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TWICE IS ENOUGH METHOD FOR $A^T A$ CONJUGATE DIRECTIONS AND FOR BICONJUGATE DIRECTIONS

JÓZSEF ABAFFY AND VITTORIO MORIGGIA

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Abstract. The aim of the article is to increase the accuracy of $A^T A$ conjugate directions and biconjugate directions by applying the twice is enough method to them [14]. The twice is enough algorithm and analysis are due to W. Kahan, cf. Parlett's book [14], pp. 115–117. It was shown that while two consecutive orthogonalization steps improved the accuracy of the computation, further orthogonalization steps failed to provide additional benefit, establishing the principle of "twice is enough". In our previous works, we have introduced the "twice is enough" type algorithms for conjugate directions of positive definite symmetric matrices, cf. [1, 4, 6] and [3]. These results were also generalized for arbitrary symmetric matrices [2]. In the paper [3], we generalized this idea to the computation of conjugate directions. Now, we show that it can be generalized to any matrices; furthermore, we give the conjugate directions of the problem $A^T A$ and the biconjugate directions of any square matrix A . With the help of intensive testing [7], we propose specialized algorithms for these problems. We compared our algorithms to four well-known biconjugate methods that we implemented to obtain the biconjugated directions as well [9]. Using the refined conjugate directions, they can be used to further refine the solution of systems of linear equations iteratively, and to solve $Ax = B$ where B contains all possible right-hand vectors b . We underline that in the computations of the $A^T A$ conjugate directions and the biconjugate directions we do not need to compute the $A^T A$ matrix directly. Another goal of the article, in addition to the applicability of the twice is enough idea to conjugate and biconjugate directions, is to determine the most accurate methods for producing conjugate and biconjugate directions. For this we will need the `vpa` option of MATLAB.

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Keywords: linear algebra, biconjugate direction

1. INTRODUCTION

In this paper, for the sake of simplicity and brevity, suppose that A is an n by n square and nonsingular matrix, that is $\text{rank}(A) = n$. We will see that without these

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conditions all our statements remain true. That is consider the problem

$$Ax = b$$

where $A \in \mathbb{R}^{n \times n}$ is an arbitrary matrix, $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. In this article, conjugate directions for $A^T A$ are defined as follows.

Definition 1. Let A be arbitrary nonsingular matrix. Then, we say that the vectors $p_1, \dots, p_n$ are $A^T A$ conjugates, if

$$p_i^T A^T A p_j = \begin{cases} 0, & \text{if } i \neq j \\ \text{non zero}, & \text{otherwise.} \end{cases}$$

Furthermore, we define the biconjugate directions as follows.

Definition 2. Let A be arbitrary nonsingular matrix. Then, we say that the vectors $p_1, \dots, p_n$ and $t_1, \dots, t_n$ are A biconjugate if

$$t_i^T A p_j = \begin{cases} 0, & \text{if } i \neq j \\ \text{non zero}, & \text{otherwise.} \end{cases}$$

Note that if we have the $P = [p_1, \dots, p_n]$ conjugate directions to $A^T A$ then we can compute the $T = [t_1, \dots, t_n]$ matrix from it as

$$t_i = A p_i, \quad i = 1, \dots, n$$

and therefore we also have the biconjugate directions of A .

Remark 1. We already noted in our earlier papers that to determine the most accurate algorithms, we need to have the exact conjugate directions. In double-precision arithmetic, we need the knowledge of exact conjugate directions in order to determine the best methods for calculating conjugate directions. To do this, we use MATLAB's VPA option for 50 digits, because we believe that its first 16 digits are accurate. We did not choose more than 50 digits because this calculation is very time-consuming and our personal computer is not very efficient. Matrix P contains the computed $p_i, i = 1, \dots, n$ vectors in columns and similarly S contains the same directions computed by vpa for 50 decimals. Then the differences B are

$$B = S^T (A^T A) S - \text{vpa}(P^T) (A^T A) \text{vpa}(P)$$

$$yB = \max_{i,j} (\text{abs}(\text{double}(B)))$$

In the two formulas, we are talking about determining the difference matrix with the help of the matrix containing the precise S conjugate directions calculated with vpa and the matrices calculated with the conjugate directions computed in double precision. If the difference were zero in double precision, then we have the best method. This difference, i.e. deviation, is calculated using the formula in the next line. The log10 value of the max deviation gives the number of exact digits, see below.

Let us now move on to checking the accuracy of the biconjugate vectors t_i . Similarly, we have to calculate the accuracy of the biconjugate vectors t_i , i.e

$$\begin{aligned} TTP &= AP \\ TT &= \text{vpa}(AS) \\ BB &= TT^T AS - \text{vpa}(TTP^T) A \text{vpa}(P) \\ yBB &= \max_{i,j} (\text{abs}(\text{double}(BB))) \end{aligned} \tag{1.1}$$

The minimum number of accurate digits of the conjugate vectors p_i and t_i , for $i = 1, \dots, n$, are computed by

$$\begin{aligned} ym &= -\log_{10}(yB) \\ tm &= -\log_{10}(yBB) \end{aligned}$$

Remark 2. Finally, we note that in the testing we do not apply any precondition method because we want to find the most accurate algorithms even for difficult problems and without manipulating them in any way before usage.

2. THE ABS CLASS AND THE "TWICE IS ENOUGH" ALGORITHM

In the next subsection 2.1 we present the ABS class, then the ABS Conjugate Direction Parlett- Kahan type (*ABS_CD_PK*) in subsection 2.2.

2.1. Description of the ABS class with the *ABS_CD_PK*

We present the general ABS class where the matrix A can even be rectangular. Let us consider the following scaled system

$$V^T Ax = V^T b$$

where $A \in \mathbb{R}^{n \times n}$ is an arbitrary matrix, $V \in \mathbb{R}^{n \times n}$ is an arbitrary non-singular matrix, $b \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

The class of the scaled ABS algorithm [5] is as follows.

ABS class with *ABS_CD_PK*

- Step 1. Set $x_1 \in \mathbb{R}^n$, $H_1 = I \in \mathbb{R}^{n \times n}$ where I is the unit matrix, $i = 1$, $\kappa = 1.25$ and $iflag = 0$.
- Step 2. Let $v_i \in \mathbb{R}^n$ be arbitrary, except that $v_1, \dots, v_i$ be linearly independent. Compute the residual error vector $r_i = Ax_i - b$. If $r_i = 0$, stop and x_i solves the system. Otherwise, compute the scalar $\tau_i = v_i^T r_i$ and the vector $s_i = H_i A^T v_i$.
- Step 3. If $s_i \neq 0$, go to Step 4; if $s_i = 0$ and $\tau_i = 0$, set $x_{i+1} = x_i$, $H_{i+1} = H_i$, $iflag = iflag + 1$, and if $i < n$, go to Step 6; otherwise stop; if $s_i = 0$ and $\tau_i \neq 0$, set $iflag = -i$ and stop.

Step 4. Compute the search direction p_i by

$$p_i = H_i^T z_i$$

where $z_i \in \mathbb{R}^n$ is arbitrary saving for $z_i^T H_i A^T v_i \neq 0$.

$$p_i = \text{ABS_CD_PK}(H_i, A, \kappa, p_i, z_i).$$

Step 5. Update the approximation of the solution by

$$x_{i+1} = x_i - \alpha_i p_i$$

where the step size α_i is given by

$$\alpha_i = \frac{\tau_i}{v_i^T A p_i}$$

If $i = n$, stop and x_{n+1} is the solution of the equations.

Step 6. Update the matrix H_i by

$$H_{i+1} = H_i - \frac{H_i A^T v_i w_i^T H_i}{w_i^T H_i A^T v_i} \quad (2.1)$$

where w_i is arbitrary, but the denominator must be non-zero.

Step 7. Set $i = i + 1$ and go to Step 2.

Some important theorems are following, cf. [5] page 95pp.

Theorem 1. *The residual vector $r_{i+1} = Ax_{i+1} - b$ is orthogonal to the first i columns of V i.e. $V^T r_{i+1} = 0$*

Theorem 2. *For $1 \leq i \leq n$ the vectors $A^T v_1, \dots, A^T v_i$ are non zero, linearly independent and a basis of the null space of H_{i+1} . The vectors $w_1, \dots, w_i$ are non-zero, linearly independent, and a basis of the null space of H_{i+1} .*

The $\text{ABS_CD_PK}()$ function returns the new p_i vector in case projection is required, see below in Section 2.2. The properties of this algorithm can be found in [5]. Hereinafter, we use the index i only when needed. Before we turn to our "twice is enough" type reprojection algorithm for the general matrix, we remark that in an earlier papers we developed it for positive definite symmetric matrices (see [1] and [2]), for symmetric matrices in [8] as well as to the computation of conjugate directions in [3]. In the paper [8], we presented an intensive testing for symmetric but singular, indefinite and singular plus indefinite test matrices. In that work the twice is enough type algorithm ABS_CD_PK was defined, like in the original twice is enough algorithm [14], to control also the linear dependency. Since the symmetry of $A^T A$ is always guaranteed and it is not necessary to care about the symmetry of A , like in the Gaussian problem. In this paper we test some non-symmetric test problems from MATLAB R2021b Gallery.

2.2. Twice is enough type reprojection in ABS class

The twice is enough algorithm and analysis are due to W. Kahan, see pages 115-117 of Parlett's book [14]. In our previous works, we have introduced the "twice is enough" type algorithms for conjugate directions of positive definite symmetric matrices, cf. [1, 3, 4, 6]. These results were also generalized for arbitrary symmetric matrices in [2]. In the paper [3], we generalized this idea to the computation of conjugate directions. Now, we show that it can be generalized to any matrices; furthermore, we give the conjugate directions of the problem $A^T A$ and the biconjugate directions of any square matrix A .

We first present the Parlett-Kahan algorithm [14]:

"Suppose that a simple subprogram *orthog* is available which, given $y \neq o$ and z , computes an approximation x' to $p \equiv z - y(y \cdot z / \|y\|^2)$. Let the error $e' (\equiv x' - p)$ satisfy $\|e'\| \leq \epsilon \|z\|$ for some tiny positive ϵ independent of y and z . Let κ any fixed value in the range $[1/(0.83 - \epsilon), 0.83/\epsilon]$.

Algorithm. First call *orthog*(y, z, x') to get x' .

- Case 1. If $\|x'\| \geq \|z\|/\kappa$ accept $x = x'$ and $e = e'$.
Otherwise call *orthog*(y, x', x'') to get x'' with error $e'' \equiv x'' - (x' - yy \cdot x' / \|y\|^2)$ satisfying $\|e''\| \leq \epsilon \|x'\|$ and proceed to Case 2.
- Case 2. If $\|x''\| \geq \|x'\|/\kappa$ accept $x = x''$, $e = x'' - p$.
- Case 3. If $\|x''\| < \|x'\|/\kappa$ accept $x = o$, $e = -p$."

together with its Lemma:

Lemma. The vector x computed by the algorithm ensure that $\|e\| \leq (1 + 1/\kappa)\epsilon\|z\|$ and $\|y \cdot x\| \leq \kappa\epsilon\|y\|\|x\|$."

Our ABS_CD_PK algorithm is as follows. Using the notation of the original Parlett-Kahan algorithm in [14], let $x = H^T z$ and κ is fixed in the range $[1/(0.83\epsilon), 0.83/\epsilon]$. Furthermore, let e' represents the error vector between the computed x' and its accurate value, see [1] and [4, pp. 41-51].

Algorithm 1. ABS Conjugate Direction of Parlett-Kahan-type for arbitrary matrices ($p = \text{ABS_CD_PK}(H, A, \kappa, p, z)$).

Compute Let $x = p$ (because we use the Parlett-Kahan notations) and $x' = Ax$.

- Case 1. If $x'^T A^T A x' > z^T A^T A z / \kappa$, accept $x = x'$ and $e = e'$, otherwise compute $x'' = H^T x'$ to get x'' with error e'' and go to Case 2.
- Case 2. If $x''^T A^T A x'' \geq x'^T A^T A x'' / \kappa$ accept $x = x''$ and $e = e''$.
- Case 3. Otherwise, accept $x = 0$.

Remark 3. Here, the indefiniteness is also possible. Just as in the proof of the algorithm, the positivity of these values was supposed, therefore no any new proof is

needed, because here the scalar products are ≥ 0 . Furthermore, we note that in the ABS algorithms and in this *ABS_CD_PK*, the singularity is well-treated.

Remark 4. We underline again that this procedure works for any matrices.

The theorem for the "twice is enough" type reprojections is as follows

Theorem 3. *The vector x computed by the *ABS_CD_PK* algorithm ensures that for the error vector e*

$$(e^T A^T A e) \leq \text{eps}(z^T A^T A z) + O(\text{eps}^2)$$

and

$$(p_0^T A^T A x) \leq \kappa \text{eps}(p_0^T A^T A p_0)(x^T A^T A x) + O(\text{eps}^2).$$

where p_0 is any other $A^T A$ conjugate direction.

For the proof, see [1],[4, pp. 41-51]. In this work we run our experiments with $\kappa = 100$ and $\kappa = 1000$ without any relevant improvement, then we accept $\kappa = 1.25$ as suggested in Parlett's book [14]. We have to note that the optimal value of κ is unknown and we believe that it is problem dependent.

3. THE CONSIDERED ABS SUBCLASS

In this paper, we consider the subclass S3 of ABS. This is because we can easily derive the conjugate directions of the matrix $A^T A$ or the biconjugate directions of A from this subclass.

3.1. The S3 subclass

In the S3 subclass, vector v_i is defined as

$$v_i = A p_i$$

Theorem 4. *Let A be square nonsingular. Then the scaled ABS class with $v_i = A p_i$ generates $A^T A$ -conjugate search vectors and x_{i+1} minimizes, over the linear variety $x_1 + \text{Span}(p_1, \dots, p_i)$ the quadratic function $F(x) = (x - x^+)^T A^T A (x - x^+)$ where x^+ is the solution of the linear system.*

For the proof see Theorem 8.11 in [5]. Again, we note that all our algorithms treat the singularity, therefore this condition of the theorem is just a formality. Also note that if the projection vector p_i is zero in a certain step (less than eps , where constant eps in MATLAB is the distance from 1.0 to the next larger double precision number) then all further p_i vectors are zeros, because in the matrix update H_i the rank one term becomes zero.

In [7] we tested 61 methods in S3, while in this article we present only three best special cases according to our tests. This means that the number of exact digits is the highest for the conjugate direction with the worst accuracy for the best algorithms.

- 1) $z_i = e_i$, $w_i = A^T v_i$ (S3e) The projection matrix H_i is symmetric.

- 2) $z_i = e_i$, $w_i = e_i$ (S3ee) The projection matrix H_i is not symmetric.
- 3) $z_i = e_i$, $w_i = p_i$ (S3ep) The projection matrix H_i is not symmetric.

3.2. Classical algorithms

As we did not find algorithms in MATLAB R2021b that produce biconjugate directions for a matrix A , nor did we find them commercially, we implemented four methods to calculate the solution of linear systems of equations with biconjugate directions: BiCG in Broyden et al. [9, 10, 13], p. 52; BiCGL in Broyden et al. [9, 10, 13], p. 53; BiCR in Broyden et al. [9, 12], p. 60 and BiCRL in Broyden et al. [9, 12], p. 61. Algorithm descriptions can also be found in [7] where all the source codes written in MATLAB can also be found in the Appendix. Note that the bicgstab algorithm and its variants do not give biconjugate directions, see for example [15] and [17].

4. TEST PROBLEMS

Before we list the problems that are being tested, we have to note two things. For all problems, we computed the quantity

$$q = \text{cond}_\infty(A^T A) \times \text{eps} \quad (4.1)$$

where cond is the condition number, ∞ means infinite norm and eps is MATLAB eps . If the value of q reaches 1 for a matrix A , then the algorithm that solves the system of linear equations cannot be criticized if it does not provide an acceptable solution. See Wilkinson [16] or Golub and Van Loan [11]. Therefore, in the sequel we compute q for different dimensions of A and the dimension where q reaches 1 defines the upper bound of the feasible interval in which we test our methods.

In the followings, the vector $x_1 = 0$ is chosen for each test problem and for each method. The right side of the system of equations b is determined by generating a solution with the built-in `rand()` function, which is valid for all methods, including the known ones. We did this for a fair comparison.

4.1. Chosen test problems from the MATLAB Gallery

In this subsection we report each problem followed by the interval $[a, b]$, where a is the smallest tested dimension and b is the largest one. In the case this value becomes larger than 3000, then we give the test interval $[100, 110]$ because we observed that q is practically the same for all of these dimensions and the memory and computation capabilities of the computer we used is limited. Detailed description about the test problems can be found in MATLAB Gallery.

We have chosen the following non-symmetric matrices, most of them from the gallery of MATLAB R2021b: *binomial* [6,27], *chebspec* [4,10], *chebvand* [5,10], *chow* [5,10], *circul* [100,110], *clement* [5,50], *cycol* [5,10], *dramadah* [5,28], *forsythe* [100,110], *frank* [5,10], *grcar* [100,110], *invhess* [100,110], *invol* [2,5], *jordbloc* [100,110], *kahan* [5,41], *krylov* [5,12], *leslie* [100,110], *lesp* [100,110], *lotkin* [3,6],

parter [100,110], *qmult* [100,110], *rand* [100,110], *randcolu* [100,110], *randhess* [100,110], *randjorth* [5,10], *rando* [100,110], *randsvd* [5,10], *riemann* [100,110], *toeppen* [100,110], *triv* [5,22]. Two more test problems are not in the Gallery of MATLAB: *gfpp* [100,110], and *makejcf* [100,107].

5. TEST RESULTS

In this section we report our test results of the above described problems for some special case only, because of the limited scope of this article, detailed tables can be found in [7], p. 13. In that paper, we present the accuracy of the projection vectors for P and T matrices for all the three cases: without any reprojection, with the *ABS_CD_PK* method and with reprojections in all steps. The linear dependency is threshold by 2ϵ s.

First, we present the results of the small-dimension problems, then those where the dimension is between 100-110. We notice again that $\kappa = 1.25$, after a short testing with others κ , can be accepted as well as proposed by Parlett and Kahan. The experiments were performed on a personal computer with Intel(R) Core(TM) i7 2.67 GHz CPU with integrated graphics, 6 GB RAM 64-bit Operating System running Microsoft Windows 10 Professional and MATLAB version R2021b.

5.1. Test results of the small dimensions problems

Here, we present test results of 4 classical methods defined above that is the so called BiCG, BiCGL, BiCR and the BiCRL methods. The accuracy of the biconjugate directions is computed by (1.1). The BiCGL method seems to be the best among them, but also this method gives worse results (see Table 1 below) than many our methods. Therefore we did not make any experiment with "high" dimension.

In Tables 1 instead of zero correct digits we often find a negative number of accurate digits. Let see an example to better understand the reason that the worst number of accurate digits can be negative in the diagonal.

Suppose that the worst number of accurate digits is in the diagonal. Furthermore, suppose that all the elements in the diagonal are accurate for all digits except one element. Let the accurate diagonal element in this case 55555555555555 be of 15 digits and the computed one differs from it in the last digits, say with a value of 1.5. that is 55555555555556.5. That is the deviation is 1.5. In this case the minus logarithm in base 10 is $-\log_{10}(1.5) = -0.1761$. On the other hand, the number of accurate digits is 14, which is more than acceptable. Therefore, it is obvious that if the positions of the negative worst-case values are in the diagonals, we need to examine the number of exact digits. In our cases, with the negative worst accuracy in the diagonal, the number of the accurate digits are not good.

In Table 1 below, only the diagonal negative numbers were left and the rest was replaced by zero. We can see that all 4 classical methods usually give a negative or small positive number for the number of exact digits while we show the same

TABLE 1. Worst number of accurate digits of the projection vector and the number of reprojections in maximum dimensions by ABS_CD_PK

<i>Methods</i>	binomial <i>dim = 6-27</i>	chebspec <i>dim = 4-10</i>	chebvand <i>dim = 5-10</i>	chow <i>dim = 5-10</i>	clement <i>dim = 5-50</i>	cycol <i>dim = 5-10</i>	dramadah <i>dim = 5-28</i>	frank <i>dim = 5-10</i>
BiCG	0.00	0.00	0.00	-1.06	9.36	11.04	0.00	1.77
BiCGL	-inf	0.15	6.60	9.18	-115.33	crash d.	-8.77	0.50
BiCR	0.00	12.59	4.50	5.96	1.41	crash d.	1.05	4.22
BiCRL	-inf	0.00	9.07	4.45	-124.36	crash d.	-10.52	0.00
S3e	3.01 (22)	14.07 (6)	14.07 (9)	16.00 (8)	12.46 (32)	16.00 (4)	14.64 (26)	14.59 (9)
S3ee	7.07 (25)	14.13 (9)	13.49 (9)	16.00 (8)	0.00 (28)	16.00 (7)	15.28 (25)	14.87 (9)
S3ep	7.11 (25)	14.03 (8)	13.45 (9)	16.00 (8)	0.00 (28)	16.00 (7)	15.24 (25)	14.81 (9)

<i>Methods</i>	invol <i>dim = 2-5</i>	kahan <i>dim = 5-41</i>	krylov <i>dim = 5-12</i>	lotkin <i>dim = 3-6</i>	randjorth <i>dim = 2-10</i>	randsvd <i>dim = 2-10</i>	triv <i>dim = 5-22</i>
BiCG	9.28	0.00	0.00	3.03	0.00	-0.74	5.64
BiCGL	7.76	3.56	-70.05	13.01	-41.99	16.00	6.26
BiCR	0.00	0.00	0.00	7.08	0.00	1.31	0.01
BiCRL	0.00	0.56	-127.39	13.05	-49.68	16.00	0.00
S3e	12.28 (4)	15.34 (34)	11.37 (9)	14.52 (5)	5.21 (8)	10.66 (9)	15.03 (19)
S3ee	12.03 (4)	15.18 (37)	12.13 (10)	14.07 (5)	5.30 (7)	11.66 (9)	16.00 (21)
S3ep	11.75 (4)	15.21 (37)	12.71 (1)	14.68 (5)	5.64 (8)	9.65 (9)	16.00 (21)

worst number of accurate digits to the projection vectors P using the ABS_CD_PK algorithm. In parentheses, we give the number of reprojections for the maximum dimensions.

In Table 1 we present the number of worst deviation in two decimal digit precision. The values are the minimum value in the worst number of accurate digits considering all dimensions of the projection vectors.

Note that in case of problem *clement* at least the solutions are not very bad for S3ee and S3ep, $1.377\text{e}-008$ and $1.174\text{e}-008$ respectively in maximum dimensions.

Moreover, the methods BiCG, BiCR and BiCRL in case *cycol* crash down by division by zero in the MATLAB function `mrdivide`, which did not happen in MATLAB R2007b with the same code, see our working paper [7].

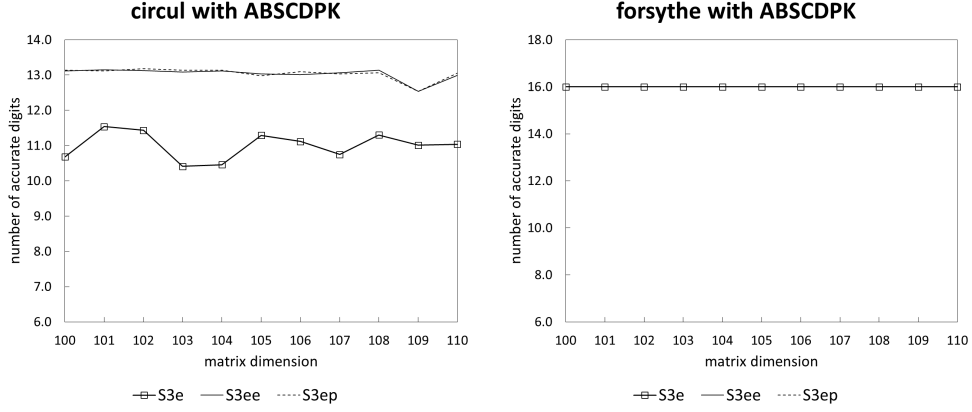
Furthermore, in problem *krylov* not only the p vectors are very bad, cf. methods BiCGL, BiCRL, but even the solutions of the linear systems are not good. In the method BiCG of case *randsvd* the maximum norm of the worst residuals equals 0.06452. The *randsvd* is the only problem where the BiCGL and BiCRL are definitely better than our chosen methods.

Note that in the method BiCGL of case *randjorth* also the maximum norm of the residuals are bad, while in the method BiCRL it is 0.0007351.

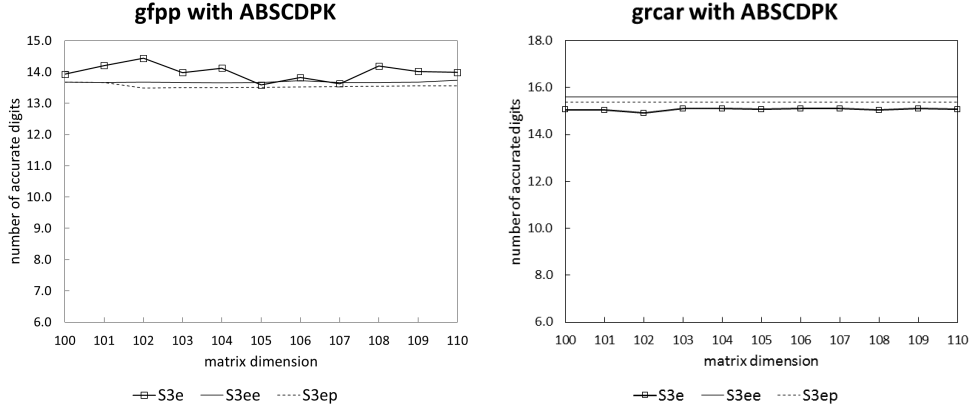
We conclude from Table 1 that the three ABS methods are definitely much better than the classical ones. Therefore, we skipped to test all the classical methods for large-scale problems.

5.2. Test results of problems for dimensions 100-110

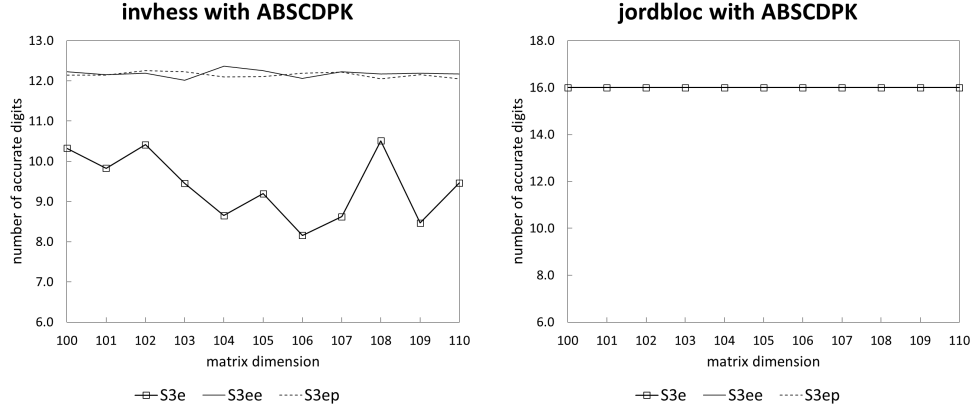
The "big" dimensional nonsymmetric problems are: *circul*, *forsythe*, *grcar*, *invhess*, *jordbloc*, *leslie*, *lesp*, *parter*, *qmult*, *rand*, *rndcolu*, *randhess*, *rando*, *riemann*, *gfpp*, *makejcf* and *toepd*.



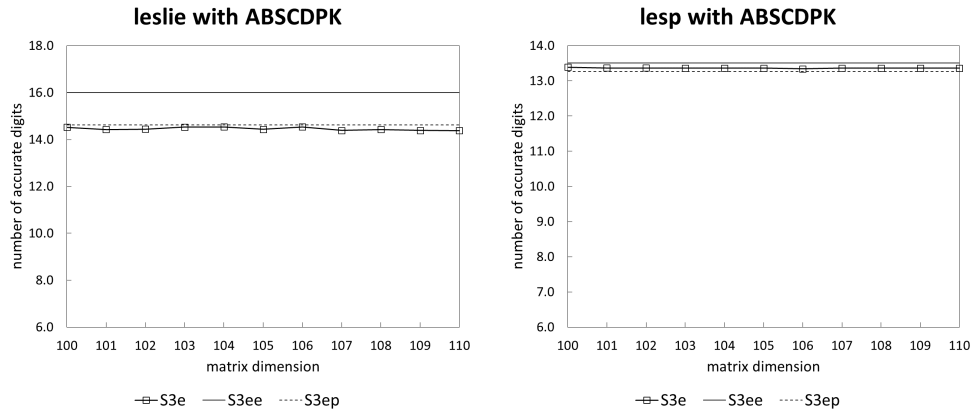
The number of reprojections in maximal dimension in S3e, S3ee, S3ep are 109, 109, 109 respectively for case *circul*, which means reprojections in all steps, and 0, 0, 0 for case *forsythe*, which means no reprojection at all, in fact all the digits are correct since the problem is too easy. For details see [7].



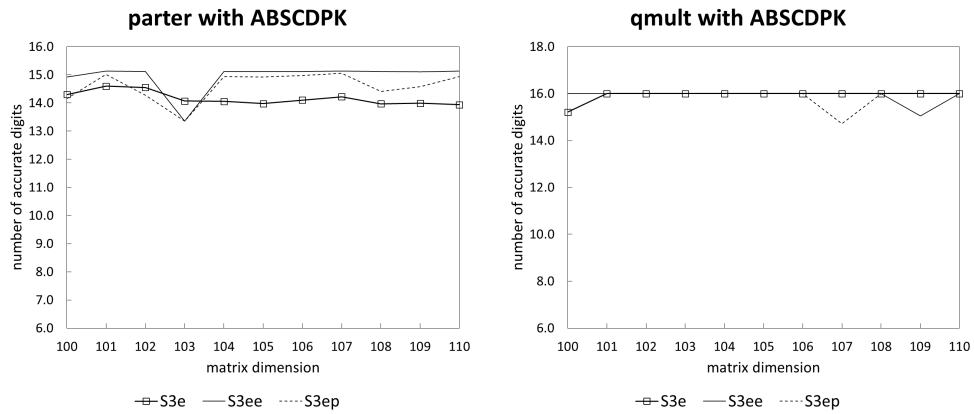
The number of reprojections in maximal dimension in S3e, S3ee, S3ep are 108, 106, 106 respectively for case *gfpp* and 107, 0, 0 for *grcar*.



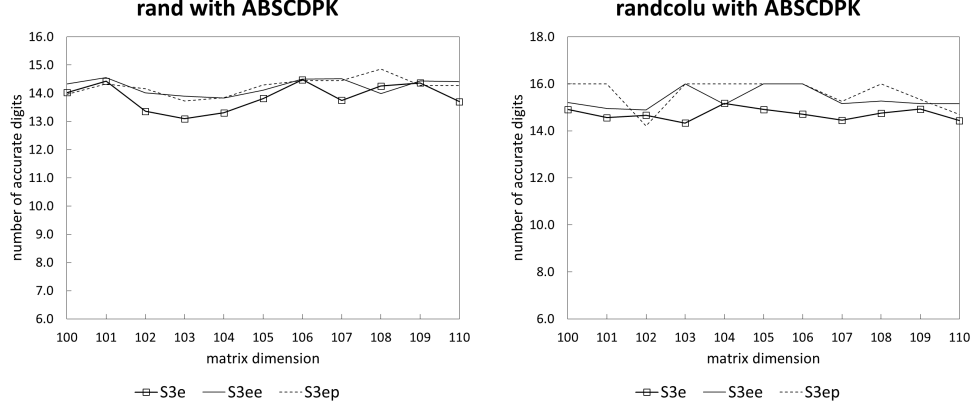
The number of reprojections in maximal dimension in S3e, S3ee, S3ep are 104, 109, 109 respectively for case *invhess* and 109, 109, 109 for case *jordbloc*.



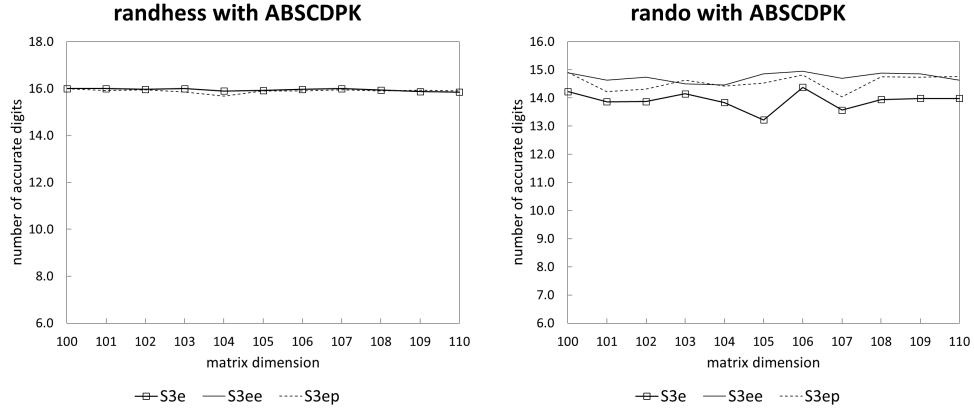
The number of reprojections in maximal dimension in S3e, S3ee, S3ep are 109, 107, 107 respectively for case *leslie* and 109, 0, 0 for case *lesp*.



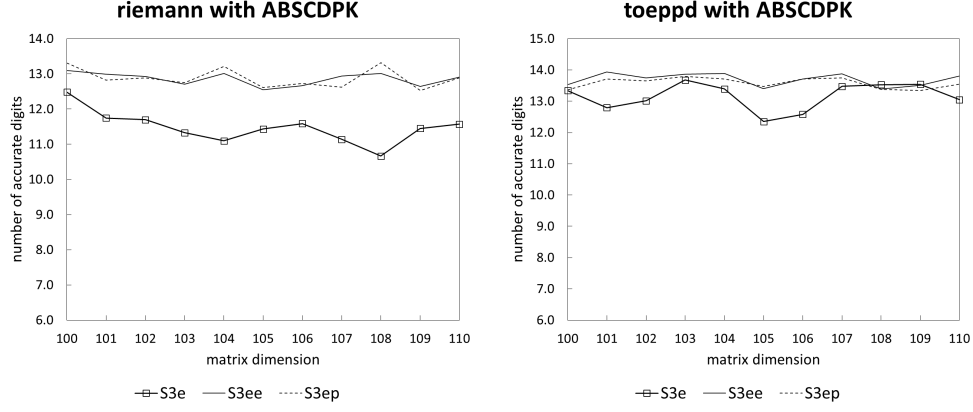
The number of reprojections in maximal dimension in S3e, S3ee, S3ep are 5, 0, 0 respectively for case *parter* and 0, 0, 0 for case *qmult*.



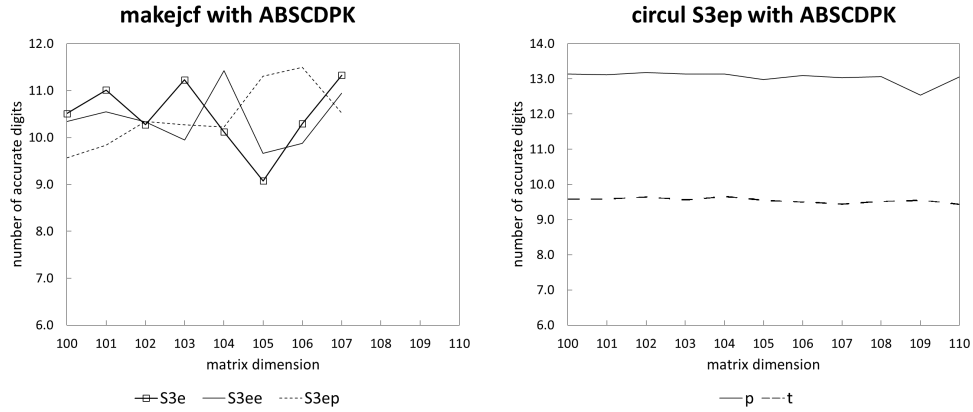
The number of reprojections in maximal dimension in S3e, S3ee, S3ep are 109, 109, 109 respectively for case *rand* and 79, 57, 55 for case *randcolu*.



The number of reprojections in maximal dimension in S3e, S3ee, S3ep are 0, 0, 0 respectively for case *randhess* and 109, 108, 108 for case *rando*.



The number of reprojections in maximal dimension in S3e, S3ee, S3ep are, respectively, 66, 14, 14 for case *riemann* and 93, 78, 78 for case *toeppd*,



For case *makejcf* the number of reprojections in maximal dimension in S3e, S3ee, S3ep are 105, 106, 106, respectively.

Figure named “circul S3ep with ABSCDPK” shows that the number of accurate digits in worst t vector can be more than 3 digits less than the p_i vectors, $i = 1, \dots, n$. When the accuracy of the t conjugate directions are important, we suggest to use our algorithms for the matrix A^T . In fact, using S3e, S3ee, S3ep the p_i vectors are more accurate and are also the t_i vectors of the original problem, $i = 1, \dots, n$. Note further that these t_i vectors are not used in the computations of the solutions of the linear system of equations.

As it can be seen, some figures show strange drop in the curves. We explained it describing a case in our working paper [7].

5.3. Conclusion

In this paper we tested three ABS algorithms (S3e, S3ee and S3ep) using 32 test-problems to decide which is the best using the twice is enough theorem for biconjugate directions. These methods were compared by 4 well-known methods (BiCG, BiCGL, BiCR and BiCRL). Based on these test problems, we found that the best two methods are S3ee and S3ep to compute the p_i , for $i = 1, \dots, n$, conjugate and biconjugate directions accurately.

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ON SOME NEW AND GENERAL q -HERMITE–HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In this paper, we establish a new version of Hermite–Hadamard type inequality for convex functions. Moreover, we establish a general version of q -integral identity involving q -differentiable functions to prove some new q -midpoint and q -trapezoidal type inequalities for q -differentiable convex functions. It is also shown that the newly established inequalities can be converted into some existing inequalities within the literature. Finally, we add some mathematical examples to show the validation of newly established inequalities.

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1. INTRODUCTION

The Hermite–Hadamard (H–H) inequality, named after Charles Hermite and Jacques Hadamard and commonly known as Hadamard’s inequality, says that if a function $f: [a, b] \rightarrow \mathbb{R}$ is convex, the following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

If f is a concave mapping, the above inequality holds in the opposite direction. There has been much research done in the direction of H–H for different kinds of convexities. For example, in [7, 10], the authors established some inequalities linked with midpoint and trapezoid formulas of numerical integration for convex functions.

On the other hand, in [3, 4], Alp et al. and Bermudo et al. used q -derivatives and integrals to prove two different versions of quantum Hermite–Hadamard (q -H–H) inequalities and some estimates. The q -H–H inequalities are described as:

Theorem 1 ([3, 4]). *For a convex mapping $f: [a, b] \rightarrow \mathbb{R}$, the following inequalities hold for $q \in (0, 1)$:*

$$f\left(\frac{qa+b}{[2]_q}\right) \leq \frac{1}{b-a} I_{a+}^q f(b) \leq \frac{qf(a) + f(b)}{[2]_q}, \quad (1.2)$$

$$f\left(\frac{a+qb}{[2]_q}\right) \leq \frac{1}{b-a} I_{b-}^q f(a) \leq \frac{f(a) + qf(b)}{[2]_q}. \quad (1.3)$$

Remark 1. It is very easy to observe that by adding (1.2) and (1.3), we have the following q -H–H inequality for $q \in (0, 1)$ (see, [4]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)} [I_{a+}^q f(b) + I_{b-}^q f(a)] \leq \frac{f(a) + f(b)}{2}. \quad (1.4)$$

Recently, Ali et al. [1] and Sitthiwirattam et al. [13] used new techniques to prove the following two different and new versions of H–H type inequalities:

Theorem 2 ([1, 13]). *For a convex mapping $f: [a, b] \rightarrow \mathbb{R}$, the following inequalities hold for $q \in (0, 1)$:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \left[I_{\frac{a+b}{2}+}^q f(b) + I_{\frac{a+b}{2}-}^q f(a) \right] \leq \frac{f(a) + f(b)}{2}, \quad (1.5)$$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \left[I_{a+}^q f\left(\frac{a+b}{2}\right) + I_{b-}^q f\left(\frac{a+b}{2}\right) \right] \leq \frac{f(a) + f(b)}{2}. \quad (1.6)$$

Remark 2. By setting the limit as $q \rightarrow 1^-$ in (1.2)–(1.6), we recapture the traditional H–H inequality (1.1).

There has been much research done in the direction of q -integral inequalities for different kinds of convexities. For example, in [5], some new midpoint and trapezoidal type inequalities for q -integrals and q -differentiable convex functions were established. The authors of [2, 6] used q -integral and established Simpson's type inequalities for q -differentiable convex and general convex functions. For more recent inequalities in q -calculus, one can consult [8, 11, 15–17].

As a result of these ongoing studies, we establish a new and general version of q -Hermite–Hadamard type inequality for convex functions. Moreover, we establish some new q -midpoint and q -trapezoidal type inequalities for q -differentiable convex functions. We also show that the newly established inequalities are generalizations of the inequalities proved in [4, 13].

2. PRELIMINARIES OF q -CALCULUS

We shall recall some basics of quantum calculus in this section. For the sake of brevity, let $q \in (0, 1)$ and we use the following notation (see, [9]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \dots + q^{n-1}.$$

Definition 1 ([16]). The left or q_a -derivative of $f: [a, b] \rightarrow \mathbb{R}$ at $x \in [a, b]$ is expressed as:

$${}_a D_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \quad (2.1)$$

Definition 2 ([4]). The right or q^b -derivative of $f: [a, b] \rightarrow \mathbb{R}$ at $x \in [a, b]$ is expressed as:

$${}_b D_q f(x) = \frac{f(qx + (1-q)b) - f(x)}{(1-q)(b-x)}, \quad x \neq b.$$

Definition 3 ([16]). The left or q_a -integral of $f: [a, b] \rightarrow \mathbb{R}$ at $x \in [a, b]$ is defined as:

$$I_{a+}^q f(x) = \int_a^x f(t) {}_a d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a).$$

Definition 4 ([3]). The right or q^b -integral of $f: [a, b] \rightarrow \mathbb{R}$ at $x \in [a, b]$ is defined as:

$$I_{b-}^q f(x) = \int_x^b f(t) {}^b d_q t = (1-q)(b-x) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b).$$

Lemma 1 ([12]). For continuous functions $f, g: [a, b] \rightarrow \mathbb{R}$, the following equality true:

$$\begin{aligned} & \int_0^c g(t) {}^b D_q f(ta + (1-t)b) d_q t \\ &= \frac{1}{b-a} \int_0^c D_q g(t) f(qta + (1-qt)b) {}_0 d_q t - \frac{g(t)f(ta + (1-t)b)}{b-a} \Big|_0^c. \end{aligned}$$

Lemma 2 ([14]). For continuous functions $f, g: [a, b] \rightarrow \mathbb{R}$, the following equality true:

$$\begin{aligned} & \int_0^c g(t) {}_a D_q f(tb + (1-t)a) d_q t \\ &= \frac{g(t)f(tb + (1-t)a)}{b-a} \Big|_0^c - \frac{1}{b-a} \int_0^c D_q g(t) f(qtb + (1-qt)a) {}_0 d_q t. \end{aligned}$$

3. q -HERMITE-HADAMARD INEQUALITY

In this section, we establish a general version of Hermite-Hadamard inequality by using convexity.

Theorem 3. Let $f: [a, b] \rightarrow \mathbb{R}$ be a convex mapping, then we have the following inequality for $\lambda \in (0, 1]$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\lambda(b-a)} [I_{\lambda a + (1-\lambda)b} f(b) + I_{\lambda b + (1-\lambda)a} f(a)] \leq \frac{f(a) + f(b)}{2}. \quad (3.1)$$

Proof. From convexity, we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2} [f(x) + f(y)]. \quad (3.2)$$

By setting $x = tb + (1-t)(\lambda a + (1-\lambda)b)$ and $y = ta + (1-t)(\lambda b + (1-\lambda)a)$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2} [f(tb + (1-t)(\lambda a + (1-\lambda)b)) + f(ta + (1-t)(\lambda b + (1-\lambda)a))]. \quad (3.3)$$

By applying q -integration with respect to t over $[0, 1]$, also from Definitions 3 and 4, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2} \left[\int_0^1 f(tb + (1-t)(\lambda a + (1-\lambda)b)) dt + \int_0^1 f(ta + (1-t)(\lambda b + (1-\lambda)a)) dt \right] \\ &= \frac{1}{2\lambda(b-a)} [I_{(\lambda a + (1-\lambda)b)+} f(b) + I_{(\lambda b + (1-\lambda)a)-} f(a)]. \end{aligned}$$

Hence the first inequality is proved.

We again use convexity to prove the second inequality as follows:

$$\begin{aligned} &f(tb + (1-t)(\lambda a + (1-\lambda)b)) + f(ta + (1-t)(\lambda b + (1-\lambda)a)) \\ &\leq tf(b) + (1-t)f(\lambda a + (1-\lambda)b) + tf(a) + (1-t)f(\lambda b + (1-\lambda)a) \\ &\leq tf(b) + (1-t)(\lambda f(a) + (1-\lambda)f(b)) + tf(a) + (1-t)(\lambda f(b) + (1-\lambda)f(a)) \\ &\leq f(a) + f(b). \end{aligned}$$

Thus, we obtain the required inequality by applying q -integration concerning t over $[0, 1]$, also from Definitions 3 and 4. $\square$

Remark 3. By setting $\lambda = 1$ in Theorem 3, then we recapture inequality (1.4).

Remark 4. If we set $\lambda = \frac{1}{2}$ in Theorem 3, then we recapture inequality (1.5).

Corollary 1. If we set $\lambda = \frac{1}{3}$ in Theorem 3, then we have the following new q -Hermite-Hadamard type inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{3}{2(b-a)} \left[I_{\frac{a+2b}{3}+} f(b) + I_{\frac{2a+b}{3}-} f(a) \right] \leq \frac{f(a) + f(b)}{2}.$$

Remark 5. It is worth mentioning here that for different choices of $\lambda \in (0, 1]$ in Theorem 3, one can obtain several different and new q -Hermite-Hadamard type inequalities.

4. PARAMETERIZED INEQUALITIES

In this section, we establish some new q -midpoint and trapezoidal type inequalities for q -differentiable functions.

Let us begin with the following lemma.

Lemma 3. Let $f: [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a left and right q -differentiable function. If the functions ${}_{\lambda a+(1-\lambda)b}D_q f$ and ${}^{\lambda b+(1-\lambda)a}D_q f$ are continuous and integrable over $[a, b]$, then we have the following new equality for $\lambda \in (0, 1)$:

$$\begin{aligned} & \frac{1}{b-a} [I_{\lambda a+(1-\lambda)b} f(b) + I_{\lambda b+(1-\lambda)a} f(a)] \\ & - \lambda [f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)] \\ & = \lambda^2 (b-a) \left[\int_0^1 (1-qt) {}_{\lambda a+(1-\lambda)b}D_q f(tb + (1-t)(\lambda a + (1-\lambda)b)) {}_0d_q t \right. \\ & \quad \left. - \int_0^1 (1-qt) {}^{\lambda b+(1-\lambda)a}D_q f(ta + (1-t)(\lambda b + (1-\lambda)a)) {}_0d_q t \right]. \end{aligned} \quad (4.1)$$

Proof. From Lemma 2, we have

$$\begin{aligned} & \int_0^1 (1-qt) {}_{\lambda a+(1-\lambda)b}D_q f(tb + (1-t)(\lambda a + (1-\lambda)b)) {}_0d_q t \\ & = \frac{(1-q)f(b)}{\lambda(b-a)} - \frac{f(\lambda a + (1-\lambda)b)}{\lambda(b-a)} \\ & \quad + \frac{q}{\lambda(b-a)} \int_0^1 f(qtb + (1-qt)(\lambda a + (1-\lambda)b)) \\ & = \frac{(1-q)f(b)}{\lambda(b-a)} - \frac{f(\lambda a + (1-\lambda)b)}{\lambda(b-a)} \\ & \quad + \frac{1}{\lambda(b-a)} \left[(1-q) \sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)(\lambda a + (1-\lambda)b)) - (1-q)f(b) \right] \\ & = \frac{1}{\lambda^2(b-a)^2} I_{\lambda a+(1-\lambda)b} f(b) - \frac{f(\lambda a + (1-\lambda)b)}{\lambda(b-a)}. \end{aligned} \quad (4.2)$$

Similarly, from Lemma 1, we have

$$\begin{aligned} & \int_0^1 (1-qt) {}^{\lambda b+(1-\lambda)a}D_q f(ta + (1-t)(\lambda b + (1-\lambda)a)) {}_0d_q t \\ & = \frac{f(\lambda b + (1-\lambda)a)}{\lambda(b-a)} - \frac{1}{\lambda^2(b-a)^2} I_{\lambda b+(1-\lambda)a} f(a). \end{aligned} \quad (4.3)$$

Thus, we obtain the required equality (4.1) by subtracting (4.3) from (4.2) and multiplying the resultant one with $\lambda^2(b-a)$. $\square$

Corollary 2. In Lemma 3, if we set $\lambda = 1$, then we obtain the following identity:

$$\begin{aligned} & \frac{1}{2(b-a)} [I_{a+} f(b) + I_{b-} f(a)] - \frac{f(a) + f(b)}{2} \\ & = \frac{b-a}{2} \left[\int_0^1 (1-qt) {}_aD_q f(tb + (1-t)a) {}_0d_q t \right. \end{aligned}$$

$$- \int_0^1 (1-qt) {}^b Df(ta + (1-t)b) {}_0 d_q t \Big].$$

This equality is new and can be used for finding the right estimates of the inequality (1.4) that have not been established earlier.

Remark 6. In Lemma 3, if we set $\lambda = \frac{1}{2}$, then we obtain the following identity:

$$\begin{aligned} & \frac{1}{b-a} \left[I_{\frac{a+b}{2}+} f(b) + I_{\frac{a+b}{2}-} f(a) \right] - f\left(\frac{a+b}{2}\right) \\ &= \frac{(b-a)}{4} \left[\int_0^1 (1-qt) {}^{\frac{a+b}{2}} D_q f \left(tb + (1-t) \left(\frac{a+b}{2} \right) \right) {}_0 d_q t \right. \\ & \quad \left. - \int_0^1 (1-qt) {}^{\frac{a+b}{2}} Df \left(ta + (1-t) \left(\frac{a+b}{2} \right) \right) {}_0 d_q t \right]. \end{aligned}$$

This equality is established by Sitthiwirattam et al. in [13].

Theorem 4. If the functions $|\lambda a + (1-\lambda)b {}^b D_q f|$ and $|\lambda b + (1-\lambda)a {}^a D_q f|$ are convex and Lemma 3 holds, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{b-a} \left[I_{(\lambda a + (1-\lambda)b)+} f(b) + I_{(\lambda b + (1-\lambda)a)-} f(a) \right] \right. \\ & \quad \left. - \lambda [f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)] \right| \\ & \leq \lambda^2 (b-a) \left[\left(\frac{(1-\lambda)q}{[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) |\lambda a + (1-\lambda)b {}^b D_q f(b)| \right. \\ & \quad + \frac{\lambda q}{[3]_q} |\lambda a + (1-\lambda)b {}^b D_q f(a)| + \left(\frac{(1-\lambda)q}{[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) |\lambda b + (1-\lambda)a {}^a D_q f(a)| \\ & \quad \left. + \frac{\lambda q}{[3]_q} |\lambda b + (1-\lambda)a {}^a D_q f(b)| \right]. \end{aligned}$$

Proof. By taking modulus in (4.1) and from convexity, we have

$$\begin{aligned} & \left| \frac{1}{b-a} \left[I_{(\lambda a + (1-\lambda)b)+} f(b) + I_{(\lambda b + (1-\lambda)a)-} f(a) \right] \right. \\ & \quad \left. - \lambda [f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)] \right| \\ & \leq \lambda^2 (b-a) \left[\int_0^1 (1-qt) |\lambda a + (1-\lambda)b {}^b D_q f(tb + (1-t)(\lambda a + (1-\lambda)b))| {}_0 d_q t \right. \\ & \quad \left. + \int_0^1 (1-qt) |\lambda b + (1-\lambda)a {}^a Df(ta + (1-t)(\lambda b + (1-\lambda)a))| {}_0 d_q t \right] \\ & \leq \lambda^2 (b-a) \left[\int_0^1 (1-qt) (t |\lambda a + (1-\lambda)b {}^b D_q f(b)| + (1-t)\lambda |\lambda a + (1-\lambda)b {}^b D_q f(a)|) {}_0 d_q t \right. \end{aligned}$$

$$\begin{aligned}
& + \int_0^1 (1-qt)(1-\lambda)(1-t) \left| {}_{\lambda a+(1-\lambda)b}D_q f(b) \right| {}_0d_q t \\
& + \int_0^1 (1-qt) \left(t \left| {}^{\lambda b+(1-\lambda)a}D_q f(a) \right| + (1-t)\lambda \left| {}^{\lambda b+(1-\lambda)a}D_q f(b) \right| \right) {}_0d_q t \\
& + \int_0^1 (1-qt)(1-\lambda)(1-t) \left| {}^{\lambda b+(1-\lambda)a}D_q f(a) \right| {}_0d_q t \Big] \\
& = \lambda^2 (b-a) \left[\left(\frac{(1-\lambda)q}{[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) \left| {}_{\lambda a+(1-\lambda)b}D_q f(b) \right| \right. \\
& + \frac{\lambda q}{[3]_q} \left| {}_{\lambda a+(1-\lambda)b}D_q f(a) \right| + \left(\frac{(1-\lambda)q}{[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) \left| {}^{\lambda b+(1-\lambda)a}D_q f(a) \right| \\
& \left. + \frac{\lambda q}{[3]_q} \left| {}^{\lambda b+(1-\lambda)a}D_q f(b) \right| \right].
\end{aligned}$$

Thus, the proof is completed. $\square$

Remark 7. If we set $\lambda = \frac{1}{2}$ in Theorem 4, then we have the following inequality:

$$\begin{aligned}
& \left| \frac{1}{b-a} \left[I_{\frac{a+b}{2}+} f(b) + I_{\frac{a+b}{2}-} f(a) \right] - f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{(b-a)}{4} \left[\left(\frac{q}{2[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) \left| {}_{\frac{a+b}{2}}D_q f(b) \right| + \frac{q}{2[3]_q} \left| {}_{\frac{a+b}{2}}D_q f(a) \right| \right. \\
& \left. + \left(\frac{q}{2[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) \left| {}^{\frac{a+b}{2}}D_q f(a) \right| + \frac{q}{2[3]_q} \left| {}^{\frac{a+b}{2}}D_q f(b) \right| \right].
\end{aligned}$$

This inequality is established by Sitthiwiratham et al. in [13].

Corollary 3. In Theorem 4, if we set $\lambda = 1$, then we have the following new trapezoidal type inequality linked with the inequality (1.4):

$$\begin{aligned}
& \left| \frac{1}{2(b-a)} \left[I_{a+} f(b) + I_{b-} f(a) \right] - \frac{f(a) + f(b)}{2} \right| \\
& \leq \frac{(b-a)}{2} \left[\frac{1}{[4]_q + q[2]_q} \left| {}_a D_q f(b) \right| + \frac{q}{[3]_q} \left| {}_a D_q f(a) \right| \right. \\
& \left. + \frac{1}{[4]_q + q[2]_q} \left| {}^b D_q f(a) \right| + \frac{q}{[3]_q} \left| {}^b D_q f(b) \right| \right].
\end{aligned}$$

Theorem 5. *If the functions $|\lambda a + (1-\lambda)b D_q f|^s$ and $|\lambda b + (1-\lambda)a D_q f|^s$, for $s \geq 1$ are convex and Lemma 3 holds, then the following inequality holds:*

$$\begin{aligned}
& \left| \frac{1}{b-a} [I_{(\lambda a + (1-\lambda)b)+} f(b) + I_{(\lambda b + (1-\lambda)a)-} f(a)] \right. \\
& \quad \left. - \lambda [f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)] \right| \\
& \leq \lambda^2 (b-a) \left(\frac{1}{[2]_q} \right)^{1-\frac{1}{s}} \left[\left(\left(\frac{(1-\lambda)q}{[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) |\lambda a + (1-\lambda)b D_q f(b)|^s \right. \right. \\
& \quad \left. \left. + \frac{\lambda q}{[3]_q} |\lambda a + (1-\lambda)b D_q f(a)|^s \right)^{\frac{1}{s}} \right. \\
& \quad \left. + \left(\left(\frac{(1-\lambda)q}{[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) |\lambda b + (1-\lambda)a D_q f(a)|^s \right. \right. \\
& \quad \left. \left. + \frac{\lambda q}{[3]_q} |\lambda b + (1-\lambda)a D_q f(b)|^s \right)^{\frac{1}{s}} \right].
\end{aligned}$$

Proof. By taking modulus in (4.1) and power mean inequality, we have

$$\begin{aligned}
& \left| \frac{1}{b-a} [I_{(\lambda a + (1-\lambda)b)+} f(b) + I_{(\lambda b + (1-\lambda)a)-} f(a)] \right. \\
& \quad \left. - \lambda [f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)] \right| \\
& \leq \lambda^2 (b-a) \left(\int_0^1 (1-qt) {}_0d_q t \right)^{1-\frac{1}{s}} \\
& \quad \times \left[\left(\int_0^1 (1-qt) |\lambda a + (1-\lambda)b D_q f(tb + (1-t)(\lambda a + (1-\lambda)b))|^s {}_0d_q t \right)^{\frac{1}{s}} \right. \\
& \quad \left. + \left(\int_0^1 (1-qt) |\lambda b + (1-\lambda)a D_q f(ta + (1-t)(\lambda b + (1-\lambda)a))|^s {}_0d_q t \right)^{\frac{1}{s}} \right].
\end{aligned}$$

By using the convexity of the functions $|\lambda a + (1-\lambda)b D_q f|^s$ and $|\lambda b + (1-\lambda)a D_q f|^s$, we have

$$\begin{aligned}
& \left| \frac{1}{b-a} [I_{(\lambda a + (1-\lambda)b)+} f(b) + I_{(\lambda b + (1-\lambda)a)-} f(a)] \right. \\
& \quad \left. - \lambda [f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)] \right| \\
& \leq \lambda^2 (b-a) \left(\frac{1}{[2]_q} \right)^{1-\frac{1}{s}} \times \left[\left(\left(\frac{(1-\lambda)q}{[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) |\lambda a + (1-\lambda)b D_q f(b)|^s \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{\lambda q}{[3]_q} \left| \lambda a + (1-\lambda)b D_q f(a) \right|^s \Big)^{\frac{1}{s}} \\
& + \left(\left(\frac{(1-\lambda)q}{[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) \left| \lambda b + (1-\lambda)a D_q f(a) \right|^s \right. \\
& \left. + \frac{\lambda q}{[3]_q} \left| \lambda b + (1-\lambda)a D_q f(b) \right|^s \right)^{\frac{1}{s}} \Big].
\end{aligned}$$

Thus, the proof is completed. $\square$

Remark 8. If we set $\lambda = \frac{1}{2}$ in Theorem 4, then we have the following inequality:

$$\begin{aligned}
& \left| \frac{1}{b-a} \left[I_{\frac{a+b}{2}+} f(b) + I_{\frac{a+b}{2}-} f(a) \right] - f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{(b-a)}{4} \left(\frac{1}{[2]_q} \right)^{1-\frac{1}{s}} \\
& \times \left[\left(\left(\frac{q}{2[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) \left| \frac{a+b}{2} D_q f(b) \right|^s + \frac{q}{2[3]_q} \left| \frac{a+b}{2} D_q f(a) \right|^s \right)^{\frac{1}{s}} \right. \\
& \left. + \left(\left(\frac{q}{2[3]_q} + \frac{1}{[4]_q + q[2]_q} \right) \left| \frac{a+b}{2} D_q f(a) \right|^s + \frac{q}{2[3]_q} \left| \frac{a+b}{2} D_q f(b) \right|^s \right)^{\frac{1}{s}} \right]
\end{aligned}$$

This inequality is established by Sitthiwiratham et al. in [13].

Corollary 4. In Theorem 4, if we set $\lambda = 1$, then we have the following new trapezoidal type inequality linked with the inequality (1.4):

$$\begin{aligned}
& \left| \frac{1}{2(b-a)} \left[I_{a+} f(b) + I_{b-} f(a) \right] - \frac{f(a) + f(b)}{2} \right| \\
& \leq \frac{(b-a)}{2} \left[\frac{1}{[4]_q + q[2]_q} \left| {}_a D_q f(b) \right| + \frac{q}{[3]_q} \left| {}_a D_q f(a) \right| \right. \\
& \left. + \frac{1}{[4]_q + q[2]_q} \left| {}^b D_q f(a) \right| + \frac{q}{[3]_q} \left| {}^b D_q f(b) \right| \right].
\end{aligned}$$

Theorem 6. If the functions $\left| \lambda a + (1-\lambda)b D_q f \right|^s$ and $\left| \lambda b + (1-\lambda)a D_q f \right|^s$, for $s > 1$ are convex and Lemma 3 holds, then the following inequality holds:

$$\left| \frac{1}{b-a} \left[I_{\lambda a + (1-\lambda)b} f(b) + I_{\lambda b + (1-\lambda)a} f(a) \right] - f\left(\frac{a+b}{2}\right) \right|$$

$$\begin{aligned}
& -\lambda [f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)] \Big| \\
& \leq \lambda^2 (b-a) \left(\frac{1-(1-q)^{r+1}}{q[r+1]_q} \right)^{\frac{1}{r}} \times \left[\left(\left(\frac{1}{[2]_q} (1+q(1-\lambda)) \right) \Big|_{\lambda a+(1-\lambda)b} D_q f(b) \right)^s \right. \\
& \quad + \frac{\lambda q}{[2]_q} \Big|_{\lambda a+(1-\lambda)b} D_q f(a) \Big|^s \Big]^{\frac{1}{s}} + \left(\left(\frac{1}{[2]_q} (1+q(1-\lambda)) \right) \Big|_{\lambda b+(1-\lambda)a} D_q f(a) \right)^s \\
& \quad + \frac{\lambda q}{[2]_q} \Big|_{\lambda b+(1-\lambda)a} D_q f(b) \Big|^s \Big]^{\frac{1}{s}} \Big].
\end{aligned}$$

where $s^{-1} + r^{-1} = 1$.

Proof. By taking modulus in (4.1) and Hölder inequality, we have

$$\begin{aligned}
& \left| \frac{1}{b-a} [I_{(\lambda a+(1-\lambda)b)+} f(b) + I_{(\lambda b+(1-\lambda)a)-} f(a)] \right. \\
& \quad \left. - \lambda [f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)] \right| \\
& \leq \frac{(b-a)}{4} \left(\int_0^1 (1-qt)^r {}_0d_q t \right)^{\frac{1}{r}} \\
& \quad \times \left[\left(\int_0^1 \left| \lambda a+(1-\lambda)b D_q f(tb + (1-t)(\lambda a + (1-\lambda)b)) \right|^s {}_0d_q t \right)^{\frac{1}{s}} \right. \\
& \quad \left. + \left(\int_0^1 \left| \lambda b+(1-\lambda)a D_q f(ta + (1-t)(\lambda b + (1-\lambda)a)) \right|^s {}_0d_q t \right)^{\frac{1}{s}} \right].
\end{aligned}$$

Since the functions $\left| \lambda a+(1-\lambda)b D_q f \right|^s$ and $\left| \lambda b+(1-\lambda)a D_q f \right|^s$ are convex, we have

$$\begin{aligned}
& \left| \frac{1}{b-a} [I_{(\lambda a+(1-\lambda)b)+} f(b) + I_{(\lambda b+(1-\lambda)a)-} f(a)] \right. \\
& \quad \left. - \lambda [f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)] \right| \\
& \leq \lambda^2 (b-a) \left(\frac{1-(1-q)^{r+1}}{q[r+1]_q} \right)^{\frac{1}{r}} \times \left[\left(\left(\frac{1}{[2]_q} (1+q(1-\lambda)) \right) \Big|_{\lambda a+(1-\lambda)b} D_q f(b) \right)^s \right. \\
& \quad + \frac{\lambda q}{[2]_q} \Big|_{\lambda a+(1-\lambda)b} D_q f(a) \Big|^s \Big]^{\frac{1}{s}} + \left(\left(\frac{1}{[2]_q} (1+q(1-\lambda)) \right) \Big|_{\lambda b+(1-\lambda)a} D_q f(a) \right)^s \\
& \quad + \frac{\lambda q}{[2]_q} \Big|_{\lambda b+(1-\lambda)a} D_q f(b) \Big|^s \Big]^{\frac{1}{s}} \Big].
\end{aligned}$$

Hence, the proof is completed. $\square$

Remark 9. In Theorem 6, if we set $\lambda = \frac{1}{2}$, then we have the following inequality:

$$\begin{aligned} & \left| \frac{1}{b-a} \left[I_{\frac{a+b}{2}+} f(b) + I_{\frac{a+b}{2}-} f(a) \right] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{(b-a)}{4} \left(\frac{1 - (1-q)^{r+1}}{q[r+1]_q} \right)^{\frac{1}{r}} \times \left[\left(\frac{2+q}{2[2]_q} \left| \frac{a+b}{2} D_q f(b) \right|^s + \frac{q}{2[2]_q} \left| \frac{a+b}{2} D_q f(a) \right|^s \right)^{\frac{1}{s}} \right. \\ & \quad \left. + \left(\frac{2+q}{2[2]_q} \left| \frac{a+b}{2} D_q f(a) \right|^s + \frac{q}{2[2]_q} \left| \frac{a+b}{2} D_q f(b) \right|^s \right)^{\frac{1}{s}} \right]. \end{aligned}$$

This inequality is established by Sitthiwiratham et al. in [13].

Corollary 5. In Theorem 6, if we set $\lambda = 1$, then we have the following new trapezoidal type inequality linked with the inequality (1.4):

$$\begin{aligned} & \left| \frac{1}{2(b-a)} [I_{a+} f(b) + I_{b-} f(a)] - \frac{f(a) + f(b)}{2} \right| \\ & \leq \frac{(b-a)}{2} \left(\frac{1 - (1-q)^{r+1}}{q[r+1]_q} \right)^{\frac{1}{r}} \left[\left(\frac{1}{[2]_q} \left| {}_a D_q f(b) \right|^s + \frac{q}{[2]_q} \left| {}_a D_q f(a) \right|^s \right)^{\frac{1}{s}} \right. \\ & \quad \left. + \left(\frac{1}{[2]_q} \left| {}_b D_q f(a) \right|^s + \frac{q}{[2]_q} \left| {}_b D_q f(b) \right|^s \right)^{\frac{1}{s}} \right]. \end{aligned}$$

5. EXAMPLES

In this part of the paper, we give some examples to show the validity of newly established inequalities.

Example 1. We consider a convex function $f(x) = x^2$. From Theorem 3 with $a = 1$, $b = 2$, $q = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, we have

$$f\left(\frac{a+b}{2}\right) = \left(\frac{3}{2}\right)^2 = \frac{9}{4},$$

$$\begin{aligned} & \frac{3}{2(b-a)} \left[I_{\frac{a+2b}{3}+} f(b) + I_{\frac{2a+b}{3}-} f(a) \right] \\ & = \frac{3}{2} \left(\left(1 - \frac{1}{2}\right) \left(2 - \frac{5}{3}\right) \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(\left(\frac{1}{2}\right)^n \times 2 + \left(1 - \left(\frac{1}{2}\right)^n\right) \frac{5}{3} \right)^2 \right. \end{aligned}$$

$$+ \left(1 - \frac{1}{2}\right) \left(\frac{4}{3} - 1\right) \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(\left(\frac{1}{2}\right)^n + \left(1 - \left(\frac{1}{2}\right)^n\right) \frac{4}{3} \right)^2 = \frac{913}{378}$$

and

$$\frac{f(a) + f(b)}{2} = \frac{1 + 2^2}{2} = \frac{5}{2}.$$

Thus,

$$\frac{9}{4} < \frac{913}{378} < \frac{5}{2}$$

and Theorem 3 is valid.

Example 2. We consider a convex function $f(x) = x^2$. From Theorem 3 with $a = 1$, $b = 2$, $q = \frac{1}{2}$ and $\lambda = \frac{1}{4}$, we have

$$f\left(\frac{a+b}{2}\right) = \left(\frac{3}{2}\right)^2 = \frac{9}{4},$$

$$\frac{2}{(b-a)} \left[I_{\frac{a+3b}{4}+} f(b) + I_{\frac{3a+b}{4}-} f(a) \right] = \frac{817}{336}$$

and

$$\frac{f(a) + f(b)}{2} = \frac{1 + 2^2}{2} = \frac{5}{2}.$$

Thus,

$$\frac{9}{4} < \frac{817}{336} < \frac{5}{2}$$

and Theorem 3 is valid.

Example 3. We consider a convex function $f(x) = x^2$. From Theorem 4 with $a = 1$, $b = 2$, $q = \frac{1}{2}$ and $\lambda = 1$, we have

$$\left| \frac{1}{2(b-a)} [I_{a+} f(b) + I_{b-} f(a)] - \frac{f(a) + f(b)}{2} \right| = \frac{2}{21}$$

and

$$\begin{aligned} & \frac{(b-a)}{2} \left[\frac{1}{[4]_q + q[2]_q} |{}_a D_q f(b)| + \frac{q}{[3]_q} |{}_a D_q f(a)| \right. \\ & \quad \left. + \frac{1}{[4]_q + q[2]_q} |{}_b D_q f(a)| + \frac{q}{[3]_q} |{}_b D_q f(b)| \right] \\ &= \frac{1}{2} \left(\frac{1}{1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \frac{1}{2}(1 + \frac{1}{2})} \times \frac{7}{2} + \frac{\frac{1}{2}}{1 + \frac{1}{2} + (\frac{1}{2})^2} \times 2 \right. \\ & \quad \left. + \frac{1}{1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \frac{1}{2}(1 + \frac{1}{2})} \times \frac{5}{2} + \frac{\frac{1}{2}}{1 + \frac{1}{2} + (\frac{1}{2})^2} \times 4 \right) = 2. \end{aligned}$$

Thus,

$$\frac{2}{21} < 2$$

and Theorem 4 is valid.

Remark 10. Similarly, one can check the validity of all the results.

6. CONCLUDING REMARKS

In this paper, we established a family of q -Hermite–Hadamard type inequalities for convex functions. Moreover, we proved some new integral inequalities for differentiable convex functions in the setting of q -calculus. The new inequalities can help us to find the error bounds for the midpoint, trapezoidal, and many other numerical integration formulas in q -calculus. It is also shown with the examples that the newly established inequalities are valid for convex functions. It is an interesting problem that the upcoming researchers can obtain similar inequalities for the functions of two variables.

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A LEGENDRE WAVELET COLLOCATION METHOD FOR SOLVING NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. The Legendre wavelet based method has been employed in this paper to investigate neutral delay differential equations. The highest order derivative is approximated by Legendre wavelet using the integral operator technique. Then integrations of Legendre wavelet are used to approximate the lower order derivatives and unknown function. To get an algebraic system of linear or nonlinear equations, approximated values of unknown function and its derivatives are substituted in neutral delay differential equations. On solving the developed system, we get unknown wavelet coefficients and subsequently the approximate solution. To analyze the theoretical usability of the approach, the upper bound of error norm is established. Moreover, the theoretical results are confirmed through few numerical experiments. A comparison of the results of presented method with method available in literature is given to conclude the superiority of the proposed method.

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1. INTRODUCTION

The term “delay” in differential equations refers to the temporal latencies between observation and control action in the mathematical formulation of natural and technological problems. So we can say that a differential equation in which delay exist in unknown variable and/or its derivatives, is known as delay differential equation. It occurs in many real-life problems, such as SARS-CoV-2, where there is a temporal lag between the period of an infected person transmits the virus, and the time susceptible person get infected. The main reason for which, neutral delay differential equations (NDDEs) is getting attention is that it plays vital role in many areas of sciences. NDDEs have wide range of applications in applied mathematics, physics, ecology,

engineering, etc. For example, in biological science, delay differential equation exhibit better picture of population fluctuation than ordinary differential equation [16]. It has many applications in the dynamical system too. As a result, it can be claimed that most of the mathematical problems have a time delay.

We shall investigate the following type of neutral delay differential equation (NDDE):

$$\begin{aligned} w''(t) &= f(t, w(t), w(t - \pi_1(t, w(t))), w'(t), w'(t - \pi_2(t, w(t))), \\ &\quad w''(t), w''(t - \pi_3(t, w(t))), \quad t \in [\alpha, \beta], \end{aligned} \quad (1.1)$$

with initial and delay conditions

$$w(t) = \phi(t), \quad t \leq \alpha, \quad (1.2)$$

and boundary conditions (BCs):

$$y(\alpha) = \xi, \quad \text{and} \quad y(\beta) = \eta, \quad (1.3)$$

where $f : [\alpha, \beta] \times \mathbb{R}^5 \rightarrow \mathbb{R}$ is a differentiable function, $\pi_1(t, w(t))$, $\pi_2(t, w(t))$, and $\pi_3(t, w(t))$ are continuous functions on $[\alpha, \beta] \times \mathbb{R}$ such that $t - \pi_1(t, w(t))$, $t - \pi_2(t, w(t))$, $t - \pi_3(t, w(t)) < \beta$. Also, $\phi(t)$ represents the initial function which is given in Vanani and Aminataei [23].

In the late 1970s, technique for solving second-order delay differential equation having constant delay with boundary conditions was given by Nevers and Schmitt [11], where they solved the problem by using Euler's method and shooting technique.

In literature review, we found various methods which have been used to solve DDEs, like finite difference, Richardson method and collocation method (see [10], [6], [15], [19], [14], [20]). For more details about the collocation methods for a variety of differential equations, please refer to ([4], [1], [2], [3], [7], [17], [18]). However very few people have obtained numerical solution of second order DDEs for non constant delay with BCs. For example, convergent method of order three was given by Backke and Jackiewicz [6] where they applied Richardson's extrapolation technique, and Cahlon and Nachman [8] solved time dependent delay problems using Atkinson's product integration technique.

In recent decades, methods based on wavelet basis function has grown manifolds and the main reason of its attraction is that, they are more efficient and gives more accurate numerical results as compared to other well known methods (see [5], [14], [21], [13]). Wavelets are a powerful and efficient mathematical tool that divides a data function into multiple frequency constituents and investigates each constituent with a resolution that corresponds to its scale [22].

This paper aims to propose integral operator approach in which the highest order derivative is approximated in terms of Legendre wavelet for solving second order NDDEs with boundary conditions. The paper's outline is presented as follows. In Section 2, we present definition of multiresolution analysis, basic definition of wavelet, Legendre wavelet and function approximation by Legendre wavelet. In Section 3, we discussed our method for solving NDDEs. We carried out the Legendre wavelet's

convergence analysis in Section 4. Section 5, consist of two linear and one nonlinear problem respectively, to demonstrate the proposed method's validation using maximum absolute errors. Moreover, we made a comparison between exact solution and existing method such as direct block method [15].

2. PRELIMINARY CONSIDERATIONS

This section gives an insight into the basic definitions that is used in the rest part of the paper.

Definition 2.1. A multiresolution analysis (MRA), which is also referred as the wavelet's 'heart', was first introduced in the year 1989. It plays a vital role in writing the wavelet in a broad sense. It gives the ability to write any arbitrary function $w \in \mathcal{L}^2(\mathbb{R})$ over the multiresolution approximation space. MRA's goal is to break down the entire function spaces into spaces, $\mathcal{V}^k$ and $\mathcal{W}^k$, namely wavelet subspace and scaling function subspace, respectively. An arbitrary function $w \in \mathcal{L}^2(\mathbb{R})$ is projectable on $\mathcal{V}^k$, if $\mathcal{V}^k$ satisfies the following conditions:

- (1) $\mathcal{V}^k \subset \mathcal{V}^{k+1}$,
- (2) $\bigcup_{k \in \mathbb{Z}} \mathcal{V}^k = \mathcal{L}^2(\mathbb{R})$, i.e., $\{\mathcal{V}^k\}$'s are dense in $\mathcal{L}^2(\mathbb{R})$,
- (3) The collection $\{\phi(t-n), n \in \mathbb{Z}\}$ forms an orthonormal basis for $\mathcal{V}^0$,
- (4) $w(\cdot) \in \mathcal{V}^k \iff w(2\cdot) \in \mathcal{V}^{k+1}$, for all $k \in \mathbb{N}$,
- (5) $\bigcap_{k \in \mathbb{Z}} \mathcal{V}^k = \{0\}$, i.e., there is nothing common in all the subspaces.

The wavelet subspace is defined in the following manner:

$$\mathcal{W}^k = \left\{ \psi_k^\lambda; k, \lambda \in \mathbb{Z} \right\},$$

where $\mathcal{W}^k$ is orthogonal complement of $\mathcal{V}^k$ in $\mathcal{V}^{k+1}$ such that

$$\mathcal{V}^{k+1} = \mathcal{V}^k \oplus \mathcal{W}^k. \quad (2.1)$$

On repeating the above steps, we get

$$\mathcal{V}^K = \mathcal{V}^{K_0} \oplus \bigoplus_{k=K_0}^{K-1} \mathcal{W}^k, \quad K > K_0. \quad (2.2)$$

If $P_{\mathcal{V}^K}$ project any arbitrary function $w(t) \in \mathcal{L}^2(\mathbb{R})$ on $\mathcal{V}^k$, we can conclude from dense criteria of MRA that

$$P_{\mathcal{V}^K} w(t) \longrightarrow w(t), \quad \text{as } K \longrightarrow \infty. \quad (2.3)$$

From equations (2.2) and (2.3), we can define scaling function projection and wavelet projection in the following way:

$$P_{\mathcal{V}^K} w(t) \approx \sum_{\lambda} \tilde{\mu}_k^\lambda \phi_k^\lambda(t), \quad (2.4)$$

$$P_{qK} w(t) \approx \sum_{\lambda} \tilde{\mu}_{K_0}^{\lambda} \phi_{K_0}^{\lambda}(t) + \sum_{\lambda} \sum_{k=K_0}^{K-1} \mu_k^{\lambda} \psi_k^{\lambda}(t), \quad (2.5)$$

where the coefficients μ_k^{λ} and $\tilde{\mu}_{K_0}^{\lambda}$ can be evaluated by applying the orthogonal property of the wavelet $\psi(t)$ and scaling function $\phi(t)$ as

$$\tilde{\mu}_{K_0}^{\lambda} = \int_{-\infty}^{\infty} w(t) \phi_{K_0}^{\lambda}(t) dt, \quad \mu_k^{\lambda} = \int_{-\infty}^{\infty} w(t) \psi_k^{\lambda}(t) dt. \quad (2.6)$$

2.1. Wavelet and Legendre Wavelet

Definition 2.2. Any orthogonal system which comes from MRA is called mother wavelet, if its total integration is exactly zero. i.e.

$$\int_{-\infty}^{\infty} \psi(t) dt = 0.$$

The dilation and translation of mother wavelet gives birth to a group of functions which is referred as wavelet, and defined as:

$$\psi_{\mathcal{D}}^{\mathcal{T}}(t) = \frac{1}{\sqrt{\mathcal{D}}} \psi\left(\frac{t - \mathcal{T}}{\mathcal{D}}\right), \quad \mathcal{D} \neq 0, \mathcal{T} \in \mathbb{R}, \quad (2.7)$$

where $\mathcal{D}$ and $\mathcal{T}$ are representing dilation and translation parameters, respectively ([9], [12]). On restricting these two parameters upto the discrete values $\mathcal{D} = \mathcal{D}_0^{-k}$, $\mathcal{T} = \lambda \mathcal{T}_0 \mathcal{D}_0^{-k}$, where $\mathcal{D}_0 > 1$ and $\mathcal{T}_0 > 1$, we get the following discrete wavelet:

$$\psi_k^{\lambda} = \left(\sqrt{\mathcal{D}_0}\right)^k (\mathcal{D}_0^k t - \lambda \mathcal{T}_0). \quad (2.8)$$

Definition 2.3. The n^{th} -order Legendre polynomials having weight function $w(t) = 1$, denoted by $\mathcal{P}_n(t)$ are orthogonal system on the interval $[-1, 1]$, it can be determined by the following recurrence relation:

$$\begin{aligned} \mathcal{P}_{m+1}(t) &= \left(\frac{2m+1}{m+1}\right) t \mathcal{P}_m(t) - \left(\frac{m}{m+1}\right) \mathcal{P}_{m-1}(t), \quad m = 1, 2, 3 \dots \\ \mathcal{P}_0(t) &= 1, \mathcal{P}_1(t) = t, \dots \end{aligned}$$

Now, we define Legendre wavelet with four arguments in such a way that $\mathcal{L}_k^{\lambda}(t) = \mathcal{L}(k, \lambda, m, t)$ over the interval $[0, 1)$ [24],

$$\mathcal{L}_k^{\lambda}(t) = \begin{cases} \sqrt{m + \frac{1}{2}} 2^{\frac{k}{2}} \mathcal{P}_m(2^k t - \lambda), & t \in \left[\frac{\lambda-1}{2^k}, \frac{\lambda+1}{2^k}\right) \\ 0, & \text{elsewhere} \end{cases} \quad (2.9)$$

where 'm' denotes the order of Legendre polynomial varying from 0 to a fixed positive value $M-1$, $\lambda = 1, 2, 3, \dots, 2^k - 1$. The family of Legendre wavelet produces an orthonormal basis of $\mathcal{L}^2(\mathbb{R})$. For each pair of m and λ , we can re-write the Legendre wavelet in single index in the form:

$$\mathcal{L}_i(t) = \mathcal{L}_k^{\lambda}(t), \quad (2.10)$$

where the wavelet number ι satisfies, $\iota = \lambda + 2^{k-1}m$.

2.1.1. Approximation of function by Legendre wavelet

An arbitrary function $w \in \mathcal{L}^2[0, 1]$ is capable of being expanded into series of Legendre wavelet as [25], from equation (2.5), we get

$$w(t) = \sum_{\lambda} \tilde{\mu}_{K_0}^{\lambda} \mathcal{L}_{K_0}^{\lambda}(t) + \sum_{\lambda} \sum_{k=K_0}^{K-1} \mu_k^{\lambda} \mathcal{L}_k^{\lambda}(t) = \sum_{\iota=1}^{\infty} \mu_{\iota} \mathcal{L}_{\iota}(t). \quad (2.11)$$

To get the best approximation, we truncate the infinite series given in equation (2.11) in the manner shown below for a fixed natural number N :

$$w(t) \approx \sum_{\iota=1}^N \mu_{\iota} \mathcal{L}_{\iota}(t) = \mu^T \mathcal{L}(t), \quad (2.12)$$

where

$$\begin{aligned} \mu^T &= [\mu_1, \mu_2, \dots, \mu_N], \\ \mathcal{L}(t) &= [\mathcal{L}_1(t), \mathcal{L}_2(t), \dots, \mathcal{L}_N(t)]^T, \end{aligned}$$

where $N = 2^{k-1}M$ and collocation points are determined by $t(p) = \frac{p-0.5}{N}$, where $1 \leq p \leq N$, $N = 2^K$, $K \in \mathbb{N}$.

2.1.2. Integration of Legendre Wavelet

Let us denote the first and second integrations of Legendre wavelet by $\mathcal{J}_1$ and $\mathcal{J}_2$. These integrations can be easily calculated using equation (2.9) and can be expressed as follows:

$$\begin{aligned} \mathcal{J}_1^1(t) &= \int_0^t \mathcal{L}_{\iota}(\tilde{t}) d\tilde{t}, \\ \mathcal{J}_1^2(t) &= \int_0^t \int_0^{\tilde{t}} \mathcal{L}_{\iota}(\tilde{\tilde{t}}) d\tilde{\tilde{t}} d\tilde{t}. \end{aligned}$$

The above integrations can be obtained as :

$$\mathcal{J}_1^1(t) = \begin{cases} \left(\frac{1}{\sqrt{2}}\right)^k \rho\left(\frac{1}{2m+1}\right) \{\mathcal{P}_{m+1}(t) - \mathcal{P}_{m-1}(t) - \mathcal{P}_{m+1}(-1) \\ \quad + \mathcal{P}_{m-1}(-1)\}, & t \in [\mathbf{v}_1, \mathbf{v}_2) \\ \left(\frac{1}{\sqrt{2}}\right)^k \rho\left(\frac{1}{2m+1}\right) \{\mathcal{P}_{m+1}(1) - \mathcal{P}_{m-1}(1) - \mathcal{P}_{m+1}(-1) \\ \quad + \mathcal{P}_{m-1}(-1)\}, & t \in [\mathbf{v}_2, 1) \end{cases}$$

$$\mathcal{J}_i^2(t) = \begin{cases} \left(\frac{1}{\sqrt{2}}\right)^{3k} \rho\left(\frac{1}{2m+1}\right) \left\{ \left(\frac{1}{2m+2}\right) \{ \mathcal{P}_{m+2}(t) - \mathcal{P}_m(t) - \mathcal{P}_{m+2}(-1) + \mathcal{P}_m(-1) \} \right. \\ \quad \left. - \left(\frac{1}{2m-2}\right) \{ \mathcal{P}_m(t) - \mathcal{P}_{m-2}(t) - \mathcal{P}_m(-1) + \mathcal{P}_{m-2}(-1) \} \right. \\ \quad \left. - 2(t+1) \{ \mathcal{P}_{m+1}(-1) + \mathcal{P}_{m-1}(-1) \} \right\}, & t \in [\mathbf{v}_1, \mathbf{v}_2) \\ \left(\frac{1}{\sqrt{2}}\right)^{3k} \rho\left(\frac{1}{2m+1}\right) \left\{ \left(\frac{1}{2m+2}\right) \{ \mathcal{P}_{m+1}(1) - \mathcal{P}_{m-1}(1) - \mathcal{P}_{m+1}(-1) \} \right. \\ \quad \left. + \mathcal{P}_{m-1}(-1) \right\} - \left(\frac{1}{2m-2}\right) \{ \mathcal{P}_m(1) - \mathcal{P}_{m-2}(1) - \mathcal{P}_m(-1) \\ \quad + \mathcal{P}_{m-2}(-1) \} - 2 \{ \mathcal{P}_{m+1}(-1) + \mathcal{P}_{m-1}(-1) \} + (t-1) \\ \quad \times \{ \mathcal{P}_{m+1}(1) - \mathcal{P}_{m-1}(1) - \mathcal{P}_{m+1}(-1) + \mathcal{P}_{m-1}(-1) \} \}, & t \in [\mathbf{v}_2, 1) \end{cases}$$

where $\mathbf{v}_1 = \frac{\lambda-1}{2^k}$, $\mathbf{v}_2 = \frac{\lambda+1}{2^k}$ and $\rho = \sqrt{\frac{2m+1}{2}}$.

3. METHOD FOR SOLUTION OF NDDE

For the sake of convenience, we use ‘ Σ ’ instead of $\sum_{i=1}^n$ throughout the paper.

Now, approximate the higher order derivative in the form of Legendre wavelet

$$w''(t) \approx \sum d_i \mathcal{L}_i(t). \quad (3.1)$$

Now, integrate equation (3.1) twice from 0 to t, we get

$$w'(t) \approx \sum d_i \mathcal{J}_i^1(t) + w'(0), \quad (3.2)$$

$$w(t) \approx \sum d_i \mathcal{J}_i^2(t) + t w'(0) + w(0). \quad (3.3)$$

On putting $t = 1$ in equation (3.3), we get

$$\begin{aligned} w(1) &\approx w(0) + w'(0) + \sum d_i \mathcal{J}_i^2(1), \\ w'(0) &\approx w(1) - w(0) - \sum d_i \mathcal{J}_i^2(1). \end{aligned} \quad (3.4)$$

On substituting $w'(0)$ in equations (3.2) and (3.3), we get the following equations:

$$w'(t) \approx \sum d_i \mathcal{J}_i^1(t) + \Lambda - \sum d_i \mathcal{J}_i^2(1), \quad (3.5)$$

$$w(t) \approx \sum d_i \mathcal{J}_i^2(t) + t(\Lambda - \sum d_i \mathcal{J}_i^2(1)) + w(0), \quad (3.6)$$

where $\Lambda = w(1) - w(0)$.

Replace t by $(t - \pi_3(t, w(t)))$, $(t - \pi_2(t, w(t)))$ and $(t - \pi_1(t, w(t)))$ in equations (3.1), (3.5) and (3.6) respectively, we get

$$w''(t - \pi_3(t, w(t))) \approx \sum d_i \mathcal{L}_i(t - \pi_3(t, w(t))), \quad (3.7)$$

$$w'(t - \pi_2(t, w(t))) \approx \sum d_i \mathcal{J}_i^1(t - \pi_2(t, w(t))) + w'(0), \quad (3.8)$$

$$\begin{aligned} w(t - \pi_1(t, w(t))) &\approx \sum d_i \mathcal{J}_i^2(t - \pi_1(t, w(t))) + (t - \pi_1(t, w(t))) \\ &\quad \times (\Lambda - \sum d_i \mathcal{J}_i^2(1)) + w(0). \end{aligned} \quad (3.9)$$

On substituting the values from equations (3.1) and ((3.5)-(3.9)), in equation (1.1) we get the following system of equation:

$$\begin{aligned} \sum d_i \mathcal{L}_i \approx & f(t, \sum d_i \mathcal{J}_i^2(t) + t(\Lambda - \sum d_i \mathcal{J}_i^2(1)) + w(0), \sum d_i \mathcal{J}_i^2(t - \pi_1(t, w(t))) \\ & + (t - \pi_1(t, w(t))) \times (\Lambda - \sum d_i \mathcal{J}_i^2(1)) + w(0), \sum d_i \mathcal{J}_i^1(t) \\ & + \Lambda - \sum d_i \mathcal{J}_i^2(1), \sum d_i \mathcal{J}_i^1(t - \pi_2(t, w(t))) \\ & + \Lambda - \sum d_i \mathcal{J}_i^2(1), \sum d_i \mathcal{L}_i(t), \sum d_i \mathcal{L}_i(t - \pi_3(t, w(t))))). \end{aligned} \quad (3.10)$$

We determine the Legendre wavelet coefficients by solving the above system of equations. Then, in order to obtain the approximate solution, we put the values of these coefficients into equation (3.6). While dealing with nonlinear NDDE, we employ Newton's method to solve the resulting system.

4. CONVERGENCE ANALYSIS

The convergence analysis of Legendre wavelet basis is covered in this section.

To demonstrate the convergence analysis of the proposed method we use the analytic version of equation (3.6) as

$$w(t) = \sum_{i=1}^{\infty} d_i \mathcal{J}_i^2(t) + t(\Lambda - \sum_{i=1}^{\infty} d_i \mathcal{J}_i^2(1)) + w(0). \quad (4.1)$$

Theorem 1. Let $w(t) \in \mathcal{L}^2[0, 1]$ be such that $|w''(t)| \leq \alpha_0, \forall t \in (0, 1)$ and $\alpha_0 > 0$. If $w''(t) = \sum_{i=1}^{\infty} d_i \mathcal{L}_i(t)$, then we have the following inequality:

$$|d_i| \leq 2^{-\frac{k}{2}} \sigma \alpha_0 \wp, \quad (4.2)$$

where $\sigma = \sqrt{m + \frac{1}{2}}$ and $\wp$ is constant defined in the proof below.

Proof. We have

$$w''(t) = \sum_{i=1}^{\infty} d_i \mathcal{L}_i(t), \quad (4.3)$$

$$|d_i| = \left| \int_0^1 w''(t) \mathcal{L}_i(t) dt \right| \leq \sup_{t \in [0, 1]} |w''(t)| \int_0^1 |\mathcal{L}_i(t)| dt \leq \alpha_0 2^{-\frac{k}{2}} \sigma \wp.$$

Taking inner product of equation (4.3) and applying orthonormality condition of $\mathcal{L}_i(t)$, we get equation (4.2). We have applied mean value theorem for integral and $\wp = \frac{\int_{-1}^1 |\mathcal{P}'_{m+1}(t) - \mathcal{P}'_{m-1}(t)| dt}{\sqrt{2m+1}}$. Therefore, we have

$$|d_i| \leq 2^{-\frac{k}{2}} \sigma \alpha_0 \wp. \quad (4.4)$$

□

Theorem 2. Let the analytic and approximate solution of equation (1.1) is denoted by $w(t)$ and $P_{\mathcal{V}^K}w(t)$ respectively and $w(t) \in \mathcal{L}^2[0, 1]$, $|w''(t)| \leq \alpha_0, \forall t \in (0, 1)$ with $\alpha_0 > 0$. If ε_K is the error of the approximation then we have following inequality:

$$\|\varepsilon_K\|_2 \leq \frac{1}{3} \rho^2 \alpha_0 \mathfrak{I}^2(\mathfrak{I} - \lambda) 2^{-K+1}.$$

Proof. Suppose $w(t) \in \mathcal{L}^2[0, 1]$ be the analytical solution of (1.1) and $P_{\mathcal{V}^K}w(t)$ is the projection of $w(t)$ on the multiresolution space $\mathcal{V}^K$. To obtain the error norm, we calculate the $\mathcal{L}^2$ norm of the difference of exact solution $w(t)$ and projection $P_{\mathcal{V}^K}w(t)$ of it. Detailed description of determining the error norm's upper bound is given below: We have

$$\begin{aligned} \|\varepsilon_K\|_2 &= \|w(t) - P_{\mathcal{V}^K}w(t)\|_2 \\ &= \left\| \sum_{k=K+1}^{\infty} \sum_{i=2^k}^{2^{K+1}-1} d_i \mathcal{J}_i^2(t) - \sum_{k=K+1}^{\infty} \sum_{i=2^k}^{2^{K+1}-1} d_i \mathcal{J}_i^2(1) \right\|_2. \end{aligned}$$

Using Minskowski inequality, we get

$$\begin{aligned} \|\varepsilon_K\|_2 &\leq \left\| \sum_{k=K+1}^{\infty} \sum_{i=2^k}^{2^{K+1}-1} d_i \mathcal{J}_i^2(t) \right\|_2 + \left\| \sum_{k=K+1}^{\infty} \sum_{i=2^k}^{2^{K+1}-1} d_i \mathcal{J}_i^2(1) \right\|_2 \\ &\leq \left\| \sum_{k=K+1}^{\infty} \sum_{i=2^k}^{2^{K+1}-1} d_i \mathcal{J}_i^2(t) \right\|_2 + \left\| \sum_{k=K+1}^{\infty} \sum_{i=2^k}^{2^{K+1}-1} d_i \mathcal{J}_i^2(1) \right\|_2 \end{aligned} \quad (4.5)$$

$$\begin{aligned} &\leq \sum_{k=K+1}^{\infty} \sum_{i=2^k}^{2^{K+1}-1} |d_i| \left(\int_0^1 |\mathcal{J}_i^2(t)|^2 dt \right)^{\frac{1}{2}} + \sum_{k=K+1}^{\infty} \sum_{i=2^k}^{2^{K+1}-1} |d_i| \\ &\quad \left(\int_0^1 |\mathcal{J}_i^2(1)|^2 dt \right)^{\frac{1}{2}}. \end{aligned} \quad (4.6)$$

Now, consider the Cauchy's formula for repeating integration as

$$I^n \mathcal{L}_i(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} \mathcal{L}_i(s) ds. \quad (4.7)$$

For $n = 2$, we get

$$I^2 \mathcal{L}_i(t) = \mathcal{J}_i^2(t) = \int_0^t (t-s)^2 \mathcal{L}_i(s) ds.$$

Now,

$$\begin{aligned} I^2 \mathcal{L}_i(t) &= \int_{v_1}^{v_2} (t-s) \rho 2^{\frac{k}{2}} \mathcal{P}_m(2^k s - \lambda) ds = \int_{-1}^1 \left(t - \frac{\varsigma + \lambda}{2^k}\right) \rho 2^{\frac{k}{2}} 2^{-k} \mathcal{P}_m(\varsigma) d\varsigma \\ &= \int_{-1}^1 \frac{(2^k t - \varsigma - \lambda)}{2^k} \rho 2^{-\frac{k}{2}} \mathcal{P}_m(\varsigma) d\varsigma \end{aligned}$$

$$\leq 2^{\frac{-3k}{2}} \rho \frac{\max_{-1 \leq t \leq 1} |\mathcal{P}'_{m+1}(t) - \mathcal{P}'_{m-1}(t)|}{2m+1} \int_{-1}^1 (2^k t - \varsigma - \lambda) d\varsigma$$

$$|I^2 \mathcal{L}_t(t)| = |\mathcal{J}_t^2(t)| \leq 2^{\frac{-3k+2}{2}} \rho \mathfrak{S} \mathfrak{F}, \quad (4.8)$$

where $\varsigma = 2^k s - \lambda$, $\mathfrak{S} = \frac{\max_{-1 \leq t \leq 1} |\mathcal{P}'_{m+1}(t) - \mathcal{P}'_{m-1}(t)|}{2m+1}$ and $\mathfrak{F} = \max_{t \in [0,1]} |(2^k t - \lambda)|$.

In similar fashion, we can get

$$|I^2 \mathcal{L}_t(1)| = |\mathcal{J}_t^2(1)| \leq 2^{\frac{-3k+2}{2}} (2^k - \lambda) \rho \mathfrak{S}, \quad (4.9)$$

where $\mathfrak{S} = \frac{\max_{-1 \leq t \leq 1} |\mathcal{P}'_{m+1}(t) - \mathcal{P}'_{m-1}(t)|}{2m+1}$. On substituting equations (4.2), (4.8) and (4.9) in equation (4.6) and after simplification, we get

$$\begin{aligned} \|\epsilon_K\|_2 &\leq \sum_{k=K+1}^{\infty} \sum_{t=2^K}^{2^{K+1}-1} 2^{-2k+1} \rho^2 \alpha_0 \mathfrak{S}^2 (\mathfrak{F} + 2^k - \lambda) \\ &\leq \sum_{k=K+1}^{\infty} 2^{-2k+1} \left\{ \rho^2 \alpha_0 \mathfrak{S}^2 (\mathfrak{F} + 2^k - \lambda) \right\} (2^{K+1} - 2^K) \\ &\leq 2^K \sum_{k=K+1}^{\infty} 2^{-2k+1} \rho^2 \alpha_0 \mathfrak{S}^2 (\mathfrak{F} + 2^k - \lambda) \\ \|\epsilon_K\|_2 &\leq \frac{1}{3} \rho^2 \alpha_0 \mathfrak{S}^2 (\mathfrak{F} - \lambda) 2^{-K+1}. \end{aligned} \quad (4.10)$$

□

From equation (4.10) we can say that the error and the resolution level K are related to each other inversely, which implies that as $K \rightarrow \infty$, then $\|\epsilon_K\| \rightarrow 0$.

5. NUMERICAL EXAMPLES

Problem 1. Let

$$w''(t) + w'(t) + \sqrt{\cos t} w'(\sqrt{t}) + (\sin(\sqrt{t}) + e^t) w(\sin t) = f(t), \quad (5.1)$$

which satisfies the BCs:

$$w(0) = 1, \quad w(1) = e.$$

The exact solution is

$$w(t) = e^t. \quad (5.2)$$

The source function $f(t)$ can be calculated with the help of exact solution.

We have applied Legendre wavelet series method (LWSM) for solving equation (5.1). The obtained maximum absolute errors (MAE) and CPU time for various convergence parameters (M , K) are reported in Table 1. The table shows clearly that as we increase the values of the convergence parameters (M , K), the MAE decreases

gradually and CPU time increases at a low rate. A comparison of exact solution with LWSM solution is displayed in Figure 1.

TABLE 1. MAE with CPU time of Problem 1

(M, K)	Exact solution	MAE	CPU time (seconds)
(8, 1)	2.5536	$2.2662e-12$	0.1180
(8, 2)	2.6346	$3.6326e-13$	0.2442
(8, 3)	2.6761	$1.7764e-15$	0.7261
(8, 4)	2.6971	$8.8818e-16$	2.5288

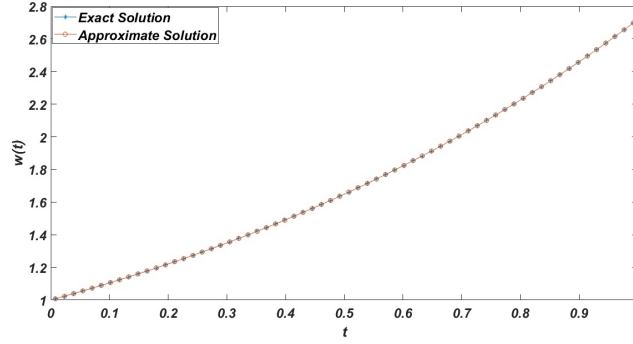


FIGURE 1. Graph of exact and LWSM solutions of Problem 1 for $M = 8$ and $K = 4$

Problem 2. Let

$$w''(t) = -2e^{-t} + \frac{w(t)}{2} + e^{\frac{-t}{2}} w\left(\frac{t}{2}\right), \quad t \in [0, 1], \quad (5.3)$$

which satisfies the BCs:

$$w(0) = 0, \quad w(1) = e^{-1}.$$

The exact solution is

$$w(t) = te^{-t}. \quad (5.4)$$

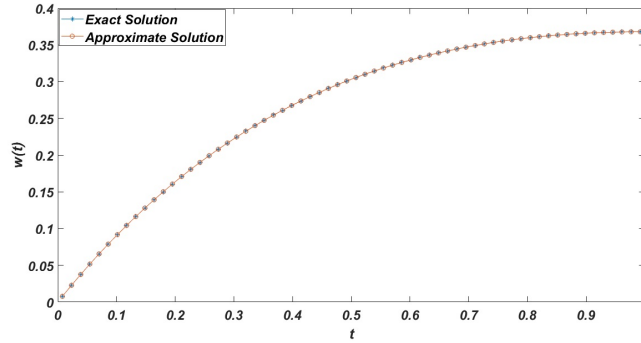
The source function $f(t)$ can be calculated with the help of the exact solution.

We have applied LWSM for solving equation (5.3). The obtained MAE and CPU time for various convergence parameters (M, K) are reported in Table 2. The table shows clearly that as we increase the values of the convergence parameters (M, K) , the MAE decreases gradually and CPU time increases at a low rate. A comparison of exact solution with LWSM solution is displayed in Figure 2.

TABLE 2. MAE with CPU time of Problem 2

(M, K)	Exact solution	MAE	CPU time (seconds)
(8, 1)	0.3671	$7.2649e-12$	0.1072
(8, 2)	0.3677	$1.0288e-12$	0.1696
(8, 3)	0.3678	$4.0523e-15$	0.4781
(8, 4)	0.3679	$1.6653e-16$	1.6053

From Table 2, we can see that the MAE of the developed method is $1.0288e-12$ where as the MAE in [15] is $2.3421e-08$, so we can say that the developed method is more accurate.

FIGURE 2. Graph of exact and LWSM solutions of Problem 2 for $M = 8$ and $K = 4$ **Problem 3.** Let

$$w(t)w''(t) + w'(t) + \sqrt{\cos t}w'(\sqrt{t}) + \sin(\sqrt{t}) + e^t w(\sin t) = f(t), \quad (5.5)$$

which satisfies the BCs:

$$w(0) = 1, \quad w(1) = e.$$

The exact solution is

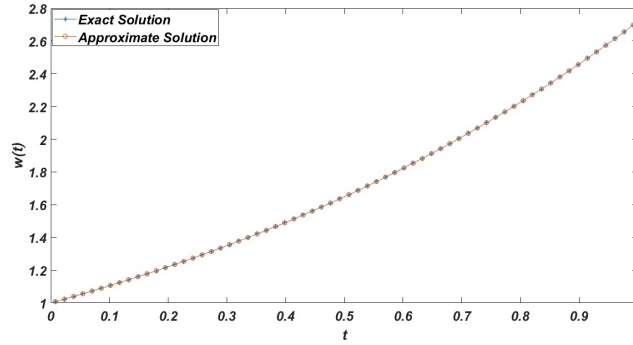
$$w(t) = e^t. \quad (5.6)$$

The source function $f(t)$ can be calculated with the help of exact solution.

We have applied LWSM for solving equation (5.5). The obtained MAE and CPU time for various convergence parameters (M, K) are reported in Table 3. The table shows clearly that as we increase the values of the convergence parameters (M, K) , the MAE decreases gradually and CPU time increases at a low rate. A comparison of exact solution with LWSM solution is displayed in Figure 3.

TABLE 3. MAE with CPU time of Problem 3

(M, K)	Exact solution	MAE	CPU time (seconds)
(8, 1)	2.5536	$2.4698e-12$	0.2279
(8, 2)	2.6346	$4.0767e-13$	0.3362
(8, 3)	2.6761	$2.2204e-15$	0.7627
(8, 4)	2.6971	$8.8818e-16$	2.4628

FIGURE 3. Graph of exact and LWSM solutions of Problem 3 for $M = 8$ and $K = 4$

CONCLUSION

In this paper, we have used LWSM to get the numerical solution of second order NDDEs. We have employed integral operator technique, i.e., the highest order derivative is approximated in terms of Legendre wavelet basis and then the integrations of Legendre wavelet are used to approximate the unknown variable and its lower order derivatives. The advantage of this technique is that we don't need to deal with the BCs separately. These conditions are automatically taken into consideration. We approximate delay term directly by using Legendre wavelet, and not by any series expansion such as Taylor series. In addition, the proposed method converges significantly fast as compared to previous method such as direct block method [15] and is easy to execute. For all computational work we have used MATLAB 2021, intel i5 and windows 10.

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SOME RESULTS ON CHARACTERS OF A CLASS OF P-POLYNOMIAL TABLE ALGEBRAS

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Abstract. In this paper, we study the character values of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$. To this end, we obtain a trigonometric polynomial to calculate the eigenvalues of the first intersection matrix of these table algebras using the $\mathbf{z}$ -transform. Finally by applying some methods for tridiagonal matrices, the character values of these table algebras are given in terms of Chebyshev polynomials.

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1. INTRODUCTION

The characters of table algebras are applied to study the properties of table algebras and can be used in association schemes and finite groups (see [7] and [16]). In particular, the eigenvalues of an association scheme which determine its algebraic structure can be obtained using the characters of table algebras (see [10]). The character values of certain table algebras have also been calculated in some articles such as [19] for lower dimensions.

In our previous work [12], we have calculated the character values of two classes of perfect P-polynomial table algebras given in [17] using the eigenstructure of some special tridiagonal matrices. Here, we intend to study the character values of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$ whose

first intersection matrix is a $(d+1) \times (d+1)$ tridiagonal matrix (cf. [4]) as follows:

$$\frac{1}{4} \begin{pmatrix} 0 & 4 & 0 & & & & \\ 4k & 3k-6 & k+2 & \ddots & & & \\ 0 & k+2 & 2k-4 & k+2 & & & \\ & \ddots & & \ddots & \ddots & & \\ & & & & k+2 & 2k-4 & k+2 \\ & & & & & k+2 & 3k-2 \end{pmatrix}, \quad (1.1)$$

where k is the valency of the table algebras. It should be noted that the adjacency algebra of a distance-regular graph is a monotonic P-polynomial table algebra [2, Proposition III.1.2]. The adjacency algebra arising from a distance-regular graph whose nontrivial distance relations all are of the same valency is also homogeneous. However, there are other homogeneous monotonic P-polynomial table algebras which do not correspond to distance-regular graphs [4].

As we know, it is possible to calculate the character values of P-polynomial table algebras by obtaining the eigenvalues of their first intersection matrix (see [4, Remark 3.1]). Additionally, the first intersection matrix of P-polynomial table algebras is tridiagonal and the eigenstructure of tridiagonal matrices has been studied in many articles such as [13, 14] and [15]. However in this work, the eigenvalues of the tridiagonal matrix (1.1) are found through developing methods using the $\mathbf{z}$ -transform which lead to a trigonometric polynomial. The roots of trigonometric polynomials can be obtained by some numerical methods such as the approach in [5] and through them, reasonable approximations of character values are found. That is of course not our goal in this work.

The structure of this paper is as follows. Section 2 is about important concepts and definitions that this work is based on. In Section 3, we study the eigenvalues of the first intersection matrix of homogeneous monotonic P-polynomial table algebras and obtain some results regarding their character values. Some concluding remarks are stated in Section 4.

Throughout this paper, $\mathbb{C}$ and $\mathbb{R}^+$ denote the complex numbers and the positive real numbers, respectively.

2. PRELIMINARY DEFINITIONS AND CONCEPTS

In this section, we review some important concepts related to this work. In particular, we introduce table algebras and specifically P-polynomial table algebras in 2.1. An overview of the $\mathbf{z}$ -transform and some of its properties is given in 2.2.

2.1. Table algebras

We first go over some concepts related to table algebras and P-polynomial table algebras. Note that some definitions and interpretations of table algebras are not

exactly the same in all references. For example, table algebras are generally non-commutative in some papers (e.g., [18]). However, we follow the definitions in [4].

Let A be a finite-dimensional associative commutative algebra with a basis $\mathbf{B} = \{x_0 = 1_A, x_1, \dots, x_d\}$. Then $(A, \mathbf{B})$ is a *table algebra* if the following conditions are satisfied:

- (i) $x_i x_j = \sum_{m=0}^d \beta_{ijm} x_m$ with $\beta_{ijm} \in \mathbb{R}^+ \cup \{0\}$, for all i, j ;
- (ii) there is an algebra automorphism of A (denoted by $\bar{}$) such that its order divides 2 and if $x_i \in \mathbf{B}$, then $\bar{x}_i \in \mathbf{B}$ and $\bar{i}$ is defined by $x_{\bar{i}} = \bar{x}_i$;
- (iii) for all i, j , we have $\beta_{ij0} \neq 0$ if and only if $j = \bar{i}$, and $\beta_{i\bar{i}0} > 0$.

In the item (i), the β_{ijm} are called the *intersection numbers* of $(A, \mathbf{B})$ and also, the elements of $\mathbf{B}$ are called the *basis elements* of $(A, \mathbf{B})$. $(A, \mathbf{B})$ is called a *real table algebra* if $i = \bar{i}$ for all $i = 0, 1, \dots, d$. Additionally, the i -th *intersection matrix* of $(A, \mathbf{B})$ is a matrix whose entries are the intersection numbers of $(A, \mathbf{B})$ in the form of:

$$B_i = \begin{pmatrix} \beta_{i00} & \beta_{i01} & \dots & \beta_{i0d} \\ \beta_{i10} & \beta_{i11} & \dots & \beta_{i1d} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{id0} & \beta_{id1} & \dots & \beta_{idd} \end{pmatrix}_{(d+1) \times (d+1)}.$$

For every table algebra $(A, \mathbf{B})$, there exists a unique algebra homomorphism such as $f : A \rightarrow \mathbb{C}$ with $f(x_i) = f(x_{\bar{i}}) \in \mathbb{R}^+$, $i = 0, 1, \dots, d$, (cf. [1, Lemma 2.9]). If $f(x_i) = \beta_{i\bar{i}0}$, $i = 0, 1, \dots, d$, then $(A, \mathbf{B})$ is called *standard*. If $d \geq 2$ and for $i > 0$, $f(x_i)$ is constant, then $(A, \mathbf{B})$ is called *homogeneous*.

A *P-polynomial table algebra* $(A, \mathbf{B})$ is a real standard table algebra for which there exist the complex coefficient polynomials $v_i(x)$ with $v_i(x_1) = x_i$, $i = 2, \dots, d$. For every P-polynomial table algebra $(A, \mathbf{B})$, there exist $b_{i-1}, a_i, c_{i+1} \in \mathbb{R}$ such that

$$x_1 x_i = b_{i-1} x_{i-1} + a_i x_i + c_{i+1} x_{i+1}, \quad (2.1)$$

with $b_i \neq 0$, $(i = 0, 1, \dots, d-1)$, $c_i \neq 0$, $(i = 1, \dots, d)$, and $b_{-1} = c_{d+1} = 0$. So, the first intersection matrix of a P-polynomial table algebra is a tridiagonal matrix as follows:

$$B_1 = \begin{pmatrix} a_0 & c_1 & & & \\ b_0 & a_1 & c_2 & & \\ & b_1 & a_2 & \ddots & \\ & & \ddots & \ddots & c_d \\ & & & b_{d-1} & a_d \end{pmatrix}_{(d+1) \times (d+1)}, \quad (2.2)$$

and the *valency* of the P-polynomial table algebra is $b_0 = \beta_{110}$. A *monotonic table algebra* is a P-polynomial table algebra for which $c_i \leq c_{i+1}$ ($i = 1, \dots, d$), and $b_i \geq b_{i+1}$ ($i = 0, \dots, d-2$).

Suppose that $(A, \mathbf{B})$ is a table algebra. It is well known that A is semisimple and the set of its primitive idempotents $\{e_0, e_1, \dots, e_d\}$ forms another basis for A (see

[2, Page 93]). Consequently, each $x_i \in \mathbf{B}$ can be written as

$$x_i = \sum_{j=0}^d p_i(j) e_j,$$

where $p_i(j) \in \mathbb{C}$. On the other hand, if we consider $\{\chi_0, \chi_1, \dots, \chi_d\}$ as the set of irreducible characters of the algebra A , we are interested in the values which each χ_i takes on the basis elements of A . These values are called the *character values* of $(A, \mathbf{B})$. The $p_i(j)$ are equal to the character values of $(A, \mathbf{B})$. More precisely, for each $i = 0, 1, \dots, d$

$$p_i(j) = \chi_j(x_i), \quad j = 0, 1, \dots, d,$$

see [3, Page 11] for more details.

Also, the $p_i(j)$ are equal to the eigenvalues of the i -th intersection matrix. If $(A, \mathbf{B})$ is a P-polynomial table algebra, then we have

$$p_i(j) = v_i(p_1(j)), \quad i = 0, 1, \dots, d, \quad (2.3)$$

where $v_i(x)$ is a complex coefficient polynomial of degree i with $v_i(x_1) = x_i$.

2.2. $\mathbf{z}$ -transform

We now give an overview of $\mathbf{z}$ -transform. The concept of $\mathbf{z}$ -transform has the same role in discrete-time signals as Laplace transform does in continuous-time signals. For a discrete-time signal which is a sequence of real or complex numbers such as $x[n]$, its $\mathbf{z}$ -transform is defined as the power series

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}, \quad (2.4)$$

where n is an integer and z is a complex variable. The function $X(z)$ in (2.4) is called the two-sided or bilateral $\mathbf{z}$ -transform of $x[n]$. The one-sided or unilateral $\mathbf{z}$ -transform of $x[n]$ is defined by

$$X(z) = \sum_{n=0}^{+\infty} x[n] z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots.$$

We use the notation $x[n] \leftrightarrow X(z)$ to show that $X(z)$ is the $\mathbf{z}$ -transform of $x[n]$.

The $\mathbf{z}$ -transform is a linear operation. This means that if we have $x[n] \leftrightarrow X(z)$ and $u[n] \leftrightarrow U(z)$, then

$$ax[n] + bu[n] \leftrightarrow aX(z) + bU(z), \quad (2.5)$$

where $a, b \in \mathbb{C}$.

Let $x[n] \leftrightarrow X(z)$ and q be a positive integer. Then we have

$$x[n - q] \leftrightarrow z^{-q} X(z), \quad (2.6)$$

and

$$x[n + q] \leftrightarrow z^q X(z) - x[0]z^q - x[1]z^{q-1} - \dots - x[q-1]z. \quad (2.7)$$

The proof of the above properties and more facts about the $\mathbf{z}$ -transform can be found in [11, Chapter 7].

3. HOMOGENEOUS MONOTONIC P-POLYNOMIAL TABLE ALGEBRAS

Throughout this section, we study the characters of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$. To do so, we concentrate on calculating the eigenvalues of the first intersection matrix of homogeneous monotonic P-polynomial table algebras which is given in (1.1).

Theorem 1. *Let $(A, \mathbf{B})$ be a homogeneous monotonic P-polynomial table algebra with $\mathbf{B} = \{x_0 = 1_A, x_1, \dots, x_d\}$ and $d \geq 5$. Then the eigenvalues of the first intersection matrix of $(A, \mathbf{B})$ are given by*

$$\eta_i = \frac{k+2}{2} \cos(\theta_i) + \frac{k-2}{2}, \quad i = 0, 1, \dots, d,$$

where k is the valency of $(A, \mathbf{B})$ and the θ_i are the roots of the following equation:

$$(k+2) \sin((d+2)\theta) - 4 \sin((d+1)\theta) - 2k \sin(d\theta) + 4 \sin((d-1)\theta) + (k-2) \sin((d-2)\theta) = 0.$$

Proof. The first intersection matrix of $(A, \mathbf{B})$ is given in (1.1). Since we wish to use right eigenvectors (i.e. column vectors) here, the transpose of the first intersection matrix should be used, so that the automatic degree map eigenvalue k clearly corresponds to the right eigenvector $[1, \dots, 1]^t$. The transpose of the first intersection matrix can be written as follows:

$$B_1^T = \frac{1}{4} \begin{pmatrix} -2k+4 & 4k & 0 & & & \\ 4 & k-2 & k+2 & \ddots & & \\ 0 & k+2 & 0 & k+2 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & k+2 & 0 & k+2 & \\ & & & k+2 & k+2 & \end{pmatrix} + \left(\frac{k-2}{2}\right) I_{(d+1) \times (d+1)}.$$

Denote the above tridiagonal matrix by P . Then the eigenvalues of B_1 are

$$\eta_i = \frac{\lambda_i}{4} + \frac{k-2}{2}, \quad i = 0, 1, \dots, d, \quad (3.1)$$

where λ_i , ($i = 0, 1, \dots, d$), are the eigenvalues of P . Let λ be an eigenvalue of P and $u = [u_1, \dots, u_{d+1}]^t$ be the eigenvector corresponding to λ . Then we can consider the eigenvector u as a sequence $\{u[i]\}_{i=0}^\infty$ with

$$u[i] = \begin{cases} u_i & \text{if } i = 1, \dots, d+1, \\ 0 & \text{otherwise,} \end{cases}$$

and $u_1 \neq 0$. Since $Pu = \lambda u$, we have

$$\begin{cases} u[0] = 0 \\ (k+2)u[0] + (k+2)u[2] = \lambda u[1] + (2k-4)u[1] + (2-3k)u[2] \\ (k+2)u[1] + (k+2)u[3] = \lambda u[2] + (k-2)u[1] + (2-k)u[2] \\ (k+2)u[2] + (k+2)u[4] = \lambda u[3] \\ \vdots \\ (k+2)u[d-1] + (k+2)u[d+1] = \lambda u[d] \\ (k+2)u[d] + (k+2)u[d+2] = \lambda u[d+1] - (k+2)u[d+1] \\ u[d+2] = 0. \end{cases} \quad (3.2)$$

Consequently, we have the following equation:

$$(k+2)u[h+2] + (k+2)u[h] = \lambda u[h+1] + f[h+1], \quad h = 0, 1, \dots \quad (3.3)$$

where

$$f[h] = \begin{cases} (2k-4)u[1] + (2-3k)u[2] & \text{if } h = 1, \\ (k-2)u[1] + (2-k)u[2] & \text{if } h = 2, \\ -(k+2)u[d+1] & \text{if } h = d+1, \\ 0 & \text{otherwise.} \end{cases}$$

The $\mathbf{z}$ -transform of $f[h]$ is

$$\begin{aligned} F[z] &= \sum_{h=0}^{\infty} f[h]z^{-h} \\ &= ((2k-4)u[1] + (2-3k)u[2])z^{-1} \\ &\quad + ((k-2)u[1] + (2-k)u[2])z^{-2} - (k+2)u[d+1]z^{-(d+1)}. \end{aligned} \quad (3.4)$$

From (2.7), we calculate the $\mathbf{z}$ -transform of (3.3) which is

$$(k+2)(z^2U(z) - u[0]z^2 - u[1]z + U(z)) = \lambda zU(z) - \lambda u[0]z + zF(z) - f[0]z$$

and since $u[0] = f[0] = 0$, we can obtain $U(z)$ as follows:

$$\begin{aligned} U(z) &= \frac{1}{(k+2)z^2 - \lambda z + k+2} (F[z] + (k+2)u[1])z \\ &= \frac{1}{(k+2)z^{-2} - \lambda z^{-1} + k+2} (F[z] + (k+2)u[1])z^{-1}. \end{aligned} \quad (3.5)$$

Note that from Remark 2.3, the eigenvalues of B_1 are real and therefore the coefficients of $(k+2)z^{-2} - \lambda z^{-1} + k+2$ are all real and two cases may arise as follows.

(i) If $(k+2)z^{-2} - \lambda z^{-1} + k+2$ has two distinct roots, then the roots are

$$\gamma_{\pm} = \frac{\lambda \pm \sqrt{\omega}}{2(k+2)},$$

where $\omega = \lambda^2 - 4(k+2)^2 \neq 0$. We can write $\gamma_{\pm} = p \pm iq$, where $p, q \in \mathbb{C}$ and $q \neq 0$. From $\gamma_+ \gamma_- = p^2 + q^2 = 1$ and $\gamma_+ + \gamma_- = 2p = \lambda/(k+2)$, it follows that

$$\gamma_{\pm} = \sqrt{p^2 + q^2} (\cos(\theta) \pm i \sin(\theta)) = e^{\pm i\theta}, \quad \cos(\theta) = \frac{\lambda}{2(k+2)}, \quad (3.6)$$

where θ is either a real or a pure imaginary number. Let

$$X(z) = \frac{1}{(k+2)z^{-2} - \lambda z^{-1} + (k+2)}.$$

From (2.6), we know that

$$z^{-q}X(z) \leftrightarrow x[j-q]H[j-q] \quad \text{where} \quad H[x] = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \quad (3.7)$$

So that, by getting the inverse $\mathbf{z}$ -transform from (3.5) and using (3.4), we have

$$\begin{aligned} u[j] &= (k+2)u[1]x[j-1]H[j-1] + ((2k-4)u[1] + (2-3k)u[2])x[j-2]H[j-2] \\ &\quad + ((k-2)u[1] + (2-k)u[2])x[j-3]H[j-3] \\ &\quad - (k+2)u[d+1]x[j-d-2]H[j-d-2]. \end{aligned} \quad (3.8)$$

We shall now find $x[j]$ and plug it into (3.8). By partial fractions decomposition of $X(z)$, we can write

$$X(z) = \frac{1}{\sqrt{\omega}} \left(\frac{1}{\gamma_- - z^{-1}} - \frac{1}{\gamma_+ - z^{-1}} \right),$$

therefore,

$$x[j] = \frac{1}{\sqrt{\omega}} \left((\gamma_-)^{-(j+1)} - (\gamma_+)^{-(j+1)} \right) = \frac{2i}{\sqrt{\omega}} \sin((j+1)\theta). \quad (3.9)$$

Note that $\{a^{-(n+1)}\}_{n=0}^{\infty} \leftrightarrow 1/(a - z^{-1})$. From (3.8) and (3.9), we have

$$\begin{aligned} u[j] &= \frac{2i}{\sqrt{\omega}} \left[(k+2)u[1] \sin(j\theta)H[j-1] \right. \\ &\quad + ((2k-4)u[1] + (2-3k)u[2]) \sin((j-1)\theta)H[j-2] \\ &\quad + ((k-2)u[1] + (2-k)u[2]) \sin((j-2)\theta)H[j-3] \\ &\quad \left. - (k+2)u[d+1] \sin((j-d-1)\theta)H[j-d-2] \right]. \end{aligned} \quad (3.10)$$

Setting $j = d+2$ in (3.10) yields

$$\begin{aligned} u[d+2] &= \frac{2i}{\sqrt{\omega}} \left[(k+2)u[1] \sin((d+2)\theta) \right. \\ &\quad + ((2k-4)u[1] + (2-3k)u[2]) \sin((d+1)\theta) \\ &\quad \left. + ((k-2)u[1] + (2-k)u[2]) \sin(d\theta) - (k+2)u[d+1] \sin(\theta) \right]. \end{aligned} \quad (3.11)$$

Similarly, when $j = d + 1$,

$$\begin{aligned} u[d+1] = \frac{2i}{\sqrt{\omega}} & \left[(k+2)u[1] \sin((d+1)\theta) \right. \\ & + ((2k-4)u[1] + (2-3k)u[2]) \sin(d\theta) \\ & \left. + ((k-2)u[1] + (2-k)u[2]) \sin((d-1)\theta) \right]. \end{aligned} \quad (3.12)$$

Moreover, we know that

$$\gamma_+ - \gamma_- = \frac{\sqrt{\omega}}{(k+2)} = 2i \sin(\theta). \quad (3.13)$$

From (3.11), (3.12) and (3.13), we conclude that

$$\begin{aligned} u[d+2] = \frac{2i}{\sqrt{\omega}} & \left[(k+2)u[1] \sin((d+2)\theta) + ((k-6)u[1] + (2-3k)u[2]) \sin((d+1)\theta) \right. \\ & + ((2-k)u[1] + 2ku[2]) \sin(d\theta) \\ & \left. + ((2-k)u[1] + (k-2)u[2]) \sin((d-1)\theta) \right]. \end{aligned}$$

From (3.2), we know that

$$u[2] = \frac{\lambda + 2k - 4}{4k} u[1], \quad (3.14)$$

and also, by (3.6), we have $\lambda = 2(k+2) \cos(\theta)$, consequently we get

$$\begin{aligned} u[d+2] = \frac{i u[1](k+2)}{2k\sqrt{\omega}} & \left[(k+2) \sin((d+2)\theta) - 4 \sin((d+1)\theta) - 2k \sin(d\theta) \right. \\ & \left. + 4 \sin((d-1)\theta) + (k-2) \sin((d-2)\theta) \right]. \end{aligned}$$

Since $u[d+2] = 0$, θ is the root of the following equation:

$$\begin{aligned} & (k+2) \sin((d+2)\theta) - 4 \sin((d+1)\theta) - 2k \sin(d\theta) + 4 \sin((d-1)\theta) \\ & + (k-2) \sin((d-2)\theta) = 0. \end{aligned} \quad (3.15)$$

(ii) Suppose that $(k+2)z^{-2} - \lambda z^{-1} + (k+2)$ has a repeated root. As such, $\omega = 0$ and $\lambda = \pm 2(k+2)$. Let $Y(z) = \frac{z^{-1}}{(k+2)z^{-2} - \lambda z^{-1} + (k+2)}$. From (3.4), (3.5) and (3.7), we get

$$\begin{aligned} u[j] = & (k+2)u[1]y[j] + ((2k-4)u[1] + (2-3k)u[2])y[j-1]H[j-1] \\ & + ((k-2)u[1] + (2-k)u[2])y[j-2]H[j-2] \\ & - (k+2)u[d+1]y[j-d-1]H[j-d-1]. \end{aligned} \quad (3.16)$$

We shall now find $y[j]$ and plug it into (3.16). We can write

$$Y(z) = \frac{z^{-1}}{(k+2)z^{-2} - \lambda z^{-1} + k+2} = \frac{z^{-1}}{(k+2)(z^{-2} - \frac{\lambda}{k+2}z^{-1} + 1)} = \frac{z^{-1}}{(k+2)(z^{-1} \pm 1)^2}.$$

That yields

$$y[j] = \frac{1}{k+2} (\pm 1)^{j+1} j. \quad (3.17)$$

Notice that the above equality is obtained due to $\{na^n\}_{n=0}^\infty \leftrightarrow az/(z-a)^2$ (cf. [11, Table 7.3]). From (3.16) and (3.17), we have

$$\begin{aligned} u[j] = \frac{1}{k+2} & \left[(k+2)u[1](\pm 1)^{j+1}j \right. \\ & + ((2k-4)u[1] + (2-3k)u[2])(\pm 1)^j(j-1)H[j-1] \\ & + ((k-2)u[1] + (2-k)u[2])(\pm 1)^{j-1}(j-2)H[j-2] \\ & \left. - (k+2)u[d+1](\pm 1)^{j-d}(j-d-1)H[j-d-1] \right]. \end{aligned} \quad (3.18)$$

If we set $j = d+2$ in (3.18), then we have

$$\begin{aligned} u[d+2] = \frac{1}{k+2} & \left[(k+2)u[1](\pm 1)^{d+3}(d+2) \right. \\ & + ((2k-4)u[1] + (2-3k)u[2])(\pm 1)^{d+2}(d+1) \\ & \left. + ((k-2)u[1] + (2-k)u[2])(\pm 1)^{d+1}d - (k+2)u[d+1] \right], \end{aligned}$$

and also, $u[d+1]$ is as follows

$$\begin{aligned} u[d+1] = \frac{1}{k+2} & \left[(k+2)u[1](\pm 1)^{(d+2)}(d+1) \right. \\ & + ((2k-4)u[1] + (2-3k)u[2])(\pm 1)^{(d+1)}d \\ & \left. + ((k-2)u[1] + (2-k)u[2])(\pm 1)^d(d-1) \right], \end{aligned}$$

so by using this and $u[d+2] = 0$, we obtain that

$$\begin{aligned} (k+2)(\pm 1)u[1](d+2) & + ((k-6)u[1] + (2-3k)u[2])(d+1) \\ & + ((2-k)u[1] + 2ku[2])(\pm 1)d - ((k-2)u[1] + (2-k)u[2])(d-1) = 0. \end{aligned} \quad (3.19)$$

We now consider two equations (3.20) and (3.21) from (3.19) as follows. The first equation is

$$\begin{aligned} (k+2)u[1](d+2) & + ((k-6)u[1] + (2-3k)u[2])(d+1) \\ & + ((2-k)u[1] + 2ku[2])d - ((k-2)u[1] + (2-k)u[2])(d-1) = 0. \end{aligned} \quad (3.20)$$

In this case, $\lambda = -2(k+2)$ and also from (3.14), we conclude that $u[2] = -2u[1]/k$. After some simplifications, we obtain from (3.20) that $d = -2$ which is a contradiction. The other equation is

$$\begin{aligned} -(k+2)u[1](d+2) & + ((k-6)u[1] + (2-3k)u[2])(d+1) \\ & - ((2-k)u[1] + 2ku[2])d - ((k-2)u[1] + (2-k)u[2])(d-1) = 0. \end{aligned} \quad (3.21)$$

In this case, $\lambda = 2a$ and from (3.14), we conclude that $u[2] = u[1]$. After some simplifications, we get $d = -1$ which is a contradiction.

All in all, we conclude that the θ_i can all be obtained from (3.15) and so, from (3.6) we have $\lambda_i = 2(k+2)\cos(\theta_i)$, where the λ_i are the eigenvalues of the matrix P . Consequently, by (3.1), the eigenvalues of B_1 are

$$\eta_i = \frac{k+2}{2}\cos(\theta_i) + \frac{k-2}{2} \quad (i = 0, 1, \dots, d),$$

as expected. $\square$

Importantly, Theorem 1 shows that the general form of the eigenvalues of the first intersection matrix of a homogeneous monotonic P-polynomial table algebra with valency k is $\cos(\theta_i)(k+2)/2 + (k-2)/2$ which means that the eigenvalues lie in the interval

$$[-2, k],$$

and when $k > 2$, this is a better boundary than $[-k, k]$ which is given in [4, Proposition 3.1].

In the following theorem, we calculate the characters of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$.

Theorem 2. *Let $(A, \mathbf{B})$ be a homogeneous monotonic P-polynomial table algebra with $\mathbf{B} = \{x_0 = 1_A, x_1, \dots, x_d\}$ and $d \geq 5$. Then the character values of $(A, \mathbf{B})$ are*

$$\begin{aligned} p_0(j) &= 1, \\ p_1(j) &= \eta_j, \\ p_i(j) &= \left(\frac{\sqrt{k+2}}{2}\right)^{i-4} \left[\left(\eta_j^2 - \frac{3k-6}{4}\eta_j - k \right) U_{i-2} \left(\frac{2\eta_j - k + 2}{2\sqrt{k+2}} \right) \right. \\ &\quad \left. - \left(\frac{\sqrt{k+2}}{2} \right)^3 \eta_j U_{i-3} \left(\frac{2\eta_j - k + 2}{2\sqrt{k+2}} \right) \right], \end{aligned}$$

where $i = 2, \dots, d$, $j = 0, 1, \dots, d$ and the η_j are the eigenvalues of B_1 which are obtained in Theorem 1.

Proof. For each i , $i = 0, 1, \dots, d$, the $p_i(j)$, $j = 0, 1, \dots, d$, are equal to the eigenvalues of the i -th intersection matrix B_i . Since $B_0 = I_{d+1}$, we have $p_0(j) = 1$ for all j . Similarly, the $p_1(j)$ are equal to the eigenvalues of B_1 which are calculated in Theorem 1. To obtain the $p_i(j)$, $i = 2, \dots, d$, we must calculate the complex coefficient polynomial $v_i(x)$, where $x_i = v_i(x_1)$. Obviously, $v_1(x) = x$, and from (1.1) and (2.1) we get

$$x_1 x_1 = k + \frac{3k-6}{4}x_1 + \frac{k+2}{4}x_2 \Rightarrow v_2(x) = \frac{4x^2 - (3k-6)x - 4k}{k+2}.$$

We claim that

$$v_i(x) = \frac{4}{k+2} \left(\left(x - \frac{k-2}{2} \right) v_{i-1}(x) - \frac{k+2}{4} v_{i-2}(x) \right), \quad (3.22)$$

where $i = 2, \dots, d$. To prove this, we use induction on i . It is fairly straightforward using (2.1) to get

$$v_3(x) = \frac{4}{k+2} \left(\left(x - \frac{k-2}{2} \right) v_2(x) - \frac{k+2}{4} v_1(x) \right).$$

Now, we assume that (3.22) holds for $i < d$. From (2.1) and the induction hypothesis, it is concluded that

$$v_d(x) = \frac{4}{k+2} \left(\left(x - \frac{k-2}{2} \right) v_{d-1}(x) - \frac{k+2}{4} v_{d-2}(x) \right).$$

We now consider the following recursive relation:

$$\varphi_n(x) = \left(x - \frac{k-2}{2} \right) \varphi_{n-1}(x) - \frac{k+2}{4} \varphi_{n-2}(x), \quad n > 2,$$

with $\varphi_1(x) = (k+2)x/4$ and $\varphi_2(x) = x^2 - (3k-6)x/4 - k$. By using [6, Lemma 1], we find that

$$\varphi_n(x) = \begin{vmatrix} \frac{k+2}{4}x & 1 & 0 & & & \\ k & \frac{4}{k+2}x - \frac{3k-6}{k+2} & 1 & \ddots & & \\ 0 & \frac{k+2}{4} & x - \frac{k-2}{2} & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & & \frac{k+2}{4} & x - \frac{k-2}{2} & 1 \\ & & & & \frac{k+2}{4} & x - \frac{k-2}{2} \end{vmatrix}_{n \times n}.$$

Laplace expansion gives

$$\varphi_n(x) = \left(x^2 - \frac{3k-6}{4}x - k \right) D_{n-2}(x) - \left(\frac{k+2}{4} \right)^2 x D_{n-3}(x), \quad (3.23)$$

where $D_n(x)$ is the characteristic polynomial of

$$\begin{pmatrix} \frac{k-2}{2} & 1 & 0 & & \\ \frac{k+2}{4} & \frac{k-2}{2} & 1 & \ddots & \\ 0 & \frac{k+2}{4} & \frac{k-2}{2} & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ & & & \frac{k+2}{4} & \frac{k-2}{2} \end{pmatrix}_{n \times n},$$

so from [8, Section 2], we can see that

$$D_n(x) = \left(\frac{\sqrt{k+2}}{2} \right)^n U_n \left(\frac{2x - k + 2}{2\sqrt{k+2}} \right). \quad (3.24)$$

Finally, from (3.22), (3.23) and (3.24) we conclude that

$$\begin{aligned} v_i(x) &= \frac{4}{k+2} \varphi_i(x) \\ &= \left(\frac{\sqrt{k+2}}{2} \right)^{i-4} \left[\left(x^2 - \frac{3k-6}{4}x - k \right) U_{i-2} \left(\frac{2x-k+2}{2\sqrt{k+2}} \right) \right. \\ &\quad \left. - \left(\frac{\sqrt{k+2}}{2} \right)^3 x U_{i-3} \left(\frac{2x-k+2}{2\sqrt{k+2}} \right) \right], \end{aligned}$$

for $i = 2, \dots, d$. Due to (2.3), the proof is now complete. $\square$

Example 1. Let $(A, \mathbf{B})$ be a homogeneous monotonic P-polynomial table algebra of valency $k = 2$ and diameter $d \geq 5$. Then from (1.1), the first intersection of $(A, \mathbf{B})$ is

$$B_1 = \begin{pmatrix} 0 & 1 & & & & \\ 2 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \end{pmatrix}_{(d+1) \times (d+1)}.$$

Now, we calculate the characters of $(A, \mathbf{B})$. From Theorem 1, we must find the roots of the following equation:

$$\sin((d+2)\theta) - \sin((d+1)\theta) - \sin(d\theta) + \sin((d-1)\theta) = 0. \quad (3.25)$$

It implies that

$$\cos((d+1)\theta) \sin(\theta) - \cos(d\theta) \sin(\theta) = 0.$$

We can assume that $\sin(\theta) \neq 0$, because otherwise, $\theta = n\pi$, ($n \in \mathbb{N}$), and $\lambda = 2a \cos(n\pi) = \pm 2a$ which leads to $k = 2$ being an eigenvalue of B_1 , i.e. case (ii) in Theorem 1, but k can be obtained from the equation (3.25) which follows from the case (i) in Theorem 1. This gives

$$p_1(j) = \eta_j = 2 \cos \left(\frac{2j\pi}{2d+1} \right), \quad j = 0, \dots, d.$$

The other characters of $(A, \mathbf{B})$ can also be found through Theorem 2. The $p_i(j)$, $i = 2, \dots, d$, are

$$p_i(j) = (\eta_j^2 - 2) U_{i-2} \left(\frac{\eta_j}{2} \right) - \eta_j U_{i-3} \left(\frac{\eta_j}{2} \right) = 2 \cos \left(\frac{ij\pi}{2d+1} \right),$$

for $j = 0, \dots, d$, where $U_i(x)$ and $T_i(x)$ are the i -th degree Chebyshev polynomials of second kind and first kind, respectively. The above equalities follow from the properties of Chebyshev polynomials which can be found in [9].

4. CONCLUDING REMARKS

In this paper, we use the $\mathbf{z}$ -transform concept along with techniques from linear algebra and matrix theory to calculate the character values of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$. Importantly, we calculate the eigenvalues of a special classes of tridiagonal matrices which may have applications in other fields. Next, we obtain the characters of homogeneous monotonic P-polynomial table algebras with finite dimension $d \geq 5$ in terms of Chebyshev polynomials. For $k > 2$, we plan to apply the results of Theorem 2 combined with some other techniques to find the character values in our future work.

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COINCIDENCE POINT RESULTS ON A METRIC SPACE ENDOWED WITH A LOCALLY T -TRANSITIVE BINARY RELATION EMPLOYING COMPARISON FUNCTIONS

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Abstract. In this paper, we prove coincidence point results for a pair of self mappings defined on a metric space employing a comparison function and a locally T -transitive binary relation. Our results extend and generalize several known results especially those are contained in Alam and Imdad [Filomat 31 (14) (2017) 4421-4439] and Arif *et al.* [Miskolc Math. Notes 23 (1) (2022) 71-83]. Finally, we construct some examples to demonstrate the accomplished improvements in our newly proved results in this paper.

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1. INTRODUCTION AND PRELIMINARIES

Banach contraction principle [or, in short BCP] (*cf.* [7]) is a pioneer and noted result of metric fixed point theory. In the course of last several years, numerous researchers extended this result by weakening the contraction conditions besides enlarging the class of underlying metric spaces. In the existing literature, one of the weaker form of contraction conditions is nonlinear contraction. A self-mapping ϕ defined on $[0, \infty)$ satisfying $\phi(s) < s$ for each $s > 0$ is control function. A self-mapping T defined on a metric space (X, d) is said to be a nonlinear contraction with respect to control function ϕ (or, in short, ϕ -contraction) if $d(Tx, Ty) \leq \phi(d(x, y))$ for all $x, y \in X$. In 1968, Browder [10] established a nonlinear version of BCP, wherein author assumed ϕ to be increasing control function and right continuous. Thereafter, numerous mathematicians generalized the BCP by utilizing different types of control functions such as: Boyd-Wong [9] and Matkowski [16] contractions. In 1986, Turinici [24] discovered the idea of order-theoretic fixed points results. In 2004, Ran and Reurings [20] reformulated a comparatively more natural order-theoretic version of classical BCP. There exists a vast literature on relation-theoretic results on fixed, coincidence and

common fixed point (e.g., Ran and Reurings [20] and Nieto and Rodríguez-López [18], Agarwal et al. [1], Alam et al. [2], O'Regan and Petruşel [19], Turinici [25], Ghods et al. [11], Ben-El-Mechaiekh [8], etc). Later, Alam and Imdad [3, 4] established relation-theoretic analogues of BCP employing amorphous relation which in turn unify the several well known relevant order-theoretic fixed point theorems. Recently, Arif et al. [6] proved a Matkowski type nonlinear fixed point result using a locally T -transitive binary relation.

Our aim in this article is to establish some coincidence and common fixed point results for Matkowski type nonlinear contractions on a metric space endowed with locally T -transitive binary relation. Newly proved results generalize and extend the main results of Arif et al. [6], Alam and Imdad [4] and several others. Moreover, a corollary to one of our main results remains a sharpened version of a corresponding earlier known result of the existing literature. In order to demonstrate the validity of the hypotheses and degree of generality of our results, we also furnish some examples.

2. PRELIMINARIES

For the sake of simplicity, we collect some basic notions and propositions for our subsequent discussion.

Definition 1 ([12, 14]). Let (T, g) be a pair of self-mappings defined on a nonempty set X . Then

- (i) a point $x \in X$ is said to be a coincidence point of the pair (T, g) if $Tx = gx$,
- (ii) a point $y \in X$ is said to be a point of coincidence of the pair (T, g) if there exists $x \in X$ such that $y = Tx = gx$,
- (iii) a coincidence point $x \in X$ of the pair (T, g) is said to be a common fixed point if $x = Tx = gx$,
- (iv) a pair (T, g) is called commuting if $T(gx) = g(Tx), \forall x \in X$,
- (v) a pair (T, g) is called weakly compatible if T and g commutes at their coincidence point.

Definition 2 ([13, 22, 23]). Let (T, g) be a pair of self-mappings defined on a metric space (X, d) . Then

- (i) (T, g) is said to be weakly commuting if for all $x \in X$, $d(T(gx), g(Tx)) \leq d(Tx, gx)$,
- (ii) (T, g) is said to be compatible if $\lim_{n \rightarrow \infty} d(T(gx_n), g(Tx_n)) = 0$ whenever $\{x_n\} \subset X$ is a sequence such that $\lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} Tx_n$,
- (iii) T is said to be g -continuous at $x \in X$ if $gx_n \xrightarrow{d} gx$, for all sequence $\{x_n\} \subset X$, we have $Tx_n \xrightarrow{d} Tx$. Moreover, T is said to be g -continuous if it is continuous at every point of X .

Definition 3 ([15]). A subset $\mathcal{R}$ of $X \times X$ is called a binary relation on X . We say that “ x relates y under $\mathcal{R}$ ” if and only if $(x, y) \in \mathcal{R}$.

Throughout this paper, $\mathcal{R}$ stands for a ‘non-empty binary relation’ (i.e., $\mathcal{R} \neq \emptyset$) instead of ‘binary relation’ while $\mathbb{N}$, the set of natural numbers (also, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$, (where $\mathcal{R}^{-1} := \{(y, x) \in X^2 : (x, y) \in \mathcal{R}\}$) and $\mathcal{R}|_Z := \mathcal{R} \cap Z^2$, $Z \subseteq X$. Indeed, $\mathcal{R}|_Z$ is a relation on Z induced by $\mathcal{R}$. Although, $\mathcal{R} = \emptyset$ denotes the null relation, while $\mathcal{R} = X^2$ the universal relation.

Definition 4. [5] Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let T be a self-mapping defined on X . Then $\mathcal{R}$ is called “ T -transitive” if for any $x, y, z \in X$,

$$(Tx, Ty), (Ty, Tz) \in \mathcal{R} \Rightarrow (Tx, Tz) \in \mathcal{R}.$$

Definition 5 ([3]). Let X be a non-empty set X endowed with a binary relation $\mathcal{R}$. Then a sequence $\{x_n\} \subset X$ is said to be $\mathcal{R}$ -preserving if $(x_n, x_{n+1}) \in \mathcal{R}$, $\forall n \in \mathbb{N}_0$.

Definition 6 ([5]). Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Then $\mathcal{R}$ is called “locally transitive” if for each (effectively) $\mathcal{R}$ -preserving sequence $\{x_n\} \subset X$ (with range $D = \{x_n : n \in \mathbb{N}\}$), such that $\mathcal{R}|_D$ is transitive.

Definition 7 ([5]). Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let T be a self-mapping defined X . Then $\mathcal{R}$ is called locally T -transitive if for each (effectively) $\mathcal{R}$ -preserving sequence $\{x_n\} \subset T(X)$ (with range $D = \{x_n : n \in \mathbb{N}\}$), such that $\mathcal{R}|_D$ is transitive.

Proposition 1. Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let (T, g) be a pair of self-mappings defined on X . Then

- (i) $\mathcal{R}$ is T -transitive $\Leftrightarrow \mathcal{R}|_{T(X)}$ is transitive,
- (ii) $\mathcal{R}$ is locally T -transitive $\Leftrightarrow \mathcal{R}|_{T(X)}$ is locally transitive,
- (iii) $\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is locally transitive $\Rightarrow \mathcal{R}$ is locally T -transitive,
- (iv) $\mathcal{R}$ is transitive $\Rightarrow \mathcal{R}$ is T -transitive $\Rightarrow \mathcal{R}$ is locally T -transitive,
- (iv) if $T(X) \subseteq g(X)$, then g -transitivity of $\mathcal{R} \Rightarrow \mathcal{R}$ is T -transitive and locally g -transitivity of $\mathcal{R} \Rightarrow \mathcal{R}$ is locally T -transitive but not conversely.

Definition 8 ([15]). Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Then $\mathcal{R}$ is called complete if for all x, y in X , either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$ which is denoted by $[x, y] \in \mathcal{R}$.

Proposition 2 ([3]). Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Then $(x, y) \in \mathcal{R}^s$ if and only if $[x, y] \in \mathcal{R}$.

Definition 9 ([3]). Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let T be a self-mapping defined X . Then $\mathcal{R}$ is called T -closed if for all $x, y \in X$, $(x, y) \in \mathcal{R} \Rightarrow (Tx, Ty) \in \mathcal{R}$.

Definition 10 ([4]). Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let (T, g) be a pair of self-mappings defined on X . Then $\mathcal{R}$ is called (T, g) -closed if for all $x, y \in X$, $(gx, gy) \in \mathcal{R} \Rightarrow (Tx, Ty) \in \mathcal{R}$.

Definition 11 ([4]). Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$. Then (X, d) is said to be $\mathcal{R}$ -complete if every $\mathcal{R}$ -preserving Cauchy sequence in X converges to a point in X .

Remark 1 ([4]). Every complete metric space is $\mathcal{R}$ -complete, where $\mathcal{R}$ denotes a binary relation. Moreover, if $\mathcal{R} = X^2$, then notions of $\mathcal{R}$ -completeness and completeness are the same.

Definition 12 ([4]). Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and let T be a self-mapping defined on X . Then T is said to be $\mathcal{R}$ -continuous at x if $x_n \xrightarrow{d} x$, for any $\mathcal{R}$ -preserving sequence $\{x_n\} \subset X$, we have $Tx_n \xrightarrow{d} Tx$. Moreover, T is said to be $\mathcal{R}$ -continuous if it is $\mathcal{R}$ -continuous at every point of X .

Definition 13 ([4]). Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and let (T, g) be a pair of self-mappings defined on X . Then T is said to be $(g, \mathcal{R})$ -continuous at x if $gx_n \xrightarrow{d} gx$, for any $\mathcal{R}$ -preserving sequence $\{gx_n\} \subset X$, we have $Tx_n \xrightarrow{d} Tx$. Moreover, T is called $(g, \mathcal{R})$ -continuous if it is $(g, \mathcal{R})$ -continuous at every point of X .

Remark 2. Every continuous mapping is $\mathcal{R}$ -continuous, where $\mathcal{R}$ denotes a binary relation. Moreover, if $\mathcal{R} = X^2$, then notions of continuity and $\mathcal{R}$ -continuity are the same.

Definition 14 ([3]). Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$. Then $\mathcal{R}$ is said to be d -self-closed if for any $\mathcal{R}$ -preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x] \in \mathcal{R}$, for all $k \in \mathbb{N}$.

Definition 15 ([4]). Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and let g be a self-mapping defined on X . Then $\mathcal{R}$ is said to be (g, d) -self-closed if for any $\mathcal{R}$ -preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[gx_{n_k}, gx] \in \mathcal{R}$, for all $k \in \mathbb{N}$.

Definition 16 ([21]). Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Then a subset D of X is said to be $\mathcal{R}$ -directed if for every pair of points x, y in D , there is w in X such that $(x, w) \in \mathcal{R}$ and $(y, w) \in \mathcal{R}$.

Definition 17 ([4]). Let X be a non-empty set endowed with a binary relation $\mathcal{R}$ and let g be a self-mapping defined on X . Then a subset D of X is said to be $(g, \mathcal{R})$ -directed if all x, y in D , there is 'w' in X such that $(x, gw) \in \mathcal{R}$ and $(y, gw) \in \mathcal{R}$.

Remark 3. On setting $g = I$, (the identity mapping on X) in Definitions 10, 13, 15 and 17 reduces to Definitions 9, 12, 14 and 16, respectively.

Definition 18 ([15]). Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Given $x, y \in X$, a path of length k (where k is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{w_0, w_1, w_2, \dots, w_k\} \subset X$ satisfying the following:

- (i) $w_0 = x$ and $w_k = y$,
- (ii) $(w_i, w_{i+1}) \in \mathcal{R}$ for each i ($0 \leq i \leq k-1$).

Moreover, a path of length k involves $k+1$ points of X , although they may or may not be distinct.

Definition 19 ([4]). Let X be a non-empty set endowed with a binary relation $\mathcal{R}$. Then a subset D of X is called $\mathcal{R}$ -connected if for each pair $x, y \in D$, there exists a path in $\mathcal{R}$ from x to y .

Definition 20 ([4]). Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and let (T, g) be a pair of self-mappings defined on X . Then (T, g) is said to be $\mathcal{R}$ -compatible if $\lim_{n \rightarrow \infty} d(g(Tx_n), T(gx_n)) = 0$, whenever $\lim_{n \rightarrow \infty} g(x_n) = \lim_{n \rightarrow \infty} T(x_n)$, for any sequence $\{x_n\} \subset X$ such that $\{Tx_n\}$ and $\{gx_n\}$ are $\mathcal{R}$ -preserving.

For a given non-empty set X together with a binary relation $\mathcal{R}$ on X and a pair of self-mappings (T, g) on X , we use the following notations:

- $X(T, g, \mathcal{R}) := \{x \in X : (gx, Tx) \in \mathcal{R}\}$;
- $C(T, g)$: the collection of all coincidence points of (T, g) ;
- $\bar{C}(T, g)$: the collection of all points of coincidence of (T, g) ;
- $F(T, g)$: the collection of all common fixed points of (T, g) ;
- $F(T)$: the collection of all fixed points of T ;
- $X(T, \mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\}$.

In fact, the idea of a comparison function was initiated by Matkowski [16] in 1975, we denote the set of all comparison functions by Φ . A mapping $\varphi: [0, \infty) \rightarrow [0, \infty)$ is said to be a comparison function if φ satisfies the following properties:

- (Φ_1) φ is increasing;
- (Φ_2) $\lim_{n \rightarrow \infty} \varphi^n(s) = 0$ for each $s > 0$, where, for each n , φ^n is the n -th iterate of φ .

Now, we propose the two main properties of the comparison function:

Proposition 3 ([17]). *If φ is a comparison function, then $\varphi(s) < s$, for each $s > 0$.*

Proof. On contrary suppose that exists $s_0 > 0$ such that $s_0 \leq \varphi(s_0)$. Since φ is increasing, $\varphi(s_0) \leq \varphi^2(s_0)$, it follows that $s_0 \leq \varphi(s_0) \leq \varphi^2(s_0)$. Thus, inductively for all $n \in \mathbb{N}$, we obtain $s_0 \leq \varphi^n(s_0)$ which on letting $n \rightarrow \infty$, gives rise, $s_0 \leq 0$, which is a contradiction. Hence $\varphi(s) < s$. $\square$

Proposition 4 ([17]). *If φ is a comparison function, then $\varphi(0) = 0$.*

Proof. On contrary suppose that $\varphi(0) = s$ for some $s > 0$. Since $0 < s$ and φ is increasing, $\varphi(0) \leq \varphi(s)$, it follows that $s \leq \varphi(s) < s$, which is contradiction, hence $\varphi(0) = 0$. $\square$

Proposition 5. *Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and let (T, g) be a pair of self-mappings defined on X and also let $\varphi \in \Phi$. Then the following conditions are equivalent:*

- (I) $d(Tx, Ty) \leq \varphi(d(gx, gy))$ with $(gx, gy) \in \mathcal{R}$;
- (II) $d(Tx, Ty) \leq \varphi(d(gx, gy))$ with $[gx, gy] \in \mathcal{R}$.

Proof. Obviously $(II) \Rightarrow (I)$. We claim that $(I) \Rightarrow (II)$, choose $x, y \in X$ such that $[gx, gy] \in \mathcal{R}$. If $(gx, gy) \in \mathcal{R}$, then (II) immediately follows from (I) . Otherwise, if $(gy, gx) \in \mathcal{R}$, then by (I) and owing to the symmetry of d (metric), we conclude the claim. $\square$

For the sake of completeness, we state the main results of Arif et al., [6] and Alam Imdad [5] respectively, which are given under:

Theorem 1 ([6]). *Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$ and let T be a self-mapping defined on X . Suppose that the following conditions hold:*

- (i) (X, d) is $\mathcal{R}$ -complete;
- (ii) $\mathcal{R}$ is T -closed and locally T -transitive;
- (iii) either T is $\mathcal{R}$ -continuous or $\mathcal{R}$ is d -self-closed;
- (iv) $X(T, \mathcal{R})$ is non-empty;
- (v) there exists a comparison function φ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then $F(T) \neq \emptyset$. Moreover, if $T(X)$ is $\mathcal{R}^s$ -connected, then $F(T)$ is singleton.

Theorem 2 ([4], Theorem 2). *Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$, let (T, g) be a pair of self-mappings defined on X and Z be an $\mathcal{R}$ -complete subspace of X . Assume that the following conditions hold:*

- (a) $T(X) \subseteq Z \cap g(X)$;
- (b) $X(T, g, \mathcal{R})$ is non-empty;
- (c) $\mathcal{R}$ is (T, g) -closed;
- (d) there exists $\alpha \in [0, 1)$ such that $d(Tx, Ty) \leq \alpha d(gx, gy)$ for all $x, y \in X$ with $(gx, gy) \in \mathcal{R}$;
- (e) (e_1) (T, g) is $\mathcal{R}$ -compatible;
- (e_2) g is $\mathcal{R}$ -continuous;
- (e_3) T is $\mathcal{R}$ -continuous or $\mathcal{R}$ is (g, d) -self-closed;

or, alternatively

- (e') (e'_1) $Z \subseteq g(X)$;
- (e'_2) either T is $(g, \mathcal{R})$ -continuous or T and g are continuous or $\mathcal{R}|_Z$ is d -self-closed.

Then $C(T, g) \neq \emptyset$.

Indeed, the main results of this paper are based on the following points:

- Theorem 1 is extended for a pair of self-mappings (T, g) defined on a non-empty set X ,
- Theorem 2 is improved by replacing relatively weaker contraction condition,
- Theorem 1 is generalized by replacing comparatively weaker notions namely $\mathcal{R}$ -completeness of any subspace $Z \subseteq X$, with $T(X) \subseteq Z$ rather than $\mathcal{R}$ completeness of whole space X ,
- some examples are constructed to demonstrate the realized improvement in the proved results in this article.

3. MAIN RESULTS

Now, we are equipped to prove our main result as follows:

Theorem 3. *Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$, let (T, g) be a pair of self-mappings defined on X and Z be an $\mathcal{R}$ -complete subspace of X . Assume that hypotheses (a), (b), (e), or $[(e') \text{ in which modified } (e'_2) : \text{either } T \text{ is } (g, \mathcal{R})\text{-continuous or } T \text{ is continuous and } g \text{ is bijective and bi-continuous or } \mathcal{R}|_Z \text{ is } d\text{-self-closed.}] \text{ of Theorem 2 along with the following conditions hold:}$*

- (k) $\mathcal{R}$ is (T, g) -closed and locally T -transitive;
- (l) there exists a comparison function ϕ such that

$$d(Tx, Ty) \leq \phi(d(gx, gy)) \text{ for all } x, y \in X \text{ with } (gx, gy) \in \mathcal{R}.$$

Then $C(T, g) \neq \emptyset$.

Proof. Let $x_0 \in X$ such that $(gx_0, Tx_0) \in \mathcal{R}$ (due to hypothesis (b)). Construct a Picard Jungck sequence $\{gx_n\}$ based at the point x_0 , i.e.,

$$g(x_{n+1}) = T(x_n) \quad \forall n \in \mathbb{N}_0. \quad (3.1)$$

Since $(gx_0, Tx_0) \in \mathcal{R}$, $\mathcal{R}$ is (T, g) -closed and using (3.1), inductively we have

$$(gx_n, gx_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0. \quad (3.2)$$

In view of (3.1) and (3.2), we have $\{Tx_n\}$ is $\mathcal{R}$ -preserving in Z (due to hypothesis (a)), i.e.,

$$(Tx_n, Tx_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0. \quad (3.3)$$

Now, if $d(gx_{n_0+1}, gx_{n_0}) = 0$ for some $n_0 \in \mathbb{N}_0$, then in view of (3.1), we have $T(x_{n_0}) = g(x_{n_0})$ so that x_{n_0} is a coincidence point of T and g and hence we are done. On the other hand, if $d(gx_{n+1}, gx_n) > 0 \quad \forall n \in \mathbb{N}_0$, then applying the contractivity condition (l) to (3.2), we deduce, for all $n \in \mathbb{N}$ that

$$d(gx_{n+1}, gx_n) \leq \phi(d(gx_n, gx_{n-1})),$$

which on using (3.2), contractive condition (I) and increasing property of φ , reduces to

$$d(gx_{n+1}, gx_n) \leq \varphi^n(d(gx_1, gx_0)). \quad (3.4)$$

Making $n \rightarrow \infty$, in (3.4) and using the definition of comparison function, we get

$$\lim_{n \rightarrow \infty} d(gx_{n+1}, gx_n) = 0. \quad (3.5)$$

Fix $\varepsilon > 0$. Then in view of (3.5), we can choose fix $n \in \mathbb{N}$ (corresponding to given $\varepsilon > 0$) such that

$$d(gx_{n+1}, gx_n) < \varepsilon - \varphi(\varepsilon). \quad (3.6)$$

Now, we claim that $\{gx_n\}$ is a Cauchy sequence. To substantiate the claim, using increasing property of φ , (3.1), (3.2) and (3.6), we obtain

$$\begin{aligned} d(gx_{n+2}, gx_n) &\leq d(gx_{n+2}, gx_{n+1}) + d(gx_{n+1}, gx_n) \\ &< d(Tx_{n+1}, Tx_n) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(d(gx_{n+1}, gx_n)) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(\varepsilon - \varphi(\varepsilon)) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(\varepsilon) + \varepsilon - \varphi(\varepsilon) \\ &= \varepsilon. \end{aligned}$$

Now, again using the increasing property of φ , (3.2) and locally T -transitivity of $\mathcal{R}$, we obtain

$$\begin{aligned} d(gx_{n+3}, gx_n) &\leq d(gx_{n+3}, gx_{n+1}) + d(gx_{n+1}, gx_n) \\ &< (d(Tx_{n+2}, Tx_n)) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(d(gx_{n+2}, gx_n)) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(\varepsilon - \varphi(\varepsilon)) + \varepsilon - \varphi(\varepsilon) \\ &\leq \varphi(\varepsilon) + \varepsilon - \varphi(\varepsilon) \\ &= \varepsilon, \end{aligned}$$

so that inductively yields,

$$d(gx_{n+k}, gx_n) < \varepsilon \text{ for all } k \in \mathbb{N}. \quad (3.7)$$

Set $m_0 := n$ and $m := n + k$ for all $k \in \mathbb{N}$ in (3.7), then the inequality (3.7) yields that $d(gx_m, gx_n) < \varepsilon \forall m, n \geq m_0$ with $m > n$, which shows that the sequence $\{gx_n\}$ is a Cauchy and $\{gx_n\}$ is $\mathcal{R}$ -preserving. By $\mathcal{R}$ -completeness of (Z, d) , there exists $z \in Z$ such that

$$\lim_{n \rightarrow \infty} g(x_n) = z. \quad (3.8)$$

On using (3.1), (3.2) and (3.8), we obtain

$$\lim_{n \rightarrow \infty} T(x_n) = z. \quad (3.9)$$

Now, to complete the proof by using (e) or (e'). Assume that (e) holds. Using (3.2), (3.8) and assumption (e₂) (i.e., $\mathcal{R}$ -continuity of g), we have

$$\lim_{n \rightarrow \infty} g(gx_n) = g(\lim_{n \rightarrow \infty} gx_n) = g(z). \quad (3.10)$$

Now, utilizing (3.1), (3.2) and (3.9) and assumption (e₂) (i.e., $\mathcal{R}$ -continuity of g), we have

$$\lim_{n \rightarrow \infty} g(Tx_n) = g(\lim_{n \rightarrow \infty} Tx_n) = g(z). \quad (3.11)$$

Since $\{Tx_n\}$ and $\{gx_n\}$ are $\mathcal{R}$ -preserving (owing to (3.2) and (3.3)) and $\lim_{n \rightarrow \infty} g(x_n) = \lim_{n \rightarrow \infty} T(x_n) = z$ (owing to (3.8) and (3.9)), using assumption (e₁) (i.e., $\mathcal{R}$ -compatibility of T and g), give rise

$$\lim_{n \rightarrow \infty} d(gTx_n, Tgx_n) = 0. \quad (3.12)$$

We need to prove that z is a coincidence point of T and g . To do this, we use assumption (e₃). Assume that T is $\mathcal{R}$ -continuous. On using (3.2), (3.9) and $\mathcal{R}$ -continuity of T , we obtain

$$\lim_{n \rightarrow \infty} T(gx_n) = T(\lim_{n \rightarrow \infty} gx_n) = T(z). \quad (3.13)$$

Applying (3.11), (3.12), (3.13) and the continuity of d , we obtain

$$d(gz, Tz) = d(\lim_{n \rightarrow \infty} gTx_n, \lim_{n \rightarrow \infty} Tgx_n) = \lim_{n \rightarrow \infty} d(gTx_n, Tgx_n) = 0,$$

so that

$$g(z) = T(z).$$

Thus $z \in C(T, g)$.

Alternatively, assume that $\mathcal{R}$ is (g, d) -self-closed. As $\{gx_n\}$ is $\mathcal{R}$ -preserving (due to (3.2)) and $g(x_n) \xrightarrow{d} z$ (in view of (3.8)), due to (g, d) -self-closedness of $\mathcal{R}$, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$[ggx_{n_k}, gz] \in \mathcal{R} \quad \forall k \in \mathbb{N}_0. \quad (3.14)$$

Since $g(x_{n_k}) \xrightarrow{d} z$, so equations (3.10)-(3.13) also hold for $\{x_{n_k}\}$ (instead of $\{x_n\}$). On using (3.14), assumption (l) and Propositions 3 and 4 (either $d(ggx_{n_k}, gz)$ is zero or non-zero), we have

$$d(Tgx_{n_k}, Tz) \leq \varphi(d(ggx_{n_k}, gz)) \leq d(ggx_{n_k}, gz) \quad \forall k \in \mathbb{N}_0. \quad (3.15)$$

Employing the triangular inequality, continuity of 'd', (3.10), (3.11), (3.12) and (3.15), we get

$$d(gz, Tz) \leq d(gz, gTx_{n_k}) + d(gTx_{n_k}, Tgx_{n_k}) + d(Tgx_{n_k}, Tz)$$

$$\begin{aligned} &\leq d(gz, gTx_{n_k}) + d(gTx_{n_k}, Tgx_{n_k}) + d(ggx_{n_k}, gz) \\ &\rightarrow 0 \text{ as } k \rightarrow \infty \end{aligned}$$

so that

$$g(z) = T(z).$$

Thus $z \in C(T, g)$ and hence again we are done.

Now, we assume that (e') holds. Owing to assumption (e'_1) (i.e., $Z \subseteq g(X)$), we can find some $u \in X$ such that $z = g(u)$. Hence, (3.8) and (3.9) respectively reduce to

$$\lim_{n \rightarrow \infty} g(x_n) = g(u). \quad (3.16)$$

$$\lim_{n \rightarrow \infty} T(x_n) = g(u). \quad (3.17)$$

Now, we need to prove that u is a coincidence point of T and g . To accomplish this, we use assumption (e'_2) . Firstly, assume that T is $(g, \mathcal{R})$ -continuous, then using (3.2) and (3.16), we get

$$\lim_{n \rightarrow \infty} T(x_n) = T(u). \quad (3.18)$$

On using (3.17) and (3.18), we get $g(u) = T(u)$. Hence, $u \in C(T, g)$.

Secondly, assume that T is continuous and g is bijective and bi-continuous. Then on using (3.16) and (3.17), we get

$$T(u) = Tg^{-1}(gu) = Tg^{-1}(\lim_{n \rightarrow \infty} gx_n) = \lim_{n \rightarrow \infty} Tg^{-1}(gx_n) = \lim_{n \rightarrow \infty} T(x_n) = g(u).$$

Thus $u \in C(T, g)$ and hence again we are through.

Finally, assume that $\mathcal{R}|_Z$ is d -self-closed. Since $\{gx_n\}$ is $\mathcal{R}|_Z$ -preserving (due to (3.2)) and $g(x_n) \xrightarrow{d} g(u) \in Z$ (due to (3.8)), using d -self-closeness of $\mathcal{R}|_Z$, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$[gx_{n_k}, gu] \in \mathcal{R}|_Z \quad \forall k \in \mathbb{N}_0. \quad (3.19)$$

Applying (3.1), (3.19), assumption (I) , Propositions 3 and 4, (whether $d(gx_{n_k}, gu)$ is zero or non-zero), we obtain

$$d(gx_{n_k+1}, Tu) = d(Tx_{n_k}, Tu) \leq \phi(d(gx_{n_k}, gu)) \leq d(gx_{n_k}, gu) \quad \forall k \in \mathbb{N}_0 \quad (3.20)$$

Applying (3.16), (3.20) and continuity of d , we get

$$d(gu, Tu) = d(\lim_{k \rightarrow \infty} gx_{n_k+1}, Tu) = \lim_{k \rightarrow \infty} d(gx_{n_k+1}, Tu) \leq \lim_{k \rightarrow \infty} d(gx_{n_k}, gu),$$

so that

$$g(u) = T(u).$$

Hence $u \in C(T, g)$. This completes the proof. $\square$

Now, we present a consequence of Theorem 3 by assuming the map $g = I$ (identity mapping on X), remains a sharpened form of Theorem 1 (due to Arif et al. [6]) in the context of $\mathcal{R}$ -complete subspace of X , which runs as:

Corollary 1. *Let (X, d) be a metric space endowed with a binary relation $\mathcal{R}$, let T be a self-mapping on X and Z be an $\mathcal{R}$ -complete subspace of X with $T(X) \subseteq Z$. Suppose that the following conditions hold:*

- (i) $\mathcal{R}$ is T -closed and locally T -transitive;
- (ii) either T is $\mathcal{R}$ -continuous or $\mathcal{R}$ is d -self-closed;
- (iii) $X(T, \mathcal{R})$ is non-empty;
- (iv) there exists a comparison function ϕ such that

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then $F(T) \neq \emptyset$.

Corollary 2. *The conclusion of Theorem 3 (also, Corollary 1) holds, if locally T -transitivity of $\mathcal{R}$ is replaced by any one of the following conditions:*

- (i) $\mathcal{R}$ is transitive;
- (ii) $\mathcal{R}$ is T -transitive;
- (iii) $\mathcal{R}$ is g -transitive;
- (iv) $\mathcal{R}$ is locally transitive.

