary 2.11], where we put $\nu = 1 - \alpha + \lambda$) holds. In particular, if the function $c_\alpha(\cdot;\lambda)$ has no finite limit and $\liminf_{t\to+\infty} c_\alpha(t;\lambda) > -\infty$, then system (1.1) is oscillatory.

In this paper, we provide non-oscillatory criteria for the case where there exists a finite limit of the function $c_\alpha(\cdot;\lambda)$, i.e.,

$$\lim_{t\to+\infty} c_\alpha(t;\lambda) =: c_\alpha^*(\lambda) \in \mathbb{R}.$$

Under this assumption, we put for any $\lambda \in]\alpha,+\infty[$ and $\mu \in [0,\alpha[$

$$Q(t;\alpha,\lambda) := \frac{1}{f^{\lambda-\alpha}(t)} \left(c_\alpha^*(\lambda) - \int_0^t p(s) f^\lambda(s)\,\mathrm{d}s \right) \quad \text{for } t \geq 0,$$

$$H(t;\alpha,\mu) := f^{\alpha-\mu}(t) \int_0^t p(s) f^\mu(s)\,\mathrm{d}s \quad \text{for } t \geq 0.$$

Moreover, let us denote

$$\begin{aligned} Q_*(\alpha,\lambda) &:= \liminf_{t\to+\infty} Q(t;\alpha,\lambda), & H_*(\alpha,\mu) &:= \liminf_{t\to+\infty} H(t;\alpha,\mu), \\ Q^*(\alpha,\lambda) &:= \limsup_{t\to+\infty} Q(t;\alpha,\lambda), & H^*(\alpha,\mu) &:= \limsup_{t\to+\infty} H(t;\alpha,\mu). \end{aligned} \tag{1.6}$$

2. Main results

This section contains formulations of the main results of the paper. Firstly, we formulate the non-oscillation criteria for system (1.1) in terms of the lower and upper limits of the function $Q(\cdot,\alpha,\lambda)$.

For any $\kappa \in \mathbb{R}$, let us denote by $A(\kappa)$ and $B(\kappa)$ the smallest and the greatest roots of the equation

$$\alpha|x|^{\frac{1+\alpha}{\alpha}} + \lambda x + (\lambda-\alpha)\kappa = 0. \tag{2.1}$$

Let us note that, the equation (2.1) has exactly two real roots if $\kappa < \left(\frac{\lambda}{1+\alpha}\right)^{\alpha} \frac{1}{\lambda-\alpha}$. Moreover, $A(\kappa) \in \left]-\infty, \left(\frac{\mu}{1+\alpha}\right)^{\alpha}\right[$, i.e. the smallest one is always negative (see Figure 1(a), where $\alpha = 2$, $\lambda = 3$, $\kappa = \frac{1}{2}$).

Theorem 1. *Let* $\lambda \in]\alpha, +\infty[$,

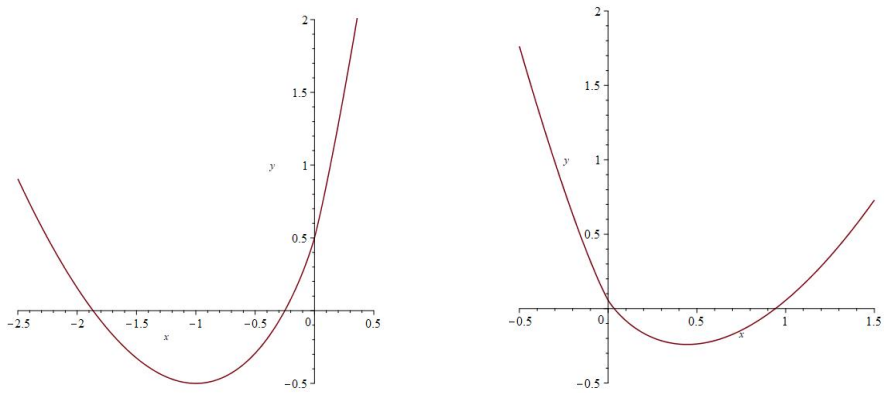
$$A(\kappa) + \kappa < Q_*(\alpha,\lambda) \quad \text{and} \quad Q^*(\alpha,\lambda) < \frac{1}{\lambda-\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} \tag{2.2}$$

be fulfilled, where $\kappa = \frac{\alpha(\lambda-\alpha)+\lambda}{(\lambda-\alpha)(1+\alpha)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$. *Then, system* (1.1) *is non-oscillatory.*

Before we formulate the following statement, we denote by $\widetilde{B}(\eta)$ the greatest root of the equation

$$\alpha|x|^{\frac{1+\alpha}{\alpha}} - \alpha x + \eta = 0. \tag{2.3}$$

Let us note that, the equation (2.3) has exactly two real roots if $\eta < \left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}$, moreover, $\widetilde{B}(\eta) \in \left]\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}, \infty\right[$, i.e. the greatest one is always positive (see Figure 1(b), where $\alpha = 2$, $\eta = \frac{1}{18}$).



(a) $f(x) = 2|x|^{\frac{3}{2}} + 3x + \frac{1}{2}$ (b) $g(x) = 2|x|^{\frac{3}{2}} - 2x + \frac{1}{18}$

FIGURE 1.

Theorem 2. *Let* $\lambda \in]\alpha, +\infty[$,

$$-\infty < Q_*(\alpha,\lambda) \le A(\kappa) + \kappa \tag{2.4}$$

and

$$Q^*(\alpha,\lambda) < Q_*(\alpha,\lambda) + \widetilde{B}(\eta) + B\left(Q_*(\alpha,\lambda) + \widetilde{B}(\eta)\right) \tag{2.5}$$

be fulfilled, where $\kappa = \frac{\alpha(\lambda-\alpha)+\lambda}{(\lambda-\alpha)(1+\alpha)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$ *and* $\eta = (\lambda-\alpha)Q_*(\alpha,\lambda)$. *Then, system* (1.1) *is non-oscillatory.*

Remark 1. Theorem 2 complements Theorem 1 in certain sense. Indeed, if the first inequality in (2.2) is not satisfied and $Q_*(\alpha,\lambda)$ is finite, then condition (2.4) holds. In such a case, it is sufficient to verify condition (2.5) and the system (1.1) is non-oscillatory according Theorem 2 (see Example 1) .

In the following theorems we established the non-oscillation criteria in terms of the lower and upper limits of the function $H(\cdot,\alpha,\mu)$. Now, we denote by $\bar{A}(\nu)$ and $\bar{B}(\nu)$ the smallest and greatest roots of the equation

$$\alpha|x|^{\frac{1+\alpha}{\alpha}}+\mu x+(\alpha-\mu)\nu=0. \tag{2.6}$$

Let us note that, the equation (2.6) has two real roots if $\nu<\frac{1}{\alpha-\mu}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha}$, moreover, $\bar{A}(\nu)\in\left]-\infty,\left(\frac{\mu}{1+\alpha}\right)^{\alpha}\right]$, i.e. the smallest one is always negative.

Theorem 3. *Let* $\mu\in[0,\alpha[$,

$$-\frac{\alpha(2\alpha+1)}{(1+\alpha)(\alpha-\mu)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}<H_*(\alpha,\mu)\quad\text{and}\quad H^*(\alpha,\mu)<\nu-\bar{A}(\nu) \tag{2.7}$$

be fulfilled with $\nu=-\frac{\alpha(\alpha+\mu)+\mu}{(\alpha-\mu)(1+\alpha)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$. *Then, system* (1.1) *is non-oscillatory.*

Finally, we formulate the statement which completes the previous one in the same sense, as it is mentioned in Remark 1 for Theorem 1 and Theorem 2.

Theorem 4. *Let* $\mu\in[0,\alpha[$,

$$-\infty<H_*(\alpha,\mu)\le-\frac{\alpha(2\alpha+1)}{(1+\alpha)(\alpha-\mu)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha} \tag{2.8}$$

and

$$H^*(\alpha,\mu)<\delta-\bar{A}(\delta) \tag{2.9}$$

be fulfilled, where $\delta=\left(\widehat{B}((\alpha-\mu)H_*(\alpha,\mu))\right)^{\frac{\alpha}{1+\alpha}}+H_*(\alpha,\mu)$ *and* $\widehat{B}(\xi)$ *is the greatest root of the equation*

$$\alpha|x|^{\frac{\alpha}{1+\alpha}}+\alpha x+\xi=0,\quad\text{for }\xi\le 0.$$

Then, system (1.1) *is non-oscillatory.*

Let us note, that the $\widehat{B}(\xi)\in\left]-\left(\frac{\alpha}{1+\alpha}\right)^{\alpha},+\infty\right[$ and $\widehat{B}(\xi)>0$ for $\xi<0$.

Remark 2. In [1, Section 3.1], there are functions Q, H and also non-oscillatory criteria defined for the equation (1.4) with the particular parameters $\lambda=\alpha+1$ and $\mu=0$. However, in this paper, they are formulated in a more general way, where $\lambda\in]\alpha,+\infty[$ and $\mu\in[0,\lambda[$. One can see (e.g. Example 2) that it is meaningful, since we can decide on non-oscillation in more cases of the system in question (1.1).

Example 1. Let $\alpha = 2$, $\lambda = 3$,

$$g(t) = \frac{1}{(1+t)^2},$$

and

$$p(t) = \left(\frac{89}{432} + \frac{5\sqrt{105}}{144}\right)\left(\sin(\ln(1+t)) + \cos(\ln(1+t)) - \frac{15\sqrt{105}-19}{89+15\sqrt{105}}\right) \cdot (1+t)$$

for $t \geq 0$.

One can verify that

$$f(t) = \int_t^{+\infty} g(s)\,\mathrm{d}s = \frac{1}{1+t} \quad \text{for } t \geq 0$$

and

$$\begin{aligned} c_2(t,3) &= \frac{1}{1+t}\int_0^t \left(\int_0^s p(\xi)\,\mathrm{d}\xi\right) ds \\ &= \frac{(-15\sqrt{105}-69)\sin(\ln(1+t)) + (-19+15\sqrt{105})\ln(t+1) + 108t}{432(t+1)} \end{aligned}$$

for $t > 0$. Hence, we get

$$c_2^*(3) = \lim_{t\to+\infty} c_2(t,3) = \frac{1}{4}.$$

Moreover,

$$\begin{aligned} Q(t;2,3) &= (t+1)\left(\frac{1}{4} - \int_0^t \frac{p(s)}{(1+s)^3} ds\right) \\ &= \frac{(89+15\sqrt{105})\left(\cos\left(\frac{\ln(1+t)}{2}\right)\right)^2 - 15\sqrt{105} - 35}{216} \quad \text{for } t \geq 0, \end{aligned}$$

therefore,

$$Q_*(2,3) = \liminf_{t\to+\infty} Q(t;2,3) = -\frac{15\sqrt{105}+35}{216}$$

and

$$Q^*(2,3) = \limsup_{t\to+\infty} Q(t;2,3) = \frac{1}{4}.$$

On the other hand, for $\alpha = 2$, $\lambda = 3$, we have $\kappa = \frac{20}{27}$ and the equation (2.1) is of the form

$$2|x|^{\frac{3}{2}} + 3x + \frac{20}{27} = 0.$$

It is not difficult to verify that

$$A\left(\frac{20}{27}\right)+\frac{20}{27}=-\frac{(5+\sqrt{105})^2}{144}+\frac{20}{27}=-\frac{15\sqrt{105}+35}{216}=Q_*(2,3).$$

Consequently, we cannot apply Theorem 1, since the first inequality in (2.2) is not satisfied. However, one can show that Theorem 2 guarantees non-oscillation of the system in question. Indeed, for $\lambda=3$ and $\alpha=2$, we have $\eta=Q_*(2,3)$ and equation (2.3) is of the form

$$2|x|^{\frac{3}{2}}-2x-\frac{15\sqrt{105}+35}{216}=0.$$

One can verify that

$$\widetilde{B}\left(-\frac{15\sqrt{105}+35}{216}\right)=\frac{65+5\sqrt{105}}{72}$$

and

$$B\left(Q_*(\alpha,\lambda)+\widetilde{B}(\eta)\right)=-\frac{4}{9}.$$

Hence,

$$Q^*(2,3)=\frac{1}{4}<\frac{8}{27}=Q_*(2,3)+\widetilde{B}\left(-\frac{15\sqrt{105}+35}{216}\right)+B\left(Q_*(\alpha,\lambda)+\widetilde{B}(\eta)\right).$$

Consequently, according to Theorem 2, system (1.1) is non-oscillatory.

Example 2. Let $\alpha=2$,

$$g(t)=\frac{1}{(1+t)^2},\quad\text{and}\quad p(t)=\left(\frac{2\cos t}{3}-\frac{7}{9(1+t)}\right)(1+t)^2\quad\text{for } t\geq 0.$$

If we put $\mu=1$, then one can calculate that

$$f(t)=\int_t^{+\infty}g(s)\,\mathrm{d}s=\frac{1}{1+t}\quad\text{for } t\geq 0$$

and

$$\begin{aligned}H(t;2,1)&=f(t)\int_0^t p(s)f(s)\,\mathrm{d}s=\frac{1}{1+t}\int_0^t\left(\frac{2(1+s)\cos s}{3}-\frac{7}{9}\right)\mathrm{d}s\\&=\frac{6(1+t)\sin t+6\cos t-7t-6}{9(1+t)}\quad\text{for } t\geq 0.\end{aligned}$$

Hence,

$$H_*(2,1)=\liminf_{t\to+\infty}H(t;2,1)=-\frac{13}{9},$$

and

$$H^*(2,1)=\limsup_{t\to+\infty}H(t;2,1)=-\frac{1}{9}.$$

For $\alpha = 2$, $\mu = 1$, we have $\nu = -\frac{8}{27}$ and the equation (2.6) is of the form

$$2|x|^{\frac{3}{2}} + x - \frac{8}{27} = 0.$$

One can verify that

$$\bar{A}\left(-\frac{8}{27}\right) = -\left(\frac{(57+4\sqrt{203})^{\frac{1}{3}}+1}{6} + \frac{1}{6(57+4\sqrt{203})^{\frac{1}{3}}}\right)^2,$$

thus,

$$\begin{aligned}\nu - \bar{A}(\nu) &= -\frac{8}{27} + \left(\frac{(57+4\sqrt{203})^{\frac{1}{3}}+1}{6} + \frac{1}{6(57+4\sqrt{203})^{\frac{1}{3}}}\right)^2 \\ &\approx -0.019 > -\frac{1}{9} = H^*(2,1).\end{aligned}$$

Clearly,

$$-\frac{\alpha(2\alpha+1)}{(1+\alpha)(\alpha-\mu)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha} = -\frac{40}{27} < -\frac{13}{9} = H_*(2,1).$$

We see that both conditions in (2.7) are satisfied and, therefore, according to Theorem 3, system (1.1) is non-oscillatory.

On the other hand, if we put $\mu = 0$, then

$$-\frac{2\alpha+1}{1+\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha} = -\frac{20}{27} > -\frac{19}{18} = H_*(2,0),$$

and now we cannot apply Theorem 3. Consequently, it is meaningful to consider our criteria with the "weight" f^{μ}.

3. Proofs of the main results

Firstly, we present an auxiliary lemma, which we use to prove the main theorems.

Lemma 1 ([11, Lemma 3.1])**.** *Let there exist a locally absolutely continuous function* $\sigma\colon [a,+\infty[\ \to \mathbb{R}$ *satisfying the inequality*

$$\sigma'(t) \le -p(t) - \alpha g(t)|\sigma(t)|^{\frac{1+\alpha}{\alpha}} \quad \text{for a. e. } t \ge a, \tag{3.1}$$

where $a \ge 0$*. Then, system* (1.1) *is non-oscillatory.*

It is not difficult to verify the next lemma by a direct calculation.

Lemma 2. *Let*

$$y(x) := \alpha|x|^{\frac{1+\alpha}{\alpha}} + \beta x + \gamma,$$

where $\alpha, \beta > 0$ *and* $\gamma \in \mathbb{R}$*. Then,*

$$y'(x) < 0 \ \text{ for }]-\infty, x_1[\,, \quad y'(x) > 0 \ \text{ for }]x_1, \infty[\,, \tag{3.2}$$

where $x_1 = -\left(\frac{\beta}{1+\alpha}\right)^{\alpha}$, *and*

$$\lim_{x\to-\infty} y(x) = +\infty, \qquad \lim_{x\to+\infty} y(x) = +\infty.$$

Proof of Theorem 1. In view of (1.6) and (2.2), there exists $t_0 > 0$ such that

$$A(\kappa) + \kappa < Q(t;\alpha,\lambda) < \frac{1}{\lambda-\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} \quad \text{for } t \geq t_0.$$

Hence,

$$A(\kappa) < Q(t;\alpha,\lambda) - \kappa < -\left(\frac{\alpha}{1+\alpha}\right)^{\alpha} \quad \text{for } t \geq t_0. \tag{3.3}$$

One can show that $x_2 = -\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$ is the root of the equation (2.1). Moreover, by virtue of the hypothesis $\lambda > \alpha$ and Lemma 2 (with $\beta = \lambda$ and $\gamma = \kappa(\lambda-\alpha)$), we get

$$A(\kappa) < x_1 < x_2, \quad \text{and} \quad \alpha|x|^{\frac{1+\alpha}{\alpha}} + \lambda x + (\lambda-\alpha)\kappa < 0 \quad \text{for } x \in \,]A(\kappa), x_2[, \tag{3.4}$$

where $x_1 = -\left(\frac{\lambda}{1+\alpha}\right)^{\alpha}$. The latter inequalities, together with (3.3), yield

$$\alpha|Q(t;\alpha,\lambda) - \kappa|^{\frac{1+\alpha}{\alpha}} + \lambda(Q(t;\alpha,\lambda) - \kappa) + (\lambda-\alpha)\kappa \leq 0 \quad \text{for } t \geq t_0. \tag{3.5}$$

Let us introduce the function σ as follows

$$\sigma(t) := \frac{1}{f^{\alpha}(t)}\left(Q(t;\alpha,\lambda) - \kappa\right) \quad \text{for a. e. } t \geq t_0. \tag{3.6}$$

It is clear that

$$\sigma'(t) = \frac{g(t)}{f^{1+\alpha}(t)}\left(\lambda(Q(t;\alpha,\lambda) - \kappa) + (\lambda-\alpha)\kappa\right) - p(t) \quad \text{for } t \geq t_0.$$

The latter equality, together with (3.5), implies

$$\sigma'(t) \leq \frac{g(t)}{f^{1+\alpha}(t)}\left(-\alpha|Q(t;\alpha,\lambda) - \kappa|^{\frac{1+\alpha}{\alpha}}\right) - p(t) \quad \text{for a. e. } t \geq 0.$$

Hence, in view of (3.6), we get that inequality (3.1) is satisfied with $a = t_0$. Consequently, according to Lemma 1, system (1.1) is non-oscillatory. □

Proof of Theorem 2. By virtue of (1.6), (2.4) and (2.5), there exist $\varepsilon > 0$ and $t_\varepsilon > 0$ such that

$$Q_*(\alpha,\lambda) - \varepsilon < Q(t;\alpha,\lambda) < Q^*(\alpha,\lambda) + \varepsilon \tag{3.7}$$

and

$$Q^*(\alpha,\lambda) + \varepsilon < Q_*(\alpha,\lambda) - \varepsilon + \widetilde{B}(\eta_\varepsilon) + B\left(Q_*(\alpha,\lambda) - \varepsilon + \widetilde{B}(\eta_\varepsilon)\right) \quad \text{for } t \geq t_\varepsilon \tag{3.8}$$

hold with $\eta_\varepsilon = (\lambda-\alpha)(Q_*(\alpha,\lambda) - \varepsilon)$.

An analysis similar to that in the proof of Theorem 1 shows that (3.4) holds, where $x_1 = -\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$ and $x_2 = -\left(\frac{\lambda}{1+\alpha}\right)^{\alpha}$. Therefore,

$$A(\kappa) < -\left(\frac{\lambda}{1+\alpha}\right)^{\alpha} \quad \text{and} \quad \alpha|x_1|^{\frac{1+\alpha}{\alpha}} + \lambda x_1 + (\lambda-\alpha)\kappa < 0.$$

The latter inequality guarantees that

$$\kappa < \frac{1}{\lambda-\alpha}\left(\frac{\lambda}{1+\alpha}\right)^{1+\alpha}.$$

Hence, in view of (2.4), we obtain

$$Q_*(\alpha,\lambda) \le A(\kappa) + \kappa < \left(\frac{\lambda}{1+\alpha}\right)^{\alpha} \frac{\alpha-\alpha\lambda+\alpha^2}{(1+\alpha)(\lambda-\alpha)}$$

and, consequently,

$$Q_*(\alpha,\lambda)(\lambda-\alpha) < \left(\frac{\lambda}{1+\alpha}\right)^{\alpha} \frac{\alpha-\alpha\lambda+\alpha^2}{1+\alpha}. \tag{3.9}$$

On the other hand, the function $z\colon x \mapsto \alpha|x|^{\frac{1+\alpha}{\alpha}} - \alpha x + \eta_\varepsilon$ is decreasing on $\left]-\infty, \left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\right[$, and increasing on $\left]\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}, \infty\right[$. Moreover, by virtue of (3.9), we get

$$z\left(\left(\frac{\lambda}{1+\alpha}\right)^{\alpha}\right) < 0.$$

Hence, $\widetilde{B}(\eta_\varepsilon) > \left(\frac{\lambda}{1+\alpha}\right)^{\alpha}$ and, consequently,

$$-\widetilde{B}(\eta_\varepsilon) < -\left(\frac{\lambda}{1+\alpha}\right)^{\alpha}. \tag{3.10}$$

If we put

$$\kappa_\varepsilon = Q_*(\alpha,\lambda) - \varepsilon + \widetilde{B}(\eta_\varepsilon), \tag{3.11}$$

then it is not difficult to verify that $-\widetilde{B}(\eta_\varepsilon)$ is the root of equation (2.1) with $\kappa = \kappa_\varepsilon$. Moreover, (3.10) and Lemma 2 (with $\beta = \lambda$ and $\gamma = (\lambda-\alpha)\kappa_\varepsilon$) imply, that $-B(\eta_\varepsilon) = A(\kappa_\varepsilon)$ and

$$\alpha|x|^{\frac{1+\alpha}{\alpha}} + \lambda x + (\lambda-\alpha)\kappa_\varepsilon < 0 \quad \text{for } x \in \left]A(\kappa_\varepsilon), B(\kappa_\varepsilon)\right[. \tag{3.12}$$

In view of (3.7), (3.8) and (3.11), we get

$$A(\kappa_\varepsilon) = -\widetilde{B}(\eta_\varepsilon) \le Q(t;\alpha,\lambda) - \kappa_\varepsilon \le B(\kappa_\varepsilon) \quad \text{for } t \ge t_\varepsilon.$$

The latter inequalities and (3.12) yield (3.5) with $\kappa = \kappa_\varepsilon$ and $t_0 = t_\varepsilon$.

Now, let the function σ be defined by formula (3.6) with $\kappa = \kappa_\varepsilon$ and $t_0 = t_\varepsilon$. Analogously, as in the proof of Theorem 1, one can verify that inequality (3.1) with $a = t_\varepsilon$ holds and, consequently, according to Lemma 1, system (1.1) is non-oscillatory.

□

Proof of Theorem 3. In view of (1.6) and (2.7), there exist $t_0 > 0$ such that

$$-\frac{\alpha(2\alpha+1)}{(1+\alpha)(\alpha-\mu)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha} < H(t;\alpha,\mu) < \nu - \bar{A}(\nu) \quad \text{for } t \geq t_0. \tag{3.13}$$

According to Lemma 2 (with $\beta = \mu$ and $\gamma = (\alpha-\mu)\nu$), one can see that function

$$y(x) := \alpha|x|^{\frac{1+\alpha}{\alpha}} + \mu x + (\alpha-\mu)\nu \quad \text{for } x \in \mathbb{R} \tag{3.14}$$

satisfies relations (3.2) with $x_1 = -\left(\frac{\mu}{\alpha+1}\right)^{\alpha}$. Moreover, it is not difficult to verify, that $\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$ is the greatest root of the equation (2.6). Hence, by virtue of (3.2), we have

$$y(x) < 0 \quad \text{for } x \in \left]\bar{A}(\nu), \left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\right[. \tag{3.15}$$

On the other hand, from (3.13), we obtain

$$\bar{A}(\nu) < \nu - H(t;\alpha,\mu) < \left(\frac{\alpha}{1+\alpha}\right)^{\alpha} = \bar{B}(\nu) \quad \text{for } t \geq t_0.$$

The latter inequalities, together with (3.14) and (3.15), yield

$$\alpha|\nu - H(t;\alpha,\mu)|^{\frac{1+\alpha}{\alpha}} + \mu(\nu - H(t;\alpha,\mu)) + (\alpha-\mu)\nu \leq 0 \quad \text{for } t \geq t_0. \tag{3.16}$$

Now, we put

$$\sigma(t) := \frac{1}{f^{\alpha}(t)}(\nu - H(t;\alpha,\mu)) \quad \text{for } t \geq t_0. \tag{3.17}$$

One can show that

$$\sigma'(t) = \frac{g(t)}{f^{1+\alpha}(t)}(\mu(\nu - H(t;\alpha,\mu)) + (\alpha-\mu)\nu) - p(t) \quad \text{for a. e. } t \geq t_0.$$

Hence, in view of (3.16), we obtain

$$\sigma'(t) \leq \frac{g(t)}{f^{1+\alpha}(t)}\left(-\alpha|\nu - H(t;\alpha,\mu)|^{\frac{1+\alpha}{\alpha}}\right) - p(t) \quad \text{for a. e. } t \geq t_0.$$

Consequently, by virtue of (3.17), we get that (3.1) holds with $a = t_0$ and, according to Lemma 1, system (1.1) is non-oscillatory. □

Proof of Theorem 4. In view of (1.6) and (2.9), there exist $\varepsilon > 0$ and $t_\varepsilon > 0$ such that

$$H_*(\alpha,\mu) - \varepsilon < H(t;\alpha,\mu) < H^*(\alpha,\mu) + \varepsilon \quad \text{for } t \geq t_\varepsilon \tag{3.18}$$

and

$$H^*(\alpha,\mu) + \varepsilon < \delta_\varepsilon - \bar{A}(\delta_\varepsilon) \quad \text{for } t \geq t_\varepsilon \tag{3.19}$$

hold, where

$$\delta_\varepsilon = \left(\widehat{B}((\alpha-\mu)(H_*(\alpha,\mu) - \varepsilon))\right)^{\frac{\alpha}{1+\alpha}} + H_*(\alpha,\mu) - \varepsilon. \tag{3.20}$$

From (2.8), we get

$$\widehat{B}((\alpha-\mu)(H_*(\alpha,\mu) - \varepsilon)) > 0.$$

Moreover, in view of the latter inequality, one can show that

$$\left(\widehat{B}((\alpha-\mu)(H_*(\alpha,\mu)-\varepsilon))\right)^{\frac{\alpha}{1+\alpha}}$$

is the greatest root of the equation (2.6) with $\nu=\delta_\varepsilon$, i.e.,

$$\bar{B}(\delta_\varepsilon)=\left(\widehat{B}((\alpha-\mu)(H_*(\alpha,\mu)-\varepsilon))\right)^{\frac{\alpha}{1+\alpha}}.$$

Consequently, from (3.18), (3.19) and (3.20), we get

$$\bar{A}(\delta_\varepsilon)<\delta_\varepsilon-H(t;\alpha,\mu)<\bar{B}(\delta_\varepsilon). \tag{3.21}$$

On the other hand, an analysis similar to that in the proof of Theorem 3 shows that function

$$y_\varepsilon(x):=\alpha|x|^{\frac{1+\alpha}{\alpha}}+\mu x+(\alpha-\mu)\delta_\varepsilon$$

satisfies relations

$$y(x)<0 \quad \text{for } x\in\left]\bar{A}(\delta_\varepsilon),\bar{B}(\delta_\varepsilon)\right[.$$

The latter inequality together with (3.21) yield (3.16) with $\nu=\delta_\varepsilon$ and $t_0=t_\varepsilon$. Analogously, as in the proof of Theorem 3, one can verify that function

$$\sigma(t):=\frac{1}{f^\alpha(t)}\left(\delta_\varepsilon-H(t;\alpha,\mu)\right) \quad \text{for } t\geq t_\varepsilon$$

satisfies inequality (3.1) with $a=t_\varepsilon$ and, consequently, according to Lemma 1, system (1.1) is non-oscillatory. □

ACKNOWLEDGEMENTS

This work was supported by the Brno University of Technology from the Specific Research Project no. FSI-S-23-8161.

REFERENCES

[1] O. Došlý and P. Řehák, *Half-linear differential equations*. Amsterdam: Elsevier, 2005.

[2] M. Dosoudilová, A. Lomtatidze, and J. Šremr, "Oscillatory properties of solutions to certain two-dimensional systems of non-linear ordinary differential equations," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 120, pp. 57–75, 2015, doi: 10.1016/j.na.2015.02.014.

[3] E. Hille, "Non-oscillation theorems," *Trans. Amer. Math. Soc.*, vol. 64, no. 2, pp. 234–252, 1948, doi: 10.2307/1990500.

[4] N. Kandelaki, A. Lomtatidze, and D. Ugulava, "Non-oscillatory theorems." *Transactions of the American Mathematical Society*, vol. 64, no. 64, pp. 234–252, 1948, doi: 10.2307/1990500.

[5] T. Kusano and Y. Naito, "Oscillation and nonoscillation criteria for second order quasilinear differential equations," *Acta Math. Hung.*, vol. 76, no. 1-2, pp. 81–99, 1997, doi: 10.1007/BF02907054.

[6] A. Lomtatidze, "Oscillation and nonoscillation of Emden-Fowler type equations of second order," *Arch. Math., Brno*, vol. 32, no. 3, pp. 181–193, 1996.

[7] A. Lomtatidze, "Oscillation and nonoscillation criteria for second-order linear differential equations," *Georgian Math. J.*, vol. 4, no. 2, pp. 129–138, 1997.

[8] A. Lomtatidze and J. Šremr, "On oscillation and nonoscillation of two-dimensional linear differential systems," *Georgian Math. J.*, vol. 20, no. 3, pp. 573–600, 2013, doi: 10.1515/gmj-2013-0025.

[9] J. D. Mirzov, "On some analogs of Sturm's and Kneser's theorems for nonlinear systems," *J. Math. Anal. Appl.*, vol. 53, pp. 418–425, 1976, doi: 10.1016/0022-247X(76)90120-7.
[10] Z. Nehari, "Oscillation criteria for second-order linear differential equations," *Trans. Amer. Math. Soc.*, vol. 85, no. 2, pp. 428–445, 1957, doi: 10.2307/1992939.
[11] Z. Opluštil, "On non-oscillation for certain system of non-linear ordinary differential equations," *Georgian Math. J.*, vol. 24, no. 2, pp. 277–285, 2017, doi: 10.1515/gmj-2016-0068.
[12] Z. Opluštil, "Oscillatory properties of certain system of non-linear ordinary differential equations," *Miskolc Math. Notes*, vol. 19, no. 1, pp. 439–459, 2018, doi: 10.18514/MMN.2018.2391.

Author's address

Zdeněk Opluštil
Brno University of Technology, Faculty of Mechanical Engineering, Institute of Mathematics, Technická 2, 616 69 Brno, Czech Republic
E-mail address: oplustil@fme.vutbr.cz

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 985–997

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4507

ON (M,P)-FUNCTIONS WITH SOME FEATURES AND NEW INEQUALITIES

ERHAN SET, ALİ KARAOĞLAN, İMDAT İŞCAN, AND NESLİHAN KILIÇ

Received 30 November, 2022

Abstract. In this study, we introduce a generalization of P-function, called (M,P)-functions, via weighted mean functions given by İşcan. Then, we prove some new inequalities for (M,P)-functions. Also, we give new properties for (M,P)-functions and present some results for the special cases of M.

2010 *Mathematics Subject Classification:* 26A51; 26D10; 26E60

Keywords: (M,P)-functions, MN-convex functions, means, weighted means, integral inequalities

1. Introduction

Historically, pedagogically and logically, the study of convex functions begins in the context of real-valued functions of real variable. Convex functions have important applications and at same time they give rise to a variety of generalizations. The geometric definition of a convex function specifies the following. A real-valued function is said to be convex if the line segment connecting two points of its graph lies above the graph. Equivalently, a real-valued function is convex if its epigraph (the set of points on or above its graph) is convex. A convex function $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is bounded and its restriction to (a,b) is continuous. Simple examples of convex functions are $f(x) = x^2$ on $(-\infty,\infty)$, $g(x) = \sin x$ on $[-\pi,0]$, $k(x) = |x|$ on $(-\infty,\infty)$. The analytic definition of a convex function is as follows.

Definition 1. The function $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

for all $x,y \in [a,b]$ and $\lambda \in [0,1]$. We say that f is concave if $(-f)$ is convex.

Definition 2 ([5])**.** A function $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ is P function or that f belongs to the class of $P(I)$, if it is nonnegative and, for all $x,y \in I$ and $\lambda \in [0,1]$, satisfies the

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

following inequality;

$$f\big(\lambda x+(1-\lambda)y\big) \leq f(x)+f(y). \tag{1.2}$$

Convex functions play an important role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by a number of convenient properties. The generalized condition of convexity, i.e. MN-convexity with respect to arbitrary means M and N, was proposed in 1933 by Aumann [4]. Recently many authors have dealt with these generalizations. In particular, Niculescu [14] compared MN-convexity with relative convexity. In [3], Anderson et al. studied certain generalizations of these notions for a positive-valued function of a positive variable as follows:

Definition 3. A function $M\colon (0,\infty)\times(0,\infty)\to(0,\infty)$ is called a mean function if the following conditions are satisfied.

(M1) $M(u,v)=M(v,u)$,
(M2) $M(u,u)=u$,
(M3) $u<M(u,v)<v$ whenever $u<v$,
(M4) $M(\lambda u,\lambda v)=\lambda M(u,v)$ for all $\lambda>0$.

Example 1. For $u,v\in(0,\infty)$

$$M(u,v)=A(u,v)=A=\frac{u+v}{2}$$

is the Arithmetic Mean,

$$M(u,v)=G(u,v)=G=\sqrt{uv}$$

is the Geometric Mean,

$$M(u,v)=H(u,v)=H=A^{-1}(u^{-1},v^{-1})=\frac{2uv}{u+v}$$

is the Harmonic Mean,

$$M(u,v)=L(u,v)=L=\begin{cases} \frac{u-v}{\ln u-\ln v} & u\neq v \\ u & u=v \end{cases}$$

is the Logarithmic Mean,

$$M(u,v)=I(u,v)=I=\begin{cases} \frac{1}{e}\left(\frac{u^u}{v^v}\right)^{\frac{1}{u-v}} & u\neq v \\ u & u=v \end{cases}$$

is the Identric Mean,

$$M(u,v)=M_p(u,v)=M_p=\begin{cases} A^{1/p}(u^p,v^p)=\left(\frac{u^p+v^p}{2}\right)^{1/p} & p\in\mathbb{R}\setminus\{0\} \\ G(u,v)=\sqrt{uv} & p=0 \end{cases}$$

is the p-Power Mean, In particular, we have the following inequality

$$M_{-1}=H\leq M_0=G\leq L\leq I\leq A=M_1.$$

In [3], Anderson et al. gave a new definition of MN-convex functions called MN-midpoint convex with the help of M and N weighted mean as follows.

Definition 4. Let M and N be two means defined on the intervals $I \subset (0,\infty)$ and $J \subset (0,\infty)$ respectively, a function $f\colon I \to J$ is called MN-midpoint convex if it satisfies

$$f(M(u,v)) \leq N(f(u), f(v))$$

for all $u, v \in I$.

In [9], İşcan gave a new definition of function called weighted mean function as follows.

Definition 5. A function $M\colon (0,\infty) \times (0,\infty) \times [0,1] \to (0,\infty)$ is called a weighted mean function if

(WM1) $M(u,v,\lambda) = M(v,u,1-\lambda)$.
(WM2) $M(u,u,\lambda) = u$.
(WM3) $u < M(u,v,\lambda) < v$ whenever $u < v$ and $\lambda \in (0,1)$. Also $\{M(u,v,0), M(u,v,1)\} = \{u,v\}$.
(WM4) $M(\alpha u, \alpha v, \lambda) = \alpha M(u,v,\lambda)$ for all $\alpha > 0$.
(WM5) Let $\lambda \in [0,1]$ be fixed. Then $M(u,v,\lambda) \leq M(w,v,\lambda)$ whenever $u \leq w$ and $M(u,v,\lambda) \leq M(u,\omega,\lambda)$ whenever $v \leq \omega$.
(WM6) Let $u, v \in (0,\infty)$ be fixed and $u \neq v$. Then $M(u,v,.)$ is a strictly monotone and continuous function on $[0,1]$.
(WM7) $M(M(u,v,\lambda), M(z,w,\lambda), s) = M(M(u,z,s), M(v,w,s), \lambda)$ for all $u,v,z,w \in (0,\infty)$ and $s, \lambda \in [0,1]$.
(WM8) $M(u,v,s\lambda_1 + (1-s)\lambda_2) = M(M(u,v,\lambda_1), M(u,v,\lambda_2), s)$ for all $u,v \in (0,\infty)$ and $s, \lambda_1, \lambda_2 \in [0,1]$.

Remark 1 ([9]). According to the above definition every weighted mean function is a mean function with $\lambda = 1/2$. Also, By (WM6) we can say that for each $x \in [u,v] \subseteq (0,\infty)$ there exists a $\lambda \in [0,1]$ such that $x = M(u,v,\lambda)$. Morever;

i) If $M(u,v,.)$ is a strictly increasing, then $M(u,v,0) = u$ and $M(u,v,1) = v$ whenever $u < v$ (i.e. $M(u,v,\lambda)$ is in the positive direction)
ii) If $M(u,v,.)$ is a strictly deccreasing, then $M(u,v,0) = v$ and $M(u,v,1) = u$ whenever $u < v$ (i.e. $M(u,v,\lambda)$ is in the negative direction) and $M(u,v,.)([0,1]) = [\min\{u,v\}, \max\{u,v\}]$.

Throughout this paper, we will assume that different weighted means have the same direction unless otherwise stated.

