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Half-linear differential equations: Regular variation, principal solutions, and asymptotic classes

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Abstract. We are interested in the structure of the solution space of second-order half-linear differential equations taking into account various classifications regarding asymptotics of solutions. We focus on an exhaustive analysis of the relations among several types of classes which include the classes constructed with respect to the values of the limits of solutions and their quasiderivatives, the classes of regularly varying solutions, the classes of principal and nonprincipal solutions, and the classes of the solutions that obey certain asymptotic formulae. Many of our observations are new even in the case of linear differential equations, and we provide also the revision of existing results.

Keywords: half-linear differential equation, regularly varying function, principal solution, asymptotic formula.

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1 Introduction

We consider the half-linear differential equation

$$(r(t)\Phi(u'))' + p(t)\Phi(u) = 0, \tag{1.1}$$

$t \in [a, \infty)$, $a > 0$, where $r(t) > 0$, $\Phi(u) = |u|^{\alpha-1} \operatorname{sgn} u$, $\alpha > 1$. By Φ^{-1} we mean the inverse of Φ. Note that $\Phi^{-1}(u) = |u|^{\beta-1} \operatorname{sgn} u$, where β is the conjugate number to α, i.e.,

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

We study asymptotic properties of equation (1.1) from several points of view. We deal with the sets of solutions classified according to the values of their limits and the limits of their quasiderivatives, the classes of regularly varying solutions (with prescribed indices), the classes of principal and nonprincipal solutions, and the classes of solutions satisfying quite

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precise asymptotic formulae. We provide an exhaustive discussion concerning the relations among these classes and, in fact, in each setting we describe the entire solution space of (1.1). A big part of our results is new even in the linear case (where such a comprehensive treatment has not been known previously). In addition, we offer a revision and completion of existing results and place them into a broader context. To be more precise, all the results where $p > 0$ (and $L > 0$) are new, with the exception of some of the inclusions involving the formulae in terms of $\mathcal{L}$, which are established in [22, Section 5]. We utilize in the proofs also another results from [22], namely Theorem 3.3 and Lemma 3.5 on regular variation of the elements of the solution space. As for the case $p < 0$, all the results where $\eta < 0$ (in Theorem 2.1 and Theorem 2.2) or $\delta + \alpha < \gamma$ (the entire Theorem 2.3) or $\eta_i < 0$ (in Theorem 2.4) are new. Moreover, the results in the case $p < 0$ are newly supplemented by the formulae in terms of $\mathcal{B}_k$, and some of the known inclusions involving $\mathcal{G}_k, \mathcal{H}_k$ are completed in sense of equalities. The known results which are included in Theorem 2.1 and Theorem 2.2 (except of those involving $\mathcal{L}$) are taken from [19, Section 6] and [23, Section 4], see also Lemma 3.19. The relations with the formulae involving $\mathcal{L}$ in Theorems 2.1, 2.2, 2.4 for the case $p < 0$ and $L < 0$ are taken from [20, Theorem 2, Theorem 4]. Thanks to the parallel analysis of the cases $p < 0$ and $p > 0$, we can see similarities and differences between these two cases. This concerns not only the statements, but also the proofs, some of them can be unified, some other require A different approach. Further relations and comparisons with existing results are spread throughout the text.

Some phenomena which can occur only in the purely half-linear case (i.e., $\alpha \neq 2$) are revealed. Recall that (1.1) arises out when studying radially symmetric solutions of certain partial differential equations with p-Laplacian, thus the results can be useful in theory of PDEs. Our observations are important also from stability point of view and can find applications in a description of Poincaré–Perron solutions which are associated to perturbations of some autonomous nonlinear differential equations.

An important role in our theory is played by the condition

$$\lim_{t\to\infty} \frac{t^{\alpha} p(t)}{r(t)} = C_{\gamma}. \tag{1.2}$$

This condition guarantees that the set of all positive solutions of (1.1) consists of regularly varying solutions of known indices which are related to the value of the limit $C_{\gamma} \in (-\infty, (|\alpha - 1 - \gamma|/\alpha)^{\alpha}]$, γ being the index of regular variation of r, see Theorem 3.3. As for the existence of a regularly varying solution of (1.1), note that there are known conditions in certain integral (more general) forms that are not only sufficient but also necessary (1.1), see [9,10]. Since we assume regular variation of p and r (as we wish to include precise asymptotic formulae into our relations among the classes), the integral conditions reduce to (1.2), and thereby (1.2) actually becomes also necessary, see Lemma 3.5. We however emphasize that thanks to Theorem 3.3 we work with the entire solution space, and there is no sign condition on p a-priori needed.