4. UNIQUENESS RESULTS

In this section, we present the corresponding uniqueness results, as follows:

Theorem 4. *If in the hypotheses of Theorem 3, the assumption $\mathcal{R}|_{g(X)}^s$ -connectedness of $T(X)$ is added, then $\bar{C}(T, g)$ is singleton.*

Proof. Owing to Theorem 3, let $\bar{x}$ and $\bar{y}$ be the two elements of $\bar{C}(T, g)$, then there exist $x, y \in X$ such that

$$\bar{x} = g(x) = T(x) \quad \text{and} \quad \bar{y} = g(y) = T(y). \quad (4.1)$$

We claim that $\bar{x} = \bar{y}$. Since $T(x), T(y) \in T(X) \subseteq g(X)$, by assumption the $\mathcal{R}|_{g(X)}^s$ -connectedness of $T(X)$, there exists a path (say $\{gw_0, gw_1, gw_2, \dots, gw_k\}$) of some finite length k in $\mathcal{R}|_{g(X)}^s$ from $T(x)$ to $T(y)$ (where $w_0, w_1, w_2, \dots, w_k \in X$). Owing to (4.1), without loss of generality, we may set $w_0 = x$ and $w_k = y$. Thus, we obtain

$$[gw_i, gw_{i+1}] \in \mathcal{R}|_{g(X)} \text{ for each } i \ (0 \leq i \leq k-1). \quad (4.2)$$

Define the constant sequences $w_n^0 = x$ and $w_n^k = y$. On using (4.1), we have $g(w_{n+1}^0) = T(w_n^0) = \bar{x}$ and $g(w_{n+1}^k) = T(w_n^k) = \bar{y} \quad \forall n \in \mathbb{N}_0$. Set $w_0^1 = w_1, w_0^2 = w_2, \dots, w_0^{k-1} = w_{k-1}$. As $T(X) \subseteq g(X)$, on the lines similar to proof of Theorem 3, we can furnish sequences $\{w_n^1\}, \{w_n^2\}, \dots, \{w_n^{k-1}\}$ in X such that $g(w_{n+1}^1) = T(w_n^1), g(w_{n+1}^2) = T(w_n^2), \dots, g(w_{n+1}^{k-1}) = T(w_n^{k-1}) \quad \forall n \in \mathbb{N}_0$. Hence, we have

$$g(w_{n+1}^i) = T(w_n^i) \quad \forall n \in \mathbb{N}_0 \text{ and for each } i \ (0 \leq i \leq k). \quad (4.3)$$

Now, we assert that

$$[gw_n^i, gw_n^{i+1}] \in \mathcal{R} \quad \forall n \in \mathbb{N}_0 \text{ and for each } i (0 \leq i \leq k-1). \quad (4.4)$$

We prove this assertion by an inductive method with respect to index n . It follows from (4.3) that (4.4) holds for $n = 0$. Assume that (4.4) holds for $n = r > 0$, i.e.,

$$[gw_r^i, gw_r^{i+1}] \in \mathcal{R} \text{ for each } i (0 \leq i \leq k-1).$$

Since $\mathcal{R}$ is (T, g) -closed and owing to Proposition 2, we obtain

$$[Tw_r^i, Tw_r^{i+1}] \in \mathcal{R} \text{ for each } i (0 \leq i \leq k-1),$$

which on utilizing (4.3), give rise

$$[gw_{r+1}^i, gw_{r+1}^{i+1}] \in \mathcal{R} \text{ for each } i (0 \leq i \leq k-1).$$

It follows that (4.4) holds for $n = r + 1$. Therefore, by induction, (4.4) holds for all $n \in \mathbb{N}_0$. Now, for all $n \in \mathbb{N}_0$ and for each $i (0 \leq i \leq k-1)$, define $\eta_n^i := d(gw_n^i, gw_n^{i+1})$. We claim that

$$\lim_{n \rightarrow \infty} \eta_n^i = 0, \text{ for each } i (0 \leq i \leq k-1). \quad (4.5)$$

With fix i , two cases arise. Firstly, assume that $\eta_{n_0}^i := d(gw_{n_0}^i, gw_{n_0}^{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, then by the assumption (I) and Proposition 4, we obtain $d(Tw_{n_0}^i, Tw_{n_0}^{i+1}) = 0$. Consequently on using (4.3), we get $\eta_{n_0+1}^i = d(gw_{n_0+1}^i, gw_{n_0+1}^{i+1}) = d(Tw_{n_0}^i, Tw_{n_0}^{i+1}) = 0$. Thus by induction, we get $\eta_n^i = 0 \forall n \geq n_0$, yielding thereby $\lim_{n \rightarrow \infty} \eta_n^i = 0$. Secondly, assume that $\eta_n^i > 0 \forall n \in \mathbb{N}_0$, then on using (4.3), (4.4), the assumption (I), increasing property of ϕ and Proposition 3, we obtain

$$\begin{aligned} \eta_n^i &= d(gw_n^i, gw_n^{i+1}) = d(Tw_{n-1}^i, Tw_{n-1}^{i+1}) \leq \phi(d(gw_{n-1}^i, gw_{n-1}^{i+1})) = \phi(\eta_{n-1}^i) \\ &\leq \phi^2(\eta_{n-2}^i) \\ &\vdots \\ &\leq \phi^n(\eta_0^i). \end{aligned}$$

Tending with $n \rightarrow \infty$, we get $\lim_{n \rightarrow \infty} \eta_n^i = 0$. Therefore, in both cases, claim (4.5) is proved for each $i (0 \leq i \leq k-1)$. On using triangular inequality and (4.5), we obtain

$$d(\bar{x}, \bar{y}) \leq \eta_n^0 + \eta_n^1 + \cdots + \eta_n^{k-1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Thus $\bar{x} = \bar{y}$, which ends the proof. $\square$

Corollary 3. *If in the hypotheses of Theorem 4, we replace the assumption $\mathcal{R}|_{g(X)}^s$ -connectedness of $T(X)$ by either completeness of $\mathcal{R}|_{T(X)}$ or $\mathcal{R}|_{g(X)}^s$ -directedness of $T(X)$, then the conclusion of Theorem 4 also holds.*

Theorem 5. *If in the hypotheses of Theorem 4, the assumption of injectivity of any one of the mappings T and g is added, then $\bar{C}(T, g)$ is singleton.*

Theorem 6. *If in the hypotheses of Theorem 4, the assumption of weakly compatibility of T and g (embodied in conditions (e')) is added, then $F(T, g)$ is singleton.*

The proofs of Corollary 3, Theorems 5 and 6 can be obtained on the lines similar to Corollary 4.6, Theorems 4.7, and 4.8 respectively, contained in [4].

Remark 4. If in addition to hypotheses of Corollary 1, $\mathcal{R}^s$ -connectedness of $T(X)$ is added, then $F(T)$ is singleton.

5. ILLUSTRATIVE EXAMPLES

Finally, we furnish some examples to demonstrate the realized improvement of our proved results.

Example 1. Let $(X = [0, \infty), d)$ (where $d(x, y) := |x - y|$, for all $x, y \in X$) be a metric space endowed with a binary relation $\mathcal{R}$, where $\mathcal{R} := \{(x, y) \in X^2 : x - y > 0, x, y \in [0, \frac{1}{2})\} \cup \{(\frac{1}{n}, \frac{1}{n+1})_{n=2}^\infty\}$. On X , define a pair of self-mappings (T, g) by $T(x) = \frac{x}{1+2x}$ and $g(x) = \frac{x}{1+x}$ for all $x \in X$. Clearly, neither $\mathcal{R}$ is transitive nor g -transitive but it is T -transitive, hence it is locally T -transitive and also $\mathcal{R}$ is (T, g) -closed. Define a comparison function ϕ by $\phi(s) = \frac{s}{s+1} \forall s \in [0, \infty)$. On choosing $\mathcal{R}$ -complete subspace $Z = [0, 1)$ of X with $T(X) \subseteq Z \cap g(X)$ (as $T(X) = [0, \frac{1}{2})$ and $g(X) = [0, 1)$). Now, for all $(gx, gy) \in \mathcal{R}$, we have

$$\begin{aligned} d(Tx, Ty) &= \left| \frac{x}{1+2x} - \frac{y}{1+2y} \right| = \left| \frac{(x-y)}{1+2x+2y+4xy} \right| \\ &\leq \left(\frac{(x-y)}{1+x+y+2xy+(x-y)} \right) \\ &= \left(\frac{(x-y)/(1+x+y+2xy)}{(1+x+y+2xy+(x-y))/(1+x+y+2xy)} \right) \\ &= \frac{\left(\frac{x}{1+x} - \frac{y}{1+y} \right)}{1 + \left(\frac{x}{1+x} - \frac{y}{1+y} \right)} \\ &= \frac{(gx - gy)}{1 + (gx - gy)} = \frac{d(gx, gy)}{1 + d(gx, gy)} = \phi(d(gx, gy)). \end{aligned}$$

Hence the pair (T, g) and ϕ satisfy the contractive condition (I) of Theorem 3. Further, it can be easily seen that $X(T, g, \mathcal{R}) \neq \emptyset$ and continuity of the pair (T, g) . Observe that in view of Theorem 3, (T, g) has a coincidence point (namely: $x = 0$).

Further, the conditions $\mathcal{R}|_{g(X)}^s$ -connectedness of $T(X)$ and weakly compatibility of (T, g) of Theorem 6 can be easily verified. Notice that $x = 0$, is the unique common fixed point of (T, g) .

But contraction condition (d) of Theorem 2 (due to Alam and Imdad [4]) is not satisfied as on choosing very small positive number ε with $(g\varepsilon, g0) \in \mathcal{R}$, we have

$$\frac{\varepsilon}{1+2\varepsilon} = d(T\varepsilon, T0) \leq \alpha d(g\varepsilon, g0) = \alpha \frac{\varepsilon}{1+\varepsilon},$$

which implies that $\alpha \geq \frac{1+\varepsilon}{1+2\varepsilon}$. However, $\alpha \geq 1$, as ε tends to zero, which is a contradiction, this shows that genuineness of our newly proved results.

Example 2. Let $(X = [0, 4], d)$ be a metric space endowed with a binary relation $\mathcal{R} = \{(0, 0), (0, 1), (1, 0), (1, 1), (3, 0)\} \cup \{(x_n, x_{n+1}) : x_n = 4 - \frac{1}{n}\}$, where d is a usual metric on X . It is easy to see that $\mathcal{R}$ is neither transitive nor g -transitive but it is locally T -transitive. On X , define a pair of self-mappings (T, g) by

$$T(x) = \begin{cases} 0, & 0 \leq x \leq 1; \\ 1, & 1 < x < 4, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} [x], & 0 \leq x \leq 1; \\ 3, & 1 < x < 4, \end{cases}$$

where $[\cdot]$ is a greatest integer function. Clearly, $\mathcal{R}$ is (T, g) -closed. Let $Z = \{0, 1\}$, then Z is $\mathcal{R}$ -complete and $T(X) = \{0, 1\} \subseteq Z \subseteq g(X) = \{0, 1, 3\}$. Define a comparison function ϕ by $\phi(s) = \frac{1}{2}s$, for all $s \in [0, \infty)$.

Let $\{x_n\}$ be any $\mathcal{R}|_Z$ -preserving sequence in Z such that $x_n \xrightarrow{d} x$. As $(x_n, x_{n+1}) \in \mathcal{R}|_Z$, for all $n \in \mathbb{N}$, there exists positive integer $N \in \mathbb{N}$ such that $x_n = x \in \{0, 1\}$ for all $n \geq N$. Hence, we can choose a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k} = x$, for all $k \in \mathbb{N}$, which amounts to saying that $[x_{n_k}, x] \in \mathcal{R}|_Z$, for all $k \in \mathbb{N}$. Hence, $\mathcal{R}|_Z$ is d -self-closed. It can be easily seen that contraction condition (l) and remaining hypotheses of Theorem 3 are also satisfied.

Observe that, in view of Theorem 3, (T, g) has a coincidence point (namely $x = 0$). Furthermore, the remaining hypotheses of Theorem 6 also holds. Thus, all the hypotheses of Theorem 6 are satisfied and hence (T, g) has a unique common fixed point (namely $x = 0$).

On setting $g = I$, identity mapping on X in the present example and further all the conditions of Corollary 1 can be easily verified. Notice that, in view of Corollary 1 and Remark 4, $x = 0$ is the unique fixed point of T . But (X, d) is not $\mathcal{R}$ -complete, therefore this example cannot be covered by Theorem 1, which substantiate the utility of Corollary 1 over Theorem 1 (due to Arif et al. [6]).

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REPRESENTATION OF GENERALIZED CIRCULAR SURFACES AS RIGID BODY MOTIONS

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Abstract. In this paper, we introduce generalized circular surfaces, a generalization of generalized tube surfaces and circular surfaces. Moreover, we define a special dual quaternion by using the moving frame along the spine curve of generalized circular surface. We then show that the screw motion obtained by this dual quaternion can be used to construct generalized circular surfaces. We also prove that these generalized circular surfaces can be expressed by homothetic motions. Finally, we provide some examples of generalized circular surfaces with figures.

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Keywords: tube surfaces, generalized tube surfaces, circular surfaces, dual quaternion surfaces, rigid body motions (screw motions)

1. INTRODUCTION

A tube surface or a pipe surface is formed by the envelope of the spheres whose centers lie on a curve in three-dimensional real vector space $\mathbb{R}^3$ (which is called a spine curve) and whose radius are constant [15]. Generalized tube surfaces [16] and circular surfaces [19] were defined by generalizing tube surfaces according to different variables, then these kind surfaces attracted attention of many researchers. There are many objects in the world take form of these surfaces.

Real quaternions are defined as a four-dimensional number system [17]. This number system has a practical method in the performing of rotation since they rotate a vector around any axis in $\mathbb{R}^3$ [24]. Recently, real quaternionic and matrix representations of surfaces have been extensively studied by many researchers. [2,3,5,6,9,23,26,27]. Similar problem has been also considered in Minkowski and (pseudo-) Galilean spaces by using split quaternions [4, 13, 21] and (split-) semi quaternions [28].

Dual quaternions are introduced as a number system isomorphic to the tensor product of real quaternions and dual numbers [10]. In the following years, this number system has been used in the rigid body motions (i.e., screw motions) [1, 7, 14, 18,

29]. Dual quaternions have been applied in dynamics, computer graphics, robotics and spacecrafts, etc.

In this paper, we introduce generalized circular surface by generalizing of a generalized tube surface and a circular surface. A generalized circular surface (i.e., a tube surface, a generalized tube surface or a circular surface) is constituted of a spine curve and a rotation part. In [3], the rotation part of a tube surface was generated by a real quaternion. We combine the spine curve and the real quaternion in a dual quaternion. To do this, we define a dual quaternion whose translation part is a spine curve in $\mathbb{R}^3$ and rotation part is a real quaternion. Then, we prove that screw motion obtained by this dual quaternion generates a tube surface, a circular surface, a generalized tube surface or a generalized circular surface in $\mathbb{R}^3$.

2. PRELIMINARIES

In this section, some basic concepts will be given to provide a background to the main results of this paper.

Symbols

a :	Scalar	A :	Dual number
a :	Real vector in $\mathbb{R}^3$	$\hat{A}$:	Dual vector
d :	Line in $\mathbb{R}^3$	q :	Real quaternion
θ :	Real angle	Q :	Dual quaternion

2.1. Real quaternions

A real quaternion can be represented as

$$q = a_0 + a_1i + a_2j + a_3k, \quad (2.1)$$

where a_0, a_1, a_2, a_3 are real numbers and i, j, k are mutually perpendicular unit vectors satisfying $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. The set of real quaternions is often represented by H .

A real quaternion q can be also expressed as $q = S(q) + V(q)$, where $S(q) = a_0$ is the scalar part and $V(q) = a_1i + a_2j + a_3k$ is the vector part of q . If $S(q) = 0$, then q is called a pure real quaternion (i.e., a real vector in $\mathbb{R}^3$). Quaternion product of real quaternions $q = S(q) + V(q)$ and $p = S(p) + V(p)$ is

$$q \star p = S(q)S(p) - \langle V(q), V(p) \rangle + S(q)V(p) + S(p)V(q) + V(q) \times V(p), \quad (2.2)$$

where $\langle, \rangle$ and $\times$ denote the usual scalar and vector products in $\mathbb{R}^3$, respectively.

The quaternionic-conjugate of q can be defined as

$$\bar{q} = S(q) - V(q) = a_0 - a_1i - a_2j - a_3k, \quad (2.3)$$

For the real quaternions p and q , the equality

$$\overline{p \star q} = \bar{q} \star \bar{p} \quad (2.4)$$

can be given. The norm of q can be defined as

$$N(q) = q \star \bar{q} = \bar{q} \star q = a_0^2 + a_1^2 + a_2^2 + a_3^2. \quad (2.5)$$

q is called a unit real quaternion if $N(q) = 1$. A unit real quaternion $q = a_0 + a_1i + a_2j + a_3k$ for $a_1^2 + a_2^2 + a_3^2 \neq 0$ can be expressed in the form $q = \cos \theta + \sin \theta v$, where $\cos \theta = a_0$, $\sin \theta = \sqrt{a_1^2 + a_2^2 + a_3^2}$ and $v = \frac{a_1i + a_2j + a_3k}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$.

Let $p = a_0 + a_1i + a_2j + a_3k$ be a unit real quaternion and w be a vector in $\mathbb{R}^3$ (i.e., a pure quaternion). Then,

$$\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3; \quad \phi(w) = p \star w \star p^{-1} = p \star w \star \bar{p}. \quad (2.6)$$

is a linear mapping. Matrix representation of this mapping can be given as

$$M = \begin{bmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & -2a_0a_3 + 2a_1a_2 & 2a_0a_2 + 2a_1a_3 \\ 2a_0a_3 + 2a_1a_2 & a_0^2 + a_2^2 - a_1^2 - a_3^2 & -2a_0a_1 + 2a_2a_3 \\ -2a_0a_2 + 2a_1a_3 & 2a_0a_1 + 2a_2a_3 & a_0^2 + a_3^2 - a_2^2 - a_1^2 \end{bmatrix}, \quad (2.7)$$

which is orthogonal and represents a rotation in $\mathbb{R}^3$. Thus, linear mapping ϕ can be given as

$$\phi(w) = p \star w \star \bar{p} = Mw. \quad (2.8)$$

If p is in the form $p = \cos \theta + \sin \theta v$, then v is the rotation axis of the rotation in $\mathbb{R}^3$. Moreover, $\phi(w) = p \star w \star \bar{p} = Mw$ rotates the vector w in $\mathbb{R}^3$ around the vector v by a real angle 2θ .

For more information about real quaternions, see [7, 17, 24].

A homothetic motion in $\mathbb{R}^3$ can be defined by

$$y(s, t) = h(s, t)M(s, t)x(s, t) + n(s, t), \quad (2.9)$$

where y is the position vector of a point in the fixed space $\mathbb{R}'$ and x is the position vector of the moving space $\mathbb{R}$. M is an orthogonal matrix, n is a translation vector and h is a scalar and s, t are real parameters [8, 20].

2.2. Dual quaternions

A dual number $A = a_0 + \epsilon a_1$ is constituted of two real numbers a_0, a_1 whose real unit is 1 and dual unit is $\epsilon \neq 0$ satisfying $\epsilon^2 = 0$. Addition and multiplication rules of $A = a_0 + \epsilon a_1$ and $B = b_0 + \epsilon b_1$ are defined by

$$A + B = (a_0 + b_0) + \epsilon(a_1 + b_1), \quad (2.10)$$

$$AB = a_0b_0 + \epsilon(a_0b_1 + a_1b_0), \quad (2.11)$$

respectively. The dual conjugate of a dual number $A = a_0 + \epsilon a_1$ is

$$A^* = a_0 - \epsilon a_1. \quad (2.12)$$

Module $\mathbb{D}^3$ on dual numbers is given by

$$\mathbb{D}^3 = \{\hat{A} = a_0 + \epsilon a_1 : a_0, a_1 \in \mathbb{R}^3\}. \quad (2.13)$$

Each element of $\mathbb{D}^3$ is called a dual vector. The scalar and vector products of $\hat{A} = a_0 + \varepsilon a_1$ and $\hat{B} = b_0 + \varepsilon b_1$ are defined respectively by

$$\langle \hat{A}, \hat{B} \rangle = \langle a_0, b_0 \rangle + \varepsilon (\langle a_0, b_1 \rangle + \langle a_1, b_0 \rangle), \quad (2.14)$$

$$\hat{A} \times \hat{B} = a_0 \times b_0 + \varepsilon (a_0 \times b_1 + a_1 \times b_0). \quad (2.15)$$

For further information about dual numbers, see [10, 22, 25, 29].

A dual quaternion can be represented as

$$Q = A_0 + A_1 i + A_2 j + A_3 k, \quad (2.16)$$

where A_0, A_1, A_2, A_3 are dual numbers and i, j, k are the same mutually perpendicular unit vectors as in real quaternions. The set of dual quaternions is often represented by $\mathbb{H}$.

An alternative representation of a dual quaternion can be given as

$$Q = S(Q) + V(Q), \quad (2.17)$$

where $S(Q) = A_0$ and $V(Q) = A_1 i + A_2 j + A_3 k$ are, respectively, the scalar and the vector parts of Q . If $S(Q) = 0$, then Q is called a pure dual quaternion (i.e., a dual vector in $\mathbb{D}^3$). Quaternion product of any two dual quaternions $Q = S(Q) + V(Q)$ and $P = S(P) + V(P)$ is introduced as

$$\begin{aligned} Q \star P &= S(Q)S(P) - \langle V(Q), V(P) \rangle + S(Q)V(P) \\ &\quad + S(P)V(Q) + V(Q) \times V(P). \end{aligned} \quad (2.18)$$

The dual conjugate, quaternionic conjugate, quaternionic-dual conjugate and norm of a dual quaternion $Q = A_0 + A_1 i + A_2 j + A_3 k = q_0 + \varepsilon q_1$, where $q_0 = a_0 + a_1 i + a_2 j + a_3 k$ and $q_1 = a_0^* + a_1^* i + a_2^* j + a_3^* k$ for $A_0 = a_0 + \varepsilon a_0^*$, $A_1 = a_1 + \varepsilon a_1^*$, $A_2 = a_2 + \varepsilon a_2^*$ and $A_3 = a_3 + \varepsilon a_3^*$, can be given as

$$Q^* = A_0^* + A_1^* i + A_2^* j + A_3^* k = q_0 - \varepsilon q_1, \quad (2.19)$$

$$\bar{Q} = A_0 - A_1 i - A_2 j - A_3 k = \bar{q}_0 + \varepsilon \bar{q}_1, \quad (2.20)$$

$$\bar{Q}^* = A_0^* - A_1^* i - A_2^* j - A_3^* k = \bar{q}_0 - \varepsilon \bar{q}_1, \quad (2.21)$$

$$N(Q) = Q \star \bar{Q} = \bar{Q} \star Q = A_0^2 + A_1^2 + A_2^2 + A_3^2, \quad (2.22)$$

respectively. If $N(Q) = 1$, then Q is called a unit dual quaternion.

For more information about dual quaternions, see [1, 7, 10, 14, 18, 29].

3. GENERALIZED CIRCULAR SURFACES

In this section, we define generalized circular surfaces as a generalization of generalized tube surfaces and circular surfaces. Then, we show that these surfaces are constituted of a spine curve and a rotation part. Moreover, we show that the rotation part is generated by a real quaternion.

Definition 1. (generalized circular surface) Let $a_1(t)$, $a_2(t)$ and $a_3(t)$ be orthonormal vector fields and $\{a_1(t), a_2(t), a_3(t) = a_1(t) \times a_2(t)\}$ be an arbitrary frame on a curve $\alpha(t)$ in $\mathbb{R}^3$ such as Frenet frame, Darboux frame, Bishop frame, moving frame, etc. Then a generalized circular surface can be parameterized as

$$\phi(t, \theta) = \alpha(t) + r(t, \theta)(\cos \theta a_1(t) + \sin \theta a_2(t)) \in \mathbb{R}^3, \quad (3.1)$$

where $r(t, \theta) \in \mathbb{R}$ is a real variable determined by the real parameters $t, \theta \in \mathbb{R}$.

A generalized circular surface can be categorized as:

- (1) If $r(t, \theta)$ is a real constant $r \in \mathbb{R}$, then

$$\phi(t, \theta) = \alpha(t) + r(\cos \theta a_1(t) + \sin \theta a_2(t)) \quad (3.2)$$

represents a tube surface in $\mathbb{R}^3$ [15].

- (2) If $r(t, \theta)$ is a real variable $r(\theta) \in \mathbb{R}$, then

$$\phi(t, \theta) = \alpha(t) + r(\theta)(\cos \theta a_1(t) + \sin \theta a_2(t)) \quad (3.3)$$

represents a generalized tube surface in $\mathbb{R}^3$ [16].

- (3) If $r(t, \theta)$ is a real variable $r(t) \in \mathbb{R}$, then

$$\phi(t, \theta) = \alpha(t) + r(t)(\cos \theta a_1(t) + \sin \theta a_2(t)) \quad (3.4)$$

represents a circular surface in $\mathbb{R}^3$ [19].

For further information about tube surfaces, generalized tube surfaces and circular surfaces, see [11, 12, 15, 16, 19].

Remark 1. A generalized circular surface is constituted of a spine curve $\alpha(t)$ and a rotation part $R(t, \theta)$ as

$$\phi(t, \theta) = \alpha(t) + r(t, \theta)R(t, \theta) \in \mathbb{R}^3, \quad r(t, \theta) \in \mathbb{R} \quad (3.5)$$

where $R(t, \theta) = \cos \theta a_1(t) + \sin \theta a_2(t)$. In [3], the rotation part were generated by the unit real quaternion $p(t, \theta) = \cos \theta + \sin \theta a_3(t)$ as

$$R(t, \theta) = p(t, \theta) \star a_1(t), \quad (3.6)$$

where $a_1(t)$ is a pure real quaternion. $R(t, \theta) = p(t, \theta) \star a_1(t)$ represents a rotation performed by the real quaternions. Thus, the generalized circular surface can be expressed as

$$\phi(t, \theta) = \alpha(t) + r(t, \theta)p(t, \theta) \star a_1(t). \quad (3.7)$$

4. REPRESENTATION OF GENERALIZED CIRCULAR SURFACES AS SCREW MOTIONS

In this section, we define a dual quaternion surface whose rotation part is a unit real quaternion surface and translation part is a spine curve $\alpha(t)$ in $\mathbb{R}^3$. We show that screw motion obtained by this dual quaternion surface constructs a generalized circular surface $\phi(t, \theta)$ in $\mathbb{R}^3$.

Let $Q_w = 1 + \varepsilon w$ be a dual quaternion corresponding to the real vector $w \in \mathbb{R}^3$. A dual quaternion P can be expressed as

$$P = p + \frac{\varepsilon}{2} \alpha \star p, \quad (4.1)$$

where p is a unit real quaternion and α is a real vector (i.e., a pure real quaternion). p and α represent rotation and translation parts of P , respectively. Then the mapping

$$\Psi : \mathbb{H} \rightarrow \mathbb{H}, \quad \Psi(Q_w) = P \star Q_w \star \bar{P}^* \quad (4.2)$$

can be given as

$$\begin{aligned} \Psi(Q_w) &= P \star Q_w \star \bar{P}^* \\ &= \left(p + \frac{\varepsilon}{2} \alpha \star p \right) \star (1 + \varepsilon w) \star \left(\bar{p} - \frac{\varepsilon}{2} \bar{p} \star \bar{\alpha} \right) \\ &= 1 + \varepsilon (p \star w \star \bar{p} + \alpha), \end{aligned} \quad (4.3)$$

where $\Psi(Q_w) = 1 + \varepsilon (p \star w \star \bar{p} + \alpha)$ is a dual quaternion corresponding to the vector $p \star w \star \bar{p} + \alpha$ in $\mathbb{R}^3$. If $p = \cos \theta + \sin \theta v$, then $p \star w \star \bar{p} + \alpha$ means that $p \star w \star \bar{p}$ (which is given by Eqs. (2.6)-(2.8)) rotates the vector w in $\mathbb{R}^3$ around the axis v by an angle 2θ , and afterwards $p \star w \star \bar{p} + \alpha$ translates the vector $p \star w \star \bar{p}$ along the same axis v by translation vector α [1, 7, 14, 18, 29].

Definition 2. Let $\{a_1(t), a_2(t), a_3(t) = a_1(t) \times a_2(t)\}$ be a moving frame on a curve $\alpha(t)$ in $\mathbb{R}^3$. Then, using unit real quaternion surface $p(t, \theta) = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} a_3(t)$, dual quaternion surface can be defined as

$$P(t, \theta) = p(t, \theta) + \frac{\varepsilon}{2} \alpha(t) \star p(t, \theta), \quad (4.4)$$

where $p(t, \theta)$ is the rotation part of $P(t, \theta)$ and $\alpha(t)$ is the translation part of $P(t, \theta)$.

Theorem 1. Let $\{a_1(t), a_2(t), a_3(t) = a_1(t) \times a_2(t)\}$ be a moving frame on a curve $\alpha(t)$ in $\mathbb{R}^3$ and $p(t, \theta) = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} a_3(t)$ be a unit real quaternion surface. For the dual quaternions $Q_{r(t, \theta) a_1(t)} = 1 + \varepsilon r(t, \theta) a_1(t)$ and $P(t, \theta) = p(t, \theta) + \frac{\varepsilon}{2} \alpha(t) \star p(t, \theta)$, the screw motion

$$\Psi(Q_{r(t, \theta) a_1(t)}) = P(t, \theta) \star Q_{r(t, \theta) a_1(t)} \star \bar{P}^*(t, \theta) \quad (4.5)$$

generates the generalized circular surface

$$\phi(t, \theta) = \alpha(t) + r(t, \theta)(\cos \theta a_1(t) + \sin \theta a_2(t)) \in \mathbb{R}^3, \quad r(t, \theta) \in \mathbb{R}. \quad (4.6)$$

Proof. We will not use the parameters t and θ for simplicity.

Using Eq. (4.3), Eq. (4.5) can be expressed as

$$\begin{aligned} \Psi(Q_{ra_1}) &= P \star Q_{ra_1} \star \bar{P}^* \\ &= P \star (1 + \varepsilon r a_1) \star \bar{P}^* \\ &= 1 + \varepsilon (r p \star a_1 \star \bar{p} + \alpha). \end{aligned} \quad (4.7)$$

Here $\Psi(Q_{ra_1}) = 1 + \varepsilon(rp \star a_1 \star \bar{p} + \alpha)$ is a dual quaternion corresponding to the following equation

$$rp \star a_1 \star \bar{p} + \alpha \in \mathbb{R}^3. \quad (4.8)$$

From Eq. (2.2), we obtain

$$\begin{aligned} p \star a_1 \star \bar{p} &= \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} a_3 \right) \star (0 + a_1) \star \bar{p} \\ &= \left(-\sin \frac{\theta}{2} \langle a_3, a_1 \rangle + \cos \frac{\theta}{2} a_1 + \sin \frac{\theta}{2} a_3 \times a_1 \right) \star \bar{p}. \end{aligned} \quad (4.9)$$

Since $\langle a_3, a_1 \rangle = 0$ and $a_3 \times a_1 = a_2$, we get

$$\begin{aligned} p \star a_1 \star \bar{p} &= \left(\cos \frac{\theta}{2} a_1 + \sin \frac{\theta}{2} a_2 \right) \star \bar{p} \\ &= \left(0 + \cos \frac{\theta}{2} a_1 + \sin \frac{\theta}{2} a_2 \right) \star \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} a_3 \right) \\ &= - \left\langle \cos \frac{\theta}{2} a_1 + \sin \frac{\theta}{2} a_2, -\sin \frac{\theta}{2} a_3 \right\rangle \\ &\quad + \cos \frac{\theta}{2} \left(\cos \frac{\theta}{2} a_1 + \sin \frac{\theta}{2} a_2 \right) \\ &\quad - \left(\cos \frac{\theta}{2} a_1 + \sin \frac{\theta}{2} a_2 \right) \times \sin \frac{\theta}{2} a_3. \end{aligned} \quad (4.10)$$

Since $\langle a_1, a_3 \rangle = \langle a_2, a_3 \rangle = 0$, $a_1 \times a_3 = -a_2$ and $a_2 \times a_3 = a_1$, we get

$$\begin{aligned} p \star a_1 \star \bar{p} &= \cos^2 \frac{\theta}{2} a_1 + \cos \frac{\theta}{2} \sin \frac{\theta}{2} a_2 \\ &\quad - \cos \frac{\theta}{2} \sin \frac{\theta}{2} a_1 \times a_3 - \sin^2 \frac{\theta}{2} a_2 \times a_3 \\ &= \cos \theta a_1 + \sin \theta a_2. \end{aligned} \quad (4.11)$$

Using this equation in Eq. (4.8), we obtain

$$rp \star a_1 \star \bar{p} + \alpha = \alpha + r(\cos \theta a_1 + \sin \theta a_2). \quad (4.12)$$

It is obvious that this equation represents the generalized circular surface

$$\phi(t, \theta) = \alpha(t) + r(t, \theta) (\cos \theta a_1(t) + \sin \theta a_2(t)). \quad (4.13)$$

This completes the proof. $\square$

Theorem 2. Let $\phi(t, \theta)$ be a generalized circular surface corresponding to the screw motion

$$\Psi(Q_{r(t, \theta)a_1(t)}) = P(t, \theta) \star Q_{r(t, \theta)a_1(t)} \star \bar{P}^*(t, \theta), \quad (4.14)$$

where $\{a_1(t), a_2(t), a_3(t) = a_1(t) \times a_2(t)\}$ is any moving frame on $\alpha(t)$, $Q_{r(t, \theta)a_1(t)} = 1 + \varepsilon r(t, \theta)a_1(t)$, $P(t, \theta) = p(t, \theta) + \frac{\varepsilon}{2}\alpha(t) \star p(t, \theta)$ and $p(t, \theta) = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} a_3(t)$.

Then, the generalized circular surface $\phi(t, \theta)$ can be expressed by the homothetic motion of $a_1(t)$ in $\mathbb{R}^3$ as

$$\phi(t, \theta) = \alpha(t) + h(t, \theta)M(t, \theta)a_1(t), \quad (4.15)$$

where $M(t, \theta)$ is an orthogonal matrix satisfying $p(t, \theta) \star a_1(t) \star \bar{p}(t, \theta) = M(t, \theta)a_1(t)$, $h(t, \theta) = r(t, \theta)$ is a homothetic scalar, $\alpha(t)$ is a translation vector and t, θ are homothetic parameters.

Proof. From Eqs. (4.12) and (4.13), surface generalization $\phi(t, \theta)$ can be expressed as

$$\phi(t, \theta) = \alpha(t) + r(t, \theta)p(t, \theta) \star a_1(t) \star \bar{p}(t, \theta). \quad (4.16)$$

Using Eqs. (2.6)-(2.8), we get

$$p(t, \theta) \star a_1(t) \star \bar{p}(t, \theta) = M(t, \theta)a_1(t). \quad (4.17)$$

Using this equation in Eq. (4.16), we obtain

$$\begin{aligned} \phi(t, \theta) &= \alpha(t) + r(t, \theta)p(t, \theta) \star a_1(t) \star \bar{p}(t, \theta) \\ &= \alpha(t) + h(t, \theta)M(t, \theta)a_1(t). \end{aligned} \quad (4.18)$$

This completes the proof. $\square$

Remark 2. If we take the real variable $r(t, \theta) \in \mathbb{R}$ as the real constant r , as the real variable $r(t)$ related to the real parameter t or as the real variable $r(\theta)$ related to the real parameter θ in Theorem 4.2, then the screw motions

$$\Psi(Q_{ra_1(t)}) = P(t, \theta) \star Q_{ra_1(t)} \star \bar{P}^*(t, \theta), \quad (4.19)$$

$$\Psi(Q_{r(t)a_1(t)}) = P(t, \theta) \star Q_{r(t)a_1(t)} \star \bar{P}^*(t, \theta), \quad (4.20)$$

$$\Psi(Q_{r(\theta)a_1(t)}) = P(t, \theta) \star Q_{r(\theta)a_1(t)} \star \bar{P}^*(t, \theta) \quad (4.21)$$

$$(4.22)$$

generates a tube surface, a circular surface and a generalized tube surface, respectively, as

$$\phi(t, \theta) = \alpha(t) + r(\cos \theta a_1(t) + \sin \theta a_2(t)), \quad (4.23)$$

$$\phi(t, \theta) = \alpha(t) + r(t)(\cos \theta a_1(t) + \sin \theta a_2(t)), \quad (4.24)$$

$$\phi(t, \theta) = \alpha(t) + r(\theta)(\cos \theta a_1(t) + \sin \theta a_2(t)). \quad (4.25)$$

Example 1. Let us take a curve in $\mathbb{R}^3$ as

$$\alpha(t) = \left(\frac{\sqrt{3}t}{2}, \sin \frac{t}{2}, \cos \frac{t}{2} \right) \quad (4.26)$$

and its moving frame vectors as

$$a_1(t) = \left(0, -\sin \frac{t}{2}, -\cos \frac{t}{2} \right), \quad (4.27)$$

$$a_2(t) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2} \cos \frac{t}{2}, -\frac{\sqrt{3}}{2} \sin \frac{t}{2} \right), \quad (4.28)$$

$$a_3(t) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2} \cos \frac{t}{2}, -\frac{1}{2} \sin \frac{t}{2} \right). \quad (4.29)$$

Thus, we get the dual quaternion surface

$$\begin{aligned} P(t, \theta) &= p(t, \theta) + \frac{\varepsilon}{2} \alpha(t) \star p(t, \theta) \\ &= \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} a_3(t) \right) \\ &\quad + \frac{\varepsilon}{2} \left(0 + \left(\frac{\sqrt{3}t}{2}, \sin \frac{t}{2}, \cos \frac{t}{2} \right) \right) \star \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} a_3(t) \right), \end{aligned} \quad (4.30)$$

where $p(t, \theta) = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} a_3(t)$. Then, screw motion $\Psi(Q_{r(t, \theta) a_1(t)})$ can be given as

$$\begin{aligned} \Psi(Q_{r(t, \theta) a_1(t)}) &= P(t, \theta) \star Q_{r(t, \theta) a_1(t)} \star \bar{P}^*(t, \theta) \\ &= 1 + \varepsilon (r(t, \theta) p(t, \theta) \star a_1(t) \star \bar{p}(t, \theta) + \alpha(t)). \end{aligned} \quad (4.31)$$

Thus, generalized circular surface corresponding to this equation can be obtained as

$$\begin{aligned} \phi(t, \theta) &= \alpha(t) + r(t, \theta) p(t, \theta) \star a_1(t) \star \bar{p}(t, \theta) \\ &= \left(\frac{\sqrt{3}t}{2}, \sin \frac{t}{2}, \cos \frac{t}{2} \right) + r(t, \theta) (\cos \theta a_1(t) + \sin \theta a_2(t)), \end{aligned} \quad (4.32)$$

where $a_1(t) = \left(0, -\sin \frac{t}{2}, -\cos \frac{t}{2} \right)$ and $a_2(t) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2} \cos \frac{t}{2}, -\frac{\sqrt{3}}{2} \sin \frac{t}{2} \right)$.

$\phi(t, \theta)$ can be given as tube surface with $r(t, \theta) = r = \frac{1}{3}$ as in Fig. 1a, generalized tube surface with $r(t, \theta) = r(\theta) = \frac{\theta}{3}$ as in Fig. 1b, circular surface with $r(t, \theta) = r(t) = \frac{t}{3}$ as in Fig. 1c and generalized circular surface with $r(t, \theta) = \frac{t\theta}{3}$ as in Fig. 1d for intervals $-10 \leq t \leq 10$ and $-5 \leq \theta \leq 5$.

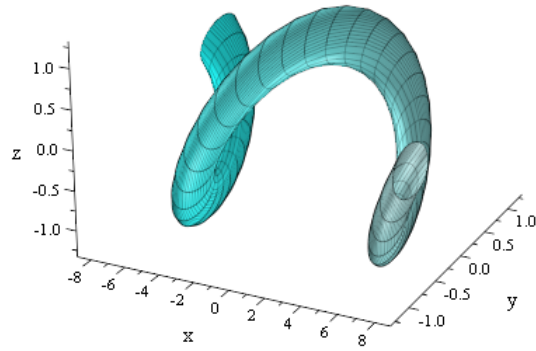


Fig. 1a. Geometric representation of a tube surface

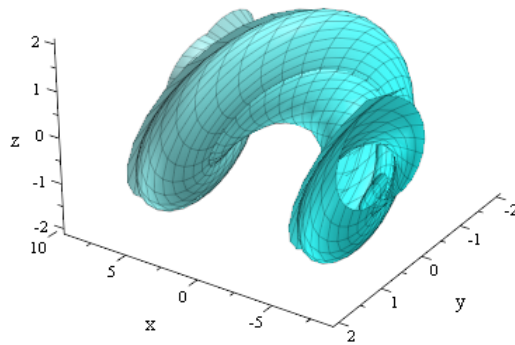


Fig. 1b. Geometric representation of a generalized tube surface

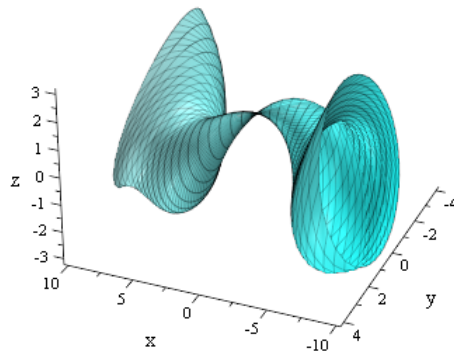


Fig. 1c. Geometric representation of a circular surface

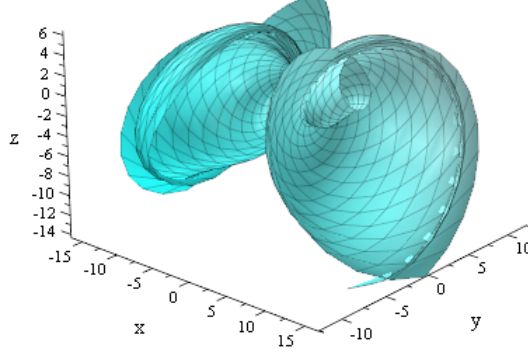


Fig. 1d. Geometric representation of a generalized circular surface

5. CONCLUSIONS

We introduce generalized circular surface as

$$\phi(t, \theta) = \alpha(t) + r(t, \theta)(\cos \theta a_1(t) + \sin \theta a_2(t)) \in \mathbb{R}^3, \quad r(t, \theta) \in \mathbb{R},$$

where $\{a_1(t), a_2(t), a_3(t) = a_1(t) \times a_2(t)\}$ is an arbitrary moving frame (i.e., Frenet frame, Darboux frame, Bishop frame, moving frame, etc.) on the spine curve $\alpha(t)$.

We show that the spine curve $\alpha(t)$ is the translation part and $R(t, \theta) = \cos \theta a_1(t) + \sin \theta a_2(t)$ is the rotation part of the generalized circular surface $\phi(t, \theta)$. The rotation part of the generalized circular surface can be generated by a real quaternion as

$$\begin{aligned} \phi(t, \theta) &= \alpha(t) + r(t, \theta)R(t, \theta) \\ &= \alpha(t) + r(t, \theta)p(t, \theta) \star a_1(t), \end{aligned}$$

where $p(t, \theta) = \cos \theta + \sin \theta a_3(t)$ is a real quaternion.

Let $\{a_1(t), a_2(t), a_3(t) = a_1(t) \times a_2(t)\}$ be a moving frame on the curve $\alpha(t)$ and let $p(t, \theta) = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} a_3(t)$ be a unit real quaternion surface. We show that screw motion obtained by using dual quaternions $P(t, \theta) = p(t, \theta) + \frac{\varepsilon}{2} \alpha(t) \star p(t, \theta)$ and $Q_{r(t, \theta)a_1(t)}$ (which is a dual quaternion corresponding to $r(t, \theta)a_1(t) \in \mathbb{R}^3$) constructs a generalized circular surface in $\mathbb{R}^3$ as

$$\begin{aligned} \Psi(Q_{r(t, \theta)a_1(t)}) &= P(t, \theta) \star Q_{r(t, \theta)a_1(t)} \star \bar{P}^*(t, \theta) \\ &= 1 + \varepsilon (\alpha(t) + r(t, \theta) (\cos \theta a_1(t) + \sin \theta a_2(t))). \end{aligned}$$

This equation represents the generalized circular surface

$$\phi(t, \theta) = \alpha(t) + r(t, \theta) (\cos \theta a_1(t) + \sin \theta a_2(t)).$$

If we take $r(t, \theta)$ as real constant r or as real variables $r(t)$ or $r(\theta)$, then the screw motion Ψ generates a tube surface, a circular surface or a generalized tube surface,

respectively, as

$$\begin{aligned}\phi(t, \theta) &= \alpha(t) + r(\cos \theta a_1(t) + \sin \theta a_2(t)), \\ \phi(t, \theta) &= \alpha(t) + r(t)(\cos \theta a_1(t) + \sin \theta a_2(t)), \\ \phi(t, \theta) &= \alpha(t) + r(\theta)(\cos \theta a_1(t) + \sin \theta a_2(t)).\end{aligned}$$

Moreover, the generalized circular surface is expressed by homothetic motion as

$$\phi(t, \theta) = \alpha(t) + h(t, \theta)M(t, \theta)a_1(t),$$

where $M(t, \theta)$ is an orthogonal matrix satisfying $p(t, \theta) \star a_1(t) \star \bar{p}(t, \theta) = M(t, \theta)a_1(t)$, $h(t, \theta) = r(t, \theta)$ is a homothetic scalar, $\alpha(t)$ is a translation vector and t, θ are homothetic parameters.

Special dual quaternion and screw motion used in this paper can be studied in the researching of some surfaces in Minkowski, Galilean or pseudo-Galilean spaces.

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M-SRIVASTAVA HYPERGEOMETRIC FUNCTIONS: INTEGRAL REPRESENTATIONS AND SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce a new extensions of Srivastava's triple hypergeometric functions H_A , H_B and H_C , by using an modified beta function, which given with a generalized M-series in its kernel. We also introduce a new extension of Appell's hypergeometric function of the first kind by using the same modified beta function. Furthermore, we give some integral representations of the new extensions of Srivastava's triple hypergeometric functions. Finally, we obtain solutions of fractional differential equations involving new extensions of Srivastava's triple hypergeometric functions.

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1. INTRODUCTION AND PRELIMINARIES

Scientists have conducted a lot of research in recent years on various generalizations of special functions (see for example [2, 3, 5, 7–10, 16, 19, 21, 28] and reference therein). Particularly, the modified gamma function for $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\kappa) > 0$ and $\xi_1, \dots, \xi_p, \eta_1, \dots, \eta_q \neq 0, -1, -2, \dots$ was introduced by Ata in [4] as follows:

$$\begin{aligned} M\Gamma_{p,q}^{(\alpha,\beta)}(\kappa; \rho) &= M\Gamma_{p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \kappa; \rho) \\ &= \int_0^\infty \Delta^{\kappa-1} {}_pM_q^\beta \left(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; -\Delta - \frac{\rho}{\Delta} \right) d\Delta. \end{aligned}$$

Also, the modified beta function for $\Re(\alpha) > 0$, $\Re(\rho) > 0$, $\Re(\kappa) > 0$, $\Re(\omega) > 0$ and $\xi_1, \dots, \xi_p, \eta_1, \dots, \eta_q \neq 0, -1, -2, \dots$ was introduced by Ata in [4] as follows:

$$MB_{p,q}^{(\alpha,\beta)}(\kappa, \omega; \rho) = MB_{p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \kappa, \omega; \rho)$$

$$= \int_0^1 \Delta^{\kappa-1} (1-\Delta)^{\omega-1} {}^\alpha_p M_q^\beta \left(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \frac{-\rho}{\Delta(1-\Delta)} \right) d\Delta. \quad (1.1)$$

If we take $\Delta = (\sin \phi)^2$ in the equation (1.1), then

$$\begin{aligned} {}^M B_{p,q}^{(\alpha,\beta)}(\kappa, \omega; \rho) &= {}^M B_{p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \kappa, \omega; \rho) \\ &= 2 \int_0^{\frac{\pi}{2}} (\sin \phi)^{2\kappa-1} (\cos \phi)^{2\omega-1} \\ &\quad \times {}^\alpha_p M_q^\beta(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; -\rho(\sec \phi)^2 (\csc \phi)^2) d\phi. \end{aligned} \quad (1.2)$$

If we take $\Delta = \frac{\Lambda}{1+\Lambda}$ in the equation (1.1), then

$$\begin{aligned} {}^M B_{p,q}^{(\alpha,\beta)}(\kappa, \omega; \rho) &= {}^M B_{p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \kappa, \omega; \rho) \\ &= \int_0^\infty \frac{\Lambda^{\kappa-1}}{(1+\Lambda)^{\kappa+\omega}} \\ &\quad \times {}^\alpha_p M_q^\beta \left(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; -2\rho - \rho \left(\Lambda + \frac{1}{\Lambda} \right) \right) d\Lambda. \end{aligned} \quad (1.3)$$

If we take $\Delta = \frac{\Lambda-u}{v-u}$ in the equation (1.1), then

$$\begin{aligned} {}^M B_{p,q}^{(\alpha,\beta)}(\kappa, \omega; \rho) &= {}^M B_{p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \kappa, \omega; \rho) \\ &= (v-u)^{1-\kappa-\omega} \int_u^v (\Lambda-u)^{\kappa-1} (v-\Lambda)^{\omega-1} \\ &\quad \times {}^\alpha_p M_q^\beta \left(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \frac{-\rho(v-u)^2}{(\Lambda-u)(v-\Lambda)} \right) d\Lambda. \end{aligned} \quad (1.4)$$

Moreover, the modified Gauss hypergeometric function for $\Re(\mu_3) > \Re(\mu_2) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$ and $\xi_1, \dots, \xi_p, \eta_1, \dots, \eta_q \neq 0, -1, -2, \dots$ was introduced by Ata in [4] as follows:

$$\begin{aligned} {}^M F_{p,q}^{(\alpha,\beta)}(\mu_1, \mu_2; \mu_3; \tau; \rho) &= {}^M F_{p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \mu_1, \mu_2; \mu_3; \tau; \rho) \\ &= \sum_{n=0}^{\infty} (\mu_1)_n \frac{{}^M B_{p,q}^{(\alpha,\beta)}(\mu_2 + n, \mu_3 - \mu_2; \rho)}{B(\mu_2, \mu_3 - \mu_2)} \frac{\tau^n}{n!}, \quad (|\tau| < 1). \end{aligned}$$

The modified special functions given above were called M-gamma, M-beta and M-Gauss hypergeometric functions by Ata, respectively. If we put $\rho = 0$ and $p = q = \xi_1 = \eta_1 = \alpha = \beta = 1$ to the M-gamma, M-beta and M-Gauss hypergeometric functions, we get the classical special functions [1], respectively, as follows:

- The gamma function for $\Re(\kappa) > 0$:

$$\Gamma(\kappa) = \int_0^\infty \Delta^{\kappa-1} \exp(-\Delta) d\Delta.$$

- The beta function for $\Re(\kappa) > 0$ and $\Re(\omega) > 0$:

$$B(\kappa, \omega) = \int_0^1 \Delta^{\kappa-1} (1-\Delta)^{\omega-1} d\Delta.$$

- The Gauss hypergeometric function for $\Re(\mu_3) > \Re(\mu_2) > 0$:

$${}_2F_1(\mu_1, \mu_2; \mu_3; \tau) = \sum_{n=0}^{\infty} (\mu_1)_n \frac{B(\mu_2+n, \mu_3-\mu_2)}{B(\mu_2, \mu_3-\mu_2)} \frac{\tau^n}{n!}, \quad (|\tau| < 1).$$

The gamma and beta functions relation [1] is as follows:

$$B(\kappa, \omega) = \frac{\Gamma(\kappa)\Gamma(\omega)}{\Gamma(\kappa+\omega)}, \quad (\Re(\kappa) > 0, \Re(\omega) > 0). \quad (1.5)$$

The function ${}_pM_q^\beta$ used above is known as the generalized M-series [23] for $\Re(\alpha) > 0$ and $\xi_1, \dots, \xi_p, \eta_1, \dots, \eta_q \neq 0, -1, -2, \dots$ which defined as:

$${}_pM_q^\beta(\tau) = {}_pM_q^\beta(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \tau) = \sum_{n=0}^{\infty} \frac{(\xi_1)_n \dots (\xi_p)_n}{(\eta_1)_n \dots (\eta_q)_n} \frac{\tau^n}{\Gamma(\alpha n + \beta)}.$$

The symbol $(\cdot)_n$ used above denotes the Pochhammer symbol [1] and is defined by

$$(\zeta)_n = \frac{\Gamma(\zeta+n)}{\Gamma(\zeta)} = \begin{cases} \zeta(\zeta+1) \dots (\zeta+n-1), & n = 1, 2, \dots, \\ 1, & n = 0. \end{cases} \quad (1.6)$$

The equation (1.6) also yields [26]

$$(\zeta)_{n+m} = (\zeta)_n (\zeta+n)_m. \quad (1.7)$$

The binomial theorem [1] is as follows:

$$(1-\Delta)^{-\zeta} = \sum_{n=0}^{\infty} (\zeta)_n \frac{\Delta^n}{n!}, \quad (|\Delta| < 1). \quad (1.8)$$

The Caputo fractional derivative operator [18] for $\Re(\varepsilon) > 0$, $m-1 < \Re(\varepsilon) < m$, ($m \in \mathbb{N}$) is given by

$${}^cD_\rho^\varepsilon \{f(\rho)\} = \frac{1}{\Gamma(m-\varepsilon)} \int_0^\rho (\rho-\omega)^{m-\varepsilon-1} f^{(m)}(\omega) d\omega, \quad (\rho > 0).$$

The Laplace and inverse Laplace transforms for $\Re(s) > 0$ in [17], respectively, are defined as:

$$\mathcal{L}\{f(\rho); s\} = F(s) = \int_0^\infty \exp(-s\rho) f(\rho) d\rho$$

and

$$\mathcal{L}^{-1}\{F(s)\} = f(\rho) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(s\rho) F(s) ds, \quad (c > 0).$$

The Laplace transform of the Caputo fractional derivative is as follows [20]:

$$\mathcal{L}\left\{{}^c D_{\rho}^{\varepsilon}\{f(\rho)\};s\right\}=s^{\varepsilon}F(s)-\sum_{k=0}^{m-1}s^{\varepsilon-k-1}f^{(k)}(0), \quad (m-1<\Re(\varepsilon)\leq m). \quad (1.9)$$

We introduce the new extended Appell's hypergeometric function of the first kind for $\Re(\mu_4) > \Re(\mu_1) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$ and $\max\{|\kappa|, |\omega|\} < 1$ as follows:

$$\begin{aligned} {}^M F_{1,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega; \rho) &= {}^M F_{1,p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega; \rho) \\ &:= \sum_{m,n=0}^{\infty} (\mu_2)_m (\mu_3)_n \frac{{}^M B_{p,q}^{(\alpha,\beta)}(\mu_1 + m + n, \mu_4 - \mu_1; \rho)}{B(\mu_1, \mu_4 - \mu_1)} \frac{\kappa^m \omega^n}{m! n!}, \end{aligned} \quad (1.10)$$

which we called as M-Appell hypergeometric function F_1 .

2. M-SRIVASTAVA HYPERGEOMETRIC FUNCTIONS

Srivastava defined triple hypergeometric functions H_A , H_B and H_C in [24, 25] and then scientists have studied on various extended of these functions [6, 11–15, 22, 27].

We introduce the new extended Srivastava's triple hypergeometric functions as follows:

$$\begin{aligned} {}^M H_{A,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) &= {}^M H_{A,p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) \\ &:= \sum_{m,n,k=0}^{\infty} \frac{(\mu_1)_{m+k} (\mu_2)_{m+n}}{(\mu_4)_m} \frac{{}^M B_{p,q}^{(\alpha,\beta)}(\mu_3 + n + k, \mu_5 - \mu_3; \rho)}{B(\mu_3, \mu_5 - \mu_3)} \frac{\kappa^m \omega^n \tau^k}{m! n! k!}, \end{aligned} \quad (2.1)$$

$$(\Re(\alpha) > 0, \Re(\rho) > 0, \mathbf{r} < 1, \mathbf{s} < 1, \mathbf{t} < (1 - \mathbf{r})(1 - \mathbf{s})),$$

$$\begin{aligned} {}^M H_{B,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \rho) &= {}^M H_{B,p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \rho) \\ &:= \sum_{m,n,k=0}^{\infty} \frac{(\mu_1 + \mu_2)_{2m+n+k} (\mu_3)_{n+k}}{(\mu_4)_m (\mu_5)_n (\mu_6)_k} \frac{{}^M B_{p,q}^{(\alpha,\beta)}(\mu_1 + m + k, \mu_2 + m + n; \rho)}{B(\mu_1, \mu_2)} \frac{\kappa^m \omega^n \tau^k}{m! n! k!}, \end{aligned}$$

$$(\Re(\alpha) > 0, \Re(\rho) > 0, \mathbf{r} + \mathbf{s} + \mathbf{t} + 2\sqrt{\mathbf{rst}} < 1),$$

and

$$\begin{aligned} {}^M H_{C,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \rho) &= {}^M H_{C,p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \rho) \\ &:= \sum_{m,n,k=0}^{\infty} \frac{(\mu_2)_{m+n} (\mu_3)_{n+k}}{(\mu_4)_n} \frac{{}^M B_{p,q}^{(\alpha,\beta)}(\mu_1 + m + k, \mu_4 + n - \mu_1; \rho)}{B(\mu_1, \mu_4 + n - \mu_1)} \frac{\kappa^m \omega^n \tau^k}{m! n! k!}, \end{aligned} \quad (2.2)$$

$$\left(\Re(\alpha) > 0, \Re(\rho) > 0, \mathbf{r} < 1, \mathbf{s} < 1, \mathbf{t} < 1, \mathbf{r} + \mathbf{s} + \mathbf{t} - 2\sqrt{(1-\mathbf{r})(1-\mathbf{s})(1-\mathbf{t})} < 2 \right),$$

where for brevity, the co-ordinates are written $(\mathbf{r}, \mathbf{s}, \mathbf{t})$ instead of $(|\kappa|, |\omega|, |\tau|)$.

Respectively, we called them as M-Srivastava hypergeometric function H_A , M-Srivastava hypergeometric function H_B , and M-Srivastava hypergeometric function H_C . Obviously for $\rho = 0$ and $p = q = \xi_1 = \eta_1 = \alpha = \beta = 1$, these functions are reduced to the Srivastava's triple hypergeometric functions H_A , H_B , and H_C .

The equations (2.1) and (2.2) can also be given with the following series representations:

$$\begin{aligned} {}^M H_{A,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) \\ = \sum_{m=0}^{\infty} \frac{(\mu_1)_m (\mu_2)_m}{(\mu_4)_m} {}^M F_{1,p,q}^{(\alpha,\beta)}(\mu_3, \mu_2 + m, \mu_1 + m; \mu_5; \omega, \tau; \rho) \frac{\kappa^m}{m!}, \\ {}^M H_{C,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \rho) \\ = \sum_{n=0}^{\infty} \frac{(\mu_2)_n (\mu_3)_n}{(\mu_4)_n} {}^M F_{1,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2 + n, \mu_3 + n; \mu_4 + n; \kappa, \tau; \rho) \frac{\omega^n}{n!}, \end{aligned}$$

where ${}^M F_{1,p,q}^{(\alpha,\beta)}$ is the extended Appell's hypergeometric function of the first kind given by (1.10).

3. INTEGRAL REPRESENTATIONS FOR M-SRIVASTAVA HYPERGEOMETRIC FUNCTION H_A

Theorem 1. Let $\Re(\mu_5) > \Re(\mu_3) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_A holds true:

$$\begin{aligned} {}^M H_{A,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) \\ = \frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5 - \mu_3)} \int_0^1 \Delta^{\mu_3-1} (1-\Delta)^{\mu_5-\mu_3-1} (1-\omega\Delta)^{-\mu_2} (1-\tau\Delta)^{-\mu_1} \\ \times {}_p M_q^{\alpha\beta} \left(\frac{-\rho}{\Delta(1-\Delta)} \right) {}_2 F_1 \left(\mu_1, \mu_2; \mu_4; \frac{\kappa}{(1-\omega\Delta)(1-\tau\Delta)} \right) d\Delta. \end{aligned}$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A , we have

$$\begin{aligned} {}^M H_{A,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) \\ = \sum_{m,n,k=0}^{\infty} \frac{(\mu_1)_{m+k} (\mu_2)_{m+n}}{(\mu_4)_m} \frac{{}^M B_{p,q}^{(\alpha,\beta)}(\mu_3 + n + k, \mu_5 - \mu_3; \rho)}{B(\mu_3, \mu_5 - \mu_3)} \frac{\kappa^m \omega^n \tau^k}{m! n! k!} \\ = \sum_{m,n,k=0}^{\infty} \frac{(\mu_1)_{m+k} (\mu_2)_{m+n}}{(\mu_4)_m} \frac{1}{B(\mu_3, \mu_5 - \mu_3)} \int_0^1 \Delta^{\mu_3+n+k-1} (1-\Delta)^{\mu_5-\mu_3-1} \end{aligned}$$

$$\times {}^{\alpha}M_p^{\beta} \left(\frac{-\rho}{\Delta(1-\Delta)} \right) \frac{\kappa^m}{m!} \frac{\omega^n}{n!} \frac{\tau^k}{k!} d\Delta.$$

By using equations (1.5) and (1.7), we get

$$\begin{aligned} & {}^MH_{A,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) \\ &= \frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)} \sum_{m,n,k=0}^{\infty} \frac{(\mu_1)_{m+k}(\mu_2)_{m+n}}{(\mu_4)_m} \int_0^1 \Delta^{\mu_3+n+k-1} (1-\Delta)^{\mu_5-\mu_3-1} \\ & \quad \times {}^{\alpha}M_p^{\beta} \left(\frac{-\rho}{\Delta(1-\Delta)} \right) \frac{\kappa^m}{m!} \frac{\omega^n}{n!} \frac{\tau^k}{k!} d\Delta \\ &= \frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)} \int_0^1 \Delta^{\mu_3-1} (1-\Delta)^{\mu_5-\mu_3-1} {}^{\alpha}M_p^{\beta} \left(\frac{-\rho}{\Delta(1-\Delta)} \right) \\ & \quad \times \sum_{m=0}^{\infty} \frac{(\mu_1)_m(\mu_2)_m}{(\mu_4)_m} \frac{\kappa^m}{m!} \sum_{n=0}^{\infty} (\mu_2+m)_n \frac{(\omega\Delta)^n}{n!} \sum_{k=0}^{\infty} (\mu_1+m)_k \frac{(\tau\Delta)^k}{k!} d\Delta. \end{aligned}$$

By using equation (1.8), we obtain

$$\begin{aligned} & {}^MH_{A,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) \\ &= \frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)} \int_0^1 \Delta^{\mu_3-1} (1-\Delta)^{\mu_5-\mu_3-1} {}^{\alpha}M_p^{\beta} \left(\frac{-\rho}{\Delta(1-\Delta)} \right) \\ & \quad \times \sum_{m=0}^{\infty} \frac{(\mu_1)_m(\mu_2)_m}{(\mu_4)_m} \frac{\kappa^m}{m!} (1-\omega\Delta)^{-\mu_2-m} (1-\tau\Delta)^{-\mu_1-m} d\Delta \\ &= \frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)} \int_0^1 \Delta^{\mu_3-1} (1-\Delta)^{\mu_5-\mu_3-1} (1-\omega\Delta)^{-\mu_2} (1-\tau\Delta)^{-\mu_1} \\ & \quad \times {}^{\alpha}M_p^{\beta} \left(\frac{-\rho}{\Delta(1-\Delta)} \right) {}_2F_1 \left(\mu_1, \mu_2; \mu_4; \frac{\kappa}{(1-\omega\Delta)(1-\tau\Delta)} \right) d\Delta, \end{aligned}$$

which completes the proof. $\square$

Theorem 2. Let $\Re(\mu_4) > \Re(\mu_2) > 0$, $\Re(\mu_5) > \Re(\mu_3) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M -Srivastava hypergeometric function H_A holds true:

$$\begin{aligned} & {}^MH_{A,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) \\ &= \frac{\Gamma(\mu_4)\Gamma(\mu_5)}{\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4-\mu_2)\Gamma(\mu_5-\mu_3)} \int_0^1 \int_0^1 \Lambda^{\mu_2-1} \Delta^{\mu_3-1} (1-\Lambda)^{\mu_4-\mu_2-1} \\ & \quad \times (1-\Delta)^{\mu_5-\mu_3-1} (1-\omega\Delta)^{\mu_1-\mu_2} [(1-\omega\Delta)(1-\tau\Delta) - \kappa\Lambda]^{-\mu_1} \\ & \quad \times {}^{\alpha}M_p^{\beta} \left(\frac{-\rho}{\Lambda(1-\Lambda)} \right) {}^{\alpha}M_p^{\beta} \left(\frac{-\rho}{\Delta(1-\Delta)} \right) d\Lambda d\Delta. \end{aligned}$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A and by making similar calculations in the proof of Theorem 1, the proof is completed. $\square$

Theorem 3. Let $\Re(\mu_4) > \Re(\mu_2) > 0$, $\Re(\mu_5) > \Re(\mu_3) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_A holds true:

$$\begin{aligned} & {}^M H_{A,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) \\ &= \frac{\Gamma(\mu_4)\Gamma(\mu_5)}{\Gamma(\mu_2)\Gamma(\mu_3)\Gamma(\mu_4 - \mu_2)\Gamma(\mu_5 - \mu_3)} \int_0^1 \int_0^1 \Lambda^{\mu_2-1} \Delta^{\mu_3-1} (1 - \Lambda)^{\mu_4 - \mu_2 - 1} \\ & \quad \times (1 - \Delta)^{\mu_5 - \mu_3 - 1} (1 - \omega\Delta)^{-\mu_2} (1 - \kappa\Lambda - \tau\Delta)^{-\mu_1} \left(1 - \frac{\kappa\omega\Lambda\Delta}{(1 - \omega\Delta)(1 - \kappa\Lambda - \tau\Delta)} \right)^{-\mu_1} \\ & \quad \times {}_p M_q^\beta \left(\frac{-\rho}{\Lambda(1 - \Lambda)} \right) {}_p M_q^\beta \left(\frac{-\rho}{\Delta(1 - \Delta)} \right) d\Lambda d\Delta. \end{aligned}$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A and by making similar calculations in the proof of Theorem 1, the proof is completed. $\square$

Theorem 4. Let $\Re(\mu_5) > \Re(\mu_3) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_A holds true:

$$\begin{aligned} & {}^M H_{A,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) \\ &= \frac{2\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5 - \mu_3)} \int_0^{\frac{\pi}{2}} (\sin \phi)^{2\mu_3-1} (\cos \phi)^{2\mu_5-2\mu_3-1} (1 - \omega(\sin \phi)^2)^{-\mu_2} \\ & \quad \times (1 - \tau(\sin \phi)^2)^{-\mu_1} {}_p M_q^\beta (-\rho(\sec \phi)^2(\csc \phi)^2) \\ & \quad \times {}_2F_1 \left(\mu_1, \mu_2; \mu_4; \frac{\kappa}{(1 - \omega(\sin \phi)^2)(1 - \tau(\sin \phi)^2)} \right) d\phi. \end{aligned}$$

Proof. By using the formula (1.2) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A and by making similar calculations in the proof of Theorem 1, the proof is completed. $\square$

Theorem 5. Let $\Re(\mu_5) > \Re(\mu_3) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_A holds true:

$$\begin{aligned} & {}^M H_{A,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) \\ &= \frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5 - \mu_3)} \int_0^\infty \Lambda^{\mu_3-1} (1 + \Lambda)^{\mu_1 + \mu_2 - \mu_5} (1 + \Lambda - \omega\Lambda)^{-\mu_2} \\ & \quad \times (1 + \Lambda - \tau\Lambda)^{-\mu_1} {}_p M_q^\beta \left(-2\rho - \rho \left(\Lambda + \frac{1}{\Lambda} \right) \right) \end{aligned}$$

$$\times {}_2F_1\left(\mu_1, \mu_2; \mu_4; \frac{\kappa(1+\Lambda)^2}{(1+\Lambda-\omega\Lambda)(1+\Lambda-\tau\Lambda)}\right) d\Lambda.$$

Proof. By using the formula (1.3) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A and by making similar calculations in the proof of Theorem 1, the proof is completed. $\square$

Theorem 6. Let $\Re(\mu_5) > \Re(\mu_3) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_A holds true:

$$\begin{aligned} & {}^M H_{A,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \rho) \\ &= \frac{\Gamma(\mu_5)}{\Gamma(\mu_3)\Gamma(\mu_5-\mu_3)} (v-u)^{1+\mu_1+\mu_2-\mu_5} \int_u^v (\Lambda-u)^{\mu_3-1} (v-\Lambda)^{\mu_5-\mu_3-1} \\ & \quad \times (v-u-\omega(\Lambda-u))^{-\mu_2} (v-u-\tau(\Lambda-u))^{-\mu_1} {}_p M_q^\beta\left(\frac{-\rho(v-u)^2}{(\Lambda-u)(v-\Lambda)}\right) \\ & \quad \times {}_2F_1\left(\mu_1, \mu_2; \mu_4; \frac{\kappa(v-u)^2}{(v-u-\omega(\Lambda-u))(v-u-\tau(\Lambda-u))}\right) d\Lambda. \end{aligned}$$

Proof. By using the formula (1.4) of the M-beta function in the definition of M-Srivastava hypergeometric function H_A and by making similar calculations in the proof of Theorem 1, the proof is completed. $\square$

4. INTEGRAL REPRESENTATIONS FOR M-SRIVASTAVA HYPERGEOMETRIC FUNCTION H_B

Theorem 7. Let $\Re(\mu_1) > 0$, $\Re(\mu_2) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_B holds true:

$$\begin{aligned} & {}^M H_{B,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \rho) \\ &= \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \int_0^1 \Delta^{\mu_1-1} (1-\Delta)^{\mu_2-1} {}_p M_q^\beta\left(\frac{-\rho}{\Delta(1-\Delta)}\right) \\ & \quad \times X_4(\mu_1+\mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa\Delta(1-\Delta), \omega(1-\Delta), \tau\Delta) d\Delta. \end{aligned}$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_B , we have

$$\begin{aligned} & {}^M H_{B,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \rho) \\ &= \sum_{m,n,k=0}^{\infty} \frac{(\mu_1+\mu_2)_{2m+n+k} (\mu_3)_{n+k}}{(\mu_4)_m (\mu_5)_n (\mu_6)_k} \frac{{}^M B_{p,q}^{(\alpha,\beta)}(\mu_1+m+k, \mu_2+m+n; \rho)}{B(\mu_1, \mu_2)} \frac{\kappa^m \omega^n \tau^k}{m! n! k!} \\ &= \frac{1}{B(\mu_1, \mu_2)} \sum_{m,n,k=0}^{\infty} \frac{(\mu_1+\mu_2)_{2m+n+k} (\mu_3)_{n+k}}{(\mu_4)_m (\mu_5)_n (\mu_6)_k} \int_0^1 \Delta^{\mu_1+m+k-1} (1-\Delta)^{\mu_2+m+n-1} \end{aligned}$$

$$\times {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)\frac{\kappa^m}{m!}\frac{\omega^n}{n!}\frac{\tau^k}{k!}d\Delta.$$

By using equation (1.5) and by making necessary arrangements, we get

$$\begin{aligned} & {}^MH_{B,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \rho) \\ &= \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \int_0^1 \Delta^{\mu_1-1} (1-\Delta)^{\mu_2-1} {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) \\ & \quad \times \sum_{m,n,k=0}^{\infty} \frac{(\mu_1 + \mu_2)_{2m+n+k} (\mu_3)_{n+k}}{(\mu_4)_m (\mu_5)_n (\mu_6)_k} \frac{(\kappa\Delta(1-\Delta))^m}{m!} \frac{(\omega(1-\Delta))^n}{n!} \frac{(\tau\Delta)^k}{k!} d\Delta \\ &= \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \int_0^1 \Delta^{\mu_1-1} (1-\Delta)^{\mu_2-1} {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) \\ & \quad \times X_4(\mu_1 + \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa\Delta(1-\Delta), \omega(1-\Delta), \tau\Delta) d\Delta, \end{aligned}$$

which completes the proof. $\square$

Remark 1. The function X_4 used above is known as the Exton's function X_4 [26] and defined by

$$\begin{aligned} X_4(\mu_1, \mu_2; \mu_3, \mu_4, \mu_5; \kappa, \omega, \tau) &= \sum_{m,n,k=0}^{\infty} \frac{(\mu_1)_{2m+n+k} (\mu_2)_{n+k}}{(\mu_3)_m (\mu_4)_n (\mu_5)_k} \frac{\kappa^m}{m!} \frac{\omega^n}{n!} \frac{\tau^k}{k!}, \\ & \quad \left(2\sqrt{\mathbf{r}} + \left(\sqrt{\mathbf{s}} + \sqrt{\mathbf{t}}\right)^2 < 1\right), \end{aligned}$$

where for brevity, the co-ordinates are written $(\mathbf{r}, \mathbf{s}, \mathbf{t})$ instead of $(|\kappa|, |\omega|, |\tau|)$.

Theorem 8. Let $\Re(\mu_1) > 0$, $\Re(\mu_2) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_B holds true:

$$\begin{aligned} & {}^MH_{B,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \rho) \\ &= \frac{2\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \int_0^{\frac{\pi}{2}} (\sin\phi)^{2\mu_1-1} (\cos\phi)^{2\mu_2-1} {}_p^{\alpha}M_q^{\beta}\left(-\rho(\sec\phi)^2(\csc\phi)^2\right) \\ & \quad \times X_4(\mu_1 + \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa(\sin\phi)^2(\cos\phi)^2, \omega(\cos\phi)^2, \tau(\sin\phi)^2) d\phi. \end{aligned}$$

Proof. By using the formula (1.2) of the M-beta function in the definition of M-Srivastava hypergeometric function H_B and by making similar calculations in the proof of Theorem 7, the proof is completed. $\square$

Theorem 9. Let $\Re(\mu_1) > 0$, $\Re(\mu_2) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for M-Srivastava hypergeometric function H_B holds true:

$$\begin{aligned} & {}^MH_{B,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \rho) \\ &= \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \int_0^{\infty} \frac{\Lambda^{\mu_1-1}}{(1+\Lambda)^{\mu_1+\mu_2}} {}_p^{\alpha}M_q^{\beta}\left(-2\rho - \rho\left(\Lambda + \frac{1}{\Lambda}\right)\right) \end{aligned}$$

$$\times X_4 \left(\mu_1 + \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \frac{\kappa\Lambda}{(1+\Lambda)^2}, \frac{\omega}{1+\Lambda}, \frac{\tau\Lambda}{1+\Lambda} \right) d\Lambda.$$

Proof. By using the formula (1.3) of the M-beta function in the definition of M-Srivastava hypergeometric function H_B and by making similar calculations in the proof of Theorem 7, the proof is completed. $\square$

Theorem 10. Let $\Re(\mu_1) > 0$, $\Re(\mu_2) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_B holds true:

$$\begin{aligned} & {}^M H_{B,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \rho) \\ &= \frac{\Gamma(\mu_1 + \mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} (v-u)^{1-\mu_1-\mu_2} \int_u^v (\Lambda-u)^{\mu_1-1} (v-\Lambda)^{\mu_2-1} {}^\alpha_p M_q^\beta \left(\frac{-\rho(v-u)^2}{(\Lambda-u)(v-\Lambda)} \right) \\ & \times X_4 \left(\mu_1 + \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \frac{\kappa(\Lambda-u)(v-\Lambda)}{(v-u)^2}, \frac{\omega(v-\Lambda)}{v-u}, \frac{\tau(\Lambda-u)}{v-u} \right) d\Lambda. \end{aligned}$$

Proof. By using the formula (1.4) of the M-beta function in the definition of M-Srivastava hypergeometric function H_B and by making similar calculations in the proof of Theorem 7, the proof is completed. $\square$

5. INTEGRAL REPRESENTATIONS FOR M-SRIVASTAVA HYPERGEOMETRIC FUNCTION H_C

Theorem 11. Let $\Re(\mu_4) > \Re(\mu_1) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_C holds true:

$$\begin{aligned} & {}^M H_{C,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \rho) \\ &= \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4 - \mu_1)} \int_0^1 \Delta^{\mu_1-1} (1-\Delta)^{\mu_4-\mu_1-1} (1-\kappa\Delta)^{-\mu_2} (1-\tau\Delta)^{-\mu_3} \\ & \times {}^\alpha_p M_q^\beta \left(\frac{-\rho}{\Delta(1-\Delta)} \right) {}_2F_1 \left(\mu_2, \mu_3; \mu_4 - \mu_1; \frac{\omega(1-\Delta)}{(1-\kappa\Delta)(1-\tau\Delta)} \right) d\Delta. \end{aligned}$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_C , we have

$$\begin{aligned} & {}^M H_{C,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \rho) \\ &= \sum_{m,n,k=0}^{\infty} \frac{(\mu_2)_{m+n}(\mu_3)_{n+k}}{(\mu_4)_n} \frac{{}^M B_{p,q}^{(\alpha,\beta)}(\mu_1 + m + k, \mu_4 + n - \mu_1; \rho)}{B(\mu_1, \mu_4 + n - \mu_1)} \frac{\kappa^m \omega^n \tau^k}{m! n! k!} \\ &= \sum_{m,n,k=0}^{\infty} \frac{(\mu_2)_{m+n}(\mu_3)_{n+k}}{(\mu_4)_n} \frac{1}{B(\mu_1, \mu_4 + n - \mu_1)} \int_0^1 \Delta^{\mu_1+m+k-1} (1-\Delta)^{\mu_4+n-\mu_1-1} \\ & \times {}^\alpha_p M_q^\beta \left(\frac{-\rho}{\Delta(1-\Delta)} \right) \frac{\kappa^m \omega^n \tau^k}{m! n! k!} d\Delta. \end{aligned}$$

By using equations (1.5), (1.6), (1.7) and multiplied by $\frac{\Gamma(\mu_4)\Gamma(\mu_4-\mu_1)}{\Gamma(\mu_4)\Gamma(\mu_4-\mu_1)}$, we get

$$\begin{aligned}
& {}^M H_{C,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \rho) \\
&= \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)} \int_0^1 \Delta^{\mu_1-1} (1-\Delta)^{\mu_4-\mu_1-1} {}_p M_q^\beta \left(\frac{-\rho}{\Delta(1-\Delta)} \right) \\
&\quad \times \sum_{m,n,k=0}^{\infty} \frac{(\mu_4)_n}{(\mu_4-\mu_1)_n} \frac{(\mu_2)_n (\mu_2+n)_m (\mu_3)_n (\mu_3+n)_k}{(\mu_4)_n} \frac{(\kappa\Delta)^m}{m!} \frac{(\omega(1-\Delta))^n}{n!} \frac{(\tau\Delta)^k}{k!} d\Delta \\
&= \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)} \int_0^1 \Delta^{\mu_1-1} (1-\Delta)^{\mu_4-\mu_1-1} {}_p M_q^\beta \left(\frac{-\rho}{\Delta(1-\Delta)} \right) \\
&\quad \times \sum_{n=0}^{\infty} \frac{(\mu_2)_n (\mu_3)_n}{(\mu_4-\mu_1)_n} \sum_{m=0}^{\infty} (\mu_2+n)_m \frac{(\kappa\Delta)^m}{m!} \sum_{k=0}^{\infty} (\mu_3+n)_k \frac{(\tau\Delta)^k}{k!} \frac{(\omega(1-\Delta))^n}{n!} d\Delta.
\end{aligned}$$

By using equation (1.8), we obtain

$$\begin{aligned}
& {}^M H_{C,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \rho) \\
&= \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)} \int_0^1 \Delta^{\mu_1-1} (1-\Delta)^{\mu_4-\mu_1-1} {}_p M_q^\beta \left(\frac{-\rho}{\Delta(1-\Delta)} \right) \\
&\quad \times \sum_{n=0}^{\infty} \frac{(\mu_2)_n (\mu_3)_n}{(\mu_4-\mu_1)_n} (1-\kappa\Delta)^{-\mu_2-n} (1-\tau\Delta)^{-\mu_3-n} \frac{(\omega(1-\Delta))^n}{n!} d\Delta \\
&= \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4-\mu_1)} \int_0^1 \Delta^{\mu_1-1} (1-\Delta)^{\mu_4-\mu_1-1} (1-\kappa\Delta)^{-\mu_2} (1-\tau\Delta)^{-\mu_3} \\
&\quad \times {}_p M_q^\beta \left(\frac{-\rho}{\Delta(1-\Delta)} \right) {}_2F_1 \left(\mu_2, \mu_3; \mu_4-\mu_1; \frac{\omega(1-\Delta)}{(1-\kappa\Delta)(1-\tau\Delta)} \right) d\Delta,
\end{aligned}$$

which completes the proof. $\square$

Theorem 12. Let $\Re(\mu_1) > 0$, $\Re(\mu_2) > 0$, $\Re(\mu_4) > 0$, $\Re(\mu_4 - \mu_1 - \mu_2) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_C holds true:

$$\begin{aligned}
& {}^M H_{C,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \rho) \\
&= \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_4-\mu_1-\mu_2)} \int_0^1 \int_0^1 \Delta^{\mu_1-1} \Lambda^{\mu_2-1} (1-\Delta)^{\mu_4-\mu_1-1} \\
&\quad \times (1-\Lambda)^{\mu_4-\mu_1-\mu_2-1} (1-\kappa\Delta)^{\mu_3-\mu_2} (1-\kappa\Delta-\omega\Lambda-\tau\Delta+\omega\Delta\Lambda+\kappa\tau\Delta^2)^{-\mu_3} \\
&\quad \times {}_p M_q^\beta \left(\frac{-\rho}{\Delta(1-\Delta)} \right) {}_p M_q^\beta \left(\frac{-\rho}{\Lambda(1-\Lambda)} \right) d\Delta d\Lambda.
\end{aligned}$$

Proof. By using the formula (1.1) of the M-beta function in the definition of M-Srivastava hypergeometric function H_C and by making similar calculations in the proof of Theorem 11, the proof is completed. $\square$

Theorem 13. Let $\Re(\mu_4) > \Re(\mu_1) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_C holds true:

$$\begin{aligned} {}^M H_{C,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \rho) \\ = \frac{2\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4 - \mu_1)} \int_0^{\frac{\pi}{2}} (\sin \phi)^{2\mu_1-1} (\cos \phi)^{2\mu_4-2\mu_1-1} (1 - \kappa(\sin \phi)^2)^{-\mu_2} \\ \times (1 - \tau(\sin \phi)^2)^{-\mu_3} {}_p M_q^\beta(-\rho(\sec \phi)^2(\csc \phi)^2) \\ \times {}_2F_1\left(\mu_2, \mu_3; \mu_4 - \mu_1; \frac{\omega(\cos \phi)^2}{(1 - \kappa(\sin \phi)^2)(1 - \tau(\sin \phi)^2)}\right) d\phi. \end{aligned}$$

Proof. By using the formula (1.2) of the M-beta function in the definition of M-Srivastava hypergeometric function H_C and by making similar calculations in the proof of Theorem 11, the proof is completed. $\square$

Theorem 14. Let $\Re(\mu_4) > \Re(\mu_1) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_C holds true:

$$\begin{aligned} {}^M H_{C,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \rho) \\ = \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4 - \mu_1)} \int_0^\infty \Lambda^{\mu_1-1} (1 + \Lambda)^{\mu_2+\mu_3-\mu_4} (1 + \Lambda - \kappa\Lambda)^{-\mu_2} \\ \times (1 + \Lambda - \tau\Lambda)^{-\mu_3} {}_p M_q^\beta\left(-2\rho - \rho\left(\Lambda + \frac{1}{\Lambda}\right)\right) \\ \times {}_2F_1\left(\mu_2, \mu_3; \mu_4 - \mu_1; \frac{\omega(1 + \Lambda)}{(1 + \Lambda - \kappa\Lambda)(1 + \Lambda - \tau\Lambda)}\right) d\Lambda. \end{aligned}$$

Proof. By using the formula (1.3) of the M-beta function in the definition of M-Srivastava hypergeometric function H_C and by making similar calculations in the proof of Theorem 11, the proof is completed. $\square$

Theorem 15. Let $\Re(\mu_4) > \Re(\mu_1) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$. The following integral representation for the M-Srivastava hypergeometric function H_C holds true:

$$\begin{aligned} {}^M H_{C,p,q}^{(\alpha,\beta)}(\mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \rho) \\ = \frac{\Gamma(\mu_4)}{\Gamma(\mu_1)\Gamma(\mu_4 - \mu_1)} (v - u)^{1+\mu_2+\mu_3-\mu_4} \int_u^v (\Lambda - u)^{\mu_1-1} (v - \Lambda)^{\mu_4-\mu_1-1} \\ \times (v - u - \kappa(\Lambda - u))^{-\mu_2} (v - u - \tau(\Lambda - u))^{-\mu_3} {}_p M_q^\beta\left(\frac{-\rho(v - u)^2}{(\Lambda - u)(v - \Lambda)}\right) \\ \times {}_2F_1\left(\mu_2, \mu_3; \mu_4 - \mu_1; \frac{\omega(v - \Lambda)(v - u)}{(v - u - \kappa(\Lambda - u))(v - u - \tau(\Lambda - u))}\right) d\Lambda. \end{aligned}$$

Proof. By using the formula (1.4) of the M-beta function in the definition of M-Srivastava hypergeometric function H_C and by making similar calculations in the proof of Theorem 11, the proof is completed. $\square$

6. SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING M-SRIVASTAVA HYPERGEOMETRIC FUNCTIONS

Example 1. Let $1 < \Re(\epsilon) \leq 2$ and $\Re(\alpha) > 0$. Assume that, the fractional differential equation

$${}^c D_{\rho}^{\epsilon} \{f(\rho)\} = {}^M H_{A,p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \epsilon \rho)$$

with the initial conditions

$$f(0) = f'(0) = 0$$

are given. By considering equation (1.9) and by applying the Laplace transform to the fractional differential equation, we have

$$\mathfrak{L} \left\{ {}^c D_{\rho}^{\epsilon} \{f(\rho)\}; s \right\} = \mathfrak{L} \left\{ {}^M H_{A,p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \epsilon \rho); s \right\}$$

and then,

$$\begin{aligned} s^{\epsilon} F(s) - s^{\epsilon-1} f(0) - s^{\epsilon-2} f'(0) \\ = \frac{{}^M H_{A,p+1,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p, 1; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \frac{\epsilon}{s})}{s}. \end{aligned}$$

By using the initial conditions, we get

$$F(s) = \frac{{}^M H_{A,p+1,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p, 1; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \frac{\epsilon}{s})}{s^{\epsilon+1}}.$$

By applying the inverse Laplace transform to the last equation and by making the necessary calculations, we obtain the solution function as:

$$f(\rho) = \frac{{}^M H_{A,p+1,q+1}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p, 1; \eta_1, \dots, \eta_q, 1 + \epsilon; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5; \kappa, \omega, \tau; \epsilon \rho)}{\Gamma(1 + \epsilon) \rho^{-\epsilon}}.$$

Example 2. Let $1 < \Re(\epsilon) \leq 2$ and $\Re(\alpha) > 0$. Assume that, the fractional differential equation

$${}^c D_{\rho}^{\epsilon} \{f(\rho)\} = {}^M H_{B,p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \epsilon \rho)$$

with the initial conditions

$$f(0) = f'(0) = 0$$

are given. By considering the equation (1.9) and by applying the Laplace transform to the fractional differential equation, we have

$$\mathfrak{L} \left\{ {}^c D_{\rho}^{\epsilon} \{f(\rho)\}; s \right\} = \mathfrak{L} \left\{ {}^M H_{B,p,q}^{(\alpha,\beta)}(\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \epsilon \rho); s \right\}$$

and then,

$$\begin{aligned} s^\varepsilon F(s) - s^{\varepsilon-1} f(0) - s^{\varepsilon-2} f'(0) \\ = \frac{{}^M H_{B,p+1,q}^{(\alpha,\beta)} \left(\xi_1, \dots, \xi_p, 1; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \frac{\varepsilon}{s} \right)}{s}. \end{aligned}$$

By using the initial conditions, we get

$$F(s) = \frac{{}^M H_{B,p+1,q}^{(\alpha,\beta)} \left(\xi_1, \dots, \xi_p, 1; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \frac{\varepsilon}{s} \right)}{s^{\varepsilon+1}}.$$

By applying the inverse Laplace transform to the last equation and by making the necessary calculations, we obtain the solution function as:

$$f(\rho) = \frac{{}^M H_{B,p+1,q+1}^{(\alpha,\beta)} (\xi_1, \dots, \xi_p, 1; \eta_1, \dots, \eta_q, 1 + \varepsilon; \mu_1, \mu_2, \mu_3; \mu_4, \mu_5, \mu_6; \kappa, \omega, \tau; \varepsilon \rho)}{\Gamma(1 + \varepsilon) \rho^{-\varepsilon}}.$$

Example 3. Let $1 < \Re(\varepsilon) \leq 2$ and $\Re(\alpha) > 0$. Assume that, the fractional differential equation

$${}^c D_\rho^\varepsilon \{f(\rho)\} = {}^M H_{C,p,q}^{(\alpha,\beta)} (\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \varepsilon \rho)$$

with the initial conditions

$$f(0) = f'(0) = 0$$

are given. By considering the equation (1.9) and by applying the Laplace transform to the fractional differential equation, we have

$$\mathfrak{L} \left\{ {}^c D_\rho^\varepsilon \{f(\rho)\}; s \right\} = \mathfrak{L} \left\{ {}^M H_{C,p,q}^{(\alpha,\beta)} (\xi_1, \dots, \xi_p; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \varepsilon \rho); s \right\}$$

and then,

$$\begin{aligned} s^\varepsilon F(s) - s^{\varepsilon-1} f(0) - s^{\varepsilon-2} f'(0) \\ = \frac{{}^M H_{C,p+1,q}^{(\alpha,\beta)} \left(\xi_1, \dots, \xi_p, 1; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \frac{\varepsilon}{s} \right)}{s}. \end{aligned}$$

By using the initial conditions, we get

$$F(s) = \frac{{}^M H_{C,p+1,q}^{(\alpha,\beta)} \left(\xi_1, \dots, \xi_p, 1; \eta_1, \dots, \eta_q; \mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \frac{\varepsilon}{s} \right)}{s^{\varepsilon+1}}.$$

By applying the inverse Laplace transform to the last equation and by making the necessary calculations, we obtain the solution function as:

$$f(\rho) = \frac{{}^M H_{C,p+1,q+1}^{(\alpha,\beta)} (\xi_1, \dots, \xi_p, 1; \eta_1, \dots, \eta_q, 1 + \varepsilon; \mu_1, \mu_2, \mu_3; \mu_4; \kappa, \omega, \tau; \varepsilon \rho)}{\Gamma(1 + \varepsilon) \rho^{-\varepsilon}}.$$

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THE μ -NEUTRAL FRACTIONAL MULTI-DELAYED DIFFERENTIAL EQUATIONS

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Abstract. The μ -neutral linear fractional multi-delayed differential nonhomogeneous system with noncommutative coefficient matrices is introduced. The novel μ -neutral multi-delayed perturbation of Mittag-Leffler type matrix function is proposed. Based on this, an explicit solution to the system is investigated step by step. The existence uniqueness of solutions to μ -neutral nonlinear fractional multi-delayed differential system is obtained with regard to the supremum norm. The notion of stability analysis in the sense of solutions to the described system is discussed on the grounds of the fixed point approach.

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1. INTRODUCTION AND PRELIMINARIES

Fractional Calculus which is seen as a generalisation of the ordinary calculus has drawn many of researchers' attention in recent years. And also it has been started to be exponentially utilized in so many different kinds of areas like finance, engineering, neurons, electric conductance, diffusion, thermodynamics, computed tomography, mechanism, mathematical physics.

Fractional (ordinary) delay differential equations [9, 10, 12–14] are of a considerable importance since they permit to describe such systems that the rate of change depends both on the present and delayed states and on the past state unlike other systems. They are very often exploited in propagation of energy or information and transport in interdependent systems (see e.g. [1, 6, 16] and reference therein). In reference [7], Khusainov and Shuklin were able to find a solution of the below linear fractional delayed differential system in terms of producing the delayed exponential matrix.

$$\begin{cases} w'(t) = Bw(t) + Fw(t-r), & t > 0 \quad r > 0 \text{ (delay)}, \\ w(t) = \phi(t), & -r \leq t \leq 0. \end{cases} \quad (1.1)$$

The drawback of system (1.1) is to assume that the coefficient matrices B and F must be commutative. In reference [10], Li and Wang consider the fractional version of system (1.1) in the case of $F = \Theta$. The drawback is also in progress because of validation of commutativity of coefficient matrices when $F = \Theta$. In reference [12], Mahmudov investigates a general version of fractional delay differential system whose coefficient matrices do not need to be either commutative or zero. Having proposed a newly delay perturbation of Mittag-Leffler type matrix function, Mahmudov gives an excellent representation of solutions of the mentioned system. In the sequel, Mahmudov extends the fractional delay differential equations in the work [12] to multi-delayed version in the work [13] and solve it.

In the neutral version of the fractional(ordinary) delay differential equations [18], [15], [5] the fractional(ordinary) derivative of the unknown function appears mostly with delays and rarely without delays. These kinds of systems are used from population growth to the motion of radiation electrons, spread of epidemic. In reference [15], Pospisil and Skripkova investigate the following linear neutral fractional delay differential equations

$$\begin{cases} w'(t) - Aw'(t-r) = Bw(t) + Fw(t-r) + f(t), & t > 0, r > 0, \\ w(t) = \phi(t), & -r \leq t \leq 0, \end{cases} \quad (1.2)$$

where f is continuous from $[0, \infty)$ to $\mathbb{R}^n$, ϕ is continuously differentiable form $[-r, 0]$ to $\mathbb{R}^n$ and r is a retardation. The coefficient matrices A, B, F are permutable, that is $AB = BA$, $AF = FA$, $BF = FB$. Zhang et al. [18] look into the representation of the solution to the neutral fractional linear differential system having a constant delay

$$\begin{cases} {}^C\mathfrak{D}_{0+}^\alpha (w(t) - Aw(t-r)) = Bw(t) + Fw(t-r) + f(t), & t > 0, \\ w(t) = \phi(t), & -r \leq t \leq 0, \end{cases} \quad (1.3)$$

where ${}^C\mathfrak{D}_{0+}^\alpha$ is Caputo fractional derivative of order α , $0 < \alpha < 1$, $r > 0$, f is continuous from $[0, \infty)$ to $\mathbb{R}^n$, $A, B, F \in \mathbb{R}^{n \times n}$, and ϕ is continuously differentiable form $[-r, 0]$ to $\mathbb{R}^n$. In an attempt to solve system (1.3), Zhang et al. [18] exploited Laplace integral transform. This produced some drawback and mistakes because the representation of power series of the fundamental solution is unknowable.

In reference [2], Almeida consider a Caputo type fractional derivative with respect to another function. Some features between the fractional derivative and integral, Fermat's Theorem, Taylor's Theorem, and a lot more are studied. In reference [3], Almeida et al. questionnaire existence and uniqueness results for the initial value problem of nonlinear fractional differential equations involving a Caputo-type fractional derivative with respect to another function with the help of the some standard fixed point theorems and develop the Picard iteration method for solving numerically the problem and obtain results on the long-term behavior of solutions.

Motivated by the above cited studies, we handle the below μ -Caputo type (more general) fractional neutral differential multi-delayed equations with noncommutative

matrices

$$\begin{cases} {}^C_{0+}\mathfrak{J}_\mu^\alpha \left[w(t) - \sum_{i=1}^d A_i w(t-r_i) \right] &= Bw(t) + \sum_{i=1}^d F_i w(t-r_i) + \mathfrak{T}(t), t \in (0, T], \\ w(t) &= \varphi(t), -r \leq t \leq 0, \end{cases} \quad (1.4)$$

where the Caputo fractional derivative ${}^C_{0+}\mathfrak{J}_\mu^\alpha$ is of order $\alpha \in (0, 1)$. For each of $i = 1, 2, 3, \dots, d$, A_i, B, C_i are square coefficient constant matrices which do not need to be permutable and $r_i > 0$ is a retardation and $r := \max\{r_i : i = 1, 2, 3, \dots, d\}$. An arbitrary vector function $\phi(x)$ is continuously differentiable and $\mathfrak{T} \in C([0, T], \mathbb{R}^n)$ with $T = ld$ for a fixed $l \in \mathbb{N}$. After finding the explicit solutions of (1.4), we obtain the explicit solutions to the below equations (1.5)

$$\begin{cases} {}^C_{0+}\mathfrak{J}_\mu^\alpha \left[w(t) - \sum_{i=1}^d A_i w(t-r_i) \right] &= Bw(t) + \sum_{i=1}^d F_i w(t-r_i) + \mathfrak{T}(t, w(t)), \\ w(t) &= \varphi(t), -r \leq t \leq 0, \end{cases} \quad (1.5)$$

where $\mathfrak{T} \in C([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$ and the others are the same as (1.4).

Remark 1. By choosing $A_i = \Theta, i = 1, 2, 3, \dots, d$ and $\mu(t) = t$, μ -Caputo fractional neutral differential multi-delayed equations with nonpermutable matrices overlaps with fractional linear multi-delay differential equations in the study [13].

In the current work,

- we introduce the μ -neutral Caputo type fractional linear(or semi-linear) multi-delayed differential equations with non-permutable constant coefficient matrices,
- we propose newly the μ -neutral multi-delayed perturbation of two parameter Mittag-Leffler type matrix function,
- we present a representation of a solution for the μ -neutral Caputo type fractional linear(or semi-linear) multi-delayed differential equations with non-permutable constant coefficient matrices by sharing the μ -neutral multi-delayed perturbation of two parameter Mittag-Leffler type matrix function,
- we examine the existence and uniqueness of solutions of the μ -neutral fractional multi-delayed differential equations' system,
- we show the stability of the μ -neutral fractional order multi-delayed differential system in the sense of Ulam-Hyers, and illustrate the theoretical findings.

Now we remind a couple of well-recognized basic notions in the literature.

For $n \in \{1, 2, 3, \dots\}$, the space $C^n([0, T], \mathbb{R}^n)$ is all of continuously n th order differentiable vector-valued functions from $[0, T]$ to $\mathbb{R}^n$ with $\|x\|_C := \sup_{t \in [0, T]} \|x(t)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$. For $n = 0$, $C([0, T], \mathbb{R}^n) = C^0([0, T], \mathbb{R}^n)$ is all of continuous vector-valued functions. Let $AC[0, T]$ be the space of functions which are absolutely

continuous on $[0, T]$. We denote by $AC^n[0, T]$ the space of complex-valued functions $f(x)$ which have continuous derivatives up to order $n - 1$ on $[0, T]$ such that $f^{(n-1)}(x) \in AC[0, T]$. Let $\mathfrak{T}$ and an increasing function μ on $[0, T]$ be integrable and continuously differentiable, respectively and let $\mu'(t) \neq 0$ $t \in [0, T]$. μ -Riemann-Liouville fractional integrals[8][2] of $\mathfrak{T} \in AC^n[0, T]$ of order $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ are given by

$$({}_{0+}^{RL}\mathfrak{I}_{\mu}^{\alpha}\mathfrak{T})(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (\mu(t) - \mu(s))^{\alpha-1} \mathfrak{T}(s) d\mu(s),$$

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$ with $Re(\alpha) > 0$. μ -Riemann-Liouville fractional derivatives[8][2] of $\mathfrak{T} \in AC^n[0, T]$ of order $\alpha > 0$ are given by

$$({}_{0+}^{RL}\mathfrak{D}_{\mu}^{\alpha}\mathfrak{T})(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{d\mu(t)} \right)^n \int_0^t (\mu(t) - \mu(s))^{n-\alpha-1} f(s) d\mu(s),$$

where $n = [\alpha] + 1$. If $\mathfrak{T} \in AC^n([0, T], \mathbb{R})$, $\mu \in C^n([0, T], \mathbb{R})$ with μ is increasing and $\mu'(t) \neq 0$ for every $t \in [0, T]$, then the μ -Caputo fractional derivatives[8][2] of $\mathfrak{T}$ of order α is defined as

$$({}_{0+}^C\mathfrak{D}_{\mu}^{\alpha}\mathfrak{T})(t) := {}_{0+}^{RL}\mathfrak{I}_{\mu}^{n-\alpha} \left(\frac{1}{\mu'(t)} \frac{d}{dt} \right)^n \mathfrak{T}(t). \quad (1.6)$$

From [8] and [2], We have, for $\Re(\alpha) \geq 0$, $\Re(\beta) > 0$, and $\alpha \in (0, 1)$,

$$({}_{0+}^C\mathfrak{D}_{\mu}^{\alpha}\mathfrak{T})(t) = {}_{0+}^{RL}\mathfrak{I}_{\mu}^{\alpha} [\mathfrak{T}(t) - \mathfrak{T}(0)], \quad {}_{0+}^C\mathfrak{D}_{\mu}^{\alpha} [\mu(t)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} [\mu(t)]^{\beta-\alpha-1}$$

2. MAIN RESULTS

In the current section, we present our findings to begin with defining the μ -neutral multi-delayed perturbation of Mittag-Leffler type matrix function. We look for a solution of system (1.4) and prove the existence and uniqueness of solutions and Ulam-Hyers stability of system (1.4).

2.1. The μ - neutral multi-delayed perturbation of Mittag-Leffler type matrix function

For now on, we exploit the μ -NMDP of ML function for the μ -neutral multi-delayed perturbation of Mittag-Leffler type matrix function.

It is clear that a generalisation of the exponential function is the ML function. Delayed and Delayed perturbed and multi-delayed perturbed versions which are called delayed Mittag-Leffler type matrix function, delayed perturbation of Mittag-Leffler type matrix function and multi-delayed perturbation of Mittag-Leffler type matrix function by the numbers are presented in the work [9, 12, 13] respectively. In the current study, the μ -NMDP of ML function is given through identifying matrix equation

for $Q_j(s)$ for $j = 0, 1, 2, \dots$

$$\begin{aligned}
 Q_{j+1}(s_1, s_2, \dots, s_d) &= BQ_j(s_1, s_2, \dots, s_d) + \sum_{i=1}^d F_i Q_j(s_1, s_2, \dots, s_i - r_i, \dots, s_d) \\
 &\quad + \sum_{i=1}^d A_i Q_{j+1}(s_1, s_2, \dots, s_i - r_i, \dots, s_d), \quad (2.1) \\
 Q_0(s_1, s_2, \dots, s_d) &= Q_j(-r_1, \dots, s_d) = Q_j(s_1, \dots, -r_d) = \Theta, \\
 Q_1(0, \dots, 0) &= I, \\
 Q_1(s_1, s_2, \dots, s_d) &= \Theta, \quad s_i \neq 0.
 \end{aligned}$$

where $s_i = 0, r_i, 2r_i, \dots$, Θ is the zero matrix, and I is the unit matrix.

In the following definition, we give the function of μ -NMDP of ML type matrix with the aid of the multivariate function $Q_{k+1}(s_1, s_2, \dots, s_d)$

Definition 1. The μ -neutral multi-delayed perturbation of the Mittag-Leffler type matrix function $\chi_\mu^{\alpha, \beta}(t, s)$ is given by

$$\chi_\mu^{\alpha, \beta}(t, s) = \begin{cases} \Theta, & -r \leq t < 0, s \geq 0, \\ \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1} \frac{[\mu(t) - \mu(s + \sum_{j=1}^d i_j r_j)]_+^{k\alpha + \beta - 1}}{\Gamma(k\alpha + \beta)}, & t \geq 0, s \geq 0, \end{cases} \quad (2.2)$$

where $Q_{k+1} := Q_{k+1}(i_1 r_1, \dots, i_d r_d)$, $[t]_+ = \max(0, t)$.

Remark 2. Let $\mu(t) = t$, and $s = 0$ in $\chi_\mu^{\alpha, \beta}(t, s)$. Then we have,

- (1) For $A_i = F_i = \Theta$, $i = 1, 2, \dots, d$, The μ -NMDP ML function reduces to the ML function [19] i.e. $\chi_\mu^{\alpha, \beta}(t, s) = t^{\beta-1} E_{\alpha, \beta}(Bt^\alpha)$.
- (2) For $A_i = \Theta$, $i = 1, 2, \dots, d$ and $F_i = \Theta$, $i = 2, \dots, d$, then the μ -NMDP ML function matches up with delay perturbation of ML function [12].
- (3) For $A_i = \Theta$, $i = 1, 2, \dots, d$ and $F_i = \Theta$, $i = 2, \dots, d$, $B = \Theta$, $\chi_\mu^{\alpha, \beta}(t, s)$ reduces to delayed ML function [9].
- (4) The μ -NMDP ML function is not equal to that one of [13, Def. 3.3] because the matrix $Q_k(s)$ for $k = 0, 1, 2, \dots$ in (2.1) is different from that of the study [13]. Under the condition $A_i = \Theta$, $i = 1, 2, \dots, d$, however, they coincide.
- (5) Since the constant matrices are commutative together with suitable selections, $\chi_\mu^{\alpha, \beta}(t, s)$ overlaps with $X(t)$ in (2.4) in the work [15].
- (6) By depending on suitable selections r_i, A_i, F_i, B , $i = 1, 2, \dots, d$, one can easily obtain $\epsilon_{\alpha, \beta}^{r_1, r_2}(\mathcal{A}, \mathcal{B}, \mathcal{F}; x)$ in Definition 3.1 in the work [5] from $\chi_\mu^{\alpha, \beta}(t, s)$.

2.2. The μ -neutral multi-delayed Caputo fractional differential equations' analytic solutions

First of all, we share beneficial theorem and lemma to be exploited in the next proofs.

Theorem 1. Let $\chi_\mu^{\alpha,\beta}(t,s)$ be as defined in (2.2). The following holds true

$${}_{0+}^C \mathfrak{I}_\mu^\alpha \left[\chi_\mu^{\alpha,1}(t,0) - \sum_{j=1}^d A_j \chi_\mu^{\alpha,1}(t,r_j) \right] = B \chi_\mu^{\alpha,1}(t,0) + \sum_{j=1}^d F_j \chi_\mu^{\alpha,1}(t,r_j)$$

Proof. Firstly, we compute the term ${}_{0+}^C \mathfrak{I}_\mu^\alpha \chi_\mu^{\alpha,1}(t,0)$.

$$\begin{aligned} {}_{0+}^C \mathfrak{I}_\mu^\alpha \chi_\mu^{\alpha,1}(t,0) &= \sum_{k=1}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) {}_{0+}^C \mathfrak{I}_\mu^\alpha \left(\frac{[\mu(t) - \mu(\sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)} \right) \\ &= \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+2}(i_1 r_1, \dots, i_d r_d) \frac{[\mu(t) - \mu(\sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)}. \end{aligned} \quad (2.3)$$

One can obtain ${}_{0+}^C \mathfrak{I}_\mu^\alpha \chi_\mu^{\alpha,1}(t, r_j)$ as follows by using the similar calculations.

$${}_{0+}^C \mathfrak{I}_\mu^\alpha \mathcal{P}_{\alpha,1}^\mu(t, r_j) := \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+2}(i_1 r_1, \dots, i_d r_d) \frac{[\mu(t) - \mu(r_j + \sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)}.$$

By combining (2.1) with (2.3), we keep calculating

$$\begin{aligned} & {}_{0+}^C \mathfrak{I}_\mu^\alpha \chi_\mu^{\alpha,1}(t,0) \\ &= B \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) \frac{[\mu(t) - \mu(\sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)} \\ &\quad + \sum_{j=1}^d F_j \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_j r_j - r_j, \dots, i_d r_d) \frac{[\mu(t) - \mu(\sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)} \\ &\quad + \sum_{j=1}^d A_j \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+2}(i_1 r_1, \dots, i_j r_j - r_j, \dots, i_d r_d) \frac{[\mu(t) - \mu(\sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)} \\ &= B \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) \frac{[\mu(t) - \mu(\sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)} \\ &\quad + \sum_{j=1}^d F_j \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) \frac{[\mu(t) - \mu(r_j + \sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)} \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^d A_j \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+2}(i_1 r_1, \dots, i_d r_d) \frac{[\mu(t) - \mu(r_j + \sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)} \\
& = B\mathcal{X}_\mu^{\alpha,1}(t, 0) + \sum_{j=1}^d F_j \mathcal{X}_\mu^{\alpha,1}(t, r_j) + \sum_{k=1}^d A_j \left({}^C_{0+} \mathfrak{I}_\mu^\alpha \mathcal{X}_\mu^{\alpha,1}(t, r_j) \right).
\end{aligned}$$

which provides the craved result. $\square$

Corollary 1. A solution of system (1.4) is

$$w(t) = \left[\mathcal{X}_\mu^{\alpha,1}(t, 0) - \sum_{j=1}^d \mathcal{X}_\mu^{\alpha,1}(t, r_j) A_j \right] \varphi(0),$$

provided that $w(t) = \varphi(t)$ with $-r \leq t \leq 0$.

Proof. With the help of the variation of constants' technique, set

$$w(t) = \left[\mathcal{X}_\mu^{\alpha,1}(t, 0) - \sum_{j=1}^d \mathcal{X}_\mu^{\alpha,1}(t, r_j) A_j \right] \kappa,$$

where a constant κ is unknown. It is clear that $\mathcal{X}_\mu^{\alpha,1}(0, 0) = I$ and $\mathcal{X}_\mu^{\alpha,1}(0, r_j) = \Theta$. As a result, $\kappa = \varphi(0)$. $\square$

Lemma 1. Let $\mathcal{X}_\mu^{\alpha,\beta}(t, s)$ be as in (2.2). The following mathematical equation is true:

$$\begin{aligned}
& \int_0^t (\mu(t) - \mu(s))^{-\alpha} \int_0^s \mathcal{X}_\mu^{\alpha,\alpha}(s, x) \mathfrak{T}(x) d\mu(x) d\mu(s) \\
& = \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) \int_0^t \frac{\Gamma(1-\alpha) [\mu(t) - \mu(x + \sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)} \mathfrak{T}(x) d\mu(x).
\end{aligned}$$

Proof. One can easily prove this theorem with the help of a simple substitution $v = \frac{\mu(s) - \mu(x + \sum_{n=1}^d i_n r_n)}{\mu(t) - \mu(x + \sum_{n=1}^d i_n r_n)}$ together with the expansion of $\mathcal{X}_\mu^{\alpha,\alpha}$, so it is ignored. $\square$

Now, the coming theorem is one of main theorems as to the desired solutions. It gives a part of the solution under the zero initial condition.

Theorem 2. The following function

$$w(t) = \int_0^t \mathcal{X}_\mu^{\alpha,\alpha}(t, s) \mathfrak{T}(s) d\mu(s), \quad t \geq 0,$$

is a solution of system (1.4) under the condition $w(t) = 0$ with $-r \leq t \leq 0$.

Proof. To see this, we consider the following expression by keeping Lemma 1 in mind

$${}^C_{0+} \mathfrak{I}_\mu^\alpha \left(\int_0^t \mathcal{X}_\mu^{\alpha,\alpha}(t, x) \mathfrak{T}(x) d\mu(x) \right)$$

$$\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\mu(t)} \int_0^t (\mu(t) - \mu(s))^{-\alpha} \int_0^s \mathcal{X}_\mu^{\alpha,\alpha}(s, x) \Upsilon(x) d\mu(x) d\mu(s) \\
&= \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} \mathcal{Q}_{k+1}(i_1 r_1, \dots, i_d r_d) \\
&\quad \times \frac{d}{d\mu(t)} \int_0^t \frac{[\mu(t) - \mu(x + \sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)} \Upsilon(x) d\mu(x) \\
&= \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} \mathcal{Q}_{k+1}(i_1 r_1, \dots, i_d r_d) \\
&\quad \times \frac{d}{d\mu(t)} \int_0^{t - \sum_{n=1}^d i_n r_n} \frac{[\mu(t) - \mu(x + \sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)} \Upsilon(x) d\mu(x) \\
&= \sum_{k=1}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} \mathcal{Q}_{k+1}(i_1 r_1, \dots, i_d r_d) \\
&\quad \times \frac{d}{d\mu(t)} \int_0^{t - \sum_{n=1}^d i_n r_n} \frac{[\mu(t) - \mu(x + \sum_{n=1}^d i_n r_n)]_+^{k\alpha}}{\Gamma(k\alpha + 1)} \Upsilon(x) d\mu(x) + \Upsilon(t) \\
&= \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} \mathcal{Q}_{k+2}(i_1 r_1, \dots, i_d r_d) \\
&\quad \times \int_0^t \frac{[\mu(t) - \mu(x + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} \Upsilon(x) d\mu(x) + \Upsilon(t). \tag{2.4}
\end{aligned}$$

One can easily obtain

$$\begin{aligned}
&C_{0+} \mathfrak{I}_\mu^\alpha \left(\int_0^t \mathcal{X}_\mu^{\alpha,\alpha}(t, x + r_j) \Upsilon(x) d\mu(x) \right) = \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} \mathcal{Q}_{k+2}(i_1 r_1, \dots, i_d r_d) \\
&\quad \times \int_0^t \frac{[\mu(t) - \mu(x + r_j + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} \Upsilon(x) d\mu(x)
\end{aligned}$$

By having Lemma 1 in mind, we combine (2.1) with (2.4) to obtain the following

$$\begin{aligned}
&C_{0+} \mathfrak{I}_\mu^\alpha \left(\int_0^t \mathcal{X}_\mu^{\alpha,\alpha}(t, x) \Upsilon(x) d\mu(x) \right) \\
&= B \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} \mathcal{Q}_{k+1}(i_1 r_1, \dots, i_d r_d) \\
&\quad \times \int_0^t \frac{[\mu(t) - \mu(x + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} \Upsilon(x) d\mu(x) + \Upsilon(t)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^d F_j \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_j r_j - r_j, \dots, i_d r_d) \\
& \times \int_0^t \frac{[\mu(t) - \mu(x + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} \Upsilon(x) d\mu(x) \\
& + \sum_{j=1}^d A_j \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+2}(i_1 r_1, \dots, i_j r_j - r_j, \dots, i_d r_d) \\
& \times \int_0^t \frac{[\mu(t) - \mu(x + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} \Upsilon(x) d\mu(x) \\
& = B \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) \\
& \times \int_0^t \frac{[\mu(t) - \mu(x + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} \Upsilon(x) d\mu(x) + \Upsilon(t) \\
& + \sum_{j=1}^d F_j \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) \\
& \times \int_0^t \frac{[\mu(t) - \mu(t + r_j + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} \Upsilon(x) d\mu(x) \\
& + \sum_{j=1}^d A_j \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+2}(i_1 r_1, \dots, i_d r_d) \\
& \times \int_0^t \frac{[\mu(t) - \mu(x + r_j + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} \Upsilon(x) d\mu(x) \\
& = B \int_0^t \mathcal{X}_\mu^{\alpha, \alpha}(t, x) \Upsilon(x) d\mu(x) + \sum_{j=1}^d F_j \int_0^t \mathcal{X}_\mu^{\alpha, \alpha}(t, x + j_k) \Upsilon(x) d\mu(x) + \Upsilon(t) \\
& + \sum_{j=1}^d A_j \left[{}_{0^+}^C \mathfrak{D}_\mu^\alpha \left(\int_0^t \mathcal{X}_\mu^{\alpha, \alpha}(t, x + r_j) \Upsilon(x) d\mu(x) \right) \right]
\end{aligned}$$

which gives the inevitable result. $\square$

The next theorem is the last one of main theorems as to a solution of the homogeneous part of system (1.4).

Theorem 3. *The following $\mathbb{R}^n$ -valued continuous function*

$$w(t) = \sum_{j=1}^d \int_{-r_j}^0 \chi_{\mu}^{\alpha, \alpha}(t, r_j + s) [F_j \varphi(s) + A_j ({}^C_{0+} \mathfrak{I}_{\mu}^{\alpha} \varphi)(s)] d\mu(s)$$

is a solution of system (1.4) with $w(t) = \varphi(t)$, $-r \leq t \leq 0$ and $\mathfrak{T} = 0$.

Proof. Now we consider

$$\begin{aligned} & {}^C_{0+} \mathfrak{I}_{\mu}^{\alpha} \left(\sum_{j=1}^d \int_{-r_j}^0 \chi_{\mu}^{\alpha, \alpha}(t, r_j + x) F_j \varphi(x) d\mu(x) \right) \\ &= {}^C_{0+} \mathfrak{I}_{\mu}^{\alpha} \left(\sum_{j=1}^d \int_{-r_j}^0 \sum_{k=1}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) \right. \\ &\quad \times \frac{[\mu(t) - \mu(x + r_j + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} F_j \varphi(x) d\mu(x) \Big) \\ &= {}^C_{0+} \mathfrak{I}_{\mu}^{\alpha} \left(\sum_{j=1}^d \int_{-r_j}^0 \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+2}(i_1 r_1, \dots, i_d r_d) \right. \\ &\quad \times \frac{[\mu(t) - \mu(x + r_j + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + 2\alpha - 1}}{\Gamma(k\alpha + 2\alpha)} F_j \varphi(x) d\mu(x) \Big) \end{aligned} \quad (2.5)$$

By applying (2.1) to (2.5), we get

$$\begin{aligned} & {}^C_{0+} \mathfrak{I}_{\mu}^{\alpha} \left(\sum_{j=1}^d \int_{-r_j}^0 \chi_{\mu}^{\alpha, \alpha}(t, x + r_j) F_j \varphi(x) d\mu(x) \right) \\ &= B \sum_{j=1}^d \int_{-r_j}^0 \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) \\ &\quad \times {}^C_{0+} \mathfrak{I}_{\mu}^{\alpha} \frac{[\mu(t) - \mu(t + r_j + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + 2\alpha - 1}}{\Gamma(k\alpha + 2\alpha)} F_j \varphi(x) d\mu(x) \\ &\quad + \sum_{m=1}^d C_m \sum_{j=1}^d \int_{-r_j}^0 \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_m r_m - r_m, \dots, i_d r_d) \\ &\quad \times {}^C_{0+} \mathfrak{I}_{\mu}^{\alpha} \frac{[\mu(t) - \mu(x + r_j + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + 2\alpha - 1}}{\Gamma(k\alpha + 2\alpha)} F_j \varphi(x) d\mu(x) \\ &\quad + \sum_{m=1}^d A_m \left[{}^C_{0+} \mathfrak{I}_{\mu}^{\alpha} \left(\sum_{j=1}^d \int_{-r_j}^0 \sum_{k=0}^{\infty} \sum_{i_1, i_2, \dots, i_d=0}^{\infty} Q_{k+2}(i_1 r_1, \dots, i_m r_m - r_m, \dots, i_d r_d) \right. \right. \\ &\quad \times \left. \frac{[\mu(t) - \mu(x + r_j + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + 2\alpha - 1}}{\Gamma(k\alpha + 2\alpha)} F_j \varphi(x) d\mu(x) \right) \Big] \end{aligned}$$

$$\begin{aligned}
&= B \sum_{j=1}^d \int_{-r_j}^0 \sum_{k=0, i_1, i_2, \dots, i_d=0}^{\infty} \sum_{k=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) \\
&\quad \times \frac{[\mu(t) - \mu(x + r_j \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} F_j \varphi(x) d\mu(x) \\
&+ \sum_{m=1}^d C_m \sum_{j=1}^d \int_{-r_j}^0 \sum_{k=0, i_1, i_2, \dots, i_d=0}^{\infty} \sum_{k=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) \\
&\quad \times \frac{[\mu(t) - \mu(x + r_j + r_m + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} F_j \varphi(x) d\mu(x) \\
&+ \sum_{m=1}^d A_m \left[C_{0+} \varpi_{\mu}^{\alpha} \left(\sum_{j=1}^d \int_{-r_j}^0 \sum_{k=0, i_1, i_2, \dots, i_d=0}^{\infty} \sum_{k=0}^{\infty} Q_{k+1}(i_1 r_1, \dots, i_d r_d) \right. \right. \\
&\quad \left. \left. \times \frac{[\mu(t) - \mu(x + r_j + r_m + \sum_{n=1}^d i_n r_n)]_+^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} F_j \varphi(x) d\mu(x) \right) \right] \\
&= B \left(\sum_{j=1}^d \int_{-r_j}^0 \chi_{\mu}^{\alpha, \alpha}(t, x + r_j) F_j \varphi(x) d\mu(x) \right) \\
&\quad + \sum_{m=1}^d C_m \left(\sum_{j=1}^d \int_{-r_j}^0 \mathcal{P}_{\alpha, \alpha}(t, x + r_j + r_m) F_j \varphi(x) d\mu(x) \right) \\
&\quad + \sum_{m=1}^d A_m \left[C_{0+} \varpi_{\mu}^{\alpha} \left(\sum_{j=1}^d \int_{-r_j}^0 \chi_{\mu}^{\alpha, \alpha}(t, x + r_j + r_m) F_j \varphi(x) d\mu(x) \right) \right].
\end{aligned}$$

which provides $\sum_{j=1}^d \int_{-r_j}^0 \chi_{\mu}^{\alpha, \alpha}(t, x + r_j) A_j (C_{0+} \varpi_{\mu}^{\alpha} \varphi)(x) d\mu(x)$. In a similar way, it can be easily shown that $\sum_{j=1}^d \int_{-r_j}^0 \chi_{\mu}^{\alpha, \alpha}(t, x + r_j) F_j \varphi(x) d\mu(x)$ is a solution. Furthermore, summation of them is also a solution due to superposition technique. This completes the proof. $\square$

So far we have found the parts of the stepwise solution, now let's put the parts together in the below corollary which stands for a whole solution of system (1.4).

Corollary 2. *The following $\mathbb{R}^n$ -valued continuous function*

$$\begin{aligned}
w(t) &= \left[\chi_{\mu}^{\alpha, 1}(t, 0) - \sum_{m=1}^d \chi_{\mu}^{\alpha, 1}(t, r_m) A_m \right] \varphi(0) + \int_0^t \chi_{\mu}^{\alpha, \alpha}(t, s) \mathbb{T}(s) d\mu(s) \\
&\quad + \sum_{m=1}^d \int_{-r_m}^0 \chi_{\mu}^{\alpha, \alpha}(t, s + r_m) [F_m \varphi(s) + A_m (C_{0+} \varpi_{\mu}^{\alpha} \varphi)(s)] d\mu(s)
\end{aligned}$$

is an exact analytical solution of system (1.4). In the light of Corollary 2, one can easily acquire a mild Volterra type integral equation's solution formula of system (1.5).

Remark 3. Here are some special cases depending on selections of the coefficient matrices. For $\mu(x) = x$ and $y = 0$

- (1) If $A_j = \Theta$, $j = 1, 2, \dots, d$ and $F_j = \Theta$, $j = 2, \dots, d$, then Corollary 2 matches up with Corollary 1 in the reference [12].
- (2) Corollary 2 with $\Upsilon = 0$ reduces to Theorem 3.2 in the work [9] providing that $a_j = \Theta$, $j = 1, 2, \dots, d$ and $F_j = \Theta$, $j = 2, \dots, d$, $B = \Theta$.
- (3) Corollary 2 overlap with Theorem 4.2 in the study [13] under the condition $A_j = \Theta$, $j = 1, 2, \dots, d$.
- (4) Even if the constant coefficient matrices are commutative, our findings also are valid. If the coefficient matrices are permutable in addition to appropriate selections, Corollary 2 reduces to Theorem 6 in the paper [15].
- (5) Corollary 2 reduces to Theorem 3.5 in the paper [5] on taking $d = 2$ and without loss of generality $A_1 = F_2 = \Theta$.

2.3. Existence and uniqueness of solutions of neutral fractional multi-delayed differential equations' system

In this subsection, we look for answers to three kinds of questions : is there a solution for system (1.5)?, is the solution unique? Subsequent to given answers, we finish discussing. When we look at features of each term in system (1.5) like $\Upsilon(t, w(t))$ is continuous, we find an explicit solution in corollary 2. Unfortunately, these features or conditions are not enough to make the explicit solution unique. So, we add one more feature to the continuous function $\Upsilon(t, w(t))$ in order to make the explicit solution satisfy the uniqueness. This feature is that the continuous function $\Upsilon(t, w(t))$ satisfies the Lipschitz condition in the second component with the Lipschitz constant L_Υ , that is

$$\|\Upsilon(t, w(t)) - \Upsilon(t, u(t))\| \leq L_\Upsilon \|w(t) - u(t)\|.$$

Prior to carrying on, we discuss an inequality about $\chi_\mu^{\alpha, \beta}$ in the following lemma.

Lemma 2. Let $\chi_\mu^{\alpha, \beta}(t, s)$ be as in (2.2).

$$\int_0^t \left\| \chi_\mu^{\alpha, \beta}(t, s) \right\| d\mu(s) \leq (\mu(t) - \mu(0)) \left\| \chi_\mu^{\alpha, \beta}(t, 0) \right\|$$

holds true.

Proof. In order to see whether the above inequality is true, it is sufficient to use $[\mu(t) - \mu(s + \sum_{j=1}^d i_j r_j)] \leq [\mu(t) - \mu(\sum_{j=1}^d i_j r_j)]$ along with the expansion $\chi_\mu^{\alpha, \beta}(t, s)$. So, it is omitted. $\square$

The following theorem is about existence and uniqueness of system (1.5).

Theorem 4. *If the continuous function $\Upsilon(t, w(t))$ satisfies the Lipschitz condition in the second component with the Lipschitz constant L_Υ with*

$$L_\Upsilon(\mu(T) - \mu(0)) \|\chi_\mu^{\alpha, \alpha}(T, 0)\| < 1,$$

then the integral equation in the corollary 2 is of a unique solution in $[-d, T]$.

Proof. Define $\mathcal{G} : C([-r, T], \mathbb{R}^n) \rightarrow C([-r, T], \mathbb{R}^n)$ by

$$\begin{aligned} \mathcal{G}w(t) = & \left[\chi_\mu^{\alpha, 1}(t, 0) - \sum_{m=1}^d \chi_\mu^{\alpha, 1}(t, r_m) A_m \right] \varphi(0) + \int_0^t \chi_\mu^{\alpha, \alpha}(t, x) \Upsilon(x, w(x)) d\mu(x) \\ & + \sum_{m=1}^d \int_{-r_m}^0 \chi_\mu^{\alpha, \alpha}(t, x + r_m) [F_m \varphi(x) + A_m ({}_{0+}^C \mathfrak{I}_\mu^\alpha \varphi)(x)] d\mu(x) \end{aligned}$$

For arbitrary $w, u \in C([-r, T], \mathbb{R}^n)$, we consider by using Lemma 2

$$\begin{aligned} \|\mathcal{G}w(t) - \mathcal{G}u(t)\| & \leq \int_0^t \|\chi_\mu^{\alpha, \alpha}(t, s)\| \|\Upsilon(s, w(s)) - \Upsilon(s, u(s))\| ds \\ & = L_\Upsilon(\mu(T) - \mu(0)) \|\chi_\mu^{\alpha, \alpha}(T, 0)\| \|w - u\|_C. \end{aligned}$$

The statements of this theorem ensure that $\mathcal{G}$ is a contraction. By the Banach Contraction principle, $\mathcal{G}$ is of a unique fixed point on $[-r, T]$, that is $\exists! w_0 \in C([-r, T], \mathbb{R}^n)$, $w_0(t) = \mathcal{G}w_0(t)$. $\square$

2.4. Stability of neutral fractional order multi-delayed differential system in the sense of Ulam-Hyers

We investigate the stability of system (1.5).

Definition 2. Let $\varepsilon > 0$. The system (1.5) is said to be Ulam-Hyers stable if for every solution $w \in C([0, T], \mathbb{R}^n)$ of inequality,

$$\left\| {}_{0+}^C \mathfrak{I}_\mu^\alpha \left[w(t) - \sum_{i=1}^d A_i w(t - r_i) \right] - Bw(t) - \sum_{i=1}^d F_i w(t - r_i) - \Upsilon(t, w(t)) \right\| \leq \varepsilon, \quad (2.6)$$

there exists a solution $w_0 \in C([0, T], \mathbb{R}^n)$ of system (1.5), and $\eta > 0$ such that

$$\|w(t) - w_0(t)\| \leq \eta \varepsilon, \quad t \in [0, T].$$

Remark 4. A function $w \in C^1([0, T], \mathbb{R}^n)$ is a solution of the inequality equation (2.6) if and only if there exists a function $z \in C^1([0, T], \mathbb{R}^n)$, such that $\|z(t)\| < \varepsilon$ and ${}_{0+}^C \mathfrak{I}_\mu^\alpha [w(t) - \sum_{i=1}^d A_i w(t - r_i)] = Bw(t) + \sum_{i=1}^d F_i w(t - r_i) + \Upsilon(t, w(t)) + z(t)$.

Theorem 5. *Suppose that all of the assumptions of Theorem 4 are hold. Then system (1.5) is Ulam-Hyers stable.*

Proof. Let $w \in C([0, T], \mathbb{R}^n)$ be a solution of the inequality (2.6), i.e.

$$\left\| {}^C_{0+}\mathfrak{I}_\mu^\alpha \left[w(t) - \sum_{i=1}^d A_i w(t - r_i) \right] - Bw(t) - \sum_{i=1}^d F_i w(t - r_i) - \mathfrak{T}(t, w(t)) \right\| \leq \varepsilon. \quad (2.7)$$

Let $w_0 \in C([0, T], \mathbb{R}^n)$ be the unique solution of system (1.5), so that

$${}^C_{0+}\mathfrak{I}_\mu^\alpha \left[w_0(t) - \sum_{i=1}^d A_i w_0(t - r_i) \right] = Bw_0(t) + \sum_{i=1}^d F_i w_0(t - r_i) + \mathfrak{T}(t, w_0(t))$$

for each $t \in [0, T]$ and $0 < \alpha < 1$, $w_0(t) = w(t)$ with $-r \leq t \leq 0$. By combining Remark 4 and equation (2.7), there exists a function $z \in C([0, T], \mathbb{R}^n)$ such that $\|z(t)\| < \varepsilon$,

$${}^C_{0+}\mathfrak{I}_\mu^\alpha \left[w(t) - \sum_{i=1}^d A_i w(t - r_i) \right] = Bw(t) + \sum_{i=1}^d F_i w(t - r_i) + \mathfrak{T}(t, w(t)) + z(t). \quad (2.8)$$

So we deduced the solution $w(t)$ from (2.8) with the aid of Corollary 2 and the function $\mathcal{G}$,

$$w(t) = \mathcal{G}w(t) + \int_0^t \mathcal{X}_\mu^{\alpha, \alpha}(t, x) z(x) d\mu(x)$$

So we have the following estimation

$$\|\mathcal{G}w(t) - w(t)\| \leq \int_0^t \|\mathcal{X}_\mu^{\alpha, \alpha}(t, s)\| \|z(s)\| d\mu(s) \leq (\mu(T) - \mu(0)) \|\mathcal{X}_\mu^{\alpha, \alpha}(T, 0)\| \varepsilon.$$

By the fixed point property of the operator $\mathcal{G}$ given in the proof of Theorem 4, we obtain

$$\begin{aligned} \|w_0(t) - w(t)\| &\leq \|\mathcal{G}w_0(t) - \mathcal{G}w(t)\| + \|\mathcal{G}w(t) - w(t)\| \\ &\leq L_{\mathfrak{T}}(\mu(T) - \mu(0)) \|\mathcal{X}_\mu^{\alpha, \alpha}(T, 0)\| \|w - u\|_C \\ &\quad + (\mu(T) - \mu(0)) \|\mathcal{X}_\mu^{\alpha, \alpha}(T, 0)\| \varepsilon. \end{aligned}$$

By taking C -norm on the left hand side and then rearranging the above equation

$$(1 - L_{\mathfrak{T}}(\mu(T) - \mu(0)) \|\mathcal{X}_\mu^{\alpha, \alpha}(T, 0)\|) \|w - u\|_C \leq (\mu(T) - \mu(0)) \|\mathcal{X}_\mu^{\alpha, \alpha}(T, 0)\| \varepsilon,$$

then we get the desired result

$$\|w - u\|_C \leq \eta \cdot \varepsilon, \quad \eta = \frac{(\mu(T) - \mu(0)) \|\mathcal{X}_\mu^{\alpha, \alpha}(T, 0)\|}{(1 - L_{\mathfrak{T}}(\mu(T) - \mu(0)) \|\mathcal{X}_\mu^{\alpha, \alpha}(T, 0)\|)} > 0.$$

□

Remark 5. It is clear as day that all findings are valid when the coefficient matrices are commutative.

3. AN ILLUSTRATED EXAMPLE

We will consider $\sqrt{t}$ -Caputo type fractional neutral differential multi-delayed equations

$$\begin{cases} {}^C_{0+}\mathfrak{D}_\mu^{0.8} [w(t) - A_1 w(t-0.3) - A_2 w(t-0.2)] = Bw(t) \\ \quad + F_1 w(t-0.3) + \frac{e^t}{4(1+e^t)} \sin(w(t)), \quad t \in (0, 0.6], \\ w(t) = (t^3 \ 2t + 1)^T, \quad -0.3 \leq t \leq 0, \end{cases} \quad (3.1)$$

where $\mu(t) = \sqrt{t}$, and

$$A_1 = \begin{pmatrix} 0.170 & 0.830 \\ 0 & 0.350 \end{pmatrix}, A_2 = \begin{pmatrix} 0.36 & 0.64 \\ 0.07 & 0.11 \end{pmatrix}, B = \begin{pmatrix} 0.33 & 0 \\ 0.03 & 0.125 \end{pmatrix},$$

and

$$F_1 = \begin{pmatrix} 0.43 & 470 \\ 0.03 & 0.125 \end{pmatrix},$$

which are pairwise noncommutative matrices, e.g., $A_1 A_2 \neq A_2 A_1$, $A_1 B \neq B A_1$, $A_2 B \neq B A_2$, $F_1 A_2 \neq A_2 F_1$, etc. With the well-known maximum absolute row sum of the matrix $\|\cdot\|_\infty$; $\|A_1\|_\infty = 1$, $\|A_2\|_\infty = 1$, $\|B\|_\infty = 0.33$, and $\|F_1\|_\infty = 1$. One can easily check that $\mathfrak{T}(t, w(t)) = \frac{e^t}{4(1+e^t)} \sin(w(t))$ is continuous in addition being the Lipschitz function with the Lipschitz constant $L_{\mathfrak{T}} = 0.25$ and

$$L_{\mathfrak{T}}(\mu(0.6) - \mu(0)) \mathcal{X}_{\mu, 0.8, 0.8}^{\|A_1 + A_2\|_\infty, \|B\|_\infty, \|F_1\|_\infty}(0.6, 0) \cong 0.0509175 < 1.$$

Hence, all of conditions of Theorem 4 and 5 holds, so system (3.1) which has an unique solution is Ulam-Hyers stable.

4. CONCLUSION

In the current paper, we firstly introduce the μ -neutral Caputo type fractional linear multi-delayed differential equations with non-permutable constant coefficient matrices. To obtain a representation of a solution for it, we newly define the μ -neutral multi-delayed perturbation of two parameter Mittag-Leffler type matrix function. We consider the existence and uniqueness of solutions and Ulam-Hyers stability of the μ -neutral fractional multi-delayed differential equations' system.

In terms of qualitative properties and fractional differential equations, this paper includes many different kinds of comprehensive studies [12, 13, 15] because for some special cases of μ , we obtain **the classical Caputo fractional derivative** [8, 17], Hadamard fractional derivative [8], the Caputo–Hadamard fractional derivative [4] and the Caputo–Erdélyi–Kober fractional derivative [11].

The next possible further work can be devoted to study asymptotic stability, exponential stability, finite time stability, and also Lyapunov type stability of the μ -neutral Caputo fractional multi-delayed differential equations with noncommutative coefficient matrices as well as its relative controllability and iterative learning controllability. Another possible direction for additional studies is to extend our system (1.4)

to μ -fractional functional evolution equations and all possibilities as noted just above can be questioned once again for this new system.

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STATISTICALLY p_τ -CONVERGENCE IN LATTICE-NORMED LOCALLY SOLID RIESZ SPACES

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Abstract. In this paper, we introduce the concept of the statistical convergence with respect to solid topology and Riesz valued norms on lattice-normed locally solid Riesz spaces. Moreover, we give the notions of statistically p_τ -bounded and statistically p_τ -dense sequence, and we introduce statistically p_τ -continuous and statistically p_τ -bounded operators. We also investigate some properties and examples of these concepts.

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1. INTRODUCTORY FACTS

Lattice-valued norms on Riesz spaces and statistical convergence of sequences provide natural and efficient tools in the theory of functional analysis. A Riesz space that was introduced by Riesz [20] is an ordered vector space having many applications in measure theory, operator theory, and economics [2, 3, 19, 23]. On the other hand, a lattice-normed space is a lattice-valued norm, and it is enough to mention the theory of lattice-normed vector lattices [11, 17, 18]. As an active area of research, statistical convergence is a generalization of the ordinary convergence of a real sequence, and the idea of statistical convergence was firstly introduced by Zygmund [16]. After then, Fast [13] and Steinhaus [21] independently improved that idea. Several applications and generalizations of the statistical convergence of sequences have been investigated by several authors [1, 5, 6, 8, 9, 14, 21, 22]. The main aim of the present paper is to introduce the concept of *statistical p_τ -convergence* on lattice-normed locally solid Riesz spaces, which attracted the attention of several authors in a series of recent papers [1, 5, 6, 9].

The concept of the statistical convergence in Riesz spaces was introduced by Erçan [12], where the notion of the statistically u -uniformly convergent sequence was

introduced in Riesz spaces. Then Albayrak and Pehlivan extended the statistical convergence to locally solid Riesz spaces with respect to solid topology [1]. Recently, Aydın et al., have investigated some studies about the statistical convergence on Riesz spaces and locally solid Riesz spaces [5, 6, 9].

We now turn our attention to some basic notions which will be used in this paper. A real-valued vector space E with a partial order relation " $\leq$ " on E (i.e. it is an antisymmetric, reflexive and transitive relation) is called an *ordered vector space* whenever, for every $x, y \in E$, we have

- (a) $x \leq y$ implies $x + z \leq y + z$ for all $z \in E$,
- (b) $x \leq y$ implies $\lambda x \leq \lambda y$ for every $0 \leq \lambda \in \mathbb{R}$.

An ordered vector space E is called a *Riesz space* or a *vector lattice* if, for any two vectors $x, y \in E$, the infimum $x \wedge y = \inf\{x, y\}$ and the supremum $x \vee y = \sup\{x, y\}$ exist in E . For an element x in a Riesz space E , the *positive part*, the *negative part*, and the *module* of x are

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \quad \text{and} \quad |x| := x \vee (-x),$$

respectively. In the present paper, the vertical bar $|\cdot|$ of elements of Riesz spaces will stand for the module of the given elements.

By a *linear topology* τ on a vector space X , we mean a topology τ on X that makes the addition and the scalar multiplication continuous. A *topological vector space* (X, τ) is a vector space X equipped with a linear topology τ . A linear topology τ on a vector space E has a base $\mathcal{N}$ for the zero neighborhoods satisfying the following:

- (i) Each $V \in \mathcal{N}$ is *balanced*, i.e., $\lambda V \subseteq V$ for all scalars $|\lambda| \leq 1$;
- (ii) Every $V \in \mathcal{N}$ is *absorbing*, i.e., for every element x , there exists a positive real $\lambda > 0$ such that $x \in \lambda V$;
- (iii) For each $V_1, V_2 \in \mathcal{N}$, there is $V \in \mathcal{N}$ such that $V \subseteq V_1 \cap V_2$;
- (iv) For every $V \in \mathcal{N}$, there exists $U \in \mathcal{N}$ with $U + U \subseteq V$;
- (v) For any scalar λ and each $V \in \mathcal{N}$, the set λV is also in $\mathcal{N}$.

In this article, unless otherwise stated, when we mention a zero neighborhood, it means that it always belongs to a base that holds the above properties. In this paper, neighborhoods of zero will often be referred to as *zero neighborhoods*.

A subset A of a vector lattice E is called *solid* if, for each $x \in A$ and $y \in E$ with $|y| \leq |x|$ it holds $y \in A$. Let E be a Riesz space and τ be a linear topology on it. Then the pair (E, τ) is said to be a *locally solid Riesz space* if τ has a base that consists of solid sets; for much more details on these notions, see [2, 3, 23].

Definition 1. Let X be a vector space and E be a Riesz space. Then $p: X \rightarrow E_+$ is called an *E -valued vector norm* whenever it satisfies the following conditions:

- (1) $p(x) = 0 \Leftrightarrow x = 0$;
- (2) $p(\lambda x) = |\lambda|p(x)$ for all $\lambda \in \mathbb{R}$;
- (3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Then the triple (X, p, E) is called a *lattice-normed space*, abbreviated as *LNS*. If, in addition, X is a Riesz space and the vector norm p is monotone (i.e., $|x| \leq |y| \Rightarrow p(x) \leq p(y)$ holds for all $x, y \in X$) then the triple (X, p, E) is called a *lattice-normed Riesz space*. We abbreviate it as *LNRS*. A subset $Y \subseteq X$ is called *p-bounded* if there exists $e \in E$ such that $p(y) \leq e$ for all $y \in Y$. A sequence (x_n) in X is called *p-convergent* to $x \in X$ (or, shortly, $x_n \xrightarrow{p} x$) whenever $p(x_n - x) \xrightarrow{o} 0$ holds in E . We refer the reader for more information on *LNRSs* [11, 17, 18]. We shall keep in mind also the following examples.

Example 1. Let X be a normed space with a norm $\|\cdot\|$. Then $(X, \|\cdot\|, \mathbb{R})$ is an *LNS*.

Example 2. Let X be a Riesz space. Then $(X, |\cdot|, X)$ is an *LNRS*.

Now, we remind some basic properties of the concept related to the statistical convergence. Consider a set K of positive integers. Then the *natural density* of K is defined by

$$\delta(K) := \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where the vertical bar of sets will stand for the cardinality of the given sets. We refer the reader to an exposition on the natural density of sets [13, 14]. In the same way, a real sequence (x_n) is called *statistically convergent* to L provided that

$$\lim_{m \rightarrow \infty} \frac{1}{m} |\{n \leq m : |x_n - L| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$.

Let X be a topological space and (x_n) be a sequence in X . Then (x_n) is said to be *statistically convergent* to $x \in X$ whenever, for each neighborhood U of x , we have $\delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0$. On the other hand, a sequence (x_n) in a locally solid Riesz space (E, τ) is *statistically τ -convergent* to $x \in E$ whenever we have $\delta(\{n \in \mathbb{N} : (x_n - x) \notin U\}) = 0$ for every zero neighborhood U [1, 5–10].

2. STATISTICALLY p_τ -CONVERGENCE

In this section, we introduce the statistically topological convergence on lattice-normed spaces. Also, we give some basic results about this concept.

Definition 2. Let (X, p, E) be an *LNRS*. If (E, τ) is a locally solid Riesz space then (X, p, E_τ) is called a *lattice-normed locally solid Riesz space*. We abbreviate it as *LNLS*.

Definition 3. Let (X, p, E_τ) be an *LNLS* and (x_n) be a sequence in X . Then (x_n) is said to be *statistically p_τ -convergent* (*st- p_τ -convergent*, for short) to x if it is provided that

$$\lim_{m \rightarrow \infty} \frac{1}{m} |\{n \leq m : p(x_n - x) \notin U\}| = 0$$

holds for every zero neighborhood U . In this case, we write $x_n \xrightarrow{\text{st-}p_\tau} x$.

Briefly, a sequence (x_n) is statistically p_τ -convergent to $x \in X$ if $\delta(K_U) = 0$ for each zero neighborhood U , where $K_U = \{n \in \mathbb{N} : p(x_n - x) \notin U\}$. Note that, in order to simplify the presentation, we take zero neighborhoods from a solid base because, for every zero neighborhood V , there exists a zero neighborhood solid set U such that $U \subseteq V$.

Example 3. Consider the $LNLS (E, |\cdot|, E_\tau)$ for an arbitrary locally solid Riesz space (E, τ) . Then statistically τ - and p_τ -convergence coincide.

Example 4. Let $\|\cdot\|$ be a lattice norm (i.e., $|x| \leq |y|$ implies $\|x\| \leq \|y\|$) on a Riesz space X . Then, it follows from [2, Theorem 2.28] that the topology τ on X generated by the norm is a solid topology. Moreover, the statistically norm convergence and p_τ -convergence agree on the $LNLS (X, |\cdot|, X_\tau)$.

Example 5. Let's consider the $LNLS (c, |\cdot|, c_\tau)$, where c is the set of all convergent real sequences and τ is the topology generated by the supremum norm on c . Take the sequence (x_n) in c such that $x_n = (0, \dots, \frac{1}{n}, 0, \dots)$ for each n . Now, consider the set of all zero neighborhoods $\mathcal{N} := \{U_r : r \in \mathbb{R}_+\}$, where U_r is the set $\{x \in c : \|x\|_\infty \leq r\}$. Thus, it follows that $x_n \xrightarrow{\text{st-}p_\tau} 0$ in c . Indeed, fix an arbitrary zero neighborhood set U in c . Then, there exist some $r > 0$ such that $U_r \subseteq U$. Consider the set

$$K_r = \{n \in \mathbb{N} : p(x_n - 0) \in U_r\} = \left\{n \in \mathbb{N} : \|x_n\|_\infty = \frac{1}{n} < r\right\}.$$

It can be seen that $\delta(K_r) = 1$. Thus, we have $x_n \xrightarrow{\text{st-}p_\tau} 0$.

Remark 1. Let (X, p, E_τ) be an $LNLS$. If $|y| \leq |x|$ for any $x, y \in X$, then we have that $p(y) \notin U$ implies $p(x) \notin U$ for an arbitrary zero neighborhood U . Indeed, assume that $p(y) \notin U$ and $p(x) \in U$ hold. It follows from the solidness of U that we have $p(y) \in U$ because $|y| \leq |x|$ implies $p(y) \leq p(x) \in U$. So, there is a contradiction, and so, we have the desired result, $p(x) \notin U$.

Proposition 1. *Let (X, p, E_τ) be an $LNLS$. Then, the following statements hold:*

- (i) *every p -convergent sequence is $\text{st-}p_\tau$ -convergent;*
- (ii) *every order convergent sequence is $\text{st-}p_\tau$ -convergent if order convergence implies p -convergence in X .*

Proof.

- (i) Let $x_n \xrightarrow{p} x$ in X . Then, there exists a sequence $q_n \downarrow 0$ in E such that $p(x_n - x) \leq q_n$ for every $n \in \mathbb{N}$. On the other hand, take an arbitrary zero neighborhood U and arbitrary $m \in \mathbb{N}$. Then, there exists a positive integer $k > 0$ such that $\frac{1}{k}q_m \in U$, and so, we have $\frac{1}{k}q_n \in U$ for all $n \geq m$ because U is solid and

absorbing set. Now, take an arbitrary index n_k such that $q_{n_k} \leq \frac{1}{k}q_m$. Hence, we have $q_n \in U$ for all $n \geq n_k$. So, we observe that

$$\delta(\{n \in \mathbb{N} : q_n \notin U\}) = \delta(\{1, 2, \dots, n_k - 1\}) = 0.$$

It follows from the inequality $p(x_n - x) \leq q_n$ for all n that we have $\delta(\{n \in \mathbb{N} : p(x_n - x) \notin U\}) = 0$, i.e., $x_n \xrightarrow{\text{st-}p_\tau} x$.

(ii) The proof is similar to the first part. □

Theorem 1. *The st- p_τ -limit is linear in LNLSs.*

Proof. Suppose that $x_n \xrightarrow{\text{st-}p_\tau} x$ and $y_n \xrightarrow{\text{st-}p_\tau} y$ in an LNLS (X, p, E_τ) . For any zero neighborhood U , there exists another zero neighborhood V such that $V + V \subseteq U$. Then, we have $\delta(\{n \in \mathbb{N} : p(x_n - x) \notin V\}) = 0$ and $\delta(\{n \in \mathbb{N} : p(y_n - y) \notin V\}) = 0$. It follows from the following inequality

$$p(x_n + y_n - x - y) \leq p(x_n - x) + p(y_n - y) \in V + V \subseteq U$$

that if $p(x_k - x) \in V$ and $p(y_k - y) \in V$ for some indexes k , then $p(x_k + y_k - x - y) \in U$. Thus, we observe that $\{n \in \mathbb{N} : p(x_n + y_n - x - y) \notin U\} \subseteq \{n \in \mathbb{N} : p(x_n - x) \notin V\} \cup \{n \in \mathbb{N} : p(y_n - y) \notin V\}$, and so, we obtain $\delta(\{n \in \mathbb{N} : p(x_n + y_n - x - y) \notin U\}) = 0$. Therefore, we get the desired result, i.e., $x_n + y_n \xrightarrow{\text{st-}p_\tau} x + y$.

Now, take a scalar $\alpha \in \mathbb{R}$ such that $|\alpha| \leq 1$ and an arbitrary zero neighborhood U . Then, it follows that $\delta(K) = 1$, where $K = \{n \in \mathbb{N} : p(x_n - x) \in U\}$. Since U is a balanced set, by the equality

$$p(\alpha x_n - \alpha x) = |\alpha|p(x_n - x) \in |\alpha|U \subseteq U,$$

we get $p(\alpha x_n - \alpha x) \in U$ for all $n \in K$. Thus, we obtain

$$\{n \in \mathbb{N} : p(x_n - x) \in U\} \subseteq \{n \in \mathbb{N} : p(\alpha x_n - \alpha x) \in U\},$$

and so, we have $\delta(\{n \in \mathbb{N} : p(\alpha x_n - \alpha x) \in U\}) = 1$, i.e., $\alpha x_n \xrightarrow{\text{st-}p_\tau} \alpha x$.

Consider the case $|\alpha| > 1$. Then, for a given zero neighborhood U , we have $\frac{1}{|\alpha|}U \subseteq U$ because of the balanced property. Fix $\gamma = \frac{1}{|\alpha|}$. Since γU is a solid zero neighborhood, there exists another zero neighborhood W such that $W \subseteq \gamma U$, and so, we have $|\alpha|W \subseteq U$. By the same consideration of the above part, it follows from

$$p(\alpha x_n - \alpha x) = |\alpha|p(x_n - x) \in |\alpha|W \subseteq U$$

that $\delta(\{n \in \mathbb{N} : p(\alpha x_n - \alpha x) \in U\}) = 1$. So, we get $\alpha x_n \xrightarrow{\text{st-}p_\tau} \alpha x$ for every $\alpha \in \mathbb{R}$. □

Theorem 2. *Let (X, p, E_τ) be an LNLS. Then,*

- (i) *if τ is a Hausdorff solid topology, then the st- p_τ -limit is uniquely determined,*
- (ii) *the statistically p_τ -version of the squeeze law holds.*

Proof.

- (i) Assume that $x_n \xrightarrow{\text{st-}p_\tau} x_1$ and $x_n \xrightarrow{\text{st-}p_\tau} x_2$ and τ has the Hausdorff property. Let U be a zero neighborhood in E . Then, there exists a zero neighborhood V satisfying $V + V \subseteq U$. Thus, we have $\delta(K_1) = \delta(K_2) = 1$ for the sets $K_1 = \{n \in \mathbb{N} : p(x_n - x_1) \in V\}$ and $K_2 = \{n \in \mathbb{N} : p(x_n - x_2) \in V\}$. It follows that

$$p(x_1 - x_2) \leq p(x_1 - x_n) + p(x_n - x_2) \in V + V \subseteq U$$

for every $n \in K_1 \cap K_2$. Then, by the solidness of U , $p(x_1 - x_2) \in U$ holds. So, we have $p(x_1 - x_2) \in U$ for every zero neighborhood. Since the intersection of all zero neighborhoods in a Hausdorff space is zero, we obtain $p(x_1 - x_2) = 0$, i.e., $x_1 = x_2$.

- (ii) Suppose that $x_n \leq y_n \leq z_n$ holds for all $n \in \mathbb{N}$, and $x_n \xrightarrow{\text{st-}p_\tau} w$ and $z_n \xrightarrow{\text{st-}p_\tau} w$. Take an arbitrary zero neighborhood U , and so, there exists another zero neighborhood V such that $V + V \subseteq U$. Hence, we have $\delta(K_1) = \delta(K_2) = 1$, where $K_1 = \{n \in \mathbb{N} : p(x_n - w) \in V\}$ and $K_2 = \{n \in \mathbb{N} : p(z_n - w) \in V\}$. By the inequality $x_n \leq y_n \leq z_n$, we have $x_n - w \leq y_n - w \leq z_n - w$, and so, $|y_n - w| \leq |x_n - w| + |z_n - w|$ holds for all n . Therefore, we obtain

$$p(y_n - w) \leq p(x_n - w) + p(z_n - w) \in V + V \subseteq U$$

for every $n \in K_1 \cap K_2$. Thus, we obtain $\delta(\{n \in \mathbb{N} : p(y_n - w) \in U\}) = 1$. So, we get the desired result, $y_n \xrightarrow{\text{st-}p_\tau} w$.

□

Definition 4. Let (X, p, E_τ) be an *LNLS* and (x_n) be a sequence in X . Then, (x_n) is said to be *statistically p_τ -bounded* if, for every zero neighborhood U , there exist some $\lambda > 0$ such that

$$\delta(\{n \in \mathbb{N} : p(x_n) \notin \lambda U\}) = 0.$$

Remark 2.

- (i) Every p -bounded sequence is statistically p_τ -bounded. Indeed, let (x_n) be a p -bounded sequence in an *LNLS* (X, p, E_τ) . It follows from [2, Theorem 2.19 (1)] that $p(x_n)$ is topologically bounded in E because it is order bounded in E . So, for each zero neighborhood U , there exists $\lambda > 0$ such that $p(x_n) \in \lambda U$ for all $n \in \mathbb{N}$. Thus, (x_n) is statistically p_τ -bounded.
- (ii) Let (X, p, E_τ) be an *LNLS* and (x_n) be an order bounded sequence in X . Then, (x_n) is statistically p_τ -bounded. Indeed, since (x_n) is order bounded, there exists $x \in X_+$ such that $|x_n| \leq x$ for all $n \in \mathbb{N}$. It follows from the monotonicity of p that $p(x_n) \leq p(x)$ for all n , i.e., $p(x_n)$ is order bounded in E . Thus, by applying [2, Theorem 2.19 (1)], $p(x_n)$ is topologically bounded in E , and so, it is statistically p_τ -bounded.

Proposition 2. Every *st- p_τ -convergent* sequence is statistically p_τ -bounded.

Proof. Let (x_n) be an st - p_τ -convergent sequence to x in an $LNLS$ (X, p, E_τ) . Take an arbitrary zero neighborhood U with another zero neighborhood V such that $V + V \subseteq U$. Then, we have $\delta(\{n \in \mathbb{N} : p(x_n - x) \notin V\}) = 0$. On the other hand, by using the absorbing property of V , there exists a positive scalar λ such that $\lambda p(x) \in V$. Choose $r := \min\{1, \lambda\}$. Then, we have $rp(x) \in V$ because of $rp(x) \leq \lambda p(x)$. Therefore, if $p(x_n - x) \in V$, then $rp(x_n - x) \in V$ by the balancing property of V . It follows from the inequality $rp(x_n) \leq rp(x_n - x) + rp(x) \in V + V \subseteq U$ that we obtain $\delta(\{n \in \mathbb{N} : rp(x_n) \notin U\}) = 0$, i.e., (x_n) is statistically p_τ -bounded. $\square$

3. STATISTICAL p_τ -DENSITY AND p_τ -LIMIT POINTS

Recall that a sublattice Y of a Riesz space X is called

- (-) *dense with respect to order convergence* if every vector in X is the order limit of a net in Y ,
- (-) *order dense* whenever for each $0 < x \in X$ there exists some $y \in Y$ with $0 < y \leq x$.

Motivated by these definitions, we give the following notions.

Definition 5. Let (X, p, E_τ) be an $LNLS$. A sublattice $Y \subseteq X$ is called

- (i) *statistically p_τ -dense* in X with respect to st - p_τ -convergence whenever for all $x \in X$ there exists a non zero sequence (y_n) in Y such that $y_n \xrightarrow{st-p_\tau} x$;
- (ii) *p_τ -dense* in X with respect to st - p_τ -boundedness if, for any $x \in X$, there exists a non zero sequence (y_n) in Y such that the sequence $(x - y_n)$ is statistically p_τ -bounded.

It is clear from Proposition 2 that statistical p_τ -density implies p_τ -density. But, the converse need not hold in general. To see this, consider the following example.

Example 6. Let X be the set of real-valued bounded functions on $[0, 1]$ denoted in the form $f := g + h$, where g is continuous and h vanishes except at finitely many points. Then, $(X, |\cdot|, X_\tau)$ is an $LNLS$, where τ is the topology generated by supremum norm on X . Then, the sublattice $Y := C[0, 1]$, all continuous functions on $[0, 1]$, is statistically p_τ -dense in X . Indeed, it is clear that, for any $f \in X$, there exists a sequence (g_n) in Y such that $g_n \xrightarrow{o} f$. It follows from [4, Remark 2.2 (2)] and Proposition 1(ii) that we obtain $g_n \xrightarrow{st-p_\tau} f$. On the other hand, we observe that Y is not p_τ -dense in X because, for the characteristic function $\chi_{\{\frac{1}{3}\}}$ in X , there is not any sequence (g_n) in Y such that $(f - g_n)$ is statistically p_τ -bounded.

Proposition 3. Let Y be a sublattice of a locally solid Riesz space (X, τ) . If Y is p_τ -dense in $LNLS$ $(X, |\cdot|, X_\tau)$, then Y is order dense.

Proof. Take a positive nonzero element $0 \neq x \in X_+$. Then, there is a sequence (y_n) in Y such that $(y_n - \frac{1}{2}x)$ is statistically p_τ -bounded. Thus, for an arbitrary zero neighborhood U , there exist some $\lambda > 0$ such that $\delta(\{n \in \mathbb{N} : \lambda|y_n - \frac{1}{2}x| \in U\}) = 1$.

Take $u \in U$ such that $|y_m - \frac{1}{2}x| = \frac{1}{\lambda}u \leq \frac{1}{3}x$ for some $m \in \mathbb{N}$. Then, it follows that $0 < y_m \leq x$, and so, Y is order dense in X . $\square$

Example 7. Let Y be the Riesz space c_0 of all convergent to zero real sequences and X be the Riesz space ℓ_∞ of all bounded real sequences. Then, for the *LNLS* $(\ell_\infty, |\cdot|, \ell_\infty)$, Y is order bounded in X . But, it is not p_τ -dense in X .

Now, we turn our attention to statistical p_τ -limit and p_τ -cluster points. The following notions are p_τ -versions of classical statistical points [10, 14].

Definition 6. Let (X, p, E_τ) be an *LNLS*. Then, a point $x \in X$ is called

- (1) an *st- p_τ -limit point* of a sequence (x_n) in X whenever there is an index set $K = \{k_1 < k_2 < \dots\} \subset \mathbb{N}$ such that $\delta(K) > 0$ and $p(x_{k_n} - x) \xrightarrow{\tau} 0$.
- (2) an *st- p_τ -cluster point* of a sequence (x_n) in X if, for each zero neighborhood U , we have $\delta(\{n \in \mathbb{N} : p(x_n - x) \in U\}) > 0$.

For a sequence (x_n) in a space X , let $\Lambda_p(x_n)$ denote the set of all *st- p_τ -limit points* of (x_n) , and let $\Theta_p(x_n)$ denote the set of all *st- p_τ -cluster points* of (x_n) .

Example 8. Let $(E, |\cdot|, E_\tau)$ be an *LNLS* and $x, y \in E$. Then, take a sequence (x_n) denoted by $x_n := x$ if n is a square and $x_n := y$ otherwise. Thus, it is clear that $\Lambda_p(x_n) = \Theta_p(x_n) = \{y\}$.

Theorem 3. For an *LNLS* (X, p, E_τ) and sequence (x_n) in X , $\Lambda_p(x_n) \subseteq \Theta_p(x_n)$.

Proof. Let $x \in \Lambda_p(x_n)$ and $p(x_{n_k} - x) \xrightarrow{\tau} 0$ holds on an index set K such that $\delta(K) = \lambda > 0$. Fix a zero neighborhood U in E . Then, it follows from $p(x_{n_k} - x) \xrightarrow{\tau} 0$ that there exists an $n_j \in K$ such that $p(x_{n_k} - x) \in U$ for $n_j \leq n_k \in K$. Also, we can easily observe the following subset

$$\{n_k : k \in \mathbb{N}\} \setminus \{n_1, n_2, \dots, n_{j-1}\} \subseteq \{n \in \mathbb{N} : p(x_n - x) \in U\}.$$

Therefore, we have

$$\delta(\{n \in \mathbb{N} : p(x_n - x) \in U\}) \geq \lambda > 0$$

Then, we get $x \in \Theta_p(x_n)$ because U is arbitrary. $\square$

Theorem 4. Let (x_n) and (y_n) be two sequences in an *LNLS* (X, p, E_τ) . If the natural density of $\{n \in \mathbb{N} : x_n \neq y_n\}$ is zero, then $\Theta_p(x_n) = \Theta_p(y_n)$ and $\Lambda_p(x_n) = \Lambda_p(y_n)$.

Proof. Assume that $w \in \Theta_p(x_n)$ and U is an arbitrary zero neighborhood. Thus, we have $\delta(\{n \in \mathbb{N} : p(x_n - w) \in U\}) > 0$. It follows from the subset

$$\{n \in \mathbb{N} : p(x_n - w) \in U\} \setminus \{n \in \mathbb{N} : x_n \neq y_n\} \subseteq \{n \in \mathbb{N} : p(y_n - w) \in U\}$$

that we obtain $\delta(\{n \in \mathbb{N} : p(y_n - w) \in U\}) > 0$. Therefore, we get $w \in \Theta_p(y_n)$. In a similar way, one proves $\Theta_p(y_n) \subseteq \Theta_p(x_n)$. Hence, we get the desired result, $\Theta_p(x_n) = \Theta_p(y_n)$.

The equality $\Lambda_p(x_n) = \Lambda_p(y_n)$ can be obtained similarly. $\square$

Theorem 5. Let (x_n) be a sequence in an LNLS (X, p, E_τ) and F be a compact subset of E satisfying $\delta(\{n \in \mathbb{N} : p(x_n) \in F\}) > 0$. Then, we have $p(\Theta_p(x_n)) \cap F \neq \emptyset$.

Proof. Assume that $p(\Theta_p(x_n)) \cap F = \emptyset$. Then, for every $w \in X$ such that $p(w) \in F$, we have $w \notin \Theta_p(x_n)$. Thus, there is a zero neighborhood U_w such that $\delta(K_w) = 0$, where $K_w := \{n \in \mathbb{N} : p(x_n - w) \in U_w\}$. On the other hand, consider an open cover $\{U_w : p(w) \in F\}$ of F . Then, we have a finite subcover $U_{w_1}, U_{w_2}, \dots, U_{w_j}$ of F . So, it follows from

$$\{n \in \mathbb{N} : p(x_n - w) \in F\} \subseteq K_{w_1} \cup K_{w_2} \cup \dots \cup K_{w_j}$$

that $\delta(\{n \in \mathbb{N} : p(x_n - w) \in F\}) = 0$, a contradiction. $\square$

4. STATISTICALLY p_τ -CONTINUOUS OPERATOR

In this section, we introduce continuous and bounded operators with respect to the statistical p_τ -convergence. Recall that an operator T between LNLS (X, p, E) and (Y, q, F) is called

- p -continuous whenever $x_n \xrightarrow{p} 0$ in X implies $T(x_n) \xrightarrow{p} 0$ in Y ,
- p -bounded if it maps p -bounded sets in X to p -bounded sets in Y .

Motivated by these definitions, we introduce the following notions.

Definition 7. Let $T : (X, p, E_\tau) \rightarrow (Y, q, F_\tau)$ be an operator between LNLSs. Then, T is said to be

- (1) statistically p_τ -continuous if $x_n \xrightarrow{\text{st-}p_\tau} x$ in X implies $T(x_n) \xrightarrow{\text{st-}p_\tau} T(x)$ in Y ,
- (2) statistically p_τ -bounded if it sends statistically p_τ -bounded sequences to statistically p_τ -bounded sequences.

It is clear that the collection of all statistically p_τ -continuous operators between LNLSs is a vector space.

Example 9. Consider the LNLS $(c_{00}, |\cdot|, \ell_\infty)$, where the solid topology on ℓ_∞ is generated by the supremum norm $\|\cdot\|_\infty$. Define an operator $S : (c_{00}, |\cdot|, \ell_\infty) \rightarrow (c_{00}, |\cdot|, c_{00})$ denoted by

$$S(x) = \left(\sum_{n=1}^{\infty} |x_n| \right) x$$

for all $x := (x_n) \in c_{00}$. Consider a sequence $x = (x_n^k) = (x_1^k, x_2^k, \dots)$ in c_{00} denoted by $x_n^k = (x_n^1, x_n^2, \dots) = (1, \dots, 1, 0, 0, \dots)$ in c_{00} . Thus, it is order bounded by the element $(1) = (1, 1, \dots)$ in ℓ_∞ , and so, it is topological bounded in ℓ_∞ by applying [2, Theorem 2.19]. Hence, it is statistically p_τ -bounded in c_{00} . Then, it follows that

$$S(x) = (1x_1^k, 2x_2^k, 3x_3^k, \dots, nx_n^k, \dots)$$

is not bounded in c_{00} , and so, S is not a statistically p_τ -bounded operator.

Theorem 6. *If an operator $(E, |\cdot|, E_\tau) \rightarrow (F, |\cdot|, F_\tau)$ between LNLSs is uniformly continuous, then it is statistically p_τ -continuous.*

Proof. Assume that $T: E \rightarrow F$ is a uniformly continuous operator. Then, we show that $T: (E, |\cdot|, E_\tau) \rightarrow (F, |\cdot|, F_\tau)$ is statistically p_τ -continuous. Let $x_n \xrightarrow{\text{st-}p_\tau} x$ in E . Take any fixed zero neighborhood V in F . Now, by applying uniform continuity of T , we have some zero neighborhood U in E so that $(a - b) \in U$ implies $T(a - b) \in V$. On the other hand, by $x_n \xrightarrow{\text{st-}p_\tau} x$, we have $\delta(K) = 1$, where $K = \{n \in \mathbb{N} : (x_n - x) \in U\}$. Also, we have $T(x_n - x) \in V$ for every $n \in K$. Then, it follows that $K \subseteq M := \{n \in \mathbb{N} : T(x_n - x) \in V\}$, and so, we obtain $\delta(M) = 1$. Therefore, $T(x_n) \xrightarrow{\text{st-}p_\tau} T(x)$. $\square$

It is well known that every order continuous operator is order bounded; see [3, Lemma 1.54]. But, a statistically p_τ -continuity as an operator between two Riesz spaces need not be order bounded. To see this, we consider Lozanovsky's example; see [3, Exercise 10. p.289].

Example 10. Take an operator $T: L_1[0, 1] \rightarrow c_0$ defined by

$$T(f) = \left(\int_0^1 f(x) \sin x \, dx, \int_0^1 f(x) \sin 2x \, dx, \dots \right).$$

Then, T is not order bounded. But, it can be shown that T is norm continuous. Therefore, by applying Theorem 6, the operator $T: (L_1[0, 1], |\cdot|, L_1[0, 1]) \rightarrow (c_0, |\cdot|, c_0)$ is statistically p_τ -continuous because the continuity and the uniform continuity are equivalent for operators between normed spaces.

Proposition 4. *Every p -bounded operator from an LNLS to another LNLS with an order bounded zero neighborhood is statistically p_τ -bounded.*

Proof. Suppose that $T: X \rightarrow Y$ is a p -bounded operator. We show that

$$T: (X, p, E_\tau) \rightarrow (Y, q, F_\tau)$$

is statistically p_τ -bounded. Let (x_n) be a statistically p_τ -bounded sequence in X . Fix an arbitrary zero neighborhood U in E . Then, there exists $\lambda > 0$ such that $\delta(K) = 1$, where $K := \{n \in \mathbb{N} : p(x_n) \in \lambda U\}$. Then, it follows from [15, Theorem 2.2] that the set $\{p(x_n) : n \in K\}$ is order bounded in E , and so, $\{x_n : n \in K\}$ is p -bounded in X . By applying p -boundedness of T , $\{T(x_n) : n \in K\}$ is p -bounded in Y , i.e., $\{q(T(x_n)) : n \in K\}$ is order bounded in F . Thus, it follows from [2, Theorem 2.19] that $\{q(T(x_n)) : n \in K\}$ is topologically bounded in F , and so, $T(x_n)$ is statistically p_τ bounded because of $\delta(K) = 1$. Therefore, we get the desired result. $\square$

Theorem 7. *A statistically p_τ -continuous operator between LNLSs is statistically p_τ -bounded.*

Proof. Let $T: (X, p, E_\tau) \rightarrow (Y, q, F_\tau)$ be a statistically p_τ -continuous operator and (x_n) be a statistically p_τ -bounded sequence in X . Take a zero neighborhood U in

E . Then, there exists $\lambda > 0$ such that $\delta(K) = 1$ for the set $K = \{n \in \mathbb{N} : p(x_n) \in \lambda U\}$. Let's consider an index set $M := \mathbb{N} \times K$ with the lexicographic order. That is, $(m, k') \leq (n, k)$ iff $m < n$ or else $m = n$ and $k' \leq k$. Take the sequence $x_{(n,k)} = \frac{1}{\lambda n} x_k$. Thus, we have $p(x_{(n,k)}) = \frac{1}{\lambda n} p(x_k)$, and so, we get $x_{(n,k)} \xrightarrow{\text{st-}p_\tau} 0$ because U is arbitrary and $\frac{1}{\lambda n} p(x_k) \leq \frac{1}{\lambda} p(x_k)$ for every n . By using the statistically p_τ -continuity of T , we obtain $T(x_{(n,k)}) \xrightarrow{\text{st-}p_\tau} 0$. It follows from Proposition 2 that $T(x_{(n,k)})$ is statistically p_τ -bounded sequence in Y . Now, take an arbitrary zero neighborhood W in F . Then, for a fixed $n_0 \in \mathbb{N}$, there is $\alpha > 0$ such that $\delta(J) = 0$, where

$$J = \left\{ k \in K : q(T(x_{n_0,k})) = \frac{1}{\lambda n_0} q(T(x_k)) \notin \alpha U \right\} = \{k \in K : q(T(x_k)) \in \alpha \lambda n_0 \notin U\}.$$

Therefore, we get $T(x_n)$ is statistically p_τ -bounded. $\square$

The lattice operations in an $LNLS$ are statistically p_τ -continuous in the following sense.