Example 2 ([9]).

$$M(u,v,\lambda) = A(u,v,\lambda) = A_\lambda = (1-\lambda)u + \lambda v$$

is the Weighted Arithmetic Mean,

$$M(u,v,\lambda) = G(u,v,\lambda) = G_\lambda = u^{1-\lambda} v^{\lambda}$$

is the Weighted Geometric Mean,

$$M(u,v,\lambda)=H(u,v,\lambda)=H_\lambda=A^{-1}(u^{-1},v^{-1},\lambda)=\frac{uv}{\lambda u+(1-\lambda)v}$$

is the Weighted Harmonic Mean,

$$M(u,v,\lambda)=M_p(u,v,\lambda)=M_{p,\lambda}=\begin{cases} A^{1/p}(u^p,v^p,\lambda)=((1-\lambda)x^p+\lambda y^p)^{1/p} & p\in\mathbb{R}\setminus\{0\} \\ G(u,v,\lambda)=u^{1-\lambda}v^\lambda & p=0 \end{cases}$$

is the p-Power Mean. In particular, we have the following inequality

$$M_{-1,\lambda}=H_\lambda\leq M_{0,\lambda}=G_\lambda\leq M_{1,\lambda}=A_\lambda\leq M_{p,\lambda}$$

for all $x,y\in(0,\infty),t\in[0,1]$ and $p\geq 1$.

İşcan [9] proved the equalities in the following proposition.

Proposition 1. *If $M\colon(0,\infty)\times(0,\infty)\times[0,1]\to(0,\infty)$ is a weighted mean function, then the following identities hold:*

$$M\left(M\left(a,M(a,b,s),\lambda\right),M\left(b,M(a,b,s),\lambda\right),s\right)=M(a,b,s), \tag{1.3}$$

$$M\left(M(a,b,\lambda),M(b,a,\lambda),1/2\right)=M(a,b,1/2). \tag{1.4}$$

Many different definitions of convexity have been made by mathematicians until now. One of these definition was given by İşcan as follows.

Definition 6 ([9])**.** Let M and N be two weighted means defined on the intervals $I\subseteq(0,\infty)$ and $J\subseteq(0,\infty)$ respectively, a function $f\colon I\to J$ is called MN-convex (concave) if it satisfies

$$f\left(M(u,v,\lambda)\right)\leq(\geq)N\left(f(u),f(v),\lambda\right)$$

for all $u,v\in I$ and $\lambda\in[0,1]$.

We note that by considering the special cases of M and N, we obtain several different results. For some recent results related to convex functions, MN-convexity and some kinds of convexity obtained by using weighted means, see [1,4,6,7,11–14,16].

Definition 7 ([9])**.** Let M and N be two weighted means defined on the intervals $[u,v]\subseteq(0,\infty)$ and $J\subseteq(0,\infty)$ respectively and $f\colon[u,v]\to J$ be a function. We say that f is symmetric with respect to $M(u,v,1/2)$, if it satisfies

$$f\left(M(u,v,\lambda)\right)=f\left(M(u,v,1-\lambda)\right)$$

for all $\lambda\in[0,1]$.

Definition 8 ([10])**.** A function f defined on $[a,b]$ is said to be of bounded variation on $[a,b]$ if its total variation $Var(f)$ on $[a,b]$ is finite, where

$$Var(f) = \sup \sum_{j=1}^{n} \left| f(t_j) - f(t_{j-1}) \right|, \tag{1.5}$$

the supremum being taken over all partitions

$$a = t_0 < t_1 < \ldots < t_n = b \tag{1.6}$$

of the interval $[a,b]$; here, $n \in \mathbb{N}$ is arbitrary and so is the choice of values $t_1, \ldots, t_{n-1}$ in $[a,b]$ which, however, must satisfy (1.6).

Obviously, all functions of bounded variation on $[a,b]$ form a vector space. A norm on this space is given by

$$\|f\| = |f(a)| + Var(f). \tag{1.7}$$

The normed space thus defined is denoted by $BV[a,b]$, where BV suggest "bounded variation".

In 1905, E. Almansi [2] proved the following theorem.

Theorem 1. *Let f and f' be continuous functions on the interval (a,b) and let $f(a) = f(b)$ and $\int_a^b f(x)dx = 0$. Then*

$$\int_a^b \left[f(x)\right]^2 dx \leq \left(\frac{b-a}{2\pi}\right)^2 \int_a^b [f'(x)]^2 dx. \tag{1.8}$$

The aim of this paper is to give a new definition called (M,P)-function of P-functions that belongs to the class of $P(I)$ via the weighted means, obtain new inequalities using (M,P)-functions and present some properties of (M,P)-functions.

2. Main Results

Definition 9. Let $I \subset (0,\infty)$ be an interval, let $M\colon I \times I \times [0,1] \to (0,\infty)$ be a weighted mean function and let $f\colon I \to \mathbb{R}$ be a function. Then f is said to be an (M,P)-function if the inequality

$$f\big(M(x,y,t)\big) \leq f(x) + f(y)$$

holds for all $x,y \in I$ and $t \in [0,1]$.

Remark 2. If we choose M as the weighted arithmetic mean in Definition 9, we obtain the class of P-function.

Remark 3. If $f\colon I \to \mathbb{R}$ is an (M,P)-function on I,

$$f(x) \geq 0 \quad \forall x \in I.$$

Theorem 2. *Every MN-convex function is an (M,P)-function.*

Proof. Let $M\colon I\times I\times[0,1]\to(0,\infty)$, $N: J\times J\times[0,1]\to(0,\infty)$ be two weighted mean functions on intervals $J\subset(0,\infty)$, $I\subset(0,\infty)$ respectively and $f\colon I\to J$ be a MN-convex function. Then we can write

$$f\big(M(x,y,t)\big)\leq N\left(f(x),f(y),t\right) \tag{2.1}$$

for all $x,y\in I$ and all $t\in[0,1]$.

On the other hand, we can write

$$f(x)\leq f(x)+f(y)$$

and

$$f(y)\leq f(x)+f(y).$$

Then, we obtain,

$$N\big(f(x),f(y),t\big)\leq N\big(f(x)+f(y),f(x)+f(y),t\big)=f(x)+f(y). \tag{2.2}$$

So, using (2.1) and (2.2), the proof is completed. □

Theorem 3 (Hermite-Hadamard's inequalities for (M,P)-functions)**.** *Let M be a weighted mean function defined on the interval $I\subset(0,\infty)$ and $f\colon I\to J$ is an (M,P)-function. If the following integral exists, then we have the following inequalities for (M,P)-functions*

$$f\big(M(x,y,1/2)\big)\leq 2\int_0^1 f\big(M(x,y,t)\big)dt\leq 2[f(x)+f(y)] \tag{2.3}$$

for all $x,y\in I$ with $x<y$.

Proof. Since f is (M,P)-function, using (WM1) and (1.4) equality, we have

$$\begin{aligned} f\big(M(x,y,1/2)\big) &= f\Big(M\big(M(x,y,t),M(x,y,1-t),1/2\big)\Big) \\ &\leq f\big(M(x,y,t)\big)+f\big(M(x,y,1-t)\big) \end{aligned} \tag{2.4}$$

for all $t\in[0,1]$. Integrating both sides of (2.4) inequality respect to t over [0,1], we obtain

$$\begin{aligned} \int_0^1 f\big(M(x,y,1/2)\big)dt &= f\big(M(x,y,1/2)\big) \\ &\leq \int_0^1 f\big(M(x,y,t)\big)dt+\int_0^1 f\big(M(x,y,1-t)\big)dt \\ &= 2\int_0^1 f\big(M(x,y,t)\big)dt. \end{aligned} \tag{2.5}$$

Otherwise, we can write

$$f\big(M(x,y,t)\big)\leq f(x)+f(y) \tag{2.6}$$

for all $t \in [0,1]$. Integrating both sides of (2.6) inequality respect to t over [0,1], we obtain

$$\int_0^1 f\big(M(x,y,t)\big)dt \leq f(x)+f(y). \tag{2.7}$$

Then, using (2.5) and (2.7) inequalities, we get the desired result. □

Remark 4. Let $I \subset (0,\infty)$ and $f\colon I \to \mathbb{R}$. If f is an (M,P)-function and $M = A$ (A is the weighted arithmetic mean), then using (2.3), we have the following Hermite-Hadamard's inequalities (see [5], Theorem 3.1).

$$\begin{aligned} f\big(A(x,y,1/2)\big) = f\left(\frac{x+y}{2}\right) &\leq 2\int_0^1 f\big(A(x,y,t)\big)dt \\ &= 2\int_0^1 f\big((1-t)x+ty\big)dt \\ &= \frac{2}{y-x}\int_x^y f(u)du \\ &\leq 2[f(x)+f(y)]. \end{aligned}$$

Remark 5. Let $I \subset (0,\infty)$ and $f\colon I \to \mathbb{R}$. If f is an (M,P)-function and $M = G$ (G is the weighted geometric mean), then using (2.3), we have the following Hermite-Hadamard's inequalities (see [15], Theorem 2.2, Corollary 2.2, for $h(t)$=1).

$$\begin{aligned} f\big(G(x,y,1/2)\big) = f\left(\sqrt{xy}\right) &\leq 2\int_0^1 f\big(G(x,y,t)\big)dt \\ &= 2\int_0^1 f\big(x^{1-t}y^t\big)dt \\ &= \frac{2}{\ln y - \ln x}\int_x^y \frac{f(u)}{u}du \\ &\leq 2[f(x)+f(y)]. \end{aligned}$$

Remark 6. Let $I \subset (0,\infty)$ and $f\colon I \to \mathbb{R}$. If f is an (M,P)-function and $M = H$ (H is the weighted harmonic mean), then using (2.3), we have the following Hermite-Hadamard's inequalities (see [8], Theorem 4).

$$\begin{aligned} f\big(H(x,y,1/2)\big) = f\left(\frac{2xy}{x+y}\right) &\leq 2\int_0^1 f\big(H(x,y,t)\big)dt \\ &= 2\int_0^1 f\left(\frac{xy}{tx+(1-t)y}\right)dt \\ &= \frac{2xy}{y-x}\int_x^y \frac{f(u)}{u^2}du \\ &\leq 2[f(x)+f(y)]. \end{aligned}$$

Theorem 4. *If $f\colon [a,b] \subset (0,\infty) \to \mathbb{R}$ is an (M,P)-function, f is bounded on $[a,b]$.*

Proof. Since f is an (M,P)-function, $f(x) \geq 0$ respect to Remark 3 for all $x \in [a,b]$. Then, f is a function bounded below. Also, we can write $x = M(a,b,t)$ for $\forall x \in [a,b]$ and $\exists t \in [0,1]$. Then, we get

$$f(x) = f\big(M(a,b,t)\big) \leq f(a) + f(b) = k$$

and f is a function bounded above. Consequently, f is a bounded function. □

Theorem 5. *Let M be weighted mean defined on the interval $I \subset (0,\infty)$. If $f\colon I \to \mathbb{R}$ is an (M,P)-function and $\alpha > 0$, then αf is an (M,P)-function.*

Proof. Since f is an (M,P)-function, we have

$$\begin{aligned}\alpha f\big(M(x,y,t)\big) &\leq \alpha\big(f(x) + f(y)\big)\\ &= \alpha f(x) + \alpha f(y).\end{aligned}$$

This shows that αf is an (M,P)-function. So, the proof of theorem is completed. □

Theorem 6. *Let M be weighted mean function defined on the interval $I \subset (0,\infty)$. If $f_\alpha : I \to \mathbb{R}$ be an arbitrary family of (M,P)-functions and let $f(x) = \sup_\alpha f_\alpha(x)$. If $K = \{u \in I : f(u) < \infty\}$ is nonempty, then K is an interval and f is an (M,P)-function on K.*

Proof. Let $t \in [0,1]$ and $x,y \in K$ be arbitrary. Also, since f_α is an (M,P)-function, f_α is bounded. Then

$$\begin{aligned}f\big(M(x,y,t)\big) &= \sup_\alpha f_\alpha\big(M(x,y,t)\big)\\ &\leq \sup_\alpha \left(f_\alpha(x) + f_\alpha(y)\right)\\ &\leq \sup_\alpha f_\alpha(x) + \sup_\alpha f_\alpha(y)\\ &= f(x) + f(y)\\ &< \infty.\end{aligned}$$

This shows simultaneously that K is an interval, since it contains every point between any two of its points and that f is an (M,P)-function on K. The proof of the theorem is completed. □

Theorem 7. *Let M be weighted mean function defined on the interval $[x,y] \subseteq (0,\infty)$. If function $f\colon [x,y] \to \mathbb{R}$ is an (M,P)-function and symmetric with respect to $M(x,y,1/2)$, then we have*

$$f\big(M(x,y,1/2)\big) \leq 2f(u) \leq 2\big[f(x) + f(y)\big] \tag{2.8}$$

for all $u \in [x,y]$.

Proof. Let $u \in [x,y]$ be arbitrary point. Then there exist a $t \in [0,1]$ such that $u = M(x,y,t)$. Since $f\colon [x,y] \to J$ is an (M,P)-function and symmetric with respect to $M(x,y,1/2)$, by using equality (1.4) we have

$$f\big(M(x,y,1/2)\big) = f\Big(M\big(M(x,y,t), M(x,y,1-t), 1/2\big)\Big)$$

$$\begin{aligned}
&\leq f\big(M(x,y,t)\big)+f\big(M(x,y,1-t)\big)\\
&=f\big(M(x,y,t)\big)+f\big(M(x,y,t)\big)\\
&=2f(u).
\end{aligned}$$

Thus, we obtain the left-hand side of inequality (2.8). Secondly, since f is an (M,P)-function and $(WM5)$ with (1.4), we get

$$\begin{aligned}
2f(u)&=f\big(M(x,y,t)\big)+f\big(M(x,y,t)\big)\\
&\leq f(x)+f(y)+f(x)+f(y)\\
&=2f(x)+2f(y)\\
&=2\left[f(x)+f(y)\right].
\end{aligned}$$

So, the proof of the theorem is completed. □

Theorem 8. *Let M be weighted mean function defined on the interval $I\subset(0,\infty)$. If the functions $f,g\colon\ \to\mathbb{R}$ are (M,P)-functions, then $f+g$ is also an (M,P)-function.*

Proof. Since f and g are (M,P)-functions, we have

$$f\big(M(x,y,t)\big)\leq f(x)+f(y)$$

and

$$g\big(M(x,y,t)\big)\leq g(x)+g(y)$$

for all $x,y\in I$ and $t\in[0,1]$. Then we can write

$$\begin{aligned}
(f+g)\big(M(x,y,t)\big)&=f\big(M(x,y,t)\big)+g\big(M(x,y,t)\big)\\
&\leq f(x)+f(y)+g(x)+g(y)\\
&=f(x)+g(x)+f(y)+g(y)\\
&=(f+g)(x)+(f+g)(y).
\end{aligned}$$

So, this completes the proof. □

Theorem 9. *Let $0<a<b$ and $M\colon[a,b]\times[a,b]\times[0,1]\to(0,\infty)$ be a weighted mean function defined on $[a,b]$, $f\colon[a,b]\to(0,\infty)$, f and f' be continuous functions on (a,b) with $f(a)=f(b)$ and $\int_0^1 f\big(M(a,b,t)\big)dt=0$. If $|f'|$ is an (M,P)-function on $[a,b]$, then the following inequality holds*

$$\int_0^1 f^2\big(M(a,b,t)\big)dt\leq\frac{\left[|f'(a)|+|f'(b)|\right]^2}{4\pi^2}\int_0^1(\varphi'(t))^2dt,$$

where $\varphi(t)=M(a,b,t)$, $\forall t\in[0,1]$.

Proof. Let $\varphi(t)=M(a,b,t)$ and $\hbar(t)=f\circ\varphi(t)$. Since φ is strictly monotone, $\varphi\in\mathrm{BV}[0,1]$, then $\varphi'\in L[0,1]$. Also, we can write $\varphi(0)=a$, $\varphi(1)=b$ and therefore

$\hbar(0)=f\big(M(a,b,0)\big)=f(a)=f(b)=f\big(M(a,b,1)\big)=\hbar(1)$. Also, since $\int_0^1 \hbar(t)dt=\int_0^1 f\big(M(a,b,t)\big)dt=0$, φ satisfies the hypothesis of Theorem 1. So that, we can write

$$\int_0^1 \hbar^2(t)dt \le \frac{1}{4\pi^2}\int_0^1 \big(\hbar'(t)\big)^2 dt. \tag{2.9}$$

Then, we have,

$$\begin{aligned}\frac{1}{4\pi^2}\int_0^1 \big(\hbar'(t)\big)^2 dt &= \frac{1}{4\pi^2}\int_0^1 \big[f'\big(\varphi(t)\big)\varphi'(t)\big]^2 dt\\ &= \frac{1}{4\pi^2}\int_0^1 |f'\big(\varphi(t)\big)|^2\big(\varphi'(t)\big)^2 dt.\end{aligned}$$

Since $|f'|$ is (M,P)-function on $[a,b]$, we get

$$\begin{aligned}\frac{1}{4\pi^2}\int_0^1 \big(\hbar'(t)\big)^2 dt &\le \frac{1}{4\pi^2}\int_0^1 \big[|f'(a)|+|f'(b)|\big]^2\big(\varphi'(t)\big)^2 dt\\ &= \frac{\big[|f'(a)|+|f'(b)|\big]^2}{4\pi^2}\int_0^1 (\varphi'(t))^2 dt.\end{aligned} \tag{2.10}$$

Using (2.9) and (2.10), we get the desired result. □

Corollary 1. *If we take $M=A$ (A is the weighted arithmetic mean) in Theorem 9, we get*

$$\int_a^b f^2(x)dx \le \frac{(b-a)^3}{4\pi^2}\big[|f'(a)|+|f'(b)|\big]^2.$$

Corollary 2. *If we take $M=G$ (G is the weighted geometric mean) in Theorem 9, we get*

$$\int_a^b \frac{f^2(x)}{x}dx \le \frac{[\ln b-\ln a]^2(b^2-a^2)}{8\pi^2}\big[|f'(a)|+|f'(b)|\big]^2.$$

Corollary 3. *If we take $M=H$ (H is the weighted harmonic mean) in Theorem 9, we get*

$$\int_a^b \frac{f^2(x)}{x^2}dx \le \frac{(b^3-a^3)(b-a)^2}{12(ab)^2\pi^2}\big[|f'(a)|+|f'(b)|\big]^2.$$

Theorem 10. *Let $0<a<b$ and $M\colon [a,b]\times[a,b]\times[0,1]\to(0,\infty)$ be a weighted mean function defined on $[a,b]$, $f\colon [a,b]\to(0,\infty)$, f and f' be continuous functions on (a,b) with $f(a)=f(b)$ and $\int_0^1 f\big(M(a,b,t)\big)dt=0$. If $|f'|^q$ is an (M,P)-function on $[a,b]$, then the following inequality holds*

$$\int_0^1 f^2\big(M(a,b,t)\big)dt \le \frac{\big[|f'(a)|^q+|f'(b)|^q\big]^{\frac{2}{q}}}{4\pi^2}\left(\int_0^1 \big|\varphi'(t)\big|^{2p}dt\right)^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{q}, q>1$, $\varphi(t)=M(a,b,t)$, $\forall t\in[0,1]$.

Proof. Let $\varphi(t) = M(a,b,t)$ and $\hbar(t) = f \circ \varphi(t)$. Since φ is strictly monotone, $\varphi \in \mathrm{BV}[0,1]$, then $\varphi' \in L[0,1]$. Also, we can write $\varphi(0) = a$, $\varphi(1) = b$ and therefore $\hbar(0) = f\big(M(a,b,0)\big) = f(a) = f(b) = f\big(M(a,b,1)\big) = \hbar(1)$. Also, since $\int_0^1 \hbar(t)dt = \int_0^1 f\big(M(a,b,t)\big)dt = 0$, φ is satisfies the hypothesis of Theorem 1. So that, we can write

$$\int_0^1 \hbar^2(t)dt \leq \frac{1}{4\pi^2}\int_0^1 \big(\hbar'(t)\big)^2 dt. \tag{2.11}$$

Using Hölder inequality, we have

$$\begin{aligned}
\frac{1}{4\pi^2}\int_0^1 \big(\hbar'(t)\big)^2 dt &= \frac{1}{4\pi^2}\int_0^1 \big[f'\big(\varphi(t)\big)\varphi'(t)\big]^2 dt \\
&= \frac{1}{4\pi^2}\int_0^1 |f'\big(\varphi(t)\big)|^2 \left|\varphi'(t)\right|^2 dt \\
&\leq \frac{1}{4\pi^2}\left(\int_0^1 \left(\left|f'\big(\varphi(t)\big)\right|^2\right)^q dt\right)^{\frac{1}{q}} \left(\int_0^1 \left(\left|\varphi'(t)\right|^2\right)^p dt\right)^{\frac{1}{p}} \\
&= \frac{1}{4\pi^2}\left(\int_0^1 \left(\left|f'\big(\varphi(t)\big)\right|^q\right)^2 dt\right)^{\frac{1}{q}} \left(\int_0^1 \left|\varphi'(t)\right|^{2p} dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $|f'|^q$ is (M,P)-function on $[a,b]$, we get

$$\begin{aligned}
\frac{1}{4\pi^2}\int_0^1 \big(\hbar'(t)\big)^2 dt &\leq \frac{1}{4\pi^2}\left(\int_0^1 \left(|f'(a)|^q + |f'(b)|^q\right)^2 dt\right)^{\frac{1}{q}} \left(\int_0^1 \left|\varphi'(t)\right|^{2p} dt\right)^{\frac{1}{p}} \\
&= \frac{\left[|f'(a)|^q + |f'(b)|^q\right]^{\frac{2}{q}}}{4\pi^2}\left(\int_0^1 \left|\varphi'(t)\right|^{2p} dt\right)^{\frac{1}{p}}.
\end{aligned} \tag{2.12}$$

Using (2.11) and (2.12), we get the desired result. □

Corollary 4. *If we take $M = A$ (A is the weighted arithmetic mean) in Theorem 10, we get*

$$\int_a^b f^2(x)dx \leq \frac{(b-a)^3}{4\pi^2}\left[|f'(a)|^q + |f'(b)|^q\right]^{\frac{2}{q}}.$$

Corollary 5. *If we take $M = G$ (G is the weighted geometric mean) in Theorem 10, we get*

$$\int_a^b \frac{f^2(x)}{x}dx \leq \frac{[\ln b - \ln a]^{3-\frac{1}{p}}(b^{2p} - a^{2p})^{\frac{1}{p}}}{4\pi^2(2p)^{\frac{1}{p}}}\left[|f'(a)|^q + |f'(b)|^q\right]^{\frac{2}{q}}.$$

Corollary 6. *If we take $M = H$ (H is the weighted harmonic mean) in Theorem 10, we get*

$$\int_a^b \frac{f^2(x)}{x^2}dx \leq \frac{(b-a)^{3-\frac{1}{p}}(ab)(b^{1-4p}-a^{1-4p})^{\frac{1}{p}}}{4\pi^2(1-4p)^{\frac{1}{p}}}\left[|f'(a)|^q+|f'(b)|^q\right]^{\frac{2}{q}}.$$

REFERENCES

[1] J. Aczél, "A generalization of the notion of convex functions," *Norske Vid. Selsk. Forhd., Trondhjem*, vol. 19, no. 24, pp. 87–90, 1947.

[2] E. Almansi, "Sopra una delle esperienze di Plateau," *Ann. Mat. Pure Appl.*, vol. 3, no. 12, pp. 1–17, 1905.

[3] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, "Generalized convexity and inequalities," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 1294–1308, nov 2007, doi: 10.1016/j.jmaa.2007.02.016.

[4] G. Aumann, "Konvexe Funktionen und Induktion bei Ungleichungenzwischen Mittelverten," *Bayer. Akad. Wiss. Math.-Natur. Kl. Abh., Math. Ann.*, vol. 109, pp. 405–413, 1933.

[5] S. S. Dragomir, J. Pečarić, and L. E. Persson, "Some inequalities of Hadamard type," *Soochow Journal of Mathematics*, vol. 21, no. 3, pp. 335–341, 1995.

[6] İ.İşcan, "Hermite-Hadamard type inequalities for harmonically convex functions," *Hacet. J. Math. Stat.*, vol. 43, no. 6, pp. 935–942, 2014.

[7] İ.İşcan, "Ostrowski type inequalities for p-convex functions," *New Trends in Mathematical Sciences*, vol. 4, no. 3, pp. 140–150, 2016, doi: 10.20852/ntmsci.2016318838.

[8] İ.İşcan, S. Numan, and K. Bekar, "Hermite-Hadamard and Simpson type inequalities for differentiable harmonically P-functions," *British Journal of Mathematics and Computer Science*, vol. 4, no. 14, pp. 1908–1920, 2014, doi: 10.9734/BJMCS/2014/10338.

[9] İ.İşcan, "On weighted means and MN-convex functions," *Turkish Journal of Inequalities*, vol. 5, no. 2, pp. 70–81, 2021.

[10] E. Kreyszig, *Introductory Functional Analysis with Applications*. University of Windsor, Newyork Santa Barbara London Sydney Toronto, 1978.

[11] J. Matkowski, "Convex functions with respect to a mean and acharacterization of quasi-arithmetic means," *Real Anal. Exchange*, vol. 29, pp. 229–246, 2003/2004.

[12] T. Z. Mirkovic, "New inequalities of Wirtinger type for convex and MN-Convex Fuctions," *Facta Universitatis, Series: Mathematics and Informatics*, vol. 34, no. 2, pp. 165–173, 2019, doi: 10.22190/FUMI1902165M.

[13] C. P. Niculescu, "Convexity according to the geometric mean," *Math. Inequal. Appl.*, vol. 3, no. 2, pp. 155–167, 2000, doi: 10.7153/mia-03-19.

[14] C. P. Niculescu, "Convexity according to means," *Math. Inequal. Appl.*, vol. 6, pp. 571–579., 2003, doi: 10.7153/mia-06-53.

[15] M. A. Noor, K. I. Noor, and M. U. Awan, "Some inequalities for geometrically-arithmetically h-convex functions," *Creat. Math. Inform.*, vol. 24, no. 2, pp. 193–200, 2015.

[16] A. W. Roberts and D. E. Varberg, *Convex functions*. Academic Press, New York, 1973.

Authors' addresses

Erhan Set

(**Corresponding author**) Ordu University, Faculty of Sciences and Arts, Department of Mathematics, Ordu, Turkey

E-mail address: erhanset@yahoo.com

Ali Karaoğlan
Ordu University, Faculty of Sciences and Arts, Department of Mathematics, Ordu, Turkey
E-mail address: alikaraoglan@odu.edu.tr

İmdat İşcan
Giresun University, Faculty of Sciences and Arts, Department of Mathematics, Giresun, Turkey
E-mail address: imdat.iscan@giresun.edu.tr

Neslihan Kılıç
Ordu University, Faculty of Sciences and Arts, Department of Mathematics, Ordu, Turkey
E-mail address: neslimavi1907@gmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 999–1008

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4433

MULTIPLICITY OF POSITIVE SOLUTIONS FOR A THIRD-ORDER BOUNDARY VALUE PROBLEM WITH NONLOCAL CONDITIONS OF INTEGRAL TYPE

SERGEY SMIRNOV

Received 03 October, 2022

Abstract. We prove the existence of multiple positive solutions for a nonlinear third-order nonlocal boundary value problem by applying Krasnosel'skii's fixed point theorem. To illustrate the applicability of the obtained results, we consider an example.

2010 *Mathematics Subject Classification:* 34B10; 34B15; 34B18

Keywords: third-order nonlinear boundary value problems, integral boundary condition, multiplicity of positive solutions, Green's function, Krasnosel'skii's fixed point theorem

1. Introduction

We study boundary value problem consisting of the nonlinear third-order differential equation

$$x''' + f(t,x) = 0, \quad t \in (0,1), \tag{1.1}$$

and the boundary conditions

$$x(0) = 0, \quad x'(0) = 0, \quad x(1) = \int_0^1 x(\xi)\,d\xi. \tag{1.2}$$

We assume that $f\colon [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

The purpose of the paper is to give results on the existence of multiple positive solutions to (1.1), (1.2) by applying Krasnosel'skii's fixed point theorem. By a positive solution of (1.1), (1.2) we understand $C^3[0,1]$ function which is positive on $0 < t \le 1$ and satisfies differential equation (1.1) for $0 < t < 1$ and boundary conditions (1.2). However, note that if $f(t,0) = 0$, then boundary value problem (1.1), (1.2) always has the trivial solution. Since $f(t,x)$ is not defined for $x < 0$, every solution of (1.1), (1.2) is nonnegative. In what follows, we will show that every nonnegative nontrivial solution of (1.1), (1.2) is positive.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

Actually, our main result states that for each given positive integer n, we can indicate f so that problem (1.1), (1.2) has at least n positive solutions. The approach that we will use to obtain this result is the often used one. First, we rewrite problem (1.1), (1.2) as an equivalent integral equation by constructing the corresponding Green's function. Then, we define an operator in the suitable cone of nonnegative continuous functions, and hence the problem reduces to find fixed points of the operator. Finally, we prove the existence of multiple fixed points in the cone using Krasnosel'skii's cone compression and expansion theorem of norm type [7, 8].

Krasnosel'skii's theorem is a very convenient and effective tool for studying the existence and multiplicity of positive solutions to boundary value problems, and therefore a lot of authors employ this technique in their research. J.R. Graef and B. Yang consider third-order three-point boundary value problem and give sufficient conditions for the existence of multiple positive solutions to this problem in [3]. The same authors, in [4], obtain existence and nonexistence results for positive solutions for a higher-order multi-point boundary value problem, and in [5], obtain sufficient conditions for the existence and nonexistence of positive solutions for a third-order two-point boundary value problem. J.R.L. Webb and G. Infante establish the existence of multiple positive solutions for a second-order nonlocal boundary value problem in [12]. The same authors, in [13], give a method of establishing the existence of multiple positive solutions for a large number of nonlinear differential equations of arbitrary order with any allowed number of nonlocal conditions. The problem we are studying here is a special case of a family of problems studied in [10], but in this paper, we use somewhat different conditions on the nonlinearity in the equation. We will explain this difference in Example 1 (see Section 4). A. Calamai and G. Infante study a parameter-dependent analogue of problem (1.1),(1.2) in the context of delay equations in [1].

Nonlocal boundary conditions make it possible to obtain more accurate models and consequently often appear in physics and various branches of applied mathematics. Much research has been done on nonlocal boundary value problems in the last decades. See, for example, the papers by J.R. Graef and J.R.L. Webb [2], J.R.L. Webb [9–11], J.R. Graef and B. Yang [6] and references therein.

Since our main tool in this paper is Krasnosel'skii's fixed point theorem, let us state this theorem for the reader's convenience.

Theorem 1 (Krasnosel'skii, [8])**.** *Let E be a Banach space and $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, $T\colon K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to K$ is a completely continuous operator such that*

(A) $\|Tx\| \le \|x\|$, $\forall x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \ge \|x\|$, $\forall x \in K \cap \partial\Omega_2$, *or*
(B) $\|Tx\| \ge \|x\|$, $\forall x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \le \|x\|$, $\forall x \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

The paper contains three sections besides the Introduction. In Section 2, we rewrite the main problem as an equivalent integral equation, by constructing the corresponding Green's function. Also, we give some inequalities for Green's function here. In Section 3, we prove our main theorem on the existence of multiple positive solutions for the problem. In conclusion, we consider an example to illustrate the applicability of our main result in Section 4.

2. Construction and estimation of the Green's function

Our first goal is to rewrite problem (1.1), (1.2) as an equivalent integral equation. So, let us consider the linear equation

$$x''' + h(t) = 0, \quad t \in (0,1), \tag{2.1}$$

together with boundary conditions (1.2).

Proposition 1. *Let $h\colon [0,1] \to \mathbb{R}$ be a continuous function. Then the function defined by*

$$x(t) = \int_0^1 G(t,s)h(s)\,ds$$

is the unique solution of boundary value problem (2.1),(1.2), where

$$G(t,s) = \frac{1}{4}\begin{cases} t^2(1-s)^2(s+2) - 2(t-s)^2, & 0 \le s \le t \le 1, \\ t^2(1-s)^2(s+2), & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

Proof. Let $x(t)$ be a solution of problem (2.1),(1.2). Integrating equation (2.1) thrice, we get

$$x(t) = x(0) + t\,x'(0) + \frac{1}{2}t^2 x''(0) - \frac{1}{2}\int_0^t (t-s)^2 h(s)\,ds,$$

and, in view of boundary conditions (1.2), we obtain

$$x(t) = \frac{1}{2}t^2 x''(0) - \frac{1}{2}\int_0^t (t-s)^2 h(s)\,ds.$$

Since $x(1) = \int_0^1 x(\xi)\,d\xi$, it follows that

$$\frac{1}{2}x''(0) - \frac{1}{2}\int_0^1 (1-s)^2 h(s)\,ds = \int_0^1 \left(\frac{1}{2}\xi^2 x''(0) - \frac{1}{2}\int_0^\xi (\xi-s)^2 h(s)\,ds\right) d\xi$$

and hence

$$x''(0) = \frac{1}{2}\int_0^1 (1-s)^2(s+2)h(s)\,ds.$$

Therefore,

$$\begin{aligned}
x(t) &= \frac{1}{4}t^2\int_0^1 (1-s)^2(s+2)h(s)\,ds - \frac{1}{2}\int_0^t (t-s)^2 h(s)\,ds \\
&= \frac{1}{4}\int_0^t t^2(1-s)^2(s+2)h(s)\,ds + \frac{1}{4}\int_t^1 t^2(1-s)^2(s+2)h(s)\,ds \\
&\quad - \frac{1}{2}\int_0^t (t-s)^2 h(s)\,ds \\
&= \frac{1}{4}\int_0^t \left(t^2(1-s)^2(s+2) - 2(t-s)^2\right)h(s)\,ds + \frac{1}{4}\int_t^1 t^2(1-s)^2(s+2)h(s)\,ds.
\end{aligned}$$

□

Hence boundary value problem (1.1), (1.2) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s)f(s,x(s))\,ds, \quad 0 \le t \le 1, \tag{2.3}$$

in the sense that x is a solution of (1.1), (1.2) iff it is a solution of (2.3). Here $G(t,s)$ denotes the Green's function for the problem $x''' = 0$, (1.2), and is explicitly given by (2.2).

Next, we prove some inequalities for the Green's function $G(t,s)$.

Proposition 2. *For all* $(t,s) \in [0,1]\times[0,1]$, *we have*

$$G(t,s) \ge 0. \tag{2.4}$$

If $(t,s) \in (0,1)\times(0,1)$, *then*

$$G(t,s) > 0.$$

Proof. For $0 \le t \le s \le 1$, it is obvious that $t^2(1-s)^2(s+2) \ge 0$.
If $0 \le s \le t \le 1$, we have

$$\begin{aligned}
&t^2(1-s)^2(s+2) - 2(t-s)^2 \ge t^2(1-s)^2(s+2) - (s+2)(t-s)^2 \\
&= (s+2)\left(t^2(1-s)^2 - (t-s)^2\right) = (s+2)\left(s^2(t-1)(t+1) - 2st(t-1)\right) \\
&= s(s+2)(t-1)(t(s-2)+s) = s(s+2)(1-t)(t(2-s)-s) \ge 0.
\end{aligned}$$

□

Proposition 3. *For all* $(t,s) \in [0,1] \times [0,1]$, *we have*

$$G(t,s) \le \frac{s(1-s)^2(s+2)}{2(3-s^2)}. \tag{2.5}$$

Proof. Let us find the maximum of $G(t,s)$ for each s with respect to t.
For $0 \le s \le t \le 1$, the maximum occurs at $t = \dfrac{2}{3-s^2}$ and is equal to $\dfrac{s(1-s)^2(s+2)}{2(3-s^2)}$.
If $0 \le t \le s \le 1$, the maximum occurs at $t = s$ and is equal to $\dfrac{s^2(1-s)^2(s+2)}{4}$.
Since for all $(t,s) \in [0,1] \times [0,1]$,

$$\frac{s(1-s)^2(s+2)}{2(3-s^2)} = \frac{s^2(1-s)^2(s+2)}{2s(3-s^2)} \ge \frac{s^2(1-s)^2(s+2)}{4},$$

we get the proof. □

Proposition 4. *For all* $(t,s) \in [1/2,1] \times [0,1]$, *we have*

$$G(t,s) \ge \frac{1}{4} \cdot \frac{s(1-s)^2(s+2)}{2(3-s^2)}. \tag{2.6}$$

Proof. For $\Lambda_1 = \{(t,s) : 1/2 \le t \le 1,\ 0 \le s \le 1,\ s \le t\}$, we have

$$\inf_{\Lambda_1} \frac{G(t,s)}{\dfrac{s(1-s)^2(s+2)}{2(3-s^2)}} = \inf_{\Lambda_1} \frac{(3-s^2)(4t-2s-t^2(3-s^2))}{2(1-s)^2(s+2)} = \frac{1}{3}.$$

If $\Lambda_2 = \{(t,s) : 1/2 \le t \le 1,\ 0 \le s \le 1,\ t \le s\}$, then

$$\inf_{\Lambda_2} \frac{G(t,s)}{\dfrac{s(1-s)^2(s+2)}{2(3-s^2)}} = \min_{\Lambda_2} \frac{t^2(3-s^2)}{2s} = \frac{1}{4}.$$

Therefore,

$$\frac{G(t,s)}{\dfrac{s(1-s)^2(s+2)}{2(3-s^2)}} \ge \frac{1}{4} \quad \text{for} \quad \frac{1}{2} \le t \le 1,\ 0 \le s \le 1.$$

□

Proposition 5. *Every nonnegative nontrivial solution* $x(t)$ *of* (1.1), (1.2) *is positive.*

Proof. Suppose that there exists $t_0 \in (0,1)$ such that $x(t_0) = 0$. Since boundary value problem (1.1), (1.2) is equivalent to integral equation (2.3) we get

$$x(t_0) = \int_0^1 G(t_0,s) f(s,x(s))\, ds = 0.$$

Since $G(t_0,s)f(s,x(s)) \geq 0$ for all $s \in [0,1]$ then

$$G(t_0,s)f(s,x(s)) = 0 \quad \text{for all } s \in [0,1].$$

Since $G(t_0,s) > 0$ for all $s \in (0,1)$ we get that $x'''(s) = -f(s,x(s)) = 0$ for all $s \in (0,1)$. Therefore $x(s)$ is a polynomial of degree at most two. Since $x(s)$ satisfies boundary conditions (1.2) it follows that $x(s) = 0$ for all $s \in [0,1]$. We get the contradiction. □

3. Existence of multiple positive solutions

In this section, we prove our main result on the existence of multiple positive solutions for boundary value problem (1.1), (1.2) by applying Krasnosel'skii's fixed point theorem.