A deeper approach to asymptotic formulae (including the critical – double root cases, see below) and related problems in the framework not only of Karamata theory, but also of de Haan theory (the classes Gamma and Pi) can be found in [19,20,22,23]. Relations of regularly varying solutions of (1.1) to Poincaré–Perron solutions are examined in [21,22]. For further results concerning asymptotics of half-linear differential equations in the framework of regular variation see [6,9–11,14–17]. A very important work which shows how the Karamata theory can be applied to study qualitative properties of various differential equations is the monograph [12] by Marić, see also [18], where the progress after the year 2000 is summarized.

Recall that by the Sturm type separation theorem which extends to half-linear equations, see [6, Chapter 1], a solution of (1.1) is oscillatory (i.e., it is not of eventually one sign) if and only if all solutions of (1.1) are oscillatory. Hence, we can classify equation (1.1) as oscillatory or nonoscillatory as in the linear case. We are interested in behavior of nonoscillatory solutions of (1.1). Since the solution space (1.1) is homogeneous, without loss of generality we may consider only the set

$$\mathcal{S} = \{y : y(t) \text{ is a positive solution of (1.1) for large } t\}.$$

Assuming that p is eventually of one sign we get that all solutions in $\mathcal{S}$ are eventually monotone, thus any such a solution belongs to one of the classes

$$\mathcal{IS} = \{y \in \mathcal{S} : y'(t) > 0 \text{ for large } t\}, \qquad \mathcal{DS} = \{y \in \mathcal{S} : y'(t) < 0 \text{ for large } t\}.$$

The classes $\mathcal{IS}, \mathcal{DS}$ can further be divided into four mutually disjoint subclasses

$$\mathcal{IS}_B = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = M_y \in (0,\infty)\right\}, \qquad \mathcal{IS}_\infty = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = \infty\right\},$$
$$\mathcal{DS}_B = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = M_y \in (0,\infty)\right\}, \qquad \mathcal{DS}_0 = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = 0\right\}.$$

The so-called quasiderivative $y^{[1]}$ of $y \in \mathcal{S}$ is defined by $y^{[1]} = r\Phi(y')$. We introduce the following convention that is pertinent to the limits of solutions and their quasiderivatives:

$$\mathcal{IS}_{uv} = \left\{y \in \mathcal{IS} : \lim_{t\to\infty} y(t) = u,\ \lim_{t\to\infty} y^{[1]}(t) = v\right\},$$
$$\mathcal{DS}_{uv} = \left\{y \in \mathcal{DS} : \lim_{t\to\infty} y(t) = u,\ \lim_{t\to\infty} |y^{[1]}(t)| = v\right\};$$

for the subscripts of $\mathcal{IS}$ and $\mathcal{DS}$, by $u = B$ and $v = B$ we mean that the value of u and v, respectively, is a positive number. Denote

$$J_p = \int_a^\infty |p(s)|\,\mathrm{d}s, \qquad J_r = \int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s, \tag{1.3}$$

Let $p < 0$. Then

$$\mathcal{S} = \mathcal{IS} \cup \mathcal{DS}, \quad \text{where } \mathcal{IS} \neq \emptyset \neq \mathcal{DS}, \tag{1.4}$$

see [5], [6, Chapter 4]. It is almost immediate (thanks to monotonicity) that

$$\mathcal{IS} = \mathcal{IS}_{\infty\infty} \cup \mathcal{IS}_{\infty B} \cup \mathcal{IS}_{B\infty} \cup \mathcal{IS}_{BB}$$

and

$$\mathcal{DS} = \mathcal{DS}_{00} \cup \mathcal{DS}_{0B} \cup \mathcal{DS}_{B0} \cup \mathcal{DS}_{BB},$$

see also [5], [6, Chapter 4]. The solutions in $\mathcal{IS}_{\infty\infty}$ are called *strongly increasing* and the solutions in $\mathcal{DS}_{00}$ are called *strongly decreasing*, together they form *extremal* solutions. The solutions in $\mathcal{IS}_{\infty B}$ are called *regularly increasing* and the solutions in $\mathcal{DS}_{0B}$ are called *regularly decreasing*.