Theorem 8. *If $x_n \xrightarrow{\text{st-}p_\tau} x$ and $y_n \xrightarrow{\text{st-}p_\tau} y$ in an $LNLS$, then $x_n \vee y_n \xrightarrow{\text{st-}p_\tau} x \vee y$.*

Proof. Suppose that $x_n \xrightarrow{\text{st-}p_\tau} x$ and $y_n \xrightarrow{\text{st-}p_\tau} y$ in an $LNLS$ (X, p, E_τ) and U be an arbitrary zero neighborhood in E with another zero neighborhood V such that $V + V \subseteq U$. Then, we have $\delta(K) = \delta(M) = 1$ for the sets

$$K = \{n \in \mathbb{N} : p(x_n - x) \in V\} \quad \text{and} \quad M = \{n \in \mathbb{N} : p(y_n - y) \in V\}.$$

Take an index set $J := K \cap M$. Then, it follows from [19, Theorem 12.4] that we observe

$$p(x_n \vee y_n - x \vee y) \leq p(x_n - x) + p(y_n - y) \in V + V \subseteq U$$

for all $n \in J$. Then, by using the solidness of U , we have $p(x_n \vee y_n - x \vee y) \in U$ for all $n \in J$. Thus, we get $\delta(\{n \in \mathbb{N} : p(x_n \vee y_n - x \vee y) \in U\}) = 1$. As a result, we have $x_n \vee y_n \xrightarrow{\text{st-}p_\tau} x \vee y$. $\square$

Corollary 1. *Let $x_n \xrightarrow{\text{st-}p_\tau} x$ and $y_n \xrightarrow{\text{st-}p_\tau} y$ hold in an $LNLS$. Then, we have the following statements:*

- (i) $x_n^+ \xrightarrow{\text{st-}p_\tau} x^+$;
- (ii) $x_n^- \xrightarrow{\text{st-}p_\tau} x^-$;
- (iii) $|x_n| \xrightarrow{\text{st-}p_\tau} |x|$;
- (iv) $x_n \wedge y_n \xrightarrow{\text{st-}p_\tau} x \wedge y$.

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RIESZ POTENTIALS IN THE LOCAL VARIABLE MORREY-LORENTZ SPACES AND SOME APPLICATIONS

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Abstract. In this paper, we prove the boundedness of the Riesz potential I_α in local variable Morrey-Lorentz spaces. Also we apply our results to particular operators such as fractional maximal operator, fractional Marcinkiewicz operator and fractional powers of some analytic semigroups in these spaces.

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1. INTRODUCTION

The Lorentz-Morrey space $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ was first defined in [22] and also considered in [15, 24, 31]. Later, the local Morrey-Lorentz spaces $\mathcal{M}_{p,q;\lambda}^{loc}(\mathbb{R}^n)$ are introduced and the basic properties of these spaces are given in [1]. These spaces are a very natural generalization of the Lorentz spaces such that $\mathcal{M}_{p,q;0}^{loc}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. Recently, in [2, 13] and [14] the authors have studied the boundedness of the Hilbert transform, the Hardy-Littlewood maximal operator M and the Calderón-Zygmund operators T , and the Riesz potential I_α on the local Morrey-Lorentz spaces $\mathcal{M}_{p,q;\lambda}^{loc}$ by using related rearrangement inequalities, respectively.

The study of function spaces with variable exponent has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [6, 8, 25, 26, 33]). Various results on non-weighted and weighted boundedness in variable exponent Lebesgue spaces have been proved for maximal, singular and fractional type operators, we refer to surveying papers [9] and [27]. In [18], authors define the variable local Morrey-Lorentz spaces and proved the boundedness of maximal operator M and Calderon-Zygmund operators T in these spaces, also they apply their results to some operators of harmonic analysis such as Bochner-Riesz operator, Marcinkiewicz operator and fractional powers of

some analytic semigroups. In [32], the authors give the definition of central Lorentz-Morrey space of variable exponent by the symmetric decreasing rearrangement. They prove the boundedness of maximal operator in these spaces and establish Sobolev's inequality for Riesz potentials.

Local variable Morrey-Lorentz spaces generalize variable exponent Lorentz spaces such that $\mathcal{M}_{p(\cdot),q(\cdot);0}^{loc} = L_{p(\cdot),q(\cdot)}$, when $\lambda = 0$. In [11] variable exponent Lorentz spaces $L_{p(\cdot),q(\cdot)}$ are introduced and the boundedness of the singular integral and fractional type operators and corresponding ergodic operators are proved in these spaces. The inclusion theorems for variable Lorentz spaces were proved in [19]. We should point out that, due to their own fine structures, Lorentz spaces appear frequently in the study on various critical or endpoint analysis problems from many different research fields and there exist enormous literatures on this subject.

In this paper, we prove the boundedness of Riesz potential I_α from the local variable Morrey-Lorentz spaces $\mathcal{M}_{p(\cdot),r(\cdot),\lambda}^{loc}(\mathbb{R}^n)$ to local variable Morrey-Lorentz spaces $\mathcal{M}_{q(\cdot),s(\cdot),\lambda}^{loc}(\mathbb{R}^n)$. We also give some applications of our main result.

The organization of this article is as follows. In Section 2, we give some notation and definitions. In Section 3, we prove the boundedness of the Riesz potential I_α from the local variable Morrey-Lorentz spaces $\mathcal{M}_{p(\cdot),r(\cdot),\lambda}^{loc}(\mathbb{R}^n)$ to local variable Morrey-Lorentz spaces $\mathcal{M}_{q(\cdot),s(\cdot),\lambda}^{loc}(\mathbb{R}^n)$. In Section 4, as applications we get the boundedness of fractional maximal operator M_α , fractional Marcinkiewicz operator $\mu_{\Omega,\alpha}$ and fractional powers of some analytic semi groups $L^{-\alpha/2}$ from $\mathcal{M}_{p(\cdot),r(\cdot),\lambda}^{loc}(\mathbb{R}^n)$ to $\mathcal{M}_{q(\cdot),s(\cdot),\lambda}^{loc}(\mathbb{R}^n)$.

Throughout the paper we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence.

2. PRELIMINARIES

The present paper deals with the boundedness of the Riesz potential I_α defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \quad f \in L_1^{loc}(\mathbb{R}^n),$$

in the local Morrey-Lorentz spaces $\mathcal{M}_{p,q;\lambda}^{loc}(\mathbb{R}^n)$.

Further we apply this result to particular operators such as a fractional maximal operator, fractional Marcinkiewicz operator and fractional powers of some analytic semigroups.

For each measurable function f on $(0, \infty)$ and each $t > 0$, the following operator

$$(S_\alpha f)(t) = t^{\frac{\alpha}{n}-1} \int_0^t f(s) ds + \int_t^\infty s^{\frac{\alpha}{n}-1} f(s) ds$$

was defined by A. P. Calderón [5]. The importance of S_α is based on the fact that it dominates the Riesz potential I_α .

Theorem 1 ([23, 28]). *If the condition*

$$(S_\alpha f^*)(1) = \int_0^1 f^*(s)ds + \int_1^\infty s^{\frac{\alpha}{n}-1} f^*(s)ds < \infty \quad (2.1)$$

holds for $f \in L_1^{loc}(\mathbb{R}^n)$, then the Riesz potential $(I_\alpha f)(x)$, $x \in \mathbb{R}^n$, exists almost everywhere. Furthermore, the inequality

$$(I_\alpha f)^*(t) \leq CS_\alpha(f^*)(t), \quad 0 < t < \infty, \quad (2.2)$$

is valid, where f^ denotes the non-increasing rearrangement of f defined by*

$$f^*(t) = \inf \{ \lambda > 0 : |\{y \in \mathbb{R}^n : |f(y)| > \lambda\}| \leq t \} \quad \text{for all } t \in (0, \infty)$$

and C is a constant independent of f and t .

Let $p(t)$ be a measurable function on $(0, \infty)$. We mainly suppose that

$$1 < p_- \leq p(t) \leq p_+ < \infty, \quad (2.3)$$

where

$$p_- := \inf_{0 < t \leq \infty} p(t), \quad p_+ := \sup_{0 < t \leq \infty} p(t).$$

We denote by $p'(\cdot) = \frac{p(t)}{p(t)-1}$. We will use the following decay conditions:

$$|p(t) - p(0)| \leq \frac{A_0}{|\ln t|}, \quad 0 < t \leq \frac{1}{2} \quad (2.4)$$

$$|p(t) - p(\infty)| \leq \frac{A_\infty}{\ln t}, \quad t \geq 2, \quad (2.5)$$

where $A_0, A_\infty > 0$ do not depend on t .

By $p \in \mathcal{P}_{0,\infty}(0, \infty)$ we denote the set of bounded measurable functions (not necessarily with values in $[1, \infty)$), which satisfy the decay conditions (2.4) and (2.5). Also, by $L_{p(\cdot)}(0, \infty)$ we denote the variable exponent Lebesgue space of measurable functions φ on $(0, \infty)$ such that

$$\mathcal{J}_{p(\cdot)}(\varphi) = \int_0^\infty |\varphi(s)|^{p(s)} ds < \infty.$$

This is a Banach function space with respect to the norm (see e.g. [10])

$$\|\varphi\|_{L_{p(\cdot)}} = \inf \left\{ \lambda > 0 : \mathcal{J}_{p(\cdot)} \left(\frac{\varphi}{\lambda} \right) \leq 1 \right\}.$$

Definition 1 ([21]). Let q satisfy the condition (2.3). We denote by $LM_{q(\cdot), \lambda} \equiv LM_{q(\cdot), \lambda}(0, \infty)$ the variable exponent local Morrey space with finite norm

$$\begin{aligned} \|\varphi\|_{LM_{q(\cdot), \lambda}} &= \sup_{t>0} t^{-\frac{\lambda}{q_*(t)}} \|\varphi\|_{L_{q(\cdot)}(0, t)} \\ &= \sup_{t>0} \inf \left\{ \eta > 0 : \int_0^t \left| \frac{\varphi(s)}{\eta t^{\frac{\lambda}{q_*(t)}}} \right|^{q(s)} ds \leq 1 \right\}, \end{aligned}$$

where $q_*(t) = q(0), 0 < t < 1$ and $q_*(t) = q(\infty), t \geq 1$.

Definition 2 ([11]). Let p, q satisfy the condition (2.3). We denote by $L_{p(\cdot), q(\cdot)}(\mathbb{R}^n)$ variable exponent Lorentz space, the space of functions f on $\mathbb{R}^n$ such that $t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^*(t) \in L_{q(\cdot)}(0, \infty)$, i.e.

$$\mathcal{J}_{p(\cdot), q(\cdot)}(f) = \int_0^\infty t^{\frac{q(t)}{p(t)} - 1} (f^*(t))^{q(t)} dt < \infty$$

and we denote

$$\|f\|_{L_{p(\cdot), q(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \sigma > 0 : \mathcal{J}_{p(\cdot), q(\cdot)} \left(\frac{f}{\sigma} \right) \leq 1 \right\} = \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^*(t) \right\|_{L_{q(\cdot)}(0, \infty)},$$

where f^* denotes the non-increasing rearrangement of f such that

$$f^*(t) = \inf \{ \lambda > 0 : |\{y \in \mathbb{R}^n : |f(y)| > \lambda\}| \leq t \}, \quad \forall t \in (0, \infty)$$

and

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$$

(see [3]). More information about variable exponent Lorentz spaces can be found in [11, 16].

Now we give the definition of local variable Morrey-Lorentz spaces, which is defined in [18].

Definition 3. Let p, q satisfy the condition (2.3) and $0 \leq \lambda < 1$. We denote by $\mathcal{M}_{p(\cdot), q(\cdot), \lambda}^{loc} \equiv \mathcal{M}_{p(\cdot), q(\cdot), \lambda}^{loc}(\mathbb{R}^n)$ the local variable Morrey-Lorentz space, the space of all measurable functions with finite quasinorm

$$\|f\|_{\mathcal{M}_{p(\cdot), q(\cdot), \lambda}^{loc}} := \sup_{t > 0} t^{-\frac{\lambda}{q_*(t)}} \left\| \tau^{\frac{1}{p(\tau)} - \frac{1}{q(\tau)}} f^*(\tau) \right\|_{L_{q(\cdot)}(0, t)}.$$

These spaces generalize variable exponent Lorentz spaces such that $\mathcal{M}_{p(\cdot), q(\cdot), 0}^{loc} = L_{p(\cdot), q(\cdot)}$, when $\lambda = 0$ (see [11]). Also, if $\lambda = 0$ and $q(\cdot) = p(\cdot)$ then $\mathcal{M}_{p(\cdot), p(\cdot), 0}^{loc} = L_{p(\cdot)}$ are variable exponent Lebesgue spaces (see [17]).

3. THE BOUNDEDNESS OF RIESZ POTENTIAL I_α IN THE LOCAL VARIABLE MORREY-LORENTZ SPACES

In this section, we prove the boundedness of Riesz potential I_α in the local variable Morrey-Lorentz spaces. We need the following two definitions about Hardy operators which are used in the proof of our main theorems. These operators are very important in analysis and have been widely studied.

Definition 4 ([11]). Let $\beta(t)$ and $\gamma(t)$ are measurable functions on $(0, \infty)$. The weighted Hardy operators $H_{\gamma(\cdot)}^{\beta(\cdot)}$ and $\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)}$ with power weight acting on φ are defined by

$$H_{\gamma(\cdot)}^{\beta(\cdot)} \varphi(t) = t^{\beta(t)+\gamma(t)-1} \int_0^t \frac{\varphi(y)}{y^{\gamma(y)}} dy$$

and

$$\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)} \varphi(t) = t^{\beta(t)+\gamma(t)} \int_t^\infty \frac{\varphi(y)}{y^{\gamma(y)+1}} dy.$$

The following lemma provides some minimal assumptions on the function $\tau^{\frac{\lambda}{q_*(\tau)}}$ under which the so-defined spaces contain "nice" functions.

Lemma 1 ([21]). Let r satisfy the condition (2.3), $r \in \mathcal{P}_0(0, \infty)$ and $0 \leq \lambda < 1$. Then the assumption

$$\sup_{\tau > 0} \frac{[\min\{1, \tau\}]^{\frac{1}{r'(0)}}}{\tau^{\frac{\lambda}{r_*(\tau)}}} < \infty \quad (3.1)$$

is sufficient for bounded functions f with compact support to belong to the local variable Morrey-Lorentz spaces $\mathcal{M}_{p(\cdot), r(\cdot), \lambda}^{loc}(\mathbb{R}^n)$.

Lemma 2 ([21]). Let r satisfy the condition (2.3), $r \in \mathcal{P}_0(0, \infty)$, $0 \leq \lambda < 1$, $\lim_{t \rightarrow 0} t^{\gamma(t)}$ exists and finite and the condition (3.1) is satisfied. Suppose that the following conditions hold.

- (i) $t^{\gamma(t)-a}$ and $t^{\frac{\lambda}{r_*(t)}-\gamma(t)-a}$ are almost decreasing for some $a \in \mathbb{R}$, in the case of operator $H_{\gamma(\cdot)}^{\beta(\cdot)}$.
- (ii) $t^{-\gamma(t)+b}$ and $t^{\frac{\lambda}{r_*(t)}-\gamma(t)+b}$ are almost increasing for some $b \in \mathbb{R}$, in the case of operator $\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)}$.

Then the conditions

$$\gamma(t) < \frac{\lambda}{r_*(t)} + \frac{1}{r'(t)}, \quad \gamma(t) > \frac{\lambda}{r_*(t)} - \frac{1}{r(\infty)}$$

are sufficient for the Hardy operators $H_{\gamma(\cdot)}^{\beta(\cdot)}$ and $\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)}$, respectively, to be defined on the space $LM_{r(\cdot), \lambda}(\mathbb{R}^n)$.

Lemma 3 ([21]). Let r satisfy the condition (2.3), $r \in \mathcal{P}_0(0, \infty)$, $0 \leq \lambda < 1$. Suppose also that the conditions (3.1) and of Lemma 2 are satisfied. Then the operators $H_{\gamma(\cdot)}^{\beta(\cdot)}$ and $\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)}$ are bounded from the space $LM_{r(\cdot), \lambda}(0, \infty)$ to the space $LM_{s(\cdot), \lambda}(0, \infty)$ if $\gamma(t) < \frac{\lambda}{r_*(t)} + \frac{1}{r'(t)}$, $\gamma(t) > \frac{\lambda}{r_*(t)} - \frac{1}{r(t)}$, respectively.

The following theorem is the main result of our paper in which we give the boundedness of Riesz potential in the local variable Morrey-Lorentz spaces.

Theorem 2. Let p, q, r, s satisfy the condition (2.3), $p, q, r, s \in \mathcal{P}_{0,\infty}(0, \infty)$, $0 \leq \lambda < 1$ and $f \in \mathcal{M}_{p(\cdot), r(\cdot), \lambda}^{loc}(\mathbb{R}^n)$. Suppose that the conditions (3.1) and of Lemma 2 are satisfied. If $\frac{r(t)}{r(t)+\lambda} < p(t) < \left(\frac{\lambda}{r(t)} + \frac{\alpha}{n}\right)^{-1}$, $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda \left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{n}$, then the Riesz potential I_α is bounded from the local variable Morrey-Lorentz spaces $\mathcal{M}_{p(\cdot), r(\cdot), \lambda}^{loc}(\mathbb{R}^n)$ to $\mathcal{M}_{q(\cdot), s(\cdot), \lambda}^{loc}(\mathbb{R}^n)$.

Proof. From the definition of local variable Morrey-Lorentz spaces and the inequality (2.2) we get

$$\begin{aligned} \|I_\alpha f\|_{\mathcal{M}_{q(\cdot), s(\cdot), \lambda}^{loc}} &= \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \|\tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)}} (I_\alpha f)^*(\tau)\|_{L_{s(\cdot)}(0, t)} \\ &\leq C \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \left\| \tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)}} \left(\tau^{\frac{\alpha}{n}-1} \int_0^\tau f^*(y) dy + \int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y) dy \right) \right\|_{L_{s(\cdot)}(0, t)} \\ &\leq C \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \left\| \tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)} + \frac{\alpha}{n}-1} \int_0^\tau f^*(y) dy \right\|_{L_{s(\cdot)}(0, t)} \\ &\quad + C \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \left\| \tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)}} \int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y) dy \right\|_{L_{s(\cdot)}(0, t)} \\ &= I_1 + I_2. \end{aligned}$$

Let us estimate I_1 :

$$I_1 = C \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \left\| \tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)} + \frac{\alpha}{n}-1} \int_0^\tau f^*(y) dy \right\|_{L_{s(\cdot)}(0, t)} = C \|H_{\gamma(\cdot)}^{\beta(\cdot)} g\|_{L_{s(\cdot), \lambda}(0, \infty)}.$$

We take $\gamma(t) = \frac{1}{p(t)} - \frac{1}{r(t)}$ and consider the Hardy operator $H_{\gamma(\cdot)}^{\beta(\cdot)}$ and $g(t) = t^{\frac{1}{p(t)} - \frac{1}{r(t)}} f^*(t)$. Therefore we get $\beta(t) = \frac{1}{q(t)} - \frac{1}{s(t)} + \frac{1}{r(t)} - \frac{1}{p(t)} + \frac{\alpha}{n}$. By Lemma 3, we have $\beta(t) = (1 - \lambda) \left(\frac{1}{r(t)} - \frac{1}{s(t)}\right)$, then we obtain $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda \left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{n}$.

Hence the operator $H_{\gamma(\cdot)}^{\beta(\cdot)}$ is bounded from the Morrey space $L_{r(\cdot), \lambda}(0, \infty)$ to $L_{s(\cdot), \lambda}(0, \infty)$ under the condition $\gamma(t) = \frac{1}{p(t)} - \frac{1}{r(t)} < \frac{1}{r'(t)} + \frac{\lambda}{r_*(t)}$. Then we get

$$\begin{aligned} I_1 &\leq C \|H_{\gamma(\cdot)}^{\beta(\cdot)} g\|_{L_{s(\cdot), \lambda}(0, \infty)} \leq C \|g\|_{L_{r(\cdot), \lambda}(0, \infty)} \\ &= C \sup_{t>0} t^{-\frac{\lambda}{r_*(t)}} \|\tau^{\frac{1}{p(\tau)} - \frac{1}{r(\tau)}} f^*(\tau)\|_{L_{r(\cdot)}(0, t)} \\ &= C \|f\|_{\mathcal{M}_{p(\cdot), r(\cdot), \lambda}^{loc}}. \end{aligned} \tag{3.2}$$

Now we consider I_2 :

$$I_2 = C \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \left\| \tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)}} \int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y) dy \right\|_{L_{s(\cdot)}(0, t)} = C \|\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)} g\|_{L_{s(\cdot), \lambda}(0, \infty)}.$$

We take $\gamma(t) = \frac{1}{p(t)} - \frac{1}{r(t)} - \frac{\alpha}{n}$ in the Hardy operator $\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)}$ and $g(t) = t^{\frac{1}{p} - \frac{1}{r}} f^*(t)$. Therefore we get $\beta(t) = \frac{1}{q(t)} - \frac{1}{s(t)} + \frac{1}{r(t)} - \frac{1}{p(t)} + \frac{\alpha}{n}$. By Lemma 3, we have $\beta(t) = (1 - \lambda) \left(\frac{1}{r(t)} - \frac{1}{s(t)} \right)$, then we obtain $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda \left(\frac{1}{r(t)} - \frac{1}{s(t)} \right) + \frac{\alpha}{n}$.

Hence the operator $\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)}$ is bounded from the Morrey space $L_{r,\lambda}(0, \infty)$ to $L_{s,\lambda}(0, \infty)$ under the condition $\frac{\lambda}{r_*(t)} - \frac{1}{r(t)} < \gamma(t) = \frac{1}{p(t)} - \frac{1}{r(t)} - \frac{\alpha}{n}$. Then we get

$$\begin{aligned} \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \|\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)}\|_{L_{s(\cdot)}(0,t)} &\leq C \|g\|_{L_{r(\cdot),\lambda}(0,\infty)} = C \sup_{t>0} t^{-\frac{\lambda}{r_*(t)}} \|\tau^{\frac{1}{p(\tau)} - \frac{1}{r(\tau)}} f^*(\tau)\|_{L_{r(\cdot)}(0,t)} \\ &= C \|f\|_{M_{p(\cdot),r(\cdot);\lambda}^{loc}}. \end{aligned} \quad (3.3)$$

From inequalities (3.2) and (3.3) we obtain the boundedness of the operator I_α from $M_{p(\cdot),r(\cdot);\lambda}^{loc}$ to $M_{q(\cdot),s(\cdot);\lambda}^{loc}$. $\square$

Remark 1. In the case $\lambda = 0$, from Theorem 2 we get the boundedness of the operator I_α in the variable exponent Lorentz spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ which is proved in [11].

4. SOME APPLICATIONS

Theorem 2 can be applied to various operators which are estimated from above by the Riesz potentials. In this section, we apply the theorem to the fractional maximal operator, fractional Marcinkiewicz operator and the fractional powers of some analytic semigroups.

4.1. Fractional maximal operator

For $0 \leq \alpha < n$, we define the fractional maximal operator

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}-1} \int_{B(x,t)} |f(y)| dy,$$

where $B(x,t)$ is the open ball centered at x of radius t for $x \in \mathbb{R}^n$ and $|B(x,t)|$ is a Lebesgue measure of $B(x,t)$ such that $|B(x,t)| = \omega_n t^n$ in which ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. The fractional maximal operator M_α is closely related to a Riesz potential operator such that

$$M_\alpha f(x) \leq \omega_n^{\frac{\alpha}{n}-1} (I_\alpha |f|)(x). \quad (4.1)$$

Corollary 1. Let p, q, r, s satisfy the condition (2.3), $p, q, r, s \in \mathcal{P}_{0,\infty}(0, \infty)$, $0 \leq \lambda < 1$, and $f \in \mathcal{M}_{p(\cdot),r(\cdot);\lambda}^{loc}(\mathbb{R}^n)$. Suppose that the conditions (3.1) and of Lemma 2 are satisfied. If $\frac{r(t)}{r(t)+\lambda} < p(t) < \left(\frac{\lambda}{r(t)} + \frac{\alpha}{n} \right)^{-1}$, $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda \left(\frac{1}{r(t)} - \frac{1}{s(t)} \right) + \frac{\alpha}{n}$, then the fractional maximal operator M_α is bounded from the local variable Morrey-Lorentz spaces $\mathcal{M}_{p(\cdot),r(\cdot);\lambda}^{loc}(\mathbb{R}^n)$ to $\mathcal{M}_{q(\cdot),s(\cdot);\lambda}^{loc}(\mathbb{R}^n)$.

Remark 2. In the case $\lambda = 0$, from Theorem 2 we get the boundedness of the operator M_α in the variable exponent Lorentz spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ which is proved in [11].

4.2. Fractional Marcinkiewicz operator

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

- (a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, i.e.,

$$\Omega(tx) = \Omega(x) \text{ for any } t > 0, x \in \mathbb{R}^n \setminus \{0\}.$$

- (b) Ω has mean zero on S^{n-1} , i.e.,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0.$$

- (c) $\Omega \in \text{Lip}_\gamma(S^{n-1})$, $0 < \gamma \leq 1$, that is, there exists a constant $C > 0$ such that

$$|\Omega(x') - \Omega(y')| \leq C|x' - y'|^\gamma \text{ for any } x', y' \in S^{n-1}.$$

In 1958, Stein [29] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,\alpha,t}(f)(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y) dy.$$

The continuity of the Marcinkiewicz operator μ_Ω was extensively studied in [7, 12, 20, 30]. Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\}$. Then it is clear that $\mu_\Omega(f)(x) = \|F_{\Omega,t}(f)(x)\|$.

By Minkowski inequality and the conditions on Ω , we get

$$\begin{aligned} \mu_{\Omega,\alpha}(f)(x) &\leq \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \\ &\leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy = I_\alpha(|f|)(x). \end{aligned}$$

Then we have the following corollary.

Corollary 2. Let p, q, r, s satisfy the condition (2.3), $p, q, r, s \in \mathcal{P}_{0,\infty}(0, \infty)$ and $f \in \mathcal{M}_{p(\cdot),r(\cdot),\lambda}^{loc}(\mathbb{R}^n)$. Suppose that the conditions (3.1) and of Lemma 2 are satisfied. If $\frac{r(t)}{r(t)+\lambda} < p(t) < \left(\frac{\lambda}{r(t)} + \frac{\alpha}{n}\right)^{-1}$, $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda \left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{n}$, then the fractional Marcinkiewicz operator $\mu_{\Omega,\alpha}$ is bounded from the local variable Morrey-Lorentz spaces $\mathcal{M}_{p(\cdot),r(\cdot),\lambda}^{loc}(\mathbb{R}^n)$ to $\mathcal{M}_{q(\cdot),s(\cdot),\lambda}^{loc}(\mathbb{R}^n)$.

Remark 3. In the case $\lambda = 0$, from Theorem 2 we get the boundedness of the operator $\mu_{\Omega,\alpha}$ in the variable exponent Lorentz spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

4.3. Fractional powers of some analytic semigroups

Suppose that L is a linear operator on L_2 which generates an analytic semigroup e^{-tL} with the kernel $p_t(x, y)$ satisfying a Gaussian upper bound, i.e.,

$$|p_t(x, y)| \leq \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \quad (4.2)$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$, where $c_1, c_2 > 0$ are independent of x, y and t .

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I_α . Property (4.2) is satisfied for large classes of differential operators. In [4], other examples of operators which are estimates from above by the Riesz potentials are given. Since the semigroup e^{-tL} has the kernel $p_t(x, y)$ which satisfies condition (4.2), it follows that

$$|L^{-\alpha/2} f(x)| \leq CI_\alpha(|f|)(x).$$

Hence we get the following corollary.

Corollary 3. *Let p, q, r, s satisfy the condition (2.3), $p, q, r, s \in \mathcal{P}_{0,\infty}(0, \infty)$, $0 \leq \lambda < 1$ and $f \in \mathcal{M}_{p(\cdot), r(\cdot), \lambda}^{loc}(\mathbb{R}^n)$. Suppose that the conditions (3.1) and of Lemma 2 are satisfied. If $\frac{r(t)}{r(t)+\lambda} < p(t) < \left(\frac{\lambda}{r(t)} + \frac{\alpha}{n}\right)^{-1}$, $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda \left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{n}$, then the operator $L^{-\alpha/2}$ is bounded from the local variable Morrey-Lorentz spaces $\mathcal{M}_{p(\cdot), r(\cdot), \lambda}^{loc}(\mathbb{R}^n)$ to $\mathcal{M}_{q(\cdot), s(\cdot), \lambda}^{loc}(\mathbb{R}^n)$.*

Remark 4. In the case $\lambda = 0$, from Theorem 2 we get the boundedness of the operator $L^{-\alpha/2}$ in the variable exponent Lorentz spaces $L_{p(\cdot), q(\cdot)}(\mathbb{R}^n)$.

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CRITICAL POINT APPROACHES TO NONLINEAR SQUARE ROOT LAPLACIAN EQUATIONS

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Abstract. This work is devoted to the study of multiplicity results of solutions for a class of nonlinear equations involving the square root of the Laplacian. Indeed, we will use variational methods for smooth functionals, defined on reflexive Banach spaces, in order to achieve the existence of at least three solutions for the equations. Moreover, assuming that the nonlinear terms are nonnegative, we will prove that the solutions are nonnegative. Finally, by presenting an example, we will ensure the applicability of our results.

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1. INTRODUCTION

In the study ahead, we discuss the existence of multiple nontrivial weak solutions for the problem involving the square root of the Laplacian

$$\begin{cases} \mathcal{A}_{1/2}y = \varepsilon\gamma(\tau)g(y) & \text{in } \mathcal{D}, \\ y = 0 & \text{on } \partial\mathcal{D}, \end{cases} \quad (1)$$

where $\mathcal{D}$ is an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary $\partial\mathcal{D}$, $\varepsilon > 0$ is a real parameter, $\gamma: \mathcal{D} \rightarrow \mathbb{R}$ is an $L^\infty(\mathcal{D})$ -function with $\text{essinf}_{t \in \Omega} \gamma(t) > 0$, and $\mathcal{A}_{1/2}$ is a fractional operator which will be introduced in Section 2 (see [4–6]).

The fractions of the Laplacian, as in the square root of the Laplacian $\mathcal{A}_{1/2}$, are the infinitesimal generators of Lévy stable diffusion processes. They emerge in irregular diffusions in plasmas, flames propagation, population dynamics, geophysical fluid dynamics, and American options in finance.

Recently, elliptic equations involving fractional powers of the Laplacian have been under investigation from a physical point of view. Nonlocal operators play a major role in describing a set of phenomena. The recent paper of Cabré and Tan [4] can

be cited here where the authors discuss the existence, nonexistence, and regularity of positive solutions of the problem (1) with power-type nonlinearities, along with a conjecture of Gidas–Spruck type and symmetry results of Gidas–Ni–Nirenberg type. Some earlier discussions can be seen in the papers [1–3, 9, 10, 12–14, 18, 19]. For instance, Molica Bisci et al. in [12] reviewed (1) concerning the existence of solutions. They established the existence of at least three weak solutions in L^∞ -bounded function space for certain values of ε requiring the nonlinear term g to be continuous, having a suitable growth, by using a variant of Caffarelli–Silvestre’s extension method. Moreover, Ambrosio et al. in [1], explained the existence and nonexistence of weak solutions for (1). They established the existence of at least two nontrivial L^1 -bounded weak solutions for large values of ε , requiring the nonlinear term g to be continuous, superlinear at zero, and sublinear at infinity by using a suitable variant of the Caffarelli–Silvestre extension method.

The present article concentrates on this issue because it is clear that in (1), there are singularities in the term $\mathcal{A}_{1/2}(y)$, which leads to problems in the proofs. This article concerns existence results for (1). Our main result gives conditions ensuring the existence of at least two and infinitely many weak solutions for (1).

Throughout this paper, we denote by $\mathcal{G}$ the class of all continuous functions $g : \mathbb{R} \rightarrow \mathbb{R}$ that are superlinear at zero, i.e., $\lim_{t \rightarrow 0} \frac{g(t)}{t} = 0$ and sublinear at infinity, i.e., $\lim_{|t| \rightarrow \infty} \frac{g(t)}{t} = 0$.

The following theorem is the main result of this paper.

Theorem 1. *Let $g \in \mathcal{G}$.*

- (a) *If g is a positive-valued function, then (1) has at least two weak solutions.*
- (b) *If g is an odd function, then (1) has infinitely many weak solutions.*

This article has the following structure. In Section 2, necessary definitions, notations, and variational theorems are reviewed. In Section 3, a proof of the main results is presented. Finally, in Section 4, an example and several comments are given.

2. PRELIMINARIES

We give the following theorems (see [11, 16]) as our main tools to prove the results of Section 3.

Theorem 2 ([11, Theorem 4.4]). *Let X be a Banach space and $\mathcal{M} : X \rightarrow \mathbb{R}$ be a differentiable functional that is bounded below. If $\mathcal{M}$ satisfies the $(PS)_c$ -condition with $c = \inf_X \mathcal{M}$, then $\mathcal{M}$ has a minimum on X .*

Obviously, the $(PS)_c$ -condition for each $c \in \mathbb{R}$ is obtained by the (PS) -condition.

Theorem 3 ([11, Theorem 4.10]). *Let $Q \in C^1(X, \mathbb{R})$ satisfy the (PS) -condition. Assume that there exist $y_0, y_1 \in X$ and a bounded neighborhood $\mathcal{D}$ of y_0 satisfying $y_1 \notin \mathcal{D}$ and $\inf_{z \in \partial \mathcal{D}} Q(z) > \max\{Q(y_0), Q(y_1)\}$. Then there exists a critical point y of Q , i.e., $Q'(y) = 0$, with $Q(y) > \max\{Q(y_0), Q(y_1)\}$.*

Theorem 4 ([16, Theorem 9.12]). *Let X be a real Banach space with infinite dimension, $Q \in C^1(X, \mathbb{R})$ be an even functional which satisfies the (PS)-condition, and $Q(0) = 0$. Assume that $X = \mathcal{V} \oplus \mathcal{W}$, where $\mathcal{V}$ has infinite dimension, there exist $\beta > 0$ and $e > 0$ so that $Q(y) \geq \beta$ for each $y \in \mathcal{W}$ with $\|y\| = e$ and for any subspace $\mathcal{Y} \subset X$ with finite dimension, there is $d = d(\mathcal{Y})$ so that $Q(y) \geq 0$ on $X \setminus B_d(\mathcal{Y})$. Then Q has an unbounded sequence of critical values.*

Readers can see [7, 20], where Theorems 3 and 4 have been successfully used to guarantee multiple solutions of degenerate nonlocal problems and nonlinear impulsive differential equations, respectively. Reference can also be made to [21], where Theorem 4 has been used to guarantee the existence of infinitely many solutions for a boundary value problem.

To produce appropriate function spaces and apply the variational method to discuss the existence of solutions for (1), we introduce the following preliminaries, which have been initially given in [4], and which we will use in the proofs of our main results. Let $\mathcal{H}_0^{1/2}(\mathcal{D})$ be the space

$$\mathcal{H}_0^{1/2}(\mathcal{D}) := \left\{ y \in L^2(\mathcal{D}) : y = \sum_{j=1}^{\infty} \kappa_j \mathcal{E}_j \quad \text{and} \quad \sum_{j=1}^{\infty} \kappa_j^2 \varepsilon_j^{1/2} < \infty \right\}$$

with the norm

$$\|y\|_{\mathcal{H}_0^{1/2}(\mathcal{D})} := \left(\sum_{j=1}^{\infty} \kappa_j^2 \varepsilon_j^{1/2} \right)^{1/2},$$

where $\{\mathcal{E}_j, \varepsilon_j\}_{j \in \mathbb{N}}$ are the eigenfunctions and eigenvalues of the usual linear problem $\mathcal{A}_{1/2}y = \varepsilon g(y)$ in $\mathcal{D}$ with Dirichlet boundary condition $y = 0$ on $\partial\mathcal{D}$. We note that

$$0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \dots < \varepsilon_j < \varepsilon_{j+1} < \dots$$

and $\varepsilon_j \rightarrow \infty$ as $j \rightarrow \infty$. Moreover, we can suppose that the eigenfunctions $\{\mathcal{E}_j\}_{j \in \mathbb{N}}$ are normalized as

$$\int_{\mathcal{D}} |\nabla \mathcal{E}_j(\tau)|^2 d\tau = \varepsilon_j \int_{\mathcal{D}} |\mathcal{E}_j(\tau)|^2 d\tau = \varepsilon_j, \quad j \in \mathbb{N}$$

and

$$\int_{\mathcal{D}} \nabla \mathcal{E}_i(\tau) \nabla \mathcal{E}_j(\tau) d\tau = \varepsilon_j \int_{\mathcal{D}} \mathcal{E}_i(\tau) \mathcal{E}_j(\tau) d\tau = 0, \quad i \neq j.$$

Remark 1. The operator $\mathcal{A}_{1/2}$ of the Laplace operator $-\Delta$ in a bounded domain with zero-boundary conditions is given via the spectral decomposition, using the powers of the eigenvalues of the primary operator. Therefore, pursuant to primeval results on positive operators in $\mathcal{D}$, if $\{(\mathcal{E}_j, \varepsilon_j)\}_{j \in \mathbb{N}}$ are the eigenfunctions and eigenvalues of the usual linear Dirichlet problem $-\Delta y = \varepsilon g(y)$ in $\mathcal{D}$ with Dirichlet boundary condition $y = 0$ on $\partial\mathcal{D}$, then $\{(\mathcal{E}_j, \varepsilon_j^{1/2})\}_{j \in \mathbb{N}}$ are the eigenfunctions and eigenvalues of the corresponding fractional problem $\{(\mathcal{E}_j, \varepsilon_j)\}_{j \in \mathbb{N}}$.

Remark 2. The fractional operator $\mathcal{A}_{1/2}$ should not be confused with the integro-differential operator $(-\Delta)^{1/2}$. In fact, these two operators, although determined in the same way, have distinct behaviors in eigenvalues, eigenfunctions and so on (see [15, 17] for more details).

Remark 3. The operator $\mathcal{A}_{1/2}$ is well defined on the space $\mathcal{H}_0^{1/2}(\mathcal{D})$, and it has the form $\mathcal{A}_{1/2}y = \sum_{j=1}^{\infty} \kappa_j^2 \varepsilon_j^{1/2} \mathcal{E}_j$, where $\kappa_j = \int_{\mathcal{D}} y(\tau) \mathcal{E}_j(\tau) d\tau$.

Remark 4. Let $\mathcal{D}$ be a bounded domain. Set $C_{\mathcal{D}} := \{(\tau, \xi) : \tau \in \mathcal{D}, \xi > 0\} \subset \mathbb{R}_+^{n+1}$ and $\partial_L C_{\mathcal{D}} := \partial \mathcal{D} \times [0, \infty)$, and for a function $y \in \mathcal{H}_0^{1/2}(\mathcal{D})$, define the harmonic extension $E(y)$ to the cylinder $C_{\mathcal{D}}$ as the solution of $\operatorname{div}(\nabla E(y)) = 0$ in $C_{\mathcal{D}}$, $E(y) = 0$ on $\partial_L C_{\mathcal{D}}$ and $\operatorname{Tr}(E(y)) = y$ on $\mathcal{D}$, where $\operatorname{Tr}(E(y))(\tau) = E(y)(\tau, 0)$ for all $\tau \in \mathcal{D}$. We note that $E(y)$ belongs to the Hilbert space

$$\begin{aligned} \mathcal{X}_0^{1/2}(\mathcal{D}) &:= \left\{ z \in L^2(\mathcal{D}) : z = 0 \quad \text{on} \quad \partial_L C_{\mathcal{D}}, \int_{\mathcal{D}} |\nabla z(\tau, \xi)|^2 d\tau d\xi < \infty \right\} \\ &= \left\{ z \in L^2(C_{\mathcal{D}}) : z = \sum_{j=1}^{\infty} \eta_j \mathcal{E}_j e^{-\varepsilon_j^{1/2} \xi} \quad \text{and} \quad \sum_{j=1}^{\infty} \eta_j^2 \varepsilon_j^{1/2} < \infty \right\}, \end{aligned}$$

with the standard norm

$$\|z\|_{\mathcal{X}_0^{1/2}(\mathcal{D})} := \left(\int_{C_{\mathcal{D}}} |\nabla z(\tau, \xi)|^2 d\tau d\xi \right)^{\frac{1}{2}}$$

(see [4, Lemma 2.10] for more details). On the other hand, the trace operator $\operatorname{Tr} : \mathcal{X}_0^{1/2}(C_{\mathcal{D}}) \rightarrow \mathcal{H}_0^{1/2}(\mathcal{D})$, given by $\operatorname{Tr}(z)(\tau) := z(\tau, \xi)$ for all $\tau \in \mathcal{D}$, is a continuous map (see [4, Lemma 2.6]). Moreover,

$$\mathcal{H}_0^{1/2}(\mathcal{D}) := \left\{ y \in L^2(\mathcal{D}) : y = \operatorname{Tr}(z) \quad \text{for some} \quad z \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}}) \right\} \subset \mathcal{H}^{1/2}(\mathcal{D})$$

and $\|\operatorname{Tr}(z)\|_{\mathcal{H}_0^{1/2}(\mathcal{D})} \leq \|z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})}$. The embedding $j : \operatorname{Tr}(\mathcal{H}_0^{1/2}(C_{\mathcal{D}})) \hookrightarrow L^p(\mathcal{D})$ is continuous for any $p \in [1, 2n/n-1]$, and it is compact whenever $p \in [1, 2n/n-1)$. Hence, if $p \in [1, 2n/n-1]$, then there is a constant $c_p > 0$ (depending on p, n and the Lebesgue measure of $\mathcal{D}$) so that

$$\|\operatorname{Tr}(z)\|_{L^p(\mathcal{D})} \leq c_p \|z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} \quad (2)$$

for every $z \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ (see [2–4]). We define the fractional operator $\mathcal{A}_{1/2}$ in $\mathcal{D}$ with

$$\mathcal{A}_{1/2}y(\tau) := - \lim_{\xi \rightarrow 0^+} \frac{\partial E(y)}{\partial \xi}(\tau, \xi) \quad \text{for all} \quad \tau \in \mathcal{D},$$

where $E(y) \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ is the extension of $y \in \mathcal{H}_0^{1/2}(\mathcal{D})$. Thus,

$$\mathcal{A}_{1/2}y(\tau) = \frac{\partial E(y)}{\partial \mathbf{v}}(\tau) \quad \text{for all} \quad \tau \in \mathcal{D},$$

where $\mathbf{v}$ is the unit outer normal to $C_{\mathcal{D}}$ at $\mathcal{D} \times \{0\}$.

A function $y = \text{Tr}(z) \in \mathcal{H}_0^{1/2}(\mathcal{D})$ is a weak solution of (1) if $z \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ is a weak solution of

$$\begin{cases} -\text{div}(\nabla z) = 0 & \text{in } C_{\mathcal{D}}, \\ z = 0 & \text{on } \partial_L C_{\mathcal{D}}, \\ \frac{\partial z}{\partial \mathbf{v}} = \varepsilon \gamma(\tau) g(\text{Tr}(\tau)) & \text{on } \mathcal{D}, \end{cases} \quad (3)$$

i.e.,

$$\int_{C_{\mathcal{D}}} \langle \nabla z, \nabla \vartheta \rangle d\tau d\xi = \varepsilon \int_{\Omega} \beta(\tau) f(\text{Tr}(z)(\tau)) \text{Tr}(\vartheta)(\tau) d\tau$$

for every $\vartheta \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$.

Now, we consider the functionals

$$\mathcal{M}(z) := \frac{1}{2} \|z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})}^2 \quad \text{and} \quad \mathcal{N}(z) := \int_{\mathcal{D}} \gamma(\tau) G(\text{Tr}(z)(\tau)) d\tau \quad (4)$$

for all $z \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$. Since

$$\lim_{\|z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} \rightarrow \infty} \mathcal{M}(z) = \infty,$$

this means that the functional $\mathcal{M} : \mathcal{X}_0^{1/2}(C_{\mathcal{D}}) \rightarrow \mathbb{R}$ is coercive. Also, $\mathcal{M}$ and $\mathcal{N}$ are Fréchet differentiable and

$$\mathcal{M}'(z)\vartheta = \int_{C_{\mathcal{D}}} \langle \nabla z, \nabla \vartheta \rangle d\tau d\xi \quad \text{and} \quad \mathcal{N}'(z)\vartheta = \int_{\mathcal{D}} \gamma(\tau) f(\text{Tr}(z)(\tau)) \text{Tr}(\vartheta)(\tau) d\tau$$

for all $z, \vartheta \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$. So, the critical points of the functional $I_{\varepsilon} : \mathcal{X}_0^{1/2}(C_{\mathcal{D}}) \rightarrow \mathbb{R}$ defined by $I_{\varepsilon}(z) := \mathcal{M}(z) - \varepsilon \mathcal{N}(z)$ are exactly the weak solutions of (3). Thus, the traces of critical points of I_{ε} are the weak solutions of (1).

Remark 5. The functional I_{ε} is weakly lower semicontinuous on $\mathcal{X}_0^{1/2}(C_{\mathcal{D}})$. Indeed, the map $z \rightarrow \int_{\mathcal{D}} \gamma(\tau) G(\text{Tr}(z)(\tau)) d\tau$ is continuous in the weak topology of $\mathcal{X}_0^{1/2}(C_{\mathcal{D}})$, and the map $z \rightarrow \int_{C_{\mathcal{D}}} |\nabla z(\tau, \xi)|^2 d\tau d\xi$ is lower semicontinuous in the weak topology of $\mathcal{X}_0^{1/2}(C_{\mathcal{D}})$.

3. PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 1. For this purpose, we first provide the following lemma.

Lemma 1 ([1, Lemma 4.1]). *Let $g \in \mathcal{G}$. Then for every $\varepsilon > 0$, the functional I_{ε} satisfies the (PS)-condition.*

3.1. Proof of Theorem 1(a)

Proof of Theorem 1(a). In our case, it is clear that $I_\varepsilon(0) = 0$. Lemma 1 gives that I_ε satisfies the (PS)-condition.

Step 1

Since $g \in \mathcal{G}$, for $\varepsilon > 0$ fixed, there exist two positive constants δ_1 and δ_2 such that $|f(t)| \leq \frac{\varepsilon}{\|\beta\|_{L^\infty(\Omega)}}|t|$ for all $t \in \mathbb{R}$ with $0 < |t| < \delta_1\lambda$, and $|g(t)| \leq \frac{\lambda}{\|\gamma\|_{L^\infty(\mathcal{D})}}|t|$ for all $t \in \mathbb{R}$ with $|t| > \delta_2\lambda$. Set $0 < \delta_\lambda < \min\{1, \delta_1\lambda, \delta_2\lambda^{-1}\}$. Then

$$|g(t)| \leq \frac{\lambda}{\|\gamma\|_{L^\infty(\mathcal{D})}}|t| \quad \text{for all } t \in \mathbb{R} \quad \text{with } 0 < |t| < \delta_\lambda \quad \text{and} \quad |t| > \delta_2\lambda.$$

Besides, the function $h(t) = \frac{|g(t)|}{|t|}$ is continuous on $[\delta_\lambda, \delta_\lambda^{-1}] \cup [-\delta_\lambda^{-1}, -\delta_\lambda]$, and therefore, there exists $0 < m_\lambda < \frac{1-\lambda}{c_2^2\|\gamma\|_{L^\infty(\mathcal{D})}}$ such that $|h(t)| \leq m_\lambda$ for all $t \in [\delta_\lambda, \delta_\lambda^{-1}] \cup [-\delta_\lambda^{-1}, -\delta_\lambda]$. Thus, we can say

$$|g(t)| \leq \left(\frac{\lambda}{\|\gamma\|_{L^\infty(\mathcal{D})}} + m_\lambda \right) |t| \quad \text{for all } t \in \mathbb{R}.$$

Now, since $\sup_{t \in \mathbb{R}} G(t) > 0$ and by using (2) for $p = 2$, we have

$$\begin{aligned} |\mathcal{N}(z)| &\leq \int_{\mathcal{D}} |\gamma(\tau)| |G(\text{Tr}(z)(\tau))| d\tau \\ &\leq \|\gamma\|_{L^\infty(\mathcal{D})} \int_{\mathcal{D}} \int_0^{\text{Tr}(z)(\tau)} \left(\frac{\lambda}{\|\gamma\|_{L^\infty(\mathcal{D})}} + m_\lambda \right) |\xi| d\xi d\tau \\ &\leq \frac{\lambda + m_\lambda \|\gamma\|_{L^\infty(\mathcal{D})}}{2} \|\text{Tr}(z)(\tau)\|_{L^2(\mathcal{D})}^2 \\ &\leq \frac{\lambda + m_\lambda \|\gamma\|_{L^\infty(\mathcal{D})}}{2} c_2^2 \|z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})}^2. \end{aligned}$$

Then, for any $y \in \mathcal{X}$, by (4), we get

$$I_\varepsilon(y) \geq \frac{\|y\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})}^2}{2} \left(1 - \lambda - c_2^2 m_\lambda \|\gamma\|_{L^\infty(\mathcal{D})} \right), \quad (5)$$

and we conclude that I_ε is a coercive functional which is bounded below. Because I_ε satisfies the (PS)-condition by Lemma 1, Theorem 2 yields that there is a minimum point z_0 of I_ε on $\mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ and $0 = I_\varepsilon(0) \geq I_\varepsilon(z_0)$ and $I'_\varepsilon(z_0) = 0$.

Step 2

Since $z_0 = \min_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} I_{\varepsilon}$, there is $\mathcal{L} > 0$ large enough so that

$$I_{\varepsilon}(z_0) \leq 0 < \inf_{z \in \partial B_{\mathcal{L}}} I_{\varepsilon}(z),$$

where $B_{\mathcal{L}} = \{z \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}}) : \|z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} < \mathcal{L}\}$. Now, we will show that there exists z_1 with $\|z_1\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} > \mathcal{L}$ such that $I_{\varepsilon}(z_1) < \inf_{\partial B_{\mathcal{L}}} I_{\varepsilon}(z)$. For this, let $\vartheta_1 \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ and $z_1 = \kappa \vartheta_1$, $\kappa > 0$, where ϑ_1 corresponding to ε_1 is the first eigenfunction of (3) and $\|\vartheta_1\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} = 1$. Since $\lim_{|t| \rightarrow \infty} \frac{g(t)}{t} = 0$, we can consider constants $a > 0$ and $\alpha > 2$ in a way that $G(t) \geq a|t|^{\alpha}$ for all $t \in \mathbb{R}$. Hence,

$$\begin{aligned} I_{\varepsilon}(z_1) &= (\mathcal{M} - \varepsilon \mathcal{N})(\kappa \vartheta_1) \\ &\leq \frac{1}{2} \|\kappa \vartheta_1\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})}^2 - \varepsilon \int_{\mathcal{D}} G(\kappa \vartheta_1(\tau)) d\tau \\ &\leq \frac{\kappa^2}{2} - \varepsilon \kappa^{\alpha} a \int_{\mathcal{D}} |\vartheta_1(\tau)|^{\alpha} d\tau. \end{aligned}$$

So by $\alpha > 2$, there is sufficiently large $\kappa > \mathcal{L} > 0$ so that $I_{\varepsilon}(\kappa \vartheta_1) < 0$. Therefore, $\max\{I_{\varepsilon}(z_0), I_{\varepsilon}(z_1)\} < \inf_{z \in \partial B_{\mathcal{L}}} I_{\varepsilon}(z)$. Then, Theorem 3 with $\mathcal{X} := \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ and $\vartheta := I_{\varepsilon}$ gives us the critical point z^* . So, y_0 and y^* are two critical points of I_{ε} and thus, are two weak solutions of (3). $\square$

3.2. Proof of Theorem 1(b)

Proof of Theorem 1(b). In order to prove this part, we set $\mathcal{X} := \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ and consider the continuously Gâteaux-differentiable functional I_{ε} , that, by (4), is an even functional and $I_{\varepsilon}(0) = 0$. We continue the proof of this part in two steps to show that the functional I_{ε} satisfies the assumptions of Theorem 4.

Step 1

The coercivity of I_{ε} is achieved by (5), and this, together with the (PS)-condition, by the minimization theorem [11, Theorem 4.4], implies that the functional I_{ε} has a minimum critical point y with $I_{\varepsilon}(y) \geq \mathfrak{t} > 0$ and $\|y\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} = \rho$ for sufficiently small $\rho > 0$.

Step 2

Let $\mathcal{K} \subset \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ be a finite-dimensional subspace. Since $\lim_{|t| \rightarrow \infty} \frac{g(t)}{t} = 0$, there exist constants $a > 0$ and $\alpha > 2$ so that $G(t) \geq a|t|^{\alpha}$ for all $t \in \mathbb{R}$. Now, for each $\kappa > 0$

and $y \in \mathcal{K} \setminus \{0\}$ with $\|y\|_{\mathcal{H}_0^{1/2}(C_D)} = 1$, we have

$$\begin{aligned} I_\varepsilon(\kappa y) &= (\mathcal{M} - \varepsilon \mathcal{N})(\kappa z) \\ &\leq \frac{1}{2} \|\kappa z\|_{\mathcal{H}_0^{1/2}(C_D)}^{2\alpha} - \varepsilon \int_D G(\kappa z(\tau)) d\tau \\ &\leq \frac{\kappa^2}{2} \|y\|_{\mathcal{H}_0^{1/2}(C_D)}^{2\alpha} - \varepsilon \kappa^\alpha a_1 \int_D |z(\tau)|^\alpha d\tau \rightarrow -\infty \quad \text{as } \kappa \rightarrow \infty. \end{aligned}$$

This yields the existence of κ_0 so that $\|\kappa z\|_{\mathcal{H}_0^{1/2}(C_D)} > \rho$ and $I_\varepsilon(\kappa z) < 0$ for every $\kappa \geq \kappa_0 > 0$. Since $\mathcal{K}$ is a finite-dimensional subspace, there exists $\mathcal{R} = \mathcal{R}(\mathcal{K}) > 0$ such that $I_\varepsilon(z) \leq 0$ on $\mathcal{K} \setminus B_{\mathcal{R}}(\mathcal{K})$. Due to Theorem 4, the functional $I_\varepsilon(z)$ has infinitely many critical points, which are the weak solutions of (3). $\square$

4. EXAMPLES AND REMARKS

Finally, we give an example and several concluding remarks.

Example 1. Let $\mathcal{D} = \{(\tau_1, \tau_2) \in \mathbb{R}^2 : \tau_1^2 + \tau_2^2 < 4\} \subset \mathbb{R}^2$ and $g(t) = t^2/(1+t^2)$ for all $t \in \mathbb{R}$. We see that $g(t) \geq 0$ for all $t \in \mathbb{R}$ is an odd function and $g \in \mathcal{G}$. Hence, the problem $\mathcal{A}_{1/2}y = \varepsilon y^2/(1+y^2)$ in $\mathcal{D}$ with boundary condition $y = 0$ on $\partial\mathcal{D}$, by applying Theorem 1, for every $\varepsilon > 0$, possesses infinitely many weak solutions in the space $\mathcal{H}_0^{1/2}(\mathcal{D})$.

Remark 6. For each $g \in \mathcal{G}$, the number $c_g := \max_{|t|>0} \frac{|g(t)|}{|t|}$ is well defined and strictly positive. Furthermore, the sublinear growth condition at infinity on the nonlinearity g complements the classical Ambrosetti and Rabinowitz assumption.

Remark 7. Example 1 shows that our existence results for (1) in Theorem 1 is different from the existence results of Molica Bisci in [12]. This is because the function g in [12] should satisfy the condition

$$|g(t)| \leq a_1 + a_2 |t|^{q-1} \quad \text{for } a_1, a_2 > 0, q \in \left(1, \frac{2n}{n-1}\right), t \in \mathbb{R}, \quad (6)$$

while in Example 1, $2n/(n-1) = 2$ and $g(t) = t^2/(1+t^2)$, and so g does not satisfy (6).

Remark 8. We note that, if $g(0) \neq 0$, then Theorem 1(a) gives the existence of two nontrivial weak solutions for (1). If we do not consider the condition $g(0) \neq 0$, then the second solution y_2 of (1) can be trivial, but the problem has at least one nontrivial solution.

Remark 9. We notice that, by Remark 8, the solutions obtained in Example 1 are nonzero because $g(0) = 1 \neq 0$.

Remark 10. If $\limsup_{\xi \rightarrow 0^+} g(\xi)/|\xi| = \infty$ and $\liminf_{\xi \rightarrow 0^+} g(\xi)/|\xi| > -\infty$, then the second weak solution obtained by Theorem 1(a) can be nontrivial even in the case $g(0) = 0$. Indeed, let $\varepsilon > 0$ and let $\mathcal{M}$ and $\mathcal{N}$ be as given in (4). Due to Theorem 2 and Lemma 1, $I_\varepsilon = \mathcal{M} - \varepsilon\mathcal{N}$ has a critical point y_ε that is a global minimum of I_ε . The function y_ε cannot be trivial. Indeed, by the same argument as [8, Remark 3.3], we can prove that $\limsup_{\|y\| \rightarrow 0^+} \mathcal{M}(y)/\mathcal{N}(y) = \infty$. So, there is a sequence $\{\zeta_n\} \subset \mathcal{X}$ that converges strongly to zero, and for sufficiently large n , $I_\varepsilon(\zeta_n) = \mathcal{M}(\zeta_n) - \varepsilon\mathcal{N}(\zeta_n) < 0$. Since y_ε is a global minimum of I_ε , we obtain $I_\varepsilon(y_\varepsilon) < 0$, so that y_ε is not trivial.

Remark 11. If g is nonnegative, then Theorem 1(a) is a bifurcation result, i.e., the pair $(0, 0) \in \mathcal{H}_g^\varepsilon \subset \mathcal{H}_0^{1/2}(\mathcal{D}) \times \mathbb{R}$ with

$$\mathcal{H}_g^\varepsilon := \left\{ (y_\varepsilon, \varepsilon) \in \mathcal{H}_0^{1/2}(\mathcal{D}) \times (0, \infty) : y_\varepsilon \text{ is a nontrivial weak solution of (1)} \right\}.$$

Practically, by the proof of Theorem 1(a), $\|y_\varepsilon\|_{\mathcal{H}_0^{1/2}(\mathcal{D})} \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence, there exist two sequences $\{y_k\}$ in $\mathcal{H}_0^{1/2}(\mathcal{D})$ and $\{\varepsilon_k\}$ in $\mathbb{R}^+$ (here $y_k = y_{\varepsilon_k}$) such that $\varepsilon_k \rightarrow 0^+$ and $\|y_k\| \rightarrow 0$ as $k \rightarrow \infty$. Moreover, since g is nonnegative, $\mathcal{M}(y) < 0$ for all $y \in \mathbb{R}$, and thus, the mapping $(0, \varepsilon^*) \ni \varepsilon \mapsto I_\varepsilon(y_\varepsilon)$ is strictly decreasing. Hence, for every $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon^*)$ with $\varepsilon_1 \neq \varepsilon_2$, the weak solutions y_{ε_1} and y_{ε_2} ensured by Theorem 1(a) are different.

Remark 12. In Theorem 1, we looked for the points of the functional I_ε naturally associated with (3). We note that, in general, I_ε can be unbounded from below in $\mathcal{X}_0^{1/2}(C_{\mathcal{D}})$. Indeed, for example, in the case when $g(t) = 1 + |t|^{a-2}t$ for all $t \in \mathbb{R}$ with $a > 2$, for any fixed $z \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}}) \setminus \{0\}$ and $\mathfrak{t} \in \mathbb{R}$, we obtain

$$\begin{aligned} I_\varepsilon(\mathfrak{t}z) &\leq \frac{1}{2} \|\mathfrak{t}z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})}^2 - \varepsilon \int_{\mathcal{D}} \gamma(\tau) G(\text{Tr}(\mathfrak{t}z(\tau))) d\tau \\ &\leq \frac{\mathfrak{t}^2}{2} \|z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})}^2 - \varepsilon \underline{\gamma} |\mathcal{D}| \left(\mathfrak{t} \|\text{Tr}(z)\|_{L^1(\mathcal{D})} + \frac{\mathfrak{t}^a}{a} \|\text{Tr}(z)\|_{L^a(\mathcal{D})}^a \right) \rightarrow -\infty \end{aligned}$$

as $\mathfrak{t} \rightarrow \infty$, where $\underline{\gamma} = \inf_{\tau \in \mathcal{D}} \gamma(\tau)$ and $|\mathcal{D}| = \int_{\mathcal{D}} d\tau$. Hence, direct minimization cannot be used to find critical points of the functional I_ε .

Remark 13. All results of this paper are still valid if we consider the more general problem

$$\begin{cases} \mathcal{A}_{m/2}y = \varepsilon\gamma(\tau)g(y) & \text{in } \mathcal{D}, \\ y = 0 & \text{on } \partial\mathcal{D} \end{cases}$$

with $k \in (0, 2)$ and $\mathcal{A}_{m/2}y(\tau) := -\kappa_m \lim_{z \rightarrow 0^+} z^{1-m} \frac{\partial E(y)}{\partial z}(\tau, z)$ for all $\tau \in \mathcal{D}$, where y belongs to the space

$$\mathcal{H}_0^{m/2}(\mathcal{D}) := \left\{ y \in L^2(\mathcal{D}) : \|y\|_{\mathcal{H}_0^{m/2}(\mathcal{D})} = \left(\sum_{j=1}^{\infty} a_j^2 \varepsilon_j^{m/2} \right)^{1/2} < \infty \right\}$$

and its m -harmonic extension $E_m(y)$ to the cylinder C is the unique solution of the local problem

$$\begin{cases} -\operatorname{div}(y^{1-m} \nabla z) = 0 & \text{in } C_{\mathcal{D}}, \\ E_m(y) = 0 & \text{on } \partial_L C_{\mathcal{D}}, \\ \operatorname{Tr}(E_m(y)) = y & \text{on } \mathcal{D}. \end{cases}$$

$E_m(y)$ as an extension lies in the space

$$\begin{aligned} \mathcal{X}_0^{m/2}(\mathcal{D}) &:= \left\{ z \in L^2(\Omega) : z = 0 \text{ on } \partial_L C_{\mathcal{D}}, \int_{\mathcal{D}} \xi^{1-m} |\nabla z(\tau, \xi)|^2 d\tau d\xi < \infty \right\} \\ &= \left\{ z \in L^2(C_{\mathcal{D}}) : z = \sum_{j=1}^{\infty} \eta_j \mathcal{E}_j e^{-\varepsilon_j^{m/2} \xi} \text{ and } \sum_{j=1}^{\infty} \eta_j^2 \varepsilon_j^{m/2} < \infty \right\} \end{aligned}$$

and the operator $E_m : \mathcal{H}_0^{1/2}(\mathcal{D}) \rightarrow \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ is an isometry.

Remark 14. Although we did not use the test function in the proof of our main results, in the approach of some manuscripts on this topic such as [12], $v_{\alpha_0}^{t_0} : \Omega \rightarrow \mathbb{R}$ as

$$v_{\alpha_0}^{t_0}(y) := \begin{cases} 0 & \text{if } y \in \overline{\Omega} - B(y_0, t_0), \\ \frac{2\alpha_0}{t_0}(t_0 - |y - y_0|) & \text{if } y \in B(y_0, t_0) - B(y_0, t_0/2), \\ \alpha_0 & \text{if } y \in B(y_0, t_0/2) \end{cases}$$

as a class of test functions in $\mathcal{X}_0^{m/2}(\mathcal{D})$ is used.

Remark 15. If g is an odd function, then we have the same result as Theorem 1(b) by placing the following conditions on g :

- ($\mathcal{G}_1$) there are constants $\rho > 0$ and $0 < \varepsilon R_1 < \frac{1}{2} \min\{1, m_0\}$ such that $G(y) \leq R_1 |y|^2$, for all $y \in \mathbb{R}$ with $|y| \leq \rho$;
- ($\mathcal{G}_2$) there are constants $\rho_1 > 0$, $\delta_1 > 0$ and $\alpha_1 > \gamma$ such that $G(y) \geq \delta_1 |y|^{\alpha_1}$ for all $y \in \mathbb{R}$ with $|y| \geq \rho$;
- ($\mathcal{G}_3$) there is a constant $\beta > 2$, $\gamma_2 \geq 0$ and $0 < \alpha_2 < 2$ such that $vG(\xi) - \xi g(\xi) \leq \gamma_2 |y|^{\alpha_2}$.

Remark 16. In the future, it may be of interest to continue the research of this paper in this line, extending the study to the case that g is not a continuous function, but discontinuous, starting from the case that g is a function of vanishing mean oscillation.

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NOTE ON BLOW UP SOLUTIONS FOR A GENERAL CLASS OF SEMILINEAR PARABOLIC EQUATIONS INVOLVING SECOND ORDER OPERATOR

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Abstract. This work deals with an initial-boundary value problem of Laplacian parabolic equation

$$\begin{aligned}(h(u))_t + \triangle_p u &= f(u(x, t)), & \text{in } \Omega \times (0, \infty), \\ u(x, t) &= 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x_0) &= u_0 \geq 0, & x \in \bar{\Omega},\end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$. Our contribution is to give a new condition on nonlinearity to obtain the blow-up solutions of the above equations.

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1. INTRODUCTION

Over the past few years, a blow-up phenomenon of global solutions for non linear reaction diffusion equations and systems has been extensively investigated by many authors. These authors studied the questions of global existence, blow-up at some finite time, blow-up rate, blow-up set, asymptotic behavior for solutions and so on as well as a variety of methods used to research these questions. Particularly, the problems of the blow-up and global solutions for non linear reaction diffusion equations under Dirichlet boundary conditions have been considered in [1, 4, 5, 7–9, 12, 13]. However, due to the explosive nature of the solutions, it is very important in applications to determine lower bounds on the blow-up time. Presently, the research on the lower bound of the blow-up time for the non local problems with Dirichlet or Neumann boundary condition had some new progress. We provide the reader to the literature [3, 6] and [11], for some recent interesting research on the local reaction-diffusion equation with non local boundary conditions we can refer to [9].

Inspired by the above mentioned papers, especially from [3] and [13], we discuss the blow-up solutions of the following initial-boundary value problem of p -Laplacian parabolic equation:

$$\begin{aligned} (h(u))_t + \Delta_p u &= f(u(x, t)), & \text{in } \Omega \times (0, \infty), \\ u(x, t) &= 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x_0) &= u_0 \geq 0, & x \in \bar{\Omega}, \end{aligned} \quad (1.1)$$

where $h(\tau), f(\tau) \in C^\infty(\mathbb{R}^+)$, $h(\tau) > 0$, $h'(\tau) \geq 1$ and $f(\tau) > 0$ for $\tau > 0$.

We borrow some ideas from the work [3] and we extend them to more general parabolic problem which involves the p -Laplacian operator.

The papers [4, 8], and [9] studied the special cases of problem (1.1). In [9], the following problem was considered by Payne and Schaefer:

$$\begin{aligned} u_t &= \Delta_p u + f(u(x, t)), & \text{in } \Omega \times (0, \infty), \\ u(x, t) &= 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x_0) &= u_0 \geq 0, & x \in \bar{\Omega}, \end{aligned}$$

where Ω is a bounded domain on $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. They used a differential inequality technique and a comparison principle to obtain a lower bound on blow-up time when blow-up occurs.

Little later, Payne, Philippin and Schaefer investigated the problem in [8]:

$$\begin{aligned} u_t &= \nabla \cdot (\rho(|\nabla u|^2) \nabla u) + f(u), & \text{in } \Omega \times (0, \infty), \\ u(x, t) &= 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x_0) &= u_0 \geq 0, & x \in \bar{\Omega}, \end{aligned}$$

where Ω is a bounded domain on $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Under appropriate assumptions on the functions f, ρ and u_0 , a lower bound on blow-up time was showed by applying a differential technique when blow-up does occur. Moreover, a criterion for blow-up and conditions which ensure that blow-up cannot occur were also obtained. Finally, the following problem was studied by Payne and Philippin in [4]:

$$\begin{aligned} u_t &= \Delta u + K(t)f(u(x, t)), & \text{in } \Omega \times (0, \infty), \\ u(x, t) &= 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x_0) &= u_0 \geq 0, & x \in \bar{\Omega}, \end{aligned}$$

where Ω is a bounded domain of $\mathbb{R}^N$, ($N \geq 2$) with smooth boundary $\partial\Omega$.

A first-order differential inequality technique and Sobolev inequality were used to give the sufficient conditions which guarantee the blow-up or the global existence of the solution. In addition, lower and upper bounds on blow-up time were also derived.

We proceed as follows. In Section 2 we establish a sufficient condition on the data of problem (1.1) to guarantee the blow-up of the solution and obtain an upper bound on blow-up time by a new condition noted (C_p) .

We introduce a condition sufficient to ensure the solution blows up at some finite time and obtain an upper bound on blow-up time.

$$(C_p) \quad \lambda F(u) \leq uf(u) + \mu u^p + \delta, \quad u > 0,$$

where $0 < \mu \leq \frac{\lambda-2}{2}\lambda_{1,p}$, and $\lambda_{1,p}$ is the first eigenvalue of the p -Laplacian Δ_p . Blow-up phenomena for this kind of problems in bounded domains have been extensively studied. Concavity method has been used so far to derive the blow-up solutions for some variants of the equations (1.1), see [1, 6, 8, 13]. By using the concavity method, Philippin and Proytcheva, in [10], obtained the blow-up solutions for problem (1.1) (with $h = id$) under the conditions:

$$(C^*) \quad (2 + \varepsilon)F(u) \leq uf(u), \quad u > 0, \varepsilon > 0.$$

We point out that the used condition (C_p) is more weaker than (C^*) , so it is more interesting. Meanwhile, local existence of classical solutions (or weak solutions) to such problems have been also established by many works. Since that not all solutions of these equations exist for all time, so many authors deal with the sufficient conditions for the local existence of solutions see for example [2–4].

2. MAIN RESULT AND PROOFS

Let consider the functional H such that:

$$H(t) = \int_0^t h'(\tau) \tau d\tau.$$

Through this paper, we suppose that:

$$(H_0) \quad H(t) \geq h'(t)t, \quad \text{for } t > 0.$$

In this section, our main result can be read as follows:

Theorem 1. *Under the hypothesis (H_0) , let a function f satisfy the condition (C_p) and $p > 2$. If the initial data $u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$ such that*

$$-\frac{1}{p} \int_\Omega |\nabla u_0(x)|^p dx + \int_\Omega [F(u_0(x)) - \delta] dx \geq 0, \quad (2.1)$$

then the nonnegative weak solution to the problem (1.1) blows up at finite time T^ , i.e*

$$\lim_{t \rightarrow T^*} \int_0^t \int_\Omega u^2(x, s) dx = +\infty,$$

where $s \geq 2$ and δ is the constant in the condition (C_p) .

Consider the Sobolev space $W_0^{1,p}(\Omega)$, which is the closure of C^∞ functions compactly supported in Ω for the norm:

$$\|u\|_{1,p}^p = \int_{\Omega} |u(x)|^p dx + \int_{\Omega} |\nabla u(x)|^p dx.$$

The p-Laplacian operator is defined by

$$\begin{aligned} \Delta_p : W_0^{1,p}(\Omega) &\longrightarrow W^{-1,q}(\Omega) \\ u &\longmapsto \Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \end{aligned}$$

where $W^{-1,q}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$ and we have

$$1 < p, q < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Let u be a function of $W_0^{1,p}(\Omega)$, not identically 0. Through the above definitions, the function u is called an eigenfunction if

$$\int_{\Omega} |\nabla u(x)|^{p-2} \nabla u \nabla \phi dx = \lambda \int_{\Omega} |u(x)|^{p-2} u \phi dx \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

The corresponding real number λ is called an eigenvalue.

Lemma 1 ([4]). *For $1 < P < \infty$, there exist $\lambda_{1,p} > 0$ and an eigenfunction $e_{1,p} \in W_0^{1,p}(\Omega)$ such that*

$$\begin{aligned} \Delta_p e_{1,p} &= \lambda_{1,p} e_{1,p} \\ e_{1,p}(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

Moreover, $\lambda_{1,p}$ is given by:

$$\lambda_{1,p} = \inf_{W_0^{1,p}(\Omega)} \frac{\int_{\Omega} |\nabla v|^p}{\int_{\Omega} |v|^p} > 0.$$

Proof of Theorem 1.

$$\begin{aligned} \int_0^t \int_{\Omega} (u_t^2(x,s))_t dx ds &\leq -\frac{1}{p} \int_{\Omega} [|\nabla u(x,t)|^p - |\nabla u_0(x)|^p] dx \\ &\quad + \int_{\Omega} [F(u(x,t)) - F(u_0(x))] dx. \end{aligned} \tag{2.2}$$

We define a function J_p by:

$$J_p(t) = -\frac{1}{p} \int_{\Omega} |\nabla u(x,t)|^p dx + \int_{\Omega} [F(u(x,t)) - \delta] dx.$$

Then it follows from (2.2) that

$$J_p(t) = -\frac{1}{p} \int_{\Omega} |\nabla u(x,t)|^p dx + \int_{\Omega} [F(u(x,t)) - \delta] dx$$

$$\begin{aligned}
&\geq -\frac{1}{p} \int_{\Omega} |\nabla u_0|^p dx + \int_{\Omega} [F(u_0) - \delta] dx + \int_0^t \int_{\Omega} h'(u) u_t^2(x, s) dx ds \\
&= J_p(0) + \int_0^t \int_{\Omega} h'(u) u_t^2(x, s) dx ds.
\end{aligned}$$

On the other hand, we define a function by

$$\Psi_p(t) = \int_0^t \int_{\Omega} H(u(x, s)) dx ds + M, \quad t \geq 0,$$

where $M > 0$ is a constant to be determined later. Accordingly, by virtue the definition of the functional H and the fact that $du = \frac{\partial u}{\partial t} dt$, it yields

$$\begin{aligned}
\Psi_p'(t) &= \int_{\Omega} H(u(x, s)) dx = \int_{\Omega} \int_0^t h'(u) u_t u dx + \int_{\Omega} H(u_0) dx \\
&= \int_0^t \int_{\Omega} (H(u(x, t)))_t dx.
\end{aligned} \tag{2.3}$$

Then

$$\begin{aligned}
\Psi_p''(t) &= \int_{\Omega} H(u(x, s))_t dx \\
&= \int_{\Omega} |\Delta_p u| u - f(u) u dx \\
&= \int_{\Omega} -|\nabla u|^p dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} u dS + \int_{\Omega} f(u(x, s)) u(x, s) dx \\
&= \int_{\Omega} -|\nabla u|^p dx + \int_{\Omega} f(u(x, s)) u(x, s) dx.
\end{aligned}$$

By using the condition (C_p) , Lemma 1, and (2.3) in turn, we obtain that

$$\begin{aligned}
\Psi_p''(t) &\geq - \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} [\lambda F(u) - \beta u^p - \lambda \delta] dx \\
&= \lambda J_p(t) + \frac{\lambda - p}{p} \left[- \int_{\Omega} |\nabla u|^p dx - \beta \int_{\Omega} u^p(x, t) p dx \right] \\
&\geq \lambda J_p(t) + \left[\frac{(\lambda - p) \lambda_{1,p}}{p} - \beta \right] \int_{\Omega} u^p dx \\
&\geq \lambda J_p(t) \\
&\geq \lambda \left[J_p(0) + \int_0^t \int_{\Omega} h'(u) u_t^2(x, s) dx ds \right].
\end{aligned} \tag{2.4}$$

Applying the Schwartz inequality, we obtain that

$$\begin{aligned}
(\Psi')^2(t) &\leq 4(1 + \delta) \left(\int_{\Omega} \int_0^t h'(u) u u_t(x, s) ds dx \right)^2 + \left(1 + \frac{1}{\delta} \right) (H(u_0(x)) dx)^2 \\
&\leq 4(1 + \delta) \left[\int_{\Omega} \left(\int_0^t (h'(u) u)^2 ds \right)^{\frac{1}{2}} \left(\int_0^t u_t^2(x, s) ds \right)^{\frac{1}{2}} \right]^2 + \left(1 + \frac{1}{\delta} \right) [H(u_0(x)) dx]^2
\end{aligned}$$

$$\begin{aligned} &\leq 4(1+\delta) \left(\int_{\Omega} \int_0^t (h'(u)u)^2 ds dx \right) \left(\int_{\Omega} \int_0^t u_t^2(x,s) ds dx \right) \\ &\quad + (1 + \frac{1}{\delta}) [H(u_0(x))dx]^2, \end{aligned} \quad (2.5)$$

where $\delta > 0$ is arbitrary. Combining the above estimates (2.4) and (2.5), we obtain that

$$\Psi_p''(t)\Psi_p(t) - (1+\sigma)\Psi_p'(t)^2 > 0$$

by choosing $\sigma = \delta = \sqrt{\frac{\lambda}{2}} - 1 > 0$ and M is large enough. This means that the solutions u blow up in finite time T^* . $\square$

Remark 1. By the same argument with slight changes, we may prove that our result is still available for the following problem

$$\begin{aligned} (h(u))_t + \Delta u &= f(u(x,t)), & \text{in } \Omega \times (0, \infty), \\ u(x,t) &= 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x_0) &= u_0 \geq 0, & x \in \bar{\Omega}, \end{aligned}$$

In fact, the above theorem can work for the case $p = 2$.

Example 1. For $p \geq 2$, where the functional $h = id$ and $f(u(x,t)) = u^p$, the weak solution to the problem (1.1) blows up at finite time T^* provided

$$-\int_{\Omega} |\nabla u_0(x)|^p dx + \int_{\Omega} [|u_0(x)|^p - \delta] dx \geq 0.$$

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POSITIVE SOLUTIONS FOR INTEGRAL BOUNDARY VALUE PROBLEMS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we consider integral boundary value problems of nonlinear fractional differential equations. Existence results of positive solutions for the problem are obtained based on the Guo-Krasnoselskii theorem and the Five functional fixed point theorem. Simple examples follow the main results in successive sections.

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1. INTRODUCTION

In this study, we consider the following IBVP

$$\begin{cases} D^\beta(\phi_p({}^c D^\alpha y(t))) + f(t, y(t)) = 0, & t \in [0, 1], \\ y(0) = y''(0) = 0, \quad y(1) = k \int_0^1 y(s) ds, \\ \phi_p({}^c D^\alpha y(0)) = [\phi_p({}^c D^\alpha y(0))]' = 0, \end{cases} \quad (1.1)$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, $0 < k < 2$, ${}^c D^\alpha$ and D^β are the Caputo and Riemann-Liouville derivatives respectively, $\phi_p(y) = |y|^{p-2}y$ such that $p > 1$, $\phi_p^{-1} = \phi_q$ with $1/p + 1/q = 1$ and $f(t, y) \in C([0, 1] \times [0, +\infty), [0, +\infty))$.

A great deal of interest has emerged in the field of fractional differential equations. To be more specific, in the scientific discipline, precisely the area of mathematical modelling of processes in polymer rheology, physics, aerodynamics and chemistry among others, has fully embraced fractional differential equations as a vital tool in the description of hereditary properties for some materials and processes. Advances in fractional calculus theories has led to its applications in engineering, mechanics, chemistry, physics, among others, see [1, 3, 9, 10, 12–17, 20, 24].

An appreciable amount of authors have intensively concentrated on the existence and multiplicity of positive solution for boundary value problems of nonlinear fractional differential equations by applying certain fixed point theorems which include the Guo-Krasnosel'skii fixed point theorem, upper and lower solutions method, the Leggett-Williams fixed point theorem and the Schauder fixed-point theorem to mention but a few [5, 6, 8, 26–28]. Some work has been covered involving integral boundary value problems, detailed approaches on integral boundary value problems can be seen in [4] and references entailed therein.

However, as far as we know, a limited number of work involving both a combination of Caputo-Riemann Liouville derivatives and integral boundary value conditions have been considered some of which include [7, 18, 19, 25]. Our work presents an in depth expansion and comprehensive approach inevitable in the field of fractional differential equations addressing the limitation outlined above.

This paper is organized in such a manner, Section 2 presents some necessary background material, lemmas and definitions. Section 3 deals with the existence of single and multiple positive solutions for the functional differential equation with fractional order (1.1) based on the Guo-Krasnoselskii theorem. Section 4 focuses on the existence of multiple positive solutions for the fractional differential equation by means of applying the Five functional fixed point theorem.

2. BASIC DEFINITIONS AND PRELIMINARIES

In this section, we introduce some necessary definitions and lemmas.

Definition 1 ([4, Definition 2.2]). The integral

$$I^\beta g(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds,$$

where $\beta > 0$, is the fractional integral of order β for a function $g(t)$.

Definition 2. The gamma function is defined by the integral formula

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

The integral converges absolutely for $\operatorname{Re}(z) > 0$. For any positive integer n , $\Gamma(n) = (n-1)!$.

Definition 3 ([4, Definition 2.3]). For a function $g(t)$ the expression

$$D_{0+}^\beta g(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\beta-1} g(s) ds,$$

is called the Riemann-Liouville fractional derivative of order β , where $n = [\beta] + 1$, and $[\beta]$ denotes the integer part of number β , provided that the right-hand side of the previous equation is point-wise defined on $(0, \infty)$.

Definition 4 ([21, Definition 2.2]). The α order Caputo fractional derivatives for a function $f(t)$ is defined as follows:

$${}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \quad n-1 < \alpha < n.$$

Definition 5 ([21, Definition 2.3]). Let $P \subseteq K$ be a nonempty, convex closed set and K a real Banach space. Then P is called a cone in K provided that

- (1) $\lambda y \in P$, for all $y \in P$ and $\lambda \geq 0$;
- (2) $y, -y \in P$ implies that $y = 0$.

Definition 6 ([21, Definition 2.4]). Let P be a cone in real Banach space K . If the map

$\vartheta: P \rightarrow [0, \infty)$ is continuous and satisfies

$$\vartheta(tx + (1-t)y) \geq t\vartheta(x) + (1-t)\vartheta(y), \quad x, y \in P, t \in [0, 1],$$

then ϑ is called a nonnegative continuous concave functional on P .

In a similar way, the map Υ is a nonnegative continuous convex function on a cone P of a real Banach space K provided that $\Upsilon: P \rightarrow [0, \infty)$ is continuous and

$$\Upsilon(tx + (1-t)y) \leq t\Upsilon(x) + (1-t)\Upsilon(y),$$

for all $x, y \in P$ and $t \in [0, 1]$.

Lemma 1 ([2, Lemma 2.2]). Assume that $g \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\beta > 0$ that belongs to $C(0, 1) \cap L(0, 1)$. Then

$$I^\beta D^\beta g(t) = g(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2} + \cdots + c_N t^{\beta-N},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \dots, N$, where N is the smallest integer greater than or equal to β .

Lemma 2 ([21, Lemma 2.1]). Assume that $\alpha > 0$ and $n = [\alpha] + 1$. If the function $y \in L[0, 1] \cap C[0, 1]$, then there exists $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, such that

$$I^\alpha ({}^c D^\alpha f(t)) = f(t) - c_1 - c_2 t - \cdots - c_n t^{n-1}.$$

Lemma 3. The problem (1.1) has a unique solution as follows:

$$y(t) = \int_0^1 G(t, s) \Phi_q(I^\beta f(s, y(s))) ds, \quad t \in [0, 1]$$

where

$$G(t, s) = \begin{cases} \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s)) - \alpha(2-k)(t-s)^{\alpha-1}}{(2-k)\Gamma(\alpha+1)}, & 0 \leq s \leq t \leq 1, \\ \frac{2t(1-s)^{\alpha-1}(\alpha-k(1-s))}{(2-k)\Gamma(\alpha+1)}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (2.1)$$

is the Green's function.

Proof. Let $u(t) = \varphi_p({}^c D^\alpha y(t))$, we now show that IBVP (1.1) can be expressed as the following IBVP:

$$\begin{cases} D^\beta u(t) + f(t, y(t)) = 0, \\ u(0) = u'(0) = 0 \end{cases} \quad (2.2)$$

and

$$\begin{cases} {}^c D^\alpha y(t) = \varphi_q(u(t)), & t \in (0, 1) \\ y(0) = y''(0) = 0, \quad y(1) = k \int_0^1 y(s) ds. \end{cases} \quad (2.3)$$

Using Lemma 1 and (2.2), we get

$$u(t) = -I^\beta f(t, y(t)) + c_1 t^{\beta-1} + c_2 t^{\beta-2},$$

since $u(0) = u'(0) = 0$, then $c_1 = c_2 = 0$ and we have

$$u(t) = -I^\beta f(t, y(t)) = \frac{-1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, y(s)) ds.$$

Also, from (2.3) and [4], we get

$$\begin{aligned} y(t) &= \int_0^t \frac{2t(1-s)^{\alpha-1}(\alpha - k(1-s)) - \alpha(t-s)^{\alpha-1}(2-k)}{(2-k)\Gamma(\alpha+1)} \varphi_q(I^\beta f(s, y(s))) ds \\ &\quad + \int_t^1 \frac{2t(1-s)^{\alpha-1}(\alpha - k(1-s))}{(2-k)\Gamma(\alpha+1)} \varphi_q(I^\beta f(s, y(s))) ds \\ &= \int_0^1 G(t, s) \varphi_q(I^\beta f(s, y(s))) ds. \end{aligned}$$

This completes the proof. $\square$

Lemma 4 ([21, Lemma 2.3]). *The function $G(t, s)$ defined in (2.1) satisfies the following properties:*

- (1) $0 < G(t, s) \leq \frac{2}{(2-k)\Gamma(\alpha)}$, for $t, s \in (0, 1)$ if and only if $0 < k < 2$.
- (2) $tG(1, s) \leq G(t, s) \leq \frac{2\alpha}{k(\alpha-2)}G(1, s)$, for all $t, s \in (0, 1)$, $2 < \alpha < 3$ and $0 < k < 2$.
- (3) $G(t, s)$ and $\frac{G(t, s)}{t}$ are two continuous functions for all $t, s \in [0, 1]$, $2 < \alpha < 3$ and $k \neq 2$.

Lemma 5 ([11, Theorem 2.3]). *Let K be a Banach space and let $X \subset K$ be a cone in K . Assume that Ω_1 and Ω_2 are open subsets of K with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T: X \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow X$ be completely continuous operator. In addition, suppose that either*

- (1) $\|Ty\| \leq \|y\|$, for all $y \in X \cap \partial\Omega_1$ and $\|Ty\| \geq \|y\|$, for all $y \in X \cap \partial\Omega_2$ or
 - (2) $\|Ty\| \leq \|y\|$, for all $y \in X \cap \partial\Omega_2$ and $\|Ty\| \geq \|y\|$, for all $y \in X \cap \partial\Omega_1$ holds.
- Then T has a fixed point in $X \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

Lemma 6 ([22, Lemma 2.6]). *Let K be a real Banach space, $P \subset K$ be a cone, $\Omega_r = \{y \in P: \|y\| \leq r\}$. Let the operator $T: P \cap \Omega_r \rightarrow P$ be completely continuous and satisfying $Tu \neq u$, $\forall u \in \partial\Omega_r$.*

Then

- (1) *if $\|Tu\| \leq \|u\|$, $\forall u \in \partial\Omega_r$, then $i(T, \Omega_r, P) = 1$,*
- (2) *if $\|Tu\| \geq \|u\|$, $\forall u \in \partial\Omega_r$, then $i(T, \Omega_r, P) = 0$.*

3. MAIN RESULTS

In this section we show the existence results for (1.1). We define the operator $T: C[0, 1] \rightarrow C[0, 1]$ as

$$Ty(t) = \int_0^1 G(t, s) \phi_q(I^\beta f(s, y(s))) ds, \quad (3.1)$$

with G defined in (2.1). Let $K = C[0, 1]$ be a Banach space endowed with norm $\|\cdot\|$ and $P \subset K$ be the cone expressed as follows

$$P = \left\{ y \in K, \frac{y(t)}{t} \in K, y(t) \geq \frac{tk(\alpha - 2)}{2\alpha} \|y\|, \text{ for all } t \in [0, 1] \right\}. \quad (3.2)$$

We set

$$f_0^* = \lim_{y \rightarrow 0^+} \left\{ \min_{t \in [0, 1]} \frac{f(t, y)}{\phi_p(y)} \right\} \text{ and } f_\infty^* = \lim_{y \rightarrow \infty} \left\{ \max_{t \in [0, 1]} \frac{f(t, y)}{\phi_p(y)} \right\}.$$

Theorem 1. *Suppose that either of the two following conditions is satisfied:*

- (1) $f_0^* = \infty$ and $f_\infty^* = 0$.
- (2) $f_0^* = 0$, $f_\infty^* = \infty$ and there exists $v > 0$ for which $f(t, \omega) \geq v \omega^\delta f(t, \omega)$ for all $t \in (0, 1]$.

Then, for all $\alpha \in (2, 3)$ and $k \in (0, 2)$, the problem (1.1) has at least one solution that belongs to the cone P defined in (3.2)

Proof. Firstly, we prove that $T: P \rightarrow P$ is completely continuous. By the continuity and the non-negativeness of functions G and f on their domains of definition, we get that if $y \in P$ then $Ty \in K$ and $Ty(t) \geq 0$ for all $t \in [0, 1]$.

Furthermore,

$$\frac{Ty(t)}{t} = \int_0^1 \frac{G(t, s)}{t} \phi_q(I^\beta f(s, y(s))) ds,$$

that is, from condition (3) in Lemma 4, a continuous function for all $y \in K$. We can see that $T(P) \subset P$. We take $y \in P$, then for all $t \in [0, 1]$ the following inequalities are satisfied

$$Ty(t) = \int_0^1 G(t, s) \phi_q(I^\beta f(s, y(s))) ds$$

$$\geq \frac{tk(\alpha-2)}{2\alpha} \max_{t \in [0,1]} \left\{ \int_0^1 G(t,s) \Phi_q(I^\beta f(s,y(s))) ds \right\} = \frac{tk(\alpha-2)}{2\alpha} \|Ty\|.$$

By the continuity of functions G and f , the operator $T: P \rightarrow P$ is continuous. Let $\Omega \subset P$ be bounded, that is, there exists a positive constant $\mathcal{M} > 0$ such that $\|y\| \leq \mathcal{M}$ for all $y \in \Omega$.

We define $\mathcal{L} = \max_{0 \leq t \leq 1, 0 \leq y \leq \mathcal{M}} |f(t,y)| + 1$.

Then, for all $y \in \Omega$, it is satisfied that

$$\begin{aligned} |Ty(t)| &\leq \int_0^1 G(t,s) \Phi_q(I^\beta f(s,y(s))) ds \\ &\leq \frac{\mathcal{L}^{q-1}}{(\Gamma(\beta+1))^{q-1}} \int_0^1 G(t,s) ds, \quad \text{for all } t \in [0,1]. \end{aligned}$$

Therefore, the set $T(\Omega)$ is bounded in K . For each $y \in \Omega$, we get

$$\begin{aligned} |(Ty)'(t)| &= \left| - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} \Phi_q(I^\beta f(s,y(s))) ds \right. \\ &\quad \left. + \int_0^1 \frac{2(1-s)^{\alpha-1}(\alpha-k+ks)}{(2-k)\alpha\Gamma(\alpha)} \Phi_q(I^\beta f(s,y(s))) ds \right| \\ &\leq \frac{\mathcal{L}^{q-1}}{(\Gamma(\beta+1))^{q-1}} \left[\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)} \right] =: \mathcal{N}. \end{aligned}$$

Consequently, for all $t_1, t_2 \in [0,1], t_1 < t_2$, we obtain

$$|(Ty)(t_2) - Ty(t_1)| \leq \int_{t_1}^{t_2} |(Ty)'(s)| ds \leq \mathcal{N}(t_2 - t_1).$$

and the set $T(\Omega)$ is equicontinuous. By the Arzela-Ascoli Theorem we conclude that $\overline{T(\Omega)}$ is compact, that is, $T: P \rightarrow P$ is a completely continuous operator. We now consider the following cases:

Case 1: In part 1 of Theorem 1, $f_0^* = \infty$ and $f_\infty^* = 0$. Since $f_0^* = \infty$, then there exists a constant $\eta_1 > 0$ such that $f(t,y) \geq \Phi_p(\vartheta_1 y)$ for all $0 < y \leq \eta_1$, where $\vartheta_1 > 0$ satisfies

$$\vartheta_1 \frac{k(\alpha-2)}{2\alpha(\Gamma(\beta+1))^{q-1}} \int_0^1 G(t,s) s^{\beta(q-1)+1} ds \geq 1. \quad (3.3)$$

We assign $y \in P$, such that $\|y\| = \eta_1$, then from (3.3), we get

$$\begin{aligned} \|Ty\| &= \max_{t \in [0,1]} \left\{ \int_0^1 G(t,s) \Phi_q(I^\beta f(s,y(s))) ds \right\} \\ &\geq \|y\| \vartheta_1 \frac{k(\alpha-2)}{2\alpha(\Gamma(\beta+1))^{q-1}} \int_0^1 G(t,s) s^{\beta(q-1)+1} ds \geq \|y\|. \end{aligned}$$

Since $f(t, \cdot)$ is a continuous function on $[0, \infty]$, we denote

$$\bar{f}(t, y) = \max_{u \in [0, y]} \{f(t, u)\}.$$

Trivially, $\bar{f}(t, \cdot)$ is non-decreasing on $[0, \infty)$. Also, since $f_\infty^* = 0$, it is easy to see from [4] that

$$\lim_{y \rightarrow \infty} \left\{ \max_{t \in [0, 1]} \frac{\bar{f}(t, y)}{y} \right\} = 0.$$

We assign $\vartheta_2 > 0$ satisfying

$$\vartheta_2 \frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \leq 1. \quad (3.4)$$

Then, there exists $\eta_2, \eta_1 > 0$ such that $\bar{f}(t, y) \leq \varphi_p(\vartheta_2 y)$ for all $y \geq \eta_2$. We let $y \in P$ be such that $\|y\| = \eta_2$, therefore, from the definition of $\bar{f}$ and (3.4), we obtain the inequalities:

$$\begin{aligned} \|Ty\| &= \max_{t \in [0, 1]} \left\{ \int_0^1 G(t, s) \varphi_q(I^\beta f(s, y(s))) ds \right\} \\ &\leq \|y\| \vartheta_2 \frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \leq \|y\|. \end{aligned}$$

Therefore, from the Guo Krasnoselskii fixed point theorem, we conclude that IBVP (1.1) has at least one positive solution.

Case 1: We let $\vartheta_2 > 0$ be as given in (3.4), since $f_0 = 0$, there exists a constant $x_1 > 0$ such that $f(t, y) \leq \varphi_p(\vartheta_2 y)$ for $0 \leq y \leq x_1$. We set $y \in P$, such that $\|y\| = x_1$. Then, we get

$$\begin{aligned} Ty(t) &= \int_0^1 G(t, s) \varphi_q(I^\beta f(s, y(s))) ds \leq \vartheta_2 \int_0^1 G(t, s) \varphi_q(I^\beta f(s, y(s))) ds \\ &\leq \vartheta_2 \|y\| \frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \\ &\leq \|y\|. \end{aligned}$$

We now consider $\vartheta_3 > 0$ which satisfies

$$v \vartheta_3 \frac{k(\alpha-2)}{2\alpha(\Gamma(\beta+1))^{q-1}} \max_{t \in [0, 1]} \left\{ \int_0^1 s^{\beta(q-1)+\delta} G(t, s) ds \right\} \geq 1,$$

where $v > 0$ and $\delta > 0$ the constants given in Condition 2 of Theorem 1. $f_\infty^* = \infty$ implies that there exists constants $x_2 > x_1 > 0$ such that $f(t, y) \geq \varphi_p(\vartheta_3 y)$ for all $y \geq x_2$. We let $y \in P$ such that $\|y\| = \frac{2\alpha(\Gamma(\beta+1))^{q-1}}{k(\alpha-2)} x_2$. Therefore, for all $t > 0$, the following inequality holds:

$$\frac{y(t)}{t} \geq \frac{k(\alpha-2)}{2\alpha(\Gamma(\beta+1))^{q-1}} \|y\| = x_2.$$

Condition 2 of Theorem 1 gives us the following property:

$$\begin{aligned} \|Ty\| &= \max_{t \in [0,1]} \left\{ \int_0^1 G(t,s) \varphi_q(I^\beta f(s, y(s))) ds \right\} \\ &\geq \nu \max_{t \in [0,1]} \left\{ \int_0^1 s^\delta G(t,s) \varphi_q \left(I^\beta f \left(s, \frac{y(s)}{s} \right) \right) ds \right\} \\ &\geq \nu \vartheta_3 \frac{k(\alpha-2)}{2\alpha(\Gamma(\beta+1))^{q-1}} \|y\| \max_{t \in [0,1]} \left\{ \int_0^1 s^{\beta(q-1)+\delta} G(t,s) ds \right\} \geq \|y\|. \end{aligned}$$

Since $\frac{y(t)}{t}$ is a continuous on $(0,1]$ and $\lim_{t \rightarrow 0^+} \frac{y(t)}{t}$ exist, then the integrals in the previous inequalities are well defined. Thus by the Guo Krasnoselskii fixed point theorem, we conclude that IBVP (1.1) has at least one positive solution. $\square$

Remark 1. Condition 2 of Theorem 1 for problem (1.1) generalizes: $f: [0, \infty) \rightarrow (0, \infty)$ is non-decreasing and there exists $\delta \in (0, 1)$ such that $f(\iota\omega) \geq \iota^\delta f(\omega)$ for all $\iota \in (0, 1)$ and $\omega \in [0, \infty)$, which is condition (A3) imposed in [4]. This condition is less restrictive if δ is greater since $\iota \in (0, 1)$. In addition, assumptions of monotonicity are not imposed on the function f .

Example 1. Consider the fractional differential equation:

$$\begin{cases} D^{\frac{3}{2}}(\varphi_2({}^c D^{\frac{5}{2}} y(t))) + 1 + \frac{1}{2} \sin(y) = 0, & t \in [0, 1], \\ y(0) = y''(0) = 0, \quad y(1) = \frac{\pi}{3+\sin(1)} \int_0^1 y(s) ds, \\ \varphi_2({}^c D^{\frac{5}{2}} y(0)) = [\varphi_2({}^c D^{\frac{5}{2}} y(0))]' = 0. \end{cases} \quad (3.5)$$

Trivially, $f_0^* = \infty$ and $f_\infty^* = 0$. Therefore, from Theorem 1, we conclude that problem (3.5) has at least one positive solution.

We now proceed to show that IBVP (1.1) has at least two positive solutions. We define

$$\begin{aligned} f^0 &= \lim_{y \rightarrow 0^+} \sup_{t \in [0,1]} \frac{f(t,y)}{\varphi_p(l_1 \|y\|)}, \quad f_0 = \lim_{y \rightarrow 0^+} \inf_{t \in [0,1]} \frac{f(t,y)}{\varphi_p(l_2 \|y\|)}, \\ f^\infty &= \lim_{y \rightarrow +\infty} \sup_{t \in [0,1]} \frac{f(t,y)}{\varphi_p(l_3 \|y\|)}, \quad f_\infty = \lim_{y \rightarrow +\infty} \inf_{t \in [0,1]} \frac{f(t,y)}{\varphi_p(l_4 \|y\|)}. \end{aligned}$$

Let

$$B = \frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \quad \text{and} \quad B_1 = \frac{1}{4(\Gamma(\beta+1))^{q-1}} \int_{s \in I} s^{\beta(q-1)} G(1,s) ds,$$

where $I \in [\frac{1}{4}, \frac{3}{4}]$.

Theorem 2. Assume that $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$, and the following conditions hold;

(N₁) $f_0 = f_\infty = +\infty$.

(N₂) There exists a constant $\rho_1 > 0$ such that $f(t, y) \leq \Phi_p(l_5 \|y\|)$ for $t \in [0, 1]$, $y \in [0, \rho_1]$.

Then, IBVP (1.1) has at least two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| < \rho_1 < \|y_2\|, \quad \text{for} \\ 0 < \frac{1}{l_2 B_1} < 1 < \frac{1}{l_5 B} < +\infty \quad \text{and} \quad 0 < \frac{1}{l_4 B_1} < 1 < \frac{1}{l_5 B} < +\infty. \quad (3.6)$$

Proof. Since

$$f_0 = \lim_{y \rightarrow 0^+} \inf_{t \in [0, 1]} \frac{f(t, y)}{\Phi_p(l_2 \|y\|)} = +\infty,$$

there is $\rho_0 \in (0, \rho_1)$ such that

$$f(t, y) \geq \Phi_p(l_2 \|y\|) \text{ for } t \in [0, 1], y \in [0, \rho_0].$$

Let

$$\Omega_{\rho_0} = \{y \in P : \|y\| \leq \rho_0\}.$$

Then, for any $y \in \partial\Omega_{\rho_0}$, it follows from Lemma 4 that

$$\begin{aligned} (Ty)(t) &= \int_0^1 G(t, s) \Phi_q \left(\int_0^1 I^\beta f(s, y(s)) \right) ds \\ &\geq \min_{t \in I} \left\{ \frac{l_2 t}{(\Gamma(\beta + 1))^{q-1}} \int_0^1 s^{\beta(q-1)} G(1, s) ds \|y\| \right\} \\ &\geq l_2 \frac{1}{4(\Gamma(\beta + 1))^{q-1}} \int_{s \in I} s^{\beta(q-1)} G(1, s) ds \|y\|. \end{aligned}$$

Therefore,

$$\|Ty\| \geq l_2 B_1 \|y\|.$$

Considering also (3.6), we get

$$\|Ty\| \geq \|y\|, \quad \forall y \in \partial\Omega_{\rho_0}.$$

By Lemma 6, we get

$$i(T, \Omega_{\rho_0}, P) = 0. \quad (3.7)$$

Also,

$$f_\infty = \lim_{y \rightarrow \infty} \inf_{t \in [0, 1]} \frac{f(t, y)}{\Phi_p(l_4 \|y\|)} = +\infty,$$

there is $\rho_0^*, \rho_0^* > \rho_1$, such that

$$f(t, y) \geq \Phi_p(l_4 \|y\|) \text{ for } t \in [0, 1], y \in [\rho_0^*, +\infty).$$

Let

$$\Omega_{\rho_0^*} = \{y \in P : \|y\| \leq \rho_0^*\}.$$

Then, for any $y \in \partial\Omega_{\rho_0^*}$, it follows from Lemma 4 that

$$\begin{aligned} (Ty)(t) &= \int_0^1 G(t,s) \Phi_q \left(\int_0^1 I^\beta f(s, y(s)) \right) ds \\ &\geq \min_{t \in I} \left\{ \frac{l_4 t}{(\Gamma(\beta+1))^{q-1}} \int_0^1 s^{\beta(q-1)} G(1,s) ds \|y\| \right\} \\ &\geq l_4 \frac{1}{4(\Gamma(\beta+1))^{q-1}} \int_{s \in I} s^{\beta(q-1)} G(1,s) ds \|y\|. \end{aligned}$$

Therefore,

$$\|Ty\| \geq l_4 B_1 \|y\|.$$

Considering also (3.6), we get

$$\|Ty\| \geq \|y\|, \quad \forall y \in \partial\Omega_{\rho_0^*}.$$

By Lemma 6, we get

$$i(T, \Omega_{\rho_0^*}, P) = 0.$$

Finally, let $\Omega_{\rho_1} = \{y \in P : \|y\| \leq \rho_1\}$ for any $y \in \partial\Omega_{\rho_1}$, it follows from Lemma 4, 6 and (N_2) that

$$\begin{aligned} (Ty)(t) &= \int_0^1 G(t,s) \Phi_q \left(\int_0^1 I^\beta f(s, y(s)) \right) ds \\ &\leq \max_{0 \leq t \leq 1} \left\{ \frac{l_5}{(\Gamma(\beta+1))^{q-1}} \int_0^1 s^{\beta(q-1)} G(t,s) ds \|y\| \right\} \\ &= l_5 \frac{2}{(2-k)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \|y\|. \end{aligned}$$

Therefore,

$$\|Ty\| \leq l_5 B \|y\|.$$

Considering also (3.6), we get

$$\|Ty\| \leq \|y\|, \quad \forall y \in \partial\Omega_{\rho_1}.$$

By Lemma 6, we get

$$i(T, \Omega_{\rho_1}, P) = 1. \tag{3.8}$$

From (3.7)-(3.8) and $\rho_0 < \rho_1 < \rho_0^*$, we get

$$i(T, \Omega_{\rho_0^*} \setminus \overline{\Omega}_{\rho_1}, P) = -1, \quad i(T, \Omega_{\rho_1} \setminus \overline{\Omega}_{\rho_0}, P) = 1.$$

Thus, T has a fixed point $y_1 \in \Omega_{\rho_1} \setminus \overline{\Omega}_{\rho_0}$ and a fixed point $y_2 \in \Omega_{\rho_0^*} \setminus \overline{\Omega}_{\rho_1}$. Trivially, y_1, y_2 are both positive solutions of IBVP (1.1) and $0 < \|y_1\| < \rho_1 < \|y_2\|$. $\square$

Similarly, we get the following results;

Corollary 1. Assume that $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$ and the following conditions hold:

- (N_1) $f^0 = f^\infty = 0$.
 (N_2) There exists a constant $\rho_2 > 0$ such that $f(t, y) \geq \varphi_p(l_6 \|y\|)$ for $t \in [0, 1]$, $y \in [0, \rho_2]$. Then IBVP (1.1) has at least two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| < \rho_2 < \|y_2\| \quad \text{for} \\ 0 < \frac{1}{l_6 B_1} < 1 < \frac{1}{l_3 B} < +\infty \quad \text{and} \quad 0 < \frac{1}{l_6 B_1} < 1 < \frac{1}{l_1 B} < +\infty.$$

Example 2. Consider the following integral boundary value problem:

$$\begin{cases} D^{\frac{9}{5}}(\varphi_2({}^c D^{\frac{7}{3}} y(t))) + \frac{1}{3}(\|y\|^{\frac{1}{2}} + \frac{2}{3}\|y\|) = 0, & t \in [0, 1], \\ y(0) = y''(0) = 0, \quad y(1) = \frac{3}{4} \int_0^1 y(s) ds, \\ \varphi_2({}^c D^{\frac{7}{3}} y(0)) = [\varphi_2({}^c D^{\frac{7}{3}} y(0))]' = 0, \end{cases} \quad (3.9)$$

where $\alpha = \frac{7}{3}$, $\beta = \frac{9}{5}$, $p = 2$, $k = \frac{3}{4}$. By computation we see that $B = 0.80159$ and $B_1 = 0.0022194$. Taking $\rho_1 = 9$, $l_5 = \frac{1}{3}$, we get

$$f(t, y) \leq \frac{1}{3} \left(3 + \frac{2}{3} \cdot 9 \right) = 3 = \varphi_p(l_5 \|y\|) = \varphi_2 \left(\frac{1}{3} \cdot 9 \right), \quad \text{for } t \in [0, 1], y \in [0, \rho_1].$$

Therefore, condition (N_2) is satisfied. It can be easily seen that condition (N_1) holds. Also, let $l_2 = 130$ and $l_4 = 121$, we get

$$0 < \frac{1}{l_2 B_1} < 1 < \frac{1}{l_5 B} < +\infty \quad \text{and} \quad 0 < \frac{1}{l_4 B_1} < 1 < \frac{1}{l_5 B} < +\infty.$$

Hence, by Theorem 2, IBVP (3.9) has at least two solutions y_1 and y_2 such that $0 < \|y_1\| < 9 < \|y_2\|$ for the given values of l_5 , l_2 and l_4 .

4. EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS

Sufficient conditions for the existence of at least three positive solutions for the p -Laplacian IBVP (1.1) are derived by means of the Five functional fixed point theorem in this section. Let γ, Υ and ω be nonnegative continuous convex functionals on P , ψ and ϑ be a nonnegative continuous concave functionals on P . Therefore, for positive real numbers h', a', b', d' and c' , we denote the following convex sets

$$\begin{aligned} P(\gamma, c') &= \{y \in P: \gamma(y) < c'\}, \\ P(\gamma, \vartheta, a', c') &= \{y \in P: a' \leq \vartheta(y); \gamma(y) \leq c'\}, \\ R(\gamma, \Upsilon, d', c') &= \{y \in P: \Upsilon(y) \leq d'; \gamma(y) \leq c'\}, \\ P(\gamma, \omega, \vartheta, a', b', c') &= \{y \in P: a' \leq \vartheta(y); \omega(y) \leq b'; \gamma(y) \leq c'\}, \end{aligned}$$

$$R(\gamma, \Upsilon, \psi, h', d', c') = \{y \in P : h' \leq \psi(y); \Upsilon(y) \leq d'; \gamma(y) \leq c'\}.$$

Theorem 3 ([23, Theorem 4.1]). *Suppose P be a cone in a real Banach space K . Let ϑ and ψ be nonnegative continuous concave functionals on P , γ, Υ and ω be a nonnegative continuous convex functionals on P , such that for some positive numbers c' and e' , $\vartheta(y) \leq \Upsilon(y)$ and $\|y\| \leq e'\gamma(y)$, for all $y \in \overline{P(\gamma, c')}$. Suppose further that*

$T : \overline{P(\gamma, c')} \rightarrow \overline{P(\gamma, c')}$ is completely continuous and there exist constants h', d', a' and $b' \geq 0$ with $0 < d' < a'$ such that each of the following is satisfied

- (C₁) $\{y \in P(\gamma, \omega, \vartheta, a', b', c') : \vartheta(y) > a'\} \neq \emptyset$ and $\vartheta(Ty) > a'$ for $y \in P(\gamma, \omega, \vartheta, a', b', c')$,
- (C₂) $\{y \in R(\gamma, \Upsilon, \psi, h', d', c') : \Upsilon(y) > d'\} \neq \emptyset$ and $\Upsilon(Ty) > d'$ for $y \in R(\gamma, \Upsilon, \psi, h', d', c')$,
- (C₃) $\vartheta(Ty) > a'$ provided $y \in P(\gamma, \vartheta, a', c')$ with $\omega(Ty) > b'$,
- (C₄) $\Upsilon(Ty) < d'$ provided $y \in R(\gamma, \Upsilon, \psi, h', d', c')$ with $\psi(Ty) < h'$.

Then, T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\gamma, c')}$ such that

$$\Upsilon(y_1) \leq d', a' < \vartheta(y_2) \text{ and } d' < \Upsilon(y_3) \text{ with } \vartheta(y_3) < a'.$$

We denote the nonnegative continuous concave functionals ϑ, ψ and the nonnegative continuous convex functionals Υ, ω, γ on P by

$$\begin{aligned} \vartheta(y) &= \min_{t \in I} y(t), \quad \psi(y) = \min_{t \in I_1} y(t), \\ \gamma(y) &= \max_{t \in [0,1]} y(t), \quad \Upsilon(y) = \max_{t \in I_1} y(t), \quad \omega(y) = \max_{t \in I} y(t), \end{aligned}$$

where $I = [\frac{1}{4}, \frac{3}{4}]$ and $I_1 = [\frac{1}{3}, \frac{2}{3}]$. For any $y \in P$,

$$\vartheta(y) = \min_{t \in I} y(t) \leq \max_{t \in I_1} y(t) = \Upsilon(y) \quad (4.1)$$

and

$$\|y\| \leq 4M \min_{t \in I} y(t) \leq 4M \max_{t \in [0,1]} y(t) = 4M\gamma(y), \quad (4.2)$$

where $M = \frac{2\alpha}{k(\alpha-2)}$.

Let

$$\begin{aligned} E &= \left[\frac{1}{4(\Gamma(\beta+1))^{q-1}} \int_{s \in I} s^{\beta(q-1)} G(1, s) ds \right]^{-1}, \\ Z &= \left[\frac{M}{(\Gamma(\beta+1))^{q-1}} \int_0^1 s^{\beta(q-1)} G(1, s) ds \right]^{-1}. \end{aligned}$$

Theorem 4. *Suppose there exist constants $0 < a' < b' < 4Mb' \leq c'$ and assume that f satisfies the following conditions:*

- (H₁) $f(t, y) < \phi_p(a'Z)$ for $(t, y) \in [0, 1] \times [\frac{1}{4M}a', a']$;
- (H₂) $f(t, y) > \phi_p(b'E)$ for $(t, y) \in [\frac{1}{4}, \frac{3}{4}] \times [b', 4Mb']$;
- (H₃) $f(t, y) < \phi_p(c'Z)$ for $(t, y) \in [0, 1] \times [0, c']$.

Then the integral boundary value problem (1.1) has at least three positive solutions y_1, y_2 and y_3 such that $\Upsilon(y_1) < a', b' < \vartheta(y_2)$ and $a' < \Upsilon(y_3)$ with $\vartheta(y_3) < b'$.

Proof. We need to show that there exist three fixed points $y_1, y_2, y_3 \in P$ of T defined in (3.1).

Since T is completely continuous and from (4.1) - (4.2), for each $y \in P$, $\vartheta(y) \leq \Upsilon(y)$ and $\|y\| \leq 4M\gamma(y)$, we now show that $T: \overline{P(\gamma, c')} \rightarrow \overline{P(\gamma, c')}$. Let $y \in \overline{P(\gamma, c')}$, then

$0 \leq y \leq c'$. We use (H_3) to get

$$\begin{aligned} \gamma(Ty) &= \max_{t \in [0,1]} \int_0^1 G(t,s) \phi_q(I^\beta f(s, y(s))) ds \\ &\leq \frac{c' Z M}{(\Gamma(\beta + 1))^{q-1}} \int_0^1 s^{\beta(q-1)} G(1,s) ds < c'. \end{aligned}$$

Thus, $T: \overline{P(\gamma, c')} \rightarrow \overline{P(\gamma, c')}$. We show that conditions (C_1) and (C_2) of Theorem 3 are satisfied. Trivially,

$$\frac{b' + 4Mb'}{2} \in \{y \in P(\gamma, \omega, \vartheta, b', 4Mb', c') : \vartheta(y) > b'\} \neq \emptyset$$

and

$$\frac{\frac{1}{4M}a' + a'}{2} \in \{y \in R(\gamma, \Upsilon, \psi, \frac{1}{4M}a', a', c') : \Upsilon(y) < a'\} \neq \emptyset.$$

We proceed by letting $y \in P(\gamma, \omega, \vartheta, b', 4Mb', c')$ or $y \in R(\gamma, \Upsilon, \psi, \frac{1}{4M}a', a', c')$. Then, $b' \leq y \leq 4Mb'$ and $\eta a' \leq y \leq a'$, where $\eta > 0$. We apply condition (H_2) to obtain

$$\begin{aligned} \vartheta(Ty) &= \min_{t \in I} \int_0^1 G(t,s) \phi_q(I^\beta f(s, y(s))) ds \\ &> \frac{1}{4(\Gamma(\beta + 1))^{q-1}} b' E \int_{s \in I} s^{\beta(q-1)} G(1,s) ds = b'. \end{aligned}$$

Intuitively, by condition (H_1) , we get

$$\begin{aligned} \Upsilon(Ty) &= \max_{t \in I_1} \int_0^1 G(t,s) \phi_q(I^\beta f(s, y(s))) ds \\ &< \frac{M a' Z}{(\Gamma(\beta + 1))^{q-1}} \int_0^1 s^{\beta(q-1)} G(1,s) ds = a'. \end{aligned}$$

To show that (C_3) is satisfied, let $y \in P(\gamma, \vartheta, b', c')$ with $\omega(Ty) > 4Mb'$. Then

$$\begin{aligned} \vartheta(Ty) &= \min_{t \in I} \int_0^1 G(t,s) \phi_q(I^\beta f(s, y(s))) ds \\ &\geq \frac{1}{4M} \max_{t \in I} \int_0^1 G(t,s) \phi_q(I^\beta f(s, y(s))) ds = \frac{1}{4M} \omega(Ty) > b'. \end{aligned}$$

Finally, we prove that (H_4) holds. Let $y \in R(\gamma, \Upsilon, a', c')$ with $\psi(Ty) < \frac{1}{4M}a'$. Then, we get

$$\begin{aligned}\Upsilon(Ty) &= \max_{t \in I_1} \int_0^1 G(t, s) \phi_q(I^\beta f(s, y(s))) ds \\ &\leq 4M \min_{t \in I_1} \int_0^1 G(t, s) \phi_q(I^\beta f(s, y(s))) ds = 4M\psi(Ty) < a'.\end{aligned}$$

We have shown that all conditions of Theorem 3 are satisfied. Therefore, the IBVP (1.1) has at least three positive solutions y_1, y_2 and y_3 such that $\Upsilon(y_1) < a', b' < \vartheta(y_2)$ and $a' < \Upsilon(y_3)$ with $\vartheta(y_3) < b'$. This completes the proof. $\square$

Example 3. Consider the fractional differential equation:

$$\begin{cases} D^{\frac{3}{2}}(\phi_2({}^c D^{\frac{5}{2}}y(t))) + f(t, y(t)) = 0, & t \in [0, 1], \\ y(0) = y''(0) = 0, \quad y(1) = \frac{1}{2} \int_0^1 y(s) ds, \\ \phi_2({}^c D^{\frac{5}{2}}y(0)) = [\phi_2({}^c D^{\frac{5}{2}}y(0))]' = 0. \end{cases} \quad (4.3)$$

where

$$f(t, y) = \begin{cases} \frac{t}{20} + y^2 + 216y^3, & y \leq 1, \\ \frac{t}{20} + y + 216, & y > 1. \end{cases}$$

We set $a' = \frac{1}{10}$, $b' = 1$ and $c' = 100$. By computation, $4Mb' = 80$, $E = 216,02$ and $Z = 4,0755$. As a result, $f(t, y)$ satisfies

$$\begin{aligned}f(t, y) &= \frac{t}{20} + y^2 + 216y^3 \leq 0,28 < \phi_2(a'Z) \approx 0,41 \text{ for } (t, y) \in [0, 1] \times \left[\frac{1}{400}, \frac{1}{10}\right], \\ f(t, y) &= \frac{t}{20} + y + 216 \geq 217,01 > \phi_2(b'E) \approx 216,02 \text{ for } (t, y) \in I \times [1, 80], \\ f(t, y) &= \frac{t}{20} + y + 216 \leq 316,05 < \phi_2(c'Z) \approx 407,55 \text{ for } (t, y) \in [0, 1] \times [0, 100].\end{aligned}$$

Since all conditions of Theorem 4 hold. Therefore, problem (4.3) has at least three positive solutions by Theorem 4.

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CERTAIN INVESTIGATIONS OF PSEUDO Z-SYMMETRIC SPACETIMES

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Abstract. The section 1 of this paper deals with the definition for a pseudo symmetric spacetime. In section 2, it is proved that a $(PZS)_4$ spacetime satisfying Codazzi type of Z-tensor does not exist. In which condition a $(PZS)_4$ spacetime can be a perfect fluid has been found as a necessary and sufficient condition. After that, special properties are obtained if the $(PZS)_4$ spacetime has harmonic conformal curvature tensor.

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1. INTRODUCTION

Symmetric spaces have been studied because of their importance in differential geometry. They are used in various branches of mathematics such as compact Lie groups, Grassmannian and bounded symmetric fields. Every symmetric space has its own special geometry such as Euclidean, elliptical and hyperbolic geometry, etc. The initial study on symmetric spaces was made by E. Cartan, [11].

On the other hand, these fields have a lot in common properties, and they have a beautiful theory. Symmetric spaces can be considered from many perspectives. They can be viewed as Riemannian manifolds with point reflection or with parallel curvature tensor or as a homogeneous space with a special isotropy or with special holonomy or special Killing vector fields or a particular Lie group involution.

Let us assume that (M, g, ∇) denotes an n -dimensional Riemannian manifold admitting the Levi-Civita connection ∇ . If the curvature tensor R of this manifold satisfies the condition $\nabla R = 0$, then this manifold is named locally symmetric. This condition is equivalent form to the fact that every point in $P \in M$ is the isometry of the local symmetry $F(P)$, [21]. Because of the importance of the symmetric manifolds in differential geometry, many authors have studied some properties of them, [12, 22, 27], etc. A pseudo symmetric spacetime is a four-dimensional time-oriented

Lorentzian manifold if the curvature tensor satisfies the condition, [26]

$$R_{hijk,l} = 2A_l R_{hijk} + A_h R_{lij k} + A_i R_{hljk} + A_j R_{hil k} + A_k R_{hij l} \quad (1.1)$$

such that A_l is a non-zero 1-form, R_{hijk} is the curvature tensor obeying the property $R_{lij k} = g_{hl} R_{ijk}^h$, and the notation “,” is the covariant derivative with respect to the metric tensor g . These manifolds are indicated by $(PS)_n$. In (1.1), for $A_l = 0$, a pseudo symmetric manifold is reduced to a symmetric manifold in the sense of Cartan.

(M, g) is called pseudo-Ricci symmetric $(PRS)_n$ manifold if its non-vanishing Ricci tensor R_{ij} obeys the condition

$$R_{ij,l} = 2A_l R_{ij} + A_i R_{lj} + A_j R_{il} \quad (1.2)$$

here A_l is a non-vanishing 1-form. As a particular case, if we consider the condition $R_{ij,k} = 0$ in (1.2) then this manifold is said to be Ricci symmetric.

In a Riemannian or a semi-Riemannian manifold of dimension n ($n > 3$), the conformal curvature tensor C_{ijk}^h of type $(1, 3)$ is given by the following form

$$\begin{aligned} C_{ijk}^h = R_{ijk}^h - \frac{1}{n-2} [R_k^h g_{ij} - R_j^h g_{ik} + R_{ij} \delta_k^h - R_{ik} \delta_j^h] \\ + \frac{R}{(n-1)(n-2)} [g_{ij} \delta_k^h - g_{ik} \delta_j^h] \end{aligned} \quad (1.3)$$

where R_{ijk}^h is the curvature tensor, R_{ij} is the Ricci tensor and R is the scalar curvature of this manifold, [28].

Assume that M is a semi-Riemannian manifold of dimension n and the metric g having the signature (p, q) such that $p + q = n$. If we choose $(p, q) = (1, n-1)$ or $(n-1, 1)$ then M is called a Lorentzian manifold, [21].

A generalized Robertson-Walker spacetime (GRW) is an n -dimensional spacetime ($n \geq 3$) written as a warped product $-I \times_\psi M^*$. Here, it is assumed that the warping function or scaling factor ($\psi > 0$) is a smooth function, M^* is considered as a Riemannian manifold with $(n-1)$ dimensional and an open interval of the real line is denoted by I , [1, 2]. If we consider M^* as a three-dimensional Riemannian manifold with constant curvature, then the GRW spacetime reduces to a RW spacetime. Examples of special spaces contained in these GRW spacetimes are de Sitter space-time, Einstein-de Sitter space-time, static Einstein space-time and Friedmann cosmological models. These spaces have been studied in many papers. Some of them are [4, 25] etc.

A perfect fluid is a four-dimensional spacetime whose non-vanishing Ricci tensor R_{ij} is of the form

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j \quad (1.4)$$

assuming that α and β are smooth functions, g is the Lorentzian metric and A_i is the velocity vector satisfying the conditions $A_i A^i = -1$ and $A^i = g^{ij} A_j$ [12, 22, 27].

We know that a *RW* spacetime is a perfect fluid spacetime [21] and also every four-dimensional *GRW* spacetime is a perfect fluid spacetime if and only if it is a *RW* spacetime. By using these results, some remarkable theorems are proved in [2–6, 8–17]. The energy-momentum tensor (briefly EMT) in general relativity theory defines the matter content. This content is considered a content that has dynamic and kinematic quantities such as density, pressure and velocity, acceleration, vortex, shear and expansion. It is known that a perfect fluid has no heat conduction terms and it has no EMT related to viscosity.

In a perfect fluid spacetime, EMT is given as

$$T_{ij} = (\sigma + p)A_i A_j + p g_{ij} \quad (1.5)$$

where σ is the energy density and p is the isotropic pressure of this spacetime, [21].

We know that the Einstein's field equations with the cosmological constant Λ are in the following form

$$R_{ij} - \frac{R}{2} + \Lambda g_{ij} = k T_{ij} \quad (1.6)$$

where k denotes the gravitational constant and T_{ij} is the energy momentum tensor satisfying the relation (1.5). And the Einstein's field equations (in short, EFE) without cosmological constant are in the following form

$$R_{ij} - \frac{R}{2} = k T_{ij}.$$

The dark energy in modern cosmology is considered as a candidate to accelerate the expansion of the universe, and the σ and p as scalar functions are hypothetically constrained by an equation of state that regulates the quality of the ideal fluid.

This is actually an equation of the form $p = p(\sigma, T_0)$, where T_0 represents the absolute temperature. To reduce the equation of state to $p = p(\sigma)$, the condition that T_0 must be constant. The perfect fluid with this property is called as isentropic [5]. If we choose as $\sigma = p$, then the perfect fluid is called as stiff matter, [7], [8], [9], [10], [13], [15], [16], [17], [18], [19], [20], [21].

If the energy density σ and the isotropic pressure p of the fluid are related by $p = w\sigma$, then w is said to be (EoS) parameter of the equation of state. (EoS) parameter plays an important role in examining the energy density and investigating the expansion of the universe. Using this parameter, different phases of the universe can be observed. The accelerating phase of the universe discussed in recent studies is portrayed when we consider the condition $w < -\frac{1}{3}$. If the conditions $-1 < w < 0$ and $w < -1$ hold, then it includes the quintessence phase and phantom regime, respectively.

The vector U is called as K -compatible vector field (where K can be the Riemann, the Weyl or a generalized tensor)

$$(U_i K_{jkl}^m + U_j K_{kil}^m + U_k K_{ijl}^m) U_m = 0. \quad (1.7)$$

And, U is a vector field which is called Riemann or Weyl permutable if the following relations hold

$$R_{kl[i}^m U_{j]} U_m = 0 \quad \text{or} \quad C_{kl[i}^m U_{j]} U_m = 0.$$

For a Riemannian manifold or a semi-Riemannian manifold, a vector field U is said to be a torse-forming if it satisfies

$$\nabla_X U = \rho X + \mu(X)U,$$

when $X \in TM$, $\mu(X)$ and ρ denote a linear form and a scalar function, respectively. If we consider the local transcription, the above relation is written as

$$U_{i,j} = \rho g_{ij} + \mu_j U_i \quad (1.8)$$

here U_i and μ_i denote the covariant components of U and μ . Considering the equation (1.8), if the vector field U_i satisfies the condition

$$U_{i,j} = \rho g_{ij} \quad (1.9)$$

then U_i is said to be a concircular vector field, [8, 15].

A generalized symmetric tensor of type $(0, 2)$ called Z -tensor was defined by Mantica and Molinari [16] in the form

$$Z_{ij} = R_{ij} + \phi g_{ij}, \quad (1.10)$$

ϕ being a scalar function. The trace of this tensor shown by Z is a scalar function and we have from (1.10)

$$Z = g^{ij} Z_{ij} = R + n\phi. \quad (1.11)$$

Taking $\phi = -\frac{R}{n}$, we obtain the classical Z -tensor. The generalized Z -tensor is called as the Z -tensor, shortly. When special cases of the Z -tensor are chosen, some well-known structures of Riemannian manifolds are found. For example,

- (i) if $Z_{ij} = 0$ (i.e, Z -flat), then this manifold reduces to an Einstein manifold [10],
- (ii) if $Z_{ij} = \lambda_i Z_{ij}$ (Z -recurrent), then this manifold reduces to a generalized Ricci recurrent manifold, where $Z_{ij} \neq 0$,
- (iii) if $Z_{ij,l} = Z_{il,j}$ (Codazzi tensor), then we find,

$$R_{ij,l} - R_{il,j} = \frac{1}{2(n-1)}(g_{ij}R_l - g_{il}R_j).$$

This result defines the nearly conformal symmetric manifold $(NCS)_n$, [23].

- (iv) By using (1.6) and (1.10), the Z -tensor is related by EMT of EFE with the cosmological constant Λ is in the form, [13]

$$Z_{ij} = kT_{ij} \quad (1.12)$$

where

$$\phi = -\frac{1}{2}R + \Lambda \quad (1.13)$$

and k is the gravitational constant and ϕ denotes an arbitrary scalar function. So, the Z -tensor may be considered as a generalized Einstein gravitational tensor. If we take $Z = 0$ which means the vacuum solution then we get an Einstein space with the condition $\Lambda = (\frac{n-2}{2n})R$ and the conservation of the total EMT ($T_{i,l}^l = 0$) gives $Z_{ij,l} = 0$. Thus, this spacetime has the conserved energy-momentum density.

These manifolds are studied in considerable details by many authors because of their importance [19, 20], etc. In section 2, we will discuss the properties of pseudo Z -symmetric spacetimes shown by $(PZS)_4$.

2. PSEUDO Z -SYMMETRIC SPACETIMES

In this section, using (1.2), we consider four-dimensional pseudo Z symmetric spacetime $(PZS)_4$ satisfying the following condition

$$Z_{ij,l} = 2A_l Z_{ij} + A_i Z_{jl} + A_j Z_{il} \quad (2.1)$$

where Z_{ij} is a symmetric tensor defined by (1.10), A_i is a non-null covector and it is called the associated 1-form of this spacetime.

If the Z -tensor satisfies EFE without cosmological constant, from (1.13), we have

$$\phi = -\frac{R}{2}. \quad (2.2)$$

Theorem 1. *Let us consider a $(PZS)_4$ spacetime. At a point P of this spacetime, if the scalar function Z is non-zero, then the Ricci tensor is of eigenvector A_k with the eigenvalue $\frac{5}{6}R$.*

Proof. Assume that M be a $(PZS)_4$ spacetime. If we multiply (2.1) by g^{il} , then we get

$$Z_{j,l}^l = 3A^m Z_{mj} + A_j Z. \quad (2.3)$$

On the other hand, differentiating the equation (1.10), it can be seen that

$$Z_{ij,l} = R_{ij,l} + \phi_l g_{ij}. \quad (2.4)$$

Now, multiplying (2.4) by g^{il} , one can obtain that

$$Z_{j,l}^l = R_{j,l}^l + \phi_j. \quad (2.5)$$

If we put the Ricci identity $R_{j,l}^l = \frac{1}{2}R_j$ in (2.5), then we find

$$Z_{j,l}^l = \frac{1}{2}R_j + \phi_j. \quad (2.6)$$

With the help of the equation (2.2), (2.6) can be reduced to

$$Z_{j,l}^l = 0. \quad (2.7)$$

Hence, if we compare the equations (2.3) and (2.7), then we can see that

$$A^m Z_{mj} = -\frac{Z}{3} A_j. \quad (2.8)$$

From (1.11) and (2.2), the equation (2.8) reduces to

$$A^m Z_{mj} = -\frac{2}{3} \phi A_j. \quad (2.9)$$

By the aid of (1.10) and (2.9), we obtain

$$A^m R_{mj} = -\frac{5}{3} \phi A_j. \quad (2.10)$$

Finally, from (2.2) and (2.10), we find

$$A^m R_{mj} = \frac{5}{6} R A_j.$$

Thus, we complete the proof. $\square$

Theorem 2. A $(PZS)_4$ spacetime with Codazzi type Z-tensor does not exist.

Proof. If we assume that $(PZS)_4$ spacetime is of Codazzi type, then we get from (2.1)

$$A_k Z_{ij} - A_j Z_{ik} = 0. \quad (2.11)$$

Multiplying (2.11) by A^k and assuming that $A^k A_k = \|A\|^2$,

$$Z_{ij} = \frac{1}{\|A\|^2} Z_{ik} A^k A_j. \quad (2.12)$$

Considering (2.9) in (2.12), we find

$$Z_{ij} = -\frac{2}{3\|A\|^2} \phi A_i A_j. \quad (2.13)$$

If we use the equation (1.10), (2.13) can be written as

$$R_{ij} = -\frac{2}{3\|A\|^2} \phi A_i A_j - \phi g_{ij}. \quad (2.14)$$

Multiplying (2.14) by g^{ij} and using the equation (2.2), one can easily seen that

$$\phi = 0.$$

From this result and the equation (2.14), we can say that $R_{ij} = 0$. Finally, we get from (1.10), $Z_{ij} = 0$. Hence, the proof is completed. $\square$

For a $(PZS)_4$ spacetime, considering the equations (1.3) and (2.1), the conformal curvature tensor is obtained as the following relation, [20]

$$C_{ijl,m}^m = \frac{1}{2} \left[A_j Z_{il} - \frac{4Z}{9} g_{il} A_j - A_i Z_{jl} + \frac{4Z}{9} g_{jl} A_i \right]. \quad (2.15)$$

In this case, assuming that $C_{jkl,m}^m = 0$ in (2.15), then we also have the following equation, [20]

$$Z_{ij} = \frac{4Z}{9} g_{ij} + \frac{7Z}{9} A_i A_j. \quad (2.16)$$

Now, considering the equations (1.10), (1.11), (2.2) and (2.16), we can write the Ricci tensor, [20]

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j \quad (2.17)$$

taking $\alpha = \frac{R}{18}$ and $\beta = -\frac{7R}{9}$ in (1.4).

Theorem 3. *A necessary and sufficient condition for a $(PZS)_4$ spacetime to be a perfect fluid is that the conformal curvature tensor is harmonic.*

Proof. Assume that our spacetime is a $(PZS)_4$ spacetime satisfying the property $C_{jkl,m}^m = 0$. Then, this spacetime is quasi-Einstein, [18] and it satisfies the condition (2.17). By using (2.2) and (2.17) in (1.10), we get

$$Z_{ij} = -\frac{4}{9} R g_{ij} - \frac{7}{9} R A_i A_j. \quad (2.18)$$

Also, we have from (1.11) and (2.2)

$$Z = -R. \quad (2.19)$$

By putting (2.18) and (2.19) in (2.15), we get $C_{ijl,m}^m = 0$. Hence, the proof is completed. $\square$

Theorem 4 ([20, Theorem 3.3]). *Let M be a $(PZS)_4$ spacetime with the property $C_{ijl,m}^m = 0$. If the condition $A_k A^k < 0$ holds, then EMT is of the perfect fluid form with the properties*

$$\sigma = \frac{T}{3}, \quad p = \frac{4T}{9}. \quad (2.20)$$

The condition $R_{ij} A^i A^j > 0$ is said to be timelike convergence condition for every timelike vector field A_i and the Ricci tensor R_{ij} of type $(0, 2)$ of a spacetime, [24].

Theorem 5. *If a $(PZS)_4$ spacetime with harmonic conformal curvature tensor admits the timelike vector field with the convergence condition, then this spacetime obeys cosmic strong energy condition.*

Proof. Assume that $(PZS)_4$ spacetime with harmonic conformal curvature tensor admits the timelike convergence condition. Multiplying (2.17) by $A^i A^j$ and using $A_i A^i = -1$, we find

$$R_{ij} A^i A^j = -\frac{5}{6}R. \quad (2.21)$$

Since A_i is a timelike vector field and convergence condition $R_{ij} A^i A^j > 0$ holds in (2.21) then we have $R < 0$. Now, if we multiply (1.12) by g^{ij} , then we obtain

$$Z = kT. \quad (2.22)$$

Hence, using (2.19) and (2.22), one can easily seen that

$$T = -\frac{R}{k}. \quad (2.23)$$

Remembering that the sign of R is negative and using (2.23), we have $T > 0$. In this case, from (2.20)

$$\sigma > 0, \quad p > 0. \quad (2.24)$$

Finally, we get that $\sigma + 3p > 0$. The inequality $\sigma + 3p > 0$ shows that this spacetime obeys cosmic strong energy condition, [5]. This result completes the proof. $\square$

Theorem 6. *A $(PZS)_4$ spacetime with harmonic conformal curvature tensor contains pure matter if this spacetime admits the timelike convergence condition.*

Proof. From the timelike convergence condition, we say that a $(PZS)_4$ spacetime has the negative scalar curvature. Also, we know from (2.24), $\sigma > 0$. Then, we can say that our spacetime under this consideration is of pure matter. $\square$

Proposition 1 ([20, Proposition 5.10]). *Let M be a $(PZS)_4$ spacetime with the property $C_{ijl,m}^m = 0$. Then the spacetime is a GRW-spacetime and we have*

$$U_{l,k} = f(g_{kl} + U_k U_l) \quad (2.25)$$

assuming that f is a suitable scalar function and U_k is a concircular vector field.

Theorem 7. *In a $(PZS)_4$ spacetime admitting harmonic conformal curvature tensor, the electric part of the conformal curvature tensor is vanishing.*

Proof. In a spacetime, the Weyl tensor is described by the symmetric and traceless tensors E_{ij} and H_{ij} and they have 10 independent components. If we consider the vector field A_i with the condition $A_i A^i = -1$, the electric and magnetic parts for the Weyl tensors are:

$$E_{ij} = A^k A^l C_{kijl}, \quad H_{ij} = A^k A^l \tilde{C}_{kijl} \quad (2.26)$$

where $\tilde{C}_{kijl} = \frac{1}{2} \epsilon_{kimn} C_{jl}^{mn}$ is the dual, [9]. E_{ij} and H_{ij} satisfy the conditions $E_{ij} A^i = 0$ and $H_{ij} A^i = 0$. Hence, each of them has 5 independent components and they completely describe the Weyl tensor.

If a $(PZS)_4$ spacetime obeys the property $C_{ijk,m}^m = 0$ then, from Proposition 1, [20], this spacetime is a GRW-spacetime. Hence, we have from (2.25)

$$A_{i,j} = f(g_{ij} + A_i A_j). \quad (2.27)$$

This equation means that A_k is in the form (1.8). Taking the covariant derivative of (2.27) and using the equation (2.27) again, we get

$$A_{i,jk} = f_k(g_{ij} + A_i A_j) + f^2 [(g_{ik} + A_i A_k)A_j + (g_{jk} + A_j A_k)A_i] \quad (2.28)$$

where $f_k = f_{,k}$. From [29], we have $f_k = \mu A_k$.

Interchanging the indices in (2.28), subtracting these two equations and assuming the following equation

$$w_k = f_k - f^2 A_k \quad (2.29)$$

where μ is a scalar function, then we obtain from (2.28) and (2.29)

$$A_{i,jk} - A_{i,kj} = w_k g_{ij} - w_j g_{ik}. \quad (2.30)$$

The equation (2.30) reduces to

$$R_{ijk}^h A_h = (\mu - f^2)[g_{ij} A_k - g_{ik} A_j]. \quad (2.31)$$

With the help of (2.26), we can write

$$\begin{aligned} E_{ij} &= R_{kijl} A^k A^l - \frac{1}{2} (g_{ij} R_{kl} - g_{il} R_{kj} + g_{kl} R_{ij} - g_{kj} R_{il}) A^k A^l \\ &\quad + \frac{R}{6} (g_{ij} g_{kl} - g_{il} g_{jk}) A^k A^l. \end{aligned} \quad (2.32)$$

From (2.17), we have

$$R_{ij} = \frac{R}{18} g_{ij} - \frac{7R}{9} A_i A_j. \quad (2.33)$$

By using (2.32), we get

$$E_{ij} = R_{kijl} A^k A^l + \frac{5R}{18} (g_{ij} + A_i A_j). \quad (2.34)$$

Considering the equations (2.31) and (2.34), it can be found that

$$E_{ij} = \left[-(\mu - f^2) + \frac{5R}{18} \right] (g_{ij} + A_i A_j). \quad (2.35)$$

On the other hand, from (2.31), we have

$$R_{hk} A^h = 3(\mu - f^2) A_k. \quad (2.36)$$

Finally, by using (2.33), (2.36) reduces to

$$\mu - f^2 = \frac{5R}{18}. \quad (2.37)$$

In this case, with the help of (2.35) and (2.37), it can be found that the electric part of the Weyl tensor is vanishing. Then, we complete the proof. $\square$

Theorem 8. *A $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor has a Riemannian compatible vector field.*

Proof. Considering a $(PZS)_4$ obeying harmonic conformal curvature tensor, using the equation (2.31) and the properties of the curvature tensor known as $R_{ijk}^h = g^{hm}R_{mijk}$, we find

$$R_{mijk}A^m = (\mu - f^2)[g_{ij}A_k - g_{ik}A_j]. \quad (2.38)$$

In this case, considering the equation (2.38) and $R_{mijk} = R_{kjim}$, we can write

$$(R_{jklm}A^m)A_i + (R_{kilm}A^m)A_j + (R_{ijlm}A^m)A_k = 0. \quad (2.39)$$

Finally, from (2.39), we can write

$$(R_{jklm}A_i + R_{kilm}A_j + R_{ijlm}A_k)A^m = 0. \quad (2.40)$$

Comparing the equations (1.7) and (2.40) gives that this spacetime has a Riemann compatible vector field A_k . $\square$

Theorem 9. *In a $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor, the magnetic part for the Weyl tensor is vanishing.*

Proof. For a $(PZS)_4$ spacetime admitting harmonic conformal curvature tensor, by the aid of the equations (1.12) and (2.18), we have EMT in the form

$$T_{ij} = \gamma g_{ij} + \eta A_i A_j$$

where $\gamma = -\frac{4R}{9k}$ and $\eta = -\frac{7R}{9k}$.

We know that for a $(PZS)_4$ spacetime, if the trace of Z-tensor is non-zero, then this spacetime is Weyl compatible and it is of Weyl compatible EMT, [19]. In addition, on a four-dimensional spacetime admitting a Weyl compatible EMT of the form $T_{jl} = \alpha g_{jl} + \beta A_j A_l$, the magnetic part for the Weyl tensor is vanishing [19].

From the above results, we complete the proof. $\square$

Theorem 10. *A $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor is conformally flat.*

Proof. Theorem 7 and Theorem 9 accomplish the proof. $\square$

Theorem 11. *A $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor is of Petrov type O.*

Proof. The proof follows from the Theorem 10. $\square$

Theorem 12. *A $(PZS)_4$ spacetime admitting harmonic conformal curvature tensor is a warped product written as $-I \times_{q^2} M^*$ such that M^* is an 3-dimensional Einstein manifold.*

Proof. Let us consider a $(PZS)_4$ with the property $C_{ijl,m}^m = 0$. Then, this spacetime is a GRW-spacetime, [20].

Since the GRW-spacetime has the proper concircular vector field, then it is the necessary and sufficient condition that a coordinate system exists with respect to which the fundamental quadratic differential form may be given in the following

$$ds^2 = -(dx^1)^2 + e^q g_{\alpha\beta}^* dx^\alpha dx^\beta \quad (2.41)$$

where $(\alpha, \beta = 2, 3, \dots, n)$ and q is a scalar function of x^1 only, i.e., $q = q(x^1) \neq \text{const}$. The first fundamental form for a GRW-spacetime is the form as in (2.41).

Since e^q is a positive function for all q , from (2.41), this spacetime is a warped product space written by

$$g = -dt^2 \oplus f^2(t)g^*$$

here we assume that g^* is the metric tensor of an 3-dimensional Riemannian manifold M^* and f is a smooth function defined by $f: I \rightarrow (0, \infty)$ on M^* . The warped product is conformally conservative if and only if M^* is an Einstein manifold, [14]. Hence, the proof is completed. $\square$

Theorem 13. *In a $(PZS)_4$ spacetime admitting harmonic conformal curvature tensor, the metric of this spacetime is conformal to a RW-spacetime.*

Proof. From Theorem 12, a $(PZS)_4$ spacetime admitting harmonic curvature tensor is a warped product spacetime. The warped product is conformally conservative if and only if M^* is an Einstein manifold, [14].

For a warped product spacetime if g^* is a 3-dimensional Riemannian manifold admitting constant curvature, then we get the manifold (M^4, g) and this manifold is called a RW-spacetime. Thus, this result completes the proof. $\square$

Theorem 14. *A $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor is a subprojective space.*

Proof. Assume that $(PZS)_4$ obeys harmonic curvature tensor. Thus, from Theorem 10, our spacetime is conformally flat. Since the associated covector A_k is a timelike vector field, then this space is subprojective, [20]. From this result, we complete the proof. $\square$

Theorem 15. *In a $(PZS)_4$ spacetime obeying harmonic conformal curvature tensor, the flow is geodesic, irrotational and has no shear.*

Proof. From [6], it is known that a four-dimensional perfect fluid spacetime admitting an equation of state given by $p = p(\sigma)$ and divergence-free Weyl conformal curvature tensor is conformally flat, and this spacetime is endowed with the RW metric. Also, the flow is irrotational, geodesic and it has no shear, [6]. From Theorem 4 and the equation (1.5), we complete the proof. $\square$

Theorem 16. *For a perfect fluid $(PZS)_4$ spacetime, if the conditions $\sigma = \frac{T}{3}$ and $p = \frac{4T}{9}$ are satisfied, then the following facts hold for this spacetime:*

It is

- (1) *conformally flat and also Petrov type O.*
- (2) *a GRW-spacetime and also a RW-spacetime.*
- (3) *a subprojective spacetime.*

Proof. We know from [20, Propositon 3.6], that in a perfect fluid $(PZS)_4$ spacetime, if the conditions $\sigma = \frac{T}{3}$ and $p = \frac{4T}{9}$ are satisfied, then $C_{ijl,m}^m = 0$ on any coordinate domain of $(PZS)_4$. Also, we know from [20], [Prop.5.10] that if an n -dimensional ($n \geq 4$) spacetime satisfies the property $C_{ijl,m}^m = 0$, then this spacetime is a GRW-spacetime.

From these results and by using Theorem 10, Theorem 11, Theorem 13 and Theorem 14, the proof is clear. $\square$

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ON A SEQUENCE OF RATIONAL NUMBERS WITH UNUSUAL DIVISIBILITY BY A POWER OF 2

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Abstract. In this note we consider the sequence of rational numbers $b_n = \sum_{k=1}^n 2^k/k$. We show that the power of 2 in the expansion of b_n is unusually large, at least $n + 1 - \log_2(n + 1)$, and that this bound is best possible. The sequence $b_n, n = 1, 2, 3, \dots$, is related to the sequence A0031449 in the On-Line Encyclopedia of Integer Sequences.

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1. INTRODUCTION

In [6] Farhi considered the following sequence $a_1 = 1$,

$$a_n = \frac{na_{n-1}}{2} + (n-1)! \quad (1.1)$$

for $n = 2, 3, \dots$. Set also

$$b_n = \sum_{k=1}^n \frac{2^k}{k} \quad (1.2)$$

for $n \in \mathbb{N}$. Then, by $b_n - b_{n-1} = 2^n/n$ and (1.1), we obtain

$$a_n = \frac{n!}{2^n} b_n \quad (1.3)$$

for each $n \in \mathbb{N}$.

In [6], expressing a_n in terms of Genocchi numbers and Stirling numbers of the first kind, Farhi showed that

$$a_n \in \mathbb{N} \quad (1.4)$$

for each $n \in \mathbb{N}$. This, according to the definition of a_n in (1.1), is nontrivial and in some sense reminds the surprising integrality conditions of so-called Somos sequences (see [14] and also some subsequent work in [8, 11, 17, 18]). The fractional parts of the sequence $\frac{2^n}{n}, n = 1, 2, 3, \dots$, were considered in [4, 5].

Of course, there are several alternative ways to prove (1.4) which are simpler than that in [6]. This was observed by Farhi in a subsequent paper [7]. For example, by the identity

$$\sum_{k=0}^n \frac{k!(n-k)!}{n!} = \sum_{k=0}^n \frac{1}{\binom{n}{k}} = \frac{n+1}{2^n} \sum_{k=0}^n \frac{2^k}{k+1} \quad (1.5)$$

(see [12, 15]), using (1.2) and (1.3) we find that

$$\sum_{k=0}^n k!(n-k)! = \frac{(n+1)!}{2^n} \sum_{k=0}^n \frac{2^k}{k+1} = \frac{(n+1)!}{2^{n+1}} b_{n+1} = a_{n+1},$$

which implies (1.4). In fact, $\sum_{k=0}^n k!(n-k)!, n = 1, 2, 3, \dots$, is the sequence A0031449 in [13].

For a prime number p and a positive integer u by $v_p(u)$ we denote the largest nonnegative integer k for which p^k divides u . Likewise, for a rational $r = u/v$, where $u, v \in \mathbb{N}$ are relatively prime integers, we set $v_p(r) = v_p(u) - v_p(v)$.

With this notation in [6, Corollary 2.5] it was also shown that

$$v_2\left(\sum_{k=1}^n \frac{2^k}{k}\right) \geq s_2(n), \quad (1.6)$$

where $s_2(n)$ is the sum of digits of n in base 2. This was improved in [7, Theorem 2.5], where it was shown that

$$v_2\left(\sum_{k=1}^n \frac{2^k}{k}\right) \geq n - \lfloor \log_2 n \rfloor \quad (1.7)$$

for each $n \in \mathbb{N}$.

Now, we will refine the estimates (1.6), (1.7) and obtain a sharp bound.

Theorem 1. *For each $n \in \mathbb{N}$ we have*

$$v_2\left(\sum_{k=1}^n \frac{2^k}{k}\right) \geq n + 1 - \log_2(n+1), \quad (1.8)$$

with equality if and only if $n = 2^k - 1$ for $k \in \mathbb{N}$.

Note that

$$n + 1 - \log_2(n+1) \geq n - \lfloor \log_2 n \rfloor. \quad (1.9)$$

Indeed, choose a unique integer $k \geq 0$ satisfying $2^k \leq n < 2^{k+1}$. Then, $1 + \lfloor \log_2 n \rfloor = k + 1$ and $\log_2(n+1) \leq \log_2(2^{k+1}) = k + 1$, which proves (1.9).

From the above proof of (1.9) we see that the right hand sides of (1.7) and (1.8) are equal only if n has the form $n = 2^k - 1$. We will derive (1.8) from the inequality (2.3) below, which is stronger than (1.8) for many $n \in \mathbb{N}$ that are not of the form $2^k - 1$. The nontrivial part is to show that for $n = 2^k - 1$ one has equality in (1.8). (The proof of (1.7) in [7] is entirely different: it uses (1.5) and some other identity.) The proof

of Theorem 1 is self-contained except that we need any version of the fact that the sequence $v_2(b_n)$, $n = 1, 2, 3, \dots$, is unbounded as $n \rightarrow \infty$.

Let D_n be the least common multiple of $1, 2, 3, \dots, n$. By (1.2), it is clear that $D_n b_n \in \mathbb{N}$ for each $n \in \mathbb{N}$. From Theorem 1 we will derive the following corollary which strengthens (1.4) (since (1.4) holds even if the factor $n!$ in (1.3) is replaced by D_n).

Corollary 1. *For each $n \in \mathbb{N}$ we have*

$$\frac{D_n b_n}{2^n} = D_n \sum_{k=1}^n \frac{2^{k-n}}{k} \in \mathbb{N}. \quad (1.10)$$

Finally, we remark that in [15] Sury proved not only (1.5) but also the identity

$$\sum_{k=1}^n \frac{2^k}{k} = 2 \left(\binom{n}{1} + \frac{1}{3} \binom{n}{3} + \frac{1}{5} \binom{n}{5} + \frac{1}{7} \binom{n}{7} + \dots \right). \quad (1.11)$$

See also [1–3, 9, 10, 16] for some related identities. For example, in [10] Meštrović showed that

$$\sum_{k=1}^n \frac{2^k - 1}{k} = \sum_{k=1}^n \frac{1}{k} \binom{n}{k}. \quad (1.12)$$

We conclude with a simple generalization of (1.11) and (1.12).

Theorem 2. *For each $n \in \mathbb{N}$ and each $z \in \mathbb{C}$ we have*

$$\sum_{k=1}^n \frac{(1+z)^k - (1-z)^k}{k} = 2 \left(z \binom{n}{1} + \frac{z^3}{3} \binom{n}{3} + \frac{z^5}{5} \binom{n}{5} + \frac{z^7}{7} \binom{n}{7} + \dots \right)$$

and

$$\sum_{k=1}^n \frac{(1+z)^k - 1}{k} = \sum_{k=1}^n \frac{z^k}{k} \binom{n}{k}. \quad (1.13)$$

In particular, we give a very simple proof of (1.13), which shows that the identities (1.11), (1.12) and [10, Corollary 1.2] are not so mysterious as it may appear from their proofs in [10]. They come by inserting $z = 1$ into the natural identities of Theorem 2.

2. PROOFS

Proof of Theorem 1. For $n \in \mathbb{N}$ put

$$d_n = v_2(b_n),$$

where b_n is defined in (1.2). Then, $b_n = 2^{d_n} u_n / v_n$, where u_n and v_n are odd coprime positive integers. Note that $d_1 = 1$, $d_2 = 2$, $\dots$, etc. We have

$$b_{n+1} - b_n = \frac{2^{n+1}}{n+1} = \frac{2^{n+1-v_2(n+1)}}{(n+1)2^{-v_2(n+1)}},$$

where $n + 1 - v_2(n + 1) > 0$. From

$$b_{n+1} = \frac{2^{d_n} u_n}{v_n} + \frac{2^{n+1-v_2(n+1)}}{(n+1)2^{-v_2(n+1)}}$$

and the fact that the integer $(n+1)2^{-v_2(n+1)}$ is odd, we find that

$$d_{n+1} = \min(d_n, n + 1 - v_2(n + 1)) \quad (2.1)$$

if

$$d_n \neq n + 1 - v_2(n + 1), \quad (2.2)$$

and

$$d_{n+1} > d_n$$

if

$$d_n = n + 1 - v_2(n + 1).$$

We claim that for each $n \in \mathbb{N}$

$$d_n \geq \min_{k \geq n} (k + 1 - v_2(k + 1)). \quad (2.3)$$

Indeed, if for some $n \in \mathbb{N}$ the inequality opposite to (2.3) holds then d_n is less than $k + 1 - v_2(k + 1)$ for $k = n, n + 1, n + 2, \dots$. Then, by (2.1) and (2.2), we should have $d_n = d_{n+1} = d_{n+2} = \dots$. Thus, the sequence d_k , $k = 1, 2, 3, \dots$, is bounded, which is impossible by (1.6) or (1.7). This proves (2.3).

Next, since $u \geq 2^{v_2(u)}$, for any $u \in \mathbb{N}$ we have $v_2(u) \leq \log_2(u)$. Hence, from (2.3) we get $d_n \geq \min_{k \geq n} (k + 1 - \log_2(k + 1))$. The function $f(x) = x + 1 - \log_2(x + 1)$ is increasing for $x \geq 1$. So, for any $n \in \mathbb{N}$, the smallest value of the function $f(x)$ in the set $x \in \{n, n + 1, n + 2, \dots\}$ is attained at $x = n$. Thus, $d_n \geq n + 1 - \log_2(n + 1)$, which is (1.8).

Further, equality in (1.8) can only hold if $\log_2(n + 1)$ is an integer. For $n \in \mathbb{N}$ this happens for $n = 2^k - 1$, where $k \in \mathbb{N}$, only. So the values $n = 2^k - 1$, $k = 1, 2, 3, \dots$, are the only values for which equality in (1.8) can possibly be attained. We will show that it is always attained, namely,

$$d_{2^k-1} = v_2(b_{2^k-1}) = 2^k - k \quad (2.4)$$

for every $k \in \mathbb{N}$.

Fix any $k \in \mathbb{N}$. For a contradiction assume that $d_{2^k-1} \neq 2^k - k$, so that (2.2) is true for $n = 2^k - 1$. Then, by (2.1), we must have

$$d_{2^k} = \min(d_{2^k-1}, 2^k - k) \leq 2^k - k.$$

However, by (1.8) and $2^k + 1 < 2^{k+1}$, it follows that

$$d_{2^k} \geq 2^k + 1 - \log_2(2^k + 1) > 2^k + 1 - (k + 1) = 2^k - k,$$

which contradicts to the previous inequality. This rules out the possibility $d_{2^k-1} \neq 2^k - k$ and so proves (2.4). $\square$

Proof of Corollary 1. Let L_n be the least common multiple of all odd integers between 1 and n . By (1.2), it is clear that $L_n b_n \in \mathbb{N}$ for every $n \in \mathbb{N}$. Furthermore, since L_n is an odd integer, this implies $L_n b_n 2^{-v_2(b_n)} \in \mathbb{N}$. Next, in view of $D_n = L_n 2^{\lfloor \log_2 n \rfloor}$ we obtain

$$D_n b_n 2^{-v_2(b_n) - \lfloor \log_2 n \rfloor} \in \mathbb{N},$$

which implies (1.10) provided that $v_2(b_n) + \lfloor \log_2 n \rfloor \geq n$. However, by Theorem 1, we already know that $v_2(b_n) \geq n + 1 - \log_2(n + 1)$. This completes the proof of the corollary by (1.9). (Of course, as observed by the referee, $v_2(b_n) + \lfloor \log_2 n \rfloor \geq n$ already holds by Farhi's inequality (1.7).) $\square$

Proof of Theorem 2. In order to prove (1.13) we fix $n \in \mathbb{N}$ and set

$$f(z) = \sum_{k=1}^n \frac{(1+z)^k - 1}{k} - \sum_{k=1}^n \frac{z^k}{k} \binom{n}{k}.$$

Then,

$$f'(z) = \sum_{k=1}^n (1+z)^{k-1} - \sum_{k=1}^n z^{k-1} \binom{n}{k} = \frac{(1+z)^n - 1}{(1+z) - 1} - \frac{(1+z)^n - 1}{z} = 0$$

for $z \neq 0$. Inserting $z = 0$ we obtain $f'(0) = n - \binom{n}{1} = 0$ as well. Hence, $f'(z) = 0$ for each $z \in \mathbb{C}$, which implies that $f(z)$ is a constant. From $f(0) = 0$ we conclude that $f(z) = 0$ for each $z \in \mathbb{C}$. This proves (1.13).

Clearly, (1.13) also implies the first identity of this theorem by subtracting (1.13) with z replaced by $-z$ from (1.13) with z itself. $\square$

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A NEW LOOK AT FINSLER SURFACES AND LANDSBERG'S PDE

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Abstract. In this paper, we introduce a new look at Finsler surfaces. Landsberg surfaces are Finsler surfaces that are solutions of a system of non-linear partial differential equations. Considering the unicorn's Landsberg problem, we reduce this system to a single non-linear PDE which we call the Landsberg's PDE. By making use of the new look of Finsler surfaces, we solve the Landsberg's PDE and get a class of solutions. Moreover, we show that these solutions and their conformal transformations are Berwaldian.

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1. INTRODUCTION

A Finsler manifold (M, F) is said to be *Berwald* if the coefficients of Berwald connection depend only on the position arguments and this is equivalent to that the Berwald parallel translation P_c along a curve $c(t)$ is linear isometry between $(T_p M, F_p)$ and $(T_q M, F_q)$ where $c(t)$ joins the points $p, q \in M$. A Finsler manifold (M, F) is said to be *Landsberg* if the horizontal covariant derivative of the metric tensor of F with respect to the Berwald connection vanishes and this is equivalent to the fact that the parallel translation P_c along c preserves the induced Riemannian metrics on the slit tangent spaces, i.e., $P_c : (T_p M \setminus \{0\}, g_p) \longrightarrow (T_q M \setminus \{0\}, g_q)$ is an isometry. There are many other characterizations for Berwald and Landsberg metrics.

It is known that every Berwald space is Landsberg, but the converse is a long-existing problem in Finsler geometry, which is still open. Matsumoto, one of the best geometers who had a very significant contribution to Finsler geometry in the last century, called the problem the most important unsolved problem in Finsler geometry. Many geometers in the Finslerian area of research looking for a regular Landsberg space which is not Berwald. Even they want to know if such spaces exists or not. From the applicable point of view, especially in Physics, G. Asanov [1] obtained class of metrics (singular or y-local) arising from Finslerian General Relativity. These metrics are non-Berwaldian Landsberg metrics. Later, Z. Shen [10], generalized this

class and classified all Landsberg (α, β) -metrics which are not Berwaldian. Due to the many unsuccessful attempts made for finding non-Berwaldian Landsberg metrics, D. Bao [2] called them “unicorns”.

There are some papers devoted to the unicorn problem in dimension two. For example, R. Bryant claimed there exists the singular Landsberg Finsler surfaces which are not Berwaldian, moreover among them there is surfaces with vanishing flag curvature (cf. [2]). Later, Zhou [15] confirmed R. Bryant’s claim by giving examples of Landsberg surfaces which are not Berwaldian. Recently in [7] it was shown that the examples obtained by Zhou are in fact Berwaldian. By the way, the spherically symmetric Landsberg surfaces are studied in [5]. In [14], G. Thompson claimed that there are Landsberg Finsler metrics in dimension two which are not Berwaldian, but no concrete examples of such surfaces are given. Also, one can see the work of S. V. Sabau in dimension two, for example, see [9]. For concrete examples and further studies of the unicorns for higher dimensions we refer to [4, 6, 8].

In this paper, we rewrite the Finsler function on any two-dimensional manifold as follows:

$$F = |y^1| f(x, \varepsilon u), \quad u = \frac{y^2}{y^1}, \quad \varepsilon := \operatorname{sgn}(y^1)$$

where $f(x, \varepsilon u) := F(x, \varepsilon, \varepsilon u)$ is a positive smooth function on $M \times \mathbb{R}$ and $|\cdot|$ is the absolute value. It should be noted that if we start by regular Finsler function F , then the Finsler function $F(x, y) = |y^1| f(x, \varepsilon u)$ is regular although the function u has a singularity at $y^1 = 0$. As an example (cf. [11, Example 1.2.2 Page 15]):

$$F(x, y) = \sqrt{(y^1)^2 + (y^2)^2} + B y^1 = |y^1| \left(\sqrt{1 + u^2} + \varepsilon B \right).$$

In this example $f(x, \varepsilon u) = \sqrt{1 + u^2} + \varepsilon B$.

We calculate the Berwald and Landsberg curvatures. We show that the Landsberg condition leads to a single PDE. By solving this PDE we show that all two-dimensional Landsberg metrics on the form

$$F = |y^1| \phi(t), \quad t := \rho(x^1, x^2)u, \quad u := y^2/y^1$$

and their conformal transformations are Berwaldian (cf. Theorems 1 and 2). As by-product, we get explicit formulae for these solutions, precisely, the classes of the Landsberg solutions given by

$$F = \sqrt{c_3 \rho^2 (y^2)^2 - (c - 2) \rho y^1 y^2 - 2c_1 (y^1)^2} e^{\frac{c}{\sqrt{c^2 - 4c_2}} \operatorname{arctanh}\left(\frac{2c_3 \rho y^2 - (c-2)y^1}{y^1 \sqrt{c^2 - 4c_2}}\right)},$$

are Berwaldian where $c := 2c_1 c_3 + c_2 + 1$ and $c_2 > 0, a, b, c_1, c_3$ are constants (cf. Theorem 3).

In case of higher dimensions $n \geq 3$, the conformal transformation of a Minkowski metrics can yield a non-Berwaldian Landsberg metrics (cf. [4]). Assuming that a manifold (M, F) is Minkowski if the Finsler function F depends on the directional

argument y only, in contrast of the higher dimensions, we prove that if the conformal transformation of any two dimensional Minkowski metric is Landsberg then it must be Berwaldian (cf. Theorem 4).

2. PRELIMINARIES

Let M be an n -dimensional manifold and (TM, π_M, M) be its tangent bundle and $(\mathcal{T}M, \pi, M)$ the subbundle of nonzero tangent vectors. We denote by (x^i) local coordinates on the base manifold M and by (x^i, y^i) the induced coordinates on TM . The vector 1-form J on TM defined, locally, by $J = \frac{\partial}{\partial y^i} \otimes dx^i$ is called the natural almost-tangent structure of TM . The vertical vector field $C = y^i \frac{\partial}{\partial y^i}$ on TM is called the canonical or the Liouville vector field.

A vector field $S \in \mathfrak{X}(\mathcal{T}M)$ is called a spray if $JS = C$ and $[C, S] = S$. Locally, a spray can be expressed as follows

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where the *spray coefficients* $G^i = G^i(x, y)$ are 2-homogeneous functions in the $y = (y^1, \dots, y^n)$ variable.

A nonlinear connection is defined by an n -dimensional distribution $H : u \in \mathcal{T}M \rightarrow H_u \subset T_u(\mathcal{T}M)$ that is supplementary to the vertical distribution, which means that for all $u \in \mathcal{T}M$, we have

$$T_u(\mathcal{T}M) = H_u(\mathcal{T}M) \oplus V_u(\mathcal{T}M). \quad (2.1)$$

Every spray S induces a canonical nonlinear connection through the corresponding horizontal and vertical projectors,

$$h = \frac{1}{2}(Id + [J, S]), \quad v = \frac{1}{2}(Id - [J, S]).$$

Equivalently, the canonical nonlinear connection induced by a spray can be expressed in terms of an almost product structure $\Gamma = [J, S] = h - v$. With respect to the induced nonlinear connection, a spray S is horizontal, which means that $S = hS$. Locally, the two projectors h and v can be expressed as follows

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes \delta y^i,$$

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j(x, y) \frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + G_i^j(x, y) dx^j, \quad G_i^j(x, y) = \frac{\partial G^j}{\partial y^i}.$$

Moreover, the coefficients of the Berwald connection is given by

$$G_{ij}^h = \frac{\partial G_j^h}{\partial y^i}.$$

$$R = \frac{1}{2}[h, h] = \frac{1}{2}R_{jk}^i \frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k, \quad R_{jk}^i = \frac{\delta G_j^i}{\delta x^k} - \frac{\delta G_k^i}{\delta x^j}$$

is called the curvature of S . From the curvature tensor one can obtain the Riemann curvature [11] (or the Jacobi endomorphism, see [3]), which is defined by

$$\Phi = R_j^i dx^j \otimes \frac{\partial}{\partial y^i}, \quad R_j^i = 2 \frac{\partial G^i}{\partial x^j} - S(G_j^i) - G_k^i G_j^k. \quad (2.2)$$

The two curvature tensors are related by

$$\Phi = i_S R, \quad 3R = [J, \Phi], \quad (2.3)$$

respectively.

From now on, for simplicity, we use the notations $\partial_i := \frac{\partial}{\partial x^i}$, $\dot{\partial}_i := \frac{\partial}{\partial y^i}$.

Definition 1. A Finsler manifold of dimension n is a pair (M, F) , where M is a differentiable manifold of dimension n and F is a map

$$F : TM \longrightarrow \mathbb{R},$$

such that

- (a) F is smooth and strictly positive on TM and $F(x, y) = 0$ if and only if $y = 0$,
- (b) F is positively homogeneous of degree 1 in the directional argument y : $\mathcal{L}_C F = F$,
- (c) The metric tensor $g_{ij} = \dot{\partial}_i \dot{\partial}_j E$ has rank n on TM , where $E := \frac{1}{2} F^2$ is the energy function.

Since the 2-form $dd_J E$ is non-degenerate, the Euler-Lagrange equation

$$i_S dd_J E = -dE$$

uniquely determines a spray S on TM . This spray is called the *geodesic spray* of the Finsler function.

Definition 2. A spray S on a manifold M is called *Finsler metrizable* if there exists a Finsler function F such that the geodesic spray of the Finsler manifold (M, F) is S .

It is known that a spray S is Finsler metrizable if and only if there exists a non-degenerate solution F for the system

$$d_h F = 0, \quad d_C F = F, \quad (2.4)$$

where h is the horizontal projector associated to S .

The local formula for the hv-curvature tensor G of Berwlad connection and the Landsbeg tensor L given, respectively, by

$$G = G_{ijk}^h dx^i \otimes dx^j \otimes dx^k \otimes \dot{\partial}_h \quad (2.5)$$

$$L = L_{ijk} dx^i \otimes dx^j \otimes dx^k, \quad (2.6)$$

where $G_{ijk}^h := \dot{\partial}_k G_{ij}^h$ and $L_{ijk} := -\frac{1}{2} F G_{ijk}^h \ell_h$.

Definition 3. A Finsler manifold (M, F) is said to be *Berwald* if and only if G_{ijk}^h vanishes identically.

Definition 4. A Finsler manifold (M, F) is called *Landsberg* if and only if L_{ijk} vanishes identically.

3. A NEW LOOK AT FINSLER SURFACES AND THE LANDSBERG'S PDE

In what follows, ∂_1 (resp. ∂_2) stands for the partial differentiation with respect to x^1 (resp. x^2) and $\dot{\partial}_1$ (resp. $\dot{\partial}_2$) stands for the partial differentiation with respect to y^1 (resp. y^2).

The following lemmas are useful for subsequent use.

Lemma 1. Let F be a Finsler function on a two-dimensional manifold M , then F can be written in the form

$$F = \begin{cases} |y^1| f(x, \varepsilon u), & u = \frac{y^2}{y^1}, y^1 \neq 0, \varepsilon := \operatorname{sgn}(y^1) \\ 0, & y^1 = y^2 = 0 \\ |y^2| f(x, \varepsilon v), & v = \frac{y^1}{y^2}, y^2 \neq 0, \varepsilon := \operatorname{sgn}(y^2) \end{cases} \quad (3.1)$$

where $f(x, \varepsilon u) := F(x, \varepsilon, \varepsilon u)$ is a positive smooth function on $M \times \mathbb{R}$ and $|\cdot|$ is the absolute value. Moreover, for the expression $F = |y^1| f(x, \varepsilon u)$, the coefficients G^1 and G^2 of the geodesic spray are given by

$$G^1 = f_1(x, u)(y^1)^2, \quad G^2 = f_2(x, u)(y^1)^2, \quad (3.2)$$

where the functions f_1 and f_2 are smooth functions on $M \times \mathbb{R}$ and given as follows

$$f_1 = \frac{(\partial_1 f + u \partial_2 f) f'' - (\partial_1 f' + u \partial_2 f' - \partial_2 f) f'}{2 f f''}, \quad (3.3)$$

$$f_2 = \frac{u(\partial_1 f + u \partial_2 f) f'' + (\partial_1 f' + u \partial_2 f' - \partial_2 f)(f - u f')}{2 f f''}, \quad (3.4)$$

where f' (resp. f'') is the first (resp. the second) derivative of f with respect to u and so on.

Before we go to prove the above lemma, let's give the following remark.