For our constructions, consider the Banach space $C[0,1]$ with the norm

$$\|x\| = \max_{0\leq t\leq 1} |x(t)|, \quad x \in C[0,1].$$

Define a cone K in $C[0,1]$ by

$$K = \left\{ x \in C[0,1] \ : \ x(t) \geq 0, \ \min_{\frac{1}{2}\leq t\leq 1} x(t) \geq \frac{1}{4}\|x\| \right\},$$

and an integral operator $T\colon K \to C[0,1]$ by

$$(Tx)(t) = \int_0^1 G(t,s)f(s,x(s))\,ds, \quad 0 \leq t \leq 1.$$

It is easy to see that boundary value problem (1.1), (1.2) has a solution x if and only if x is a fixed point of the operator T. Also, it is well known that $T\colon K \to C[0,1]$ is a completely continuous operator.

Proposition 6. $T(K) \subset K$.

Proof. From inequality (2.4), it follows that for $x \in K$, $(Tx)(t) \geq 0$ on $[0,1]$. Also, for $x \in K$, we have from (2.5) that

$$(Tx)(t) = \int_0^1 G(t,s)f(s,x(s))\,ds \leq \int_0^1 \frac{s(1-s)^2(s+2)}{2(3-s^2)} f(s,x(s))\,ds,$$

so that

$$\|Tx\| \leq \int_0^1 \frac{s(1-s)^2(s+2)}{2(3-s^2)} f(s,x(s))\,ds. \tag{3.1}$$

And next, if $x \in K$, we have by (2.6) and (3.1),

$$\min_{\frac{1}{2}\le t\le 1}(Tx)(t) = \min_{\frac{1}{2}\le t\le 1}\int_0^1 G(t,s)f(s,x(s))\,ds$$
$$\ge \frac{1}{4}\int_0^1 \frac{s(1-s)^2(s+2)}{2(3-s^2)}f(s,x(s))\,ds \ge \frac{1}{4}\|Tx\|.$$

□

We shall use the following notations:

$$I_1 = \left(\max_{0\le t\le 1}\int_0^1 G(t,s)\,ds\right)^{-1} < \left(\max_{0\le t\le 1}\int_{1/2}^1 G(t,s)\,ds\right)^{-1} = I_2.$$

The next two propositions will be used in the proof of our main result.

Proposition 7. *Suppose that there exists $r > 0$ such that $f(t,x) \le I_1 r$ for $(t,x) \in [0,1]\times[0,r]$. If $x \in K$ with $\|x\| = r$, then $\|Tx\| \le r$.*

Proof. If $x \in K$ with $\|x\| = r$, then for $t \in [0,1]$ we have

$$(Tx)(t) = \int_0^1 G(t,s)f(s,x(s))\,ds \le I_1 r\int_0^1 G(t,s)\,ds \le I_1 r\max_{0\le t\le 1}\int_0^1 G(t,s)\,ds = r,$$

or $\|Tx\| \le r$.

□

Proposition 8. *Suppose that there exists $r > 0$ such that $f(t,x) \ge I_2 r$ for $(t,x) \in [0,1]\times[r/4,r]$. If $x \in K$ with $\|x\| = r$, then $\|Tx\| \ge r$.*

Proof. If $x \in K$ with $\|x\| = r$, then for every $s \in [1/2,1]$ we have $\min_{\frac{1}{2}\le s\le 1} x(s) \ge \frac{1}{4}\|x\| = \frac{1}{4}r$ and $x(s) \in [r/4,r]$. Therefore,

$$\|Tx\| = \max_{0\le t\le 1}\int_0^1 G(t,s)f(s,x(s))\,ds$$
$$\ge \max_{0\le t\le 1}\int_{1/2}^1 G(t,s)f(s,x(s))\,ds \ge I_2 r\max_{0\le t\le 1}\int_{1/2}^1 G(t,s)\,ds = r.$$

□

Theorem 2. *Suppose that there exist* $2m$ *constants* $0 < r_1 < r_2 < \cdots < r_{2m-1} < r_{2m}$, *and let* $\alpha(n) = 2n - \dfrac{1-(-1)^n}{2}$, $\beta(n) = 2n - \dfrac{1+(-1)^n}{2}$, *where* $1 \le n \le m$. *If*

$$f(t,x) \le I_1\, r_{\alpha(n)} \quad \text{for } (t,x) \in [0,1] \times [0, r_{\alpha(n)}]$$

and

$$f(t,x) \ge I_2\, r_{\beta(n)} \quad \text{for } (t,x) \in [0,1] \times [r_{\beta(n)}/4, r_{\beta(n)}],$$

then boundary value problem (1.1),(1.2) *has at least* m *positive solutions* $x_n(t)$ *such that* $r_{2n-1} \le \|x_n\| \le r_{2n}$.

Proof. If $\Omega_k = \{x \in C[0,1] : \|x\| < r_k\}$, $1 \le k \le 2m$, then, from Proposition 7 and Proposition 8, we have

$$\|Tx\| \le \|x\| \quad \text{for } x \in K \cap \partial\Omega_{\alpha(n)},$$

and

$$\|Tx\| \ge \|x\| \quad \text{for } x \in K \cap \partial\Omega_{\beta(n)}.$$

From Theorem 1, we see that T has fixed point in each of the sets $K \cap \left(\overline{\Omega}_{2n} \setminus \Omega_{2n-1}\right)$. Thus, boundary value problem (1.1), (1.2) has at least m positive solutions. □

4. Example

Example 1. Consider boundary value problem (1.1),(1.2) with

$$f(t,x) = \begin{cases} 28x^2, & 0 \le x \le 1, \\ 622(x-1)^2 + 28, & 1 \le x \le 2, \\ \dfrac{60}{2\sqrt{6}-1}\left(\sqrt{x-1} - 1\right) + 650, & 2 \le x \le 25, \\ g(x),\ g(25) = 710, & 25 \le x, \end{cases}$$

where $g\colon [25,\infty) \to [0,\infty)$ is a continuous function.

We have $I_1 = \dfrac{256}{9} \approx 28.44$ and $I_2 = \dfrac{24}{1057}\left(7537 - 405\sqrt{105}\right) \approx 76.9$.

If we choose $r_1 = 1$, $r_2 = 8$, $r_3 = 8.4$, $r_4 = 25$, we get

$$\begin{aligned} f(t,x) &\le I_1\, r_1 \quad \text{for } (t,x) \in [0,1] \times [0, r_1], \\ f(t,x) &\ge I_2\, r_2 \quad \text{for } (t,x) \in [0,1] \times [r_2/4, r_2], \\ f(t,x) &\ge I_2\, r_3 \quad \text{for } (t,x) \in [0,1] \times [r_3/4, r_3], \\ f(t,x) &\le I_1\, r_4 \quad \text{for } (t,x) \in [0,1] \times [0, r_4]. \end{aligned}$$

Therefore, by Theorem 2, the boundary value problem has at least two positive solutions $x_1(t)$ and $x_2(t)$ such that

$$1 \le \|x_1\| \le 8, \quad 8.4 \le \|x_2\| \le 25.$$

Solutions $x_1(t)$ and $x_2(t)$ are depicted in Figure 1 and Figure 2. These figures were obtained by using the program Wolfram Mathematica 11.1. Each solution has a double

zero at $t = 0$ and is equal to its antiderivative at $t = 1$. The initial conditions for these solutions are $x_1(0) = 0$, $x_1'(0) = 0$, $x_1''(0) \approx 6.27314$ and $x_2(0) = 0$, $x_2'(0) = 0$, $x_2''(0) \approx 170.0105$.

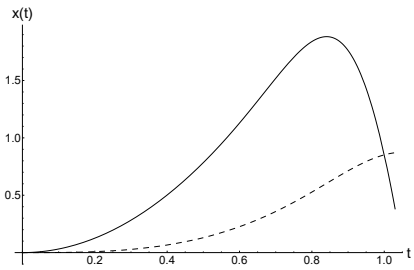


FIGURE 1. Solution x_1 (solid) with its antiderivative (dashed).

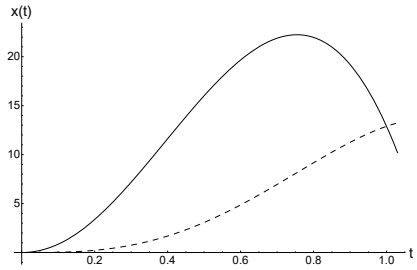


FIGURE 2. Solution x_2 (solid) with its antiderivative (dashed).

Now, let us discuss the difference between conditions on nonlinearity f in Theorem 2 and in [10] (Theorem 2.2). According to [10], the problem has at least two positive solutions if certain conditions are fulfilled, two of which are the following inequalities

$$0 \le \lim_{x \to 0+} \frac{f(x)}{x} < \mu_1, \quad 0 \le \lim_{x \to \infty} \frac{f(x)}{x} < \mu_1,$$

where μ_1 is called the principal characteristic value of operator T or the principal eigenvalue of the corresponding boundary value problem. We see, that the first inequality is fulfilled, but the second one is not fulfilled, because $\lim_{x \to \infty} \frac{f(x)}{x}$ can be every nonnegative number in our example. Also, the author would like to mention, that our conditions allow us to get an estimate of the norm for positive solutions to the problem.

REFERENCES

[1] A. Calamai and G. Infante, "An affine Birkhoff-Kellogg-type result in cones with applications to functional differential equations." *Math. Meth. Appl. Sci.*, pp. 1–9, 2022, doi: 10.1002/mma.8665.

[2] J. R. Graef and J. R. L. Webb, "Third order boundary value problems with nonlocal boundary conditions." *Nonlinear Anal.*, vol. 71, pp. 1542–1551, 2009, doi: 10.1016/j.na.2008.12.047.

[3] J. R. Graef and B. Yang, "Multiple positive solutions to a three point third order boundary value problem." *Discrete Contin. Dyn. Syst.*, vol. suppl., pp. 337–344, 2005.

[4] J. R. Graef and B. Yang, "Positive solutions to a multi-point higher order boundary value problem." *J. Math. Anal. Appl.*, vol. 316, pp. 409–421, 2006.

[5] J. R. Graef and B. Yang, "Existence and nonexistence of positive solutions of a nonlinear third order boundary value problem." *Electron. J. Qual. Theory Differ. Equ.*, vol. 9, pp. 1–13, 2008, doi: 10.14232/ejqtde.2008.1.31.

[6] J. R. Graef and B. Yang, "Positive solutions of a third order nonlocal boundary value problem." *Discrete Contin. Dyn. Syst. Ser. S*, vol. 1, pp. 89–97, 2008, doi: 10.3934/dcdss.2008.1.89.

[7] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones.* San Diego: Academic Press, 1988.

[8] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations.* Groningen: Noordhoff, 1964.

[9] J. R. L. Webb, "Multiple positive solutions of some nonlinear heat flow problems." *Discrete Contin. Dyn. Syst. Ser.*, vol. suppl., pp. 895–903, 2005.

[10] J. R. L. Webb, "Higher order non-local $(n-1,1)$ conjugate type boundary value problems." *Mathematical models in engineering, biology and medicine*, vol. 1124, pp. 332–341, 2009.

[11] J. R. L. Webb, "Non-local second-order boundary value problems with derivative-dependent nonlinearity." *Phil. Trans. R. Soc. A*, vol. 379, pp. 1–12, 2021, doi: 10.1098/rsta.2019.0383.

[12] J. R. L. Webb and G. Infante, "Positive solutions of nonlocal boundary value problems involving integral conditions." *NoDEA Nonlinear Differential Equations Appl.*, vol. 15, pp. 45–67, 2008, doi: 10.1007/s00030-007-4067-7.

[13] J. R. L. Webb and G. Infante, "Non-local boundary value problems of arbitrary order." *J. London Math. Soc.*, vol. 79, no. 2, pp. 238–258, 2009, doi: 10.1112/jlms/jdn066.

Author's address

Sergey Smirnov
Institute of Mathematics and Computer Science, University of Latvia, Raina bulvaris 29, Riga, LV-1459, Latvia

Faculty of Physics, Mathematics and Optometry, University of Latvia, Jelgavas iela 3, Riga, LV-1004, Latvia
E-mail address: srgsm@inbox.lv

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 1009–1024

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4309

ON GENERALIZED M-PROJECTIVE CURVATURE TENSOR OF PARA-SASAKIAN MANIFOLD

SWATI JAIN, TEERATHRAM RAGHUWANSHI, MANOJ KUMAR PANDEY, AND ANIL GOYAL

Received 27 June, 2022

Abstract. The purpose of the present paper is to study some properties of generalized M-projective curvature tensor of a para-Sasakian manifold admitting Zamkovoy connection. The generalized M-projective is obtained with the help of a new generalized (0,2) symmetric tensor $\mathcal{Z}$ introduced by Mantica and Suh [10]. It is shown that para-Sasakian manifold satisfying the condition $R(X,Y)\cdot\tilde{M}^{**}=0$ is an η-Einstein manifold. Also, we found that a para-Sasakian manifold satisfying $\tilde{M}^{**}(X,Y)\cdot S=0$ is either an Einstein manifold or $\psi=1$ on it.

2010 *Mathematics Subject Classification:* 53C15; 53C25

Keywords: M-projective curvature tensor, generalized M-projective curvature tensor, Zamkovoy connection, para-Sasakian manifold, Einstein manifold, η-Einstein manifold

1. Introduction

The notion of the almost para-contact structure on a differentiable manifold is defined by I. Sato [14, 15]. The para-contact metric manifolds have been studied by many authors in recent years. The structure is an analogue of the almost contact structure [5, 13] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). Every differentiable manifold with almost para-contact structure defined by I. Sato has a compatible Riemannian metric.

An almost para-contact structure on a pseudo-Riemannian manifold M of dimension $(2n+1)$ defined is by S. Kaneyuki and M. Konzai [18] and they constructed the almost paracomplex structure on $M\times R$. Recently, S. Zamkovoy [22] has associated the almost para-contact structure given in [18] to a pseudo-Riemannian metric of signature $(n+1,n)$ and showed that any almost para-contact structure admits such a pseudo-Riemannian metric.

The study of M-projective curvature tensor has been a very attractive field for investigations in the past many decades. M-projective curvature tensor was introduced

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

by G. P. Pokhariyal and R. S. Mishra [12] in 1971. Also, in 1986, R. H. Ojha [11] extended some properties of the M-projective curvature tensor in Sasakian and Kähler manifolds. In 2010, the study of M-projective curvature tensor in Riemannian manifolds and also in Kenmotsu manifolds was resumed by S. K. Chaubey and R. H. Ojha [6]. Further, R. N. Singh and S. K. Pandey [19] have studied various geometric properties of M-projective curvature tensor on $N(k)$-contact metric manifolds. In 2020, A. Mandal and A. Das [8] studied some properties of the M-projective curvature tensor in Sasakian manifolds. Afterwards, several researchers have carried out the study of M-projective curvature tensor in a variety of directions such as [9, 16–18]. The M-projective curvature tensor defined by G. P. Pokhariyal and R. S. Mishra [12] is given as below;

$$M^*(X,Y,U) = R(X,Y,U) - \frac{1}{2(n-1)}[S(Y,U)X - S(X,U)Y + g(Y,U)QX - g(X,U)QY],$$

for all $X,Y,U \in \chi(M)$, where $\chi(M)$ is the set of all vector field of manifold M, $R(X,Y)U$ is the Riemannian curvature tensor of type $(0,3)$ and S is the Ricci tensor, i.e,

$$S(X,Y) = g(QX,Y),$$

where Q is a Ricci operator of type $(1,1)$.

Also, the type $(0,4)$ M-projective curvature tensor field $'M^*$ is given by

$$\begin{aligned} 'M^*(X,Y,U,V) = {} & 'R(X,Y,U,V) \\ & - \frac{1}{2(n-1)}[S(Y,U)g(X,V) - S(X,U)g(Y,V) \\ & + g(Y,U)S(X,V) - g(X,U)S(Y,V)], \end{aligned} \tag{1.1}$$

where

$$'M^*(X,Y,U,V) = g(M^*(X,Y,U),V)$$

and

$$'R(X,Y,U,V) = g(R(X,Y,U),V)$$

for the arbitrary vector fields $X,Y,U,V \in \chi(M)$.

A new generalized $(0,2)$ symmetric tensor $\mathcal{Z}$, defined by Mantica and Suh [10], is given by the following relation

$$\mathcal{Z}(X,Y) = S(X,Y) + \psi g(X,Y), \tag{1.2}$$

where ψ is an arbitrary scalar function.

From equation (1.2), we have

$$\mathcal{Z}(\phi X,\phi Y) = S(\phi X,\phi Y) + \psi g(\phi X,\phi Y),$$

By using equation (1.2) in equation (1.1), we get

$$\begin{aligned} {}'M^*(X,Y,U,V) = {}'R(X,Y,U,V) \\ -\frac{1}{2(n-1)}[\mathcal{Z}(Y,U)g(X,V)-\mathcal{Z}(X,U)g(Y,V) \\ +g(Y,U)\mathcal{Z}(X,V)-g(X,U)\mathcal{Z}(Y,V)] \\ -\frac{\psi}{(n-1)}[g(Y,V)g(X,U)-g(Y,U)g(X,V)]. \end{aligned} \tag{1.3}$$

If we denote the first five terms of above equation by

$$\begin{aligned} {}'M^{**}(X,Y,U,V) = {}'R(X,Y,U,V) \\ -\frac{1}{2(n-1)}[\mathcal{Z}(Y,U)g(X,V)-\mathcal{Z}(X,U)g(Y,V) \\ +g(Y,U)\mathcal{Z}(X,V)-g(X,U)\mathcal{Z}(Y,V)], \end{aligned} \tag{1.4}$$

then the equation (1.3) reduces to

$$\begin{aligned} {}'M^{**}(X,Y,U,V) = ({}'M^*)(X,Y,U,V)+\frac{\psi}{(n-1)}[g(Y,V)g(X,U) \\ -g(X,V)g(Y,U)]. \end{aligned}$$

We call this new tensor field ${}'M^{**}$ defined by equation (1.4), generalized M-projective curvature tensor of para-Sasakian manifold.

In 2008, The notion of Zamkovoy connection was introduced by S. Zamkovoy [22] for a para-contact manifold. And this connection is defined as a canonical para-contact connection whose torsion is the obstruction of para-contact manifold to be a para-Sasakian manifold [1]. For an n-dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ,ξ,η,g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, the Zamkovoy connection is defined by [22]

$$\tilde{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y \tag{1.5}$$

for all $X,Y,U \in \chi(M)$.

This connection was further studied by A. M. Blaga in para Kenmotsu manifolds [4] and A. Biswas, K. K. Baishya in Sasakian manifolds [2,3].

In this paper, we study some properties of the generalized M-projective curvature tensor of para-Sasakian manifold with respect to the Zamkovoy connection. The present paper is organized as follows: Section 2 is devoted to preliminaries and we give some relations between curvature tensor (resp. Ricci tensor) with respect to Zamkovoy connection and curvature tensor (resp. Ricci tensor) with respect to Levi-Civita connection. In Section 3, we describe briefly the generalized M-projective curvature tensor on para-Sasakian manifold with respect to the connection. In Section 4, we show that a generalized M-projectively semi-symmetric para-Sasakian

manifold is an η-Einstein manifold. Further in Section 5, the goal is to examine implication of the condition $\tilde{M}^{**}(X,Y)\cdot S = 0$ and we show that the para-Sasakian manifold is either an Einstein manifold or $\psi = 1$ on it. In the last section, we show that $\phi^2((\nabla_V \tilde{M}^{**})(X,Y,U)) = 0$ is an η-Einstein manifold.

2. Preliminaries

An $(2n+1)$-dimensional differentiable manifold M is said to have almost para-contact structure (ϕ,ξ,η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field known as characteristic vector field and η is a 1-form on M satisfying the following relations [18]

$$\phi^2 = I - \eta\otimes\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\phi(\xi) = 0, \qquad \eta(\phi X) = 0,$$

and

$$\operatorname{rank}(\phi) = 2n,$$

where I denotes the identity transformation, a differentiable manifold with almost para-contact structure (ϕ,ξ,η) is called an almost para-contact manifold [1].

Moreover, the tensor field ϕ induces an almost paracomplex structure on the para-contact distribution D = ker (η), i.e, the eigendistributions $D^{\pm}$ corresponding to the eigenvalues ± 1 of ϕ are both n-dimensional.

If an almost para-contact manifold M with an almost para-contact structure (ϕ,ξ,η) admits a pseudo-Riemannian metric g such that [22]

$$g(\phi X,\phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

for all $X,Y \in \chi(M)$, then we say that M is an almost para-contact metric manifold with an almost para-contact metric structure (ϕ,ξ,η,g) and such metric g is called compatible metric. Any compatible metric g is necessarily of signature $(n+1,n)$.

From (2.3) one can see that [22]

$$g(X,\phi Y) = -g(\phi X,Y),$$

and also we take

$$\eta(X) = g(X,\xi) \tag{2.4}$$

for any $X,Y \in \chi(M)$. The fundamental 2-form of M is defined by

$$\alpha(X,Y) = g(X,\phi Y).$$

The structure (ϕ,ξ,η,g) satisfying conditions (2.1) to (2.4) is called an almost para-contact metric structure and the manifold M with such a structure is called an almost para-contact Riemannian manifold [14].

An almost para-contact metric structure becomes a para-contact metric structure [22] if

$$g(X,\phi Y) = d\eta(X,Y),$$

for all vector field $X,Y \in \chi(M)$, where

$$d\eta(X,Y) = \frac{1}{2}[X\eta(Y) - Y\eta(X) - \eta([X,Y])].$$

For a $(2n+1)$ dimensional manifold M with the structure (ϕ,ξ,η,g), one can also construct a local orthonormal basis which is called a ϕ-basis $(X_i,\phi X_i,\xi)$, $(i = 1,2,\ldots,n)$ [22]

An almost para-contact metric manifold structure (ϕ,ξ,η,g) is para-Sasakian manifold if and only if the Levi-Civita connection ∇ of g satisfies [22]

$$(\nabla_X\phi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.5}$$

for any $X,Y \in \chi(M)$.

From (2.5), it can be seen that

$$(\nabla_X\xi) = -\phi X. \tag{2.6}$$

Example 1. [1]. Let $M = R^{2n+1}$ be the $(2n+1)$- dimensional real number space with $(x_1,y_1,x_2,y_2,\ldots,x_n,y_n,z)$ standard coordinate system. Defining

$$\phi\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha}, \quad \phi\frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial x_\alpha}, \quad \phi\frac{\partial}{\partial z} = 0,$$
$$\xi = \frac{\partial}{\partial z}, \quad \eta = dz,$$
$$g = \eta\otimes\eta + \sum_{\alpha=1}^{n} dx_\alpha\otimes dx_\alpha - \sum_{\alpha=1}^{n} dy_\alpha\otimes dy_\alpha,$$

where $\alpha = 1,2,\ldots,n$, then the set (M,ϕ,ξ,η,g) is an almost para-contact metric manifold.

In a para-Sasakian manifold, the following relations also hold [22]:

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.7}$$
$$R(X,Y,\xi) = \eta(X)Y - \eta(Y)X,$$
$$R(X,\xi,Y) = -R(\xi,X,Y) = -g(X,Y)\xi + \eta(Y)X, \tag{2.8}$$
$$R(\xi,X,\xi) = X - \eta(X)\xi,$$
$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y),$$

and

$$S(X,\xi) = -2n\eta(X), \tag{2.9}$$

for any vector fields $X,Y,Z \in \chi(M)$. Here, R is Riemannian curvature tensor and S is the Ricci tensor defined by $g(QX,Y) = S(X,Y)$, where Q is the Ricci operator.

In view of (2.6), the equation (1.5) becomes

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)\phi X + g(X,\phi Y)\xi. \tag{2.10}$$

On a para-Sasakian manifold, the connection $\tilde{\nabla}$ has the following properties [1]:

$$\tilde{\nabla}\eta = 0, \quad \tilde{\nabla}g = 0, \quad \tilde{\nabla}\xi = 0,$$

and

$$(\tilde{\nabla}_X\phi)Y = (\nabla_X\phi)Y + g(X,Y)\xi - \eta(Y)X.$$

for any vector fields $X,Y, \in \chi(M)$.

It is known that the curvature tensor $\tilde{R}$ of a para-Sasakian manifold M with respect to the Zamkovoy connection $\tilde{\nabla}$ defined by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z$$

satisfies the following [1]

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X \\ &\quad - \eta(X)\eta(Z)Y + 2g(X,\phi Y)\phi Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X, \\ \tilde{S}(X,Y) &= S(X,Y) - 2g(X,Y) + (2n+2)\eta(X)\eta(Y),\end{aligned} \tag{2.11}$$

and

$$\tilde{\mathrm{r}} = r - 2n$$

for any $X,Y,Z \in \chi(M)$, where R, S and r are curvature tensor, Ricci tensor and scalar curvature relative to ∇ respectively and $\tilde{R}$, $\tilde{S}$ and $\tilde{r}$ are curvature tensor, Ricci tensor and scalar curvature relative to $\tilde{\nabla}$. From (2.11), it is easy to note that $\tilde{S}$ is symmetric.

Further, it is known that [1] on a para-Sasakian manifold, the following relations hold

$$g(\tilde{R}(X,Y)Z,\xi) = \eta(\tilde{R}(X,Y)Z) = 0, \tag{2.12}$$

$$\tilde{R}(X,Y)\xi = \tilde{R}(\xi,X)Y = \tilde{R}(\xi,X)\xi = 0, \tag{2.13}$$

and

$$\tilde{S}(X,\xi) = 0$$

for any $X,Y,U \in \chi(M)$.

Definition 1. An $(2n+1)$-dimensional para-Sasakian manifold M is said to be η-Einstein manifold if the Ricci tensor of type $(0,2)$ is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

for any $X,Y \in \chi(M)$ where a and b are scalars.

From (2.11), it can also be noted that if $\tilde{S}(X,Y)=0$. then

$$S(X,Y)=2g(X,Y)+(-2n-2)\eta(X)\eta(Y).$$

which proves that if a para- Sasakian manifold M is Ricci-flat with respect to the Zamkovoy connection, then it is an η-Einstein manifold.

3. Generalized M-projective curvature tensor of para-Sasakian manifold

In this section, we study generalized M-projective curvature tensor of the para-Sasakian manifold with respect to the Zamkovoy connection and state some of its properties. The M-projective curvature tensor $\tilde{M}$ with respect to the Zamkovoy connection $\tilde{\nabla}$ is given by

$$\begin{aligned}\tilde{M}^*(X,Y,U)=&\ \tilde{R}(X,Y,U)\\ &-\frac{1}{2(n-1)}[\tilde{S}(Y,U)X-\tilde{S}(X,U)Y\\ &\qquad+g(Y,U)\tilde{Q}X-g(X,U)\tilde{Q}Y],\end{aligned}\tag{3.1}$$

Also, the type $(0,4)$ M-projective curvature tensor field $'\tilde{M}^*$ is given by

$$\begin{aligned}'\tilde{M}^*(X,Y,U,V)=&\ '\tilde{R}(X,Y,U,V)\\ &-\frac{1}{2(n-1)}[\tilde{S}(Y,U)g(X,V)-\tilde{S}(X,U)g(Y,V)\\ &\qquad+g(Y,U)\tilde{S}(X,V)-g(X,U)\tilde{S}(Y,V)],\end{aligned}\tag{3.2}$$

where

$$'\tilde{M}^*(X,Y,U,V)=g(\tilde{M}^*(X,Y,U),V)$$

and

$$'\tilde{R}(X,Y,U,V)=g(\tilde{R}(X,Y,U),V)$$

for the arbitrary vector fields $X,Y,U,V\in\chi(M)$.

Now, differentiating covariantly equation (3.1) with respect to V, we get

$$\begin{aligned}(\nabla_V\tilde{M}^*)(X,Y)U=&\ (\nabla_V\tilde{R})(X,Y)U\\ &-\frac{1}{2(n-1)}[(\nabla_V\tilde{S})(Y,U)X-(\nabla_V\tilde{S})(X,U)Y\\ &\qquad+g(Y,U)(\nabla_V\tilde{Q})X-g(X,U)(\nabla_V\tilde{Q})Y].\end{aligned}\tag{3.3}$$

The generalized M-projective curvature tensor of para-Sasakian manifold with respect to the Zamkovoy connection is defined by,

$$\begin{aligned} {}'\tilde{M}^*(X,Y,U,V) = {}'\tilde{R}(X,Y,U,V) \\ -\frac{1}{2(n-1)}[\mathcal{Z}(Y,U)g(X,V)-\mathcal{Z}(X,U)g(Y,V) \\ +g(Y,U)\mathcal{Z}(X,V)-g(X,U)\mathcal{Z}(Y,V)] \\ -\frac{\psi}{(n-1)}[g(Y,V)g(X,U)-g(Y,U)g(X,V)]. \end{aligned}$$

If we denote the first five terms of above equation by

$$\begin{aligned} {}'\tilde{M}^{**}(X,Y,U,V) = {}'\tilde{R}(X,Y,U,V) \\ -\frac{1}{2(n-1)}[\mathcal{Z}(Y,U)g(X,V)-\mathcal{Z}(X,U)g(Y,V) \\ +g(Y,U)\mathcal{Z}(X,V)-g(X,U)\mathcal{Z}(Y,V)], \end{aligned} \tag{3.4}$$

then the equation (3.4) reduces to

$$\begin{aligned} {}'\tilde{M}^{**}(X,Y,U,V) = {}'\tilde{M}^*(X,Y,U,V)+\frac{\psi}{(n-1)}[g(Y,V)g(X,U) \\ -g(X,V)g(Y,U)]. \end{aligned} \tag{3.5}$$

We call this new tensor field ${}'\tilde{M}^{**}$ defined by equation (3.4), generalized M-projective curvature tensor of para-Sasakian manifold with respect to the Zamkovoy connection.

If ψ=0, then from equation (3.5), we have

$$ {}'\tilde{M}^{**}(X,Y,U,V) = {}'\tilde{M}^*(X,Y,U,V). $$

Lemma 1. *If the scalar function ψ vanishes on para-Sasakian manifold, then the M-projective curvature tensor and generalized M-projective curvature tensor are identical.*

Lemma 2. *Generalized M-projective curvature tensor of para-Sasakian manifold with respect to the Zamkovoy connection satisfies Bianchi's first identity.*

Remark 1. Generalized M-projective curvature tensor ${}'\tilde{M}^{**}$ of para-Sasakian manifold with respect to the Zamkovoy connection is:

(a) skew-symmetric in the first two slots,
(b) skew-symmetric in the last two slots

and

(c) symmetric in pair of slots.

Proposition 1. *Generalized M-projective curvature tensor of para-Sasakian manifold satisfies the following identities:*

$$(a)\ \tilde{M}^{**}(\xi,Y,U) = -\tilde{M}^{**}(Y,\xi,U) = \left[\frac{(1-\psi)}{(n-1)}\right][g(Y,U)\xi - \eta(U)Y] - \left[\frac{1}{2(n-1)}\right][S(Y,U)\xi - \eta(U)QY], \tag{3.6}$$

$$(b)\ \tilde{M}^{**}(X,Y,\xi) = -\left[\frac{(1-\psi)}{(n-1)}\right][\eta(X)Y - \eta(Y)X] + \left[\frac{1}{2(n-1)}\right][\eta(Y)QX - \eta(X)QY], \tag{3.7}$$

$$(c)\ \eta(\tilde{M}^{**}(U,V,Y)) = \left[\frac{(1-\psi)}{(n-1)}\right][g(V,Y)\eta(U) - g(U,Y)\eta(V)] + \left[\frac{1}{2(n-1)}\right][S(U,Y)\eta(V) - S(V,Y)\eta(U)]. \tag{3.8}$$

4. Generalized M-projectively semi-symmetric para-Sasakian manifold with respect to the Zamkovoy connection

Definition 2. A para-Sasakian manifold is said to be semi-symmetric [20] if it satisfies the condition

$$R(X,Y)\cdot R = 0,$$

where $R(X,Y)$ is considered as the derivation of the tensor algebra at each point of the manifold.

Definition 3. A para-Sasakian manifold is said to be generalized M-projectively semi-symmetric if it satisfies the condition

$$R(X,Y)\cdot \tilde{M}^{**} = 0,$$

where $\tilde{M}^{**}$ is generalized M-projective curvature tensor relative to $\tilde{\nabla}$ and $R(X,Y)$ is considered as the derivation of the tensor algebra at each point of the manifold.

Theorem 1. *A generalized M-projectively semi-symmetric para-Sasakian manifold with respect to the Zamkovoy connection is an η-Einstein manifold.*

Proof. Consider

$$R(X,Y)\cdot \tilde{M}^{**} = 0,$$

Now, we put $X = \xi$ in above equation, we get

$$(R(\xi,X)\cdot \tilde{M}^{**})(U,V,Y) = 0,$$

for any $X,Y,U,V \in \chi(M)$, where $\tilde{M}^{**}$ is generalized M-projective curvature tensor, which gives

$$\begin{aligned}0 &= R(\xi,X,\tilde{M}^{**}(U,V,Y)) - \tilde{M}^{**}(R(\xi,X,U),V,Y)\\ &\quad - \tilde{M}^{**}(U,R(\xi,X,V),Y) - \tilde{M}^{**}(U,V,R(\xi,X,Y).\end{aligned}$$

In view of the equation (2.8), the above equation takes the form

$$\begin{aligned}0 &= \eta(\tilde{M}^{**}(U,V,Y))X -' \tilde{M}^{**}(U,V,Y,X)\xi\\ &\quad + g(X,U)\eta(\tilde{M}^{**}(\xi,V,Y) - \eta(V)\eta(\tilde{M}^{**}(U,X,Y)\\ &\quad + g(X,V)\eta(\tilde{M}^{**}(U,\xi,Y) - \eta(Y)\eta(\tilde{M}^{**}(U,V,X)\\ &\quad + g(X,Y)\eta(\tilde{M}^{**}(U,V,\xi) - \eta(U)\eta(\tilde{M}^{**}(X,V,Y).\end{aligned}$$

Taking inner product of above equation with ξ and using equations (2.2), (2.3), (2.12), (2.13), (3.5), (3.6), (3.7) and (3.8), we get

$$\begin{aligned}0 &= -\,'\tilde{M}^{**}(U,V,Y,X)\\ &\quad - \frac{1}{2(n-1)}\left[S(U,X)\eta(Y)\eta(V) - S(V,X)\eta(U)\eta(Y)\right]\\ &\quad + \frac{(1-\psi)}{(n-1)}\left[g(X,U)g(Y,V) - g(X,V)g(Y,U)\right]\\ &\quad - \frac{1}{2(n-1)}\left[S(V,Y)g(X,U) - S(Y,U)g(X,V)\right]\\ &\quad - \frac{n}{(n-1)}\left[g(X,U)\eta(Y)\eta(V) - g(X,V)\eta(U)\eta(Y)\right].\end{aligned}$$

By virtue of the equations (2.11) and (3.2), the above equation reduces to

$$\begin{aligned}'\tilde{R}(U,V,Y,X) &= \frac{1}{2(n-1)}[\tilde{S}(V,Y)g(U,X) - \tilde{S}(U,Y)g(V,X)\\ &\qquad + \tilde{S}(U,X)g(V,Y) - \tilde{S}(V,X)g(U,Y)]\\ &\quad - \frac{1}{2(n-1)}\left[S(U,X)\eta(Y)\eta(V) - S(V,X)\eta(U)\eta(Y)\right]\\ &\quad + \frac{(1-\psi)}{(n-1)}\left[g(X,U)g(Y,V) - g(X,V)g(Y,U)\right]\\ &\quad - \frac{1}{2(n-1)}\left[S(V,Y)g(X,U) - S(Y,U)g(X,V)\right]\\ &\quad - \frac{n}{(n-1)}\left[g(X,U)\eta(Y)\eta(V) - g(X,V)\eta(U)\eta(Y)\right].\end{aligned}$$

Let $\{e_i : i = 1,2\ldots,n\}$ be an orthonormal basis. Putting $X = U = e_i$ in above equation and taking summation over i, we get

$$S(Y,V) = \left[\frac{r+2n-4n\psi-2}{(2n-1)}\right] g(Y,V) + \left[\frac{r+4n^2-2}{(2n-1)}\right] \eta(Y)\eta(V).$$

This shows that generalized M-projectively semi-symmetric para-Sasakian manifold is an η-Einstein manifold. □

5. PARA-SASAKIAN MANIFOLD SATISFYING $\tilde{M}^{**}(X,Y)\cdot S = 0$

In this section, we consider para-Sasakian manifold with Zamkovoy connection [7] satisfying the condition

$$\tilde{M}^{**}(X,Y)\cdot S = 0,$$

for all $X,Y \in \chi(M)$, where $\tilde{M}^{**}$ is generalized M-projective curvature tensor of para-Sasakian manifold.

Theorem 2. *A para-Sasakian manifold admitting Zamkovoy connection satisfying $\tilde{M}^{**}(X,Y)\cdot S = 0$ is either an Einstein manifold or $\psi = 1$.*

Proof. Consider

$$(\tilde{M}^{**}(\xi,X)\cdot S)(U,V) = 0,$$

which gives

$$S(\tilde{M}^{**}(\xi,X,U),V) + S(U,\tilde{M}^{**}(\xi,X,V)) = 0.$$

Using equations (2.9), (2.10) and (3.6) in above equation, we get

$$\begin{aligned} 0 = &\left[\frac{-2n(\psi-1)}{(n-1)}\right][g(X,U)\eta(V) + g(X,V)\eta(U)] \\ &- \left[\frac{1}{(n-1)}\right][S(X,V)\eta(U) + S(X,U)\eta(V)] \\ &+ \left[\frac{\psi}{(n-1)}\right][S(X,V)\eta(U) + S(X,U)\eta(V)]. \end{aligned}$$

Putting $U = \xi$ in the above equation and using the equations (2.3), (2.4) and (2.9), we get

$$\left[\frac{(\psi-1)}{(n-1)}\right][S(X,V) - 2ng(X,V)] = 0,$$

which gives either $\psi = 1$ or

$$S(X,V) = 2ng(X,V).$$

This shows that generalized M-projectively Ricci semi-symmetric para-Sasakian manifold with respect to the Zamkovoy connection is either an Einstein manifold or $\psi = 1$ on it. □

6. A PARA-SASAKIAN MANIFOLD SATISFYING $\phi^2((\nabla_V R)(X,Y,U)) = 0$

Below we present the definition given by Takahashi [21]

Definition 4. A para-Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_V R)(X,Y,U)) = 0, \tag{6.1}$$

for all vector fields X,Y,U,V orthogonal to ξ.

Definition 5. A para-Sasakian manifold is said to be ϕ-symmetric if

$$\phi^2((\nabla_V R)(X,Y,U)) = 0, \tag{6.2}$$

for arbitrary vector fields X,Y,U,V.