Let $p > 0$. If $J_r = \infty$, then $\mathcal{DS} = \emptyset$ while if $J_p = \infty$, then $\mathcal{IS} = \emptyset$, see [6, Chapter 4]. Note that if $J_r = \infty = J_p$, then $\mathcal{S} = \emptyset$ since (1.1) is oscillatory by the Leighton–Wintner type criterion, see [6, Theorem 1.2.9]. Moreover, it is easy to show that if $J_r = \infty$ (and $J_p < \infty$), then

$$\mathcal{S} = \mathcal{IS} = \mathcal{IS}_{\infty B} \cup \mathcal{IS}_{\infty 0} \cup \mathcal{IS}_{B0}, \tag{1.5}$$

while if $J_p = \infty$ (and $J_r < \infty$), then

$$\mathcal{S} = \mathcal{DS} = \mathcal{DS}_{B\infty} \cup \mathcal{DS}_{0\infty} \cup \mathcal{DS}_{0B}, \tag{1.6}$$

see [4]. The solutions in $\mathcal{IS}_{\infty B}$ and $\mathcal{DS}_{B\infty}$ are called *dominant*, the solutions in $\mathcal{IS}_{\infty 0}$ and $\mathcal{DS}_{0\infty}$ are called *intermediate*, the solutions in $\mathcal{IS}_{B0}$ and $\mathcal{DS}_{0B}$ are called *subdominant*. An important role in studying (non)emptiness of the subclasses $\mathcal{IS}_{uv}$ and $\mathcal{DS}_{uv}$ and related problems is played by the integral conditions (3.1). Some of these relations will be used in our proofs. For more information in this direction, see [2–6].

If (1.1) is nonoscillatory, then there exists a nontrivial solution y of (1.1) such that for every nontrivial solution u of (1.1) with $u \neq \lambda y$, $\lambda \neq 0$, we have

$$\frac{y'(t)}{y(t)} < \frac{u'(t)}{u(t)} \quad \text{for large } t,$$

see, e.g., [6, Section 4.2]. Such a solution is said to be *principal solution*. Solutions of (1.1) which are not principal are called *nonprincipal solutions*. Principal solutions are unique up to a constant multiple. We denote

$$\mathcal{P} = \{y \in \mathcal{S} : y \text{ is principal}\}.$$

Some characterizations of principal solutions are presented in Theorems 3.20–3.26 for the purposes of our later use, see also [2,3,13]. Note that the situation concerning a description of principal solutions is substantially more complicated in the case $p > 0$ than in the case $p < 0$ for half-linear equations.

A measurable function $f : [a, \infty) \to (0, \infty)$ is called *regularly varying (at infinity) of index ϑ* if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\vartheta \quad \text{for every } \lambda \in (0, \infty); \tag{1.7}$$

we write $f \in \mathcal{RV}(\vartheta)$. If $\vartheta = 0$, we speak about *slowly varying* functions; we write $f \in \mathcal{SV}$, thus $\mathcal{SV} = \mathcal{RV}(0)$. If $f \in \mathcal{RV}(\vartheta)$, then relation (1.7) holds uniformly on each compact λ-set in $(0, \infty)$ (the so-called *Uniform Convergence Theorem*, see, e.g., [1]). It follows that $f \in \mathcal{RV}(\vartheta)$ if and only if there exists a function $L \in \mathcal{SV}$ such that $f(t) = t^\vartheta L(t)$ for every t. The slowly varying component of $f \in \mathcal{RV}(\vartheta)$ will be denoted by L_f, i.e.,

$$L_f(t) := \frac{f(t)}{t^\vartheta}, \tag{1.8}$$

unless stated otherwise. We adopt notation (1.8) also for negative functions f such that $|f| \in \mathcal{RV}(\vartheta)$. The so-called *Representation Theorem* (see, e.g., [1]) says the following: $f \in \mathcal{RV}(\vartheta)$ if and only if

$$f(t) = \varphi(t) t^\vartheta \exp\left\{\int_a^t \frac{\psi(s)}{s}\,\mathrm{d}s\right\}, \tag{1.9}$$

$t \geq a$, for some $a > 0$, where φ, ψ are measurable with $\lim_{t\to\infty} \varphi(t) = C \in (0, \infty)$ and $\lim_{t\to\infty} \psi(t) = 0$. A function $f \in \mathcal{RV}(\vartheta)$ can alternatively be represented as

$$f(t) = \varphi(t) \exp\left\{\int_a^t \frac{\omega(s)}{s}\,\mathrm{d}s\right\}, \tag{1.10}$$