Remark 1. It should be noted that if we start by regular Finsler function F , then the Finsler function $F(x, y) = |y^1| f(x, \varepsilon u)$ is regular although the function u has a singularity at $y^1 = 0$. As an example (cf. [11, Example 1.2.2 Page 15]):

$$F(x, y) = \sqrt{(y^1)^2 + (y^2)^2} + B y^1 = |y^1| \left(\sqrt{1 + u^2} + \varepsilon B \right).$$

In this example $f(x, \varepsilon u) = \sqrt{1 + u^2} + \varepsilon B$.

Proof of Lemma 1. Since $F = 0$ only on the zero section, then away from the zero section at each $x \in M$, at least one of the y 's is non zero. Without loss of generality, we assume that $y^1 \neq 0$. Then by using the fact that the Finsler function F is positively homogeneous of degree 1 in y , we have

$$F(x, y^1, y^2) = F\left(x, |y^1| \frac{y^1}{|y^1|}, |y^1| \frac{y^2}{|y^1|}\right) = |y^1| F(x, \varepsilon, \varepsilon u) = |y^1| f(x, \varepsilon u),$$

where $f(x, \varepsilon u) := F(x, \varepsilon, \varepsilon u)$.

Now, assume that $F = |y^1| f(x^1, x^2, \varepsilon u)$. Since the coefficients G^i of the geodesic spray of F are homogeneous of degree 2 in y , then we can write

$$G^1 = f_1(x^1, x^2, u)(y^1)^2, \quad G^2 = f_2(x^1, x^2, u)(y^1)^2.$$

Keeping the facts that $\frac{\partial f}{\partial(\varepsilon u)} = \varepsilon \frac{\partial f}{\partial u} = \varepsilon f'$ and $\varepsilon^2 = 1$ in mind, then straightforward calculations lead to

$$\ell_1 := \dot{\partial}_1 F = \varepsilon f + |y^1| f'(-y^2/(y^1)^2) = \varepsilon f - \varepsilon u f' = \varepsilon(f - u f'),$$

$$\ell_2 := \dot{\partial}_2 F = |y^1| f'(1/y^1) = \frac{|y^1|}{y^1} f' = \varepsilon f'.$$

The coefficients N_j^i of the non-linear connection can be calculated in the form

$$\begin{aligned} N_1^1 &= \dot{\partial}_1 G^1 = 2y^1 f_1 - y^2 f_1', & N_2^1 &= \dot{\partial}_2 G^1 = y^1 f_1', \\ N_1^2 &= \dot{\partial}_1 G^2 = 2y^1 f_2 - y^2 f_2', & N_2^2 &= \dot{\partial}_2 G^2 = y^1 f_2'. \end{aligned} \quad (3.5)$$

Now, we have to determine the coefficients G^1 and G^2 of the geodesic spray of F . Since S is the geodesic spray of F , then $d_h F = 0$ that is $\partial_i F - N_i^h \dot{\partial}_h F = 0$. Then we have the system

$$\begin{aligned} |y^1| \partial_1 f - (2y^1 f_1 - y^2 f_1')(\varepsilon f - \varepsilon u f') - (2y^1 f_2 - y^2 f_2')\varepsilon f' &= 0, \\ |y^1| \partial_2 f - y^1 f_1'(\varepsilon f - \varepsilon u f') - y^1 f_2' \varepsilon f' &= 0. \end{aligned}$$

Since we assume that $y^1 \neq 0$, then dividing the above system by $|y^1|$, we get

$$\partial_1 f - 2f_1(f - u f') + u f_1'(f - u f') - 2f_2 f' + u f_2' f' = 0, \quad (3.6)$$

$$\partial_2 f - f_1'(f - u f') - f_2' f' = 0. \quad (3.7)$$

Multiplying (3.7) by u and then adding it to (3.6), we have

$$\partial_1 f + u \partial_2 f - 2f_1(f - u f') - 2f_2 f' = 0. \quad (3.8)$$

Differentiating (3.6) and (3.7) with respect to u yields the following

$$\partial_1 f' + 2u f_1 f'' - f_1'(f - u f') + u(f_1'(f - u f'))' - f_2' f' - 2f_2 f'' + u(f_2' f')' = 0 \quad (3.9)$$

$$\partial_2 f' - (f_1'(f - u f'))' - (f_2' f')' = 0. \quad (3.10)$$

Multiplying (3.10) by u and then adding it to (3.9) and using (3.7), we get

$$\partial_1 f' + u \partial_2 f' - \partial_2 f + 2u f_1 f'' - 2f_2 f'' = 0. \quad (3.11)$$

Multiplying (3.8) by f'' and (3.11) by f' and then by subtraction, we obtain the required formula for f_1 . Finally, substituting by f_1 into (3.11), we get the formula of f_2 . $\square$

Lemma 2. *The coefficients G_{jk}^h of Berwald connection are given by*

$$\begin{aligned} G_{11}^1 &= 2f_1 - 2uf_1' + u^2 f_1'', & G_{12}^1 &= f_1' - uf_1'', & G_{22}^1 &= f_1'', \\ G_{11}^2 &= 2f_2 - 2uf_2' + u^2 f_2'', & G_{12}^2 &= f_2' - uf_2'', & G_{22}^2 &= f_2''. \end{aligned} \quad (3.12)$$

Proof. The proof comes from the fact that $G_{jk}^h = \dot{\partial}_k N_j^h$ together with (3.5). $\square$

Proposition 1. *The components R_{jk}^i of the curvature tensor are given by*

$$R_{12}^1 = y^1 (uf_1'^2 - f_1' f_2' - 2uf_1 f_1'' + 2f_2 f_1'' - u \partial_2 f_1' + 2\partial_2 f_1 - \partial_1 f_1'), \quad (3.13)$$

$$R_{12}^2 = y^1 (-2uf_1 f_2'' + 2f_2 f_2'' + uf_1' f_2' - 2f_2 f_1' + 2f_1 f_2' - f_2'^2 - u \partial_2 f_2' + 2\partial_2 f_2 - \partial_1 f_2'). \quad (3.14)$$

Proof. Using the the fact that $R_{jk}^h = \delta_k N_j^h - \delta_k N_j^h$, we have

$$R_{12}^1 = \delta_2 N_1^1 - \delta_1 N_2^1 = \partial_2 N_1^1 - N_2^1 \dot{\partial}_1 N_1^1 - N_2^2 \dot{\partial}_2 N_1^1 - \partial_1 N_2^1 + N_1^1 \dot{\partial}_1 N_2^1 + N_1^2 \dot{\partial}_2 N_2^1,$$

$$R_{12}^2 = \delta_2 N_1^2 - \delta_1 N_2^2 = \partial_2 N_1^2 - N_2^1 \dot{\partial}_1 N_1^2 - N_2^2 \dot{\partial}_2 N_1^2 - \partial_1 N_2^2 + N_1^1 \dot{\partial}_1 N_2^2 + N_1^2 \dot{\partial}_2 N_2^2.$$

Now by substituting from (3.5) and (2) into the above formulae we get the result. $\square$

Example 1. Let $M = \mathbb{R}^2$ and F be a Finsler function given by

$$F = \sqrt{e^{x^2}(y^1)^2 + (y^2)^2}.$$

Then we have

$$f(x, u) = \sqrt{e^{x^2} + u^2}.$$

By substituting into (3.3) and (3.4), we get $f_1 = \frac{1}{2}u$, $f_2 = -\frac{1}{4}e^{x^2}$. Hence, the spray coefficients are given by

$$G^1 = (y^1)^2 \frac{1}{2}u = \frac{1}{2}y^1 y^2, \quad G^2 = -\frac{1}{4}e^{x^2}(y^1)^2 = -\frac{1}{4}e^{x^2}(y^1)^2.$$

Making use of (3.13), (3.14), we have

$$R_{12}^1 = y^1 \left(\frac{1}{4}u \right) = \frac{1}{4}y^2, \quad R_{12}^2 = -y^1 \left(\frac{1}{4}e^{x^2} \right) = -\frac{1}{4}e^{x^2}y^1.$$

Now, for any surface (M, F) , we can calculate the components of the Berwald and Landsberg curvatures as follows.

Lemma 3. *The components G_{ijk}^h of Berwald curvature are given by*

$$\begin{aligned} G_{111}^1 &= -\frac{u^3}{y^1} f_1''', & G_{111}^2 &= -\frac{u^3}{y^1} f_2''', & G_{211}^1 &= \frac{u^2}{y^1} f_1''', & G_{211}^2 &= \frac{u^2}{y^1} f_2''', \\ G_{221}^1 &= -\frac{u}{y^1} f_1''', & G_{221}^2 &= -\frac{u}{y^1} f_2''', & G_{222}^1 &= \frac{1}{y^1} f_1''', & G_{222}^2 &= \frac{1}{y^1} f_2'''. \end{aligned} \quad (3.15)$$

The components L_{ijk} of the Landsberg curvature are given by

$$\begin{aligned} L_{111} &= \frac{u^3 f}{2} (f_1''' \ell_1 + f_2''' \ell_2), & L_{112} &= -\frac{u^2 f}{2} (f_1''' \ell_1 + f_2''' \ell_2), \\ L_{122} &= -\frac{u f}{2} (f_1''' \ell_1 + f_2''' \ell_2), & L_{222} &= -\frac{f}{2} (f_1''' \ell_1 + f_2''' \ell_2). \end{aligned} \quad (3.16)$$

Proof. We do the calculations for one component and the rest can be calculated in a similar manner. By using (2) and the definition of the Berwald tensor $G_{ijk}^h = \dot{\partial}_k G_{ij}^h$, we can calculate, for example, G_{111}^1 as follows

$$G_{111}^1 = \dot{\partial}_1 G_{11}^1 = (2f_1 - 2uf_1' + u^2 f_1'')' \frac{-y^2}{(y^1)^2} = -\frac{u^3}{y^1} f_1'''. \quad \square$$

By the same way, one gets the component $G_{111}^2 = -\frac{u^3}{y^1} f_2'''$. Hence, the first component L_{111} of the Landsberg curvature is given by

$$L_{111} = -\frac{F}{2} (G_{111}^1 \ell_1 + G_{111}^2 \ell_2) = \frac{u^3 f}{2} (f_1''' \ell_1 + f_2''' \ell_2).$$

□

The above lemma gives rise to the following proposition.

Proposition 2. *Any two dimensional Finsler manifold (M, F) in the form (3.1) is Landsbergian if and only if the following PDE*

$$f_1''' \ell_1 + f_2''' \ell_2 = 0 \quad (3.17)$$

is satisfied. The above PDE will be called the two-dimensional Landsberg's PDE.

Proof. Since in the Landsberg space all the components of the Landsberg tensor L_{ijk} vanishes identically, then by making use of (3.16) the result follows. □

4. SOLUTIONS TO THE LANDSBERG'S PDE

We are going to solve the Landsberg's PDE in a special case, so we focus our attention to the Finsler functions on the form

$$F = |y^1| \phi(t), \quad t := \rho(x^1, x^2)u, \quad (4.1)$$

where we consider, without loss of generality, that $y^1 \neq 0$.

Remark 2. It should be noted that one of our goals is to solve the Landsberg's PDE (3.17), so when $\phi(t) = t$ in (4.1) then we use the separation of variables method to solve (3.17). Therefore, the assumption (4.1) is a more general method than the separation of variables.

Let's define the function Q as follows

$$Q := \frac{\phi_t}{\phi - t\phi_t},$$

where the subscript t refers to the differentiation with respect to t . Moreover, in this case, ϕ is given by

$$\phi = \exp\left(\int \frac{Q}{1+tQ} dt\right). \quad (4.2)$$

Lemma 4. *The spray coefficients of the Finsler functions given in (4.1), are determined by the functions*

$$f_1 = -\frac{Q^2}{Q_t} \frac{\partial_1 \rho}{2\rho}, \quad (4.3)$$

$$f_2 = \frac{u\partial_1 \rho + u^2\partial_2 \rho}{2\rho} + \frac{Q}{Q_t} \frac{\partial_1 \rho}{2\rho^2}. \quad (4.4)$$

Proof. Let F be given by (4.1) and $f(x, u) = \phi(t)$. We have the following formulas

$$f' = \rho \phi_t, \quad f'' = \rho^2 \phi_{tt},$$

$$\partial_1 f = u \phi_t \partial_1 \rho, \quad \partial_2 f = u \phi_t \partial_2 \rho,$$

$$\partial_1 f' = \phi_t \partial_1 \rho + \rho u \phi_{tt} \partial_1 \rho, \quad \partial_2 f' = \phi_t \partial_2 \rho + \rho u \phi_{tt} \partial_2 \rho.$$

Substituting by the above equalities into (3.3) and (3.4) and by making use of the facts that $Q = \frac{\phi_t}{\phi - t\phi_t}$ and $Q_t = \frac{\phi\phi_{tt}}{(\phi - t\phi_t)^2}$, we get the required formulae for f_1 and f_2 . $\square$

Focusing more on Q , we have the following lemma.

Lemma 5. *Let the Finsler function be given in the form $F = |y^1| \phi(t)$. Then, we have the following assertions*

- (a) *If $Q_{tt} = 0$, then the manifold (M, F) is Berwaldian.*
- (b) *If $Q = \theta(x)$, then the Finsler function is degenerate.*

Proof. To prove (a), if $Q_{tt} = 0$, then $Q = at + b$, where a and b are arbitrary constants. Moreover, we have the following

$$\frac{Q^2}{Q_t} = \frac{a^2 t^2 + 2abt + b^2}{a}, \quad \frac{Q}{Q_t} = \frac{at + b}{a}.$$

Making use of the above quantities together with (4.3) and (4.4), we conclude that the coefficients of the spray are quadratic and hence the metric is Berwaldain.

To prove (b), one can see that, if $Q = \theta(x)$, then we have

$$\frac{Q}{1+tQ} = \frac{\theta(x)}{1+t\theta(x)}.$$

Therefore, by using (3.1) and (4.2), we have

$$F = |y^1| \exp(\ln(1+t\theta(x))) = \varepsilon(y^1 + \rho\theta(x)y^2).$$

This means that the Finsler function is linear and hence the metric tensor is degenerate. $\square$

Now, we are in a position to announce the main result in this work.

Theorem 1. *All two-dimensional Landsberg metrics in the form (4.1), that is, $F = |y^1| f(x^1, x^2, u)$, $f(x^1, x^2, u) = \phi(t)$, $t := \rho(x^1, x^2)u$, $u = y^2/y^1$, $y^1 \neq 0$, are Berwaldian.*

Proof. Let (M, F) be two-dimensional Landsbergian, then by Proposition 2, the Landsberg's PDE is satisfied, that is

$$f_1''' \ell_1 + f_2''' \ell_2 = 0.$$

Now, assume that the solutions of the Landsberg's PDE is in the form

$$f(x^1, x^2, u) = \phi(t), \quad t = \rho(x)u.$$

Since $\ell_1 = \varepsilon(f - uf') = \varepsilon(\phi - t\phi_t)$ and $\ell_2 = \varepsilon f' = \varepsilon \rho \phi_t$, the above PDE can be rewritten in the form

$$f_1''' + \rho Q f_2''' = 0. \quad (4.5)$$

Now, we are going to find analytic solution for (4.5). By making use of (4.3) and (4.4), we have

$$\begin{aligned} f_1''' &= \frac{Q \partial_1 \rho}{2 \rho Q_t^4} (Q Q_t^2 Q_{ttt} - 6 Q Q_t Q_{tt} Q_{ttt} + 6 Q Q_{tt}^3 + 4 Q_t^3 Q_{ttt} - 6 Q_t^2 Q_{tt}^2), \\ f_2''' &= -\frac{\partial_1 \rho}{2 \rho^2 Q_t^4} (Q Q_t^2 Q_{ttt} - 6 Q Q_t Q_{tt} Q_{ttt} + 6 Q Q_{tt}^3 + 2 Q_t^3 Q_{ttt} - 3 Q_t^2 Q_{tt}^2). \end{aligned}$$

Now, using Lemma 5 (a) together with the facts that the functions Q , ρ , $\partial_1 \rho$ and Q_t should be non-zero, the PDE (4.5) transformed to the ODE

$$3 Q_{tt}^2 - 2 Q_t Q_{ttt} = 0.$$

Making use of Lemma 5 (b), $Q_{tt} \neq 0$. Hence, the above ODE can be rewritten in the form

$$1 + 2 \left(\frac{Q_t}{Q_{tt}} \right)_t = 0.$$

Moreover, the above ODE has the solution

$$\frac{Q_t}{Q_{tt}} = -\frac{1}{2}t + c_1,$$

where c_1 is arbitrary constant. Furthermore, we can find Q_t , since

$$\frac{Q_{tt}}{Q_t} = \frac{2}{2c_1 - t}.$$

Which gives easily the formula of Q_t as follows

$$Q_t = \frac{c_2}{(2c_1 - t)^2}, \quad c_2 > 0. \quad (4.6)$$

By solving the above differential equation, we get

$$Q = \frac{c_2}{2c_1 - t} + c_3, \quad (4.7)$$

where c_1, c_2, c_3 are arbitrary constants such that $c_2 > 0$.

Now, substituting by Q and Q_t into f_1 and f_2 given in Lemma 4 and then substituting into the coefficients G^1 and G^2 given by (3.2), we conclude that the coefficients of the spray are quadratic and hence the metric is Berwaldain. $\square$

Consider the conformal transformation of the metrics on the form (4.1), precisely,

$$\bar{F} = \sigma(x^1, x^2)F = |y^1| \sigma \phi(t) \quad (4.8)$$

Now, to get information about the transformed spray (the geodesic spray of $\bar{F}$), we have to calculate $\bar{f}_1$ and $\bar{f}_2$.

Lemma 6. *Under the conformal transformation (4.8), the functions f_1 and f_2 transform as follows:*

$$\begin{aligned} \bar{f}_1 &= f_1 + \frac{\partial_1 \sigma + u \partial_2 \sigma}{2\sigma} + \frac{\partial_2 \sigma}{2\sigma \rho} \frac{Q}{Q_t} - \frac{\partial_1 \sigma}{2\sigma} \frac{Q^2}{Q_t}, \\ \bar{f}_2 &= f_2 + \frac{u(\partial_1 \sigma + u \partial_2 \sigma)}{2\sigma} + \frac{\partial_1 \sigma}{2\rho \sigma} \frac{Q}{Q_t} - \frac{\partial_2 \sigma}{2\sigma \rho^2} \frac{1}{Q_t}. \end{aligned}$$

Proof. Consider the conformal transformation (4.8), precisely,

$$\bar{F} = \sigma(x^1, x^2)F = |y^1| \sigma \phi(t), \quad t = \rho(x^1, x^2)u, \quad u = y^2/y^1.$$

Since $\bar{F} = |y^1| \bar{f}(x, u) = |y^1| \sigma(x^1, x^2) f(x, u)$, where

$$\bar{f} = \sigma f.$$

Now, we have to calculate $\bar{f}_1$ and $\bar{f}_2$. For this purpose, we have the following

$$\begin{aligned} \bar{f}' &= \sigma f', \quad \bar{f}'' = \sigma f'', \\ \partial_1 \bar{f} &= \sigma \partial_1 f + f \partial_1 \sigma, \quad \partial_2 \bar{f} = \sigma \partial_2 f + f \partial_2 \sigma, \\ \partial_1 \bar{f}' &= \sigma \partial_1 f' + f' \partial_1 \sigma, \quad \partial_2 \bar{f}' = \sigma \partial_2 f' + f' \partial_2 \sigma. \end{aligned}$$

By making use of the above relations together with the help of the quantities $Q = \frac{\phi_t}{\phi - t\phi_t}$, $Q_t = \frac{\phi\phi_{tt}}{(\phi - t\phi_t)^2}$, $f' = \rho\phi_t$ and $f'' = \rho^2\phi_{tt}$, then (3.3) and (3.4) lead to

$$\begin{aligned}\bar{f}_1 &= f_1 + \frac{\partial_1\sigma + u\partial_2\sigma}{2\sigma} + \frac{\partial_2\sigma}{2\sigma\rho} \frac{Q}{Q_t} - \frac{\partial_1\sigma}{2\sigma} \frac{Q^2}{Q_t}, \\ \bar{f}_2 &= f_2 + \frac{u(\partial_1\sigma + u\partial_2\sigma)}{2\sigma} + \frac{\partial_1\sigma}{2\rho\sigma} \frac{Q}{Q_t} - \frac{\partial_2\sigma}{2\sigma\rho^2} \frac{1}{Q_t}.\end{aligned}$$

As to be shown. $\square$

Theorem 2. *The conformal transformations of the Finsler functions given in (4.1) by any positive smooth function $\sigma(x^1, x^2)$ are Berwaldain.*

Proof. By the formulae (4.6) and (4.7), we have

$$Q_t = \frac{c_2}{(2c_1 - t)^2}, \quad Q = \frac{c_2}{2c_1 - t} + c_3.$$

Taking the fact that the geodesic spray of the class (4.1) is quadratic into account. Then, substituting by Q and Q_t into the formulae of $\bar{f}_1$ and $\bar{f}_2$ which are given in Lemma 6, we conclude that the geodesic spray of $\bar{F}$ is quadratic and hence the space is Berwaldian. $\square$

As by-product, we get the following explicit formulae for two dimensional Landsberg metrics which are Berwaldain.

Theorem 3. *The Landsberg metric in the form (4.1) is given by*

$$F = \sqrt{c_3\rho^2(y^2)^2 - (c-2)\rho y^1 y^2 - 2c_1(y^1)^2} e^{\frac{c}{\sqrt{c^2-4c_2}} \operatorname{arctanh}\left(\frac{2c_3\rho y^2 - (c-2)y^1}{y^1\sqrt{c^2-4c_2}}\right)}. \quad (4.9)$$

As a special case, let $\rho(x) = 1$ so that $\phi(t) = \phi(u)$. Then by making use of Theorem 2, we have

$$f(x^1, x^2, u) = \sigma(x^1, x^2)\phi(u)$$

and we get the class

$$F = \sigma(x) \sqrt{c_3(y^2)^2 - (c-2)y^1 y^2 - 2c_1(y^1)^2} e^{\frac{c}{\sqrt{c^2-4c_2}} \operatorname{arctanh}\left(\frac{2c_3 y^2 - (c-2)y^1}{y^1\sqrt{c^2-4c_2}}\right)}. \quad (4.10)$$

In addition, the metrics that satisfy that $Q_{tt} = 0$ are given by

$$F = \sqrt{a\rho^2(y^2)^2 + b\rho y^1 y^2 + (y^1)^2} e^{-\frac{b}{\sqrt{b^2-4a}} \operatorname{arctanh}\left(\frac{2a\rho y^2 + by^1}{y^1\sqrt{b^2-4a}}\right)}. \quad (4.11)$$

Where $t = \rho(x^1, x^2)u$, $u = \frac{y^2}{y^1}$, $c := 2c_1c_3 + c_2 + 1$ and $c_2 > 0, a, b, c_1, c_3$ are constants.

Proof. To calculate the first class, by (4.7), one can show that

$$\frac{Q}{1+tQ} = \frac{-c_3t + 2c_1c_3 + c_2}{-c_3t^2 + (2c_1c_3 + c_2 - 1)t + 2c_1}$$

which can be rewritten in the following useful form

$$\frac{Q}{1+tQ} = \frac{1}{2} \frac{2c_3t - c + 2}{c_3t^2 - (c-2)t - 2c_1} + \frac{2cc_3}{(c^2 - 4c_2) - (2c_3t - c + 2)^2}.$$

Hence, we have

$$\int \frac{Q}{1+tQ} dt = \frac{1}{2} \ln(c_3t^2 - (c-2)t - 2c_1) + \frac{c}{\sqrt{c^2 - 4c_2}} \operatorname{arctanh} \left(\frac{2c_3t - (c-2)}{\sqrt{c^2 - 4c_2}} \right).$$

Now, using (4.2), the function $\phi(t)$ can be calculated and so we get the Finsler function.

To find the third class of the Finsler functions, since $Q_{tt} = 0$, then $Q = at + b$. Now, we have

$$\frac{Q}{1+tQ} = \frac{at+b}{at^2+bt+1} = \frac{2at+b}{2(at^2+bt+1)} - \frac{2ab}{b^2-4a-(2at+b)^2}.$$

Hence, by the help of (3.1) and (4.2), the required formula can be obtained. $\square$

As a direct consequence, we can make use of the surprising result due to Z. Szabó [12] and, for more details, we refer to [13].

Corollary 1. *All two-dimensional Landsberg metrics on the form (4.1) and their conformal transformations are flat (locally Minkowski) or Riemannian.*

In case of higher dimensions $n \geq 3$, the conformal transformation of a Minkowski metrics can yield a non-Berwaldian Landsberg metrics (cf. [4]). Assume that a manifold (M, F) is Minkowski if the Finsler function F depends on the directional argument y only. In contrast of the higher dimensions, we have the following result.

Theorem 4. *If the conformal transformation of any two dimensional Minkowski metric is Landsberg, then it is Berwaldian.*

Proof. Let (M, F) be a Minkowski space, then one can write

$$F = |y^1| f(u).$$

In this case F can be considered a special case of the form (4.1) by setting $\rho(x) = 1$. Now, the conformal transformation of F takes the form

$$\bar{F} = \sigma(x)F = |y^1| \sigma(x)f(u).$$

Since $\bar{F}$ is Landsbergian then $f(u)$ is given by (4.10), that is $f(u) = \phi(u)$. Consequently, $\bar{F}$ is Berwaldian. $\square$

5. CONCLUDING REMARKS

We end this work by the following remarks.

- If we consider the Landsberg surface in the form

$$F = |y^2| \varphi(\rho(x)v), \quad v := y^1/y^2,$$

then we get a similar formula of $\varphi(\rho(x)v)$ as the formula giving by (4.9). In fact this corresponds a change of coordinates by switching x^1 and x^2 and this implies switching y^1 and y^2 . Moreover, if we need to fix a coordinate system and work on $F = |y^2| \varphi(\rho(x)v)$, then $u = \frac{1}{v}$. To get the function φ , we have to substitute by $u = \frac{1}{v}$ in (4.9), that is $\varphi(\rho(x)v) = \varphi(\frac{\rho(x)}{v})$. That is, we obtain the same formula of the Finsler function F .

- If there is a singularity in the surfaces given by (4.9) or (4.11) at certain direction, then this singularity does not come from the assumption that $F = |y^1| \phi(\rho(x)u)$ rather it comes from being F a solution of the Landsberg's PDE. Moreover, the choice of the constants c_1, c_2, c_3 play an essential rule to determine whether the resultant metric is regular. Many, but not all, Berwaldian surfaces among them the Riemannian ones can be obtained by specific choice of these constants. For example the choice $c_1 = c_2 = 1$ and $c_3 = -1$ leads to $c = 0$ and hence we get the Riemannian metrics

$$F = \sqrt{a(x)(y^1)^2 + b(x)y^1y^2 + c(x)(y^2)^2},$$

for some functions $a(x), b(x), c(x)$.

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A NUMERICAL COMPARATIVE STUDY FOR THE SINGULARLY PERTURBED NONLINEAR VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS ON LAYER-ADAPTED MESHES

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Abstract. This article deals with the singularly perturbed nonlinear Volterra-Fredholm integro-differential equations. Firstly, some priori bounds are presented. Then, the finite difference scheme is constructed on non-uniform mesh by using interpolating quadrature rules [5] and composite numerical integration formulas. The error estimates are derived in the discrete maximum norm. Finally, theoretical results are performed on two examples and they are compared for both Bakhvalov (B-type) and Shishkin (S-type) meshes.

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1. INTRODUCTION

Nonlinear integro-differential equations (NIDEs) are effective tools for modelling of many real life situations. Their applications appear in physical processes [14], biological events [27], population dynamics [20], financial problems [1] and other areas [31].

In the literature, many papers have been written about NIDEs. In general, solving of such problems is hard and even sometimes impossible with classical analytical techniques. Thus, reliable numerical methods have been developed such as Bernstein polynomial method [8], collocation techniques [32], Nyström discretization [24], Legendre wavelets [22], finite difference methods [11, 29] and Galerkin finite element approach [18].

The above-mentioned studies have only dealt with the regular cases (absent the layer behavior). This article concerns with the following initial-value problem of the

singularly perturbed nonlinear Volterra-Fredholm integro-differential equation:

$$Lu := \varepsilon u'(t) + f(t, u(t)) + \int_0^t K_1(t, s, u(s))ds + \lambda \int_0^T K_2(t, s, u(s))ds = 0, \quad (1.1)$$

$$u(0) = A, \quad t \in I = (0, T], \quad (1.2)$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, λ is an arbitrary parameter, $\bar{I} = [0, T]$ and A is a given constant. $f(t, u(t))$ ($(t, u) \in \bar{I} \times \mathbb{R}$) and $K_1(t, s, u(s))$, $K_2(t, s, u(s))$ ($(t, s, u) \in \bar{I} \times \bar{I} \times \mathbb{R}$) are sufficiently smooth functions and $0 < \alpha \leq \left| \frac{\partial f}{\partial u} \right| \leq p^* < \infty$. In [16], A. A. Hamoud and K. P. Ghadle have discussed the existence and uniqueness results of the problem (1.1)-(1.2) without singular perturbation. In [31], different variational techniques have been proposed for Volterra-Fredholm integro-differential equations. Furthermore, the linear form of the problem (1.1)-(1.2) has been considered by using finite difference schemes in [9, 13].

Singular perturbation phenomena is classified by a small parameter ε multiplying highest order derivative terms in the differential equation. The class of such problems generally involves the boundary layers. The solution of the problem changes rapidly within layer region whereas it behaves slowly and regularly outside of the layer region. To overcome this complexity, robust discretizations are needed [12, 21, 23, 26]. Some existence and uniqueness results about singularly perturbed problems have been given in [15, 23]. In recent times, notable techniques and various numerical schemes have been presented for singularly perturbed integro-differential equations (see [2, 6, 9, 10, 13, 17, 19, 25, 29, 33–35]). Our aim in this paper is to present a uniform numerical method for solving singularly perturbed nonlinear integro-differential equations and compare the obtained results on Bakhvalov and Shishkin type meshes. In addition to the subject, Bakhvalov mesh was developed by N. S. Bakhvalov in 1969 [7] and G. I. Shishkin designed the piecewise equidistant mesh [30].

The rest of the paper is as follows: The properties of the analytical solution of the problem (1.1)-(1.2) are given in Section 2. In Section 3, the difference scheme is established on non-uniform mesh. The stability and convergence of the method are investigated in Section 4. In Section 5, some numerical experiments are presented.

2. PRELIMINARIES

This section is devoted to the asymptotic bounds of the problem (1.1)-(1.2).

Lemma 1 (see [19, Lemma 2.1]). *Take into account the following initial-value problem:*

$$\varepsilon v'(t) + a(t)v(t) = F(t), \quad t \in I, \quad (2.1)$$

$$v(0) = A. \quad (2.2)$$

Let $a(t) \geq \alpha > 0$, $F(t) \in C(\bar{I})$, $|F(t)| \leq \mathcal{F}(t)$ and $\mathcal{F}(t)$ is a nondecreasing function. Then, the solution of the problem (2.1)-(2.2) holds that

$$|v(t)| \leq |A| + \alpha^{-1} \mathcal{F}(t), \quad t \in I.$$

Lemma 2. Suppose that

$$p, q \in C^1[0, T], \quad \frac{\partial}{\partial t} G_1(t, s) \in C^1[0, T]^2, \quad \frac{\partial}{\partial t} G_2(t, s) \in C^1[0, T]^2 \quad (2.3)$$

and

$$\gamma = e^{\alpha^{-1} \bar{G}_1 T} \alpha^{-1} |\lambda| \max_{0 \leq t \leq T} \int_0^t |G_2(t, s)| ds < 1.$$

Then, the solution $u(t)$ of the problem (1.1)-(1.2) satisfies that

$$\|u\|_\infty \leq C_0 \quad (2.4)$$

and

$$|u'(t)| \leq C \left\{ 1 + \frac{1}{\varepsilon} e^{\frac{-at}{\varepsilon}} \right\}, \quad t \in [0, T], \quad (2.5)$$

where

$$C_0 = (1 - \gamma)^{-1} (|A| + \alpha^{-1} \|q\|_\infty) e^{\alpha^{-1} \bar{G}_1 T}$$

and

$$\bar{G}_1 = \max_{\bar{I} \times \bar{I}} |G_1(t, s)|.$$

Here, for any continuous function $v(t)$ defined on the corresponding interval, we use the maximum norm $\|v\|_\infty = \max_{[0, T]} |v(t)|$ and C (in some cases subscripted) is a generic positive constant [2, 29].

Proof. Firstly, we prove the relation (2.4). Applying the mean value theorem to the functions in the equation (1.1), we find

$$\varepsilon u'(t) + p(t)u(t) + \int_0^t G_1(t, s)u(s)ds + \lambda \int_0^T G_2(t, s)u(s)ds = q(t), \quad t \in I \quad (2.6)$$

where

$$\begin{aligned} p(t) &= \frac{\partial}{\partial u} f(t, \tilde{u}), \quad \tilde{u} = \rho u, \quad 0 < \rho < 1, \\ G_1(t, s) &= \frac{\partial}{\partial u} K_1(t, s, \bar{u}), \quad \bar{u} = \theta u, \quad 0 < \theta < 1, \\ G_2(t, s) &= \frac{\partial}{\partial u} K_2(t, s, \check{u}), \quad \check{u} = \zeta u, \quad 0 < \zeta < 1, \end{aligned}$$

and

$$q(t) = -f(t, 0) - \int_0^t K_1(t, s, 0)ds - \lambda \int_0^T K_2(t, s, 0)ds.$$

From here, we have

$$\varepsilon u'(t) + p(t)u(t) = F(t), \quad t \in I, \quad (2.7)$$

where

$$F(t) = q(t) - \int_0^t G_1(t, s)u(s)ds - \lambda \int_0^T G_2(t, s)u(s)ds. \quad (2.8)$$

Then, we estimate (2.8) as

$$|F(t)| \leq \|q\|_\infty + \bar{G}_1 \int_0^t |u(s)|ds + |\lambda| \int_0^T |G_2(t, s)| |u(s)|ds.$$

Considering Lemma 1 for the equation (2.7), we get

$$|u(t)| \leq \delta + \alpha^{-1} \bar{G}_1 \int_0^t |u(s)|ds, \quad (2.9)$$

where

$$\delta = |A| + \alpha^{-1} \|q\|_\infty + \alpha^{-1} |\lambda| \int_0^T |G_2(t, s)| |u(s)|ds.$$

Applying the Gronwall's inequality to the relation (2.9), we obtain

$$\|u\|_\infty \leq \frac{(|A| + \alpha^{-1} \|q\|_\infty) \exp(\alpha^{-1} \bar{G}_1 T)}{\left(1 - \alpha^{-1} |\lambda| \exp(\alpha^{-1} \bar{G}_1 T) \max_{t \in [0, T]} \int_0^t |G_2(t, s)|ds\right)},$$

which points to the proof of (2.4). Now, we show the proof of the relation (2.5) (see [9, 11, 13, 29]). From (2.6), we write

$$|u'(0)| \leq \frac{1}{\varepsilon} \left(|q(0)| + |p(0)| |A| + |\lambda| \int_0^T |G_2(0, s)| |u(s)|ds \right) \leq \frac{C}{\varepsilon}. \quad (2.10)$$

Differentiating the equation (2.7), we find

$$\varepsilon v' + p(t)v = H(t), \quad v(t) = u'(t), \quad (2.11)$$

where

$$H(t) = q'(t) - p'(t)u(t) - \int_0^t \frac{\partial}{\partial t} G_1(t, s)u(s)ds - G_1(t, t)u(t) - \lambda \int_0^T \frac{\partial}{\partial t} G_2(t, s)u(s)ds.$$

Because of (2.3) and (2.4), we have

$$|H(t)| \leq C. \quad (2.12)$$

From (2.11), it can be seen that

$$u'(t) = u'(0)e^{-\frac{1}{\varepsilon} \int_0^t p(\eta) d\eta} + \frac{1}{\varepsilon} \int_0^t H(\xi) e^{-\frac{1}{\varepsilon} \int_\xi^t p(\eta) d\eta} d\xi.$$

By taking into account (2.10) and (2.12), it is obtained:

$$|u'(t)| \leq \frac{C}{\varepsilon} e^{-\frac{\alpha}{\varepsilon}} + \alpha^{-1} \|H\|_{\infty} \left(1 - e^{-\frac{\alpha}{\varepsilon}}\right),$$

which shows the validity of (2.5). $\square$

3. DISCRETE SCHEME

In this section, we construct the difference scheme on the non-uniform mesh.

Notation 1 ([28]). Let ω_N be any non-uniform mesh on $[0, T]$:

$$\omega_N = \{0 < t_1 < t_2 < \dots < t_{N-1} < t_N = T, \tau_i = t_i - t_{i-1}, i = 1, 2, \dots, N\}$$

and

$$\bar{\omega}_N = \omega_N \cup \{t_0 = 0\}.$$

For any mesh function on $\bar{\omega}_N$, we use [28]

$$v_i = v(t_i), \quad v_{\bar{i}, i} = \frac{v_i - v_{i-1}}{\tau_i}, \quad \|v\|_{\infty} \equiv \|v\|_{\infty, \bar{\omega}_N} := \max_{0 \leq i \leq N} |v_i|.$$

Definition 1 ([3, 21, 23, 26, 30, Shishkin-Type Mesh]). For an even number N , we divide each of the subintervals $[0, \sigma]$ and $[\sigma, T]$ into $\frac{N}{2}$ equidistant subintervals. The transition parameter σ is stated as

$$\sigma = \min \left\{ \frac{T}{2}, \alpha^{-1} \varepsilon \ln N \right\}.$$

We use the notation $\tau^{(1)}$ for the mesh width in $[0, \sigma]$ and the notation $\tau^{(2)}$ for the width in $[\sigma, T]$. Hence, the mesh stepsizes hold

$$\tau^{(1)} = \frac{2\sigma}{N}, \quad \tau^{(2)} = \frac{2(T - \sigma)}{N},$$

$$\tau^{(1)} \leq TN^{-1}, \quad TN^{-1} \leq \tau^{(2)} \leq 2TN^{-1}, \quad \tau^{(1)} + \tau^{(2)} = 2TN^{-1}.$$

t_i mesh points are shown by

$$\bar{\omega}_N = \begin{cases} t_i = i\tau^{(1)}, & i = 0, 1, \dots, \frac{N}{2}, t_i \in [0, \sigma]; \\ t_i = \sigma + \left(i - \frac{N}{2}\right)\tau^{(2)}, & i = \frac{N}{2} + 1, \dots, N, t_i \in [\sigma, T]. \end{cases}$$

Definition 2 ([4, 7, 21, 26, Bakhvalov-Type Mesh]). We divide each of the subintervals $[0, \sigma]$ and $[\sigma, T]$ into $\frac{N}{2}$ equidistant subintervals. The transition point σ is expressed by

$$\sigma = \min \left\{ \frac{T}{2}, \alpha^{-1} \varepsilon |\ln \varepsilon| \right\}.$$

t_i node points are described as follows:

If $\sigma < \frac{T}{2}$,

$$t_i = \begin{cases} -\alpha^{-1} \varepsilon \ln [1 - (1 - \varepsilon) 2i/N], & i = 0, 1, \dots, \frac{N}{2}, \\ \sigma + (i - \frac{N}{2}) \tau, & i = \frac{N}{2} + 1, \dots, N, \end{cases}$$

and if $\sigma = \frac{T}{2}$,

$$t_i = \begin{cases} -\alpha^{-1} \varepsilon \ln [1 - (1 - \exp(\frac{-\alpha T}{2\varepsilon})) 2i/N], & i = 0, 1, \dots, \frac{N}{2}, \\ \sigma + (i - \frac{N}{2}) \tau, & i = \frac{N}{2} + 1, \dots, N, \end{cases}$$

where $\tau = 2(T - \sigma)/N$.

Now, we establish the finite difference scheme. To design the difference method, we utilize that

$$\begin{aligned} \tau_i^{-1} \int_{t_{i-1}}^{t_i} \varepsilon u'(t) dt + \tau_i^{-1} \int_{t_{i-1}}^{t_i} f(t, u) dt + \tau_i^{-1} \int_{t_{i-1}}^{t_i} \int_0^t K_1(t, s, u(s)) ds dt \\ + \tau_i^{-1} \lambda \int_{t_{i-1}}^{t_i} \int_0^T K_2(t, s, u(s)) ds dt = 0. \end{aligned} \quad (3.1)$$

Applying interpolating quadrature rules [5] and some manipulations in [11, 29] for the first term and the second term of the relation (3.1), we find

$$\tau_i^{-1} \int_{t_{i-1}}^{t_i} [\varepsilon u'(t) + f(t, u)] dt = \varepsilon u_{\bar{t}, i} + f(t_i, u_i) + R_i^{(1)} \quad (3.2)$$

where

$$R_i^{(1)} = -\tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \frac{\partial}{\partial \xi} f(\xi, u(\xi)) d\xi. \quad (3.3)$$

For the third and fourth terms of (3.1), using interpolating quadrature rules again [5, 11, 29] and right side rectangle formula, we get

$$\begin{aligned} \tau_i^{-1} \int_{t_{i-1}}^{t_i} \int_0^t K_1(t, s, u(s)) ds dt + \tau_i^{-1} \lambda \int_{t_{i-1}}^{t_i} \int_0^T K_2(t, s, u(s)) ds dt \\ = \sum_{j=1}^i \tau_j K_{1, ij}(t_i, t_j, u_j) + \lambda \sum_{j=1}^N \tau_j K_{2, ij}(t_i, t_j, u_j) + R_i^{(2)} + R_i^{(3)} + R_i^{(4)} + R_i^{(5)} \end{aligned} \quad (3.4)$$

where remainder terms

$$R_i^{(2)} = -\tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \left(\int_0^t \frac{\partial}{\partial \xi} K_1(\xi, s, u(s)) ds \right) d\xi, \quad (3.5)$$

$$R_i^{(3)} = \sum_{j=1}^i \int_{t_{j-1}}^{t_j} (\xi - t_{j-1}) \frac{\partial}{\partial \xi} K_1(t_i, \xi, u(\xi)) d\xi, \quad (3.6)$$

$$R_i^{(4)} = -\lambda \tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \left(\int_0^T \frac{\partial}{\partial \xi} K_2(\xi, s, u(s)) ds \right) d\xi \quad (3.7)$$

and

$$R_i^{(5)} = \lambda \sum_{j=1}^N \int_{t_{j-1}}^{t_j} (\xi - t_{j-1}) \frac{\partial}{\partial \xi} K_2(t_i, \xi, u(\xi)) d\xi. \quad (3.8)$$

From the relations (3.2) and (3.4), the following difference problem is written:

$$\varepsilon u_{\bar{i},i} + f(t_i, u_i) + \sum_{j=1}^i \tau_j K_{1,i,j}(t_i, t_j, u_j) + \lambda \sum_{j=1}^N \tau_j K_{2,i,j}(t_i, t_j, u_j) + R_i = 0, \quad (3.9)$$

$$u_0 = A, \quad i = 1, 2, \dots, N,$$

where the error term is expressed by

$$R_i = \sum_{k=1}^5 R_i^{(k)}. \quad (3.10)$$

Omitting the error term R_i in (3.9), we can present that

$$\varepsilon y_{\bar{i},i} + f(t_i, y_i) + \sum_{j=1}^i \tau_j K_{1,i,j}(t_i, t_j, y_j) + \lambda \sum_{j=1}^N \tau_j K_{2,i,j}(t_i, t_j, y_j) = 0, \quad (3.11)$$

$$y_0 = A, \quad i = 1, 2, \dots, N.$$

4. CONVERGENCE ANALYSIS

Let the error function $z_i = y_i - u_i$, $i = 0, 1, 2, \dots, N$ be the solution of the following discrete problem:

$$\begin{aligned} l z_i &= \varepsilon z_{\bar{i},i} + [f(t_i, y_i) - f(t_i, u_i)] + \sum_{j=1}^i \tau_j [K_{1,i,j}(t_i, t_j, y_j) - K_{1,i,j}(t_i, t_j, u_j)] \\ &\quad + \lambda \sum_{j=1}^N \tau_j [K_{2,i,j}(t_i, t_j, y_j) - K_{2,i,j}(t_i, t_j, u_j)] = R_i, \end{aligned} \quad (4.1)$$

$$i = 1, 2, \dots, N, \quad z_0 = 0, \quad (4.2)$$

where R_i is given by (3.10).

Lemma 3 (see [19, Lemma 4.1]). *Consider the following difference problem:*

$$\varepsilon v_{\bar{i},i} + p_i v_i = F_i, \quad i = 0, 1, 2, \dots, N, \quad v_0 = A. \quad (4.3)$$

Let $|F_i| \leq \mathcal{F}_i$ and $\mathcal{F}_i$ be nondecreasing function. Then, the solution of (4.3) satisfies

$$|v_i| \leq |A| + \alpha^{-1} \mathcal{F}_i, \quad i = 0, 1, 2, \dots, N.$$

Lemma 4. *Let z_i be the solution of the problem (4.1)-(4.2). If*

$$\bar{\gamma} = \alpha^{-1} |\lambda| e^{\alpha^{-1} \bar{G}_1 t_i} \max_{1 \leq i \leq N} \sum_{j=1}^N \tau_j |G_{2,ij}| < 1,$$

it is held that

$$\|z\|_{\infty, \bar{\omega}_N} \leq \alpha^{-1} (1 - \bar{\gamma})^{-1} e^{\alpha^{-1} \bar{G}_1 t_i} \|R\|_{\infty}.$$

Proof. From (4.1), applying the mean value theorem, we have

$$\varepsilon z_{i,i} + p_i z_i + \sum_{j=1}^i \tau_j G_{1,ij} z_j + \lambda \sum_{j=1}^N \tau_j G_{2,ij} z_j = R_i, \quad i = 1, 2, \dots, N, \quad (4.4)$$

where

$$\begin{aligned} p_i &= \frac{\partial}{\partial u} f(t_i, u_i + \mu_1 z_i), & 0 < \mu_1 < 1, \\ G_{1,ij} &= \frac{\partial}{\partial u} K_1(t_i, s_j, u_i + \mu_2 z_i), & 0 < \mu_2 < 1 \end{aligned}$$

and

$$G_{2,ij} = \frac{\partial}{\partial u} K_2(t_i, s_j, u_i + \mu_3 z_i), \quad 0 < \mu_3 < 1.$$

Then, the difference equation (4.4) can be written in the form

$$\varepsilon z_{i,i} + p_i z_i = F_i$$

where

$$F_i = R_i - \sum_{j=1}^i \tau_j G_{1,ij} z_j - \lambda \sum_{j=1}^N \tau_j G_{2,ij} z_j. \quad (4.5)$$

From (4.5), we get

$$|F_i| \leq \|R\|_{\infty} + \sum_{j=1}^i \tau_j |G_{1,ij}| |z_j| + |\lambda| \sum_{j=1}^N \tau_j |G_{2,ij}| |z_j|. \quad (4.6)$$

Furthermore, considering Lemma 3 and the inequality (4.6), we have

$$|z_i| \leq \bar{\delta} + \alpha^{-1} \bar{G}_1 \sum_{j=1}^i \tau_j |z_j|, \quad (4.7)$$

where

$$\bar{\delta} = \alpha^{-1} \|R\|_{\infty} + \alpha^{-1} |\lambda| \sum_{j=1}^N \tau_j |G_{2,ij}| |z_j|.$$

Applying the difference analogue of Gronwall's inequality to the relation (4.7), it can be written obviously that

$$\|z\|_{\infty} \leq \frac{e^{\alpha^{-1}\bar{G}_1 t_i} \alpha^{-1} \|R\|_{\infty}}{1 - \alpha^{-1} |\lambda| e^{\alpha^{-1}\bar{G}_1 t_i} \max_{1 \leq i \leq N} \sum_{j=1}^N \tau_j |G_{2,ij}|},$$

which shows the proof of the lemma (see [9, 11, 13, 29]). $\square$

Lemma 5. *Under the conditions of Lemma 2, the following estimate is satisfied on Shishkin-type mesh*

$$\|R\|_{\infty, \omega_N} \leq CN^{-1} \ln N$$

and for the Bakhvalov-type mesh, it is written that

$$\|R\|_{\infty, \omega_N} \leq CN^{-1}.$$

Proof.

a) Here, we present a parallel approach to [11, 29]. Firstly, we consider the term (3.3). From the relation (3.3), we obtain

$$\begin{aligned} |R_i^{(1)}| &\leq \tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \left| \frac{\partial}{\partial \xi} f(\xi, u(\xi)) + \frac{\partial}{\partial u} f(\xi, u(\xi)) u'(\xi) \right| d\xi \\ &\leq C \left\{ \tau_i + \int_{t_{i-1}}^{t_i} |u'(\xi)| d\xi \right\}, \quad i = 1, 2, \dots, N. \end{aligned} \quad (4.8)$$

For the remainder term (3.5), since $\left| \frac{\partial K_1}{\partial \xi} \right| \leq C$ and $|u(t)| \leq C$, it follows that

$$\begin{aligned} |R_i^{(2)}| &\leq \tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \left| K_1(\xi, t, u(t)) + \int_0^t \frac{\partial}{\partial \xi} K_1(\xi, s, u(s)) ds \right| d\xi \\ &\leq C \tau_i, \quad i = 1, 2, \dots, N. \end{aligned} \quad (4.9)$$

Next, from the remainder term (3.6), we get

$$\begin{aligned} |R_i^{(3)}| &\leq \sum_{j=1}^i \int_{t_{j-1}}^{t_j} (\xi - t_{j-1}) \left| \frac{\partial}{\partial \xi} K_1(t, \xi, u(\xi)) + \frac{\partial}{\partial u} K_1(t, \xi, u(\xi)) u'(\xi) \right| d\xi \\ &\leq C \left\{ \tau_i + \int_{t_{i-1}}^{t_i} |u'(\xi)| d\xi \right\}. \end{aligned} \quad (4.10)$$

For the term (3.7), we can write

$$\begin{aligned} |R_i^{(4)}| &\leq \lambda \tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \left| K_2(\xi, t, u(t)) + \int_0^T \frac{\partial}{\partial \xi} K_2(\xi, s, u(s)) ds \right| d\xi \\ &\leq C \tau_i, \quad i = 1, 2, \dots, N. \end{aligned} \quad (4.11)$$

Next, for the remainder term (3.8), we have

$$\begin{aligned} |R_i^{(5)}| &\leq \lambda \sum_{j=1}^N \int_{t_{j-1}}^{t_j} (\xi - t_{j-1}) \left| \frac{\partial}{\partial \xi} K_2(t, \xi, u(\xi)) + \frac{\partial}{\partial u} K_2(t, \xi, u(\xi)) u'(\xi) \right| d\xi \\ &\leq C \left\{ \tau_i + \int_{t_{i-1}}^{t_i} |u'(\xi)| d\xi \right\}. \end{aligned} \quad (4.12)$$

Substituting the inequalities (4.8), (4.9), (4.10), (4.11) and (4.12) in (3.10), and considering (2.5), we obtain

$$|R_i| \leq C \left\{ \tau_i + \frac{1}{\varepsilon} \int_{t_{i-1}}^{t_i} e^{-\frac{\omega}{\varepsilon}} dt \right\}. \quad (4.13)$$

Now, we estimate the error approximations according to the node points of Shishkin type mesh. Firstly, we consider the case $\sigma = \frac{T}{2}$. Then $\frac{T}{2} \leq \alpha^{-1} \varepsilon \ln N$, $\tau^{(1)} = \tau^{(2)} = \tau = TN^{-1}$. From (4.13), we get

$$|R_i| \leq C \{N^{-1} + \varepsilon^{-1} TN^{-1}\} \leq C \{N^{-1} + \alpha^{-1} N^{-1} \ln N\}.$$

Then, we obtain

$$|R_i| \leq CN^{-1} \ln N, \quad i = 1, 2, \dots, N.$$

Now, we consider the case $\sigma = \alpha^{-1} \varepsilon \ln N$, so that $\alpha^{-1} \varepsilon \ln N < \frac{T}{2}$. We estimate separately R_i on $[0, \sigma]$ and $[\sigma, T]$. In the layer region $[0, \sigma]$, the relation (4.13) degrades as

$$|R_i| \leq C (1 + \varepsilon^{-1}) \tau^{(1)} \leq C (1 + \varepsilon^{-1}) \frac{2\alpha^{-1} \varepsilon \ln N}{N} \leq CN^{-1} \ln N, \quad i = 1, 2, \dots, \frac{N}{2}.$$

To estimate R_i in the layer region $[\sigma, T]$, the inequality (4.13) can be written in the form

$$|R_i| \leq C \left\{ \tau^{(2)} + \alpha^{-1} \left(e^{-\frac{\omega_{i-1}}{\varepsilon}} - e^{-\frac{\omega_i}{\varepsilon}} \right) \right\}, \quad i = \frac{N}{2} + 1, \dots, N. \quad (4.14)$$

On account of $t_i = \alpha^{-1} \varepsilon \ln N + (i - \frac{N}{2}) \tau^{(2)}$, it is found that

$$e^{-\frac{\omega_{i-1}}{\varepsilon}} - e^{-\frac{\omega_i}{\varepsilon}} = \frac{1}{N} e^{\frac{-\alpha(i-1-\frac{N}{2})\tau^{(2)}}{\varepsilon}} \left(1 - e^{-\frac{\alpha\tau^{(2)}}{\varepsilon}} \right) \leq N^{-1}.$$

This relation and (4.14) yield the estimate $|R_i| \leq CN^{-1}$ [3, 6, 19, 29, 33].

b) Now, we consider the node points of Bakhvalov type mesh. For the interval $[0, \sigma]$, we find

$$\begin{aligned} \tau_i = t_i - t_{i-1} &= \alpha^{-1} \varepsilon \left\{ -\ln \left[1 - (1 - \varepsilon) \frac{2i}{N} \right] + \ln \left[1 - (1 - \varepsilon) \frac{2(i-1)}{N} \right] \right\} \\ &\leq 2\alpha^{-1} (1 - \varepsilon) N^{-1}. \end{aligned}$$

Considering (4.13), we have

$$|R_i| \leq C \left\{ \tau_i + \int_{t_{i-1}}^{t_i} |u'(t)| dt \right\} \leq C \left\{ \tau_i + \alpha^{-1} \left[e^{-\frac{\alpha_{i-1}}{\varepsilon}} - e^{-\frac{\alpha_i}{\varepsilon}} \right] \right\}.$$

From here, it can be seen clearly that

$$e^{-\frac{\alpha_{i-1}}{\varepsilon}} - e^{-\frac{\alpha_i}{\varepsilon}} = 2(T - \varepsilon)N^{-1}. \quad (4.15)$$

By virtue of (4.15), it is obtained $|R_i| \leq 4\alpha^{-1}CN^{-1}$. For the interval $[\sigma, T]$, using $\tau = 2(T - \sigma)/N = TN^{-1}$, $\sigma = \frac{T}{2}$ and $\frac{T}{2} < \alpha^{-1}\varepsilon \ln \varepsilon$, we get

$$\max_{t_{i-1}}^{t_i} \int \frac{1}{\varepsilon} e^{-\frac{\alpha}{\varepsilon}} dt \leq \frac{2e^{-1}}{\alpha T} \tau = 2e^{-1}\alpha^{-1}CN^{-1},$$

$$|R_i| \leq C \left(T + \frac{2}{e\alpha} \right) N^{-1}.$$

Hence, we can show that $|R_i| \leq CN^{-1}$ [4, 10, 29].

□

5. NUMERICAL EXPERIMENTS

TABLE 1. Error approximations e^N and order of convergence p^N on S-Mesh

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	0.01514971 0.8076	0.00865547 0.8147	0.00492091 0.8296	0.00276883 0.8403	0.00154652
2^{-2}	0.02951389 0.7890	0.01708056 0.8188	0.00968299 0.8205	0.00548280 0.8276	0.00308931
2^{-3}	0.05611796 0.7540	0.03327546 0.7862	0.01929541 0.8138	0.01097674 0.8257	0.00619335
2^{-4}	0.08652760 0.7079	0.05297076 0.7411	0.03169212 0.7942	0.01827619 0.8102	0.01042275
2^{-5}	0.10183001 0.7223	0.06172058 0.7699	0.03619636 0.8021	0.02075972 0.8208	0.01175285
2^{-6}	0.10948122 0.7280	0.06609549 0.7704	0.03874848 0.8035	0.02220149 0.8232	0.01254775
2^{-7}	0.11330682 0.7306	0.06828295 0.7833	0.03967454 0.8127	0.02258739 0.8324	0.01268525
...	...	...	...	...	...
2^{-10}	0.11665423 0.7327	0.07019697 0.7891	0.04062234 0.8172	0.02305427 0.8349	0.01292431
e^N	0.11665423	0.07019697	0.04062234	0.02305427	0.01292431
p^N	0.7079	0.7411	0.7942	0.8102	

In this section, we present the numerical results. Because of the existence of non-linear terms in the discretization (3.11), we apply the quasilinearization technique [11, 29]. Hence, we obtain

$$\begin{aligned} & \varepsilon y_{i,i}^{(n)} + f(t_i, y_i^{(n-1)}) + \frac{\partial}{\partial y} f(t_i, y_i^{(n-1)}) (y_i^{(n)} - y_i^{(n-1)}) \\ & + \sum_{j=1}^i \tau_j \left[K_{1,ij}(t_i, t_j, y_j^{(n-1)}) + \frac{\partial}{\partial y} K_{1,ij}(t_i, t_j, y_j^{(n-1)}) (y_j^{(n)} - y_j^{(n-1)}) \right] \\ & + \lambda \sum_{j=1}^N \tau_j \left[K_{2,ij}(t_i, t_j, y_j^{(n-1)}) + \frac{\partial}{\partial y} K_{2,ij}(t_i, t_j, y_j^{(n-1)}) (y_j^{(n)} - y_j^{(n-1)}) \right] = 0, \\ & y_0^{(n)} = A, \quad i = 1, 2, \dots, N. \end{aligned}$$

TABLE 2. Error approximations e^N and order of convergence p^N on B-Mesh

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	0.00822594 0.9205	0.00434577 0.9309	0.00227957 0.9360	0.00119146 0.9514	0.00061615
2^{-2}	0.01627748 0.9129	0.00864498 0.9268	0.00454739 0.9341	0.00237997 0.9493	0.00123259
2^{-3}	0.02598728 0.9267	0.01367026 0.9480	0.00708582 0.9529	0.00366041 0.9660	0.00187382
2^{-4}	0.03670097 0.9381	0.01915414 0.9549	0.00988129 0.9635	0.00506712 0.9681	0.00259023
2^{-5}	0.05057400 0.9395	0.02636907 0.9577	0.01357648 0.9669	0.00694575 0.9777	0.00352693
2^{-6}	0.06205511 0.9404	0.03233567 0.9545	0.01668547 0.9741	0.00849394 0.9838	0.00429500
2^{-7}	0.07308114 0.9379	0.03814559 0.9568	0.01965310 0.9901	0.00989449 0.9942	0.00496710
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
2^{-10}	0.09727740 0.9266	0.05117653 0.9515	0.02646298 0.9703	0.01350693 0.9757	0.00686821
e^N	0.09727740	0.05117653	0.02646298	0.01350693	0.00686821
p^N	0.9129	0.9268	0.9341	0.9493	

Example 1. Consider the first problem

$$\begin{aligned} & \varepsilon u'(t) + u^3(t) + u(t) + e^{-3t/\varepsilon} \left(\frac{\varepsilon - 6}{6} \right) \\ & + \frac{\varepsilon}{4} e^{\frac{-2}{\varepsilon}} - \frac{5\varepsilon}{12} + \frac{1}{2} \int_0^t u^3(s) ds + \frac{1}{2} \int_0^1 u^2(s) ds = 0, \\ & u(0) = 1. \end{aligned}$$

whose exact solution is $u(t) = e^{-t/\epsilon}$. Maximum pointwise errors are determined as $e^N = |y_i - u_i|$. Also, the convergence rates are calculated with $p^N = \frac{\ln(e^N/e^{2N})}{\ln 2}$. The computed results are presented in Tables 1-2.

TABLE 3. Error approximations e^N and order of convergence p^N on S-Mesh

ϵ	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	0.03373796 0.8286	0.01899680 0.8424	0.01059502 0.8488	0.00588290 0.8531	0.00325679
2^{-2}	0.06498652 0.8032	0.03724032 0.8268	0.02099584 0.8442	0.01169480 0.8573	0.00645516
2^{-3}	0.11815619 0.7218	0.07164096 0.7968	0.04123813 0.8433	0.02298536 0.8315	0.01291668
2^{-4}	0.17177238 0.7971	0.09885501 0.8186	0.05604937 0.8064	0.03204937 0.8327	0.01799512
2^{-5}	0.17288072 0.7757	0.10097806 0.8092	0.05762941 0.8209	0.03262288 0.8419	0.01820037
2^{-6}	0.17959856 0.7559	0.10635639 0.7978	0.06117739 0.8020	0.03508942 0.8332	0.01969521
2^{-7}	0.18295747 0.7544	0.10845881 0.7969	0.06242714 0.8146	0.03549438 0.8365	0.01987633
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
2^{-10}	0.18589653 0.7238	0.11255785 0.8062	0.06436884 0.8214	0.03642489 0.8469	0.02025114
e^N	0.18589653	0.11255785	0.06436884	0.03642489	0.02025114
p^N	0.7218	0.7968	0.8020	0.8315	

Example 2. Take into account the second problem

$$\epsilon u'(t) + 2u(t) - e^{-u(t)} + \int_0^t e^{\sin(u(s))} ds + \frac{1}{4} \int_0^1 \sin(u(s)) ds = 0$$

with $u(0) = 1$. The exact solution of this problem is unknown. Since the exact solution is unknown, we use the double-mesh principle [6, 12, 13]. The error estimates are denoted by $e^N = |y_i^N - y_i^{2N}|$ and the order of convergence is computed as $p^N = \frac{\ln(e^N/e^{2N})}{\ln 2}$. The obtained results are shown in Tables 3-4.

From Tables 1-4, we observe that the presented method produces better results on B-mesh. Because of the effect of logarithmic factor on S-mesh, the rate of the convergence on S-mesh is lower than B-mesh. In spite of the fact that the proposed method provides reliable results on layer-adapted meshes, it is almost the first-order convergent.

6. CONCLUSION

In this paper, we have designed a new difference scheme for the nonlinear integro-differential equations with layer behavior. Error approximations have been analyzed

TABLE 4. Error approximations e^N and order of convergence p^N on B-Mesh

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	0.02940317 0.9104	0.01564298 0.9325	0.00819606 0.9417	0.00426697 0.9634	0.00218826
2^{-2}	0.05665633 0.8854	0.03066933 0.9722	0.01563324 0.9859	0.00789362 0.9929	0.00396636
2^{-3}	0.10560228 0.8395	0.05901346 0.8540	0.03264993 0.8892	0.01762837 0.9238	0.00929240
2^{-4}	0.15490687 0.8253	0.08742185 0.8628	0.04807278 0.9047	0.02567700 0.9269	0.01350556
2^{-5}	0.15916537 0.8776	0.08662951 0.9044	0.04628234 0.8958	0.02487489 0.9366	0.01299633
2^{-6}	0.16095246 0.8670	0.08824835 0.9027	0.04720203 0.9127	0.02507356 0.9255	0.01320159
2^{-7}	0.16772393 0.8473	0.09322601 0.9074	0.04970150 0.9190	0.02628564 0.9325	0.01377238
...	...	...	...	...	...
2^{-10}	0.17329378 0.8685	0.09491387 0.9119	0.05044516 0.9211	0.02664005 0.9356	0.01392799
e^N	0.17329378	0.09491387	0.05044516	0.02664005	0.01392799
p^N	0.8253	0.8540	0.8892	0.9238	

on both B-type and S-type meshes. Because of the nonlinear terms in the iteration, the quasilinearization technique has been used. To show the suitability of the method, we have tested it on two examples and the computed results have been summarized in Tables 1-4. We can deduce that for majority values of N , exact errors decrease and the order of convergence is close to 1. Numerical investigations can be sustained for more sophisticated types such as partial integro-differential equations, delay form, higher dimensional, etc.

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COMPARISON OF BLUPS UNDER MULTIPLE PARTITIONED LINEAR MODEL AND ITS CORRECTLY-REDUCED MODELS

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Abstract. In this study, we investigate the relations between a multiple partitioned linear model and its correctly-reduced models. We consider the comparison problem of covariance matrices of the best linear unbiased predictors (BLUPs) of all unknown vectors including partial parameters under these models by using the block matrix rank and inertia formulas. We derive various inequalities and equalities for covariance matrices of BLUPs under some general assumptions. Also, results for special cases are given.

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1. INTRODUCTION

In the present paper, we consider the comparison problem of covariance matrices of predictors in the context of a multiple partitioned linear model and its reduced models by using the inertia and rank formulas together with elementary block matrix operations. Before proceeding, we introduce the notations used in this paper. $\mathbb{R}^{m \times n}$ stands for the set of all $m \times n$ real matrices. $\mathbf{A}'$, $r(\mathbf{A})$, $\mathcal{C}(\mathbf{A})$ and $\mathbf{A}^+$ denote the transpose, the rank, the column space, and the Moore–Penrose generalized inverse of $\mathbf{A} \in \mathbb{R}^{m \times n}$, respectively. $\mathbf{I}_m$ denotes the identity matrix of order m . $\mathbf{E}_\mathbf{A} = \mathbf{A}^\perp = \mathbf{I}_m - \mathbf{A}\mathbf{A}^+$ stands for the orthogonal projector. $i_+(\mathbf{A})$ and $i_-(\mathbf{A})$ denote the positive and the negative inertias of symmetric matrix $\mathbf{A}$, respectively, and for both $i_\pm(\mathbf{A})$ and $i_\mp(\mathbf{A})$ are used. The inequality $\mathbf{A}_1 - \mathbf{A}_2 \prec (\preceq, \succ, \succeq) \mathbf{0}$ or $\mathbf{A}_1 \prec (\preceq, \succ, \succeq) \mathbf{A}_2$ mean that the difference $\mathbf{A}_1 - \mathbf{A}_2$ is negative definite (negative semi-definite, positive definite, positive semi-definite) matrix in the Löwner partial ordering for the symmetric matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ of same size.

Linear regression models are one of the most commonly used tools in statistical theory and their applications to analyze data and to develop new methods. These models may need to be converted to their different forms such as certain partitioned

forms to meet the requirements of the analysis. In this way, by using various linear transformations in linear partitioned models, reduced linear models of original models can be obtained for making statistical inferences of general parametric functions of partial parameters. In this case, original linear models and their reduced models become competing linear regression models for estimation and prediction problems on all unknown vectors with general parametric functions of partial parameters. In this paper, we consider the following linear regression model with its multiple partitioned form:

$$\begin{aligned} \mathcal{M} : \mathbf{y} = \mathbf{X}\alpha + \varepsilon &= [\mathbf{X}_1, \dots, \mathbf{X}_t] [\alpha'_1, \dots, \alpha'_t]' + \varepsilon = \mathbf{X}_1\alpha_1 + \dots + \mathbf{X}_t\alpha_t + \varepsilon \\ \text{with } E(\varepsilon) &= \mathbf{0} \text{ and } \text{cov}(\varepsilon, \varepsilon) = D(\varepsilon) = \sigma^2 \Sigma, \end{aligned} \quad (1.1)$$

where $\mathbf{y} \in \mathbb{R}^{n \times 1}$ is a vector of observable response variables, $\mathbf{X}_i \in \mathbb{R}^{n \times k_i}$ is a known matrix of arbitrary rank with $\mathbf{X} = [\mathbf{X}_1, \dots, \mathbf{X}_t] \in \mathbb{R}^{n \times k}$, $\alpha_i \in \mathbb{R}^{k_i \times 1}$ is a vector of fixed but unknown parameters with $\alpha = [\alpha'_1, \dots, \alpha'_t]' \in \mathbb{R}^{k \times 1}$, $\varepsilon \in \mathbb{R}^{n \times 1}$ is an unobservable vector of random errors, σ^2 is a positive unknown parameter, and $\Sigma \in \mathbb{R}^{n \times n}$ is a known positive semi-definite matrix of arbitrary rank, $k_1 + \dots + k_t = k$ and $i = 1, \dots, t$. The matrix $\mathbf{X}$ in (1.1) can also be written as

$$\mathbf{X} = \widehat{\mathbf{X}}_i + \mathbf{R}_i = \widehat{\mathbf{X}}_1 + \dots + \widehat{\mathbf{X}}_t,$$

where $\widehat{\mathbf{X}}_i = [\mathbf{0}, \dots, \mathbf{X}_i, \dots, \mathbf{0}]$ and $\mathbf{R}_i = [\mathbf{X}_1, \dots, \mathbf{X}_{i-1}, \mathbf{0}, \mathbf{X}_{i+1}, \dots, \mathbf{X}_t]$, $i = 1, \dots, t$, and thereby $\mathcal{M}$ can be written as $\mathbf{y} = (\widehat{\mathbf{X}}_i + \mathbf{R}_i)\alpha + \varepsilon = \mathbf{X}_i\alpha_i + \mathbf{R}_i\alpha + \varepsilon$. By pre-multiplying $\mathbf{R}_i^\perp$ on the both sides of $\mathcal{M}$,

$$\mathcal{R}_i : \mathbf{R}_i^\perp \mathbf{y} = \mathbf{R}_i^\perp \mathbf{X}_i \alpha_i + \mathbf{R}_i^\perp \varepsilon \quad (1.2)$$

is obtained, $i = 1, \dots, t$. The model $\mathcal{R}_i$ in (1.2) encompasses one of the partial unknown parameters in the multiple partitioned linear model $\mathcal{M}$. This model is a reduced model of $\mathcal{M}$, which is also known as correctly-reduced models of $\mathcal{M}$; see, e.g., [2, 4] among others.

Consideration of $\mathcal{M}$ and $\mathcal{R}_i$ together is meaningful for obtaining the results separately or simultaneously for making estimation or prediction on joint unknown vectors α_i and ε . Thus, we construct the following vector that consists all unknown vectors with partial parameters in the considered models:

$$\mathbf{r}_i = \mathbf{K}_i \alpha_i + \mathbf{H} \varepsilon = [\mathbf{0}, \dots, \mathbf{K}_i, \dots, \mathbf{0}] \alpha + \mathbf{H} \varepsilon := \widehat{\mathbf{K}}_i \alpha + \mathbf{H} \varepsilon \quad (1.3)$$

in accordance with the partition considered in (1.1) for given matrices $\mathbf{K}_i \in \mathbb{R}^{s \times k_i}$ and $\mathbf{H} \in \mathbb{R}^{s \times n}$, where $\widehat{\mathbf{K}}_i = [\mathbf{0}, \dots, \mathbf{K}_i, \dots, \mathbf{0}]$, $i = 1, \dots, t$. According to assumptions in (1.1), we can write

$$\begin{aligned} D(\mathbf{y}) &= \sigma^2 \Sigma, \quad D(\mathbf{R}_i^\perp \mathbf{y}) = \sigma^2 \mathbf{R}_i^\perp \Sigma \mathbf{R}_i^\perp, \quad D(\mathbf{r}_i) = \sigma^2 \mathbf{H} \Sigma \mathbf{H}', \\ \text{cov}(\mathbf{r}_i, \mathbf{y}) &= \sigma^2 \mathbf{H} \Sigma, \quad \text{and} \quad \text{cov}(\mathbf{r}_i, \mathbf{R}_i^\perp \mathbf{y}) = \sigma^2 \mathbf{H} \Sigma \mathbf{R}_i^\perp. \end{aligned} \quad (1.4)$$

The best linear unbiased predictors (BLUPs) defined from the minimum covariance matrix requirement in the Löwner partial ordering are one of the fundamental predictors in statistical analysis. Their covariance matrices are usually used as comparison criteria to determine optimal predictors among other types of unbiased predictors. In this paper, we investigate relations between multiple partitioned linear model $\mathcal{M}$ and its correctly-reduced models $\mathcal{R}_i$, which can be considered as competing models for making statistical inference on partial parameters, by comparing the covariance matrices of BLUPs. We establish variety of inequalities and equalities for covariance matrices of BLUPs of $\mathbf{r}_i$ in (1.3) by considering the general assumptions given in (1.4) under models $\mathcal{M}$ and $\mathcal{R}_i$. Statistical inference problems on BLUPs involve complicated matrix operations. Therefore, we use formulas of inertia and rank for block matrices with elementary block matrix operations while derivation of BLUPs, characterizations of their properties and establish inequalities and equalities for covariance matrices of BLUPs under considered models. These kinds of mathematical tools simplify matrix expressions when we face with heavy mathematical computations with matrix expressions including Moore-Penrose inverses of matrices. For more details on inertias and ranks of symmetric matrices and relations between inertias and Löwner partial ordering of symmetric matrices, see, e.g., [15, 18, 19, 24]. As further reference for comparison of covariance matrix of predictors/estimators by using matrix rank/inertia method, we may mention [6, 22, 23] among others. More related work on prediction/estimation problems under partitioned linear models can be found in; see, e.g., [5, 7–12, 14, 17, 25].

2. PRELIMINARY RESULTS ON BLUPS

In this section, we review the concepts related the BLUPs of unknown vectors under $\mathcal{M}$ and $\mathcal{R}_i$, $i = 1, \dots, t$. We give the following definitions regarding many fundamental results and facts about consistency of the models, predictability and estimability of unknown parameters under the models, matrix equations and formulas for BLUPs; see e.g., [1, 3, 16, 21].

Definition 1. The consistency requirements are given as follows:

- (1) The model $\mathcal{M}$ is said to be consistent if and only if $\mathbf{y} \in \mathcal{C}[\mathbf{X}, \Sigma]$ holds with probability 1.
- (2) $\mathcal{R}_i$ is said to be consistent if and only if $\mathbf{R}_i^\perp \mathbf{y} \in \mathcal{C}[\mathbf{R}_i^\perp \mathbf{X}_i, \mathbf{R}_i^\perp \Sigma \mathbf{R}_i^\perp]$ holds with probability 1, $i = 1, \dots, t$.

Note that $\mathcal{R}_i$ is consistent under the assumption of consistency of $\mathcal{M}$. In what follows, we will assume that the considered models are consistent.

Definition 2. The predictability requirements are given as follows for $i = 1, \dots, t$.

- (1) $\mathbf{r}_i$ is predictable under $\mathcal{M} \Leftrightarrow \mathcal{C}(\hat{\mathbf{K}}_i') \subseteq \mathcal{C}(\mathbf{X}') \Leftrightarrow r \begin{bmatrix} \mathbf{X} \\ \hat{\mathbf{K}}_i \end{bmatrix} = r(\mathbf{X}) \Leftrightarrow \hat{\mathbf{K}}_i \alpha$ is estimable under $\mathcal{M}$.

- (2) $\widehat{\mathbf{X}}_i \alpha + \varepsilon$ is predictable under $\mathcal{M} \Leftrightarrow \mathcal{C}(\widehat{\mathbf{X}}'_i) \subseteq \mathcal{C}(\mathbf{X}') \Leftrightarrow r(\mathbf{X}_i) + r(\mathbf{R}_i) = r(\mathbf{X})$
 $\Leftrightarrow \widehat{\mathbf{X}}_i \alpha$ is estimable under $\mathcal{M}$, where $\widehat{\mathbf{X}}_i = [\mathbf{0}, \dots, \mathbf{X}_i, \dots, \mathbf{0}]$.
- (3) $\mathbf{R}_i^\perp \widehat{\mathbf{X}}_i \alpha + \varepsilon$ is always predictable under $\mathcal{M}$, $\mathbf{R}_i^\perp \widehat{\mathbf{X}}_i \alpha$ is always estimable under $\mathcal{M}$, and ε is always predictable under $\mathcal{M}$.
- (4) $r(\mathbf{R}_i^\perp \widehat{\mathbf{X}}_i) = k_i \Leftrightarrow \alpha_i$ is estimable under $\mathcal{M}$.
- (5) $\mathbf{r}_i$ is predictable under $\mathcal{R}_i \Leftrightarrow \mathcal{C}(\mathbf{K}'_i) \subseteq \mathcal{C}(\mathbf{X}'_i \mathbf{R}_i^\perp) \Leftrightarrow r \begin{bmatrix} \mathbf{R}_i^\perp \mathbf{X}_i \\ \mathbf{K}_i \end{bmatrix} = r(\mathbf{R}_i^\perp \mathbf{X}_i) \Leftrightarrow$
 $\mathbf{K}_i \alpha_i$ is estimable under $\mathcal{R}_i$.
- (6) $\mathbf{X}_i \alpha_i + \varepsilon$ is predictable under $\mathcal{R}_i \Leftrightarrow \mathcal{C}(\mathbf{X}'_i) \subseteq \mathcal{C}(\mathbf{X}'_i \mathbf{R}_i^\perp) \Leftrightarrow r(\mathbf{X}_i) = r(\mathbf{R}_i^\perp \mathbf{X}_i)$
 $\Leftrightarrow \mathbf{X}_i \alpha_i$ is estimable under $\mathcal{R}_i$.
- (7) $\mathbf{R}_i^\perp \mathbf{X}_i \alpha_i + \varepsilon$ is always predictable under $\mathcal{R}_i$, $\mathbf{R}_i^\perp \mathbf{X}_i \alpha_i$ is always estimable under $\mathcal{R}_i$, and ε is always predictable under $\mathcal{R}_i$.
- (8) $r(\mathbf{R}_i^\perp \mathbf{X}_i) = k_i \Leftrightarrow \alpha_i$ is estimable under $\mathcal{R}_i$.

Note that if $\mathbf{r}_i$ is predictable under $\mathcal{R}_i$, $i = 1, \dots, t$, then it is predictable under $\mathcal{M}$.

Definition 3. The BLUP and the best linear unbiased estimator (BLUE) expressions for models $\mathcal{M}$ and $\mathcal{R}_i$ are given as follows, $i = 1, \dots, t$.

- (1) Let $\mathbf{r}_i$ be predictable under $\mathcal{M}$. If there exists $\mathbf{L}_i \mathbf{y}$ such that

$$D(\mathbf{L}_i \mathbf{y} - \mathbf{r}_i) = \min \text{ s.t. } E(\mathbf{L}_i \mathbf{y} - \mathbf{r}_i) = \mathbf{0}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{L}_i \mathbf{y}$ is defined to be the BLUP of $\mathbf{r}_i$ and is denoted by $\mathbf{L}_i \mathbf{y} = \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i) = \text{BLUP}_{\mathcal{M}}(\widehat{\mathbf{K}}_i \alpha + \mathbf{H} \varepsilon)$. If $\mathbf{H} = \mathbf{0}$ in $\mathbf{r}_i$, $\mathbf{L}_i \mathbf{y}$ corresponds the BLUE of $\widehat{\mathbf{K}}_i \alpha$, denoted by $\text{BLUE}_{\mathcal{M}}(\widehat{\mathbf{K}}_i \alpha)$, under $\mathcal{M}$.

- (2) Let $\mathbf{r}_i$ be predictable under $\mathcal{R}_i$. If there exists $\mathbf{G}_i \mathbf{R}_i^\perp \mathbf{y}$ such that

$$D(\mathbf{G}_i \mathbf{R}_i^\perp \mathbf{y} - \mathbf{r}_i) = \min \text{ s.t. } E(\mathbf{G}_i \mathbf{R}_i^\perp \mathbf{y} - \mathbf{r}_i) = \mathbf{0}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{G}_i \mathbf{R}_i^\perp \mathbf{y}$ is defined to be the BLUP of $\mathbf{r}_i$ and is denoted by $\mathbf{G}_i \mathbf{R}_i^\perp \mathbf{y} = \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i) = \text{BLUP}_{\mathcal{R}_i}(\mathbf{K}_i \alpha_i + \mathbf{H} \varepsilon)$. If $\mathbf{H} = \mathbf{0}$ in $\mathbf{r}_i$, $\mathbf{G}_i \mathbf{R}_i^\perp \mathbf{y}$ corresponds the best linear unbiased estimator (BLUE) of $\mathbf{K}_i \alpha_i$, denoted by $\text{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i \alpha_i)$, under $\mathcal{R}_i$.

The fundamental BLUP equations of $\mathbf{r}_i$ and their covariance matrices under $\mathcal{M}$ and $\mathcal{R}_i$, $i = 1, \dots, t$, are collected in the following theorem.

Theorem 1. Let $\mathcal{M}$ and $\mathcal{R}_i$ be as given in (1.1) and (1.2), respectively, and assume that $\mathbf{r}_i$ is predictable under $\mathcal{R}_i$ (also predictable under $\mathcal{M}$), $i = 1, \dots, t$. Then the following results hold.

- (1) BLUP of $\mathbf{r}_i$ under $\mathcal{M}$ is

$$\text{BLUP}_{\mathcal{M}}(\mathbf{r}_i) = \mathbf{L}_i \mathbf{y} = \left(\begin{bmatrix} \widehat{\mathbf{K}}_i & \mathbf{H} \Sigma \mathbf{X}^\perp \end{bmatrix} \mathbf{W}^+ + \mathbf{U} \mathbf{W}^\perp \right) \mathbf{y}, \quad (2.1)$$

where $\mathbf{U} \in \mathbb{R}^{s \times n}$ is an arbitrary matrix and $\mathbf{W} = [\mathbf{X}, \Sigma \mathbf{X}^\perp]$. In particular,

- (a) $\mathbf{L}_i$ is unique $\Leftrightarrow r[\mathbf{X}, \Sigma\mathbf{X}^\perp] = n$.
- (b) $\text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)$ is unique $\Leftrightarrow \mathcal{M}$ is consistent.
- (c) $r[\mathbf{X}, \Sigma\mathbf{X}^\perp] = r[\mathbf{X}, \Sigma] = r[\mathbf{X}, \mathbf{X}^\perp\Sigma]$.
- (d) $\mathcal{C}[\mathbf{X}, \Sigma\mathbf{X}^\perp] = \mathcal{C}[\mathbf{X}, \Sigma] = \mathcal{C}[\mathbf{X}, \mathbf{X}^\perp\Sigma]$.
- (e) The following covariance matrix equalities hold.

$$D[\text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] = \sigma^2 [\hat{\mathbf{K}}_i, \mathbf{H}\Sigma\mathbf{X}^\perp] \mathbf{W}^+ \Sigma ([\hat{\mathbf{K}}_i, \mathbf{H}\Sigma\mathbf{X}^\perp] \mathbf{W}^+)', \quad (2.2)$$

$$\begin{aligned} D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] &= \sigma^2 ([\hat{\mathbf{K}}_i, \mathbf{H}\Sigma\mathbf{X}^\perp] \mathbf{W}^+ - \mathbf{H}) \Sigma \\ &\quad \times ([\hat{\mathbf{K}}_i, \mathbf{H}\Sigma\mathbf{X}^\perp] \mathbf{W}^+ - \mathbf{H})'. \end{aligned} \quad (2.3)$$

(2) BLUP of $\mathbf{r}_i$ under $\mathcal{R}_i$ is

$$\begin{aligned} \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i) &= \mathbf{G}_i \mathbf{R}_i^\perp \mathbf{y} \\ &= ([\mathbf{K}_i, \mathbf{H}\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp] \mathbf{W}_i^+ + \mathbf{U}_i \mathbf{W}_i^\perp) \mathbf{R}_i^\perp \mathbf{y}, \end{aligned} \quad (2.4)$$

where $\mathbf{U}_i \in \mathbb{R}^{s \times n}$ is an arbitrary matrix and $\mathbf{W}_i = [\mathbf{R}_i^\perp\mathbf{X}_i, \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp]$.
In particular,

- (a) $\mathbf{G}_i$ is unique $\Leftrightarrow r[\mathbf{R}_i^\perp\mathbf{X}_i, \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp] = n$.
- (b) $\text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)$ is unique $\Leftrightarrow \mathcal{R}_i$ is consistent.
- (c) $r[\mathbf{R}_i^\perp\mathbf{X}_i, \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp] = r[\mathbf{R}_i^\perp\mathbf{X}_i, \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp]$
 $= r[\mathbf{R}_i^\perp\mathbf{X}_i, (\mathbf{R}_i^\perp\mathbf{X}_i)^\perp\mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp]$.
- (d) $\mathcal{C}[\mathbf{R}_i^\perp\mathbf{X}_i, \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp] = \mathcal{C}[\mathbf{R}_i^\perp\mathbf{X}_i, \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp]$
 $= \mathcal{C}[\mathbf{R}_i^\perp\mathbf{X}_i, (\mathbf{R}_i^\perp\mathbf{X}_i)^\perp\mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp]$.
- (e) The following covariance matrix equalities hold.

$$\begin{aligned} D[\text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] &= \sigma^2 [\mathbf{K}_i, \mathbf{H}\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp] \mathbf{W}_i^+ \mathbf{R}_i^\perp \Sigma \mathbf{R}_i^\perp \\ &\quad \times ([\mathbf{K}_i, \mathbf{H}\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp] \mathbf{W}_i^+)', \end{aligned} \quad (2.5)$$

$$\begin{aligned} D[\mathbf{r}_i - \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] &= \sigma^2 ([\mathbf{K}_i, \mathbf{H}\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp] \mathbf{W}_i^+ \mathbf{R}_i^\perp - \mathbf{H}) \Sigma \\ &\quad \times ([\mathbf{K}_i, \mathbf{H}\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp] \mathbf{W}_i^+ \mathbf{R}_i^\perp - \mathbf{H})'. \end{aligned} \quad (2.6)$$

3. MAIN RESULTS

In this section, some results on the comparison of covariance matrices of predictors under multiple partitioned linear model $\mathcal{M}$ and its reduced models $\mathcal{R}_i$, $i = 1, \dots, t$, are derived and related conclusions are established for special cases by using block matrices' rank and inertia formulas with elementary block matrix operations.

Theorem 2. Let $\mathcal{M}$ and $\mathcal{R}_i$ be as given in (1.1) and (1.2), respectively, and assume that $\mathbf{r}_i$ is predictable under $\mathcal{R}_i$ (also predictable under $\mathcal{M}$), $i = 1, \dots, t$. Denote

$$\mathbf{M} = \begin{bmatrix} \Sigma & \Sigma \mathbf{R}_i^\perp & \Sigma \mathbf{H}' & \mathbf{0} & \mathbf{X} \\ \mathbf{R}_i^\perp \Sigma & \mathbf{0} & \mathbf{0} & \mathbf{R}_i^\perp \mathbf{X}_i & \mathbf{0} \\ \mathbf{H} \Sigma & \mathbf{0} & \mathbf{0} & \mathbf{K}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i' \mathbf{R}_i^\perp & \mathbf{K}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then

$$\begin{aligned} i_+(D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] - D[\mathbf{r}_i - \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)]) \\ = i_+(\mathbf{M}) - r[\mathbf{X}, \Sigma] - r(\mathbf{R}_i^\perp \mathbf{X}_i), \end{aligned} \quad (3.1)$$

$$\begin{aligned} i_-(D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] - D[\mathbf{r}_i - \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)]) \\ = i_-(\mathbf{M}) - r[\mathbf{R}_i^\perp \mathbf{X}_i, \mathbf{R}_i^\perp \Sigma \mathbf{R}_i^\perp] - r(\mathbf{X}), \end{aligned} \quad (3.2)$$

$$\begin{aligned} r(D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] - D[\mathbf{r}_i - \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)]) \\ = r(\mathbf{M}) - r[\mathbf{X}, \Sigma] - r(\mathbf{R}_i^\perp \mathbf{X}_i) - r[\mathbf{R}_i^\perp \mathbf{X}_i, \mathbf{R}_i^\perp \Sigma \mathbf{R}_i^\perp] - r(\mathbf{X}). \end{aligned} \quad (3.3)$$

In consequence, the following results hold.

- (1) $D[\mathbf{r}_i - \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] \succ D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$
 $\Leftrightarrow i_-(\mathbf{M}) = r[\mathbf{R}_i^\perp \mathbf{X}_i, \mathbf{R}_i^\perp \Sigma \mathbf{R}_i^\perp] + r(\mathbf{X}) + s.$
- (2) $D[\mathbf{r}_i - \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] \prec D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$
 $\Leftrightarrow i_+(\mathbf{M}) = r[\mathbf{X}, \Sigma] + r(\mathbf{R}_i^\perp \mathbf{X}_i) + s.$
- (3) $D[\mathbf{r}_i - \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] \succcurlyeq D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$
 $\Leftrightarrow i_+(\mathbf{M}) = r[\mathbf{X}, \Sigma] + r(\mathbf{R}_i^\perp \mathbf{X}_i).$
- (4) $D[\mathbf{r}_i - \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] \preccurlyeq D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$
 $\Leftrightarrow i_-(\mathbf{M}) = r[\mathbf{R}_i^\perp \mathbf{X}_i, \mathbf{R}_i^\perp \Sigma \mathbf{R}_i^\perp] + r(\mathbf{X}).$
- (5) $D[\mathbf{r}_i - \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] = D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$
 $\Leftrightarrow r(\mathbf{M}) = r[\mathbf{X}, \Sigma] + r(\mathbf{R}_i^\perp \mathbf{X}_i) + r[\mathbf{R}_i^\perp \mathbf{X}_i, \mathbf{R}_i^\perp \Sigma \mathbf{R}_i^\perp] + r(\mathbf{X}).$

Many consequences can be derived from Theorem 2 for different choices of the matrices $\mathbf{K}_i$ and $\mathbf{H}$ in $\mathbf{r}_i$, $i = 1, \dots, t$. By setting special choices of the matrices $\mathbf{K}_i$ and $\mathbf{H}$ in this theorem, the block matrix $\mathbf{M}$ in Theorem 2 can be reduced to some simpler forms. Some of them are given below.

Corollary 1. Let $\mathcal{M}$ and $\mathcal{R}_i$ be as given in (1.1) and (1.2), respectively, $i = 1, \dots, t$.

- (1) Assume that $\mathbf{K}_i \alpha_i$ is predictable under $\mathcal{R}_i$ (also predictable under $\mathcal{M}$). Denote

$$\mathbf{M} = \begin{bmatrix} \Sigma & \Sigma \mathbf{R}_i^\perp & \mathbf{0} & \mathbf{0} & \mathbf{X} \\ \mathbf{R}_i^\perp \Sigma & \mathbf{0} & \mathbf{0} & \mathbf{R}_i^\perp \mathbf{X}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i' \mathbf{R}_i^\perp & \mathbf{K}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then, the following results hold.

- (a) $D[\text{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i)] \succ D[\text{BLUE}_{\mathcal{M}}(\mathbf{K}_i\alpha_i)]$
 $\Leftrightarrow i_-(\mathbf{M}) = r[\mathbf{R}_i^\perp \mathbf{X}_i, \mathbf{R}_i^\perp \Sigma \mathbf{R}_i^\perp] + r(\mathbf{X}) + s.$
 - (b) $D[\text{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i)] \prec D[\text{BLUE}_{\mathcal{M}}(\mathbf{K}_i\alpha_i)]$
 $\Leftrightarrow i_+(\mathbf{M}) = r[\mathbf{X}, \Sigma] + r(\mathbf{R}_i^\perp \mathbf{X}_i) + s.$
 - (c) $D[\text{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i)] \succcurlyeq D[\text{BLUE}_{\mathcal{M}}(\mathbf{K}_i\alpha_i)]$
 $\Leftrightarrow i_+(\mathbf{M}) = r[\mathbf{X}, \Sigma] + r(\mathbf{R}_i^\perp \mathbf{X}_i).$
 - (d) $D[\text{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i)] \preccurlyeq D[\text{BLUE}_{\mathcal{M}}(\mathbf{K}_i\alpha_i)]$
 $\Leftrightarrow i_-(\mathbf{M}) = r[\mathbf{R}_i^\perp \mathbf{X}_i, \mathbf{R}_i^\perp \Sigma \mathbf{R}_i^\perp] + r(\mathbf{X}).$
 - (e) $D[\text{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i)] = D[\text{BLUE}_{\mathcal{M}}(\mathbf{K}_i\alpha_i)]$
 $\Leftrightarrow r(\mathbf{M}) = r[\mathbf{X}, \Sigma] + r(\mathbf{R}_i^\perp \mathbf{X}_i) + r[\mathbf{R}_i^\perp \mathbf{X}_i, \mathbf{R}_i^\perp \Sigma \mathbf{R}_i^\perp] + r(\mathbf{X}).$
- (2) Assume that $\mathbf{X}_i\alpha_i$ is predictable under $\mathcal{R}_i$ (also predictable under $\mathcal{M}$). Then

$$i_\pm(D[\text{BLUE}_{\mathcal{M}}(\mathbf{X}_i\alpha_i)] - D[\text{BLUE}_{\mathcal{R}_i}(\mathbf{X}_i\alpha_i)]) \\ = r(D[\text{BLUE}_{\mathcal{M}}(\mathbf{X}_i\alpha_i)] - D[\text{BLUE}_{\mathcal{R}_i}(\mathbf{X}_i\alpha_i)]) = 0.$$

- (3) $\mathbf{R}_i^\perp \mathbf{X}_i\alpha_i$ is always predictable under $\mathcal{R}_i$ (also always predictable under $\mathcal{M}$). Then

$$i_\pm(D[\text{BLUE}_{\mathcal{M}}(\mathbf{R}_i^\perp \mathbf{X}_i\alpha_i)] - D[\text{BLUE}_{\mathcal{R}_i}(\mathbf{R}_i^\perp \mathbf{X}_i\alpha_i)]) \\ = r(D[\text{BLUE}_{\mathcal{M}}(\mathbf{R}_i^\perp \mathbf{X}_i\alpha_i)] - D[\text{BLUE}_{\mathcal{R}_i}(\mathbf{R}_i^\perp \mathbf{X}_i\alpha_i)]) = 0.$$

4. CONCLUDING REMARKS

In this study, we consider comparison problems of predictors under a multiple partitioned linear model and its correctly-reduced models. We present inertia and rank relations between covariance matrices of BLUPs of unknown vectors under considered models by using various inertia and rank formulas of block matrices with elementary matrix operations. In order to establish the general results on the predictors, we consider the general linear function of all unknown vectors under general assumptions. The results obtained in this paper can present useful aspects for determining relation between multiple partitioned linear model and its correctly-reduced models regarding statistical inference on partial parameters by comparing performances of BLUPs/BLEs under the considered models.

APPENDIX

We collect some fundamental results of block matrices in the following lemmas used in the proofs of the main results in the paper.

Lemma 1 ([18]). Let $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{m \times n}$, or, let $\mathbf{A}_1 = \mathbf{A}'_1, \mathbf{A}_2 = \mathbf{A}'_2 \in \mathbb{R}^{m \times m}$. Then,

- (1) $\mathbf{A}_1 = \mathbf{A}_2 \Leftrightarrow r(\mathbf{A}_1 - \mathbf{A}_2) = 0.$
- (2) $\mathbf{A}_1 \succ \mathbf{A}_2 \Leftrightarrow i_+(\mathbf{A}_1 - \mathbf{A}_2) = m$ and $\mathbf{A}_1 \prec \mathbf{A}_2 \Leftrightarrow i_-(\mathbf{A}_1 - \mathbf{A}_2) = m.$
- (3) $\mathbf{A}_1 \succcurlyeq \mathbf{A}_2 \Leftrightarrow i_-(\mathbf{A}_1 - \mathbf{A}_2) = 0$ and $\mathbf{A}_1 \preccurlyeq \mathbf{A}_2 \Leftrightarrow i_+(\mathbf{A}_1 - \mathbf{A}_2) = 0.$

Lemma 2 ([18]). *Let $\mathbf{A}_1 = \mathbf{A}'_1 \in \mathbb{R}^{m \times m}$, $\mathbf{A}_2 = \mathbf{A}'_2 \in \mathbb{R}^{n \times n}$, $\mathbf{Q} \in \mathbb{R}^{m \times n}$, and $k \in \mathbb{R}$. Then,*

$$r(\mathbf{A}_1) = i_+(\mathbf{A}_1) + i_-(\mathbf{A}_1). \quad (4.1)$$

$$i_{\pm}(k\mathbf{A}_1) = \begin{cases} i_{\pm}(\mathbf{A}_1) & \text{if } k > 0 \\ i_{\mp}(\mathbf{A}_1) & \text{if } k < 0 \end{cases}. \quad (4.2)$$

$$i_{\pm} \begin{bmatrix} \mathbf{A}_1 & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{A}_2 \end{bmatrix} = i_{\pm} \begin{bmatrix} \mathbf{A}_1 & -\mathbf{Q} \\ -\mathbf{Q}' & \mathbf{A}_2 \end{bmatrix} = i_{\mp} \begin{bmatrix} -\mathbf{A}_1 & \mathbf{Q} \\ \mathbf{Q}' & -\mathbf{A}_2 \end{bmatrix}. \quad (4.3)$$

$$i_{\pm} \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} = i_{\pm}(\mathbf{A}_1) + i_{\pm}(\mathbf{A}_2), \quad i_{+} \begin{bmatrix} \mathbf{0} & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{0} \end{bmatrix} = i_{-} \begin{bmatrix} \mathbf{0} & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{0} \end{bmatrix} = r(\mathbf{Q}). \quad (4.4)$$

Lemma 3 ([18]). *Let $\mathbf{A}_1 = \mathbf{A}'_1 \in \mathbb{R}^{m \times m}$, $\mathbf{B} = \mathbf{B}' \in \mathbb{R}^{n \times n}$, and $\mathbf{A}_2 \in \mathbb{R}^{m \times n}$. Then,*

$$i_{\pm} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}'_2 & \mathbf{0} \end{bmatrix} = r(\mathbf{A}_2) + i_{\pm}(\mathbf{E}_{\mathbf{A}_2} \mathbf{A}_1 \mathbf{E}_{\mathbf{A}_2}). \quad (4.5)$$

$$i_{+} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}'_2 & \mathbf{0} \end{bmatrix} = r \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \end{bmatrix} \quad \text{and} \quad i_{-} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}'_2 & \mathbf{0} \end{bmatrix} = r(\mathbf{A}_2) \quad \text{if } \mathbf{A}_1 \succcurlyeq \mathbf{0}. \quad (4.6)$$

$$i_{\pm} \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}'_2 & \mathbf{B} \end{bmatrix} = i_{\pm}(\mathbf{A}_1) + i_{\pm}(\mathbf{B} - \mathbf{A}'_2 \mathbf{A}_1^+ \mathbf{A}_2) \quad \text{if } \mathcal{C}(\mathbf{A}_2) \subseteq \mathcal{C}(\mathbf{A}_1). \quad (4.7)$$

Lemma 4 ([13]). *The linear matrix equation $\mathbf{A}\mathbf{X} = \mathbf{B}$ is consistent if and only if $r \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix} = r(\mathbf{A})$, or equivalently, $\mathbf{A}\mathbf{A}^+ \mathbf{B} = \mathbf{B}$. In this case, the general solution of $\mathbf{A}\mathbf{X} = \mathbf{B}$ can be written in the following form $\mathbf{X} = \mathbf{A}^+ \mathbf{B} + (\mathbf{I} - \mathbf{A}^+ \mathbf{A})\mathbf{U}$, where $\mathbf{U}$ is an arbitrary matrix.*

Constrained quadratic matrix-valued function optimization problem related to minimization problems on covariance matrices of predictors is given in the following lemma.

Lemma 5 ([19]). *Let $\mathbf{A} \in \mathbb{R}^{n \times p}$, $\mathbf{B} \in \mathbb{R}^{m \times p}$ be given matrices, and $\mathbf{P} \in \mathbb{R}^{n \times n}$ symmetric positive semi-definite matrix. Also assume that there exists $\mathbf{X}_0 \in \mathbb{R}^{m \times n}$ such that $\mathbf{X}_0 \mathbf{A} = \mathbf{B}$. Then the maximal positive inertia of $\mathbf{X}_0 \mathbf{P} \mathbf{X}'_0 - \mathbf{X} \mathbf{P} \mathbf{X}'$ subject to all solutions of $\mathbf{X} \mathbf{A} = \mathbf{B}$ is*

$$\max_{\mathbf{X} \mathbf{A} = \mathbf{B}} i_{+}(\mathbf{X}_0 \mathbf{P} \mathbf{X}'_0 - \mathbf{X} \mathbf{P} \mathbf{X}') = r \begin{bmatrix} \mathbf{X}_0 \mathbf{P} \\ \mathbf{A}' \end{bmatrix} - r(\mathbf{A}) = r(\mathbf{X}_0 \mathbf{P} \mathbf{A}^{\perp}). \quad (4.8)$$

Hence there exists solution $\mathbf{X}_0$ of $\mathbf{X}_0 \mathbf{A} = \mathbf{B}$ such that $\mathbf{X}_0 \mathbf{P} \mathbf{X}'_0 \preccurlyeq \mathbf{X} \mathbf{P} \mathbf{X}'$ holds for all solutions of $\mathbf{X} \mathbf{A} = \mathbf{B}$ if and only if $\mathbf{X}_0$ satisfies both $\mathbf{X}_0 \mathbf{A} = \mathbf{B}$ and $\mathbf{X}_0 \mathbf{P} \mathbf{A}^{\perp} = \mathbf{0}$.

Characterization of fundamental BLUP properties can be found in the statistical literature. The similar approach used in proof of Theorem 1 given below was used for transformed linear mixed models in [6], for the different approaches; see, e.g., [15, 21].

Proof of Theorem 1. Let $\mathbf{L}_i\mathbf{y}$ and $\mathbf{T}_i\mathbf{y}$ be two unbiased linear predictors for $\mathbf{r}_i$, $i = 1, \dots, t$. Then, according to Definition 3, we can write the following expressions

$$\mathbf{E}(\mathbf{L}_i\mathbf{y} - \mathbf{r}_i) = \mathbf{0} \Leftrightarrow \mathbf{L}_i\mathbf{X} = \widehat{\mathbf{K}}_i, \text{ i.e., } [\mathbf{L}_i, -\mathbf{I}_s] \begin{bmatrix} \mathbf{X} \\ \widehat{\mathbf{K}}_i \end{bmatrix} = \mathbf{0}, \quad (4.9)$$

$$\mathbf{D}(\mathbf{L}_i\mathbf{y} - \mathbf{r}_i) = \sigma^2(\mathbf{L}_i - \mathbf{H})\Sigma(\mathbf{L}_i - \mathbf{H})' = \sigma^2 [\mathbf{L}_i, -\mathbf{I}_s] \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix} \Sigma \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix}' [\mathbf{L}_i, -\mathbf{I}_s]',$$

$$\mathbf{D}(\mathbf{T}_i\mathbf{y} - \mathbf{r}_i) = \sigma^2(\mathbf{T}_i - \mathbf{H})\Sigma(\mathbf{T}_i - \mathbf{H})' = \sigma^2 [\mathbf{T}_i, -\mathbf{I}_s] \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix} \Sigma \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix}' [\mathbf{T}_i, -\mathbf{I}_s]'. \quad (4.10)$$

Then the matrix minimization problem for finding the BLUP under $\mathcal{M}$ characterized in Definition 3 can be accordingly expressed as to find solution $\mathbf{L}_i$ of the consistent linear matrix equation $\mathbf{L}_i\mathbf{X} = \widehat{\mathbf{K}}_i$ such that $\mathbf{D}(\mathbf{L}_i\mathbf{y} - \mathbf{r}_i) \preceq \mathbf{D}(\mathbf{T}_i\mathbf{y} - \mathbf{r}_i)$ s.t. $\mathbf{T}_i\mathbf{X} = \widehat{\mathbf{K}}_i$, i.e.,

$$\begin{aligned} & [\mathbf{L}_i, -\mathbf{I}_s] \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix} \Sigma \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix}' [\mathbf{L}_i, -\mathbf{I}_s]' \preceq [\mathbf{T}_i, -\mathbf{I}_s] \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix} \Sigma \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix}' [\mathbf{T}_i, -\mathbf{I}_s]' \\ & \text{s.t. } \mathbf{T}_i\mathbf{X} = \widehat{\mathbf{K}}_i. \end{aligned} \quad (4.10)$$

(4.10) is a standard constrained quadratic matrix-valued function optimization problem in the Löwner partial ordering as given in Lemma 5. Applying (4.8) to (4.10), the maximal positive inertia of $\mathbf{D}(\mathbf{L}_i\mathbf{y} - \mathbf{r}_i) - \mathbf{D}(\mathbf{T}_i\mathbf{y} - \mathbf{r}_i)$ subject to $\mathbf{T}_i\mathbf{X} = \widehat{\mathbf{K}}_i$ is obtained as follows:

$$\begin{aligned} & \max_{\mathbf{E}(\mathbf{T}_i\mathbf{y} - \mathbf{r}_i) = \mathbf{0}} i_+(\mathbf{D}(\mathbf{L}_i\mathbf{y} - \mathbf{r}_i) - \mathbf{D}(\mathbf{T}_i\mathbf{y} - \mathbf{r}_i)) \\ & = r \left[\begin{bmatrix} [\mathbf{L}_i, -\mathbf{I}_s] \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix} \Sigma \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix}' \\ \begin{bmatrix} \mathbf{X} \\ \widehat{\mathbf{K}}_i \end{bmatrix}' \end{bmatrix} - r \begin{bmatrix} \mathbf{X} \\ \widehat{\mathbf{K}}_i \end{bmatrix} \right] \\ & = r \left([\mathbf{L}_i, -\mathbf{I}_s] \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix} \Sigma \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix}' \begin{bmatrix} \mathbf{X} \\ \widehat{\mathbf{K}}_i \end{bmatrix}^\perp \right). \end{aligned} \quad (4.11)$$

Combining (4.9) with (4.11), we conclude that $\mathbf{D}(\mathbf{L}_i\mathbf{y} - \mathbf{r}_i) = \min \Leftrightarrow$ there exists $\mathbf{L}_i$ satisfying both

$$\mathbf{L}_i\mathbf{X} = \widehat{\mathbf{K}}_i \text{ and } [\mathbf{L}_i, -\mathbf{I}_s] \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix} \Sigma \begin{bmatrix} \mathbf{I}_n \\ \mathbf{H} \end{bmatrix}' \begin{bmatrix} \mathbf{X} \\ \widehat{\mathbf{K}}_i \end{bmatrix}^\perp = \mathbf{0},$$

i.e., $\mathbf{L}_i\mathbf{y} = \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i) \Leftrightarrow \mathbf{L}_i \begin{bmatrix} \mathbf{X} & \Sigma\mathbf{X}^\perp \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{K}}_i & \mathbf{H}\Sigma\mathbf{X}^\perp \end{bmatrix}$. This matrix equation is consistent. By using Lemma 4, we obtain (2.1). Results (a) and (b) follow directly

from (2.1) and for (c) and (d), we refer [20, Lemma 2.1(a)]. (2.2) is seen from (1.4) and (2.1). We obtain

$$\text{cov}\{\text{BLUP}_{\mathcal{M}}(\mathbf{r}_i), \mathbf{r}_i\} = [\widehat{\mathbf{K}}_i, \mathbf{H}\Sigma\mathbf{X}^\perp] [\mathbf{X}, \Sigma\mathbf{X}^\perp]^\perp \Sigma\mathbf{H}' \quad (4.12)$$

by using (1.4) and (2.1). (2.3) is seen from (2.2) and (4.12). Thus, the first part of the proof is completed.

The second part of the theorem is obtained in a similar way to the first part. $\square$

Proof of Theorem 2. By using (2.6), and applying (4.7) to the difference between $D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$ and $D[\mathbf{r}_i - \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)]$, we obtain

$$\begin{aligned} & i_{\pm}(D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] - D[\mathbf{r}_i - \text{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)]) \\ &= i_{\pm} \left(D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] - (\mathbf{Z}_i \mathbf{W}_i^\perp \mathbf{R}_i^\perp - \mathbf{H}) \Sigma (\mathbf{Z}_i \mathbf{W}_i^\perp \mathbf{R}_i^\perp - \mathbf{H})' \right) \\ &= i_{\pm} \left[\begin{array}{cc} \Sigma & \Sigma (\mathbf{Z}_i \mathbf{W}_i^\perp \mathbf{R}_i^\perp - \mathbf{H})' \\ (\mathbf{Z}_i \mathbf{W}_i^\perp \mathbf{R}_i^\perp - \mathbf{H}) \Sigma & D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] \end{array} \right] - i_{\pm}(\Sigma) \\ &= i_{\pm} \left(\begin{bmatrix} \Sigma & -\Sigma\mathbf{H}' \\ -\mathbf{H}\Sigma & D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] \end{bmatrix} + \begin{bmatrix} \Sigma\mathbf{R}_i^\perp & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_i \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{W}_i \\ \mathbf{W}_i' & \mathbf{0} \end{bmatrix}^\perp \begin{bmatrix} \mathbf{R}_i^\perp \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_i' \end{bmatrix} \right) \\ & \quad - i_{\pm}(\Sigma), \end{aligned} \quad (4.13)$$

where $\mathbf{Z}_i = [\mathbf{K}_i, \mathbf{H}\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp]$ and $\mathbf{W}_i = [\mathbf{R}_i^\perp\mathbf{X}_i, \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp]$. We can reapply (4.7) to (4.13) since $\mathcal{C}(\mathbf{R}_i^\perp\Sigma) = \mathcal{C}(\mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp) \subseteq \mathcal{C}(\mathbf{W}_i)$ and $\mathcal{C}(\mathbf{Z}_i') \subseteq \mathcal{C}(\mathbf{W}_i')$ hold. Then (4.13) is equivalently written as follows by setting the matrices $\mathbf{Z}_i$ and $\mathbf{W}_i$.

$$\begin{aligned} & i_{\pm} \left[\begin{array}{ccccc} \mathbf{0} & -\mathbf{R}_i^\perp\mathbf{X}_i & -\mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp & \mathbf{R}_i^\perp\Sigma & \mathbf{0} \\ -\mathbf{X}_i'\mathbf{R}_i^\perp & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_i' \\ -(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp\mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{R}_i^\perp\mathbf{X}_i)^\perp\mathbf{R}_i^\perp\Sigma\mathbf{H}' \\ \Sigma\mathbf{R}_i^\perp & \mathbf{0} & \mathbf{0} & \Sigma & -\Sigma\mathbf{H}' \\ \mathbf{0} & \mathbf{K}_i & \mathbf{H}\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp & -\mathbf{H}\Sigma & D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] \end{array} \right] \\ & \quad - r[\mathbf{R}_i^\perp\mathbf{X}_i, \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp] - i_{\pm}(\Sigma) \\ &= i_{\pm} \left[\begin{array}{cccc} -\mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp & -\mathbf{R}_i^\perp\mathbf{X}_i & -\mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp & \mathbf{R}_i^\perp\Sigma\mathbf{H}' \\ -\mathbf{X}_i'\mathbf{R}_i^\perp & \mathbf{0} & \mathbf{0} & \mathbf{K}_i' \\ -(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp\mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp & \mathbf{0} & \mathbf{0} & (\mathbf{R}_i^\perp\mathbf{X}_i)^\perp\mathbf{R}_i^\perp\Sigma\mathbf{H}' \\ \mathbf{H}\Sigma\mathbf{R}_i^\perp & \mathbf{K}_i & \mathbf{H}\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp & D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] - \mathbf{H}\Sigma\mathbf{H}' \end{array} \right] \\ & \quad - r[\mathbf{R}_i^\perp\mathbf{X}_i, \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp] \\ &= i_{\pm} \left[\begin{array}{ccc} -\mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp & -\mathbf{R}_i^\perp\mathbf{X}_i & \mathbf{R}_i^\perp\Sigma\mathbf{H}' \\ -\mathbf{X}_i'\mathbf{R}_i^\perp & \mathbf{0} & \mathbf{K}_i' \\ \mathbf{H}\Sigma\mathbf{R}_i^\perp & \mathbf{K}_i & D[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] - \mathbf{H}\Sigma\mathbf{H}' \end{array} \right] \\ & \quad + i_{\pm}((\mathbf{R}_i^\perp\mathbf{X}_i)^\perp\mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp(\mathbf{R}_i^\perp\mathbf{X}_i)^\perp) - r[\mathbf{R}_i^\perp\mathbf{X}_i, \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp] \end{aligned}$$

$$\begin{aligned}
&= i_{\mp} \left(\begin{bmatrix} \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp} & \mathbf{R}_i^{\perp} \Sigma \mathbf{H}' & \mathbf{R}_i^{\perp} \mathbf{X}_i \\ \mathbf{H} \Sigma \mathbf{R}_i^{\perp} & \mathbf{H} \Sigma \mathbf{H}' & \mathbf{K}_i \\ \mathbf{X}_i' \mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_s \\ \mathbf{0} \end{bmatrix} \mathbf{D}[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] \begin{bmatrix} \mathbf{0} & \mathbf{I}_s & \mathbf{0} \end{bmatrix} \right) \\
&\quad - r[\mathbf{R}_i^{\perp} \mathbf{X}_i, \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp}] + i_{\pm}((\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp} \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp} (\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp}). \tag{4.14}
\end{aligned}$$

We can apply (4.7) to (4.14) after setting $\mathbf{D}[\mathbf{r}_i - \text{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$ in (2.3). Then in a similar way to obtaining (4.13), (4.14) is equivalently written as

$$\begin{aligned}
&i_{\mp} \left(\begin{bmatrix} \Sigma & \mathbf{0} & -\Sigma \mathbf{H}' & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp} & \mathbf{R}_i^{\perp} \Sigma \mathbf{H}' & \mathbf{R}_i^{\perp} \mathbf{X}_i \\ -\mathbf{H} \Sigma & \mathbf{H} \Sigma \mathbf{R}_i^{\perp} & \mathbf{H} \Sigma \mathbf{H}' & \mathbf{K}_i \\ \mathbf{0} & \mathbf{X}_i' \mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\widehat{\mathbf{K}}_i, \mathbf{H} \Sigma \mathbf{X}^{\perp}] \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right. \\
&\quad \times \left. \begin{bmatrix} \mathbf{0} & \mathbf{W}' \end{bmatrix}^+ \begin{bmatrix} \Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [\widehat{\mathbf{K}}_i, \mathbf{H} \Sigma \mathbf{X}^{\perp}]' & \mathbf{0} \end{bmatrix} \right) \\
&\quad + i_{\pm}((\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp} \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp} (\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp}) - r[\mathbf{R}_i^{\perp} \mathbf{X}_i, \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp}] - i_{\mp}(\Sigma). \tag{4.15}
\end{aligned}$$

We can apply (4.7) to (4.15) since $\mathcal{C}([\widehat{\mathbf{K}}_i, \mathbf{H} \Sigma \mathbf{X}^{\perp}]') \subseteq \mathcal{C}(\mathbf{W}')$ and $\mathcal{C}(\Sigma) \subseteq \mathcal{C}(\mathbf{W})$, where $\mathbf{W} = [\mathbf{X}, \Sigma \mathbf{X}^{\perp}]$. From Lemma 2 and 3, and by using elementary block matrix operations, (4.15) is equivalently written as

$$\begin{aligned}
&i_{\mp} \left[\begin{array}{ccccccc} \mathbf{0} & -\mathbf{X} & -\Sigma \mathbf{X}^{\perp} & \Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} \\ -\mathbf{X}^{\perp} \Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}^{\perp} \Sigma \mathbf{H}' & \mathbf{0} \\ \Sigma & \mathbf{0} & \mathbf{0} & \Sigma & \mathbf{0} & -\Sigma \mathbf{H}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp} & \mathbf{R}_i^{\perp} \Sigma \mathbf{H}' & \mathbf{R}_i^{\perp} \mathbf{X}_i \\ \mathbf{0} & \widehat{\mathbf{K}}_i & \mathbf{H} \Sigma \mathbf{X}^{\perp} & -\mathbf{H} \Sigma & \mathbf{H} \Sigma \mathbf{R}_i^{\perp} & \mathbf{H} \Sigma \mathbf{H}' & \mathbf{K}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i' \mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} \end{array} \right] \\
&\quad - r[\mathbf{R}_i^{\perp} \mathbf{X}_i, \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp}] + i_{\pm}((\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp} \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp} (\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp}) - i_{\mp}(\Sigma) - r[\mathbf{X}, \Sigma \mathbf{X}^{\perp}] \\
&= i_{\mp} \left[\begin{array}{ccccccc} -\Sigma & -\mathbf{X} & -\Sigma \mathbf{X}^{\perp} & \mathbf{0} & \Sigma \mathbf{H}' & \mathbf{0} & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} & \mathbf{0} \\ -\mathbf{X}^{\perp} \Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}^{\perp} \Sigma \mathbf{H}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp} & \mathbf{R}_i^{\perp} \Sigma \mathbf{H}' & \mathbf{R}_i^{\perp} \mathbf{X}_i & \mathbf{0} \\ \mathbf{H} \Sigma & \widehat{\mathbf{K}}_i & \mathbf{H} \Sigma \mathbf{X}^{\perp} & \mathbf{H} \Sigma \mathbf{R}_i^{\perp} & \mathbf{0} & \mathbf{K}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i' \mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} & \mathbf{0} \end{array} \right] \\
&\quad - r[\mathbf{R}_i^{\perp} \mathbf{X}_i, \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp}] + i_{\pm}((\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp} \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp} (\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp}) - r[\mathbf{X}, \Sigma]
\end{aligned}$$

$$\begin{aligned}
&= i_{\mp} \begin{bmatrix} -\Sigma & -\mathbf{X} & \mathbf{0} & \Sigma \mathbf{H}' & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp} & \mathbf{R}_i^{\perp} \Sigma \mathbf{H}' & \mathbf{R}_i^{\perp} \mathbf{X}_i \\ \mathbf{H} \Sigma & \widehat{\mathbf{K}}_i & \mathbf{H} \Sigma \mathbf{R}_i^{\perp} & \mathbf{0} & \mathbf{K}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_i' \mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} - r[\mathbf{R}_i^{\perp} \mathbf{X}_i, \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp}] \\
&\quad + i_{\pm}((\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp} \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp} (\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp}) - r[\mathbf{X}, \Sigma] + i_{\mp}(\mathbf{X}^{\perp} \Sigma \mathbf{X}^{\perp}) \\
&= i_{\pm} \begin{bmatrix} \Sigma & \Sigma \mathbf{R}_i^{\perp} & \Sigma \mathbf{H}' & \mathbf{0} & \mathbf{X} \\ \mathbf{R}_i^{\perp} \Sigma & \mathbf{0} & \mathbf{0} & \mathbf{R}_i^{\perp} \mathbf{X}_i & \mathbf{0} \\ \mathbf{H} \Sigma & \mathbf{0} & \mathbf{0} & \mathbf{K}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i' \mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} - r[\mathbf{R}_i^{\perp} \mathbf{X}_i, \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp}] \\
&\quad + i_{\pm}((\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp} \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp} (\mathbf{R}_i^{\perp} \mathbf{X}_i)^{\perp}) - r[\mathbf{X}, \Sigma] + i_{\mp}(\mathbf{X}^{\perp} \Sigma \mathbf{X}^{\perp}). \tag{4.16}
\end{aligned}$$

In consequence, by using (4.5) and (4.6), we obtain (3.1) and (3.2). According to (4.1), adding the equalities in (3.1) and (3.2) yields (3.3). Applying Lemma 1 to (3.1)-(3.3) yields (a)-(e). $\square$

Proof of Corollary 1. The first part of the corollary is an immediate consequence of Theorem 2 by setting $\mathbf{H} = \mathbf{0}$ in the matrix $\mathbf{M}$.

For the second part, we set $\mathbf{K}_i = \mathbf{X}_i$ and $\mathbf{H} = \mathbf{0}$ in the matrix $\mathbf{M}$ and then this matrix reduces the simpler form. In this case, we obtain

$$\begin{aligned}
&i_{+}(\mathbf{D}[\text{BLUE}_{\mathcal{M}}(\mathbf{X}_i \alpha_i)] - \mathbf{D}[\text{BLUE}_{\mathcal{R}_i}(\mathbf{X}_i \alpha_i)]) \\
&= i_{+} \begin{bmatrix} \Sigma & \Sigma \mathbf{R}_i^{\perp} & \mathbf{X} \\ \mathbf{R}_i^{\perp} \Sigma & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} \end{bmatrix} + r(\mathbf{X}_i) - r[\mathbf{X}, \Sigma] - r(\mathbf{R}_i^{\perp} \mathbf{X}_i)
\end{aligned}$$

and

$$\begin{aligned}
&i_{-}(\mathbf{D}[\text{BLUE}_{\mathcal{M}}(\mathbf{X}_i \alpha_i)] - \mathbf{D}[\text{BLUE}_{\mathcal{R}_i}(\mathbf{X}_i \alpha_i)]) \\
&= i_{-} \begin{bmatrix} \Sigma & \Sigma \mathbf{R}_i^{\perp} & \mathbf{X} \\ \mathbf{R}_i^{\perp} \Sigma & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} \end{bmatrix} + r(\mathbf{X}_i) - r[\mathbf{R}_i^{\perp} \mathbf{X}_i, \mathbf{R}_i^{\perp} \Sigma \mathbf{R}_i^{\perp}] - r(\mathbf{X}).
\end{aligned}$$

By using (4.6) and the predictability requirement of $\mathbf{X}_i \alpha_i$ under the models given in Definition 2, and in view of well-known rank equality $r[\mathbf{A}, \mathbf{B}] = r(\mathbf{A}) + r(\mathbf{E}_A \mathbf{B})$ (for any conformable $\mathbf{A}, \mathbf{B}$), we obtain the required result.

The third part is obtained in a similar way to the second part by setting $\mathbf{K}_i = \mathbf{R}_i^{\perp} \mathbf{X}_i$ and $\mathbf{H} = \mathbf{0}$ in the matrix $\mathbf{M}$ in Theorem 2. $\square$

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ON MODIFIED INTUITIONISTIC FUZZY SOFT METRIC SPACES AND APPLICATION

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Abstract. This paper introduces self maps satisfying (CLR_α) and $(CLR_{\alpha\beta})$ property in Modified Intuitionistic Fuzzy Soft Metric Space (MIFSMS). Moreover, we have also extended common fixed point theorems for weakly compatible maps satisfying common limit range property in the setting of MIFSMS. We have also given an application utilising our new results.

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Keywords: soft set, modified intuitionistic fuzzy metric space, weakly compatible map, fixed point

1. INTRODUCTION

To deal with uncertainty Zadeh [15] presented Fuzzy Sets that includes the membership function only. Further K.T. Atanassov [1] gave Intuitionistic Fuzzy Set to overcome the limitations of Fuzzy sets. Moreover, in case of data consisting of parameters, Molodstov [13] gave Soft Sets to deal with the uncertainties. Das and Samanta [4, 5] applied the concept of soft sets to metric spaces and hence presented Soft Metric Spaces utilizing soft points of soft sets. Maji et al [12], in 2001, introduced Fuzzy Soft Sets. Beaula and Gunaseli [2] applied the metric space concept to Fuzzy Soft Sets and introduced Fuzzy Soft Metric Spaces using fuzzy soft point and defined some of its properties.

Saadiat et al [14] gave another important concept of Modified Intuitionistic Fuzzy Metric Spaces by using continuous t —representable norm. Saadiat et al [14] also gave fixed point results for compatible and weakly compatible maps in Modified Intuitionistic Fuzzy Metric Spaces. The soft metric space (in short, SMS) was a result of employing the soft set theory to metric space by Das and Samanta [4–6]. Later on, various FPT's were stated and proved by different researchers in SMS. Vishal and Aanchal [9] introduced Modified Intuitionistic Fuzzy Soft Metric Space (MIFSMS) and proved the fixed point theorems in its settings. S. Chauhan et al. [3]

uses the common limit range property to gave some fixed point theorems for weakly compatible maps in modified intuitionistic fuzzy metric spaces.

The main objective of our paper is to introduce common limit range property and fixed point theorems for two pairs of self maps satisfying the above said property in the setting of MIFSMS.

2. PRELIMINARIES

The following section includes the basic definitions and results already proved in the literature that will be the foundation of our new results.

In this section, X is taken as universe, U represents the parameter set, $\bar{U}$ is taken as the absolute soft set and $SP(\bar{X})$ denotes the collection of all the soft points of $\bar{X}$.

Definition 1 ([13, Definition 2.1]). Soft set is a pair (S, U) on a universe X , where U represents the parameter set and S defines the map from U to power set of X i.e. $S : U \rightarrow P(X)$.

Definition 2 ([5, Definition 2.2.1]). A soft metric space is a 3-tuple $(\bar{X}, \bar{\mu}, U)$, where the soft metric $\bar{\mu}$ is defined from $SP(\bar{X}) \times SP(\bar{X})$ to $R(U)$. Here, $R(U)$ is the set containing non-negative soft real numbers and $\bar{\mu}$ satisfies the given conditions for all $\bar{u}_{e_1}, \bar{v}_{e_2}, \bar{w}_{e_3} \in SP(\bar{X})$:

- (i) $\bar{\mu}(\bar{u}_{e_1}, \bar{v}_{e_2}) > \bar{0}$,
- (ii) $\bar{\mu}(\bar{u}_{e_1}, \bar{v}_{e_2}) = \bar{0}$ iff $\bar{u}_{e_1} = \bar{v}_{e_2}$,
- (iii) $\bar{\mu}(\bar{u}_{e_1}, \bar{v}_{e_2}) = \bar{\mu}(\bar{v}_{e_2}, \bar{u}_{e_1})$,
- (iv) $\bar{\mu}(\bar{u}_{e_1}, \bar{v}_{e_2}) \leq \bar{\mu}(\bar{u}_{e_1}, \bar{w}_{e_3}) + \bar{\mu}(\bar{w}_{e_3}, \bar{v}_{e_2})$.

Definition 3 ([12, Definition 2.1]). Fuzzy soft set is a pair (S, U) over a universe X , where U represents the parameter set and S defines the map from U to $F(X)$, where latter is the set containing fuzzy subsets in universe X i.e. $S : U \rightarrow F(X)$.

Definition 4 ([8, Definition 3.5]). A soft fuzzy metric space is the 3-tuple $(\bar{X}, S, *)$, where soft fuzzy metric on $\bar{X}$ is given by map $S : SP(\bar{X}) \times SP(\bar{X}) \times (0, \infty) \rightarrow [0, 1]$ satisfying the given condition for all $\bar{u}_{e_1}, \bar{v}_{e_2}, \bar{w}_{e_3} \in SP(\bar{X}), s, t > 0$:

- (i) $S(\bar{u}_{e_1}, \bar{v}_{e_2}, t) > 0$,
- (ii) $S(\bar{u}_{e_1}, \bar{v}_{e_2}, t) = 1$ iff $\bar{u}_{e_1} = \bar{v}_{e_2}$,
- (iii) $S(\bar{u}_{e_1}, \bar{v}_{e_2}, t) = S(\bar{v}_{e_2}, \bar{u}_{e_1}, t)$,
- (iv) $S(\bar{u}_{e_1}, \bar{v}_{e_2}, t + s) \geq S(\bar{u}_{e_1}, \bar{w}_{e_3}, t) * S(\bar{w}_{e_3}, \bar{v}_{e_2}, s)$,
- (v) $S(\bar{u}_{e_1}, \bar{v}_{e_2}, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Definition 5 ([14, Definition 1.6]). A *modified intuitionistic fuzzy metric space* is given by 3-tuple $(X, \mathfrak{M}_{M,N}, \Theta)$, where X is any arbitrary set, M, N are fuzzy sets from $X^2 \times (0, \infty)$ to $[0, 1]$ satisfying $M(p, q, t) + N(p, q, t) \leq 1$ for all $p, q \in X$ and $t > 0$, continuous t -representable norm is given as Θ and $\mathfrak{M}_{M,N}$ is a function from $X^2 \times (0, \infty)$ to L^* fulfilling the given conditions for all $p, q, r \in X$ and $t, s > 0$:

- (i) $\mathfrak{W}_{M,N}(p, q, t) >_{L^*} 0_{L^*}$;
- (ii) $\mathfrak{W}_{M,N}(p, q, t) = 1_{L^*}$ iff $p = q$;
- (iii) $\mathfrak{W}_{M,N}(p, q, t) = \mathfrak{W}_{M,N}(q, p, t)$;
- (iv) $\mathfrak{W}_{M,N}(p, q, t + s) \geq_{L^*} \Theta(\mathfrak{W}_{M,N}(p, r, t), \mathfrak{W}_{M,N}(r, q, s))$;
- (v) $\mathfrak{W}_{M,N}(p, q, \cdot) : (0, \infty) \rightarrow L^*$ is continuous.

Here, *modified intuitionistic fuzzy metric* $\mathfrak{W}_{M,N}$ is given as

$$\mathfrak{W}_{M,N}(p, q, t) = (M(p, q, t), N(p, q, t)).$$

Definition 6 ([7, Definition 2.1]). A map $S : X \rightarrow IF^U$, where X is an arbitrary set and IF^U is the set consisting of all the intuitionistic fuzzy subsets of U , then S is a function defined for every $a \in X$ as $S(a) = \{ \langle u, \mu_{S(a)}, \nu_{S(a)} \rangle : u \in U \}$, where degree of association and non association is given by $\mu_{S(a)}$ and $\nu_{S(a)}$ respectively.

Definition 7 ([7, Definition 4.1]). A map Θ defined from $(L^*)^2$ to L^* is a triangular norm (t – norm) if it satisfies the given conditions:

- (i) $\Theta(p, 1_{L^*}) = p$, for all $p \in L^*$;
- (ii) $\Theta(p, q) = \Theta(q, p)$, for all $(p, q) \in (L^*)^2$;
- (iii) $\Theta(p, \Theta(q, r)) = \Theta(\Theta(p, q), r)$, for all $(p, q, r) \in (L^*)^3$;
- (iv) $p \leq p'$ and $q \leq q' \Rightarrow \Theta(p, q) \leq_{L^*} \Theta(p', q')$, for all $(p, p', q, q') \in (L^*)^4$.

Definition 8 ([7, Definition 4.3]). A continuous t –representable norm is a continuous t – norm Θ on L^* iff it implies the existence of a t – conorm $\diamond$ on $[0, 1]$, which is continuous, so that

$$\Theta(p, q) = (p_1 * q_1, p_2 \diamond q_2),$$

for all $p = (p_1, p_2)$ and $q = (q_1, q_2) \in L^*$.

Definition 9 ([9, Definition 3.3]). A MIFSMS is a 3-tuple $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$, where $\bar{X}$ is any set, M and N are soft fuzzy sets, Θ is a continuous t –representable norm, $\mathfrak{W}_{M,N}$ is a mapping from $SP(\bar{X}) \times SP(\bar{X}) \times (0, \infty)$ to L^* so that the given assertions are satisfied for all $\bar{p}_{e_1}, \bar{q}_{e_2}, \bar{r}_{e_3} \in SP(\bar{X})$ and $s, t > 0$:

- (i) $\mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t) >_{L^*} 0_{L^*}$;
- (ii) $\mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t) = 1_{L^*}$ iff $\bar{p}_{e_1} = \bar{q}_{e_2}$;
- (iii) $\mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t) = \mathfrak{W}_{M,N}(\bar{q}_{e_2}, \bar{p}_{e_1}, t)$;
- (iv) $\mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t + s) \geq_{L^*} \Theta(\mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{r}_{e_3}, t), \mathfrak{W}_{M,N}(\bar{r}_{e_3}, \bar{q}_{e_2}, s))$;
- (v) $\mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, \cdot) : (0, \infty) \rightarrow L^*$ is continuous.

Then, $\mathfrak{W}_{M,N}$ is called MIFSM on $\bar{X}$. Here the level of closeness, non closeness between $\bar{p}_{e_1}, \bar{q}_{e_2}$ w.r.t. t is given by the maps $M(\bar{p}_{e_1}, \bar{q}_{e_2}, t)$ and $N(\bar{p}_{e_1}, \bar{q}_{e_2}, t)$ respectively and metric $\mathfrak{W}_{M,N}$ is given as

$$\mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t) = (M(\bar{p}_{e_1}, \bar{q}_{e_2}, t), N(\bar{p}_{e_1}, \bar{q}_{e_2}, t)).$$

Remark 1 ([9, Remark 3.1]). The function $M(\bar{p}_{e_1}, \bar{q}_{e_2}, t)$ is increasing and the function $N(\bar{p}_{e_1}, \bar{q}_{e_2}, t)$ is decreasing in a MIFSMS $\bar{X}$, for all $\bar{p}_{e_1}, \bar{q}_{e_2} \in SP(\bar{X})$.

Example 1 ([9, Example 3.1]). Consider $(\bar{X}, \mu)$ be a soft metric space and M, N be soft fuzzy sets on $SP(\bar{X}) \times SP(\bar{X}) \times (0, \infty)$ as given below:

$$\begin{aligned}\mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t) &= (M(\bar{p}_{e_1}, \bar{q}_{e_2}, t), N(\bar{p}_{e_1}, \bar{q}_{e_2}, t)) \\ &= \left(\frac{ht^n}{ht^n + m\mu(\bar{p}_{e_1}, \bar{q}_{e_2})}, \frac{m\mu(\bar{p}_{e_1}, \bar{q}_{e_2})}{ht^n + m\mu(\bar{p}_{e_1}, \bar{q}_{e_2})} \right)\end{aligned}$$

for all $h, t, m, n \in R^+$; $\Theta(\bar{r}, \bar{s}) = (\bar{r}_1 \bar{s}_1, \min(\bar{r}_2 + \bar{s}_2, 1))$ for all $\bar{r} = (\bar{r}_1, \bar{r}_2)$ and $\bar{s} = (\bar{s}_1, \bar{s}_2) \in L^*$. The above $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ serves as an example of a MIFSMS.

Lemma 1 ([9, Lemma 3.3]). Consider $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ be a MIFSMS. Then $\mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t)$ is increasing with respect to $t > 0$ in $(L^*, \leq_{L^*})$ for all $\bar{p}_{e_1}, \bar{q}_{e_2} \in SP(\bar{X})$.

Definition 10 ([9, Definition 3.4]). Consider $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ be a MIFSMS. $\mathfrak{W}_{M,N}$ is called continuous on $SP(\bar{X}) \times SP(\bar{X}) \times (0, \infty)$ if

$$\lim_{n \rightarrow \infty} \mathfrak{W}_{M,N}(\bar{p}_{e_n}, \bar{q}_{e_n}, t_n) = \mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t),$$

where $\{(\bar{p}_{e_n}, \bar{q}_{e_n}, t_n)\}$ is a sequence converging to $(\bar{p}_{e_1}, \bar{q}_{e_2}, t)$.

Lemma 2 ([9, Lemma 3.4]). For $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ be a MIFSMS, then $\mathfrak{W}_{M,N}$ is continuous on $SP(\bar{X}) \times SP(\bar{X}) \times (0, \infty)$.

Definition 11 ([9, Definition 4.1]). The open ball in a MIFSMS $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ is defined by $B(\bar{p}_{e_1}, \alpha, t)$, having center $\bar{p}_{e_1} \in \bar{X}$ and radius $0 < \alpha < 1$ for any $t > 0$,

$$B(\bar{p}_{e_1}, \alpha, t) = \{\bar{q}_{e_2} \in \bar{X} : \mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t) >_{L^*} (N_s(\alpha), \alpha)\}.$$

Theorem 1 ([9, Theorem 4.1]). Every open ball $B(\bar{p}_{e_1}, \alpha, t)$ in MIFSMS is an open set.

Remark 2 ([9, Remark 4.1]). The topology procured by MIFSMS $\mathfrak{W}_{M,N}$ on $\bar{X}$ in MIFSMS $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ is defined as

$$\begin{aligned}\tau_{M,N} &= \{Y \subseteq \bar{X} : \text{for every } \bar{p}_{e_1} \in Y, \text{ there exist } t > 0 \text{ and } \alpha \in (0, 1) \mid \\ &\text{so that } B(\bar{p}_{e_1}, \alpha, t) \subseteq Y\}.\end{aligned}$$

Theorem 2 ([9, Theorem 4.2]). A MIFSMS $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ is a Hausdorff space.

Theorem 3 ([9, Theorem 4.4]). If $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ be MIFSMS and $\tau_{M,N}$ is a topology on $\bar{X}$. Then $\bar{p}_{e_n} \rightarrow \bar{p}_{e_1}$ iff

$$\lim_{n \rightarrow \infty} \mathfrak{W}_{M,N}(\bar{p}_{e_n}, \bar{p}_{e_1}, t) = 1_{L^*}$$

for sequence $\{\bar{p}_{e_n}\}$ in $\bar{X}$.

Definition 12 ([9, Definition 4.3]). Consider $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ be MIFSMS and $\{\bar{p}_{e_n}\}$ be any sequence in $\bar{X}$, then

1. the sequence is Cauchy iff for every $t > 0$, there exist $n_o \in N$ so that

$$\lim_{n_o \rightarrow \infty} \mathfrak{W}_{M,N}(\bar{p}_{e_n}, \bar{p}_{e_{n+m}}, t) = 1_{L^*}$$

for each $n, m \geq n_o$.

2. the sequence converges to $\bar{p}$ iff for every $t > 0$,

$$\lim_{n \rightarrow \infty} \mathfrak{W}_{M,N}(\bar{p}_{e_n}, \bar{p}, t) = 1_{L^*}.$$

Lemma 3 ([10, Lemma 3.1]). Consider $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ be a MIFSMS and for all $\bar{p}_{e_1}, \bar{q}_{e_2} \in SP(\bar{X})$, $t > 0$, $0 < k < 1$, (2.1) holds, ie.

$$\mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, kt) \geq_{L^*} \mathfrak{W}_{M,N}(\bar{p}_{e_1}, \bar{q}_{e_2}, t). \quad (2.1)$$

Then $\bar{p}_{e_1} = \bar{q}_{e_2}$.

Definition 13 ([10, Definition 3.2]). Consider two self maps A and S on MIFSMS $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$. The maps are weakly compatible if

$$A\bar{p}_{e_1} = S\bar{p}_{e_1} \Rightarrow AS\bar{p}_{e_1} = SA\bar{p}_{e_1}$$

for some $\bar{p}_{e_1} \in SP(\bar{X})$.

3. MAIN RESULTS

In this section we are defining common limit range property in the setting of MIFSMS as follows:

Definition 14. A pair of self maps (ϕ, α) in a MIFSMS $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ satisfies CLR_α property, known as common limit range property in relation to α , if there exists a sequence $\{\bar{p}_{e_n}\}$ in $\bar{X}$ so that (3.1) holds for some $\bar{p} \in \alpha(\bar{X})$ and $t > 0$,

$$\lim_{n \rightarrow \infty} \mathfrak{W}_{M,N}(\phi\bar{p}_{e_n}, \bar{p}, t) = \lim_{n \rightarrow \infty} \mathfrak{W}_{M,N}(\alpha\bar{p}_{e_n}, \bar{p}, t) = 1_{L^*}. \quad (3.1)$$

Definition 15. Two pairs of self maps (ϕ, α) and (η, β) in a MIFSMS $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ satisfies $(CLR_{\alpha\beta})$ property, known as common limit range property in relation to α and β , if there exists two sequences $\{\bar{p}_{e_n}\}$ and $\{\bar{q}_{e_n}\}$ in $\bar{X}$ so that (3.2) holds for some $\bar{h} \in \alpha(\bar{X}) \cap \beta(\bar{X})$ and $t > 0$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathfrak{W}_{M,N}(\phi\bar{p}_{e_n}, \bar{h}, t) &= \lim_{n \rightarrow \infty} \mathfrak{W}_{M,N}(\alpha\bar{p}_{e_n}, \bar{h}, t) = \lim_{n \rightarrow \infty} \mathfrak{W}_{M,N}(\eta\bar{q}_{e_n}, \bar{h}, t) \\ &= \lim_{n \rightarrow \infty} \mathfrak{W}_{M,N}(\beta\bar{q}_{e_n}, \bar{h}, t) = 1_{L^*}. \end{aligned} \quad (3.2)$$

Example 2. Consider $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ be a MIFSMS, where $\bar{X} = [0, \infty)$ and

$$\mathfrak{W}_{M,N}(\bar{l}, \bar{k}, \rho) = (M(\bar{l}, \bar{k}, \rho), N(\bar{l}, \bar{k}, \rho)) = \left(\frac{\rho}{\rho + |\bar{l} - \bar{k}|}, \frac{|\bar{l} - \bar{k}|}{\rho + |\bar{l} - \bar{k}|} \right),$$

for every $\bar{l}, \bar{k} \in \bar{X}$ and $\rho > 0$.

The self maps ϕ and α on $\bar{X}$ are defined as $\phi(\bar{l}) = \bar{l} + 1$ and $\alpha(\bar{l}) = 3\bar{l}$, for every $\bar{l} \in \bar{X}$.

Take a sequence $\{\bar{\imath}_{e_n}\} = \{1 + \frac{1}{n}\}$ in $\bar{X}$, then

$\varpi_{M,N}(\phi\bar{\imath}_{e_n}, 3, \rho) = \varpi_{M,N}(\alpha\bar{\imath}_{e_n}, 3, \rho) = 1_{L^*}$, where $3 \in \alpha(\bar{X})$ and $\rho > 0$. Thus the maps ϕ and α satisfies (CLR_α) property.

Example 3. Consider $(\bar{X}, \varpi_{M,N}, \Theta)$ be a MIFSMS, where $\bar{X} = [3, 27)$, $\Theta(\bar{p}, \bar{q}) = (\bar{p}_1\bar{q}_1, \min\{\bar{p}_2 + \bar{q}_2, 1\})$, for every $\bar{p} = (\bar{p}_1, \bar{p}_2)$ and $\bar{q} = (\bar{q}_1, \bar{q}_2) \in L^*$ with MIFSM ϖ defined as follows for every $\bar{\imath}, \bar{\kappa} \in \bar{X}$,

$$\varpi(\bar{\imath}, \bar{\kappa}, \rho) = \left(\frac{\rho}{\rho + |\bar{\imath} - \bar{\kappa}|}, \frac{|\bar{\imath} - \bar{\kappa}|}{\rho + |\bar{\imath} - \bar{\kappa}|} \right).$$

Consider $\Upsilon : L^* \rightarrow L^*$ defined as $\Upsilon(\rho_1, \rho_2) = (\sqrt{\rho_1}, 0)$ for every $\rho = (\rho_1, \rho_2) \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$. Let the self maps ϕ, η, α and β defined as

$$\begin{aligned} \phi(\bar{\imath}) &= \begin{cases} 3 & \text{for } \bar{\imath} \in \{3\} \cup (11, 27), \\ 21 & \text{for } 3 < \bar{\imath} \leq 11. \end{cases} & \eta(\bar{\imath}) &= \begin{cases} 3 & \text{for } \bar{\imath} \in \{3\} \cup (11, 27), \\ 8 & \text{for } 3 < \bar{\imath} \leq 11. \end{cases} \\ \alpha(\bar{\imath}) &= \begin{cases} 3 & \text{for } \bar{\imath} = 3, \\ 12 & \text{for } 3 < \bar{\imath} \leq 11, \\ \frac{\bar{\imath}+1}{4} & \text{for } 11 < \bar{\imath} < 27. \end{cases} & \beta(\bar{\imath}) &= \begin{cases} 3 & \text{for } \bar{\imath} = 3, \\ 20 & \text{for } 3 < \bar{\imath} \leq 11, \\ \bar{\imath} - 8 & \text{for } 11 < \bar{\imath} < 27. \end{cases} \end{aligned}$$

Take sequences $\{\bar{\imath}_{e_n} = 11 + \frac{1}{n}\}$, $\{\bar{\kappa}_{e_n} = 3\}$, then we have

$$\lim_{n \rightarrow \infty} \phi(\bar{\imath}_{e_n}) = \lim_{n \rightarrow \infty} \alpha(\bar{\imath}_{e_n}) = \lim_{n \rightarrow \infty} \eta(\bar{\kappa}_{e_n}) = \lim_{n \rightarrow \infty} \beta(\bar{\kappa}_{e_n}) = 3 \in \alpha(\bar{X}) \cap \beta(\bar{X}).$$

Hence, (ϕ, α) and (η, β) satisfies $CLR_{\alpha\beta}$ property.

Definition 16. Two families of self mappings $\{\phi_i\}_{i \in \{1, 2, 3, \dots, m\}}$ and $\{\alpha_k\}_{k \in \{1, 2, 3, \dots, n\}}$ are said to be pairwise commuting if they satisfies the following conditions:

1. $\phi_i\phi_j = \phi_j\phi_i$ for every $i, j \in \{1, 2, 3, \dots, m\}$,
2. $\alpha_k\alpha_l = \alpha_l\alpha_k$ for every $k, l \in \{1, 2, 3, \dots, n\}$,
3. $\phi_i\alpha_k = \alpha_k\phi_i$ for every $i \in \{1, 2, 3, \dots, m\}$ and $k \in \{1, 2, 3, \dots, n\}$.

4. FIXED POINT THEOREMS

Before giving the fixed point theorems, we are going to state and prove a lemma that will be used in our new theorems.

Lemma 4. Consider $\phi, \eta, \alpha, \beta$ be self maps on a MIFSMS $(\bar{X}, \varpi_{M,N}, \Theta)$, satisfies the following conditions:

1. pair (ϕ, α) or (η, β) exhibits common limit range property with respect to α or β respectively,
2. $\phi(\bar{X})$ is strictly contained in $\beta(\bar{X})$, or $\eta(\bar{X})$ is strictly contained in $\alpha(\bar{X})$,
3. $\beta(\bar{X})$ or $\alpha(\bar{X})$ is a closed subset of $\bar{X}$,
4. for sequences $\{\bar{p}_{e_n}\}$ and $\{\bar{q}_{e_n}\}$ in $\bar{X}$, either $\eta(\bar{q}_{e_n})$ is convergent whenever $\beta(\bar{q}_{e_n})$ converges, or $\phi(\bar{p}_{e_n})$ is convergent whenever $\alpha(\bar{p}_{e_n})$ converges,

5. for a constant $\varepsilon \in (0, 1)$, (4.1) holds for every $\bar{p}, \bar{q} \in \bar{X}$ and $t > 0$,

$$\mathfrak{W}_{M,N}(\phi\bar{p}, \eta\bar{q}, \varepsilon t) \geq_{L^*} \min\{\mathfrak{W}_{M,N}(\alpha\bar{p}, \beta\bar{q}, t), \mathfrak{W}_{M,N}(\phi\bar{p}, \alpha\bar{p}, t), \mathfrak{W}_{M,N}(\eta\bar{q}, \beta\bar{q}, t), \mathfrak{W}_{M,N}(\phi\bar{p}, \beta\bar{q}, t), \mathfrak{W}_{M,N}(\eta\bar{q}, \alpha\bar{p}, t)\}. \quad (4.1)$$

Then pairs (ϕ, α) and (η, β) exhibits common limit range property with respect to α and β , more precisely known as $(CLR_{\alpha\beta})$ property.

Proof. Consider the pair (ϕ, α) exhibits (CLR_{α}) property, that implies the existence of a sequence $\{\bar{p}_{e_n}\}$ in $\bar{X}$ so that (4.2) holds for $\bar{h} \in \alpha(\bar{X})$,

$$\lim_{n \rightarrow \infty} \phi\bar{p}_{e_n} = \lim_{n \rightarrow \infty} \alpha\bar{p}_{e_n} = \bar{h}. \quad (4.2)$$

As $\phi(\bar{X}) \subset \beta(\bar{X})$ and $\beta(\bar{X})$ is a closed subset of $\bar{X}$, thus for every sequence $\{\bar{p}_{e_n}\}$ in $\bar{X}$ implies the existence of a sequence $\{\bar{q}_{e_n}\}$ in $\bar{X}$ so that $\phi\bar{p}_{e_n} = \beta\bar{q}_{e_n}$. Thus (4.3) holds for $\bar{h} \in \alpha(\bar{X}) \cap \beta(\bar{X})$,

$$\lim_{n \rightarrow \infty} \beta\bar{q}_{e_n} = \lim_{n \rightarrow \infty} \phi\bar{p}_{e_n} = \bar{h}. \quad (4.3)$$

Overall, we have $\phi\bar{p}_{e_n} \rightarrow \bar{h}$, $\alpha\bar{p}_{e_n} \rightarrow \bar{h}$ and $\beta\bar{q}_{e_n} \rightarrow \bar{h}$. Now to claim that $\eta\bar{q}_{e_n} \rightarrow \bar{h}$. Substituting $\bar{p} = \bar{p}_{e_n}$, $\bar{q} = \bar{q}_{e_n}$ in inequality (4.1), we have

$$\begin{aligned} \mathfrak{W}_{M,N}(\phi\bar{p}_{e_n}, \eta\bar{q}_{e_n}, \varepsilon t) &\geq_{L^*} \min\{\mathfrak{W}_{M,N}(\alpha\bar{p}_{e_n}, \beta\bar{q}_{e_n}, t), \mathfrak{W}_{M,N}(\phi\bar{p}_{e_n}, \alpha\bar{p}_{e_n}, t), \\ &\quad \mathfrak{W}_{M,N}(\eta\bar{q}_{e_n}, \beta\bar{q}_{e_n}, t), \mathfrak{W}_{M,N}(\phi\bar{p}_{e_n}, \beta\bar{q}_{e_n}, t), \\ &\quad \mathfrak{W}_{M,N}(\eta\bar{q}_{e_n}, \alpha\bar{p}_{e_n}, t)\}. \end{aligned}$$

Let $\lim_{n \rightarrow \infty} \mathfrak{W}_{M,N}(\eta\bar{q}_{e_n}, \bar{l}, t) = 1_{L^*}$, where $\bar{l} (\neq \bar{h})$ for all $t > 0$. Then, taking $n \rightarrow \infty$, we get

$$\begin{aligned} \mathfrak{W}_{M,N}(\bar{h}, \bar{l}, \varepsilon t) &\geq_{L^*} \min\{\mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t), \mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t), \mathfrak{W}_{M,N}(\bar{l}, \bar{h}, t), \mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t), \\ &\quad \mathfrak{W}_{M,N}(\bar{l}, \bar{h}, t)\}, \\ &\geq_{L^*} \min\{1_{L^*}, 1_{L^*}, \mathfrak{W}_{M,N}(\bar{l}, \bar{h}, t), 1_{L^*}, \mathfrak{W}_{M,N}(\bar{l}, \bar{h}, t)\}, \\ &\geq_{L^*} \mathfrak{W}_{M,N}(\bar{h}, \bar{l}, t). \end{aligned}$$

Thus, by Lemma 3 $\bar{h} = \bar{l}$. So, pairs (ϕ, α) and (η, β) satisfies $(CLR_{\alpha\beta})$ property. $\square$

Now we are ready to prove our fixed point results, stated and proved as follows:

Theorem 4. Consider $\phi, \eta, \alpha, \beta$ be self maps on a MIFSMS $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ satisfying inequality (4.1) of Lemma 4. The pair of self maps (ϕ, α) and (η, β) possess a coincidence point if they exhibit $(CLR_{\alpha\beta})$ property. Moreover, maps ϕ, α, η and β possess a unique common fixed point if pairs (ϕ, α) and (η, β) are weakly compatible.

Proof. Since pair (ϕ, α) and (η, β) exhibit $(CLR_{\alpha\beta})$ property, this implies the existence of two sequences $\{\bar{p}_{e_n}\}$ and $\{\bar{q}_{e_n}\}$ in $\bar{X}$ so that (4.4) holds for $\bar{h} \in \alpha(\bar{X}) \cap \beta(\bar{X})$,

$$\lim_{n \rightarrow \infty} \phi\bar{p}_{e_n} = \lim_{n \rightarrow \infty} \alpha\bar{p}_{e_n} = \lim_{n \rightarrow \infty} \eta\bar{q}_{e_n} = \lim_{n \rightarrow \infty} \beta\bar{q}_{e_n} = \bar{h}. \quad (4.4)$$

Thus, $\alpha\bar{u} = \bar{h}$ and $\beta\bar{u} = \bar{h}$, where $\bar{u}, \bar{v} \in \bar{X}$. Now to claim that $\phi\bar{u} = \alpha\bar{u}$, substituting $\bar{p} = \bar{u}$ and $\bar{q} = \bar{q}_{e_n}$ in (4.1), we have

$$\begin{aligned} \mathfrak{W}_{M,N}(\phi\bar{u}, \eta\bar{q}_{e_n}, \varepsilon t) \geq_{L^*} \min\{\mathfrak{W}_{M,N}(\alpha\bar{u}, \beta\bar{q}_{e_n}, t), \mathfrak{W}_{M,N}(\phi\bar{u}, \alpha\bar{u}, t), \mathfrak{W}_{M,N}(\eta\bar{q}_{e_n}, \beta\bar{q}_{e_n}, t), \\ \mathfrak{W}_{M,N}(\phi\bar{u}, \beta\bar{q}_{e_n}, t), \mathfrak{W}_{M,N}(\eta\bar{q}_{e_n}, \alpha\bar{u}, t)\}. \end{aligned}$$

Taking $n \rightarrow \infty$, we get

$$\begin{aligned} \mathfrak{W}_{M,N}(\phi\bar{u}, \bar{h}, \varepsilon t) \geq_{L^*} \min\{\mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t), \mathfrak{W}_{M,N}(\phi\bar{u}, \bar{h}, t), \mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t), \\ \mathfrak{W}_{M,N}(\phi\bar{u}, \bar{h}, t), \mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t)\}, \end{aligned}$$

and so

$$\begin{aligned} \mathfrak{W}_{M,N}(\phi\bar{u}, \bar{h}, \varepsilon t) \geq_{L^*} \min\{1_{L^*}, \mathfrak{W}_{M,N}(\phi\bar{u}, \bar{h}, t), 1_{L^*}, \mathfrak{W}_{M,N}(\phi\bar{u}, \bar{h}, t), 1_{L^*}\}, \\ \mathfrak{W}_{M,N}(\phi\bar{u}, \bar{h}, \varepsilon t) \geq_{L^*} \mathfrak{W}_{M,N}(\phi\bar{u}, \bar{h}, t). \end{aligned}$$

Thus, we have $\phi\bar{u} = \bar{h}$ and so $\phi\bar{u} = \alpha\bar{u} = \bar{h}$, this implies that pair (ϕ, α) has $\bar{u}$ as its coincidence point.

Claim that $\eta\bar{v} = \beta\bar{v}$, substituting $\bar{p} = \bar{u}$ and $\bar{q} = \bar{v}$ in (4.1), we have

$$\begin{aligned} \mathfrak{W}_{M,N}(\phi\bar{u}, \eta\bar{v}, \varepsilon t) \geq_{L^*} \min\{\mathfrak{W}_{M,N}(\alpha\bar{u}, \beta\bar{v}, t), \mathfrak{W}_{M,N}(\phi\bar{u}, \alpha\bar{u}, t), \mathfrak{W}_{M,N}(\eta\bar{v}, \beta\bar{v}, t), \\ \mathfrak{W}_{M,N}(\phi\bar{u}, \beta\bar{v}, t), \mathfrak{W}_{M,N}(\eta\bar{v}, \alpha\bar{u}, t)\}, \end{aligned}$$

so we get

$$\begin{aligned} \mathfrak{W}_{M,N}(\bar{h}, \eta\bar{v}, \varepsilon t) \geq_{L^*} \min\{\mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t), \mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t), \mathfrak{W}_{M,N}(\eta\bar{v}, \bar{h}, t), \\ \mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t), \mathfrak{W}_{M,N}(\eta\bar{v}, \bar{h}, t)\}, \end{aligned}$$

and

$$\begin{aligned} \mathfrak{W}_{M,N}(\bar{h}, \eta\bar{v}, \varepsilon t) \geq_{L^*} \min\{1_{L^*}, 1_{L^*}, \mathfrak{W}_{M,N}(\eta\bar{v}, \bar{h}, t), 1_{L^*}, \mathfrak{W}_{M,N}(\eta\bar{v}, \bar{h}, t)\}, \\ \geq_{L^*} \mathfrak{W}_{M,N}(\bar{h}, \eta\bar{v}, t). \end{aligned}$$

Thus, we have $\bar{h} = \eta\bar{v}$ and hence $\eta\bar{v} = \beta\bar{v} = \bar{h}$, this implies that pair (η, β) has $\bar{v}$ as its coincidence point.

Since (ϕ, α) is weakly compatible and $\phi\bar{u} = \alpha\bar{u}$, hence $\phi\bar{h} = \phi\alpha\bar{u} = \alpha\phi\bar{u} = \alpha\bar{h}$. Now to claim that $\bar{h}$ is a common fixed point of (ϕ, α) , substituting $\bar{p} = \bar{h}$ and $\bar{q} = \bar{v}$ in (4.1), we have

$$\begin{aligned} \mathfrak{W}_{M,N}(\phi\bar{h}, \eta\bar{v}, \varepsilon t) \geq_{L^*} \min\{\mathfrak{W}_{M,N}(\alpha\bar{h}, \beta\bar{v}, t), \mathfrak{W}_{M,N}(\phi\bar{h}, \alpha\bar{h}, t), \mathfrak{W}_{M,N}(\eta\bar{v}, \beta\bar{v}, t), \\ \mathfrak{W}_{M,N}(\phi\bar{h}, \beta\bar{v}, t), \mathfrak{W}_{M,N}(\eta\bar{v}, \alpha\bar{h}, t)\}, \end{aligned}$$

so we get

$$\begin{aligned} \mathfrak{W}_{M,N}(\phi\bar{h}, \bar{h}, \varepsilon t) \geq_{L^*} \min\{\mathfrak{W}_{M,N}(\phi\bar{h}, \bar{h}, t), \mathfrak{W}_{M,N}(\phi\bar{h}, \phi\bar{h}, t), \mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t), \\ \mathfrak{W}_{M,N}(\phi\bar{h}, \bar{h}, t), \mathfrak{W}_{M,N}(\bar{h}, \phi\bar{h}, t)\}, \end{aligned}$$

and

$$\begin{aligned}\mathfrak{W}_{M,N}(\phi\bar{h}, \bar{h}, \varepsilon t) &\geq_{L^*} \min\{\mathfrak{W}_{M,N}(\phi\bar{h}, \bar{h}, t), 1_{L^*}, 1_{L^*}, \mathfrak{W}_{M,N}(\phi\bar{h}, \bar{h}, t), \mathfrak{W}_{M,N}(\phi\bar{h}, \bar{h}, t)\}, \\ &\geq_{L^*} \mathfrak{W}_{M,N}(\phi\bar{h}, \bar{h}, t).\end{aligned}$$

Thus, we have $\phi\bar{h} = \bar{h} = \alpha\bar{h}$, this implies, the pair (ϕ, α) has $\bar{h}$ as its common fixed point.

As pair (η, β) is weakly compatible, hence $\eta\bar{h} = \eta\beta\bar{h} = \beta\eta\bar{h} = \beta\bar{h}$. Substituting $\bar{p} = \bar{u}$ and $\bar{q} = \bar{h}$ in (4.1), we have

$$\begin{aligned}\mathfrak{W}_{M,N}(\phi\bar{u}, \eta\bar{h}, \varepsilon t) &\geq_{L^*} \min\{\mathfrak{W}_{M,N}(\alpha\bar{u}, \beta\bar{h}, t), \mathfrak{W}_{M,N}(\phi\bar{u}, \alpha\bar{u}, t), \mathfrak{W}_{M,N}(\eta\bar{h}, \beta\bar{h}, t), \\ &\quad \mathfrak{W}_{M,N}(\phi\bar{u}, \beta\bar{h}, t), \mathfrak{W}_{M,N}(\eta\bar{h}, \alpha\bar{u}, t)\},\end{aligned}$$

so, we get

$$\begin{aligned}\mathfrak{W}_{M,N}(\bar{h}, \eta\bar{h}, \varepsilon t) &\geq_{L^*} \min\{\mathfrak{W}_{M,N}(\bar{h}, \eta\bar{h}, t), \mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t), \mathfrak{W}_{M,N}(\eta\bar{h}, \eta\bar{h}, t), \\ &\quad \mathfrak{W}_{M,N}(\bar{h}, \eta\bar{h}, t), \mathfrak{W}_{M,N}(\eta\bar{h}, \bar{h}, t)\},\end{aligned}$$

and

$$\begin{aligned}\mathfrak{W}_{M,N}(\bar{h}, \eta\bar{h}, \varepsilon t) &\geq_{L^*} \min\{\mathfrak{W}_{M,N}(\bar{h}, \eta\bar{h}, t), 1_{L^*}, 1_{L^*}, \mathfrak{W}_{M,N}(\bar{h}, \eta\bar{h}, t), \mathfrak{W}_{M,N}(\eta\bar{h}, \bar{h}, t)\}, \\ &\geq_{L^*} \mathfrak{W}_{M,N}(\bar{h}, \eta\bar{h}, t).\end{aligned}$$

Thus, we have $\eta\bar{h} = \bar{h} = \beta\bar{h}$, this implies that pair (η, β) has $\bar{h}$ as their common fixed point. Hence, maps ϕ, η, α and β have $\bar{h}$ as their common fixed point.

Uniqueness: Consider maps ϕ, η, α and β have another common fixed point say $\bar{w} (\neq \bar{h})$. Substitute $\bar{p} = \bar{w}$ and $\bar{q} = \bar{h}$ in (4.1), we have

$$\begin{aligned}\mathfrak{W}_{M,N}(\phi\bar{w}, \eta\bar{h}, \varepsilon t) &\geq_{L^*} \min\{\mathfrak{W}_{M,N}(\alpha\bar{w}, \beta\bar{h}, t), \mathfrak{W}_{M,N}(\phi\bar{w}, \alpha\bar{w}, t), \mathfrak{W}_{M,N}(\eta\bar{h}, \beta\bar{h}, t), \\ &\quad \mathfrak{W}_{M,N}(\phi\bar{w}, \beta\bar{h}, t), \mathfrak{W}_{M,N}(\eta\bar{h}, \alpha\bar{w}, t)\},\end{aligned}$$

so, we get

$$\begin{aligned}\mathfrak{W}_{M,N}(\bar{w}, \bar{h}, \varepsilon t) &\geq_{L^*} \min\{\mathfrak{W}_{M,N}(\bar{w}, \bar{h}, t), \mathfrak{W}_{M,N}(\bar{w}, \bar{w}, t), \mathfrak{W}_{M,N}(\bar{h}, \bar{h}, t), \\ &\quad \mathfrak{W}_{M,N}(\bar{w}, \bar{h}, t), \mathfrak{W}_{M,N}(\bar{h}, \bar{w}, t)\},\end{aligned}$$

and

$$\begin{aligned}\mathfrak{W}_{M,N}(\bar{w}, \bar{h}, \varepsilon t) &\geq_{L^*} \min\{\mathfrak{W}_{M,N}(\bar{w}, \bar{h}, t), 1_{L^*}, 1_{L^*}, \mathfrak{W}_{M,N}(\bar{w}, \bar{h}, t), \mathfrak{W}_{M,N}(\bar{w}, \bar{h}, t)\}, \\ &\geq_{L^*} \mathfrak{W}_{M,N}(\bar{w}, \bar{h}, t).\end{aligned}$$

Thus, we have $\bar{w} = \bar{h}$. Therefore ϕ, η, α and β possess a unique common fixed point. $\square$

Theorem 5. Consider $\phi, \eta, \alpha, \beta$ be self maps on a MIFSMS $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ satisfying conditions 1-5 of Lemma 4. The maps ϕ, α, η , and β possess a unique common fixed point if the pairs (ϕ, α) and (η, β) are weakly compatible.

Proof. The pairs (ϕ, α) and (η, β) exhibit $(CLR_{\alpha\beta})$ property by Lemma 4, implies the existence of two sequences $\{\bar{p}_{e_n}\}$ and $\{\bar{q}_{e_n}\}$ in $\bar{X}$ so that (4.5) holds for $\bar{h} \in \alpha(\bar{X}) \cap \beta(\bar{X})$,

$$\lim_{n \rightarrow \infty} \phi \bar{p}_{e_n} = \lim_{n \rightarrow \infty} \alpha \bar{p}_{e_n} = \lim_{n \rightarrow \infty} \eta \bar{q}_{e_n} = \lim_{n \rightarrow \infty} \beta \bar{q}_{e_n} = \bar{h}. \quad (4.5)$$

The rest can be proved on the similar lines to Theorem 4. $\square$

Following are the two illustration making use of recently proved theorems:

Example 4. Consider $\bar{X} = [1, 15]$ in MIFSMS $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$, t -representable norm is given by $\Theta(\bar{a}, \bar{b}) = (\bar{a}_1 \bar{b}_1, \min\{\bar{a}_2 + \bar{b}_2, 1\})$ for all $\bar{a} = (\bar{a}_1, \bar{a}_2)$ and $\bar{b} = (\bar{b}_1, \bar{b}_2) \in L^*$ so that for all $\bar{p}, \bar{q} \in \bar{X}$ and $t > 0$, MIFSM is defined as

$$\mathfrak{W}_{M,N}(\bar{p}, \bar{q}, t) = \left(\frac{t}{t + |\bar{p} - \bar{q}|}, \frac{|\bar{p} - \bar{q}|}{t + |\bar{p} - \bar{q}|} \right).$$

Define self maps $\phi, \eta, \alpha, \beta$ by

$$\begin{aligned} \phi(\bar{p}) &= \begin{cases} 1 & \text{if } \bar{p} \in \{1\} \cup (3, 15); \\ 2 & \text{if } \bar{p} \in (1, 3]. \end{cases} & \eta(\bar{p}) &= \begin{cases} 1 & \text{if } \bar{p} \in \{1\} \cup (3, 15); \\ 3 & \text{if } \bar{p} \in (1, 3]. \end{cases} \\ \alpha(\bar{p}) &= \begin{cases} 1 & \text{if } \bar{p} = 1; \\ 5 & \text{if } \bar{p} \in (1, 3]; \\ \frac{\bar{p}+1}{4} & \text{if } \bar{p} \in (3, 15). \end{cases} & \beta(\bar{p}) &= \begin{cases} 1 & \text{if } \bar{p} = 1; \\ 9 + \bar{p} & \text{if } \bar{p} \in (1, 3]; \\ \bar{p} - 2 & \text{if } \bar{p} \in (3, 15). \end{cases} \end{aligned}$$

Taking $\{\bar{p}_{e_n} = 3 + \frac{1}{n}\}$, $\{\bar{q}_{e_n} = 1\}$ or $\{\bar{p}_{e_n} = 1\}$, $\{\bar{q}_{e_n} = 3 + \frac{1}{n}\}$, it is clear that pairs (ϕ, α) and (η, β) exhibits $(CLR_{\alpha\beta})$ property.

$$\lim_{n \rightarrow \infty} \phi \bar{p}_{e_n} = \lim_{n \rightarrow \infty} \alpha \bar{p}_{e_n} = \lim_{n \rightarrow \infty} \eta \bar{q}_{e_n} = \lim_{n \rightarrow \infty} \beta \bar{q}_{e_n} = 1 \in \alpha(\bar{X}) \cap \beta(\bar{X}).$$

It is also clear that $\phi(\bar{X}) \subset \beta(\bar{X})$ and $\eta(\bar{X}) \subset \alpha(\bar{X})$. Thus, Theorem 4 is applicable and hence (ϕ, α) and (η, β) have 1 as their unique common fixed point, which is also a point of coincidence.

Theorem 5 cannot be applied in the above example, as $\alpha(\bar{X}) = [1, 4) \cup \{5\}$ and $\beta(\bar{X}) = [1, 13)$ are not closed subsets of $\bar{X}$.

Example 5. If maps α, β in Example 4 are replaced by the following maps, keeping rest the same:

$$\begin{aligned} \alpha(\bar{p}) &= \begin{cases} 1 & \text{if } \bar{p} = 1; \\ 4 & \text{if } \bar{p} \in (1, 3]; \\ \frac{\bar{p}+1}{4} & \text{if } \bar{p} \in (3, 15). \end{cases} & \beta(\bar{p}) &= \begin{cases} 1 & \text{if } \bar{p} = 1; \\ 13 & \text{if } \bar{p} \in (1, 3); \\ \bar{p} - 2 & \text{if } \bar{p} \in [3, 15). \end{cases} \end{aligned}$$

Now, $\phi(\bar{X}) \subset \beta(\bar{X})$ and $\eta(\bar{X}) \subset \alpha(\bar{X})$ also $\alpha(\bar{X}) = [1, 4]$ and $\beta(\bar{X}) = [1, 13]$ are closed subsets of $\bar{X}$. Thus Theorem 5 is applicable here, hence (ϕ, α) and (η, β) have 1 as their unique common fixed point.

Next, we are giving a corollary:

Corollary 1. Consider ϕ and α be self maps on a MIFSMS $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ and suppose that

1. (ϕ, α) satisfies (CLR_α) property.
2. there exist $\varepsilon \in (0, 1)$ so that (4.6) holds for all $\bar{p}, \bar{q} \in \bar{X}$ and $t > 0$,

$$\mathfrak{W}_{M,N}(\phi\bar{p}, \phi\bar{q}, \varepsilon t) \geq_{L^*} \min\{\mathfrak{W}_{M,N}(\alpha\bar{p}, \alpha\bar{q}, t), \mathfrak{W}_{M,N}(\phi\bar{p}, \alpha\bar{p}, t), \mathfrak{W}_{M,N}(\phi\bar{q}, \alpha\bar{q}, t), \mathfrak{W}_{M,N}(\phi\bar{p}, \alpha\bar{q}, t), \mathfrak{W}_{M,N}(\phi\bar{q}, \alpha\bar{p}, t)\}. \quad (4.6)$$

Then the pair (ϕ, α) possess a coincidence point. Moreover, (ϕ, α) possess a unique common fixed point in $\bar{X}$ if they are weakly compatible.

Proof. Take $\phi = \eta$ and $\alpha = \beta$ in Theorem 4, the result will be proved on the similar lines. $\square$

5. APPLICATION

In this section we are going to give an application utilising our new results.