Analogous to the conditions (6.1) and (6.2), we consider a para-Sasakian manifold satisfying

$$\phi^2((\nabla_V \tilde{M}^{**})(X,Y,U)) = 0, \tag{6.3}$$

for arbitary vector fields X,Y,U,V.

Theorem 3. *A para-Sasakian manifold admitting Zamkovoy connection satisfying* $\phi^2((\nabla_V \tilde{M}^{**})(X,Y,U)) = 0$ *is an Einstein manifold.*

Proof. Taking covariant derivative of equation (3.5) with respect to vector field V, we obtain

$$(\nabla_V \tilde{M}^{**})(X,Y,U) = (\nabla_V \tilde{M}^{*})(X,Y,U) + \frac{dr(\psi)}{(n-1)}[g(X,U)Y - g(Y,U)X)].$$

Using equation (3.3) in the above equation, we get

$$\begin{aligned}(\nabla_V \tilde{M}^{**})(X,Y,U) = (\nabla_V \tilde{R})(X,Y,U) &+ \frac{dr(\psi)}{(n-1)}[g(X,U)Y - g(Y,U)X)] \\ &- \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U)X - (\nabla_V \tilde{S})(X,U)Y \\ &\qquad + g(Y,U)(\nabla_V \tilde{Q})X - g(X,U)(\nabla_V \tilde{Q})Y].\end{aligned} \tag{6.4}$$

Assume that the manifold is generalized M-projectively ϕ-symmetric, then from equation (6.3), we have

$$\phi^2((\nabla_V \tilde{M}^{**})(X,Y,U)) = 0,$$

which on using equation (2.1), gives

$$(\nabla_V \tilde{M}^{**})(X,Y,U) = \eta((\nabla_V \tilde{M}^{**})(X,Y,U))\xi.$$

Using equation (6.4) in above equation, we get

$$\begin{aligned}
&(\nabla_V \tilde{R})(X,Y,U) + \frac{dr(\psi)}{(n-1)}[g(X,U)Y - g(Y,U)X)] \\
&- \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U)X - (\nabla_V \tilde{S})(X,U)Y \\
&\qquad + g(Y,U)(\nabla_V \tilde{Q})X - g(X,U)(\nabla_V \tilde{Q})Y] \\
&= \eta((\nabla_V \tilde{R})(X,Y,U))\xi + \frac{dr(\psi)}{(n-1)}[g(X,U)\eta(Y) - g(Y,U)\eta(X))]\xi \\
&- \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U)\eta(X) - (\nabla_V \tilde{S})(X,U)\eta(Y) \\
&\qquad + g(Y,U)\eta((\nabla_V \tilde{Q})X) - g(X,U)\eta((\nabla_V \tilde{Q})Y)]\xi.
\end{aligned}$$

Taking inner product of the above equation with W, we get

$$\begin{aligned}
&g((\nabla_V \tilde{R})(X,Y,U),W) + \frac{dr(\psi)}{(n-1)}[g(X,U)g(Y,W) - g(Y,U)g(X,W)] \\
&- \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U)g(X,W) - (\nabla_V \tilde{S})(X,U)g(Y,W) \\
&\qquad + g(Y,U)g((\nabla_V \tilde{Q})X,W) - g(X,U)g((\nabla_V \tilde{Q})Y,W)] \\
&= \eta((\nabla_V \tilde{R})(X,Y,U))\eta(W) + \frac{dr(\psi)}{(n-1)}[g(X,U)\eta(Y)\eta(W) \\
&\qquad - g(Y,U)\eta(X)\eta(W)] \\
&- \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U)\eta(X)\eta(W) - (\nabla_V \tilde{S})(X,U)\eta(Y)\eta(W) \\
&\qquad + g(Y,U)\eta((\nabla_V \tilde{Q})X)\eta(W) - g(X,U)\eta((\nabla_V \tilde{Q})Y)\eta(W)].
\end{aligned}$$

Putting $X = W = e_i$ and taking summation over i, we obtain

$$\begin{aligned}
0 = &-\frac{1}{(n-1)}(\nabla_V \tilde{S})(Y,U) - \frac{dr(\psi)}{(n-1)}[\eta(Y)\eta(U) - g(Y,U)] \\
&- \frac{1}{2(n-1)}[g(Y,U)g((\nabla_V \tilde{Q})e_i,e_i) - g((\nabla_V \tilde{Q})Y,U)] \\
&- \frac{2n}{(n-1)}dr(\psi)g(Y,U) - \eta((\nabla_V \tilde{R})(e_i,Y,U))\eta(e_i) \\
&+ \frac{1}{2(n-1)}[(\nabla_V \tilde{S})(Y,U) - (\nabla_V \tilde{S})(e_i,U)\eta(Y)\eta(e_i) \\
&\qquad + g(Y,U)\eta((\nabla_V \tilde{Q})e_i)\eta(e_i) - \eta((\nabla_V \tilde{Q})Y)\eta(U)].
\end{aligned}$$

Taking $U = \xi$ in the above equation, we have

$$\begin{aligned} 0 = &-\frac{1}{2(n-1)}(\nabla_V \tilde{S})(Y,\xi) - \eta((\nabla_V \tilde{R})(e_i,Y,\xi))\eta(e_i) \\ &-\frac{1}{2(n-1)}[dr(\tilde{V})\eta(Y) - (\nabla_V \tilde{S})(e_i,\xi)\eta(e_i)\eta(Y) \\ &\qquad + \eta((\nabla_V \tilde{Q})e_i)\eta(e_i)\eta(Y)] \\ &-\frac{2n}{(n-1)}dr(\psi)\eta(Y). \end{aligned} \tag{6.5}$$

Now

$$\eta((\nabla_V \tilde{R})(e_i,Y,\xi)\eta(e_i) = g((\nabla_V \tilde{R})(e_i,Y,\xi),\xi)g(e_i,\xi).$$

Also

$$\begin{aligned} g((\nabla_V \tilde{R})(e_i,Y,\xi),\xi) = &\, g(\nabla_V \tilde{R}(e_i,Y,\xi),\xi) - g(\tilde{R}(\nabla_V e_i,Y,\xi),\xi) \\ &- g(\tilde{R}(e_i,\nabla_V Y,\xi),\xi) - g(\tilde{R}(e_i,Y,\nabla_V \xi),\xi). \end{aligned} \tag{6.6}$$

Since $\{e_i\}$ is an orthonormal basis, so $\nabla_X e_i = 0$ and using equation (2.8), we get

$$g(\tilde{R}(e_i,\nabla_V Y,\xi),\xi) = 0,$$

Since

$$g(\tilde{R}(e_i,Y,\xi),\xi) + g(\tilde{R}(\xi,\xi,Y),e_i) = 0.$$

Therefore, we have

$$g(\nabla_V \tilde{R}(e_i,Y,\xi),\xi) + g(\tilde{R}e_i,Y,\xi),\nabla_V \xi) = 0.$$

Using this fact in equation (6.6), we get

$$g((\nabla_V \tilde{R})(e_i,Y,\xi),\xi) = 0. \tag{6.7}$$

Also

$$\eta((\nabla_V \tilde{Q})e_i)\eta(e_i) = g((\nabla_V \tilde{Q})e_i,\xi)g(e_i,\xi) = g((\nabla_V \tilde{Q})\xi,\xi).$$

Using equations (2.6) and (2.9), we get

$$\eta((\nabla_V \tilde{Q})e_i)\eta(e_i) = 0. \tag{6.8}$$

Using equations (6.7) and (6.8) in (6.5), we have

$$(\nabla_V \tilde{S})(Y,\xi) = -4ndr(\psi)\eta(Y) + dr(\tilde{V})\eta(Y). \tag{6.9}$$

Taking $Y = \xi$ in above equation and using equations (2.7) and (2.10), we get

$$dr(\psi) = -\frac{dr(\tilde{V})}{4n}, \tag{6.10}$$

which shows that r is constant.

Now, we have

$$(\nabla_V \tilde{S})(Y,\xi) = \nabla_V \tilde{S}(Y,\xi) - \tilde{S}(\nabla_V Y,\xi) - \tilde{S}(Y,\nabla_V \xi),$$

then by using (2.5), (2.6), (2.10) in the above equation, it follows that

$$\tilde{S}(Y,\phi V) = 0. \tag{6.11}$$

Putting $Y = \phi Y$ in above equation and using (2.11), (6.9), (6.10) and (6.11), we obtain

$$S(Y,V) = -2g(Y,V) - 2n\eta(V)\eta(Y).$$

which shows that M^{2n+1} is an η-Einstein manifold. □

ACKNOWLEDGEMENTS

The authors would like to express their thanks to the referees for their constructive advices and comments that helped to improve this paper.

REFERENCES

[1] B. E. Acet, E. Kılıç, and S. Y. Perktaş, "Some curvature conditions on a para-Sasakian manifold with canonical paracontact connection." *Int. J. Math. Math. Sci.*, vol. 2012, p. 24, 2012, doi: 10.1155/2012/395462.

[2] A. Biswas and K. K. Baishya, "A general connection on Sasakian manifolds and the case of almost pseudo symmetric Sasakian manifolds." *Sci. Stud. Res., Ser. Math. Inform.*, vol. 29, no. 1, pp. 59–72, 2019.

[3] A. Biswas and K. K. Baishya, "Study on generalised pseudo (Ricci) symmetric Sasakian manifold admitting general connection." *Bull. Transilv. Univ. Braşov, Ser. III, Math. Inform. Phys.*, vol. 12, no. 2, pp. 233–246, 2019, doi: 10.31926/but.mif.2019.12.61.2.4.

[4] A. M. Blaga, "Canonical connections on para-Kenmotsu manifolds." *Novi Sad J. Math.*, vol. 45, no. 2, pp. 131–142, 2015, doi: 10.30755/NSJOM.2014.050.

[5] D. E. Blair, *Contact manifolds in Riemannian geometry.* Springer, Cham, 1976. doi: 10.1007/BFb0079307.

[6] S. K. Chaubey and R. H. Ojha, "On the M-projective curvature tensor of a Kenmotsu manifold." *Differ. Geom. Dyn. Syst.*, vol. 12, pp. 52–60, 2010.

[7] P. Majhi and U. C. De, "Classifications of $N(k)$-contact metric manifolds satisfying certain curvature conditions." *Acta Math. Univ. Comen., New Ser.*, vol. 84, no. 1, pp. 167–178, 2015.

[8] A. Mandal and A. Das, "On M-projective curvature tensor of Sasakian manifolds admitting zamkovoy connection." *Adv. Math. Sci. J.*, vol. 9, no. 10, pp. 8929–8940, 2020.

[9] A. Mandal and A. Das, "Pseudo projective curvature tensor on Sasakian manifolds admitting zamkovoy connection." *Bull. Cal. Math. Soc.*, vol. 112, no. 5, pp. 431–450, 2020.

[10] C. A. Mantica and Y. J. Suh, "Pseudo Z symmetric Riemannian manifolds with harmonic curvature tensors," *Int. J. Geom. Methods Mod. Phys.*, vol. 9, no. 1, pp. 1 250 004, 21, 2012, doi: 10.1142/S0219887812500041.

[11] R. H. Ojha, "M-projectively flat Sasakian manifolds," *Indian J. Pure Appl. Math.*, vol. 17, pp. 481–484, 1986.

[12] G. P. Pokhariyal and R. S. Mishra, "Curvature tensors and their relativistics significance." *Yokohama Math. J.*, vol. 18, pp. 105–108, 1970.

[13] S. Sasaki, "On differentiable manifolds with certain structures which are closely related to almost contact structure. I." *Tôhoku Math. J. (2)*, vol. 12, pp. 459–476, 1960, doi: 10.2748/tmj/1178244407.

[14] I. Sato, "On a structure similar to the almost contact structure II." *Tensor. New Series*, vol. 31, pp. 199–205, 1970.

[15] I. Sato, "On a structure similar to the almost contact structure." *Tensor, New Ser.*, vol. 30, pp. 219–224, 1976.

[16] J. P. Singh, "On m-projectively flat almost pseudo Ricci symmetric manifolds." *Acta Math. Univ. Comen., New Ser.*, vol. 86, no. 2, pp. 335–343, 2017.

[17] R. N. Singh, M. K. Pandey, and D. Gautam, "On nearly quasi Einstein manifold." *Int. J. Math. Anal., Ruse*, vol. 5, no. 33-36, pp. 1767–1773, 2011. [Online]. Available: www.m-hikari.com/ijma/ijma-2011/ijma-33-36-2011/index.html

[18] R. N. Singh, S. K. Pandey, and G. Pandey, "On a type of Kenmotsu manifold," *Bull. Math. Anal. Appl.*, vol. 4, no. 1, pp. 117–132, 2012. [Online]. Available: www.bmathaa.org/repository/docs/BMAA4_1_11.pdf

[19] R. N. Singh and S. K. Pandey, "On the M-projective curvature tensor of $N(\kappa)$-contact metric manifolds." *ISRN Geom.*, vol. 2013, p. 6, 2013, id/No 932564, doi: 10.1155/2013/932564.

[20] Z. I. Szabó, "Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R = 0$. I: The local version." *J. Differ. Geom.*, vol. 17, pp. 531–582, 1982, doi: 10.4310/jdg/1214437486.

[21] T. Takahashi, "Sasakian manifold with pseudo-Riemannian metric." *Tôhoku Math. J. (2)*, vol. 21, pp. 271–290, 1969, doi: 10.2748/tmj/1178242996.

[22] S. Zamkovoy, "Canonical connections on paracontact manifolds." *Ann. Global Anal. Geom.*, vol. 36, no. 1, pp. 37–60, 2009, doi: 10.1007/s10455-008-9147-3.

Authors' addresses

Swati Jain

(**Corresponding author**) University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya, Department of Mathematics, 462033, Bhopal, Madhya Pradesh, India

E-mail address: swatijain2884@gmail.com

Teerathram Raghuwanshi

University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya, Department of Mathematics, 462033, Bhopal, Madhya Pradesh, India

E-mail address: teerathramsgs@gmail.com

Manoj Kumar Pandey

University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya, Department of Mathematics, 462033, Bhopal, Madhya Pradesh, India

E-mail address: mkp_apsu@rediffmail.com

Anil Goyal

University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya, Department of Mathematics, 462033, Bhopal, Madhya Pradesh, India

E-mail address: anil_goyal03@rediffmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 1025–1041

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4439

SOLVING CAPUTO FRACTIONAL ORDER DIFFERENTIAL EQUATIONS VIA FIXED POINT RESULTS OF α-GERAGHTY CONTRACTION TYPE MAPPINGS

ABDELLAH TAQBIBT, M'HAMED EL OMARI, AND SAID MELLIANI

Received 05 October, 2022

Abstract. In this paper, by the fixed point results of α–Geraghty contraction type mappings, we prove the existence and uniqueness results of solutions for some Caputo fractional order differential equations in metric spaces. In addition, we conclude the applicability of these results with an illustrative example.

2010 *Mathematics Subject Classification:* 35G05;35G55;46N20

Keywords: fractional differential equation, α-admissible, Caputo fractional derivative, α-Geraghty contraction type mappings, fixed point, metric spaces

1. Introduction

The theory of fractional order differential equations has emerged as an interesting area to explore in recent years. Note that this theory has a lot of applications in the description of many events in the real world. For example, fractional order differential equations are often applicable in engineering, chemistry, physics, biology, etc. For examples and details see [2, 3, 11–13, 15]. Currently in the mathematical literature, studies on the existence, uniqueness, multiplicity of solutions of nonlinear fractional problems use nonlinear analysis techniques such as fixed point theorems. Fixed point theorems are the basic mathematical tools, to show solutions in different types of equations. [4, 6]. The theory of fixed points is at the heart of nonlinear analysis since it provides the tools for having existence theorems in a lot of different nonlinear problems, tools to have existence theorems in many different nonlinear problems. The fixed point theorems are often based on certain properties (such as complete continuity, monotony, contraction...) that the application considered must satisfy [1, 9, 16, 19, 20]. In this work, we are studying the existence, uniqueness of the solution for some of the nonlinear fractional differential equations with integral boundary conditions, in particular, we have presented the existence and uniqueness

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

for some classes of differential equations of fractional order in the sense of Caputo. For this we used fixed point results of α-Geraghty contraction type mappings.

2. Preliminaries

We start with the definitions of triangular α-admissible and α-admissible mappings, which was introduced by Karapinar, Samet et al [17, 18, 20].

Definition 1 ([20, Definition 2.2])**.** Let $\mathcal{S}\colon \Xi \longrightarrow \Xi$ be a mapping and a function $\alpha\colon \Xi^2 \longrightarrow \mathbb{R}^+$. $\mathcal{S}$ is called a α-admissible if

$$\alpha(\vartheta,\eta) \geq 1 \Longrightarrow \alpha(\mathcal{S}\vartheta,\mathcal{S}\eta) \geq 1 \qquad \text{for all } \vartheta,\eta \in \Xi.$$

Example 1 ([20, Example 2.2]). Let $\Xi = \mathbb{R}^*_+$.
Define $\alpha\colon \Xi^2 \longrightarrow \mathbb{R}^+$ and $\mathcal{S}\colon \Xi \longrightarrow \Xi$, as follows $\mathcal{S}\vartheta = \ln(\vartheta)$ for all $\vartheta \in \Xi$, and

$$\alpha(\vartheta,\eta) = \begin{cases} 0 & \text{if } \vartheta < \eta, \\ 2 & \text{otherwise.} \end{cases}$$

Then, $\mathcal{S}$ is α-admissible.

Definition 2 ([17, Definition 1])**.** A $\mathcal{S}$ map is called a triangular α-admissible if it is α-admissible and α satisfies the following condition:

$$\begin{cases} \alpha(\vartheta,\sigma) \geq 1 \\ \alpha(\sigma,\eta) \geq 1 \end{cases} \quad \text{implies } \alpha(\vartheta,\eta) \geq 1, \text{ for all } \vartheta,\eta,\sigma \in \Xi.$$

Definition 3 ([14, Definition, p. 606])**.** Let $\psi\colon \mathbb{R}^+ \to]0,1[$ be a function. ψ is said to be strong Geraghty function if $\{y_n\} \subset [0,\infty)$ and $\lim\limits_{n\to\infty} \psi(y_n) = 1$ implies $\lim\limits_{n\to\infty} y_n = 0$.

The set of all Geraghty functions is denoted by $\mathcal{G}$.

Definition 4 ([5, Definition 1.8])**.** Let (Ξ,d) be a metric space and a function $\alpha\colon \Xi^2 \longrightarrow [0,\infty)$. Ξ is called a α-regular

$$\text{if for every sequence } (\mu_m)_{m\in\mathbb{N}} \subset \Xi, \text{ such that } \begin{cases} \alpha(\mu_m,\mu_{m+1}) \geq 1, \\ \lim\limits_{m\to\infty} \mu_m = \mu, \end{cases} \quad \text{for all } m \in \mathbb{N}$$

there exist a subsequence $(\mu_{m_l})_{l\in\mathbb{N}}$ of $(\mu_m)_{m\in\mathbb{N}}$, where $\alpha(\mu_{m_l},\mu) \geq 1$, $\forall l \in \mathbb{N}$.

Cho et al. [10] proved the fixed point theorem via α-Geraghty contraction.

Definition 5 ([10, Definition 4])**.** Let (Ξ,d) be a metric space and a function $\alpha\colon \Xi^2 \longrightarrow [0,\infty)$. A map $\mathcal{S}\colon \Xi \longrightarrow \Xi$ is said to be α-Geraghty contraction if there exist $\psi \in \mathcal{G}$, such that

$$\alpha(\vartheta,\eta)d(\mathcal{S}\vartheta,\mathcal{S}\eta) \leq \psi(d(\vartheta,\eta))d(\vartheta,\eta),$$

for all $\vartheta,\eta \in \Xi$.

The Following result plays a key role in our main results.

Theorem 1 ([10, Theorem 4]). *Let (Ξ, d) be a complete metric space and $\alpha\colon \Xi^2 \to \mathbb{R}^+$ be a function. Define a map $\mathcal{S}\colon \Xi \longrightarrow \Xi$. Assume that*

1) *$\mathcal{S}$ is a triangular α-admissible;*
2) *$\mathcal{S}$ is a α-Geraghty contraction;*
3) *either Ξ is α-regular or $\mathcal{S}$ is continous;*
4) *there exists $\theta_0 \in \Xi$ with $\alpha(\theta_0, \mathcal{S}\theta_0) \geq 1$.*

Then $\mathcal{S}$ has a fixed point. Moreover, if

5) *for all fixed point ϑ, η of $\mathcal{S}$, either $\alpha(\vartheta, \eta) \geq 1$ or $\alpha(\eta, \vartheta) \geq 1$,*

then $\mathcal{S}$ has a uniques fixed point.

Definition 6. For x a strictly positive real number, the gamma function is defined by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\, dt.$$

Now, we gives the definitions of the Caputo fractional order integral and derivative.

Definition 7. The Caputo fractional order derivative is defined by

$$\mathfrak{D}^{\beta} g(s) = \int_0^s \frac{1}{\Gamma(p-\beta)} \frac{g^{(p)}(\sigma)}{(s-\sigma)^{\beta+1-p}}\, d\sigma, \quad p-1 < \beta < p, \quad p \in \mathbb{N}^*,$$

and the fractional order integral is given by

$$\mathrm{J}^{\beta} g(s) = \int_0^s \frac{(s-\sigma)^{\beta-1} g(\sigma)}{\Gamma(\beta)}\, d\sigma, \quad \beta > 0.$$

3. Main Results

Let $(\mathrm{C}(I), \mathrm{d})$ be a metric space and $\mathrm{d}\colon \mathrm{C}(I)^2 \to \mathbb{R}_+$ is defined by

$$\mathrm{d}(y,z) = \sup_{\xi \in I} |y(\xi) - z(\xi)|.$$

Then, $(\mathrm{C}(I), \mathrm{d})$ is a complete metric space.

Now, we consider the following problem

$$\begin{cases} (\mathfrak{D}^{\beta}\, \mathsf{u})(\xi) = \mathsf{f}\big(\xi, \mathsf{u}(\xi)\big), \quad \xi \in I = [0,T], \quad T > 0, \\ a\mathsf{u}(0) + b\mathsf{u}(T) = c, \end{cases} \tag{3.1}$$

where $\mathfrak{D}^{\beta}$ is a Caputo fracional derivative of order $0 < \beta < 1$, $\mathsf{f} \in \mathrm{C}(I \times \mathbb{R})$ and $a, b, c \in \mathbb{R}$ such that $a + b \neq 0$.

Lemma 1 ([7, Lemma 3.2])**.** *The problem* (3.1) *is equivalent with following equation*

$$\mathsf{u}(\xi) = M_c + \frac{1}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}\mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma - \frac{N_b}{\Gamma(\beta)}\int_0^{T}(T-\sigma)^{\beta-1}\mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma, \quad (3.2)$$

where $M_c = \frac{c}{a+b}$ *and* $N_b = \frac{b}{a+b}$.

Denote by Φ the class of all continuous and nondecreasing functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying

$$\varphi(b\xi) \le b\varphi(\xi) \le b\xi \quad \text{for } b > 1.$$

Theorem 2. *Let* $\beta \in (0,1)$. *Suppose that*

(I) There exists a function $\delta\colon \mathrm{C}(I)\times\mathrm{C}(I) \to \mathbb{R}$, $r \in (0,\beta)$ *and* $m \in L^{1/r}$ *such that*

$$|\mathsf{f}(\xi,x) - \mathsf{f}(\xi,y)| \le m(\xi)\frac{|x-y|}{1+\|x-y\|_{L^\infty}},$$

where

$$\left|\Big(\int_0^T (\xi-\sigma)^{\frac{\beta-1}{1-r}}\,d\sigma\Big)^{1-r}\frac{M(1+N_b)}{\Gamma(\beta)}\right| \le \varphi(\|x-y\|_{L^\infty}),$$

$M = \|m\|_{L^{1/r}}$, $\varphi \in \Phi$, *and for* $x,y \in \mathrm{C}(I)$ *with* $\delta(x,y) \ge 0$.

(II) There exist $\gamma_0 \in \mathrm{C}(I)$ *such that,* $\delta\big(\gamma_0(\xi), \aleph_{\gamma_0}(\xi)\big)$ *where*

$$\aleph_{\gamma_0}(\xi) = M_c + \frac{1}{\Gamma(\beta)}\left[\int_0^{\xi}(\xi-s)^{\beta-1}\mathsf{f}(s,\gamma_0(s))\,ds - N_b\int_0^T (T-\zeta)^{\beta-1}\mathsf{f}(\zeta,\gamma_0(\zeta))\,d\zeta\right].$$

(III) For each $x,y \in \mathrm{C}(I)$, *we get*

$$\delta\big(x(\xi),y(\xi)\big) \ge 0 \Rightarrow \delta\big(A_x,A_y\big) \ge 0,$$

where

$$A_x = M_c + \frac{1}{\Gamma(\beta)}\left[\int_0^{\xi}(\xi-s)^{\beta-1}\mathsf{f}(s,x(s))ds - N_b\int_0^T (T-\zeta)^{\beta-1}\mathsf{f}(\zeta,x(\zeta))\,d\zeta\right],$$

$$A_y = M_c + \frac{1}{\Gamma(\beta)}\left[\int_0^{\xi}(\xi-s)^{\beta-1}\mathsf{f}(s,y(s))ds - N_b\int_0^T (T-\zeta)^{\beta-1}\mathsf{f}(\zeta,y(\zeta))\,d\zeta\right].$$

(V) For each $(z_n)_{n\in\mathbb{N}} \subset \mathrm{C}(I)$ *such that* $\delta(z_n,z_{n+1}) \ge 0$ *and* $\lim_{n\to\infty} z_n = z$, *then* $\delta(z_n,z) \ge 0$.

Then, the problem (3.1) *has at least one Solution.*

Proof. According to Lemma 1, then $u \in \mathrm{C}^1(I)$ is a soulution of problem (3.1) if only if it is a solution of equation (3.2).

Now, we define $\mathcal{T}\colon \mathrm{C}(I) \to \mathrm{C}(I)$ by

$$\mathcal{T}\mathsf{u}(\xi) = M_c + \frac{1}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}\mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma - \frac{N_b}{\Gamma(\beta)}\int_0^T (T-\sigma)^{\beta-1}\mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma,$$

where $M_c = \frac{c}{a+b}$ and $N_b = \frac{b}{a+b}$.

For this purpose, the problem reduces to finding a fixed point of $\mathcal{T}$.

Lets $u, v \in \mathrm{C}(I)$ be such that $\delta(x,y) \geq 0$, we have

$$\begin{aligned}\mathcal{T}\mathsf{u}(\xi) - \mathcal{T}\mathsf{v}(\xi) =& \frac{1}{\Gamma(\beta)} \int_0^{\xi} (\xi-\sigma)^{\beta-1} \left[\mathsf{f}(\sigma,\mathsf{u}(\sigma)) - \mathsf{f}(\sigma,\mathsf{v}(\sigma))\right] d\sigma \\ &- \frac{N_b}{\Gamma(\beta)} \int_0^{T} (T-\sigma)^{\beta-1} \left[\mathsf{f}(\sigma,\mathsf{u}(\sigma)) - \mathsf{f}(\sigma,\mathsf{v}(\sigma))\right] d\sigma.\end{aligned}$$

And thus,

$$\begin{aligned}|\mathcal{T}\mathsf{u}(\xi) - \mathcal{T}\mathsf{v}(\xi)| \leq& \frac{1}{\Gamma(\beta)} \int_0^{\xi} (\xi-\sigma)^{\beta-1} |\mathsf{f}(\sigma,\mathsf{u}(\sigma)) - \mathsf{f}(\sigma,\mathsf{v}(\sigma))| \, d\sigma \\ &+ \frac{N_b}{\Gamma(\beta)} \int_0^{T} (T-\sigma)^{\beta-1} |\mathsf{f}(\sigma,\mathsf{u}(\sigma)) - \mathsf{f}(\sigma,\mathsf{v}(\sigma))| \, d\sigma.\end{aligned}$$

We know that

$$\max\left\{\int_0^{\xi} (\xi-\sigma)^{\beta-1} \, d\sigma, \int_0^{T} (T-\sigma)^{\beta-1} \, d\sigma\right\} \leq \int_0^{T} (\xi-\sigma)^{\beta-1} \, d\sigma.$$

Then,

$$|\mathcal{T}\mathsf{u}(\xi) - \mathcal{T}\mathsf{v}(\xi)| \leq \frac{1+N_b}{\Gamma(\beta)} \int_0^{T} (\xi-\sigma)^{\beta-1} |\mathsf{f}(\sigma,\mathsf{u}(\sigma)) - \mathsf{f}(\sigma,\mathsf{v}(\sigma))| \, d\sigma.$$

By using (I), we obtain

$$\begin{aligned}|\mathcal{T}\mathsf{u}(\xi) - \mathcal{T}\mathsf{v}(\xi)| \leq& \frac{1+N_b}{\Gamma(\beta)} \\ &\times \int_0^{T} (\xi-\sigma)^{\beta-1} m(\sigma) \frac{|\mathsf{u}(\sigma) - \mathsf{u}(\sigma)|}{1+\|\mathsf{u}-\mathsf{v}\|_{L^\infty}} \, d\sigma \\ \leq& \frac{\|\mathsf{u}-\mathsf{v}\|_{L^\infty}}{1+\|\mathsf{u}-\mathsf{v}\|_{L^\infty}} \\ &\times \quad \frac{1+N_b}{\Gamma(\beta)} \int_0^{T} (\xi-\sigma)^{\beta-1} \frac{m(\sigma)}{2} \, d\sigma \\ \leq& \frac{\|\mathsf{u}-\mathsf{v}\|_{L^\infty}}{\|\mathsf{u}-\mathsf{v}\|_{L^\infty}+1} \\ &\times \left[\frac{1+N_b}{\Gamma(\beta)} \int_0^{T} (\xi-\sigma)^{\beta-1} m(\sigma) \, d\sigma\right].\end{aligned}$$

On the other hand, it is clear that

$$g(s) = (\xi - s)^{-1+\beta} \in L^{\frac{1}{-q+1}}[0,T], \qquad \text{for } q \in [0,\beta).$$

Let

$$b = \frac{-1+\beta}{1-q}, \text{ then } b \in]-1,0[.$$

According to Hölder inequality, we can obtain

$$\begin{aligned}\left|\int_0^{\xi}(\xi-\sigma)^{\beta-1}m(\sigma)\,d\sigma\right| \leq & \left(\int_0^{\xi}(\xi-\sigma)^{\frac{\beta-1}{1-q}}\,d\sigma\right)^{1-q}\left(\int_0^{\xi}|m^{1/q}(\sigma)|\,d\sigma\right)^{q}\\ \leq & M\left(\int_0^{\xi}(\xi-\sigma)^{\frac{\beta-1}{1-q}}\,d\sigma\right)^{1-q}.\end{aligned}$$

So,

$$\begin{aligned}|\mathcal{T}\mathsf{u}(\xi)-\mathcal{T}\mathsf{v}(\xi)| \leq & \frac{\|\mathsf{u}-\mathsf{v}\|_{L^\infty}}{\|\mathsf{u}-\mathsf{v}\|_{L^\infty+1}}\\ & \times\frac{1}{2}\left[\frac{M(1+N_b)}{\Gamma(\beta)}\left(\int_0^{\xi}(\xi-\sigma)^{\frac{\alpha-1}{1-q}}\,d\sigma\right)^{1-q}\right].\end{aligned}$$

From (I), we get

$$|\mathcal{T}\mathsf{u}(\xi)-\mathcal{T}\mathsf{v}(\xi)| \leq \varphi(\|\mathsf{u}-\mathsf{v}\|_{L^\infty})\frac{\|\mathsf{u}-\mathsf{v}\|_{L^\infty}}{\|\mathsf{u}-\mathsf{v}\|_{L^\infty+1}}.$$

So,

$$\mathrm{d}(\mathcal{T}\mathsf{u},\mathcal{T}\mathsf{v}) \leq \frac{\mathrm{d}(\mathsf{u},\mathsf{v})}{\mathrm{d}(\mathsf{u},\mathsf{v})+1}\varphi(\mathrm{d}(\mathsf{u},\mathsf{v})).$$

Let a function $\alpha\colon \mathrm{C}(I)^2 \to \mathbb{R}^+$ defined by

$$\alpha(\mathsf{u},\mathsf{v}) = \begin{cases} 1 & \text{if } \delta(\mathsf{u}(\xi),\mathsf{v}(\xi)) \geq 0 \quad \xi \in I,\\ 0 & \text{otherwise.}\end{cases}$$

Then,

$$\begin{aligned}\mathrm{d}(\mathcal{T}\mathsf{u},\mathcal{T}\mathsf{v})\alpha(\mathsf{u},\mathsf{v}) \leq & \frac{\mathrm{d}(\mathsf{u},\mathsf{v})}{\mathrm{d}(\mathsf{u},\mathsf{v})+1}\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))\alpha(\mathsf{u},\mathsf{v})\\ \leq & \frac{\mathrm{d}(\mathsf{u},\mathsf{v})}{\mathrm{d}(\mathsf{u},\mathsf{v})+1}\varphi(\mathrm{d}(\mathsf{u},\mathsf{v})).\end{aligned}$$

Since $\varphi \in \Phi$, we have

$$\begin{aligned}\alpha(\mathsf{u},\mathsf{v})\mathrm{d}(\mathcal{T}\mathsf{u},\mathcal{T}\mathsf{v}) \leq & \frac{\mathrm{d}(\mathsf{u},\mathsf{v})}{\mathrm{d}(\mathsf{u},\mathsf{v})+1}\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))\\ \leq & \frac{\mathrm{d}(\mathsf{u},\mathsf{v})}{\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))+1}\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))\\ \leq & \frac{\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))}{\varphi(\mathrm{d}(\mathsf{u},\mathsf{v}))+1}\mathrm{d}(\mathsf{u},\mathsf{v}).\end{aligned}$$

Hence,

$$\alpha(u,v)d(\mathcal{T}u,\mathcal{T}v) \leq d(u,v)\psi\big(d(u,v)\big),$$

where $\psi(\xi) = \frac{\varphi(\xi)}{\varphi(\xi)+1} \in \mathcal{G}, \quad \xi > 0$.

Then, $\mathcal{T}$ is a α-Geraghty contraction.

On the other hand according to (III) and for each $u,v \in C(I)$, we have

$$\begin{aligned} \alpha(u,v) \geq 1 \text{ implies } \delta(u,v) &\geq 0 \\ \Longrightarrow \delta(\mathcal{T}u,\mathcal{T}v) &\geq 0 \\ \Longrightarrow \alpha(\mathcal{T}u,\mathcal{T}v) &\geq 1. \end{aligned}$$

So, $\mathcal{T}$ is α-admissible. Mooreover, $\mathcal{T}$ is a triangular α-admissible.

By (II), we can obtain

$$\delta\left(\gamma_0(\xi), M_c + \frac{1}{\Gamma(\beta)}\left[\int_0^{\xi}(\xi-s)^{\beta-1}f(s,\gamma_0(s))\,ds - N_b\int_0^{T}(T-\zeta)^{\beta-1}f(\zeta,\gamma_0(\zeta))\,d\zeta\right]\right) \geq 0.$$

That implies,

$$\delta(\gamma_0,\mathcal{T}\gamma_0) \geq 0.$$

And thus,

$$\alpha(\gamma_0,\mathcal{T}\gamma_0) \geq 1.$$

So,

$$\alpha(\gamma_0,\mathcal{T}\gamma_0) \geq 1.$$

From (IV), we have $(z_n)_{n\in\mathbb{N}} \subset C(I)$ with $\alpha(z_n,z_{n+1}) \geq 1$, which gives

$$\delta(z_n,z_{n+1}) \geq 0.$$

Then, $\delta(z_n,z) \geq 0$. And thus,

$$\delta(z_n,z) \geq 0.$$

That implies,

$$\alpha(z_n,b) \geq 1.$$

So, using Theorem 1, then $\mathcal{T}$ has a fixed point in $C(I)$ is a solution of problem (3.1). □

We denote by $\mathrm{Fix}(\mathcal{T})$ the set of fixed points of $\mathcal{T}$.

(CD) We have $\delta(u,v) \geq 0$, for each $u,v \in \mathrm{Fix}(\mathcal{T})$.

Theorem 3. *Adding (CD) to the hypotheses of Theorem 2, we obtain the uniqueness of the fixed point of $\mathcal{T}$.*

Proof. By the contion (CD), if u and v are two solutions of problem (3.1), implies that $u,v \in \mathrm{Fix}(\mathcal{T})$ and thus

$$\delta(u,v) \geq 0.$$

So,

$$\alpha(\mathsf{u},\mathsf{v}) \geq 1.$$

According Theorem 1 and Theorem 2, then the mapping $\mathcal{T}$ has a unique fixed point. □

Now, we are discuss the existence, uniqueness of solutions for the following caputo fractional problem

$$\begin{cases} (\mathfrak{D}^{\beta}\mathsf{u})(\xi) = \mathsf{f}\big(\xi,\mathsf{u}(\xi)\big), & \xi \in I = [0,1], \\ a\mathsf{u}'(t_1) + b\mathsf{u}'(t_2) = \theta \displaystyle\int_0^{\eta} \mathsf{u}(s)\,ds, & \mathsf{u}(0) = 0, \end{cases} \tag{3.3}$$

where $\mathfrak{D}^{\beta}$ is a Caputo fracional order derivative of $0 < \beta < 1$, $\mathsf{f} \in \mathrm{C}(I \times \mathbb{R})$ and $a,b,\theta \in \mathbb{R}$ with $\theta\frac{\eta^2}{2} - at_1 - bt_2 \neq 0$. The boundary conditions in (3.3) implies that the linear combination of the values of the derivative of the unknown function at nonlocal positions t_1 and t_2 are proportional to the continuous distribution of the unknown function over a strip of an arbitrary length η.

Lemma 2 ([8, Lemma 2.2])**.** *For all $\xi \in I$. The problem (3.3) is equivalent with following equation*

$$\begin{aligned} \mathsf{u}(\xi) = &\int_0^{\xi} \frac{(\xi-\sigma)^{\beta-1}}{\Gamma(\beta)} \mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma \\ &+ \frac{\xi}{A_\theta}\Bigg(\int_0^{t_1} \frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma + \int_0^{t_2} \frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma \\ &\qquad - \theta \int_0^{\eta} \left[\frac{1}{\Gamma(\beta)} \int_0^{\sigma} (\sigma-\tau)^{\beta-1} \mathsf{f}(\tau,\mathsf{u}(\tau))d\tau \right] d\sigma \Bigg), \end{aligned}$$

where $A_\theta = \theta\frac{\eta^2}{2} - at_1 - bt_2 \neq 0$.