$t \geq a$, for some $a > 0$, where φ, ω are measurable with $\lim_{t\to\infty} \varphi(t) = C \in (0, \infty)$ and $\lim_{t\to\infty} \omega(t) = \vartheta$. A regularly varying function f is said to be *normalized regularly varying*, we write $f \in \mathcal{NRV}(\vartheta)$, if $\varphi(t) \equiv C$ in (1.9) or in (1.10). If (1.9) holds with $\vartheta = 0$ and $\varphi(t) \equiv C$, we say that f is *normalized slowly varying*, we write $f \in \mathcal{NSV}$. We denote

$$\mathcal{S}_{\mathcal{SV}} = \mathcal{S} \cap \mathcal{SV}, \qquad \mathcal{S}_{\mathcal{RV}}(\vartheta)c = \mathcal{S} \cap \mathcal{RV}(\vartheta),$$
$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{S} \cap \mathcal{NSV}, \qquad \mathcal{S}_{\mathcal{NRV}}(\vartheta) = \mathcal{S} \cap \mathcal{NRV}(\vartheta);$$

a similar convention is used when $\mathcal{S}$ is replaced by $\mathcal{DS}$ or $\mathcal{IS}$. Some properties of regularly varying functions are gathered in Proposition 3.1 and Theorem 3.2; for more information see [1,8].

The condition

$$|p| \in \mathcal{RV}(\delta), \qquad r \in \mathcal{RV}(\gamma), \tag{1.11}$$

which plays an important role in our theory, in fact is not needed for showing regular variation of solutions to (1.1), but it enables us to provide a precise asymptotic description. We will assume that $\delta \neq -1$ and $\gamma \neq \alpha - 1$ which leads to avoiding the critical (double-root – see (2.3)) setting. The critical setting (which is considered in connection with searching precise asymptotic formulae in [20,22] and requires a more refined approach) could be treated also in the framework of our topic – a finer classification would however be needed. Denote

$$G(t) = \Phi^{-1}\left(\frac{tp(t)}{r(t)}\right), \quad J = \int_a^\infty |G(t)|\,\mathrm{d}t, \quad H(t) = \frac{t^{\alpha-1}p(t)}{r(t)}, \quad R = \int_a^\infty |H(t)|\,\mathrm{d}t. \tag{1.12}$$

If (1.11) holds and $\delta + \alpha = \gamma$, then

$$G(t) = \frac{1}{t}\Phi^{-1}\left(\frac{L_p(t)}{L_r(t)}\right) \quad \text{and} \quad H(t) = \frac{L_p(t)}{tL_r(t)} \tag{1.13}$$

by Proposition 3.1. Observe that if $\alpha \neq 2$, then the situation where $J = \infty$ and $R < \infty$ (or vice versa) can occur under the conditions (1.11) and $\delta + \alpha = \gamma$. An example can easily be constructed via the relations in (1.13). This fact substantially affects the structure of the solution space of (1.1) which turns out to be more complex than in the linear case. Lemma 3.7 describes a connection of J, R with the integrals in (3.1) which play a central role in studying the existence problems in the classes $\mathcal{IS}_{uv}, \mathcal{DS}_{uv}$. To simplify writing asymptotic formulae, we adopt the notation

$$\mathfrak{E}(\sigma, \tau, K, f) = \exp\left\{\int_\sigma^\tau (1 + o(1))Kf(s)\,\mathrm{d}s\right\},$$

where $o(1)$ is meant either as $\tau \to \infty$ when $\tau < \infty$ or as $\sigma \to \infty$ when $\tau = \infty$. As usually, for f, g which are either both positive or both negative, the relation $f(t) \sim g(t)$ as $t \to \infty$ means $\lim_{t\to\infty} f(t)/g(t) = 1$, while $f(t) = o(g(t))$ as $t \to \infty$ means $\lim_{t\to\infty} f(t)/g(t) = 0$. The sets presented below are introduced for purposes of an easy and synoptic incorporation of asymptotic formulae to other classifications; the constants M_y, N_y are defined by

$$M_y = \lim_{t\to\infty} y(t), \qquad N_y = \lim_{t\to\infty} y^{[1]}(t).$$

The sets $\mathcal{G}_1, \mathcal{G}_2, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ are pertinent to the solutions in the classes $\mathcal{SV}$ and $\mathcal{RV}(\varrho)$, respectively, where

$$\varrho = \frac{\alpha - 1 - \gamma}{\alpha - 1}, \tag{1.14}$$

under the condition $C_\gamma = 0$, and are defined by:

$$\mathcal{G}_1 = \left\{ y \in \mathcal{S} : y(t) = \mathfrak{E}(a, t, -1/\Phi^{-1}(\delta+1), G) \right\},$$
$$\mathcal{G}_2 = \left\{ y \in \mathcal{S} : y(t) = M_y \mathfrak{E}(t, \infty, 1/\Phi^{-1}(\delta+1), G) \right\},$$