Theorem 6. Consider $(\bar{X}, \mathfrak{W}_{M,N}, \Theta)$ be MIFSMS. Consider $\phi, \eta, \alpha, \beta$ are self maps satisfies the following conditions:

- (i) pairs (ϕ, α) and (η, β) exhibit $(CLR_{\alpha\beta})$ property.
- (ii) for $\bar{p}, \bar{q} \in \bar{X}$, $\rho > 0$,

$$\int_0^{s_1} \Upsilon(\rho) d\rho \geq_{L^*} \int_0^{s_2} \Upsilon(\rho) d\rho,$$

where $\Upsilon(\rho)$ is Lebesgue integrable function and

$$\begin{aligned} s_1 &= \mathfrak{W}_{M,N}(\phi\bar{p}, \eta\bar{q}, \varepsilon\rho), \\ s_2 &= \min\{\mathfrak{W}_{M,N}(\phi\bar{p}, \beta\bar{q}, \rho), \mathfrak{W}_{M,N}(\eta\bar{q}, \alpha\bar{p}, \rho), \\ &\quad \mathfrak{W}_{M,N}(\phi\bar{p}, \alpha\bar{p}, \rho), \mathfrak{W}_{M,N}(\eta\bar{q}, \beta\bar{q}, \rho), \\ &\quad \mathfrak{W}_{M,N}(\alpha\bar{p}, \beta\bar{q}, \rho)\}. \end{aligned}$$

Then, pairs (ϕ, α) and (η, β) possess a coincidence point. Moreover, maps ϕ, α, η and β possess a unique common fixed point if (ϕ, α) and (η, β) are weakly compatible.

Proof. Since, (ϕ, α) and (η, β) satisfies $(CLR_{\alpha\beta})$ property, thus equation (4.4) of Theorem 4 is satisfied. From condition (ii) we have

$$\int_0^{\mathfrak{W}_{M,N}(\phi\bar{p}_{e_n}, \eta\bar{h}, \rho)} \Upsilon(\rho) d\rho \geq_{L^*} \int_0^{s_2} \Upsilon(\rho) d\rho,$$

where

$$s_2 = \min\{\mathfrak{W}_{M,N}(\phi\bar{p}_{e_n}, \beta\bar{h}, \rho), \mathfrak{W}_{M,N}(\eta\bar{h}, \alpha\bar{p}_{e_n}, \rho), \\ \mathfrak{W}_{M,N}(\phi\bar{p}_{e_n}, \alpha\bar{p}_{e_n}, \rho), \mathfrak{W}_{M,N}(\eta\bar{h}, \beta\bar{h}, \rho), \\ \mathfrak{W}_{M,N}(\alpha\bar{p}_{e_n}, \beta\bar{h}, \rho)\}.$$

Taking $n \rightarrow \infty$, we have

$$\eta\bar{h} = \beta\bar{h}. \quad (5.1)$$

From condition (ii) and taking $n \rightarrow \infty$, we have

$$\int_0^{\mathfrak{W}_{M,N}(\phi\bar{h}, \eta\bar{q}_{e_n}, \rho)} \Upsilon(\rho) d\rho \geq_{L^*} \int_0^{s_2} \Upsilon(\rho) d\rho,$$

where

$$s_2 = \min\{\mathfrak{W}_{M,N}(\phi\bar{h}, \beta\bar{q}_{e_n}, \rho), \mathfrak{W}_{M,N}(\eta\bar{q}_{e_n}, \alpha\bar{h}, \rho), \\ \mathfrak{W}_{M,N}(\phi\bar{h}, \alpha\bar{h}, \rho), \mathfrak{W}_{M,N}(\eta\bar{q}_{e_n}, \beta\bar{q}_{e_n}, \rho), \\ \mathfrak{W}_{M,N}(\alpha\bar{h}, \beta\bar{q}_{e_n}, \rho)\}.$$

Thus, we have

$$\phi\bar{h} = \alpha\bar{h}. \quad (5.2)$$

From equations (5.1) and (5.2), we have

$$\phi\bar{h} = \alpha\bar{h} = \eta\bar{h} = \beta\bar{h}. \quad (5.3)$$

Thus pairs (ϕ, α) and (η, β) possess a common coincidence point.

Now, let us consider

$$\phi\bar{h} = \alpha\bar{h} = \eta\bar{h} = \beta\bar{h} = \bar{u}. \quad (5.4)$$

The weak compatibility of pairs (ϕ, α) and (η, β) implies

$$\phi\bar{u} = \alpha\bar{u}, \eta\bar{u} = \beta\bar{u}. \quad (5.5)$$

By condition (ii) and equations (5.3), (5.4) and (5.5), we get

$$\int_0^{\mathfrak{W}_{M,N}(\phi\bar{u}, \eta\bar{h}, \rho)} \Upsilon(\rho) d\rho \geq_{L^*} \int_0^{s_2} \Upsilon(\rho) d\rho,$$

where

$$s_2 = \min\{\mathfrak{W}_{M,N}(\phi\bar{u}, \beta\bar{h}, \rho), \mathfrak{W}_{M,N}(\eta\bar{h}, \alpha\bar{u}, \rho), \\ \mathfrak{W}_{M,N}(\phi\bar{u}, \alpha\bar{u}, \rho), \mathfrak{W}_{M,N}(\eta\bar{h}, \beta\bar{h}, \rho), \\ \mathfrak{W}_{M,N}(\alpha\bar{u}, \beta\bar{h}, \rho)\}.$$

So, we have

$$\mathfrak{W}_{M,N}(\phi\bar{u}, \bar{u}, \rho) >_{L^*} \mathfrak{W}_{M,N}(\phi\bar{u}, \bar{u}, \rho).$$

Hence, we obtain

$$\bar{u} = \phi\bar{u} = \alpha\bar{u}.$$

Using condition (ii), we have

$$\int_0^{\mathfrak{W}_{M,N}(\phi\bar{h}, \eta\bar{u}, \rho)} \Upsilon(\rho) d\rho \geq_{L^*} \int_0^{s_2} \Upsilon(\rho) d\rho,$$

where

$$s_2 = \min\{\mathfrak{W}_{M,N}(\phi\bar{h}, \beta\bar{u}, \rho), \mathfrak{W}_{M,N}(\eta\bar{u}, \alpha\bar{h}, \rho), \\ \mathfrak{W}_{M,N}(\phi\bar{h}, \alpha\bar{h}, \rho), \mathfrak{W}_{M,N}(\eta\bar{u}, \beta\bar{u}, \rho), \\ \mathfrak{W}_{M,N}(\alpha\bar{h}, \beta\bar{u}, \rho)\}.$$

Thus, we get

$$\bar{u} = \phi\bar{u} = \alpha\bar{u} = \eta\bar{u} = \beta\bar{u}.$$

Hence, ϕ, α, η and β possess $\bar{u}$ as their common fixed point. The uniqueness of fixed point is easy to prove by using condition (ii). $\square$

Following example validates the Theorem 6.

Example 6. Replacing maps α and β in example (3) by

$$\alpha(\bar{t}) = \begin{cases} 3 & \text{for } \bar{t} = 3, \\ 3 + \bar{t} & \text{for } 3 < \bar{t} \leq 11, \\ \frac{\bar{t}+1}{4} & \text{for } 11 < \bar{t} < 27. \end{cases} \quad \beta(\bar{t}) = \begin{cases} 3 & \text{for } \bar{t} = 3, \\ 11 + \bar{t} & \text{for } 3 < \bar{t} \leq 11, \\ \bar{t} - 8 & \text{for } 11 < \bar{t} < 27. \end{cases}$$

Then, $\phi(\bar{X}) = \{3, 21\} \subset [3, 22] = \beta(\bar{X})$ and $\eta(\bar{X}) = \{3, 8\} \subset [3, 14] = \alpha(\bar{X})$.

Therefore, every assertion of Theorem 6 is satisfied with $\Upsilon(\rho) = 1$. Thus, maps ϕ, α, η and β possess 3 as their common invariant point.

6. CONCLUSION

We have defined self maps satisfying $(CLR_{\alpha\beta})$ property in Modified Intuitionistic Fuzzy Soft Metric Space in this paper. Results of Saadati et al. [14] and Imdad et al. [11] have been generalized to MIFSMS. Examples have been given to prove the applicability of our new results. An application to integral type contraction have also been proved.

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SOLVABILITY OF A THIRD-ORDER SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS VIA A GENERALIZED FIBONACCI SEQUENCE

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Abstract. In this paper, we solve in closed-form the following third-order system of nonlinear difference equations

$$x_{n+1} = \frac{y_n y_{n-1} x_{n-1}^p}{x_n (a_n y_{n-2}^q + b_n y_n y_{n-1})}, \quad y_{n+1} = \frac{x_n x_{n-1} y_{n-1}^q}{y_n (c_n x_{n-2}^p + d_n x_n x_{n-1})}, \quad p, q \in \mathbb{N}, n \in \mathbb{N}_0$$

where the initial values $x_{-i}, y_{-i}, i = 0, 1, 2$ and the parameters $(a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0}, (c_n)_{n \in \mathbb{N}_0}, (d_n)_{n \in \mathbb{N}_0}$ are non-zero real numbers. The form of the solutions of the one dimensional case of our system and a more general system defined by one to one functions are also presented.

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Keywords: systems of difference equations, closed form of the solutions, generalized Fibonacci sequences

1. INTRODUCTION

Difference equations represent an important topic from discrete mathematics nowadays, which generally describes the discrete quantities in phenomena in different disciplines, such as biology and computer sciences. Recently, there has been a noticeable development in the study of nonlinear difference equations and their systems, and within this area, an essential part ought to be reserved for the investigations of the closed-form formulas of solutions, although it is often difficult to achieve. We refer the interested reader to [1–7, 9–16, 18, 19]. In this context, as a generalization of the system

$$x_{n+1} = \frac{x_{n-1} y_n}{y_n \pm y_{n-2}}, \quad y_{n+1} = \frac{y_{n-1} x_n}{x_n \pm x_{n-2}}, \quad n \in \mathbb{N}_0. \quad (1.1)$$

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studied in [17], the authors of [8] considered the following system of difference equations

$$x_{n+1} = \frac{x_{n-k+1}^p y_n}{a y_{n-k}^p + b y_n}, \quad y_{n+1} = \frac{y_{n-k+1}^p x_n}{\alpha x_{n-k} + \beta x_n}, \quad n, p \in \mathbb{N}_0, k \in \mathbb{N}, \quad (1.2)$$

where the coefficients a, b, α, β and the initial values $x_{-i}, y_{-i}, i \in 0, 1, \dots, k$ are real numbers.

Our goal in this paper is to find the solution form of the following system of difference equations

$$x_{n+1} = \frac{y_n y_{n-1} x_{n-1}^p}{x_n (a_n y_{n-2}^q + b_n y_n y_{n-1})}, \quad y_{n+1} = \frac{x_n x_{n-1} y_{n-1}^q}{y_n (c_n x_{n-2}^p + d_n x_n x_{n-1})}, \quad n \in \mathbb{N}_0, p, q \in \mathbb{N} \quad (1.3)$$

where the parameters $(a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0}, (c_n)_{n \in \mathbb{N}_0}, (d_n)_{n \in \mathbb{N}_0}$ and the initial values $x_{-i}, y_{-i}, i = 0, 1, 2$, are non-zero real numbers. We will also determine the form of the solution of the following system which is a generalization of System (1.3).

$$\begin{aligned} x_{n+1} &= f^{-1} \left(\frac{g(y_n) g(y_{n-1}) (f(x_{n-1}))^p}{f(x_n) [a_n (g(y_{n-2}))^q + b_n g(y_n) g(y_{n-1})]} \right), \\ y_{n+1} &= g^{-1} \left(\frac{f(x_n) f(x_{n-1}) (g(y_{n-1}))^q}{g(y_n) [c_n (f(x_{n-2}))^p + d_n f(x_n) f(x_{n-1})]} \right), \quad n \in \mathbb{N}_0, p, q \in \mathbb{N}, \end{aligned} \quad (1.4)$$

where $f, g : D \rightarrow \mathbb{R}$ are one to one continuous functions on $D \subseteq \mathbb{R}$, the initial values $x_{-i}, y_{-i}, i = 0, 1, 2$, are real numbers in D and the parameters $(a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0}, (c_n)_{n \in \mathbb{N}_0}, (d_n)_{n \in \mathbb{N}_0}$ are non-zero real numbers.

Let us recall the following basic and well known lemma.

Lemma 1. *Let $(a_n)_{n \in \mathbb{N}_0}$ and $(b_n)_{n \in \mathbb{N}_0}$ be two sequences of real numbers. Consider the linear difference equation*

$$y_{n+k} = a_n y_n + b_n, \quad k = 2, 3, \quad n \in \mathbb{N}_0.$$

Then,

$$y_{kn+i} = \left[\prod_{j=0}^{n-1} a_{kj+j} \right] y_i + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} a_{kj+j} \right] b_{kr+i}, \quad \text{for } i = 0, 1, \dots, k-1.$$

Moreover, if $(a_n)_{n \in \mathbb{N}_0}$ and $(b_n)_{n \in \mathbb{N}_0}$ are constants (i.e. $a_n = a, b_n = b$), then

$$y_{kn+i} = \begin{cases} y_i + bn, & a = 1, \\ a^n y_i + \left(\frac{a^n - 1}{a - 1} \right) b, & \text{otherwise,} \end{cases} \quad \text{for } i = 0, 1, \dots, k-1, \quad n \in \mathbb{N}_0.$$

where, as usual, $\prod_{j=i}^k A_j = 1$ and $\sum_{j=i}^k A_j = 0$, for all $k < i$.

We will write the formulas of the solutions of our System (1.3) using the terms of the sequence $\{F_n\}_{n=0}^\infty$ defined by the second-order linear difference equation

$$F_{n+2} = F_{n+1} + rF_n, \quad r = p, q, \quad F_0 = F_1 = 1, \quad n \in \mathbb{N}_0. \quad (1.5)$$

This is one of the generalizations of the Fibonacci sequence and it is called the r -Fibonacci sequence. In the following, we give the first thirteen terms of it.

$$\begin{aligned} F_0 &= 1, \\ F_1 &= 1, \\ F_2 &= 1 + r, \\ F_3 &= 1 + 2r, \\ F_4 &= 1 + 3r + r^2, \\ F_5 &= 1 + 4r + 3r^2, \\ F_6 &= 1 + 5r + 6r^2 + r^3, \\ F_7 &= 1 + 6r + 10r^2 + 4r^3, \\ F_8 &= 1 + 7r + 15r^2 + 10r^3 + r^4, \\ F_9 &= 1 + 8r + 21r^2 + 20r^3 + 5r^4, \\ F_{10} &= 1 + 9r + 28r^2 + 35r^3 + 15r^4 + r^5, \\ F_{11} &= 1 + 10r + 36r^2 + 56r^3 + 35r^4 + 6r^5, \\ F_{12} &= 1 + 11r + 45r^2 + 84r^3 + 70r^4 + 21r^5 + r^6. \end{aligned}$$

2. THE SOLUTIONS OF SYSTEM (1.3)

In this part, we show the solvability of our System (1.3). In fact we will give the closed form of the well-defined solutions of our system.

Definition 1. A solution $\{x_n, y_n\}_{n \geq -2}$ of System (1.3) is said to be well-defined if

$$x_n (a_n y_{n-2}^q + b_n y_n y_{n-1}) y_n (c_n x_{n-2}^p + d_n x_n x_{n-1}) \neq 0, \quad n \in \mathbb{N}_0.$$

Remark 1. We want to note that the sequence $\{F_n\}_{n=0}^\infty$ will be defined by

$$F_{n+2} = F_{n+1} + pF_n, \quad F_0 = F_1 = 1, \quad n \in \mathbb{N}_0$$

for the formulas of the x_n -component of the solutions and

$$F_{n+2} = F_{n+1} + qF_n, \quad F_0 = F_1 = 1, \quad n \in \mathbb{N}_0$$

for the formulas of the y_n -component of the solutions.

Now, we will start the resolution of our system. Let $\{x_n, y_n\}_{n \geq -2}$ be a well-defined solution of System (1.3). We have

$$\begin{aligned} x_{n+1} &= \frac{y_n y_{n-1} x_{n-1}^p}{x_n (a_n y_{n-2}^q + b_n y_n y_{n-1})}, & y_{n+1} &= \frac{x_n x_{n-1} y_{n-1}^q}{y_n (c_n x_{n-2}^p + d_n x_n x_{n-1})}, \\ \frac{x_{n+1} x_n}{x_{n-1}^p} &= \frac{y_n y_{n-1}}{a_n y_{n-2}^q + b_n y_n y_{n-1}}, & \frac{y_{n+1} y_n}{y_{n-1}^q} &= \frac{x_n x_{n-1}}{c_n x_{n-2}^p + d_n x_n x_{n-1}}, \\ \frac{x_{n-1}^p}{x_{n+1} x_n} &= \frac{a_n y_{n-2}^q + b_n y_n y_{n-1}}{y_n y_{n-1}}, & \frac{y_{n-1}^q}{y_{n+1} y_n} &= \frac{c_n x_{n-2}^p + d_n x_n x_{n-1}}{x_n x_{n-1}}, \\ \frac{x_{n-1}^p}{x_{n+1} x_n} &= a_n \frac{y_{n-2}^q}{y_n y_{n-1}} + b_n, & \frac{y_{n-1}^q}{y_{n+1} y_n} &= c_n \frac{x_{n-2}^p}{x_n x_{n-1}} + d_n. \end{aligned}$$

Taking the change of variables

$$u_n = \frac{x_{n-2}^p}{x_n x_{n-1}}, \quad v_n = \frac{y_{n-2}^q}{y_n y_{n-1}}, \quad (2.1)$$

System (1.3) can be written as

$$u_{n+1} = a_n v_n + b_n, \quad v_{n+1} = c_n u_n + d_n, \quad n \in \mathbb{N}_0. \quad (2.2)$$

Hence, we have

$$\begin{aligned} u_{n+2} &= a_{n+1} v_{n+1} + b_{n+1} = a_{n+1} [c_n u_n + d_n] + b_{n+1} = a_{n+1} c_n u_n + (a_{n+1} d_n + b_{n+1}), \\ v_{n+2} &= c_{n+1} u_{n+1} + d_{n+1} = c_{n+1} [a_n v_n + b_n] + d_{n+1} = c_{n+1} a_n v_n + (c_{n+1} b_n + d_{n+1}). \end{aligned}$$

From this, we get, for all $n \in \mathbb{N}_0$, the following linear second order nonhomogeneous difference equations,

$$\begin{cases} u_{n+2} = a_{n+1} c_n u_n + (a_{n+1} d_n + b_{n+1}), \\ v_{n+2} = c_{n+1} a_n v_n + (c_{n+1} b_n + d_{n+1}). \end{cases} \quad (2.3)$$

From Lemma 1, we have for all $n \in \mathbb{N}_0$ and for $i = 0, 1$, the solutions of equations in (2.3) are

$$u_{2n+i} = \left[\prod_{j=0}^{n-1} a_{2j+i+1} c_{2j+i} \right] u_i + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} a_{2j+i+1} c_{2j+i} \right] (a_{2r+i+1} d_{2r+i} + b_{2r+i+1}), \quad (2.4)$$

$$v_{2n+i} = \left[\prod_{j=0}^{n-1} c_{2j+i+1} a_{2j+i} \right] v_i + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} c_{2j+i+1} a_{2j+i} \right] (c_{2r+i+1} b_{2r+i} + d_{2r+i+1}). \quad (2.5)$$

From (2.1) and Equations (2.4) and (2.5), it follows that for all $n \in \mathbb{N}_0$

$$u_{2n} = \left[\prod_{j=0}^{n-1} a_{2j+1} c_{2j} \right] \frac{x_{-2}^p}{x_0 x_{-1}} + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} a_{2j+1} c_{2j} \right] (a_{2r+1} d_{2r} + b_{2r+1}), \quad (2.6)$$

$$u_{2n+1} = \frac{\left[\prod_{j=0}^{n-1} a_{2j+2} c_{2j+1} \right] [a_0 y_{-2}^q + b_0 y_0 y_{-1}]}{y_0 y_{-1}} + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} a_{2j+2} c_{2j+1} \right] (a_{2r+2} d_{2r+1} + b_{2r+2}) \quad (2.7)$$

$$v_{2n} = \frac{\left[\prod_{j=0}^{n-1} c_{2j+1} a_{2j} \right] y_{-2}^q}{y_0 y_{-1}} + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} c_{2j+1} a_{2j} \right] (c_{2r+1} b_{2r} + d_{2r+1}), \quad (2.8)$$

and

$$v_{2n+1} = \frac{\left[\prod_{j=0}^{n-1} c_{2j+2} a_{2j+1} \right] [c_0 x_{-2}^p + d_0 x_0 x_{-1}]}{x_0 x_{-1}} + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} c_{2j+2} a_{2j+1} \right] (c_{2r+2} b_{2r+1} + d_{2r+2}). \quad (2.9)$$

Now we give the solution form of Equations (2.3) when all the coefficients in System (1.3) are constant. To do this, we suppose that $a_n = a, b_n = b, c_n = c$ and $d_n = d$, for every $n \in \mathbb{N}_0$. Then Equations (2.3) becomes

$$\begin{cases} u_{n+2} = acu_n + ad + b, \\ v_{n+2} = cav_n + cb + d. \end{cases} \quad (2.10)$$

From Lemma 1, we have for all $n \in \mathbb{N}_0$ and for $i = 0, 1$, the solutions of equations in (2.10) are

$$u_{2n+i} = \begin{cases} u_i + (ad + b)n, & ac = 1, \\ (ac)^n u_i + \left(\frac{(ac)^n - 1}{ac - 1} \right) (ad + b), & \text{otherwise}, \end{cases} \quad (2.11)$$

and

$$v_{2n+i} = \begin{cases} v_i + (csb + ct + d)n, & ac = 1, \\ (ca)^n v_i + \left(\frac{(ca)^n - 1}{ca - 1} \right) (cb + d), & \text{otherwise}. \end{cases} \quad (2.12)$$

From (2.1) and Equations (2.11) and (2.12), it follows that for all $n \in \mathbb{N}_0$

$$u_{2n} = \begin{cases} \frac{x_{-2}^p}{x_0 x_{-1}} + (ad + b)n, & ac = 1, \\ \frac{(ac)^n x_{-2}^p}{x_0 x_{-1}} + \left(\frac{(ac)^n - 1}{ac - 1} \right) (ad + b), & \text{otherwise}, \end{cases} \quad (2.13)$$

$$u_{2n+1} = \begin{cases} \frac{ay_{-2}^q + by_0y_{-1}}{y_0y_{-1}} + (ad+b)n, & ac = 1, \\ \frac{(ac)^n(ay_{-2}^q + by_0y_{-1})}{y_0y_{-1}} + \left(\frac{(ac)^n - 1}{ac - 1}\right)(ad+b), & \text{otherwise}, \end{cases} \quad (2.14)$$

$$v_{2n} = \begin{cases} \frac{y_{-2}^q}{y_0y_{-1}} + (cb+d)n, & ac = 1, \\ \frac{(ca)^ny_{-2}^q}{y_0y_{-1}} + \left(\frac{(ca)^n - 1}{ca - 1}\right)(cb+d), & \text{otherwise}, \end{cases} \quad (2.15)$$

$$v_{2n+1} = \begin{cases} \frac{cx_{-2}^p + dx_0x_{-1}}{x_0x_{-1}} + (cb+d)n, & ac = 1, \\ (ca)^n \frac{cx_{-2}^p + dx_0x_{-1}}{x_0x_{-1}} + \left(\frac{(ca)^n - 1}{ca - 1}\right)(cb+d), & \text{otherwise}. \end{cases} \quad (2.16)$$

Now, from (2.1) it follows that

$$x_n = \frac{x_{n-2}^p}{u_n x_{n-1}}, \quad y_n = \frac{y_{n-2}^q}{v_n y_{n-1}}, \quad (2.17)$$

So we have

$$x_0 = \frac{x_{-2}^p}{u_0 x_{-1}},$$

hence

$$x_0 = \frac{x_{-2}^{pF_0}}{u_0^{F_0} x_{-1}^{F_1}}.$$

Moreover,

$$x_1 = \frac{x_{-1}^p}{u_1 x_0} = \frac{u_0 x_{-1} x_{-1}^p}{u_1 x_{-2}^p} = \frac{u_0 x_{-1}^{p+1}}{u_1 x_{-2}^p},$$

so

$$x_1 = \frac{u_0^{F_1} x_{-1}^{F_2}}{u_1^{F_0} x_{-2}^{pF_1}}$$

and

$$x_2 = \frac{x_0^p}{u_2 x_1} = \frac{x_{-2}^{p^2} u_1 x_{-2}^p}{u_0^p x_{-1}^p u_2 u_0 x_{-1}^{p+1}} = \frac{u_1 x_{-2}^{p^2+p}}{u_2 u_0^{p+1} x_{-1}^{2p+1}},$$

thus

$$x_2 = \frac{u_1^{F_1} x_{-2}^{pF_2}}{u_2^{F_0} u_0^{F_2} x_{-1}^{F_3}}.$$

Similarly, we obtain

$$\begin{aligned}
 x_3 &= \frac{u_2 u_0^{2p+1} x_{-1}^{p^2+3p+1}}{u_3 u_1^{p+1} x_{-2}^{2p^2+p}} = \frac{u_2^{F_1} u_0^{F_3} x_{-1}^{F_4}}{u_3^{F_0} u_1^{F_2} x_{-2}^{pF_3}}, \\
 x_4 &= \frac{u_3 u_1^{2p+1} x_{-2}^{p^3+3p^2+p}}{u_4 u_2^{p+1} u_0^{p^2+3p+1} x_{-1}^{3p^2+4p+1}} = \frac{u_3^{F_1} u_1^{F_3} x_{-2}^{pF_4}}{u_4^{F_0} u_2^{F_2} u_0^{F_4} x_{-1}^{F_5}}, \\
 x_5 &= \frac{u_4 u_2^{2p+1} u_0^{3p^2+4p+1} x_{-1}^{p^3+6p^2+5p+1}}{u_5 u_3^{p+1} u_1^{p^2+3p+1} x_{-2}^{3p^3+4p^2+p}} = \frac{u_4^{F_1} u_2^{F_3} u_0^{F_5} x_{-1}^{F_6}}{u_5^{F_0} u_3^{F_2} u_1^{F_4} x_{-2}^{pF_5}}, \\
 x_6 &= \frac{u_5 u_3^{2p+1} u_1^{3p^2+4p+1} x_{-2}^{p^4+6p^3+5p^2+p}}{u_6 u_4^{p+1} u_2^{p^2+3p+1} u_0^{p^3+6p^2+5p+1} x_{-1}^{4p^3+10p^2+6p+1}} = \frac{u_5^{F_1} u_3^{F_3} u_1^{F_5} x_{-2}^{pF_6}}{u_6^{F_0} u_4^{F_2} u_2^{F_4} u_0^{F_6} x_{-1}^{F_7}}, \\
 x_7 &= \frac{u_6^{F_1} u_4^{F_3} u_2^{F_5} u_0^{F_7} x_{-1}^{F_8}}{u_7^{F_0} u_5^{F_2} u_3^{F_4} u_1^{F_6} x_{-2}^{pF_7}} = \frac{\prod_{i=0}^3 u_{2i}^{F_{2(3-i)+1}} x_{-1}^{F_8}}{\prod_{i=0}^3 u_{2i+1}^{F_{2(3-i)}} x_{-2}^{pF_7}}, \\
 x_8 &= \frac{u_7^{F_1} u_5^{F_3} u_3^{F_5} u_1^{F_7} x_{-2}^{pF_8}}{u_8^{F_0} u_6^{F_2} u_4^{F_4} u_2^{F_6} u_0^{F_8} x_{-1}^{F_9}} = \frac{\prod_{i=0}^3 u_{2i+1}^{F_{2(4-i)-1}} x_{-2}^{pF_8}}{\prod_{i=0}^4 u_{2i}^{F_{2(4-i)}} x_{-1}^{F_9}}, \\
 x_9 &= \frac{u_8^{F_1} u_6^{F_3} u_4^{F_5} u_2^{F_7} u_0^{F_9} x_{-1}^{F_{10}}}{u_9^{F_0} u_7^{F_2} u_5^{F_4} u_3^{F_6} u_1^{F_8} x_{-2}^{pF_9}} = \frac{\prod_{i=0}^4 u_{2i}^{F_{2(4-i)+1}} x_{-1}^{F_{10}}}{\prod_{i=0}^4 u_{2i+1}^{F_{2(4-i)}} x_{-2}^{pF_9}}, \\
 x_{10} &= \frac{u_9^{F_1} u_7^{F_3} u_5^{F_5} u_3^{F_7} u_1^{F_9} x_{-2}^{pF_{10}}}{u_{10}^{F_0} u_8^{F_2} u_6^{F_4} u_4^{F_6} u_2^{F_8} u_0^{F_{10}} x_{-1}^{F_{11}}} = \frac{\prod_{i=0}^4 u_{2i+1}^{F_{2(5-i)-1}} x_{-2}^{pF_{10}}}{\prod_{i=0}^5 u_{2i}^{F_{2(5-i)}} x_{-1}^{F_{11}}}.
 \end{aligned}$$

By induction, it follows that

$$x_{2n} = \frac{\prod_{i=0}^{n-1} u_{2i+1}^{F_{2(n-i)-1}} x_{-2}^{pF_{2n}}}{\prod_{i=0}^n u_{2i}^{F_{2(n-i)}} x_{-1}^{F_{2n+1}}}, \quad x_{2n+1} = \frac{\prod_{i=0}^n u_{2i}^{F_{2(n-i)+1}} x_{-1}^{F_{2(n+1)}}}{\prod_{i=0}^n u_{2i+1}^{F_{2(n-i)}} x_{-2}^{pF_{2n+1}}}.$$

Similarly we have

$$y_0 = \frac{y_{-2}^q}{v_0 y_{-1}},$$

hence

$$y_0 = \frac{y_{-2}^{qF_0}}{v_0^{F_0} y_{-1}^{F_1}}.$$

Moreover,

$$y_1 = \frac{y_{-1}^q}{v_1 y_0} = \frac{v_0 y_{-1} y_{-1}^q}{v_1 y_{-2}^q} = \frac{v_0 y_{-1}^{q+1}}{v_1 y_{-2}^q},$$

so

$$y_1 = \frac{v_0^{F_1} y_{-1}^{F_2}}{v_1^{F_0} y_{-2}^{qF_1}}$$

and

$$y_2 = \frac{y_0^q}{v_2 y_1} = \frac{y_{-2}^{q^2} v_1 y_{-2}^q}{v_0^q y_{-1}^q v_2 v_0 y_{-1}^{q+1}} = \frac{v_1 y_{-2}^{q^2+q}}{v_2 v_0^{q+1} y_{-1}^{2q+1}},$$

thus

$$y_2 = \frac{v_1^{F_1} y_{-2}^{qF_2}}{v_2^{F_0} v_0^{F_2} y_{-1}^{F_3}}.$$

Similarly, we obtain

$$\begin{aligned} y_3 &= \frac{v_2 v_0^{2q+1} y_{-1}^{q^2+3q+1}}{v_3 v_1^{q+1} y_{-2}^{2q^2+q}} = \frac{v_2^{F_1} v_0^{F_3} y_{-1}^{F_4}}{v_3^{F_0} v_1^{F_2} y_{-2}^{qF_3}}, \\ y_4 &= \frac{v_3 v_1^{2q+1} y_{-2}^{q^3+3q^2+q}}{v_4 v_2^{q+1} v_0^{q^2+3q+1} y_{-1}^{3q^2+4q+1}} = \frac{v_3^{F_1} v_1^{F_3} y_{-2}^{qF_4}}{v_4^{F_0} v_2^{F_2} v_0^{F_4} y_{-1}^{F_5}}, \\ y_5 &= \frac{v_4 v_2^{2q+1} v_0^{3q^2+4q+1} y_{-1}^{q^3+6q^2+5q+1}}{v_5 v_3^{q+1} v_1^{q^2+3q+1} y_{-2}^{3q^3+4q^2+q}} = \frac{v_4^{F_1} v_2^{F_3} v_0^{F_5} y_{-1}^{F_6}}{v_5^{F_0} v_3^{F_2} v_1^{F_4} y_{-2}^{qF_5}}, \\ y_6 &= \frac{v_5 v_3^{2q+1} v_1^{3q^2+4q+1} y_{-2}^{q^4+6q^3+5q^2+q}}{v_6 v_4^{q+1} v_2^{q^2+3q+1} v_0^{q^3+6q^2+5q+1} y_{-1}^{4q^3+10q^2+6q+1}} = \frac{v_5^{F_1} v_3^{F_3} v_1^{F_5} y_{-2}^{qF_6}}{v_6^{F_0} v_4^{F_2} v_2^{F_4} v_0^{F_6} y_{-1}^{F_7}}, \\ y_7 &= \frac{v_6^{F_1} v_4^{F_3} v_2^{F_5} v_0^{F_7} y_{-1}^{F_8}}{v_7^{F_0} v_5^{F_2} v_3^{F_4} v_1^{F_6} y_{-2}^{qF_7}} = \frac{\prod_{i=0}^3 v_{2i}^{F_{2(3-i)+1}} y_{-1}^{F_8}}{\prod_{i=0}^3 v_{2i+1}^{F_{2(3-i)}} y_{-2}^{qF_7}}, \end{aligned}$$

$$y_8 = \frac{v_7^{F_1} v_5^{F_3} v_3^{F_5} v_1^{F_7} y_{-2}^{pF_8}}{v_8^{F_0} v_6^{F_2} v_4^{F_4} v_2^{F_6} v_0^{F_8} y_{-1}^{F_9}} = \frac{\prod_{i=0}^3 v_{2i+1}^{F_{2(4-i)-1}} y_{-2}^{qF_8}}{\prod_{i=0}^4 v_{2i}^{F_{2(4-i)}} y_{-1}^{F_9}},$$

$$y_9 = \frac{v_8^{F_1} v_6^{F_3} v_4^{F_5} v_2^{F_7} v_0^{F_9} y_{-1}^{F_{10}}}{v_9^{F_0} v_7^{F_2} v_5^{F_4} v_3^{F_6} v_1^{F_8} y_{-2}^{qF_9}} = \frac{\prod_{i=0}^4 v_{2i}^{F_{2(4-i)+1}} y_{-1}^{F_{10}}}{\prod_{i=0}^4 v_{2i+1}^{F_{2(4-i)}} y_{-2}^{qF_9}},$$

$$y_{10} = \frac{v_9^{F_1} v_7^{F_3} v_5^{F_5} v_3^{F_7} v_1^{F_9} y_{-2}^{qF_{10}}}{v_{10}^{F_0} v_8^{F_2} v_6^{F_4} v_4^{F_6} v_2^{F_8} v_0^{F_{10}} y_{-1}^{F_{11}}} = \frac{\prod_{i=0}^4 v_{2i+1}^{F_{2(5-i)-1}} y_{-2}^{qF_{10}}}{\prod_{i=0}^5 v_{2i}^{F_{2(5-i)}} y_{-1}^{F_{11}}}.$$

By induction, it follows that

$$y_{2n} = \frac{\prod_{i=0}^{n-1} v_{2i+1}^{F_{2(n-i)-1}} y_{-2}^{qF_{2n}}}{\prod_{i=0}^n v_{2i}^{F_{2(n-i)}} y_{-1}^{F_{2n+1}}}, \quad y_{2n+1} = \frac{\prod_{i=0}^n v_{2i}^{F_{2(n-i)+1}} y_{-1}^{F_{2(n+1)}}}{\prod_{i=0}^n v_{2i+1}^{F_{2(n-i)}} y_{-2}^{qF_{2n+1}}}.$$

From the above calculations, we summarize in the following theorem the form of the solutions of System (1.3).

Theorem 1. *Let $\{x_n, y_n\}_{n \geq -2}$ be a well-defined solution of System (1.3). Then, for all $n \in \mathbb{N}_0$, we have*

$$x_{2n} = \left(\frac{\prod_{i=0}^{n-1} u_{2i+1}^{F_{2(n-i)-1}}}{\prod_{i=0}^n u_{2i}^{F_{2(n-i)}}} \right) \frac{x_{-2}^{pF_{2n}}}{x_{-1}^{F_{2n+1}}}, \quad x_{2n+1} = \left(\frac{\prod_{i=0}^n u_{2i}^{F_{2(n-i)+1}}}{\prod_{i=0}^n u_{2i+1}^{F_{2(n-i)}}} \right) \frac{x_{-1}^{F_{2(n+1)}}}{x_{-2}^{pF_{2n+1}}},$$

$$y_{2n} = \left(\frac{\prod_{i=0}^{n-1} v_{2i+1}^{F_{2(n-i)-1}}}{\prod_{i=0}^n v_{2i}^{F_{2(n-i)}}} \right) \frac{y_{-2}^{qF_{2n}}}{y_{-1}^{F_{2n+1}}} \quad \text{and} \quad y_{2n+1} = \left(\frac{\prod_{i=0}^n v_{2i}^{F_{2(n-i)+1}}}{\prod_{i=0}^n v_{2i+1}^{F_{2(n-i)}}} \right) \frac{y_{-1}^{F_{2(n+1)}}}{y_{-2}^{qF_{2n+1}}},$$

where the terms of the sequences $(u_n)_{n \in \mathbb{N}_0}$ and $(v_n)_{n \in \mathbb{N}_0}$ are given by formulas (2.6) - (2.9) in the case of variable coefficients and (2.13) - (2.16) in the case of constant coefficients.

Remark 2. If we take $c_n = a_n, d_n = b_n, n \in \mathbb{N}_0$ $q = p$ and $y_{-i} = x_{-i}, i = 0, 1, 2$, then, we obtain the one dimensional version of System (1.3), that is the equation

$$x_{n+1} = \frac{x_{n-1}^{p+1}}{a_n x_{n-2}^p + b_n x_n x_{n-1}}, \quad p \in \mathbb{N}, n \in \mathbb{N}_0. \quad (2.18)$$

As a consequence the solutions of Equations (2.18) can be obtained from Theorem 1, and their formulas are given in the the following result.

Corollary 1. *Let $\{x_n\}_{n \geq -2}$ be a well-defined solution of Equation (2.18), then for $n \in \mathbb{N}_0$ we have*

$$x_{2n} = \left(\frac{\prod_{i=0}^{n-1} u_{2i+1}^{F_{2(n-i)-1}}}{\prod_{i=0}^n u_{2i}^{F_{2(n-i)}}} \right) \frac{x_{-2}^{pF_{2n}}}{x_{-1}^{F_{2n+1}}}, \quad x_{2n+1} = \left(\frac{\prod_{i=0}^n u_{2i}^{F_{2(n-i)+1}}}{\prod_{i=0}^n u_{2i+1}^{F_{2(n-i)}}} \right) \frac{x_{-1}^{F_{2(n+1)}}}{x_{-2}^{pF_{2n+1}}},$$

the terms of the sequence $(u_n)_{n \in \mathbb{N}_0}$ are given by formulas (2.6), (2.7) in the case of variables coefficients and (2.13), (2.14) in the case of constant coefficients.

3. THE FORM OF THE SOLUTIONS OF A MORE GENERAL SYSTEM DEFINED BY ONE TO ONE FUNCTIONS

In this part, we will show the solvability of the following system

$$\begin{aligned} x_{n+1} &= f^{-1} \left(\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})]} \right), \\ y_{n+1} &= g^{-1} \left(\frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)[c_n(f(x_{n-2}))^p + d_n f(x_n)f(x_{n-1})]} \right), \quad n \in \mathbb{N}_0, p, q \in \mathbb{N}, \end{aligned} \quad (3.1)$$

where $f, g : D \rightarrow \mathbb{R}$ are one to one continuous functions on $D \subseteq \mathbb{R}$, the initial values $x_{-i}, y_{-i}, i = 0, 1, 2$, are real numbers in D and the parameters $(a_n)_{n \in \mathbb{N}_0}, (b_n)_{n \in \mathbb{N}_0}, (c_n)_{n \in \mathbb{N}_0}, (d_n)_{n \in \mathbb{N}_0}$ are non-zero real numbers.

Definition 2. A solution $\{x_n, y_n\}_{n \geq -2}$ of System (3.1) is said to be well-defined if for all $n \in \mathbb{N}_0$, we have

$$\begin{aligned} f(x_n)[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})] &\neq 0, \\ g(y_n)[c_n(f(x_{n-2}))^p + d_n f(x_n)f(x_{n-1})] &\neq 0, \\ \frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})]} &\in D_{f^{-1}} \end{aligned}$$

and

$$\frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)[c_n(f(x_{n-2}))^p + d_n f(x_n)f(x_{n-1})]} \in D_{g^{-1}}.$$

Since f and g are one to one continuous functions, then we get

$$f(x_{n+1}) = \frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)(a_n(g(y_{n-2}))^q + b_ng(y_n)g(y_{n-1}))},$$

$$g(y_{n+1}) = \frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)(c_n(f(x_{n-2}))^p + d_nf(x_n)f(x_{n-1}))}.$$

Taking the change of variables

$$X_n = f(x_n), \quad Y_n = g(y_n), \quad n \in \mathbb{N}_0 \quad (3.2)$$

it follows that System (3.1) can be transformed to the following system

$$X_{n+1} = \frac{Y_n Y_{n-1} X_{n-1}^p}{X_n(a_n Y_{n-2}^q + b_n Y_n Y_{n-1})}, \quad Y_{n+1} = \frac{X_n X_{n-1} Y_{n-1}^q}{Y_n(c_n X_{n-2}^p + d_n X_n X_{n-1})}, \quad n \in \mathbb{N}_0,$$

which is in the form (1.3). So, using (3.2), the fact that

$$x_n = f^{-1}(X_n), \quad y_n = g^{-1}(Y_n), \quad n \in \mathbb{N}_0$$

and Theorem (1), we get the following result which describes the form of the solutions of System (3.1).

Theorem 2. *Let $\{x_n, y_n\}_{n \geq -2}$ be a well-defined solution of System (3.1). Then, for all $n \in \mathbb{N}_0$, we have*

$$x_{2n} = f^{-1} \left(\left(\frac{\prod_{i=0}^{n-1} u_{2i+1}^{F_{2(n-i)-1}}}{\prod_{i=0}^n u_{2i}^{F_{2(n-i)}}} \right) \frac{(f(x_{-2}))^{pF_{2n}}}{(f(x_{-1}))^{F_{2n+1}}} \right),$$

$$x_{2n+1} = f^{-1} \left(\left(\frac{\prod_{i=0}^n u_{2i}^{F_{2(n-i)+1}}}{\prod_{i=0}^n u_{2i+1}^{F_{2(n-i)}}} \right) \frac{(f(x_{-1}))^{F_{2(n+1)}}}{(f(x_{-2}))^{pF_{2n+1}}} \right),$$

$$y_{2n} = g^{-1} \left(\left(\frac{\prod_{i=0}^{n-1} v_{2i+1}^{F_{2(n-i)-1}}}{\prod_{i=0}^n v_{2i}^{F_{2(n-i)}}} \right) \frac{(g(y_{-2}))^{qF_{2n}}}{(g(y_{-1}))^{F_{2n+1}}} \right)$$

and

$$y_{2n+1} = g^{-1} \left(\left(\frac{\prod_{i=0}^n v_{2i}^{F_{2(n-i)+1}}}{\prod_{i=0}^n v_{2i+1}^{F_{2(n-i)}}} \right) \frac{(g(y_{-1}))^{F_{2(n+1)}}}{(g(y_{-2}))^{qF_{2n+1}}} \right),$$

where the terms of the sequences $(u_n)_{n \in \mathbb{N}_0}$ and $(v_n)_{n \in \mathbb{N}_0}$ are given by the following formulas

$$u_{2n} = \frac{\left[\prod_{j=0}^{n-1} a_{2j+1} c_{2j} \right] (f(x_{-2}))^p}{f(x_0)f(x_{-1})} + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} a_{2j+1} c_{2j} \right] (a_{2r+1} d_{2r} + b_{2r+1}), \quad (3.3)$$

$$\begin{aligned} u_{2n+1} = & \frac{\left[\prod_{j=0}^{n-1} a_{2j+2} c_{2j+1} \right] [a_0(g(y_{-2}))^q + b_0 g(y_0)g(y_{-1})]}{g(y_0)g(y_{-1})} \\ & + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} a_{2j+2} c_{2j+1} \right] (a_{2r+2} d_{2r+1} + b_{2r+2}), \end{aligned} \quad (3.4)$$

$$v_{2n} = \frac{\left[\prod_{j=0}^{n-1} c_{2j+1} a_{2j} \right] (g(y_{-2}))^q}{g(y_0)g(y_{-1})} + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} c_{2j+1} a_{2j} \right] (c_{2r+1} b_{2r} + d_{2r+1}), \quad (3.5)$$

$$\begin{aligned} v_{2n+1} = & \frac{\left[\prod_{j=0}^{n-1} c_{2j+2} a_{2j+1} \right] [c_0(f(x_{-2}))^p + d_0 f(x_0)f(x_{-1})]}{f(x_0)f(x_{-1})} \\ & + \sum_{r=0}^{n-1} \left[\prod_{j=r+1}^{n-1} c_{2j+2} a_{2j+1} \right] (c_{2r+2} b_{2r+1} + d_{2r+2}), \end{aligned} \quad (3.6)$$

if the coefficients are variables and by the following formulas

$$u_{2n} = \begin{cases} \frac{(f(x_{-2}))^p}{f(x_0)f(x_{-1})} + (ad + b)n, & ac = 1, \\ \frac{(ac)^n (f(x_{-2}))^p}{f(x_0)f(x_{-1})} + \left(\frac{(ac)^n - 1}{ac - 1} \right) (ad + b), & \text{otherwise,} \end{cases} \quad (3.7)$$

$$u_{2n+1} = \begin{cases} \frac{a(g(y_{-2}))^q + bg(y_0)g(y_{-1})}{g(y_0)g(y_{-1})} + (ad + b)n, & ac = 1, \\ \frac{(ac)^n (a(g(y_{-2}))^q + bg(y_0)g(y_{-1}))}{g(y_0)g(y_{-1})} + \left(\frac{(ac)^n - 1}{ac - 1} \right) (ad + b), & \text{otherwise,} \end{cases} \quad (3.8)$$

$$v_{2n} = \begin{cases} \frac{(g(y_{-2}))^q}{g(y_0)g(y_{-1})} + (cb+d)n, & ac = 1, \\ \frac{(ca)^n(g(y_{-2}))^q}{g(y_0)g(y_{-1})} + \left(\frac{(ca)^n - 1}{ca - 1}\right)(cb+d), & \text{otherwise,} \end{cases} \quad (3.9)$$

$$v_{2n+1} = \begin{cases} \frac{c(f(x_{-2}))^p + df(x_0)f(x_{-1})}{f(x_0)f(x_{-1})} + (cb+d)n, & ac = 1, \\ \frac{(ca)^n(c(f(x_{-2}))^p + df(x_0)f(x_{-1}))}{f(x_0)f(x_{-1})} + \left(\frac{(ca)^n - 1}{ca - 1}\right)(cb+d), & \text{otherwise,} \end{cases} \quad (3.10)$$

if the coefficients are constants, again the sequence $\{F_n\}_{n=0}^\infty$ is defined by

$$F_{n+2} = F_{n+1} + pF_n, F_0 = F_1 = 1, n \in \mathbb{N}_0$$

for the formulas of the x_n -component of the solutions and

$$F_{n+2} = F_{n+1} + qF_n, F_0 = F_1 = 1, n \in \mathbb{N}_0$$

for the formulas of the y_n -component of the solutions.

Remark 3. Clearly if we take the functions f and g such that $f(x) = x$ and $g(x) = x$, then System (3.1) will be nothing other than System (1.3).

4. CONCLUSION

In this paper, a third-order system of nonlinear difference equations was proposed. Using some changes of variables and p -Fibonacci numbers, the solutions of this system were written in closed-form. It is showed also that a more general system defined by one to one functions is also solvable in closed form.

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NOTE ON THE MOMENT GENERATING FUNCTION OF THE MULTIVARIATE NORMAL DISTRIBUTION

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To the memory of Michael Cichon

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Abstract. We present a streamlined proof of a formula for the derivatives of the moment generating function of the multivariate normal distribution. We formulate it in terms of the summation of the contractions by pairings, which encodes a combinatorial computation procedure. We give two applications. First, we provide a simple proof of Isserlis' theorem and derive a formula for the moments of the multivariate normal distribution. Second, we calculate the moments of the product of a finite number of correlated normally and lognormally distributed random variables.

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1. INTRODUCTION

In this note, we present a streamlined proof of a formula for arbitrary derivatives of the moment generating function of the multivariate normal distribution. As applications of the formula, we provide a simple proof of Isserlis' theorem and derive a formula for the moments of the multivariate normal distribution. Then we calculate the moments of the product of a finite number of correlated normally and lognormally distributed random variables.

Throughout this note, we let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of nonnegative integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$ the set of positive real numbers. Also, we let N denote a positive integer.

2. DERIVATIVES OF THE MOMENT GENERATING FUNCTION OF THE MULTIVARIATE NORMAL DISTRIBUTION

2.1. Preliminaries

Let $\mathbf{X} = (X_1, \dots, X_N)^T$ denote an N -dimensional normally distributed random vector with mean vector $\mathbf{m} = (m_i) \in \mathbb{R}^N$ and covariance matrix $C = (c_{ij}) \in \mathbb{R}^{N \times N}$. We have $\mathbb{E}[X_i] = m_i$, $\text{Cov}(X_i, X_j) = c_{ij}$, and in particular $\text{Var}[X_i] = \sigma_i^2 = c_{ii}$. With these notations, the probability density function of $\mathbf{X}$ is given by

$$f_N(\mathbf{x}; \mathbf{m}, C) = (2\pi)^{-\frac{n}{2}} (\det C)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T C^{-1}(\mathbf{x} - \mathbf{m})\right) \quad \text{for } \mathbf{x} \in \mathbb{R}^N,$$

and the moment generating function of $\mathbf{X}$ is given by

$$M_{\mathbf{X}}(\mathbf{t}) = \mathbb{E}[e^{\mathbf{t} \cdot \mathbf{X}}] = \exp\left(\mathbf{m}^T \mathbf{t} + \frac{1}{2} \mathbf{t}^T C \mathbf{t}\right) \quad \text{for } \mathbf{t} = (t_1, \dots, t_N)^T$$

where $\mathbf{t} \cdot \mathbf{X} = \mathbf{t}^T \mathbf{X}$ (see [5, Chapter 45]).

By differentiating $M_{\mathbf{X}}(\mathbf{t})$, we have the following expectation value

$$\mathbb{E}[X_1^{\alpha_1} \dots X_N^{\alpha_N} e^{\mathbf{t} \cdot \mathbf{X}}] = \left(\frac{\partial}{\partial t_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial t_N}\right)^{\alpha_N} M_{\mathbf{X}}(\mathbf{t}).$$

Put

$$Q(\mathbf{t}) = \mathbf{m}^T \mathbf{t} + \frac{1}{2} \mathbf{t}^T C \mathbf{t} = \sum_{i=1}^N m_i t_i + \frac{1}{2} \sum_{i,j=1}^N c_{ij} t_i t_j,$$

so that $M_{\mathbf{X}}(\mathbf{t}) = e^{Q(\mathbf{t})}$. Also put $\boldsymbol{\xi}(\mathbf{t}) = \mathbf{m} + C\mathbf{t}$, namely, $\boldsymbol{\xi}(\mathbf{t}) = (\xi_1(\mathbf{t}), \dots, \xi_N(\mathbf{t}))^T$ where $\xi_k(\mathbf{t}) = m_k + \sum_{j=1}^N c_{kj} t_j$ ($1 \leq k \leq N$). Then, for $k, i \in \{1, \dots, N\}$ we have

$$\frac{\partial Q(\mathbf{t})}{\partial t_k} = \xi_k(\mathbf{t}), \tag{2.1}$$

$$\frac{\partial \xi_k(\mathbf{t})}{\partial t_i} = c_{ki}. \tag{2.2}$$

In other words, $\boldsymbol{\xi}(\mathbf{t})$ is the gradient vector of $Q(\mathbf{t})$ and C is the Hessian matrix of $Q(\mathbf{t})$.

Given any differentiable function $F(\mathbf{t})$, we apply the product rule and Eq.(2.1) to get

$$\frac{\partial}{\partial t_k} \left(e^{Q(\mathbf{t})} F(\mathbf{t}) \right) = e^{Q(\mathbf{t})} \left(\frac{\partial Q(\mathbf{t})}{\partial t_k} F(\mathbf{t}) + \frac{\partial F(\mathbf{t})}{\partial t_k} \right) = e^{Q(\mathbf{t})} \left(\xi_k(\mathbf{t}) F(\mathbf{t}) + \frac{\partial F(\mathbf{t})}{\partial t_k} \right). \tag{2.3}$$

By successively applying Eq.(2.3) and Eq.(2.2), one can calculate the derivatives of $M_{\mathbf{X}}(\mathbf{t}) = e^{Q(\mathbf{t})}$.

Example 1. For brevity, we write $Q = Q(\mathbf{t})$ and $\xi_k = \xi_k(\mathbf{t})$.

$$(i) \quad \mathbb{E}[X_1 e^{\mathbf{t} \cdot \mathbf{X}}] = \frac{\partial}{\partial t_1} M_{\mathbf{X}}(\mathbf{t}) = e^{Q(\mathbf{t})} \xi_1.$$

- (ii) $\mathbb{E}[X_1 X_2 e^{\mathbf{t} \cdot \mathbf{X}}] = \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} M_{\mathbf{X}}(\mathbf{t}) = e^{\mathcal{Q}}(\xi_1 \xi_2 + c_{12})$. By setting $X_2 = X_1$, we have $\mathbb{E}[X_1^2 e^{\mathbf{t} \cdot \mathbf{X}}] = e^{\mathcal{Q}}(\xi_1^2 + \sigma_1^2)$.
- (iii) $\mathbb{E}[X_1 X_2 X_3 e^{\mathbf{t} \cdot \mathbf{X}}] = \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_3} M_{\mathbf{X}}(\mathbf{t}) = e^{\mathcal{Q}}(\xi_1 \xi_2 \xi_3 + c_{12} \xi_3 + c_{13} \xi_2 + c_{23} \xi_1)$. By setting $X_3 = X_1$, we have $\mathbb{E}[X_1^2 X_2 e^{\mathbf{t} \cdot \mathbf{X}}] = e^{\mathcal{Q}}(\xi_1^2 \xi_2 + 2c_{12} \xi_1 + \sigma_1^2 \xi_2)$. Further, by setting $X_2 = X_1$, we have $\mathbb{E}[X_1^3 e^{\mathbf{t} \cdot \mathbf{X}}] = e^{\mathcal{Q}}(\xi_1^3 + 3\sigma_1^2 \xi_1)$.
- (iv) $\mathbb{E}[X_1 X_2 X_3 X_4 e^{\mathbf{t} \cdot \mathbf{X}}] = \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_3} \frac{\partial}{\partial t_4} M_{\mathbf{X}}(\mathbf{t}) = e^{\mathcal{Q}}(\xi_1 \xi_2 \xi_3 \xi_4 + c_{12} \xi_3 \xi_4 + c_{13} \xi_2 \xi_4 + c_{14} \xi_2 \xi_3 + c_{23} \xi_1 \xi_4 + c_{24} \xi_1 \xi_3 + c_{34} \xi_1 \xi_2 + c_{12} c_{34} + c_{13} c_{24} + c_{14} c_{23})$. By setting $X_3 = X_1$ and $X_4 = X_2$, we have $\mathbb{E}[X_1^2 X_2^2 e^{\mathbf{t} \cdot \mathbf{X}}] = e^{\mathcal{Q}}(\xi_1^2 \xi_2^2 + \sigma_1^2 \xi_2^2 + \sigma_2^2 \xi_1^2 + 4c_{12} \xi_1 \xi_2 + 2c_{12}^2 + \sigma_1^2 \sigma_2^2)$. Further, by setting $X_2 = X_1$, we have $\mathbb{E}[X_1^4 e^{\mathbf{t} \cdot \mathbf{X}}] = e^{\mathcal{Q}}(\xi_1^4 + 6\sigma_1^2 \xi_1^2 + 3\sigma_1^4)$.

2.2. The key formula

By generalizing Example 1, we prove a formula for arbitrary derivatives of the moment generating function of the multivariate normal distribution. Our proof is adapted from [1, Lemma 5.2.], but we slightly extend it to treat high-order derivatives and introduce combinatorial notions to describe the computation procedure. As we shall see later, these small improvements turn out to be useful in the applications.

For an N -dimensional multi-index $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, put $|\alpha| = \sum_{i=1}^N \alpha_i$ and $\alpha! = \prod_{i=1}^N \alpha_i!$. For $\mathbf{t} = (t_1, \dots, t_N)^T$, write $\mathbf{t}^\alpha = t_1^{\alpha_1} \dots t_N^{\alpha_N}$ and denote the differential operator of index α with respect to $\mathbf{t}$ by $\partial^\alpha = \left(\frac{\partial}{\partial t_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial t_N}\right)^{\alpha_N}$. For $\alpha = (\alpha_i), \beta = (\beta_i) \in \mathbb{N}^N$, write $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ for all i . If $\beta \leq \alpha$, then $\alpha - \beta = (\alpha_i - \beta_i) \in \mathbb{N}^N$.

For $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, define its corresponding multiset by $V_\alpha = \{(i, j) \mid 1 \leq i \leq N, 1 \leq j \leq \alpha_i\} \subset \{1, \dots, N\} \times \mathbb{N}$. For $1 \leq i \leq N$, write $V_\alpha(i) = \{(i, j) \mid 1 \leq j \leq \alpha_i\} \subset V_\alpha$. Then $\bigcup_{i=1}^N V_\alpha(i) = V_\alpha$. Note that $\#V_\alpha(i) = \alpha_i$ and $\#V_\alpha = |\alpha|$. Conversely, for a finite subset E of $\{1, \dots, N\} \times \mathbb{N}$, define $\mathbf{v}(E) = (\mathbf{v}_1, \dots, \mathbf{v}_N) \in \mathbb{N}^N$ by $\mathbf{v}_i = \#(\pi^{-1}(i) \cap E) \in \mathbb{N}$. Here π denotes the projection from $\{1, \dots, N\} \times \mathbb{N}$ onto $\{1, \dots, N\}$, namely $\pi(i, j) = i$ for $(i, j) \in \{1, \dots, N\} \times \mathbb{N}$. Then we have $\mathbf{v}(V_\alpha) = \alpha$ and $\mathbf{v}(W) \leq \alpha$ for any subset W of V_α .

In general, given a finite set E and an integer $k \geq 0$, we call a k -pairing of E a set of k (unordered) pairs formed by $2k$ distinct elements of E . We denote the set of k -pairings of V_α by $\mathcal{P}_k(\alpha)$, namely

$$\mathcal{P}_k(\alpha) = \left\{ \sigma = \{W_i\}_{i=1}^k \mid W_i\text{'s are distinct 2-element subsets of } V_\alpha \right\}.$$

Note that $\mathcal{P}_k(\alpha)$ is nonempty only if $0 \leq k \leq \lfloor |\alpha|/2 \rfloor$, where the symbol $\lfloor x \rfloor$ denotes the greatest integer which does not exceed $x \in \mathbb{R}$. In fact, if $0 \leq k \leq \lfloor |\alpha|/2 \rfloor$, then we have

$$\#\mathcal{P}_k(\alpha) = \frac{1}{k!} \prod_{j=0}^{k-1} \binom{|\alpha| - 2j}{2} = \frac{|\alpha|!}{2^k k! (|\alpha| - 2k)!}. \quad (2.4)$$

Put $\mathcal{P}(\alpha) = \bigcup_{k=0}^{\lfloor |\alpha|/2 \rfloor} \mathcal{P}_k(\alpha)$, the set of all possible pairings of V_α .

Given a vector $\boldsymbol{\eta} = (\eta_1, \dots, \eta_N)^T \in \mathbb{R}^N$ and a k -pairing $\sigma = \{W_i\}_{i=1}^k \in \mathcal{P}_k(\alpha)$, we define the contraction of $\boldsymbol{\eta}$ by σ , denoted by $\widehat{\boldsymbol{\eta}}_\sigma$, by

$$\widehat{\boldsymbol{\eta}}_\sigma = c_\sigma \boldsymbol{\eta}^{\mathfrak{t}(V_\alpha \setminus \bigcup \sigma)} \in \mathbb{R}$$

where $c_\sigma = \prod_{i=1}^k c_{\pi(W_i)} = \prod_{i=1}^k c_{\pi(a_i), \pi(b_i)}$, $W_i = \{a_i, b_i\} \subset V_\alpha$, $a_i \neq b_i$, $\bigcup \sigma = \bigcup_{i=1}^k W_i$. Note that c_σ is well-defined because the covariance matrix $C = (c_{ij})$ is symmetric. For $\sigma \in \mathcal{P}_k(\alpha)$, write $\bar{\sigma} = \mathfrak{t}(\bigcup \sigma) \in \mathbb{N}^N$. Then we have $\bar{\sigma} \leq \alpha$, $|\bar{\sigma}| = 2k$ and $\mathfrak{t}(V_\alpha \setminus \bigcup \sigma) = \alpha - \bar{\sigma}$. We can thus write $\widehat{\boldsymbol{\eta}}_\sigma = c_\sigma \boldsymbol{\eta}^{\alpha - \bar{\sigma}}$. Note also that for $\sigma \in \mathcal{P}_k(\alpha)$, $\tau \in \mathcal{P}_l(\alpha)$ such that $\bigcup \sigma \cap \bigcup \tau = \emptyset$, we have $\sigma \cup \tau \in \mathcal{P}_{k+l}(\alpha)$ and $\widehat{\boldsymbol{\eta}}_{\sigma \cup \tau} = c_\sigma c_\tau \boldsymbol{\eta}^{\mathfrak{t}(V_\alpha \setminus \bigcup (\sigma \cup \tau))}$.

To illustrate the procedure of contraction, let $N \geq 3$ and $\alpha = (3, 3, 2, 0, \dots, 0)$. Figure 1 depicts an illustrative 2-pairing of V_α consisting of $\{(1, 1), (1, 2)\}$ and $\{(2, 2), (3, 1)\}$. The contraction of $\boldsymbol{\eta} = (\eta_1, \dots, \eta_N)^T$ by this 2-pairing equals $c_{11}c_{23}\eta_1\eta_2^2\eta_3$.

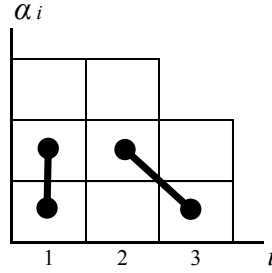


FIGURE 1. An illustrative contraction by a 2-pairing

Finally, for $\boldsymbol{\eta} \in \mathbb{R}^N$ and $\alpha \in \mathbb{N}^N$, we introduce the following notation for the sum of the contractions of $\boldsymbol{\eta}$ over all possible pairings of V_α , namely

$$R_\alpha(\boldsymbol{\eta}) = \sum_{\sigma \in \mathcal{P}(\alpha)} \widehat{\boldsymbol{\eta}}_\sigma = \sum_{\sigma \in \mathcal{P}(\alpha)} c_\sigma \boldsymbol{\eta}^{\mathfrak{t}(V_\alpha \setminus \bigcup \sigma)} \in \mathbb{R}.$$

The following key formula will serve as our main tool.

Lemma 1. *Let $\mathbf{X}$ denote an N -dimensional normally distributed random vector with mean vector $\mathbf{m} = (m_i) \in \mathbb{R}^N$ and covariance matrix $C = (c_{ij}) \in \mathbb{R}^{N \times N}$, and let $M_{\mathbf{X}}(\mathbf{t})$ denote its moment generating function. Then, for $\alpha = (\alpha_i) \in \mathbb{N}^N$, we have*

$$\mathbb{E}[\mathbf{X}^\alpha e^{\mathbf{t} \cdot \mathbf{X}}] = \partial^\alpha M_{\mathbf{X}}(\mathbf{t}) = e^{Q(\mathbf{t})} R_\alpha(\boldsymbol{\xi}(\mathbf{t})) \quad (2.5)$$

where $Q(\mathbf{t}) = \mathbf{m}^T \mathbf{t} + (1/2) \mathbf{t}^T C \mathbf{t}$ and $\boldsymbol{\xi}(\mathbf{t}) = \mathbf{m} + C \mathbf{t}$.

Proof. We need to prove the second equality in Eq.(2.5). We proceed by induction on α . First, when $\alpha = 0 = (0, \dots, 0)$, then $\mathcal{P}(\alpha)$ consists only of the 0-pairing, namely $\mathcal{P}(0) = \{\emptyset\}$. Thus, we have $R_0(\boldsymbol{\xi}(\mathbf{t})) = \boldsymbol{\xi}(\mathbf{t})^0 = 1$. Hence, Eq.(2.5) holds trivially in this case.

Next, assuming that the second equality in Eq.(2.5) is true for $\alpha = (\alpha_i) \in \mathbb{N}^N$, we show that it holds for $\tilde{\alpha} = (\alpha_1, \dots, \alpha_j + 1, \dots, \alpha_N)$ for $1 \leq j \leq N$. Differentiating Eq.(2.5) with respect to t_j and using Eq.(2.3), we have

$$\partial^{\tilde{\alpha}} M_X(\mathbf{t}) = \frac{\partial}{\partial t_j} \left(e^{Q(\mathbf{t})} R_\alpha(\xi(\mathbf{t})) \right) = e^{Q(\mathbf{t})} \left(\xi_j(\mathbf{t}) R_\alpha(\xi(\mathbf{t})) + \frac{\partial R_\alpha(\xi(\mathbf{t}))}{\partial t_j} \right).$$

Put $S(\mathbf{t}) = \xi_j(\mathbf{t}) R_\alpha(\xi(\mathbf{t})) + \frac{\partial R_\alpha(\xi(\mathbf{t}))}{\partial t_j}$. Also, put $\omega = (j, \alpha_j + 1)$ so that $V_{\tilde{\alpha}} = V_\alpha \cup \{\omega\}$. We want to show $S(\mathbf{t}) = R_{\tilde{\alpha}}(\xi(\mathbf{t}))$.

Concerning the first term of $S(\mathbf{t})$, we have

$$\xi_j(\mathbf{t}) R_\alpha(\xi(\mathbf{t})) = \sum_{\sigma \in \mathcal{P}(\alpha)} c_\sigma \xi(\mathbf{t})^{\mathbf{t}(V_{\tilde{\alpha}} \setminus \cup \sigma)}.$$

If we regard $\sigma \in \mathcal{P}(\alpha)$ as an element of $\mathcal{P}(\tilde{\alpha})$, then $\xi_j(\mathbf{t}) R_\alpha(\xi(\mathbf{t}))$ is the sum of the contractions of $\xi(\mathbf{t})$ over all the pairings of $V_{\tilde{\alpha}}$ that do not involve ω .

Next we calculate the second term of $S(\mathbf{t})$. For $\sigma \in \mathcal{P}_k(\alpha)$, let $\mathbf{t}(V_\alpha \setminus \cup \sigma) = (v_1, \dots, v_N)$. Then, by Eq.(2.2) we have

$$\frac{\partial}{\partial t_j} \xi(\mathbf{t})^{\mathbf{t}(V_\alpha \setminus \cup \sigma)} = \frac{\partial}{\partial t_j} \xi(\mathbf{t})^{(v_1, \dots, v_N)} = \sum_{i: v_i > 0} v_i c_{ij} \xi(\mathbf{t})^{(v_1, \dots, v_i - 1, \dots, v_N)}.$$

Observe that the rightmost term is the sum of the contractions of $\xi(\mathbf{t})^{\mathbf{t}(V_\alpha \setminus \cup \sigma)}$ over all the 1-pairings of $V_{\tilde{\alpha}}$ matching ω with each $s \in V_\alpha \setminus \cup \sigma$. Note that ω and s are distinct since $\omega \notin V_\alpha$. For $\sigma \in \mathcal{P}_k(\alpha)$ and $s \in V_\alpha \setminus \cup \sigma$, define $\sigma_s \in \mathcal{P}_{k+1}(\tilde{\alpha})$ by $\sigma_s = \sigma \cup \{s, \omega\}$. Then we have

$$\frac{\partial}{\partial t_j} \widehat{\xi(\mathbf{t})}_\sigma = \sum_{s \in V_\alpha \setminus \cup \sigma} \widehat{\xi(\mathbf{t})}_{\sigma_s}.$$

Hence $\frac{\partial R_\alpha(\xi(\mathbf{t}))}{\partial t_j}$ is the sum of the contractions of $\xi(\mathbf{t})$ over all the pairings of $V_{\tilde{\alpha}}$ involving ω .

The sum of the both terms of $S(\mathbf{t})$ exhausts the contractions of $\xi(\mathbf{t})$ by all pairings of $V_{\tilde{\alpha}}$. Hence, the proof is complete. $\square$

3. MOMENTS OF THE MULTIVARIATE NORMAL DISTRIBUTION

As an application of the key formula (2.5), we provide a simple proof of Isserlis' theorem and derive a formula for the moments of the multivariate normal distribution (see [9]). In fact, evaluating formula (2.5) at $\mathbf{t} = \mathbf{0}$, we have

$$\mathbb{E} \left[\prod_{i=1}^N X_i^{\alpha_i} \right] = e^{Q(\mathbf{0})} R_\alpha(\xi(\mathbf{0})) = R_\alpha(\mathbf{m}) = \sum_{\sigma \in \mathcal{P}(\alpha)} c_\sigma \mathbf{m}^{\mathbf{t}(V_\alpha \setminus \cup \sigma)}. \quad (3.1)$$

This immediately implies Isserlis' theorem ([4]) as follows.

Proposition 1. Let $\mathbf{X} = (X_1, \dots, X_N)^T$ denote an N -dimensional normally distributed random vector with zero mean. Then we have

$$\mathbb{E} \left[\prod_{i=1}^N X_i \right] = \begin{cases} \sum_{\sigma \in \mathcal{P}_{N/2}(N)} \prod_{\{i,j\} \in \sigma} \text{Cov}(X_i, X_j) & \text{if } N \text{ is even,} \\ 0 & \text{if } N \text{ is odd,} \end{cases}$$

where $\mathcal{P}_{N/2}(N)$ denotes the set of $(N/2)$ -pairings of $\{1, \dots, N\}$.

Proof. In Eq.(3.1) choose $\alpha = (1, \dots, 1)$ and identify V_α with $\{1, \dots, N\}$. When $\mathbf{m} = \mathbf{0}$, the term under summation in Eq.(3.1) is zero except when $V_\alpha \setminus \bigcup \sigma = \emptyset$, namely $\bigcup \sigma = \{1, \dots, N\}$. This condition holds only when N is even; and, in this case we have

$$\mathbb{E} \left[\prod_{i=1}^N X_i \right] = \sum_{\sigma \in \mathcal{P}_{N/2}(\alpha)} c_\sigma = \sum_{\sigma \in \mathcal{P}_{N/2}(N)} \prod_{\{i,j\} \in \sigma} c_{ij}.$$

In the last step, we identified $\mathcal{P}_{N/2}(\alpha) = \mathcal{P}_{N/2}(N)$ under the above identification $V_\alpha = \{1, \dots, N\}$. This completes the proof. $\square$

Next we want to express $\mathbb{E} [\prod_{i=1}^N X_i^{\alpha_i}]$ as a polynomial in c_{ij} 's and m_i 's. From Eq.(3.1), we have

$$\mathbb{E} \left[\prod_{i=1}^N X_i^{\alpha_i} \right] = R_\alpha(\mathbf{m}) = \sum_{k=0}^{\lfloor |\alpha|/2 \rfloor} \sum_{\sigma \in \mathcal{P}_k(\alpha)} c_\sigma \mathbf{m}^{\alpha - \bar{\sigma}}. \quad (3.2)$$

Here $\bar{\sigma} = (\bar{\sigma}_i) = \mathbf{1}(\bigcup \sigma) \in \mathbb{N}^N$. Note that $|\bar{\sigma}| = 2k$ and $\bar{\sigma} \leq \alpha$ for $\sigma \in \mathcal{P}_k(\alpha)$. Let $\sigma = \{W_s\}_{s=1}^k \in \mathcal{P}_k(\alpha)$. Then we can write

$$c_\sigma = \prod_{s=1}^k c_{\pi(W_s)} = \prod_{i \leq j} c_{ij}^{n_{ij}} = C^{\Delta(\sigma)}$$

where $\Delta(\sigma) = (n_{ij}) \in \mathbb{N}^{N \times N}$ is determined by

$$n_{ij} = \begin{cases} \#\{s \mid \pi(W_s) = \{i, j\}, 1 \leq s \leq k\} & \text{for } i \leq j, \\ 0 & \text{for } i > j. \end{cases}$$

Here we used the notation $A^B = \prod_{i,j} a_{ij}^{b_{ij}} \in \mathbb{R}$ for two real matrices $A = (a_{ij}), B = (b_{ij})$ of the same size (let $0^0 = 1$ by convention). These n_{ij} 's satisfy $\sum_{i,j=1}^N n_{ij} = k$ and $\sum_{i=1}^N n_{ih} + \sum_{j=1}^N n_{hj} = \bar{\sigma}_h \leq \alpha_h$ for $1 \leq h \leq N$.

Conversely, for $\alpha \in \mathbb{N}^N$ and $0 \leq k \leq \lfloor |\alpha|/2 \rfloor$, consider an upper triangular, non-negative integer matrix $L = (l_{ij}) \in \mathbb{N}^{N \times N}$ satisfying the following set of conditions, which we call $\mathcal{C}(\alpha, k)$:

$$\begin{cases} l_{ij} = 0 & \text{for } i > j, \\ \sum_{i,j=1}^N l_{ij} = k, \\ \sum_{i=1}^N l_{ih} + \sum_{j=1}^N l_{hj} \leq \alpha_h & \text{for } 1 \leq h \leq N. \end{cases} \quad (3.3)$$

For such L , put $\lambda(L) = (\lambda(L)_h) \in \mathbb{N}^N$ where $\lambda(L)_h = \sum_{i=1}^N l_{ih} + \sum_{j=1}^N l_{hj}$ for $1 \leq h \leq N$. Namely, put $\lambda(L) = (|\tilde{l}_1|, \dots, |\tilde{l}_N|)$ where $\tilde{l}_1, \dots, \tilde{l}_N \in \mathbb{N}^N$ are defined in column vector notation by $(\tilde{l}_1, \dots, \tilde{l}_N) = L + L^T$. Then the second and third conditions in (3.3) are equivalent to $|\lambda(L)| = 2k$ and $\lambda(L) \leq \alpha$, respectively. Note that if $\sigma \in \mathcal{P}_k(\alpha)$ satisfies $\Delta(\sigma) = L$, then $\bar{\sigma} = \mathfrak{i}(\cup \sigma) = \lambda(L)$.

Given a matrix $L = (l_{ij}) \in \mathbb{N}^{N \times N}$ satisfying the condition $\mathcal{C}(\alpha, k)$ for some $\alpha \in \mathbb{N}^N$ and $0 \leq k \leq \lfloor |\alpha|/2 \rfloor$, a k -pairing $\sigma \in \mathcal{P}_k(\alpha)$ such that $\Delta(\sigma) = L$ consists of l_{ij} distinct pairs matching between the elements of $V_\alpha(i)$ and $V_\alpha(j)$ for each $i, j \in \{1, \dots, N\}$. Let $d_L(\alpha, k) \in \mathbb{N}$ denote the number of k -pairings $\sigma \in \mathcal{P}_k(\alpha)$ such that $\Delta(\sigma) = L$. Then we have

$$d_L(\alpha, k) = \frac{1}{2^{\sum_{i=1}^N l_{ii}}} \frac{1}{\prod_{i,j=1}^N l_{ij}!} \frac{\prod_{i=1}^N \alpha_i!}{\prod_{i=1}^N (\alpha_i - \lambda(L)_i)!} = \frac{1}{2^{\text{Tr}L}} \frac{1}{L!} \frac{\alpha!}{(\alpha - \lambda(L))!} \quad (3.4)$$

where $\text{Tr}L = \sum_{i=1}^N l_{ii}$ and $L! = \prod_{i,j=1}^N l_{ij}!$.

To illustrate the correspondence between matrices satisfying $\mathcal{C}(\alpha, k)$ and pairings of V_α , let $N = 2$, $\alpha = (3, 1)$ and $L = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\lambda(L) = (3, 1)$ and $|\lambda(L)| = 4$. Thus L satisfies the condition $\mathcal{C}(\alpha, 2)$. Hence, $d_L(\alpha, 2) = 3$. Figure 2 depicts the three 2-pairings σ of V_α such that $\Delta(\sigma) = L$.

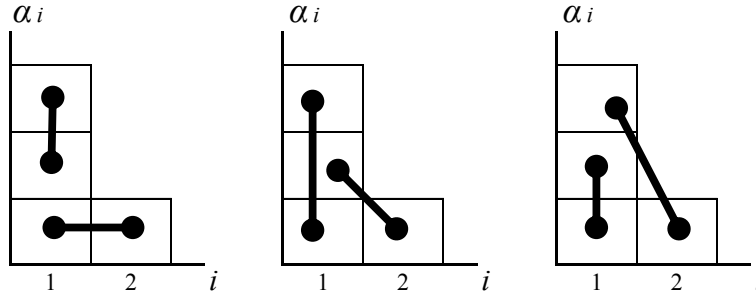


FIGURE 2. Three 2-pairings $\sigma \in \mathcal{P}_2((3, 1))$ such that $\Delta(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

Theorem 1. Let $\mathbf{X} = (X_1, \dots, X_N)^T$ denote an N -dimensional normally distributed random vector with mean vector $\mathbf{m} = (m_i) \in \mathbb{R}^N$ and covariance matrix $C = (c_{ij}) \in \mathbb{R}^{N \times N}$. Then, for $\alpha = (\alpha_i) \in \mathbb{N}^N$, we have

$$\mathbb{E} \left[\prod_{i=1}^N X_i^{\alpha_i} \right] = \sum_{k=0}^{\lfloor |\alpha|/2 \rfloor} \sum_{L: \mathcal{C}(\alpha, k)} \frac{\alpha!}{2^{\text{Tr}L} L! (\alpha - \lambda(L))!} C^L \mathbf{m}^{\alpha - \lambda(L)}$$

where the second summation is taken over $L = (l_{ij}) \in \mathbb{N}^{N \times N}$ satisfying the condition $\mathcal{C}(\alpha, k)$ given in (3.3).

Proof. From Eq.(3.2) and the observations made in the above, we have

$$\mathbb{E} \left[\prod_{i=1}^N X_i^{\alpha_i} \right] = \sum_{k=0}^{\lfloor |\alpha|/2 \rfloor} \sum_{\sigma \in \mathcal{P}_k(\alpha)} c_{\sigma} \mathbf{m}^{\alpha - \bar{\sigma}} = \sum_{k=0}^{\lfloor |\alpha|/2 \rfloor} \sum_L d_L(\alpha, k) C^L \mathbf{m}^{\alpha - \lambda(L)}$$

where the second summation in the last term is taken over $L \in \mathbb{N}^{N \times N}$ such that $L = \Delta(\sigma)$ for some $\sigma \in \mathcal{P}_k(\alpha)$. Hence, from Eq.(3.4) we have the desired formula. This completes the proof. $\square$

A result equivalent to Theorem 1 is stated in [7] and a proof based on mathematical induction is presented in [6]. Our combinatorial proof gives an alternative proof.

Corollary 1. For $\alpha \in \mathbb{N}^N$ and $0 \leq k \leq \lfloor |\alpha|/2 \rfloor$, we have

$$\sum_{L: C(\alpha, k)} \frac{\alpha!}{2^{\text{Tr} L} L! (\alpha - \lambda(L))!} = \frac{|\alpha|!}{2^k k! (|\alpha| - 2k)!}$$

where the summation over $L \in \mathbb{N}^{N \times N}$ is taken in the same manner as in Theorem 1.

Proof. Indeed, $\#\mathcal{P}_k(\alpha)$ is the sum of $d_L(\alpha, k)$ over L . The above relation thus follows from Eq.(2.4) and Eq.(3.4). This completes the proof. $\square$

4. THE NORMAL-LOGNORMAL PRODUCT DISTRIBUTION AND ITS MOMENTS

Consider a random variable expressed as the product of a finite number of correlated normally and lognormally distributed random variables. As another application of the key formula (2.5), we calculate the moments of this random variable.

4.1. The normal-lognormal product distribution

This subsection summarizes basic properties of the multivariate distribution composed of normal and lognormal distributions, as well as the univariate distribution defined as the product of normal and lognormal distributions. Note that these distributions were considered independently by researchers in various fields ([2],[3],[10]).

Let $N = n + l$ where $n, l \in \mathbb{N}$. We say that an N -dimensional random vector $\mathbf{Y} = (Y_1, \dots, Y_N)^T$ follows joint normal-lognormal distribution of index (n, l) if $\tilde{\mathbf{Y}} = (Y_1, \dots, Y_n, \log Y_{n+1}, \dots, \log Y_N)^T$ is normally distributed. This definition is equivalent to say that $\mathbf{Y}$ is of the form $\mathbf{Y} = (X_1, \dots, X_n, e^{X_{n+1}}, \dots, e^{X_N})^T$ where $\mathbf{X} = (X_1, \dots, X_N)^T$ is normally distributed. We say that a random variable Z follows normal-lognormal product distribution of index (n, l) if Z is of the form $Z = \prod_{i=1}^N Y_i$ where $\mathbf{Y} = (Y_1, \dots, Y_N)^T$ follows joint normal-lognormal distribution of index (n, l) . Equivalently, Z is of the form $Z = (\prod_{i=1}^n X_i) e^{\sum_{j=1}^l X_{n+j}}$ where $\mathbf{X} = (X_1, \dots, X_N)^T$ is normally distributed. We call $\mathbf{X}$ the underlying normal distribution of $\mathbf{Y}$ and Z .

Proposition 2. Let $\mathbf{Y} = (Y_1, \dots, Y_N)^T$ denote an N -dimensional random vector following joint normal-lognormal distribution of index (n, l) with $N = n + l$, and

let $\mathbf{X} = (X_1, \dots, X_N)^T$ denote its underlying normal distribution with mean vector $\mathbf{m} = (m_i) \in \mathbb{R}^N$ and covariance matrix $C = (c_{ij}) \in \mathbb{R}^{N \times N}$. Let $I = \{1, \dots, n\}$ and $J = \{n+1, \dots, N\}$.

(1) The probability density function of $\mathbf{Y}$ is given by

$$g_{n,l}(\mathbf{y}; \mathbf{m}, C) = (2\pi)^{-\frac{n}{2}} (\det C)^{-\frac{1}{2}} \left(\prod_{i \in J} y_i \right)^{-1} \exp \left(-\frac{1}{2} (\tilde{\mathbf{y}} - \mathbf{m})^T C^{-1} (\tilde{\mathbf{y}} - \mathbf{m}) \right)$$

where $\mathbf{y} = (y_1, \dots, y_n, y_{n+1}, \dots, y_N)^T \in \mathbb{R}^n \times \mathbb{R}_+^l$ and $\tilde{\mathbf{y}} = (y_1, \dots, y_n, \log y_{n+1}, \dots, \log y_N)^T \in \mathbb{R}^N$.

(2) The mean of Y_i is given by

$$\mathbb{E}[Y_i] = \begin{cases} m_i & \text{for } i \in I, \\ e^{m_i + \frac{\sigma_i^2}{2}} & \text{for } i \in J. \end{cases}$$

(3) The covariance between Y_i and Y_j is given by

$$\text{Cov}(Y_i, Y_j) = \begin{cases} c_{ij} & \text{for } i, j \in I, \\ e^{m_j + \frac{\sigma_j^2}{2}} c_{ij} & \text{for } i \in I, j \in J, \\ e^{m_i + m_j + \frac{\sigma_i^2 + \sigma_j^2}{2}} (e^{c_{ij}} - 1) & \text{for } i, j \in J. \end{cases}$$

From this it follows that $\text{Cov}(Y_i, Y_j) = 0$ if and only if $\text{Cov}(X_i, X_j) = 0$ for $i, j = 1, \dots, N$.

Proof.

(1) Let $f_N(x_1, \dots, x_N; \mathbf{m}, C)$ be the probability density function of $\mathbf{X}$. By change of variables $(X_1, \dots, X_N) = (Y_1, \dots, Y_n, \log Y_{n+1}, \dots, \log Y_N)$, we have

$$\begin{aligned} g_{n,l}(y_1, \dots, y_N; \mathbf{m}, C) &= f_N(x_1, \dots, x_N; \mathbf{m}, C) \frac{\partial(x_1, \dots, x_N)}{\partial(y_1, \dots, y_N)} \\ &= f_N(y_1, \dots, y_n, \log y_{n+1}, \dots, \log y_N; \mathbf{m}, C) \prod_{i=1}^l \frac{1}{y_{n+i}}. \end{aligned}$$

(2) For $i \in I$, we have $\mathbb{E}[Y_i] = \mathbb{E}[X_i] = m_i$. For $i \in J$, we have $\mathbb{E}[Y_i] = \mathbb{E}[e^{X_i}] = M_{X_i}(1) = e^{m_i + \frac{\sigma_i^2}{2}}$.

(3) For $i, j \in I$, we have $\text{Cov}(Y_i, Y_j) = \text{Cov}(X_i, X_j) = c_{ij}$.

For $i \in I, j \in J$, recall Stein's lemma ([8]. See also [3, Chapter 9, Appendix 2].) which asserts that $\text{Cov}(X, \phi(Y)) = \mathbb{E}[\phi'(Y)] \text{Cov}(X, Y)$ for a bivariate normal distribution (X, Y) and a differentiable function $\phi(x)$ such that (i) $\lim_{x \rightarrow \pm\infty} \phi(x) e^{-ax^2} = 0$ for any $a > 0$, and (ii) $\mathbb{E}[\phi'(Y)]$ exists. Noting that (X_i, X_j) is normally distributed, we have $\text{Cov}(Y_i, Y_j) = \text{Cov}(X_i, e^{X_j}) = \mathbb{E}[e^{X_j}] \text{Cov}(X_i, X_j) = e^{m_j + \frac{\sigma_j^2}{2}} c_{ij}$. An alternative proof is given in Example 2 (i) below.

For $i, j \in J$, we have

$$\text{Cov}(Y_i, Y_j) = \text{Cov}(e^{X_i}, e^{X_j}) = \mathbb{E}[e^{X_i+X_j}] - \mathbb{E}[e^{X_i}]\mathbb{E}[e^{X_j}].$$

Without loss of generality, we may fix $i = 1, j = 2$. Thus, we have

$$\mathbb{E}[e^{t_1 X_1 + t_2 X_2}] = \exp\left(m_1 t_1 + m_2 t_2 + \frac{1}{2}(c_{11} t_1^2 + 2c_{12} t_1 t_2 + c_{22} t_2^2)\right).$$

Evaluating this function at $(t_1, t_2) = (1, 1)$, we have

$$\mathbb{E}[e^{X_1+X_2}] = e^{m_1+m_2+\frac{1}{2}(\sigma_1^2+2c_{12}+\sigma_2^2)}.$$

Thus

$$\begin{aligned} \text{Cov}(Y_1, Y_2) &= e^{m_1+m_2+\frac{1}{2}(\sigma_1^2+2c_{12}+\sigma_2^2)} - e^{m_1+\frac{\sigma_1^2}{2}} e^{m_2+\frac{\sigma_2^2}{2}} \\ &= e^{m_1+m_2+\frac{\sigma_1^2+\sigma_2^2}{2}} (e^{c_{12}} - 1). \end{aligned}$$

□

When $n = 1$, we have the following integral representation of the probability density function of the normal-lognormal product distribution of index $(1, l)$.

Proposition 3. *Let Z denote a random variable following normal-lognormal product distribution of index $(1, l)$ derived by the joint normal-lognormal distribution $\mathbf{Y}$ of the same index. Let $\mathbf{m} = (m_i) \in \mathbb{R}^{l+1}$ and $C = (c_{ij}) \in \mathbb{R}^{(l+1) \times (l+1)}$ denote, respectively, the mean vector and the covariance matrix of the underlying normal distribution. Then, the probability density function of Z is given by*

$$p_{1,l}(z) = \int_{\mathbb{R}^l} g_{1,l}(ze^{-w_1-w_2-\dots-w_l}, e^{w_1}, \dots, e^{w_l}; \mathbf{m}, C) dw_1 \dots dw_l$$

where $g_{1,l}(y_1, \dots, y_{l+1}; \mathbf{m}, C)$ is the probability density function of $\mathbf{Y}$ given in Proposition 2 (1).

Proof. Since $Z = Y_1 Y_2 \dots Y_{l+1}$ and $Y_2 \dots Y_{l+1} > 0$, we have

$$p_{1,l}(z) = \int_{\mathbb{R}_+^l} g_{1,l}\left(\frac{z}{y_2 \dots y_{l+1}}, y_2, \dots, y_{l+1}; \mathbf{m}, C\right) \frac{dy_2}{y_2} \dots \frac{dy_{l+1}}{y_{l+1}}.$$

By putting $w_i = \log y_{i+1}$ ($1 \leq i \leq l$), we have the desired expression. □

4.2. Moments of the normal-lognormal product distribution

Theorem 2. *Let Z denote a random variable following normal-lognormal product distribution of index (n, l) with $N = n + l$, and let $\mathbf{X} = (X_1, \dots, X_N)^T$ denote its underlying normal distribution with mean vector $\mathbf{m} = (m_i) \in \mathbb{R}^N$ and covariance matrix $C = (c_{ij}) \in \mathbb{R}^{N \times N}$. Then the d -th raw moment of Z is given by*

$$\mathbb{E}[Z^d] = \mathbb{E}\left[(X_1 \dots X_n e^{X_{n+1} + \dots + X_N})^d\right] = e^{Q(d, l)} R_{d, 0}(\boldsymbol{\zeta}) \quad (4.1)$$

where $d_{n,0} = (\overbrace{d, \dots, d}^n, 0, \dots, 0)$, $\mathbf{d}_{0,l} = (0, \dots, 0, \overbrace{d, \dots, d}^l)^T$ and $\boldsymbol{\zeta} = (\zeta_i) = \boldsymbol{\xi}(\mathbf{d}_{0,l})$. Explicitly, $Q(\mathbf{d}_{0,l}) = d \sum_{j=1}^l m_{n+j} + \frac{d^2}{2} \sum_{i,j=1}^l c_{n+i,n+j}$ and $\zeta_i = m_i + d \sum_{j=1}^l c_{i,n+j}$ ($1 \leq i \leq N$).

Proof. For $\boldsymbol{\alpha} = (\alpha_i) \in \mathbb{N}^N$ where $N = n + l$, put $\boldsymbol{\alpha}_{n,0} = (\alpha_1, \dots, \alpha_n, 0, \dots, 0)$ and $\boldsymbol{\alpha}_{0,l} = (0, \dots, 0, \alpha_{n+1}, \dots, \alpha_N)^T$. Substituting $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{n,0}$ in the key formula (2.5) and evaluating it at $\mathbf{t} = \boldsymbol{\alpha}_{0,l}$, we have

$$\mathbb{E}[X_1^{\alpha_1} \dots X_n^{\alpha_n} e^{\alpha_{n+1}X_{n+1} + \dots + \alpha_N X_N}] = \partial^{\boldsymbol{\alpha}_{n,0}} M_{\mathbf{X}}(\boldsymbol{\alpha}_{0,l}) = e^{Q(\boldsymbol{\alpha}_{0,l})} R_{\boldsymbol{\alpha}_{n,0}}(\boldsymbol{\xi}(\boldsymbol{\alpha}_{0,l})). \quad (4.2)$$

Setting $\alpha_1 = \dots = \alpha_N = d \in \mathbb{N}$ in Eq.(4.2), we have Eq.(4.1). This completes the proof. $\square$

When $d = 1$, we have the mean of the normal-longnormal product distribution of index (n, l) .

Corollary 2. *With the notation as above, when $d = 1$, we have*

$$\mathbb{E}[Z] = \mathbb{E}[X_1 \dots X_n e^{X_{n+1} + \dots + X_N}] = e^{Q_0} \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\sigma \in \mathcal{P}_k(n)} \left(\prod_{\{i,j\} \in \sigma} c_{ij} \prod_{i \in \{1, \dots, n\} \setminus \cup \sigma} \zeta_i \right)$$

where $Q_0 = \sum_{j=1}^l m_{n+j} + \frac{1}{2} \sum_{i,j=1}^l c_{n+i,n+j}$, $\zeta_i = m_i + \sum_{j=1}^l c_{i,n+j}$ ($1 \leq i \leq n$) and $\mathcal{P}_k(n)$ denotes the set of k -pairings of $\{1, \dots, n\}$.

Proof. When $d = 1$, $Q(\mathbf{d}_{0,l}) = Q_0$. For notational simplicity put $\mathbf{v} = 1_{n,0} = (\overbrace{1, \dots, 1}^n, 0, \dots, 0)$. Then we have

$$\begin{aligned} R_{\mathbf{v}}(\boldsymbol{\zeta}) &= \sum_{\sigma \in \mathcal{P}(\mathbf{v})} c_{\sigma} \boldsymbol{\zeta}^{\mathbf{v} - \bar{\sigma}} = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\sigma \in \mathcal{P}_k(\mathbf{v})} c_{\sigma} \boldsymbol{\zeta}^{\mathbf{v} - \bar{\sigma}} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\sigma \in \mathcal{P}_k(n)} \left(\prod_{\{i,j\} \in \sigma} c_{ij} \prod_{i \in \{1, \dots, n\} \setminus \cup \sigma} \zeta_i \right). \end{aligned}$$

In the last step, we identified $\mathcal{P}_k(\mathbf{v})$ with $\mathcal{P}_k(n)$ by identifying $V_{\mathbf{v}} = \{1, \dots, n\}$. Thus, from Eq.(4.1) the corollary follows. $\square$

When $n = 1$, we have the following explicit formula for arbitrary moments of the normal-longnormal product distribution of index $(1, l)$.

Corollary 3. *With the notation as above, when $n = 1$, we have*

$$\mathbb{E}[Z^d] = \mathbb{E}[(X_1 e^{X_2 + \dots + X_{l+1}})^d] = e^{Q_1} \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{d!}{2^k k! (d-2k)!} \sigma_1^{2k} \zeta_1^{d-2k}$$

where $\sigma_1^2 = c_{11}$, $Q_1 = d \sum_{j=1}^l m_{j+1} + \frac{d^2}{2} \sum_{i,j=1}^l c_{i+1,j+1}$ and $\zeta_1 = m_1 + d \sum_{j=1}^l c_{1,j+1}$.

Proof. When $n = 1$, $Q(\mathbf{d}_{0,l}) = Q_1$. By definition, for $d_{1,0} = (d, 0, \dots, 0) \in \mathbb{N}^{l+1}$ and $\boldsymbol{\zeta} = (\zeta_i) \in \mathbb{R}^{l+1}$, we have

$$\begin{aligned} R_{d_{1,0}}(\boldsymbol{\zeta}) &= \sum_{k=0}^{\lfloor d/2 \rfloor} \sum_{\boldsymbol{\sigma} \in \mathcal{P}_k(d_{1,0})} c_{\boldsymbol{\sigma}} \boldsymbol{\zeta}^{d_{1,0} - \boldsymbol{\sigma}} = \sum_{k=0}^{\lfloor d/2 \rfloor} \# \mathcal{P}_k(d_{1,0}) c_{11}^k \zeta_1^{d-2k} \\ &= \sum_{k=0}^{\lfloor d/2 \rfloor} \frac{d!}{2^k k! (d-2k)!} \sigma_1^{2k} \zeta_1^{d-2k}. \end{aligned}$$

Here we used Eq.(2.4) in the last step. Therefore, from Eq.(4.1) the corollary follows. $\square$

By applying the idea of the proof of Theorem 2 to the results of Example 1, we can directly calculate the first few moments of the normal-lognormal product distribution for $n, l \in \{1, 2\}$ and $d \leq 8/(n+l)$.

Example 2. For subsequent use, we evaluate $\xi_k(\mathbf{t})$ and $Q(\mathbf{t})$ at $\mathbf{t} = a\mathbf{e}_i + b\mathbf{e}_j$ (namely, $t_i = a, t_j = b$, and $t_h = 0$ for $h \neq i, j$):

$$\begin{aligned} \xi_k(a\mathbf{e}_i + b\mathbf{e}_j) &= m_k + ac_{ki} + bc_{kj}, \\ Q(a\mathbf{e}_i + b\mathbf{e}_j) &= am_i + bm_j + \frac{1}{2}(a^2\sigma_i^2 + 2abc_{ij} + b^2\sigma_j^2). \end{aligned}$$

(i) Index $(1, 1)$ case: Evaluating $\mathbb{E}[X_1 e^{\mathbf{t} \cdot \mathbf{X}}]$ at $\mathbf{t} = \mathbf{e}_2$, we have

$$\mathbb{E}[Z] = \mathbb{E}[X_1 e^{X_2}] = e^{m_2 + \frac{\sigma_2^2}{2}} (m_1 + c_{12}).$$

Evaluating $\mathbb{E}[X_1^2 e^{\mathbf{t} \cdot \mathbf{X}}]$ at $\mathbf{t} = 2\mathbf{e}_2$, we have

$$\mathbb{E}[Z^2] = \mathbb{E}[X_1^2 e^{2X_2}] = e^{2m_2 + 2\sigma_2^2} ((m_1 + 2c_{12})^2 + \sigma_1^2).$$

Hence, we have

$$\text{Var}[Z] = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 = e^{2m_2 + \sigma_2^2} \left[e^{\sigma_2^2} ((m_1 + 2c_{12})^2 + \sigma_1^2) - (m_1 + c_{12})^2 \right].$$

Also, from the above results, we have

$$\text{Cov}(X_1, e^{X_2}) = \mathbb{E}[X_1 e^{X_2}] - \mathbb{E}[X_1] \mathbb{E}[e^{X_2}] = e^{m_2 + \frac{\sigma_2^2}{2}} c_{12}.$$

This gives an alternative proof of the second case of Proposition 2 (3).

Evaluating $\mathbb{E}[X_1^3 e^{\mathbf{t} \cdot \mathbf{X}}]$ at $\mathbf{t} = 3\mathbf{e}_2$, and $\mathbb{E}[X_1^4 e^{\mathbf{t} \cdot \mathbf{X}}]$ at $\mathbf{t} = 4\mathbf{e}_2$, we have

$$\begin{aligned} \mathbb{E}[Z^3] &= \mathbb{E}[X_1^3 e^{3X_2}] = e^{3m_2 + (9/2)\sigma_2^2} ((m_1 + 3c_{12})^3 + 3\sigma_1^2 (m_1 + 3c_{12})), \\ \mathbb{E}[Z^4] &= \mathbb{E}[X_1^4 e^{4X_2}] = e^{4m_2 + 4\sigma_2^2} ((m_1 + 4c_{12})^4 + 6\sigma_1^2 (m_1 + 4c_{12})^2 + 3\sigma_1^4). \end{aligned}$$

(ii) Index $(1, 2)$ case: Evaluating $\mathbb{E}[X_1 e^{\mathbf{t} \cdot \mathbf{X}}]$ at $\mathbf{t} = \mathbf{e}_2 + \mathbf{e}_3$, we have

$$\mathbb{E}[Z] = \mathbb{E}[X_1 e^{X_2 + X_3}] = e^{m_2 + m_3 + \frac{1}{2}(\sigma_2^2 + 2c_{23} + \sigma_3^2)} (m_1 + c_{12} + c_{13}).$$

Evaluating $\mathbb{E}[X_1^2 e^{t \cdot X}]$ at $t = 2e_2 + 2e_3$, we have

$$\mathbb{E}[Z^2] = \mathbb{E}[X_1^2 e^{2X_2 + 2X_3}] = e^{2(m_2 + m_3) + 2(\sigma_2^2 + 2c_{23} + \sigma_3^2)}((m_1 + 2c_{12} + 2c_{13})^2 + \sigma_1^2).$$

(iii) Index $(2, 1)$ case: Evaluating $\mathbb{E}[X_1 X_2 e^{t \cdot X}]$ at $t = e_3$, we have

$$\mathbb{E}[Z] = \mathbb{E}[X_1 X_2 e^{X_3}] = e^{m_3 + \frac{\sigma_3^2}{2}}((m_1 + c_{13})(m_2 + c_{23}) + c_{12}).$$

Evaluating $\mathbb{E}[X_1^2 X_2^2 e^{t \cdot X}]$ at $t = 2e_3$, we have

$$\begin{aligned} \mathbb{E}[Z^2] &= \mathbb{E}[X_1^2 X_2^2 e^{2X_3}] \\ &= e^{2m_3 + 2\sigma_3^2} [(m_1 + 2c_{13})^2 (m_2 + 2c_{23})^2 + \sigma_2^2 (m_1 + 2c_{13})^2 \\ &\quad + \sigma_1^2 (m_2 + 2c_{23})^2 + 4c_{12}(m_1 + 2c_{13})(m_2 + 2c_{23}) + 2c_{12}^2 + \sigma_1^2 \sigma_2^2]. \end{aligned}$$

(iv) Index $(2, 2)$ case: Evaluating $\mathbb{E}[X_1 X_2 e^{t \cdot X}]$ at $t = e_3 + e_4$, we have

$$\begin{aligned} \mathbb{E}[Z] &= \mathbb{E}[X_1 X_2 e^{X_3 + X_4}] \\ &= e^{m_3 + m_4 + \frac{1}{2}(\sigma_3^2 + 2c_{34} + \sigma_4^2)}((m_1 + c_{13} + c_{14})(m_2 + c_{23} + c_{24}) + c_{12}). \end{aligned}$$

Evaluating $\mathbb{E}[X_1^2 X_2^2 e^{t \cdot X}]$ at $t = 2e_3 + 2e_4$, we have

$$\begin{aligned} \mathbb{E}[Z^2] &= \mathbb{E}[X_1^2 X_2^2 e^{2X_3 + 2X_4}] \\ &= e^{2(m_3 + m_4) + 2(\sigma_3^2 + 2c_{34} + \sigma_4^2)} [(m_1 + 2c_{13} + 2c_{14})^2 (m_2 + 2c_{23} + 2c_{24})^2 \\ &\quad + \sigma_2^2 (m_1 + 2c_{13} + 2c_{14})^2 + \sigma_1^2 (m_2 + 2c_{23} + 2c_{24})^2 \\ &\quad + 4c_{12}(m_1 + 2c_{13} + 2c_{14})(m_2 + 2c_{23} + 2c_{24}) + 2c_{12}^2 + \sigma_1^2 \sigma_2^2]. \end{aligned}$$

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ON MODELS OF THE LIE ALGEBRA $\mathcal{K}_5$ AND LAURICELLA FUNCTIONS USING AN INTEGRAL TRANSFORMATION

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Abstract. We construct new $(n+1)$ -variable models of irreducible representations of the Lie algebra $\mathcal{K}_5$. An n -fold integral transformation is used to obtain a new set of models of $\mathcal{K}_5$ in terms of difference-differential operators. These models are further exploited to obtain recurrence relations, generating functions and addition theorems.

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Keywords: Lie algebra $\mathcal{K}_5$, hypergeometric functions, Lauricella functions

1. INTRODUCTION

There is a close connection between models of Lie algebras and identities in special function theory. Miller [8] and Vilenkin [19] have developed this connection systematically from its basics. Manocha [7], Govil and Manocha [3] have studied this relationship via certain integral transformations thereby obtaining various recurrence relations and identities in the theory of special functions. Sahai [11, 12], Sahai and Srivastava [13] have extended this approach to obtain numerous results in the theory of special functions of one and several variables. The Lie algebras considered for obtaining these results mainly include the special linear complex algebra $sl(2, \mathbb{C})$, the oscillator algebra $\mathcal{G}(0, 1)$ and the 3-dimensional Euclidean algebra $\mathcal{T}_3$. In the present paper, we extend this study to the 5-dimensional Lie algebra $\mathcal{K}_5$. We construct new $(n+1)$ -variable models of irreducible representations of $\mathcal{K}_5$ in terms of differential operators. Using an integral transformation, we obtain models of $\mathcal{K}_5$ which are explored for recurrence relations, generating functions and addition theorems. These results are believed to be new.

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Earlier, notable contributions in relating the Lie algebra $\mathcal{K}_5$ and special functions have come from the works of Khan and Ali [6], Pathan et al [9], Srivastava et al [18] and Yadav and Rani [20]. The multi-variable Lauricella series as well as their Srivastava-Daoust generalizations are extensively described in the monograph by Srivastava and Karlsson [16]. Some developments in the usage of Lie algebraic techniques to study the theory of multi-variable and multi-index Hermite polynomials have been studied in [2]. Decomposition formulas, linearization problems and certain interesting properties associated with generalized Lauricella functions also exist in literature, see [4, 5, 15]. Moreover, the incomplete Lauricella functions and their associated properties including integral representations, finite summation formulas, transformation and derivative formulas have been discussed in [1]. Recently, Srivastava [14] has listed some important developments in the theory and applications of hypergeometric and related functions.

The paper is organized as follows. In Section 2, we give a brief review of the Lie algebra $\mathcal{K}_5$ and its representations $R'(\omega, m_0, \mu)$ and $\uparrow'_{\omega, \mu}$. One variable models of representations $R'(0, m_0, \mu)$ and $\uparrow'_{\omega, \mu}$ are given following Miller [8]. An r -fold integral transformation is defined and transforms of certain operator expressions are given that are needed for our discussion [3]. In Section 3, we give $(n+1)$ -variable models of representations of $\mathcal{K}_5$ in which the representation spaces have basis functions appearing as ${}_1F_0\left(-\lambda; -; \sum_{i=1}^n u_i x_i\right) t^\lambda$ and ${}_1F_0\left(-\lambda - \omega; -; \sum_{i=1}^n u_i x_i\right) t^{\lambda+\omega}$, respectively. In Section 4, we obtain the transformed models of Section 3 in terms of difference-differential operators with basis functions as Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$, respectively. Further, we utilize these models to obtain some recurrence relations and generating functions. Finally, in Section 5, all the models of Sections 3 are utilized to obtain interesting addition theorems.

2. PRELIMINARIES

The Lie algebra $\mathcal{K}_5$ is the Lie algebra of 5-dimensional complex Lie group K_5 , given by:

$$K_5 = \left\{ g(q, a, b, c, \tau) = \begin{pmatrix} 1 & ce^\tau & be^{-\tau} & 2a - bc & \tau \\ 0 & e^\tau & 2qe^{-\tau} & b - 2qc & 0 \\ 0 & 0 & e^{-\tau} & -c & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| q, a, b, c, \tau \in \mathbb{C} \right\},$$

where the group operation is matrix multiplication [8], satisfying the multiplication law:

$$\begin{aligned} & g_1(q, a, b, c, \tau) g_2(q', a', b', c', \tau') \\ &= g(q + q'e^{2\tau}, a + a' + e^\tau cb' + c^2 q' e^{2\tau}, b + b'e^\tau + 2cq' e^{2\tau}, c + c'e^{-\tau}, \tau + \tau'), \end{aligned} \quad (2.1)$$

where $q, a, b, c, \tau, q', a', b', c', \tau' \in \mathbb{C}$ and $g, g_1, g_2 \in K_5$.

The identity element of K_5 is $g(0, 0, 0, 0, 0)$ and the inverse of $g(q, a, b, c, \tau)$ is $g(-qe^{-2\tau}, -a + bc - c^2q, -be^{-\tau} + 2cqe^{-\tau}, -ce^{\tau}, -\tau)$. The 5-dimensional complex Lie algebra $\mathcal{K}_5$ has basis $\{\mathcal{J}^3, \mathcal{J}^{\pm}, \mathcal{E}, \mathcal{Q}\}$ satisfying the commutation relations:

$$\begin{aligned} [\mathcal{J}^3, \mathcal{J}^{\pm}] &= \pm \mathcal{J}^{\pm}, & [\mathcal{J}^3, \mathcal{Q}] &= 2\mathcal{Q}, & [\mathcal{J}^-, \mathcal{J}^+] &= \mathcal{E}, \\ [\mathcal{J}^-, \mathcal{Q}] &= 2\mathcal{J}^+, & [\mathcal{J}^+, \mathcal{Q}] &= 0, \\ [\mathcal{J}^{\pm}, \mathcal{E}] &= [\mathcal{J}^3, \mathcal{E}] = [\mathcal{Q}, \mathcal{E}] = \mathcal{O}, \end{aligned} \quad (2.2)$$

where $\mathcal{O}$ is the 5×5 zero matrix.

Let ρ be an irreducible representation of $\mathcal{K}_5$ on a vector space V and let

$$J^{\pm} = \rho(\mathcal{J}^{\pm}), \quad J^3 = \rho(\mathcal{J}^3), \quad E = \rho(\mathcal{E}), \quad Q = \rho(\mathcal{Q})$$

be operators on V . Clearly, these operators obey the commutation relations identical to (2.2).

Let S denote the spectrum of J^3 and let the irreducible representation ρ satisfies the following conditions:

- (1) Each eigenvalue of J^3 has multiplicity equal to one.
- (2) There is a enumerable basis for V consisting of all the eigenvectors of J^3 .

This ensures that S is enumerable and that there exists a basis for V consisting of vectors $\{f_m \mid m \in S\}$ such that $J^3 f_m = m f_m$. It is well known that the oscillator algebra $\mathcal{G}(0, 1)$ is a subalgebra of Lie algebra $\mathcal{K}_5$ and an irreducible representation of $\mathcal{G}(0, 1)$ satisfying (1)–(2) is isomorphic to one of the irreducible representations $R(\omega, m_0, \mu)$ or $\uparrow'_{\omega, \mu}$ of $\mathcal{G}(0, 1)$. In the following, we study such representations of $\mathcal{K}_5$. Indeed, we have [8].

Theorem 1. *Every irreducible representation ρ of $\mathcal{K}_5$ is isomorphic to a representation in the following list:*

- (1) *The representation $R'(\omega, m_0, \mu)$ defined for all $\omega, m_0, \mu \in \mathbb{C}$ such that $\mu \neq 0$, $0 \leq \operatorname{Re} m_0 < 1$ and $\omega + m_0$ is not an integer. The spectrum of J^3 is the set $S = \{m_0 + n : n \text{ is an integer}\}$.*
- (2) *The representation $\uparrow'_{\omega, \mu}$ defined for all $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0$. The spectrum of J^3 is the set $S = \{-\omega + n : n \text{ is non-negative integer}\}$.*

For each of the above cases, there is a basis of V consisting of vectors $\{f_m \mid m \in S\}$ such that

$$\begin{aligned} J^3 f_m &= m f_m, & E f_m &= \mu f_m, & Q f_m &= \mu f_{m+2}, \\ J^+ f_m &= \mu f_{m+1}, & J^- f_m &= (m + \omega) f_{m-1}. \end{aligned} \quad (2.3)$$

On the right hand side of (2.3), $f_m = 0$ if $m \notin S$.

One variable models of the irreducible representations of $\mathcal{K}_5$ are given by:

Representation $R'(0, m_0, \mu)$:

$$J^3 = z \frac{d}{dz}, \quad J^+ = \mu z, \quad J^- = \frac{d}{dz},$$

$$E = \mu, \quad Q = \mu z^2, \quad f_\lambda(z) = z^\lambda,$$

where $\lambda \in S = \{m_0 + n \mid m_0 \in \mathbb{C} - \{0\}, 0 \leq \operatorname{Re} m_0 < 1, n = 0, \pm 1, \dots\}$.

Representation $\uparrow'_{\omega, \mu}$:

$$J^3 = -\omega + z \frac{d}{dz}, \quad J^+ = \mu z, \quad J^- = \frac{d}{dz},$$

$$E = \mu, \quad Q = \mu z^2, \quad f_\lambda(z) = z^{\lambda + \omega},$$

where $\lambda \in S = \{-\omega + n \mid n = 0, 1, \dots\}$.

2.1. Integral Transformation

Let V be a complex vector space consisting of all analytic functions $f(z_1, \dots, z_r)$, analytic at $(z_1, \dots, z_r) = (0, \dots, 0)$. We define

$$h(\beta_i, \gamma_j, \gamma, x_i) = I[f(z_1, \dots, z_r)]$$

$$= \left(\prod_{i=1}^k \frac{\Gamma(\gamma_i)}{\Gamma(\beta_i) \Gamma(\gamma_i - \beta_i)} \right) \frac{\Gamma(\gamma)}{\Gamma(\beta_{k+1}) \dots \Gamma(\beta_r) \Gamma(\gamma - \sum_{i=k+1}^m \beta_i)}$$

$$\times \underbrace{\int \dots \int}_{r\text{-fold}} \prod_{i=1}^r u_i^{\beta_i - 1} \prod_{i=1}^k (1 - u_i)^{\gamma_i - \beta_i - 1} \left(1 - \sum_{j=k+1}^m u_j \right)^{\gamma - \sum_{j=k+1}^m \beta_j - 1}$$

$$\times f(z_1, \dots, z_r) du_1 \dots du_r,$$

where,

$$z_i = u_i x_i, \quad i = 1, \dots, r;$$

$$\operatorname{Re} \gamma_i > \operatorname{Re} \beta_i > 0, \quad i = 1, \dots, k;$$

$$\operatorname{Re} \beta_j > 0, \quad j = k+1, \dots, m;$$

$$\operatorname{Re} \left(\gamma - \sum_{j=k+1}^m \beta_j \right) > 0;$$

and the path of integration is $0 \leq u_i \leq 1, i = 1, \dots, k, u_j \geq 0, j = k+1, \dots, m, \dots, r$ and $\sum_{j=k+1}^m u_j \leq 1$ [3]. Then $W = IV$ is an isomorphic image of V under the transformation $I: f(z_1, \dots, z_r) \rightarrow h(\beta_i, \gamma_j, \gamma, x_i)$. Next, we obtain transforms of certain expressions under the transformation I in terms of difference operators and differential operators defined as follows:

$$E_{\gamma_i} h(\beta_i, \gamma_i) = h(\beta_i, \gamma_i + 1),$$

$$\begin{aligned} L_{\gamma_i} h(\beta_i, \gamma_i) &= h(\beta_i, \gamma_i - 1), \\ \Delta_{\gamma_i} h(\beta_i, \gamma_i) &= (E_{\gamma_i} - 1)h(\beta_i, \gamma_i), \\ E_{\beta_i \gamma_i} h(\beta_i, \gamma_i) &= E_{\beta_i} [E_{\gamma_i} h(\beta_i, \gamma_i)], \end{aligned}$$

where $h(\beta_i, \gamma_i \pm 1) = h(\beta_1, \gamma_1, \dots, \beta_{i-1}, \gamma_{i-1}, \beta_i, \gamma_i \pm 1, \dots, \beta_r, \gamma_r)$.

We obtain the following transforms under I :

$$\begin{aligned} I[u_i f] &= \begin{cases} \frac{\beta_i}{\gamma_i} E_{\beta_i \gamma_i} h, & 1 \leq i \leq k; \\ \frac{\beta_i}{\gamma} E_{\beta_i \gamma} h, & k+1 \leq i \leq m. \end{cases} \\ I \left[u_i \frac{\partial}{\partial u_i} f \right] &= \beta_i \Delta_{\beta_i} h = x_i \frac{\partial}{\partial x_i} h, \quad 1 \leq i \leq r. \end{aligned}$$

Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$ are defined as follows [17]:

$$\begin{aligned} F_A^{(n)} &:= F_A^{(n)}(a; b_1, \dots, b_n; c_1, \dots, c_n; z_1, \dots, z_n) \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} \dots (b_n)_{m_n}}{(c_1)_{m_1} \dots (c_n)_{m_n}} \frac{z_1^{m_1}}{m_1!} \dots \frac{z_n^{m_n}}{m_n!}, \\ &|z_1 + \dots + z_n| < 1; \\ F_D^{(n)} &:= F_D^{(n)}(a; b_1, \dots, b_n; c; z_1, \dots, z_n) \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} \dots (b_n)_{m_n}}{(c)_{m_1+\dots+m_n}} \frac{z_1^{m_1}}{m_1!} \dots \frac{z_n^{m_n}}{m_n!}, \\ &|z_i| < 1, \quad i = 1, \dots, n; \end{aligned} \tag{2.4}$$

and $F_A^{(n)}(a+i)$ stands for $F_A^{(n)}$ in (2.4) with a replaced by $(a+i)$, etc..

The integral representations of Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$ are given by [3]:

$$\begin{aligned} &F_A^{(n)}[\alpha; \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; x_1, \dots, x_n] \\ &= \prod_{i=1}^n \frac{\Gamma(\gamma_i)}{\Gamma(\beta_i) \Gamma(\gamma_i - \beta_i)} \underbrace{\int_0^1 \dots \int_0^1}_{n\text{-fold}} \prod_{i=1}^n u_i^{\beta_i-1} (1-u_i)^{\gamma_i-\beta_i-1} \\ &\quad \times \left(1 - \sum_{i=1}^n u_i x_i\right)^{-\alpha} du_1 \dots du_n, \end{aligned}$$

$$\operatorname{Re} \gamma_i > \operatorname{Re} \beta_i > 0, \quad i = 1, \dots, n, \quad \sum_{i=1}^n |x_i| < 1;$$

$$F_D^{(n)}[\alpha; \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n]$$

$$\begin{aligned}
&= \prod_{i=1}^n \frac{\Gamma(\gamma)}{\Gamma(\beta_i) \Gamma(\gamma - \beta_i)} \underbrace{\int \dots \int}_{n\text{-fold}} \prod_{i=1}^n u_i^{\beta_i-1} (1 - \sum_{i=1}^n u_i)^{\gamma - \sum_{i=1}^n \beta_i - 1} \\
&\quad \times (1 - \sum_{i=1}^n u_i x_i)^{-\alpha} du_1 \dots du_n,
\end{aligned}$$

$\operatorname{Re} \beta_i > 0$, $i = 1, \dots, n$; $\operatorname{Re}(\gamma - \sum_{i=1}^n \beta_i) > 0$, $|x_i| < 1$, $i = 1, \dots, n$; and the path of integration is $u_i \geq 0$, $i = 1, \dots, n$; $\sum_{i=1}^n u_i \leq 1$.

3. $(n+1)$ -VARIABLE MODELS

Theorem 2. Let $u = \phi(z)$ and $z = t(1 - \sum_{i=1}^n z_i)$ then $z \frac{du}{dz} = t \frac{\partial u}{\partial t}$ and $\frac{du}{dz} = \frac{\partial u}{\partial t} - t^{-1} \sum_{i=1}^n z_i \frac{\partial u}{\partial z_i}$.

Proof. We have $z = t(1 - \sum_{i=1}^n z_i)$, which leads to

$$\begin{aligned}
\frac{\partial u}{\partial t} &= (1 - \sum_{i=1}^n z_i) \frac{du}{dz}, \\
\frac{\partial u}{\partial z_i} &= -t \frac{du}{dz}, \quad i = 1, \dots, n.
\end{aligned}$$

From this follows the conclusion of the theorem. $\square$

Directed by Theorems 1 and 2, we give below $(n+1)$ -variable models of the representations $R'(0, m_0, \mu)$ and $\uparrow'_{\omega, \mu}$ of the Lie algebra $\mathcal{K}_5$. Let $\mathcal{F}$ be the space of all analytic and single-valued functions for all $z_i = u_i x_i \neq 1$, $i = 1, 2, \dots, n$ and $t \neq 0$. Multiplier representations of the Lie group K_5 induced by the J -operators are also given.

Model IA:

Representation $R'(0, m_0, \mu)$:

$$\begin{aligned}
J^3 &= t \frac{\partial}{\partial t}, \quad J^+ = \mu t (1 - \sum_{i=1}^n u_i x_i), \quad J^- = t^{-1} \left[t \frac{\partial}{\partial t} - \sum_{i=1}^n u_i \frac{\partial}{\partial u_i} \right], \\
E &= \mu, \quad Q = \mu t^2 (1 - \sum_{i=1}^n u_i x_i)^2, \quad f_\lambda(u_1, \dots, u_n, t) = t^\lambda (1 - \sum_{i=1}^n u_i x_i)^\lambda, \quad (3.1)
\end{aligned}$$

where $\lambda \in S = \{m_0 + n \mid m_0 \in \mathbb{C} - \{0\}, 0 \leq \operatorname{Re} m_0 < 1, n = 0, \pm 1, \dots\}$.

The multiplier representation $T_1(g)f$ of the Lie group K_5 induced by the operators (3.1) on $\mathcal{F}$ is

$$[T_1(g)f](u_1, \dots, u_n, t) = \exp \left\{ \mu \left[qt^2 (1 - \sum_{i=1}^n u_i x_i)^2 + a + bt (1 - \sum_{i=1}^n u_i x_i) \right] \right\}$$

$$\times f \left[\frac{u_1}{(1+\frac{c}{t})}, \dots, \frac{u_n}{(1+\frac{c}{t})}, t e^{\tau} \left(1 + \frac{c}{t} \right) \right], \quad (3.2)$$

where $|\frac{c}{t}| < 1$ and $g = g(q, a, b, c, \tau) \in K_5$.

Model IIA:

Representation $\uparrow'_{\omega, \mu}$:

$$\begin{aligned} J^3 &= -\omega + t \frac{\partial}{\partial t}, \quad J^+ = \mu t \left(1 - \sum_{i=1}^n u_i x_i \right), \quad J^- = t^{-1} \left[t \frac{\partial}{\partial t} - \sum_{i=1}^n u_i \frac{\partial}{\partial u_i} \right], \\ E &= \mu, \quad Q = \mu t^2 \left(1 - \sum_{i=1}^n u_i x_i \right)^2, \quad f_{\lambda}(u_1, \dots, u_n, t) = t^{\lambda+\omega} \left(1 - \sum_{i=1}^n u_i x_i \right)^{\lambda+\omega}, \end{aligned} \quad (3.3)$$

where $\lambda \in S = \{-\omega + n \mid n = 0, 1, \dots\}$.

The multiplier representation $T_2(g)f$ of the Lie group K_5 induced by the operators (3.3) on $\mathcal{F}$ is

$$\begin{aligned} [T_2(g)f](u_1, \dots, u_n, t) &= \exp \left\{ \mu \left[q t^2 \left(1 - \sum_{i=1}^n u_i x_i \right)^2 + a + b t \left(1 - \sum_{i=1}^n u_i x_i \right) \right] - \omega \tau \right\} \\ &\times f \left[\frac{u_1}{(1+\frac{c}{t})}, \dots, \frac{u_n}{(1+\frac{c}{t})}, t e^{\tau} \left(1 + \frac{c}{t} \right) \right], \end{aligned}$$

where $|\frac{c}{t}| < 1$ and $g = g(q, a, b, c, \tau) \in K_5$.

4. TRANSFORMED $(n+1)$ -VARIABLE MODELS OF $\mathcal{K}_5$

To obtain models in terms of difference-differential operators with the basis functions appearing as Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$ respectively, we utilize a theorem from Govil and Manocha [3]:

Theorem 3. Let ρ be an irreducible representation of the Lie algebra $\mathcal{K}_5$ in terms of operators $\{J^3, J^{\pm}, E, Q\}$ on a representation space V with basis functions $\{f_{\lambda} \mid \lambda \in S\}$. Then the transformation I induces another irreducible representation σ of $\mathcal{K}_5$ on the representation space $W = IV$ having basis functions $\{h_{\lambda} \mid \lambda \in S\}$ in terms of operators $\{K^3, K^{\pm}, E', Q'\}$, where

$$\begin{aligned} K^3 &= IJ^3I^{-1}, \quad K^{\pm} = IJ^{\pm}I^{-1}, \quad E' = IEI^{-1}, \quad Q' = IQI^{-1}, \\ h_{\lambda} &= If_{\lambda}, \quad \lambda \in S. \end{aligned}$$

That is, ρ and σ are isomorphic. Indeed, the behavior of the commutation relations satisfied by K -operators is same as that of J -operators.

We give below the transforms of Models IA and IIA discussed above. The new models are in terms of difference-differential operators.

Model IB(i):

$$K^3 = t \frac{\partial}{\partial t}, \quad K^+ = \mu t \left(1 - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma_i} E_{\beta_i \gamma_i} \right), \quad K^- = t^{-1} \left(t \frac{\partial}{\partial t} - \sum_{i=1}^n \beta_i \Delta_{\beta_i} \right),$$

$$E' = \mu, \quad Q' = \mu t^2 \left(1 - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma_i} E_{\beta_i \gamma_i} \right)^2,$$

$$h_\lambda(x_1, \dots, x_n, t) = F_A^{(n)}(-\lambda, \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; x_1, \dots, x_n) t^\lambda,$$

where $\lambda \in S = \{m_0 + n \mid m_0 \in \mathbb{C} - \{0\}, 0 \leq \operatorname{Re} m_0 < 1, n = 0, \pm 1, \dots\}$.

Model IB(ii):

$$K^3 = t \frac{\partial}{\partial t}, \quad K^+ = \mu t \left(1 - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma} E_{\beta_i \gamma} \right), \quad K^- = t^{-1} \left(t \frac{\partial}{\partial t} - \sum_{i=1}^n \beta_i \Delta_{\beta_i} \right),$$

$$E' = \mu, \quad Q' = \mu t^2 \left(1 - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma} E_{\beta_i \gamma} \right)^2,$$

$$h_\lambda(x_1, \dots, x_n, t) = F_D^{(n)}(-\lambda, \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n) t^\lambda,$$

where $\lambda \in S = \{m_0 + n \mid m_0 \in \mathbb{C} - \{0\}, 0 \leq \operatorname{Re} m_0 < 1, n = 0, \pm 1, \dots\}$.

Model IIB(i):

$$K^3 = -\omega + t \frac{\partial}{\partial t}, \quad K^+ = \mu t \left(1 - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma_i} E_{\beta_i \gamma_i} \right), \quad K^- = t^{-1} \left(t \frac{\partial}{\partial t} - \sum_{i=1}^n \beta_i \Delta_{\beta_i} \right),$$

$$E' = \mu, \quad Q' = \mu t^2 \left(1 - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma_i} E_{\beta_i \gamma_i} \right)^2,$$

$$h_\lambda(x_1, \dots, x_n, t) = F_A^{(n)}(-\lambda - \omega, \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; x_1, \dots, x_n) t^{\lambda + \omega},$$

where $\lambda \in S = \{-\omega + n \mid n = 0, 1, \dots\}$.

Model IIB(ii):

$$K^3 = -\omega + t \frac{\partial}{\partial t}, \quad K^+ = \mu t \left(1 - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma} E_{\beta_i \gamma} \right), \quad K^- = t^{-1} \left(t \frac{\partial}{\partial t} - \sum_{i=1}^n \beta_i \Delta_{\beta_i} \right),$$

$$E' = \mu, \quad Q' = \mu t^2 \left(1 - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma} E_{\beta_i \gamma} \right)^2,$$

$$h_\lambda(x_1, \dots, x_n, t) = F_D^{(n)}(-\lambda - \omega, \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n) t^{\lambda + \omega},$$

where $\lambda \in S = \{-\omega + n \mid n = 0, 1, \dots\}$.

The models given above satisfy the following:

$$\begin{aligned} [K^3, K^\pm] &= \pm K^\pm, \quad [K^3, Q'] = 2Q', \quad [K^-, K^+] = E', \\ [K^-, Q'] &= 2K^+, \quad [K^+, Q'] = 0, \quad [K^\pm, E'] = [K^3, E'] = [Q', E'] = 0, \end{aligned}$$

and thus lead to a representation of $\mathcal{K}_5$. Also, Model IB(i), Model IB(ii) satisfy

$$\begin{aligned} K^3 f_\lambda &= \lambda f_\lambda, \quad K^+ f_\lambda = \mu f_{\lambda+1}, \quad K^- f_\lambda = \lambda f_{\lambda-1}, \\ E' f_\lambda &= \mu f_\lambda, \quad Q' f_\lambda = \mu f_{\lambda+2}, \end{aligned}$$

and Model IIB(i), Model IIB(ii) satisfy

$$\begin{aligned} K^3 f_\lambda &= \lambda f_\lambda, \quad K^+ f_\lambda = \mu f_{\lambda+1}, \quad K^- f_\lambda = (\lambda + \omega) f_{\lambda-1}, \\ E' f_\lambda &= \mu f_\lambda, \quad Q' f_\lambda = \mu f_{\lambda+2}. \end{aligned}$$

4.1. Recurrence relations and generating functions

We shall be utilizing Models IA and IIA for obtaining generating functions. Transformed Models IB(i), IB(ii) and IIB(i), IIB(ii) are further exploited for obtaining recurrence relations. To obtain generating functions, we follow the method given in Sahai [12]. We leave details and present the results only as follows:

4.1.1. Recurrence relations

We obtain the following two term and three term recurrence relations using raising and lowering operators, respectively of Model IB(i):

$$\begin{aligned} F_A^{(n)} - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma_i} F_A^{(n)}(\beta_i + 1, \gamma_i + 1) - F_A^{(n)}(-\lambda - 1) &= 0, \\ (\lambda + \sum_{i=1}^n \beta_i) F_A^{(n)} - \sum_{i=1}^n \beta_i F_A^{(n)}(\beta_i + 1) - \lambda F_A^{(n)}(-\lambda + 1) &= 0, \\ F_A^{(n)} - \sum_{i=1}^n \frac{2\beta_i x_i}{\gamma_i} F_A^{(n)}(\beta_i + 1, \gamma_i + 1) + \sum_{j=1}^n \frac{\beta_j^2 x_j^2}{\gamma_j^2} F_A^{(n)}(\beta_j + 2, \gamma_j + 2) \\ + \sum_{1 \leq i \neq j \leq n} \left(\frac{\beta_i x_i}{\gamma_i} \right) \left(\frac{\beta_j x_j}{\gamma_j} \right) F_A^{(n)}(\beta_i + 1, \gamma_i + 1, \beta_j + 1, \gamma_j + 1) - F_A^{(n)}(-\lambda - 2) &= 0, \\ F_A^{(n)}(-\lambda - 1) - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma_i} F_A^{(n)}(\beta_i + 1, \gamma_i + 1) + \sum_{j=1}^n \frac{\beta_j^2 x_j^2}{\gamma_j^2} F_A^{(n)}(\beta_j + 2, \gamma_j + 2) \\ + \sum_{1 \leq i \neq j \leq n} \left(\frac{\beta_i x_i}{\gamma_i} \right) \left(\frac{\beta_j x_j}{\gamma_j} \right) F_A^{(n)}(\beta_i + 1, \gamma_i + 1, \beta_j + 1, \gamma_j + 1) - F_A^{(n)}(-\lambda - 2) &= 0. \end{aligned}$$

Similarly, we obtain the following recurrence relations using Model IB(ii):

$$F_D^{(n)} - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma} F_D^{(n)}(\beta_i + 1, \gamma + 1) - F_D^{(n)}(-\lambda - 1) = 0,$$

$$\begin{aligned}
& (\lambda + \sum_{i=1}^n \beta_i) F_D^{(n)} - \sum_{i=1}^n \beta_i F_D^{(n)}(\beta_i + 1) - \lambda F_D^{(n)}(-\lambda + 1) = 0, \\
& F_D^{(n)} - \sum_{i=1}^n \frac{2\beta_i x_i}{\gamma} F_D^{(n)}(\beta_i + 1, \gamma + 1) + \sum_{j=1}^n \frac{\beta_j^2 x_j^2}{\gamma^2} F_D^{(n)}(\beta_j + 2, \gamma + 2) \\
& + \sum_{1 \leq i \neq j \leq n} \frac{(\beta_i x_i)(\beta_j x_j)}{\gamma^2} F_D^{(n)}(\beta_i + 1, \beta_j + 1, \gamma + 2) - F_D^{(n)}(-\lambda - 2) = 0, \\
& F_D^{(n)}(-\lambda - 1) - \sum_{i=1}^n \frac{\beta_i x_i}{\gamma} F_D^{(n)}(\beta_i + 1, \gamma + 1) + \sum_{j=1}^n \frac{\beta_j^2 x_j^2}{\gamma^2} F_D^{(n)}(\beta_j + 2, \gamma + 2) \\
& + \sum_{1 \leq i \neq j \leq n} \frac{(\beta_i x_i)(\beta_j x_j)}{\gamma^2} F_D^{(n)}(\beta_i + 1, \beta_j + 1, \gamma + 2) - F_D^{(n)}(-\lambda - 2) = 0.
\end{aligned}$$

Similarly, other recurrence relations can be obtained using Models IIB(i) and IIB(ii).

4.1.2. Generating functions

The matrix elements $A_{lk}(g)$ of $T_1(g)f$ of the Lie group K_5 with respect to the basis functions $\{f_{m_0+k} \mid m_0 \in \mathbb{C} \setminus \{0\}, 0 \leq \operatorname{Re} m_0 < 1, k = 0, \pm 1, \dots\}$ are defined by:

$$\begin{aligned}
[T_1(g)f_{m_0+k}](u_1, \dots, u_n, t) &= \sum_{l=-\infty}^{\infty} A_{lk}(g) f_{m_0+l}(u_1, \dots, u_n, t), \quad (4.1) \\
g &= g(q, a, b, c, \tau) \in K_5.
\end{aligned}$$

The special case of (4.1) for the particular case $q = 0$ is obtained by putting an expression for $T_1(g)$ from (3.2) in (4.1) and then computing the matrix elements $A_{lk}(g)$ by comparing the coefficients of t^l on both sides of (4.1). This leads to the following generating function:

$$\begin{aligned}
& \exp \left[\mu b t \left(1 - \sum_{i=1}^n u_i x_i \right) \right] \left(1 - \sum_{i=1}^n \frac{u_i x_i}{1 + \frac{c}{t}} \right)^{m_0+k} \left(1 + \frac{c}{t} \right)^{m_0+k} t^k \quad (4.2) \\
&= \sum_{l=-\infty}^{\infty} c^{k-l} L_{m_0+l}^{k-l}(-\mu bc) \left(1 - \sum_{i=1}^n u_i x_i \right)^{m_0+l} t^l,
\end{aligned}$$

where $k = 0, \pm 1, \dots$ such that $k \geq l$, $|\sum_{i=1}^n u_i x_i| < 1$, $|\sum_{i=1}^n \frac{u_i x_i}{1 + \frac{c}{t}}| < 1$ and $L_{m_0+l}^{k-l}(-\mu bc)$ are Laguerre polynomials defined by [8, 10]:

$$L_v^\alpha(z) = \frac{\Gamma(v + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(v + 1)} {}_1F_1(-v; \alpha + 1; z). \quad (4.3)$$

Similarly, the following generating function is obtained by putting $a = c = \tau = 0$ in (4.1):

$$\begin{aligned} & \exp \left\{ \mu \left[qt^2 \left(1 - \sum_{i=1}^n u_i x_i \right)^2 + bt \left(1 - \sum_{i=1}^n u_i x_i \right) \right] \right\} \left(1 - \sum_{i=1}^n u_i x_i \right)^k t^k \\ &= \sum_{l=-\infty}^{\infty} \frac{(-\mu q)^{(l-k)/2}}{(l-k)!} H_{l-k} \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right) \left(1 - \sum_{i=1}^n u_i x_i \right)^l t^l, \end{aligned} \quad (4.4)$$

where $k = 0, \pm 1, \dots$ such that $l \geq k$, $|\sum_{i=1}^n u_i x_i| < 1$ and $H_{l-k} \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right)$ are Hermite polynomials defined by [8, 10]:

$$\exp(2xy - y^2) = \sum_{l=0}^{\infty} \frac{y^l}{l!} H_l(x). \quad (4.5)$$

Another set of generating function is obtained by putting $a = b = \tau = 0$ in (4.1):

$$\begin{aligned} & \exp \left\{ \mu q t^2 \left(1 - \sum_{i=1}^n u_i x_i \right)^2 \right\} \left(1 + \frac{c}{t} \right)^{m_0+k} \left(1 - \sum_{i=1}^n \frac{u_i x_i}{1 + \frac{c}{t}} \right)^{m_0+k} t^k \\ &= \sum_{l=-\infty}^{\infty} c^{k-l} \Gamma(m_0 + k + 1) \sum_m \frac{(\mu q c^2)^m}{m! (2m + k - l)! \Gamma(m_0 + l - 2m + 1)} \\ & \quad \times \left(1 - \sum_{i=1}^n u_i x_i \right)^{m_0+l} t^l, \end{aligned} \quad (4.6)$$

where m ranges over all integral values so that summand makes sense, $k = 0, \pm 1, \dots$, $|\sum_{i=1}^n u_i x_i| < 1$ and $\left| \sum_{i=1}^n \frac{u_i x_i}{1 + \frac{c}{t}} \right| < 1$.

Similarly, the following generating function is obtained by putting $q = a = c = \tau = 0$ in (4.1):

$$\exp \left[\mu b t \left(1 - \sum_{i=1}^n u_i x_i \right) \right] \left(1 - \sum_{i=1}^n u_i x_i \right)^k t^k = \sum_{l=-\infty}^{\infty} \frac{(\mu b)^{l-k}}{(l-k)!} \left(1 - \sum_{i=1}^n u_i x_i \right)^l t^l, \quad (4.7)$$

where $k = 0, \pm 1, \dots$ such that $l \geq k$ and $|\sum_{i=1}^n u_i x_i| < 1$. We also obtain the following generating function by putting $q = a = b = \tau = 0$ in (4.1):

$$\begin{aligned} & \left(1 + \frac{c}{t} \right)^{m_0+k} \left(1 - \sum_{i=1}^n \frac{u_i x_i}{1 + \frac{c}{t}} \right)^{m_0+k} t^k \\ &= \sum_{l=-\infty}^{\infty} c^{k-l} \binom{m_0+k}{m_0+l} \left(1 - \sum_{i=1}^n u_i x_i \right)^{m_0+l} t^l, \end{aligned} \quad (4.8)$$

where $k = 0, \pm 1, \dots$ such that $k \geq l$, $|\sum_{i=1}^n u_i x_i| < 1$ and $\left| \sum_{i=1}^n \frac{u_i x_i}{1 + \frac{c}{t}} \right| < 1$.

Similarly, the following generating function is obtained by putting $b = c = \tau = 0$ in (4.1):

$$\begin{aligned} & \exp \left[\mu q t^2 \left(1 - \sum_{i=1}^n u_i x_i \right)^2 \right] \left(1 - \sum_{i=1}^n u_i x_i \right)^k t^k \\ &= \sum_{l=-\infty}^{\infty} \frac{(\mu q)^{(l-k)/2}}{\Gamma\left(\frac{l-k}{2} + 1\right)} \left(1 - \sum_{i=1}^n u_i x_i \right)^l t^l, \end{aligned} \quad (4.9)$$

where $k = 0, \pm 1, \dots$ such that $l \geq k$ and $|\sum_{i=1}^n u_i x_i| < 1$.

Another set of generating function is obtained by putting $a = \tau = 0$ in (4.1):

$$\begin{aligned} & \exp \left\{ \mu \left[q t^2 \left(1 - \sum_{i=1}^n u_i x_i \right)^2 + b t \left(1 - \sum_{i=1}^n u_i x_i \right) \right] \right\} \left(1 + \frac{c}{t} \right)^{m_0+k} \\ & \times \left(1 - \sum_{i=1}^n \frac{u_i x_i}{1 + \frac{c}{t}} \right)^{m_0+k} t^k \\ &= \sum_{l=-\infty}^{\infty} c^{k-l} \binom{m_0+k}{m_0+l} \sum_{m=0}^{\infty} \frac{[-(-\mu q)^{1/2} c]^m}{m!} \frac{(-m_0-l)_m}{(k-l+1)_m} H_m \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right) \\ & \times \left(1 - \sum_{i=1}^n u_i x_i \right)^{m_0+l} t^l, \end{aligned} \quad (4.10)$$

where $k = 0, \pm 1, \dots$ such that $k \geq l$, $|\sum_{i=1}^n u_i x_i| < |1 + \frac{c}{t}|$, $|\frac{c}{t}| < 1$ and $|\sum_{i=1}^n u_i x_i| < 1$.

The matrix elements $B_{lk}(g)$ of the $T_2(g)f$ of the Lie group K_5 with respect to the basis functions $\{f_k = t^k (1 - \sum_{i=1}^n u_i x_i)^k \mid k = 0, 1, \dots\}$ are defined by:

$$[T_2(g)f_k](u_1, \dots, u_n, t) = \sum_{l=0}^{\infty} B_{lk}(g) f_l(u_1, \dots, u_n, t), \quad g = g(q, a, b, c, \tau) \in K_5. \quad (4.11)$$

We obtain the following generating functions under special cases of (4.11), similar to (4.2)–(4.10). We list a few of them and remaining can be obtained similarly.

$$\begin{aligned} & \exp \left[\mu b t \left(1 - \sum_{i=1}^n u_i x_i \right) \right] \left(1 - \sum_{i=1}^n \frac{u_i x_i}{1 + \frac{c}{t}} \right)^k \left(1 + \frac{c}{t} \right)^k t^k \\ &= \sum_{l=0}^{\infty} c^{k-l} L_l^{k-l} (-\mu b c) \left(1 - \sum_{i=1}^n u_i x_i \right)^l t^l, \end{aligned}$$

where $k = 0, 1, \dots$ such that $k \geq l \geq 0$, $|\sum_{i=1}^n u_i x_i| < |1 + \frac{c}{t}|$, $|\frac{c}{t}| < 1$;

$$\exp \left\{ \mu \left[q t^2 \left(1 - \sum_{i=1}^n u_i x_i \right)^2 + b t \left(1 - \sum_{i=1}^n u_i x_i \right) \right] \right\} \left(1 - \sum_{i=1}^n u_i x_i \right)^k t^k$$

$$= \sum_{l=0}^{\infty} \frac{(-\mu q)^{(l-k)/2}}{(l-k)!} H_{l-k} \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right) \left(1 - \sum_{i=1}^n u_i x_i \right)^l t^l,$$

where $k = 0, 1, \dots$ such that $l \geq k \geq 0$, $|\sum_{i=1}^n u_i x_i| < 1$.

5. ADDITION THEOREMS

The matrix elements $A_{lk}(g)$ of the multiplier representations $T_1(g)f$ of the Lie group K_5 satisfy the addition theorem:

$$A_{lk}(g_1 g_2) = \sum_{j=-\infty}^{\infty} A_{lj}(g_1) A_{jk}(g_2), \quad l, k = 0, \pm 1, \pm 2, \dots \quad (5.1)$$

We now enumerate some special cases of (5.1) when $g_1 = g_1(q, a, b, c, \tau)$ and $g_2 = g_2(q', a', b', c', \tau')$ are chosen suitably. For example, if we assign $q = a = b = \tau = q' = a' = c' = \tau' = 0$ and $b' = b$ then (5.1) leads to

$$e^{\mu b c} c^{k-l} L_{m_0+l}^{k-l}(-\mu b c) = \sum_{j=k}^{\infty} c^{j-l} \binom{m_0+j}{m_0+l} \frac{(\mu b)^{j-k}}{(j-k)!}. \quad (5.2)$$

Similarly, assigning $b = c = \tau = q' = a' = c' = \tau' = 0$ and $b' = b$ in (2.1) gives the following addition theorem

$$\frac{(-\mu q)^{(l-k)/2}}{(l-k)!} H_{l-k} \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right) = \sum_{j=k}^{\infty} \frac{(\mu q)^{(l-j)/2}}{\Gamma(\frac{l-j}{2} + 1)} \frac{(\mu b)^{j-k}}{(j-k)!}. \quad (5.3)$$

Another addition theorem can be obtained by putting $a = \tau = q' = a' = c' = \tau' = 0$ and $b' = b$. This yields

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{[-(-\mu q)^{1/2} c]^n}{n!} \frac{(-m_0-l)_n}{(k-l+1)_n} H_n \left(\frac{\mu b}{(-\mu q)^{1/2}} \right) \\ &= \sum_{j=k}^{\infty} (\mu b c)^{j-k} \binom{m_0+j}{m_0+k} \frac{(k-l)!}{(j-l)!} \sum_{m=0}^{\infty} \frac{[-(-\mu q)^{1/2} c]^m}{m!} \frac{(-m_0-l)_m}{(j-l+1)_m} \\ & \quad \times H_m \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right). \end{aligned} \quad (5.4)$$

Finally, we put $a = \tau = q' = a' = c' = \tau' = 0$ and $c' = c$ resulting in the following addition theorem

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{[-2(-\mu q)^{1/2} c]^n}{n!} \frac{(-m_0-l)_n}{(k-l+1)_n} 2^{k-l} H_n \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right) \\ &= \sum_{j=l}^k \binom{k-l}{j-l} \sum_{m=0}^{\infty} \frac{[-(-\mu q)^{1/2} c]^m}{m!} \frac{(-m_0-l)_m}{(j-l+1)_m} H_m \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right). \end{aligned} \quad (5.5)$$

Further, the matrix elements $B_{lk}(g)$ of the multiplier representations $T_2(g)f$ of the Lie group K_5 satisfy the addition theorem:

$$B_{lk}(g_1 g_2) = \sum_{j=0}^{\infty} B_{lj}(g_1) B_{jk}(g_2), \quad l, k = 0, 1, 2, \dots \quad (5.6)$$

Using the same choices for $g_1 = g_1(q, a, b, c, \tau)$ and $g_2 = g_2(q', a', b', c', \tau')$ as in (5.2)–(5.5), we obtain the following addition theorems from (5.6)

$$\begin{aligned} e^{\mu bc} c^{k-l} L_l^{k-l}(-\mu bc) &= \sum_{j=k}^{\infty} c^{j-l} \binom{j}{l} \frac{(\mu b)^{j-k}}{(j-k)!}, \text{ provided } j \geq l \geq 0; \\ \frac{(-\mu q)^{n/2}}{n!} H_n \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right) &= \sum_{m=0}^n \frac{(\mu q)^{(n-m)/2}}{\Gamma(\frac{n-m}{2} + 1)} \frac{(\mu b)^m}{m!}, \\ \sum_{n=0}^{\infty} \frac{[-(-\mu q)^{1/2} c]^n}{n!} \frac{(-l)_n}{(k-l+1)_n} H_n \left(\frac{\mu b}{(-\mu q)^{1/2}} \right) \\ &= \sum_{j=k}^{\infty} (\mu bc)^{j-k} \binom{j}{k} \frac{(k-l)!}{(j-l)!} \sum_{m=0}^{\infty} \frac{[-(-\mu q)^{1/2} c]^m}{m!} \frac{(-l)_m}{(j-l+1)_m} H_m \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right), \\ \sum_{n=0}^{\infty} \frac{[-2(-\mu q)^{1/2} c]^n}{n!} \frac{(-l)_n}{(k-l+1)_n} 2^{k-l} H_n \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right) \\ &= \sum_{j=l}^k \binom{k-l}{j-l} \sum_{m=0}^{\infty} \frac{[-(-\mu q)^{1/2} c]^m}{m!} \frac{(-l)_m}{(k-l+1)_m} H_m \left(\frac{\mu b}{2(-\mu q)^{1/2}} \right), \end{aligned}$$

respectively.

6. CONCLUSION

We have obtained recurrence relations and generating functions involving basis functions as Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$ respectively, for $(n+1)$ -variable models of $\mathcal{H}_5$. Results for $n=1$ and $n=2$ will lead to the corresponding relations involving basis functions as Gauss hypergeometric function ${}_2F_1$ and Appell functions F_2, F_1 respectively.

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ATTRACTING SETS OF IMPULSE-PERTURBED HEAT EQUATION IN THE SPACE OF CONTINUOUS FUNCTIONS

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Abstract. The paper deals with the impulsive infinite-dimensional problem generated by solutions of the thermal conductivity equation under the condition of impulse "pumping" of heat. Moments of impulses are not fixed and are determined by the amount of total heat in the system. It is proved that such a problem generates impulsive dynamical system in the space of continuous functions and its ω -limits sets are investigated.

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1. INTRODUCTION

Many papers [1, 3, 10–13, 16] are devoted to the qualitative behavior of impulsive dynamical systems, in particular to the study of ω -boundary sets. The number of papers establish the solvability and properties of solutions for various types systems such that Navier-Stokes systems [9], linear differential equation systems with a deviating argument [14], systems of functional differential equations in both linear and nonlinear cases [15, 17], nonlinear systems of viscoelasticity with a memory term [6].

In infinite-dimensional phase spaces, such studies are recently carried out in papers [2, 4, 5, 8] and concerned the study of uniform attraction sets for evolutionary impulse-perturbed systems in Hilbert spaces. In particular, using L^2 -theory, in [4, 5] it is investigated qualitative behaviour of some parabolic impulse-perturbed system, where impulsive set was given as seminorm in phase space. As authors know, only in paper [16] the limit modes of impulse-perturbed heat equation in the space of continuous functions are studied.

In this paper we treat the impulsive infinite-dimensional problem generated by solutions of the heat equation under the condition of impulsive "pumping" of heat. Moments of impulse are not fixed and are determined by the amount of total heat in

the system. We prove that such a problem generates impulsive dynamical system in the space of continuous functions and its ω -limit sets are investigated.

2. PROBLEM SETTING

Let a function $u = u(t, x)$, $t > 0$, $x \in (0, \pi)$ be defined from the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + h(x), \\ u|_{x=0} = u|_{x=\pi} = 0, \\ u|_{t=0} = u_0(x), \end{cases} \quad (2.1)$$

where $h \in C([0, \pi])$ is given, $u_0 \in X = \{u \in C([0, \pi]) | u(0) = u(\pi) = 0\}$ is a phase space of (2.1) with the norm $\|u\| = \max_{x \in [0, \pi]} |u(x)|$.

Let's consider a functional $\psi : X \rightarrow \mathbb{R}$ in form

$$\psi(u) = \int_0^\pi u(x) dx,$$

that determines the total amount of heat in the system.

We formulate the following impulse problem [13]: if at time t the functional ψ reaches a fixed value $\psi_0 > 0$ on the non-negative solution of problem (2.1), then at this moment there is an instantaneous "pumping" of heat by α , and system (2.1) is in a new position

$$u(t, x) + \alpha(x), \quad (2.2)$$

where $\alpha(x) \geq 0$ is given, and $\int_0^\pi \alpha(x) dx = \alpha_0 > 0$.

We show that problem (2.1), (2.2) generates an impulsive dynamical system [13], i.e. its dynamics is given by a continuous semigroup $V : \mathbb{R}_+ \times X \rightarrow X$, the trajectories of which, when a fixed set $M \subset X$ (impulse set) is reached, are transferred by mapping $I : M \rightarrow X$ (impulse mapping) to the new position Iu . In this case, for the correct setting of such a system requires that the following conditions are met

$$\begin{cases} V : \mathbb{R}_+ \times X \rightarrow X \text{ is continuous semigroup,} \\ M \subset X \text{ is closed, } M \cap IM = \emptyset, \\ \forall u_0 \in M \exists \tau = \tau(u_0) > 0 \forall t \in (0, \tau) V(t, u_0) \notin M. \end{cases} \quad (2.3)$$

Introduce the following notations:

$$u^+ := Iu \text{ for } u \in M,$$

$$M^+(u_0) = \left(\bigcup_{t>0} V(t, u_0) \right) \cap M \text{ for } u_0 \in X.$$

If the property (2.3) holds and $M^+(u_0) \neq \emptyset$, then

$$\exists s = s(u_0) > 0 \forall t \in (0, s) V(t, u_0) \notin M, V(s, u_0) \in M. \quad (2.4)$$

Then for every $u_0 \in X$ the impulse trajectory $\{\tilde{V}(t, u_0), t \geq 0\}$ is constructed in such way [5]:

- if $M^+(u_0) = \emptyset$, then $\tilde{V}(t, u_0) = V(t, u_0)$, $t \geq 0$,
- if $M^+(u_0) \neq \emptyset$ then for $s_0 := s(u_0)$, $u_1 = V(s_0, u_0)$ and

$$\tilde{V}(t, u_0) = \begin{cases} V(t, u_0), & t \in [0, s_0), \\ u_1^+, & t = s_0, \end{cases}$$

- if $M^+(u_1^+) = \emptyset$, then $\tilde{V}(t, u_0) = V(t - s_0, u_1^+)$, $t \geq s_0$,
- if $M^+(u_1^+) \neq \emptyset$, then for $s_1 := s(u_1^+)$, $u_2 = V(s_1, u_1^+)$ and

$$\tilde{V}(t, u_0) = \begin{cases} V(t - s_0, u_1^+), & t \in [s_0, s_0 + s_1), \\ u_2^+, & t = s_0 + s_1, \end{cases}$$

and etc. As a result we obtain finite or infinite number of impulse points $\{u_{n+1}^+ = IV(s_n, u_n^+), u_0^+ = u_0\}$ and corresponding time moments

$$T_{n+1} := \sum_{k=1}^n s_k, \quad T_0 := 0,$$

and $\tilde{V}$ is defined by formula

$$\tilde{V}(t, u_0) = \begin{cases} V(t - T_n, u_n^+), & t \in [T_n, T_{n+1}), \\ u_{n+1}^+, & t = T_{n+1}. \end{cases} \quad (2.5)$$

There is another condition

$$\forall u_0 \in X \text{ trajectory } t \mapsto \tilde{V}(t, u_0) \text{ is defined on } [0, +\infty), \quad (2.6)$$

i.e. either the number of impulses is not more than finite or

$$T_n \rightarrow \infty, \quad n \rightarrow \infty.$$

When the conditions (2.3), (2.6) are fulfilled, $\tilde{V}: \mathbb{R}_+ \times X \rightarrow X$ is a semigroup with the right-continuous trajectories [7], which is called impulsive dynamical system. In this paper we prove that problem (2.1), (2.2) generates impulsive dynamical system (2.5) and study its ω -limit sets

$$\omega(u_0) = \bigcap_{T>0} \overline{\bigcup_{t>T} \tilde{V}(t, u_0)}.$$

3. RESULTS

The solutions of problem (2.1) generate the continuous semigroup $V: \mathbb{R}_+ \times X \rightarrow X$ by formula

$$V(t, u_0) = \sum_{k=1}^{\infty} \left\{ u_k^0 e^{-k^2 t} + \frac{h_k}{k^2} (1 - e^{-k^2 t}) \right\} \sin kx, \quad (3.1)$$

where $u_k^0 = \frac{2}{\pi} \int_0^\pi u_0(x) \sin kx dx$.

Impulsive set has a form

$$M = \{u \in X | u(x) \geq 0, \Psi(u) = \Psi_0\}. \quad (3.2)$$

Impulsive mapping $I: M \rightarrow X$ has a form

$$(Iu)(x) = u(x) + \alpha(x). \quad (3.3)$$

Theorem 1. *Let*

$$h(x) = \sum_{k=1}^{\infty} h_k \sin kx, \quad h_{2k-1} \leq 0, \quad \sum_{k=1}^{\infty} |h_k| < H, \quad (3.4)$$

$$\alpha(x) = O(x), \quad x \rightarrow 0, \quad \alpha(x) = O(\pi - x), \quad x \rightarrow \pi.$$

Then problem (3.1)-(3.3) generates impulsive dynamical system, i.e. the conditions (2.3), (2.6) hold.

Proof. Since the semigroup $V: \mathbb{R}_+ \times X \rightarrow X$ is continuous, the set $M \subset X$ is closed, for $u \in M$

$$\int_0^\pi (Iu)(x) dx = \Psi_0 + \alpha_0 > \Psi_0,$$

then $M \cap IM = \emptyset$.

For further thinking, we will show that for $u_0(x) \geq 0$ (but $u_0(x) \not\equiv 0$) the function $t \mapsto \Psi(V(t, u_0))$ decreases at interval $[0, +\infty)$.

Let us denote $u(t, x) = V(t, u_0)$. Then

$$u(t, x) = V(t, u_0) = v(t, x) + \omega(t, x),$$

where

$$v(t, x) = \sum_{k=1}^{\infty} u_k^0 e^{-k^2 t} \sin kx \text{ is the solution of (2.1) with } h \equiv 0,$$

$$\omega(t, x) = \sum_{k=1}^{\infty} \frac{h_k}{k^3} (1 - e^{-k^2 t}) \sin kx \text{ is the solution of (2.1) with } u_0 \equiv 0.$$

Since

$$\Psi(\omega(t)) = \sum_{k=1}^{\infty} \frac{h_k}{k^2} (1 - e^{-k^2 t}) (1 - (-1)^k) = 2 \sum_{k=1}^{\infty} \frac{h_{2k-1}}{(2k-1)^3} (1 - e^{-(2k-1)^2 t}),$$

then for $t_2 > t_1$

$$\Psi(\omega(t_2)) - \Psi(\omega(t_1)) = 2 \sum_{k=1}^{\infty} \frac{h_{2k-1}}{(2k-1)^3} (e^{-(2k-1)^2 t_1} - e^{-(2k-1)^2 t_2}) \leq 0.$$

So, it's sufficient to prove that $\forall t_2 > t_1 \quad \Psi(v(t_2)) < \Psi(v(t_1))$.

If this is not the case, then for some $t_2 > t_1$

$$\int_0^{\pi} v(t_2, x) dx = \int_0^{\pi} v(t_1, x) dx.$$

Then, by integrating the equation $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$, we obtain

$$\begin{aligned} v(t_2, x) - v(t_1, x) &= \int_{t_1}^{t_2} \frac{\partial^2 v}{\partial x^2} dt, \\ \int_0^{\pi} (v(t_2, x) - v(t_1, x)) dx &= \int_{t_1}^{t_2} (v_x(t, \pi) - v_x(t, 0)) dt. \end{aligned} \quad (3.5)$$

Since the maximum principle the inequality $v(t, x) \geq 0$ holds $\forall t \geq 0$, then

$$v(t, 0) = v(t, \pi) = 0 \Rightarrow v_x(t, \pi) \leq 0, \quad v_x(t, 0) \geq 0.$$

Therefore the left part of equality (3.5) equal to zero:

$$0 = \int_0^{\pi} (v(t_2, x) - v(t_1, x)) dx,$$

and for the difference from the right side of (3.5), the inequality holds

$$v_x(t, \pi) - v_x(t, 0) \leq 0.$$

Hence

$$\begin{aligned} \forall t \in [t_1, t_2] \quad v_x(t, \pi) = v_x(t, 0) &\Rightarrow \\ v_x(t, \pi) = 0, \quad v_x(t, 0) = 0 &\quad \forall t \in [t_1, t_2] \quad \Rightarrow u_0(x) \equiv 0. \end{aligned}$$

Thus, the function $t \mapsto \Psi(V(t, u_0))$ strictly decreases.

Then, if $u_0 \in M$, then $\forall t > 0$

$$\Psi(V(t, u_0)) < \Psi(u_0) = \Psi_0 \Rightarrow V(t, u_0) \notin M.$$

Thus, the conditions (2.3) hold. Let us prove the condition (2.6).

If $M^+(u_0) = \emptyset$ (in particular, if $u_0 \in M$), then trajectory is not subject to impulse perturbations, and hence it exists for $\forall t \geq 0$. The same conclusion about global existence can be made for trajectories with a finite number of perturbations. Therefore, we will assume that the trajectory has an infinite number of perturbations.

Suppose that $M^+(u_0) \neq \emptyset$. Then $\exists s_0 > 0: u(s_0) \in M$. Therefore for $u_1^+ = IV(s_0, u_0)$ we obtain

$$u_1^+(x) = u(s_0) + \alpha(x), \quad \Psi(u_1^+) = \Psi_0 + \alpha_0 > \Psi_0.$$

Let $s_1 = s(u_1^+)$. Due to (3.4) we get

$$\exists A > 0 : \alpha(x) \leq A \sin x, \quad x \in [0, \pi].$$

Further, $\forall t \geq 0 \quad \omega(t, \cdot) \in \mathbb{C}^1([0, \pi])$, moreover

$$\forall t \geq 0 \quad \forall x \in [0, \pi] \quad |\omega(t, x)| \leq \sum_{k=1}^{\infty} \frac{|h_k|}{k} (1 - e^{-k^2 t}) \leq H.$$

So, $\forall s \geq 0$

$$|\omega(s, x)| \leq \bar{H} \cdot \sin x, \quad \text{where } \bar{H} \geq H. \quad (3.6)$$

Finally, since $v(s_0, \cdot) \in \mathbb{C}^1([0, \pi])$, then there exists $C_0 = C_0(u_0) > 0$ that

$$v(s_0, x) \leq C_0 \cdot \sin x. \quad (3.7)$$

therefore

$$u_1^+(x) \leq (C_0 + \bar{H} + A) \cdot \sin x. \quad (3.8)$$

From the maximum principle for solution of problem

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \\ v|_{x=0} = v|_{x=\pi} = 0, \\ v|_{t=0} = u_1^+(x) \end{cases}$$

we obtain

$$v(t, x) \leq (C_0 + \bar{H} + A)e^{-t} \cdot \sin x.$$

Then

$$\max\{v_x(t, 0) - v_x(t, \pi)\} \leq (C_0 + \bar{H} + A)e^{-t}.$$

So, for $t > 0$ and for $u(t, x) = v(t, x) + \omega(t, x)$

$$\begin{aligned} \frac{d}{dt} \Psi(u(t)) &= v_x(t, \pi) - v_x(t, 0) + 2 \sum_{k=1}^{\infty} \frac{h_{2k-1}}{2k-1} e^{-(2k-1)^2 t} \\ &\geq -2(C_0 + \bar{H} + A)e^{-t} - 2\bar{H}. \end{aligned} \quad (3.9)$$

If $u(s_1) \in M$, then from inequality (3.9) we deduce

$$\begin{aligned} \Psi_0 &\geq \Psi_0 + \alpha_0 - 2(C_0 + \bar{H} + A)(1 - e^{-s_1}) - 2\bar{H} \cdot s_1 \\ (C_0 + \bar{H} + A)(1 - e^{-s_1}) + \bar{H} \cdot s_1 &\geq \frac{\alpha_0}{2}. \end{aligned} \quad (3.10)$$

From (3.10) we deduce that

$$s_1 \geq \tau_1,$$

where τ_1 is the solution of equation

$$(C_0 + \bar{H} + A)(1 - e^{-\tau}) + \bar{H} \cdot \tau = \frac{\alpha_0}{2}. \quad (3.11)$$

Considering the estimate $1 - e^{-x} \leq x$, $x \geq 0$, we get from the formula (3.11) that

$$s_1 \geq \frac{\alpha_0}{2(C_0 + \bar{H} + A)}.$$

Then

$$\begin{aligned} u_2^+ &= IV(s_1, u_1^+) = v(s_1) + \omega(s_1) + \alpha(x) \\ u_2^+(x) &\leq (C_0 + \bar{H} + A)e^{-s_1} \cdot \sin x + \bar{H} \sin x + A \sin x \\ &\leq (C_0 + 2(\bar{H} + A)) \sin x. \end{aligned}$$

Then, repeating the previous considerations, we obtain

$$s_2 \geq \frac{\alpha_0}{2(C_0 + \bar{H} + 2(\bar{H} + A))}.$$

In the n th step we get an estimate

$$s_n \geq \frac{\alpha_0}{2(C_0 + \bar{H} + n(\bar{H} + A))}, \quad (3.12)$$

which means that $\sum_{n=0}^{\infty} s_n = \infty$. Thus, the condition 2.6 holds. $\square$

The following theorem is the main result of this article.

Theorem 2. *Let the conditions of Theorem 1 hold and moreover, for the function*

$$\alpha(x) = \sum_{k=1}^{\infty} \alpha_k \sin kx$$

we get

$$\forall n \geq 2 \quad \alpha_{2n-1} \leq 0, \quad \sum_{n=2}^{\infty} \frac{1}{2n-1} |\alpha_{2n-1}| < \alpha_1.$$

Then for $\forall u_0 \in X$ ω -limit set

$$\omega(u_0) = \bigcap_{T>0} \overline{\bigcup_{t \geq T} \tilde{V}(t, u_0)}$$

is nonempty, compact subset of X and

$$\text{dist}(\tilde{V}(t, u_0), \omega(u_0)) \rightarrow 0, \quad t \rightarrow \infty. \quad (3.13)$$

Proof. We will prove that

$$\forall t_m \nearrow \infty \quad \text{a sequence } \xi_m = \tilde{V}(t_m, u_0) \text{ is precompact in } X. \quad (3.14)$$

From (3.14) it evidently follows that $\omega(u_0) \neq \emptyset$, it is compact and satisfies the condition (3.13). To do this, we first show that for the sequence $\{s_n\}_{n \geq 0}$, except the estimate (3.12), the following estimate is valid:

$$\exists \bar{s} = \bar{s}(u_0) > 0: \quad \inf_{k \geq 1} s_k \geq \bar{s}. \quad (3.15)$$

Based on the formula (3.1), for $T_{k+1} = \sum_{i=0}^k s_i$ we have equality

$$\Psi_0 = 2 \sum_{j=1}^{\infty} \frac{1}{2j-1} \left\{ \alpha_{2j-1} \cdot \sum_{i=0}^k e^{-(2j-1)^2(T_{k+1}-T_i)} + u_{2j-1}^0 \cdot e^{-(2j-1)^2 T_{k+1}} \right\}$$

$$+ 2 \sum_{j=1}^{\infty} \frac{h_{2j-1}}{(2j-1)^3} (1 - e^{-(2j-1)^2 T_{k+1}}). \quad (3.16)$$

According [13], we will rewrite the formula (3.16) in the form

$$\begin{aligned} \frac{\Psi_0}{2} = & \sum_{j=1}^{\infty} \frac{1}{2j-1} \left\{ \alpha_{2j-1} \cdot \sum_{i=0}^k e^{-(2j-1)^2 \sum_{r=i}^k s_r} + u_{2j-1}^0 \cdot e^{-(2j-1)^2 \sum_{r=i}^k s_r} \right\} \\ & + \sum_{j=1}^{\infty} \frac{h_{2j-1}}{(2j-1)^3} (1 - e^{-(2j-1)^2 \sum_{r=i}^k s_r}). \end{aligned}$$

With this value $k \geq 2$, we multiply this equality by a multiplier e^{s_k} and subtract equality (3.16) with k reduced by 1. Then we get

$$\begin{aligned} \frac{\Psi_0}{2} (e^{s_k} - 1) = & \alpha_1 + \sum_{j=2}^{\infty} \frac{1}{2j-1} \alpha_{2j-1} \left\{ e^{-4j(j-1)s_k} - (1 - e^{-4j(j-1)s_k}) \times \right. \\ & \times \sum_{i=1}^{k-1} e^{-(2j-1)^2 \sum_{r=i}^{k-1} s_r} \left. \right\} - \sum_{j=1}^{\infty} \frac{1}{2j-1} u_{2j-1}^0 \cdot e^{-(2j-1)^2 T_k} (1 - e^{-4j(j-1)s_k}) \\ & + \sum_{j=1}^{\infty} \frac{h_{2j-1}}{(2j-1)^3} \cdot \left((1 - e^{-(2j-1)^2 T_{k+1}}) \cdot e^{s_k} - (1 - e^{-(2j-1)^2 T_k}) \right). \end{aligned} \quad (3.17)$$

Suppose that up to some subsequence there is convergence $s_k \rightarrow 0+$, $k \rightarrow \infty$. Then the left side of the equation (3.17) tends to 0+. According that under Theorem 1 $T_k \rightarrow \infty$, we deduce that the term with the value u_{2j-1}^0 tends to zero. For the same reasons, the last term with the value h_{2j-1}^0 also tends to zero. Let's analyze the second term. Here

$$\begin{aligned} \beta_j^k &:= e^{-4j(j-1)s_k} - (1 - e^{-4j(j-1)s_k}) \cdot \sum_{i=1}^{k-1} e^{-(2j-1)^2 \sum_{r=i}^{k-1} s_r} \\ &\leq e^{-4j(j-1)s_k} - (1 - e^{-4j(j-1)s_k}) \cdot e^{-(2j-1)^2 \cdot s_{k-1}}. \end{aligned}$$

Hence, $\lim_{k \rightarrow \infty} \beta_j^k \leq 1$. If $\lim_{k \rightarrow \infty} \beta_j^k \in [0, 1]$, then

$$\lim_{k \rightarrow \infty} \left(\alpha_1 + \sum_{j=2}^{\infty} \frac{1}{2j-1} \cdot \alpha_{2j-1} \cdot \beta_j^k \right) \geq \alpha_1 - \sum_{j=2}^{\infty} \frac{1}{2j-1} \cdot |\alpha_{2j-1}| > 0$$

and we have a contradiction.

If $\lim_{k \rightarrow \infty} \beta_j^k \leq 0$, then

$$\lim_{k \rightarrow \infty} \left(\alpha_1 + \sum_{j=2}^{\infty} \frac{1}{2j-1} \cdot \alpha_{2j-1} \cdot \beta_j^k \right) \geq \alpha_1 > 0$$

and also we have a contradiction. Thus, estimate (3.15) holds.