We make the following hypotheses will be used in the sequel.

(H1) There exists a function $\rho\colon \mathrm{C}(I)^2 \to \mathbb{R}$, $r \in (0,\beta)$ and $\lambda \in L^{1/r}([0,T])$, $T > 0$ with

$$|\mathsf{f}(\xi,x) - \mathsf{f}(\xi,y)| \leq \lambda(\xi)|x-y|,$$

where

$$\left| \lambda_r(\xi) \left[\frac{B_\xi}{\Gamma(\beta)} + \frac{1}{|A_\theta|} \Big(\frac{B_{t_1} + B_{t_2}}{\Gamma(\beta-1)} + \theta \int_0^{\eta} \frac{B_\sigma}{\Gamma(\beta)}\,d\sigma \Big) \right] \right| \leq \psi(\|x-y\|_{L^\infty}),$$

$\psi \in \mathcal{G}$, $\lambda_r(\xi) = \|\lambda\|_{L^{1/r}([0,\xi])}$, and for $x,y \in \mathrm{C}(I)$ with $\rho(x,y) \geq 0$, such that

$$B_\xi = \int_0^{\xi} (\xi-\sigma)^{\beta-1}\,d\sigma, \quad B_{t_1} = \int_0^{t_1} (t_1-\sigma)^{\beta-2}\,d\sigma,$$

$$B_{t_2} = \int_0^{t_2} (t_2 - \sigma)^{\beta-2}\, d\sigma, \quad B_\sigma = \int_0^{\sigma} (\sigma - \tau)^{\beta-1}\, d\tau.$$

(H2) There exist $\gamma_0 \in \mathrm{C}(I)$ such that, $\rho\big(\gamma_0(\xi), \aleph_{\gamma_0}(\xi)\big) \geq 0$ where

$$\begin{aligned}\aleph_{\gamma_0}(\xi) = &\int_0^{\xi} \frac{(\xi-\sigma)^{\beta-1}}{\Gamma(\beta)} \mathsf{f}(\sigma, \gamma_0(\sigma))\, d\sigma \\ &+ \frac{\xi}{A_\theta}\Bigg(\int_0^{t_1} \frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, \gamma_0(\sigma))\, d\sigma \\ &\qquad + \int_0^{t_2} \frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, \gamma_0(\sigma))\, d\sigma \\ &\qquad - \theta \int_0^{\eta} \left[\frac{1}{\Gamma(\beta)} \int_0^{\sigma} (\sigma-\tau)^{\beta-1} \mathsf{f}(\tau, \gamma_0(\tau)) d\tau \right] d\sigma \Bigg).\end{aligned}$$

(H3) For each $x, y \in \mathrm{C}(I)$, we get

$$\rho\big(x(\xi), y(\xi)\big) \geq 0 \Rightarrow \delta\big(A_x, A_y\big) \geq 0,$$

where

$$\begin{aligned}A_x = &\int_0^{\xi} \frac{(\xi-\sigma)^{\beta-1}}{\Gamma(\beta)} \mathsf{f}(\sigma, x(\sigma))\, d\sigma \\ &+ \frac{\xi}{A_\theta}\Bigg(\int_0^{t_1} \frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, x(\sigma))\, d\sigma \\ &\qquad + \int_0^{t_2} \frac{((t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, x(\sigma))\, d\sigma \\ &\qquad - \theta \int_0^{\eta} \left[\frac{1}{\Gamma(\beta)} \int_0^{\sigma} (\sigma-\tau)^{\beta-1} \mathsf{f}(\tau, x(\tau)) d\tau \right] d\sigma \Bigg), \\ A_y = &\int_0^{\xi} \frac{(\xi-\sigma)^{\beta-1}}{\Gamma(\beta)} \mathsf{f}(\sigma, y(\sigma))\, d\sigma \\ &+ \frac{\xi}{A_\theta}\Bigg(\int_0^{t_1} \frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, y(\sigma))\, d\sigma \\ &\qquad + \int_0^{t_2} \frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)} \mathsf{f}(\sigma, y(\sigma))\, d\sigma \\ &\qquad - \theta \int_0^{\eta} \left[\frac{1}{\Gamma(\beta)} \int_0^{\sigma} (\sigma-\tau)^{\beta-1} \mathsf{f}(\tau, y(\tau)) d\tau \right] d\sigma \Bigg).\end{aligned}$$

(H4) For each $(y_n)_{n\in\mathbb{N}} \subset \mathrm{C}(I)$ such that $\rho(y_n, y_{n+1}) \geq 0$ and $\lim_{n\to\infty} y_n = y$, then $\delta(y_n, y) \geq 0$.

Theorem 4. *Assume that (H1)-(H4) hold. then the problem (3.3) has at least one Solution. Morover, if $\delta(x,y) \geq 0$, for each $u,v \in S_{(3.3)}$, where $S_{(3.3)}$ denotes the set of solutions of problem (3.3). Then the problem (3.3) has a unique solution.*

Proof. From Lemma 2, we define an map $\mathcal{S}\colon \mathrm{C}(I) \to \mathrm{C}(I)$ by

$$\begin{aligned}\mathcal{S}\mathsf{u}(\xi) =& \frac{1}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}\mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma \\ &+\frac{\xi}{A_\theta}\Bigg(\frac{1}{\Gamma(\beta-1)} \\ &\qquad\times\left[\int_0^{t_1}(t_1-\sigma)^{\beta-2}\mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma+\int_0^{t_2}(t_2-\sigma)^{\beta-2}\mathsf{f}(\sigma,\mathsf{u}(\sigma))\,d\sigma\right] \\ &\qquad-\theta\int_0^{\eta}\left[\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}\mathsf{f}(\tau,\mathsf{u}(\tau))d\tau\right]d\sigma\Bigg).\end{aligned}$$

We find a fixed point of $\mathcal{S}$. Let $v,w \in \mathrm{C}(I)$, we have

$$\begin{aligned}&\mathcal{S}\mathsf{v}(\xi)-\mathcal{S}\mathsf{w}(\xi) \\ &\quad=\frac{1}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}\mathsf{f}(\sigma,\mathsf{v}(\sigma))\,d\sigma \\ &\qquad+\frac{\xi}{A_\theta}\Bigg(\frac{1}{\Gamma(\beta-1)} \\ &\qquad\qquad\times\left[\int_0^{t_1}(t_1-\sigma)^{\beta-2}\mathsf{f}(\sigma,\mathsf{v}(\sigma))\,d\sigma+\int_0^{t_2}(t_2-\sigma)^{\beta-2}\mathsf{f}(\sigma,\mathsf{v}(\sigma))\,d\sigma\right] \\ &\qquad\qquad-\theta\int_0^{\eta}\left[\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}\mathsf{f}(\tau,\mathsf{v}(\tau))d\tau\right]d\sigma\Bigg) \\ &\qquad-\frac{1}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}\mathsf{f}(\sigma,\mathsf{w}(\sigma))\,d\sigma \\ &\qquad-\frac{\xi}{A_\theta}\Bigg(\frac{1}{\Gamma(\beta-1)} \\ &\qquad\qquad\times\left[\int_0^{t_1}(t_1-\sigma)^{\beta-2}\mathsf{f}(\sigma,\mathsf{w}(\sigma))\,d\sigma+\int_0^{t_2}(t_2-\sigma)^{\beta-2}\mathsf{f}(\sigma,\mathsf{w}(\sigma))\,d\sigma\right] \\ &\qquad\qquad-\theta\int_0^{\eta}\left[\int_0^{\sigma}\frac{(\sigma-\tau)^{\beta-1}}{\Gamma(\beta)}\,\mathsf{f}(\tau,\mathsf{w}(\tau))d\tau\right]d\sigma\Bigg) \\ &\quad=\frac{1}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}\Big[-\mathsf{f}(\sigma,\mathsf{w}(\sigma))+\mathsf{f}(\sigma,\mathsf{v}(\sigma))\Big]\,d\sigma\end{aligned}$$

$$+\frac{\xi}{A_\theta}\Bigg(\int_0^{t_1}\frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)}\big(-\mathsf{f}(\sigma,\mathsf{w}(\sigma))+\mathsf{f}(\sigma,\mathsf{v}(\sigma))\big)\,d\sigma$$
$$+\int_0^{t_2}\frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)}\big(-\mathsf{f}(\sigma,\mathsf{w}(\sigma))+\mathsf{f}(\sigma,\mathsf{v}(\sigma))\big)\,d\sigma$$
$$-\theta\int_0^{\eta}\left[\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}\big(\mathsf{f}(\tau,\mathsf{v}(\tau))-\mathsf{f}(\tau,\mathsf{w}(\tau))\big)\,d\tau\right]d\sigma\Bigg).$$

Then,

$$|\mathcal{S}\mathsf{v}(\xi)-\mathcal{S}\mathsf{w}(\xi)|\leq\frac{1}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}|\mathsf{f}(\sigma,\mathsf{v}(\sigma))-\mathsf{f}(\sigma,\mathsf{w}(\sigma))|\,d\sigma$$
$$+\frac{\xi}{|A_\theta|}\Bigg(\int_0^{t_1}\frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|-\mathsf{f}(\sigma,\mathsf{w}(\sigma))+\mathsf{f}(\sigma,\mathsf{v}(\sigma))|\,d\sigma$$
$$+\int_0^{t_2}\frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|-\mathsf{f}(\sigma,\mathsf{w}(\sigma))+\mathsf{f}(\sigma,\mathsf{v}(\sigma))|\,d\sigma$$
$$+\theta\int_0^{\eta}\left[\int_0^{\sigma}\frac{(\sigma-\tau)^{\beta-1}}{\Gamma(\beta)}|\mathsf{f}(\tau,\mathsf{v}(\tau))-\mathsf{f}(\tau,\mathsf{w}(\tau))|\,d\tau\right]d\sigma\Bigg).$$

According to (I), we get

$$|\mathcal{S}\mathsf{v}(\xi)-\mathcal{S}\mathsf{w}(\xi)|\leq\int_0^{\xi}\frac{(\xi-\sigma)^{\beta-1}}{\Gamma(\beta)}|\lambda(\sigma)||\mathsf{v}(\sigma)-\mathsf{w}(\sigma)|\,d\sigma$$
$$+\frac{\xi}{|A_\theta|}\Bigg(\int_0^{t_1}\frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|\lambda(\sigma)||\mathsf{v}(\sigma)-\mathsf{w}(\sigma)|\,d\sigma$$
$$+\int_0^{t_2}\frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|\lambda(\sigma)||\mathsf{v}(\sigma)-\mathsf{w}(\sigma)|\,d\sigma$$
$$+\theta\int_0^{\eta}\left[\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}|\lambda(\tau)||\mathsf{v}(\tau)-\mathsf{w}(\tau)|\,d\tau\right]d\sigma\Bigg).$$

So,

$$|\mathcal{S}\mathsf{v}(\xi)-\mathcal{S}\mathsf{w}(\xi)|\leq\frac{\|\mathsf{v}-\mathsf{w}\|_\infty}{\Gamma(\beta)}\int_0^{\xi}(\xi-\sigma)^{\beta-1}|\lambda(\sigma)|\,d\sigma$$
$$+\frac{\|\mathsf{v}-\mathsf{w}\|_\infty}{|A_\theta|}\Bigg(\int_0^{t_1}\frac{(t_1-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|\lambda(\sigma)|\,d\sigma$$
$$+\int_0^{t_2}\frac{(t_2-\sigma)^{\beta-2}}{\Gamma(\beta-1)}|\lambda(\sigma)|\,d\sigma$$

$$+\theta\int_0^{\eta}\left[\frac{1}{\Gamma(\beta)}\int_0^{\sigma}(\sigma-\tau)^{\beta-1}|\lambda(\tau)|\,d\tau\right]d\sigma\Bigg).$$

Let

$$b=\frac{\beta-1}{1-q},\ \text{it is clear that } b\in]-1,0[,$$

for $q\in[0,\beta)$. On the other hand, we have

$$h(\sigma)=(\xi-\sigma)^{\beta-2}\in L^{\frac{1}{1-q}}[0,\xi].$$

From Hölder inequality, we get

$$\begin{aligned}|\mathcal{S}\mathsf{v}(\xi)-\mathcal{S}\mathsf{w}(\xi)|\leq&\|\mathsf{v}-\mathsf{w}\|_\infty\frac{\lambda_r}{\Gamma(\beta)}\Big(\int_0^{\xi}(\xi-\sigma)^{\frac{-1+\beta}{-q+1}}\,d\sigma\Big)^{-q+1}\\&+\frac{\|\mathsf{v}-\mathsf{w}\|_\infty}{|A_\theta|}\Bigg(\frac{\lambda_r}{\Gamma(-1+\beta)}\Bigg[\Big(\int_0^{t_1}(t_1-\sigma)^{\frac{-2+\beta}{1q+1}}\,d\sigma\Big)^{-q+1}\\&\qquad+\Big(\int_0^{t_2}(t_2-\sigma)^{\frac{-2+\beta}{-q+1}}\,d\sigma\Big)^{-q+1}\Bigg]\\&\qquad+\theta\int_0^{\eta}\Big[\frac{\lambda_r}{\Gamma(\beta)}\Big(\int_0^{\sigma}(\sigma-\tau)^{\frac{-1+\beta}{-q+1}}\,d\tau\Big)^{-q+1}d\sigma\Bigg)\\\leq&\lambda_r\|\mathsf{v}-\mathsf{w}\|_\infty\Bigg[\frac{B_\xi}{\Gamma(\beta)}\frac{1}{|A_\theta|}\Big(\frac{B_{t_1}+B_{t_2}}{\Gamma(\beta-1)}+\theta\int_0^{\eta}\frac{B_\sigma}{\Gamma(\beta)}\,d\sigma\Big)\Bigg],\end{aligned}$$

where $\lambda_r=\sup\limits_{\xi\in I}\lambda_r(\xi)$.

By (I), we obtain

$$|\mathcal{S}\mathsf{v}(\xi)-\mathcal{S}\mathsf{w}(\xi)|\leq\|\mathsf{v}-\mathsf{w}\|_\infty\varphi\big(\|\mathsf{v}-\mathsf{w}\|_\infty\big).$$

So,

$$\mathrm{d}(\mathcal{S}\mathsf{v}-\mathcal{S}\mathsf{w})\leq\mathrm{d}(\mathsf{v}-\mathsf{w})\varphi\big(\mathrm{d}(\mathsf{v}-\mathsf{w})\big).$$

Let $\alpha\colon \mathrm{C}(I)^2\to\mathbb{R}^+$ such that

$$\alpha(\mathsf{v},\mathsf{w})=\begin{cases}1 & \text{if } \delta(\mathsf{v}(\xi),\mathsf{w}(\xi))\geq 0\quad \xi\in I,\\ 0 & \text{otherwise.}\end{cases}$$

Then,

$$\begin{aligned}\alpha(\mathsf{v},\mathsf{w})\mathrm{d}(\mathcal{S}\mathsf{v},\mathcal{S}\mathsf{w})&\leq\alpha(\mathsf{v},\mathsf{w})\mathrm{d}(\mathsf{v},\mathsf{w})\frac{1}{3}\varphi\big(\mathrm{d}(\mathsf{v}-\mathsf{w})\big)\\&\leq\mathrm{d}(\mathsf{v},\mathsf{w})\varphi\big(\mathrm{d}(\mathsf{v},\mathsf{w})\big).\end{aligned}$$

Hence,

$$\alpha(\mathsf{v},\mathsf{w})\mathrm{d}(\mathcal{S}\mathsf{v},\mathcal{S}\mathsf{w})\leq\mathrm{d}(\mathsf{v}-\mathsf{w})\varphi\big(\mathrm{d}(\mathsf{v},\mathsf{w})\big),$$

where $\psi(\xi) = \varphi(\xi)$, $\xi \in]0,1[$.

Then, $\mathcal{S}$ is a α-Geraghty contraction.

From (III) and for each $\mathsf{v},\mathsf{w} \in \mathrm{C}(I)$, we get

$$\begin{aligned} \alpha(\mathsf{v},\mathsf{w}) \geq 1 \quad \text{implies that } \rho(\mathsf{v},\mathsf{w}) &\geq 0 \\ \rho(\mathcal{S}\mathsf{v},\mathcal{S}\mathsf{w}) &\geq 0 \\ \alpha(\mathcal{S}\mathsf{v},\mathcal{S}\mathsf{w}) &\geq 1. \end{aligned}$$

Then, $\mathcal{S}$ is α-admissible. Moreover, $\mathcal{S}$ is a triangular α-admissible

By (II), we have

$$\rho(\gamma_0,\mathcal{S}\gamma_0) \geq 0.$$

And thus,

$$\rho(\gamma_0,\mathcal{S}\gamma_0) \geq 1.$$

So,

$$\rho(\gamma_0,\mathcal{S}\gamma_0) \geq 1.$$

According to (IV), we have $(y_n)_{n\in\mathbb{N}} \subset \mathrm{C}(I)$ with $\alpha(y_n,y_{n+1}) \geq 1$, which gives

$$\delta(y_n,y_{n+1}) \geq 0.$$

Then, $\rho(y_n,y) \geq 0$.

And thus,

$$\rho(y_n,y) \geq 0.$$

That implies ,

$$\alpha(y_n,b) \geq 1.$$

So, by Theorem 1, then $\mathcal{S}$ has a fixed point in $\mathrm{C}(I)$ is a solution of problem (3.3).

Finally, if w and v are two solutions of problem (3.3), then

$$\rho(\mathsf{v},\mathsf{w}) \geq 0.$$

So,

$$\alpha(\mathsf{v},\mathsf{w}) \geq 1.$$

From Theorem 1, then the problem (3.3) has a unique solution. □

4. Example

Consider the following fractional boundary value problem

$$\begin{cases} (\mathfrak{D}^{\beta}\,\mathsf{u})(\xi) = \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\,\frac{\sin|\mathsf{u}(\xi)|}{1+|\mathsf{u}(\xi)|}, & \xi \in I = [0,1], \\ \mathsf{u}(0) = 0, \\ \mathsf{u}'(1/4) + 3\mathsf{u}'(1/2) = 2\int_0^{1/3} \mathsf{u}(\zeta)\,d\zeta, \end{cases} \tag{4.1}$$

where $\mathfrak{D}^{\beta}$ is a Caputo fracional order derivative with $\beta = 1/2$ and

$$\mathsf{f}\big(\xi, \mathsf{u}(\xi)\big) = \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \frac{\sin|\mathsf{u}(\xi)|}{\big(1+|\mathsf{u}(\xi)|\big)}.$$

Let $(\mathrm{C}(I), \mathrm{d})$ be a metric space where $\mathrm{d}\colon \mathrm{C}(I)^2 \to \mathbb{R}_+$ is given by

$$\mathrm{d}(y,z) = \|y-z\|_\infty.$$

Lets $\mathsf{u}, \mathsf{v} \in \mathrm{C}(I)$, we have

$$\mathsf{f}\big(\xi, \mathsf{u}(\xi)\big) - \mathsf{f}\big(\xi, \mathsf{v}(\xi)\big) = \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \frac{\sin|\mathsf{u}(\xi)|}{1+|\mathsf{u}(\xi)|} - \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \frac{\sin|\mathsf{v}(\xi)|}{1+|\mathsf{v}(\xi)|}.$$

Then,

$$\begin{aligned}
|\mathsf{f}\big(\xi, \mathsf{u}(\xi)\big) - \mathsf{f}\big(\xi, \mathsf{v}(\xi)\big)| &= \left| \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \frac{\sin|\mathsf{u}(\xi)|}{1+|\mathsf{u}(\xi)|} - \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \frac{\sin|\mathsf{v}(\xi)|}{1+|\mathsf{v}(\xi)|} \right| \\
&\leq \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \left| \frac{\sin|\mathsf{u}(\xi)|}{1+|\mathsf{u}(\xi)|} - \frac{\sin|\mathsf{v}(\xi)|}{1+|\mathsf{v}(\xi)|} \right| \\
&\leq \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \\
&\quad \times \left| \frac{\sin|\mathsf{u}| + |\mathsf{v}|\sin|\mathsf{u}| - \sin|\mathsf{v}| - |\mathsf{u}|\sin|\mathsf{v}|}{\big(1+|\mathsf{u}|\big)\big(1+|\mathsf{v}|\big)} \right|.
\end{aligned}$$

Case-1: If $|\mathsf{v}(\xi)| \leq |\mathsf{u}(\xi)|$, we get

$$\begin{aligned}
|\mathsf{f}\big(\xi, \mathsf{u}(\xi)\big) - \mathsf{f}\big(\xi, \mathsf{v}(\xi)\big)| &\leq \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \\
&\quad \times \left| \frac{\sin|\mathsf{u}| - \sin|\mathsf{v}| + |\mathsf{v}(\xi)||\sin|\mathsf{u}| - \sin|\mathsf{v}|}{\big(1+|\mathsf{u}(\xi)|\big)\big(1+|\mathsf{v}(\xi)|\big)} \right| \\
&\leq \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} \\
&\quad \times \left| \sin|\mathsf{u}| - \sin|\mathsf{v}| \frac{1+|\mathsf{v}(\xi)|}{\big(1+|\mathsf{u}(\xi)|\big)\big(1+|\mathsf{v}(\xi)|\big)} \right| \\
&\leq \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}} 2\sin\left(\frac{|\mathsf{u}|-|\mathsf{v}|}{2}\right) \cos\left(\frac{|\mathsf{u}|+|\mathsf{v}|}{2}\right) \\
&\quad \times \frac{1+|\mathsf{v}(\xi)|}{\big(1+|\mathsf{u}(\xi)|\big)\big(1+|\mathsf{v}(\xi)|\big)}.
\end{aligned}$$

Since $\sin u \leq u$ for all $u \geq 0$, then

$$\begin{aligned}|\mathsf{f}\big(\xi,\mathsf{u}(\xi)\big)-\mathsf{f}\big(\xi,\mathsf{v}(\xi)\big)| \leq & \frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\big(|\mathsf{u}|-|\mathsf{v}|\big)\\ & \times \frac{1+|\mathsf{v}(\xi)|}{\big(1+|\mathsf{u}(\xi)|\big)\big(1+|\mathsf{v}(\xi)|\big)}\\ \leq & |\mathsf{u}-\mathsf{v}|\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\\ & \times \frac{1+|\mathsf{v}(\xi)|}{\big(1+|\mathsf{u}(\xi)|\big)\big(1+|\mathsf{v}(\xi)|\big)}\\ \leq & |\mathsf{u}-\mathsf{v}|\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{1}{\big(1+|\mathsf{u}(\xi)|\big)}.\end{aligned}$$

Case-2: If $|\mathsf{v}(\xi)| \geq |\mathsf{u}(\xi)|$, we can obtain

$$\begin{aligned}|\mathsf{f}\big(\xi,\mathsf{u}(\xi)\big)-\mathsf{f}\big(\xi,\mathsf{v}(\xi)\big)| \leq & |\mathsf{u}-\mathsf{v}|\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\\ & \times \frac{1+|\mathsf{u}(\xi)|}{\big(1+|\mathsf{u}(\xi)|\big)\big(1+|\mathsf{v}(\xi)|\big)}\\ \leq & |\mathsf{u}-\mathsf{v}|\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{1}{\big(1+|\mathsf{v}(\xi)|\big)}.\end{aligned}$$

Finally, we have

$$|\mathsf{f}\big(\xi,\mathsf{u}(\xi)\big)-\mathsf{f}\big(\xi,\mathsf{v}(\xi)\big)| \leq |\mathsf{u}-\mathsf{v}|\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{1}{\big(1+\max\{|\mathsf{u}(\xi)|,|\mathsf{v}(\xi)|\}\big)}.$$

Then, the hypothesis (H1) is satisfied with

$$|\mathsf{f}\big(\xi,\mathsf{u}(\xi)\big)-\mathsf{f}\big(\xi,\mathsf{v}(\xi)\big)| \leq \lambda(\xi)|\mathsf{u}-\mathsf{v}|,$$

where $\lambda(\xi)=\frac{\exp\{-\xi^2\}}{\sqrt{64+\xi}}\frac{1}{\big(1+\max\{|\mathsf{u}(\xi)|,|\mathsf{v}(\xi)|\}\big)}$,

$$B_\xi=-2\sqrt{\xi}, \qquad B_{t_1}=4, \qquad B_{t_2}=2\sqrt{2}, \qquad B_\sigma=-2\sqrt{\sigma},$$

and

$$\left\|\lambda_r(\xi)\big[\frac{B_t}{\Gamma(\beta)}+\frac{1}{|A_\theta|}\big(\frac{B_{t_1}+B_{t_2}}{\Gamma(\beta-1)}+\theta\int_0^\eta\frac{B_\sigma}{\Gamma(\beta)}\,d\sigma\big)\big]\right\|_\infty \leq \psi(\|u-v\|_{L^\infty})$$

with $\psi(\xi)=\frac{1}{1+\xi}$, for all $t>0$.

Next, we define the function $\alpha\colon C(I)^2 \to [0,\infty)$ by

$$\alpha(u(\xi),v(\xi)) = \begin{cases} 1 & \text{if } \rho(u(\xi),v(\xi)) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

with $\rho(u(\xi),v(\xi)) = \sup_{\xi\in I} |u(\xi)-v(\xi)|$.

Then, the hypotheses (H2) and (H3) are satisfied, where $\gamma_0(\xi) = u(0)$. Moreover from the defnition of the ρ , then (H4) holds.

Finally, by Theorem 4, we get the existence of solutions and the uniqueness of problem (4.1).

ACKNOWLEDGEMENTS

The authors would like to express their sincere apreciation to the referes for their very helpful suggestions and many comments.

REFERENCES

[1] M. I. Abbas and M. A. Ragusa, "Nonlinear fractional differential inclusions with non-singular Mittag-Leffler kernel." *AIMS Mathematics*, vol. 7, no. 11, pp. 20 328–20 340, 2022, doi: 10.3934/math.20221113.

[2] S. Abbas, M. Benchohra, M.Greaf, and J. Henderson, "Implicit fractional differential and integral equations: existence and stability." *De Gruyter Series in Nonlinear Analysis and Applications.*, vol. 26, 2018, doi: 10.1515/9783110553819.

[3] S. Abbas, M. Benchohra, and G. M. N'guérékata, "Topics in fractional differential equations." *Springer, New Yor.*, 2012.

[4] S. Abbas, S. Benchohra, and D. Krim, "Initial value proplems for Caputo-Fabrizio implicit fractional dierential equations in *b*-metrice spaces." *Bulletin of the Transilvania University of Brasov. Series III*, vol. 63, no. 1, pp. 1–12, 2021, doi: 10.31926/but.mif.2021.1.63.1.1.

[5] H. Afshari, H. Aydii, and E. Karapinar, "On generalized $\alpha-\psi$-Geraghty contractions on metric spaces." *Georgian Math. J.* , vol. 27, no. 1, pp. 9–21, 2018, doi: 10.1515/gmj-2017-0063.

[6] H. Afsharit, S. Kalantari, and D. Baleanul, "Solution of fractional differential equations via α-φ-geraghty type mappings." *Advances in Difference Equations*, vol. 2018, no. 1, pp. 2017–347, 2018, doi: 10.1186/s13662-018-1807-4.

[7] R. Agrawl, M. Benchohra, and S. Hamani, "A survey on existence results for boundary value problems of non linear fractional differential equations and inclusions." *Acta Applicandae Mathematicae* , vol. 109, pp. 973–1033, 2010, doi: 10.1007/s10440-008-9356-6.

[8] S. Akhter, "Existence of weak solutions for Caputo's fractional derivative in Banach spaces." *Journal of Bangladesh Mathematical Society*, vol. 39, pp. 111–118, 2019, doi: 10.3329/ganit.v39i0.44166.

[9] M. Awadalla, M. Subramanian, M.Manigandan, and K. Abuasbeh, "Existence and H-U stability of solution for coupled system of fractional-order with integral conditions involving Caputo-Hadamard derivatives, Hadamard integrals." *Journal of Function Spaces*, vol. 2022, p. 9471590, 2022, doi: 10.1155/2022/9471590.

[10] S. H. Cho, J. S. Bae, and E. Karapinar, "Fixed point theorems for α-Geraghty contraction type maps in metric spaces." *Fixed Point Theory and Applcation*, vol. 2013, no. 1, pp. 1–11, 2013, doi: 10.1186/1687-1812-2013-329.

[11] K. Diethelm, "The analysis of fractional diffrential Equations." *Lecture Notes in Mathematics.*, vol. 2004, 2010.

[12] K. Diethelm and A. D. Freed, "On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity." *Scientific computing in chemical engineering II*, vol. 220, pp. 217–224, 1999, doi: 10.1007/978-3-642-60185-9_24.

[13] V. Gafiychuk, B. Datsun, and V. Meleshko, "Mathematical modeling of time fractional reactiondiffusion systems." *Journal of Computational and Applied Mathematics*, vol. 220, pp. 215–225, 2008, doi: 10.1016/j.cam.2007.08.011.

[14] M. Geraghty, "On contractive mappings." *Proceedings of the American Mathematical Society*, vol. 40, no. 2, pp. 604–608, 1973.

[15] R. Hilfer, "Applications of fractional calculus in physics." *World Scientific*, 2000.

[16] E. Karapinar, "α-φ-Geraghty contraction type mappings and some related fixed point resultss." *Filomat*, pp. 37–48, 2014, doi: 10.2298/FIL1401037K.

[17] E. Karapinar, P. Kumam, and P. Salimi, "On α-φ-Meir-Keeler contractive mappings." *Fixed Point Theory and Applcation*, vol. 2013, no. 1, pp. 1–12, 2013, doi: 10.1186/1687-1812-2013-94.

[18] E. Karapinar and B. Samet, "Generalized α-ψ-contractive type mappings and related fixed point theorems with applications." *Abstract and Applied Analysis*, vol. 17, pp. 793–486, 2012, doi: 10.1155/2012/793486.

[19] A. Lachouri, A. Ardjouni, and A. Djoudi, "Existence results for nonlinear sequential Caputo and Caputo-Hadamard fractional differential equations with three-point boundary conditions in Banach spaces." *Filomat*, vol. 36, no. 14, pp. 4717–4727, 2022, doi: 10.2298/FIL2214717L.

[20] B. Samet, C. Vetro, and P. Vetrol, "Fixed point for α-φ-contractive type mappings." *Nonlinear Analysis: Theory Methods Application*, vol. 75, no. 4, pp. 2145–2165, 2014, doi: 10.1016/j.na.2011.10.014.

Authors' addresses

Abdellah Taqbibt

(**Corresponding author**) Sultan Moulay Slimane University, Laboratory of Applied Mathematics and Scientific Calculus, FST, PO Box 523, Beni Mellal, Morocco

E-mail address: taqbibt.gmi@gmail.com

M'Hamed El Omari

Sultan Moulay Slimane University, Laboratory of Applied Mathematics and Scientific Calculus, FST, PO Box 523, Beni Mellal, Morocco

E-mail address: m.elomari@usms.ma

Said Melliani

Sultan Moulay Slimane University, Laboratory of Applied Mathematics and Scientific Calculus, FST, PO Box 523, Beni Mellal, Morocco

E-mail address: said.melliani@gmail.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 1043–1059

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4372

OPTIMAL CASH HOLDING MODEL BASED ON MINIMIZATION OF RUIN PROBABILITY

ZHENGYAN WANG, YAN ZHANG, AND PEIBIAO ZHAO

Received 22 September, 2019

Abstract. The management of optimal cash holding is particularly important for the sustainability of enterprises' survival and business decision-making. It is not good for enterprises to have more or less cash holding. In the meantime, the probability of enterprise bankruptcy is also one of the most important research topics in the field of economic finance and management science, such as the early warning of investment risk and financial decision support. In this paper, an optimal cash holding model is established. It is assumed that the objective of optimal cash holding is to minimize the probability of bankruptcy and the security area of cash holding is the constraint condition. By using the dynamic stochastic programming method, the optimal conversion strategy and the analytic expressions of the value function are obtained, and the relevant economic explanations and numerical examples are given. The effects of capital market parameters and consumption function parameters on the optimal conversion strategy and optimal cash holding are discussed.

2010 *Mathematics Subject Classification:* 90B50; 91G10

Keywords: ruin probability, the optimal cash holding, safe area, dynamic stochastic programming

1. Introduction

Capital is the blood of enterprises. The health of enterprises is closely related to the successful operation of funds. Since the financial crisis took place in 2008, Chinese enterprises, especially small and medium-sized enterprises have paid more and more attention to their cash holding decisions due to China's macroeconomic policies. This is especially true after the breakout of COVID-19 in 2020. This paper holds that enterprises will put survival in the first place and then consider income in some circumstances. Take financial institutions, such as insurance companies as an example, due to the particularity of their business, the enterprises usually pay more

The first author was supported in part by the National Natural Science Foundation of China Fund, Grant No. 11871257.

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

attention to the problem of ruin probability. Another example is that for enterprises in the process of recession avoiding bankrupty is the first thing to do. Also, for enterprises under some special background, no bankruptcy means victory, for example, companies in manufacture, tourism, and catering industries have to struggle to survival after being hit by COVID-19. At this time, the decision of cash holding is particularly important.

In the field of economic finance and management science, the ruin probability of enterprises is also one of the most important research topics. It plays a significant role in the early warning of investment risk and financial decision support. In finace, when a company's net assets is negative, that is, when the value of assets is less than the value of liabilities, stock bankruptcy will happen. When its operating cash flow is not enough to meet the existing debts due, flow bankruptcy will happen, which means that an enterprise cannot pay its debts. The possibility of enterprises' bankruptcy is an indisputable fact. The possibility of bankruptcy can be measured by ruin probability. Different companies and industries should have different bankruptcy probability. [14] defined the moment of investors' bankruptcy is when remaining wealth reaches zero for the first time. In order to fully consider the safety of the company's operations, this article defines the moment of bankruptcy as the first time that company's residual wealth falls below a threshold m instead of zero, ie $\tau = \inf\{t > 0; 0 < X_t \leq m\}$, the ruin probability is $P_{x_t}\{\tau < +\infty\}$, X_t is the residual assets of an enterprise. When the company's remaining wealth is less than m, it will be considered bankrupt. However, in practice, it does not really mean that the company is going bankrupt. It may just indicate that the company needs to adjust its cash holding strategy and maintain the liquidity of the company's normal operations, or the company's financial situation is not very bad after having considered many other aspects of the company. But studying the ruin probability of companies is still of great theoretical and practical significance. The ruin probability can be used as an indicator of the stability and development of a company as a comprehensive residual wealth management process. Taking insurance companies as an example, it is supposed that bankruptcy occurs when its surplus is zero for the first time in most cases. Although this does not mean it will happen in reality.

The ruin probability was first proposed by [6]. Under the discrete-time and discrete-state model, the optimal investment strategy problem that minimizes the ruin probability is considered for general investors. Long afterwards, [4] first considered the problem of minimizing the ruin probability for insurance companies. Different from previous studies on bankruptcy, [5] adopted the dynamic programming method provided by [7] and [11] to obtain the explicit expression of the minimum ruin probability and the corresponding optimal strategy. This is the first application of dynamic programming principle and HJB equation theory in bankruptcy problems. Since then, they have become the main tools to study the minimization of bankruptcy probability. Then [5] studied the maximization survival strategy of companies with fixed debt.

[9] considered the problem of minimizing the ruin probability of allowed investment under the classical Cramer-Lundberg model. And after that, [17] introduced linear reinsurance control based on their model. For more models, see the results of [[16], [8], [18], [20], [2], [1], [13], [3], [12], [10], [22]].

Assuming that the objective function pursued by an enterprise is the pursuit of minimizing ruin probability, the main problem to be studied in this paper can be summarized as follows: for an enterprise, how to determine the optimal cash holding in a safe area to achieve the minimum ruin probability.

2. Modeling

In this paper, we assume that the residual assets of an enterprise is X_t at time t, and the ratio of the residual assets in the form of cash is $0 \leq \beta_t \leq 1$. As a result, the cash holding amount is $C_t = \beta_t X_t$, and the holding amount of risky assets is $R_t = (1-\beta_t)X_t$. According to Robichek's [15] trade-off theory, it is known that there is a cash holding target, and the cash holding level will be adjusted when the actual cash holding deviates from the target. [19] pointed out that there is a safe area $[L,H]$ for cash holding. When the cash holding level of enterprisess is not in the safe area, enterprises need to adjust cash assets and risky assets to make it return to the safe area. If there is a parameter μ_t, and $\mu_t(1-\beta_t)X_t$ is the conversion amount. When $\mu_t > 0$, it means that risky assets are converted into cash assets, and when $\mu_t < 0$, the case is just the opposite. There is no need to discuss the case when $\mu_t = 0$.

In addition, it is assumed that enterprises have a consumption function on which wealth depends. Under the condition of income flow, [20] mentioned how to convert the proportional consumption rate into linear consumption, but did not make a detailed discussion. However, if there is no income flow, bankruptcy will never happen under the proportional consumption model. [2] defined bankruptcy under the proportional consumption pattern as the reduction of wealth to a predetermined level (which must be greater than zero). Based on the above references, we assumed that the consumption function is a linear function $c(X_t) = c + \theta X_t$, which not only reflects the basic consumption expenditure of enterprises, but also reflects the general state of enterprises' increasing expenditure with increasing wealth. Where $c > 0$ is the minimum consumption of enterprises to meet daily expenses, and $0 \leq \theta \leq r_0$ is the consumption ratio that increases with wealth (such as dividends paid, rewards given to employees, etc.).

The existence of the safe area requires the cash holding after conversion $\beta_t X_t + \mu_t(1-\beta_t)X_t - (c+\theta X_t)$ must be in the safe area $[L,H]$. As a result, μ_t must be in the interval $\left[\frac{L+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}, \frac{H+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}\right]$. Let

$$D_1 = [a_1, b_1] = \left[\frac{L+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}, \frac{H+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}\right]$$

represent the set of all feasible strategies μ_t.

Let $B(t)$ denote the price of risky-free assets at time t and $S(t)$ denote the price of risky assets at time t, then $B(t)$, $S(t)$ can be characterized by the following stochastic processes:

$$dB(t) = r_0 B(t)dt \tag{2.1}$$

$$dS(t) = S(t)(r_1 dt + \sigma_1 dW(t)) \tag{2.2}$$

Where r_0, r_1 are the expected rate of return of risky-free assets and risky assets respectively, σ_1 is the volatility and $W(t)$ is the one-dimensional Standard Brownian motion.