and

$$\mathcal{H}_1 = \left\{ y \in \mathcal{S} : y(t) = y(t_0) + \int_{t_0}^{t} r^{1-\beta}(s) \mathfrak{E}(a, s, -(\beta-1)/\Phi(\varrho), H)\, \mathrm{d}s \right\},$$
$$\mathcal{H}_2 = \left\{ y \in \mathcal{S} : y(t) = \int_{t}^{\infty} r^{1-\beta}(s) \mathfrak{E}(a, s, -(\beta-1)/\Phi(\varrho), H)\, \mathrm{d}s \right\},$$
$$\mathcal{H}_3 = \left\{ y \in \mathcal{S} : y(t) = y(t_0) + \int_{t_0}^{t} r^{1-\beta}(s) \Phi^{-1}(N_y) \mathfrak{E}(s, \infty, (\beta-1)/\Phi(\varrho), H)\, \mathrm{d}s \right\},$$
$$\mathcal{H}_4 = \left\{ y \in \mathcal{S} : y(t) = \int_{t}^{\infty} r^{1-\beta}(s) \Phi^{-1}(-N_y) \mathfrak{E}(s, \infty, (\beta-1)/\Phi(\varrho), H)\, \mathrm{d}s \right\}.$$

If $\int_a^\infty |H(s)|\, \mathrm{d}s = \infty$, then $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_0$ (see Lemma 3.14), where

$$\mathcal{H}_0 = \left\{ y \in \mathcal{S} : y(t) = t r^{1-\beta}(s) \mathfrak{E}(a, t, -(\beta-1)/\Phi(\varrho), H) \right\}.$$

The sets $\mathcal{L}_1, \mathcal{L}_2$ which are designed for the case $C_\gamma \neq 0$ and for an alternative description in the case $C_\gamma = 0$ with $\mathcal{RV}(\varrho)$ solutions, are given by:

$$\mathcal{L}_1(\vartheta, \eta) = \left\{ y \in \mathcal{S} : y(t) = t^\vartheta \mathfrak{E}\left(a, t, \frac{1-\beta}{\Phi(\vartheta) - C_\gamma/\vartheta}, L(\vartheta, \eta, \cdot) \right) \right\},$$
$$\mathcal{L}_2(\vartheta, \eta) = \left\{ y \in \mathcal{S} : y(t) = D t^\vartheta \mathfrak{E}\left(t, \infty, \frac{\beta-1}{\Phi(\vartheta) - C_\gamma/\vartheta + \eta|\vartheta|^{\alpha-2}}, L(\vartheta, \eta, \cdot) \right) \right\},$$

where

$$L(\vartheta, \eta, t) = \frac{1}{t} \left[\frac{t^\alpha p(t)}{r(t)} - C_\gamma + \Phi(\vartheta) \left(\frac{t r'(t)}{r(t)} - \gamma \right) \right], \quad \text{with } |L(\vartheta, \eta, \cdot)| \in \mathcal{RV}(\eta - 1),$$

and $D = \lim_{t\to\infty} y(t)/t^\vartheta$. If A is a set, then by the equality $A = \mathcal{L}(\vartheta, \eta)$ we mean that

$$A = \begin{cases} \mathcal{L}_1(\vartheta, \eta) & \text{if } \int_a^\infty |L(\vartheta, \eta, s)|\, \mathrm{d}s = \infty, \\ \mathcal{L}_2(\vartheta, \eta) & \text{if } \int_a^\infty |L(\vartheta, \eta, s)|\, \mathrm{d}s < \infty. \end{cases} \tag{1.15}$$

In view of Proposition 3.1, if $\eta < 0$ and $A = \mathcal{L}(\vartheta, \eta)$, then $A = \mathcal{L}_2(\vartheta, \eta)$. Note that in our results we actually have $\lim_{t\to\infty} L(\vartheta, \eta, t) = 0$, thus by the Representation Theorem (1.9), we get $\mathcal{L}(\vartheta, \eta) \subset \mathcal{RV}(\vartheta)$, $\vartheta \in \mathbb{R}, \eta \leq 0$. If $C_\gamma = 0$, then

$$L(0, \eta, t) = H(t) \quad \text{and} \quad L(\varrho, \eta, t) = \frac{L_p(t)}{t L_r(t)} - \Phi(\varrho) \frac{L_r'(t)}{L_r(t)}.$$