Now we show that $\forall u_0 \in X$ the set $\bigcup_{t \geq 0} \tilde{V}(t, u_0)$ is bounded. From the estimate

$$\forall t \geq s \geq 0 \quad \|V(t, u_0)\|_\infty \leq M_1 \|V(s, u_0)\| e^{-(t-s)} + M_2 \quad (3.18)$$

we obtain the required statement for non-impulsive trajectories, or for trajectories with a finite number of impulsive perturbations. Hence, let the trajectory with an infinite number of perturbations start from the point u_0 . Repeating the arguments (3.6) - (3.8) from the proof of Theorem 1, as well as using the estimate (3.15), we obtain

$$\begin{aligned} u_1^+(x) &\leq (C_0 + \bar{H} + A) \cdot \sin x \Rightarrow \|u_1^+\| \leq C_0 + \bar{H} + A, \\ u(s_1, x) &\leq ((C_0 + \bar{H} + A)e^{-\bar{s}} + \bar{H}) \sin x, \\ u_2^+(x) &\leq ((C_0 + \bar{H} + A)e^{-\bar{s}} + H + A) \sin x \Rightarrow \|u_2^+\| \leq C_0 e^{-\bar{s}} + (H + A)(1 + e^{-\bar{s}}), \\ u(s_2, x) &\leq (((C_0 + \bar{H} + A)e^{-\bar{s}} + H + A)e^{-\bar{s}} + H) \sin x, \\ u_3^+(x) &\leq (((C_0 + \bar{H} + A)e^{-\bar{s}} + H + A)e^{-\bar{s}} + H + A) \sin x \Rightarrow \\ \|u_3^+\| &\leq C_0 e^{-2\bar{s}} + (H + A)(1 + e^{-\bar{s}} + e^{-2\bar{s}}). \end{aligned}$$

Continuing this process, at the n th step we obtain that

$$\begin{aligned} \|u_3^+\| &\leq C_0 e^{-n\bar{s}} + (H + A)(1 + e^{-\bar{s}} + e^{-2\bar{s}} + \dots + e^{-n\bar{s}}) \\ &\leq C_0 e^{-n\bar{s}} + (H + A)(1 - e^{-\bar{s}})^{-1}. \end{aligned} \quad (3.19)$$

From estimates (3.18) and (3.19) we derive

$$\exists M = M(u_0) > 0 \quad \sup_{t \geq 0} \|\tilde{V}(t, u_0)\| \leq M. \quad (3.20)$$

Now we prove (3.14). It is known [7] that the following estimate holds:

$$\|V(t, u_0)\|_{C^1} \leq \frac{K_1}{t^\delta} \|u_0\| + \frac{K_2 t^{1-\delta}}{1-\delta} \|h\|, \quad (3.21)$$

where constants $\delta \in (0, 1)$, $K_1 > 0$, $K_2 > 0$ don't depend on t, u_0 . Then from (3.15), (3.20), (3.21) we obtain that

$$\exists K = K(u_0) \quad \sup_{n \geq 1} \|u_0(s_n)\|_{C^1} \leq K. \quad (3.22)$$

This means that sequence $\{u_n^+ = u(s_n) + \alpha\}_{n=1}^\infty$ is precompact in X .

Then for $t_m \nearrow +\infty \quad \exists n = n(m) \geq 1, \quad n(m) \rightarrow \infty, \quad m \rightarrow \infty$ such that $t_m \in [T_{n(m)}, T_{n(m)+1})$.

Therefore, according to (2.5), $\xi_m = \tilde{V}(t_m, u_0) = V(t_m - T_{n(m)}, u_{n(m)}^+)$.

If $t_m - T_{n(m)} \rightarrow \infty, \quad m \rightarrow \infty$, then for sufficient large m

$$\xi_m = V(1, V(t_m - T_{n(m)} - 1, u_{n(m)}^+)) = V(1, \tilde{V}(t_m - T_{n(m)} - 1, u_{n(m)}^+)),$$

thus,

$$\|\xi_m\|_{\mathbb{C}^1} \leq K \cdot M + \frac{K_2}{1-\delta} \|h\|,$$

that means the precompactness of $\{\xi_m\}$.

Otherwise up to some subsequence $t_m - T_{n(m)} \rightarrow \tau \geq 0$ and then precompactness of $\{\xi_m\}$ follows from precompactness of sequence $\{u_{n(m)}^+\}$ and continuity of semigroup V . Theorem is proven. $\square$

Remark 1. In the case of condition $M^+(u_0) \neq \emptyset$ the structure of the function $\omega(u_0)$ can be complicated. In particular, [13] states that under conditions of Theorems 1 and 2 there is a discontinuous impulsive cycle in the system (2.1), (2.2).

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FRACTIONAL INTEGRAL INEQUALITIES FOR PREINVEX FUNCTIONS VIA ATANGANA-BALEANU INTEGRAL OPERATORS

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Abstract. Different types of functions have been used in many various areas of mathematics until today, and today they are used to obtain new equality and inequalities in fractional analysis, which is one of the important areas of mathematics. In this study, we obtained integral inequalities for preinvex functions with the help of Atangana Baleanu fractional integral operator.

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1. INTRODUCTION

In this section, let us recall well known definitions and concepts as follows.

Firstly, we recall the definition of invex set that has a clear geometric interpretation as follows.

Definition 1 (see [9]). Let $\mu : \mathbb{X} \times \mathbb{X} \neq \emptyset \rightarrow \mathbb{R}$ be a real valued function, then $\mathbb{X}$ is said to be invex with respect to $\mu(.,.)$ if $k_1 + t\mu(k_2, k_1) \in \mathbb{X}$, $\forall k_1, k_2 \in \mathbb{X}$ and $t \in [0, 1]$.

Note that, every convex set is also invex with respect to $\mu(k_2, k_1) = k_2 - k_1$, but every invex set is not necessarily convex (see [9]).

The concept of function, whose emergence in the world of mathematics dates back to ancient times, is seen as one of the distinguishing features between classical and modern mathematics. The concept of function, which was defined in various ways and developed by mathematicians, was first introduced in the 17th century, when the basic objects of mathematics were taken as geometric curves. Functions are also used in science branches such as physics, biology, engineering, apart from mathematics. There are many types of functions and one of these types of functions is preinvex functions. This interesting class of functions is defined as follows.

Definition 2 (see, e.g, [34]). Let $\mathbb{X} \neq \emptyset \subseteq \mathbb{R}$ be an invex set with respect to $\mu : \mathbb{X} \times \mathbb{X} \neq \emptyset \rightarrow \mathbb{R}$. Then the function $f : \mathbb{X} \rightarrow \mathbb{R}$ is said to be preinvex with respect to μ if

$$f(k_1 + t\mu(k_2, k_1)) \leq (1-t)f(k_1) + tf(k_2), \quad \forall k_1, k_2 \in \mathbb{X}, t \in [0, 1]. \quad (1.1)$$

We say that f is preincave if $(-f)$ is preinvex. If we take $\mu(k_2, k_1) = k_2 - k_1$ in (1.1), preinvex functions reduce to convex functions in the classical sense. Then, it is obvious that every convex function is a preinvex function but every preinvex function is not convex. For example, the function $f(x) = -|x|$ is not a convex function, but it is a preinvex function with respect to μ , where

$$\mu(k_2, k_1) = \begin{cases} k_2 - k_1, & \text{if } k_1 \leq 0, k_2 \leq 0 \text{ and } k_1 \geq 0, k_2 \geq 0 \\ k_1 - k_2, & \text{otherwise.} \end{cases}$$

Mohan and Neogy [22] introduced a condition defined as follows:

Condition 1. Let $A \subseteq \mathbb{R}$ be an invex set with respect to $\mu : A \times A \rightarrow \mathbb{R}$. We say that the function μ satisfies Condition 1, if for any $x, y \in A$ and any $t \in [0, 1]$,

$$\begin{aligned} \mu(y, y + t\mu(x, y)) &= -t\mu(x, y) \\ \mu(x, y + t\mu(x, y)) &= (1-t)\mu(x, y). \end{aligned} \quad (1.2)$$

Note that for every $x, y \in A$ and every $t_1, t_2 \in [0, 1]$ from Condition 1, we have

$$\mu(y + t_2\mu(x, y), y + t_1\mu(x, y)) = (t_2 - t_1)\mu(x, y). \quad (1.3)$$

We will use the condition in our main results.

Convex functions take place in many areas of mathematics and these functions have attracted the attention of mathematicians working in the field of inequality theory. Therefore, studies in the theory of inequality have provided to the emergence of many new inequalities. One of the most important of these inequalities is the Hermite-Hadamard inequality obtained by Charles Hermite and Jacques Hadamard and many generalizations of this inequality have been obtained. This inequality, which has attracted the attention of many mathematicians is below.

Assume that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function defined on the interval I of $\mathbb{R}$ where $a < b$. The following statement

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2} \quad (1.4)$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.

To provide detail information on convexity, some different type inequalities and more, see the papers [5, 7, 11, 19–21, 30–32, 35, 36].

Many new version of Hermite-Hadamard inequality is proved for different functions and inequalities. One of these proved by Noor [23] for preinvex functions as follows.

Theorem 1. Let $f : I = [k_1, k_1 + \mu(k_2, k_1)] \rightarrow (0, \infty)$ be a preinvex function on the interval of real numbers I° and $k_1, k_2 \in I^\circ$ with $k_1 < k_1 + \mu(k_2, k_1)$. Then the following inequality holds.

$$f\left(\frac{2k_1 + \mu(k_2, k_1)}{2}\right) \leq \frac{1}{\mu(k_2, k_1)} \int_{k_1}^{k_1 + \mu(k_2, k_1)} f(x) dx \leq \frac{f(k_1) + f(k_2)}{2}.$$

Fractional calculus is a generalization of ordinary calculus with more than 300 years of history. Fractional calculus was started in 1965 by Leibniz and L'Hospital as a result of a correspondence which continued several months and the question Leibniz asked to L'Hospital of "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?". And then, this question attracted the interest of many well known mathematicians. The concepts of fractional calculus were examined further over the course of the 18th and 19th centuries. In today's world, the applications of fractional calculus are very wide at fields as viscoelasticity, rheology, acoustics, optics, chemical and control theory, statistical physics, robotics, electrical and mechanical engineering, bio engineering, etc. Many mathematicians with the development of fractional calculus have defined many fractional derivative and integral operators to find solutions to real-world problems. Some of them are as follows.

Firstly, we recall the Caputo-Fabrizio fractional integral operators.

Definition 3 ([3]). Let $f \in H^1(0, b)$, $b > a$, $\alpha \in [0, 1]$ then, the definition of the left and right side of Caputo-Fabrizio fractional integral is:

$$({}^{CF}I_a^\alpha)(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \int_a^t f(y) dy,$$

and

$$({}^{CF}I_b^\alpha)(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \int_t^b f(y) dy,$$

where $M(\alpha)$ is normalization function.

In the sequel of the paper, we will denote normalization function as $B(\alpha)$ with $B(0) = B(1) = 1$.

Muhammad Tariq et.al [33] provided an integral inequality of Hermite-Hadamard type for preinvex functions via Caputo-Fabrizio fractional integral inequality as follows.

Theorem 2. Let $f : I = [k_1, k_1 + \mu(k_2, k_1)] \rightarrow (0, \infty)$ be a preinvex function on I° and $f \in L[k_1, k_1 + \mu(k_2, k_1)]$. If $\alpha \in [0, 1]$, then the following inequality holds

$$f\left(\frac{2k_1 + \mu(k_2, k_1)}{2}\right) \leq \frac{B(\alpha)}{\alpha \mu(k_2, k_1)} \left[{}^{CF}I_{k_1}^\alpha \{f(k)\} + {}^{CF}I_{k_1 + \mu(k_2, k_1)}^\alpha \{f(k)\} - \frac{2(1-\alpha)}{B(\alpha)} f(k) \right]$$

$$\leq \frac{f(k_1) + f(k_2)}{2},$$

where $k \in [k_1, k_1 + \mu(k_2, k_1)]$.

Atangana and Baleanu produced a new derivative operators using Mittag-Leffler function in Caputo-Fabrizio derivative operator as follows.

Definition 4 ([10]). Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1)$ then, the definition of the new fractional derivative is given:

$${}^{ABC}D_t^\alpha [f(t)] = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(x) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{(1-\alpha)} \right] dx. \quad (1.5)$$

Definition 5 ([10]). Let $f \in H^1(a, b)$, $b > a$, $\alpha \in [0, 1)$. Then the definition of the new fractional derivative is given by

$${}^{ABR}D_t^\alpha [f(t)] = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t f(x) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{(1-\alpha)} \right] dx. \quad (1.6)$$

Equations (1.5) and (1.6) have a non-local kernel. Also in equation (1.5), when the function is constant, we get zero.

The related fractional integral operator has been defined by Atangana-Baleanu as follows.

Definition 6 ([10]). The fractional integral associate to the new fractional derivative with non-local kernel of a function $f \in H^1(a, b)$ as defined:

$${}^{AB}I_a^\alpha \{f(t)\} = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t f(y)(t-y)^{\alpha-1} dy,$$

where $b > a, \alpha \in (0, 1]$.

In [2], Abdeljawad and Baleanu introduced the right hand side of integral operator as follows. The right fractional new integral with ML kernel of order $\alpha \in (0, 1]$ is defined by

$${}^{AB}I_b^\alpha \{f(t)\} = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_t^b f(y)(y-t)^{\alpha-1} dy.$$

For more information related to different kinds of fractional operators, we recommend the following papers to the readers [1, 3, 4, 6, 12–15, 24–29].

The purpose of this paper is to provide some new inequalities for preinvex functions that includes the Atangana-Baleanu integral operators. Some special cases are also considered.

2. HERMITE-HADAMARD TYPE INEQUALITIES FOR PREINVEX FUNCTIONS VIA ATANGANA-BALEANU FRACTIONAL INTEGRAL OPERATORS

Theorem 3. Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\mu : I \times I \neq \emptyset \rightarrow \mathbb{R}$ and $k_1, k_2 \in I$ with $k_1 < k_1 + \mu(k_2, k_1)$. If $f : [k_1, k_1 + \mu(k_2, k_1)] \rightarrow \mathbb{R}$ is a preinvex function, $f \in L[k_1, k_1 + \mu(k_2, k_1)]$ and μ satisfies Condition 1, the following inequalities for Atangana-Baleanu fractional integral operators hold

$$\begin{aligned} f\left(\frac{2k_1 + \mu(k_2, k_1)}{2}\right) &\leq \frac{B(\alpha)\Gamma(\alpha)}{2[\mu(k_2, k_1)]^\alpha} \left[{}^{AB}I_{k_1}^\alpha \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{f(k_1)\} \right] \\ &\quad - \frac{(1-\alpha)\Gamma(\alpha)}{2[\mu(k_2, k_1)]^\alpha} [f(k_1) + f(k_1 + \mu(k_2, k_1))] \\ &\leq \frac{f(k_1) + f(k_2)}{2} \end{aligned}$$

where $\alpha \in (0, 1]$, $B(\alpha) > 0$ is normalization function and $\Gamma(\cdot)$ is the Gamma function.

Proof. Since f is preinvex function on $[k_1, k_1 + \mu(k_2, k_1)]$, we can write (see, e.g., [17, 18]):

$$2f\left(\frac{2k_1 + \mu(k_2, k_1)}{2}\right) \leq f(k_1 + t\mu(k_2, k_1)) + f(k_1 + (1-t)\mu(k_2, k_1)) \quad (2.1)$$

Multiplying both sides of (2.1) by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned} &\frac{2}{B(\alpha)\Gamma(\alpha)}f\left(\frac{2k_1 + \mu(k_2, k_1)}{2}\right) \\ &\leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha-1} f(k_1 + t\mu(k_2, k_1)) dt \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha-1} f(k_1 + (1-t)\mu(k_2, k_1)) dt \\ &= \frac{\alpha}{B(\alpha)\Gamma(\alpha)[\mu(k_2, k_1)]^\alpha} \int_{k_1}^{k_1 + \mu(k_2, k_1)} (x - k_1)^{\alpha-1} f(x) dx \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)[\mu(k_2, k_1)]^\alpha} \int_{k_1}^{k_1 + \mu(k_2, k_1)} (k_1 + \mu(k_2, k_1) - y)^{\alpha-1} f(y) dy. \end{aligned}$$

Then we can write

$$\begin{aligned} &\frac{2}{B(\alpha)\Gamma(\alpha)}f\left(\frac{2k_1 + \mu(k_2, k_1)}{2}\right) \\ &\leq \frac{1}{[\mu(k_2, k_1)]^\alpha} \left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{k_1}^{k_1 + \mu(k_2, k_1)} (x - k_1)^{\alpha-1} f(x) dx + \frac{(1-\alpha)}{B(\alpha)} f(k_1) \right] \\ &\quad - \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^\alpha} f(k_1) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{[\mu(k_2, k_1)]^\alpha} \left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{k_1}^{k_1+\mu(k_2, k_1)} (k_1 + \mu(k_2, k_1) - y)^{\alpha-1} f(y) dy \right. \\
& \left. + \frac{(1-\alpha)}{B(\alpha)} f(k_1 + \mu(k_2, k_1)) \right] - \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^\alpha} f(k_1 + \mu(k_2, k_1)).
\end{aligned}$$

So, using Atangana-Baleanu fractional integral operators, we get

$$\begin{aligned}
& \frac{2}{B(\alpha)\Gamma(\alpha)} f\left(\frac{2k_1 + \mu(k_2, k_1)}{2}\right) \\
& \leq \frac{1}{[\mu(k_2, k_1)]^\alpha} \left[{}^{AB}I_{k_1}^\alpha \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1+\mu(k_2, k_1)}^\alpha \{f(k_1)\} \right] \\
& \quad - \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^\alpha} [f(k_1) + f(k_1 + \mu(k_2, k_1))].
\end{aligned}$$

and the first inequality is proved.

For the proof of the second inequality in (2.1), we first note that if f is a preinvex function, then we can write

$$f(k_1 + t\mu(k_2, k_1)) \leq (1-t)f(k_1) + tf(k_2)$$

and

$$f(k_1 + (1-t)\mu(k_2, k_1)) \leq tf(k_1) + (1-t)f(k_2).$$

By adding these inequalities side by side, we have

$$f(k_1 + t\mu(k_2, k_1)) + f(k_1 + (1-t)\mu(k_2, k_1)) \leq f(k_1) + f(k_2). \quad (2.2)$$

Then, multiplying both sides of (2.2) by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)} t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned}
& \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha-1} f(k_1 + t\mu(k_2, k_1)) dt + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^1 t^{\alpha-1} f(k_1 + (1-t)\mu(k_2, k_1)) dt \\
& \leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)} [f(k_1) + f(k_2)] \int_0^1 t^{\alpha-1} dt.
\end{aligned}$$

Then, we can write

$$\begin{aligned}
& \frac{1}{[\mu(k_2, k_1)]^\alpha} \left[{}^{AB}I_{k_1}^\alpha \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1+\mu(k_2, k_1)}^\alpha \{f(k_1)\} \right] \\
& \quad - \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^\alpha} [f(k_1) + f(k_1 + \mu(k_2, k_1))] \\
& \leq \frac{f(k_1) + f(k_2)}{B(\alpha)\Gamma(\alpha)}.
\end{aligned}$$

So, the proof of this theorem is completed. $\square$

Remark 1. Setting $\mu(k_2, k_1) = k_2 - k_1$ in Theorem 3 gives the same result as in [16], Proposition 2.1, inequality (13).

Remark 2. Setting $\alpha = 1$ in Theorem 3 gives the same result as in Theorem 1.

Theorem 4. Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\mu : I \times I \neq \emptyset \rightarrow \mathbb{R}$ and $k_1, k_2 \in I$ with $k_1 < k_1 + \mu(k_2, k_1)$. If $f, g : [k_1, k_1 + \mu(k_2, k_1)] \rightarrow \mathbb{R}$ is preinvex functions, $f, g \in L[k_1, k_1 + \mu(k_2, k_1)]$, then the following inequalities for Atangana-Baleanu fractional integral operators hold:

$$\begin{aligned} & \frac{1}{[\mu(k_2, k_1)]^\alpha} \left[{}^{AB}I_{k_1}^\alpha \{fg(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{fg(k_1)\} \right] \\ & \leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[[f(k_1)g(k_1) + f(k_2)g(k_2)] \left(\frac{2}{\alpha(\alpha+1)(\alpha+2)} + \frac{1}{\alpha+2} \right) \right. \\ & \quad \left. + 2 \frac{[f(k_1)g(k_2) + f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)} \right] \\ & \quad + \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^\alpha} [f(k_1)g(k_1) + f(k_1 + \mu(k_2, k_1))g(k_1 + \mu(k_2, k_1))], \end{aligned}$$

where $\alpha \in (0, 1]$, $B(\alpha) > 0$ is a normalization function and $\Gamma(\cdot)$ is the Gamma function.

Proof. Since f and g are preinvex functions on $[k_1, k_1 + \mu(k_2, k_1)]$, we get

$$f(k_1 + t\mu(k_2, k_1)) \leq (1-t)f(k_1) + tf(k_2)$$

and

$$g(k_1 + t\mu(k_2, k_1)) \leq (1-t)g(k_1) + tg(k_2).$$

By multiplying both inequalities side by side, we get

$$\begin{aligned} & f(k_1 + t\mu(k_2, k_1))g(k_1 + t\mu(k_2, k_1)) \\ & \leq (1-t)^2 f(k_1)g(k_1) + t^2 f(k_2)g(k_2) + t(1-t) [f(k_1)g(k_2) + f(k_2)g(k_1)]. \end{aligned} \quad (2.3)$$

By multiplying both sides of (2.3) with $(1-t)^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 (1-t)^{\alpha-1} f(k_1 + t\mu(k_2, k_1))g(k_1 + t\mu(k_2, k_1)) dt \\ & \leq \int_0^1 (1-t)^{\alpha-1} \left[(1-t)^2 f(k_1)g(k_1) + t^2 f(k_2)g(k_2) + t(1-t) [f(k_1)g(k_2) + f(k_2)g(k_1)] \right] dt \\ & = \frac{f(k_1)g(k_1)}{\alpha+2} + 2 \frac{f(k_2)g(k_2)}{\alpha(\alpha+1)(\alpha+2)} + \frac{[f(k_1)g(k_2) + f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)}. \end{aligned} \quad (2.4)$$

By changing the variable $k_1 + t\mu(k_2, k_1) = x$, we can write the inequality in (2.4) as

$$\begin{aligned} & \frac{1}{[\mu(k_2, k_1)]^\alpha} \int_{k_1}^{k_1 + \mu(k_2, k_1)} (k_1 + \mu(k_2, k_1) - x)^{\alpha-1} f(x)g(x)dx \\ & \leq \frac{f(k_1)g(k_1)}{\alpha+2} + 2\frac{f(k_2)g(k_2)}{\alpha(\alpha+1)(\alpha+2)} + \frac{[f(k_1)g(k_2) + f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)}. \end{aligned} \quad (2.5)$$

Multiplying both sides of (2.5) by $\frac{\alpha}{B(\alpha)\Gamma(\alpha)}$ and then adding the term

$$\frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^\alpha} f(k_1 + \mu(k_2, k_1))g(k_1 + \mu(k_2, k_1))$$

to both sides of (2.5) and finally using Atangana-Baleanu fractional integral operators, we get

$$\begin{aligned} & \frac{1}{[\mu(k_2, k_1)]^\alpha} \left[{}^{AB}I_{k_1}^\alpha \{fg(k_1 + \mu(k_2, k_1))\} \right] \\ & \leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[\frac{f(k_1)g(k_1)}{\alpha+2} + 2\frac{f(k_2)g(k_2)}{\alpha(\alpha+1)(\alpha+2)} + \frac{[f(k_1)g(k_2) + f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)} \right] \\ & \quad + \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^\alpha} f(k_1 + \mu(k_2, k_1))g(k_1 + \mu(k_2, k_1)). \end{aligned} \quad (2.6)$$

Similarly, by multiplying both sides of (2.3) with $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 t^{\alpha-1} f(k_1 + t\mu(k_2, k_1))g(k_1 + t\mu(k_2, k_1))dt \\ & \leq \int_0^1 t^{\alpha-1} \left[(1-t)^2 f(k_1)g(k_1) + t^2 f(k_2)g(k_2) + t(1-t)[f(k_1)g(k_2) + f(k_2)g(k_1)] \right] dt \\ & = 2\frac{f(k_1)g(k_1)}{\alpha(\alpha+1)(\alpha+2)} + \frac{f(k_2)g(k_2)}{\alpha+2} + \frac{[f(k_1)g(k_2) + f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)}. \end{aligned}$$

By making similar calculations to those in the proof of (2.6), we obtain

$$\begin{aligned} & \frac{1}{[\mu(k_2, k_1)]^\alpha} \left[{}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{fg(k_1)\} \right] \\ & \leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[2\frac{f(k_1)g(k_1)}{\alpha(\alpha+1)(\alpha+2)} + 2\frac{f(k_2)g(k_2)}{\alpha+2} + \frac{[f(k_1)g(k_2) + f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)} \right] \\ & \quad + \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^\alpha} f(k_1)g(k_1). \end{aligned} \quad (2.7)$$

Adding (2.6) and (2.7) side by side, we get

$$\frac{1}{[\mu(k_2, k_1)]^\alpha} \left[{}^{AB}I_{k_1}^\alpha \{fg(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{fg(k_1)\} \right]$$

$$\begin{aligned} &\leq \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \left[[f(k_1)g(k_1) + f(k_2)g(k_2)] \left(\frac{2}{\alpha(\alpha+1)(\alpha+2)} + \frac{1}{\alpha+2} \right) \right. \\ &\quad \left. + 2 \frac{[f(k_1)g(k_2) + f(k_2)g(k_1)]}{(\alpha+1)(\alpha+2)} \right] \\ &\quad + \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^\alpha} [f(k_1)g(k_1) + f(k_1 + \mu(k_2, k_1))g(k_1 + \mu(k_2, k_1))] \end{aligned}$$

and the proof is completed. $\square$

3. SOME FURTHER RESULTS FOR PREINVEX FUNCTIONS VIA ATANGANA-BALEANU FRACTIONAL INTEGRAL OPERATORS

Lemma 1 ([8]). *Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\mu: I \times I \setminus \emptyset \rightarrow \mathbb{R}$ and $k_1, k_2 \in I$ with $k_1 < k_1 + \mu(k_2, k_1)$. Suppose that $f: I \rightarrow \mathbb{R}$ be a differentiable function. If $f' \in L[k_1, k_1 + \mu(k_2, k_1)]$, the following identity for Atangana-Baleanu fractional integral operators holds:*

$$\begin{aligned} &\frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^\alpha \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{f(k_1)\} \right] \\ &\quad - \left(\frac{[\mu(k_2, k_1)]^\alpha + (1-\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \\ &= \int_0^1 (1-t)^\alpha f'(k_1 + t\mu(k_2, k_1)) dt - \int_0^1 t^\alpha f'(k_1 + t\mu(k_2, k_1)) dt, \end{aligned} \quad (3.1)$$

where $\alpha \in (0, 1]$, $t \in [0, 1]$, $B(\alpha)$ is a normalization function and $\Gamma(\cdot)$ is the Gamma function.

Proof. By using integration, we have

$$\begin{aligned} &\int_0^1 (1-t)^\alpha f'(k_1 + t\mu(k_2, k_1)) dt \\ &= \frac{(1-t)^\alpha f(k_1 + t\mu(k_2, k_1))}{\mu(k_2, k_1)} \Big|_0^1 + \frac{\alpha}{\mu(k_2, k_1)} \int_0^1 f(k_1 + t\mu(k_2, k_1)) (1-t)^{\alpha-1} dt \\ &= -\frac{f(k_1)}{\mu(k_2, k_1)} + \frac{\alpha}{\mu(k_2, k_1)} \int_0^1 (1-t)^{\alpha-1} f(k_1 + t\mu(k_2, k_1)) dt \\ &= -\frac{f(k_1)}{\mu(k_2, k_1)} + \frac{\alpha}{[\mu(k_2, k_1)]^{\alpha+1}} \int_{k_1}^{k_1 + \mu(k_2, k_1)} (k_1 + \mu(k_2, k_1) - x)^{\alpha-1} f(x) dx. \end{aligned} \quad (3.2)$$

If we multiply both sides of (3.2) by $\frac{1}{B(\alpha)\Gamma(\alpha)}$, we get

$$\frac{1}{B(\alpha)\Gamma(\alpha)} \int_0^1 (1-t)^\alpha f'(k_1 + t\mu(k_2, k_1)) dt$$

$$\begin{aligned}
&= -\frac{f(k_1)}{B(\alpha)\Gamma(\alpha)\mu(k_2, k_1)} \\
&\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)[\mu(k_2, k_1)]^{\alpha+1}} \int_{k_1}^{k_1+\mu(k_2, k_1)} (k_1 + \mu(k_2, k_1) - x)^{\alpha-1} f(x) dx.
\end{aligned}$$

Then, we can write

$$\begin{aligned}
&\frac{1}{B(\alpha)\Gamma(\alpha)} \int_0^1 (1-t)^\alpha f'(k_1 + t\mu(k_2, k_1)) dt \\
&= -\frac{f(k_1)}{B(\alpha)\Gamma(\alpha)\mu(k_2, k_1)} \\
&\quad + \frac{1}{[\mu(k_2, k_1)]^{\alpha+1}} \left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{k_1}^{k_1+\mu(k_2, k_1)} (k_1 + \mu(k_2, k_1) - x)^{\alpha-1} f(x) dx \right. \\
&\quad \left. + \frac{(1-\alpha)}{B(\alpha)} f(k_1 + \mu(k_2, k_1)) \right] - \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^{\alpha+1}} f(k_1 + \mu(k_2, k_1)).
\end{aligned}$$

Using Atangana-Baleanu fractional integral operators, we have

$$\begin{aligned}
&\frac{1}{B(\alpha)\Gamma(\alpha)} \int_0^1 (1-t)^\alpha f'(k_1 + t\mu(k_2, k_1)) dt \\
&= -\frac{f(k_1)}{B(\alpha)\Gamma(\alpha)\mu(k_2, k_1)} + \frac{1}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^\alpha \{f(k_1 + \mu(k_2, k_1))\} \right] \\
&\quad - \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^{\alpha+1}} f(k_1 + \mu(k_2, k_1)).
\end{aligned} \tag{3.3}$$

Similarly, using integration, we get

$$\begin{aligned}
&\int_0^1 t^\alpha f'(k_1 + t\mu(k_2, k_1)) dt \\
&= \frac{t^\alpha f(k_1 + t\mu(k_2, k_1))}{\mu(k_2, k_1)} \Big|_0^1 - \frac{\alpha}{\mu(k_2, k_1)} \int_0^1 f(k_1 + t\mu(k_2, k_1)) t^{\alpha-1} dt \\
&= \frac{f(k_1 + \mu(k_2, k_1))}{\mu(k_2, k_1)} - \frac{\alpha}{\mu(k_2, k_1)} \int_0^1 t^{\alpha-1} f(k_1 + t\mu(k_2, k_1)) dt \\
&= \frac{f(k_1 + \mu(k_2, k_1))}{\mu(k_2, k_1)} - \frac{\alpha}{[\mu(k_2, k_1)]^{\alpha+1}} \int_{k_1}^{k_1+\mu(k_2, k_1)} (u - k_1)^{\alpha-1} f(u) du.
\end{aligned} \tag{3.4}$$

If we multiply both sides of (3.4) by $-\frac{1}{B(\alpha)\Gamma(\alpha)}$, we have

$$-\frac{1}{B(\alpha)\Gamma(\alpha)} \int_0^1 t^\alpha f'(k_1 + t\mu(k_2, k_1)) dt$$

$$= -\frac{f(k_1 + \mu(k_2, k_1))}{B(\alpha)\Gamma(\alpha)\mu(k_2, k_1)} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)[\mu(k_2, k_1)]^{\alpha+1}} \int_{k_1}^{k_1 + \mu(k_2, k_1)} (u - k_1)^{\alpha-1} f(u) du.$$

Then we can write

$$\begin{aligned} & -\frac{1}{B(\alpha)\Gamma(\alpha)} \int_0^1 t^\alpha f'(k_1 + t\mu(k_2, k_1)) dt \\ &= -\frac{f(k_1 + \mu(k_2, k_1))}{B(\alpha)\Gamma(\alpha)\mu(k_2, k_1)} + \frac{1}{[\mu(k_2, k_1)]^{\alpha+1}} \left[\frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{k_1}^{k_1 + \mu(k_2, k_1)} (u - k_1)^{\alpha-1} f(u) du \right. \\ & \quad \left. + \frac{(1-\alpha)}{B(\alpha)} f(k_1) \right] - \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^{\alpha+1}} f(k_1). \end{aligned}$$

Using Atangana-Baleanu fractional integral operators, we have

$$\begin{aligned} & -\frac{1}{B(\alpha)\Gamma(\alpha)} \int_0^1 t^\alpha f'(k_1 + t\mu(k_2, k_1)) dt \\ &= -\frac{f(k_1 + \mu(k_2, k_1))}{B(\alpha)\Gamma(\alpha)\mu(k_2, k_1)} + \frac{1}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{f(k_1)\} \right] \\ & \quad - \frac{(1-\alpha)}{B(\alpha)[\mu(k_2, k_1)]^{\alpha+1}} f(k_1). \end{aligned} \quad (3.5)$$

By adding identities (3.3) and (3.5), we obtain desired result. So, the proof is completed. $\square$

Remark 3. Setting $\mu(k_2, k_1) = k_2 - k_1$ in Lemma 1 gives the same result as in [16], Theorem 3.1, equality (29).

Theorem 5. Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\mu : I \times I \neq \emptyset \rightarrow \mathbb{R}$ and $k_1, k_2 \in I$ with $k_1 < k_1 + \mu(k_2, k_1)$. Suppose that $f : I \rightarrow \mathbb{R}$ be a differentiable function and $f' \in L[k_1, k_1 + \mu(k_2, k_1)]$. If $|f'|$ is a preinvex function, we have the following inequality for Atangana-Baleanu fractional integral operators:

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^\alpha \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{[\mu(k_2, k_1)]^\alpha + (1-\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \right| \\ & \leq \frac{|f'(k_1)| + |f'(k_2)|}{\alpha + 1}, \end{aligned}$$

where $\alpha \in (0, 1]$, $B(\alpha)$ is a normalization function and $\Gamma(\cdot)$ is the Gamma function.

Proof. By using the identity that is given in Lemma 1 and properties of modulus, we can write

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^{\alpha} \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^{\alpha} \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{[\mu(k_2, k_1)]^{\alpha} + (1 - \alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \right| \\ &= \left| \int_0^1 (1-t)^{\alpha} f'(k_1 + t\mu(k_2, k_1)) dt - \int_0^1 t^{\alpha} f'(k_1 + t\mu(k_2, k_1)) dt \right| \\ &\leq \int_0^1 (1-t)^{\alpha} |f'(k_1 + t\mu(k_2, k_1))| dt + \int_0^1 t^{\alpha} |f'(k_1 + t\mu(k_2, k_1))| dt. \end{aligned}$$

Since $|f'|$ is a preinvex function, we obtain

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^{\alpha} \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^{\alpha} \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{[\mu(k_2, k_1)]^{\alpha} + (1 - \alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \right| \\ &\leq \int_0^1 (1-t)^{\alpha} [(1-t)|f'(k_1)| + t|f'(k_2)|] dt + \int_0^1 t^{\alpha} [(1-t)|f'(k_1)| + t|f'(k_2)|] dt \\ &= \frac{|f'(k_1)| + |f'(k_2)|}{\alpha + 1}. \end{aligned}$$

So, the proof is completed. $\square$

Corollary 1. In Theorem 5, if we choose $\mu(k_2, k_1) = k_2 - k_1$ we obtain

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{(k_2 - k_1)^{\alpha+1}} \left[{}^{AB}I_{k_1}^{\alpha} \{f(k_2)\} + {}^{AB}I_{k_2}^{\alpha} \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{(k_2 - k_1)^{\alpha} + (1 - \alpha)\Gamma(\alpha)}{(k_2 - k_1)^{\alpha+1}} \right) [f(k_1) + f(k_2)] \right| \\ &\leq \frac{|f'(k_1)| + |f'(k_2)|}{\alpha + 1}. \end{aligned}$$

Theorem 6. Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\mu : I \times I \neq \emptyset \rightarrow \mathbb{R}$ and $k_1, k_2 \in I$ with $k_1 < k_1 + \mu(k_2, k_1)$. Suppose that $f : I \rightarrow \mathbb{R}$ be a differentiable function and $f' \in L[k_1, k_1 + \mu(k_2, k_1)]$. If $|f'|^q$ is a preinvex function, we have the following inequality for Atangana-Baleanu fractional integral operators:

$$\left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^{\alpha} \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^{\alpha} \{f(k_1)\} \right] \right|$$

$$\begin{aligned}
& - \left(\frac{[\mu(k_2, k_1)]^\alpha + (1 - \alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \Bigg| \\
& \leq 2 \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left(\frac{|f'(k_1)|^q + |f'(k_2)|^q}{2} \right)^{\frac{1}{q}},
\end{aligned}$$

where $p^{-1} + q^{-1} = 1$, $q > 1$, $\alpha \in (0, 1]$, $B(\alpha)$ is a normalization function and $\Gamma(\cdot)$ is the Gamma function.

Proof. By using Lemma 1, we get

$$\begin{aligned}
& \left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^\alpha \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{f(k_1)\} \right] \right. \\
& \quad \left. - \left(\frac{[\mu(k_2, k_1)]^\alpha + (1 - \alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \right| \\
& \leq \int_0^1 (1-t)^\alpha |f'(k_1 + t\mu(k_2, k_1))| dt + \int_0^1 t^\alpha |f'(k_1 + t\mu(k_2, k_1))| dt.
\end{aligned}$$

By applying Hölder inequality, we get

$$\begin{aligned}
& \left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^\alpha \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{f(k_1)\} \right] \right. \\
& \quad \left. - \left(\frac{[\mu(k_2, k_1)]^\alpha + (1 - \alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \right| \\
& \leq \left(\int_0^1 (1-t)^{\alpha p} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(k_1 + t\mu(k_2, k_1))|^q dt \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 t^{\alpha p} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(k_1 + t\mu(k_2, k_1))|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

By using the preinvexity of $|f'|^q$, we obtain

$$\begin{aligned}
& \left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^\alpha \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{f(k_1)\} \right] \right. \\
& \quad \left. - \left(\frac{[\mu(k_2, k_1)]^\alpha + (1 - \alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \right| \\
& \leq \left(\int_0^1 (1-t)^{\alpha p} dt \right)^{\frac{1}{p}} \left(\int_0^1 [(1-t)|f'(k_1)|^q + t|f'(k_2)|^q] dt \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 t^{\alpha p} dt \right)^{\frac{1}{p}} \left(\int_0^1 [(1-t)|f'(k_1)|^q + t|f'(k_2)|^q] dt \right)^{\frac{1}{q}}.
\end{aligned}$$

By calculating the integrals given in the above inequality, we get the desired result. $\square$

Corollary 2. *In Theorem 6, if we choose $\mu(k_2, k_1) = k_2 - k_1$ we obtain*

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{(k_2 - k_1)^{\alpha+1}} \left[{}^{AB}I_{k_1}^\alpha \{f(k_2)\} + {}^{AB}I_{k_2}^\alpha \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{(k_2 - k_1)^\alpha + (1 - \alpha)\Gamma(\alpha)}{(k_2 - k_1)^{\alpha+1}} \right) [f(k_1) + f(k_2)] \right| \\ & \leq 2 \left(\frac{1}{\alpha p + 1} \right)^{\frac{1}{p}} \left(\frac{|f'(k_1)|^q + |f'(k_2)|^q}{2} \right)^{\frac{1}{q}}. \end{aligned}$$

Theorem 7. *Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\mu : I \times I \neq \emptyset \rightarrow \mathbb{R}$ and $k_1, k_2 \in I$ with $k_1 < k_1 + \mu(k_2, k_1)$. Suppose that $f : I \rightarrow \mathbb{R}$ be a differentiable function and $f' \in L[k_1, k_1 + \mu(k_2, k_1)]$. If $|f'|^q$ is a preinvex function, we have the following inequality for Atangana-Baleanu fractional integral operators:*

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^\alpha \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{[\mu(k_2, k_1)]^\alpha + (1 - \alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \right| \\ & \leq \left(\frac{1}{\alpha + 1} \right)^{1 - \frac{1}{q}} \left[\left(\frac{|f'(k_1)|^q}{\alpha + 2} + \frac{|f'(k_2)|^q}{(\alpha + 1)(\alpha + 2)} \right)^{\frac{1}{q}} + \left(\frac{|f'(k_1)|^q}{(\alpha + 1)(\alpha + 2)} + \frac{|f'(k_2)|^q}{\alpha + 2} \right)^{\frac{1}{q}} \right], \end{aligned}$$

where $\alpha \in (0, 1]$, $q \geq 1$, $B(\alpha)$ is a normalization function and $\Gamma(\cdot)$ is the Gamma function.

Proof. By using Lemma 1 and applying power mean inequality, we have

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^\alpha \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^\alpha \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{[\mu(k_2, k_1)]^\alpha + (1 - \alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \right| \\ & \leq \int_0^1 (1 - t)^\alpha |f'(k_1 + t\mu(k_2, k_1))| dt + \int_0^1 t^\alpha |f'(k_1 + t\mu(k_2, k_1))| dt \\ & \leq \left(\int_0^1 (1 - t)^\alpha dt \right)^{1 - \frac{1}{q}} \left(\int_0^1 (1 - t)^\alpha |f'(k_1 + t\mu(k_2, k_1))|^q dt \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 t^\alpha dt \right)^{1 - \frac{1}{q}} \left(\int_0^1 t^\alpha |f'(k_1 + t\mu(k_2, k_1))|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

By using the preinvexity of $|f'|^q$, we have

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^{\alpha} \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^{\alpha} \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{[\mu(k_2, k_1)]^{\alpha} + (1 - \alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \right| \\ & \leq \left(\int_0^1 (1-t)^{\alpha} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^{\alpha} [(1-t)|f'(k_1)|^q + t|f'(k_2)|^q] dt \right)^{\frac{1}{q}} \\ & \quad + \left(\int_0^1 t^{\alpha} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\alpha} [(1-t)|f'(k_1)|^q + t|f'(k_2)|^q] dt \right)^{\frac{1}{q}} \\ & = \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left[\left(\frac{|f'(k_1)|^q}{\alpha+2} + \frac{|f'(k_2)|^q}{(\alpha+1)(\alpha+2)} \right)^{\frac{1}{q}} + \left(\frac{|f'(k_1)|^q}{(\alpha+1)(\alpha+2)} + \frac{|f'(k_2)|^q}{\alpha+2} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

So, the proof is completed. $\square$

Corollary 3. In Theorem 7, if we choose $\mu(k_2, k_1) = k_2 - k_1$ we obtain

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{(k_2 - k_1)^{\alpha+1}} \left[{}^{AB}I_{k_1}^{\alpha} \{f(k_2)\} + {}^{AB}I_{k_2}^{\alpha} \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{(k_2 - k_1)^{\alpha} + (1 - \alpha)\Gamma(\alpha)}{(k_2 - k_1)^{\alpha+1}} \right) [f(k_1) + f(k_2)] \right| \\ & \leq \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left[\left(\frac{|f'(k_1)|^q}{\alpha+2} + \frac{|f'(k_2)|^q}{(\alpha+1)(\alpha+2)} \right)^{\frac{1}{q}} + \left(\frac{|f'(k_1)|^q}{(\alpha+1)(\alpha+2)} + \frac{|f'(k_2)|^q}{\alpha+2} \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Theorem 8. Let $I \subseteq \mathbb{R}$ be an open invex subset with respect to $\mu : I \times I \neq \emptyset \rightarrow \mathbb{R}$ and $k_1, k_2 \in I$ with $k_1 < k_1 + \mu(k_2, k_1)$. Suppose that $f : I \rightarrow \mathbb{R}$ be a differentiable function and $f' \in L[k_1, k_1 + \mu(k_2, k_1)]$. If $|f'|^q$ is a preinvex function, we have the following inequality for Atangana-Baleanu fractional integral operators:

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^{\alpha} \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^{\alpha} \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{[\mu(k_2, k_1)]^{\alpha} + (1 - \alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \right| \\ & \leq \frac{2}{p(\alpha p + 1)} + \frac{|f'(k_1)|^q + |f'(k_2)|^q}{q}, \end{aligned}$$

where $p^{-1} + q^{-1} = 1$, $q > 1$, $\alpha \in (0, 1]$, $B(\alpha)$ is a normalization function and $\Gamma(\cdot)$ is the Gamma function.

Proof. By using identity that is given in Lemma 1 and applying the Young inequality as $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$, we get

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \left[{}^{AB}I_{k_1}^{\alpha} \{f(k_1 + \mu(k_2, k_1))\} + {}^{AB}I_{k_1 + \mu(k_2, k_1)}^{\alpha} \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{[\mu(k_2, k_1)]^{\alpha} + (1 - \alpha)\Gamma(\alpha)}{[\mu(k_2, k_1)]^{\alpha+1}} \right) [f(k_1) + f(k_1 + \mu(k_2, k_1))] \right| \\ & \leq \int_0^1 (1-t)^{\alpha} |f'(k_1 + t\mu(k_2, k_1))| dt + \int_0^1 t^{\alpha} |f'(k_1 + t\mu(k_2, k_1))| dt \\ & \leq \frac{1}{p} \int_0^1 (1-t)^{\alpha p} dt + \frac{1}{q} \int_0^1 |f'(k_1 + t\mu(k_2, k_1))|^q dt \\ & \quad + \frac{1}{p} \int_0^1 t^{\alpha p} dt + \frac{1}{q} \int_0^1 |f'(k_1 + t\mu(k_2, k_1))|^q dt. \end{aligned}$$

By using the preinvexity of $|f'|^q$ and by a simple computation, we have the desired result. $\square$

Corollary 4. In Theorem 8, if we choose $\mu(k_2, k_1) = k_2 - k_1$ we obtain

$$\begin{aligned} & \left| \frac{B(\alpha)\Gamma(\alpha)}{(k_2 - k_1)^{\alpha+1}} \left[{}^{AB}I_{k_1}^{\alpha} \{f(k_2)\} + {}^{AB}I_{k_2}^{\alpha} \{f(k_1)\} \right] \right. \\ & \quad \left. - \left(\frac{(k_2 - k_1)^{\alpha} + (1 - \alpha)\Gamma(\alpha)}{(k_2 - k_1)^{\alpha+1}} \right) [f(k_1) + f(k_2)] \right| \\ & \leq \frac{2}{p(\alpha p + 1)} + \frac{|f'(k_1)|^q + |f'(k_2)|^q}{q}. \end{aligned}$$

4. CONCLUSION

In this study, firstly, a new inequalities of Hermite-Hadamard type for preinvex functions via Atangana-Baleanu fractional integral operators was provided. And then an identity including Atangana-Baleanu integral operators has been proved and some integral inequalities are established by using preinvex functions, Hölder inequality, power-mean inequality, Young inequality with the help of this identity. Some results in this study are the generalizations and refinements of the existing results. Researchers can produce new equalities such as the integral identity in this study and obtain similar inequalities of these identity-based inequalities.

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ON IDEAL CONVERGENCE OF SEQUENCES VIA CATALAN MATRIX OPERATOR

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Abstract. Catalan sequence initially introduced by Euler but named after Eugene Catalan has found abundant applications in the fields of probability theory, number theory, analysis, combinatorics etc. Recently İlkan formulated the matrix associated with Catalan numbers and studied its domain in $c_0(\tilde{C})$ and $c(\tilde{C})$. In this paper, by employing Catalan matrix defined by Catalan sequence and convergence via ideal, we establish Catalan sequence spaces $c_0^I(\tilde{C})$, $c^I(\tilde{C})$ and $\ell_\infty^I(\tilde{C})$, study the properties of these corresponding spaces and establish inclusion relations.

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1. INTRODUCTION

A linear subspace of the set of all real or complex sequences denoted by ω is termed as a sequence space. The standard sequence spaces namely null, convergent and bounded sequence spaces are denoted by c_0, c and ℓ_∞ respectively. Establishment of sequence spaces to study their algebraic and topological properties have a substantial relevance in summability theory. Every so often when the sequences fail to approach a finite limit, matrix transformation comes in handy by acting as a linear operator between two sequence spaces and assigning a limit after the transformation. If γ and Υ are two sequence spaces and $M = (m_{nk})$ is an infinite matrix such that $(m_{nk}) \in \mathbb{R}$ then M defines a matrix transformation from γ into Υ denoted by $M: \gamma \rightarrow \Upsilon$ if for every sequence $u = (u_k) \in \gamma$, the sequence $Mu = \{M_n(u)\}$, the M transform of u is in Υ , where

$$M_n(u) = \sum_{k=0}^{\infty} m_{nk} u_k \quad \text{for all } n \in \mathbb{N}. \quad (1.1)$$

The classification of all matrices M such that $M: \gamma \rightarrow \Upsilon$ is denoted by (γ, Υ) . Therefore $M \in (\gamma, \Upsilon)$ iff the series in equation (1.1) on the right hand side converges $\forall n \in \mathbb{N}$, every $u \in \gamma$ and we obtain $Mu = \{M_n(u)\} \in \Upsilon$.

Over the course of time the domains of various triangular matrices have been investigated as a means of assigning a finite number to the sequences' limit. A handful of them have been explored in these literatures, see [1, 4, 5, 11–13]. In addition to the mentioned references, more can be found on applications of summability theory and matrix transformations in [2, 3, 6, 9, 17].

Classified among such pre-existing notions of matrix domain, a novel matrix namely Catalan matrix was defined by İlkhani in [10] whose terms are comprised of Catalan numbers. It is an integer sequence with its first few terms as 1, 1, 2, 5, 14, 42, 132, 429 The sequence named after Catalan was originally found in Euler's work, which he encountered while attempting to find the number of ways a convex polygon can be divided into triangles with no diagonals intersecting each other. It was Catalan who found an analogy between this sequence and the parantheses problem, see [8]. The answer to innumerable problems from every field of mathematics, one of them being able to find all possible arrangements which allow $2n$ number of people sitting across a round table to shake hands with each other without crossing hands; was this interesting sequence of numbers, see [15]. The standard formula for finding n th Catalan number was given by Catalan as

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

and the recurrence with the initial condition given as

$$C_{n+1} = \sum_{k=0}^n C_k C_{n-k}, \quad C_0 = 1 \text{ is satisfied by Catalan numbers.}$$

Recently the Catalan matrix associated with the Catalan numbers $\tilde{C} = (\tilde{c}_{nk})$ was defined by Merve İlkhani in [10] as,

$$(\tilde{c}_{nk}) = \begin{cases} \frac{C_k C_{n-k}}{C_{n+1}} & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases} \quad (1.2)$$

Equivalently

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and its corresponding inverse $\tilde{C}^{-1} = (\tilde{c}_{nk}^{-1})$ is given by,

$$\tilde{c}_{nk}^{-1} = \begin{cases} (-1)^{n-k} \frac{C_{k+1}}{C_0^{n-k+1} C_n} \mathcal{H}_{n-k} & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

where $\mathcal{H}_0 = 1$ and $\mathcal{H}_n$ is given by

$$\mathcal{H}_n = \begin{vmatrix} C_1 & C_0 & 0 & 0 & 0 & \dots & 0 \\ C_2 & C_1 & C_0 & 0 & 0 & \dots & 0 \\ C_3 & C_2 & C_1 & C_0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_n & C_{n-1} & C_{n-2} & C_{n-3} & C_{n-4} & \dots & C_1 \end{vmatrix}$$

$\forall n \in \mathbb{N} \setminus 0$.

İlkhani in [10] defined new Banach sequence spaces using the Catalan matrix, namely $c_0(\tilde{C})$ and $c(\tilde{C})$ as following,

$$c_0(\tilde{C}) = \{u = (u_k) \in \omega : \lim_{n \rightarrow \infty} \frac{1}{C_{n+1}} \sum_{k=0}^n C_k C_{n-k} u_k = 0\},$$

$$c(\tilde{C}) = \{u = (u_k) \in \omega : \lim_{n \rightarrow \infty} \frac{1}{C_{n+1}} \sum_{k=0}^n C_k C_{n-k} u_k \text{ exists}\}.$$

and showed some promising results of these spaces such as linearity, BK spaces with respect to norms defined on them, linear isomorphism and moved on to define compact operators too.

İlkhani and Kara later in [14] came up with two novel spaces defined manoeuvring the Catalan matrix as,

$$\ell_p(\tilde{C}) = \{u = (u_k) \in \omega : \sum_{n=0}^{\infty} \left| \frac{1}{C_{n+1}} \sum_{k=0}^n C_k C_{n-k} u_k \right|^p < \infty\},$$

$$\ell_\infty(\tilde{C}) = \{u = (u_k) \in \omega : \sup_n \left| \frac{1}{C_{n+1}} \sum_{k=0}^n C_k C_{n-k} u_k \right| < \infty\}.$$

In the entire paper the $\tilde{C}$ -transform of a sequence $u = (u_k)$ will be denoted and defined as,

$$v = \tilde{C}_n(u) = \frac{1}{C_{n+1}} \sum_{k=0}^n C_k C_{n-k} u_k \quad \text{for all } n \in \mathbb{N}.$$

Inquisitive readers about Catalan sequences and their properties can look further in the literature [15, 18, 22].

2. PRELIMINARIES

The problem of assigning a novel version of limit to sequences whose limits did not exist in the usual sense was addressed by Steinhaus [19]. He came up with the idea of defining density of the set $\mathcal{L}_\varepsilon(c) = \{i \in \mathbb{N} : |c_i - c| \geq \varepsilon\}$ to be 0 for the sequence (c_i) to be statistically convergent. Later Kostyrko et al. [16] put forward the notion of ideal convergence of a sequence as a sequel to statistical convergence. He stated that for a sequence (c_i) to be ideal convergent to c , the set $\{i \in \mathbb{N} : |c_i - c| \geq \varepsilon\}$ belongs to I , where I is a family of subsets of $\mathbb{N}$ such that $\emptyset \in I$, A and B in $I \implies A \cup B \in I$ and $A \subseteq B, B \in I \implies A \in I$. A sequence $c = (c_i) \in \omega$ is I -bounded if there exists $L > 0$ such that $\{i \in \mathbb{N} : |c_i| > L\} \in I$. The concept of a Cauchy sequence was generalised to I -Cauchy sequence in [20] stating if for a sequence $c = (c_i) \in \omega$ and $\forall \varepsilon > 0, \exists K = K(\varepsilon) \in \mathbb{N}$ such that the set $\{k \in \mathbb{N} : |c_i - c_K| \geq \varepsilon\}$ belongs to I , then the sequence is I -Cauchy.

On parallel grounds of being motivated to initiate different other forms of convergence method; summability method using infinite matrix as an operator was defined to study the sequences' convergence. The idea was to use the matrix with certain conditions in a way that it sums convergent series to its Cauchy sums and eventually would appoint a limit to the diverging ones too. In [21], an infinite matrix $M = (m_{nk})_{n,k \in \mathbb{N}}$ was deemed to be regular iff all the terms of the sequence after operating matrix is bounded, $\lim_{n \rightarrow \infty} m_{nk} = 0 \forall k \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \sum_k m_{nk} = 1$.

Following are certain terms and definitions that will be required further in the literature.

Definition 1 ([1, Definition 8.3.1]). Two sequences $c = (c_i)$ and $d = (d_i)$ are said to be equal for almost all i relative to I , if the set $\{i \in \mathbb{N} : c_i \neq d_i\} \in I$.

Definition 2 ([20, Definition 2.1]). A sequence space S is said to be solid (or normal), if $(\beta_i c_i) \in S$ whenever $(c_i) \in S$ and (β_i) be a sequence of scalar in ω such that $|\beta_i| < 1$ and $i \in \mathbb{N}$.

Definition 3 ([20, Definition 2.3]). A sequence space is said to be monotone, if for all infinite $K \subseteq \mathbb{N}$ and (c_i) belonging to the space, the sequence $(\alpha_i c_i)$, where

$$\alpha_i = \begin{cases} 1, & i \in K, \\ 0 & \text{otherwise;} \end{cases}$$

also lies in the same sequence space.

Lemma 1 ([20, Lemma 2.6]). Let $K \in \mathcal{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.

Definition 4 ([1, Definition 4.1.1]). Matrix domain of an infinite matrix $M = (m_{nk})$ is the sequence space defined as $\mathcal{X}_M = \{c = (c_i) \in \omega : Mc \in \mathcal{X}\}$.

3. CATALAN STATISTICAL CONVERGENCE

Definition 5. A sequence $u = (u_k)$ is defined to be Catalan statistically convergent (or $\tilde{C}$ -statistically convergent), if for every $\varepsilon > 0$, $\exists$ a number u_0 , such that the density of the following set,

$$\mathcal{K}_\varepsilon(\tilde{C}) = \{k \in \mathbb{N} : |\tilde{C}(u_k) - u_0| \geq \varepsilon\},$$

is 0. The space of all $\tilde{C}$ -statistically convergent sequences will be designated by $\mathcal{S}(\tilde{C})$.

Example 1. Classic example of $\mathcal{S}(\tilde{C})$ is the space of all Catalan convergent sequences $c(\tilde{C})$.

Example 2.

$$\tilde{C}(u_k) = \begin{cases} k & \text{if } k = 3^n \text{ or } 5^n \\ \frac{1}{k} & \text{otherwise.} \end{cases}$$

The above mentioned sequence $(u_k) \in \mathcal{S}(\tilde{C})$ which statistically converges to 0.

Definition 6. A sequence $u = (u_k)$ is defined to be $\tilde{C}$ -statistically Cauchy, if $\exists \mathcal{N} = \mathcal{N}(\varepsilon)$, such that the density of the set,

$$\{k \in \mathbb{N} : |\tilde{C}u_k - \tilde{C}u_{\mathcal{N}}| \geq \varepsilon\},$$

is 0 $\forall \varepsilon > 0$.

Theorem 1. If u is a $\tilde{C}$ -statistically convergent sequence, then u is $\tilde{C}$ -statistically Cauchy.

Proof. Let $\varepsilon > 0$. Assume that $u_k \xrightarrow{\mathcal{S}(\tilde{C})} u_0$. This implies $|\tilde{C}u_k - u_0| < \frac{\varepsilon}{2}$ for almost all k . Choose $\mathcal{N}$ such that $|\tilde{C}u_{\mathcal{N}} - u_0| < \frac{\varepsilon}{2}$ then

$$|\tilde{C}u_k - \tilde{C}u_{\mathcal{N}}| < |\tilde{C}u_k - u_0| + |\tilde{C}u_{\mathcal{N}} - u_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for almost all } k.$$

Thus u is a $\tilde{C}$ -statistically Cauchy sequence. $\square$

Theorem 2. *If u is a sequence for which there is a $\tilde{C}$ -statistically convergent sequence v such that $\tilde{C}u_k = \tilde{C}v_k$ for almost all k , then u is a $\tilde{C}$ -statistically convergent sequence.*

Proof. Assume that $\tilde{C}u_k = \tilde{C}v_k$ for almost all k and $v_k \xrightarrow{S(\tilde{C})} l$. Then for $\varepsilon > 0$ and for each k ,

$$\{k \leq n : |\tilde{C}u_k - l| \geq \varepsilon\} \subseteq \{k \leq n : \tilde{C}u_k \neq \tilde{C}v_k\} \cup \{k \leq n : |\tilde{C}v_k - l| \geq \varepsilon\}.$$

Since $\tilde{C}u_k = \tilde{C}v_k$, the set on the right hand side of the equation contains a finite number of integers, say $h = h(\varepsilon)$. Thus for $\tilde{C}u_k = \tilde{C}v_k$ for almost all k ,

$$\lim_n \frac{|\{k \leq n : |\tilde{C}u_k - l| \geq \varepsilon\}|}{n} \leq \lim_k |n \leq k : \tilde{C}u_k \neq \tilde{C}v_k| + \lim_n \frac{h}{n} = 0$$

Hence $u_k \xrightarrow{S(\tilde{C})} l$. □

Definition 7. A sequence (u_k) is affirmed to be $\tilde{C}$ -stastically bounded provided the existence of some number $u_0 \geq 0$ guarantees the density of the set $\{k : |u_k| > u_0\}$ is null.

Corollary 1. $\tilde{C}$ -statistically convergence $\implies \tilde{C}$ -statistically boundedness.

Corollary 2. The set of $\tilde{C}$ -statistically convergent sequences is dense in ω .

4. I -CONVERGENT CATALAN SEQUENCE SPACE

In the current section we put forward some novel sequence spaces as an assembly of sequences whose $\tilde{C}$ transforms lie in the spaces c_0^I, c^I and ℓ_∞ . Advancing, we study some compelling results of the spaces and derive a handful of inclusion relations too. Throughout the paper we take I to be an admissible ideal of subset of $\mathbb{N}$. We delineate

$$\begin{aligned} c_0^I(\tilde{C}) &= \{u = (u_k) \in \omega : \{n \in \mathbb{N} : |\tilde{C}_n(u_k)| \geq \varepsilon\} \in I\}, \\ c^I(\tilde{C}) &= \{u = (u_k) \in \omega : \{n \in \mathbb{N} : |\tilde{C}_n(u_k) - l| \geq \varepsilon\} \in I\}, \\ \ell_\infty^I(\tilde{C}) &= \{u = (u_k) \in \omega : \exists M > 0 \text{ s.t. } \{n \in \mathbb{N} : |\tilde{C}_n(u_k)| > M\} \in I\}. \end{aligned}$$

We set down the following spaces too,

$$m_0^I(\tilde{C}) := c_0^I(\tilde{C}) \cap \ell_\infty(\tilde{C}),$$

and

$$m^I(\tilde{C}) := c^I(\tilde{C}) \cap \ell_\infty(\tilde{C}).$$

A general Catalan sequence space denoted by $\mathcal{X}(\tilde{C})$ where $\mathcal{X}$ is any sequence space can be given by

$$\mathcal{X}(\tilde{C}) = \{u = (u_k) \in \omega : (\tilde{C}u_k) \in \mathcal{X}\}.$$

Definition 8. If $u = (u_k)$ is a sequence in ω then u is said to be Catalan I -Cauchy if, for every $\varepsilon > 0$, $\exists$ a number $N = N(\varepsilon) \in \mathbb{N}$ such that $\{k \in \mathbb{N} : |\tilde{C}_n(u) - \tilde{C}_N(u)| \geq \varepsilon\} \in I$.

Corollary 3. Given I a finite admissible ideal of subset of $\mathbb{N}$, the space $c^I(\tilde{C})$ coincides with $c(\tilde{C})$ i.e. for $I = I_f$, $c^I(\tilde{C}) = c(\tilde{C})$.

Corollary 4. For a non-trivial ideal of subset of $\mathbb{N}$ such that $I_d = \{K \subseteq \mathbb{N} : d(K) = 0\}$, $c^{I_d}(\tilde{C}) = \mathcal{S}(\tilde{C})$.

Theorem 3. The sequence spaces $c_0^I(\tilde{C})$, $c^I(\tilde{C})$ and $\ell_\infty^I(\tilde{C})$ are linear spaces over $\mathbb{R}$.

Proof. Let $u = (u_k)$ and $v = (v_k)$ be random elements in $c^I(\tilde{C})$ and λ and μ be scalars. Then for a given $\varepsilon > 0$ there exist $l_1, l_2 \in \mathbb{R}$ such that

$$\begin{aligned} \left\{n \in \mathbb{N} : |\tilde{C}_n(u) - l_1| \geq \frac{\varepsilon}{2\lambda}\right\} &\in I, \\ \left\{n \in \mathbb{N} : |\tilde{C}_n(v) - l_2| \geq \frac{\varepsilon}{2\mu}\right\} &\in I, \end{aligned}$$

Let

$$\begin{aligned} \mathcal{D}_1 &= \left\{n \in \mathbb{N} : |\tilde{C}_n(u) - l_1| < \frac{\varepsilon}{2\lambda}\right\} \in \mathcal{F}(I), \\ \mathcal{D}_2 &= \left\{n \in \mathbb{N} : |\tilde{C}_n(v) - l_2| < \frac{\varepsilon}{2\mu}\right\} \in \mathcal{F}(I). \end{aligned}$$

be such that $\mathcal{D}_1^c, \mathcal{D}_2^c \in I$. Then

$$\begin{aligned} \mathcal{D}_3 &= \{n \in \mathbb{N} : |(\lambda\tilde{C}_n(u) + \mu\tilde{C}_n(v)) - (\lambda l_1 + \mu l_2)| < \varepsilon\} \\ &\supseteq \left[\left\{n \in \mathbb{N} : |\tilde{C}_n(u) - l_1| < \frac{\varepsilon}{2|\lambda|}\right\} \right] \left[\left\{n \in \mathbb{N} : |\tilde{C}_n(v) - l_2| < \frac{\varepsilon}{2|\mu|}\right\} \right]. \end{aligned}$$

Clearly $\mathcal{D}_3 \in \mathcal{F}(I)$ and hence $\mathcal{D}_3^c = \mathcal{D}_1^c \cup \mathcal{D}_2^c \in I$. Following which we can conclude that $(\lambda u_k + \mu v_k) \in c^I(\tilde{C})$. $\square$

Theorem 4. The spaces $\mathcal{X}(\tilde{C})$ are normed spaces with norm

$$\|u\|_{\mathcal{X}(\tilde{C})} = \sup_n |\tilde{C}_n(u)| \text{ where } C_n(u) = \sum_{k=0}^{\infty} c_{nk} u_k \text{ and } \mathcal{X} \in \{m^I, m_0^I\}. \quad (4.1)$$

Proof. Since the above defined spaces are linear spaces and it can be easily verified that equation (4.1) defines a norm. We therefore omit the proof since it is very straight forward. $\square$

Theorem 5. Consider the normed space $\mathcal{X}(\tilde{C})$ and let I be a non-trivial ideal. If for a sequence (u_k) we have $I\text{-}\lim_{n \rightarrow \infty} \tilde{C}_n(u_k) = l_1$ and $I\text{-}\lim_{n \rightarrow \infty} \tilde{C}_n(u_k) = l_2$, then $l_1 = l_2$.

Proof. Omitted. □

Example 3. Following is an example that depicts the convergence of a sequence via Catalan matrix which does not converge in the usual sense. Consider the sequence

$$u_k = \begin{cases} 1 & \text{if } k = 5^n, \forall n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The terms of the sequence can be written as $u = (u_k) = \{0, 1, 0, 1, 0, 0, 0, 1, \dots\}$. Operating Catalan matrix we get $C_n(u) = \{0, 0, 0, 0, 0.33, 0, 0, 0, \dots\}$. Thus the sequence eventually converges to 0. Thus $(u_k) \in c^I(\tilde{C}) \setminus c$, where c denotes the space of all convergent sequences.

Theorem 6. *The absolute property does not hold on the spaces $c^I(\tilde{C})$, $c_0^I(\tilde{C})$ and $\ell_\infty^I(\tilde{C})$, that is $\|u\|_{X(\tilde{C})} \neq \| |u| \|_{X(\tilde{C})}$. Following is an example of a sequence in $c^I(\tilde{C})$ in support of our claim. Consider a sequence $(u_k) = (-1)^k$, then,*

$$\sum_{k=0}^{\infty} c_{nk} |u_k| = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$$

Thus $\| |u| \|_{X(\tilde{C})} = \sup_n \{1, 1, 1, \dots\} = 1$ when $n \rightarrow \infty$ and

$$\| u \|_{X(\tilde{C})} = \sup_n \left| \sum_{k=0}^{\infty} c_{nk} u_k \right|$$

$$\sum_{k=0}^{\infty} c_{nk} u_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 & \dots \\ \frac{5}{14} & \frac{2}{14} & \frac{2}{14} & \frac{5}{14} & 0 & 0 & 0 & \dots \\ \frac{14}{42} & \frac{5}{42} & \frac{4}{42} & \frac{5}{42} & \frac{14}{42} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0.6 \\ 0 \\ 0.52 \\ \vdots \end{bmatrix}$$

Thus $\|u\|_{X(\tilde{C})} = \sup_n \{1, 0, 0.6, 0, 0.52, 0, 0.49, \dots\} = 0$ when $n \rightarrow \infty$. Clearly $\|u\|_{X(\tilde{C})} \neq \|u\|_{X(\tilde{C})}$.

Theorem 7. The inclusion relations $c_0^I(\tilde{C}) \subset c^I(\tilde{C}) \subset \ell_\infty^I(\tilde{C})$ are strict.

Proof. The forward relation is quite evident. The following examples are proposed to back our claim.

Example 4. Consider the sequence $(u_k) = (1 - \frac{1}{k})$. Now, for a non-trivial finite ideal $I = I_f$, $\tilde{C}_n(u) = \{0, 0.25, 0.366, 0.452, \dots\}$. One can clearly see that the sequence eventually approaches 1. Thus $(u_k) \in c^I(\tilde{C}) \setminus c_0^I(\tilde{C})$.

To support our claim of strictness relation between $c^I(\tilde{C})$ and $\ell_\infty^I(\tilde{C})$, we put forward the following example.

Example 5. Consider the sequence

$$u_k = \begin{cases} 1 - \frac{1}{k} & \text{if } k \text{ is odd,} \\ 10(1 + \frac{1}{k}) & \text{if } k \text{ is even.} \end{cases}$$

Evidently for a non-trivial finite ideal $I = I_f$, since (u_k) is bounded therefore $(u_k) \in \ell_\infty^I(\tilde{C})$. Now, $\tilde{C}_n(u) = \{0, 7.5, 0.567, 7.56, 0.657, 7.6, \dots\}$. Therefore the odd and even subsequences converge to different limits namely 1 and 10 respectively. Thus $(u_k) \in \ell_\infty^I(\tilde{C}) \setminus c^I(\tilde{C})$.

□

Theorem 8. The inclusion relation $c(\tilde{C}) \subset c^I(\tilde{C})$ is strict where I is a non trivial ideal.

Proof. It is well established that for $I = I_f$, $c^I = c$ and in general $c \subset c^I$, therefore for given sequence spaces X and Y , $X \subseteq Y$ implies $X(\tilde{C}) \subseteq Y(\tilde{C})$. Thus it is easily observable that $c(\tilde{C}) \subset c^I(\tilde{C})$.

In support of our claim of strictness we construct the following example.

Example 6. Define the sequence $u = (u_k)$ as following

$$\tilde{C}_k(u) = \begin{cases} k^{\frac{1}{3}} & \text{if } k \text{ is a cube root,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the sequence $u \in S(\tilde{C}) \setminus c(\tilde{C})$.

Example 7. Let I be a non-trivial ideal in $\mathbb{N}$. For

$$\tilde{C}_k(u) = (0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 2, 2, 2, 0, 0, 2, \dots),$$

we have

$$\mathcal{A}_e = \{k \in \mathbb{N} : \tilde{C}_k(u) \neq 0\} = \{7, 11, 12, 13, 16, 18, 19, 21, 23, 25, 27, \dots\}$$

From the looks of it we can clearly conclude that the set $\mathcal{A}$ eventually contains all the odd numbers and that $\tilde{C}_k(u) \in c^I(\tilde{C})$. But clearly it's asymptotic density is $\frac{1}{2}$ and therefore $\tilde{C}_k(u) \notin S(\tilde{C})$.

□

Theorem 9. The spaces $m_0^I(\tilde{C})$ and $m^I(\tilde{C})$ are Banach spaces normed by (4.1).

Proof. Consider a Cauchy sequence $(u_k^{(l)})$ in $m^I(\tilde{C}) \subset l_\infty(\tilde{C})$. Then $(u_k^{(l)})$ is convergent in $l_\infty(\tilde{C})$, i.e. $\lim_{l \rightarrow \infty} \tilde{C}_k^{(l)}(u) = \tilde{C}_k(u)$. Let $I\text{-}\lim \tilde{C}_k^{(l)}(u) = c_l$, $l \in \mathbb{N}$. To conclude the result, we verify that,

- (i) (c_l) is convergent to a number c .
- (ii) $I\text{-}\lim \tilde{C}_k(u) = c$.
- (i) $(u_k^{(l)})$ being a Cauchy sequence, for any $\varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that

$$\left| \tilde{C}_k^{(l)}(u) - \tilde{C}_k^{(m)}(u) \right| < \frac{\varepsilon}{3}, \text{ for all } l, m \geq n_0. \quad (4.2)$$

Now let $\mathcal{B}_l$ and $\mathcal{B}_m$ be two sets in I . Thus,

$$\mathcal{B}_l = \left\{ k \in \mathbb{N} : |\tilde{C}_k^{(l)}(u) - c_l| \geq \frac{\varepsilon}{3} \right\} \quad (4.3)$$

and

$$\mathcal{B}_m = \left\{ k \in \mathbb{N} : |\tilde{C}_k^{(m)}(u) - c_m| \geq \frac{\varepsilon}{3} \right\}. \quad (4.4)$$

Now suppose that $l, m \geq n_0$ and $k \notin \mathcal{B}_l \cap \mathcal{B}_m$. Then by using (4.2), (4.3) and (4.4) we have

$$|c_l - c_m| \leq |\tilde{C}_k^{(l)}(u) - c_l| + |\tilde{C}_k^{(m)}(u) - c_m| + |\tilde{C}_k^{(l)}(u) - \tilde{C}_k^{(m)}(u)| < \varepsilon.$$

Thus (c_l) is a Cauchy sequence and is thus convergent to c , hence $\lim_{l \rightarrow \infty} c_l = c$.

(ii) Given $\delta > 0$, we can chose j_0 as

$$|c_l - c| < \frac{\delta}{3} \quad \text{for each } l > j_0. \quad (4.5)$$

Since $(u_k^{(l)}) \rightarrow u_k$ as $l \rightarrow \infty$. Thus

$$|\tilde{C}_k^{(l)}(u) - \tilde{C}_k(u)| < \frac{\delta}{3} \quad \text{for each } l > j_0. \quad (4.6)$$

Since $(\tilde{C}_k^{(m)})$ is I -convergent to c_l , there exists $\mathcal{A} \in I$ such that for each $k \notin \mathcal{A}$, we have

$$|\tilde{C}_k^{(m)}(u) - c_m| < \frac{\delta}{3}. \quad (4.7)$$

Let $m > j_0$, then with aid of (4.5), (4.6) and (4.7) for all $k \notin \mathcal{A}$, we have

$$|\tilde{C}_k(u) - c| \leq |\tilde{C}_k(u) - \tilde{C}_k^{(m)}(u)| + |\tilde{C}_k^{(m)}(u) - c_m| + |c_m - c| < \delta.$$

This implies that (u_k) is Catalan I -convergent to ℓ . Thus $m^I(\tilde{C})$ is a Banach space. □

Corollary 5. Every I -Cauchy sequence in $m_0^I(\tilde{C}), m^I(\tilde{C})$ is I -convergent.

Theorem 10. Let $\mathcal{A}$ be a closed subset of a Banach space X . Then $\mathcal{A}(\tilde{C})$ is also closed in $X(\tilde{C})$.

Proof. The proof is trivial. □

Corollary 6. $X(\tilde{C})$ is a separable space under the restriction of X being a separable space too.

Theorem 11. The spaces $c_0^I(\tilde{C}), c^I(\tilde{C})$ and $\ell_\infty^I(\tilde{C})$ are BK spaces according to their norms defined by (4.1).

Proof. Since the given matrix in (1.2) is a triangular matrix and definition (4) holds; therefore by using Wilansky Theorem 4.3.12 [21] we can conclude our claim. □

Theorem 12. The sequence spaces $m^I(\tilde{C})$ and $m_0^I(\tilde{C})$ are Fréchet space defined by the metric

$$d(u, v) = \sup_n |\tilde{C}_n(u) - \tilde{C}_n(v)|. \quad (4.8)$$

Proof. Following Theorem 9 we conclude that the space is a complete metric space with the induced metric via norm. The metric (4.8) is a translation invariant metric

since for any $u, v, w \in m^I(\tilde{C})$

$$\begin{aligned} d(u+w, v+w) &= \sup_n |\tilde{C}_n(u+w) - \tilde{C}_n(v+w)| \\ &= \sup_n |\tilde{C}_n(u) + \tilde{C}_n(w) - \tilde{C}_n(v) - \tilde{C}_n(w)| \\ &= d(u, v) \end{aligned}$$

Evidently the spaces have coordinates continuity since $u^{(j)} \rightarrow u$ via the induced metric then $u_k^{(j)} \rightarrow u_k$ when $j \rightarrow \infty \forall k$. $\square$

Since the spaces's inclusions $m^I(\tilde{C}) \subset \ell_\infty(\tilde{C})$ and $m_0^I(\tilde{C}) \subset \ell_\infty(\tilde{C})$ are strict therefore in view of Theorem 9, we conclude the following result.

Theorem 13. *The spaces $m^I(\tilde{C})$ and $m_0^I(\tilde{C})$ are nowhere dense subsets of $\ell_\infty(\tilde{C})$.*

Proof. Since $m^I(\tilde{C}) = c^I(\tilde{C}) \cap \ell_\infty(\tilde{C})$ is a closed subset of $\ell_\infty(\tilde{C})$ therefore $\overline{m^I(\tilde{C})} = m^I(\tilde{C})$. Consider an arbitrary point $(u_k) \in m^I(\tilde{C})$. Thus for a given radius $r > 0$ there exists (v_k) in the neighbourhood of $\mathcal{B}(u, r)$ such that $(v_k) \notin c^I(\tilde{C})$ and $(v_k) \notin \ell_\infty(\tilde{C})$. Hence $\mathcal{B}(u, r) \not\subset m^I(\tilde{C})$. Since (u_k) was arbitrary and is not an interior point, therefore $m^I(\tilde{C})^\circ = \emptyset$. $\square$

Lemma 2. *If all the subsequences of a sequence (u_k) are matrix summable via regular matrix $\tilde{C} = (\tilde{c}_{nk})$ then the sequence (u_k) converges.*

Lemma 3 ([7, Theorem 1]). *Consider the finite ideal $I = I_f$ and let (u_k) be an I_f -bounded sequence in $X = m_0^{I_f}(\tilde{C}), m^{I_f}(\tilde{C})$ such that X satisfies Banach-Saks property with respect to the Catalan matrix summability $(\tilde{c}_{nk})$, then there exists a subsequence (u_{k_l}) of u_k whose all the subsequences will be $(\tilde{c}_{nk})$ summable.*

Theorem 14. *A sequence $u = (u_k) \in \omega$ is Catalan I -convergent if and only if for every $\varepsilon > 0$, there exists $\mathcal{N} = \mathcal{N}(\varepsilon) \in \mathcal{N}$, such that*

$$\{n \in \mathbb{N} : |\tilde{C}_n(u) - \tilde{C}_N(u)| < \varepsilon\} \in \mathcal{F}(I). \quad (4.9)$$

Proof. Consider the sequence $u = (u_k) \in \omega$ which is Catalan I -convergent to some number $m \in \mathbb{R}$, then for any $\varepsilon > 0$, we have

$$\mathcal{B}_\varepsilon = \left\{n \in \mathbb{N} : |\tilde{C}_n(u) - m| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(I).$$

For a fixed integer $\mathcal{N} = \mathcal{N}(\varepsilon) \in \mathcal{B}_\varepsilon$, the following inequality,

$$|\tilde{C}_n(u) - \tilde{C}_\mathcal{N}(u)| \leq |\tilde{C}_n(u) - m| + |m - \tilde{C}_\mathcal{N}(u)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

holds for all $n \in \mathcal{B}_\varepsilon$. Hence (4.9) holds. Conversely, let (4.9) be true for all $\varepsilon > 0$. Then

$$\mathcal{B}'_\varepsilon = \{n \in \mathbb{N} : \tilde{C}_n(u) \in \mathcal{I}_\varepsilon\} \in \mathcal{F}(I) \quad \text{for all } \varepsilon > 0,$$

where $\mathcal{I}_\varepsilon = [\tilde{C}_n(u) - \varepsilon, \tilde{C}_n(u) + \varepsilon]$. By taking $\varepsilon > 0$, we have $\mathcal{B}'_\varepsilon \in \mathcal{F}(I)$ and $\mathcal{B}'_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$. Hence $\mathcal{B}'_\varepsilon \cap \mathcal{B}'_{\frac{\varepsilon}{2}} \in \mathcal{F}(I)$. This implies that

$$\mathcal{J} = \mathcal{I}_\varepsilon \cap \mathcal{I}_{\frac{\varepsilon}{2}} \neq \emptyset,$$

that is,

$$\{n \in \mathbb{N} : \tilde{C}_n(u) \in \mathcal{J}\} \in \mathcal{F}(I)$$

and thus

$$\text{diam}(\mathcal{J}) \leq \frac{1}{2} \text{diam}(\mathcal{I}_\varepsilon),$$

where the diam depicts length of the interval. A sequence of compact intervals are thus obtained by applying induction as follows,

$$\mathcal{I}_\varepsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that

$$\text{diam}(I_n) \leq \frac{1}{2} \text{diam}(I_{n-1}) \quad \text{for } n = (2, 3, \dots)$$

and

$$\{n \in \mathbb{N} : \tilde{C}_n(u) \in I_n\} \in \mathcal{F}(I).$$

Then there exists a number $m \in \bigcap_{n \in \mathbb{N}} I_n$ such that $m = I\text{-}\lim \tilde{C}_n(u)$ observing that $u = (u_k) \in \omega$ is Catalan I -convergent. $\square$

Theorem 15. *Let I be an admissible ideal. The following statements are equivalent.*

- (i) $(u_k) \in c^I(\tilde{C})$;
- (ii) $\exists (v_k) \in c^I(\tilde{C})$ so that $u_k = v_k$ for a.a.k.r.I;
- (iii) $\exists (v_k) \in c^I(\tilde{C})$ and $w_k \in c_0^I(\tilde{C})$ such that $u_k = v_k + w_k \forall n \in \mathbb{N}$ and

$$\{n \in \mathbb{N} : |\tilde{C}_n(u) - l| \geq \varepsilon\} \in I;$$

- (iv) there exists a subset $\mathcal{A} = \{n_k : k \in \mathbb{N}, n_k < n_{k+1}\}$ of $\mathbb{N}$ such that $\mathcal{A} \in \mathcal{F}(I)$ and $\lim_{k \rightarrow \infty} |\tilde{C}_{n_k}(u) - l| = 0$.

Proof. Omitted. $\square$

Theorem 16. *The spaces $c_0^I(\tilde{C})$ and $m_0^I(\tilde{C})$ are solid and monotone.*

Proof. Consider the sequence $u = (u_k) \in c_0^I(\tilde{C}) \implies \{n \in \mathbb{N} : |\tilde{C}_n(u)| \geq \varepsilon\} \in I$ for $\varepsilon > 0$. Let $\xi = (\xi_n)$ be a sequence of scalars such that $-1 \leq \xi \leq 1$ for all $n \in \mathbb{N}$. Then,

$$|\tilde{C}_n(\xi u_k)| = |\xi \tilde{C}_n(u_k)| \leq |\xi| |\tilde{C}_n(u_k)| \leq |\tilde{C}_n(u_k)| \quad \text{for all } n \in \mathbb{N}.$$

Implementing inclusion relation

$$\{n \in \mathbb{N} : |\tilde{C}_n(\xi u_k)| \geq \varepsilon\} \subseteq \{n \in \mathbb{N} : |\tilde{C}_n(u_k)| \geq \varepsilon\} \in I,$$

implies

$$\{n \in \mathbb{N} : |\tilde{C}_n(\xi u_k)| \geq \varepsilon\} \in I.$$

Therefore, $(\xi u_k) \in c_0^I(\tilde{C}_n)$. Hence the space $c_0^I(\tilde{C}_n)$ is solid and therefore monotone. $\square$

Corollary 7. *For a non-maximal ideal $I \neq I_f$, the spaces $c_0^I(\tilde{C}), m_0^I(\tilde{C})$ are neither solid nor monotone.*

Theorem 17. *The spaces $c_0^I(\tilde{C})$ and $c^I(\tilde{C})$ are sequence algebra.*

Proof. Let $\varepsilon > 0$ be given. Then

$$\begin{aligned}\mathcal{A}\left(\frac{\varepsilon}{2}\right) &= \left\{n \in \mathbb{N} : |\tilde{C}_n(u) - l_1| \geq \frac{\varepsilon}{2}\right\} \in I, \\ \mathcal{B}\left(\frac{\varepsilon}{2}\right) &= \left\{n \in \mathbb{N} : |\tilde{C}_n(v) - l_2| \geq \frac{\varepsilon}{2}\right\} \in I.\end{aligned}$$

All that is left to be shown is for the following set $\mathcal{C}$ to be contained in union of $\mathcal{A}$ and $\mathcal{B}$.

$$\mathcal{C}(\varepsilon) = \left\{n \in \mathbb{N} : |\tilde{C}_n(u \cdot v) - l_1 l_2| \geq \varepsilon\right\}.$$

Now

$$\begin{aligned}\{n \in \mathbb{N} : |\tilde{C}_n(u \cdot v) - l_1 l_2| \geq \varepsilon\} &\subseteq \left\{n \in \mathbb{N} : |\tilde{C}_n(u) - l_1| \geq \frac{\varepsilon}{2}\right\} \\ &\cup \left\{n \in \mathbb{N} : |\tilde{C}_n(v) - l_2| \geq \frac{\varepsilon}{2}\right\}.\end{aligned}$$

Since

$$\mathcal{C}(\varepsilon) \subset \left\{n \in \mathbb{N} : |\tilde{C}_n(u) - l_1| \geq \frac{\varepsilon}{2}\right\} \cup \left\{n \in \mathbb{N} : |\tilde{C}_n(v) - l_2| \geq \frac{\varepsilon}{2}\right\}.$$

Thus $\mathcal{C}(\varepsilon) \subseteq \mathcal{A}\left(\frac{\varepsilon}{2}\right) \cup \mathcal{B}\left(\frac{\varepsilon}{2}\right) \Rightarrow (u \cdot v) \in c^I(\tilde{C})$. $\square$

Theorem 18. *The function $h: m^I(\tilde{C}) \rightarrow \mathbb{R}$ defined by $h(u) = I\text{-}\lim \tilde{C}_n(u)$, where $m^I(\tilde{C}) = \ell_\infty(\tilde{C}) \cap c^I(\tilde{C})$, is a Lipschitz function and hence uniformly continuous.*

Proof. The function is well defined since for any two sequences $u, v \in m^I(\tilde{C})$, such that

$$\begin{aligned}u = v &\Rightarrow I\text{-}\lim \tilde{C}_n(u) = I\text{-}\lim \tilde{C}_n(v) \\ &\Rightarrow h(u) = h(v).\end{aligned}$$

Let $(u_k), (v_k) \in m^I(\tilde{C})$, $u \neq v$. Then

$$\mathcal{A}_u = \left\{n \in \mathbb{N} : |\tilde{C}_n(u) - h(u)| \geq |u - v|_*\right\} \in I \quad (4.10)$$

and

$$\mathcal{A}_v = \left\{n \in \mathbb{N} : |\tilde{C}_n(v) - h(v)| \geq |u - v|_*\right\} \in I, \quad (4.11)$$

where $|u - v|_* = \sup_n |\tilde{C}_n(u) - \tilde{C}_n(v)|$. Thus equations (4.10), (4.11) imply

$$\mathcal{B}_u = \{n \in \mathbb{N} : |\tilde{C}_n(u) - h(u)| < |u - v|_*\} \in \mathcal{F}(I)$$

and

$$\mathcal{B}_v = \{n \in \mathbb{N} : |\tilde{C}_n(v) - h(v)| < |u - v|_*\} \in \mathcal{F}(I).$$

Hence $\mathcal{B} = \mathcal{B}_u \cap \mathcal{B}_v \in \mathcal{F}(I)$, so that $\mathcal{B} \neq \emptyset$. Therefore, selecting $n \in \mathcal{B}$, we have

$$\begin{aligned} |h(u) - h(v)| &\leq |h(u) - \tilde{C}_n(u)| + |\tilde{C}_n(u) - \tilde{C}_n(v)| + |\tilde{C}_n(v) - h(v)| \\ &\leq 3|u - v|_*. \end{aligned}$$

Thus, h is a Lipschitz function and hence uniformly continuous. $\square$

Theorem 19. If $u = (u_k), v = (v_k) \in m^I(\tilde{C})$ such that $\tilde{C}_n(u \cdot v) = \tilde{C}_n(u) \cdot \tilde{C}_n(v)$, then $(u \cdot v) \in m^I(\tilde{C})$ and $h(u \cdot v) = h(u) \cdot h(v)$, where $h: m^I(\tilde{C}) \rightarrow \mathbb{R}$ is defined by $h(u) = I\text{-}\lim \tilde{C}_n(u)$.

Proof. Since $u, v \in m^I(\tilde{C})$, therefore taking $\varepsilon = \sup_n |\tilde{C}_n(u) - \tilde{C}_n(v)| > 0$,

$$\mathcal{B}_u = \{n \in \mathbb{N} : |\tilde{C}_n(u) - h(u)| < \varepsilon\} \in \mathcal{F}(I) \quad (4.12)$$

and

$$\mathcal{B}_v = \{n \in \mathbb{N} : |\tilde{C}_n(v) - h(v)| < \varepsilon\} \in \mathcal{F}(I), \quad (4.13)$$

Now, we have

$$\begin{aligned} |\tilde{C}_n(u \cdot v) - h(u)h(v)| &= |\tilde{C}_n(u)\tilde{C}_n(v) - \tilde{C}_n(u)h(v) + \tilde{C}_n(u)h(v) - h(u)h(v)| \\ &\leq |\tilde{C}_n(u)| |\tilde{C}_n(v) - h(v)| + |h(v)| |\tilde{C}_n(u) - h(u)|. \end{aligned} \quad (4.14)$$

As $m^I(\tilde{C}) \subseteq \ell_\infty(\tilde{C})$, there exists $\mathcal{M} \in \mathbb{R}$ such that $|\tilde{C}_n(u)| < \mathcal{M}$. Therefore, from the equations (4.12), (4.13) and (4.14) we have

$$\begin{aligned} |\tilde{C}_n(u \cdot v) - h(u)h(v)| &= |\tilde{C}_n(u) \cdot \tilde{C}_n(v) - h(u)h(v)| \\ &\leq \mathcal{M}\varepsilon + |h(v)|\varepsilon = \varepsilon_1, \text{ (say)} \end{aligned}$$

for all $n \in \mathcal{B}_u \cap \mathcal{B}_v \in \mathcal{F}(I)$. Hence $(u \cdot v) \in m^I(\tilde{C})$ and $h(u \cdot v) = h(u) \cdot h(v)$.

Example 8. Consider the sequences,

$$u_k = \begin{cases} 2 & \text{if } k = 2^k, \\ 0 & \text{if } k \neq 2^k; \end{cases} \quad v_k = \begin{cases} 0 & \text{if } k = 2^k, \\ 1 & \text{if } k \neq 2^k. \end{cases}$$

Clearly the sequences do not converge in the usual sense but the sequence $w = (u \cdot v) = u_k \cdot v_k = \{0, 0, 0, 0, \dots\}$ converges to 0 which further implies $\tilde{C}_n(u \cdot u) \rightarrow 0$. Now

$$\tilde{C}_n(u) = \{0, 1, 0.4, 1, 0.48, 0.36, 0.31, \dots\} \text{ and } \tilde{C}_n(v) = \{1, 0, 1, 0, 1, 1, 1, 0, 1, \dots\}$$

which converge to 0 and 1 respectively therefore $\tilde{C}_n(u) \cdot \tilde{C}_n(v) = 0 = \tilde{C}_n(u \cdot v) \implies h(u \cdot v) = 0$. $\square$

Theorem 20. *For a non-trivial admissible ideal I and a sequence $u = (u_k) \in \omega$, if there exists a sequence $v = (v_k) \in c^I(\tilde{C})$ such that $\tilde{C}_n(u) = \tilde{C}_n(v)$, then $u \in c^I(\tilde{C})$.*

Proof. Omitted. □

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AMPLE SOCLE SUPPLEMENTED MODULES

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Abstract. In this work, amply socle supplemented modules are defined and some properties of these modules are investigated. We prove that every π -projective and socle supplemented module is amply socle supplemented. We also prove that every factor module and every homomorphic image of an amply socle supplemented module are amply socle supplemented. Let M be a projective and socle supplemented R -module. Then every finitely M -generated R -module is amply socle supplemented.

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1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R -module. We will denote a submodule N of M by $N \leq M$. Let M be an R -module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. Let M be an R -module. M is called a *hollow* module if every proper submodule of M is small in M . M is called a *local* module if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. A submodule N of an R -module M is called an *essential* submodule of M and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an R -module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M . M is called a *supplemented* module if every submodule of M has a supplement in M . If every essential submodule of M has a supplement in M , then M is called an *essential supplemented* (or briefly, *e-supplemented*) module. Let M be an R -module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement V' with $V' \leq V$, we say U has *ample supplements* in M . If every submodule of M has ample supplements in

M , then M is called an *amply supplemented* module. If every essential submodule of M has ample supplements in M , then M is called an *amply essential supplemented* (or briefly, *amply e-supplemented*) module. The intersection of all maximal submodules of an R -module M is called the *radical* of M and denoted by $\text{Rad}M$. If M have no maximal submodules, then we denote $\text{Rad}M = M$. The sum of all simple submodules of an R -module M is called the *socle* of M and denoted by $\text{Soc}M$. Let M be an R -module. It is defined the relation ' β^* ' on the set of submodules of an R -module M by $X\beta^*Y$ if and only if $Y + K = M$ for every $K \leq M$ such that $X + K = M$ and $X + T = M$ for every $T \leq M$ such that $Y + T = M$. Let M be an R -module and $K \leq V \leq M$. We say V lies above K in M if $V/K \ll M/K$ (see [15, 16]).

More informations about (amply) supplemented modules are in [2, 13–16]. More details about (amply) essential supplemented modules are in [6, 7, 9–11]. The definition of β^* relation and some properties of this relation are in [1].

Lemma 1. *Let M be an R -module. The following statements hold.*

- (i) $\text{Soc}M = \bigcap_{L \trianglelefteq M} L$.
- (ii) For $K \leq M$, $\text{Soc}K = K \cap \text{Soc}M$.
- (iii) $\text{Soc}M \trianglelefteq M$ if and only if $\text{Soc}K \neq 0$ for every nonzero submodule K of M .
- (iv) Let N be an R -module and $f : M \rightarrow N$ be an R -module homomorphism. Then $f(\text{Soc}M) \subset \text{Soc}f(M)$.
- (v) For $K \leq M$, $(\text{Soc}M + K)/K \subset \text{Soc}(M/K)$.
- (vi) If $M = \bigoplus_{\Lambda} M_{\lambda}$, then $\text{Soc}M = \bigoplus_{\Lambda} \text{Soc}M_{\lambda}$.

Proof. See [16, 21.2]. □

Definition 1 ([3–5]). Let M be an R -module. If every $U \leq M$ with $\text{Soc}M \leq U$ has a supplement in M , then M is called a *socle supplemented* (or briefly, *s-supplemented*) module.

Definition 2 ([4, 5]). Let M be an R -module and $X \leq M$. If X is a supplement of a submodule U of M with $\text{Soc}M \leq U$, then X is called a *s-supplement submodule* in M .

Lemma 2 ([4, 5]). *Let M be a socle supplemented module. Then every finitely M -generated R -module is socle supplemented.*

2. AMPLY SOCLE SUPPLEMENTED MODULES

Definition 3. Let M be an R -module. If every submodule of M which contains $\text{Soc}M$ has ample supplements in M , then M is called an *amply socle supplemented* (or briefly, *amply s-supplemented*) module. (See also [8])

Clearly, every amply socle supplemented module is socle supplemented.

Lemma 3. *Let M be an amply s-supplemented module. Then M is amply e-supplemented.*

Proof. Let $U \trianglelefteq M$. By Lemma 1, $\text{Soc}M \leq U$ and since M is amply s -supplemented, U has ample supplements in M . Hence M is amply e -supplemented. $\square$

Corollary 1. *Let M be an amply s -supplemented module. Then M is e -supplemented.*

Proof. Clear from Lemma 3. $\square$

Proposition 1. *Let M be an amply e -supplemented module and $\text{Soc}M \trianglelefteq M$. Then M is amply s -supplemented.*

Proof. Let $\text{Soc}M \leq U \leq M$. Since $\text{Soc}M \trianglelefteq M$, $U \trianglelefteq M$. Since M is amply e -supplemented, U has ample supplements in M . Hence M is amply s -supplemented. $\square$

Proposition 2. *Let M be an amply s -supplemented module. Then $M/\text{Rad}M$ have no proper essential submodules.*

Proof. Since M is amply s -supplemented, by Corollary 1, M is e -supplemented. Then by [11, Proposition 2.5], $M/\text{Rad}M$ have no proper essential submodules. $\square$

Lemma 4. *Let M be an amply s -supplemented module. Then every factor module of M is amply s -supplemented.*

Proof. Let M/K be any factor module of M . Let $\text{Soc}(M/K) \leq U/K \leq M/K$ and $M/K = U/K + V/K$. By Lemma 1, $(\text{Soc}M + K)/K \subset \text{Soc}(M/K)$. Hence $(\text{Soc}M + K)/K \leq U/K$ and $\text{Soc}M \leq U$. Since $M/K = U/K + V/K$, $M = U + V$. Since M is amply s -supplemented, U has a supplement X in M with $X \leq V$. Since $K \leq U$, by [16, 41.1], $(X + K)/K$ is a supplement of U/K in M/K . Moreover, $(X + K)/K \leq V/K$. Hence M/K is amply s -supplemented. $\square$

Corollary 2. *Every homomorphic image of an amply s -supplemented module is amply s -supplemented.*

Proof. Clear from Lemma 4. $\square$

Let M be an R -module. If for every $U, V \leq M$ with $M = U + V$, there exists an R -module homomorphism $f : M \rightarrow M$ with $f(M) \leq U$ and $(1 - f)(M) \leq V$, then M is called a π -projective module. (See [12, 16].)

Lemma 5. *If M is a π -projective and s -supplemented module, then M is an amply s -supplemented module.*

Proof. Let $\text{Soc}M \leq U \leq M$, $M = U + V$ and X be a supplement of U in M . Since M is π -projective and $M = U + V$, there exists an R -module homomorphism $f : M \rightarrow M$ such that $\text{Im}f \subset V$ and $\text{Im}(1 - f) \subset U$. So, we have $M = f(M) + (1 - f)(M) = f(U) + f(X) + U = U + f(X)$. Suppose that $a \in U \cap f(X)$. Since $a \in f(X)$, then there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x) - x + x = x - (1 - f)(x)$ and $(1 - f)(x) \in U$, we have $x = a + (1 - f)(x) \in U$. Thus $x \in U \cap X$ and so,

$a = f(x) \in f(U \cap X)$. Therefore we have $U \cap f(X) \leq f(U \cap X) \ll f(X)$. This means that $f(X)$ is a supplement of U in M . Moreover, $f(X) \subset V$. Therefore M is amply s-supplemented. $\square$

Corollary 3. *If M is projective and s-supplemented, then M is amply s-supplemented.*

Proof. Clear from Lemma 5. $\square$

Lemma 6. *Let M be an R -module. If every submodule of M which contains $\text{Soc}M$ is β^* equivalent to a s-supplement submodule in M , then M is s-supplemented.*

Proof. Let $\text{Soc}M \leq U \leq M$. By hypothesis, there exists a s-supplement submodule X in M such that $U\beta^*X$. Since X is a s-supplement submodule in M , there exists a submodule Y of M such that $\text{Soc}M \leq Y$ and X is a supplement of Y in M . Since $\text{Soc}M \leq Y$, by hypothesis, there exists a s-supplement submodule V in M such that $Y\beta^*V$. Since X is a supplement of Y in M and $Y\beta^*V$, by [1, Theorem 2.6 (ii)], X is a supplement of V in M . Since V is a supplement submodule in M , we can see that V is a supplement of X in M and since $U\beta^*X$, by [2, Theorem 2.6 (ii)], V is a supplement of U in M . $\square$

Lemma 7. *Let M be a π -projective R -module. If every submodule of M which contains $\text{Soc}M$ is β^* equivalent to a s-supplement submodule in M , then M is amply s-supplemented.*

Proof. By Lemma 6, M is s-supplemented. Then by Lemma 5, M is amply s-supplemented. $\square$

Corollary 4. *Let M be a projective R -module. If every submodule of M which contains $\text{Soc}M$ is β^* equivalent to a s-supplement submodule in M , then M is amply s-supplemented.*

Proof. Clear from Lemma 7. $\square$

Corollary 5. *Let M be a π -projective R -module. If every submodule of M which contains $\text{Soc}M$ lies above a s-supplement submodule in M , then M is amply s-supplemented.*

Proof. Clear from Lemma 7. $\square$

Lemma 8. *Let Λ be a finite index set and $\{M_\lambda\}_\Lambda$ be a family of projective R -modules. If M_λ is s-supplemented for every $\lambda \in \Lambda$, then $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is amply s-supplemented.*

Proof. Since M_λ is s-supplemented for every $\lambda \in \Lambda$, we can see that $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is s-supplemented. Since M_λ is projective for every $\lambda \in \Lambda$, by [16, 18.1], $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is projective. Since $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is projective and s-supplemented, by Corollary 3, $\bigoplus_{\lambda \in \Lambda} M_\lambda$ is amply s-supplemented. $\square$

Corollary 6. *Let M be a projective R -module. If M is s -supplemented, then $M^{(\Lambda)}$ is amply s -supplemented for every finite index set Λ .*

Proof. Clear from Lemma 8. □

Corollary 7. *Let M be a projective R -module. If M is s -supplemented, then every finitely M -generated R -module is amply s -supplemented.*

Proof. Let N be a finitely M -generated R -module. Then there exist a finite index set Λ and an R -module epimorphism $f : M^{(\Lambda)} \rightarrow N$. Since M is projective and s -supplemented, by Corollary 6, $M^{(\Lambda)}$ is amply s -supplemented. Then by Corollary 2, N is amply s -supplemented. □

Lemma 9. *Let M be an R -module. If every submodule of M is s -supplemented, then M is amply s -supplemented.*

Proof. Let $\text{Soc}M \leq U \leq M$ and $M = U + V$ with $V \leq M$. Since $\text{Soc}M \leq U$, by Lemma 1, $\text{Soc}V = V \cap \text{Soc}M \leq U \cap V$. By hypothesis, V is s -supplemented. Then $U \cap V$ has a supplement X in V . By this, $V = U \cap V + X$ and $U \cap X = U \cap V \cap X \ll X$. Then $M = U + V = U + U \cap V + X = U + X$ and $U \cap X \ll X$. Hence M is amply s -supplemented. □

Lemma 10. *Let R be any ring. Then every R -module is s -supplemented if and only if every R -module is amply s -supplemented.*

Proof. ($\implies$) Let M be any R -module. Since every R -module is s -supplemented, every submodule of M is s -supplemented. Then by Lemma 9, M is amply s -supplemented.

($\impliedby$) Clear. □

Proposition 3. *Let R be a ring. The following assertions are equivalent.*

- (i) ${}_R R$ is s -supplemented.
- (ii) ${}_R R$ is amply s -supplemented.
- (iii) Every finitely generated R -module is s -supplemented.
- (iv) Every finitely generated R -module is amply s -supplemented.

Proof. (i) $\iff$ (ii) Clear from Corollary 3, since ${}_R R$ is projective.

(i) $\implies$ (iii) Clear from Lemma 2.

(iii) $\implies$ (iv) Let M be a finitely generated R -module. Then there exist a finite index set Λ and an R -module epimorphism $f : R^{(\Lambda)} \rightarrow M$. Since every finitely generated R -module is s -supplemented, $R^{(\Lambda)}$ is s -supplemented. Since ${}_R R$ is projective, by [16, 18.1], $R^{(\Lambda)}$ is also projective. Then by Corollary 3, $R^{(\Lambda)}$ is amply s -supplemented. Since $f : R^{(\Lambda)} \rightarrow M$ is an R -module epimorphism, by Corollary 2, M is also amply s -supplemented.

(iv) $\implies$ (i) Clear. □

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THE GROUP INVERTIBILITY OF MATRICES OVER BÉZOUT DOMAINS

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Abstract. Let R be a Bézout domain, and let $A, B, C \in R^{n \times n}$ with $ABA = ACA$. If AB and CA are group invertible, we prove that AB is similar to CA . Moreover, we have $(AB)^\#$ is similar to $(CA)^\#$. This generalize the main result of Cao and Li (Group inverses for matrices over a Bézout domain, *Electronic J. Linear Algebra*, **18** (2009), 600–612).

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1. INTRODUCTION

Let R be a associate ring with an identity 1. An element $a \in R$ is Drazin invertible if there exists an element $x \in R$ such that

$$ax = xa, \quad xax = x, \quad a^{k+1}x = a^k$$

for some nonnegative integer k . The preceding x is unique if it exists. The element x is called Drazin inverse of a , and denote x by a^D . The least nonnegative integer k satisfying these equations above is the Drazin index $\text{ind}(a)$ of a . When $\text{ind}(a) = 1$, we say that a is group invertible, that is

$$ax = xa, \quad x = xax, \quad a = axa.$$

Denote x by $a^\#$. We use $R^\#$ to stand for the set of all group invertible elements in R .

Two elements $a, b \in R$ are similar, written $a \sim b$, if there exists an invertible element s such that $a = s^{-1}bs$.

It is important for Similarity to solve linear equations. However, it is difficulty to consider similarity for usual ring. In view of this, we will investigate the similarity over Bézout domain. An integral domain is called a Bézout domain if every its finitely

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generated ideal is principal. Some authors have discussed the similarity of products of matrices on background of generalized inverse. For example, in [1, Theorem 3.6], Cao and Li proved that $(AB)^\#$ and $(BA)^\#$ exist implies that $AB \sim BA$ for any $n \times n$ matrices A and B over a Bézout domain R .

This paper is partially motivated by Flanders' a classic theorem which states that the elementary divisors of AB which do not have zero as a root coincide with those of BA (see [4, Theorem 2]). In [5, Theorem 2], for strongly π -regular ring R , Hartwig proved that R is unit-regular if and only if R is regular and $(ab)^D \sim (ba)^D$ for every $a, b \in R$.

Inspired by the extension of Jacobson's lemma under the condition $ABA = ACA$ [3, 7–9], we shall improve Flanders, Cao and Li's results. In Section 2, we present a new characterization of the similarity. Under the condition $ABA = ACA$, we prove that AB and CA are similar if AB and CA are both group invertible for any $n \times n$ matrices A, B and C over a Bézout domain R . An example is provided to illustrate that our result is a nontrivial generalization of [1, Theorem 3.6]. In Section 3, we turn to the similarity of the group inverses of products of matrices over Bézout domains. It is shown that $(AB)^\#$ and $(CA)^\#$ are similar if AB and CA are both group invertible for any $n \times n$ matrices A, B and C over a Bézout domain R .

Throughout this paper, R is a Bézout domain. $R^{m \times n}$ denotes the set of all $m \times n$ matrices over R . Let $A \in R^{m \times n}$, then $R_r(A)$ is the space spanned by the columns of A :

$$R_r(A) = \{Ax | x \in R^{n \times 1}\} \subseteq R^{m \times 1}.$$

Likewise, $R_l(A)$ is the space spanned by the rows of A :

$$R_l(A) = \{yA | y \in R^{1 \times m}\} \subseteq R^{1 \times n}.$$

The column rank (respectively, row rank) of A is defined as the dimension of $R_r(A)$ (respectively, $R_l(A)$). The inner rank of A is determined as the least r such that $A = BC$, where $B \in R^{m \times r}$, $C \in R^{r \times n}$. Over a Bézout domain, it is proved that the column rank, row rank and inner rank of any matrix A coincide with each other (see [6, Proposition 2.3.4] and [2, Page 245]), the common number is called the rank of A , written $\text{rank}(A)$. Define the rank of a zero matrix is 0. The rank of $n \times n$ non-singular matrix over R is n . An $n \times n$ matrix A over a Bézout domain R is invertible in case $AB = I_n = BA$ for an $n \times n$ matrix B . Obviously, B is unique if it exists, denote A^{-1} .

2. THE SIMILARITY OF PRODUCT OF MATRICES

The purpose of this section is concerned with the similarity of product of matrices over Bézout domains. We now derive

Theorem 1. *Let R be a Bézout domain, and let $A, B, C \in R^{n \times n}$ with $ABA = ACA$. If AB and CA are group invertible, then AB is similar to CA .*

Proof. For $A, B, C \in R^{n \times n}$, by [1, Lemma 3.5], there exist invertible matrices P and Q such that

$$A = P \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} Q \text{ and } B = Q^{-1} \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 0 & 0 \\ C_1 & C_2 & C_3 \\ 0 & 0 & 0 \end{bmatrix} P^{-1},$$

also there exists an invertible matrix Q' together with the invertible matrix P above such that

$$A = P \begin{bmatrix} A'_1 & 0 \\ 0 & 0 \end{bmatrix} Q' \text{ and } C = Q'^{-1} \begin{bmatrix} B'_1 & B'_2 & B'_3 \\ 0 & 0 & 0 \\ C'_1 & C'_2 & C'_3 \\ 0 & 0 & 0 \end{bmatrix} P^{-1},$$

where $A_1, A'_1 \in R^{r \times r}$, $Q = \text{diag}(Q_1, Q_2)N$, $Q' = \text{diag}(Q'_1, Q'_2)N$, $A_1 Q_1 = \Delta = A'_1 Q'_1 \in R^{r \times r}$, $\text{rank}(\Delta) = \text{rank}(A_1) = \text{rank}(A'_1) = \text{rank}(A) = r$, Q_1, Q_2, Q'_1, Q'_2 and N are invertible, $B_1, B'_1 \in R^{s \times s}$, $C_1, C'_1 \in R^{t \times s}$. Then

$$AB = P \begin{bmatrix} A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} & A_1 \begin{bmatrix} B_3 \\ 0 \end{bmatrix} \end{bmatrix} P^{-1},$$

$$CA = Q'^{-1} \begin{bmatrix} \begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 & 0 \\ \begin{bmatrix} C'_1 & C'_2 \\ 0 & 0 \end{bmatrix} A'_1 & 0 \end{bmatrix} Q'.$$

By [1, Corollary 2.4 and 2.5], we get

$$(AB)^{\#} \text{ exists } \iff \left(A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \right)^{\#} \text{ exists, and}$$

$$R_r \left(A_1 \begin{bmatrix} B_3 \\ 0 \end{bmatrix} \right) \subseteq R_r \left(A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \right).$$

$$(CA)^{\#} \text{ exists } \iff \left(\begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 \right)^{\#} \text{ exists, and}$$

$$R_l \left(\begin{bmatrix} C'_1 & C'_2 \\ 0 & 0 \end{bmatrix} A'_1 \right) \subseteq R_l \left(\begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 \right).$$

Therefore, there exist E, F such that

$$A_1 \begin{bmatrix} B_3 \\ 0 \end{bmatrix} = A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} E, \quad \begin{bmatrix} C'_1 & C'_2 \\ 0 & 0 \end{bmatrix} A'_1 = F \begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1.$$

Then

$$AB = P \begin{bmatrix} I & -E \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & E \\ 0 & I \end{bmatrix} P^{-1},$$

$$CA = Q'^{-1} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} Q'.$$

Let

$$\begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} = R \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} S, \quad \begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} = R' \begin{bmatrix} D' & 0 \\ 0 & 0 \end{bmatrix} S'$$

for some invertible matrices R, S, R', S' , where $D \in R^{r_1 \times r_1}$, $D' \in R^{s_1 \times s_1}$, $\text{rank}(D) = r_1$, $\text{rank}(D') = s_1$. We obtain

$$A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} = A_1 R \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} S, \quad \begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 = R' \begin{bmatrix} D' & 0 \\ 0 & 0 \end{bmatrix} S' A'_1.$$

Let $SA_1R = C = \begin{bmatrix} C_4 & C_5 \\ C_6 & C_7 \end{bmatrix}$, $S'A'_1R' = C' = \begin{bmatrix} C'_4 & C'_5 \\ C'_6 & C'_7 \end{bmatrix}$. Then $\text{rank}(C) = \text{rank}(C') = r$, and

$$A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} = S^{-1} \begin{bmatrix} C_4D & 0 \\ C_6D & 0 \end{bmatrix} S,$$

$$\begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 = R' \begin{bmatrix} D'C'_4 & D'C'_5 \\ 0 & 0 \end{bmatrix} R'^{-1}.$$

Since $\left(A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}\right)^\#$ exists, $\begin{bmatrix} C_4D & 0 \\ C_6D & 0 \end{bmatrix}$ is group invertible. Furthermore, $(C_4D)^\#$ exists and $R_l(C_6D) \subset R_l(C_4D)$, then $C_6D = GC_4D$ for some matrix G . Hence

$$\begin{bmatrix} C_4D & 0 \\ C_6D & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ G & I \end{bmatrix} \begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix},$$

$$\text{rank}(C_4) = \text{rank}(C_4D) = r_1.$$

Therefore,

$$A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix}.$$

Analogously, $D'C'_4$ is group invertible, and

$$\begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 \sim \begin{bmatrix} D'C'_4 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\text{rank}(C'_4) = \text{rank}(D'C'_4) = s_1.$$

By $(C_4D)^\#$ exists and $\text{rank}(C_4) = \text{rank}(D) = r_1$, there exists X such that $C_4DXC_4D = C_4D$, hence $DXC_4D = D$ and $C_4DXC_4 = C_4$. Since $\text{rank}(C_4) = \text{rank}(D) = r_1$, we get

$DXC_4 = I = XC_4D$ and $C_4DX = I = DXC_4$, which means C_4 and D are invertible. Thus C_4D is invertible. Likewise, C'_4 , D' and $D'C'_4$ are also invertible.

On the other hand, we have

$$\begin{aligned}
ABA &= P \begin{bmatrix} I & -E \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & E \\ 0 & I \end{bmatrix} P^{-1} P \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} Q \\
&= P \begin{bmatrix} A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} N \\
&= P \begin{bmatrix} S^{-1} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix} \begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} SA_1Q_1 & 0 \\ 0 & 0 \end{bmatrix} N, \\
ACA &= P \begin{bmatrix} A'_1 & 0 \\ 0 & 0 \end{bmatrix} Q'Q'^{-1} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} Q' \\
&= P \begin{bmatrix} A'_1 \begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q'_1 & 0 \\ 0 & Q'_2 \end{bmatrix} N \\
&= P \begin{bmatrix} A'_1R' \begin{bmatrix} D' & 0 \\ 0 & 0 \end{bmatrix} S'A'_1Q'_1 & 0 \\ 0 & 0 \end{bmatrix} N \\
&= P \begin{bmatrix} S'^{-1} \begin{bmatrix} C'_4D' & 0 \\ C'_6D' & 0 \end{bmatrix} S'A'_1Q'_1 & 0 \\ 0 & 0 \end{bmatrix} N \\
&= P \begin{bmatrix} S'^{-1} \begin{bmatrix} I & 0 \\ C'_6C'^{-1}_4 & I \end{bmatrix} \begin{bmatrix} C'_4D' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -C'_6C'^{-1}_4 & I \end{bmatrix} S'A'_1Q'_1 & 0 \\ 0 & 0 \end{bmatrix} N.
\end{aligned}$$

In view of $ABA = ACA$ and the invertibility of P and N , we get

$$\begin{aligned}
&S^{-1} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix} \begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} S \\
&= S'^{-1} \begin{bmatrix} I & 0 \\ C'_6C'^{-1}_4 & I \end{bmatrix} \begin{bmatrix} C'_4D' & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -C'_6C'^{-1}_4 & I \end{bmatrix} S',
\end{aligned}$$

it follows that

$$\begin{bmatrix} C_4D & 0 \\ 0 & 0 \end{bmatrix} Y = Y \begin{bmatrix} C'_4D' & 0 \\ 0 & 0 \end{bmatrix},$$

in which $Y = \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} SS'^{-1} \begin{bmatrix} I & 0 \\ C'_6 C'_4{}^{-1} & I \end{bmatrix}$. Obviously, Y is invertible. Partition $Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$ conformally with $\begin{bmatrix} C_4 D & 0 \\ 0 & 0 \end{bmatrix}$. We have the three identities:

$$C_4 D Y_{11} = Y_{11} C'_4 D', \quad C_4 D Y_{12} = 0, \quad Y_{21} C'_4 D' = 0.$$

Combining the last two identities with the invertibility of $C_4 D$ and $C'_4 D'$ ensure that $Y_{12} = 0$ and $Y_{21} = 0$. Then $Y = \text{diag}(Y_{11}, Y_{22})$ implies that Y_{11} is invertible. So $C_4 D = Y_{11} C'_4 D' Y_{11}^{-1}$, which reveals that

$$C_4 D \sim C'_4 D' = C'_4 D' C'_4 C'_4{}^{-1}.$$

Therefore,

$$C_4 D \sim D' C'_4.$$

That is,

$$\begin{bmatrix} C_4 D & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} D' C'_4 & 0 \\ 0 & 0 \end{bmatrix}.$$

We have the conclusion $AB \sim CA$. $\square$

Corollary 1. Let $A, B, C \in R^{n \times n}$ with $ABA = ACA$. If $R_r(A) = R_r(ABA)$, then $AB \sim CA$.

Proof. By $R_r(A) = R_r(ABA)$, $(AB)^\#$ exists. $R_r(A) = R_r(ACA)$ implies $(CA)^\#$ exists. Then $AB \sim CA$. $\square$

Corollary 2. Let $A, B, C \in R^{n \times n}$ with $ABA = ACA$. If $R_r(A) = R_r(AB)$, $R_r(B) = R_r(BA)$, then $AB \sim CA$.

Proof. Since $R_r(A) = R_r(AB) = AR_r(B) = AR_r(BA) = R_r(ABA)$, by Corollary 1, $AB \sim CA$. $\square$

Theorem 2. Let $A, B, C \in R^{n \times n}$ with $ABA = ACA$. If $R_r(AB) = R_r(ABA)$, $R_r(CA) = R_r(CAB)$, then $AB \sim CA$.

Proof. By the column space of a matrix, we have the formulas $R_r(AB) = R_r(ABA) = R_r(ACA) = A \cdot R_r(CA) = A \cdot R_r(CAB) = R_r(ACAB) = R_r(ABAB)$ and $R_r(CA) = R_r(CAB) = C \cdot R_r(AB) = C \cdot R_r(ABAB) = C \cdot R_r(ACAB) = R_r(CACAB) \subseteq R_r(CACA)$. Obviously, $R_r(CA) \supseteq R_r(CACA)$. by [1, Theorem 2.1], AB and CA are group invertible. Then the conclusion follows from Theorem 1. $\square$

As an immediate consequence, we have

Corollary 3. Let $A, B, C \in R^{n \times n}$ with $ABA = ACA$. If $R_r(A) = R_r(AC) = R_r(ABA)$, then $AC \sim BA$.

Proof. Since $R_r(A) = R_r(AB) = AR_r(B) = AR_r(BA) = R_r(ABA)$, by Corollary 1, $AB \sim CA$. $\square$

Example 1. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{2 \times 2}$, then $ABA = ACA$. Since AB and CA are idempotent, $(AB)^\# = AB$ and $(CA)^\# = CA$. By virtue of Theorem 1, we have $AB \sim CA$. In fact, $AB = P(CA)P^{-1}$, where $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

3. EXTENSIONS

In this section, we turn to the similarity of group inverses. We have

Theorem 3. *Let R be a Bézout domain, and let $A, B, C \in R^{n \times n}$ with $ABA = ACA$. If AB and CA are group invertible, then $(AB)^\#$ is similar to $(CA)^\#$.*

Proof. As in the proof of Theorem 1,

$$A_1 \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} = S^{-1} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix} \begin{bmatrix} C_4 D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -G & I \end{bmatrix} S.$$

Similarly, there exists G' such that

$$\begin{bmatrix} B'_1 & B'_2 \\ 0 & 0 \end{bmatrix} A'_1 = R' \begin{bmatrix} I & -G' \\ 0 & I \end{bmatrix} \begin{bmatrix} D' C'_4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & G' \\ 0 & I \end{bmatrix} R'^{-1}.$$

So

$$AB = H \begin{bmatrix} C_4 D & 0 \\ 0 & 0 \end{bmatrix} H^{-1} \text{ and } CA = K \begin{bmatrix} D' C'_4 & 0 \\ 0 & 0 \end{bmatrix} K^{-1},$$

where

$$H = P \begin{bmatrix} I & -E \\ 0 & I \end{bmatrix} \begin{bmatrix} S^{-1} \begin{bmatrix} I & 0 \\ G & I \end{bmatrix} \\ I \end{bmatrix}$$

and

$$K = Q'^{-1} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \begin{bmatrix} R' \begin{bmatrix} I & -G' \\ 0 & I \end{bmatrix} \\ I \end{bmatrix}.$$

It is easy to check that AB and CA are group invertible, and

$$(AB)^\# = H \begin{bmatrix} D^{-1} C_4^{-1} & 0 \\ 0 & 0 \end{bmatrix} H^{-1},$$

$$(CA)^\# = K \begin{bmatrix} C_4'^{-1} D'^{-1} & 0 \\ 0 & 0 \end{bmatrix} K^{-1}.$$

In view of the similarity of $\begin{bmatrix} C_4 D & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} D' C'_4 & 0 \\ 0 & 0 \end{bmatrix}$ which has been proved in Theorem 1, we derive that $(AB)^\#$ and $(CA)^\#$ are similar. $\square$

Corollary 4. *Let $A, B, C \in \mathbb{C}^{n \times n}$ with $ABA = ACA$. If $(AC)^\#$ and $(BA)^\#$ exist, then $(AC)^\#$ is similar to $(BA)^\#$.*

Proof. This is obvious by Theorem 3. $\square$

Corollary 5. *Let R be a Bézout domain, and let $A, B, C \in R^{n \times n}$ with $ABA = ACA$. If AB and CA are group invertible, then $(AB)(AB)^\#$ is similar to $(CA)(CA)^\#$.*

Proof. From the proof of Theorem 3,

$$\begin{aligned} AB(AB)^\# &= H \begin{bmatrix} C_4 D & 0 \end{bmatrix} H^{-1} H \begin{bmatrix} (C_4 D)^{-1} & 0 \end{bmatrix} H^{-1} \\ &= H \begin{bmatrix} I & 0 \end{bmatrix} H^{-1}, \\ CA(CA)^\# &= K \begin{bmatrix} D' C'_4 & 0 \end{bmatrix} K^{-1} K \begin{bmatrix} (D' C'_4)^{-1} & 0 \end{bmatrix} K^{-1} \\ &= K \begin{bmatrix} I & 0 \end{bmatrix} K^{-1}. \end{aligned}$$

This implies that $(AB)(AB)^\#$ and $(CA)(CA)^\#$ are similar. $\square$

Cline proved that ba is Drazin invertible if ab has Drazin inverse. In this case, $(ba)^D = b[(ab)^D]^2 a$. We now derive.

Theorem 4. *Let R be a Bézout domain, and let $A, B, C \in R^{n \times n}$ with $ABA = ACA$. If AB has Drazin inverse, then there exists $k \in \mathbb{N}$ such that $(AB)^s$ is similar to $(CA)^s$ for any $s \geq k$.*

Proof. Suppose that AB has Drazin inverse with $\text{ind}(AB) = k$. By [9, Theorem 2.7], CA is Drazin invertible with $\text{ind}(CA) \leq k + 1$, and $(CA)^D = C[(AB)^D]^2 A$.

Set $s > k$. Since $AB(AB)^D$ is idempotent, we have

$$\begin{aligned} (AB)^s [(AB)^D]^s &= AB(AB)^D = [(AB)^D]^s (AB)^s, \\ ([(AB)^D]^s)^2 (AB)^s &= [(AB)^D]^{s+1} AB = [(AB)^D]^s, \\ [(AB)^s]^2 [(AB)^D]^s &= (AB)^{s+1} (AB)^D = (AB)^s. \end{aligned}$$

Accordingly, $(AB)^s$ is group invertible and $[(AB)^s]^\# = [(AB)^D]^s$. Similarly, $(CA)^s$ is group invertible and $[(CA)^s]^\# = [(CA)^D]^s$.

Let $B' = B(AB)^{s-1}$ and $C' = (CA)^{s-1} C$. Then $AB'A = AC'A$, where AB' and $C'A$ are group invertible. Therefore, the result of theorem follows by Theorem 1. $\square$

Corollary 6. *Let $A, B \in M_n(\mathbb{C})$. Then there exists $k \in \mathbb{N}$ such that $(AB)^s$ is similar to $(BA)^s$ for any $s \geq k$.*

Proof. This is obvious by choosing $B = C$ in Theorem 4. $\square$

Corollary 7. *Let R be a Bézout domain, and let $A, B, C \in R^{n \times n}$ with $ABA = ACA$. If AB and CA are group invertible, then $(AB)^2(AB)^D$ is similar to $(CA)^2(CA)^D$.*

Proof. Since AB and CA are group invertible, $(AB)^D = (AB)^\#$, $(CA)^D = (CA)^\#$. By the proof of Corollary 5,

$$(AB)^2(AB)^D = H \begin{bmatrix} C_4 D & \\ & 0 \end{bmatrix} H^{-1},$$

$$(CA)^2(CA)^D = K \begin{bmatrix} D' C'_4 & \\ & 0 \end{bmatrix} K^{-1}.$$

According to the proof of Theorem 1, $\begin{bmatrix} C_4 D & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} D' C'_4 & 0 \\ 0 & 0 \end{bmatrix}$ are similar. Then the similarity of $(AB)^2(AB)^D$ and $(CA)^2(CA)^D$ follows. $\square$

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NEWTON-TYPE INEQUALITIES ASSOCIATED WITH CONVEX FUNCTIONS VIA QUANTUM CALCULUS

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Abstract. In this paper, we firstly establish an identity by using the notions of quantum derivatives and integrals. Using this quantum identity, quantum Newton-type inequalities associated with convex functions are proved. We also show that the newly established inequalities can be recaptured into some existing inequalities by taking $q \rightarrow 1^-$. Finally, we give mathematical examples of convex functions to verify the newly established inequalities.

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1. INTRODUCTION

A function $f: [a, b] \rightarrow \mathbb{R}$ is convex if it satisfies an inequality:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

where $x, y \in [a, b]$ and $t \in [0, 1]$.

The most famous inequalities related to the integral inequalities for convex functions are Simpson- and Newton-type inequalities. Simpson's rules, famous techniques for numerical integration and approximations of definite integrals, were discovered by Thomas Simpson (1710-1761). These techniques are also known as Kepler's rule because Johannes Kepler used a similar estimation about 100 years ago. Simpson's rule consists of three-point Newton-Cotes quadrature rule, so estimations based on three steps quadratic kernel are sometimes called Newton-type inequalities.

(1) Simpson's quadrature formula (Simpson's 1/3 rule) is as follows:

$$\int_a^b f(x)dx \approx \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right],$$

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see [6] for more details.

- (2) Newton-Cotes quadrature formula or Simpson's second formula (Simpson's 3/8 rule) is as follows:

$$\int_a^b f(x)dx \approx \frac{b-a}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right],$$

see [15] for more details.

The estimations of Simpson- and Newton-type inequalities are as follows:

Theorem 1 ([6]). Suppose that $f: [a, b] \rightarrow \mathbb{R}$ is a four times continuously differentiable function on (a, b) and $\|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$, then

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^4.$$

Theorem 2 ([15]). Suppose that $f: [a, b] \rightarrow \mathbb{R}$ is a four times continuously differentiable function on (a, b) and $\|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$, then

$$\left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{6480} \|f^{(4)}\|_\infty (b-a)^4.$$

Currently, many researchers have focused on the Newton-type inequalities, see [4, 9–12, 16–18] and the references cited therein. Particularly, some researchers have studied on the Newton-type inequalities by using quantum calculus.

Quantum calculus, also known as q -calculus, gains q -analogues of mathematical objects which can be recaptured by letting $q \rightarrow 1^-$. The q -calculus has wide applications in various fields of physics and mathematics such as relativity theory, mechanics, quantum theory, orthogonal polynomials, number theory, and hypergeometric functions [8, 14]. In the beginning study of the q -calculus, the concept was revealed by renowned mathematician Euler (1707-1783), who introduced the q -parameter in Newton's infinite series. In 1910, Jackson [13] studied the concept of Euler to define q -integral and q -derivative of continuous functions over the interval $(0, \infty)$, also known as calculus without limits. In 1966, Al-Salam [1] studied the concepts of q -fractional integral inequalities and q -Riemann-Liouville fractional integral inequalities. In particular, in 2013, Tariboon and Ntouyas [20] presented the q -integral and the q -derivative of continuous functions over finite intervals. Some new results of q -calculus in Newton-type inequalities can be found in [2, 3, 5, 7, 19, 21, 22] and the references cited therein.

Inspired by the ongoing studies, we propose to prove new versions of quantum Newton-type inequalities associated with convex functions. We also prove that the newly established inequalities are the generalization of the existing Newton-type inequalities.

2. PRELIMINARIES

The definitions and fundamental concepts of q -calculus are presented in this section. Throughout this paper, let q be a constant with $0 < q < 1$ and $[a, b] \subseteq \mathbb{R}$ be an interval with $a < b$. The q -number of n is given by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbb{N}.$$

Definition 1 ([20]). For a continuous function $f: [a, b] \rightarrow \mathbb{R}$, the q -derivative on $[a, b]$ is defined as:

$${}_a D_q f(x) = \begin{cases} \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, & \text{if } x \neq a; \\ \lim_{x \rightarrow a} {}_a D_q f(x), & \text{if } x = a. \end{cases} \quad (2.1)$$

The function f is called a q_a -differentiable function if ${}_a D_q f(x)$ exists.

In Definition 1, if $a = 0$, then (2.1) is recaptured as follows:

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)(x)},$$

which is the q -Jackson derivative, see [13] for more details.

Definition 2 ([20]). For a continuous function $f: [a, b] \rightarrow \mathbb{R}$, the q_a -integral on $[a, b]$ is defined as:

$$\int_a^x f(t) {}_a d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \quad (2.2)$$

for $x \in [a, b]$. The function f is called a q_a -integrable function if $\int_a^x f(t) {}_a d_q t$ for all $x \in [a, b]$ exists.

In Definition 2, if $a = 0$, then (2.2) is recaptured as follows:

$$\int_0^x f(t) d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x), \quad (2.3)$$

which is the q -Jackson integral, see [13] for more details. Moreover, Jackson [13] gave the q -Jackson integral on the interval $[a, b]$ as follows:

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Lemma 1 ([19]). For continuous functions $f, g \rightarrow \mathbb{R}$, the following expression holds:

$$\begin{aligned} \int_0^c g(t) {}_a D_q f(tb + (1-t)a) d_q t &= \left. \frac{g(t)f(tb + (1-t)a)}{b-a} \right|_0^c \\ &\quad - \frac{1}{b-a} \int_0^c D_q g(t) f(qtb + (1-qt)a) d_q t. \end{aligned} \quad (2.4)$$

Lemma 2 ([21]). *The following expression holds:*

$$\int_a^b (x-a)^\alpha {}_a d_q x = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_q},$$

where $\alpha \in \mathbb{R} - \{-1\}$.

3. MAIN RESULTS

In this section, we will derive Newton-type inequalities for convex functions by using the q -derivative and q -integral. The following lemma is required to obtain the main results.

Lemma 3. Suppose that $f: [a, b] \rightarrow \mathbb{R}$ is a q_a -differentiable function on (a, b) such that ${}_a D_q f$ is continuous and integrable on $[a, b]$. Then, we have the following identity:

$$\begin{aligned} & \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \\ & - \frac{1}{(b-a)} \left[\int_a^{\frac{2a+b}{3}} f(x) {}_a d_q x + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x) {}_{\frac{2a+b}{3}} d_q x + \int_{\frac{a+2b}{3}}^b f(x) {}_{\frac{a+2b}{3}} d_q x \right] \\ & = \frac{b-a}{9} \left[\int_0^1 \left(qt - \frac{3}{8} \right) {}_a D_q f \left(t \frac{2a+b}{3} + (1-t)a \right) d_q t \right. \\ & \quad + \int_0^1 \left(qt - \frac{1}{2} \right) {}_a D_q f \left(t \frac{a+2b}{3} + (1-t)\frac{2a+b}{3} \right) d_q t \\ & \quad \left. + \int_0^1 \left(qt - \frac{5}{8} \right) {}_a D_q f \left(tb + (1-t)\frac{a+2b}{3} \right) d_q t \right]. \end{aligned} \quad (3.1)$$

Proof. Let

$$\begin{aligned} & \frac{b-a}{9} \left[\int_0^1 \left(qt - \frac{3}{8} \right) {}_a D_q f \left(t \frac{2a+b}{3} + (1-t)a \right) d_q t \right. \\ & \quad + \int_0^1 \left(qt - \frac{1}{2} \right) {}_a D_q f \left(t \frac{a+2b}{3} + (1-t)\frac{2a+b}{3} \right) d_q t \\ & \quad \left. + \int_0^1 \left(qt - \frac{5}{8} \right) {}_a D_q f \left(tb + (1-t)\frac{a+2b}{3} \right) d_q t \right] \\ & = \frac{(b-a)}{9} [I_1 + I_2 + I_3]. \end{aligned} \quad (3.2)$$

Using Lemma 1, we have

$$\int_0^1 \left(qt - \frac{3}{8} \right) {}_a D_q f \left(t \frac{2a+b}{3} + (1-t)a \right) d_q t$$

$$\begin{aligned}
 &= \frac{3}{b-a} \left(qt - \frac{3}{8} \right) f \left(t \frac{2a+b}{3} + (1-t)a \right) \Big|_0^1 \\
 &\quad - \frac{3}{b-a} \int_0^1 q f \left(qt \frac{2a+b}{3} + (1-qt)a \right) d_q t \\
 &= \frac{3}{b-a} \left(q - \frac{3}{8} \right) f \left(\frac{2a+b}{3} \right) + \frac{9f(a)}{8(b-a)} \\
 &\quad - \frac{3}{b-a} (1-q) \sum_{n=0}^{\infty} q^{n+1} f \left(q^{n+1} \frac{2a+b}{3} + (1-q^{n+1})a \right) \\
 &= \frac{3}{b-a} \left(q - \frac{3}{8} \right) f \left(\frac{2a+b}{3} \right) + \frac{9f(a)}{8(b-a)} \\
 &\quad - \frac{3}{b-a} (1-q) \sum_{n=1}^{\infty} q^n f \left(q^n \frac{2a+b}{3} + (1-q^n)a \right) \\
 &= \frac{3}{b-a} \left(q - \frac{3}{8} \right) f \left(\frac{2a+b}{3} \right) + \frac{9f(a)}{8(b-a)} \\
 &\quad - \frac{3}{b-a} (1-q) \left[\sum_{n=0}^{\infty} q^n f \left(q^n \frac{2a+b}{3} + (1-q^n)a \right) - f \left(\frac{2a+b}{3} \right) \right] \\
 &= \frac{9f(a)}{8(b-a)} + \frac{15}{8(b-a)} f \left(\frac{2a+b}{3} \right) - \frac{9}{(b-a)^2} \int_a^{\frac{2a+b}{3}} f(x) {}_a d_q x. \quad (3.3)
 \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
 &\int_0^1 \left(qt - \frac{1}{2} \right) {}_a D_q f \left(t \frac{a+2b}{3} + (1-t) \frac{2a+b}{3} \right) d_q t \\
 &= \frac{3}{2(b-a)} f \left(\frac{2a+b}{3} \right) + \frac{3}{2(b-a)} f \left(\frac{a+2b}{3} \right) - \frac{9}{(b-a)^2} \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x) {}_{\frac{2a+b}{3}} d_q x, \quad (3.4)
 \end{aligned}$$

and

$$\begin{aligned}
 &\int_0^1 \left(qt - \frac{5}{8} \right) {}_a D_q f \left(t b + (1-t) \frac{a+2b}{3} \right) d_q t \\
 &= \frac{15}{8(b-a)} f \left(\frac{a+2b}{3} \right) + \frac{9}{8(b-a)} f(b) - \frac{9}{(b-a)^2} \int_{\frac{a+2b}{3}}^b f(x) {}_{\frac{a+2b}{3}} d_q x. \quad (3.5)
 \end{aligned}$$

Substituting the inequalities (3.3) - (3.5) in inequality (3.2), we get the required inequality (3.1). Hence, the proof is accomplished. $\square$

Theorem 3. Under the conditions of Lemma 3, if $|{}_aD_qf|$ is a convex function on $[a, b]$, then we have the following inequality:

$$\begin{aligned} & \left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right. \\ & \quad \left. - \frac{1}{(b-a)} \left[\int_a^{\frac{2a+b}{3}} f(x) {}_a d_q x + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x) \frac{2a+b}{3} d_q x + \int_{\frac{a+2b}{3}}^b f(x) \frac{a+2b}{3} d_q x \right] \right| \\ & \leq \frac{b-a}{9} [(\Lambda_1(q) + \Lambda_3(q) + \Lambda_5(q)) |{}_aD_qf(a)| \\ & \quad + (\Lambda_2(q) + \Lambda_4(q) + \Lambda_6(q)) |{}_aD_qf(b)|], \end{aligned} \quad (3.6)$$

where $\Lambda_i(q), i = 1, 2, \dots, 6$ are defined by

$$\begin{aligned} \Lambda_1(q) &= \begin{cases} \frac{6-q-q^2-15q^3}{24[2]_q[3]_q}, & 0 < q < \frac{3}{8}; \\ \frac{480q^3+248q^2+248q-3}{768[2]_q[3]_q}, & \frac{3}{8} \leq q < 1, \end{cases} \\ \Lambda_2(q) &= \begin{cases} \frac{3-5q-5q^2}{24[2]_q[3]_q}, & 0 < q < \frac{3}{8}; \\ \frac{160q^2+160q-69}{768[2]_q[3]_q}, & \frac{3}{8} \leq q < 1, \end{cases} \\ \Lambda_3(q) &= \begin{cases} \frac{1+q+q^2-2q^3}{6[2]_q[3]_q}, & 0 < q < \frac{1}{2}; \\ \frac{4q^3+2q^2+2q+1}{12[2]_q[3]_q}, & \frac{1}{2} \leq q < 1, \end{cases} \\ \Lambda_4(q) &= \begin{cases} \frac{2-q-q^2-q^3}{6[2]_q[3]_q}, & 0 < q < \frac{1}{2}; \\ \frac{2q^3+4q^2+4q-1}{12[2]_q[3]_q}, & \frac{1}{2} \leq q < 1, \end{cases} \\ \Lambda_5(q) &= \begin{cases} \frac{5q+5q^2-3q^3}{24[2]_q[3]_q}, & 0 < q < \frac{5}{8}; \\ \frac{96q^3+40q^2-160q+275}{768[2]_q[3]_q}, & \frac{5}{8} \leq q < 1, \end{cases} \\ \Lambda_6(q) &= \begin{cases} \frac{15+q+q^2-6q^3}{24[2]_q[3]_q}, & 0 < q < \frac{5}{8}; \\ \frac{192q^3+368q^2+368q+45}{768[2]_q[3]_q}, & \frac{5}{8} \leq q < 1. \end{cases} \end{aligned}$$

Proof. By taking modulus in Lemma 3, we have

$$\begin{aligned} & \left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{(b-a)} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{b-a}{9} \left[\int_0^1 \left| qt - \frac{3}{8} \right| \left| {}_aD_qf\left(t\frac{2a+b}{3} + (1-t)a\right) \right| d_q t \right. \\ & \quad \left. + \int_0^1 \left| qt - \frac{1}{2} \right| \left| {}_aD_qf\left(t\frac{a+2b}{3} + (1-t)\frac{2a+b}{3}\right) \right| d_q t \right] \end{aligned}$$

$$\begin{aligned}
 & + \int_0^1 \left| qt - \frac{5}{8} \right| \left| {}_aD_q f \left(tb + (1-t) \frac{a+2b}{3} \right) \right| d_q t \\
 \leq & \frac{b-a}{9} \left[\int_0^1 \left| qt - \frac{3}{8} \right| \left(\frac{3-t}{3} |{}_aD_q f(a)| + \frac{t}{3} |{}_aD_q f(b)| \right) d_q t \right. \\
 & + \int_0^1 \left| qt - \frac{1}{2} \right| \left(\frac{2-t}{3} |{}_aD_q f(a)| + \frac{1+t}{3} |{}_aD_q f(b)| \right) d_q t \\
 & \left. + \int_0^1 \left| qt - \frac{5}{8} \right| \left(\frac{1-t}{3} |{}_aD_q f(a)| + \frac{2+t}{3} |{}_aD_q f(b)| \right) d_q t \right].
 \end{aligned}$$

Using Lemma 2, it can easily compute the integrals as follows:

$$\begin{aligned}
 \Lambda_1(q) &= \int_0^1 \left| qt - \frac{3}{8} \right| \frac{3-t}{3} d_q t = \begin{cases} \frac{6-q-q^2-15q^3}{24[2]_q[3]_q}, & 0 < q < \frac{3}{8}; \\ \frac{480q^3+248q^2+248q-3}{768[2]_q[3]_q}, & \frac{3}{8} \leq q < 1, \end{cases} \\
 \Lambda_2(q) &= \int_0^1 \left| qt - \frac{3}{8} \right| \frac{t}{3} d_q t = \begin{cases} \frac{3-5q-5q^2}{24[2]_q[3]_q}, & 0 < q < \frac{3}{8}; \\ \frac{160q^3+160q-69}{768[2]_q[3]_q}, & \frac{3}{8} \leq q < 1, \end{cases} \\
 \Lambda_3(q) &= \int_0^1 \left| qt - \frac{1}{2} \right| \frac{2-t}{3} d_q t = \begin{cases} \frac{1+q+q^2-2q^3}{6[2]_q[3]_q}, & 0 < q < \frac{1}{2}; \\ \frac{4q^3+2q^2+2q+1}{12[2]_q[3]_q}, & \frac{1}{2} \leq q < 1, \end{cases} \\
 \Lambda_4(q) &= \int_0^1 \left| qt - \frac{1}{2} \right| \frac{1+t}{3} d_q t = \begin{cases} \frac{2-q-q^2-q^3}{6[2]_q[3]_q}, & 0 < q < \frac{1}{2}; \\ \frac{2q^3+4q^2+4q-1}{12[2]_q[3]_q}, & \frac{1}{2} \leq q < 1, \end{cases} \\
 \Lambda_5(q) &= \int_0^1 \left| qt - \frac{5}{8} \right| \frac{1-t}{3} d_q t = \begin{cases} \frac{5q+5q^2-3q^3}{24[2]_q[3]_q}, & 0 < q < \frac{5}{8}; \\ \frac{96q^3+40q^2-160q+275}{768[2]_q[3]_q}, & \frac{5}{8} \leq q < 1, \end{cases} \\
 \Lambda_6(q) &= \int_0^1 \left| qt - \frac{5}{8} \right| \frac{2+t}{3} d_q t = \begin{cases} \frac{15+q+q^2-6q^3}{24[2]_q[3]_q}, & 0 < q < \frac{5}{8}; \\ \frac{192q^3+368q^2+368q+45}{768[2]_q[3]_q}, & \frac{5}{8} \leq q < 1. \end{cases}
 \end{aligned}$$

Hence, the proof is accomplished. $\square$

Remark 1. If we take the limit $q \rightarrow 1^-$ in Theorem 3, then inequality (3.6) becomes

$$\begin{aligned}
 & \left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{(b-a)} \int_a^b f(x) dx \right| \\
 & \leq \frac{25(b-a)}{576} [|f'(a)| + |f'(b)|],
 \end{aligned}$$

which is proven in [12].

Theorem 4. Under the conditions of Lemma 3 and $r > 1$, if $|{}_aD_qf|^r$ is a convex function on $[a, b]$, then we have the following inequality:

$$\begin{aligned} & \left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right. \\ & \quad \left. - \frac{1}{(b-a)} \left[\int_a^{\frac{2a+b}{3}} f(x) {}_a d_q x + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x) {}_{\frac{2a+b}{3}} d_q x + \int_{\frac{a+2b}{3}}^b f(x) {}_{\frac{a+2b}{3}} d_q x \right] \right| \\ & \leq \frac{b-a}{9} \left[(\Lambda_7(q))^{1-\frac{1}{r}} (\Lambda_1(q) |{}_aD_qf(a)|^r + \Lambda_2(q) |{}_aD_qf(b)|^r)^{\frac{1}{r}} \right. \\ & \quad + (\Lambda_8(q))^{1-\frac{1}{r}} (\Lambda_3(q) |{}_aD_qf(a)|^r + \Lambda_4(q) |{}_aD_qf(b)|^r)^{\frac{1}{r}} \\ & \quad \left. + (\Lambda_9(q))^{1-\frac{1}{r}} (\Lambda_5(q) |{}_aD_qf(a)|^r + \Lambda_6(q) |{}_aD_qf(b)|^r)^{\frac{1}{r}} \right], \end{aligned} \quad (3.7)$$

where $\Lambda_i(q), i = 1, 2, \dots, 6$ are given in Theorem 3, and $\Lambda_i(q), i = 7, 8, 9$ are defined by

$$\begin{aligned} \Lambda_7(q) &= \int_0^1 \left| qt - \frac{3}{8} \right| d_q t = \begin{cases} \frac{3-5q}{8[2]_q}, & 0 < q < \frac{3}{8}; \\ \frac{20q-3}{32[2]_q}, & \frac{3}{8} \leq q < 1, \end{cases} \\ \Lambda_8(q) &= \int_0^1 \left| qt - \frac{1}{2} \right| d_q t = \begin{cases} \frac{1-q}{2[2]_q}, & 0 < q < \frac{1}{2}; \\ \frac{q}{2[2]_q}, & \frac{1}{2} \leq q < 1, \end{cases} \\ \Lambda_9(q) &= \int_0^1 \left| qt - \frac{5}{8} \right| d_q t = \begin{cases} \frac{5-3q}{8[2]_q}, & 0 < q < \frac{5}{8}; \\ \frac{12q+5}{32[2]_q}, & \frac{5}{8} \leq q < 1. \end{cases} \end{aligned}$$

Proof. By taking modulus in Lemma 3, applying the power mean inequality, and using the convexity of $|{}_aD_qf|^r$, we have

$$\begin{aligned} & \left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{(b-a)} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{b-a}{9} \left[\int_0^1 \left| qt - \frac{3}{8} \right| \left| {}_aD_qf \left(t \frac{2a+b}{3} + (1-t)a \right) \right| d_q t \right. \\ & \quad + \int_0^1 \left| qt - \frac{1}{2} \right| \left| {}_aD_qf \left(t \frac{a+2b}{3} + (1-t)\frac{2a+b}{3} \right) \right| d_q t \\ & \quad \left. + \int_0^1 \left| qt - \frac{5}{8} \right| \left| {}_aD_qf \left(tb + (1-t)\frac{a+2b}{3} \right) \right| d_q t \right] \\ & \leq \frac{b-a}{9} \left[\left(\int_0^1 \left| qt - \frac{3}{8} \right| d_q t \right)^{1-\frac{1}{r}} \right. \\ & \quad \left. \times \left(\int_0^1 \left| qt - \frac{3}{8} \right| \left| {}_aD_qf \left(t \frac{2a+b}{3} + (1-t)a \right) \right|^r d_q t \right)^{\frac{1}{r}} \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{b-a}{9} \left[\left(\int_0^1 \left| qt - \frac{1}{2} \right| d_q t \right)^{1-\frac{1}{r}} \right. \\
 & \times \left(\int_0^1 \left| qt - \frac{1}{2} \right| \left| {}_a D_q f \left(t \frac{a+2b}{3} + (1-t) \frac{2a+b}{3} \right) \right|^r d_q t \right)^{\frac{1}{r}} \Big] \\
 & + \frac{b-a}{9} \left[\left(\int_0^1 \left| qt - \frac{5}{8} \right| d_q t \right)^{1-\frac{1}{r}} \right. \\
 & \times \left(\int_0^1 \left| qt - \frac{5}{8} \right| \left| {}_a D_q f \left(tb + (1-t) \frac{a+2b}{3} \right) \right|^r d_q t \right)^{\frac{1}{r}} \Big] \\
 & \leq \frac{b-a}{9} \left[\left(\int_0^1 \left| qt - \frac{3}{8} \right| d_q t \right)^{1-\frac{1}{r}} \right. \\
 & \times \left(\int_0^1 \left| qt - \frac{3}{8} \right| \left(\frac{3-t}{3} |{}_a D_q f(a)|^r + \frac{t}{3} |{}_a D_q f(b)|^r \right) d_q t \right)^{\frac{1}{r}} \Big] \\
 & + \frac{b-a}{9} \left[\left(\int_0^1 \left| qt - \frac{1}{2} \right| d_q t \right)^{1-\frac{1}{r}} \right. \\
 & \times \left(\int_0^1 \left| qt - \frac{1}{2} \right| \left(\frac{2-t}{3} |{}_a D_q f(a)|^r + \frac{1+t}{3} |{}_a D_q f(b)|^r \right) d_q t \right)^{\frac{1}{r}} \Big] \\
 & + \frac{b-a}{9} \left[\left(\int_0^1 \left| qt - \frac{5}{8} \right| d_q t \right)^{1-\frac{1}{r}} \right. \\
 & \times \left(\int_0^1 \left| qt - \frac{5}{8} \right| \left(\frac{1-t}{3} |{}_a D_q f(a)|^r + \frac{2+t}{3} |{}_a D_q f(b)|^r \right) d_q t \right)^{\frac{1}{r}} \Big].
 \end{aligned}$$

Using Lemma 2, the integrals can be easily computed as follows:

$$\begin{aligned}
 \Lambda_7(q) &= \int_0^1 \left| qt - \frac{3}{8} \right| d_q t = \begin{cases} \frac{3-5q}{8[2]_q}, & 0 < q < \frac{3}{8}; \\ \frac{20q-3}{32[2]_q}, & \frac{3}{8} \leq q < 1, \end{cases} \\
 \Lambda_8(q) &= \int_0^1 \left| qt - \frac{1}{2} \right| d_q t = \begin{cases} \frac{1-q}{2[2]_q}, & 0 < q < \frac{1}{2}; \\ \frac{q}{2[2]_q}, & \frac{1}{2} \leq q < 1, \end{cases} \\
 \Lambda_9(q) &= \int_0^1 \left| qt - \frac{5}{8} \right| d_q t = \begin{cases} \frac{5-3q}{8[2]_q}, & 0 < q < \frac{5}{8}; \\ \frac{12q+5}{32[2]_q}, & \frac{5}{8} \leq q < 1. \end{cases}
 \end{aligned}$$

Thus, the proof is accomplished. $\square$

Remark 2. If we take the limit $q \rightarrow 1^-$ in Theorem 4, then inequality (3.7) becomes

$$\begin{aligned} & \left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{36} \left\{ \left(\frac{17}{16} \right)^{1-1/r} \left(\frac{251|f'(a)|^r + 937|f'(b)|^r}{1152} \right)^{1/r} + \left(\frac{|f'(a)|^r + |f'(b)|^r}{2} \right)^{1/r} \right. \\ & \quad \left. + \left(\frac{17}{16} \right)^{1-1/r} \left(\frac{937|f'(a)|^r + 251|f'(b)|^r}{1152} \right)^{1/r} \right\}, \end{aligned}$$

which is proven in [18].

Theorem 5. Under the conditions of Lemma 3 and $r > 1$ with $s^{-1} + r^{-1} = 1$, if $|{}_aD_q f|^r$ is a convex function on $[a, b]$, then we have the following inequality:

$$\begin{aligned} & \left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right. \\ & \quad \left. - \frac{1}{(b-a)} \left[\int_a^{\frac{2a+b}{3}} f(x) {}_a d_q x + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x) {}_{\frac{2a+b}{3}} d_q x + \int_{\frac{a+2b}{3}}^b f(x) {}_{\frac{a+2b}{3}} d_q x \right] \right| \\ & \leq \frac{b-a}{9} \left[\frac{5}{8} \left(\frac{(3q+2) |{}_aD_q f(a)|^r + |{}_aD_q f(b)|^r}{3[2]_q} \right)^{\frac{1}{r}} \right. \\ & \quad + \frac{1}{2} \left(\frac{(2q+1) |{}_aD_q f(a)|^r + (q+2) |{}_aD_q f(b)|^r}{3[2]_q} \right)^{\frac{1}{r}} \\ & \quad \left. + \frac{3}{8} \left(\frac{q |{}_aD_q f(a)|^r + (2q+3) |{}_aD_q f(b)|^r}{3[2]_q} \right)^{\frac{1}{r}} \right]. \end{aligned} \quad (3.8)$$

Proof. By taking modulus in Lemma 3, applying the Hölder's inequality, and using the convexity of $|{}_aD_q f|^r$, we have

$$\begin{aligned} & \left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{1}{(b-a)} \int_a^b f(x) {}_a d_q x \right| \\ & \leq \frac{b-a}{9} \left[\int_0^1 \left| qt - \frac{3}{8} \right| \left| {}_aD_q f \left(t \frac{2a+b}{3} + (1-t)a \right) \right| d_q t \right. \\ & \quad + \int_0^1 \left| qt - \frac{1}{2} \right| \left| {}_aD_q f \left(t \frac{a+2b}{3} + (1-t) \frac{2a+b}{3} \right) \right| d_q t \\ & \quad \left. + \int_0^1 \left| qt - \frac{5}{8} \right| \left| {}_aD_q f \left(tb + (1-t) \frac{a+2b}{3} \right) \right| d_q t \right] \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{b-a}{9} \left[\left[\left(\int_0^1 \left| qt - \frac{3}{8} \right|^s d_q t \right)^{\frac{1}{s}} \left(\int_0^1 \left| {}_a D_q f \left(t \frac{2a+b}{3} + (1-t)a \right) \right|^r d_q t \right)^{\frac{1}{r}} \right] \right. \\
 &\quad + \left[\left(\int_0^1 \left| qt - \frac{1}{2} \right|^s d_q t \right)^{\frac{1}{s}} \left(\int_0^1 \left| {}_a D_q f \left(t \frac{a+2b}{3} + (1-t) \frac{2a+b}{3} \right) \right|^r d_q t \right)^{\frac{1}{r}} \right] \\
 &\quad \left. + \left[\left(\int_0^1 \left| qt - \frac{5}{8} \right|^s d_q t \right)^{\frac{1}{s}} \left(\int_0^1 \left| {}_a D_q f \left(tb + (1-t) \frac{a+2b}{3} \right) \right|^r d_q t \right)^{\frac{1}{r}} \right] \right] \\
 &\leq \frac{b-a}{9} \left[\left[\left(\int_0^1 \left| qt - \frac{3}{8} \right|^s d_q t \right)^{\frac{1}{s}} \left(\int_0^1 \left(\frac{3-t}{3} |{}_a D_q f(a)|^r + \frac{t}{3} |{}_a D_q f(b)|^r \right) d_q t \right)^{\frac{1}{r}} \right] \right. \\
 &\quad + \left[\left(\int_0^1 \left| qt - \frac{1}{2} \right|^s d_q t \right)^{\frac{1}{s}} \left(\int_0^1 \left(\frac{2-t}{3} |{}_a D_q f(a)|^r + \frac{1+t}{3} |{}_a D_q f(b)|^r \right) d_q t \right)^{\frac{1}{r}} \right] \\
 &\quad \left. + \left[\left(\int_0^1 \left| qt - \frac{5}{8} \right|^s d_q t \right)^{\frac{1}{s}} \left(\int_0^1 \left(\frac{1-t}{3} |{}_a D_q f(a)|^r + \frac{2+t}{3} |{}_a D_q f(b)|^r \right) d_q t \right)^{\frac{1}{r}} \right] \right].
 \end{aligned}$$

By using inequality (2.3), we have

$$\begin{aligned}
 \int_0^1 \left| qt - \frac{3}{8} \right|^s d_q t &= (1-q) \sum_{n=0}^{\infty} q^n \left| q^{n+1} - \frac{3}{8} \right|^s \leq (1-q) \sum_{n=0}^{\infty} q^n \left| 1 - \frac{3}{8} \right|^s \\
 &= (1-q) \frac{5^s}{8^s (1-q)} = \frac{5^s}{8^s}.
 \end{aligned}$$

So, we find that

$$\begin{aligned}
 &\frac{b-a}{9} \left[\left(\int_0^1 \left| qt - \frac{3}{8} \right|^s d_q t \right)^{\frac{1}{s}} \left(\int_0^1 \left(\frac{3-t}{3} |{}_a D_q f(a)|^r + \frac{t}{3} |{}_a D_q f(b)|^r \right) d_q t \right)^{\frac{1}{r}} \right] \\
 &= \frac{b-a}{9} \left[\left(\frac{5^s}{8^s} \right)^{\frac{1}{s}} \left(\frac{(3q+2) |{}_a D_q f(a)|^r + |{}_a D_q f(b)|^r}{3[2]_q} \right)^{\frac{1}{r}} \right].
 \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
 &\frac{b-a}{9} \left[\left(\int_0^1 \left| qt - \frac{1}{2} \right|^s d_q t \right)^{\frac{1}{s}} \left(\int_0^1 \left(\frac{2-t}{3} |{}_a D_q f(a)|^r + \frac{1+t}{3} |{}_a D_q f(b)|^r \right) d_q t \right)^{\frac{1}{r}} \right] \\
 &= \frac{b-a}{9} \left[\left(\frac{1}{2^s} \right)^{\frac{1}{s}} \left(\frac{(2q+1) |{}_a D_q f(a)|^r + (q+2) |{}_a D_q f(b)|^r}{3[2]_q} \right)^{\frac{1}{r}} \right],
 \end{aligned}$$

and

$$\begin{aligned} & \frac{b-a}{9} \left[\left(\int_0^1 \left| qt - \frac{5}{8} \right|^s d_q t \right)^{\frac{1}{s}} \left(\int_0^1 \left(\frac{1-t}{3} |{}_a D_q f(a)|^r + \frac{2+t}{3} |{}_a D_q f(b)|^r \right) d_q t \right)^{\frac{1}{r}} \right] \\ &= \frac{b-a}{9} \left[\left(\frac{3^s}{8^s} \right)^{\frac{1}{s}} \left(\frac{q |{}_a D_q f(a)|^r + (2q+3) |{}_a D_q f(b)|^r}{3[2]_q} \right)^{\frac{1}{r}} \right]. \end{aligned}$$

Thus, the proof is accomplished. $\square$

4. EXAMPLES

In this section, we give examples to support the main results.

Example 1. Let $f: [1, 5] \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$. From Theorem 3 with $q = \frac{3}{4}$, the left-hand side of inequality (3.6) becomes

$$\begin{aligned} & \left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right. \\ & \quad \left. - \frac{1}{(b-a)} \left[\int_a^{\frac{2a+b}{3}} f(x) {}_a d_{\frac{3}{4}} x + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x) {}_{\frac{2a+b}{3}} d_{\frac{3}{4}} x + \int_{\frac{a+2b}{3}}^b f(x) {}_{\frac{a+2b}{3}} d_{\frac{3}{4}} x \right] \right| \\ &= \left| \frac{1}{8} \left[f(1) + 3f\left(\frac{7}{3}\right) + 3f\left(\frac{11}{3}\right) + f(5) \right] \right. \\ & \quad \left. - \frac{1}{4} \left[\int_1^{\frac{7}{3}} x^2 {}_1 d_{\frac{3}{4}} x + \int_{\frac{7}{3}}^{\frac{11}{3}} x^2 {}_{\frac{7}{3}} d_{\frac{3}{4}} x + \int_{\frac{11}{3}}^5 x^2 {}_{\frac{11}{3}} d_{\frac{3}{4}} x \right] \right| \\ &\approx 0.6206, \end{aligned}$$

and the right-hand side of inequality (3.6) becomes

$$\begin{aligned} & \frac{b-a}{9} [(\Lambda_1(q) + \Lambda_3(q) + \Lambda_5(q)) |{}_a D_q f(a)| + (\Lambda_2(q) + \Lambda_4(q) + \Lambda_6(q)) |{}_a D_q f(b)|] \\ &= \frac{4}{9} \left[\left(\Lambda_1\left(\frac{3}{4}\right) + \Lambda_3\left(\frac{3}{4}\right) + \Lambda_5\left(\frac{3}{4}\right) \right) |{}_1 D_{\frac{3}{4}} f(1)| \right. \\ & \quad \left. + \left(\Lambda_2\left(\frac{3}{4}\right) + \Lambda_4\left(\frac{3}{4}\right) + \Lambda_6\left(\frac{3}{4}\right) \right) |{}_1 D_{\frac{3}{4}} f(5)| \right] \\ &\approx 1.6945. \end{aligned}$$

It is clear that

$$0.6206 \leq 1.6945,$$

which shows that inequality (3.6) is valid.

Example 2. Let $f: [1, 5] \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$. From Theorem 4 with $q = \frac{3}{4}$, the left-hand side of inequality (3.7) becomes

$$\begin{aligned} & \left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right. \\ & \quad \left. - \frac{1}{(b-a)} \left[\int_a^{\frac{2a+b}{3}} f(x) {}_a d_q x + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x) {}_{\frac{2a+b}{3}} d_q x + \int_{\frac{a+2b}{3}}^b f(x) {}_{\frac{a+2b}{3}} d_q x \right] \right| \\ &= \left| \frac{1}{8} \left[f(1) + 3f\left(\frac{7}{3}\right) + 3f\left(\frac{11}{3}\right) + f(5) \right] \right. \\ & \quad \left. - \frac{1}{4} \left[\int_1^{\frac{7}{3}} x^2 {}_1 d_{\frac{3}{4}} x + \int_{\frac{7}{3}}^{\frac{11}{3}} x^2 {}_{\frac{7}{3}} d_{\frac{3}{4}} x + \int_{\frac{11}{3}}^5 x^2 {}_{\frac{11}{3}} d_{\frac{3}{4}} x \right] \right| \\ &\approx 0.6206, \end{aligned}$$

and the right-hand side of inequality (3.7) becomes

$$\begin{aligned} & \frac{b-a}{9} \left[(\Lambda_7(q))^{1-\frac{1}{r}} (\Lambda_1(q) |{}_a D_q f(a)|^r + \Lambda_2(q) |{}_a D_q f(b)|^r)^{\frac{1}{r}} \right. \\ & \quad + (\Lambda_8(q))^{1-\frac{1}{r}} (\Lambda_3(q) |{}_a D_q f(a)|^r + \Lambda_4(q) |{}_a D_q f(b)|^r)^{\frac{1}{r}} \\ & \quad \left. + (\Lambda_9(q))^{1-\frac{1}{r}} (\Lambda_5(q) |{}_a D_q f(a)|^r + \Lambda_6(q) |{}_a D_q f(b)|^r)^{\frac{1}{r}} \right] \\ &= \frac{4}{9} \left[\left(\Lambda_7\left(\frac{3}{4}\right) \right)^{1-\frac{1}{2}} \left(\Lambda_1\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(1)|^2 + \Lambda_2\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(5)|^2 \right)^{\frac{1}{2}} \right. \\ & \quad + \left(\Lambda_8\left(\frac{3}{4}\right) \right)^{1-\frac{1}{2}} \left(\Lambda_3\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(1)|^2 + \Lambda_4\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(5)|^2 \right)^{\frac{1}{2}} \\ & \quad \left. + \left(\Lambda_9\left(\frac{3}{4}\right) \right)^{1-\frac{1}{2}} \left(\Lambda_5\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(1)|^2 + \Lambda_6\left(\frac{3}{4}\right) |{}_1 D_{\frac{3}{4}} f(5)|^2 \right)^{\frac{1}{2}} \right] \\ &\approx 3.3900. \end{aligned}$$

It is clear that

$$0.6206 \leq 3.3900,$$

which shows that inequality (3.7) is valid.

Example 3. Let $f: [1, 5] \rightarrow \mathbb{R}$ be defined by $f(x) = x^2$. From Theorem 5 with $q = \frac{3}{4}$, the left-hand side of inequality (3.8) becomes

$$\left| \frac{1}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] \right|$$

$$\begin{aligned}
& -\frac{1}{(b-a)} \left[\int_a^{\frac{2a+b}{3}} f(x) {}_a d_q x + \int_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} f(x) {}_{\frac{2a+b}{3}} d_q x + \int_{\frac{a+2b}{3}}^b f(x) {}_{\frac{a+2b}{3}} d_q x \right] \Bigg| \\
& = \left| \frac{1}{8} \left[f(1) + 3f\left(\frac{7}{3}\right) + 3f\left(\frac{11}{3}\right) + f(5) \right] \right. \\
& \quad \left. - \frac{1}{4} \left[\int_1^{\frac{7}{3}} x^2 {}_1 d_{\frac{3}{4}} x + \int_{\frac{7}{3}}^{\frac{11}{3}} x^2 {}_{\frac{7}{3}} d_{\frac{3}{4}} x + \int_{\frac{11}{3}}^5 x^2 {}_{\frac{11}{3}} d_{\frac{3}{4}} x \right] \right| \\
& \approx 0.6206,
\end{aligned}$$

and the right-hand side of inequality (3.8) becomes

$$\begin{aligned}
& \frac{b-a}{9} \left[\frac{5}{8} \left(\frac{(3q+2) |{}_a D_q f(a)|^r + |{}_a D_q f(b)|^r}{3[2]_q} \right)^{\frac{1}{r}} \right. \\
& \quad + \frac{1}{2} \left(\frac{(2q+1) |{}_a D_q f(a)|^r + (q+2) |{}_a D_q f(b)|^r}{3[2]_q} \right)^{\frac{1}{r}} \\
& \quad \left. + \frac{3}{8} \left(\frac{q |{}_a D_q f(a)|^r + (2q+3) |{}_a D_q f(b)|^r}{3[2]_q} \right)^{\frac{1}{r}} \right] \\
& = \frac{4}{9} \left[\frac{5}{8} \left(\frac{\left(\frac{9}{4}+2\right) |{}_1 D_{\frac{3}{4}} f(1)|^2 + |{}_1 D_{\frac{3}{4}} f(5)|^2}{3[2]_{\frac{3}{4}}} \right)^{\frac{1}{2}} \right. \\
& \quad + \frac{1}{2} \left(\frac{\left(\frac{3}{2}+2\right) |{}_1 D_{\frac{3}{4}} f(1)|^2 + \left(\frac{3}{4}+2\right) |{}_1 D_{\frac{3}{4}} f(5)|^2}{3[2]_{\frac{3}{4}}} \right)^{\frac{1}{2}} \\
& \quad \left. + \frac{3}{8} \left(\frac{\frac{3}{4} |{}_1 D_{\frac{3}{4}} f(1)|^2 + \left(\frac{3}{2}+3\right) |{}_1 D_{\frac{3}{4}} f(5)|^2}{3[2]_{\frac{3}{4}}} \right)^{\frac{1}{2}} \right] \\
& \approx 4.0741.
\end{aligned}$$

It is clear that

$$0.6206 \leq 4.0741,$$

which shows that inequality (3.8) is valid.

5. CONCLUSIONS

In this work, we proved new versions of quantum Newton-type inequalities associated with convex functions. We also demonstrated that the newly established inequalities can be recaptured into classical Newton-type inequalities by taking the limit $q \rightarrow 1^-$. Mathematical examples were given to verify the newly established inequalities. In future works, researchers can obtain similar inequalities of Newton-type inequalities associated with convex functions by using post quantum calculus.

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TWO-POINT BOUNDARY VALUE PROBLEMS FOR 4TH ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The new optimal efficient sufficient conditions are established for solvability and uniqueness of a solution of the linear and nonlinear fourth order ordinary differential equations

$$u^{(4)}(t) = p(t)u(t) + q(t) \quad \text{for } t \in [a, b],$$

$$u^{(4)}(t) = p(t)u(t) + f(t, u(t)) \quad \text{for } t \in [a, b],$$

under the following two-point boundary conditions

$$u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0 \quad (i = 0, 1),$$

and

$$u^{(i)}(a) = 0 \quad (i = 0, 1, 2), \quad u(b) = 0,$$

where $p \in L([a, b]; \mathbb{R})$ is a nonconstant sign function and $f \in K([a, b] \times \mathbb{R}; \mathbb{R})$.

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1. INTRODUCTION

Consider on the interval $I := [a, b]$ the fourth order ordinary differential equations

$$u^{(4)}(t) = p(t)u(t) + q(t), \tag{1.1}$$

and

$$u^{(4)}(t) = p(t)u(t) + f(t, u(t)), \tag{1.2}$$

under the boundary conditions

$$u^{(j)}(a) = 0, \quad u^{(j)}(b) = 0 \quad (j = 0, 1), \tag{1.3_1}$$

$$u^{(j)}(a) = 0 \quad (j = 0, 1, 2), \quad u(b) = 0, \tag{1.3_2}$$

where $p \in L(I; \mathbb{R})$, $f \in K(I \times \mathbb{R}; \mathbb{R})$.

By a solution of problem (1.2), (1.3₁), (1.3₂) we understand a function $u \in \tilde{C}^3(I; \mathbb{R})$, which satisfies equation (1.2) a. e. on I , and conditions (1.3₁), (1.3₂).

The following notations are used throughout the paper:

- $\mathbb{N}$ is the set of all natural numbers;
- $\mathbb{R} =]-\infty, +\infty[, \mathbb{R}_0^+ = [0, +\infty[, \mathbb{R}^+ =]0, +\infty[, \mathbb{R}_0^- = \mathbb{R} \setminus \mathbb{R}^+;$
- $C(I; \mathbb{R})$ is the Banach space of continuous functions $u : I \rightarrow \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\};$
- $\tilde{C}^{(3)}(I; \mathbb{R})$ is the set of functions $u : I \rightarrow \mathbb{R}$ which are absolutely continuous together with their third derivatives;
- $L(I; \mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : I \rightarrow \mathbb{R}$ with the norm $\|p\|_L = \int_a^b |p(s)| ds;$
- $K(I \times \mathbb{R}; \mathbb{R})$ is the set of functions $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the Carathéodory conditions, i.e., $f(\cdot, x) : I \rightarrow \mathbb{R}$ is a measurable function for all $x \in \mathbb{R}$, $f(t, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function for almost all $t \in I$, and for arbitrary $r > 0$ the inclusion

$$\sup\{|f(t, x)| : |x| \leq r\} \in L(I; \mathbb{R}_+)$$

holds;

- For arbitrary $x, y \in L(I; \mathbb{R})$, the notation

$$x(t) \preceq y(t) \quad (x(t) \succeq y(t)) \quad \text{for } t \in I,$$

means that $x \leq y$ ($x \geq y$) and $x \neq y$;

- We also use the notation $[x]_{\pm} = (|x| \pm x)/2.$

The fourth order equations appear as model equations for a large class of higher order differential equations, arising for example, in hydrodynamics, suspension bridge models, etc. and have become the subject of many fundamental works. These studies particularity focus on the questions of oscillatory properties and Green's function sign for some two-point boundary value problems (see, e.g., [2–5], [11–13]) which give the basis for the study of questions of solvability and unique solvability of the fourth order differential equations under different boundary conditions (see, e.g., [6]–[7], [10], [14], [16], [17]).

The aim of our paper is to study the solvability of the above mentioned problems. We have proved the unimprovable sufficient conditions of the unique solvability for the linear problem, which show that the solvability of problem (1.1), (1.3₁), ((1.1), (1.3₂)) depends only on the nonnegative (nonpositive) part of the coefficient p if this nonnegative (nonpositive) part is small enough. Based on these results, for the nonlinear problems, sufficient conditions of solvability have been proved, which in some sense improves previously known results.