Then the random variable X_t satisfies the following stochastic equation:

$$dX_t = \frac{\beta_t X_t + \mu_t(1-\beta_t)X_t}{B_t} dB_t + \frac{(1-\mu_t)(1-\beta_t)X_t}{S_t} dS_t - (c+\theta X_t)dt \tag{2.3}$$

By substituting (2.1) and (2.2) into (2.3), the following process can be obtained:

$$\begin{aligned} dX_t = {} & [r_0\beta_t X_t + r_0\mu_t(1-\beta_t)X_t + r_1(1-\mu_t)(1-\beta_t)X_t - (c+\theta X_t)]dt \\ & + (1-\mu_t)(1-\beta_t)X_t\sigma_1 dW_t \end{aligned}$$

Defining the value function $R(x) = \inf\limits_{\mu_t \in D_1} P_{X_t}(\tau < +\infty)$, it's boundary conditions are

$$R(m) = 1, R(+\infty) = 0 \tag{2.4}$$

So far, the problem to be solved in this paper can be expressed by the following model [M-1]:

$$[M-1]: \begin{cases} \inf\limits_{\mu_t \in D_1} P_{X_t}(\tau < +\infty) \\ s.t. \begin{cases} dX_t & = [r_0\beta_t X_t + r_0\mu_t(1-\beta_t)X_t + r_1(1-\mu_t)(1-\beta_t)X_t \\ & \quad -(c+\theta X_t)]dt + (1-\mu_t)(1-\beta_t)X_t\sigma_1 dW_t \\ \mu_t \in D_1 \end{cases} \end{cases}$$

3. Model solving

In order to solve the model, the threshold value of m should be set first. In the previous literature [21], no specific value of m was set. And setting $m = f(L, H, c, \theta, r_1, r_0)$ is more reasonable than setting it as a fixed value. For the convenience of discussion, this paper supposes $m = \frac{(H+c)(r_1-r_0)+c}{r_1-\theta-r_1\theta+r_0\theta}$. $(H+C)(R_1-r_0)$ is the risk premium for maximum cash holding after taking daily minimum consumption. The denominator part can be regarded as the comprehensive interest rate after considering the risky assets interest rate, risky-free asset interest rate and consumption rate, so it is reasonable and practical to regard it as the pricing formula of m.

When the surplus wealth x is less than m, bankruptcy must occur, and $R(m) = 1$. Therefore, we only discuss the solution of the model when $x > m$. Since the value

function $R(x) = \inf_{\mu_t \in D_1} P_{X_t}(\tau < \infty)$ is independent on time parameter, our goal is to obtain the minimum ruin probability $R(x)$ and the optimal strategy μ_t^*. In order to solve this problem, the following HJB equation with boundary conditions (2.4) is obtained according to Krylov's [11] theorem 1.4.5:

$$\begin{aligned} R_t + \inf\{&[r_0\beta_t x + r_0\mu_t(1-\beta_t)x + r_1(1-\mu_t)(1-\beta_t)x \\ &-(c+\theta x)]R_x + \frac{1}{2}x^2(1-\beta_t)^2(1-\mu_t)^2\sigma_1^2 R_{xx}\} = 0 \end{aligned} \tag{3.1}$$

The constraint of the security area requires the cash holding after conversion should be in the safe area. That is, $\mu_t \in D_1$. And then we can get the following verification theorem:

Theorem 1. *If $V(x)$ is a monotonically decreasing second order continuous differentiable convex solution of HJB equation (3.1) on $(0,+\infty)$ and satisfies boundary value condition (2.4), then $V(x)$ and $R(x)$ are consistent. Furthermore, let μ_t^* satisfy the following equation for all $x \in (0,+\infty)$*

$$\begin{aligned} V_t + [r_0\beta_t x + r_0\mu_t(1-\beta_t)x + r_1(1-\mu_t)(1-\beta_t)x - (c+\theta x)]V_x \\ + \frac{1}{2}x^2(1-\beta_t)^2(1-\mu_t)^2\sigma_1^2 V_{xx} = 0 \end{aligned} \tag{3.2}$$

then $\mu_s^* = \mu^*(X_s^{\mu^*})$, where $X_s^{\mu^*}$ is the solution of equation (2.3), $\mu^*(x)$ is called the optimal feedback control function, that is $V(x) = R(x) = R_{\mu_t^*}(x)$.

Proof. The proof of this theorem can be seen in Appendix 1 of [8]. □

For any monotone decreasing second order continuous differentiable convex function V. For convenience, we note that $\mu_{V(x)} = 1 + \frac{r_1 - r_0}{x(1-\beta_t)\sigma_1^2}\frac{V_x}{V_{xx}}$, $l_1 = \frac{(L+c)(r_1-r_0)+2c}{(1-\theta)(r_1-r_0)+2(r_0-\theta)}$ and $l_2 = \frac{(H+c)(r_1-r_0)+2c}{(1-\theta)(r_1-r_0)+2(r_0-\theta)}$. Then we need to discuss the problem in three cases: $m < l_1 < l_2$, $l_1 < m < l_2$ and $l_1 < l_2 < m$.

Case 1. When $m < l_1 < l_2$, we define the set firstly

$$\begin{aligned} K_1 &= \{x : x \in (m,+\infty), \mu_{V(x)} \le a_1\} \\ K_2 &= \{x : x \in (m,+\infty), a_1 \le \mu_{V(x)} \le b_1\} \\ K_3 &= \{x : x \in (m,+\infty), \mu_{V(x)} \ge b_1\} \end{aligned}$$

Lemma 1. *If $V(x)$ is a monotonically decreasing differentiable convex function on $(m,+\infty)$, and $V(x)$ satisfies the following equation on K_1*

$$\frac{V_{xx}}{V_x} = g_1(x) \triangleq -\frac{2[r_1 x - \theta x - (L+c+\theta x)(r_1-r_0) - c]}{(x-L-c-\theta x)^2\sigma_1^2} \tag{3.3}$$

then V is the solution of HJB equation (3.1) on K_1.

Proof. Because of $x > \frac{(H+c)(r_1-r_0)+c}{r_1-\theta-r_1\theta+r_0\theta} > \frac{(L+c)(r_1-r_0)+c}{r_1-\theta-r_1\theta+r_0\theta}$ and $r_1-\theta-r_1\theta+r_0\theta > r_0-\theta > 0$, we can get $(r_1-\theta-r_1\theta+r_0\theta)x > (L+c)(r_1-r_0)+c$. That is $r_1x-\theta x-(L+c+\theta x)(r_1-r_0)-c > 0$. Therefore (3.3) satisfies $\frac{V_{xx}}{V_x} < 0$. Taking (3.3) into (3.1),

$$\mu_t^* = \frac{L+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}$$

In this case, μ_t^* takes the boundary value of the security area constraint, so μ_t^* is the optimal solution of (3.1) left-hand optimization problem, and from (3.3) we know that its optimal solution is 0, then V is the solution of HJB equation (3.1) on K_1. □

Lemma 2. *If $V(x)$ is a monotonically decreasing differentiable convex function on $(m,+\infty)$, and $V(x)$ satisfies the following equation on K_2*

$$\frac{V_{xx}}{V_x} = g_2(x) \triangleq \frac{(r_1-r_0)^2}{2\sigma_1^2[(r_0-\theta)x-c]} \tag{3.4}$$

then V is the solution of HJB equation (3.1) on K_2.

Proof. Let $x \in K_2$, Because the optimal solution of optimization problem

$$\begin{aligned}\inf\{[r_0\beta_t x+r_0\mu_t(1-\beta_t)x+r_1(1-\mu_t)(1-\beta_t)x-(c+\theta x)]V_x \\ +\frac{1}{2}x^2(1-\beta_t)^2(1-\mu_t)^2\sigma_1^2V_{xx}\} = 0\end{aligned}$$

without any constraints is $\mu_t^* = 1-2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x}$.

So

$$\begin{aligned}K_2 &= \{x : x \in (m,+\infty), a_1 \le \mu_{v(x)} \le b_1\} \\ &= \{x : x \in (m,+\infty), l_1 \le x \le l_2\}\end{aligned}$$

Because $l_2 > m$, then we can have $\frac{(H+c)(r_1-r_0)+2c}{(1-\theta)(r_1-r_0)+2(r_0-\theta)} > \frac{(H+c)(r_1-r_0)+c}{r_1-\theta-r_1\theta+r_0\theta}$. By simplifying, we can get $(H+c)(r_1-r_0)(r_0-\theta) < c(r_1-r_0)(1-\theta)$. So there is $Hr_0-H\theta+cr_0-c < 0$, therefore $\frac{(H+c)(r_1-r_0)+2c}{(1-\theta)(r_1-r_0)+2(r_0-\theta)} - \frac{c}{r_0-\theta} = \frac{(r_1-r_0)(Hr_0-H\theta+cr_0-c)}{(r_0-\theta)[(1-\theta)(r_1-r_0)+2(r_0-\theta)]} < 0$. That is $\frac{(H+c)(r_1-r_0)+2c}{(1-\theta)(r_1-r_0)+2(r_0-\theta)} < \frac{c}{r_0-\theta}$, then $x < l_2 < \frac{c}{r_0-\theta}$. As a result (3.4) is less than 0, satisfying $\frac{V_{xx}}{V_x} < 0$.

According to the definition of K_2, when $x \in K_2$, $a_1 \le \mu_{V(x)} \le b_1$, so μ_t^* is the optimal solution of the left-hand optimization problem of (3.1), and the optimal solution is 0. According to equation (3.4), V is the solution of HJB equation (3.1) on K_2. □

Lemma 3. *If $V(x)$ is a monotonically decreasing differentiable convex function on $(m,+\infty)$, and $V(x)$ satisfies the following equation on K_3*

$$\frac{V_{xx}}{V_x} = g_3(x) \triangleq -\frac{2[r_1x-\theta x-(H+c+\theta x)(r_1-r_0)-c]}{(x-H-c-\theta x)^2\sigma_1^2} \tag{3.5}$$

then V is the solution of HJB equation (3.1) on K_3.

Proof. Because $x > \frac{(H+c)(r_1-r_0)+c}{r_1-\theta-r_1\theta+r_0\theta}$ and $r_1-\theta-r_1\theta+r_0\theta > r_0-\theta > 0$, so $r_1x-\theta x-(H+c+\theta x)(r_1-r_0)-c > 0$. Then (3.5) satisfies $\frac{V_{xx}}{V_x} < 0$. By substituting (3.5) into (3.1), we can get $\mu_t^* = \frac{H+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}$.

In this case, μ_t^* takes the right bound value of the security zone constraint, so μ_t^* is the optimal solution of the left-hand optimization problem of (3.1), and the optimal solution is 0. According to (3.5), V is the solution of HJB equation (3.1) on K_3. □

According to the above three lemmas, we can get the following three sets:

$$K_1 = \{x : x \in (m,+\infty), \mu_{V(x)} \le a_1\} = \{x : x \in (m,+\infty), x \le l_1\}$$
$$K_2 = \{x : x \in (m,+\infty), a_1 \le \mu_{V(x)} \le b_1\} = \{x : x \in (m,+\infty), l_1 \le x \le l_2\}$$
$$K_3 = \{x : x \in (m,+\infty), \mu_{V(x)} \ge b_1\} = \{x : x \in (m,+\infty), x \ge l_2\}$$

Next, we will discuss the solutions of equation (3.1) on these three sets. The key is to deal with the continuity of value function V at $x = l_1$ and $x = l_2$.

(1): When $x \in K_1$, according to Lemma 1, if we want to get the solution of (3.1) on K_1, we need to solve the following equation on K_1

$$\frac{V_1''}{V_1'} = g_1(x), V_1(m) = 1$$

By solving the above equation, we can get

$$V_1(x) = 1 - C_1 \int_m^x exp\left(\int_m^t g_1(s)ds\right) dt, \forall x \in K_1,$$

where C_1 is the undetermined coefficient.

(2): When $x \in K_2$, according to Lemma 2, if we want to get the solution of (3.1) on K_2, we need to solve the following equation on K_2

$$\frac{V_2''}{V_2'} = g_2(x)$$

By solving the above equation, we can get

$$V_2(x) = C_2 + C_3 \int_x^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right) dt, \forall x \in K_2,$$

where C_2, C_3 are the undetermined coefficient.

(3): When $x \in K_3$, according to Lemma 3, if we want to get the solution of (3.1) on K_3, we need to solve the following equation on K_3

$$\frac{V_3''}{V_3'} = g_3(x), V_3(+\infty) = 0$$

By solving the above equation, we can get

$$V_3(x) = C_4 \int_x^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt, \forall x \in K_3,$$

where C_4 is the undetermined coefficient.

Next, we require $V(x)$ to be continuously differentiable at $x = l_1$, i.e

$$1 - C_1 \int_m^{l_1} exp\left(\int_m^t g_1(s)ds\right) dt = C_2 + C_3 \int_{l_1}^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right) dt \tag{3.6}$$

$$C_1 exp\left(\int_m^{l_1} g_1(s)ds\right) = C_3 \tag{3.7}$$

$$C_1 exp\left(\int_m^{l_1} g_1(s)ds\right) g_1(l_1) = C_3 g_2(l_1) \tag{3.8}$$

Meanwhile, we require $V(x)$ to be continuously differentiable at $x = l_2$, i.e

$$C_2 = C_4 \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt \tag{3.9}$$

$$C_3 exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) = C_4 \tag{3.10}$$

$$C_3 exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) g_2(l_2) = C_4 g_3(l_2) \tag{3.11}$$

By solving equations (3.6)-(3.11), we can get

$$\begin{aligned}
C_1 = &\Bigg[\int_m^{l_1} exp\left(\int_m^t g_1(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) \\
&\times \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) \int_{l_1}^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right) dt\Bigg]^{-1} \\
C_2 = &\Bigg[\int_m^{l_1} exp\left(\int_m^t g_1(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) \\
&\times \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) \int_{l_1}^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right) dt\Bigg]^{-1} \\
&\times exp\left(\int_m^{l_1} g_1(s)ds\right) exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt \\
C_3 = &\Bigg[\int_m^{l_1} exp\left(\int_m^t g_1(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) \\
&\times exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt \\
&+ exp\left(\int_m^{l_1} g_1(s)ds\right) \int_{l_1}^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right) dt\Bigg]^{-1} exp\left(\int_m^{l_1} g_1(s)ds\right) \\
C_4 = &\Bigg[\int_m^{l_1} exp\left(\int_m^t g_1(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) \\
&\times \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right)
\end{aligned}$$

$$\times \int_{l_1}^{l_2} exp\left(\int_{l_1}^{t} g_2(s)ds\right)dt\Bigg]^{-1} exp\left(\int_{m}^{l_1} g_1(s)ds\right) exp\left(\int_{l_1}^{l_2} g_2(s)ds\right)$$

Substituting C_1, C_2, C_3, C_4 into (3.8) and (3.11) can verify the second-order continuous differentiability. Next, we will still give the solution of the model [M-1] in the form of theorem when $m < l_1 < l_2$.

Theorem 2. *When $m < l_1 < l_2$, μ_t^* of model [M-1] is as follows:*

$$\mu_t^* = \begin{cases} \frac{L+c+\theta x-\beta_t x}{(1-\beta_t)x} & x \in (m, l_1]; \\ 1-2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x} & x \in [l_1, l_2]; \\ \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x} & x \in [l_2, +\infty]. \end{cases}$$

the corresponding value function is as follows:

$$V(x) = \begin{cases} 1-C_1\int_m^x exp\left(\int_m^t g_1(s)ds\right)dt & x \in (m, l_1]; \\ C_2+C_3\int_x^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right)dt & x \in [l_1, l_2]; \\ C_4\int_x^{\infty} exp\left(\int_{l_2}^t g_3(s)ds\right)dt & x \in [l_2, +\infty]. \end{cases}$$

Proof. The proof process of the theorem is the above calculation and derivation process, which is omitted here. □

According to Theorem 2, it can be deduced that when $m < l_1 < l_2$, the optimal cash holding C^* is

$$C^* = \begin{cases} L & x \in (m, l_1]; \\ (1-\theta+2\frac{r_0-\theta}{r_1-r_0})x-(c+\frac{2c}{r_1-r_0}) & x \in [l_1, l_2]; \\ H & x \in [l_2, +\infty]. \end{cases}$$

and it is not difficult to verify that when $x \in [l_1, l_2]$, the value of $(1-\theta+2\frac{r_0-\theta}{r_1-r_0})x-(c+\frac{2c}{r_1-r_0})$ is in $[L, H]$.

Case 2. When $l_1 < m < l_2$, if $l_1 < x < m$, then "bankruptcy" must occur and the model has no solution. Therefore, we only need to discuss the solution of the model with the value of x in the following two sets.

$$I_1 = \{x : x \in (m, \infty), m \le x \le l_2\} \tag{3.12}$$

$$I_2 = \{x : x \in (m, \infty), x > l_2\} \tag{3.13}$$

According to the proof process of Lemma 2, the solution $V(x)$ of HJB equation (3.1) on I_1 also satisfies the following equation:

$$\frac{V''}{V'} = g_2(x), x \in I_1 \tag{3.14}$$

At the same time, the solution $V(x)$ of HJB equation (3.1) on I_2 also satisfies the following equation:

$$\frac{V''}{V'} = g_3(x), x \in I_2 \tag{3.15}$$

Next, we also need to discuss the solutions of the HJB equation (3.1) on I_1, I_2. At this time, the key problem is to deal with the value function $V(x)$ at $x = l_2$.

From equation (3.14) and $V(m) = 1$, we can get

$$V(x) = 1 - D_1 \int_m^x exp\left(\int_m^t g_2(s)ds\right) dt.$$

By (3.15) and $V(+\infty) = 0$, then $V(x) = D_2 \int_x^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt$.

Because $V(x)$ is continuously differentiable of the second order at $x = l_2$, according to the continuity and the continuous differentiability of the first order, there is

$$1 - D_1 \int_m^{l_2} exp\left(\int_m^t g_2(s)ds\right) dt = D_2 \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt$$

$$D_1 exp\left(\int_m^{l_2} g_2(s)ds\right) = D_2$$

By combining the above two equations, we can get

$$\begin{aligned} D_1 = &\left[\int_m^{l_2} exp\left(\int_m^t g_2(s)ds\right) dt + exp\left(\int_m^{l_2} g_2(s)ds\right)\right. \\ &\left.\times \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt\right]^{-1} \\ D_2 = &\left[\int_m^{l_2} exp\left(\int_m^t g_2(s)ds\right) dt + exp\left(\int_m^{l_2} g_2(s)ds\right)\right. \\ &\left.\times \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt\right]^{-1} exp\left(\int_m^{l_2} g_2(s)ds\right) \end{aligned}$$

and it's not hard to verify that D_1 and D_2 satisfy $D_1 exp\left(\int_m^{l_2} g_2(s)ds\right) g_2(l_2) = D_2 g_3(l_2)$. That is to say, the second order function is also continuously differentiable. According to the above deduction, the following theorem can be obtained:

Theorem 3. *When $l_1 < m < l_2$, μ_t^* of model [M-1] is as follows:*

$$\mu_t^* = \begin{cases} 1 - 2\frac{[c-(r_0-\theta)x]}{(1-\beta_t)(r_1-r_0)x} & x \in (m, l_2] \\ \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x} & x \in [l_2, +\infty) \end{cases}$$

and the corresponding value function is:

$$V(x) = \begin{cases} 1 - D_1 \int_m^x exp\left(\int_m^t g_2(s)ds\right) dt & x \in (m, l_2] \\ D_2 \int_x^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt & x \in [l_2, +\infty) \end{cases}$$

Proof. The proof process of the theorem is the above calculation and derivation process, which is omitted here. □

According to Theorem 3, it can also be calculated that when $l_1 < m < l_2$, the optimal cash holding amount of C^* is

$$C^* = \begin{cases} (1-\theta+2\frac{r_0-\theta}{r_1-r_0})x-(c+\frac{2c}{r_1-r_0}) & x \in (m, l_2] \\ H & x \in [l_2, +\infty). \end{cases}$$

Case 3. When $l_1 < l_2 < m$, if $x < m$, then bankruptcy must happen, and the model has no solution. Because $x \geq m$ and $l_1 < l_2 < m$, so $x > l_2$. According to the discussion of the former two cases, the optimal solution of the model [M-1] is $\mu_t^* = \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x}$. In this case, the corresponding value function $V(x)$ satisfies the following equation:

$$\frac{V''(x)}{V'(x)} = g_3(x) = -\frac{2[r_1 x-\theta x-(H+c+\theta x)(r_1-r_0)-c]}{(x-H-c-\theta x)^2\sigma_1^2}$$

By solving this equation, we have $V(x) = E_1 + E_2 \int_x^{+\infty} exp\left(\int_m^t g_3(s)ds\right)dt$, $\forall x \in (m, +\infty)$, where E_1 and E_2 are the undetermined coefficient. According to $V(m) = 1, V(+\infty) = 0$, we can get $E_1 + E_2 \int_m^{+\infty} exp\left(\int_m^t g_3(s)ds\right)dt = 1, E_1 = 0$. By solving this equation, we can get $E_1 = 0, E_2 = \frac{1}{\int_m^{+\infty} exp\left(\int_m^t g_3(s)ds\right)dt}$.

Similarly, according to the above derivation, the following theorem can be obtained:

Theorem 4. *When $l_1 < l_2 < m$, μ_t^* of model [M-1] is:*

$$\mu_t^* = \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x}, \forall x \in (m, +\infty)$$

and the corresponding value function is:

$$V(x) = E_1 + E_2 \int_x^{+\infty} exp\left(\int_m^t g_3(s)ds\right)dt, \forall x \in (m, +\infty)$$

4. Numerical examples

4.1. *Examples of model solutions*

In Combination with the actual economic situation, it is assumed that the capital market parameters are $r_0 = 0.002, r_1 = 0.1, \sigma = 0.4$. At the same time we assume that an enterprise has a surplus asset of 1 million yuan at time t and has a consumption function $c(x) = 12000 + 0.01x$. Meanwhile the company can estimate the safety area $[L, H]$ is $[20, 30]$ (the unit is ten thousand yuan). Based on the above hypothesis, it can be calculated that $m = 43.05$, $l_1 = 44.82$ and $l_2 = 54.63$, then we can get $m < l_1 < l_2$. According to Theorem 2, we can have the following conclusions.

If $\beta_t = 0.44$, then $C_t = 44 \in (m, l_1]$, $u_t^* = -0.41$, and it is not difficult to verify that the optimal cash holding is L, which is the lower limit of the safe area. If $\beta_t = 0.5$,

then $C_t = 50 \in [l_1, l_2]$, $u_t^* = -0.55$, and the optimal cash holding is $21.15 \in [20, 30]$. If $\beta_t = 0.6$, then $C_t = 60 \in [l_2, +\infty)$, and the optimal cash holding is H, which is the upper limit of the safe area. In the above three cases, u_t^* are all negative indicates that some cash holding need to be converted into risky assets, which is consistent with $X_t > H$.

4.2. *The influence of capital market parameters on the optimal cash holding*

In this part, we assume that $X_t = 435000, c = 12000, \theta = 0.001, L = 200000, H = 300000, \beta_t = 0.8, r_1 = 0.1$ at time t.

First, we discuss the influence of r_0 on μ_t^* and the optimal cash holding. It can be

TABLE 1. μ_t^* and the optimal cash holding under different r_0 values

r_0	m	l_1	l_2	μ_t^*	the optimal cash holding
0.002	43.05	44.82	54.63	-1.56	20
0.003	42.73	44.17	53.78	-1.56	20
0.004	42.42	43.52	52.94	-1.56	20
0.005	42.10	42.89	52.13	-1.48	20.66
0.006	41.79	42.28	51.32	-1.40	21.35
0.007	41.47	41.67	50.54	-1.32	22.06
0.008	41.15	41.08	49.76	-1.24	22.79
0.009	40.84	40.49	49.01	-1.15	23.53
0.01	40.52	39.92	48.26	-1.07	24.29
0.02	37.36	34.74	41.52	-0.41	30
0.03	34.21	30.36	35.83	-0.41	30

seen from Table 1 that when $r_0 \in [0.002, 0.004]$, we can get $m < l_1 < l_2$ and $m < x < l_1$. According to Theorem 2, $\mu_t^* = -1.56$. At this time, r_0 has no effect on μ_t^* and the optimal cash holding but has affect on m, l_1, l_2. We can get $m < l_1 < l_2$ and $l_1 < x < l_2$ when $r_0 \in [0.005, 0.007]$. According to Theorem 2, $\mu_t^* = 1 - 2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x}$. When $r_0 \in [0.008, 0.01]$ then we can get $l_1 < m < l_2$ and $m < x < l_2$, μ_t^* can be obtained according to Theorem 3. In both cases, μ_t^* and the optimal cash holding are increasing functions of r_0. Finally, we can get $l_1 < m < l_2$ and $x > l_2$ when $r_0 \in [0.02, 0.03]$. According to Theorem 3, $\mu_t^* = -0.41$. r_0 has no effect on μ_t^* and the optimal cash holding, and the optimal cash holding is H. At the same time, it is assumed that the initial cash holding is $\beta_t x = 34.8 > H$, so we need to convert cash assets to risky assets. This is illustrated by the negative value of μ_t^* in Table 1. Such process can also be shown in Figure 1 and Figure 2. Figure 2 also verifies that under Theorem 2 and Theorem 3, μ_t^* obtained by the model can make the optimal cash holding in the safe area. The discussion of the impact of r_1 on μ_t^* and the optimal cash holding is similar to that of r_0, and will not be repeated here.

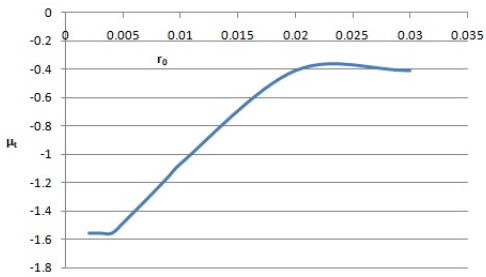


FIGURE 1. The influence of r_0 on μ_t^*

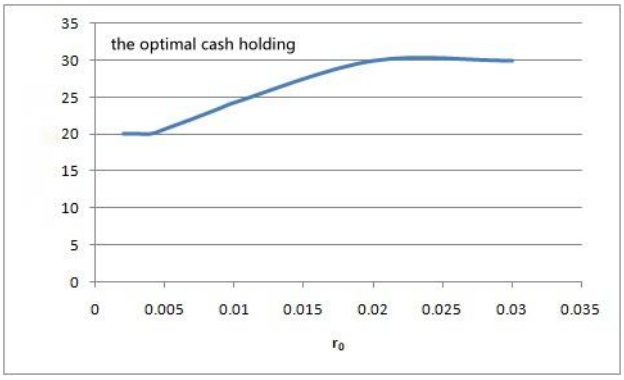


FIGURE 2. The influence of r_0 on the optimal cash holding

4.3. *The influence of c, θ on μ_t^* and the optimal cash holding*

For enterprises, c represents the minimum payment to meet the normal operation needs, and its different values affect m, l_1, l_2 and the size relationship among them. Supposing that $X_t = 50$, $\theta = 0.001$, $L = 200000$, $H = 300000$, $\beta_t = 0.3$, $r_0 = 0.002$, $r_1 = 0.1$. As can be seen from Table 2, when $c \in [0.1, 0.5]$, $l_1 < m < l_2$ and $x > l_2$. According to Theorem 3, $\mu_t^* = \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x}$. c affects μ_t^*, but has no effect on the optimal cash holding. When $c \in [1.1, 1.4]$, we can get $m < l_1 < l_2$ and $l_1 < x < l_2$. According to Theorem 2, $\mu_t^* = 1 - 2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x}$. In this case, μ_t^* and the optimal cash holding decrease with the increase of c. When $c \in [1.5, 1.8]$, $m < l_1 < l_2$ and $m < x < l_1$. According to Theorem 2, then $\mu_t^* = \frac{L+c+\theta x-\beta_t x}{(1-\beta_t)x}$. In this case, μ_t^* increases with the increase of c. However c does not affect the optimal cash holding at this time, and the optimal cash holding is L. When $c = 1.9$, $m < l_1 < l_2$ and $x < m$, bankruptcy has occurred.

TABLE 2. μ_t^* and the optimal cash holding under different c values

$c(\times 10^4)$	m	l_1	l_2	μ_t^*	the optimal cash holding
0.10	30.84	21.72	31.53	-0.99	30
0.50	35.28	30.12	39.93	-0.95	30
1.10	41.94	42.72	52.53	-1.14	27.42
1.20	43.05	44.82	54.63	-1.34	25.28
1.30	44.16	46.92	56.73	-1.55	23.14
1.40	45.27	49.02	58.83	-1.76	20.99
1.50	46.38	51.12	60.93	-1.85	20
1.60	47.49	53.22	63.03	-1.84	20
1.70	48.60	55.32	65.13	-1.83	20
1.80	49.71	57.42	67.23	-1.82	20
1.90	50.82	59.52	69.33	-	-

Next, we will discuss the influence of θ on μ_t^* and the optimal cash holding. At the beginning of this article, we assume that $0 \leq \theta < r_0$. The influence of θ on μ_t^* and optimal cash holding is shown in Table 3.

TABLE 3. μ_t^* and the optimal cash holding under different θ values

θ	m	l_1	l_2	μ_t^*	the optimal cash holding
0.001	30.84	21.72	31.53	-0.99	30
0.0011	35.32	30.18	40.01	-0.94	30
0.0012	42.03	42.90	52.75	-1.16	27.21
0.0013	43.09	45.10	54.98	-1.38	24.96
0.0014	44.36	47.32	57.21	-1.59	22.71
0.0015	45.52	49.54	59.45	-1.81	20.46
0.0016	46.69	51.77	61.71	-1.84	20
0.0017	47.86	54.01	63.97	-1.83	20
0.0018	49.04	56.27	66.24	-1.82	20
0.0019	50.21	58.53	69.53	-	-

When $\theta =\in [0.001, 0.0011]$, we can get $l_1 < m < l_2$ and $x > l_2$. According to Theorem 3, $\mu_t^* = \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x}$. In this case, μ_t^* is an increasing function of θ, but θ has no effect on the optimal cash holding. The optimal cash holding is H. When $\theta \in [0.0012, 0.0015]$, we can get $m < l_1 < l_2$ and $l_1 < x < l_2$. According to Theorem 2, $\mu_t^* = 1 - 2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x}$. At this time, μ_t^* and the optimal cash holding are decreasing functions of θ, and it is not difficult to verify that the optimal cash holding is in the safe area. When $\theta \in [0.0016, 0.0018]$, we can get $m < l_1 < l_2$ and $m < x < l_1$. According to Theorem 2, $\mu_t^* = \frac{L+c+\theta x-\beta_t x}{(1-\beta_t)x}$. In this case θ has no effect on the optimal cash holding. The optimal cash holding is L. When $\theta = 0.0019$, we can get $m < l_1 < l_2$ and $x < m$, bankruptcy has occurred.

As in Section 4.2, the effects of c and θ can also be shown graphically, but are omitted here due to space constraints.

4.4. *The influence of initial cash holding ratio β_t on the optimal cash holding*

β_t is the proportion of surplus wealth held in the form of cash assets at time t. It is not difficult to find that m, l_1, l_2 and the relationship among them have nothing to do with β_t. Supposing that $m < l_1 < l_2$ and $l_1 < x < l_2$, we can get $\mu_t^* = 1 - 2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x}$ according to Theorem 2. As can be seen from Table 4, μ_t^* is a positive value when

TABLE 4. μ_t^* and the optimal cash holding under different β_t values

β_t	m	l_1	l_2	μ_t^*	the optimal cash holding
0.1	43.05	44.82	54.63	0.48	25.28
0.2	43.05	44.82	54.63	0.41	25.28
0.3	43.05	44.82	54.63	0.33	25.28
0.4	43.05	44.82	54.63	0.22	25.28
0.5	43.05	44.82	54.63	0.06	25.28
0.6	43.05	44.82	54.63	-0.17	25.28
0.7	43.05	44.82	54.63	-0.56	25.28
0.8	43.05	44.82	54.63	-1.35	25.28
0.9	43.05	44.82	54.63	-3.69	25.28

$\beta_t \in [0.1, 0.5]$, which indicates that part of the risky assets have been converted into cash assets. When $\beta_t \in [0.6, 0.9]$, μ_t^* is negative, which means that part of the cash assets have been converted into risky assets. μ_t^* is a decreasing function of β_t, which can be seen from the calculation formula, Table 4 or Figure 4.4.

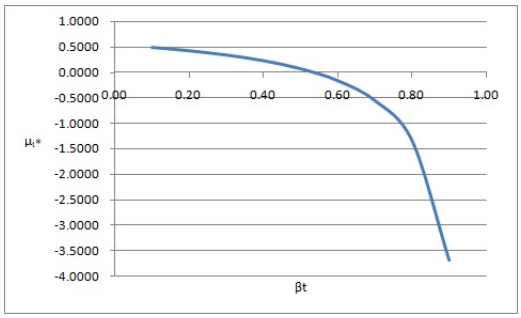


FIGURE 3. The influence of β_t on μ_t^*

5. Conclusion

In this paper, aiming at minimizing the ruin probability, we define the "bankruptcy" time as the time when the residual wealth first reaches the bankruptcy threshold m, and we also study the optimal cash holding decision problem under the security area constraint. The results of this model show that the optimal conversion strategy and the optimal cash holding are related to the consumption function parameters, capital market parameters and the initial cash holding ratio. The model can be regarded as a standard management model. On one hand, it describes the relationship between bankruptcy probability, residual wealth, cash and venture assets of investment objects involved in enterprise management. On the other hand, it can be used to describe the relationship between people, money and things in enterprise management. Although the model is established based on certain assumptions, it does not affect the management significance of the model itself.

References

[1] P. Azcue and N. Muler, "Optimal investment strategy to minimize the ruin probability of an insurance company under borrowing constraints," *Insur. Math. Econ.*, vol. 44, no. 1, pp. 26–34, 2009, doi: 10.1016/j.insmatheco.2008.09.006. [Online]. Available: citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.571.4224

[2] E. Bayraktar and V. R. Young, "Minimizing the probability of lifetime ruin under borrowing constraints," *Insur. Math. Econ.*, vol. 41, no. 1, pp. 196–221, 2007, doi: 10.1016/j.insmatheco.2006.10.015.

[3] X. Bi and S. Zhang, "Minimizing the risk of absolute ruin under a diffusion approximation model with reinsurance and investment," *J. Syst. Sci. Complex.*, vol. 28, no. 1, pp. 144–155, 2015, doi: 10.1007/s11424-015-2084-x.

[4] S. Browne, "Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin," *Math. Oper. Res.*, vol. 20, no. 4, pp. 937–958, 1995, doi: 10.1287/moor.20.4.937. [Online]. Available: semanticscholar.org/paper/3982ae7a8a1f182f07cd45e6ba7a31e3f0fa4a21

[5] S. Browne, "Survival and growth with a liability: optimal portfolio strategies in continuous time," *Math. Oper. Res.*, vol. 22, no. 2, pp. 468–493, 1997, doi: 10.1287/moor.22.2.468. [Online]. Available: semanticscholar.org/paper/fd7fc11b13119f0192b2ea341cfd3467b9db3325

[6] T. S. Ferguson, "Betting systems which minimize the probability of ruin," *J. Soc. Ind. Appl. Math.*, vol. 13, pp. 795–818, 1965, doi: 10.1137/0113051.

[7] W. H. Fleming and R. W. Rishel, *Deterministic and stochastic optimal control*, ser. Appl. Math. (N. Y.). Springer, New York, 1975, vol. 1.

[8] J. Gaier, P. Grandits, and W. Schachermayer, "Asymptotic ruin probabilities and optimal investment," *Ann. Appl. Probab.*, vol. 13, no. 3, pp. 1054–1076, 2003, doi: 10.1214/aoap/1060202834.

[9] C. Hipp and M. Plum, "Optimal investment for insurers," *Insur. Math. Econ.*, vol. 27, no. 2, pp. 215–228, 2000, doi: 10.1016/S0167-6687(00)00049-4.

[10] O. Kichmarenko and O. Stanzhytskyi, "Optimal control problems for some classes of functional-differential equations on the semi-axis," *Miskolc Math. Notes*, vol. 20, no. 2, pp. 1021–1037, 2019, doi: 10.18514/MMN.2019.2739.

[11] N. V. Krylov, *Controlled diffusion processes. Transl. by A. B. Aries*, ser. Appl. Math. (N. Y.). Springer, New York, 1980, vol. 14.

[12] Z. Liu and S. Zeng, "Differential variational inequalities in infinite Banach spaces," *Acta Math. Sci., Ser. B, Engl. Ed.*, vol. 37, no. 1, pp. 26–32, 2017, doi: 10.1016/S0252-9602(16)30112-6.
[13] S. Luo and M. Taksar, "On absolute ruin minimization under a diffusion approximation model," *Insur. Math. Econ.*, vol. 48, no. 1, pp. 123–133, 2011, doi: 10.1016/j.insmatheco.2010.10.004.
[14] Y. Luo and Z. Yang, "Research on optimal investment and utility indifference pricing model," 2011.
[15] A. A. Robichek and S. C. Myers, "Optimal financing decisions," 1965.
[16] H. Schmidli, "Optimal proportional reinsurance policies in a dynamic setting," *Scand. Actuar. J.*, vol. 2001, no. 1, pp. 55–68, 2001, doi: 10.1080/034612301750077338.
[17] H. Schmidli, "On minimizing the ruin probability by investment and reinsurance," *Ann. Appl. Probab.*, vol. 12, no. 3, pp. 890–907, 2002, doi: 10.1214/aoap/1031863173.
[18] M. I. Taksar and C. Markussen, "Optimal dynamic reinsurance policies for large insurance portfolios," *Finance Stoch.*, vol. 7, no. 1, pp. 97–121, 2003, doi: 10.1007/s007800200073.
[19] Z. Wang, G. Xu, P. Zhao, and Z. Lu, "The optimal cash holding models for stochastic cash management of continuous time," *J. Ind. Manag. Optim.*, vol. 14, no. 1, pp. 1–17, 2018, doi: 10.3934/jimo.2017034.
[20] V. R. Young, "Optimal investment strategy to minimize the probability of lifetime ruin." *N. Am. Actuar. J.*, vol. 8, no. 4, pp. 105–126, 2004, doi: 10.1080/10920277.2004.10596174.
[21] Y. Zeng and Z. Li, "Optimal investment strategy for insurers under linear constraint," *OR Trans.*, vol. 14, no. 2, pp. 106–118, 2010.
[22] T. Zhou, X. Liu, M. Hou, and C. Liu, "Numerical solution for ruin probability of continuous time model based on neural network algorithm," *Neurocomputing*, vol. 331, no. FEB.28, pp. 67–76, 2019.