The sets $\mathcal{B}_1, \ldots, \mathcal{B}_6$ are pertinent to the situations where y and/or $y^{[1]}$ have a real nonzero limit and are defined as follows:

$$\mathcal{B}_1 = \left\{ y \in \mathcal{S} : M_y - y(t) \sim \frac{\Phi^{-1}(N_y)}{\varrho} t r^{1-\beta}(t) \text{ as } t \to \infty \right\},$$
$$\mathcal{B}_2 = \left\{ y \in \mathcal{S} : N_y - y^{[1]}(t) \sim \frac{\Phi(M_y)}{\delta+1} t p(t) \text{ as } t \to \infty \right\},$$
$$\mathcal{B}_3 = \left\{ y \in \mathcal{S} : M_y - y(t) \sim \frac{M_y(\alpha-1)}{\Phi^{-1}(\delta+1)(\delta+\alpha-\gamma)} t G(t) \text{ as } t \to \infty \right\},$$
$$\mathcal{B}_4 = \left\{ y \in \mathcal{S} : N_y - y^{[1]}(t) \sim \frac{-N_y}{\Phi(\varrho)(\delta+\alpha-\gamma)} t H(t) \text{ as } t \to \infty \right\},$$
$$\mathcal{B}_5 = \left\{ y \in \mathcal{S} : t|G(t)| = o(|M_y - y(t)|) \text{ as } t \to \infty \right\},$$
$$\mathcal{B}_6 = \left\{ y \in \mathcal{S} : t|H(t)| = o(|N_y - y^{[1]}(t)|) \text{ as } t \to \infty \right\}.$$

2 Main results

In this section we present the main results that are formulated as four theorems; we distinguish, in particular, whether C_γ is zero or not and whether γ is equal to $\delta + \alpha$ or not.

First note that under the assumptions of Theorems 2.1–2.4, we have, for a given $\vartheta \in \mathbb{R}$,

$$\mathcal{S}_{\mathcal{RV}}(\vartheta) = \mathcal{S}_{\mathcal{NRV}}(\vartheta), \tag{2.1}$$

see Remark 3.4. Therefore we omit writing this relation in formulations of the theorems since it holds in each case. It is worthy of noting that because of the properties of principal solutions, in the sets that are equal to $\mathcal{P}$, we have uniqueness up to a constant multiple. This, in particular, means that, for example, in the case (i-a) of Theorem 2.1, there is only one slowly varying solution provided we fix its value at a point.

In Theorems 2.1-2.3, we need to take $\delta \neq -1$, $\gamma \neq \alpha - 1$; Theorem 2.4 does not require an inequality. In fact, the equality in the settings of Theorems 2.1-2.3 would lead to somehow critical cases (which correspond with double roots in (2.3) and/or border-line version of the Karamata integration theorem). Actually, the critical cases can be treated, but a more sophisticated approach is needed and introducing new special asymptotic subclasses is necessary. The main ingredients in analyzing these cases are suitable transformations to non-critical cases and applications of existing results (including the new ones in this paper). We will not go further in this direction. For some considerations concerning the critical case see [20,22].

The first two theorems deal with $\mathcal{SV}$ and $\mathcal{RV}(\varrho)$ solutions under the condition $\gamma = \delta + \alpha$. Recall that ϱ is defined in (1.14).

Theorem 2.1. *Let $C_\gamma = 0$ and (1.11) hold, where $\gamma = \delta + \alpha$. For the relations involving the class $\mathcal{L}(\varrho, \eta)$ assume, in addition, $|L(\varrho, \eta, \cdot)| \in \mathcal{RV}(\eta - 1)$, $\eta \le 0$, and if the condition $\delta < -1$ is supposed, let, in addition, $\delta < -1 + \eta(\alpha - 1)$. Then $\mathcal{S} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho)$, $\mathcal{S}_{\mathcal{NSV}} \neq \emptyset$, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \neq \emptyset$, and the following hold:*

(i) Assume that $J = \infty$ and $R = \infty$.

(i-a) If $p < 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{G}_1 = \mathcal{P}, \quad \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{H}_1 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta).$$

(i-b) If $p < 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{G}_1, \quad \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{H}_2 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

(i-c) If $p > 0$ and $\delta < -1$, then

$$\mathcal{S} = \mathcal{IS} = \mathcal{IS}_{\infty 0} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho), \text{ with } \mathcal{S}_{\mathcal{NSV}} = \mathcal{G}_1 = \mathcal{P}, \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{H}_1 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta).$$

(i-d) If $p > 0$ and $\delta > -1$, then

$$\mathcal{S} = \mathcal{DS} = \mathcal{DS}_{0\infty} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho), \text{ with } \mathcal{S}_{\mathcal{NSV}} = \mathcal{G}_1, \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{H}_2 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

(ii) Assume that $J < \infty$ and $R < \infty$.