Below we present some definitions and results from the works [1], [15], on which our paper is based.

Definition 1. Equation

$$u^{(4)}(t) = p(t)u(t) \quad \text{for } t \in I \quad (1.3)$$

is said to be disconjugate (non-oscillatory) on I if every nontrivial solution u has less than four zeros on I , the multiple zeros being counted according to their multiplicity.

Definition 2. We will say that $p \in D_+(I)$ if $p \in L(I; \mathbb{R}_0^+)$, and problem (1.3), (1.3₁) has a solution u such that

$$u(t) > 0 \quad \text{for } t \in]a, b[. \quad (1.4)$$

Definition 3. We will say that $p \in D_-(I)$ if $p \in L(I; \mathbb{R}_0^-)$, and problem (1.3), (1.3₂) has a solution u such that inequality (1.4) holds.

Remark 1. Let $p \in L(I; \mathbb{R}_0^+)$ ($p \in L(I; \mathbb{R}_0^-)$), and consider the equation

$$u^{(4)}(t) = \lambda^4 p(t)u(t) \quad \text{for } t \in I. \quad (1.5)$$

In [15] it is shown that the set $D_+(I)$ ($D_-(I)$) can be interpreted as a set of the functions for which $\lambda = 1$ is the first eigenvalue of problem (1.5), (1.3₁) ((1.5), (1.3₂)). Also the fact that $\lambda > 0$ is the first eigenvalue of problem (1.5), (1.3₁) ((1.5), (1.3₂)) is equivalent to the inclusion $\lambda^4 p \in D_+(I)$ ($\lambda^4 p \in D_-(I)$).

The importance of the classes D_+ and D_- for our investigation follows from the following propositions.

Proposition 1. ([15], Theorem 2) Let $p \in L(I; \mathbb{R}_0^+)$. Then equation (1.3) is disconjugate on I if and only if there exists $p^* \in D_+(I)$ such that $p(t) \preceq p^*(t)$ on I .

Proposition 2. ([15], Theorem 4) Let $p \in L(I; \mathbb{R}_0^-)$. Then equation (1.3) is disconjugate on I if and only if there exists $p_* \in D_-(I)$ such that $p(t) \succcurlyeq p_*(t)$ on I .

Also we need the following propositions:

Proposition 3. ([1], Theorem 5) Let $p \in L(I, \mathbb{R})$, and Green's function G of the equation

$$u^{(4)}(t) = [p(t)]_+ u(t), \quad (1.6_1)$$

under boundary conditions (1.3₁) be totally positive kernel. Then problem (1.1), (1.3₁) is uniquely solvable for an arbitrary $q \in L(I, \mathbb{R})$.

Proposition 4. ([1], Theorem 4) Let $p \in L(I, \mathbb{R})$, and Green's function G of the equation

$$u^{(4)}(t) = -[p(t)]_- u(t), \quad (1.6_2)$$

under boundary conditions (1.3₂) be such that $-G$ is totally positive kernel. Then problem (1.1), (1.3₂) is uniquely solvable for an arbitrary $q \in L(I, \mathbb{R})$.

Proposition 5. (Gantmacher-Krein, see [13]) Let $i \in \{1, 2\}$, $p \in L(I, \mathbb{R})$ be such that equation (1.3) is disconjugate and G is Green's function of problem (1.3), (1.3₁), ((1.3), (1.3₂)). Then $(-1)^{i-1}G$ is the totally positive kernel.

2. MAIN RESULTS

2.1. Linear Problem

Theorem 1. Let $i \in \{1, 2\}$ and the function $p_0 \in L(I; \mathbb{R})$ be such that the equation

$$\begin{aligned} u^{(4)}(t) &= [p_0(t)]_+ u(t) & \text{if } i = 1, \\ u^{(4)}(t) &= -[p_0(t)]_- u(t) & \text{if } i = 2, \end{aligned}$$

is diconjugate on I . Then if the inequality

$$(-1)^{i-1} [p(t) - p_0(t)] \leq 0 \quad \text{for } t \in I \quad (2.1_i)$$

holds, problem (1.1), (1.3₁) ((1.3₂)) is uniquely solvable.

From the last theorem with $p_0 = [p]_+$, by Proposition 1 immediately follows the corollary.

Corollary 1. Let there exist $p^* \in D_+(I)$ such that the inequality

$$[p(t)]_+ \preceq p^*(t) \quad \text{for } t \in I \quad (2.2_1)$$

holds. Then problem (1.1), (1.3₁) is uniquely solvable.

Analogously, from the last theorem with $p_0 = -[p]_-$, by Proposition 2 we get

Corollary 2. Let there exist $p_* \in D_-(I)$ such that the inequality

$$-[p(t)]_- \succcurlyeq p_*(t) \quad \text{for } t \in I \quad (2.2_2)$$

holds. Then problem (1.1), (1.3₂) is uniquely solvable.

Remark 2. Condition (2.2₁) ((2.2₂)) in Corollary 1 (2) is optimal in the sense that the inequality $\preceq$ ($\succcurlyeq$) can not be replaced by the inequality $\leq$ ($\geq$).

Corollary 3. Let there exist $M \in \mathbb{R}_0^+$ such that

$$M \frac{b-a}{2} + \int_a^b [[p(s)]_+ - M]_+ ds \leq \frac{192}{(b-a)^3}. \quad (2.3_1)$$

Then problem (1.1), (1.3₁) is uniquely solvable.

Corollary 4. Let $p \in L(I; \mathbb{R})$ and there exist $M \in \mathbb{R}_0^+$ such that

$$M \frac{495}{1024} (b-a) + \int_a^b [[p(s)]_- - M]_+ ds \leq \frac{110}{(b-a)^3}. \quad (2.3_2)$$

Then problem (1.1), (1.3₂) is uniquely solvable.

The validity of Corollary 3 (4) follows from Theorem 1 and the fact that condition (2.3₁) ((2.3₂)) guarantees the diconjugacy of equation (1.6₁) ((1.6₂)) (see [15], Theorems 3, 5).

2.2. Nonlinear Problem

Theorem 2. Let $i \in \{1, 2\}$ and there exist $r \in \mathbb{R}^+$ and $g \in L(I; \mathbb{R}_0^+)$ such that a. e. on I the inequality

$$-g(t)|x| \leq (-1)^{i-1} f(t, x) \operatorname{sgn} x \leq \delta(t, |x|) \quad \text{for} \quad |x| > r \quad (2.4_i)$$

holds, where the function $\delta \in K(I \times \mathbb{R}_0^+; \mathbb{R}_0^+)$ is nondecreasing in the second argument and

$$\lim_{\rho \rightarrow +\infty} \frac{1}{\rho} \int_a^b \delta(s, \rho) ds = 0. \quad (2.5)$$

Then if equation (1.6₁) ((1.6₂)) is disconjugate, problem (1.2), (2.3₁) ((2.3₂)) has at least one solution.

Corollary 5. Let $i \in \{1, 2\}$ and conditions (2.4_i), (2.5) of Theorem 2 hold. Then if condition (2.3₁) ((2.3₂)) is fulfilled, problem (1.2), (2.3₁) ((2.3₂)) has at least one solution.

Theorem 3. Let $p^* \in D_+(I)$ and a. e. on I the inequality

$$[f(t, x_1) - f(t, x_2)] \operatorname{sgn}(x_1 - x_2) < [p^*(t) - p(t)]|x_1 - x_2| \quad (2.6_1)$$

holds for $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$. Then problem (1.2), (1.3₁) has at most one solution.

Theorem 4. Let $p_* \in D_-(I)$ and a. e. on I the inequality

$$[f(t, x_1) - f(t, x_2)] \operatorname{sgn}(x_1 - x_2) > [p_*(t) - p(t)]|x_1 - x_2| \quad (2.6_2)$$

holds for $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$. Then problem (1.2), (1.3₂) has at most one solution.

Example 1. Consider on I the equations

$$u^{(4)}(t) = p(t)u(t) + f_0(t)|u(t)|^\alpha \operatorname{sgn} u(t) + h(t), \quad (2.7)$$

$$u^{(4)}(t) = p(t)u(t) + f_0(t) \frac{u(t)}{(1 + |u(t)|)^\beta} + h(t), \quad (2.8)$$

where $p, f_0 \in L(I; \mathbb{R})$, $\alpha \in [0, 1[$, and $\beta \in]0, 1[$.

Then if

$$\| [p]_+ \|_L \leq \frac{192}{(b-a)^3} \quad \text{for} \quad i = 1, \quad \| [p]_- \|_L \leq \frac{110}{(b-a)^3} \quad \text{for} \quad i = 2,$$

from Corollary 5 it follows that for an arbitrary $h \in L(I; \mathbb{R})$ problem (2.7) (1.3₁) ((1.3₂)) has at least one solution.

Also, if a.e. on I the inequalities

$$f_0(t) < p^*(t) - p(t) \quad \text{for} \quad i = 1, \quad f_0(t) > p^*(t) - p(t) \quad \text{for} \quad i = 2,$$

hold, from Corollary 1 if $i = 1$, and Corollary 2 if $i = 2$, it follows that for an arbitrary $h \in L(I; \mathbb{R})$ problem (2.8) (1.3₁) ((1.3₂)) has at most one solution.

3. AUXILIARY PROPOSITIONS

Lemma 1. Let $i \in \{1, 2\}$, $p \in L(I; \mathbb{R})$, $g \in L(I; \mathbb{R}_0^+)$ be the functions defined in Theorem 2, and let the function $\tilde{p} \in L(I; \mathbb{R})$ admit the inequalities

$$\begin{aligned} p(t) - g(t) &\leq \tilde{p}(t) \leq p(t) \quad \text{for } t \in I, \quad i = 1, \\ p(t) &\leq \tilde{p}(t) \leq p(t) + g(t) \quad \text{for } t \in I, \quad i = 2, \end{aligned} \quad (3.1)$$

Then there exists a number $\rho_0 > 0$ such that an arbitrary solution u of the equation

$$u^{(4)}(t) = \tilde{p}(t)u(t) + q(t), \quad (3.2)$$

under boundary condition (1.3₁) ((1.3₂)) admits the estimate

$$\|u\|_C \leq \rho_0 \|q\|_L. \quad (3.3)$$

To prove this proposition, we need Lemma 1.1 from [9].

Lemma 2. Let $\tilde{p}, p_k \in L(I; \mathbb{R})$, $v_0, v_k \in C(I; \mathbb{R})$ ($k \in \mathbb{N}$),

$$\lim_{k \rightarrow +\infty} \|v_k - v_0\|_C = 0, \quad \limsup_{k \rightarrow +\infty} \|p_k\|_L < +\infty,$$

and

$$\lim_{k \rightarrow +\infty} \int_a^t p_k(s) ds = \int_a^t \tilde{p}(s) ds \quad \text{uniformly on } I.$$

Then

$$\lim_{k \rightarrow +\infty} \int_a^t p_k(s) v_k(s) ds = \int_a^t \tilde{p}(s) v_0(s) ds \quad \text{uniformly on } I. \quad (3.4)$$

Proof. We will prove our lemma only for $i = 1$, for $i = 2$ the proof is similar. Assume that Lemma 1 is not true. Then for all $k \in \mathbb{N}$ there exist functions $p_k, q_k \in L(I; \mathbb{R})$ such that

$$h_0(t) \leq p_k(t) \leq p(t) \quad \text{for } t \in I, \quad (3.5)$$

where $h_0 := p - g$, and the problem

$$u_k^{(4)}(t) = p_k(t)u_k(t) + q_k(t), \quad u_k^{(j)}(a) = 0, \quad u_k^{(j)}(b) = 0 \quad (j = 0, 1),$$

has a solution u_k such that $\|u_k\|_C \geq k\|q_k\|_L$. Then if we suppose that

$$v_k(t) = u_k(t)/\|u_k\|_C, \quad \tilde{q}_k(t) = q_k(t)/\|u_k\|_C,$$

we obtain

$$\|v_k\|_C = 1, \quad \|\tilde{q}_k\|_L \leq \frac{1}{k}, \quad (3.6)$$

and

$$\begin{aligned} v_k^{(4)}(t) &= p_k(t)v_k(t) + \tilde{q}_k(t) \quad \text{for } t \in I, \\ v_k^{(j)}(a) &= 0, \quad v_k^{(j)}(b) = 0 \quad (j = 0, 1), \end{aligned} \quad (3.7)$$

from which due to (3.5) and (3.6) we have

$$\|v_k^{(4)}\|_L \leq \|h\|_L + \frac{b-a}{k} \quad \text{for } t \in I, \quad (3.8)$$

where $h(t) = \max(|h_0(t)|, |p(t)|)$.

In view of the equality $\|v_k\|_C = 1$, (3.8), and the boundary conditions (3.7), it is clear that sequences $\{v_k^{(j)}\}_{k=1}^{+\infty}$ ($j = \overline{0, 3}$) are uniformly bounded and equicontinuous on I . By the Arzela-Ascoli lemma, without loss of generality it can be assumed that these sequences are uniformly convergent on I . Therefore there exists a function $v_0 \in \widetilde{C}^3(I; R)$ such that

$$v_0^{(j)}(t) = \lim_{k \rightarrow +\infty} v_k^{(j)}(t) \quad (j = \overline{0, 3}) \quad \text{uniformly on } I, \quad (3.9)$$

and due to (3.6) we have

$$\|v_0\|_C = 1. \quad (3.10)$$

Set $P_k(t) = \int_a^t p_k(s) ds$. Then from (3.5) we get

$$P_k(a) = 0, \quad \int_{t_1}^{t_2} h_0(s) ds \leq P_k(t_2) - P_k(t_1) \leq \int_{t_1}^{t_2} p(s) ds \quad (3.11)$$

for $a \leq t_1 \leq t_2 \leq b$, and therefore the sequence $\{P_k\}_{k=1}^{+\infty}$ is uniformly bounded and equicontinuous on I . Thus by the Arzela-Ascoli lemma, without loss of generality it can be assumed that this sequence uniformly converges, i.e., there exists the function $P \in C(I; R)$ such that

$$\lim_{k \rightarrow +\infty} P_k(t) = P(t) \quad \text{uniformly on } I, \quad (3.12)$$

for which from (3.11) it follows that

$$\int_{t_1}^{t_2} h_0(s) ds \leq P(t_2) - P(t_1) \leq \int_{t_1}^{t_2} p(s) ds.$$

Consequently, the function P is absolutely continuous, and there exists a function $\tilde{p} \in L(I; R)$ such that $P(t) = \int_a^t \tilde{p}(s) ds$, and the inequality

$$h_0(t) \leq \tilde{p}(t) \leq p(t) \quad \text{for } t \in I \quad (3.13)$$

holds. Then due to (3.9) and (3.12), from Lemma 2 follows the validity of equality (3.4). Therefore if we integrate equation (3.7) from a to t , and pass to the limit as $k \rightarrow +\infty$, due to conditions (3.6), (3.7), (3.9), and (3.4), we find that v_0 is a solution of the problem

$$v_0^{(4)}(t) = \tilde{p}(t)v_0(t), \quad v_0^{(j)}(a) = 0, \quad v_0^{(j)}(b) = 0 \quad (j = \overline{0, 1}).$$

On the other hand, from the disconjugacy of equation (1.6₁) and inequality (3.13), in view of Theorem 1 we have $v_0 \equiv 0$, which is the contradiction with (3.10). Thus our assumption is invalid and estimation (3.3) holds. $\square$

4. PROOF OF THE MAIN RESULTS

Proof of Theorem 1. Let $i = 1$. From disconjugacy of the equation $u^{(4)} = [p_0]_+ u$ on I , by Proposition 1 follows the existence of $p^* \in D_+(I)$ such that $[p_0(t)]_+ \preccurlyeq p^*(t)$ for $t \in I$. From the last inequality and (2.1_i) it is clear that

$$[p(t)]_+ \leq [p_0(t)]_+ \preccurlyeq p^*(t) \quad \text{for } t \in I,$$

and therefore due to Proposition 1 equation (1.6₁) is disconjugate on I . But due to Proposition 5 from the disconjugacy of equation (1.6₁) it follows that Green's function of problem (1.6₁), (1.3₁) is totally positive kernel. Now the validity of our theorem immediately follows from Proposition 3.

For $i = 2$ the proof is similar with the only difference that instead of Propositions 1 and 3, Propositions 2 and 4 are used. $\square$

Proof of Theorem 2. Set $i = 1$, and for $t \in I$ define the functions

$$H(t, x) = \begin{cases} 0 & \text{if } |x| > r \\ f(t, x) & \text{if } |x| \leq r \end{cases}, \quad q^*(t) = \max\{|H(t, x)| : x \in \mathbb{R}\},$$

and

$$F_-(t, x) = \begin{cases} \frac{[f(t, x) \operatorname{sgn} x]_-}{|x|} & \text{if } |x| > r \\ 0 & \text{if } |x| \leq r \end{cases},$$

$$F_+(t, x) = \begin{cases} [f(t, x) \operatorname{sgn} x]_+ \operatorname{sgn} x & \text{if } |x| > r \\ 0 & \text{if } |x| \leq r \end{cases}.$$

Then it is clear that

$$f(t, x) = F_+(t, x) - F_-(t, x)x + H(t, x) \quad \text{for } t \in I, x \in \mathbb{R}, \quad (4.1)$$

where due to condition (2.4_i) we have the estimations

$$0 \leq F_-(t, x) \leq g(t), \quad |F_+(t, x)| \leq \delta(t, |x|) \quad \text{for } t \in I, x \in \mathbb{R}. \quad (4.2)$$

Also due to condition (2.5), there exists a constant $r_0 > r$ such that

$$\rho_0 \left(\|q^*\|_L + \int_a^b \delta(s, \rho) ds \right) < \rho \quad \text{for } \rho \geq r_0, \quad (4.3)$$

where ρ_0 is the constant defined in Lemma 1. Now consider the equation

$$u(t) = F(u)(t) \quad \text{for } t \in I, \quad (4.4)$$

where

$$F(u)(t) = \chi(\|u\|_C) \int_a^b G(t, s) f(s, u(s)) ds,$$

G is Green's function of problem (1.3), (1.3₁), and the function $\chi : R_0^+ \rightarrow [0, 1]$ is defined by the equality

$$\chi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq r_0 \\ 2 - \frac{x}{r_0} & \text{if } r_0 < x < 2r_0 \\ 0 & \text{if } x \geq 2r_0 \end{cases}.$$

From the continuity of the functions $G, \partial G / \partial t$ it follows that F maps $C(I, \mathbb{R})$ into the set $\{z \in C(I, \mathbb{R}) : \|z\|_C \leq r_1\}$, and is the compact operator, where

$$r_1 = \max_{a \leq s, t \leq b} |G(t, s)| \int_a^b f^*(s) ds, \quad f^*(t) = \sup\{|f(t, x)| : |x| \leq 2r_0\}.$$

Then according to Schauder's principle [8] equation (4.4) has a solution u . But from the definition of Green's function G , it is clear that u satisfies the boundary conditions (1.3₁) and is the solution of the equation

$$u^{(4)}(t) = \chi(\|u\|_C)[p(t)u(t) + f(t, u(t))] \quad \text{for } t \in I, \quad (4.5)$$

which in view of (4.1) can be rewritten as equation (3.2) with

$$\tilde{p}(t) = \chi(\|u\|_C)[p(t) - F_-(t, u(t))], \quad q(t) = \chi(\|u\|_C)[F_+(t, u(t)) + H(t, u(t))].$$

Also from (4.2) and the fact that the function δ is nondecreasing in the second argument, estimations (3.1) and

$$|q(t)| \leq \delta(t, \|u\|_C) + q^*(t)$$

follow. Therefore all the assumptions of Lemma 1 are fulfilled and the estimation

$$\|u\|_C \leq \rho_0 \left(\|q^*\|_L + \int_a^b \delta(s, \|u\|_C) ds \right)$$

is valid. Now if we assume that $\|u\|_C > r_0$, we get the contradiction with inequality (4.3). Consequently, $\|u\|_C \leq r_0$, and from (4.4) by the definition of the function χ we get that u is a solution of problem (1.2), (1.3₁).

Let now $i = 2$. If we redefine the functions $F_{\pm}, \tilde{p}, q$ by the following way

$$\begin{aligned} F_-(t, x) &= \begin{cases} [f(t, x) \operatorname{sgn} x]_- \operatorname{sgn} x & \text{if } |x| > r \\ 0 & \text{if } |x| \leq r \end{cases}, \\ F_+(t, x) &= \begin{cases} \frac{[f(t, x) \operatorname{sgn} x]_+}{|x|} & \text{if } |x| > r \\ 0 & \text{if } |x| \leq r \end{cases}, \\ \tilde{p}(t) &= \chi(\|u\|_C)[p(t) + F_+(t, u(t))], \\ q(t) &= \chi(\|u\|_C)[F_-(t, u(t)) + H(t, u(t))], \end{aligned}$$

then

$$f(t, x) = F_+(t, x)x - F_-(t, x) + H(t, x) \quad \text{for } t \in I, x \in \mathbb{R}, \quad (4.6)$$

and due to condition (2.4_i) we have the estimations

$$0 \leq F_+(t, x) \leq g(t), \quad |F_-(t, x)| \leq \delta(t, |x|) \quad \text{for } t \in I, x \in \mathbb{R}. \quad (4.7)$$

Now we can prove the theorem in a similar way as for the case $i = 1$ with these new definitions (4.6) and (4.7). $\square$

Proof of Theorem 3 (4). Assume that problem (1.2), (1.3₁) ((1.2), (1.3₂)) has solutions u_1 and u_2 such that $u_1 \not\equiv u_2$. Let now $u = u_1 - u_2$, and

$$p_0(t) = \begin{cases} \frac{f(t, x_1) - f(t, x_2)}{u(t)} & \text{if } u(t) \neq 0 \\ -p(t) & \text{if } u(t) = 0 \end{cases}.$$

Then u admits conditions (1.3₁) ((1.3₂)) and is a solution of the equation

$$u^{(4)}(t) = (p(t) + p_0(t))u(t). \quad (4.8)$$

where due to inequality (2.6₁) ((2.6₂)), the estimations

$$p(t) + p_0(t) < p^*(t) \quad (p(t) + p_0(t) > p_*(t)) \quad \text{a. e. on } I$$

hold. Therefore from Corollary 1 (2) it follows that problem (4.8), (1.3₁) ((4.8), (1.3₂)) has only the trivial solution, and thus $u_1 \equiv u_2$. $\square$

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EXISTENCE AND UNIQUENESS RESULTS FOR A COUPLED SYSTEM OF NONLINEAR Ψ -FRACTIONAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we establish some new existence results for certain coupled systems of nonlinear Ψ -fractional differential equations with four-point fractional integral boundary conditions. The proofs are derived from Banach fixed point theorem. As an application, a nontrivial example is presented to illustrate our theoretical results.

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1. INTRODUCTION

In this article, we are concerned with the study of the following Ψ -Caputo fractional coupled system

$$\begin{cases} {}^C D_{0+}^{q_1, \Psi} x(t) = f_1(t, x(t), y(t)) \\ {}^C D_{0+}^{q_2, \Psi} y(t) = f_2(t, x(t), y(t)), \end{cases} \quad (1.1)$$

with conditions

$$\begin{cases} x(0) = \varphi_1(x), \\ y(0) = \varphi_2(y), \\ x(1) = I_{0+}^{\alpha, \Psi} x(\theta); \theta \in]0, 1[; \alpha > 0, \\ y(1) = I_{0+}^{\beta, \Psi} y(\delta); \delta \in]0, 1[; \beta > 0, \end{cases} \quad (1.2)$$

where, ${}^C D_{0+}^{q_1, \Psi}$ and ${}^C D_{0+}^{q_2, \Psi}$ are Ψ -Caputo derivatives at order $q_1 \in (0, 1)$ and $q_2 \in (0, 1)$ of x and y respectively, $I_{0+}^{\alpha, \Psi}$ and $I_{0+}^{\beta, \Psi}$ are Ψ -fractional integrals and ϕ_1 and ϕ_2 are two continuous given functions. This type of derivative was introduced by R. Almeida in [2]. In our survey, to prove the existence of result, we exploit the topological method, which have proven to be useful in the investigation of a variety of nonlinear analysis problems. The a priori estimate method has been used to verify the existence of solutions for some nonlinear differential equations or nonlinear partial differential equations boundary value problems. Due to Isaia [9], we employ the fixed point theorem, which was determined using coincidence degree theory for condensing maps. In [9] the author investigate the problem $x(t) = \phi(t, x(t)) + \int_a^b \Psi(t, s, x(s)) ds$ by using the "a priori estimate method" which derived from the invariance under homotopy of the degree defined for α -condensing perturbations of the identity. The authors of [5] analyze a nonlinear fractional q -difference subject to nonlinear more generic four-point boundary conditions using two approaches: a prior estimate method for condensing maps and the Banach contraction principle fixed point theorem. The fractional Cauchy problem results for non-nonlinear Ψ -Caputo fractional differential equations with non-local conditions are investigated by El Mfadel et al. [7]. They used fixed point theorems, topological degree theory for condensing maps, and fractional analysis techniques to develop some novel existence theorems. The significance of fractional calculus can be seen in the modulation of a variety of phenomena in electromagnetic, electrochemistry and viscoelasticity. The choice of a Ψ function under certain conditions is not arbitrary. However, it is a solution for unifying the kernels of several forms of fractional operators, namely, Riemann-Liouville, Hadamard and Erdelyi-Kober. The purpose of this research is to investigate a coupled Ψ -fraction system with boundary conditions, which generalizes the Dirichlet problem, for two mixed situations.

Our paper is organized as follows. In Section 2, we give preliminaries concerning Ψ -fractional integral and Ψ -Caputo fractional derivative which are necessary in this study. In Section 3, we will study the existence of solutions for (1.1) by using the concept Lipschitz and Carathéodory conditions. As application, we will apply the theoretical results on an example in Section 4 followed by a conclusion in Section 5.

2. PRELIMINARIES

We assume the following notations throughout the rest of our paper.

Let $\Delta = [0, 1]$ be an interval of $\mathbb{R}$. We denote the set of continuous functions from Δ into $\mathbb{R}$ by $C = C(\Delta, \mathbb{R})$ endowed the the supremum norm $\|x\| = \sup_{t \in \Delta} |x(t)|$.

In this section, we give some definitions and properties of Ψ -fractional derivatives and Ψ -fractional integrals, (for more details see [1, 2, 4, 6, 8] and references therein).

Definition 1 ([3]). Let $q > 0$, $g \in L^1(\Delta, \mathbb{R})$ and $\Psi \in C^n(\Delta, \mathbb{R})$ such that $\Psi'(t) > 0$ for all $t \in \Delta$. The Ψ -Riemann-Liouville fractional integral at order q of the function

g is given by

$$I_{0+}^{q,\Psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{q-1} g(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function defined by $\Gamma(\lambda) = \int_0^\infty t^{\lambda-1} e^{-t} dt$, $\lambda > 0$.

Definition 2 ([3]). Let $\Psi \in C^n(\Delta, \mathbb{R})$ such that $\Psi'(t) > 0$ for all $t \in \Delta$. The Ψ -Caputo fractional derivative at order $q > 0$ of the function $g \in C^{n-1}(\Delta, \mathbb{R})$ is given by

$${}^C D_{0+}^{q,\Psi} g(t) = \frac{1}{\Gamma(n-q)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{n-q-1} g_{\Psi}^{[n]}(s) ds,$$

where

$$g_{\Psi}^{[n]}(s) = \left(\frac{1}{\Psi'(s)} \frac{d}{ds} \right)^n g(s) \text{ and } n = [q] + 1,$$

with $[q]$ is the integer part of the real number q .

Remark 1. In particular, if $q \in]0, 1[$, then we have

$${}^C D_{0+}^{q,\Psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t (\Psi(t) - \Psi(s))^{q-1} g'(s) ds$$

and

$${}^C D_{0+}^{q,\Psi} g(t) = I_{0+}^{1-q,\Psi} \left(\frac{g'(t)}{\Psi'(t)} \right).$$

Proposition 1 ([3]). Let $q > 0$, if $g \in C^{n-1}(\Delta, \mathbb{R})$, then we have

- 1) ${}^C D_{0+}^{q,\Psi} I_{0+}^{q,\Psi} g(t) = g(t)$.
- 2) $I_{0+}^{q,\Psi} {}^C D_{0+}^{q,\Psi} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g_{\Psi}^{[k]}(0)}{k!} (\Psi(t) - \Psi(0))^k$.
- 3) $I_{a+}^{q,\Psi}$ is linear and bounded from $C(\Delta, \mathbb{R})$ to $C(\Delta, \mathbb{R})$.

Lemma 1. The Ψ -Riemann-Liouville fractional integral at order q of the constant function 1 is given by:

$$I_{0+}^{q,\Psi}(1) = \frac{1}{\Gamma(q+1)} (\Psi(t) - \Psi(0))^q.$$

Proof. We have:

$$I_{0+}^{q,\Psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{q-1} g(s) ds.$$

Then,

$$\begin{aligned} I_{0+}^{q,\Psi}(1) &= \frac{1}{\Gamma(q)} \int_0^t \Psi'(s) (\Psi(t) - \Psi(s))^{q-1} ds = \frac{1}{\Gamma(q+1)} \int_0^t ((\Psi(t) - \Psi(s))^q)' ds \\ &= \frac{1}{\Gamma(q+1)} (\Psi(t) - \Psi(0))^q. \end{aligned}$$

□

3. MAIN RESULTS

For the existence of solution for the problem (1.1)-(1.2), we need the following auxiliary lemmas.

Lemma 2. For each $i = 1, 2$, let f_i be a continuous function from $[0, 1] \times \mathbb{R}^2$ into $\mathbb{R}$. So the problem (1.1)-(1.2) is equivalent to the general solution:

$$\begin{cases} x(t) = I_{0+}^{q_1, \Psi} f_1(t, x(t), y(t)) + \left(1 - \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)}\right) (\varphi_1(x) - f_1(0, \varphi_1(x), \varphi_2(y))) \\ \quad + \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} \left(I_{0+}^{\alpha, \Psi} x(\theta) - f_1(1, x(1), y(1))\right), \\ y(t) = I_{0+}^{q_2, \Psi} f_2(t, x(t), y(t)) + \left(1 - \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)}\right) (\varphi_2(y) - f_2(0, \varphi_1(y), \varphi_2(y))) \\ \quad + \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} \left(I_{0+}^{\beta, \Psi} y(\delta) - f_2(1, x(1), y(1))\right). \end{cases} \quad (3.1)$$

Proof. For some constants $c_1, c_2, m_1, m_2 \in \mathbb{R}$ and for $i = 1, 2$, $q_i > 0$, the general solution of

$$\begin{cases} {}^C D_{0+}^{q_1, \Psi} x(t) = f_1(t, x(t), y(t)) \\ {}^C D_{0+}^{q_2, \Psi} y(t) = f_2(t, x(t), y(t)) \end{cases} \quad (3.2)$$

is given by

$$\begin{aligned} x(t) &= I_{0+}^{q_1, \Psi} f_1(t, x(t), y(t)) + c_1 + (\Psi(t) - \Psi(0)) c_2, \\ y(t) &= I_{0+}^{q_2, \Psi} f_2(t, x(t), y(t)) + m_1 + (\Psi(t) - \Psi(0)) m_2. \end{aligned}$$

Using conditions (1.2) in this form we obtain

$$x(0) = \varphi_1(x) = I_{0+}^{q_1, \Psi} f_1(0, x(0), y(0)) + c_1,$$

thus

$$\begin{aligned} c_1 &= \varphi_1(x) - I_{0+}^{q_1, \Psi} f_1(0, x(0), y(0)) = \varphi_1(x) - I_{0+}^{q_1, \Psi} f_1(0, \varphi_1(x), \varphi_2(y)), \\ x(1) &= I_{0+}^{\alpha, \Psi} x(\theta) = I_{0+}^{q_1, \Psi} f_1(1, x(1), y(1)) + c_1 + (\Psi(1) - \Psi(0)) c_2. \end{aligned}$$

We obtain also $c_2 = \frac{1}{\Psi(1) - \Psi(0)} \left(I_{0+}^{\alpha, \Psi} x(\theta) - I_{0+}^{q_1, \Psi} f_1(1, x(1), y(1)) - c_1 \right)$.

Hence

$$x(t) = I_{0+}^{q_1, \Psi} f_1(t, x(t), y(t)) + \varphi_1(x) - I_{0+}^{q_1, \Psi} f_1(0, \varphi_1(x), \varphi_2(y))$$

$$\begin{aligned}
& + \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} I_{0+}^{\alpha, \Psi} x(\theta) - \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} I_{0+}^{q_1, \Psi} f_1(1, x(1), y(1)) \\
& - \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} \phi_1(x) + \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} I_{0+}^{q_1, \Psi} f_1(0, \phi_1(x), \phi_2(y)),
\end{aligned}$$

therefore

$$\begin{aligned}
x(t) &= I_{0+}^{q_1, \Psi} f_1(t, x(t), y(t)) + \left(1 - \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)}\right) (\phi_1(x) - f_1(0, \phi_1(x), \phi_2(y))) \\
& + \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} \left(I_{0+}^{\alpha, \Psi} x(\theta) - f_1(1, x(1), y(1))\right).
\end{aligned}$$

In the same way, we check the formula of $y(t)$ as follows

$$\begin{aligned}
y(t) &= I_{0+}^{q_2, \Psi} f_2(t, x(t), y(t)) + \left(1 - \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)}\right) (\phi_2(y) - f_2(0, \phi_1(x), \phi_2(y))) \\
& + \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} \left(I_{0+}^{\beta, \Psi} y(\delta) - f_2(1, x(1), y(1))\right).
\end{aligned}$$

□

We need the following hypothesis in the rest of our paper.

(H₁) For $t \in [0, 1]$ and $x_i, y_i \in \mathcal{C}$ for $i = 1, 2$, there exist two strictly positive constants $\mathcal{L}_1$ and $\mathcal{L}_2$ such that:

$$\begin{cases} \|f_1(t, x_1(t), y_1(t)) - f_1(t, x_2(t), y_2(t))\| \leq \mathcal{L}_1 (\|x_1 - x_2\| + \|y_1 - y_2\|), \\ \|f_2(t, x_1(t), y_1(t)) - f_2(t, x_2(t), y_2(t))\| \leq \mathcal{L}_2 (\|x_1 - x_2\| + \|y_1 - y_2\|). \end{cases} \quad (3.3)$$

(H₂) There exist three constants $K_i, M_i, N_i > 0$, $i = 1, 2$ and $n \in [0, 1]$ such that, for $t \in [0, 1]$ and $\forall x, y \in \mathcal{C}$ we have

$$\begin{aligned}
\|f_1(t, x(t), y(t))\| &\leq K_1 \|x\|^n + M_1 \|y\|^n + N_1 \\
\|f_2(t, x(t), y(t))\| &\leq K_2 \|x\|^n + M_2 \|y\|^n + N_2.
\end{aligned}$$

Moreover, we put

$$\varepsilon_i = 1 + \left(\frac{(\Psi(1) - \Psi(0))^\alpha}{\Gamma(\alpha + 1)}\right)^{(2-i)} + \left(\frac{(\Psi(1) - \Psi(0))^\beta}{\Gamma(\beta + 1)}\right)^{(i-1)} + 2 \frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)}.$$

Now, we consider two operators $\mathcal{P}, \mathcal{R}: \mathcal{C} \rightarrow \mathcal{C}$ as follows:

$$\begin{aligned}
\mathcal{P}(x(t)) &= I_{0+}^{q_1, \Psi} f_1(t, x(t), y(t)), \quad t \in [0, 1]. \\
\mathcal{R}(x(t)) &= \left(1 - \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)}\right) (\phi_1(x) - I_{0+}^{q_1, \Psi} f_1(0, \phi_1(x), \phi_2(y))) \\
& + \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} \left(I_{0+}^{\alpha, \Psi} x(\theta) - I_{0+}^{q_1, \Psi} f_1(1, x(1), y(1))\right), \quad t \in [0, 1].
\end{aligned}$$

Then, the general solution in Lemma 2 can be written using these operators as

$$\mathcal{H}(x(t)) = \mathcal{P}(x(t)) + \mathcal{R}(x(t)),$$

$$\mathcal{H}(y(t)) = \mathcal{P}(y(t)) + \mathcal{R}(y(t)).$$

For continuity of f_i for $i = 1, 2$, let us show that the operator $\mathcal{P}$ is well defined and admits a fixed point, to deduce the existence of the solution which is the fixed point.

Lemma 3. *The operator $\mathcal{P} : \mathcal{C} \rightarrow \mathcal{C}$ is Lipschitz with the constant*

$$\sum_{i=1}^2 k_i = \sum_{i=1}^2 \frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} \mathcal{L}_i.$$

Moreover, $\mathcal{P}$ satisfies the following inequality:

$$\| \mathcal{P}(x, y) \| \leq \sum_{i=1}^2 \frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} \mathcal{L}_i (K_i \|x\|^n + M_i \|y\|^n + N_i),$$

for every $(x, y) \in \mathcal{C}$.

Proof. Let $x_i, y_i \in \mathcal{C}$, for $i = 1, 2$, we have:

$$\begin{aligned} | \mathcal{P}(x_1(t)) - \mathcal{P}(x_2(t)) | &= | I_{0+}^{q_1, \Psi} f_1(t, x_1(t), y(t)) - I_{0+}^{q_1, \Psi} f_1(t, x_2(t), y(t)) | \\ &\leq I_{0+}^{q_1, \Psi} | f_1(t, x_1(t), y(t)) - f_1(t, x_2(t), y(t)) | \\ &\leq I_{0+}^{q_1, \Psi} (1) \mathcal{L}_1 (\|x_1 - x_2\| + \|y - y\|) \\ &\leq \frac{(\Psi(t) - \Psi(0))^{q_1}}{\Gamma(q_1 + 1)} \mathcal{L}_1 \|x_1 - x_2\| \\ &\leq \frac{(\Psi(1) - \Psi(0))^{q_1}}{\Gamma(q_1 + 1)} \mathcal{L}_1 \|x_1 - x_2\|. \end{aligned}$$

In a similar way,

$$| \mathcal{P}(y_1(t)) - \mathcal{P}(y_2(t)) | \leq \frac{(\Psi(1) - \Psi(0))^{q_2}}{\Gamma(q_2 + 1)} \mathcal{L}_2 \|y_1 - y_2\|.$$

Hence, $\mathcal{P}$ is a Lipschitz with the constant $k_i = \frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} \mathcal{L}_i$.

Moreover, for the condition, we have

$$\begin{aligned} | \mathcal{P}(x(t)) | &\leq I_{0+}^{q_1, \Psi} | f_1(t, x(t), y(t)) | \\ &\leq I_{0+}^{q_1, \Psi} (1) (K_1 \|x\|^n + M_1 \|y\|^n + N_1) \\ &\leq \frac{(\Psi(1) - \Psi(0))^{q_1}}{\Gamma(q_1 + 1)} (K_1 \|x\|^n + M_1 \|y\|^n + N_1) \end{aligned}$$

and

$$| \mathcal{P}(y(t)) | \leq \frac{(\Psi(1) - \Psi(0))^{q_2}}{\Gamma(q_2 + 1)} (K_2 \|x\|^n + M_2 \|y\|^n + N_2),$$

which gives

$$\| \mathcal{P}(x(t), y(t)) \| \leq \sum_{i=1}^{i=2} \frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} (K_i \|x\|^n + M_i \|y\|^n + N_i).$$

□

Lemma 4. $\mathcal{R}$ is a continuous operator and satisfies the growth condition defined by

$$\| \mathcal{R}(x(t)) \| \leq \left(1 + \frac{(\Psi(1) - \Psi(0))^\alpha}{\Gamma(\alpha + 1)} + 2 \frac{(\Psi(1) - \Psi(0))^{q_1}}{\Gamma(q_1 + 1)} \right) (K_1 \|x\|^n + M_1 \|y\|^n + N_1),$$

and

$$\| \mathcal{R}(y(t)) \| \leq \left(1 + \frac{(\Psi(1) - \Psi(0))^\beta}{\Gamma(\beta + 1)} + 2 \frac{(\Psi(1) - \Psi(0))^{q_2}}{\Gamma(q_2 + 1)} \right) (K_2 \|x\|^n + M_2 \|y\|^n + N_2),$$

for every $x, y \in \mathcal{C}$.

Proof. Consider a subset $\mathcal{D}_p = \{(x, y) \in \mathcal{C}, \| (x, y) \| \leq p\} \subset \mathcal{C}$, and consider a sequence $\{z_n = (x_n, y_n)\} \in \mathcal{D}_p$ such that $z_n \rightarrow z = (x, y)$.

Now to show that $\mathcal{R}$ is continuous just show that $\| \mathcal{R}(z_n) - \mathcal{R}(z) \| \rightarrow 0$. From the continuity of f_i and ϕ_i , we have $f_{i,n} \rightarrow f_i$, $\phi_1(x_n) \rightarrow \phi_1(x)$ and $\phi_2(y_n) \rightarrow \phi_2(y)$ when $n \rightarrow \infty$.

According to the hypothesis (H_1) , we obtain

$$\begin{aligned} & \Psi'(s) (\Psi(t) - \Psi(s))^{q_1-1} \| f_1(t, x_n, y_n) - f_1(t, x, y) \| \\ & \leq \Psi'(s) (\Psi(t) - \Psi(s))^{q_1-1} \mathcal{L}_1 (\|x_n - x\| + \|y_n - y\|), \end{aligned}$$

and

$$\begin{aligned} & \Psi'(s) (\Psi(t) - \Psi(s))^{q_2-1} \| f_2(t, x_n, y_n) - f_2(t, x, y) \| \\ & \leq \Psi'(s) (\Psi(t) - \Psi(s))^{q_2-1} \mathcal{L}_2 (\|x_n - x\| + \|y_n - y\|). \end{aligned}$$

By the Lebesgue dominated convergent theorem, we obtain

$$I_{0+}^{q_1, \Psi} | f_1(t, x_n, y_n) - f_1(t, x, y) | \rightarrow 0,$$

$$I_{0+}^{q_2, \Psi} | f_2(t, x_n, y_n) - f_2(t, x, y) | \rightarrow 0,$$

$$I_{0+}^{\alpha, \Psi} | x_n - x | \rightarrow 0,$$

$$I_{0+}^{\beta, \Psi} | y_n - y | \rightarrow 0,$$

as $n \rightarrow \infty$.

It follows that $\| \mathcal{R}(z_n) - \mathcal{R}(z) \| \rightarrow 0$. Hence, the operator $\mathcal{R}$ is continuous. Using (H2) to show the condition, we have

$$| \mathcal{R}(x(t)) | \leq \left(1 - \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} \right) \phi_1(x) |$$

$$\begin{aligned}
& + \left| \left(1 - \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} \right) I_{0+}^{q_1, \Psi} f_1(0, \varphi_1(x), \varphi_2(y)) \right| \\
& + \left| \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} I_{0+}^{\alpha, \Psi} x(\theta) \right| + \left| \frac{\Psi(t) - \Psi(0)}{\Psi(1) - \Psi(0)} I_{0+}^{q_1, \Psi} f_1(1, x(1), y(1)) \right| \\
& \leq | \varphi_1(x) | + I_{0+}^{q_1, \Psi}(1) | f_1(0, \varphi_1(x), \varphi_2(y)) | + I_{0+}^{\alpha, \Psi}(1) | x(\theta) | \\
& \quad + I_{0+}^{q_1, \Psi}(1) | f_1(1, x(1), y(1)) | \\
& \leq \left(1 + I_{0+}^{\alpha, \Psi}(1) \right) \| x \| + 2 I_{0+}^{q_1, \Psi}(1) (K_1 \| x \|^n + M_1 \| y \|^n + N_1) \\
& \leq \left(1 + \frac{(\Psi(1) - \Psi(0))^\alpha}{\Gamma(\alpha + 1)} \right) (K_1 \| x \|^n + M_1 \| y \|^n + N_1) \\
& \quad + 2 \frac{(\Psi(1) - \Psi(0))^{q_1}}{\Gamma(q_1 + 1)} (K_1 \| x \|^n + M_1 \| y \|^n + N_1) \\
& \leq \left(1 + \frac{(\Psi(1) - \Psi(0))^\alpha}{\Gamma(\alpha + 1)} + 2 \frac{(\Psi(1) - \Psi(0))^{q_1}}{\Gamma(q_1 + 1)} \right) \\
& \quad \cdot (K_1 \| x \|^n + M_1 \| y \|^n + N_1).
\end{aligned}$$

The same for,

$$\| \mathcal{R}(y(t)) \| \leq \left(1 + \frac{(\Psi(1) - \Psi(1))^\beta}{\Gamma(\beta + 1)} + 2 \frac{(\Psi(1) - \Psi(1))^{q_2}}{\Gamma(q_2 + 1)} \right) (K_2 \| x \|^n + M_2 \| y \|^n + N_2).$$

Which implies that

$$\| \mathcal{R}(x(t), y(t)) \| \leq \sum_{i=1}^{i=2} (K_i \| x \|^n + M_i \| y \|^n + N_i) \varepsilon_i.$$

□

Lemma 5. *If $\mathcal{R}$ is a compact operator, then $\mathcal{R}$ is Lipschitz with zero constant.*

Proof. Let $\Omega \subset \mathcal{B}_r$ a bounded set. Show that $\mathcal{R}(\Omega)$ is relatively compact in $\mathcal{C}$, which implies that $\mathcal{R}$ is compact.

For $x_i \in \Omega$ we have

$$\| \mathcal{R}(x_i) \| \leq (K_i \| x \|^n + M_i \| y \|^n + N_i) \varepsilon_i.$$

According to this inequality, the set $\mathcal{R}(\Omega)$ is uniformly bounded. For equi-continuity of $\mathcal{R}$, take $s_1, s_2 \in [0, 1]$ with $s_1 < s_2$ and let $x \in \Omega$, then we have

$$\begin{aligned}
& | \mathcal{R}(x(s_1)) - \mathcal{R}(x(s_2)) | \\
& \leq (\Psi(s_2) - \Psi(s_1)) \\
& \quad \cdot \left(| \varphi_1(x) | + | I_{0+}^{q_1, \Psi} f_1(0, \varphi_1(x), \varphi_2(y)) | + | I_{0+}^{\alpha, \Psi} x(\theta) | + | I_{0+}^{q_1, \Psi} f_1(1, x(1), y(1)) | \right) \\
& \leq (\Psi(s_2) - \Psi(s_1))
\end{aligned}$$

$$\cdot \left(1 + \frac{(\Psi(1) - \Psi(0))^\alpha}{\Gamma(\alpha + 1)} + 2 \frac{(\Psi(1) - \Psi(0))^{q_1}}{\Gamma(q_1 + 1)} \right) (K_1 \|x\|^n + M_1 \|y\|^n + N_1).$$

Which implies that

$$|\mathcal{R}((x, y)(s_1)) - \mathcal{R}((x, y)(s_2))| \leq \sum_{i=1}^{i=2} \varepsilon_i (K_i \|x\|^n + M_i \|y\|^n + N_i) (s_2 - s_1).$$

We deduce that $\|\mathcal{R}((x, y)(s_1)) - \mathcal{R}((x, y)(s_2))\| \rightarrow 0$ when $s_2 \rightarrow s_1$.

Therefore, $\mathcal{R}$ is equi-continuous. Thus, by Ascoli-Arzelà theorem, $\mathcal{R}$ is a compact operator, then $\mathcal{R}$ is Lipschitz with zero constant. $\square$

Theorem 1. *The problem (1.1) – (1.2) has at least one solution $(x, y) \in \mathcal{C}$, if $(H_1) - (H_2)$ are satisfied and $\sum_{i=1}^{i=2} k_i < 1$. And the set of solutions is bounded in $\mathcal{C}$.*

Proof. Let the operators $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{H}$ defined previously are continuous and bounded, and we have that $\mathcal{P}$ and $\mathcal{R}$ are Lipschitz with zero constant.

Hence, $\mathcal{H}$ is Lipschitz with constant k_i , then $\mathcal{H}$ is strict κ -contraction with constant k_i . Since, $\sum_{i=1}^{i=2} k_i < 1$, so $\mathcal{H}$ is κ -condensing. Define the set

$$\pi = \{(x, y) \in \mathcal{C}, \text{ there exists } \lambda \in [0, 1] \text{ such that } x = \lambda \mathcal{H}(x) \text{ and } y = \lambda \mathcal{H}(y)\},$$

and show that π is bounded.

For $x \in \pi$, we have $x = \lambda \mathcal{H}(x) = \lambda(\mathcal{P}(x) + \mathcal{R}(x))$, which implies that

$$\begin{aligned} \|x\| &\leq \lambda (\|\mathcal{P}(x)\| + \|\mathcal{R}(x)\|) \\ &\leq \left(\frac{(\Psi(1) - \Psi(0))^{q_1}}{\Gamma(q_1 + 1)} + \varepsilon_1 \right) (K_1 \|x\|^n + M_1 \|y\|^n + N_1), \end{aligned}$$

hence we get:

$$\begin{aligned} \|(x, y)\| &\leq \lambda (\|\mathcal{P}(x, y)\| + \|\mathcal{R}(x, y)\|) \\ &\leq \sum_{i=1}^{i=2} \left(\frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} + \varepsilon_i \right) (K_i \|x\|^n + M_i \|y\|^n + N_i). \end{aligned}$$

Then, π is bounded in $\mathcal{C}$.

On the other hand, suppose that π is not bounded, then we divide the inequality by $b = \|x\|$ and letting $b \rightarrow \infty$, arriving at

$$1 \leq \sum_{i=1}^{i=2} \left(\frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} + \varepsilon_i \right) \lim_{b \rightarrow \infty} \frac{K_i b^n + M_i b^n + N_i}{b} = 0,$$

which is a contradiction.

Then, π is a bounded set in $\mathcal{C}$ and the operator $\mathcal{H}$ admits at least one fixed point and it is the solution of problem (1.1) – (1.2). $\square$

Now show the existence and uniqueness of the solution by the following theorem.

Theorem 2. *The problem (1.1) – (1.2) has a unique solution if (H_1) is satisfied and*

$$\sum_{i=1}^{i=2} \left(\frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} + \varepsilon_i \right) \mathcal{L}_i < 1.$$

Proof. Let $x_1, x_2 \in \mathcal{C}$ and $t \in [0, 1]$, we have:

$$\begin{aligned} |\mathcal{H}(x_1(t)) - \mathcal{H}(x_2(t))| &\leq |\mathcal{P}(x_1(t)) - \mathcal{P}(x_2(t))| + |\mathcal{R}(x_1(t)) - \mathcal{R}(x_2(t))| \\ &\leq \frac{(\Psi(1) - \Psi(0))^{q_1}}{\Gamma(q_1 + 1)} \mathcal{L}_1 \|x_1 - x_2\| + \varepsilon_1 \mathcal{L}_1 \|x_1 - x_2\| \\ &\leq \left(\frac{(\Psi(1) - \Psi(0))^{q_1}}{\Gamma(q_1 + 1)} + \varepsilon_1 \right) \mathcal{L}_1 \|x_1 - x_2\|. \end{aligned}$$

If $\sum_{i=1}^{i=2} \left(\frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} + \varepsilon_i \right) \mathcal{L}_i < 1$, hence $\mathcal{H}$ is contraction. Then, by the Banach contraction principle, the problem (1.1) – (1.2) admits a unique solution, it is the only fixed point of the operator $\mathcal{H}$. $\square$

4. ILLUSTRATIVE EXAMPLE

In this section, we will apply the previous results to the following example. Consider the following equation:

$$\begin{cases} {}^C D_{0+}^{\frac{4}{3}, t^2} x(t) = \frac{e^{-\pi t}}{10} + \frac{\sin |x(t)| + \sin |y(t)|}{9+t}, & t \in I = [0, 1] \\ {}^C D_{0+}^{\frac{7}{5}, t^2} y(t) = \frac{e^{-15t}}{13} + \frac{\sin |x(t)| + \sin |y(t)|}{16+(t+1)^2}. \end{cases} \quad (4.1)$$

with the conditions:

$$\begin{cases} x(0) = \frac{2}{5} \sin(x) + \frac{1}{2}, \\ y(0) = \frac{3}{5} \cos(y) + \frac{1}{3}, \\ x(1) = I_{0+}^{\frac{1}{4}, t^2} x(\frac{1}{5}), \\ y(1) = I_{0+}^{\frac{4}{5}, t^2} y(\frac{2}{5}), \end{cases} \quad (4.2)$$

with the constant in problem (1.1) – (1.2):

$$q_1 = \frac{4}{3}, \quad q_2 = \frac{7}{5}, \quad \alpha = \frac{1}{4}, \quad \beta = \frac{4}{5}, \quad \theta = \frac{1}{5}, \quad \delta = \frac{2}{5},$$

and with functions

$$f_1(t, x(t), y(t)) = \frac{e^{-\pi t}}{10} + \frac{\sin |x(t)| + \sin |y(t)|}{9+t},$$

$$f_2(t, x(t), y(t)) = \frac{e^{-15t}}{13} + \frac{\sin |x(t)| + \sin |y(t)|}{16 + (t+1)^2},$$

$$\Psi(t) = t^2, \quad \varphi_1(x) = \frac{2}{5}\sin(x) + \frac{1}{2}, \quad \varphi_2(y) = \frac{3}{5}\cos(y) + \frac{1}{3}.$$

Using the parameter values provided, we get

$$\sum_{i=1}^{i=2} \left(\frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} + \varepsilon_i \right) = 9,1114.$$

We can easily show that

$$\|f_1(t, x_1(t), y_1(t)) - f_1(t, x_2(t), y_2(t))\| \leq \frac{1}{9} (\|x_1 - x_2\| + \|y_1 - y_2\|),$$

$$\|f_2(t, x_1(t), y_1(t)) - f_2(t, x_2(t), y_2(t))\| \leq \frac{1}{16} (\|x_1 - x_2\| + \|y_1 - y_2\|).$$

Hence the condition (H_1) holds with $\mathcal{L}_1 = \frac{1}{9}$, $\mathcal{L}_2 = \frac{1}{16}$.

In addition we have

$$\sum_{i=1}^{i=2} k_i = \sum_{i=1}^{i=2} \frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} \mathcal{L}_i = 0,1436.$$

For any $t \in [0, 1]$ and $x, y \in \mathbb{R}$, we have

$$|f_1(t, x(t), y(t))| \leq \frac{1}{9} |x| + \frac{1}{9} |y| + 1,$$

$$|f_2(t, x(t), y(t))| \leq \frac{1}{16} |x| + \frac{1}{16} |y| + 2.$$

Condition (H_2) holds with $M_1 = K_1 = \frac{1}{9}$, $M_2 = K_2 = \frac{1}{16}$, $n = N_1 = 1$ and $N_2 = 2$. In view of Theorem 1, the solution set:

$$\pi = \{(x, y) \in \mathcal{C}, \text{ there exists } \lambda \in [0, 1] \text{ such that } x = \lambda \mathcal{H}(x) \text{ and } y = \lambda \mathcal{H}(y)\}.$$

Then

$$\begin{aligned} \|(x, y)\| &\leq \lambda (\|\mathcal{P}(x, y)\| + \|\mathcal{R}(x, y)\|) \\ &\leq \sum_{i=1}^{i=2} \left(\frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} + \varepsilon_i \right) (K_i \|x\| + M_i \|y\| + N_i). \end{aligned}$$

By using Theorem 1, the problem (4.1) – (4.2) has at least a solution $(x, y) \in \mathcal{C}$.

Furthermore, we have:

$$\sum_{i=1}^{i=2} \left(\frac{(\Psi(1) - \Psi(0))^{q_i}}{\Gamma(q_i + 1)} + \varepsilon_i \right) \mathcal{L}_i = 0,77 < 1.$$

Hence from Theorem 2, it follows that (4.1) – (4.2) has a unique solution.

5. CONCLUSION

In this paper, we studied the existence results for a coupled system of nonlinear fractional differential equations with four-point fractional integral boundary conditions involving Ψ -Caputo fractional derivatives. The existence theorems are proved by using some fixed point theorems. As an application, an example is presented to illustrate the applicability of our main result.

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COUNTEREXAMPLE TO A CONJECTURE ABOUT DIHEDRAL QUANDLE

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Abstract. It was conjectured that the augmentation ideal of a dihedral quandle of even order $n > 2$ satisfies $|\Delta^k(\mathbf{R}_n)/\Delta^{k+1}(\mathbf{R}_n)| = n$ for all $k \geq 2$. In this article we provide a counterexample against this conjecture.

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Keywords: quandle rings, augmentation ideal

1. INTRODUCTION

A *quandle* is a pair $(A, \cdot)$ such that ‘ $\cdot$ ’ is a binary operation satisfying the following two conditions:

- (1) the map $S_a : A \rightarrow A$, defined as $S_a(b) = b \cdot a$ is an automorphism for all $a \in A$,
- (2) for all $a \in A$, we have $S_a(a) = a$.

To have a better understanding of the structure, a theory parallel to group rings was introduced by Bardakov, Passi and Singh in [1]. Let $\mathbb{Z}_n$ denote the cyclic group of order n . Then defining $a \cdot b = 2b - a$ gives a quandle structure on $A = \mathbb{Z}_n$. This is known as *dihedral quandle*. For other examples see [1]. The quandle ring of a quandle A is defined as follows. Let R be a commutative ring. Consider

$$R[A] := \left\{ \sum_i r_i a_i : r_i \in R, a_i \in A, r_i = 0 \text{ for all but finitely many } i \right\}.$$

Then this is an additive group in usual way. Define multiplication as

$$\left(\sum_i r_i a_i \right) \cdot \left(\sum_j s_j a_j \right) := \sum_{i,j} r_i s_j (a_i \cdot a_j).$$

The *augmentation ideal* of $R[A]$, $\Delta_R(A)$ is defined as the kernel of the augmentation map

$$\varepsilon : R[A] \rightarrow R, \sum_i r_i a_i \mapsto \sum_i r_i.$$

The powers $\Delta_R^k(A)$ are defined as $(\Delta_R(A))^k$. When $R = \mathbb{Z}$, we will be omitting the subscript R . The following proposition gives a basis for $\Delta_R(X)$.

Proposition 1. [1, Proposition 3.2, Page 6] A basis of $\Delta_R(X)$ as an R -module is given by $\{a - a_0 : a \in A \setminus \{a_0\}\}$, where $a_0 \in A$ is a fixed element.

The following has been conjectured in [1, Conjecture 6.5, Page 20].

Conjecture 1. Let $R_n = \{a_0, a_1, \dots, a_{n-1}\}$ denote the dihedral quandle of order n . Then we have the following statements.

- (1) For an odd integer $n > 1$, $\Delta^k(R_n) / \Delta^{k+1}(R_n) \cong \mathbb{Z}_n$ for all $k \geq 1$.
- (2) For an even integer $n > 2$, $|\Delta^k(R_n) / \Delta^{k+1}(R_n)| = n$ for $k \geq 2$.

The first statement has been confirmed by Elhamdadi, Fernando and Tsvetikhovskiy in [2, Theorem 6.2, Page 182]. The second statement holds true for $n = 4$, see [1]. Here we have given a counterexample in **Theorem 1** to show that the conjecture is not true in general.

2. COUNTEREXAMPLE

Theorem 1. Let R_8 be the dihedral quandle of order 8. Then

$$|\Delta^2(R_8) / \Delta^3(R_8)| = 16.$$

From Proposition 1, we get that $\{e_i = a_i - a_0 : i = 1, 2, \dots, n-1\}$ is a basis for $\Delta(R_n)$. We will be using this notation in the subsequent computation.

Lemma 1. Let R_{2n} denote the dihedral quandle of order $2n$ ($n \geq 2$). Then $e_i \cdot e_n = 0$ for all $i = 1, 2, \dots, 2n-1$.

Proof. Observe that

$$e_i \cdot e_n = (a_i - a_0) \cdot (a_n - a_0) = a_{2n-i} - a_{2n-i} - a_0 + a_0 = 0.$$

□

Lemma 2. Let R_{2n} denote the dihedral quandle of order $2n$ ($n \geq 2$). Then $e_i \cdot e_j = e_i \cdot e_{n+j}$ for all $j = 1, 2, \dots, n-1$ and for all $i = 1, 2, \dots, 2n-1$.

Proof. Note that

$$\begin{aligned} e_i \cdot e_{n+j} &= a_i a_{n+j} - a_i a_0 - a_0 a_{k+j} + a_0 \\ &= a_i a_j - a_i a_0 - a_0 a_j + a_0 = e_i \cdot e_j. \end{aligned}$$

□

We will use Lemma 1 and Lemma 2 to simplify the multiplication tables.

Proof of Theorem 1. Recall that a basis of $\Delta(R_8)$ is given by $\mathcal{B}_1 = \{e_1, e_2, \dots, e_7\}$. The multiplication table for the $e_i \cdot e_j$ is given as follows:

	e_1	e_2	e_3
e_1	$e_1 - e_2 - e_7$	$e_3 - e_4 - e_7$	$e_5 - e_6 - e_7$
e_2	$-e_2 - e_6$	$e_2 - e_4 - e_6$	$-2e_6$
e_3	$-e_2 - e_5 + e_7$	$e_1 - e_4 - e_5$	$e_3 - e_5 - e_6$
e_4	$-e_2 - e_4 + e_6$	$-2e_4$	$e_2 - e_4 - e_6$
e_5	$-e_2 - e_3 + e_5$	$-e_3 - e_4 + e_7$	$e_1 - e_3 - e_6$
e_6	$-2e_2 + e_4$	$-e_2 - e_4 + e_6$	$-e_2 - e_6$
e_7	$-e_1 - e_2 + e_3$	$-e_1 - e_4 + e_5$	$-e_1 - e_6 + e_7$

Since $\Delta^2(R_8)$ is generated by $e_i \cdot e_j$ as a $\mathbb{Z}$ -module, using row reduction over $\mathbb{Z}$ one can show that a $\mathbb{Z}$ -basis is given by

$$\mathcal{B}_2 = \{u_1 = e_1 - e_2 - e_7, u_2 = e_2 + e_6, u_3 = e_3 - e_4 - e_7, \\ u_4 = e_4 + 2e_6, u_5 = e_5 - e_6 - e_7, u_6 = 4e_6\}.$$

We now want to express a $\mathbb{Z}$ -basis of $\Delta^3(R_8)$ in terms of $\mathcal{B}_2$. First we calculate the products $u_i \cdot e_j$. This is presented in the following table.

	e_1	e_2	e_3
u_1	$2e_1 + e_2 - e_3 + e_6 - e_7$	$e_1 - e_2 + e_3 + e_4 - e_5 + e_6 - e_7$	$e_1 - e_4 + e_5 + 2e_6 - 2e_7$
u_2	$-3e_2 + e_4 - e_6$	$-2e_4$	$-e_2 + e_4 - 3e_6$
u_3	$e_1 + e_2 - e_3 + e_4 - e_5 - e_6 + e_7$	$2e_1 + 2e_4 - 2e_5$	$e_1 - e_2 + e_3 + e_4 - e_5 + e_6 - e_7$
u_4	$-5e_2 - e_4 + e_6$	$-2e_2 - 4e_4 + 2e_6$	$-e_2 - e_4 - 3e_6$
u_5	$e_1 + 2e_2 - 2e_3 - e_4 + e_5$	$e_1 + e_2 - e_3 + e_4 - e_5 - e_6 + e_7$	$2e_1 + e_2 - e_3 + e_6 - e_7$
u_6	$-8e_2 + 4e_4$	$-4e_2 - 4e_4 + 4e_6$	$-4e_2 - 4e_6$

Hence, a $\mathbb{Z}$ -basis for $\Delta^3(R_8)$ is given by

$$\mathcal{B}_3 = \{v_1 = e_1 - e_2 + e_3 + e_4 - e_5 + e_6 - e_7, v_2 = e_2 - e_3 - 2e_4 + 2e_5 + e_6 - e_7, \\ v_3 = -e_3 - e_4 + 2e_5 - 2e_6 - e_7, v_4 = -2e_4, v_5 = -4e_5 - 4e_6 + 4e_7, v_6 = 8e_6\}.$$

Now we will present the elements of $\mathcal{B}_3$ in terms of $\mathcal{B}_2$. We have the following presentation.

$$\begin{array}{rcll} v_1 & = & u_1 & +2u_4 - u_5 - u_6 \\ v_2 & = & u_2 - u_3 - u_4 & +2u_5 + u_6 \\ v_3 & = & -u_3 - 2u_4 & +2u_5 + u_6 \\ v_4 & = & 2u_4 & - u_6 \\ v_5 & = & & -4u_5 \\ v_6 & = & & 2u_6. \end{array}$$

Note that we can alter the basis $\mathcal{B}_2$ of $\Delta^2(\mathbf{R}_8)$ as follows:

$$\{u_1 + 2u_4 - u_5 - u_6, u_2 - u_3 - u_4 + 2u_5 + u_6, u_3 + 2u_4 - 2u_5 - u_6, u_4, u_5, u_6\}.$$

Hence,

$$\begin{aligned} \frac{\Delta^2(\mathbf{R}_8)}{\Delta^3(\mathbf{R}_8)} &\cong \frac{\mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \mathbb{Z}v_3 \oplus \mathbb{Z}u_4 \oplus \mathbb{Z}u_5 \oplus \mathbb{Z}u_6}{\mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \mathbb{Z}v_3 \oplus \mathbb{Z}(2u_4 - u_6) \oplus \mathbb{Z}(-4u_5) \oplus \mathbb{Z}(2u_6)} \\ &\cong \mathbb{Z}_4 \oplus \frac{\mathbb{Z}u_4 \oplus \mathbb{Z}u_6}{\mathbb{Z}(2u_4 - u_6) \oplus \mathbb{Z}(2u_6)} \cong \mathbb{Z}_4 \oplus \frac{\mathbb{Z}u_4 \oplus \mathbb{Z}u_6}{\mathbb{Z}u_4 \oplus \mathbb{Z}(4u_6)} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4. \end{aligned}$$

□

3. FURTHER REMARKS

We have calculated that for $k = 2, 3, 4$ and $n = 6, 8, 10$, the isomorphism

$$\frac{\Delta^k(\mathbf{R}_{2n})}{\Delta^{k+1}(\mathbf{R}_{2n})} \cong \mathbb{Z}_n \oplus \mathbb{Z}_n.$$

holds. Hence, we propose the following improved version of the main conjecture given in [1].

Conjecture 2. Let $\mathbf{R}_{2n}$ denotes the dihedral quandle of order $2n$ for $n \geq 2$. Then for $k > 1$,

$$\frac{\Delta^k(\mathbf{R}_{2n})}{\Delta^{k+1}(\mathbf{R}_{2n})} \cong \mathbb{Z}_n \oplus \mathbb{Z}_n.$$

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ON COFINITELY FLAT QUADRATIC O_K -MODULES

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Abstract. In this paper, we considered quadratic O_K -modules over the number fields and we defined the new concepts "*cofinitely quadratic O_K -module*" and "*cofinitely flat quadratic O_K -module*" for the integral ring O_K of the quadratic number fields. We described tensor product for these modules, and we extended them to the cofinitely quadratic O_K -modules. Finally we provided a main theorem by using these definitions.

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Keywords: number field, finite quadratic modul, cofinitely flat module

1. INTRODUCTION

Quadratic modules play a ubiquitous role in real algebra (see [4, 6–8] and the references therein). Many algebraic structures in real algebra and algebraic geometry are associated with quadratic modules. For instance, a quadratic module in a commutative ring A (with unit element) is a subset Q of A containing the unit element 1, which is closed under addition and under multiplication with squares and the ring A contains a smallest quadratic module, namely, the set $\sum A^2$ consisting of all sums of squares of elements of A . Quadratic modules of groups are algebraic models for homotopy connected 3-types introduced by Baues [3]. Baues in [3] constructed a functor from the category of simplicial groups to the category of quadratic modules. In [9], Lie algebra versions of quadratic modules was also defined, and the connections between 2-crossed modules, quadratic modules and simplicial Lie algebras were explored by using simplicial properties in [1].

In recent studies, quadratic modules defined on algebraic number fields has also gained importance in real algebra. For this purpose, in this work, the quadratic modules defined on such fields will be examined. Namely, quadratic modules belonging to integral rings of algebraic number fields which are finite extensions of $\mathbb{Q}$ will be

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discussed and new definitions and algebraic structions related to these modules will be given.

Let K be a number field over $\mathbb{Q}$ such that $[K : \mathbb{Q}] = n$. We shall use denotation O_K for the ring of integers of K where O_K is a finitely generated $\mathbb{Z}$ -module.

In the theory of finite O_K -modules, these modules are identified in equipped with a quadratic form $q : O_K \rightarrow K/\sigma^{-1}$ where $\sigma^{-1} = \{x \in K \mid T_{r_K/\mathbb{Q}}(xy) \in \mathbb{Z}, \forall x, y \in O_K\}$ is called *different* of K and σ^{-1} is also a fractional ideal of O_K .

We notice that we have a function

$$T_r : K/\sigma^{-1} \rightarrow \mathbb{Q}/\mathbb{Z}, \quad T_r(a + \sigma^{-1}) = t_r(a) + \mathbb{Z}.$$

Let $b \in a + \sigma^{-1}$. If we take $b = a + t$ for some $t \in \sigma^{-1}$ then $T_r(t) \in \mathbb{Z}$ holds.

2. FINITE QUADRATIC O_K -MODULES

Definition 1. Let M be a finite O_K -module, then a quadratic form on M is a function $q : M \rightarrow K/\sigma^{-1}$ which satisfies the following:

- (i) $q(ax) = a^2q(x); \forall a \in O_K, x \in M$, q is a non-degenerate quadratic form.
- (ii) The form $\beta_q = M \times M \rightarrow K/\sigma^{-1}$ defined by

$$\beta_q(x, y) = q(x + y) - q(x) - q(y)$$

is O_K -bilinear and symmetric.

β_q is non-degenerate which means

$$\begin{aligned} \beta_q(x, y) = 0 & \quad \text{if and only if} \quad x = 0 \quad \text{for all} \quad y \in M, \\ \beta_q(x, y) = 0 & \quad \text{if and only if} \quad y = 0 \quad \text{for all} \quad x \in M, \end{aligned}$$

where a finite quadratic O_K -module can be briefly shown as $\underline{M} = (M, q)$ with a pair (M, q) .

It is clear that β_q is also an O_K -balanced form on M . Therefore, we can give the following definition for $\underline{M}$ and $\underline{N}$.

Definition 2. Let $\underline{M} = (M, q)$ and $\underline{N} = (N, q')$ be finite quadratic O_K -modules with associated bilinear forms β_q and $\gamma_{q'}$ respectively. The form $\beta_q \otimes \gamma_{q'}$ defined by

$$\beta_q \otimes \gamma_{q'} : (M \otimes N) \times (M \otimes N) \rightarrow K/\sigma^{-1} \times K/\sigma^{-1}$$

$(x \otimes x', y \otimes y') \mapsto \beta_q(x, y) \gamma_{q'}(x', y')$ is bilinear and symmetric. Since the form $\underline{M} \otimes \underline{N} = (M \otimes N, q \otimes q')$ is finite quadratic O_K -module associated with O_K -bilinear symmetric form $\beta_q \otimes \gamma_{q'}$ then $\underline{M} \otimes \underline{N}$ is called "*tensor product*" of the finite quadratic O_K -modules $\underline{M} = (M, q)$ and $\underline{N} = (N, q')$.

From the definitions which are given in [10] and [5, p. 44] for the modules over the commutative rings (with unit) we can give the following definition for the finite O_K -modules.

Definition 3. Let M be an O_K -module and U be a finite O_K -submodule of M . If M/U is finitely generated, then U is called *cofinite O_K -submodule* of M .

Definition 4. Let $\underline{M} = (M, q)$ be a finite quadratic O_K -module, U be a finite O_K -submodule of M and $0 \rightarrow U^\# \xrightarrow{f} M$ be an exact sequence of finite O_K -modules such that $M/f(U^\#)$ is finitely generated. If the sequence $0 \rightarrow U \otimes U^\# \xrightarrow{I_U \otimes f} U \otimes M$ is exact, then U is called *cofinitely flat quadratic O_K -module*, where $U^\#$ is a dual group of U and it is defined by $U^\# = \{y \in \underline{M} \mid B_q(U, y) = 0\}$. We notice that $U^\#$ is also an O_K -submodule of M .

Proposition 1 ([2]). $\underline{M} = (M, q)$ be a finite quadratic O_K -module associated with the bilinear form β_q and U be an O_K -submodule of $\underline{M}$. The application $x \mapsto \beta_q(x, \cdot)$ defines an exact sequence of O_K -modules:

$$0 \rightarrow U^\# \rightarrow \underline{M} \rightarrow \text{Hom}(U, K/\sigma^{-1}) \rightarrow 0.$$

Here $\text{Hom}(U, K/\sigma^{-1})$ denotes the group of O_K -module homomorphism of U into K/σ^{-1} . In particular, one has $|U| \cdot |U^\#| = |M|$ and $(U^\#)^\# = U$.

3. MAIN THEOREM

Theorem 1. Let $\underline{M}$ be a finite quadratic O_K -module and U be an O_K -submodule of $\underline{M}$. Let $i : T \rightarrow \underline{M}$ be an inclusion homomorphism where T is any cofinite O_K -submodule of $\underline{M}$. U is a cofinitely flat O_K -module if and only if $I_U \otimes i : U \otimes T \rightarrow U \otimes \underline{M}$ is injective.

Proof. $\Rightarrow$ Since U is a cofinitely flat O_K -module from the hypothesis of our theorem then it is clear that $U^\#$ is also a cofinitely flat O_K -module from Definition 4. Hence if we take $U^\#$ instead of T then the sequence $0 \rightarrow U \otimes U^\# \xrightarrow{I_U \otimes f} U \otimes \underline{M}$ is exact which gives the injectivity of $I_U \otimes i$.

$\Leftarrow$: Let U be a cofinitely flat O_K -module and $i : U^\# \rightarrow \underline{M}$ be inclusion map for $T = U^\#$ and let $I_U \otimes i : U \otimes U^\# \rightarrow U \otimes \underline{M}$ be injective. If we take the sequence $0 \rightarrow U \xrightarrow{f} \underline{M}$ of the finite O_K -modules such that $\underline{M}/f(U)$ is finitely generated then $\underline{M}/U^\#$ is finitely generated for $f(U) = U^\#$. If we get $h : U \rightarrow U^\#, y \mapsto h(y) = f(y) (\forall y \in U)$ then it is clear that h is a homomorphism of finite O_K -modules.

Since $f = i \circ h$ then we can write $I_U \otimes f = (I_U \circ I_U) \otimes (i \circ h) = (I_U \otimes i) \circ (I_U \otimes h)$ where $I_U \otimes h$ is isomorphism of finite O_K -modules from properties (3) and (4) in [10, p. 92].

Since $(I_U \otimes i)$ and $(I_U \otimes h)$ are injective then $I_U \otimes f$ is injective and so the sequence $0 \rightarrow U \otimes U^\# \xrightarrow{I_U \otimes f} U \otimes \underline{M}$ is exact. Therefore, U is a cofinitely flat O_K -module. $\square$

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LIE TRIPLE DERIVATION AND LIE BI-DERIVATION ON QUATERNION RINGS

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Abstract. In this study, we prove the existence of the central Lie bi-derivation for the ring with identity on the quaternion ring. We also describe the triple Lie derivation using the Jordan derivation on the aforementioned ring. An example is provided to demonstrate that our result is theoretically viable.

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1. INTRODUCTION

Recently, the study of differential operators on quaternion rings has increasingly piqued the interest of researchers. Numerous scholars have investigated the formation of Lie derivations on rings. An essential question arising in this context is; ‘On a quaternion ring what are the circumstances that allow the decomposition of a Lie derivation into the sum of a derivation and an additive central map that vanishes on all commutators of rings?’ A Lie derivation of certain primitive rings was originally established to be in standard form by Martindale [15]. Miers [16, 17] considered similar findings on von Neumann algebras. In [8], Hvala demonstrated that each generalized Lie derivation of a prime ring $\mathcal{W}$ whose characteristic is not 2 is the sum of a generalized derivation and an additive central map that vanishes on all commutators of $\mathcal{W}$. Liao and Liu [12] have extended Hvala’s finding to a prime algebra’s Lie ideal. In 2020, Ghahramani et al. [6] evaluated Lie derivations on quaternion rings and demonstrated that every Lie derivation is in standard form on acceptable quaternion rings. Again in 2021, Ghahramani et al. [7] demonstrated that every Jordan derivation of $\mathcal{W}$ is a derivation, and that every derivation of $\mathcal{W}$ decomposes into the sum of an inner derivation and a derivation induced by a derivation on $\mathcal{W}$.

On the other hand in 2021, Asif et al. [3] provided an in-depth specification of all Lie triple derivations of $A \otimes F$, where A is a finite dimensional commutative algebra

and L is a simple Lie algebra over an algebraically closed field. Inspired by the present ongoing contribution in this direction, it is unsurprising to discuss Lie triple derivation and Lie bi-derivation on quaternion rings and we prove that the existence of the central Lie bi-derivation on the quaternion ring. We also describe the triple Lie derivation using the Jordan derivation on the aforementioned ring. An example is provided to demonstrate that our result is theoretically viable.

Before presenting our results, let's recall some known definitions and related theories established by eminent ring theorist for the sake of completeness. Let $\mathcal{W}$ be a ring and $\mathcal{Z}(\mathcal{W})$ be the center of $\mathcal{W}$. For each $x, y \in \mathcal{W}$, denote the Lie product (or the commutator) of x, y by $[x, y] = xy - yx$, and the Jordan product of x, y by $x \circ y = xy + yx$. An additive map Ω on $\mathcal{W}$ is called as derivation if $\Omega(xy) = \Omega(x)y + x\Omega(y)$ is always true for every $x, y \in \mathcal{W}$. An additive map Ω on $\mathcal{W}$ is called as Lie derivation if $\Omega([x, y]) = [\Omega(x), y] + [x, \Omega(y)]$ holds for all $x, y \in \mathcal{W}$. An additive map Ω on $\mathcal{W}$ is called as Lie triple derivation if $\Omega([x, y], z) = [[\Omega(x), y], z] + [[x, \Omega(y)], z] + [[x, y], \Omega(z)]$ holds for all $x, y, z \in \mathcal{W}$. Furthermore, An additive map Ω on $\mathcal{W}$ is called as Jordan derivation if $\Omega(x \circ y) = \Omega(x) \circ y + x \circ \Omega(y)$ is always true for every $x, y \in \mathcal{W}$. Every derivation is usually both a Lie and Jordan derivation, and also every Lie derivation is a Lie triple derivation. However, the contrary arguments are not true. An essential expression $[[x, y], z] = x \circ (y \circ z) - y \circ (x \circ z)$ for every $x, y, z \in \mathcal{W}$ demonstrates that every Jordan derivation is also a Lie triple derivation. Hence, working on Lie triple derivations allows treating both the major classes, viz., Jordan and Lie derivations at the same time. Previously, numerous authors have contributed significantly to the related concerns (see [1, 2, 4, 5, 9, 10, 13, 14, 17, 18]) for the Lie triple derivation. In fact, usually the researchers obtained that a Lie triple derivation on a unital algebra is proper, meaning it could be represented as the sum of additive derivation and linear functional that vanishes on all Lie triple products.

A bilinear map $\delta : \mathcal{W} \times \mathcal{W} \rightarrow \mathcal{W}$ is said to be a bi-derivation (resp. Lie bi-derivation), if it is a derivation (resp. Lie derivation) with respect to both components, that means

$$\begin{aligned} \delta(xy, z) &= \delta(x, z)y + x\delta(y, z) \quad \text{and} \quad \delta(x, yz) = \delta(x, y)z + y\delta(x, z) \\ (\text{resp. } \delta([x, y], z) &= [\delta(x, z), y] + [x, \delta(y, z)] \quad \text{and} \quad \delta(x, [y, z]) = [\delta(x, y), z] + [y, \delta(x, z)]) \end{aligned}$$

for all $x, y, z \in \mathcal{W}$. If $\mathcal{W}$ is a noncommutative ring, then the map $\delta(x, y) = \lambda[x, y]$ for all $x, y \in \mathcal{W}$, where $\lambda \in \mathcal{Z}(\mathcal{W})$, is called an inner bi-derivation. In [7] authors studied the bi-derivation on quaternion rings. In continuation, we study the Lie bi-derivation on Quaternion rings in the present article.

Let $\mathcal{W}$ be a ring with identity. Consider

$$\mathcal{Q} = \mathcal{H}(\mathcal{W}) = \{s_0 + s_1i + s_2j + s_3k \mid s_0, s_1, s_2, s_3 \in \mathcal{W}\} = \mathcal{W} \oplus \mathcal{W}i \oplus \mathcal{W}j \oplus \mathcal{W}k,$$

where $i^2 = j^2 = k^2 = ijk = -1$, and $ij = -ji$. Upon componentwise addition and multiplication with respect to the given condition, and the conventions that i, j, k commute with $\mathcal{W}$ elementwise, $\mathcal{H}(\mathcal{W})$ is a ring called the quaternion ring over $\mathcal{W}$.

Generally $\mathcal{H}(\mathcal{W})$ is noncommutative except for $\mathcal{W}$ is commutative of $\text{char}(\mathcal{W}) = 2$ (see [6, Lemma 2.1]). It is significant to mention that the quaternion ring $Q = \mathcal{H}(\mathcal{W})$ turns out to be (isomorphic to) a 2×2 full matrix ring in some specific cases. Assuming that Q has elements with the properties $a^2 = f^2 = 0$ and $af + fa = 1$, it follows that Q is isomorphic to a 2×2 complete matrix ring according to [11, Theorem 17.10]. Assume, for instance, that $\mathcal{W}$ is an algebra over the field $\mathbb{Z}_p$, where p is an odd prime integer. According to a familiar result in number theory, there exists $x, y \in \mathbb{Z}_p$ such that $1 + x^2 + y^2 = 0$. Assume now that Q has $a = \frac{1}{2}(-yi + xj - kp)$, $f = \frac{1}{2}(-yi + xj + k)$. A simple calculation establishes that $a^2 = f^2 = 0$ and $af + fa = 1$. Thus, according to the aforementioned theorem, Q is a 2×2 matrix ring. In these circumstances, any characterization of a map on Q has an analogous specification when such map is examined on the corresponding matrix ring and vice versa.

2. LIE BI-DERIVATION

In this section, we start with some basic facts about Lie bi-derivations as follows.

Lemma 1. *Let $\delta : Q \times Q \rightarrow Q$ be a Lie bi-derivation on Q . Then δ satisfies*

$$[\delta(x, y), [u, v]] + [\delta(u, v), [x, y]] = [\delta(x, v), [u, y]] + [\delta(u, y), [x, v]] \text{ for all } x, y, u, v \in Q.$$

Proof. For any $x, y, u, v \in Q$, we have

$$\begin{aligned} \delta([x, u], [y, v]) &= [\delta(x, [y, v]), u] + [x, \delta(u, [y, v])] \\ &= [[\delta(x, y), v], u] + [[y, \delta(x, v)], u] + [x, [\delta(u, y), v]] + [x, [y, \delta(u, v)]]. \end{aligned}$$

On the other way,

$$\begin{aligned} \delta([x, u], [y, v]) &= [\delta([x, u], y), v] + [y, \delta([x, u], v)] \\ &= [[\delta(x, y), u], v] + [[x, \delta(u, y)], v] + [y, [\delta(x, v), u]] + [y, [x, \delta(u, v)]]. \end{aligned}$$

On comparing above two expressions, it is easy to obtain that

$$[\delta(x, y), [u, v]] - [[x, y], \delta(u, v)] = [\delta(x, v), [u, y]] - [[x, v], \delta(u, y)].$$

for all $x, y, u, v \in Q$. □

Theorem 1. *Let $\mathcal{W}$ be a ring with identity in which 2 is invertible and let $Q = \mathcal{H}(\mathcal{W})$. Then every Lie bi-derivation of Q is central.*

Proof. Let $X = x_1 + y_1i + z_1j + w_1k$ and $Y = x_2 + y_2i + z_2j + w_2k$ be arbitrary elements of Q and let $\delta : Q \times Q \rightarrow Q$ be a Lie bi-derivation. Since δ is biadditive, we have

$$\begin{aligned} \delta(X, Y) &= \delta(x_1, x_2) + \delta(x_1, y_2i) + \delta(x_1, z_2j) + \delta(x_1, w_2k) \\ &\quad + \delta(y_1i, x_2) + \delta(y_1i, y_2i) + \delta(y_1i, z_2j) + \delta(y_1i, w_2k) \\ &\quad + \delta(z_1j, x_2) + \delta(z_1j, y_2i) + \delta(z_1j, z_2j) + \delta(z_1j, w_2k) \end{aligned}$$

$$+ \delta(w_1 k, x_2) + \delta(w_1 k, y_2 i) + \delta(w_1 k, z_2 j) + \delta(w_1 k, w_2 k).$$

Now we describe all parts of the above expansion. For any $x \in Q$, we have

$$\begin{aligned}\delta(0, x) &= \delta([x, x], x) \\ &= [x, \delta(x, x)] + [\delta(x, x), x] = 0.\end{aligned}$$

Similarly, we can have

$$\delta(0, i) = \delta(0, j) = \delta(0, k) = 0 = \delta(i, 0) = \delta(j, 0) = \delta(k, 0) = \delta(x, 0). \quad (2.1)$$

In the similar manner,

$$\delta(0, xi) = \delta(0, xj) = \delta(0, xk) = 0 = \delta(xi, 0) = \delta(xj, 0) = \delta(xk, 0). \quad (2.2)$$

By Lemma 1, we know that

$$[\delta(x, y), [u, v]] + [\delta(u, v), [x, y]] = [\delta(x, v), [u, y]] + [\delta(u, y), [x, v]] \quad (2.3)$$

for all $x, y, u, v \in Q$.

Put $u = i = y$ & $v = j$ and suppose that $\delta(x, i) = a + bi + cj + dk$, we get

$$\begin{aligned}[\delta(x, i), [i, j]] + [\delta(i, j), [x, i]] &= [\delta(x, j), [i, i]] + [\delta(i, i), [x, j]] \\ [\delta(x, i), 2k] &= 0\end{aligned}$$

$$(a + bi + cj + dk)k - k(a + bi + cj + dk) = 0.$$

Hence $\delta(x, i) = a + dk$. Again, put $y = i, u = j$ & $v = k$ in (2.3), we get

$$\begin{aligned}[\delta(x, i), [j, k]] + [\delta(j, k), [x, i]] &= 0 \\ [\delta(x, i), 2i] &= 0 \\ (a + dk)i - i(a + dk) &= 0.\end{aligned}$$

This implies that $\delta(x, i) = a$.

In similar manner, $\delta(x, j) = \delta(x, k) = a = \delta(i, x) = \delta(j, x) = \delta(k, x)$. Also, suppose that $\delta(xi, y) = u + vi + wj + zk$, then we get

$$\begin{aligned}\delta(0, x) &= \delta([xi, i], y) \\ 0 &= [xi, \delta(i, y)] + [\delta(xi, y), i] \\ &= [xi, a] + [u + vi + wj + zk, i] \\ &= [x, a]i - 2wk + 2zj,\end{aligned}$$

which implies to $\delta(xi, y) = u + vi$ and $\delta(i, y) = a \in Z(\mathcal{W})$. Note that

$$\begin{aligned}\delta(0, j) &= \delta([x, i], j) \\ 0 &= [x, \delta(i, j)] + [\delta(x, j), i] \\ &= [x, \delta(i, j)]\end{aligned}$$

it follows that $\delta(i, j) \in Z(\mathcal{W})$. Also with $\delta(x, y) = u' + v'i + w'j + z'k$, we get

$$\delta(0, y) = \delta([i, x], y)$$

$$\begin{aligned} 0 &= [i, \delta(x, y)] + [\delta(i, y), x] \\ &= [i, u' + v'i + w'j + z'k] \end{aligned}$$

and hence $\delta(x, y) = u' + v'i$. Further we obtain

$$\begin{aligned} \delta(0, y) &= \delta([j, x], y) \\ 0 &= [j, \delta(x, y)] + [\delta(j, y), x] \\ &= [j, u' + v'i]. \end{aligned}$$

This implies $\delta(x, y) = u'$. From (2.3), we arrive at

$$\begin{aligned} [\delta(xi, y), [i, j]] + [\delta(i, j), [xi, y]] &= [\delta(xi, j), [i, y]] + [\delta(i, y), [xi, j]] \\ [\delta(xi, y), 2k] &= 0 \\ [u + vi, 2k] &= 0 \end{aligned}$$

and hence $\delta(xi, y) = u$. From (2.1), we have

$$\begin{aligned} \delta(0, y) &= \delta([x, xi], y) \\ 0 &= [x, \delta(xi, y)] + [\delta(x, y), xi] \\ &= [x, u] + [u', x]i \end{aligned}$$

it follows that $\delta(xi, y) = u \in \mathcal{Z}(\mathcal{W})$ and $\delta(x, y) \in \mathcal{Z}(\mathcal{W})$.

In the similar way we can find

$$\delta(x, yi), \delta(xj, y), \delta(x, yj), \delta(xk, y), \delta(x, yk) \in \mathcal{Z}(\mathcal{W}).$$

Also, from (2.2)

$$\begin{aligned} \delta(0, xi) &= \delta([y, i], xi) \\ 0 &= [y, \delta(i, xi)] + [\delta(y, xi), i] \\ &= [y, \delta(i, xi)] \end{aligned}$$

and hence $\delta(i, xi) \in \mathcal{Z}(\mathcal{W})$. Again, we see that

$$\begin{aligned} \delta(0, xi) &= \delta([y, j], xi) \\ 0 &= [y, \delta(j, xi)] + [\delta(y, xi), j] \\ &= [y, \delta(j, xi)] \end{aligned}$$

implies that $\delta(j, xi) \in \mathcal{Z}(\mathcal{W})$.

Assume that $\delta(xi, yi) = u + iv + wj + zk$, then we get

$$\begin{aligned} \delta(0, yi) &= \delta([i, xi], yi) \\ 0 &= [i, \delta(xi, yi)] + [\delta(i, yi), xi] \\ &= [i, \delta(xi, yi)] \end{aligned}$$

and it leads to $\delta(xi, yi) = u + iv$. Further, we have

$$\begin{aligned}\delta(0, yi) &= \delta([j, xi], yi) \\ 0 &= [j, \delta(xi, yi)] + [\delta(j, yi), xi] \\ &= [j, u + vi]\end{aligned}$$

implies that $\delta(xi, yi) = u$. Again, we have

$$\begin{aligned}\delta(0, yi) &= \delta([x, xi], yi) \\ 0 &= [x, \delta(xi, yi)] \\ &= [x, u],\end{aligned}$$

this implies $\delta(xi, yi) \in \mathcal{Z}(\mathcal{W})$.

Similarly, we have $\delta(xj, yj), \delta(xk, yk) \in \mathcal{Z}(\mathcal{W})$. With (2.2), we have

$$\begin{aligned}\delta(0, yj) &= \delta([x, xi], yj) \\ 0 &= [x, \delta(xi, yj)] + [\delta(x, yj), xi] \\ &= [x, u]\end{aligned}$$

and hence $\delta(xi, yj) \in \mathcal{Z}(\mathcal{W})$. Likewise

$$\delta(xi, yk), \delta(xj, yi), \delta(xj, yk), \delta(xk, yi), \delta(xk, yj) \in \mathcal{Z}(\mathcal{W}).$$

Thus, we see that $\delta(X, Y) \in \mathcal{Z}(\mathcal{W})$ and by [7, Lemma 2.4], we have $\delta(X, Y) \in \mathcal{Z}(Q)$. $\square$

3. LIE TRIPLE DERIVATION

In this section, we present one of the main results of this article as follows:

Theorem 2. *Let $f : Q \rightarrow Q$ be a Lie triple derivation. Then there exists an element A in Q , a Lie triple derivation α on $\mathcal{W}$ and a Jordan derivation β on $\mathcal{W}$ such that $f(t) = \alpha(x) + \beta(y)i + \beta(z)j + \beta(w)k + I_A(t)$ for every element $t = x + yi + zj + wk \in Q$.*

Proof. Let f be a Lie triple derivation of Q and assume $f(i) = a + bi + cj + dk$ and $f(j) = x + yi + zj + wk$, we find

$$\begin{aligned}f(j) &= \frac{1}{4}f([i, j], i) \\ &= \frac{1}{4}[[f(i), j], i] + \frac{1}{4}[[i, f(j)], i] + \frac{1}{4}[[i, j], f(i)] \\ &= \frac{1}{2}[bk - di, i] + \frac{1}{2}[zk - wj, i] + \frac{1}{2}[k, a + bi + cj + dk] \\ &= bj + zj + wk + bj - ci \\ &\implies x + yi + zj + wk = -ci + (2b + z)j + wk\end{aligned}$$

and hence $x = 0, y = -c, 2b = 0$, this leads to $f(i) = a + cj + dk$ and $f(j) = yi + zj + wk$. Again, we see that

$$\begin{aligned}
 f(i) &= \frac{1}{4}f([j, i], j) \\
 &= \frac{1}{4}[[f(j), i], j] + \frac{1}{4}[[j, f(i)], j] + \frac{1}{4}[[j, i], f(j)] \\
 &= \frac{1}{4}[[yi + zj + wk, i], j] + \frac{1}{4}[[j, a + cj + dk], j] + \frac{1}{4}[[j, i], yi + zj + wk] \\
 &= \frac{1}{2}[-zk + wj, j] + \frac{1}{2}[di, j] + \frac{1}{2}[yi + zj + wk, k] \\
 &\implies a + cj + dk = 2zi - yj + dk,
 \end{aligned}$$

this implies that $a = 0, 2z = 0, c = -y$ and hence $f(i) = cj + dk$ and $f(j) = -ci + wk$. In similar manner, we get $f(k) = -di - wj$. Let us suppose $f(1) = a + bi + cj + dk$

$$\begin{aligned}
 0 &= f([j, k], 1) \\
 &= [[f(j), k], 1] + [[j, f(k)], 1] + [[j, k], f(1)] \\
 &= [i, f(1)] \\
 &= ai + bi^2 + cij + dik - ai - bi^2 - cji - dki \\
 &= 2ck - 2dj
 \end{aligned}$$

$\implies d = 0, c = 0$ that's why $f(1) = a + bi$. Similarly with $[i, k], 1 = 0$, we have $b = 0$. Then $f(1) = a$.

Applying f on $[sj, k], 1 = 0$, we find that

$$\begin{aligned}
 0 &= f([sj, k], 1) \\
 &= [[f(sj), k], 1] + [[sj, f(k)], 1] + [[sj, k], f(1)] \\
 &\implies f(1) \in \mathcal{Z}(\mathcal{W}).
 \end{aligned}$$

Now let us take $f(si) = x + yi + zj + wk$ and using $f(i) = cj + dk, f(j) = -ci + bk$ and $f(k) = -di - bj$

$$\begin{aligned}
 0 &= f([si, i], j) \\
 &= [[f(si), i], j] + [[si, f(i)], j] + [[si, i], f(j)] \\
 &= [[x + yi + zj + wk, i], j] + [[si, cj + dk], j] \\
 &= [2wj - 2zk, j] + [(s \circ c)k - (s \circ d)j, j] \\
 &= 4zi - 2(s \circ c)i.
 \end{aligned}$$

Also, we have

$$\begin{aligned}
 0 &= f([si, i], k) \\
 &= [[f(si), i], k] + [[si, f(i)], k] + [[si, i], f(k)]
 \end{aligned}$$

$$\begin{aligned}
&= [[x + yi + zj + wk, i], k] + [[si, cj + dk], k] \\
&= [2wj - 2zk, k] + [(s \circ c)k - (s \circ d)j, k] \\
&= 4wi - 2(s \circ d)i.
\end{aligned}$$

Thus, we conclude that $z = \frac{1}{2}(s \circ c)$ and $w = \frac{1}{2}(s \circ d)$ and this leads to

$$f(si) = x + yi + \frac{1}{2}(s \circ c)j + \frac{1}{2}(s \circ d)k.$$

Now

$$\begin{aligned}
f(si) &= \frac{1}{4}f([[k, si], k]) \\
&= \frac{1}{4}[[f(k), si], k] + \frac{1}{4}[[k, f(si)], k] + \frac{1}{4}[[k, si], f(k)] \\
x + yi + \frac{1}{2}(s \circ c)j + \frac{1}{2}(s \circ d)k &= \frac{1}{4}[[-di - bj, si], k] + \frac{1}{4}[[k, x + yi \\
&\quad + \frac{1}{2}(s \circ c)j + \frac{1}{2}(s \circ d)k], k] + \frac{1}{4}[[k, si], -di - bj],
\end{aligned}$$

this implies $x = \frac{1}{2}I_b(s)$ and by taking $y = \beta(s)$, we obtain

$$f(si) = \frac{1}{2}I_b(s) + \beta(s)i + \frac{1}{2}(s \circ c)j + \frac{1}{2}(s \circ d)k, \quad (3.1)$$

where $\beta : \mathcal{W} \rightarrow \mathcal{W}$ is an additive map uniquely determined by f . Similarly, we can find that

$$f(sj) = -\frac{1}{2}I_d(s) - \frac{1}{2}(s \circ c)i + \beta(s)j + \frac{1}{2}(s \circ b)k \quad (3.2)$$

$$f(sk) = \frac{1}{2}I_c(s) - \frac{1}{2}(s \circ d)i - \frac{1}{2}(s \circ b)j + \beta(s)k. \quad (3.3)$$

For $s \in \mathcal{W}$ and let us take $f(s) = x + yi + zj + wk$. Now applying f on $[[i, j], s] = 0$, we find that $z = \frac{1}{2}I_d(s)$ and $w = -\frac{1}{2}I_c(s)$. Also with $[[k, i], s]$, we obtain that $y = -\frac{1}{2}I_a(s)$. Therefore,

$$f(s) = \alpha(s) - \frac{1}{2}I_a(s)i + \frac{1}{2}I_d(s)j - \frac{1}{2}I_c(s)k, \quad (3.4)$$

where $\alpha : \mathcal{W} \rightarrow \mathcal{W}$ is an additive map uniquely determined by f . Let $s_1, s_2 \in \mathcal{W}$. Replacing s with $[[s_1, s_2], s_3]$ in (3.4), we infer that α is a Lie triple derivation of $\mathcal{W}$. Moreover, applying f to the identities

$$\begin{aligned}
[[s_1j, k], s_2j] &= 2(s_1 \circ s_2)k \\
[[s_1j, k], s_2i] &= 2[s_2, s_1] \\
[[s_1j, k], s_2] &= 2[s_1, s_2]i
\end{aligned}$$

and using the foregoing calculations, we find, respectively, that

$$\begin{aligned}\beta(s_1 \circ s_2) &= \beta(s_1) \circ s_2 + s_1 \circ \beta(s_2) \\ \alpha([s_1, s_2]) &= [\beta(s_1), s_2] + [s_1, \beta(s_2)] \\ \beta([s_1, s_2]) &= [\alpha(s_1), s_2] + [s_1, \beta(s_2)].\end{aligned}$$

Now let $t = x + yi + zj + wk \in Q$ be an arbitrary element. Using (3.1), (3.2), (3.3) and (3.4), we find that

$$f(t) = \alpha(x) + \beta(y)i + \beta(z)j + \beta(w)k + h(t),$$

where

$$\begin{aligned}h(t) &= \frac{1}{2}(I_b(y) - I_d(z) + I_c(w)) + \frac{i}{2}(-I_b(x) - z \circ c - w \circ d) \\ &\quad + \frac{j}{2}(I_d(x) + y \circ c - w \circ b) + \frac{k}{2}(-I_c(x) + y \circ d + z \circ b).\end{aligned}$$

It is easily seen $h(t) = I_A(t)$, where

$$A = \frac{1}{2}(-bi + dj - ck) = \frac{1}{4}(f(i)i + f(j)j + f(k)k).$$

Consequently,

$$f(t) = \alpha(x) + \beta(y)i + \beta(z)j + \beta(w)k + I_A(t).$$

This proves the required result. $\square$

Here is an example describing how to format some very long formula:

Example 1. Let $\mathcal{W} = \mathbb{Z}_4$ and $Q = H(\mathbb{Z}_4)$. It is simple to confirm that the mapping $f : Q \rightarrow Q$ provided by $f(t) = 2t$ is a Lie triple derivation. We assert that f does not have a decomposition as in the theorem. Suppose that $f(t) = \alpha(x) + \beta(y)i + \beta(z)j + \beta(w)k + I_A(t)$ for all $t = x + yi + zj + wk \in Q$, where β is Jordan derivation of $\mathcal{W}$, α is Lie triple derivation of $\mathcal{W}$ and $A = a + bi + cj + dk$ is an element of Q . Obviously, $\beta = 0$. Furthermore, because α is additive, it is obvious that there exists an element $r \in \mathcal{W}$ provided by $\alpha(x) = rx$ for every $x \in \mathcal{W}$. So as a conclusion, for any $t = x + yi + zj + wk \in Q$, $2t = f(t) = rx + I_A(t)$. Assuming $t = i$, we get $2i = -2dj + 2ck$, a contradiction.

As a direct consequence of the above theorem, we can conclude

Corollary 1. *Suppose that the ring $\mathcal{W}$ is commutative. Then Lie triple derivation f is in standard form.*

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1D INVERSE AND D1 INVERSE OF SQUARE MATRICES

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Abstract. In this paper we introduce two new classes of inverses for square matrices, which are called 1Drazin (in short, 1D) inverse and Drazin1 (in short, D1) inverse. Next, we investigated the existence and uniqueness of 1D inverse and its dual D1 inverse. Some representation and characterizations of these inverses are derived. In addition to this, we obtain some properties of 1D and D1 inverses through idempotent and binary relations.

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1. INTRODUCTION

The generalized inverse of matrices and its applications was studied by Rao et. al (1971) in [22]. The Drazin inverse in ring theory was established in [9] and the applications of Drazin inverse in solving singular linear control systems were discussed in [3]. The authors of [2, 3] have pointed out the importance of Drazin inverse in matrix theory and computations. Various applications of Drazin inverses can be found in [1]. The Drazin and Moore-Penrose (DMP) inverse was introduced in [13]. An integral representation for the DMP inverse was established in [23]. The representation of DMP inverse for maximal classes of matrices was developed in [7]. In [24], the DMP inverse for an element of a ring with involution was discussed. The DMP inverse for a Hilbert space operator was established as an extension of the corresponding case of DMP on matrices in [19]. Further, the binary relation given by the DMP inverse in relation with some other partial orders are investigated in [5]. Its extension to finite potent endomorphisms on arbitrary vector spaces was studied in [21]. Some extensions of DMP inverse can be found in [19]. Several generalized

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inverses for arbitrary index and square matrix have been discussed in [13–15, 18]. In [17], generalized inverse for matrices of an arbitrary index has been discussed by considering the adjoint matrix. From projectors, 1MP and MP1 generalized inverses along with their induced partial orders were discussed in [10].

Motivated by the work of Hernández et al. [10], in this paper we introduce and study the 1D and D1 inverses. The main contributions of this paper are the following:

- Two new classes of generalized inverses, named 1D inverse and D1 inverse, are introduced. Existence and uniqueness of these inverses of matrices are established.
- We have studied some characterizations and representations of 1D inverse and D1 inverse for square matrices.
- Maximal classes and binary relations based on these inverses are introduced along with some results.

1.1. Preliminary considerations

In this section, we present a few notations and definitions, which will be used in the subsequent sections. Throughout this paper, we denote the set of complex matrices of order $m \times n$ by $\mathbb{C}^{m \times n}$, while the range space, null space and the conjugate transpose of a matrix $A \in \mathbb{C}^{n \times n}$ are denoted, respectively, by $R(A)$, $N(A)$, and A^* . The smallest nonnegative integer k that satisfies $\text{rank}(A^k) = \text{rank}(A^{k+1})$ is called the index of the matrix $A \in \mathbb{C}^{n \times n}$ and denoted $\text{ind}(A)$. For a matrix $A \in \mathbb{C}^{m \times n}$, if a matrix X satisfies $AXA = A$ then X is called the generalized or inner inverse of A and it is denoted by A^- . Similarly, if a matrix X satisfies $XAX = X$, then X is called outer inverse of A and is denoted by $A^{(2)}$. If X is an outer inverse of $A \in \mathbb{C}^{m \times n}$ with $R(X) = T$ and $N(X) = S$ (for some T and S subspaces adequately prescribed), then we denote it by $A_{T,S}^{(2)}$. The outer inverses with a prescribed range and null-space are very important in Matrix Theory. It has been extensively used in solving the nonlinear equations [1, 20] as well as in statistics [8].

A matrix $X \in \mathbb{C}^{n \times m}$ is called the Moore-Penrose inverse of the matrix $A \in \mathbb{C}^{m \times n}$ if it satisfies $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$. It is denoted by $A^\dagger$. The 1MP inverse [10] of a rectangular matrix $A \in \mathbb{C}^{m \times n}$ is denoted as $A^{-,\dagger}$ and defined by $A^{-,\dagger} := A^-AA^\dagger$ for a fixed inner inverse A^- of A . Similarly, its dual called MP1 inverse, is defined as $A^{\dagger,-} := A^\dagger AA^-$ for some fixed inner inverse A^- of A . The Drazin inverse [6] was introduced for an element in associative rings and semigroups. Furthermore, it has been extended from square to rectangular matrices [4]. We now recall the definition of Drazin inverse for square matrices.

Definition 1 ([6]). Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. A matrix $X \in \mathbb{C}^{n \times n}$ is called the Drazin inverse of A if it satisfies

$$XA^{k+1} = A^k, XAX = X, AX = XA.$$

The Drazin inverse of A is denoted by A^D . The combination of Drazin (D) inverse and Moore-Penrose (MP) inverse is renamed as DMP inverse, which is recalled below.

Definition 2 ([13]). Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. A matrix $X \in \mathbb{C}^{n \times n}$ is called the DMP inverse of A if it satisfies $XAX = X$, $XA = AA^D$ and $A^k X = A^k A^\dagger$. Similarly, Y is called the dual DMP inverse of A if $YAY = Y$, $AY = AA^D$, $YA^k = A^\dagger A^k$.

The following results will be used in what follows.

Lemma 1 ([12], Theorem 2.2). Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$. Then, $(BA)^D = B((AB)^D)^2 A$.

2. 1D INVERSES

In this section, we introduce 1D inverse for square matrices. In addition, we discuss a few characterizations of these inverses along with its interconnection with other generalized inverses.

2.1. Existence and uniqueness of 1D inverses

Proposition 1. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and let A^- be a fixed inner inverse of A . Then $A^-AA^D \in \mathbb{C}^{n \times n}$ is the unique solution of the following system of matrix equations

$$XAX = X, \quad XA^k = A^-A^k \text{ and } AX = AA^D. \quad (2.1)$$

Proof. Let $X = A^-AA^D$. Then, we have

$$XAX = A^-AA^DAA^-AA^D = A^-AA^DAA^D = A^-AA^D = X,$$

$$XA^k = A^-AA^DA^k = A^-A^DA^{k+1} = A^-A^k,$$

and $AX = AA^-AA^D = AA^D$. Thus X is a solution of (2.1). Next, we will show the uniqueness of X . Suppose there is another solution, say Y , which satisfies equation (2.1). Now,

$$Y = YAY = YAA^D = YA^k(A^D)^k = A^-A^k(A^D)^k = A^-AA^D = X.$$

□

In view of Proposition 1, we define the following representation of 1D inverse for square matrices.

Definition 3. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and let A^- be a fixed inner inverse of A . A matrix $X \in \mathbb{C}^{n \times n}$ is called the 1D inverse of A if it satisfies

$$XAX = X, \quad XA^k = A^-A^k \text{ and } AX = AA^D.$$

We denote the 1D inverse of A by $A^{-,D}$. Clearly, $A^{-,D} = A^-AA^D$.

Remark 1. It is important to observe that every fixed inner inverse A^- of A may produce a different 1D inverse of A . From now on, when we refer to the 1D inverse of A , we are assuming that an inner A^- of A has been previously fixed and will be not be explicitly indicated.

Example 1. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We can show that $\text{ind}(A) = 2$ and $A^D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Now, if we fix $A^- = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, we get $A^-AA^D = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

On the other hand one can verify that the Moore-Penrose Drazin (MPD) inverse of A , that is, $A^{\dagger,D} = A^{\dagger}AA^D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is different from the previous one.

If we consider another inner inverse of A , say $A^- = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, then we obtain

$A^-AA^D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq A^-AA^D$. Clearly, this example shows that the 1D inverse $A^{-,D} = A^-AA^D$ is different from the MPD inverse $A^{\dagger,D}$, and it depends on the selected A^- for its computation.

The characterization of 1D inverse through outer inverse with a prescribed range and null space is presented below.

Theorem 1. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^- be a fixed inner inverse of A . Then the following statements are valid.

- (i) $AA^{-,D}$ is a projector onto $R(A^k)$ along $N(A^k)$.
- (ii) $A^{-,D} = A_{R(A^-A^k), N(A^k)}^{(2)}$.

Proof. (i) From Definition 3, we obtain

$$(AA^{-,D})^2 = A(A^{-,D}AA^{-,D}) = AA^{-,D}$$

and

$$AA^{-,D} = AA^D. \quad (2.2)$$

Using equation (2.2), we get $R(AA^{-,D}) = R(AA^D) = R(A^k)$ and $N(AA^{-,D}) = N(AA^D) = N(A^k)$.

(ii) Clearly $A^{-,D}AA^{-,D} = A^{-,D}$. From the expressions

$$A^{-,D} = A^{-}AA^D = A^{-}A^k(A^D)^k, \text{ and}$$

$$A^{-}A^k = A^{-}A^{k+1}A^D = A^{-}A^kA^DA = A^{-}A^k(A^D)^kA^k = A^{-}AA^DA^k = A^{-,D}A^k,$$

we obtain $R(A^{-,D}) = R(A^{-}A^k)$. Using the proved in (i), we can verify the null space from the following expression:

$$N(A^{-,D}) = N(AA^{-,D}) = N(AA^D) = N(A^k).$$

□

2.2. Characterizations of 1D inverses

Next, we discuss a few characterizations for a matrix to be a 1D inverse from a geometrical point of view.

Theorem 2. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^{-} be a fixed inner inverse of A . Then the following statements are equivalent:

- (i) $X = A^{-,D}$.
- (ii) $XA^k = A^{-}A^k$ and $R(X^*) = R((A^k)^*)$.
- (iii) $R(X^*) = R((A^k)^*)$, $N(X^*) = N((A^{-}A^k)^*)$ and $A^DX = (A^D)^2$.
- (iv) $A^{k+1}X = A^k$ and $N(X^*) = N((A^{-}A^k)^*)$.
- (v) $A^DAX = A^D$, $R(X^*) = R((A^k)^*)$ and $XA^k = A^{-}A^k$.
- (vi) $A^DAX = A^D$ and $N(X^*) = N((A^{-}A^k)^*)$.
- (vii) $XA^k = A^{-}A^k$ and $N(X) = N(A^k)$.

Proof. (i) $\Rightarrow$ (ii) Let $X = A^{-,D}$. Then $XA^k = A^{-}A^k$ holds by Definition 3. Moreover,

$$\begin{aligned} X^* &= (A^{-}AA^D)^* = (A^{-}(A^D)^kA^k)^* = (A^k)^*((A^D)^k)^*(A^{-})^*, \text{ and} \\ (A^k)^* &= (A^{k+1}A^D)^* = (A^kAA^D)^* = (A^kAX)^* = X^*A^*(A^k)^*. \end{aligned}$$

Thus, $R(X^*) = R((A^k)^*)$.

(ii) $\Rightarrow$ (i) From $R(X^*) = R((A^k)^*)$, we have $X = ZA^k$ for some $Z \in \mathbb{C}^{n \times n}$. Now, using (ii),

$$X = ZA^k = ZA^{k+1}A^D = (ZA^k)(AA^D) = XA^k(A^D)^k = A^{-}A^k(A^D)^k = A^{-}AA^D = A^{-,D}.$$

(i) $\Rightarrow$ (iii) Let $X = A^{-,D}$. The equality $R(X^*) = R((A^k)^*)$ can be shown as before.

On the other hand, $X^* = (A^{-}AA^D)^* = (A^{-}A^k(A^D)^k)^* = ((A^D)^k)^*(A^{-}A^k)^*$ and $(A^{-}A^k)^* = (A^{-}AA^DA^k)^* = (XA^k)^* = (A^k)^*X^*$. Thus, $N(X^*) = N((A^{-}A^k)^*)$. Finally, $A^DX = A^DA^{-}AA^D = A^DAA^DA^{-}AA^D = (A^D)^2AA^D = (A^D)^2$.

(iii) $\Rightarrow$ (ii) It is sufficient to show $A^{-}A^k = XA^k$. From (iii), we assume that $R(X^*) = R((A^k)^*)$. Then, there exists $Z \in \mathbb{C}^{n \times n}$ such that $X = ZA^k = (ZA^k)(AA^D) = XA^2(A^D)^2 = XA^2A^DX$. Hence, by (iii), $R(I - XA^2A^D)^* \subseteq N(X^*) = N((A^{-}A^k)^*)$, which implies $(I - XA^2A^D)(A^{-}A^k) = O$ and hence $A^{-}A^k = XA^2A^DA^{-}A^k = XAA^D(AA^{-}A)A^{k-1} = XA^k$. Observe that this last equality is valid for any $k \geq 0$.

(iii) $\Rightarrow$ (iv) Pre-multiplying $A^DX = (A^D)^2$ by A^{k+2} , it follows that $A^{k+1}X =$

$AA^{k+1}A^D X = A^{k+2}A^D X = A^{k+2}(A^D)^2 = A^k$. The other condition is evident.

(iv) $\Rightarrow$ (iii) Let $A^{k+1}X = A^k$. Then $R((A^k)^*) \subseteq R(X^*)$ and

$$(A^D)^2 = (A^D)^3 A = (A^D)^{k+2} A^k = (A^D)^{k+2} A^{k+1} X = A^D ((A^D)^{k+1} A^{k+1}) X = A^D X.$$

From $(A^- A^k)^* = (A^k)^* (A^-)^*$, we have $R((A^- A^k)^*) \subseteq R((A^k)^*) \subseteq R(X^*)$. Using that $N(X^*) = N((A^- A^k)^*)$ and the dimension theorem, we have $\dim(R(X^*)) = \dim(R((A^- A^k)^*))$. Hence, we arrive at $R((A^k)^*) = R((A^- A^k)^*) = R(X^*)$.

(i) $\Rightarrow$ (v) It is clear that $A^D A X = A^D (A A^{-,D}) = A^D A A^D = A^D$.

(v) $\Rightarrow$ (ii) It is trivial.

(iv) $\Rightarrow$ (vi) Let $A^{k+1}X = A^k$. Then $A^D = (A^D)^2 A = (A^D)^{k+1} A^k = (A^D)^{k+1} A^{k+1} X = A^D A X$.

(vi) $\Rightarrow$ (iv) Let $A^D = A^D A X$. Then $A^k = A^{k+1} A^D = A^{k+1} A^D A X = A^{k+1} X$.

(ii) $\Leftrightarrow$ (vii) It is trivial since $R(X^*) = N(X)^\perp$. Hence, the proof is complete. $\square$

From an algebraic and geometrical point of view we have the following characterizations.

Theorem 3. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^- be a fixed inner inverse of A . Then the following statements are equivalent:

- (i) $X = A^{-,D}$.
- (ii) $XA = A^- A^D A^2$ and $N(A^k) \subseteq N(X)$.
- (iii) $AXA = AA^D A$, $N(A^k) \subseteq N(X)$ and $R(X) \subseteq R(A^- A^k)$.
- (iv) $AX = AA^D$ and $R(X) \subseteq R(A^- A^k)$.
- (v) $X^2 A = X$ and $XA^k = A^- A^k$.
- (vi) $X^2 A = X$ and $XA = A^- A^D A^2$.

Proof. (i) $\Rightarrow$ (ii) Let $X = A^{-,D}$. Then $XA = A^- A A^D A = A^- A^D A^2$ and the null space inclusion $N(A^k) \subseteq N(X)$ follows from Theorem 2.

(ii) $\Rightarrow$ (i) From $N(A^k) \subseteq N(X)$, we have that $X = ZA^k$ for some $Z \in \mathbb{C}^{n \times n}$. Now, from $XA = A^- A^D A^2$ we get

$$X = ZA^k = ZA^{k+1} A^D = (ZA^k) A A^D = X A A^D = A^- A^D A^2 A^D = A^- A A^D = A^{-,D}.$$

(ii) $\Rightarrow$ (iii) Clearly $AXA = AA^D A$. From the equivalence of (ii) and (i), we have $X = A^- A A^D$. Thus $R(X) \subseteq R(A^- A^k)$ follows from $X = A^- A A^D = A^- A^k (A^D)^k$.

(iii) $\Rightarrow$ (ii) For $k \geq 1$, let $R(X) \subseteq R(A^- A^k)$. Then $X = A^- A^k Z$, for some $Z \in \mathbb{C}^{n \times n}$. Now, from $A^- A^k = A^- A A^{k-1} = A^- (A A^- A) A^{k-1} = A^- A A^- A^k$, we get

$$XA = A^- A^k Z A = A^- A A^- A^k Z A = A^- A X A = A^- A A^D A = A^- A^D A^2. \quad (2.3)$$

If k is either 0, then the result is trivially true.

(iii) $\Rightarrow$ (iv) It is trivially true.

(iv) $\Rightarrow$ (iii) From $R(X) \subseteq R(A^- A^k)$, we have $X = A^- A^k Z$ for some $Z \in \mathbb{C}^{n \times n}$. For $k \geq 1$, we obtain $X = A^- A A^- A^k Z = A^- A X = A^- A A^D = A^- (A^D)^k A^k$. Thus $N(A^k) \subseteq$

$N(X)$. It can be proved similarly for $k = 0$. The other conditions are trivial.

(i) $\Rightarrow$ (v) Let $X = A^{-,D}$. By definition, we obtain $XA^k = A^{-,D}A^k$. Then,

$$X^2A = (A^{-,D})^2A = A^{-}AA^DA^{-}AA^DA = A^{-}AA^D = X.$$

(v) $\Rightarrow$ (i) From $X^* = (X^2A)^* = (X^3A^2)^* = \dots = (X^{k+1}A^k)^* = (A^k)^*(X^{k+1})^*$, we obtain $R(X^*) \subseteq R((A^k)^*)$. By using $R(X^*) \subseteq R((A^k)^*)$ and $XA^k = A^{-,D}A^k$ we verify that

$$X = XA^k(A^D)^k = A^{-,D}A^k(A^D)^k = A^{-}AA^D = A^{-,D}.$$

(v) $\Rightarrow$ (vi) It follow from the equivalence of (v) $\Leftrightarrow$ (i).

(vi) $\Rightarrow$ (v) Let $XA = A^{-,D}A^2$. Then $XA^k = A^{-,D}A^{k+1} = A^{-,D}A^k$.

Now, the implication's chain is closed as (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) and (i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi). $\square$

The 1D inverse can be obtained from the Drazin inverse as given in the next theorem.

Theorem 4. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^{-} be a fixed inner inverse of A . Then $A^{-,D} = (A^{-}A^2)^D$.

Proof. In the view of the Lemma 1, we have

$$\begin{aligned} (A^{-}A^2)^D &= (A^{-}AA)^D = A^{-}A[(AA^{-}A)^D]^2A = A^{-}A(A^D)^2A = A^{-}AA^DA^DA^D \\ &= A^{-}AA^D = A^{-,D}. \end{aligned}$$

$\square$

Proposition 2. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^{-} be a fixed inner inverse of A . Then $A^{-,D} = A^D$ if and only if $A^DA^k = A^{-}A^k$.

Proof. Only if part follows from $A^DA^k = A^{-,D}A^k = A^{-}AA^DA^k = A^{-}A^k$. Conversely, we have

$$A^{-,D} = A^{-}AA^D = A^{-}A^k(A^D)^k = A^DA^k(A^D)^k = A^DAA^D = A^D.$$

$\square$

We now discuss the idempotent property for 1D inverses.

Proposition 3. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^{-} be a fixed inner inverse of A . Then

$$(i) \quad (A^{-,D})^2 = A^{-}A^D.$$

$$(ii) \quad A^{-,D} \text{ is idempotent if and only if } A^{-,D} = A^{-}A^D \text{ if and only if } A^{-,D} = A^{-,D}A.$$

Proof. (i)

$$(A^{-,D})^2 = A^{-,D}A^{-,D} = A^{-}AA^DA^{-}AA^D = A^{-}A^D(AA^{-}A)A^D = A^{-}(A^DA^DA^D) = A^{-}A^D.$$

(ii) The first equivalence is trivial from (i). Let's prove that $A^{-,D}$ is idempotent if and only if $A^{-,D} = A^{-,D}A$. In fact, let $A^{-,D}$ is idempotent. Then, by item (i), we have

$A^{-,D} = A^{-}A^D$. Now, $A^{-,D} = A^{-}AA^D = A^{-}A^DA = A^{-,D}A$. Conversely, let $A^{-,D} = A^{-,D}A$. Then

$$\begin{aligned} A^{-,D} &= A^{-,D}A = A^{-}AA^DA = A^{-}AA^DAA^DA = A^{-}AA^DAA^{-}AA^DA = A^{-,D}AA^{-,D}A \\ &= A^{-,D}A^{-,D}. \end{aligned}$$

□

Theorem 5. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^{-} be a fixed inner inverse of A . If $A^{-,D}$ is idempotent then:

- (i) $A^k = A^{k+1}$. Consequently, $A^DA^k = A^k$.
- (ii) $(A^{-,D})^k = (A^{-,D})^kA$. More generally, $(A^{-,D})^m = (A^{-,D})^mA$ for every $m \in \mathbb{N}$.
- (iii) $A^{-,D} = (A^{-,D})^mA^m$ for every $m \in \mathbb{N}$
- (iv) $A^kA^{-,D} = A^k$.

Proof. (i) Using Proposition 3 (i), we obtain

$$\begin{aligned} A^k &= A^{k+1}A^D = A^kAA^{-}AA^D = A^kAA^{-}A^DA = A^kAA^{-,D}A = A^kAA^{-}AA^DA \\ &= A^{k+1}A^DA = A^{k+1}. \end{aligned}$$

(ii) By Proposition 3 (ii), we get $(A^{-,D})^k = A^{-,D} = A^{-,D}A = (A^{-,D})^kA$.

(iii) Applying Proposition 3 (ii), repetitively, we have

$$\begin{aligned} A^{-,D} &= (A^{-,D})^m = (A^{-,D})^{m-1}A^{-,D} = (A^{-,D})^{m-1}A^{-,D}A = (A^{-,D})^{m-1}A^{-,D}A^2 \\ &= (A^{-,D})^{m-1}A^{-,D}A^3 = \dots = (A^{-,D})^mA^m. \end{aligned}$$

(iv) From part (i), $A^kA^{-,D} = A^{k+1}A^{-}AA^D = A^{k+1}A^D = A^k$. □

Next, we characterize 1D inverses by using matrix equalities.

Theorem 6. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^{-} be a fixed inner inverse of A . Then the following statements are equivalent:

- (i) $A^{-,D} = X$.
- (ii) $XAX = X$, $XA^k = A^{-}A^k$, $AXA = AA^DA$ and $AX = AA^D$.
- (iii) $A^{-}AX = X$, $XA^k = A^{-}A^k$ and $X = XAA^D$.
- (iv) $A^{-}AXAA^D = X$ and $AXA^k = A^k$.
- (v) $A^{-}AA^DA = XA$, $A^kX = A^kA^D$ and $X = XAA^D$.
- (vi) $AA^DAX = A^DA$, $XAA^DA = A^{-}AA^DA$, $XAA^DAX = X$, and $AA^DAXAA^DA = AA^DA$.

Proof. (i) $\Rightarrow$ (ii) Let $X = A^{-,D}$. In view of Proposition 1, it is sufficient to show $AXA = AA^DA$. Since $AXA = AA^{-,D}A = AA^{-}AA^DA = AA^DA$, the proof is complete.

(ii) $\Rightarrow$ (iii) It follows from $XAA^D = XAX = X$ and

$$A^{-}AX = A^{-}AA^D = A^{-}A^k(A^D)^k = XA^k(A^D)^k = XAA^D = X.$$

(iii) $\Rightarrow$ (iv) The implication is true since $A^-AX = X$ implies $AXA^k = A^k$ trivially and $A^-AXAA^D = XAA^D = X$.

(iv) $\Rightarrow$ (i) Assuming that $A^-AXAA^D = X$ and $AXA^k = A^k$, we get

$$X = A^-AXAA^D = A^-AXA^k(A^D)^k = A^-A^k(A^D)^k = A^-AA^D = A^{-,D}.$$

(i) $\Rightarrow$ (v) The first equality is trivial. The following identities hold from

$$XAA^D = A^-AA^DAA^D = A^-AA^D = X$$

and

$$A^kX = A^kA^{-,D} = A^kA^-AA^D = A^kA^D.$$

(v) $\Rightarrow$ (i) It follows from $X = XAA^D = A^-AA^DAA^D = A^-AA^D = A^{-,D}$

(i) $\Rightarrow$ (vi) We verify that $AA^DAX = AA^DAA^{-,D} = AA^DAA^-AA^D = AA^D$,

$$XAA^DA = A^{-,D}AA^DA = A^-AA^DAA^DA = A^-AA^DA,$$

$$XAA^DAX = A^{-,D}AA^DAA^{-,D} = A^-AA^DAA^DAA^-AA^D = A^-AA^D = A^{-,D} = X,$$

and

$$AA^DAXAA^DA = (AA^DAX)AA^DA = (AA^D)AA^DA = AA^DA.$$

(vi) $\Rightarrow$ (i) It is clear from $X = XAA^DAX = A^-AA^DAX = A^-AA^D = A^{-,D}$. $\square$

Next results shows that, among other things, commutative property can not be extended from Drazin inverse to 1D inverses, in general. This is an interesting difference between both classes of matrices. Furthermore, when index of A is 1, the expression $A^{-,D}A$ is the same projector as the one given by the inner inverse: A^-A .

Theorem 7. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^- be a fixed inner inverse of A . Then

- (i) $A^{-,D}A = A^-A$ if and only if $AA^DA = A$.
- (ii) $A^{-,D}A = A^DA$ if and only if $A^{-,D} = A^D$.
- (iii) $A^kA^-A^k = A^k$ if and only if $A^kA^{-,D}A^k = A^k$.
- (iv) $A^{-,D} = A^D$ if and only if $A^{-,D}A = AA^{-,D}$.

Proof. (i) Let $A^{-,D}A = A^-A$. Then $A = AA^-A = AA^{-,D}A = AA^-AA^DA = AA^DA$. Converse is trivial.

(ii) It follows directly from definition.

(iii) Let $A^kA^-A^k = A^k$. Then

$$A^k = A^kA^-A^k = A^kA^-A^DA^{k+1} = A^kA^-AA^DA^k = A^kA^{-,D}A^k.$$

Converse part follows from $A^kA^-A^k = A^kA^-A^DA^{k+1} = A^kA^-AA^DA^k = A^kA^{-,D}A^k = A^k$.

(iv) Let $A^{-,D} = A^D$. Then $A^{-,D}A = A^DA = AA^D = AA^-AA^D = AA^{-,D}$. Conversely, let $A^{-,D}A = AA^{-,D}$. By definition we have $A^{-,D}AA^{-,D} = A^{-,D}$. On the other hand, $A^{k+1}A^{-,D} = A^kAA^-AA^D = A^{k+1}A^D = A^k$. Hence, by uniqueness of Drazin inverse, $A^D = A^{-,D}$. $\square$

Theorem 5 (ii) can be refined as in the following Theorem.

Theorem 8. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$, let A^- be a fixed inner inverse of A and let $m \in \mathbb{N} \setminus \{1\}$. Then*

$$(A^{-,D})^m = \begin{cases} (A^-A^D)^{\frac{m}{2}} & \text{if } m \text{ is even.} \\ A^-(A^D)^{\frac{m+1}{2}} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. If m is even, then $m = 2t$ for some positive integer t . Now, using Proposition 3 (i),

$$(A^{-,D})^m = ((A^{-,D})^2)^t = (A^-A^D)^t = (A^-A^D)^{\frac{m}{2}}.$$

If m is odd, then $m = 2l + 1$ for some positive integer l . Again, by Proposition 3 (a),

$$\begin{aligned} (A^{-,D})^m &= ((A^{-,D})^2)^l A^{-,D} = (A^-A^D)^l A^-AA^D = (A^-A^D)^{l-1} A^-A^D A^-AA^D \\ &= (A^-A^D)^{l-1} A^-(A^D)^2 AA^-AA^D = (A^-A^D)^{l-1} A^-(A^D)^2 = \dots = A^-(A^D)^{l+1} \\ &= A^-(A^D)^{\frac{m+1}{2}}. \end{aligned}$$

□

We close this subsection by providing a canonical form for 1D inverses. Moreover, this result allows us to compare 1D inverse to Drazin inverse.

Theorem 9. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$, let A^- be a fixed inner inverse of A and let $A = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1}$ be a core-nilpotent decomposition of A with $C \in \mathbb{C}^{s \times s}$ nonsingular and $N \in \mathbb{C}^{(n-s) \times (n-s)}$ nilpotent. For an inner inverse of A of the form $A^- = P \begin{bmatrix} C^{-1} & Y \\ Z & T \end{bmatrix} P^{-1}$ (with $YN = 0$, $NZ = 0$ and $NTN = N$), the 1D inverse of A is given by*

$$A^{-,D} = P \begin{bmatrix} C^{-1} & 0 \\ Z & 0 \end{bmatrix} P^{-1} = A^D + P \begin{bmatrix} 0 & 0 \\ Z & 0 \end{bmatrix} P^{-1}.$$

Consequently, $A^{-,D} = A^D$ if and only if $Z = 0$.

Proof. It is a straightforward computation by calculating $A^{-,D} = A^-AA^D$. □

2.3. Maximal classes for 1D inverses

Maximal classes are an interesting topic in order to determine the whole class of all matrices that can be used instead of A^D in the expression $A^{-,D} = A^-AA^D$. Maximal classes for 1D inverses are investigated in the following result.

Theorem 10. *Let $X \in \mathbb{C}^{n \times n}$ and $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^- be a fixed inner inverse of A . Then the following statements are equivalent:*

- (i) $A^{-,D} = A^-AX$.
- (ii) $AA^D = AX$.

- (iii) $AA^DA = AXA$ and $N(A^k) = N(AX)$.
 (iv) $X = A^D + (I - A^-A)Z$ for any $Z \in \mathbb{C}^{n \times n}$.

Proof. (i) $\Rightarrow$ (ii) Since $A^{-,D} = A^-AX$, we get $AA^D = AA^-AA^D = AA^{-,D} = AA^-AX = AX$.

(ii) $\Rightarrow$ (iii) The null space equality $N(A^k) = N(AX)$ follows from

$$A^k = A^{k+1}A^D = A^kA^DA = A^kAX, \text{ and} \\
AX = AA^D = (A^D)^kA^k.$$

(iii) $\Rightarrow$ (i) From $N(A^k) = N(AX)$, we obtain $AX = ZA^k$ for some $Z \in \mathbb{C}^{n \times n}$.

Pre-multiplying by A^- , we have

$$A^-AX = A^-ZA^k = A^-ZA^{k+1}A^D = A^-ZA^kAA^D = A^-AXAA^D = A^-AA^DAA^D \\
= A^-AA^D = A^{-,D}.$$

(ii) $\Rightarrow$ (iv), The general solution of $AX = 0$ is given by $X = (I - A^-A)Z$ for some $Z \in \mathbb{C}^{n \times n}$. Since A^D is a particular solution of $AX = AA^D$, the general solution of $AX = AA^D$ is given by $A^D + (I - A^-A)Z$, for arbitrary $Z \in \mathbb{C}^{n \times n}$.

(iv) $\Rightarrow$ (i) Pre-multiplying both sides of the equality in (iv) by A^-A , we obtain $A^-AX = A^-AA^D + A^-AZ - A^-AA^-AZ = A^{-,D}$. $\square$

Theorem 11. Let $A, X \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^- be a fixed inner inverse of A . Then

- (i) $A^{-,D} = XAA^D$ if and only if $X = A^- + Z(I - AA^D)$ for some $Z \in \mathbb{C}^{n \times n}$.
 (ii) $A^{-,D} = XAY$ if and only if $X = A^- + Z(I - AA^D)$ and $Y = A^D + (I - A^-A)W$ for some $Y, W \in \mathbb{C}^{n \times n}$.

Proof. (i) If $X = A^- + Z(I - AA^D)$, then $XAA^D = A^{-,D}$. Conversely, let $A^{-,D} = XAA^D$. Clearly A^- is a particular solution of $XAA^D = A^{-,D}$. If Z is any solution of $XAA^D = 0$, then $ZAA^D = 0$. Thus we can express Z as $Z = Z - ZAA^D = Z(I - AA^D)$. Hence the general solution of $XAA^D = 0$ is given by $X = Z(I - AA^D)$. Consequently, $X = A^- + Z(I - AA^D)$, $Z \in \mathbb{C}^{n \times n}$, is the general solution of $XAA^D = A^{-,D}$.

(ii) It follows from part (i) and Theorem 10 (iv). $\square$

2.4. A binary relation based on 1D inverses

In view of 1MP partial order [10], we introduce the following binary relation for 1D inverse.

Definition 4. Let $A, B \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. We will say that A is below B under the relation $\leq^{-,D}$ if $A^{-,D}A = A^{-,D}B$ and $AA^{-,D} = BA^{-,D}$, and it is denoted by $A \leq^{-,D} B$.

Remark 2. Clearly, the relation $\leq^{-,D}$ is reflexive but is neither symmetric nor anti-symmetric.

The above remark is validated in the following example.

Example 2. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and $B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

We can verify that $A^D = 0 = A^{-,D}$ and $B^D = 0 = B^{-,D}$. This leads $A^{-,D}A = A^{-,D}B$ and $AA^{-,D} = BA^{-,D}$. Hence $A \leq^{-,D} B$. Similarly we can verify $B \leq^{-,D} A$ but $A \neq B$.

Next we find $B_1^D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and setting $B_1^- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$, we get $B_1^{-,D} = \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$. Clearly $A \leq^{-,D} B_1$ but $B_1 B_1^{-,D} \neq A B_1^{-,D}$.

Thus, the relation $\leq^{-,D}$ is not symmetric.

In general, the relation $\leq^{-,D}$ is not transitive as shown in the below example.

Example 3. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $A^- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ and $B^- = \begin{bmatrix} -0.5 & 1 & -0.5 \\ 0 & 0 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$. Then, we evaluate

$$A^D = A^\# = A, \quad A^{-,D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$B^D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B^{-,D} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we can verify that $A^{-,D}A = A^{-,D} = A^{-,D}B$, $AA^{-,D} = A = BA^{-,D}$, $B^{-,D}B = B^{-,D} = B^{-,D}C$, and $BB^{-,D} = B^D = CB^{-,D}$. Thus $A \leq^{-,D} B$, and $B \leq^{-,D} C$.

However, $A \not\leq^{-,D} C$ since

$$AA^{-,D} - CA^{-,D} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Next, we investigate conditions under which the binary relation is true. Observe that, by definition, $AA^{-,D} = BA^{-,D}$ if and only if $AA^D = BA^{-,D}$ for any $A, B \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$.

Proposition 4. Let $A, B \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^- be a fixed inner inverse of A . Then

- (i) $A^{-,D}A = A^{-,D}B$ if and only if $A^DA = A^DB$.
- (ii) $AA^{-,D} = BA^{-,D}$ if and only if $A^D = BA^-A^D$.

Proof.

- (i) Let $A^{-,D}A = A^{-,D}B$. Then $AA^DA = AA^-AA^DA = AA^{-,D}A = AA^{-,D}B = AA^DB$. Pre-multiplying by A^D both sides of the last equality we get the result. The converse is trivial.
- (ii) By the previous observation, it is sufficient to prove that $AA^D = BA^{-,D}$ if and only if $A^D = BA^-A^D$. Using the assumption $AA^D = BA^{-,D}$, we get

$$A^D = A^DAA^D = AA^DA^D = BA^{-,D}A^D = BA^-A^D.$$

Conversely, $AA^D = BA^-A^DA = BA^-AA^D = BA^{-,D}$. $\square$

Corollary 1. Let $A, B \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^- be a fixed inner inverse of A . Then the following statements are equivalent:

- (i) $A \leq^{-,D} B$.
- (ii) $AA^DA = BA^{-,D}A = AA^DB$.
- (iii) $AA^D = BA^{-,D}$ and $A^DA = A^DB$.

Proof. (i) $\Rightarrow$ (ii) Let $A \leq^{-,D} B$. Then $AA^DA = AA^-AA^DA = AA^{-,D}A = BA^{-,D}A$ and

$$AA^DB = AA^-AA^DB = AA^{-,D}B = AA^{-,D}A = AA^DA.$$

(ii) $\Rightarrow$ (iii) Let $AA^DA = BA^{-,D}A = AA^DB$. Then $AA^D = AA^DA A^D = BA^{-,D}AA^D = BA^{-,D}$ and $AA^D = AA^DA A^D = A^DAA^DA = A^DB$.

(iii) $\Rightarrow$ (i) It follows from Proposition 4 (i) and $BA^{-,D} = AA^D = AA^-AA^D = AA^{-,D}$. $\square$

Remark 3. On the class of $n \times n$ EP matrices we have that 1D inverses are the same as 1MP inverses (since $A^D = A^\dagger$). Hence, over that class, the 1D relation is a partial order.

Remark 4. Notice that the behaviour of this new binary relation seems to have similarities with the following comparison: if we change from $A^\dagger A = A^\dagger B$ and $AA^\dagger = BA^\dagger$ to $A^D A = A^D B$ and $AA^D = BA^D$, we loss properties and pass from a partial order to a preorder on $\mathbb{C}^{n \times n}$. We will complete this discussion further, and we will see that under certain conditions something similar occurs in this case.

The result given in Theorem 9 allows us to provide all elements B such that $A \leq^{-,d} B$ for a given matrix A .

Theorem 12. Let $A, B \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$, let A^- be a fixed inner inverse of A and let $A = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1}$ be a core-nilpotent decomposition of A with

$C \in \mathbb{C}^{s \times s}$ nonsingular and $N \in \mathbb{C}^{(n-s) \times (n-s)}$ nilpotent. For an inner inverse of A of the form $A^- = P \begin{bmatrix} C^{-1} & Y \\ Z & T \end{bmatrix} P^{-1}$ (with $YN = 0$, $NZ = 0$ and $NTN = N$), the following conditions are equivalent:

- (a) $A \leq^{-,D} B$.
- (b) $B = P \begin{bmatrix} C & 0 \\ -B_4 ZC & B_4 \end{bmatrix} P^{-1}$.

Proof. We consider the partition $B = P \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} P^{-1}$ according to the partition of A . By using the expression of $A^{-,D}$ given in Theorem 9, from $A^{-,D}A = A^{-,D}B$, some easy computations yield $B_1 = C$ and $B_2 = 0$. Similarly, from $AA^{-,D} = BA^{-,D}$ we get $B_3 = -CB_4Z$. Then, $B = P \begin{bmatrix} C & 0 \\ -B_4ZC & B_4 \end{bmatrix} P^{-1}$, with $B_4 \in \mathbb{C}^{(n-s) \times (n-s)}$. The converse is straightforward. $\square$

An interesting result to understand a little bit better this binary relation is to relate it to some known partial order.

Theorem 13. Let $A, B \in \mathbb{C}^{n \times n}$ be matrices of index at most 1 and let A^- be a fixed inner inverse of A . If $A \leq^{-,D} B$ then $A\# \leq B$, where $\# \leq$ is the left sharp partial order.

Proof. Suppose that $A \leq^{-,D} B$ for $A, B \in \mathbb{C}^{n \times n}$ being matrices of index at most 1. Then $A^{-,D}A = A^{-,D}B$ and $AA^{-,D} = BA^{-,D}$. Since matrix A has index at most 1, $A^D = A^\#$, where $A^\#$ denotes the group inverse of A (that is, $AA^\#A = A$, $A^\#AA^\#$ and $AA^\# = A^\#A$). Now, from $A^{-,D}A = A^{-,D}B$, we get $A^-AA^\#A = A^-AA^\#B$ and pre-multiplying it by $A^\#A^3$ we arrive at $A^2 = AB$. On the other hand, similarly from $AA^{-,D} = BA^{-,D}$, we get $A = BA^-A$, and then $R(A) \subseteq R(B)$. By [16, Definition 6.3.1] we have that $A\# \leq B$, that is, A is a predecessor of B under the left sharp partial order. $\square$

This last result leads us to believe that the binary relation $\leq^{-,D}$ can have a good behaviour for at most index matrices. However, the binary relation $\leq^{-,D}$ is not a partial order even on the set of matrices of index at most 1 as shown in the next example.

Example 4. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.
Then, $A^- = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$, $A^D = A$, $A^{-,D} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$, $B^- = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

$$B^D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & -3 & -2 \end{bmatrix}, B^{-,D} = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 3 & 2 \end{bmatrix}. \text{ Further, we calculate}$$

$$A^{-,D}A = A^{-,D} = A^{-,D}B, AA^{-,D} = A = BA^{-,D}, B^{-,D}B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} = B^{-,D}C$$

$$\text{and } BB^{-,D} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} = CB^{-,D}.$$

$$\text{However, } AA^{-,D} = A \neq \begin{bmatrix} 2/3 & 2/3 & 2/3 \\ -1/3 & -1/3 & -1/3 \\ 2/3 & 2/3 & 2/3 \end{bmatrix} = CA^{-,D}.$$

Hence, we conclude that $A \leq^{-,D} B, B \leq^{-,D} C$ but $A \not\leq^{-,D} C$.

After all these considerations, finally, we are able to present an interesting class of matrices over which transitivity for the relation $\leq^{-,D}$ hold. Consider

$$\mathcal{PO}_k^\ell = \left\{ A \in \mathbb{C}^{n \times n} : \text{ind}(A) = k, \|A^D\| \leq 1, \lim_{m \rightarrow \infty} \|A^m A^D - A^{-} A^m\| = 0 \right\}.$$

Theorem 14. Let $A, B \in \mathcal{PO}_k^\ell$ and let A^{-} be a fixed inner inverse of A . Then $\leq^{-,D}$ is a pre-order on $\mathcal{PO}_k^\ell$.

Proof. We are going to prove that $A \leq^{-,D} B$ if and only if $A \leq^D B$ for matrices in $\mathcal{PO}_k^\ell$.

Let $A \leq^{-,D} B$ and $m \geq k$. Then $AA^D = BA^{-}AA^D$ and $A^{-}AA^DA = A^{-}AA^DB$. Now,

$$A^DA = A^DAA^DA = A^DAA^{-}AA^DA = A^DAA^{-}AA^DB = A^DAA^DB = A^DB.$$

On the other hand,

$$\begin{aligned} AA^D - BA^D &= BA^{-}AA^D - BA^D = BA^{-}A^{m+1}(A^D)^{m+1} - BA^m(A^D)^{m+1} \\ &= B(A^{-}A^{m+1}A^D - A^mA^D)(A^D)^m = B(A^{-}A^m - A^mA^D)(A^D)^m \\ &= B(A^{-}A^m - A^DA^m)(A^D)^m. \end{aligned}$$

Therefore, $\|AA^D - BA^D\| \leq \|B\| \|A^{-}A^m - A^DA^m\| \|A^D\|^m \rightarrow 0$ as $m \rightarrow \infty$, and hence $A \leq^D B$.

Conversely, let $A \leq^D B$ and $m \geq k$. Then $A^DB = A^DA = AA^D = BA^D$. Further, $A^{-,D}A = A^{-}AA^DA = A^{-}AA^DB = A^{-,D}B$ and

$$\begin{aligned} AA^{-,D} - BA^{-,D} &= AA^{-}AA^D - BA^{-}AA^D = AA^D - BA^{-}AA^D \\ &= BA^D - BA^{-}AA^D = B(A^D)^{m+1}A^m - BA^{-}A^m(A^D)^m \\ &= B(A^mA^D - A^{-}A^m)(A^D)^m. \end{aligned}$$

Therefore, $\|AA^{-,D} - BA^{-,D}\| \leq \|B\| \|A^m A^D - A^{-} A^m\| \|(A^D)^m\| \rightarrow 0$ as $m \rightarrow \infty$. Hence, $A \leq^{-,D} B$. The fact that $\leq^D$ is a pre-order completes the proof. $\square$

3. D1 INVERSES

In this section, we will introduce the dual inverses (called D1 inverses) of 1D inverses of square matrices. The detailed results will be not included but only we shall present its definition and some comments since all of them can be directly obtained similarly to the previous ones.

Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and let A^{-} be a fixed inner inverse of A . Then $A^D A A^{-} \in \mathbb{C}^{n \times n}$ is the unique solution of the system of matrix equations $XAX = X$, $A^k X = A^k A^{-}$, and $XA = A^D A$. Such a matrix will be called the D1 inverse of A and denoted by $A^{D,-} = A^D A A^{-}$.

As well as for 1D inverses, it is easy to verify that the DMP inverse, GDMP inverse [11], and D1 inverse all are different in general. We recall that a matrix $X \in \mathbb{C}^{n \times n}$ is called a G-Drazin of $A \in \mathbb{C}^{n \times n}$ if $AXA = A$ and $A^k X = XA^k$. As before, it can be found examples where $A^{D,-} \neq A^{D,\dagger}$ and $A^{GD\dagger} = A^{GD} A A^{\dagger} \neq A^{D,-}$.

A dual result similar to Theorem 1, related to outer inverses with prescribed range and null space, can be stated. Moreover, Theorems 2, 3, and 4, corresponding to characterizations and representations, can be deduced for D1 inverses. Results related to the equality with the Drazin inverses and to the idempotent property for D1 inverses can be similarly obtained as in Propositions 2 and 3 and Theorem 5, respectively. New characterizations and new properties can be obtained similarly to those obtained in Theorems 6, 7, and 8 can be obtained also in this case. A canonical form for D1 inverses can be obtained as in Theorem 9. From those results we conclude: $A^{-,D} = A^D = A^{D,-} \Leftrightarrow AA^{-,D} = A^{-,D}A$ and $AA^{D,-} = A^{D,-}A$.

Now, a binary relation $A \leq^{D,-} B$ can be defined as $AA^{D,-} = BA^{D,-}$ and $A^{D,-}A = A^{D,-}B$.

A similar study may be done for this binary relation. For instance, if $A, B \in \mathbb{C}^{n \times n}$ are matrices of index at most 1 such that $A \leq^{D,-} B$ then $A \leq \# B$, where $\leq \#$ is the right sharp partial order.

Consider the matrix class

$$\mathcal{PO}_k^r = \left\{ A \in \mathbb{C}^{n \times n} : \text{ind}(A) = k, \|A^D\| \leq 1, \lim_{m \rightarrow \infty} \|A^m A^{-} - A^m A^D\| = 0 \right\}.$$

If $A, B \in \mathcal{PO}_k^r$ then the relation $\leq^{D,-}$ is a pre-order on $\mathcal{PO}_k^r$. Now, we combine both 1D and D1 inverses to relate them to the common inner inverse used in their definitions.

Theorem 15. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^{-} be a fixed inner inverse of A , both expressed as in (9). The following statements are valid.*

- (a) *If $A^{-}A = AA^{-}$ then $k \leq 1$ and $A^{-,D} = A^{D,-} = A^{\#}$.*
- (b) *If $A^{-,D} = A^{D,-}$ and $NT = TN$ (in decomposition given in (9)) then $A^{-}A = AA^{-}$ (and $k \leq 1$).*

3.1. An application: Solving linear systems of equations

This last subsection is about linear systems whose solution is given by the D1 inverse introduced before.

Theorem 16. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and A^- be a fixed inner inverse of A . For any $\ell \geq k$, the following equation $A^\ell x = A^\ell A^- b$, $b \in \mathbb{C}^n$, is consistent and its general solution is given by $x = A^{D,-} b + (I - A^D A)y$, for arbitrary $y \in \mathbb{C}^n$.

Proof. Let $x = A^{D,-} b + (I - A^D A)y$, with $y \in \mathbb{C}^n$. Then,

$$\begin{aligned} A^\ell x &= A^\ell (A^{D,-} b + (I - A^D A)y) = A^\ell (A^D A A^- b + (I - A^D A)y) = A^\ell (A^D)^\ell A^\ell A^- b \\ &= A^\ell A^- b. \end{aligned}$$

Therefore, x is a solution of $A^\ell x = A^\ell A^- b$.

Let z be any arbitrary solution of $A^\ell x = A^\ell A^- b$. Next, we will claim that $z = A^{D,-} b + (I - A^D A)y$ for some $y \in \mathbb{C}^n$. In fact, from $A^D A z = (A^D)^\ell A^\ell z = (A^D)^\ell A^\ell A^- b$, we obtain $z = A^{D,-} b + z - A^{D,-} b = A^{D,-} b + z - A^D A z = A^{D,-} b + (I - A^D A)z$. This completes the proof. $\square$

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THE DENSITY OF TUPLES RESTRICTED BY RELATIVELY r -PRIME CONDITIONS

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Abstract. In order to consider j -wise relative r -primality conditions that do not necessarily require all j -tuples of elements in a Dedekind domain to be relatively r -prime, we define the notion of j -wise relative r -primality with respect to a fixed j -uniform hypergraph H . This allows us to provide further generalisations to several results on natural densities not only for a ring of algebraic integers O , but also for the ring $\mathbb{F}_q[x]$.

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1. INTRODUCTION

In 1976, Benkoski proved that the natural density of the set of relatively r -prime m -tuples of positive integers (with $rm > 1$) equals $1/\zeta(rm)$, where ζ is the Riemann zeta function [1]. We note that an m -tuple of positive integers is relatively r -prime if their greatest common r th power divisor is equal to 1.

This acted as a culmination of the work of Mertens [8], Lehmer [7], and Gegenbauer [3]. Thereafter, Tóth [14, 15] and Hu [5] found the natural density of the set of j -wise relatively prime m -tuples of positive integers (where $j \leq m$). Extensions of these results have been made to ideals in a ring of algebraic integers O by Sittinger [11, 13] and subsequently to elements in a ring of algebraic integers as well by Micheli [2] and Sittinger [12]. Moreover, Morrison and Dong [9] as well as Guo, Hou, and Liu [4] gave analogous results for elements in $\mathbb{F}_q[x]$.

We can further generalise the notion of j -wise relative primality by considering relative primality conditions that require some but not all j -tuples to be relatively prime. A first step in this direction was investigated by Hu [6], who used graphs to notate which pairs of integers are to be relatively prime.

Definition 1. Let D be a Dedekind domain. Fix $r, m \in \mathbb{N}$. We say that $\beta_1, \dots, \beta_m \in D$ are **relatively r -prime** if $\mathfrak{p}^r \nmid \langle \beta_1, \dots, \beta_m \rangle$ for any prime ideal $\mathfrak{p} \subseteq D$.

In order to properly generalise the notion of G -wise relative primality, we use the concept of a j -uniform hypergraph H , in which any edge connects exactly j vertices.

Definition 2. Let D be a Dedekind domain. Fix $r, j, m \in \mathbb{N}$ where $j \leq m$, and let H be a simple undirected j -uniform hypergraph whose m vertices are $\beta_1, \dots, \beta_m \in D$. We say that $\beta_1, \dots, \beta_m \in D$ are **H -wise relatively r -prime** if any j adjacent vertices of H are relatively r -prime.

A few remarks are now in order. First, although we state the definitions in this generality, we are in particular interested in the cases of a ring of algebraic integers as well the polynomial rings $\mathbb{F}_q[x]$. Next, suppose we take $D = \mathbb{Z}$, $j = 2$, and $r = 1$. Then our hypergraph is a graph G , and Definition 2 reduces to m integers are G -wise relatively prime as defined in [6]. Moreover when $D = O$ and $H = K_m^{(j)}$, the complete j -uniform hypergraph on m vertices, this definition reduces to m elements being j -wise relatively r -prime as defined in [12].

Definition 3. Given a j -uniform hypergraph H , we say that a subset S of vertices from H is an **independent vertex set** if S does not contain any hyperedge of H . Moreover for any non-negative integer k , we let $i_k(H)$ denote the number of independent sets of k vertices in H .

We now state the main results of this article, starting with the algebraic integer case.

Theorem 1. Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$, and let K be an algebraic number field over $\mathbb{Q}$ with ring of integers O . Then, the density of the set of H -wise relatively r -prime ordered m -tuples of elements in O equals

$$\prod_{\mathfrak{p}} \left[\sum_{k=0}^m i_k(H) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\right)^{m-k} \left(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\right)^k \right],$$

where the product is over all nonzero prime ideals in O .

After setting up the pertinent notation in Section 2, we prove Theorem 1.

Since the arithmetic in the rings $\mathbb{Z}$ and $\mathbb{F}_q[x]$ have striking similarities (for further details, see [10]), we would expect that we can derive a H -wise relatively r -prime density statement for $\mathbb{F}_q[x]$. In Section 3, we state and prove an analogue of Theorem 1 for the function field case $\mathbb{F}_q[x]$.

Theorem 2. Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$. Then the density of the set of H -wise relatively r -prime ordered m -tuples of polynomials in $\mathbb{F}_q[x]$ equals

$$\prod_{f \text{ irred.}} \left[\sum_{k=0}^m i_k(H) \left(1 - \frac{1}{q^{r \deg f}}\right)^{m-k} \left(\frac{1}{q^{r \deg f}}\right)^k \right],$$

where it is understood that the product is over all monic irreducible polynomials in $\mathbb{F}_q[x]$.

Remark 1. By noting that $\mathfrak{N}(f) = |\mathbb{F}_q[x]/\langle f \rangle| = q^{\deg f}$, the analogy between this latter density statement and the one given in the algebraic number ring case is made clear.

2. DENSITY OF H -WISE RELATIVELY r -PRIME ELEMENTS IN O

Let K be an algebraic number field of degree n over $\mathbb{Q}$ with O as its ring of integers having integral basis $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$. As a way to generalise the notion of all positive integers less than or equal to some positive constant M , we define

$$O_{\mathcal{B}}[M] = \left\{ \sum_{i=1}^n c_i \alpha_i : c_i \in [-M, M) \cap \mathbb{Z} \right\}.$$

The goal of this section is to derive a H -wise relatively prime density statement in O by using the methods developed by [2] and [12]. First, we define a notion of density for a subset T of O^m that reduces to the classic notion of density over $\mathbb{Z}$ as follows.

Definition 4. Let $T \subseteq O^m$ and fix an integral basis $\mathcal{B}$ of O . The **upper and lower densities of T with respect to $\mathcal{B}$** are respectively defined as

$$\overline{\mathbb{D}}_{\mathcal{B}}(T) = \limsup_{M \rightarrow \infty} \frac{|T \cap O_{\mathcal{B}}[M]^m|}{|O_{\mathcal{B}}[M]^m|} \text{ and } \underline{\mathbb{D}}_{\mathcal{B}}(T) = \liminf_{M \rightarrow \infty} \frac{|T \cap O_{\mathcal{B}}[M]^m|}{|O_{\mathcal{B}}[M]^m|}.$$

If $\overline{\mathbb{D}}_{\mathcal{B}}(T) = \underline{\mathbb{D}}_{\mathcal{B}}(T)$, we say that its common value is called the **density of T with respect to $\mathcal{B}$** and denote this as $\mathbb{D}_{\mathcal{B}}(T)$. Whenever this density is independent of the chosen integral basis $\mathcal{B}$, we denote this density as $\mathbb{D}(T)$.

Although the manner in which we cover O could potentially depend on the choice of the given integral basis $\mathcal{B}$, it is a direct corollary to Theorem 1 that the density of the set of H -wise relatively r -prime elements in O is actually independent of the integral basis used.

For the remainder of this section, let S be a finite set of rational primes, and fix positive integers r, j, m such that $j \leq m$. Fix a j -uniform hypergraph H , and define E_S to be the set of m -tuples $z = (z_1, \dots, z_m)$ in O^m such that any ideal generated by j entries of z is H -wise relatively r -prime with respect to all $\mathfrak{p} \mid \langle p \rangle$ for each $p \in S$. That is, E_S consists of the H -wise relatively r -prime m -tuples of algebraic integers from O with respect to S .

In order to aid us in analysing E_S , let

$$\pi: O^m \rightarrow \left(\prod_{\substack{\mathfrak{p} \mid \langle p \rangle \\ p \in S}} O/\mathfrak{p}^r \right)^m$$

be the surjective homomorphism induced by the family of natural projections

$$\pi_{\mathfrak{p}^r}: O \rightarrow O/\mathfrak{p}^r \text{ for all } \mathfrak{p} \mid \langle p \rangle \text{ where } p \in S.$$

From the definition of H -wise relative r -primality of algebraic integers, we immediately deduce the following lemma.

Lemma 1. For a given prime ideal $\mathfrak{p} \mid \langle p \rangle$ where $p \in S$ and $k \in \{1, 2, \dots, m\}$, let $A_k^{(\mathfrak{p})}$ denote the set of elements in $(O/\mathfrak{p}^r)^m$ where exactly k of their m components are 0, and these k components form an independent vertex set in H . Then,

$$E_S = \pi^{-1} \left(\prod_{\substack{\mathfrak{p} \mid \langle p \rangle \\ p \in S}} \bigcup_{k=0}^m A_k^{(\mathfrak{p})} \right).$$

Proposition 1. Suppose that $\mathfrak{p}$ is a prime ideal in O that lies above a fixed rational prime p , and let $D_p = \sum_{\mathfrak{p} \mid \langle p \rangle} f_p$ where f_p denotes the inertial degree of $\mathfrak{p}$. If we fix $q \in \mathbb{N}$ and set $N = \prod_{p \in S} p^r$, then

$$|E_S \cap O_B[qN]^m| = (2q)^{mn} \prod_{\substack{\mathfrak{p} \mid \langle p \rangle \\ p \in S}} p^{rm(n-D_p)} \left[\sum_{k=0}^m i_k(H) (\mathfrak{N}(\mathfrak{p}^r) - 1)^{m-k} \mathfrak{N}(\mathfrak{p}^r)^k \right].$$

Proof. We first examine the map π . For brevity, we set $R_p = \prod_{\mathfrak{p} \mid \langle p \rangle} O/\mathfrak{p}^r$. Then we let π_N denote the reduction modulo N homomorphism, and $\Psi = (\Psi_p)_{p \in S}$ where $\Psi_p : (O/\langle p \rangle^r)^m \rightarrow R_p^m$ is the homomorphism induced by the projection maps $O/\langle p \rangle^r \rightarrow R_p$. Finally, let $\bar{\Psi}$ be its extension to $(O/\langle N \rangle)^m$ (by applying the Chinese Remainder Theorem to the primes in S). These maps are related to each other through the following diagram

$$\begin{array}{ccccc} O^m & \xrightarrow{\pi_N} & (O/\langle N \rangle)^m & \xrightarrow{\bar{\Psi}} & (\prod_{p \in S} R_p)^m \\ & & \cong \downarrow & & \downarrow = \\ & & (\prod_{p \in S} O/\langle p^r \rangle)^m & \xrightarrow{\Psi} & (\prod_{p \in S} R_p)^m \end{array}$$

and it follows that $\pi = \bar{\Psi} \circ \pi_N$.

To prove this proposition, we start by examining Ψ^{-1} . Since for each rational prime p the mapping $\Psi_p : (O/\langle p^r \rangle)^m \rightarrow R_p^m$ is a surjective free $\mathbb{Z}_{p^r}$ -module homomorphism, we have for all $y \in (\prod_{p \in S} R_p)^m$:

$$|\bar{\Psi}^{-1}(y)| = \prod_{p \in S} |\Psi_p^{-1}(y_p)| = \prod_{p \in S} |\ker \Psi_p| = \prod_{p \in S} p^{rm(n-D_p)}.$$

Next, we compute $|\pi_N^{-1}(z) \cap O_B[qN]^m|$. Given $\bar{z} = (\bar{z}_1, \dots, \bar{z}_m) \in (O/\langle N \rangle)^m$, observe that since $O/\langle N \rangle$ is a free $\mathbb{Z}_N$ -module with basis $\{\pi(\alpha_1), \dots, \pi(\alpha_n)\}$, there exist unique $c_t^j \in [0, N) \cap \mathbb{Z}$ such that

$$\bar{z}_j = \sum_{t=1}^n c_t^j \pi(\alpha_t).$$

Then for $z = (z_1, \dots, z_m) \in O^m$, it follows that $\pi_N(z) = \bar{z}$ if and only if

$$z_j = \sum_{t=1}^n (c_t^j + l_t^j N) \alpha_t$$

for some $l_t^j \in \mathbb{Z}$. Moreover, since we need $l_t^j \in [-q, q] \cap \mathbb{Z}$ for each pair of indices j and t , we deduce that

$$\left| \pi_N^{-1}(z) \cap O_{\mathcal{B}}[qN]^m \right| = (2q)^{mn}.$$

We are ready to compute $\left| E_S \cap O_{\mathcal{B}}[qN]^m \right|$. By the definition of $A_k^{(\mathfrak{p})}$, we have for any fixed k and $\mathfrak{p}$:

$$|A_k^{(\mathfrak{p})}| = i_k(H) (\mathfrak{N}(\mathfrak{p}^r) - 1)^{m-k} \mathfrak{N}(\mathfrak{p}^r)^k.$$

Since we know from the last lemma that $E_S = \pi^{-1}(J)$, where

$$J = \psi^{-1} \left(\prod_{\substack{\mathfrak{p} | \langle p \rangle \\ p \in S}} \bigcup_{k=0}^m A_k^{(\mathfrak{p})} \right),$$

it immediately follows that

$$|J| = \prod_{\substack{\mathfrak{p} | \langle p \rangle \\ p \in S}} p^{rm(n-D_p)} \sum_{k=0}^m i_k(H) (\mathfrak{N}(\mathfrak{p}^r) - 1)^{m-k} \mathfrak{N}(\mathfrak{p}^r)^k.$$

Therefore, we conclude that

$$\begin{aligned} \left| E_S \cap O_{\mathcal{B}}[qN]^m \right| &= (2q)^{mn} |J| \\ &= (2q)^{mn} \prod_{\substack{\mathfrak{p} | \langle p \rangle \\ p \in S}} p^{rm(n-D_p)} \left[\sum_{k=0}^m i_k(H) (\mathfrak{N}(\mathfrak{p}^r) - 1)^{m-k} \mathfrak{N}(\mathfrak{p}^r)^k \right], \end{aligned}$$

as desired. $\square$

We now compute the density of E_S .

Lemma 2. *Using the previous notation, we have for any integral basis $\mathcal{B}$ of O ,*

$$\mathbb{D}(E_S) = \mathbb{D}_{\mathcal{B}}(E_S) = \prod_{\substack{\mathfrak{p} | \langle p \rangle \\ p \in S}} \left[\sum_{k=0}^m i_k(H) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)} \right)^{m-k} \left(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)} \right)^k \right].$$

Proof. Define the sequence $\{a_j\}$ by $a_j = \frac{|E_S \cap O_{\mathcal{B}}[j]^m|}{|O_{\mathcal{B}}[j]^m|}$, and let D denote the value of the density in question.

First, we consider the subsequence $\{a_{qN}\}_{q \in \mathbb{N}}$, where $N = \prod_{p \in S} p^r$. We claim that this subsequence is constant. By the previous proposition along with the definitions for N and D_p ,

$$\begin{aligned} a_{qN} &= \frac{1}{(2qN)^{mn}} \left[(2q)^{mn} \cdot \prod_{\substack{\mathfrak{p} | \langle p \rangle \\ p \in S}} p^{rm(n-D_p)} \sum_{k=0}^m i_k(H) \left(\mathfrak{N}(\mathfrak{p}^r) - 1 \right)^{m-k} \mathfrak{N}(\mathfrak{p}^r)^k \right] \\ &= \prod_{\substack{\mathfrak{p} | \langle p \rangle \\ p \in S}} \left[\sum_{k=0}^m i_k(H) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)} \right)^{m-k} \left(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)} \right)^k \right]. \end{aligned}$$

Hence, $\{a_{qN}\}$ is a constant subsequence and converges to D .

Next, we show that $\{a_{c+qN}\}$ also converges to D for any $c \in \{1, 2, \dots, N-1\}$, we first find bounds for a_{c+qN} . To this end, note that

$$a_{qN} \left(\frac{2qN}{2c+2qN} \right)^{mn} \leq a_{c+qN} \leq a_{(q+1)N} \left(\frac{2(q+1)N}{2c+2qN} \right)^{mn}.$$

By letting $q \rightarrow \infty$ and applying the Squeeze Theorem, we conclude that $\{a_{c+qN}\}$ converges to D for any $c \in \{1, 2, \dots, N-1\}$. Finally, since $\{a_{c+qN}\}$ converges to D for any $c \in \{0, 1, \dots, N-1\}$, we conclude that $\{a_j\}$ converges to D . $\square$

Note that the density in Lemma 2 is independent of the integral basis $\mathcal{B}$ used. Now we are ready to establish the main theorem of this section. For convenience, we restate it here before proving it.

Theorem 3. Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$, and let K be an algebraic number field over $\mathbb{Q}$ with ring of integers $\mathcal{O}$. Then, the density of the set E consisting of H -wise relatively r -prime ordered m -tuples of elements in $\mathcal{O}$ equals

$$\prod_{\mathfrak{p}} \left[\sum_{k=0}^m i_k(H) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)} \right)^{m-k} \left(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)} \right)^k \right],$$

where the product is over all nonzero prime ideals in $\mathcal{O}$.

Proof. Fix $t \in \mathbb{N}$ and let S_t denote the set of the first t rational primes. For brevity, we write $E_t = E_{S_t}$. Since $E_t \supseteq E$,

$$\overline{\mathbb{D}}_{\mathcal{B}}(E) \leq \overline{\mathbb{D}}_{\mathcal{B}}(E_t) = \mathbb{D}(E).$$

Observe that the last equality is due to the existence of $\mathbb{D}(E)$. Letting $t \rightarrow \infty$,

$$\mathbb{D}_{\mathcal{B}}(E) \leq \prod_{\mathfrak{p}} \left[\sum_{k=0}^m i_k(H) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)} \right)^{m-k} \left(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)} \right)^k \right].$$

It remains to show the opposite inequality. Noting that $\mathbb{D}_{\mathcal{B}}(E_t) - \overline{\mathbb{D}}_{\mathcal{B}}(E_t \setminus E) \leq \mathbb{D}_{\mathcal{B}}(E)$, it suffices to show that $\lim_{t \rightarrow \infty} \overline{\mathbb{D}}_{\mathcal{B}}(E_t \setminus E) = 0$.

To this end, we introduce the following notation. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}$, p_t be the t^{th} rational prime, and M be a positive integer.

- (1) We write $\mathfrak{p} \succ M$ iff $\mathfrak{p}$ lies over a rational prime greater than M .
- (2) We write $M \succ \mathfrak{p}$ iff the rational prime lying under $\mathfrak{p}$ is less than M .

Using this notation, we can write

$$E_t \setminus E \subseteq \bigcup_{\mathfrak{p} \succ p_t} \left(\prod_{j=1}^m \mathfrak{p}^r \right) \subseteq \mathcal{O}^m,$$

where it is understood that $\prod_{j=1}^m \mathfrak{p}^r$ is the subset of $\mathcal{O}^m$ such that each entry of the m -tuple is an element of $\mathfrak{p}^r$. Then, we see that

$$(E_t \setminus E) \cap \mathcal{O}_{\mathcal{B}}[M]^m \subseteq \bigcup_{CM^n \succ \mathfrak{p} \succ p_t} \prod_{j=1}^m (\mathfrak{p}^r \cap \mathcal{O}_{\mathcal{B}}[M])$$

for some constant $C > 0$ dependent only on $\mathcal{B}$, and thus

$$\overline{\mathbb{D}}_{\mathcal{B}}(E_t \setminus E) \leq \limsup_{M \rightarrow \infty} \sum_{CM^n \succ \mathfrak{p} \succ p_t} |(\mathfrak{p}^r \cap \mathcal{O}_{\mathcal{B}}[M])^m| \cdot (2M)^{-mn}.$$

By [2, Proposition 13], there exist constants $c, d > 0$ independent of M and $\mathfrak{p}$ such that

$$|(\mathfrak{p}^r \cap \mathcal{O}_{\mathcal{B}}[M])^m| \leq \frac{(2M)^{mn}}{\mathfrak{N}(\mathfrak{p}^r)^m} + c \left(\frac{2M}{d \mathfrak{N}(\mathfrak{p}^r)^{1/n}} + 1 \right)^{mn-1}.$$

Using this bound along with the facts that $\mathfrak{N}(\mathfrak{p}) \geq p$ for every $\mathfrak{p}$ lying above a fixed rational prime p , and at most n prime ideals lie above a fixed rational prime, we obtain

$$\begin{aligned} \overline{\mathbb{D}}_{\mathcal{B}}(E_t \setminus E) &\leq \limsup_{M \rightarrow \infty} \sum_{CM^n \succ \mathfrak{p} \succ p_t} \left[\frac{1}{\mathfrak{N}(\mathfrak{p}^r)^m} + c \left(\frac{2M}{d \mathfrak{N}(\mathfrak{p}^r)^{1/n}} + 1 \right)^{mn-1} (2M)^{-mn} \right] \\ &\leq \limsup_{M \rightarrow \infty} \sum_{CM^n \succ \mathfrak{p} \succ p_t} \left[\frac{n}{p^{rm}} + cn \left(\frac{2M}{d p^{r/n}} + 1 \right)^{mn-1} (2M)^{-mn} \right]. \end{aligned}$$

It remains to show that the right side goes to 0 as $t \rightarrow \infty$. First, observe that for all sufficiently large M , we have $2M/dp^{r/n} > 1$ and thus

$$\left(\frac{2M}{d p^{r/n}} + 1 \right)^{mn-1} (2M)^{-mn} < \left(\frac{2}{d} \right)^{mn} \cdot \frac{1}{p^{rm}}.$$

Then, by writing $A = n + cn(2/d)^{mn}$ which is a constant independent of M and p , we deduce that

$$\overline{\mathbb{D}}_{\mathcal{B}}(E_t \setminus E) \leq \limsup_{M \rightarrow \infty} \sum_{CM^n \succ \mathfrak{p} \succ p_t} \frac{A}{p^{rm}} \leq \sum_{k=p_t}^{\infty} \frac{A}{k^{rm}}$$

for all sufficiently large M .