Authors' addresses

Zhengyan Wang
(**Corresponding author**) School of Economics and Management, Yancheng Institute of Technology, Jiangsu, 224056 Yancheng, China
E-mail address: 85215774@qq.com

Yan Zhang
Department of General Education, Army Engineering University of PLA, Jiangsu, 211101 Nanjing, China
E-mail address: sdzyjyw@126.com

Peibiao Zhao
School of mathematcis and statistics, Nanjing University of Science and Technology, Jiangsu, 210094 Nanjing, China
E-mail address: pbzhao@mail.njust.edu.cn

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 1061–1073

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4505

THE AUSLANDER-REITEN CORRESPONDENCE ABOUT ∞-COTILTING OBJECTS IN EXTRIANGULATED CATEGORIES

ZHEN ZHANG AND SHANCE WANG

Received 29 November, 2022

Abstract. In this paper, we introduce ∞-cotilting objects in an extriangulated category. The definition given unifies those of Wakamatsu cotilting modules and of cotilting objects in an extriangulated category. We give a similar Bazzoni characterization and a partial Auslander-Reiten correspondence between ∞-cotilting objects and resolving subcategories in an extriangulated category, which recovers several different results from the literature. More importantly, extensions of the known results for Wakamatsu cotilting modules and for cotilting objects are very natural, but nontrivial, and we use a new method to prove them. At the same time, we give a dual version respect to the ∞-tilting objects.

2010 *Mathematics Subject Classification:* 18E30; 16D90; 16G10

Keywords: extriangulated category, ∞-cotilting object, Auslander-Reiten correspondence, Bazzoni characterization

1. Introduction

Let R be an artin algebra. A finitely generated (left) R-module W is called (n-)cotilting if $Ext_R^i(W,W) = 0$ for all $i \geq 1$, the injective dimension of W is at most n, and there is a long exact sequence $W_n \to \cdots \to W_1 \to W_0 \to DR \to 0$ with each W_i in $addW$ (the closure of W under finite direct sums and summands). The celebrated 'Auslander-Reiten correspondence' of [1] establishes a very natural bijection between (basic) cotilting modules and contravariantly finite resolving subcategories of the category of finitely generated R-modules. Note that there is also a dual version of this correspondence for tilting modules and covariantly finite coresolving subcategories.

One can also define (n-)tilting objects in the (big) module category of an arbitrary ring R. One of the many celebrated results in this direction is due to Bazzoni [2,

The first author was supported in part by the National Natural Science Foundation of China (No.12101344) and the Shan Dong Provincial Natural Science Foundation of China (No.ZR2015PA001).

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

Theorem 3.11], who showed that an R-module W is (n-)cotilting if and only if the following are equivalent for all R-modules M:

(1) There is an exact sequence $0 \to M \to W^0 \to \cdots \to W^n$;
(2) $\mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(\mathrm{M},\mathrm{W}) = 0$ for all $i \geq 1$.

In the decades since they first appeared, there have been many successful attempts to extend the 'Auslander-Reiten correspondence' and/or 'Bazzoni characterization' to various new contexts and levels of generality. One example are the Wakamatsu cotilting modules, where the injective dimension of W and resolution of injective cogenerator E by $addW$ are both allowed to be infinite, see [5, Theorem 2.10]. Another are cotilting objects in triangulated categories, see [3, Theorems 3.7 and 3.15].

In the recent seminal work [6], Nakaoka and Palu unified the axioms of exact (so as a special case abelian) and triangulated categories to define extriangulated categories. The theory they developed has inspired many efforts at unifying constructions which have been made independently in exact (or abelian) and triangulated categories. One example has been tilting objects (and more generally subcategories), for which a generalized definition, an Auslander-Reiten correspondence, and a Bazzoni characterization are all successfully given in [8, Theorems 1and 2].

Motivated by this idea, in this paper, we takes this a step further to introduce ∞-cotilting (and dually ∞-tilting) objects in extriangulated categories. The definition given unifies those of Wakamatsu cotilting modules and of cotilting objects in an extriangulated category. Our main aim is to establish an Auslander-Reiten correspondence and Bazzoni characterization for these new objects.

The definition and main results in this paper are very natural, but nontrivial, extensions of the known results for Wakamatsu cotilting modules and for cotilting objects in extriangulated categories. In particular, the proof techniques are new.

2. Preliminaries

Throughout the article, $\mathscr{C}$ denotes an additive category. All subcategories considered are full additive subcategories closed under isomorphisms. We denote by $\mathscr{C}(A,B)$ the set of morphisms from A to B in $\mathscr{C}$. If $f \in \mathscr{C}(A,B)$, $g \in \mathscr{C}(B,C)$, we denote composition of f and g by gf. We recall the definition and some basic properties of extriangulated categories from [6] and [8].

Suppose that $\mathscr{C}$ is equipped with a biadditive functor $\mathbb{E} : \mathscr{C}^{op} \times \mathscr{C} \to Ab$, where Ab is the category of abelian groups. For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C,A)$ is called an $\mathbb{E}$-*extension*. Zero element $\delta \in \mathbb{E}(C,A)$ is called the spilt $\mathbb{E}$-extension.

For any $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$, since $\mathscr{C}$ and $\mathbb{E}$ are additive, we can define the $\mathbb{E}$-extension

$$\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$$

Since $\mathbb{E}$ is a bifunctor, for any $a \in \mathscr{C}(A,A')$ and $c \in \mathscr{C}(C',C)$, we have $\mathbb{E}$-extensions

$$\mathbb{E}(C,a)(\delta) \in \mathbb{E}(C,A'), \qquad \mathbb{E}(c,A)(\delta) \in \mathbb{E}(C',A.$$

We abbreviate $\mathbb{E}(C,a)(\delta)$ and $\mathbb{E}(c,A)(\delta)$ to $a_*\delta$ and $c^*\delta$, respectively.

Definition 2.1. *[6, Definition 2.3] A morphism $(a,c) : \delta \to \delta'$ of $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$ is a pair of morphisms $a \in \mathscr{C}(A,A')$ and $c \in \mathscr{C}(C,C')$ satisfying $a_*\delta = c^*\delta'$.*

Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ in $\mathscr{C}$ are said to be equivalent if there exists an isomorphism $b \in \mathscr{C}(B,B')$ which makes the following diagram commutative.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, equal] & B \arrow[r, "y"] \arrow[d, "b", "\cong"'] & C \arrow[d, equal] \\ A \arrow[r, "x'"] & B' \arrow[r, "y'"] & C \end{tikzcd}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

For any $A,C \in \mathscr{C}$, we denote as $0 = [A \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} A \oplus C \xrightarrow{[0\ 1]} C]$.

Definition 2.2. *[6, Definition 2.9] Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C,A)$. This $\mathfrak{s}$ is called a* realization *of $\mathbb{E}$ if it satisfies the following condition.*

Let $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$ be any two $\mathbb{E}$-extension with $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. Then, for any morphism $(a,c) : \delta \to \delta'$, there exists a morphism $b \in \mathscr{C}(B,B')$ which makes the following diagram commutative.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \arrow[d, "c"'] \\ A \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \end{tikzcd}$$

In this case, we say the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes δ.

Remark that this condition does not depend on the choices of the representatives of the equivalence classes. In the above situation, we say the triplet (a,b,c) realizes (a,c).

Definition 2.3. *[6, Definition 2.10] Let $\mathscr{C}$, $\mathbb{E}$ be as above. A realization $\mathfrak{s}$ of $\mathbb{E}$ is said to be additive, if it satisfies the following conditions.*

(1) *for any $A,C \in \mathscr{C}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C,A)$ satisfies $\mathfrak{s}(0) = 0$;*
(2) *for any pair of $\mathbb{E}$-extension $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$, $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.*

Definition 2.4. *[6, Definition 2.12] We call the triplet $(\mathscr{C},\mathbb{E},\mathfrak{s})$ an externally triangulated category, or for short, extriangulated category if it satisfies the following conditions:*

(ET1) $\mathbb{E}:\mathscr{C}^{op}\times\mathscr{C}\to Ab$ *is a biadditive functor.*

(ET2) $\mathfrak{s}$ *is an additive realization of* $\mathbb{E}$.

(ET3) *Let* $\delta\in\mathbb{E}(C,A)$ *and* $\delta'\in\mathbb{E}(C',A')$ *be any pair of* $\mathbb{E}$*-extensions, realized as* $\mathfrak{s}(\delta)=[A\xrightarrow{x}B\xrightarrow{y}C]$ *and* $\mathfrak{s}(\delta')=[A'\xrightarrow{x'}B'\xrightarrow{y'}C']$. *For any commutative square in* $\mathscr{C}$,

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "a"] & B \arrow[r, "y"] \arrow[d, "b"] & C \\ A' \arrow[r, "x'"] & B' \arrow[r, "y'"] & C' \end{tikzcd}$$

there exists a morphism $(a,c):\delta\to\delta'$ *which is realized by* (a,b,c).

(ET3)^{op} *Dual of* $(ET3)$.

(ET4) *Let* (A,δ,D) *and* (B,δ',F) *be two* $\mathbb{E}$*-extensions realized by* $A\xrightarrow{f}B\xrightarrow{f'}D$ *and* $B\xrightarrow{g}C\xrightarrow{g'}F$, *respectively. Then there exist an object* $\mathbb{E}\in\mathscr{C}$, *a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "f'"] \arrow[d, "g"] & D \arrow[d, "d"] \\ A \arrow[r, "h"] & C \arrow[r, "h'"] \arrow[d, "g'"] & E \arrow[d, "e"] \\ & F \arrow[r, equal] & F \end{tikzcd}$$

in $\mathscr{C}$ *and an* $\mathbb{E}$*-extension* $\delta''\in\mathscr{C}(E,A)$ *realized by* $A\xrightarrow{h}C\xrightarrow{h'}E$, *which satisfy the following compatibilities:*

(i) $D\xrightarrow{d}E\xrightarrow{e}F$ *realizes* $\mathbb{E}(F,f')(\delta')$,

(ii) $\mathbb{E}(d,A)(\delta'')=\delta$,

(iii) $\mathbb{E}(E,f)(\delta'')=\mathbb{E}(e,B)(\delta')$.

$(ET4^{op})$ *Dual of* $(ET4)$.

For an extriangulated category $\mathscr{C}$, we use the following notation [6, 8].

- A sequence $A\xrightarrow{a}B\xrightarrow{b}C$ is called *conflation* if it realizes some $\mathbb{E}$-extension $\delta\in\mathbb{E}(C,A)$, in which case, the morphism a is called an *inflation*, the morphism b is called an *deflation* and we call $A\xrightarrow{a}B\xrightarrow{b}C\overset{\delta}{\dashrightarrow}$ an $\mathbb{E}$*-triangle*.
- Let $A\xrightarrow{a}B\xrightarrow{b}C\overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle, A is called the *CoCone* of the deflation b: $B\to C$, and denote it by $\mathrm{CoCone}(b)$; C is called the *Cone* of the inflation

$a\colon A \to B$, and denote it by $\mathrm{Cone}(a)$. Note that the *CoCone* of a deflation and the *Cone* of an inflation are well-defined by [6, Remark 3.10].

Remark 2.5. *Let $\mathscr{C}$ be an extriangulated category.*

(1) *[6, Remark 2.16] Both inflations and deflations are closed under composition.*
(2) *A subcategory $\mathscr{T}$ of $\mathscr{C}$ is called extension-closed if for any $\mathbb{E}$-triangle $A \xrightarrow{a} B \xrightarrow{b} C \overset{\delta}{\dashrightarrow}$ with $A,\ C \in \mathscr{T}$ we have $B \in \mathscr{T}$.*

The following condition is analogous to the weak idempotent completeness in exact category (see [6, Condition 5.8]).

(Condition (WIC)) Consider the following conditions.

(1) Let $f \in \mathscr{C}(A,B)$, $g \in \mathscr{C}(B,C)$ be any composable pair of morphisms. If gf is an inflation, then so is f.
(2) Let $f \in \mathscr{C}(A,B)$, $g \in \mathscr{C}(B,C)$ be any composable pair of morphisms. If gf is an deflation, then so is g.

Definition 2.6. *[6, Definition 3.23] Let $\mathscr{C}$ be an extriangulated category. An object I is called* injective *if for any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and any morphism $c \in \mathscr{C}(A,I)$, there exists a morphism $b \in \mathscr{C}(B,I)$, satisfying $b \circ x = c$.*

The subcategory consisting of injective (resp., projective) objects in $\mathscr{C}$ is denoted by $\mathscr{I}$ (resp., $\mathscr{P}$).

Definition 2.7. *Let $\mathscr{X}$ be a subcategories of $\mathscr{C}$ and E an object of $\mathscr{X}$. We call E is* an $\mathbb{E}$-injective cogenerator *of $\mathscr{X}$ if $\mathbb{E}(\mathscr{X},E) = 0$ and for any object $X \in \mathscr{X}$, there is an $\mathbb{E}$-triangle $X \to E^0 \to X^1 \dashrightarrow$ with $X^1 \in \mathscr{X}$ and $E^0 \in \mathrm{add}E$.*

The $\mathbb{E}$-projective generator *can be defined dually.*

In particular, we know that an $\mathbb{E}$-injective cogenerator of $\mathscr{C}$ is an injective object and an $\mathbb{E}$-projective generator of $\mathscr{C}$ is an projective object.

It is easy to show $\mathbb{E}$-injective cogenerator (resp., $\mathbb{E}$-projective generator) of $\mathscr{X}$ is unique under isomorphism.

Definition 2.8. *A subcategory $\mathscr{X} \subset \mathscr{C}$ is called* coresolving *if it contains $\mathscr{I}$, closed under extensions and Cones of inflations.* Resolving subcategory *can be defined dually.*

Definition 2.9. *[6, Definition 3.25] Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category. If for any object $A \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle $A \to I \to A_1 \overset{\delta}{\dashrightarrow}$, with $I \in \mathscr{I}$, then we say the extriangulated category $(\mathscr{C},\mathbb{E},\mathfrak{s})$ has enough injectives. Dually, if for any object $C \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle $C_1 \to P \to C \overset{\sigma}{\dashrightarrow}$, with $P \in \mathscr{P}$, then we say the extriangulated category $(\mathscr{C},\mathbb{E},\mathfrak{s})$ has enough projectives.*

Liu and Nakaoka [4, 5.1 and 5.2] defined the higher extension groups in an extriangulated category having enough projectives and injectives. They showed the following result.

Lemma 2.10. *[4, Proposition 5.2] Let $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle. For any object $X \in \mathscr{C}$, there are long exact sequences*

$$\begin{aligned}\cdots \to \mathbb{E}^i(X,A) &\xrightarrow{f_*} \mathbb{E}^i(X,B) \xrightarrow{g_*} \mathbb{E}^i(X,C)\\ &\to \mathbb{E}^{i+1}(X,A) \xrightarrow{f_*} \mathbb{E}^{i+1}(X,B) \xrightarrow{g_*} \mathbb{E}^{i+1}(X,C) \to \cdots\\ \cdots \to \mathbb{E}^i(C,X) &\xrightarrow{f^*} \mathbb{E}^i(B,X) \xrightarrow{g^*} \mathbb{E}^i(A,X)\\ &\to \mathbb{E}^{i+1}(C,X) \xrightarrow{f^*} \mathbb{E}^{i+1}(B,X) \xrightarrow{g^*} \mathbb{E}^{i+1}(C,X) \to \cdots\end{aligned}$$

In particularly, there exist long exact sequences

$$\mathscr{C}(X,A) \xrightarrow{\mathscr{C}(X,f)} \mathscr{C}(X,B) \xrightarrow{\mathscr{C}(X,g)} \mathscr{C}(X,C) \xrightarrow{(\delta_\sharp)_X} \mathbb{E}(X,A) \xrightarrow{f_*} \mathbb{E}(X,B) \xrightarrow{g_*} \mathbb{E}(X,C) \to \cdots$$

$$\mathscr{C}(C,X) \xrightarrow{\mathscr{C}(g,X)} \mathscr{C}(B,X) \xrightarrow{\mathscr{C}(f,X)} \mathscr{C}(A,X) \xrightarrow{(\delta^\sharp)_X} \mathbb{E}(C,X) \xrightarrow{g^*} \mathbb{E}(B,X) \xrightarrow{f^*} \mathbb{E}(A,X) \to \cdots$$

For a subcategory $\mathscr{X} \subseteq \mathscr{C}$, we define

$$\mathscr{X}^{\perp} = \{Y \in \mathscr{C} \mid \mathbb{E}^i(X,Y) = 0, \forall i \geq 1,\ X \in \mathscr{X}\}.$$

and

$${}^{\perp}\mathscr{X} = \{Y \in \mathscr{C} \mid \mathbb{E}^i(Y,X) = 0, \forall i \geq 1,\ X \in \mathscr{X}\}.$$

Definition 2.11. *A subcategory $\mathscr{X} \subseteq \mathscr{C}$ is called self-orthogonal provided that*

$$\mathscr{X} \subseteq \mathscr{X}^{\perp}$$

Definition 2.12. *[8, lemma 2] An $\mathbb{E}$-triangle sequence in $\mathscr{C}$ is defined as a sequence*

$$\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

such that for any n, there are $\mathbb{E}$-triangles $K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \overset{\delta_n}{\dashrightarrow}$ and the differential $d_n = g_{n-1}f_n$, where g_n is an inflation and f_n is a deflation.

The symbol $\hat{\mathscr{X}}_n$ (resp., $\check{\mathscr{X}}_n$) denotes the subcategory of objects $A \in \mathscr{C}$ such that there exists an E-triangle sequence

$$X_n \to X_{n-1} \to \cdots \to X_0 \to A (\text{resp.}, A \to X^0 \to X^1 \to \cdots \to X^n)$$

with each X_i (resp., X^i) is contained in $\mathcal{X}$.

We denote the union of all $\hat{\mathscr{X}}_n$ (resp., $\check{\mathscr{X}}_n$) for all nonnegative n by $\hat{\mathscr{X}}$ (resp., $\check{\mathscr{X}}$).

3. ∞-COTILTING (RESP., TILTING) OBJECTS

In this section, we always assume that

(1) $\mathscr{C} = (\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category which admits products and direct sums;
(2) $\mathscr{C}$ admits an $\mathbb{E}$-injective cogenerator E and an $\mathbb{E}$-projective generator Q, see Definition 2.7;
(3) $\mathscr{C}$ satisfies Condition(WIC).

Let W be an object. Denote $addW$ by the closure of W under finite direct sums and summands. Firstly, we introduce two important subcategories which are contained in ${}^{\perp}W$ and $W^{\perp}$, respectively. Denote by ${}_W\mathcal{X}$ the subcategory of all objects $A \in W^{\perp}$ such that there is an infinite $\mathbb{E}$-triangle sequence in $\mathscr{C}$

$$\cdots \to W_2 \xrightarrow{d_2} W_1 \xrightarrow{d_1} W_0 \xrightarrow{d_0} A$$

with $W_i \in addW$ and $\mathrm{CoCone}(f_i) \in W^{\perp}$,for $i \geq 0$, where $f_0 = d_0$ and f_i is defined in Definition 2.12. Dually, denote by $\mathcal{X}_W$ the subcategory of all objects $A \in {}^{\perp}W$ such that there is an infinite $\mathbb{E}$-triangle sequence in $\mathscr{C}$

$$A \xrightarrow{d^0} W^0 \xrightarrow{d^1} W^1 \xrightarrow{d^2} W^2 \to \cdots$$

with $W^i \in addW$ and $\mathrm{Cone}(g^i) \in {}^{\perp}W$ for $i \geq 0$, where $g^0 = d^0$ and g^i is defined in Definition 2.12.

Lemma 3.1. *If W is a self-orthogonal object. Then*

(1) *the subcategory ${}_W\mathcal{X}$ is closed under extensions, direct summands , and Cones of inflations;*
(2) *the subcategory $\mathcal{X}_W$ is closed under extensions, direct summands , and CoCones of deflations.*

Proof. It follows from [8, Lemma 8]. □

Now, we give the definition of ∞-cotilting (resp., tilting) object in an extriangulated category.

Definition 3.2. *Let W be a non-zero object and E an $\mathbb{E}$-injective cogenerator of $\mathscr{C}$. We call W an ∞-cotilting object if the following conditions are satisfied:*

(1) *W is a self-orthogonal object in $\mathscr{C}$;*
(2) *$E \in {}_W\mathcal{X}$.*

If, moreover, W has injective dimension n for some positive integer n and $E \in (ad\hat{d}W)_n$, W is called *n-cotilting object*.

Definition 3.3. *Let T be a non-zero object and Q an $\mathbb{E}$-projective generator of $\mathscr{C}$. We call T an ∞-tilting object if the following conditions are satisfied:*

(1) *T is a self-orthogonal object in $\mathscr{C}$;*

(2) $Q \in \mathcal{X}_T$.

If, moreover, T has projective dimension n for some positive integer n and $Q \in (\check{addT})_n$, T is called *n-tilting object*.

Example 3.4. (1) *Let $\mathscr{C}$ be a triangulated category and W a n-cotilting (resp., n-tilting) object, then W is obvilusely an ∞-cotilting (resp. ∞-tilting) object with finite injective (resp., projective) dimension in our sense.*

(2) *Let $\mathscr{C}$ be an extriangulated category. If $addW$ is a cotilting (resp.,tilting) subcategory in the sense of [8, Definitions 8 and 7], then W is an ∞-cotilting (resp. ∞-tilting) object object with finite projective (resp., injective) dimension in our sense.*

(3) *Let Λ be an artin algebra. Consider the finitely generated left Λ-module category. Then W is Wakamatsu cotilting (resp., Wakamatsu tilting) in the sense of [5, section 2] if and only if W is ∞-cotilting (resp., ∞-tilting) object in our sense.*

Proof. We only prove (3). Let W be an ∞-cotilting object in our sense. Since $D\Lambda$ is an injective module, $D\Lambda \in {}_W\mathcal{X}$ by Definition 3.2. So W is a Wakamatsu cotilting module. Conversely, let W be a Wakamatsu cotilting module. By [5, Propositions 2.1 and 2.2], we have ${}_W\mathcal{X} = S^{\perp}$ for some left Λ module S. So for any injective Λ module E, we have $E \in S^{\perp} = {}_W\mathcal{X}$. Hence W is an ∞-cotilting object in our sense.

Dually, let W be an ∞-tilting object in our sense. Clearly $\Lambda \in \mathcal{X}_W$. So So W is a Wakamatsu tilting module. Conversely, let W be a Wakamatsu tilting module. Since $\mathscr{P} \subseteq add\Lambda$, $\mathscr{P} \subseteq \mathcal{X}_W$ by lemma 3.1. So W is an ∞-tilting object in our sense. $\square$

By Definition 3.2, if W is an ∞-cotilting object, there exists an $\mathbb{E}$-triangle sequence

$$\cdots \to W_1 \xrightarrow{d_1} W_0 \xrightarrow{d_0} E \qquad (\dagger)$$

with $W_i \in addW$.

By Definition 2.12 we can break it into several $\mathbb{E}$-triangle:

$$K_i \xrightarrow{g_i} W_i \xrightarrow{f_i} K_{i-1} \dashrightarrow$$

where $d_i = g_{i-1}f_i$ for all $i \geq 1$.

In the following we always denote $\mathrm{CoCone}(d_i)$ by K_i for each $i \geq 1$ and denote $\mathrm{CoCone}(d_0)$ by K_0, then $K_i \in W^{\perp}$ for all $i \geq 0$. Dually, if T is an ∞-tilting object, then there exists an $\mathbb{E}$-triangle sequence

$$Q \xrightarrow{d^0} T^0 \xrightarrow{d^1} T^1 \to \cdots \qquad (\ddagger)$$

with $T^i \in addT$.

In the following, we always denote $\mathrm{Cone}(d^i)$ by L^{i+1} for each $i \geq 0$ and denote $\mathrm{Cone}(d^0)$ by L^0, then $L^i \in {}^{\perp}T$ for all $i \geq 0$. Let $\mathscr{X}$ be a subcategory of $\mathscr{C}$ and A an object, a morphism $f: X_A \to A$ is called *right $\mathscr{X}$-approximation* of A if $X_A \in \mathscr{X}$ and

the morphism $\mathscr{C}(X,f):\ \mathscr{C}(X,X_A)\to\mathscr{C}(X,A)$ is epic for any object $X\in\mathscr{X}$. A *left $\mathscr{X}$-approximation* of A is defined dually.

From here to the end of this section, we always assume that any $M\in{}^{\perp}W$ admits a left $addW$-approximation and any $M\in W^{\perp}$ admits a right $addW$-approximation.

Lemma 3.5. *Let W be an object in $\mathscr{C}$.*

(1) *If W is an ∞-cotilting object. Then ${}^{\perp}(\prod_{i\in\mathbb{N}}K_i\oplus W)\subseteq\mathcal{X}_W$ for $i\geq 0$.*

(2) *If W is an ∞-tilting object. Then $(\bigoplus_{i\in\mathbb{N}}L^i\oplus W)^{\perp}\subseteq{}_W\mathcal{X}$ for $i\geq 0$.*

Proof. We only prove (1) and the proof of (2) is dually. Let $M\in{}^{\perp}(\prod_{i\in\mathbb{N}}K_i\oplus W)$, then $M\in{}^{\perp}W$ and $M\in{}^{\perp}K_i$ for each $i\geq 0$. Since E is the injective cogenerator for $\mathscr{C}$, there exists an $\mathbb{E}$-triangle $M\to E^m\to M_1\overset{\delta_M}{\dashrightarrow}$ for some nonnegative integer m by Definition 2.7. Since W is an ∞-tilting object, $E\in{}_W\mathcal{X}$. So there exist an $\mathbb{E}$-triangle sequence following from the $\mathbb{E}$-triangle sequence (†):

$$\cdots\to W_1^m\to W_0^m\to E^m$$

with $W_i\in addW$.

So by ET(3) we have the following commutative diagram:

$$\begin{tikzcd}
K_1^m \arrow[r, equal] \arrow[d] & K_1^m \arrow[d] & & \\
S \arrow[r] \arrow[d] & W_0^m \arrow[r] \arrow[d, "f_0"] & M_1 \arrow[r, dashed] \arrow[d, equal] & {} \\
M \arrow[r] \arrow[d, dashed] & E^m \arrow[r] \arrow[d, dashed, "\delta_0"] & M_1 \arrow[r, dashed, "\delta_M"] & {} \\
{} & {} & &
\end{tikzcd}$$

Since $M\in{}^{\perp}K_1$, the $\mathbb{E}$-triangle in the first column splits. So there exists an inflation $M\to S$. By the Condition (WIC)), there exists an inflation $y:\ M\to W_0^m$. By the assumption, M admits a left $addW$-approximation. Let $x:\ M\to X$ be a left $addW$-approximation. By [4, Proposition 1.20], we have an $\mathbb{E}$-triangle $M\overset{\binom{-x}{y}}{\to}X\oplus W_0^m\to M_1\dashrightarrow$. Since x is a left $addW$-approximation, we can easily show $\binom{-x}{y}$ is a left $addW$-approximation of M. So $M_1\in{}^{\perp_1}W$. Since $M\in{}^{\perp}W$ and $X\oplus W^m\in addW$, $M_1\in{}^{\perp}W$. Applying the functor $\mathscr{C}(M_1,-)$ to the $\mathbb{E}$-triangles $K_{i+1}\to W_i\to K_i$ which is following from (†) for each $i\geq 1$, respectively, we get

$$(1)\qquad \mathbb{E}^k(M_1,K_i)\cong\mathbb{E}^{k+1}(M_1,K_{i+1})$$

for any $k\geq 1$ and $i\geq 0$ by Lemma 2.10. Applying the functor $\mathscr{C}(-,K_{i+1})$ to the $\mathbb{E}$-triangle $M\to X\oplus W_0^m\to M_1\dashrightarrow$. Since $K_i\in W^{\perp}$ and $X\oplus W_0^m\in addW$, we get

$$(2)\qquad \mathbb{E}^{k+1}(M_1,K_{i+1})\cong\mathbb{E}^k(M,K_{i+1})$$

for any $k \geq 1$ and $i \geq 0$. Hence $\mathbb{E}^k(M_1, K_i) \cong \mathbb{E}^k(M, K_{i+1}) = 0$ for each $i \geq 0$ and $k \geq 1$. So $M_1 \in {}^{\perp}K_i$ and $M_1 \in {}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W)$. Repeating the same argument for M_1 in the $\mathbb{E}$-triangle $M \to X \oplus W_0^m \to M_1 \dashrightarrow$ and so on, we obtain that $M \in \mathcal{X}_W$. $\square$

By Lemma 3.5, we can get that ∞-cotilting object actually generalize at the same time both cotilting and tilting objects.

Corollary 3.6. *W is an ∞-cotilting object if and only if it is an ∞-tilting object.*

Proof. If W is an ∞-cotilting object, then W is self-orthogonal and ${}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W) \subseteq \mathcal{X}_W$ by Lemma 3.5. But $\mathscr{P} \subseteq {}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W)$. So $\mathscr{P} \subseteq \mathcal{X}_W$ and thus W is ∞-tilting. Conversely, when W is an ∞-tilting object, we have W is self-orthogonal and $(\bigoplus_{i\in\mathbb{N}} L^i \oplus W)^{\perp} \subseteq {}_W\mathcal{X}$. But $\mathscr{I} \subseteq (\bigoplus_{i\in\mathbb{N}} L^i \oplus W)^{\perp}$. So $\mathscr{I} \subseteq {}_W\mathcal{X}$ and thus W is ∞-cotilting. $\square$

When W is an n-cotilting object for some positive integer, there exist an $\mathbb{E}$-triangle sequence $W_n \to W_{n-1} \to \cdots \to W_0 \to E$ with $W_i \in addW$. Applying $\mathscr{C}(M, -)$ for any $M \in {}^{\perp}W$ to every $\mathbb{E}$-triangle $K_{i+1} \to W_{i+1} \to K_i \dashrightarrow$ for $0 \leq i \leq n-1$ and $K_n = W_n$. We get $\mathbb{E}^j(M, K_i) = 0$ for any $0 \leq i \leq n$ and $j \geq 1$. So ${}^{\perp}(\prod_{i=0}^{n} K_i \oplus W) = {}^{\perp}W$.

Hence it is reasonable to use ${}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W)$ instead of ${}^{\perp}W$ when W is an ∞-cotilting object. The dual statement holds for n-tilting objects.

So we can prove the similar Bazzoni characterization of ∞-cotilting (tilting) objects in an extriangulated category.

Theorem 3.7. *Let W be a self-orthogonal object. Then*

(1) *W is an ∞-cotilting object if and only if $\mathcal{X}_W = {}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W)$;*
(2) *T is an ∞-tilting object if and only if $(\bigoplus_{i\in\mathbb{N}} L^i \oplus T)^{\perp} = {}_T\mathcal{X}$.*

Proof. We only prove (1) and the proof of (2) is dually. For the if part, clearly $\mathscr{P} \subseteq {}^{\perp}(\prod_{i\in\mathbb{N}} K_i \prod W)$. Hence $\mathscr{P} \subseteq \mathcal{X}_W$. So W is an ∞-tilting object by Definition 3.3. And thus W is ∞-cotilting by Corollary 3.6.

For the only if part, since W is an ∞-cotilting object, ${}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W) \subseteq \mathcal{X}_W$ by Lemma 3.5. Let $M \in \mathcal{X}_W$, then $M \in {}^{\perp}W$ and there exists an $\mathbb{E}$-triangle $M \xrightarrow{h^0} X^0 \to M^1 \dashrightarrow$ with $X^0 \in addW$ and $M^1 \in \mathcal{X}_W$ by the argument at the beginning of Section 3. We only need to show $M \in {}^{\perp}K_i$ for each $i \geq 0$.

Since W is ∞-cotilting, there exist an $\mathbb{E}$-triangle sequence $\cdots \to W_1 \xrightarrow{d_1} W_0 \xrightarrow{d_0} E$ with $W_i \in addW$ and $K_i = \mathrm{CoCone}(d_i) \in W^{\perp}$ for any $i \geq 0$ by Definition 3.1. For any morphism $g \in \mathscr{C}(M, E)$. Since E is an $\mathbb{E}$-injective object, there exists a morphism $a \in \mathscr{C}(X^0, E)$ such that $g = ah^0$. Applying the functor $\mathscr{C}(X^0, -)$ to the $\mathbb{E}$-triangle $K_0 \to W_0 \to E \dashrightarrow$, we get an exact sequence

$$\mathscr{C}(X^0, W_0) \xrightarrow{\mathscr{C}(X^0, d_0)} \mathscr{C}(X^0, E) \to \mathbb{E}(X^0, K_0).$$

Since $K_0 \in W^{\perp}$, the morphism $\mathscr{C}(X^0, d_0)$ is epic. Thus there exists a morphism $b \in \mathscr{C}(X^0, W_0)$ such that $a = d_0 b$. So we get a morphism $h^0 b \in \mathscr{C}(M, W^0)$ such that

$g = (d_0 b)h^0$. Thus $\mathscr{C}(M, d_0)$ is epic.

$$\begin{tikzcd}
 & M \arrow[r, "h^0"] \arrow[d, dashed, "h^0 b"'] \arrow[dr, "g"] & X^0 \arrow[r] \arrow[d, dashed, "\exists a"] & M^1 \arrow[r, dashed] & {} \\
K_0 \arrow[r] & W_0 \arrow[r, "d_0"] \arrow[ur, dashed, "\exists b"] & E \arrow[r, dashed] & {} &
\end{tikzcd}$$

Applying the functor $\mathscr{C}(M, -)$ to the $\mathbb{E}$-triangle $K_0 \to W_0 \to E \dashrightarrow$, we get an exact sequence

$$\mathscr{C}(M, W_0) \xrightarrow{\mathscr{C}(M, d_0)} \mathscr{C}(M, E) \to \mathbb{E}(M, K_0) \to \mathbb{E}(M, W_0).$$

Since $M \in {}^{\perp}W$, we get $\mathbb{E}(M, K_0) = 0$, i.e. $M \in {}^{\perp_1}K_0$. Thus $M \in {}^{\perp}K_0$. Since $M^1 \in \mathcal{X}_W \subseteq {}^{\perp}W$, we can use the same argument to M_1. So we get $M_1 \in {}^{\perp}K_0$. Moreover, by applying the functor $\mathscr{C}(-, K_{t+1})$ to the $\mathbb{E}$-triangle $M \to X^0 \to M^1 \dashrightarrow$ and the functor $\mathscr{C}(M^1, -)$ to the $\mathbb{E}$-triangle $K_{t+1} \to W_{t+1} \to K_t \dashrightarrow$, we get

$$\mathbb{E}^j(M^1, K_t) \cong \mathbb{E}^{j+1}(M^1, K_{t+1}) \cong \mathbb{E}^j(M, K_{t+1})$$

for any $t \geq 0$ and $j \geq 1$. So $\mathbb{E}^j(M^1, K_t) \cong \mathbb{E}^j(M, K_{t+1})$ for any $t \geq 0$ and $j \geq 1$. By the induction, we conclude that $\mathbb{E}^j(M, K_t) = 0$ for any $t \geq 0$ and $j \geq 1$. Hence $M \in {}^{\perp}K_t$. So $M \in {}^{\perp}(\prod_{i \geq 0} K_i)$. And the result follows. □

In [3, Theorem 3.15], the authors obtained the Auslander-Reiten correspondence for tilting objects in an triangulated category. In that setting, $\mathscr{X}$ is a coresolving subcategory with an $\mathbb{E}$-projective generator such that $\mathscr{C} = \check{\mathscr{X}}_n$ and the latter condition is actually essential to prove the Auslander-Reiten correspondence. In the tilting case, the equality $\mathscr{C} = \check{\mathscr{T}}_n$ follows from the fact that $\mathscr{T}$ has finite projective dimension, which fails in general for the case of the projective dimension is infinite, see [7, Example 3.1] in mod$R$, where R is an artin algebra.

Conscious of this central difference between the two contexts, we can only prove a partial Auslander-Reiten correspondence for ∞-tilting (cotilting) objects in an ex-triangulated category.

Proposition 3.8. *Assume that W is an object of $\mathscr{C}$. Then the association $\phi : W \longrightarrow \mathcal{X}_W$ is an injective map between the isomorphism classes of ∞-cotilting objects and resolving subcategories with an $\mathbb{E}$-injective cogenerator.*

Proof. By the definition of the notation $\mathcal{X}_W$ at the beginning of Section 2 and Definition 2.7, we know W is an $\mathbb{E}$-injective cogenerator of $\mathcal{X}_W$. By Lemma 3.5, $\mathscr{P} \subseteq \mathcal{X}_W$. Since W is an ∞-cotilting object, $\mathcal{X}_W$ is closed under extensions and CoCone of deflations by Lemma 3.1. So $\mathcal{X}_W$ is a resolving subcategory by Definition 2.8. Since $\mathbb{E}$-injective cogenerator is unique under isomorphism, ϕ is injective. □

Proposition 3.9. *There exists a correspondence ψ between the class of resolving subcategories $\mathscr{X}$ with an $\mathbb{E}$-injective cogenerator and the isomorphism classes of ∞-cotilting objects, given by $\psi : \mathscr{X} \longrightarrow W$, where $\mathrm{add}W = \mathscr{X} \cap \mathscr{X}^{\perp}$.*

Proof. Assume W_1 is an $\mathbb{E}$-injective cogenerator of $\mathscr{X}$. Then $\mathbb{E}^i(X,W_1)=0$ for any $X\in\mathscr{X}$ and $i\geq 1$. Thus $addW_1\subseteq\mathscr{X}\cap\mathscr{X}^{\perp}$. Conversely, let $A\in\mathscr{X}\cap\mathscr{X}^{\perp}$. Then there exists an $\mathbb{E}$-triangle $A\to W_1^0\to B\overset{\delta}{\dashrightarrow}$ with $W_1^0\in addW_1$ and $B\in\mathscr{X}$. Since $A\in\mathscr{X}^{\perp}$, $\mathbb{E}(B,A)=0$. So $A\in addW_1$. Hence $addW_1=\mathscr{X}\cap\mathscr{X}^{\perp}=addW$. Thus W is an $\mathbb{E}$-injective cogenerator of $\mathscr{X}$. So $\mathscr{X}\subseteq\mathcal{X}_W$. Since $\mathscr{X}$ is a resolving subcategory, $\mathscr{P}\subseteq\mathscr{X}$. And thus $\mathscr{P}\subseteq\mathcal{X}_W$. Also W is self-orthogonal, W is an ∞-tilting objects by Definition 3.3. So W is an ∞-cotilting object by Corollary 3.6. □

We can collect the results in Propositions 3.8 and 3.9 and obtain the following partial Auslander-Reiten correspondence for ∞-cotilting (tilting) objects in an extriangulated category.

Theorem 3.10.