(ii-a) If $p < 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{B0} = \mathcal{G}_2 = \mathcal{B}_5 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty B} = \mathcal{H}_3 = \mathcal{B}_6 = \mathcal{L}(\varrho, \eta).$$

(ii-b) If $p < 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_5, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

(ii-c) If $p > 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS}_{B0} = \mathcal{G}_2 = \mathcal{B}_5 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS}_{\infty B} = \mathcal{H}_3 = \mathcal{B}_6 = \mathcal{L}(\varrho, \eta).$$

(ii-d) If $p > 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_5, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

Observe that Theorem 2.1 and Theorem 2.2 have the same general assumptions. They differ in the conditions regarding mutual behavior of J and R. We emphasize that the combinations $J = \infty \wedge R < \infty$ and $J < \infty \wedge R = \infty$, which are assumed in Theorem 2.2, can occur only in the purely half-linear case (i.e., $\alpha \neq 2$), and that is why we separate them into a particular theorem. In view of equalities in (1.13), it is easy to find a suitable example illustrating this setting. Indeed, take $L_r(t) = 1$ and $L_p(t) = 1/\ln^{\omega} t$, where $1 < \omega < \alpha - 1$ or $\alpha - 1 < \omega < 1$. It so arises out that the structure of the solution space in the half-linear case is generally more complex than in the linear one under our setting. In particular, under the conditions of Theorem 2.2, there can coexist strongly monotone solutions with non-extremal ones or intermediate solutions with dominant or subdominant ones. See also [4,5] where the problem of coexistence and non-linear setting is discussed in a more general context.

Theorem 2.2. *Let (1.11) hold, where $\gamma = \delta + \alpha$, and $C_\gamma = 0$. For the relations involving the class $\mathcal{L}(\varrho, \eta)$ assume, in addition, $|L(\varrho, \eta, \cdot)| \in \mathcal{RV}(\eta - 1)$, $\eta \leq 0$, and if the condition $\delta < -1$ is supposed, let, in addition, $\delta < -1 + \eta(\alpha - 1)$. Then $\mathcal{S} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho)$, $\mathcal{S}_{\mathcal{NSV}} \neq \emptyset$, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \neq \emptyset$, and the following hold:*

(i) Assume that $J = \infty$ and $R < \infty$.

(i-a) If $p < 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{G}_1 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho, \eta).$$

(i-b) If $p < 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{G}_1, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

(i-c) If $p > 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS}_{\infty 0} = \mathcal{G}_1 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS}_{\infty B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho, \eta).$$

(i-d) If $p > 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS}_{0\infty} = \mathcal{G}_1, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_6 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

(ii) Assume that $J < \infty$ and $R = \infty$.

(ii-a) If $p < 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{B0} = \mathcal{G}_2 = \mathcal{B}_5 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{H}_1 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta).$$

(ii-b) If $p < 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_5, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{H}_2 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

(ii-c) If $p > 0$ and $\delta < -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS}_{B0} = \mathcal{G}_2 = \mathcal{B}_5 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS}_{\infty\infty} = \mathcal{H}_1 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta).$$

(ii-d) If $p > 0$ and $\delta > -1$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_5, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_2 = \mathcal{H}_0 = \mathcal{L}(\varrho, \eta) = \mathcal{P}.$$

The next theorem can be seen as a complement of Theorems 2.1 and 2.2 in the sense that the condition $\delta + \alpha = \gamma$ will not be satisfied. We assume $\delta + \alpha < \gamma$ which implies $C_\gamma = 0, J < \infty, R < \infty$; this can be seen from Proposition 3.1 (see the proof of Theorem 2.3). On the other hand, in contrast to the case of equality $\delta + \alpha = \gamma$, the strict inequality allows us to consider a richer variety of combinations of conditions $\delta < -1, \delta > -1, \gamma < \alpha - 1, \gamma > \alpha - 1$. Observe that under the setting of Theorem 2.3, there are no extremal or intermediate solutions. The case $\delta + \alpha > \gamma$ is not considered since then there are no regularly varying solutions. Indeed, by Proposition 3.1, we then have $|C_\gamma| = \infty$. If $p < 0$, then by [23], the set $\mathcal{S}$ is nonempty and consists entirely of the solutions in the de Haan classes Γ and Γ_-, which are subsets of rapidly varying functions. If $p > 0$, then equation (1.1) is oscillatory by Hille–Nehari type criteria, see [6, Chapter 3], and so $\mathcal{S}$ is empty. In fact, to show that there are no $\mathcal{RV}$ solutions, we can argue in a alternative way, namely that the necessary condition is not fulfilled, see Lemma 3.5.