Finally since $\sum_{k=1}^{\infty} \frac{1}{k^{rm}}$ is convergent, we conclude that $\overline{\mathbb{D}}_{\mathcal{B}}(E_t \setminus E) = 0$. $\square$

To conclude this section, we now state a corollary that indicates how this main result provides a generalisation of the work from [12].

Corollary 1. Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$, and let K be an algebraic number field over $\mathbb{Q}$ with ring of integers $\mathcal{O}$. Then the density of the set of j -wise relatively r -prime ordered m -tuples of elements in $\mathcal{O}$ equals

$$\prod_{\mathfrak{p}} \left[\sum_{k=0}^{j-1} \binom{m}{k} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\right)^{m-k} \left(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\right)^k \right].$$

Proof. Take $H = K_m^{(j)}$ as the hypergraph, and observe that

$$i_k(H) = \begin{cases} \binom{m}{k} & \text{if } 0 \leq k \leq j-1 \\ 0 & \text{otherwise.} \end{cases}$$

Applying Theorem 3 immediately yields the desired result. $\square$

3. DENSITY OF H -WISE RELATIVELY r -PRIME ELEMENTS IN $\mathbb{F}_q[x]$

Let $\mathbb{F}_q[x]$ be the ring of polynomials over the finite field $\mathbb{F}_q$ where $q = p^k$ for some prime p and $k \in \mathbb{N}$. The goal of this section is to derive a H -wise density statement in $\mathbb{F}_q[x]$ by using methods developed in [4].

In order to define a suitable definition of density in $\mathbb{F}_q[x]$, we begin by giving an enumeration of the polynomials in $\mathbb{F}_q[x]$. Denoting the elements of $\mathbb{F}_q$ as $a_0 = 0, a_1, \dots, a_{q-1}$, let Σ be the set of all $(a_{d_0}, a_{d_1}, a_{d_2}, \dots)$ whose entries are in $\mathbb{F}_q$ and $d_i = 0$ for all sufficiently large i . Then since non-negative integers have a unique expansion base q , where q is a positive integer greater than 1, we have a bijection $\Phi: \Sigma \rightarrow \mathbb{Z}_{\geq 0}$ defined by

$$\Phi(a_{d_0}, a_{d_1}, \dots) = \sum_{i=0}^{\infty} d_i q^i.$$

Using this bijection, we define for each $j \in \mathbb{Z}_{\geq 0}$

$$f_j(x) = \sum_{i=0}^{\infty} a_{d_i} x^i, \text{ where } j = \Phi(a_{d_0}, a_{d_1}, \dots).$$

Note that $\mathbb{F}_q[x] = \{f_j(x) : j \in \mathbb{Z}_{\geq 0}\}$, thereby giving an ordering of the elements in $\mathbb{F}_q[x]$. Now, we are able to define a density in this ring.

Definition 5. Fix a positive integer $m \geq 2$, and let $\mathcal{M}_N$ be the subset of $(\mathbb{F}_q[x])^m$ consisting of m -tuples of elements in $\mathbb{F}_q[x]$ whose entries are taken from $\{f_0, f_1, \dots, f_N\}$. For any subset $T \subseteq (\mathbb{F}_q[x])^m$, we define the **upper and lower densities of T** are respectively defined as

$$\overline{\mathbb{D}}(T) = \limsup_{N \rightarrow \infty} \frac{|T \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \text{ and } \underline{\mathbb{D}}(T) = \liminf_{N \rightarrow \infty} \frac{|T \cap \mathcal{M}_N|}{|\mathcal{M}_N|}.$$

If $\overline{\mathbb{D}}(T) = \underline{\mathbb{D}}(T)$, we say that its common value is called the **density of T** and denote this as $\mathbb{D}(T)$.

Let S be a finite set of irreducible polynomials in $\mathbb{F}_q[x]$, and fix $r, j, m \in \mathbb{N}$ satisfying $j \leq m$. Fix a j -uniform hypergraph H , and let E_S denote the set of m -tuples of polynomials from $\mathbb{F}_q[x]$ that are H -wise relatively r -prime with respect to all irreducible polynomials in S .

For the following lemma and proposition, let

$$\pi: (\mathbb{F}_q[x])^m \rightarrow \left(\prod_{f \in S} \mathbb{F}_q[x] / \langle f^r \rangle \right)^m$$

be the surjective homomorphism induced by the family of natural projections

$$\pi_{f^r}: \mathbb{F}_q[x] \rightarrow \mathbb{F}_q[x] / \langle f^r \rangle \text{ for each } f \in S.$$

As in the algebraic integer case, the following lemma follows immediately from the definition of H -wise relative r -primality of elements in $\mathbb{F}_q[x]$.

Lemma 3. *For a given irreducible polynomial $f \in S$, let $A_k^{(f)}$ denote the set of elements in $(\mathbb{F}_q[x] / \langle f^r \rangle)^m$ where exactly k of their m components are 0, and these k components form an independent vertex set in H . Then,*

$$E_S = \pi^{-1} \left(\prod_{f \in S} \bigcup_{k=0}^m A_k^{(f)} \right).$$

Proposition 2. *Let $N = bq^{\deg F} - 1$ where $b \in \mathbb{N}$, and $F = \prod_{f \in S} f^r$. Then,*

$$|E_S \cap \mathcal{M}_N| = (bq^{\deg F})^m \prod_{f \in S} q^{-rm \deg f} \cdot \sum_{k=0}^m i_k(H) (q^{r \deg f} - 1)^{m-k} (q^{r \deg f})^k.$$

Proof. Let π_F denote the reduction modulo F homomorphism, and let

$$\psi: (\mathbb{F}_q[x] / \langle F \rangle)^m \rightarrow \left(\prod_{f \in S} (\mathbb{F}_q[x] / \langle f^r \rangle) \right)^m \rightarrow \prod_{f \in S} (\mathbb{F}_q[x] / \langle f^r \rangle)^m,$$

where the first part of ψ is induced by the Chinese Remainder Theorem and the second part is an obvious isomorphism of free $\mathbb{F}_q[x]$ -modules.

Now we compute $|\pi_F^{-1}(h(x)) \cap \mathcal{M}_N|$. By the Division Algorithm, we have that

$$\{f_l(x)\}_{l=0}^N = \{f_s(x) \cdot x^{\deg F} + f_t(x) \mid 0 \leq t \leq q^{\deg F} - 1 \text{ and } 0 \leq s \leq b - 1\}.$$

Then for any fixed $s \in \{0, 1, \dots, b - 1\}$, the map π_F restricted to

$$\{f_s(x) \cdot x^{\deg F} + f_t(x)\}_{t=0}^{q^{\deg F}-1} \rightarrow \mathbb{F}_q[x] / \langle F \rangle$$

is one-to-one. Since $|\ker(\pi_F)| = b^m$, we conclude that $|\pi_F^{-1}(h(x)) \cap \mathcal{M}_N| = b^m$.

We are now ready to compute $|E_S \cap \mathcal{M}_N|$. We know that $E_S = \pi^{-1}(J)$, where

$$J = \psi^{-1} \left(\prod_{f \in S} \bigcup_{k=0}^m A_k^{(f)} \right).$$

Since for any fixed $k \in \{0, 1, \dots, m\}$ and $f \in S$ we have

$$|A_k^{(f)}| = i_k(H)(q^{r \deg f} - 1)^{m-k}(q^{r \deg f})^k,$$

we deduce that

$$|J| = q^{m \deg F} \prod_{f \in S} q^{-rm \deg f} \cdot \sum_{k=0}^m i_k(H)(q^{r \deg f} - 1)^{m-k}(q^{r \deg f})^k.$$

Therefore,

$$\begin{aligned} |E_S \cap \mathcal{M}_N| &= b^m \cdot |J| \\ &= (bq^{\deg F})^m \prod_{f \in S} q^{-rm \deg f} \cdot \sum_{k=0}^m i_k(H)(q^{r \deg f} - 1)^{m-k}(q^{r \deg f})^k. \end{aligned}$$

□

We now find the density of E_S .

Lemma 4. *Using the notation from Proposition 2,*

$$\mathbb{D}(E_S) = \prod_{f \in S} \left[\sum_{k=0}^m i_k(H) \left(1 - \frac{1}{q^{r \deg f}}\right)^{m-k} \left(\frac{1}{q^{r \deg f}}\right)^k \right].$$

Proof. Let $a_j = \frac{|E_S \cap \mathcal{M}_j|}{|\mathcal{M}_j|}$ and let D be the value of the density in question. For notational brevity, we let $n = q^{\deg F}$.

We first consider the subsequence $\{a_{bn-1}\}_{b \in \mathbb{N}}$. By Proposition 2, we find that

$$\frac{|E_S \cap \mathcal{M}_{bn-1}|}{|\mathcal{M}_{bn-1}|} = \prod_{f \in S} \left[\sum_{k=0}^m i_k(H) \left(1 - \frac{1}{q^{r \deg f}}\right)^{m-k} \left(\frac{1}{q^{r \deg f}}\right)^k \right].$$

Hence, $\{a_{bn-1}\}$ trivially converges to D .

Next, we show $\{a_{bn+c}\}$ converges to D as well for each $c \in \{0, 1, \dots, n-2\}$. In a manner reminiscent of the proof to Lemma 4, we find that

$$\left(\frac{bn}{bn+c+1}\right)^m a_{bn-1} \leq a_{bn+c} \leq \left(\frac{(b+1)n}{(b+1)n+c+1}\right)^m a_{(b+1)n-1}.$$

Letting $b \rightarrow \infty$, the Squeeze Theorem implies that $\{a_{bn+c}\}$ converges to D for each $c \in \{0, 1, \dots, n-2\}$. Finally, since $\{a_{bn+c}\}$ converges to D for each $c \in \{0, 1, \dots, n-1\}$, we conclude that $\{a_j\}$ converges to D , as desired. □

Now we are ready to state and prove the main theorem of this section.

Theorem 4. *Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$. Then the density of the set of H -wise relatively r -prime ordered m -tuples of polynomials in $\mathbb{F}_q[x]$ equals*

$$\prod_{f \text{ irred.}} \left[\sum_{k=0}^m i_k(H) \left(1 - \frac{1}{q^{r \deg f}}\right)^{m-k} \left(\frac{1}{q^{r \deg f}}\right)^k \right],$$

where it is understood that the product is over all monic irreducible polynomials in $\mathbb{F}_q[x]$.

Proof. Fix a monic irreducible polynomial $f \in \mathbb{F}_q[x]$ and let K_f denote the set of ordered m -tuples $(g_1, \dots, g_m)$ such that f divides the gcd of k of the entries from $(g_1, \dots, g_m)$ whenever these k entries form an independent vertex set. Then by Lemma 4, we have

$$\mathbb{D}(K_f) = 1 - \sum_{k=0}^m i_k(H) \left(1 - \frac{1}{q^{r \deg f}}\right)^{m-k} \left(\frac{1}{q^{r \deg f}}\right)^k.$$

However for any $x \in [0, 1]$, Bernoulli's Inequality implies that

$$\begin{aligned} \sum_{k=0}^m i_k(H) x^k (1-x)^{m-k} &\geq (1-x)^m + mx(1-x)^{m-1} \\ &= (1-x)^{m-1} (1 + (m-1)x) \\ &\geq (1 - (m-1)x)(1 + (m-1)x) \\ &= 1 - (m-1)^2 x^2. \end{aligned}$$

Therefore, letting $x = q^{-\deg f}$ yields

$$\mathbb{D}(K_f) \leq \left(\frac{m-1}{q^{r \deg f}}\right)^2.$$

Next, let S_t be the set of monic irreducible polynomials of a degree greater or equal to t where $t \in \mathbb{N}$, and set $E_t = E_{S_t}$. Moreover, let $\hat{S}$ be the set of all monic irreducible polynomials in $\mathbb{F}_q[x]$. Then,

$$\begin{aligned} \overline{\mathbb{D}}(E_t \setminus E) &\leq \limsup_{N \rightarrow \infty} \frac{|\bigcup_{f \in \hat{S} \setminus S_t} K_f \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \\ &\leq \limsup_{N \rightarrow \infty} \frac{\sum_{f \in \hat{S} \setminus S_t} |K_f \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \\ &\leq \sum_{f \in \hat{S} \setminus S_t} \overline{\mathbb{D}}(K_f). \end{aligned}$$

Since $\overline{\mathbb{D}}(K_f) = \mathbb{D}(K_f)$, we obtain

$$\begin{aligned} \overline{\mathbb{D}}(E_t \setminus E) &\leq \sum_{f \in \hat{S} \setminus S_t} \mathbb{D}(K_f) \\ &\leq \sum_{f \in \hat{S} \setminus S_t} \left(\frac{m-1}{q^{r \deg f}}\right)^2 \\ &= \sum_{j=t+1}^{\infty} \frac{(m-1)^2}{q^{2rj}} \cdot \varphi(j), \end{aligned}$$

where $\varphi(j)$ denotes the number of monic irreducible polynomials of degree j in $\mathbb{F}_q[x]$.

Since any irreducible polynomial over $\mathbb{F}_q[x]$ with degree j divides $x^{q^j} - x$ (which has no multiple roots), we have $j \cdot \phi(j) \leq q^j$. Therefore

$$\overline{\mathbb{D}}(E_t \setminus E) \leq \sum_{j=t+1}^{\infty} \frac{(m-1)^2}{jq^{(2r-1)j}} \leq \frac{(m-1)^2}{q^t(q-1)},$$

in which the last inequality follows from

$$\begin{aligned} \sum_{j=t+1}^{\infty} \frac{1}{jq^{(2r-1)j}} &= \frac{1}{q^{(2r-1)(t+1)}} \cdot \sum_{j=0}^{\infty} \frac{1}{(j+t+1)q^{(2r-1)j}} \\ &\leq \frac{1}{q^{(2r-1)(t+1)}} \cdot \sum_{j=0}^{\infty} \frac{1}{q^{(2r-1)j}} \\ &\leq \frac{1}{q^t(q-1)}. \end{aligned}$$

Next, since $E \cap \mathcal{M}_N \subseteq E_t \cap \mathcal{M}_N$, it follows that

$$\overline{\mathbb{D}}(E) \leq \overline{\mathbb{D}}(E_t) \leq \mathbb{D}(E_t).$$

Similarly, since $E \cap \mathcal{M}_N = (E_t \cap \mathcal{M}_N) - ((E_t \setminus E) \cap \mathcal{M}_N)$, we obtain

$$\begin{aligned} \mathbb{D}(E) &\geq \mathbb{D}(E) - \overline{\mathbb{D}}(E \setminus E_t) \\ &\geq \mathbb{D}(E_t) - \frac{(m-1)^2}{q^t(q-1)}. \end{aligned}$$

Finally noting that $\mathbb{D}(E_t)$ exists, we conclude by letting $t \rightarrow \infty$ that

$$\begin{aligned} \mathbb{D}(E) &= \lim_{t \rightarrow \infty} \mathbb{D}(E_t) \\ &= \lim_{t \rightarrow \infty} \prod_{f \in S_t} \left[\sum_{k=0}^m i_k(H) \left(1 - \frac{1}{q^{r \deg f}}\right)^{m-k} \left(\frac{1}{q^{r \deg f}}\right)^k \right] \\ &= \prod_{f \text{ irred.}} \left[\sum_{k=0}^m i_k(H) \left(1 - \frac{1}{q^{r \deg f}}\right)^{m-k} \left(\frac{1}{q^{r \deg f}}\right)^k \right], \end{aligned}$$

and this concludes the proof. $\square$

In a manner reminiscent of the previous section, we conclude by giving without proof the analogue of Corollary 2 for $\mathbb{F}_q[x]$ as originally given in [4].

Corollary 2. *Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$. Then the density of the set of j -wise relatively r -prime ordered m -tuples of elements in $\mathbb{F}_q[x]$ equals*

$$\prod_{f \text{ irred.}} \left[\sum_{k=0}^{j-1} \binom{m}{k} \left(1 - \frac{1}{q^{r \deg f}}\right)^{m-k} \left(\frac{1}{q^{r \deg f}}\right)^k \right],$$

where it is understood that the product is over all monic irreducible polynomials in $\mathbb{F}_q[x]$.

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SECOND HANKEL DETERMINANT OF THE LOGARITHMIC COEFFICIENTS FOR A SUBCLASS OF UNIVALENT FUNCTIONS

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Abstract. In the present paper, we give the bounds for the second Hankel determinant of the logarithmic coefficients of a certain subclass of normalized univalent functions, which we have introduced here. Relevant connections of the results, which we have presented here, with those available in the existing literature are also described briefly.

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1. INTRODUCTION

Let $\mathcal{A}$ stand for the normalized class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}). \quad (1.1)$$

Also let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

A function $f(z) \in \mathcal{A}$ is called starlike if $f(\mathbb{U})$ is starlike with respect to origin. Let $\mathcal{S}^*(\alpha)$ denote the class of functions in $\mathcal{A}$, which are starlike of order α ($0 \leq \alpha < 1$) in $\mathcal{U}$. It is well known that

$$f \in \mathcal{S}^*(\alpha) \iff \Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1).$$

A function $f \in \mathcal{A}$ is called convex if $f(\mathbb{U})$ is a convex domain. Let $\mathcal{K}(\alpha)$ denote the class of all functions in $\mathcal{A}$, which are convex of order α ($0 \leq \alpha < 1$) in $\mathcal{A}$. It is also known that $f \in \mathcal{K}(\alpha)$ if and only if

$$f \in \mathcal{K}(\alpha) \iff \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1).$$

The function classes

$$\mathcal{S}^*(0) =: \mathcal{S}^* \quad \text{and} \quad \mathcal{K}(0) =: \mathcal{K}$$

consist of starlike and convex functions in $\mathbb{U}$, respectively. Moreover, the function f is convex if and only if $zf'(z)$ is starlike.

Let $\mathcal{P}$ denote the class of analytic functions $p(z)$ in $\mathbb{U}$ satisfying the following conditions:

$$p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0.$$

Thus, if $p \in \mathcal{P}$, then the function $p(z)$ has the following form:

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \quad (z \in \mathbb{U}). \quad (1.2)$$

Functions in the class $\mathcal{P}$ are known as Carathéodory functions. In fact, the coefficients of functions in $\mathcal{S}^*$ and $\mathcal{K}$ are known to have suitable representations in terms of the coefficients of functions in the Carathéodory class $\mathcal{P}$.

Associated with each function $f \in \mathcal{S}$ are its logarithmic coefficients γ_k defined by

$$\mathcal{F}_f := \log \left(\frac{f(z)}{z} \right) = 2 \sum_{k=1}^{\infty} \gamma_k z^k \quad (z \in \mathbb{U}). \quad (1.3)$$

In particular, the logarithmic coefficients of the Koebe function $\mathfrak{K}(z) = z(1-z)^{-2}$ are $\gamma_k = \frac{1}{k}$. Because of the extremal properties of the Koebe function $\mathfrak{K}(z)$, one could expect that $\gamma_k \leq \frac{1}{k}$ for each $f \in \mathcal{S}$. But this conjecture is false even in the case when $k = 2$. For the whole class $\mathcal{S}$, the sharp estimates of single logarithmic coefficients are known only for

$$|\gamma_1| \leq 1 \quad \text{and} \quad |\gamma_2| \leq \frac{1}{2} + \frac{1}{e^2} = 0.6353\dots$$

and are not known for $k \geq 3$.

Recently, logarithmic coefficients were studied by many authors and the upper bounds of the logarithmic coefficients of functions in some important subclasses of the univalent function class $\mathcal{S}$ were found (see, for example, [1, 2, 4, 6, 16, 22, 25]). For a summary of some of the significant results concerning the bounds of the logarithmic coefficients for univalent functions, we refer to [24].

Next, for $q, n \in \mathbb{N} = \{1, 2, 3, \dots\}$, the Hankel determinant $H_{q,n}(f)$ of $f \in \mathcal{A}$ of form (1.1) is defined as follows.

Definition 1. The Hankel determinant $H_{q,n}(f)$ ($q, n \in \mathbb{N}$) is defined for a function $f \in \mathcal{A}$ of form (1.1) by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

In particular, the Hankel determinant $H_{2,1}(f) = a_3 - a_2^2$ is the well-known Fekete-Szegő functional. The second Hankel determinant $H_{2,2}(f)$ is given by $H_{2,2}(f) = a_2a_4 - a_3^2$.

The problem of computing the upper bound of $H_{q,n}$ for functions in various subfamilies of the analytic function class $\mathcal{A}$ is interesting and widely studied in the literature on Geometric Function Theory of Complex Analysis. Sharp upper bounds of $H_{2,2}$ and $H_{3,1}$ for subclasses of analytic functions were obtained by several authors (see, for details, [3, 7, 8, 11–13, 18]). Srivastava *et al.* [21] considered a family of normalized analytic functions with bounded turnings in the open unit disk $\mathbb{U}$ and obtained the estimates of the fourth Hankel determinant. On the other hand, Srivastava *et al.* [19] investigated the upper bound of the third Hankel determinant for the subclass of close-to-convex functions related to the right half of the lemniscate of Bernoulli. Moreover, the bounds of the Hankel determinant of order three for the certain subfamilies of univalent functions associated with exponential functions, which are symmetric along the real axis in the region of open unit disk, were considered by Shi *et al.* [17] (see also the developments reported by Srivastava *et al.* [20] on the Fekete-Szegő type and other coefficient estimates for subclasses of analytic functions which satisfy some subordination conditions and are associated with the Gegenbauer polynomials).

Quite recently, Kowalczyk and Lecko [9] introduced the following Hankel determinant $H_{q,n}\left(\frac{F_f}{2}\right)$ in which the entries are logarithmic coefficients of f .

Definition 2. The Hankel determinant $H_{q,n}\left(\frac{F_f}{2}\right)$ involving the logarithmic coefficients of f is defined by

$$H_{q,n}\left(\frac{F_f}{2}\right) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

For a function $f \in \mathcal{S}$ given by (1.1), by differentiating (1.3) one can obtain the following logarithmic coefficients:

$$\gamma_1 = \frac{1}{2}a_2, \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) \quad \text{and} \quad \gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \quad (1.4)$$

Therefore, the second Hankel determinant of $F_f/2$ can be obtained by

$$H_{2,1}\left(\frac{F_f}{2}\right) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right). \quad (1.5)$$

Furthermore, if $f \in \mathcal{S}$, then we find for

$$f_\theta(z) = e^{-i\theta}f(e^{i\theta}z) \quad (\theta \in \mathbb{R})$$

that (see [10])

$$H_{2,1}\left(\frac{F_{f_\theta}}{2}\right) = e^{4i\theta} H_{2,1}\left(\frac{F_{f_\theta}}{2}\right).$$

In their sequel to [9], Kowalczyk and Lecko [10] derived sharp bounds for $H_{2,1}\left(\frac{F_f}{2}\right)$ for the classes of starlike and convex functions of order α in $\mathbb{U}$. The problem of computing the sharp bounds of $H_{2,1}\left(\frac{F_f}{2}\right)$ for starlike and convex functions with respect to symmetric points in the open unit disk $\mathbb{U}$ was considered by Allu and Arora [23].

Motivated essentially by the above-mentioned recent developments, we continue here to deal further with

$$H_{2,1}\left(\frac{F_f}{2}\right) = \gamma_1 \gamma_3 - \gamma_2^2$$

for functions in the class $\mathcal{R}(\lambda, \alpha)$ defined as follows.

Definition 3. For $0 \leq \alpha < 1$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\lambda, \alpha)$ ($\lambda \geq 0$) if it satisfies the following condition:

$$\Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U}). \quad (1.6)$$

2. A SET OF LEMMAS

In order to establish our main results, we will require the following lemmas.

Lemma 1. (see [5]) *If the function $p \in \mathcal{P}$ is given by the series (1.2), then*

$$|c_k| \leq 2 \quad (k \in \mathbb{N}). \quad (2.1)$$

This inequality is sharp for each k .

Lemma 2. (see [14, 15]) *Let the function $p \in \mathcal{P}$ be given by the series (1.2). Then there exist $x, z \in \mathbb{C}$ with $|x| \leq 1$ and $|z| \leq 1$ such that*

$$2c_2 = c_1^2 + x(4 - c_1^2) \quad (2.2)$$

and

$$\begin{aligned} 4c_3 = & c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 \\ & + 2(4 - c_1^2)(1 - |x|^2)z. \end{aligned} \quad (2.3)$$

3. RESULTS

Our main results on the bounds involving the logarithmic coefficients are stated as the following theorem.

Theorem 1. Let $f(z) \in \mathcal{R}(\lambda, \alpha)$. Then

$$|\gamma_1 \gamma_3 - \gamma_2^2| \leq \begin{cases} 12T(1-\alpha)^2(1+2\lambda)^3(8\lambda^2+6\lambda+1) \\ \quad \left(0 \leq \lambda \leq \frac{-3+\sqrt{9+24(1-\alpha)}}{12}\right) \\ 4T(1-\alpha)^2 \left[3(1+2\lambda)^4(1+4\lambda) - \frac{2B^2}{A}\right] \\ \quad \left(\lambda > \frac{-3+\sqrt{9+24(1-\alpha)}}{12}\right), \end{cases}$$

where

$$A := 2(1-\alpha)^2\lambda^2(5+12\lambda) - (1+2\lambda)^3(12\lambda^2+6\lambda+1), \quad (3.1)$$

$$B := (1+2\lambda)^2\lambda(6\lambda^2+3\lambda+\alpha-1) \quad (3.2)$$

and

$$T = [48(1+2\lambda)^4(1+3\lambda)^2(1+4\lambda)]^{-1}. \quad (3.3)$$

Proof. Let $f(z) \in \mathcal{R}(\lambda, \alpha)$. Then, in view of (1.6), there exists a $p \in \mathcal{P}$ such that

$$zf'(z) + \lambda z^2 f''(z) = ((1-\alpha)p(z) + \alpha)f(z) \quad (z \in \mathbb{U}). \quad (3.4)$$

Thus, by using the representations (1.1) and (1.2) in (3.4), we obtain

$$\begin{aligned} z + \sum_{k=2}^{\infty} k[1+\lambda(k-1)]a_k z^k \\ = \left(1 + (1-\alpha) \sum_{k=1}^{\infty} c_k z^k\right) \left(z + \sum_{k=2}^{\infty} a_k z^k\right) \end{aligned} \quad (3.5)$$

Upon comparing the coefficients on both sides of (3.5), we find that

$$a_2 = \frac{1-\alpha}{1+2\lambda}c_1, \quad (3.6)$$

$$a_3 = \frac{(1-\alpha)^2 c_1^2 + (1-\alpha)(1+2\lambda)c_2}{2(1+3\lambda)(1+2\lambda)} \quad (3.7)$$

and

$$\begin{aligned} a_4 = [(1-\alpha)^3 c_1^3 + (1-\alpha)^2(3+8\lambda)c_1 c_2 + 2(1-\alpha)(1+3\lambda)(1+2\lambda)c_3] \\ \cdot [6(1+4\lambda)(1+3\lambda)(1+2\lambda)]^{-1}. \end{aligned} \quad (3.8)$$

We now substitute the above expressions for the coefficients a_2 , a_3 and a_4 into the equation (1.5). After further simplification, we have

$$\begin{aligned} H_{2,1} \left(\frac{F_f}{2} \right) &= \gamma_1 \gamma_3 - \gamma_2^2 = \frac{1}{4} \left(a_2 a_4 - a_3^2 + \frac{1}{12} a_2^4 \right) \\ &= T [d_1 c_1^4 + d_2 c_1^2 c_2 + d_3 c_1 c_3 + d_4 c_2^2], \end{aligned} \quad (3.9)$$

where T is given in (3.3) and

$$d_1 = (1 - \alpha)^4 \lambda^2 (5 + 12\lambda), \quad (3.10)$$

$$d_2 = -2(1 - \alpha)^3 \lambda (1 + 2\lambda)^2, \quad (3.11)$$

$$d_3 = 4(1 - \alpha)^2 (1 + 2\lambda)^3 (1 + 3\lambda)^2 \quad (3.12)$$

and

$$d_4 = -3(1 - \alpha)^2 (1 + 2\lambda)^4 (1 + 4\lambda). \quad (3.13)$$

Since $p(z) \in \mathcal{P}$, by Lemma 1, we have $|c_1| \leq 2$. By letting $c_1 = c$, we may assume without loss of generality that $c \in [0, 2]$. Also, upon substituting the values of c_2 and c_3 from (2.2) and (2.3) into the equation (3.9), we have

$$\begin{aligned} \left| H_{2,1} \left(\frac{F_f}{2} \right) \right| &= \frac{T}{4} \left| c^4 (4d_1 + 2d_2 + d_3 + d_4) + 2c^2 x (4 - c^2) (d_2 + d_3 + d_4) \right. \\ &\quad \left. + x^2 (4 - c^2) (-d_3 c^2 + d_4 (4 - c^2)) + 2d_3 c (4 - c^2) (1 - |x|^2) z \right|. \end{aligned}$$

Applying the triangle inequality along with $|z| \leq 1$, replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (3.10) to (3.13), we get

$$\begin{aligned} \left| H_{2,1} \left(\frac{F_f}{2} \right) \right| &\leq \frac{T}{4} \left(Kc^4 + 2c^2 \mu (4 - c^2) (1 - \alpha)^2 (1 + 2\lambda)^2 \right. \\ &\quad \cdot [2\alpha\lambda + 1 + 6\lambda(1 + 2\lambda)^2] + \mu^2 (4 - c^2) (1 - \alpha)^2 (1 + 2\lambda)^3 \\ &\quad \cdot [c^2 (12\lambda^2 + 6\lambda + 1) - 8c(1 + 3\lambda)^2 + 12(8\lambda^2 + 6\lambda + 1)] \\ &\quad \left. + 8c(4 - c^2) (1 - \alpha)^2 (1 + 2\lambda)^3 (1 + 3\lambda)^2 \right) \\ &=: \mathcal{F}_\lambda(c, \mu), \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} K &:= 4(1 - \alpha)^4 \lambda^2 (5 + 12\lambda) - 4(1 - \alpha)^3 \lambda (1 + 2\lambda)^2 \\ &\quad + 4(1 - \alpha)^2 (1 + 2\lambda)^3 (1 + 3\lambda)^2 - 3(1 - \alpha)^2 (1 + 2\lambda)^4 (1 + 4\lambda). \end{aligned}$$

We next maximize the function $\mathcal{F}_\lambda(c, \mu)$ on the closed region $[0, 2] \times [0, 1]$. Differentiating $\mathcal{F}_\lambda(c, \mu)$ defined in (3.14) partially with respect to μ , we have

$$\begin{aligned} \frac{\partial \mathcal{F}_\lambda(c, \mu)}{\partial \mu} &= \frac{T}{2} (4 - c^2) (1 - \alpha)^2 (1 + 2\lambda)^2 \left(c^2 [2\alpha\lambda + 1 + 6\lambda(1 + 2\lambda)^2] \right. \\ &\quad \left. + \mu(1 + 2\lambda) [c^2 (12\lambda^2 + 6\lambda + 1) - 8c(1 + 3\lambda)^2 + 12(8\lambda^2 + 6\lambda + 1)] \right). \end{aligned} \quad (3.15)$$

For $0 < \mu < 1$, and for a fixed c with $0 < c < 2$ and $\lambda \geq 0$, we observe from (3.15) that

$$\frac{\partial \mathcal{F}_\lambda(c, \mu)}{\partial \mu} > 0 \quad (0 < \mu < 1; 0 < c < 2; \lambda \geq 0).$$

Consequently, $\mathcal{F}_\lambda(c, \mu)$ is an increasing function of μ and hence it cannot have a maximum value at any point in the interior on the closed region $[0, 2] \times [0, 1]$. Further, for fixed $c \in [0, 2]$, we have

$$\max_{0 \leq \mu \leq 1} \{\mathcal{F}_\lambda(c, \mu)\} = \mathcal{F}_\lambda(c, 1) = \mathcal{G}_\lambda(c). \quad (3.16)$$

Now, upon simplifying the relations (3.14) and (3.16), we obtain

$$\mathcal{G}_\lambda(c) = \frac{T(1-\alpha)^2}{2} \left(Ac^4 + 8Bc^2 + 24(1+2\lambda)^3(8\lambda^2 + 6\lambda + 1) \right), \quad (3.17)$$

$$\mathcal{G}'_\lambda(c) = 2Tc(1-\alpha)^2(Ac^2 + 4B) \quad (3.18)$$

and

$$\mathcal{G}''_\lambda(c) = 2T(1-\alpha)^2(3Ac^2 + 4B), \quad (3.19)$$

where A , B and T are given in (3.1), (3.2) and (3.3), respectively. We consider the following cases.

Case 1. If $\lambda = 0$, $\mathcal{G}_\lambda(c)$ is a constant and its value is given by

$$\mathcal{G}_0(c) = \frac{(1-\alpha)^2}{4}.$$

Case 2. Since $0 \leq \alpha < 1$, if we take

$$0 < \lambda \leq \frac{-3 + \sqrt{9 + 24(1-\alpha)}}{12},$$

then we conclude that $A < 0$ and $B \leq 0$. Therefore, we observe that $\mathcal{G}'_\lambda(c) < 0$ for $0 < c < 2$. Therefore, in this case, $\mathcal{G}_\lambda(c)$ is a decreasing function of c on $[0, 2]$, and we have

$$\max \{\mathcal{G}_\lambda(c)\} = \mathcal{G}_\lambda(0) = 12T(1-\alpha)^2(1+2\lambda)^3(8\lambda^2 + 6\lambda + 1). \quad (3.20)$$

Case 3. Since $0 \leq \alpha < 1$, if we choose

$$\lambda > \frac{-3 + \sqrt{9 + 24(1-\alpha)}}{12},$$

then $A < 0$ and $B > 0$. From the equation $\mathcal{G}'_\lambda(c) = 0$, it is easy to verify that the function $\mathcal{G}_\lambda(c)$ has only one critical point in the open interval $(0, 2)$ given by

$$c = 2\sqrt{-\frac{B}{A}}. \quad (3.21)$$

Substituting the value of c from (3.21) into (3.19), and after simplifying, we get

$$\mathcal{G}_\lambda''\left(2\sqrt{-\frac{B}{A}}\right) = -16T(1-\alpha)^2B < 0.$$

Therefore, by the second derivative test, $\mathcal{G}_\lambda(c)$ has its maximum value at c , where c is given by (3.21). Substituting the value of c in (3.17), and upon simplification, we obtain $\max\{\mathcal{G}_\lambda(c)\}$, given by

$$\max\{\mathcal{G}_\lambda(c)\} = 4T(1-\alpha)^2 \left[3(1+2\lambda)^4(1+4\lambda) - \frac{2B^2}{A} \right]. \quad (3.22)$$

By summarizing the conclusions in Cases 1, 2 and 3, we arrive at the following inequality:

$$|\gamma_1\gamma_3 - \gamma_2^2| \leq \begin{cases} 12T(1-\alpha)^2(1+2\lambda)^3(8\lambda^2+6\lambda+1) & \left(0 \leq \lambda \leq \frac{-3+\sqrt{9+24(1-\alpha)}}{12}\right) \\ 4T(1-\alpha)^2 \left[3(1+2\lambda)^4(1+4\lambda) - \frac{2B^2}{A} \right] & \left(\lambda > \frac{-3+\sqrt{9+24(1-\alpha)}}{12}\right). \end{cases}$$

This completes the proof of the theorem. $\square$

In its special case when $\lambda = 0$, the above theorem would yield the corresponding estimate for functions in the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$

Corollary 1. (see [10]) *Let $f(z) \in \mathcal{S}^*(\alpha)$. Then*

$$|\gamma_1\gamma_3 - \gamma_2^2| \leq \frac{(1-\alpha)^2}{4}.$$

4. CONCLUSION

The present investigation is motivated essentially by several recent developments. We have given the bounds for the second Hankel determinant:

$$H_{2,1}\left(\frac{F_f}{2}\right) = \gamma_1\gamma_3 - \gamma_2^2$$

of the logarithmic coefficients of functions belonging to a certain subclass $\mathcal{R}(\lambda, \alpha)$ of normalized univalent functions, which we have introduced here. We have also briefly described relevant connections of the results, which we have presented here, with those available in the existing literature in Geometric Function Theory of Complex Analysis.

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A POLYNOMIAL SYSTEM OF DEGREE FOUR WITH AN INVARIANT TRIANGLE CONTAINING AT LEAST FOUR SMALL AMPLITUDE LIMIT CYCLES

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Abstract. In this work, the existence of a polynomial system of degree four with an invariant triangle containing at least four small-amplitude limit cycles is proved. This result improves the result obtained in [2].

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Keywords: polynomial systems, limit cycles, invariant algebraic curves

1. INTRODUCTION

In this paper, we discuss the particular problem for the systems that have three real invariant straight lines and they form a triangle, called invariant triangle. As the triangle is a graph, inside there is at least one singularity, we will assume that is a fine focus type singularity.

In 1954, N. N Bautin in [1], show that for quadratic system ($n = 2$), the system

$$\dot{x} = x(a + bx + cy), \quad \dot{y} = y(d + ex + fy),$$

with two invariant straight lines has no limit cycles since a Dulac function can be given. Center cases may occur. So for a quadratic polynomial system with an invariant triangle, the system has no limit cycles.

For $n = 3$, Ye Yanqian in [5] considers a kind of cubic system with an invariant triangle whose sides are invariant straight lines and whose vertices are saddles.

$$\begin{aligned} \dot{x} &= x(a_0(1 - x^2) + a_1x(1 - x) + a_2y + a_4xy + a_5y^2) \\ \dot{y} &= y(b_0(1 - y^2) + b_1x + b_2y(1 - y) + b_3x^2 + b_4xy) \end{aligned}$$

Under certain conditions on the coefficients, the relative positions of other critical points of the cubic system and its invariant straight lines with respect to the invariant triangle are determined. In all of the cases, no limit cycles are found.

Z.H.Liu et. al in [3] consider a class of cubic system ($n = 3$) given by

$$X_\mu : \begin{cases} \dot{x} &= (-1 + abx)(-cx - (a^2 + ab - abc - b^2c)x^2 + y + (2a^2 - ab + a^2b - b^2 + ab^2 - abc)xy + (ab - 2a^2b)y^2) \\ \dot{y} &= (-1 + aby)(-x + (a^2 + ab^2 + b^3)xy - (a^2b + ab^2)y^2) \end{cases}$$

where $\mu = (a, b, c) \in \mathbb{R}^3$

This system have three real invariant straight lines forming an invariant triangle surrounding at least one limit cycle, where the small amplitude limit cycle are limit cycles which bifurcate out of a non hyperbolic focus.

Still remain an open problem of the existence of a cubic system with an invariant triangle containing more than one limit cycle.

For $n = 4$, Z.H.Liu et. al in [2] considers a class of systems of degree four with three real invariant straight lines forming an invariant triangle and at least three small amplitude limit cycles in the interior, the system is given by

$$X_\mu : \begin{cases} \dot{x} &= P(x, y) + bR(x, y) \\ \dot{y} &= Q(x, y) + bS(x, y) \end{cases}$$

where

$$\begin{aligned} P(x, y) &= (1 + x)(a^3\lambda x - a^3y - 4xy + a^2xy + a^3\lambda xy - 4y^2 - 4ay^2 + a^2y^2) \\ Q(x, y) &= (1 + y)(a^3x - a^2x^2 - a^2\lambda x^2 - a^3\lambda x^2 + 4xy + 4axy - a^2xy - a^2\lambda xy + 4y^2) \end{aligned}$$

and

$$\begin{aligned} R(x, y) &= (1 + x)y^2(1 + y) \\ S(x, y) &= (1 + y)(ax^2 - x^3 - axy - y^2), \end{aligned}$$

with $\mu = (a, \lambda, b) \in \mathbb{R}^3, a > 0$.

2. MAIN RESULTS

In this work, the existence of a polynomial system of degree four with an invariant triangle containing at least four small-amplitude limit cycles is proved. This result improves the result obtained in [2]. Let us consider a class of systems of degree four ($n = 4$)

$$X_\mu : \begin{cases} \dot{x} &= (1 + 2x)(\lambda x - (1 + 2\lambda)x^2 - (a - b + c)x^3 - y - 2(1 + \lambda)xy - 2cx^2y - 2y^2 - 4bxy^2 - 8ay^3) \\ \dot{y} &= (1 + 2y)(x + 2(a - b + c)x^3 + 3xy - (3a - 3b - c)x^2y + 2y^2 + 2(3a + b)xy^2 + 4ay^3) \end{cases} \quad (2.1)$$

where $\mu = (a, b, c, \lambda) \in \mathbb{R}^4$.

Lemma 1. *The system (2.1) has three real invariant straight lines, namely,*

$$x = -\frac{1}{2}, \quad y = -\frac{1}{2}, \quad \text{and } x + y = \frac{1}{2}.$$

and the above straight lines forming a triangle surrounding the origin.

Proof. It is clear that $x = -\frac{1}{2}$ and $y = -\frac{1}{2}$ are invariant straight lines, and for the line $x + y = \frac{1}{2}$, we have

$$\begin{aligned} \dot{x} + \dot{y} = & (-1 + 2x + 2y)(-x - \lambda x - x^2 - 2\lambda x^2 - ax^3 + bx^3 - cx^3 \\ & + y - xy + 3ax^2y - 3bx^2y + cx^2y + 2y^2 - 6axy^2 + 2bxy^2 + 4ay^3). \end{aligned}$$

and the lines forming a triangle surrounding the origin and this proves the Lemma. $\square$

Theorem 1. *If $\lambda = 0, a = \frac{26551 + 11\sqrt{226537}}{21064}, b = -\frac{44187 + 31\sqrt{226537}}{21064}$ and $c = \frac{-3331 + \sqrt{226537}}{10532}$, the system (2.1), at the singularity $(0,0)$ has a repelling fine focus of order four.*

Proof. The linear part of (2.1) at the singularity $(0,0)$ is

$$DX_\mu(0,0) = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}$$

If $\lambda = 0$, we consider $\tilde{\mu} = (a, b, c, \lambda)$, and we have that $\text{div} X_{\tilde{\mu}}(0,0) = 0$ and $\det DX_{\tilde{\mu}}(0,0) = 1$, then the critical point $(0,0)$ is a fine focus.

Using Mathematica Software [4], we are able to compute the Lyapunov constants $L_k, k = 0, 1, 2, 3, 4$ and then to determinate the topological type of the singular point at the origin.

If $\lambda = 0$ then

$$L_0 = 0, \quad L_1 = \frac{b - c + 3a - 2}{4}.$$

If $b = c - 3a + 2$ we have

$$L_1 = 0, \quad L_2 = \frac{6 - 2a + 11c}{6}.$$

If $a = \frac{6 + 11c}{2}$, we have

$$L_1 = 0, \quad L_2 = 0, \quad L_3 = -\frac{516 + 3331c + 5266c^2}{48}.$$

$$\text{If } c = \frac{3331 + \sqrt{226537}}{10532},$$

$$L_1 = 0, \quad L_2 = 0, \quad L_3 = 0, \quad L_4 = \frac{98979}{1250}$$

and this prove that the system (2.1), at the singularity $(0,0)$ has a repelling fine focus of order four. $\square$

Theorem 2. *In the parameters space $\mathbb{R}^4$, there exists an open set $\mathcal{N}$, such that for all $(a, b, c, \lambda) \in \mathcal{N}$, the system (2.1) has four small-amplitude limit cycles coexisting with an invariant triangle.*

Proof. We already know that the systems X_μ has a invariant triangle surrounding the origin. By Theorem 1, the system (2.1) has at the singularity $(0,0)$ a repelling weak focus of order four if $\lambda = 0$, $a = \frac{26551+11\sqrt{226537}}{21064}$, $b = -\frac{44187+31\sqrt{226537}}{21064}$ and $c = \frac{-3331+\sqrt{226537}}{10532}$. Perturbing the system in such way that the stability of the origin is reverse, a limit cycle is created (Hopf Bifurcation). Following the same process and finally Hyperbolizing the origin, four hyperbolic small amplitude limit cycles are created. $\square$

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ON ALMOST η -RICCI-BOURGUIGNON SOLITONS

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Abstract. We investigate a Riemannian manifold with almost η -Ricci-Bourguignon soliton structure. We use the Hodge-de Rham decomposition theorem to make a link with the associated vector field of an almost η -Ricci-Bourguignon soliton. Moreover, we show that a nontrivial, compact almost η -Ricci-Bourguignon soliton of constant scalar curvature is isometric to the Euclidean sphere. Using some results obtaining from almost η -Ricci Bourguignon soliton, we give the integral formulas for compact orientable almost η -Ricci-Bourguignon soliton.

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1. INTRODUCTION

The notion of Ricci solitons correspond to self-similar of Ricci flow introduced in [15] by R.S. Hamilton. Perelman [19] proved that any compact Ricci solitons is gradient. In the compact case there are nontrivial Ricci solitons [10]. Most of the proofs for compact case are found in [12] or [13]. Moreover, there do not exist gradient Ricci solitons in the noncompact case [3] and [17]. On the other hand, Naber [18] showed that noncompact shrinking solitons are gradient in some special cases.

Recently, Pigola et al. [21] introduced the notion of almost Ricci soliton. By adding the condition on the parameter λ to be a variable function, they modified the definition of Ricci soliton. Likewise, many authors studied the almost η -Ricci solitons, for example, Blaga [1] investigated almost η -Ricci solitons in $(LCS)_n$ manifolds, Siddiqi [22] studied η -Ricci Yamabe solitons on Riemannian submersions from Riemannian manifold. Generalizing for this notion Dwivedi studied in [14] the almost Ricci-Bourguignon solution and Soylu [23] examined Ricci-Bourguignon soliton and almost soliton with concurrent vector field. Blaga and Taştan [7] also studied almost Ricci-Bourguignon solitons with some special potential vector fields and almost η -Ricci-Bourguignon solitons on a doubly warped product. Dwivedi [14] derived integral formulas for compact Ricci-Bourguignon solitons and Ricci-Bourguignon almost solitons. In addition, Aquino et al. [2] and Barros and Riberio

[5, 6] presented integral formula for the compact almost Ricci solitons and generalized m -quasi Einstein metrics.

In the present paper, we give basic background of an almost η -Ricci-Bourguignon solitons and definitions of gradient solitons in section 2. In section 3, we investigated compact almost η -Ricci-Bourguignon solitons using Hodge-de Rham potential decomposition. Moreover, we study gradient η -Ricci-Bourguignon soliton and compact almost η -Ricci-Bourguignon soliton when the potential vector field is conformal. We proved that the potential vector field of a compact almost η -Ricci-Bourguignon soliton is a Killing vector field under some conditions. In section 4, we derived the integral formulas for gradient compact almost η -Ricci-Bourguignon soliton.

2. PRELIMINARIES

In this section, we recall the fundamental definitions and notions for the further study.

On an n -dimensional Riemannian manifold (M^n, g) *Ricci-Bourguignon solitons* are self-similar solutions to *Ricci-Bourguignon flow* [8]

$$\frac{\partial}{\partial t}g(t) = -2(\text{Ric} - \rho Rg), \quad (2.1)$$

where Ric is the Ricci tensor of the metric, R is the scalar curvature of the Riemannian metric g and $\rho \in \mathbb{R}$ is a real constant.

A Riemannian manifold (M^n, g) is called *Ricci-Bourguignon soliton* if the metric g satisfies the following equation

$$\text{Ric} + \frac{1}{2}\mathcal{L}_\xi g = (\lambda + \rho R)g. \quad (2.2)$$

where $\mathcal{L}_\xi g$ denotes the Lie derivative of the metric g along a vector field ξ , Ric is a Ricci tensor, R is a curvature tensor, ρ and λ are constant. Considering $\eta = df(X)$ is a 1-form, the Riemannian manifold (M^n, g) is called *η -Ricci-Bourguignon soliton* if there exist a vector field ξ , a smooth function f and $\lambda \in \mathbb{R}$ a constant such that

$$\text{Ric} + \frac{1}{2}\mathcal{L}_\xi g = (\lambda + \rho R)g + \mu df \otimes df, \quad (2.3)$$

It is called expanding, steady or shrinking, respectively, if $\lambda < 0$, $\lambda = 0$, $\lambda > 0$. The manifold is called a gradient *η -Ricci-Bourguignon soliton* when the vector field $\xi = \nabla f$ is a gradient of a differentiable function $f : M^n \rightarrow \mathbb{R}$ such that

$$\text{Ric} + \nabla^2 f = (\lambda + \rho R)g + \mu df \otimes df, \quad (2.4)$$

where $\nabla^2 f$ stands for the Hessian of f . The η -Ricci-Bourguignon soliton is called trivial when either the vector field ξ is trivial or the potential f is constant. Hence the vector field is called *Killing vector field*, i.e $\mathcal{L}_\xi g = 0$. If $n \geq 3$ and ξ is a Killing vector field, η -Ricci-Bourguignon soliton becomes trivial soliton. Then we get an *Einstein manifold* in that case, since λ is constant. If λ is a smooth function in

(2.3), then (M^n, g) is called *almost η -Ricci-Bourguignon soliton* and is denoted by $(M^n, g, \xi, \lambda, \mu)$.

Using the Hodge-de Rham decomposition theorem (see [2]), we shall decompose the vector field ξ over a compact oriented Riemannian manifold as a sum of the gradient of a function h and a free divergence vector field Y , i.e.

$$\xi = \nabla h + Y,$$

where $\operatorname{div} Y = 0$. We may indicate a proof of this decomposition for the understanding of his completeness. In fact, we consider the 1-form $\xi^\flat$. We decompose $\xi^\flat$ with the help of the Hodge-de Rham decomposition theorem as follows

$$\xi^\flat = d\alpha + \delta\beta + \gamma. \quad (2.5)$$

Considering $Y = (\delta\beta + \gamma)^\sharp$ and $(d\alpha)^\sharp = \nabla h$ to arrive at the desired result. For more simplicity let us call h the Hodge-de Rham potential.

3. MAIN RESULTS

In this section we investigate a compact almost η -Ricci-Bourguignon soliton and we give a characterization for a gradient η -Ricci-Bourguignon soliton.

We remark that the same result obtained in [2] for compact Ricci solitons also works for compact almost η -Ricci-Bourguignon solitons. We give the next theorem for more explicitly.

Theorem 1. *Let $(M^n, g, \xi, \lambda, \mu)$ be a compact almost η -Ricci-Bourguignon soliton. If M^n is also a gradient almost η -Ricci-Bourguignon soliton with potential f , then, up to a constant, it agrees with the Hodge-de Rham potential h .*

Proof. For an almost η -Ricci-Bourguignon soliton $(M^n, g, \xi, \lambda, \mu)$, we have

$$(1 - n\rho)R + \operatorname{div} \xi = n\lambda + \mu|\nabla f|^2. \quad (3.1)$$

The Hodge-de Rham decomposition allows us to write $\operatorname{div} \xi = \Delta h$. Hence we get

$$(1 - n\rho)R + \Delta h = n\lambda + \mu|\nabla f|^2. \quad (3.2)$$

If $(M^n, g, \xi, \lambda, \mu)$ is also a compact gradient almost η -Ricci-Bourguignon soliton, then from equation (2.4) we have

$$(1 - n\rho)R + \Delta f = n\lambda + \mu|\nabla f|^2. \quad (3.3)$$

Substracting equations (3.2) and (3.3), we deduce $\Delta(f - h) = 0$. Using Hopf's theorem we conclude that $f = h + c$, hence the proof is completed. $\square$

On a Riemannian manifold (M, g) , consider the function $u = e^{-\mu f}$, then we have $\nabla u = -\mu e^{-\mu f} \nabla f$, which can be found in [11]. Hence we get

$$\nabla^2 f - \mu df \otimes df = -\frac{\nabla^2 u}{\mu u} \quad (3.4)$$

and

$$\frac{\Delta u}{u\mu} = ((1 - n\rho)R - n\lambda). \quad (3.5)$$

Then using (2.4), we get

$$\text{Ric} - \frac{\nabla^2 u}{\mu u} = \lambda g + \rho Rg. \quad (3.6)$$

Recall that a vector field ∇u on a Riemannian manifold (M, g) is called a *conformal vector field* if there exists a smooth function $\psi : M \rightarrow \mathbb{R}$ such that $\frac{1}{2}\mathcal{L}_{\nabla u}g = \psi g$. The conformal vector field is nontrivial if $\psi \neq 0$. Suppose ∇u is nontrivial conformal vector field, then we can write $\frac{1}{2}\mathcal{L}_{\nabla u}g = \nabla^2 u = \frac{\Delta u}{n}g$. Putting (3.5) in (3.6), we get

$$\text{Ric} = \frac{R}{n}g, \quad (3.7)$$

where R is constant scalar curvature. Therefore, we deduce that M^n is an Einstein manifold if and only if ∇u is a conformal vector field. For more detail, (see [6]).

Suppose ∇u is nontrivial conformal vector field, i.e. $\mathcal{L}_{\nabla u}g = 2\psi g$, then (3.6) becomes

$$\text{Ric} = (\lambda + \rho R + \frac{\Psi}{\mu u})g. \quad (3.8)$$

Taking the trace of (3.8) and covariant derivative we get

$$(1 - n\rho)\nabla R = n\nabla \left(\lambda + \frac{\Psi}{\mu u} \right). \quad (3.9)$$

Now taking the divergence of (3.8) and using $\nabla R = 2\text{div Ric}$, we have

$$\left(\frac{1}{2} - \rho \right) \nabla R = \nabla \left(\lambda + \frac{\Psi}{\mu u} \right). \quad (3.10)$$

Thus (3.9) and (3.10) imply

$$(1 - n\rho)\nabla R = n \left(\frac{1}{2} - \rho \right) \nabla R, \quad (3.11)$$

which implies that R and $\lambda + \frac{\Psi}{\mu u}$ are constant.

The next theorem is a characterization for gradient almost η -Ricci-Bourguignon soliton when ξ is a conformal vector field and generalizes Theorem 3 of [2].

Theorem 2. *Let $(M^n, g, \xi, \lambda, \mu)$, $n \geq 3$, be a gradient η -Ricci-Bourguignon soliton and $\xi = \nabla u$ is a conformal vector field. Then the following conditions holds:*

- (1) *If M is compact then ∇u is a Killing vector field, so that $(M^n, g, \xi, \lambda, \mu)$ is trivial soliton,*
- (2) *If M is noncompact gradient η -Ricci-Bourguignon soliton then $(M^n, g, \xi, \lambda, \mu)$ is isometric to a Euclidean space or ∇u is Killing vector field.*

Proof. If ∇u is a conformal vector field, then there exist a smooth function ψ on M such that

$$\mathcal{L}_{\nabla u} g = 2\psi g. \quad (3.12)$$

Therefore, taking the trace of (3.12), we get

$$\operatorname{div} \nabla u = n\psi. \quad (3.13)$$

Since M is compact, integrating (3.13), we obtain

$$0 = \int_M 2 \operatorname{div} \nabla u dM = n \operatorname{Vol}(M) \psi, \quad (3.14)$$

which implies that $\psi = 0$. Then ∇u is Killing vector field. Hence, the first assertion is proved.

For the second, we have $\frac{1}{2} \mathcal{L}_{\nabla u} g = \nabla^2 u = \psi g$, since ψ constant from (3.11). Then if $\psi \neq 0$, we may use a result of Tashiro ([24], Theorem 2) to conclude that M^n is isometric to the Euclidean space. If $\psi = 0$, thus ∇u is a Killing vector field and the proof is completed. $\square$

With the help of Theorem 4.2 of [25], we obtain the following theorem for compact almost η -Ricci-Bourguignon soliton.

Theorem 3. *Let $(M^n, g, \xi, \lambda, \mu)$, $n \geq 3$, be a compact almost η -Ricci-Bourguignon soliton with $n \geq 3$. If $\xi = \nabla u$ is a nontrivial conformal vector field, then M^n is isometric to a Euclidean sphere.*

Proof. If $\xi = \nabla u$ is nontrivial conformal vector field, then $\mathcal{L}_{\nabla u} g = 2\psi g$, where $\psi \neq 0$. Then from (3.11), R and $\lambda + \frac{\Psi}{\mu u}$ are constant. We may use [22, Lemma 2.3, pp. 52] to conclude that $R \neq 0$, otherwise $\psi = 0$. Hence from (3.6), we get

$$\mathcal{L}_{\nabla u} \operatorname{Ric} = 2 \left(\lambda + \rho R + \frac{\Psi}{\mu u} \right) \psi g.$$

Since $\lambda + \rho R + \frac{\Psi}{\mu u}$ is constant, we have

$$\mathcal{L}_{\nabla u} \operatorname{Ric} = \left(\lambda + \rho R + \frac{\Psi}{\mu u} \right) \mathcal{L}_{\nabla u} g = 2 \left(\lambda + \rho R + \frac{\Psi}{\mu u} \right) \psi g.$$

Now we may apply Theorem 4.2 (pp. 54 of [25]) to conclude that M^n is isometric to a Euclidean sphere. Hence the proof is completed. $\square$

Remark 1. M^n is isometric to a Euclidean sphere if it is trivial or ∇u is conformal vector field.

On a Riemannian manifold (M^n, g) the following formulas hold [20]:

$$\operatorname{div}(\mathcal{L}_X g)(X) = \frac{1}{2} \Delta |X|^2 - |\nabla X|^2 + \operatorname{Ric}(X, X) + \nabla_X \operatorname{div} X, \quad (3.15)$$

or in $(1, 1)$ -tensor notation

$$\operatorname{div} \nabla^2 f = \operatorname{Ric}(\nabla f) + \nabla \Delta f \quad (3.16)$$

and

$$\frac{1}{2} \Delta |\nabla f|^2 = |\nabla^2 f|^2 + \operatorname{Ric}(\nabla f, \nabla f + g(\nabla \Delta f, \nabla f)). \quad (3.17)$$

These previous formulas allows us to obtain the following lemma which is a generalization of an almost η -Ricci-Bourguignon soliton.

Lemma 1. *Let $(M^n, g, \xi, \lambda, \mu)$ be an almost η -Ricci-Bourguignon soliton. Then the following equations hold:*

$$\begin{aligned} \frac{(1-n\rho)}{2} \Delta |\xi|^2 &= (1-n\rho) |\nabla \xi|^2 + (n\rho-1) \operatorname{Ric}(\xi, \xi) + n\rho \nabla_\xi \operatorname{div} \xi \\ &\quad + 2\rho(1-n\rho)g(\nabla R, \xi) - (n(2\rho+1)-2)g(\nabla \lambda, \xi) \\ &\quad + 2\mu(1-n\rho)|\xi|^2 \operatorname{div} \xi - \mu n\rho \xi(|\xi|^2) \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} \frac{(1-n\rho)}{2} (\Delta - \nabla_\xi) |\xi|^2 &= (1-n\rho) |\nabla \xi|^2 + \lambda(n\rho-1) |\xi|^2 \\ &\quad + \rho(n\rho-1)R|\xi|^2 + \mu(n\rho-1) |\xi|^4 + n\rho \nabla_\xi \operatorname{div} \xi \\ &\quad + 2\rho(1-n\rho)g(\nabla R, \xi) - (n(2\rho+1)-2)g(\nabla \lambda, \xi) \\ &\quad + 2\mu(1-n\rho) |\xi|^2 \operatorname{div} \xi - \mu n\rho \xi(|\xi|^2). \end{aligned} \quad (3.19)$$

Proof. From (2.3), we have

$$2 \operatorname{div} \operatorname{Ric} + \operatorname{div}(\mathcal{L}_\xi g) = 2\nabla \lambda + 2\rho \nabla R + 2\mu \operatorname{div}(df \otimes df). \quad (3.20)$$

Using $\operatorname{div}(df \otimes df) = \xi \operatorname{div} \xi + \nabla_\xi \xi$ and taking the trace of (2.3), we get $(1-n\rho)R + \operatorname{div} \xi = n\lambda + \mu |\xi|^2$. With the help of covariant derivative operator, we have

$$(1-n\rho) \nabla_\xi R + \nabla_\xi (\operatorname{div} \xi) = n \nabla_\xi \lambda + \mu \nabla_\xi |\xi|^2. \quad (3.21)$$

Using the contracted second Bianchi identity $\nabla R = 2 \operatorname{div} \operatorname{Ric}$ and (3.15), (3.20), (3.21), we get

$$\begin{aligned} \nabla_\xi (\operatorname{div} \xi) &= ng(\nabla \lambda, \xi) + (n\rho-1) \nabla_\xi R + \mu \xi(|\xi|^2) \\ &= ng(\nabla \lambda, \xi) + 2(n\rho-1) \operatorname{div} \operatorname{Ric}(\xi) + \mu \xi(|\xi|^2) \\ &= ng(\nabla \lambda, \xi) + \mu \xi(|\xi|^2) - (n\rho-1) \operatorname{div}(\mathcal{L}_\xi g)(\xi) + 2(n\rho-1)g(\nabla \lambda, \xi) \\ &\quad + 2\rho(n\rho-1)g(\nabla R, \xi) + 2\mu(n\rho-1) |\xi|^2 \operatorname{div} \xi + \mu(n\rho-1) \xi(|\xi|^2) \\ &= (1-n\rho) \left[\frac{1}{2} \Delta |\xi|^2 - |\nabla \xi|^2 + \operatorname{Ric}(\xi, \xi) + \nabla_\xi \operatorname{div} \xi \right] \\ &\quad + 2\rho(n\rho-1)g(\nabla R, \xi) + (n(2\rho+1)-2)g(\nabla \lambda, \xi) + \mu \xi(|\xi|^2) \end{aligned}$$

$$\begin{aligned}
& + 2\mu(n\rho - 1)|\xi|^2 \operatorname{div} \xi + \mu(n\rho - 1)\xi(|\xi|^2) \\
& = \frac{(1-n\rho)}{2} \Delta|\xi|^2 - (1-n\rho)|\nabla \xi|^2 + (1-n\rho) \operatorname{Ric}(\xi, \xi) \\
& \quad + (1-n\rho) \nabla_\xi \operatorname{div} \xi + 2\rho(n\rho - 1)g(\nabla R, \xi) + (n(2\rho + 1) - 2)g(\nabla \lambda, \xi) \\
& \quad + 2\mu(n\rho - 1)|\xi|^2 \operatorname{div} \xi + \mu n\rho \xi(|\xi|^2).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\frac{(1-n\rho)}{2} \Delta|\xi|^2 & = (1-n\rho)|\nabla \xi|^2 + (n\rho - 1) \operatorname{Ric}(\xi, \xi) + n\rho \nabla_\xi \operatorname{div} \xi \\
& \quad + 2\rho(1-n\rho)g(\nabla R, \xi) - (n(2\rho + 1) - 2)g(\nabla \lambda, \xi) \quad (3.22) \\
& \quad + 2\mu(1-n\rho)|\xi|^2 \operatorname{div} \xi - \mu n\rho \xi(|\xi|^2).
\end{aligned}$$

Thus (3.18) is proved.

Next using the fundamental equation to write

$$\operatorname{Ric}(\xi, \xi) = \lambda|\xi|^2 + \rho R|\xi|^2 + \mu|\xi|^4 - \frac{1}{2} \mathcal{L}_\xi g(\xi, \xi),$$

then we get

$$\begin{aligned}
\frac{(1-n\rho)}{2} \Delta|\xi|^2 & = (1-n\rho)|\nabla \xi|^2 + (n\rho - 1) \left[\lambda|\xi|^2 + \rho R|\xi|^2 + \mu|\xi|^4 \right. \\
& \quad \left. - \frac{1}{2} \mathcal{L}_\xi g(\xi, \xi) \right] + n\rho \nabla_\xi \operatorname{div} \xi + 2\rho(1-n\rho)g(\nabla R, \xi) \\
& \quad - (n(2\rho + 1) - 2)g(\nabla \lambda, \xi) + 2\mu(1-n\rho)|\xi|^2 \operatorname{div} \xi - \mu n\rho \xi(|\xi|^2) \\
& = (1-n\rho)|\nabla \xi|^2 + \frac{(1-n\rho)}{2} \nabla_\xi |\xi|^2 + \lambda(n\rho - 1)|\xi|^2 + \rho R(n\rho - 1)|\xi|^2 \\
& \quad + \mu(n\rho - 1)|\xi|^4 + n\rho \nabla_\xi \operatorname{div} \xi + 2\rho(1-n\rho)g(\nabla R, \xi) \\
& \quad - (n(2\rho + 1) - 2)g(\nabla \lambda, \xi) + 2\mu(1-n\rho)|\xi|^2 \operatorname{div} \xi - \mu n\rho \xi(|\xi|^2).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\frac{(1-n\rho)}{2} (\Delta - \nabla_\xi) |\xi|^2 & = (1-n\rho)|\nabla \xi|^2 + \lambda(n\rho - 1)|\xi|^2 \\
& \quad + \rho R(n\rho - 1)|\xi|^2 + \mu(n\rho - 1)|\xi|^4 + n\rho \nabla_\xi \operatorname{div} \xi \quad (3.23) \\
& \quad + 2\rho(1-n\rho)g(\nabla R, \xi) - (n(2\rho + 1) - 2)g(\nabla \lambda, \xi) \\
& \quad + 2\mu(1-n\rho)|\xi|^2 \operatorname{div} \xi - \mu n\rho \xi(|\xi|^2),
\end{aligned}$$

which completes the proof of the lemma. $\square$

Using the diffusion operator $\Delta_\xi = \Delta - \nabla_\xi$ (see [[4], pp. 143]) and taking $\xi = \nabla f$ in the previous lemma, namely, $\Delta f = \Delta - \nabla_{\nabla f}$, we get the next corollary with the help of (3.19).

Corollary 1. *Let $(M^n, g, \nabla f, \lambda, \mu)$ be a gradient almost η -Ricci-Bourguignon soliton. Then we have*

$$\begin{aligned} \frac{(1-n\rho)}{2} \Delta f |\nabla f|^2 &= (1-n\rho) |\nabla^2 f|^2 + \lambda(n\rho-1) |\nabla f|^2 \\ &\quad + \rho(n\rho-1) R |\nabla f|^2 + \mu(n\rho-1) |\nabla f|^4 + n\rho \nabla_{\nabla f} (\Delta f) \\ &\quad + 2\rho(1-n\rho) g(\nabla R, \nabla f) - (n(2\rho+1)-2) g(\nabla \lambda, \nabla f) \quad (3.24) \\ &\quad + 2\mu(1-n\rho) \Delta f |\nabla f|^2 - \mu n\rho \nabla f(|\nabla f|^2). \end{aligned}$$

Remark 2. The similar results of (3.18) and (3.19) for η -Ricci-Bourguignon soliton $(M^n, g, \xi, \lambda, \mu)$ are

$$\begin{aligned} \frac{(1-n\rho)}{2} \Delta |\xi|^2 &= (1-n\rho) |\nabla \xi|^2 + (n\rho-1) \text{Ric}(\xi, \xi) + n\rho \nabla_{\xi} \text{div} \xi \quad (3.25) \\ &\quad + 2\rho(1-n\rho) g(\nabla R, \xi) + 2\mu(1-n\rho) |\xi|^2 \text{div} \xi - \mu n\rho \xi(|\xi|^2) \end{aligned}$$

and

$$\begin{aligned} \frac{(1-n\rho)}{2} (\Delta - \nabla_{\xi}) |\xi|^2 &= (1-n\rho) |\nabla \xi|^2 + \lambda(n\rho-1) |\xi|^2 + \rho R(n\rho-1) |\xi|^2 \\ &\quad + \mu(n\rho-1) |\xi|^4 + n\rho \nabla_{\xi} \text{div} \xi + 2\rho(1-n\rho) g(\nabla R, \xi) \\ &\quad + 2\mu(1-n\rho) |\xi|^2 \text{div} \xi - \mu n\rho \xi(|\xi|^2). \quad (3.26) \end{aligned}$$

Proof. The proofs are the same as Lemma 1 with $\nabla \lambda = 0$. $\square$

In Theorem 3 of [5], the authors proved that for a compact almost Ricci soliton (M^n, g, ξ, λ) , $n \geq 3$ satisfying that $\int_M (\text{Ric}(\xi, \xi) + (n-2)g(\nabla \lambda, \xi)) dM \leq 0$, the potential vector field ξ is Killing and the soliton is trivial. Now, we give a similar result for compact almost η -Ricci-Bourguignon soliton as follows:

Theorem 4. *Let $(M^n, g, \xi, \lambda, \mu)$, $n \geq 3$, be a compact almost η -Ricci-Bourguignon soliton. If $\rho \neq \frac{1}{n}$ and*

$$\begin{aligned} \int_M \left(\text{Ric}(\xi, \xi) + \frac{n\rho}{n\rho-1} \nabla_{\xi} \text{div} \xi - 2\rho g(\nabla R, \xi) - \frac{(n(2\rho+1)-2)}{n\rho-1} g(\nabla \lambda, \xi) \right. \\ \left. - 2\mu |\xi|^2 \text{div} \xi - \frac{\mu n\rho}{n\rho-1} \xi(|\xi|^2) \right) dM \leq 0, \quad (3.27) \end{aligned}$$

then ξ is a Killing vector field and M^n is a trivial soliton.

Proof. It sufficient to integrate (3.18) of Lemma 1 with $\rho \neq \frac{1}{n}$. So we get

$$\begin{aligned} \int_M |\nabla \xi|^2 &= \int_M \left(\text{Ric}(\xi, \xi) + \frac{n\rho}{n\rho-1} \nabla_{\xi} \text{div} \xi - 2\rho g(\nabla R, \xi) \right. \\ &\quad \left. - \frac{(n(2\rho+1)-2)}{n\rho-1} g(\nabla \lambda, \xi) - 2\mu |\xi|^2 \text{div} \xi - \frac{\mu n\rho}{n\rho-1} \xi(|\xi|^2) \right) dM \leq 0. \quad (3.28) \end{aligned}$$

As we are assuming that the right-hand side of (3.18) is less than or equal to zero, we get $\nabla \xi = 0$, therefore we have $\mathcal{L}_\xi g = 0$, which yields that ξ is a Killing vector field. Thus, M^n is trivial, which completes the proof of the theorem. $\square$

As a consequence of this theorem, we give the following corollary when $\nabla \lambda = 0$.

Corollary 2. *Let $(M^n, g, \xi, \lambda, \mu)$, $n \geq 3$, be a compact η -Ricci-Bourguignon soliton. If $\rho \neq \frac{1}{n}$ and*

$$\int_M \left(\text{Ric}(\xi, \xi) + \frac{n\rho}{n\rho - 1} \nabla_\xi \text{div} \xi - 2\rho g(\nabla R, \xi) - 2\mu |\xi|^2 \text{div} \xi - \frac{\mu n\rho}{n\rho - 1} \xi(|\xi|^2) \right) dM \leq 0. \quad (3.29)$$

Then ξ is a Killing vector field and M^n is a trivial.

Proof. It is sufficient to take $\nabla \lambda = 0$. $\square$

Remark 3. Corollary 2 is an analog of Theorem 1.1 in [20] which was for the case of compact Ricci solitons. We obtain Petersen-Wylie's result from our result by taking $\rho = 0$ and ξ is a conformal vector field.

4. INTEGRAL FORMULAS FOR GRADIENT ALMOST η -RICCI-BOURGUIGNON SOLITON

In this section, we derive some integral formulas for a compact almost η -Ricci-Bourguignon soliton $(M^n, g, \xi, \lambda, \mu)$ which are the generalization formula of a natural extension obtained for an almost Ricci-Bourguignon soliton in [14], as well as a similar one in [21].

Proposition 1. *Let $(M^n, g, \nabla f, \lambda, \mu)$ be a gradient almost η -Ricci-Bourguignon soliton, then the following equations hold:*

$$(1 - n\rho)R + \Delta f = n\lambda + \mu |\nabla f|^2, \quad (4.1)$$

$$(1 - 2\rho(n - 1)) \nabla R = 2(1 - \mu) \text{Ric}(\nabla f) + 2(n - 1) \nabla \lambda + 2\mu(R - (n - 1)(\lambda + \rho R)) \nabla f, \quad (4.2)$$

$$\begin{aligned} & (\nabla_Y \text{Ric})(X, Z) - (\nabla_X \text{Ric})(Y, Z) - g(\mathcal{R}(X, Y)Z, \nabla f) \\ &= Y[\lambda]g(X, Z) - X[\lambda]g(Y, Z) + \rho((\nabla_Y R)g(X, Z) - (\nabla_X R)g(Y, Z)) \\ &+ \mu(\nabla_Y(df \otimes df)(X, Y) - \nabla_X(df \otimes df)(X, Y)), \end{aligned} \quad (4.3)$$

for any vector fields X, Y, Z on M and $\mathcal{R}$ is the Riemannian curvature tensor of M .

$$\begin{aligned} & \nabla((1 - 2\rho(n - 1))R + |\nabla f|^2 - 2(n - 1)\lambda) \\ &= 2(\lambda + \rho R) \nabla f + 2\mu(\nabla_{\nabla f} \nabla f + (|\nabla f|^2 - \Delta f) \nabla f). \end{aligned} \quad (4.4)$$

Proof. Taking the trace of equation (2.3), we obtain (4.1). For proving (4.2), taking the divergence of (2.4), we get

$$\operatorname{div} \operatorname{Ric} + \operatorname{div}(\nabla^2 f) = \mu \Delta f \nabla f + \mu \nabla_{\nabla f} \nabla f + \nabla(\lambda + \rho R). \quad (4.5)$$

From equation (4.1), we have $\Delta f = -R + n\lambda + n\rho R + \mu|\nabla f|^2$, remembering that $\nabla|\nabla f|^2 = 2\nabla_{\nabla f} \nabla f$ and $\nabla_{\nabla f} \nabla f = \lambda \nabla f + \rho R \nabla f + \mu|\nabla f|^2 \nabla f - \operatorname{Ric}(\nabla f)$ with the help of equation (3.16), we get

$$\begin{aligned} \frac{1}{2} \nabla R &= -\operatorname{Ric}(\nabla f) - \nabla(-R + n(\lambda + \rho R) + \mu|\nabla f|^2) \\ &\quad + \mu \Delta f \nabla f + \mu \nabla_{\nabla f} \nabla f + \nabla(\lambda + \rho R) \\ &= -\operatorname{Ric}(\nabla f) + \nabla R - \mu \nabla_{\nabla f} \nabla f + \mu \Delta f \nabla f - (n-1) \nabla(\lambda + \rho R). \end{aligned}$$

Hence

$$\frac{1}{2} \nabla R = \operatorname{Ric}(\nabla f) - \mu \Delta f \nabla f + \mu \nabla_{\nabla f} \nabla f + (n-1) \nabla(\lambda + \rho R). \quad (4.6)$$

Using (2.3),

$$\nabla_{\nabla f} \nabla f = (\lambda + \rho R) \nabla f + \mu|\nabla f|^2 \nabla f - \operatorname{Ric}(\nabla f). \quad (4.7)$$

Combining (4.6) and (4.7), we obtain

$$\begin{aligned} (1 - 2\rho(n-1)) \nabla R &= (1 - \mu) \operatorname{Ric}(\nabla f) + \mu((\lambda + \rho R) + \mu|\nabla f|^2 - \Delta f) \nabla f + (n-1) \nabla \lambda \\ &= (1 - \mu) \operatorname{Ric}(\nabla f) + \mu(R - (n-1)(\lambda + \rho R) \nabla f) + (n-1) \nabla \lambda. \end{aligned}$$

which gives the second assertion. To get the equation (4.3), using equation (2.4) and covariant derivatives of $\operatorname{Ric}(X, Z)$ and $\operatorname{Ric}(Y, Z)$ where X, Y, Z are any vector fields on M , we get

$$\begin{aligned} (\nabla_Y \operatorname{Ric})(X, Z) - (\nabla_X \operatorname{Ric})(Y, Z) &= (\nabla_Y \nabla_X \nabla_Z f - \nabla_X \nabla_Y \nabla_Z f) \\ &\quad + (\nabla_Y \lambda)g(X, Z) - (\nabla_X \lambda)g(Y, Z) \\ &\quad + \rho((\nabla_Y R)g(X, Z) - (\nabla_X R)g(Y, Z)) \\ &\quad + \mu(\nabla_Y(df \otimes df)(X, Y) - \nabla_X(df \otimes df)(X, Y)) \\ &= g(\mathcal{R}(X, Y)Z, \nabla f) + Y[\lambda]g(X, Z) - X[\lambda]g(Y, Z) \\ &\quad + \rho(Y[R]g(X, Z) - X[R]g(Y, Z)) \\ &\quad + \mu(\nabla_Y(df \otimes df)(X, Y) - \nabla_X(df \otimes df)(X, Y)). \end{aligned}$$

Hence (4.3) is proved. For the last equation, using (4.2), we have

$$\begin{aligned} \frac{1}{2}(1 - 2\rho(n-1)) \nabla R + \frac{1}{2} \nabla|\nabla f|^2 - (n-1) \nabla \lambda \\ = (1 - \mu) \operatorname{Ric}(\nabla f) - \operatorname{Ric}(\nabla f) + \mu(R - (n-1)(\lambda + \rho R)) \nabla f \\ + \mu|\nabla f|^2 \nabla f + (\lambda + \rho R) \nabla f. \end{aligned}$$

Then

$$\begin{aligned}
& \nabla(1 - 2\rho(n-1)R + |\nabla f|^2 - 2(n-1)\lambda) - 2(\lambda + \rho R)\nabla f \\
&= 2\mu(|\nabla f|^2 + R - (n-1)(\lambda + \rho R)\nabla f) - \text{Ric}(\nabla f) \\
&= 2\mu(|\nabla f|^2 + R - n(\lambda + \rho R)\nabla f) + (\lambda + \rho R)\nabla f - \text{Ric}(\nabla f) \\
&= 2\mu(|\nabla f|^2 + \mu|\nabla f|^2 - \Delta f + (\lambda + \rho R)\nabla f) - \text{Ric}(\nabla f) \\
&= 2\mu(\nabla_{\nabla f}\nabla f + (|\nabla f|^2 - \Delta f)\nabla f).
\end{aligned}$$

Which completes the proof of the proposition. $\square$

Corollary 3. *We have the following equations for the gradient η -Ricci-Bourguignon solitons $(M^n, g, \nabla f, \lambda, \mu)$.*

$$(1 - n\rho)R + \Delta f = n\lambda + \mu|\nabla f|^2, \quad (4.8)$$

$$(1 - 2\rho(n-1))\nabla R = 2(1 - \mu)\text{Ric}(\nabla f) + 2\mu(R - (n-1)(\lambda + \rho R))\nabla f, \quad (4.9)$$

$$\begin{aligned}
& (\nabla_Y \text{Ric})(X, Z) - (\nabla_X \text{Ric})(Y, Z) - g(\mathcal{R}(X, Y)Z, \nabla f) \\
&= \rho((\nabla_Y R)g(X, Z) - (\nabla_X R)g(Y, Z)) \\
&\quad + \mu(\nabla_Y(df \otimes df)(X, Y) - \nabla_X(df \otimes df)(X, Y)).
\end{aligned} \quad (4.10)$$

$$(4.11)$$

$$\nabla(1 - 2\rho(n-1)R + |\nabla f|^2 - 2\lambda f) = 2\rho R\nabla f + 2\mu(\nabla_{\nabla f}\nabla f + (|\nabla f|^2 - \Delta f)\nabla f). \quad (4.12)$$

Proof. The proof is the same as Proposition 1 taking $\nabla\lambda = 0$. $\square$

Lemma 2. *Let $(M^n, g, \nabla f, \lambda, \mu)$ be a gradient almost η -Ricci-Bourguignon soliton. Then we have*

$$\begin{aligned}
\left(\frac{1 - 2\rho(n-1)}{2}\right)\Delta R &= -\left|\nabla^2 f - \frac{\Delta f}{n}g\right|^2 - \left\{\frac{1 + n\mu}{n}\right\}(\Delta f)^2 - \frac{n}{2}g(\nabla f, \nabla\lambda) \\
&\quad - \frac{n}{2}\rho g(\nabla f, \nabla R) + \left\{\frac{1 - 2\mu}{2}\right\}g(\nabla f, \nabla\Delta f) \\
&\quad + \mu\text{div}(\nabla_{\nabla f}\nabla f) + (n-1)\Delta\lambda + \lambda\Delta f + \rho R\Delta f.
\end{aligned} \quad (4.13)$$

Proof. First we take the divergence of (4.4) in Proposition 1 to get

$$\begin{aligned}
& \left(\frac{1 - 2\rho(n-1)}{2}\right)\Delta R + \Delta|\nabla f|^2 - (n-1)\Delta\lambda \\
&= \lambda\Delta f + \rho R\Delta f + \mu(g(\nabla(|\nabla f|^2 - \Delta f), \nabla f) \\
&\quad + (|\nabla f|^2 - \Delta f)\Delta f + \text{div}(\nabla_{\nabla f}\nabla f)).
\end{aligned}$$

Since $\left|\nabla^2 f - \frac{\Delta f}{n}g\right|^2 = |\nabla^2 f|^2 - \frac{1}{n}(\Delta f)^2$ with the help of Bochner's formula, we deduce from the last relation:

$$\begin{aligned} \left(\frac{1-2\rho(n-1)}{2}\right)\Delta R &= -\text{Ric}(\nabla f, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n}g\right|^2 - \frac{1}{n}(\Delta f)^2 - g(\nabla\Delta f, \nabla f) \\ &\quad + (n-1)\Delta\lambda + \lambda\Delta f + \rho R\Delta f + 2\mu g(\nabla_{\nabla f}\nabla f, \nabla f) \\ &\quad + \mu(|\nabla f|^2 - \Delta f)\Delta f - g(\nabla\Delta f, \nabla f) + \text{div}(\nabla_{\nabla f}\nabla f). \end{aligned}$$

Thereby, using equation (4.1) to write $g(\nabla\Delta f, \nabla f) = g(\nabla(n(\lambda + \rho R) + \mu|\nabla f|^2 - R), \nabla f)$, then the we have

$$\begin{aligned} \left(\frac{1-2\rho(n-1)}{2}\right)\Delta R &= -\text{Ric}(\nabla f, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n}g\right|^2 - \frac{1+n\mu}{n}(\Delta f)^2 + (n-1)\Delta\lambda \\ &\quad - g(\nabla(\mu|\nabla f|^2 - R + (\lambda + \rho R)n), \nabla f) + 2\mu g(\nabla_{\nabla f}\nabla f, \nabla f) \\ &\quad + \lambda\Delta f + \rho R\Delta f + \mu(|\nabla f|^2\Delta f - g(\nabla\Delta f, \nabla f) \\ &\quad + \text{div}(\nabla_{\nabla f}\nabla f)) \\ &= -(\text{Ric}(\nabla f, \nabla f) + (n-1)g(\nabla\lambda, \nabla f)) - \left|\nabla^2 f - \frac{\Delta f}{n}g\right|^2 \\ &\quad - \frac{1+n\mu}{n}(\Delta f)^2 + (n-1)\Delta\lambda + (\lambda + \rho R)\Delta f + g(\nabla R, \nabla f) \\ &\quad - n\rho g(\nabla R, \nabla f) + \rho g(\nabla R, \nabla f) + \mu(|\nabla f|^2\Delta f \\ &\quad - g(\nabla\Delta f, \nabla f) + \text{div}(\nabla_{\nabla f}\nabla f)). \end{aligned}$$

Hence, using (4.2) and putting into the last equation, we deduce that

$$\begin{aligned} \left(\frac{1-2\rho(n-1)}{2}\right)\Delta R &= \frac{1}{2}g(\nabla R, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n}g\right|^2 - \frac{1+n\mu}{n}(\Delta f)^2 \\ &\quad + (n-1)\Delta\lambda + (\lambda + \rho R)\Delta f + \frac{\mu}{2}g(\nabla|\nabla f|^2, \nabla f) \\ &\quad + \mu[-g(\nabla\Delta f, \nabla f) + \text{div}(\nabla_{\nabla f}\nabla f)] \\ &= \frac{1}{2}g(\nabla R, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n}g\right|^2 - \frac{1+n\mu}{n}(\Delta f)^2 + (n-1)\Delta\lambda \\ &\quad + (\lambda + \rho R)\Delta f + \mu(g(\nabla_{\nabla f}\nabla f, \nabla f) - g(\nabla\Delta f, \nabla f) \\ &\quad + \text{div}\nabla_{\nabla f}\nabla f) \\ &= g(\nabla R, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n}g\right|^2 - \frac{1+n\mu}{n}(\Delta f)^2 + (n-1)\Delta\lambda \\ &\quad + (\lambda + \rho R)\Delta f + \frac{1}{2}g(\nabla f, \nabla\Delta f) - \frac{n}{2}g(\nabla\lambda, \nabla f) \end{aligned}$$

$$-\frac{n\rho}{2}g(\nabla R, \nabla f) - \mu g(\nabla \Delta f, \nabla f) + \mu \operatorname{div}(\nabla_{\nabla f} \nabla f).$$

Since $\nabla_{\nabla f} \nabla f = \frac{1}{2}g(\nabla R, \nabla f) + \frac{1}{2}g(\nabla \Delta f, \nabla f) - \frac{n}{2}g(\nabla \lambda, \nabla f) - \frac{n\rho}{2}g(\nabla R, \nabla f)$, then substituting this equation into the above formula, we get the expression in the statement, which completes the proof of the lemma. $\square$

As a consequence of this lemma, we give the following integral formula.

Theorem 5. *Let $(M^n, g, \xi = \nabla f, \lambda, \mu)$ be a compact orientable almost η -Ricci-Bourguignon soliton. Then we have*

- (1) M^n is trivial provided $\int_M (\rho g(\nabla R, \nabla f) + g(\nabla f, \nabla \lambda)) dM \geq 0$,
- (2) $\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 d\mu = -\frac{n+2}{2n} \int_M (g(\nabla f, \nabla R) + \mu \Delta f |\nabla f|^2) dM$,
- (3) If $\int_M (g(\nabla f, \nabla R) + \mu \Delta f |\nabla f|^2) dM \geq 0$ then M^n is conformally equivalent to a unit sphere $\mathbb{S}^n$.

Proof. Since M^n is compact orientable, then using Lemma 2 and Stokes' formula to infer

$$\begin{aligned} \int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 dM &= -\left(\frac{1+n\mu}{n}\right) \int_M (\Delta f)^2 dM - \left(\frac{1-2\mu}{2}\right) \int_M (\Delta f)^2 dM \\ &\quad - \frac{n}{2} \int_M (g(\nabla \lambda, \nabla f) + \rho g(\nabla R, \nabla f)) dM \\ &\quad - \rho \int_M g(\nabla f, \nabla R) dM - \int_M g(\nabla \lambda, \nabla f) dM. \end{aligned}$$

Hence, we get

$$\begin{aligned} \int_M \left(\left| \nabla^2 f - \frac{\Delta f}{n} g \right|^2 + \left(\frac{n+2}{2n} \right) (\Delta f)^2 \right) dM & \quad (4.14) \\ &= -\left(\frac{n+2}{2} \right) \int_M (g(\nabla \lambda, \nabla f) + \rho g(\nabla R, \nabla f)) dM. \end{aligned}$$

Then we have

$$\int_M (\rho g(\nabla R, \nabla f) + g(\nabla \lambda, \nabla f)) dM \geq 0,$$

it implies that if R and λ are constant, we deduce from the first assertion

$$\int_M \left(\left| \nabla^2 f - \frac{\Delta f}{n} g \right|^2 + \left(\frac{n+2}{2n} \right) (\Delta f)^2 \right) dM = 0,$$

which implies that $\nabla^2 f = \frac{\Delta f}{n} g$ and $\Delta f = 0$. So, f is constant, then M^n is trivial. Hence the first statement is proved.

For the second assertion, from (4.1), we can write

$$\int_M g(\nabla f, \nabla \lambda) dM = \frac{1}{n} \int_M g(\nabla f, \nabla((1-n\rho)R + \Delta f - \mu |\nabla f|^2)) dM.$$

Therefore, using equation (4.14), we infer

$$\begin{aligned} & \int_M \left(\left| \nabla^2 f - \frac{\Delta f}{n} g \right|^2 + \left(\frac{n+2}{2n} \right) (\Delta f)^2 \right) dM \\ &= -\frac{n+2}{2n} \int_M g(\nabla f, \nabla R) dM + \frac{n+2}{2n} \int_M (\Delta f)^2 dM \\ & \quad + \frac{\mu(n+2)}{2} \int_M g(\nabla f, \nabla |\nabla f|^2) dM. \end{aligned}$$

Therefore, after some calculations and using Stokes' formula, we deduce

$$\int_M \left| \nabla^2 f - \frac{\Delta f}{n} g \right|^2 dM = -\frac{n+2}{2n} \int_M (g(\nabla f, \nabla R) + \mu \Delta f |\nabla f|^2) dM.$$

Hence the second item is proved.

For the last item, if $\int_M (g(\nabla f, \nabla R) + \mu \Delta f |\nabla f|^2) dM \geq 0$, then we have

$$\int_M \left| \nabla^2 f - \frac{\Delta f}{n} g \right|^2 dM = 0.$$

If f is constant, the solution is trivial otherwise $\nabla^2 f = \frac{\Delta f}{n} g$. We may invoke a theorem due to Ishihara and Tashiro [16] to conclude that M^n is conformally equivalent to a sphere $\mathbb{S}^n$, which completes the proof of the theorem. $\square$

For a conformal vector field ξ on a compact orientable Riemannian manifold M^n we have $\int_M \xi_{\sharp} R dM = \int_M g(\xi, \nabla R) dM = 0$, see [9] and $\int_M |\xi|^2 \operatorname{div} \xi dM = 0$ from Lemma 1 of [6]. From Theorem 5, we give the following corollary.

Corollary 4. *Let $(M^n, g, \xi = \nabla f, \lambda, \mu)$ be a compact orientable almost η -Ricci-Bourguignon soliton. Then we have*

- (1) *If $n \geq 0$, $\int_M g(\nabla f, \nabla R) dM = 0$ and $\int_M \Delta f |\nabla f|^2 dM = 0$, then ∇f is a conformal vector field.*
- (2) *If $n = 2$ and $\int_M \Delta f |\nabla f|^2 dM = 0$, then f is constant.*

Proof. Using the last item of Theorem 5, we deduce that $\nabla^2 f = \frac{\Delta f}{n} g$, which allows us to say ∇f is conformal. Hence the first statement is proved. Moreover for $n = 2$, and supposing $\int_M \Delta f |\nabla f|^2 dM = 0$, we conclude that f is constant. $\square$

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ON n -1-ABSORBING PRIMARY IDEALS

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Abstract. The paper aims to present a new primary ideal in a commutative ring R with nonzero identity element. We introduce the new primary ideal as an n -1-absorbing primary ideal that is a generalization of both primary and 1-absorbing primary ideals. We propose to achieve two goals with this paper. Firstly, we study and characterize some essential properties of an n -1-absorbing primary ideals and figure out the relations between the other types of ideals such as prime, 1-absorbing primary and irreducible ideals. Then, we classify some special rings that admit an n -1-absorbing primary ideal. We provide the results by introducing some examples.

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1. INTRODUCTION

Throughout this study, we assume that R is a commutative ring with identity ($1 \neq 0$) and n is a positive integer.

Firstly, we introduce some well-known facts for ideals in a commutative ring that we need to use in this paper: An ideal I is called proper, if $I \neq R$. The radical of I , denoted by $\sqrt{I}$, is defined by $\{r \in R : r^n \in I \text{ for some positive integer } n\}$. A proper ideal I of R and $x \in R - I$, the residual of I by x , $(I :_R x)$ is defined by $\{r \in R : rx \in I\}$. A proper ideal I of a commutative ring is called irreducible, if I cannot be represented as an intersection of two ideals, properly contained by I . A prime ideal of a ring is called a divided prime ideal if it is comparable to every principal ideal of a ring. Moreover, we need to introduce some well-known rings that we use to study in this paper: A ring R is called a chained ring if $a|b$ or $b|a$ for every $a, b \in R$. Note that every chained ring is a divided ring. An integral domain R is called a divided domain if every prime ideal of R is a divided prime ideal. An integral domain R is called a valuation domain if either $x|y$ or $y|x$ for all $0 \neq x, y \in R$, [2].

Prime ideals in commutative algebra always deserve more attention. Obtaining some important new prime ideals provides the concepts and tools to effect a treatment of rings in a direct and elegant way so that the structures of rings are significantly figured out. This not only gives a better understanding of the structure of rings but opens a vista comparable to the developments of commutative algebra. In [2], Anderson and Badawi have introduced and studied the n -absorbing ideals, namely, the generalization of the prime ideals. According to the definition in [2], a proper ideal I of R is called an n -absorbing ideal, if $x_1, \dots, x_{n+1} \in R$ and $x_1 \cdots x_{n+1} \in I$, then there are n of the x_i 's whose product is in I . The mentioned study has led to be introduced the new ideal which is a generalization of both n -absorbing ideals and primary ideals. It has been defined by Becker and it is called n -absorbing primary ideal, [4]. Moreover, in [6], Ulucak, Koc, and Tekir have defined an n -1-absorbing prime ideal. We define a new ideal which is called an n -1-absorbing primary ideal that is a generalization of n -1-absorbing prime ideal as well. A proper ideal I of R is defined as n -absorbing primary ideal if $x_1, \dots, x_{n+1} \in R$ and $x_1 \cdots x_{n+1} \in I$ imply either $x_1 \cdots x_n \in I$ or there are n of the x_i 's whose product (except for $x_1 \cdots x_n$) is in $\sqrt{I}$, [4]. Very recently, in [3], Badawi and Celikel have introduced another primary ideals that is called 1-absorbing primary ideal. A proper ideal I of R is called a 1-absorbing primary ideal if non-units $x_1, x_2, x_3 \in R$ and $x_1 x_2 x_3 \in I$ imply either $x_1 x_2 \in I$ or $x_3 \in \sqrt{I}$, [3].

This study allows us to classify and construct a new class of prime ideals which is generalization of the primary ideals. A proper ideal I of R is called an n -1-absorbing primary ideal if $x_1 \cdots x_{n+1} \in I$ for some non-units $x_1, \dots, x_{n+1} \in R$, then, either $x_1 x_2 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$.

In the first part the focus will be on the discussion to characterize the n -1-absorbing primary ideals introducing some useful and essential properties to figure out the behaviour of the n -1-absorbing primary ideals in a commutative ring R . Moreover, we provide the results by finding some examples.

The final goal of this paper is to study on the n -1-absorbing primary ideals to characterize and find out the structure of some special rings ,e.i., Dedekind domain, valuation domain and quasi local rings. Explicitly, with the obtaining examples we will explain the structure of rings in commutative algebra.

2. CHARACTERIZATION OF n -1-ABSORBING PRIMARY IDEALS

In this section of this study we give the definition of an n -1-absorbing primary ideal. Moreover, we characterize and study some essential properties of n -1-absorbing primary ideal in a commutative ring R to understand the structure of R . Finally, we explain the structure of a commutative ring R presenting some examples for the n -1-absorbing primary ideal of R .

Definition 1. Let n be a positive integer. A proper ideal I of a ring R is called an n -1-absorbing primary ideal if $x_1 \cdots x_{n+1} \in I$ for some non-units $x_1, \dots, x_{n+1} \in R$, then either $x_1 x_2 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$.

By the definition of n -1-absorbing primary ideal, it can be easily seen that every primary ideal of R is an n -1-absorbing primary ideal, and every 1-absorbing primary ideal of R is n -1-absorbing primary ideal for $n \geq 2$. The converses of these implications can not be true generally. Therefore, to make the implication clear we give the following example. Obviously, the concepts of 1-1-absorbing primary ideal and primary ideal of R coincide for $n = 1$. Also, when we take $n = 2$, then by the definition of n -1-absorbing primary ideal, we see that the definition allows us to interpret that a 1-absorbing primary ideal and a 2-1-absorbing primary ideal coincide. Consequently, an n -1-absorbing primary ideal assert its importance that it is a generalization of both primary ideals and 1-absorbing primary ideals.

Example 1. Let $S = F[x, y]$, where F is a field. Consider $M = (x, y)S$ and $R = S_M$. It can be seen that R is a quasi local ring with maximal ideal M_M . Then $I = yM_M = (xy, y^2)R$ is a 2-1-absorbing primary ideal of R , [3, Theorem 5]. However, I is not a primary ideal of R since $xy \in I$, but neither $y \in I$ nor $x \in \sqrt{I} = yR$.

Now, we give some basic and useful results of an n -1-absorbing primary ideal as follows:

Theorem 1. Let I be a proper ideal of a ring R and n be a positive integer. Then, the following statements are satisfied:

- (1) Every n -1-absorbing prime ideal of R is an n -1-absorbing primary ideal.
- (2) Every n -1-absorbing primary ideal of R is an n -absorbing primary ideal.
- (3) Let $m \geq n$ be a positive integer. If I is an n -1-absorbing primary ideal of R , then I is an m -1-absorbing ideal of R .
- (4) Let I be an n -1-absorbing primary ideal of R . Then $\sqrt{I}$ is a prime ideal of R .

Proof.

- (1) It is clear from the definitions of n -1-absorbing prime ideal and n -1-absorbing primary ideal.
- (2) Assume that I is an n -1-absorbing primary ideal of R and $x_1 \cdots x_{n+1} \in I$ for some $x_1, \dots, x_{n+1} \in R$. If at least one of the x_i 's is unit, then the proof is done. So, we assume that $x_1, \dots, x_{n+1} \in R$ are non-unit. Since I is an n -1-absorbing primary ideal of R , then we have either $x_1 x_2 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$. This implies that $x_1 x_2 \cdots x_n \in I$ or a product of n of the x_i 's which contains x_{n+1} is in $\sqrt{I}$. Thus, I is an n -absorbing primary ideal of R .
- (3) We use mathematical induction on n, m . To show that the claim is true, it is sufficient to prove that I is an $(n+1)$ -1-absorbing primary ideal if I is an n -1-absorbing primary ideal of R . Hence, we assume that I is an n -1-absorbing primary ideal of R and $x_1 x_2 \cdots x_{n+1} x_{n+2} = (x_1 x_2) x_3 \cdots x_{n+1} x_{n+2} \in I$ for some

non-units $x_1, x_2, \dots, x_{n+2} \in R$. Since I is an n -1-absorbing primary ideal of R , we have $(x_1x_2)x_3 \cdots x_{n+1} \in I$ or $x_{n+2} \in \sqrt{I}$. As a result, I is an $(n+1)$ -1-absorbing primary ideal of R .

- (4) Let $xy \in \sqrt{I}$ and $x \notin \sqrt{I}$ for some $x, y \in R$. Then, there exists a positive integer k such that $x^k y^k \in I$. If $k \leq n$, then we get $x^n y^k = x^{n-k} x^k y^k \in I$. Since $x \notin \sqrt{I}$, we conclude that $x^n \notin I$ and hence $y^k \in \sqrt{I}$, namely, $y \in \sqrt{I}$. Now, we assume $k \geq n$. By (3), I is k -1-absorbing primary ideal and since $x^k \notin I$, we conclude that $y^k \in \sqrt{I}$ and so $y \in \sqrt{I}$. Then, $\sqrt{I}$ is a prime ideal of R . □

Now, we give the following examples to show that the converses of Theorem 1 (1), Theorem 1 (2) are not true generally. Moreover, we provide the Example 1 verifies that the converse of the Theorem 1 (4) is not true in general.

Consider the Example 1 for the converse of Theorem 1 (3): $I = yM_M = (xy, y^2)R$ is a 2-1-absorbing primary ideal of R , however it is not a 1-1-absorbing primary.

Example 2. Consider the ideal (8) of the ring $\mathbb{Z}$. It is clear that (8) is a 2-1-absorbing primary ideal but not a 2-1-absorbing prime ideal since $2 \cdot 2 \cdot 2 \in (8)$, but neither $2 \cdot 2 \in (8)$ nor $2 \in (8)$.

Example 3. Consider the ideal (15) of the ring $\mathbb{Z}$. It is clear that (15) is a 2-absorbing primary ideal but not a 2-1-absorbing primary ideal since $2 \cdot 3 \cdot 5 \in (15)$, but neither $2 \cdot 3 \in (15)$ nor $5 \in \sqrt{(15)}$.

Theorem 2. Let I be an n -1-absorbing primary ideal of R and $x \in R - I$ be a non-unit element. Then $(I :_R x)$ is an $(n-1)$ -1-absorbing primary ideal of R .

Proof. Let $x_1 \cdots x_n \in (I :_R x)$ for some non-units $x_1, \dots, x_n \in R$. Then $xx_1 \cdots x_n \in I$. By the assumption, we have either $xx_1 \cdots x_{n-1} \in I$ or $x_n \in \sqrt{I} \subseteq \sqrt{(I :_R x)}$ and so $x_1x_2 \cdots x_{n-1} \in (I :_R x)$ or $x_n \in \sqrt{(I :_R x)}$. Therefore, $(I :_R x)$ is an $(n-1)$ -1-absorbing primary ideal of R . □

Theorem 3. Let I be an n -1-absorbing primary ideal of a ring R with $I \subset \sqrt{I}$ and $n \geq 2$. If $m \geq 2$ is the least positive integer with $x^m \in I$ for $x \in \sqrt{I} - I$, then $(I :_R x^{m-1})$ is an $(n-m+1)$ -1-absorbing primary ideal of R .

Proof. We see that $m \leq n$ since I is an n -1-absorbing primary ideal of R and thus it must be $n-m+2 \geq 2$. Let $x_1x_2 \cdots x_{n-m+2} \in (I :_R x^{m-1})$ for some non-units $x_1, \dots, x_{n-m+2} \in R$. Then, $x^{m-1}x_1x_2 \cdots x_{n-m+2} \in I$. Hence, by the assumption, we obtain that $x^{m-1}x_1x_2 \cdots x_{n-m+1} \in I$ or $x_{n-m+2} \in \sqrt{I} \subseteq \sqrt{(I :_R x^{m-1})}$. Therefore, we have $x_1x_2 \cdots x_{n-m+1} \in (I :_R x^{m-1})$ or $x_{n-m+2} \in \sqrt{(I :_R x^{m-1})}$. Thus, $(I :_R x^{m-1})$ is an $(n-m+1)$ -1-absorbing primary ideal of R . □

Corollary 1. Let I be an n -1-absorbing primary ideal of a ring R with $I \subset \sqrt{I}$ and $n \geq 2$ be the smallest positive integer with $x^n \in I$ for $x \in \sqrt{I} - I$. Then $(I :_R x^{n-1})$ is a primary ideal of R .

Proof. Let us choose $n = m$ in Theorem 3. So $(I :_R x^{n-1})$ is an $(n - n + 2)$ -1-absorbing primary ideal of R , namely, $(I :_R x^{n-1})$ is a 1-1-absorbing primary ideal of R . Thus, $(I :_R x^{n-1})$ is a primary ideal of R . $\square$

Theorem 4. *A proper ideal I of a ring R is an n -1-absorbing primary ideal if and only if $\sqrt{(I :_R x_1 \cdots x_n)} = \sqrt{I}$ where $x_1 \cdots x_n \notin I$ for all non-units $x_1, \dots, x_n \in R$.*

Proof.

- ($\Rightarrow$): Let I be an n -1-absorbing primary ideal of a ring R and $x_1 \cdots x_n \notin I$ for some non-units $x_1, \dots, x_n \in R$. It is obvious that $\sqrt{I} \subseteq \sqrt{(I :_R x_1 \cdots x_n)}$. Thus, we assume that $y \in \sqrt{(I :_R x_1 \cdots x_n)}$. Then, there is a positive integer k with $y^k \in (I :_R x_1 \cdots x_n)$. Since $x_1 \cdots x_n y^k \in I$ and $x_1 \cdots x_n \notin I$, then, by the assumption, we have $y^k \in \sqrt{I}$. Thus, $y \in \sqrt{I}$ and so we have that $\sqrt{(I :_R x_1 \cdots x_n)} \subseteq \sqrt{I}$ and thus we obtain $\sqrt{(I :_R x_1 \cdots x_n)} = \sqrt{I}$.
- ($\Leftarrow$): Let $y_1 \cdots y_{n+1} \in I$ and $y_1 \cdots y_n \notin I$ for some non-units $y_1, \dots, y_{n+1} \in R$. Since $y_{n+1} \in (I :_R y_1 \cdots y_n) \subseteq \sqrt{(I :_R y_1 \cdots y_n)}$, by the assumption we get $y_{n+1} \in \sqrt{I}$. Thus, I is an n -1-absorbing primary ideal. $\square$

By the Theorem 4, we give the following corollary:

Corollary 2. *Let I be an n -1-absorbing primary ideal of a ring R and $x \in R - \sqrt{I}$. Then $\sqrt{(I :_R x^n)} = \sqrt{I}$.*

Theorem 5. *Let I be a proper ideal of R . The following statements are equivalent.*

- (1) *I is an n -1-absorbing primary ideal of R .*
- (2) *If $x_1 x_2 \cdots x_n J \subseteq I$ for some non-units $x_1, x_2, \dots, x_n \in R$ and a proper ideal J of R , then $x_1 x_2 \cdots x_n \in I$ or $J \subseteq \sqrt{I}$.*
- (3) *If $I_1 I_2 \cdots I_{n+1} \subseteq I$ for some proper ideals $I_1, I_2, \dots, I_{n+1}$ of R , then $I_1 I_2 \cdots I_n \subseteq I$ or $I_{n+1} \subseteq \sqrt{I}$.*

Proof.

- (1) $\Rightarrow$ (2): Let I be an n -1-absorbing primary ideal of R . Assume $x_1 x_2 \cdots x_n J \subseteq I$ and $x_1 x_2 \cdots x_n \notin I$ for some non-units $x_1, x_2, \dots, x_n \in R$ and a proper ideal J of R . Then $x_1 x_2 \cdots x_n j \subseteq I$ and $x_1 x_2 \cdots x_n \notin I$ for every $j \in J$. By the assumption, $j \in \sqrt{I}$ and so $J \subseteq \sqrt{I}$. Thus, we complete the proof.
- (2) $\Rightarrow$ (3): Suppose that $I_1 I_2 \cdots I_{n+1} \subseteq I$ and $I_1 I_2 \cdots I_n \not\subseteq I$ for some proper ideals $I_1, I_2, \dots, I_{n+1}$ of R . Then, there are $x_1 \in I_1, x_2 \in I_2, \dots, x_n \in I_n$ such that $x_1 x_2 \cdots x_n \notin I$. Since $x_1 x_2 \cdots x_n I_{n+1} \subseteq I$, then, by the assumption, we have $I_{n+1} \subseteq \sqrt{I}$.
- (3) $\Rightarrow$ (1): It is straightforward. $\square$

Theorem 6. Let $f : R \rightarrow R'$ be a ring homomorphism such that $f(x)$ is a non-unit in R' for all non-unit elements x in R and $f(1) = 1$. Then the followings are satisfied:

- (1) If J is an n -1-absorbing primary ideal of R' , then $f^{-1}(J)$ is an n -1-absorbing primary ideal of R .
- (2) If f is an epimorphism and I is a proper ideal of R with $\ker(f) \subseteq I$, then I is an n -1-absorbing primary ideal of R if and only if $f(I)$ is an n -1-absorbing primary ideal of R' .

Proof.

- (1) Let $x_1 \cdots x_{n+1} \in f^{-1}(J)$ for some non-unit elements $x_1, \dots, x_{n+1} \in R$. Then, $f(x_1 \cdots x_{n+1}) = f(x_1) \cdots f(x_{n+1}) \in \sqrt{J}$. By the assumption, we obtain $f(x_1) \cdots f(x_n) \in J$ or $f(x_{n+1}) \in \sqrt{J}$. Hence $x_1 \cdots x_n \in f^{-1}(J)$ or $x_{n+1} \in f^{-1}(\sqrt{J}) = \sqrt{f^{-1}(J)}$. Consequently, $f^{-1}(J)$ is an n -1-absorbing primary ideal of R .
- (2) Let f be a surjective homomorphism and I is a proper ideal of R with $\ker(f) \subseteq I$. Then $I = f^{-1}(f(I))$. So, we assume that $f(I)$ is an n -1-absorbing primary ideal of R' . Thus, I is an n -1-absorbing primary ideal of R by (1). For the converse of (2), assume that I is an n -1-absorbing primary ideal of R and $y_1 y_2 \cdots y_{n+1} \in f(I)$ for some non-unit elements $y_1, y_2, \dots, y_{n+1} \in R$. By the assumption, we get $f(x_i) = y_i$ for each $1 \leq i \leq n$ and so $f(x_1) \cdots f(x_{n+1}) \in f(I)$. Then, we have $x_1 \cdots x_{n+1} \in I$ since $\ker(f) \subseteq I$. Since I is an n -1-absorbing primary ideal, we deduce that either $x_1 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$ and so, $y_1 \cdots y_n \in f(I)$ or $y_{n+1} \in f(\sqrt{I}) = \sqrt{f(I)}$, thus the proof is done. $\square$

Corollary 3. Let I and J be proper ideals of a ring R with $I \subseteq J$ and $u(R/I) = \{x + I : x \in u(R)\}$. Then the followings are satisfied:

- (1) J is an n -1-absorbing primary ideal of R if and only if J/I is an n -1-absorbing primary ideal of R/I .
- (2) Let S be a multiplicatively closed subset of R . If I is an n -1-absorbing primary ideal of R with $I \cap S = \emptyset$, then I_S is an n -1-absorbing primary ideal of R_S .

Proof.

- (1) Consider the natural homomorphism $\pi : R \rightarrow R/I$, defined by $\pi(x) = x + I$ for each $x \in R$. Then (1) is obtained by Theorem 6 (2).
- (2) Consider the injection $i : S \rightarrow R$, defined by $i(x) = x$ for each $x \in S$. Then, the result follows by Theorem 6 (2). $\square$

The converse of Corollary 3 (2) is not true generally. See the following example:

Example 4. Let $R = \mathbb{Z}$ and $I = p_1 p_2 \mathbb{Z}$ for some distinct prime numbers p_1, p_2 . One can see that I is not an n -1-absorbing primary ideal since $p_1^n p_2 \in I$, $p_1^n \notin I$ and

$p_2 \notin \sqrt{I} = I$. Let $S = R - p_1\mathbb{Z}$. Thus, we get $I_S = (p_1\mathbb{Z})_S$ is a prime ideal of R_S and so I_S is an n -1-absorbing primary ideal of R_S .

Corollary 4. *Let S be a multiplicatively closed subset of R , and I be a proper ideal of R . If I_S is an n -1-absorbing primary ideal of R_S and $S \cap \mathbf{Z}_I(R) = \emptyset$, then I is an n -1-absorbing primary ideal of R .*

Proof. Let $x_1 \cdots x_{n+1} \in I$ for some non-units $x_1, \dots, x_n \in R$. Then $\frac{x_1 \cdots x_{n+1}}{1} = \frac{x_1}{1} \cdots \frac{x_{n+1}}{1} \in I_S$. By the assumption, we obtain that either $\frac{x_1}{1} \cdots \frac{x_n}{1} \in I_S$ or $\frac{x_{n+1}}{1} \in \sqrt{I_S} = (\sqrt{I})_S$. If $\frac{x_1 \cdots x_n}{1} = \frac{x_1}{1} \cdots \frac{x_n}{1} \in I_S$, there is a $u \in S$ such that $ux_1 \cdots x_n \in I$. Since $u \notin \mathbf{Z}_I(R)$, we have $x_1 \cdots x_n \in I$. Now, we suppose that $\frac{x_{n+1}}{1} \in (\sqrt{I})_S$. Then, there is a $v \in S$ such that $vx_{n+1} \in \sqrt{I}$ and so $(vx_{n+1})^m \in I$ for some positive integer m . Since $v^m \notin \mathbf{Z}_I(R)$, we have $x_{n+1}^m \in I$ and thus, we have $x_{n+1} \in \sqrt{I}$, which implies that I is an n -1-absorbing primary ideal of R . $\square$

Definition 2. Let I be an n -1-absorbing primary ideal of a ring R . We know that $P = \sqrt{I}$ is a prime ideal of R . Thus, I is defined as a P - n -1-absorbing primary ideal of R .

Using the definition, we give the following theorem:

Theorem 7. *Let $I_1, \dots, I_m$ be P - n -1-absorbing primary ideals of a ring R and $I = \bigcap_{i=1}^m I_i$. Then, I is a P - n -1-absorbing primary ideal of R .*

Proof. Note that $\sqrt{I} = P$. Let $x_1 \cdots x_{n+1} \in I$ and $x_1 \cdots x_n \notin I$ for some non-units $x_1, \dots, x_{n+1} \in R$. Without loss of generality, we suppose that $x_1 \cdots x_n \notin I_k$ for some $k \in \{1, 2, \dots, m\}$. Since I_k is an n -1-absorbing primary ideal, then $x_{n+1} \in \sqrt{I_k} = P$. So, it implies that I is a P - n -1-absorbing primary ideal. $\square$

Let I_1 and I_2 be n -1-absorbing primary ideals of a ring. However, when $\sqrt{I_1} \neq \sqrt{I_2}$, $I_1 \cap I_2$ might not be an n -1-absorbing primary ideal of R . For that reason, we give the following basic example.

Example 5. Consider the ideals $I_1 = (2)$ and $I_2 = (3)$ of the ring $\mathbb{Z}$. The ideals I_1 and I_2 are 2-1-absorbing primary ideals of $\mathbb{Z}$ but $I_1 \cap I_2 = (6)$ is not a 2-1-absorbing primary ideal since $2 \cdot 2 \cdot 3 \in (6)$, $2 \cdot 2 \notin (6)$ and $3 \notin \sqrt{(6)}$.

Recall that a proper ideal of a commutative ring is irreducible if it cannot be stated as an intersection of two ideals properly containing it. Then, we can give the following theorem:

Theorem 8. *Let I be an irreducible ideal of a ring R . Then, I is an n -1-absorbing primary ideal of R if and only if $(I :_R y^{n-1}) = (I :_R y^n)$ for all $y \in R - \sqrt{I}$.*

Proof.

- ($\Rightarrow$): Let I be an n -1-absorbing primary ideal of R and $y \in R - \sqrt{I}$. It is obvious that $(I :_R y^{n-1}) \subseteq (I :_R y^n)$. So, we assume $x \in (I :_R y^n)$. Then, $xy^n \in I$ and since $y \notin \sqrt{I}$, then $xy^{n-1} \in I$ and so $x \in (I :_R y^{n-1})$. Thus, $(I :_R y^{n-1}) = (I :_R y^n)$.
- ($\Leftarrow$): Let $(I :_R y^{n-1}) = (I :_R y^n)$ for all $y \in R - \sqrt{I}$ and $x_1 \cdots x_{n+1} \in I$ for some non-units x_i 's of R . Assume that $x_1 \cdots x_n \notin I$ and $x_{n+1} \notin \sqrt{I}$. By the assumption, we get $(I :_R x_{n+1}^{n-1}) = (I :_R x_{n+1}^n)$. Note that $I \subseteq (I + x_{n+1}^{n-1}R) \cap (I + x_1 \cdots x_n R)$. Suppose $a \in (I + x_{n+1}^{n-1}R) \cap (I + x_1 \cdots x_n R)$. Then, $a = i + r_1 x_{n+1}^{n-1} = j + r_2 x_1 \cdots x_n$ for some $i, j \in I$. Since $ax_{n+1} = i + r_1 x_{n+1}^n = j + r_2 x_1 \cdots x_n \cdot x_{n+1}$, then $r_1 x_{n+1}^n = ax_{n+1} - ix_{n+1} \in I$ and so $r_1 \in (I :_R x_{n+1}^n) = (I :_R x_{n+1}^{n-1})$. Thus $a = i + r_1 x_{n+1}^{n-1} \in I$ as $r_1 x_{n+1}^{n-1} \in I$. Therefore, we get $I = (I + x_{n+1}^{n-1}R) \cap (I + x_1 \cdots x_n R)$. Since I is an irreducible ideal, then either $I = (I + x_{n+1}^{n-1}R)$ or $I = (I + x_1 \cdots x_n R)$ and so $x_{n+1}^{n-1} \in I$, that is $x_{n+1} \in \sqrt{I}$, or $x_1 \cdots x_n \in I$. It leads to a contradiction. So we omit the claim " $x_1 \cdots x_n \notin I$ and $x_{n+1} \notin \sqrt{I}$ ". Thus, it must be $x_1 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$. Hence, I is an n -1-absorbing primary ideal of R . □

Theorem 9. *Let $S = R \times R'$ be a decomposable ring and $I = I \times I'$ be an ideal of S . Then $J = I \times I'$ is an n -1-absorbing primary ideal of S if and only if I is an n -1-absorbing primary ideal of R and $I' = R'$ or I' is an n -1-absorbing primary ideal of R' and $I = R$.*

Proof.

- ($\Rightarrow$): Assume that $J = I \times I'$ is an n -1-absorbing primary ideal of S . By Theorem 1(4), we have $\sqrt{I \times I'} = \sqrt{I} \times \sqrt{I'}$ which is a prime ideal of S . Thus $\sqrt{I} = R$ or $\sqrt{I'} = R'$ and so $I = R$ or $I' = R'$. Assume that $I' = R'$. Let $x_1 \cdots x_{n+1} \in I$ for some non-units $x_1, \dots, x_{n+1} \in R$. Then $(x_1, 0) \cdots (x_{n+1}, 0) \in I \times I'$ and $I \times I'$ is an n -1-absorbing primary ideal of S and so $(x_1, 0) \cdots (x_n, 0) \in I \times I'$ or $(x_{n+1}, 0) \in \sqrt{I \times I'} = \sqrt{I} \times \sqrt{I'}$. Therefore, $x_1 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$. It implies that I is an n -1-absorbing primary ideal of R . In this case $I = R$, by a similar argument we can show that I' is an n -1-absorbing primary ideal of R' .
- ($\Leftarrow$): Now, we assume that I is an n -1-absorbing primary ideal of R and $I' = R'$. Let $(x_1, y_1) \cdots (x_{n+1}, y_{n+1}) \in J = I \times R'$. Then $x_1 \cdots x_{n+1} \in I$ and since I is an n -1-absorbing primary, then $x_1 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$. Thus $(x_1, y_1) \cdots (x_n, y_n) \in J = I \times R'$ or $(x_{n+1}, y_{n+1}) \in \sqrt{J} = \sqrt{I} \times R'$ and so J is an n -1-absorbing primary ideal of R and $I' = R'$. The other case is similar. □

Theorem 10. *Let $S = R_1 \times \cdots \times R_k$ be a decomposable ring and $I = I_1 \times \cdots \times I_k$ be an ideal of S , where $(k \geq 2)$. Then, $J = I_1 \times I_2 \times \cdots \times I_k$ is an n -1-absorbing primary ideal of S if and only if there exists i , where $1 \leq i \leq k$ that I_i is an n -1-absorbing primary ideal of R_i and $I_j = R_j$, for all $j \neq i$.*

Proof. Assume that J is an n -1-absorbing primary ideal of S . We prove the claim by induction on k . Let $k = 2$. Then the result follows the Theorem 9. Now assume that $k \geq 2$ and the conclusion is true for all positive integer $t < k$. Let $S = R_1 \times R_2 \times \cdots \times R_k$ and $J = I_1 \times I_2 \times \cdots \times I_k$ is an n -1-absorbing primary ideal of R . Then, $\sqrt{J} = \sqrt{I_1 \times I_2 \times \cdots \times I_{k-1}} \times \sqrt{I_k}$. Thus $\sqrt{I_1 \times I_2 \times \cdots \times I_{k-1}} = R_1 \times R_2 \times \cdots \times R_{k-1}$ or $\sqrt{I_k} = R_k$. We conclude that the first case is done, and the case follows the induction. The converse is similar of the proof of Theorem 9. $\square$

3. n -1-ABSORBING PRIMARY IDEALS ON SOME SPECIAL RINGS

In this section, we characterize some special rings that admit an n -1-absorbing primary ideals: We determine the conditions for a ring R that admits an n -1-absorbing primary to be a quasi local ring. Moreover, we classify the quasi local rings that admit an n -1-absorbing primary ideals. We provide the results with some example. Afterwards, we classify a ring R which admits n -1-absorbing primary ideals to be a chain ring, a divided ring, a valuation domain, and a Dedekind domain. We present the results providing some examples.

Theorem 11. *Assume that I is an n -1-absorbing primary ideal of R that is not an $(n-1)$ -1-absorbing primary. Then R is a quasi local ring.*

Proof. Assume that I is an n -1-absorbing primary ideal of R that is not an $(n-1)$ -1-absorbing primary. Then, there are some non-units $x_1, \dots, x_n \in R$ such that $x_1 \cdots x_n \in I$, $x_1 \cdots x_{n-1} \notin I$ and $x_n \notin \sqrt{I}$. Let y be a non-unit in R . Then $yx_1 \cdots x_n \in I$ and so $yx_1 \cdots x_{n-1} \in I$ since I is an n -1-absorbing primary ideal and $x_n \notin \sqrt{I}$. Consider a unit element u of R . Suppose that $y + u$ is not a unit in R . Then, we have $(y + u)x_1 \cdots x_n \in I$ and thus, $(y + u)x_1 \cdots x_{n-1} = yx_1 \cdots x_{n-1} + ux_1 \cdots x_{n-1} \in I$ since $x_n \notin \sqrt{I}$. Hence, $ux_1 \cdots x_{n-1} \in I$ and so $x_1 \cdots x_{n-1} \in I$, which implies a contradiction. Therefore, $y + u$ is a unit element in R and thus, R is a quasi local ring [3, Lemma 1]. $\square$

By Theorem 11, we give the following corollary:

Corollary 5. *Let a ring R be not a quasi local ring. Then, a proper ideal I of R is an n -1-absorbing primary ideal if and only if I is a primary ideal.*

Theorem 12. *Let R be a Noetherian integral domain (not a field) and I a proper ideal of R . Then the following statements are equivalent:*

- (1) R is a Dedekind domain.

- (2) I is an n -1-absorbing primary ideal of R if and only if $I = P^m$ for some prime ideal P of R and some positive integer m .

Proof.

- (1) $\Rightarrow$ (2): Assume that I is an n -1-absorbing primary ideal of R . By Theorem 1 (4), $\sqrt{I} = P$ is a non-zero prime ideal. By assumption " R is a Dedekind domain", we get that P is maximal and thus, P is primary. Consequently, $I = P^m$ for some positive integer m by [5, Theorem 6.20]. Now, assume the converse holds, that is, $I = P^m$ for some prime ideal P of R and some positive integer m . Then, we obtain that P is maximal since R is a Dedekind domain. Thus, I is primary and so I is an n -1-absorbing primary ideal.
- (2) $\Rightarrow$ (1): Assume that a non-zero ideal I of R is an n -1-absorbing primary ideal if and only if $I = P^m$ for some prime ideal P of R and some positive integer m . Let M be a maximal ideal of R . Assume that there is an ideal J of R such that $M^2 \subset J \subset M$. Then J is a primary ideal. By assumption, J is an n -1-absorbing primary ideal. From (2), we get $J = M^m$ for some positive integer m , a contradiction. Because there doesn't exist such an ideal between M^2 and M . By [5, Theorem 6.20], R is a Dedekind domain. □

We have the following result by Theorem 12 since a principal ideal domain is a Dedekind domain.

Corollary 6. *Let R be a principal ideal domain. Then, a non-zero proper ideal I of R is an n -1-absorbing primary ideal if and only if $I = p^m$ for some prime element p of R and some positive integer m .*

We prove the following theorems that allow us to give some examples of n -1-absorbing primary ideals of R .

Theorem 13. *Let R be a quasi local domain with a maximal ideal M . Assume x is a prime element of R and $xR \subset M$. Then, $x^{n-1}M$ is an n -1-absorbing primary ideal of R which is not an $(n-1)$ -1-absorbing primary ideal.*

Proof. We consider the non-unit elements $x_1, x_2, \dots, x_{n+1} \in R$, $x_1 x_2 \cdots x_{n+1} \in x^{n-1}M$ and $x_1 x_2 \cdots x_n \notin x^{n-1}M$. If $x \nmid x_i$ for all $1 \leq i \leq n$, then $x \mid x_{n+1}$ and so $x_{n+1} \in xR = \sqrt{x^{n-1}M}$, it follows that $x^{n-1}M$ is an n -1-absorbing primary. Thus, we assume that $x^k \mid x_1 x_2 \cdots x_n$ and $x^{k+1} \nmid x_1 x_2 \cdots x_n$ for some integer k . Since R is a domain and quasi local, we have $k < n-1$. Hence, $x_1 x_2 \cdots x_n = x^k z$ for some $z \in R$ that $x \nmid z$. Then, we have $x_1 x_2 \cdots x_{n+1} \in x^{n-1}M$ and so $x_1 x_2 \cdots x_{n+1} = x^{n-1}a$ for some $a \in M$. Therefore, $x^{n-1}a = x^k z x_{n+1}$. Thus $x^k(z x_{n+1} - x^{n-k-1}a) = 0$. Since R is a domain, we conclude that $z x_{n+1} = x^{n-k-1}a$ and so $x \mid z x_{n+1}$ and then $x \mid x_{n+1}$. Therefore $x^{n-1}M$ is an n -1-absorbing primary ideal of R . So $x^{n-1}M$ is n -1-absorbing primary. Now, we show that $x^{n-1}M$ is not an $(n-1)$ -1-absorbing primary ideal: Let $m \in M \setminus xR$. Then

$x^{n-1}m \in x^{n-1}M$ when $x^{n-1} \notin x^{n-1}M$ and $m \notin xR = \sqrt{x^{n-1}M}$. Thus, $x^{n-1}M$ is not an $(n-1)$ -1-absorbing primary ideal. $\square$

Theorem 14. *Let (R, M) be a quasi local. Assume that $P \subseteq M$ is a prime ideal of R and $(P^i M : a) = P^{i-1}M$ for all positive integer i and $a \in P - P^{i-1}$. Then $P^{i-1}M$ is an i -1-absorbing primary ideal of R .*

Proof. We prove by induction on i . Let $i = 2$. Then, Theorem 8 in [3] implies that PM is a 2-1-absorbing primary ideal of R . Assume that the claim is true for all positive integer $< i$. Then, $P^{i-2}M$ is $(i-1)$ -1-absorbing primary ideal of R . We show that $P^{i-1}M$ is i -1-absorbing primary ideal of R . Let the non-unit elements $x_1, x_2, \dots, x_{i+1} \in R$, $x_1 x_2 \cdots x_{i+1} \in P^{i-1}M$ and $x_{i+1} \notin P = \sqrt{P^{i-1}M}$. Since $x_1 x_2 \cdots x_{i+1} \in P^{i-1}M \subseteq P$ and P is a prime ideal of R , we conclude that $x_k \in P$ for some $1 \leq k \leq i$. Without loss of the generality we can assume $x_1 \in P$. If $x_1 \in P^{i-1}$, then $x_1 x_2 \cdots x_i \in P^{i-1}M$ and so $P^{i-1}M$ is an i -1-absorbing primary ideal of R . Then, suppose that $x_1 \in P - P^{i-1}$. Then by the assumption, we obtain $(P^{i-1}M : a) = P^{i-2}M$. Thus, $x_2 \cdots x_{i+1} \in P^{i-2}M$ and by induction $P^{i-2}M$ is an $(i-1)$ -1-absorbing primary ideal of R . Then $x_2 \cdots x_i \in P^{i-2}M$. Therefore, $x_1 x_2 \cdots x_i \in PP^{i-2}M = P^{i-1}M$ implies the proof. $\square$

Example 6. Let $S = F[x, y, z]$, where F is a field. Consider $M = (x, y, z)S$ and $R = S_M$. It can be seen that R is a quasi local ring with maximal ideal M_M . Let $P = (x/1, y/1)S$ and $I = P^2 M_M$. It is easy to see that we have $(P^2 M : f) = PM$ for all $f \in P$. Thus I is a 3-1-absorbing primary ideal of R by Theorem 14.

Example 7. Let $R = F[x, y]$, where F is a field. Therefore R is a domain. Let $S = R - (x, y)R$. Then $S^{-1}R$ is a quasi local domain. $x/1$ is an irreducible element of S . Then we Consider that $I = \frac{x^{n-1}}{1}S^{-1}(x, y)$. So, Theorem 13 implies that I is an n -1-absorbing primary ideal and that is not an $(n-1)$ -1-absorbing primary.

Note that $\sqrt{I} = \frac{x}{1}S^{-1}R$ and it is a prime ideal of $S^{-1}R$ and then it becomes an n -1-absorbing prime ideal for all $n \geq 1$. While it is not k -1-absorbing primary ideal for $k = 1, 2, \dots, n-1$. It shows that the converse of Theorem 1(4) is not correct.

An ideal I of a ring R is called as semi-primary if $\sqrt{I}$ is prime ideal. A semi-primary ideal may not be a primary ideal (See [7, p.154]).

Theorem 15. *Let I be an n -1-absorbing primary ideal of a ring R . Then I is a semi-primary ideal.*

Proof. Let $xy \in \sqrt{I}$ and $x \notin \sqrt{I}$ for some $x, y \in R$. Then there exists a positive ineteger k such that $x^k y^k \in I$ and also $x^m \notin I$ for all positive integer m . Assume $k \leq n$. Then we get $x^n y^n = x \cdots x \cdot y^n \in I$ and since $x \cdots x = x^n \notin I$, then $y^n \in \sqrt{I}$, that is, $y \in \sqrt{I}$. Now, assume $k \geq n$. We have $\underbrace{x \cdots x}_{n-1 \text{ times}} x^{k-n+1} y^k \in I$. Since $\underbrace{x \cdots x}_{n-1 \text{ times}} x^{k-n+1} \notin I$, then $y^k \in \sqrt{I}$ and so $y \in \sqrt{I}$. Therefore, $\sqrt{I}$ is prime and thus I is semi-primary. $\square$

Theorem 16. *Let R be a Dedekind domain and I be a proper ideal of R . Then I is an n -1-absorbing primary ideal if and only if $\sqrt{I}$ is prime ideal.*

Proof.

($\Rightarrow$): It is obvious, by Theorem 1(4).

($\Leftarrow$): Assume that I is a semi-primary ideal of R , namely, $\sqrt{I}$ is a prime ideal.

We know that every nonzero prime ideal of R is maximal since R is a Dedekind domain. Hence, $\sqrt{I}$ is a maximal ideal and so I is a primary ideal. Therefore, I is an n -1-absorbing primary ideal. □

Theorem 17. *Let R be a divided ring and I a proper ideal of R . Then, I is an n -1-absorbing primary ideal of R if and only if I is a primary ideal of R .*

Proof. The part "if I is a primary ideal of R , then I is an n -1-absorbing primary ideal" is straightforward. Thus, we assume that I is an n -1-absorbing primary ideal of R . Let $xy \in I$ and $x \notin \sqrt{I}$ for some $x, y \in R$. x and y might be accepted as non-units of R . Since $\sqrt{I}$ is prime, then $y \in \sqrt{I}$ and $y^m \notin \sqrt{I}$ for all positive integers m , and then, $y^{n-1} \notin \sqrt{I}$. As R is a divided ring, then $\sqrt{I} \subset (x^{n-1})R$. Hence, $y \in (x^{n-1})R$ and so, there is an element r in R with $y = rx^{n-1}$ since $y \in \sqrt{I}$. Since I is n -1-absorbing primary ideal, $xy = xrx^{n-1} \in I$ and $x \notin \sqrt{I}$, then $rx^{n-1} \in I$ and so $y \in I$. Therefore, I is a primary ideal of R . □

A ring R is called as chained ring if $a|b$ or $b|a$ for every $a, b \in R$. Note that every chained ring is divided. Thus, we give the following corollary as a result of Theorem 17:

Corollary 7. *Let R be a chain ring and I a proper ideal of R . Then, I is an n -1-absorbing primary ideal of R if and only if I is a primary ideal of R .*

Recall that an integral domain R is a valuation domain if R is a chained ring and then we obtain that any valuation domain is a divided domain.

Theorem 18. *Let I be a proper ideal of a valuation domain R with $\sqrt{I} = P$. Then, the following are equivalent:*

- (1) I is an n -1-absorbing primary ideal of R .
- (2) I is a primary ideal of R .
- (3) $I = P^m$ for some positive integer $m \geq 1$ where $P \neq P^2$.

Proof. Firstly, note that $\sqrt{I} = P$ is a prime ideal of R .

(1) $\Rightarrow$ (2): By Theorem 17, It is trivial.

(2) $\Rightarrow$ (3): By [5, Theorem 5.11], we have obviously the result.

(3) $\Rightarrow$ (1): By [3, Theorem 11], It is obvious. □

Let M be an R -module. The idealization defined by $R(+)M = \{(a, m) : a \in R, m \in M\}$. Assume that $a_1, a_2 \in R$ and $m_1, m_2 \in M$, then $R(+)M$ is a commutative ring by the addition operator $(a_1, m_1) + (a_2, m_2) = (a_1 + a_2, m_1 + m_2)$ and the multiplication operator $(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_2 m_1 + a_1 m_2)$. Let I be an ideal of R and N be a submodule of M . Then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$, this ideal is called homogeneous. For more properties of the idealization refer to [1].

Theorem 19. Assume that $I(+)N$ is a homogeneous ideal of $R(+)M$. Then

- (1) If $I(+)N$ is an n -1-absorbing primary ideal of $R(+)M$, then I is an n -1-absorbing primary ideal of R .
- (2) If $\sqrt{I}M \subseteq N$, $(N :_M a) = N$ for all $a \notin \sqrt{I}$ and I is an n -1-absorbing primary ideal of R , Then $I(+)N$ is an n -1-absorbing primary ideal of $R(+)M$.

Proof.

- (1) Let $I(+)N$ be an n -1-absorbing primary ideal of $R(+)M$. Assume that $a_1, \dots, a_{n+1} \in R$ and $a_1 \cdots a_{n+1} \in I$. Then $(a_1, 0) \cdots (a_{n+1}, 0) \in I(+)N$ and therefore $(a_1, 0) \cdots (a_n, 0) = (a_1 \cdots a_n, 0) \in I(+)N$ or $(a_{n+1}, 0) \in \sqrt{I(+)N} = \sqrt{I}(+)N$. Thus $a_1 \cdots a_n \in I$ or $a_{n+1} \in \sqrt{I}$. It implies that I is an n -1-absorbing primary ideal of R .
- (2) Assume that $\sqrt{I}M \subseteq N$, $(N :_M a) = N$ for all $a \notin \sqrt{I}$ and I is an n -1-absorbing primary ideal of R . Let $(a_1, m_1) \cdots (a_{n+1}, m_{n+1}) \in I(+)N$ for some elements $a_1, \dots, a_{n+1} \in R$ and $m_1 \cdots m_{n+1} \in I$. It implies that

$$a_1 \cdots a_{n+1} \in I, \quad \sum_{i=1}^{n+1} a_1 \cdots \hat{a}_i \cdots a_{n+1} m_i \in N.$$

Then we have $a_1 \cdots a_n \in I$ or $a_{n+1} \in \sqrt{I}$. Firstly, we assume that $a_{n+1} \in \sqrt{I}$, then $a_{n+1}^k \in I$ for some integer k . Therefore $(a_{n+1}, m_{n+1})^{(k+1)} = (a_{n+1}^{(k+1)}, (k+1)a_{n+1}^k m_{n+1}) \in I(+)N$. It means that $(a_{n+1}, m_{n+1}) \in \sqrt{I(+)N}$. Thus, we assume that $a_{n+1} \notin \sqrt{I}$. Then $a_1 \cdots a_n \in I$ implies that $a_1 \cdots a_n m_{n+1} \in IM \subseteq N$. Hence $\sum_{i=1}^n a_1 \cdots \hat{a}_i \cdots a_n a_{n+1} m_i \in N$ and so $\sum_{i=1}^n a_1 \cdots \hat{a}_i \cdots a_n m_i \in N : a_{n+1}$. Now, since $N : a_{n+1} = N$, we conclude that $\sum_{i=1}^n a_1 \cdots \hat{a}_i \cdots a_n m_i \in N$. It shows that $(a_1, m_1) \cdots (a_n, m_n) \in I(+)N$ and therefore $I(+)N$ is an n -1-absorbing primary ideal of $R(+)M$.

□

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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A $p(x)$ -KIRCHHOFF TYPE EQUATION WITH STEKLOV BOUNDARY

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Abstract. In this paper we study a class of $p(x)$ -Kirchhoff type problem with Steklov boundary value in variable exponent Sobolev spaces. Precisely, we show the existence of at least three solutions and a nontrivial weak solution.

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1. INTRODUCTION

The purpose of this article is to establish the existence of at least three weak solutions for the following equation Steklov boundary value involving $p(x)$ -Laplacian

$$\begin{cases} M\left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)}\right) \Delta_{p(x)} u = |u|^{p(x)-2} u & \text{in } x \in \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} = \lambda f(x, u) & \text{on } x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a smooth bounded domain, λ is a positive parameter, ν is the outward normal vector on $\partial\Omega$, $p \in C(\overline{\Omega})$, $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ is a $p(x)$ -Laplacian operator, $f : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function and $M : (0, \infty) \rightarrow (0, \infty)$ is a continuous Kirchhoff function.

Problem (1.1) is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff [21]. To be more precise, Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.2)$$

where ρ , P_0 , h , E , L are constants. This equation is an extension of the classical D’Alembert’s wave equation, by considering the effects of the changes in the length

of the strings during the vibrations. For $p(x)$ -Kirchhoff type problems see, for example, [10, 11, 28].

Differential and partial differential problems with variable exponent growth condition have been received considerable attention in recent years. Moreover, these type of problems are used interesting topic like electrorheological fluids [23], image processing and stationary thermo-rheological viscous flows of non-Newtonian fluids [9], elastic mechanics [31], the mathematical description of the processes filtration of an idea barotropic gas through a porous medium [6]. Many results have been obtained on this kind of problems, for example, we refer to [13, 14, 16, 17, 20, 22, 27].

The Steklov problems involving $p(x)$ -Laplacian have been intensively studied using variational methods by many authors [3, 4, 12, 19, 24, 25, 30]. Especially, some of the authors have studied the problems of type (1.1) in the case of $M(t) = 1$. For example, In [7], the author investigated the existence and multiplicity of solutions for Steklov problem standard growth condition without Ambrosetti-Rabinowitz type condition by means of critical point theorems with Cerami condition under weaker conditions. In [5], the authors considered the existence and multiplicity of solutions for the nonlinear Steklov boundary-value problem, by using Mountain Pass, Fountain and Ricceri three-critical-points theorems. In [4], the authors obtained the existence and multiplicity of solutions of problem (1.2) when $f(x, u) = |u|^{q(x)-2}u$, by applying two versions of the mountain pass theorem and Ekeland's variational principle.

Moreover, in [1], under suitable assumptions on the nonlinearity, they studied the existence and multiplicity of solutions for a Steklov problem involving the $p(x)$ -Laplacian using variational methods. In [29], the authors proved the existence and multiplicity of a-harmonic solutions for the elliptic Steklov problem with variable exponents. Their approach is based on variational methods. In [18], the authors dealt with nontrivial weak solution a weighted Steklov problem involving the $p(x)$ -Laplacian operator, by variational method and Ekeland's variational principle. In [2], the author proved some results on the existence of weak solutions for non-linear Steklov boudary value problem in Sobolev spaces with variable exponents, by using min-max method and Ekeland's variational principle.

Inspired by the papers above mentioned, we studied problem (1.1). The present article is composed of two sections. In Section 2, we recall the definition of variable exponent Lebesgue-Sobolev spaces. In Section 3, we give the main results and their proofs are presented. We approach is based on the Ricceri Theorem and the Mountain Pass Theorem.

2. PRELIMINARIES

In order to discuss problem (1.1), we need some definitions and basic properties of variable exponent Lebesgue-Sobolev spaces which will be used later [16, 22, 28].

Set

$$C_+(\overline{\Omega}) = \{p : p \in C(\overline{\Omega}), p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $p(x) \in C_+(\overline{\Omega})$, we denote

$$1 < p^- := \inf_{x \in \overline{\Omega}} p(x) \leq p \leq p^+ := \sup_{x \in \overline{\Omega}} p(x) < \infty.$$

We define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u | u : \Omega \rightarrow \mathbb{R} \text{ is a measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.$$

We recall the following so-called Luxemburg norm on this space defined by the formula

$$|u|_{L^{p(x)}(\Omega)} := |u|_{p(x)} = \inf \left\{ \rho > 0 : \int_{\Omega} \left| \frac{u(x)}{\rho} \right|^{p(x)} dx \leq 1 \right\}.$$

Similarly, define the space $L^{p(x)}(\partial\Omega)$ for any $p(x) \in C_+(\partial\Omega)$

$$L^{p(x)}(\partial\Omega) = \left\{ u | u : \partial\Omega \rightarrow \mathbb{R} \text{ is a measurable and } \int_{\partial\Omega} |u(x)|^{p(x)} d\sigma < \infty \right\},$$

with the norm

$$|u|_{L^{p(x)}(\partial\Omega)} = |u|_{p(x), \partial\Omega} := \inf \left\{ \zeta > 0 : \int_{\partial\Omega} \left| \frac{u(x)}{\zeta} \right|^{p(x)} d\sigma \leq 1 \right\},$$

where $d\sigma$ is the measure on the boundary. Then, $L^{p(x)}(\partial\Omega)$ is a Banach space.

Proposition 1 ([22, Theorem 2.1]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ and $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'} \right) |u|_{p(x)} |v|_{p'(x)}.$$

The modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\Psi : L^{p(x)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\Psi(u) = \int_{\Omega} |u(x)|^{p(x)} dx, \quad \forall u \in L^{p(x)}(\Omega).$$

Proposition 2 ([17, Theorem 1.3]). *If $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \dots$, then we have*

- (i) $|u|_{p(x)} > 1 (= 1, < 1) \Leftrightarrow \Psi(u) > 1 (= 1, < 1)$,
- (ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \Psi(u) \leq |u|_{p(x)}^{p^+}$,
- (iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \Psi(u) \leq |u|_{p(x)}^{p^-}$,
- (iv) $|u_n - u|_{p(x)} \rightarrow 0 (\rightarrow \infty) \Leftrightarrow \Psi(u_n - u) \rightarrow 0 (\rightarrow \infty)$.

Proposition 3 ([12, Proposition 2.4]; [25, Proposition 2.5]). *Set $\Phi(u) = \int_{\partial\Omega} |u(x)|^{p(x)} d\sigma$. For $u, u_n \in L^{p(x)}(\partial\Omega)$, $n = 1, 2, \dots$, we have*

- (i) $|u|_{L^{p(x)}(\partial\Omega)} \geq 1 \Rightarrow |u|_{L^{p(x)}(\partial\Omega)}^{p^-} \leq \Phi(u) \leq |u|_{L^{p(x)}(\partial\Omega)}^{p^+}$,

- (ii) $|u|_{L^{p(x)}(\partial\Omega)} \leq 1 \Rightarrow |u|_{L^{p(x)}(\partial\Omega)}^{p^+} \leq \Phi(u) \leq |u|_{L^{p(x)}(\partial\Omega)}^{p^-}$,
 (iii) $|u_n - u|_{p(x)} \rightarrow 0 \ (\rightarrow \infty) \Leftrightarrow \Phi(u_n - u) \rightarrow 0 \ (\rightarrow \infty)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

and equipped with the norm

$$\|u\|_{1,p(x)} := \inf \left\{ \kappa > 0 : \int_{\Omega} \left(\left| \frac{\nabla u(x)}{\kappa} \right|^{p(x)} + \left| \frac{u(x)}{\kappa} \right|^{p(x)} \right) dx \leq 1 \right\},$$

or

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \forall u \in W^{1,p(x)}(\Omega).$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. We can define an equivalent norm

$$\|u\| = |\nabla u|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Proposition 4 ([14, Lemma 2.1]). *Let $p(x)$ and $q(x)$ be measurable functions such that $1 \leq p(x)q(x) \leq \infty$ and $p(x) \in L^\infty(\Omega)$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$\min \left(|u|_{p(x)q(x)}^{p^-}, |u|_{p(x)q(x)}^{p^+} \right) \leq |u|_{q(x)}^{p(x)} \leq \max \left(|u|_{p(x)q(x)}^{p^-}, |u|_{p(x)q(x)}^{p^+} \right).$$

In particular, if $p(x) = p$ is constant, then

$$||u|^p|_{q(x)} = |u|_{pq(x)}^p.$$

Proposition 5 ([12, Theorem 2.1]; [14, Lemma 3.1]; [16, Theorem 1.1]). ; ([22, Theorem 3.1]; [25, Proposition 2.2])

- (i) If $1 < p^- \leq p^+ < \infty$ then, the spaces $L^{p(x)}(\Omega)$, $L^{p(x)}(\partial\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces,
 (ii) if $q(x) \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)}, & N > p(x), \\ \infty, & N \leq p(x), \end{cases}$$

- (iii) if $q(x) \in C_+(\partial\Omega)$ and $q(x) < p^\partial(x)$ for all $x \in \partial\Omega$, then the trace embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$ is compact and continuous, where

$$p^\partial(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & N > p(x), \\ \infty, & N \leq p(x), \end{cases}$$

(iv) (Poincaré inequality) There is a positive constant $C > 0$ such that

$$|u|_{p(x)} \leq C \|u\| \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Remark 1. If $N < p^- \leq p(x)$ for any $x \in \overline{\Omega}$, by Theorem 2.2 in [17], we deduce that $W_0^{1,p^-}(\Omega)$ is compact and continuous embedded in $C(\overline{\Omega})$. Thus, we deduce that $W_0^{1,p(x)}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Defining $\|u\|_{C(\overline{\Omega})} = \sup_{x \in \overline{\Omega}} |u(x)|$, we find that there exists a positive constant $c > 0$ such that

$$|u|_{C(\overline{\Omega})} \leq c \|u\| \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Throughout this paper, we assume that M and f satisfy the following conditions:
(M₀) $M : (0, \infty) \rightarrow (0, \infty)$ is a continuous function such that

$$m_1 t^{\alpha-1} \leq M(t) \leq m_2 t^{\alpha-1} \text{ for all } t > 0$$

where m_1, m_2 and α are real numbers such that $0 < m_1 \leq m_2$ and $\alpha > 1$.

(F1) $f : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory condition and

$$|f(x, t)| \leq a(x) + b|t|^{s(x)-1} \text{ for all } (x, t) \in \partial\Omega \times \mathbb{R},$$

where $a(x) \in L^{\frac{s(x)}{s(x)-1}}(\partial\Omega)$, $b \geq 0$ is a constant and $s(x) \in C_+(\partial\Omega)$ such that

$$1 < s^- := \inf_{x \in \overline{\Omega}} s(x) \leq s(x) \leq s^+ := \sup_{x \in \overline{\Omega}} s(x) < p^- \text{ and } N < p^-.$$

(F2) If $|t| \in (0, 1)$, then $F(x, t) < 0$ and $t \in (t_0, \infty)$ for $t_0 > 1$, then $F(x, t) > \vartheta > 0$.

(F3) $F : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory condition and

$$F(x, tu) = |t|^{q(x)} F(x, u) \text{ for all } t > 0 \text{ and } (x, u) \in \partial\Omega \times \mathbb{R}$$

where $q(x) \in C(\partial\Omega)$ and $F(x, t) = \int_0^t f(x, k) dk$ for all $t > 0$ and $(x, k) \in \partial\Omega \times \mathbb{R}$.

(F4) There exists $\Omega_0 \subset \subset \partial\Omega$ with $\text{meas}_\sigma(\Omega_0) > 0$ such that $F(x, t) > 0$ for all $(x, t) \in \Omega_0 \times \mathbb{R}$.

(F5) $1 < q^+ < p^-$ and $p^- < p^\partial(x)$ for all $x \in \partial\Omega$.

Theorem 1 ([8, Theorem 1]). *Let X be a separable and reflexive real Banach space; $\Phi : X \rightarrow \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^* ; $\Psi : X \rightarrow \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that,*

- (i) $\lim_{\|u\| \rightarrow \infty} J(u) = \lim_{\|u\| \rightarrow \infty} (\Phi(u) + \lambda \Psi(u)) = \infty$ for all $\lambda > 0$,
- (ii) There are $r \in \mathbb{R}$ and $u_0, u_1 \in X$ such that $\Phi(u_0) < r < \Phi(u_1)$,
- (iii) $\inf_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) > \frac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}.$

Then there exist an open interval $\Lambda \subset (0, \infty)$ and a positive real number ρ such that for each $\lambda \in \Lambda$ the equation

$$\Phi'(u) + \lambda \Psi'(u) = 0$$

has at least three solutions in X whose norms are less than ρ .

3. MAIN RESULTS AND PROOFS

Let X denote the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$. Our main results in this paper are the proofs of the following theorems, which are based on results showed in the papers by Bonanno and Candito [8, Theorem 1] and by Willem [26].

Theorem 2. Assume that (M_0) , $(F1)$ and $(F2)$ hold. Then there exist an open interval $\Lambda \subset (0, \infty)$ and a constant $\rho > 0$ such that for any $\lambda \in \Lambda$, problem (1.1) has at least three weak solutions in X whose norms are less than ρ .

Theorem 3. Assume that (M_0) , $(F3)$, $(F4)$ and $(F5)$ hold. Then there exists $\lambda^* > 0$ such that $\lambda \in (0, \lambda^*)$, problem (1.1) has a nontrivial weak solution in X .

Proposition 6 ([5, Theorem 2.9]). Let $f : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying $(F1)$. For each $u \in X$, set $\kappa(u) = \int_{\partial\Omega} f(x, u) d\sigma$. Then $\kappa(u) \in C^1(X, \mathbb{R})$ and

$$\langle \kappa'(u), v \rangle = \int_{\partial\Omega} f(x, u) v d\sigma,$$

for all $v \in X$. Moreover, the operator $\kappa : X \rightarrow X^*$ compact.

Proposition 7 ([30, Theorem 3.1]). If one denotes

$$\varphi(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx$$

then, $\varphi \in C^1(X, \mathbb{R})$ and the derivative operator of φ , denoted by φ' , is

$$\langle \varphi'(u), v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} uv dx,$$

for all $u, v \in X$ and one has

- (i) $\varphi' : X \rightarrow X^*$ is a continuous, bounded, strictly monotone operator and homeomorphism,
- (ii) φ' is a mapping of (S_+) type, that is,
if $u_n \rightharpoonup u$ in X and $\limsup_{n \rightarrow \infty} \langle \varphi'(u_n), u_n - u \rangle \leq 0$ implies, then $u_n \rightarrow u$ in X .

In order to apply Theorem 1, we define the functionals $\Phi, \Psi : X \rightarrow \mathbb{R}$

$$\Phi(u) = \widehat{M} \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx$$

$$\Psi(u) = - \int_{\partial\Omega} F(x, u) d\sigma,$$

where $\widehat{M}(t)$ and $F(x, t)$ are denoted by

$$\widehat{M}(t) = \int_0^t M(k) dk, \quad F(x, t) = \int_0^t f(x, k) dk \text{ for all } t > 0 \text{ and } (x, k) \in \partial\Omega \times \mathbb{R}.$$

Then energy functional associated to the problem (1.1) is $J_\lambda(u) = \Phi(u) + \lambda\Psi(u)$. Moreover, from (F1), (M₀), Proposition 6 and Proposition 7, it is easy to see that the functional $\Phi, \Psi \in C^1(X, \mathbb{R})$ and we can infer that critical points of functional J are the weak solutions for problem (1.1). Then, we have

$$\begin{aligned} \langle \Phi'(u), v \rangle &= M \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\Omega} |u|^{p(x)-2} uv dx \\ \langle \Psi'(u), v \rangle &= - \int_{\partial\Omega} f(x, u) v d\sigma, \end{aligned}$$

for any $u, v \in X$.

We say that $u \in X$ is a weak solution of (1.1) if

$$\begin{aligned} M \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\Omega} |u|^{p(x)-2} uv dx \\ = \lambda \int_{\partial\Omega} f(x, u) v d\sigma \end{aligned}$$

where $v \in X$.

Proof of Theorem 2. To prove this theorem, we first show the condition (i) of Theorem 1. For any $u \in X$ with $\|u\| > 1$, using (M₀) and Proposition 2, we have

$$\begin{aligned} \Phi(u) &= \widehat{M} \left(\int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx \right) + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha} \|u\|^{\alpha p^-} + \frac{1}{p^+} \|u\|^{p^-} \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha} \|u\|^{\alpha p^-}. \end{aligned} \tag{3.1}$$

On the other hand, by (F₁) and Proposition 1, we get

$$\begin{aligned} \lambda\Psi(u) &= -\lambda \int_{\partial\Omega} F(x, u) d\sigma = -\lambda \int_{\partial\Omega} \left(\int_0^{u(x)} f(x, t) dt \right) d\sigma \\ &\geq -\lambda c_0 \int_{\partial\Omega} \left(a(x) |u(x)| + \frac{b}{s(x)} |u(x)|^{s(x)} \right) d\sigma \end{aligned}$$

$$\geq -\lambda c_0 \|a(x)\|_{\frac{s(x)}{s(x)-1}, \partial\Omega} \|u(x)\|_{s(x), \partial\Omega} - \frac{\lambda b}{s^-} \int_{\partial\Omega} |u(x)|^{s(x)} d\sigma. \quad (3.2)$$

Since we have that X is continuous embedding $L^{s^+}(\partial\Omega)$ from Proposition 4, there exist positive constant c_1 such that

$$\|u\|_{L^{s^+}(\partial\Omega)} \leq c_1 \|u\|. \quad (3.3)$$

If we use (3.2) and (3.3), we get

$$\lambda \Psi(u) \geq -\lambda c_0 \|a(x)\|_{\frac{s(x)}{s(x)-1}, \partial\Omega} \|u(x)\|_{s(x), \partial\Omega} - \frac{\lambda b c_1}{s^-} \|u\|^{s^+}.$$

Combining (3.1) and (3.3) for any $\lambda > 0$, we obtain

$$J_\lambda(u) \geq \frac{m_1}{\alpha(p^+)^{\alpha}} \|u\|^{p^-} - \lambda c_0 \|a(x)\|_{\frac{s(x)}{s(x)-1}, \partial\Omega} \|u(x)\|_{s(x), \partial\Omega} - \frac{\lambda b c_1}{s^-} \|u\|^{s^+}.$$

Since $p^- > s^+$, it follows that

$$\lim_{\|u\| \rightarrow \infty} J_\lambda(u) = \lim_{\|u\| \rightarrow \infty} (\Phi(u) + \lambda \Psi(u)) = \infty.$$

Then, condition (i) of Theorem 1 is satisfied.

Now, we show the condition (ii) of Theorem 1. By $f(x, t) = \frac{\partial F(x, t)}{\partial t}$ and the condition (F2), it is obtained that $F(x, t)$ is increasing for $t \in (t_0, \infty)$ and decreasing for $t \in (0, 1)$ uniformly for $x \in \partial\Omega$, and $F(x, 0) = 0$ is obvious, $F(x, t) \rightarrow \infty$ when $t \rightarrow \infty$, since $F(x, t) \geq \vartheta t$ uniformly on x . Then, there exists a real number $\delta > t_0$ such that

$$F(x, t) \geq 0 = F(x, 0) \geq F(x, \eta), \quad \forall u \in X, t > \delta, \eta \in (0, 1). \quad (3.4)$$

Let β, γ be two real numbers such that $0 < \beta < \min\{1, c\}$, with c given in Remark 1. Also, we choose $\gamma > \delta$ satisfying $\gamma^{p^-} |\Omega| > 1$. If we use (3.4), we have

$$F(x, t) \leq F(x, 0) = 0 \text{ for } t \in [0, \beta].$$

Then

$$\int_{\partial\Omega} \sup_{0 < t < \beta} F(x, t) d\sigma \leq \int_{\partial\Omega} F(x, 0) d\sigma = 0. \quad (3.5)$$

Moreover, since $\gamma > \delta$, we write

$$\int_{\partial\Omega} F(x, \gamma) d\sigma > 0$$

and

$$\frac{1}{c^{p^+}} \cdot \frac{\beta^{p^+}}{\gamma^{p^-}} \int_{\partial\Omega} F(x, \gamma) d\sigma > 0. \quad (3.6)$$

Combining (3.5) and (3.6), we obtain

$$\int_{\partial\Omega} \sup_{0 < t < \beta} F(x, t) d\sigma \leq 0 < \frac{1}{c_2^{p^+}} \cdot \frac{\beta^{p^+}}{\gamma^{p^-}} \int_{\partial\Omega} F(x, \gamma) d\sigma.$$

Let $u_0, u_1 \in X$, $u_0(x) = 0$ and $u_1(x) = \gamma$ for any $x \in \overline{\Omega}$. If we define $r = \frac{1}{p^+} \cdot \left(\frac{\beta}{c}\right)^{p^+}$, then $r \in (0, 1)$, $\Phi(u_0) = \Psi(u_0) = 0$,

$$\Phi(u_1) \geq \int_{\Omega} \frac{1}{p(x)} \gamma^{p(x)} dx \geq \frac{1}{p^+} \cdot \gamma^{p^+} |\Omega| > \frac{1}{p^+} \cdot 1 > \frac{1}{p^+} \cdot \left(\frac{\beta}{c}\right)^{p^+} = r$$

and

$$\Psi(u_1) = - \int_{\partial\Omega} F(x, u_1(x)) d\sigma = - \int_{\partial\Omega} F(x, \gamma) d\sigma < 0.$$

So, we deduce that $\Phi(u_0) < r < \Phi(u_1)$, that is, condition (ii) of Theorem 1 is satisfied.

Finally, we show the condition (iii) of Theorem 1, we have

$$\begin{aligned} - \frac{(\Phi(u_1) - r) \Psi(u_0) + (r - \Phi(u_0)) \Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} &= - \frac{r \Psi(u_1)}{\Phi(u_1)} \\ &= r \frac{\int_{\partial\Omega} F(x, \gamma) d\sigma}{\int_{\Omega} \frac{1}{p(x)} \gamma^{p(x)} dx} > 0. \end{aligned}$$

Let $u \in X$ such that $\Phi(u) \leq r < 1$. Since $\frac{1}{p^+} \Psi_{p(x)}(u) \leq \Phi(u) \leq r$ for $u \in X$, we obtain

$$\Psi_{p(x)}(u) \leq p^+ r = \left(\frac{\beta}{c}\right)^{p^+} < 1.$$

Then by Proposition 2 for $\|u\| < 1$, we have

$$\frac{1}{p^+} \|u\|^{p^+} \leq \frac{1}{p^+} \Psi_{p(x)}(u) \leq \Phi(u) \leq r.$$

On the other hand, from Remark 1 and (3.5), we get

$$\|u\|_{C(\overline{\Omega})} \leq c \|u\| \leq c (p^+ r)^{\frac{1}{p^+}} = \beta,$$

for all $u \in X$ and $x \in \overline{\Omega}$ with $\Phi(u) \leq r$.

The above inequality shows that

$$- \inf_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) = \sup_{u \in \Phi^{-1}((-\infty, r])} -\Psi(u) = \int_{\partial\Omega} \sup_{0 < t < \beta} F(x, t) \leq 0.$$

Thus we have

$$- \inf_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) < r \frac{\int_{\partial\Omega} F(x, \gamma) d\sigma}{\int_{\Omega} \frac{1}{p(x)} \gamma^{p(x)} dx}$$

or

$$\inf_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) > \frac{(\Phi(u_1) - r) \Psi(u_0) + (r - \Phi(u_0)) \Psi(u_1)}{\Phi(u_1) - \Phi(u_0)}.$$

So, condition (iii) of Theorem 1 is obtained. Since all conditions of Theorem 1 are verified, there exists an open interval $\Lambda \subset (0, \infty)$ and a positive real number ρ such that for each $\lambda \in \Lambda$ the equation

$$\Phi'(u) + \lambda \Psi'(u) = 0$$

has at least three solutions in X whose norms are *less than* ρ . The proof of Theorem 2 is complete. $\square$

Lemma 1. Suppose (M_0) , (F3) and (F5) hold. Then, there exist two positive real numbers τ, r and $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, we have $J_\lambda(u) \geq r > 0$, $u \in X$ with $\|u\| = \tau$ for all $\tau \in (0, 1)$.

Proof. Since we have the continuous embedding $X \hookrightarrow L^{q(x)}(\partial\Omega)$ from Proposition 4, there exist positive constants c_2 such that

$$\|u\|_{L^{q(x)}(\partial\Omega)} \leq c_2 \|u\| \text{ for all } u \in X. \quad (3.7)$$

On the other hand, using assumption (F3), there exist c_3 and c_4 are positive constant such that

$$|F(x, u)| \leq c_3 |u|^{q(x)} \text{ and } f(x, u) \leq c_4 |u|^{q(x)-1} \text{ for all } x \in \partial\Omega. \quad (3.8)$$

By taking into account (M_0) , (F3), (3.7), (3.8) and Proposition 4, we get

$$\begin{aligned} J_\lambda(u) &\geq \frac{m_1}{\alpha(p^+)^\alpha} \|u\|^{\alpha p^+} + \frac{1}{p^+} \int_{\Omega} |u|^{p(x)} dx - \lambda c_3 \int_{\partial\Omega} |u|^{q(x)} d\sigma \\ &\geq \frac{m_1}{\alpha(p^+)^\alpha} \|u\|^{\alpha p^+} - \lambda c_3 c_2^{q^-} \|u\|^{q^-} \\ &\geq \left(\frac{m_1}{\alpha(p^+)^\alpha} \tau^{\alpha p^+ - q^-} - \lambda c_3 c_2^{q^-} \right) \tau^{q^-}. \end{aligned}$$

By the above inequality, we remark that $q^- < \alpha p^+$ from (F5) and if we choose

$$\lambda^* = \frac{m_1}{2c_3 c_2^{q^-} \alpha(p^+)^\alpha} \tau^{\alpha p^+ - q^-},$$

then, there exist two positive real numbers τ and r such that $\lambda \in (0, \lambda^*)$, we obtain

$$J_\lambda(u) \geq r > 0, \forall u \in X \text{ with } \|u\| = \tau \in (0, 1).$$

The proof of Lemma 1 is complete. $\square$

Lemma 2. Assume that (M_0) , (F3), (F4) and (F5). Then, there exist $\omega \in X$ such that $\omega \geq 0$, $\omega \neq 0$ and $J_\lambda(t\omega) < 0$ for $t > 0$ small enough.

Proof. From assumption (F5), we know that $q^- < p^-$. Let $\varepsilon_0 > 0$ be such that $q^- + \varepsilon_0 < p^-$. On the other hand, since $q \in C(\overline{\Omega})$ it follows that there exists an open

set $\Omega_0 \subset \Omega \subset \subset \partial\Omega$ such that $|q(x) - q^-| < \varepsilon_0$ for all $x \in \overline{\Omega}_0$. Hence, we conclude that

$$q(x) \leq q^- + \varepsilon_0 < p^- \text{ for all } x \in \overline{\Omega}_0.$$

Let $\omega \in C_0^\infty(\Omega)$ be such that $\text{supp}(\omega) \supset \overline{\Omega}_0$, $\omega(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \leq \omega(x) \leq 1$ in Ω . Then, by the above information for any $t \in (0, 1)$, we have

$$\begin{aligned} J_\lambda(t\omega) &= \widehat{M} \left(\int_\Omega \frac{|\nabla t\omega|^{p(x)}}{p(x)} dx \right) + \int_\Omega \frac{1}{p(x)} |t\omega|^{p(x)} dx - \lambda \int_{\partial\Omega} F(x, t\omega) d\sigma \\ &\leq \frac{m_2 t^{\alpha p^-}}{\alpha(p^-)^\alpha} \int_\Omega |\nabla \omega|^{p(x)} dx + \frac{t^{p^-}}{p^-} \int_\Omega |\omega|^{p(x)} dx - \lambda \frac{t^{q^- + \varepsilon_0}}{q^+} \int_{\Omega_0} F(x, \omega) d\sigma \\ &\leq \frac{m_2 t^{p^-}}{p^-} \int_\Omega \left(|\nabla \omega|^{p(x)} + |\omega|^{p(x)} \right) dx - \lambda t^{q^- + \varepsilon_0} \int_{\Omega_0} F(x, \omega) d\sigma. \end{aligned}$$

Thus

$$J_\lambda(t\omega) < 0,$$

for $t < \xi^{1/(p^- - q^- - \varepsilon_0)}$ with

$$0 < \xi < \min \left\{ 1, \frac{\lambda p^-}{m_2} \cdot \frac{\int_{\Omega_0} F(x, \omega) d\sigma}{\int_\Omega \left(|\nabla \omega|^{p(x)} + |\omega|^{p(x)} \right) dx} \right\}.$$

On the other hand, we point out that $\int_\Omega \left(|\nabla \omega|^{p(x)} + |\omega|^{p(x)} \right) dx > 0$ from Proposition 2. If $\int_\Omega \left(|\nabla \omega|^{p(x)} + |\omega|^{p(x)} \right) dx = 0$, then $\|\omega\| = 0$ i.e. $\omega = 0$ in Ω which is a contradiction. Moreover, we know that $\int_{\Omega_0} F(x, \omega) d\sigma > 0$ from (F4). The proof of Lemma 2 is complete. $\square$

Proposition 8.

$$\lim_{n \rightarrow \infty} \int_{\partial\Omega} f(x, u_n) u_n (u_n - u) d\sigma = 0.$$

Proof. By Proposition 1, Proposition 4, Proposition 5 and inequality (3.8), we obtain

$$\begin{aligned} \left| \int_{\partial\Omega} f(x, u_n) (u_n - u) d\sigma \right| &\leq c_4 \left| \int_{\partial\Omega} \left(|u_n|^{q(x)-1} \right) (u_n - u) d\sigma \right| \\ &\leq c_5 \left| |u_n|^{q(x)-1} \right|_{L^{q'(x)}} \|u_n - u\|_{L^{q(x)}(\partial\Omega)}. \end{aligned}$$

where $c_5 > 0$ is a constant. If we consider the compact embedding $X \hookrightarrow L^{q(x)}(\partial\Omega)$, that is, $\|u_n - u\|_{L^{q(x)}(\partial\Omega)} \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\int_{\partial\Omega} f(x, u_n) (u_n - u) d\sigma \rightarrow 0.$$

$\square$

Proof of Theorem 3. By Lemma 1, we infer that there exists on the boundary of the ball centered at the origin and of radius ρ in X such that $\inf_{\partial B_\rho(0)} J_\lambda > 0$. Moreover, from (3.6), there exists $\omega \in X$ such that $J_\lambda(t\omega) < 0$, for all $t > 0$ small enough. Thus, take into account inequality (3.7), we obtain the following

$$-\infty < c_6 := \inf_{B_\rho(0)} J_\lambda < 0.$$

Let choose $\varepsilon > 0$. Then, it follows

$$0 < \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda.$$

Applying Ekeland's variational principle [15] to the functional $J_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$, we can find $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$J_\lambda(u_\varepsilon) < \inf_{B_\rho(0)} J_\lambda + \varepsilon,$$

$$J_\lambda(u_\varepsilon) < J_\lambda(u) + \varepsilon \|u - u_\varepsilon\|, \quad u \neq u_\varepsilon.$$

By the fact that

$$J_\lambda(u_\varepsilon) < \inf_{B_\rho(0)} J_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

we can infer that $u_\varepsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$ by $I_\lambda(u) = J_\lambda(u) + \varepsilon \|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of I_λ , and thus

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} \geq 0,$$

for $t > 0$ small enough and any $v \in B_1(0)$. By the above relation, we have

$$\frac{J_\lambda(u_\varepsilon + tv) - J_\lambda(u_\varepsilon)}{t} + \varepsilon \|v\| \geq 0.$$

Letting $t \rightarrow 0$, we have that $\langle J'_\lambda(u_\varepsilon), v \rangle + \varepsilon \|v\| > 0$ and we infer that $\|J'_\lambda(u_\varepsilon)\| \leq \varepsilon$. We show that there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$J_\lambda(u_n) \rightarrow c = \inf_{B_\rho(0)} J_\lambda < 0 \text{ and } J'_\lambda(u_n) \rightarrow 0.$$

Hence, we have that the sequence $\{u_n\}$ is bounded in X . Thus, there exists $u \in X$ such that, up to a subsequence, $\{u_n\}$ converges weakly to $u \in X$. So $\langle J'_\lambda(u_n), u_n - u \rangle \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, we have

$$\begin{aligned} \langle J'_\lambda(u_n), u_n - u \rangle &= M \left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)} dx \right) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \\ &\quad + \int_\Omega |u_n|^{p(x)-2} u_n (u_n - u) dx - \lambda \int_{\partial\Omega} f(x, u_n) u_n (u_n - u) d\sigma \rightarrow 0. \end{aligned} \quad (3.9)$$

By Proposition 1, Proposition 4, Proposition 5, we have

$$\int_{\Omega} |u_n|^{p(x)-2} u_n (u_n - u) dx \rightarrow 0. \quad (3.10)$$

From (3.9), (3.10) and Proposition 8, we obtain

$$M \left(\int_{\Omega} \frac{|\nabla u_n|^{p(x)}}{p(x)} dx \right) \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \rightarrow 0.$$

Moreover, from $(\mathbf{M}_0)$, we conclude that

$$\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) \rightarrow 0.$$

Eventually, from Proposition (7), we get $u_n \rightarrow u$ in $W_0^{1,p(x)}(\Omega)$, so we conclude that u is a nontrivial weak solution for problem (1.1). The proof of Theorem 3 is complete. $\square$

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