(1) *There is an inverse bijection between classes of ∞-cotilting objects W and resolving subcategories $\mathscr{X}$ with an $\mathbb{E}$-injective cogenerator, maximal among those with the same $\mathbb{E}$-injective cogenerator, and the assignments are ϕ: $W\mapsto\mathcal{X}_W$ and $\psi:\mathscr{X}\mapsto W$ with $addW=\mathscr{X}\cap\mathscr{X}^{\perp}$.*

(2) *There is an inverse bijection between classes of ∞-tilting objects T and coresolving subcategories $\mathscr{Y}$ with an $\mathbb{E}$-projective generator, maximal among those with the same $\mathbb{E}$-projective generator, and the assignments are ϕ: $T\mapsto {}_T\mathcal{X}$ and ψ: $\mathscr{Y}\mapsto T$ with $addT={}^{\perp}\mathscr{Y}\cap\mathscr{Y}$.*

Proof. We only prove (1) and the proof of (2) is dually.

Let $\mathscr{X}$ be any resolving subcategory with an $\mathbb{E}$-injective cogenerator W, then $\mathscr{X}\subseteq\mathcal{X}_W=(\phi\circ\psi)(\mathscr{X})$ by Propositions 3.8 and 3.9. Thus, for any ∞-cotilting objects W, $\phi(W)$ is maximal among those resolving subcategories with the same $\mathbb{E}$-injective cogenerator W.

Conversely, if $\mathscr{X}$ is a subcategory maximal among those with the previous properties, then W with $addW={}^{\perp}\mathscr{X}\cap\mathscr{X}$ is an $\mathbb{E}$-injective cogenerator of $\mathscr{X}$ and $\psi(\mathscr{X})=W$ is an ∞-cotilting object by Proposition 3.9. So W is an $\mathbb{E}$-injective cogenerator of $\mathcal{X}_W$. Hence $\mathcal{X}_W\subseteq\mathscr{X}$ and $\mathscr{X}=(\phi\circ\psi)(\mathscr{X})$. □

References

[1] M. Auslander and I. Reiten, "Applications of contravariantly finite subcategories," *Adv. Math.*, vol. 86, no. 1, pp. 111–152, 1991, doi: 10.1016/0001-8708(91)90037-8.

[2] S. Bazzoni, "A characterization of n-cotilting and n-tilting modules." *J. Algebra*, vol. 273, no. 1, pp. 359–372, 2004, doi: 10.1016/S0021-8693(03)00432-0.

[3] Y. Hu, H. Yao, and X. Fu, "Tilting objects in triangulated categories," *Commun. Algebra*, vol. 48, no. 1, pp. 410–429, 2020, doi: 10.1080/00927872.2019.1648646.

[4] Y. Liu and H. Nakaoka, "Hearts of twin cotorsion pairs on extriangulated categories," *J. Algebra*, vol. 528, pp. 96–149, 2019, doi: 10.1016/j.jalgebra.2019.03.005.

[5] F. Mantese and I. Reiten, "Wakamatsu tilting modules." *J. Algebra*, vol. 278, no. 2, pp. 532–552, 2004, doi: 10.1016/j.jalgebra.2004.03.023.

[6] H. Nakaoka and Y. Palu, "Extriangulated categories, Hovey twin cotorsion pairs and model structures," *Cah. Topol. Géom. Différ. Catég.*, vol. 60, no. 2, pp. 117–193, 2019. [Online]. Available: cahierstgdc.com/wp-content/uploads/2019/04/Nakaoka_Palu-LX-2.pdf

[7] T. Wakamatsu, "Stable equivalence for self-injective algebras and a generalization of tilting modules," *J. Algebra*, vol. 134, no. 2, pp. 298–325, 1990, doi: 10.1016/0021-8693(90)90055-S.

[8] B. Zhu and X. Zhuang, "Tilting subcategories in extriangulated categories," *Front. Math. China*, vol. 15, no. 1, pp. 225–253, 2020, doi: 10.1007/s11464-020-0811-7.

Authors' addresses

Zhen Zhang
Qilu Normal university, Department of Mathematics, Ji Nan 250200, China
E-mail address: zhangzhenbiye@163.com

Shance Wang
(**Corresponding author**) Qilu Normal university, Department of Mathematics, Ji Nan 250200, China
E-mail address: 157642043@qq.com

Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 1075–1084

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4397

NONDECREASING BOUNDED CONTINUOUS SOLUTIONS OF A q-DIFFERENCE EQUATION

HOU YU ZHAO AND SHAN SHAN GUO

Received 03 September, 2022

Abstract. Schauder's fixed point theorem and Banach contraction principle are used to study a q-difference equation. We give sufficient conditions for the existence, uniqueness, and stability of the nondecreasing bounded continuous solutions. We also give the approximate sequences for the corresponding solutions. Finally, some examples are considered for our results.

2010 *Mathematics Subject Classification:* 39B12; 37E05

Keywords: q-difference equation, nondecreasing bounded continuous solutions, fixed point

1. Introduction

The study about q-difference equations has a long history. For example, the linear ordinary q-difference equations have been investigated in the beginning of the 20 century by Birkhoff [4, 5], Carmichael [6], Jackson [8, 9], Adams [1], Trjitzinsky [16], Mason [11], and other authors [2, 3, 7, 10, 12, 13, 17, 18]. However, since the late 1940s, the theory has been relatively little researched. In the last 20 years the field has recovered its original vitality and the theory of q-difference equations or more generally functional equations has witnessed substantial advances. See, for example, [15, 19]. Recently, Si and Zhang [14] studied the existence of analytic solutions of the nonlinear q-difference equation

$$G\Big(f(x), f(qx), \dots, f(q^n x)\Big) = 0,$$

where f is an unknown function and $G(x)$ is given function.

This work was partially supported by Science and Technology Research Program of Chongqing Municipal Education Commission (Grant No. KJQN 202000536), the Natural Science Foundation of Chongqing(Grant No. cstc2020jcyj-msxmX0606), Research Project of Chongqing Education Commission (Grant No. CXQT21014), Innovation and Entrepreneurship Support Program of Chongqing (Grant No. cx2023094).

© 2024 The Author(s). Published by Miskolc University Press. This is an open access article under the license CC BY 4.0.

In this note, we will be concerned with

$$G\Big(f(x), f(qx), \dots, f(q^n x)\Big) = F(x), \tag{1.1}$$

where f is an unknown function, $F(x)$ and $G(x)$ are given functions. By means of the Schauder's fixed point theorem and Banach contraction principle, we discuss the existence, uniqueness and stability of nondecreasing bounded continuous solutions of equation (1.1). Furthermore, we consider the approximate solutions sequence for the corresponding solutions.

Let $M \geq 1 \geq m \geq 0$ and $C(I)$ consist of all continuous functions on $I = [a,b]$. Define

$$\begin{aligned}\Phi(I;m,M) = \{f \in C(I) : f(a) = a, f(b) = b, a \leq f(x) \leq b,\\ m(x-y) \leq f(x) - f(y) \leq M(x-y),\ \forall x,y \in I, x \geq y\}.\end{aligned}$$

Clearly, $C(I)$ is a real Banach space with respect to the uniform norm

$$\|f\| = \max\{|f(x)| : x \in I\} \quad \text{for } f \in C(I).$$

In fact, it is easy to check that $\Phi(I;m,M)$ is a metric space under the uniform norm $\|f\|$. Furthermore, suppose that the sequence $\{f_n\}_{n=1}^{\infty}$ in $\Phi(I;m,M)$ has a limit $\widetilde{f} \in C(I)$. Noting

$$\widetilde{f}(x) - \widetilde{f}(y) = (\widetilde{f}(x) - f_n(x)) + (f_n(x) - f_n(y)) + (f_n(y) - \widetilde{f}(y)), \qquad x \geq y,$$

taking $n \to \infty$, we have

$$m(x-y) \leq \widetilde{f}(x) - \widetilde{f}(y) \leq M(x-y), \qquad \forall x,y \in I,\ x \geq y$$

and

$$\widetilde{f}(a) = \lim_{n\to\infty} f_n(a) = a, \qquad \widetilde{f}(b) = \lim_{n\to\infty} f_n(b) = b.$$

Thus, we have $\widetilde{f} \in \Phi(I;m,M)$ and $\Phi(I;m,M)$ is a a complete metric space. In the class of C^1 functions the conditions in the definition of $\Phi(I;m,M)$ coincide with $m \leq f'(x) \leq M$.

The rest of the paper is organized as follows. In Section 2, we give the existence of nondecreasing bounded continuous solutions of Eq. (1.1) under the monotonicity assumption. Section 3 deals with the uniqueness and stability of those solutions. The final section presents some examples.

2. Nondecreasing bounded continuous solutions

In this section, the existence of a nondecreasing bounded continuous solutions of Eq. (1.1) will be proved. Let us give some lemmas which will be used to prove our theorem.

Lemma 1 ([21, Lemma 1]). *$\Phi(I;m,M)$ is a compact convex subset of $C(I)$.*

Lemma 2 ([20, Lemma 2]). *Suppose that $f,g \in \Phi(I;m,M)$, where $M \geq 1 \geq m > 0$, then the following inequalities hold:*

(i) $\|f^k - g^k\| \leq \sum_{j=0}^{k-1} M^j \|f-g\|$, $k \in \mathbb{Z}^+$.
(ii) $\|f-g\| \leq M \left\|f^{-1} - g^{-1}\right\|$.
(iii) $\left\|f^{-1} - g^{-1}\right\| \leq m^{-1} \|f-g\|$.
(iv) $f^{-1} \in \Phi(I;M^{-1},m^{-1})$.

Now, we shall consider (1.1) on $I = [0,b]$ under the following assumptions.

(H1) $G(x_1,x_2,\ldots,x_{n+1}) \in C(I^{n+1},I)$, $G(0,x_2,\ldots,x_{n+1}) = 0, G(b,x_2,\ldots,\ x_{n+1}) = b$.
(H2) There exist $L_i \geq l_i \geq 0$ such that, for all $x_i,y_i \in I, x_i \geq y_i, i = 1,2,\ldots,n+1$,

$$\sum_{i=1}^{n+1} l_i(x_i - y_i) \leq G(x_1,x_2,\ldots,x_{n+1}) - G(y_1,y_2,\ldots,y_{n+1}) \leq \sum_{i=1}^{n+1} L_i(x_i - y_i).$$

Remark 1. Taking $x_i = b, y_i = 0, i = 1,2,\ldots,n+1$, we see

$$b\sum_{i=1}^{n+1} l_i \leq G(b,\ldots,b) - G(0,\ldots,0) \leq b\sum_{i=1}^{n+1} L_i,$$

i.e., $\sum_{i=1}^{n} l_i \leq 1 \leq \sum_{i=1}^{n} L_i$.

Theorem 1. *Suppose that* (H1) *and* (H2) *hold, $F \in \Phi(I;m_F,M_F)$, $0 < m_F \leq 1 \leq M_F$. If inequality*

$$0 < m \leq \frac{m_F}{L_1 + M_0} \leq 1 \leq \frac{M_F}{l_1 - M_1} \leq M \tag{2.1}$$

holds for constants $M_0 = \frac{M}{m}\sum_{i=1}^{n} q^i L_{i+1}, M_1 = \frac{m}{M}\sum_{i=1}^{n} q^i l_{i+1}$, where $q \in (0,1)$. Then Eq. (1.1) *has a solution $f \in \Phi(I;m,M)$ with $0 < m \leq 1 \leq M$.*

Proof. Define a mapping $T\colon \Phi(I;m,M) \to C(I)$ by

$$Tf(x) = G\Big(x, f(qf^{-1}(x)), \ldots, f(q^n f^{-1}(x))\Big), \qquad \forall x \in I.$$

By (H1) and (H2) we know Tf is nondecreasing and $Tf(0)=0, Tf(b)=b$. For any $x,y\in I$ with $x\geq y$, from Lemma 2, we can check that

$$\begin{aligned}
&Tf(x)-Tf(y)\\
&=G\Big(x,f(qf^{-1}(x)),\dots,f(q^nf^{-1}(x))\Big)-G\Big(y,f(qf^{-1}(y)),\dots,f(q^nf^{-1}(y))\Big)\\
&\leq L_1(x-y)+L_2\Big(f(qf^{-1}(x))-f(qf^{-1}(y))\Big)+\cdots\\
&\quad+L_{n+1}\Big(f(q^nf^{-1}(x))-f(q^nf^{-1}(y))\Big)\\
&\leq L_1(x-y)+\frac{qM}{m}L_2(x-y)+\cdots+\frac{q^nM}{m}L_{n+1}(x-y)\\
&=\Big(L_1+\frac{M}{m}\sum_{i=1}^{n}q^iL_{i+1}\Big)(x-y)=(L_1+M_0)\,(x-y)
\end{aligned}$$

and

$$\begin{aligned}
&Tf(x)-Tf(y)\\
&\geq l_1(x-y)+l_2\Big(f(qf^{-1}(x))-f(qf^{-1}(y))\Big)+\cdots\\
&\quad+l_{n+1}\Big(f(q^nf^{-1}(x))-f(q^nf^{-1}(y))\Big)\\
&\geq l_1(x-y)-\frac{qm}{M}l_2(x-y)-\cdots-\frac{q^nm}{M}l_{n+1}(x-y)\\
&=\Big(l_1-\frac{m}{M}\sum_{i=1}^{n}q^il_{i+1}\Big)(x-y)=(l_1-M_1)(x-y),
\end{aligned}$$

where $M_0=\frac{M}{m}\sum_{i=1}^{n}q^iL_{i+1}, M_1=\frac{m}{M}\sum_{i=1}^{n}q^il_{i+1}$.

Theeqrefore,

$$0<(l_1-M_1)(x-y)\leq Tf(x)-Tf(y)\leq(L_1+M_0)(x-y),\qquad\forall x\geq y\in I.$$

So Tf is invertible and

$$\frac{1}{L_1+M_0}(x-y)\leq(Tf)^{-1}(x)-(Tf)^{-1}(y)\leq\frac{1}{l_1-M_1}(x-y),\qquad\forall x\geq y\in I, \tag{2.2}$$

$(Tf)^{-1}(0)=0, (Tf)^{-1}(b)=b, 0\leq(Tf)^{-1}(x)\leq b, \forall x\in I$ and $(Tf)^{-1}$ is increasing. Obviously, Tf is a homeomorphism from $I=[0,b]$ to itself.

Define a mapping $\mathcal{T}\colon\Phi(I;m,M)\to C(I)$ by

$$\mathcal{T}f(x)=(Tf)^{-1}\circ F(x). \tag{2.3}$$

Clearly, $\mathcal{T}f(0)=0, \mathcal{T}f(b)=b$ and $0\le \mathcal{T}f(x)\le b$. From (2.2) and (2.1), for $x\ge y\in I$, we have

$$\mathcal{T}f(x)-\mathcal{T}f(y)\le \frac{M_F}{l_1-M_1}(x-y)\le M(x-y)$$

and

$$\mathcal{T}f(x)-\mathcal{T}f(y)\ge \frac{m_F}{L_1+M_0}(x-y)\ge m(x-y),$$

implying that $\mathcal{T}$ is a self-mapping on $\Phi(I;m,M)$. For $f,g\in\Phi(I;m,M)$, by Lemma 2,

$$\begin{aligned}
\|\mathcal{T}f-\mathcal{T}g\| &\le \left\|(Tf)^{-1}-(Tg)^{-1}\right\| \\
&\le \frac{1}{l_1-M_1}\|Tf-Tg\| \\
&\le \frac{1}{l_1-M_1}\Bigg(L_2\Big(f(qf^{-1}(x))-g(qg^{-1}(x))\Big)+\cdots \\
&\qquad\qquad +L_{n+1}\Big(f(q^nf^{-1}(x))-g(q^ng^{-1}(x))\Big)\Bigg) \\
&\le \Big(\sum_{i=1}^{n}L_{i+1}\Big)\|f-g\|+M\Big(\sum_{i=1}^{n}q^iL_{i+1}\Big)\|f^{-1}-g^{-1}\| \\
&\le \left(\sum_{i=1}^{n}L_{i+1}+\frac{M}{m}\sum_{i=1}^{n}q^iL_{i+1}\right)\|f-g\| \\
&= (M_0+\sum_{i=1}^{n}L_{i+1})\|f-g\|, \qquad (2.4)
\end{aligned}$$

implying the continuity of $\mathcal{T}$. By Lemma 1 and Schauder's fixed-point theorem, $\mathcal{T}$ has a fixed point $f\in\Phi(I;m,M)$, which gives the desired solution. □

Theorem 2. *In addition to the assumption of Theorem 1, suppose that*

$$M_0+\sum_{i=1}^{n}L_{i+1}<1. \qquad (2.5)$$

Then for any $\varphi_0\in\Phi(I;m,M)$, there exists a sequence $(\varphi_k)_{k=0}^{\infty}\subset\Phi(I;m,M)$ which is defined by $\varphi_k=\mathcal{T}\varphi_{k-1}, k=1,2,\ldots$, convergent to φ^* which is a solution of Eq. (1.1).

Proof. Consider a mapping $\mathcal{T}$ on $\Phi(I;m,M)$ as in (2.3). Furthermore, set

$$\varphi_k=\mathcal{T}\varphi_{k-1}, \qquad \varphi_0\in\Phi(I;m,M)$$

for $k \in \mathbb{N}$. Noting that $\mathcal{T}$ is a self-mapping on $\Phi(I;m,M)$, we have $(\varphi_k)_{k=0}^{\infty}$ is a subset of $\Phi(I;m,M)$ and from (2.4),

$$\sup_{x\in[a,b]} |\mathcal{T}\varphi_{k+1}(x) - \mathcal{T}\varphi_k(x)| \leq \Big(M_0 + \sum_{i=1}^{n} L_{i+1}\Big) \|\varphi_{k+1} - \varphi_k\|,$$

i.e.,

$$\|\mathcal{T}\varphi_{k+1} - \mathcal{T}\varphi_k\| \leq \Gamma \|\varphi_{k+1} - \varphi_k\|,$$

where $\Gamma = M_0 + \sum_{i=1}^{n} L_{i+1}$. Theeqrefore,

$$\|\varphi_{k+1} - \varphi_k\| = \|\mathcal{T}\varphi_k - \mathcal{T}\varphi_{k-1}\| \leq \Gamma^k \|\varphi_1 - \varphi_0\|.$$

Let

$$\varphi_s(x) = \varphi_0(x) + \sum_{k=0}^{s-1} (\varphi_{k+1}(x) - \varphi_k(x)),$$

we now show that $\sum_{k=0}^{s-1}(\varphi_{k+1}(x) - \varphi_k(x))$ converges on the interval $[0,b]$. This would imply that $\varphi_s(x)$ has a limit on this interval as $s \to \infty$. Clearly, to establish the convergence of $\sum_{k=0}^{\infty} \Big(\varphi_{k+1}(x) - \varphi_k(x)\Big)$, we note that, in view of (2.5), the series

$$\sum_{k=0}^{\infty} \|\varphi_{k+1} - \varphi_k\| \leq \sum_{k=0}^{\infty} \Gamma^k \|\varphi_1 - \varphi_0\| = \frac{1}{1-\Gamma} \|\varphi_1 - \varphi_0\|$$

converges.

This means that $(\varphi_k)_{k=0}^{\infty}$ is a Cauchy sequence under the supreme norm and, theeqrefore, uniformly converges to a continuous function φ^* on $[0,b]$. But we already know that $\Phi(I;m,M)$ is compact, so $(\varphi_k)_{k=0}^{\infty}$ converges to φ^* in $\Phi(I;m,M)$. From ((2.4)) we see that $\mathcal{T}$ is continuous, thus $\varphi^* \leftarrow \varphi_{k+1} = \mathcal{T}\varphi_k \to \mathcal{T}\varphi^*$, we obtain $\mathcal{T}\varphi^* = \varphi^*$. Noting that $\varphi_k \in \Phi(I;m,M)$ for any $\varphi_k = \mathcal{T}\varphi_{k-1}$, $\varphi_0 \in \Phi(I;m,M)$, $k = 1,2,\ldots$. Thus $\|\varphi_k\| = \|\mathcal{T}\varphi_{k-1}\| \leq b$ and we see that $\|\varphi^*\| = \|\mathcal{T}\varphi^*\| \leq b$. Theeqrefore, the sequence of functions given by $S = (\varphi_0(x), \varphi_1(x), \ldots, \varphi_k(x), \ldots)$ can be regarded as approximate solutions of Eq. (1.1). Theorem (2) is proved. □

3. Uniqueness and stability

In this section, we consider the uniqueness and stability of the nondecreasing bounded continuous solutions of (1.1).

Theorem 3. *In addition to the assumption of Theorem 1, suppose that (2.5) holds. Then Eq. (1.1) has a unique solution in $\Phi(I;m,M)$, and the unique solution depends continuously on the given functions G and F. Furthermore, the unique solution can be obtained by the sequence $(\varphi_k)_{k=0}^{\infty}$, here $\varphi_0 \in \Phi(I;m,M)$, $\varphi_{k+1} = \mathcal{T}\varphi_k, k = 0,1,\ldots$ and $\mathcal{T}$ is defined as in (2.3).*

Proof. From the proof of Theorem (1), the map $\mathcal{T}\colon \Phi(I;m,M) \to \Phi(I;m,M)$ in (2.3). Moreover, by (2.4), we get

$$\|\mathcal{T}f - \mathcal{T}g\| \le (M_0 + \sum_{i=1}^{n} L_{i+1})\,\|f-g\|,$$

where M_0 is defined as in Theorem (1). By (2.5),

$$\Gamma = M_0 + \sum_{i=1}^{n} L_{i+1} < 1.$$

So the fixed point must be unique by the Banach fixed point theorem.

Given G_1, G_2 satisfy (H1)-(H2), $F_1, F_2 \in \Phi(I;m_F,M_F)$, we consider the corresponding operators $\mathcal{T}, \widetilde{\mathcal{T}}$ defined by (2.3). Assuming the corresponding conditions (2.1) and (2.5), there are two unique corresponding functions f_1 and f_2 in $\Phi(I;m,M)$ such that

$$f_1 = \mathcal{T}f_1, \qquad f_2 = \widetilde{\mathcal{T}}f_2.$$

Then we have

$$\begin{aligned}\|f_1 - f_2\| &\le \|\mathcal{T}f_1 - \mathcal{T}f_2\| + \|\mathcal{T}f_2 - \widetilde{\mathcal{T}}f_2\| \\ &\le \Gamma\|f_1 - f_2\| + \|\mathcal{T}f_2 - \widetilde{\mathcal{T}}f_2\|,\end{aligned}$$

which implies

$$\|f_1 - f_2\| \le \frac{1}{1-\Gamma}\|\mathcal{T}f_2 - \widetilde{\mathcal{T}}f_2\|. \tag{3.1}$$

Using (2.2),

$$\begin{aligned}\|\mathcal{T}f_2 - \widetilde{\mathcal{T}}f_2\| &= \|(G_1)^{-1}\circ F_1 - (G_2)^{-1}\circ F_2\| \\ &\le \|(G_1)^{-1}\circ F_1 - (G_1)^{-1}\circ F_2\| + \|(G_1)^{-1}\circ F_2 - (G_2)^{-1}\circ F_2\| \\ &\le \frac{1}{l_1 - M_1}\,\|F_1 - F_2\| + \|G_1^{-1} - G_2^{-1}\|,\end{aligned}$$

by (3.1), we get

$$\|f_1 - f_2\| \le \frac{1}{(1-\Gamma)(l_1 - M_1)}\,\|F_1 - F_2\| + \frac{1}{1-\Gamma}\|G_1^{-1} - G_2^{-1}\|.$$

This proves the continuous dependence of solution f upon G and F, otherwise eqreferred to as stability. From Theorem 2, we can finish the proof. □

4. Examples

In this section, some examples are provided to illustrate that the assumptions of Theorem 1 is not self-contradictory.

Example 1. First, we show that the conditions in Theorem 1 are not self-contradictory. Consider the following equation:

$$f(x)+f(x)\left(\frac{1}{2}-f(x)\right)f\left(\frac{1}{5}x\right)=x^2+\frac{x}{2},\qquad x\in\left[0,\frac{1}{2}\right],\tag{4.1}$$

where $G(x_1,x_2)=x_1+\left(\frac{1}{2}-x_1\right)x_1x_2, G(0,x_2)=0, G\left(\frac{1}{2},x_2\right)=\frac{1}{2}$. We can take $L_1=\frac{5}{4}, L_2=\frac{1}{16}, l_1=\frac{3}{4}, l_2=0$ and $F(x)=x^2+\frac{x}{2}$ and $F\in\Phi\left(\left[0,\frac{1}{2}\right];\frac{1}{2},\frac{3}{2}\right)$. Taking $m=\frac{1}{5}, M=10$, then $M_0=\frac{5}{8}, M_1=0$, and a simple calculation yields

$$0\le m=\frac{1}{5}\le\frac{m_F}{L_1+M_0}=\frac{4}{15}\le 1\le 2=\frac{M_F}{l_1-M_1}\le 10=M,$$

thus (2.1) is satisfied. Theorem 1 gives a nondecreasing bounded continuous solution f of Eq. (4.1) in $\Phi\left(\left[0,\frac{1}{2}\right];\frac{1}{5},10\right)$. Noting $M_0+L_2=\frac{11}{16}<1$, (2.5) is satisfied, hence by Theorem 3, we know that the nondecreasing continuous solution is the unique one in $\Phi\left(\left[0,\frac{1}{2}\right];\frac{1}{5},10\right)$. Furthermore, for any $\varphi_0\in\Phi\left(\left[0,\frac{1}{2}\right];\frac{1}{5},10\right)$, the unique solution of (4.1) in $\Phi\left(\left[0,\frac{1}{2}\right];\frac{1}{5},10\right)$ can be approximated by the sequence $(\varphi_k)_{k=0}^{\infty}$, $\varphi_k=\mathcal{T}\varphi_{k-1}$, $\mathcal{T}$ is defined as in (2.3), $k=1,2,\ldots$.

Example 2. Consider the following equation:

$$f(x)+f(x)\left(\frac{1}{2}-f(x)\right)f(qx)=x^2+\frac{x}{2},\qquad x\in\left[0,\frac{1}{2}\right],\tag{4.2}$$

where $q\in(0,1)$, as in Example 6.1,

$$G(x_1,x_2)=x_1+(\frac{1}{2}-x_1)x_1x_2,\qquad G(0,x_2)=0,\qquad G(\frac{1}{2},x_2)=\frac{1}{2}.$$

We can take

$$L_1=\frac{5}{4},\quad L_2=\frac{1}{16},\quad l_1=\frac{3}{4},\quad l_2=0,\quad F(x)=x^2+\frac{x}{2},\quad F\in\Omega\left(\left[0,\frac{1}{2}\right];\frac{1}{2},\frac{3}{2}\right).$$

We will consider the existence of solution $f\in\Phi\left(\left[0,\frac{1}{2}\right];\delta,M\right)$ for (4.2). Noting that $M_0=\frac{M}{8}qL_2=\frac{qM}{16\delta}, M_1=0$. In order to apply (2.1) in Theorem 1, we need

$$0<\delta\le\frac{m_F}{L_1+M_0}=\frac{8\delta}{20m+qM}\le 1\le 2=\frac{M_F}{l_1-M_1}\le M.$$

then

$$qM\le 8-20\delta\quad\text{and}\quad 2\le M.\tag{4.3}$$

Theeqrefore

$$0<q\le 4-10\delta,$$

then

$$0 < \delta < \frac{2}{5}.$$

Furthermore, in order to apply Theorem 3, we need

$$M_0 + L_2 = \frac{qM}{16\delta} + \frac{1}{16} < 1,$$

from (4.3), we have

$$2 \leq M < \frac{1}{q} \min\left\{8 - 20\delta, 15\delta\right\}. \tag{4.4}$$

From (4.3) and (4.4), we know that Eq. (4.2) has an unique nondecreasing bounded continuous solution $f \in \Omega\left(\left[0, \frac{1}{2}\right]; \delta, M\right)$ with $0 < q \leq 4 - 10\delta$ and $2 \leq M < \frac{1}{q} \min\left\{8 - 20\delta, 15\delta\right\}$ for $0 \leq \delta < \frac{2}{5}$.

By Theorem 3, for any $\varphi_0 \in \Omega\left(\left[0, \frac{1}{2}\right]; \delta, M\right)$, the unique nondecreasing continuous solution of (4.2) can be approximated by the sequence $(\varphi_k)_{k=0}^{\infty}$, $\varphi_k = \mathcal{T}\varphi_{k-1}$, $\mathcal{T}$ as in (2.3), $k = 1, 2, \ldots$.

It is easy to check that $q = \frac{1}{5}, m = \frac{1}{5}, M = 10$ in Example 1 which is a special case for Example 2.

REFERENCES

[1] C. R. Adams, "On the linear ordinary q-difference equation." *Annals of Mathematics*, vol. 30, no. 2, pp. 195–205, 1929, doi: 10.2307/1968274.

[2] Y. Andre, "Series Gevrey de type arithmetique. ii." *Annals of Mathematics*, vol. 151, no. 2, pp. 741–756, 2000, doi: 10.2307/121046.

[3] J.-P. Bezivin and A. Boutabaa, "Sur lesequations fonctionelles p-adiques aux q-differences." *Collectanea Mathematica*, vol. 43, no. 2, pp. 125–140, 1992, doi: 10.1007/BF01835698.

[4] G. Birkhoff, "The generalized Riemann problem for linear differential equations and the allied problem for linear difference and q-difference equations." *Bull. Am. Math. Soc.*, vol. 49, no. 2, pp. 508–509, 1913, doi: 10.1137/050641867.

[5] G. Birkhoff and P. Guenther, "Note on canonical form for the linear q-difference system." *Proc. Natl. Acad Sci. USA*, vol. 27, no. 4, pp. 218–222, 1941, doi: 10.1073/pnas.27.4.218.

[6] R. D. Carmichael, "The general theory of linear q-difference equations." *Am. J. Math.*, vol. 34, no. 2, pp. 147–168, 1912, doi: 10.2307/2369887.

[7] G. M. Golusin, "Solution of the fundamental plance problems of mathematical physics for the Laplace equation and multi-connected domains bounded by circumferences." *Mat. Sb.*, vol. 41, pp. 246–283[in Russian], 1934.

[8] F. H. Jackson, "On q-functions and a certain difference operator." *Trans. R. Soc. Edinb.*, vol. 46, pp. 253–281, 1908, doi: 10.1017/S0080456800002751.

[9] F. H. Jackson, "q-difference equations." *Am. J. Math.*, vol. 32, no. 4, pp. 305–314, 1910, doi: 10.2307/2370183.

[10] D. Levi, J. Negro, and M. A. DelOlmo, "Discrete q-derivatives and symmetries of q-difference equations." *J. Phys. A, Math. Gen.*, vol. 37, no. 10, pp. 3459–3473, 2004, doi: 10.1088/0305-4470/37/10/010.

[11] T. E. Mason, "On properties of the solution of linear q-difference equations with entire fucntion coefficients." *Am. J. Math.*, vol. 37, no. 4, pp. 439–444, 1915, doi: 10.2307/2370216.

[12] V. V. Mityushev, *Some mathematical problems of heat conduction 1.*, Moscow, [in Russian], 1987.

[13] J. Sauloy, "Systemes aux q-differences singuliers reguliers: classification, matrice de connexion et monodromie." *Ann. Inst. Fourier*, vol. 50, no. 4, pp. 1021–101, 2000, doi: 10.5802/aif.1784.

[14] J. G. Si and W. N. Zhang, "Analytic solutions of a q-difference equation and applications to iterative equations." *J. Differ Equ. Appl.*, vol. 10, no. 11, pp. 955–962, 2004, doi: 10.1080/10236190412331272607.

[15] J. G. Si and H. Y. Zhao, "Small divisor problem in dynamical systems and analytic solutions of a q-difference equation with a singularity at the origin." *Results. Math.*, vol. 58, no. 3-4, pp. 337–353, 2010, doi: 10.1007/s00025-010-0060-2.

[16] W. J. Trjitzinsky, "Analytic theory of linea q-difference equations." *Acta Math.*, vol. 61, no. 2, pp. 1–38, 1933, doi: 10.1007/BF02547785.

[17] M. van der Put and M. F. Singer, *Galois theory of difference equations. No. 1666*. Berlin: Springer, 1997. doi: 10.1007/BFb0096118.

[18] L. D. Vizio, "An ultrametric version of the Maillet-Malgrange theorem for nonlinear q-difference equations." *Proc. Am. Math. Soc.*, vol. 136, no. 8, pp. 2803–2814, 2008, doi: 10.1090/S0002-9939-08-09352-0.

[19] B. Xu, , and W. N. Zhang, "Small divisor problem for an analytic q-difference equation." *J. Math. Anal. Appl.*, vol. 342, no. 1, pp. 694–703, 2008, doi: 10.1016/j.jmaa.2007.12.010.

[20] W. N. Zhang and J. A. Baker, "Continuous solutions of a polynomial-like iterative equation with variable coefficients." *Ann. Pol. Math.*, vol. 73, no. 1, pp. 29–36, 2000, doi: 10.4064/ap-73-1-29-36.

[21] W. N. Zhang, K. Nikodem, and B. Xu, "Convex solutionsof polynomial-like interative equations." *J. Math. Anal. Appl.*, vol. 315, pp. 29–40, 2006, doi: 10.1016/j.jmaa.2005.10.006.

Authors' addresses

Hou Yu Zhao

(**Corresponding author**) Chongqing Normal University, School of Mathematics, 401331 Chongqing, P.R.China

E-mail address: houyu19@gmail.com

Shan Shan Guo

Chongqing Normal University, School of Mathematics, 401331 Chongqing, P.R.China

E-mail address: guoshanshan0516@163.com

EDITORS-IN-CHIEF

Miklós Rontó (Founding Editor), Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matronto@uni-miskolc.hu

Imre Juhász, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; agtji@uni-miskolc.hu

EDITORS

Irina Astashova, Faculty of Mechanics and Mathematics, Chair of Differential Equations, Lomonosov Moscow State University; The Main Building, Vorobiovy Gory, 119899 Moscow, Russia; ast.diffiety@gmail.com

Mihály Bessenyei, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; mihaly.bessenyei@uni-miskolc.hu

Mikhail A. Botchev, Keldysh Institute of Applied Mathematics, Russian Academy of Sciences; Miusskaya Sq. 4, Moscow 125047, Russia; botchev@ya.ru

Mátyás Domokos, A. Rényi Institute of Mathematics, Hungarian Academy of Sciences; P. O. Box 127, H-1053 Budapest, Hungary; matyas.domokos@mta.renyi.hu

Hemen Dutta, Department of Mathematics, Gauhati University; Guwahati-781014, India; duttah@gauhati.ac.in

Michal Fečkan, Comenius University, Department of Mathematical Analysis; Mlyn-ská dolina, 842 15 Bratislava, Slovakia; Michal.Feckan@fmph.uniba.sk

Sándor Fegyverneki, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matfs@uni-miskolc.hu

Stephan Foldes, University of Miskolc, Institute of Information Technology; H-3515 Miskolc-Egyetemváros, Hungary; mmn.foldes@gmail.com

Aurél Galántai, Óbuda University, John von Neumann Faculty of Informatics; H-1034 Budapest, Bécsi u. 96/b, Hungary; galantai.aurel@nik.uni-obuda.hu

Attila Házy, Institute of Mathematics and Informatics, Eszterházy Károly Catholic University; H-3300 Eger, Leányka u. 12, Hungary; hazy.attila@uni-eszterhazy.hu

Zhenhai Liu, Guangxi University for Nationalities; East University Road, 188, Nanning, China; zhhliu@hotmail.com

Judit Makó, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matjudit@uni-miskolc.hu

Nino Partsvania, Andrea Razmadze Mathematical Institute; 2 University St., Tbilisi 0186 Georgia; ninopa@rmi.ge

Adrian Petrusel, Department of Mathematics, Babes-Bolyai University of Cluj-Napoca; No. 1, 400084, Cluj-Napoca, Romania; petrusel@math.ubbcluj.ro

Sándor Radeleczki, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matradi@uni-miskolc.hu

Maria Alessandra Ragusa, University of Catania, Italy Viale Andrea Doria; 6-95125, Catania, Italy; maragusa@dmi.unict.it

Csaba Rakaczki, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matrcs@uni-miskolc.hu

András Rontó, Brno University of Technology, Faculty of Business and Management; Kolejni 4, Brno, 61200, Czech Republic; rontoa@fbm.vutbr.cz

Jenő Szigeti, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matjeno@uni-miskolc.hu

Jiří Šremr, Brno University of Technology, Faculty of Mechanical Engineering, Department of Mathematical Analysis; Technicka 2896/2, Kralovo Pole, Brno, 61669, Czech Republic; sremr@fme.vutbr.cz

Hari Mohan Srivastava, Department of Mathematics and Statistics, University of Victoria; Victoria, British Columbia V8W 3R4, Canada; harimsri@math.uvic.ca

Sergei Trofimchuk, Instituto de Matemática y Física, Universidad de Talca; 2 Norte 685, Talca, Chile; trofimch@inst-mat.utalca.cl

Leon van Wyk, Department of Mathematical Sciences, Stellenbosch University; P/Bag X1, Matieland 7602, Stellenbosch, South Africa; lvw@sun.ac.za

TECHNICAL EDITORS

Attila Házy, Institute of Mathematics and Informatics, Eszterházy Károly Catholic University; H-3300 Eger, Leányka u. 12, Hungary; hazy.attila@uni-eszterhazy.hu

Attila József Földvári, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matfaj@uni-miskolc.hu

Attila Körei, Institute of Mathematics, University of Miskolc; H-3515 Miskolc-Egyetemváros, Hungary; matka@uni-miskolc.hu

Miskolc Mathematical Notes

Abstracted in *Zentralblatt für Mathematik*, *Mathematical Reviews*, Реферативный журнал „Математика”

Indexed in *CNKI*, *Dimensions*, *EBSCO Academic Search Ultimate*, *EBSCO Academic Search Complete, EBSCO Academic Search Premier, EBSCO Academic Search Elite, EBSCO STM Source*, *Google Scholar, Naver Acadenic, ProQuest Central, ProQuest SciTech Premium Collection, ProQuest Technology Collection, ROAD, SCImago, SCOPUS, Web of Science Core Collection: Science Citation Index Expanded Additional, Web of Science Indexes: Essential Science Indicators, UGC-CARE List*

Responsible for publication: Prof. Dr. Zita Horváth, rector
Published by the Miskolc University Press under the leadership of Attila Szendi
Responsible for duplication: Works manager Erzsébet Pásztor
Number of copies printed: 40
Put to the Press on 29 October, 2024
Number of permission: MERT 2024-280 ME **HU e-ISSN 1787-2413**