Theorem 2.3. *Let (1.11) hold, where $\gamma > \delta + \alpha$. For the relations involving the class $\mathcal{L}(\varrho, \eta)$ assume, in addition, $|L(\varrho, \eta, \cdot)| \in \mathcal{RV}(\eta - 1)$, $\eta \leq 0$, and if the condition $\gamma < \alpha - 1$ is supposed, let, in addition, $\gamma < (\alpha - 1)(1 + \eta)$. Then $\mathcal{S} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho)$, $\mathcal{S}_{\mathcal{NSV}} \neq \emptyset$, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \neq \emptyset$, and the following hold:*

(i) Assume that $\delta < -1$ and $\gamma < \alpha - 1$.

(i-a) If $p < 0$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{B0} = \mathcal{G}_2 = \mathcal{B}_3 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty B} = \mathcal{H}_3 = \mathcal{B}_4 = \mathcal{L}(\varrho, \eta).$$

(i-b) If $p > 0$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS}_{B0} = \mathcal{G}_2 = \mathcal{B}_3 = \mathcal{P}, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS}_{\infty B} = \mathcal{H}_3 = \mathcal{B}_4 = \mathcal{L}(\varrho, \eta).$$

(ii) Assume that $\delta > -1$ and $\gamma > \alpha - 1$.

(ii-a) If $p < 0$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{B0} = \mathcal{G}_2 = \mathcal{B}_3, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_4 = \mathcal{L}(\varrho,\eta) = \mathcal{P}.$$

(ii-b) If $p > 0$, then

$$\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_3, \ \ \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_4 = \mathcal{L}(\varrho,\eta) = \mathcal{P}.$$

(iii) Assume that $\delta < -1$ and $\gamma > \alpha - 1$.

(iii-a) If $p < 0$, then

$$\begin{aligned} &\mathcal{IS}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{BB} = \mathcal{B}_1 = \mathcal{B}_2 \neq \emptyset, \\ &\mathcal{DS}_{\mathcal{NSV}} = \mathcal{DS}_{B0} \cup \mathcal{DS}_{BB}, \ \ \mathcal{DS}_{B0} = \mathcal{G}_2 = \mathcal{B}_3 \neq \emptyset, \ \ \mathcal{DS}_{BB} = \mathcal{B}_1 = \mathcal{B}_2 \neq \emptyset, \\ &\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_4 = \mathcal{L}(\varrho,\eta) = \mathcal{P}. \end{aligned}$$

(iii-b) If $p > 0$, then

$$\begin{aligned} &\mathcal{IS}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{B0} \cup \mathcal{IS}_{BB}, \ \ \mathcal{IS}_{B0} = \mathcal{G}_2 = \mathcal{B}_3 \neq \emptyset, \ \ \mathcal{IS}_{BB} = \mathcal{B}_1 = \mathcal{B}_2 \neq \emptyset, \\ &\mathcal{DS}_{\mathcal{NSV}} = \mathcal{DS}_{BB} = \mathcal{B}_1 = \mathcal{B}_2 \neq \emptyset, \\ &\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{\mathcal{NRV}}(\varrho) = \mathcal{DS}_{0B} = \mathcal{H}_4 = \mathcal{B}_4 = \mathcal{L}(\varrho,\eta) = \mathcal{P}. \end{aligned}$$

One can see that the case $\delta > -1$ and $\gamma < \alpha - 1$ is not considered in the previous theorem. This is quite natural because there are no regularly varying solutions; the reasons are almost the same as in the case $\alpha + \delta > \gamma$ (discussed before Theorem 2.3). Indeed, if $p < 0$, there are solutions only in the de Haan classes Γ and Γ_-, see [23]. If $p > 0$, then (1.1) is oscillatory by Hille–Wintner type criterion, see [6]. Alternatively, we can again argue by Lemma 3.5 since $\delta + \alpha > -1 + \gamma + 1 = \gamma$.

The next theorem can be seen as a complement to the previous ones in the sense that previously was assumed (or was guaranteed) $C_\gamma = 0$ and now we take $C_\gamma \neq 0$. Note that $C_\gamma \neq 0$ and $r \in \mathcal{RV}(\gamma)$ imply $|p| \in \mathcal{RV}(\gamma - \alpha)$. Indeed, from (1.2) and Proposition 3.1, we have $|p(t)| \sim |C_\gamma| t^{-\alpha} r(t) \in \mathcal{RV}(-\alpha + \gamma)$ as $t \to \infty$. In general, we do not need to exclude the critical case $\gamma = \alpha - 1$. However, if we take $C_\gamma > 0$, then necessarily $\gamma \neq \alpha - 1$ since we assume $C_\gamma \leq K_\gamma$, where

$$K_\gamma = \left(\frac{|\alpha - 1 - \gamma|}{\alpha} \right)^\alpha. \tag{2.2}$$

We denote

$$\vartheta_i = \Phi(\lambda_i), \qquad \vartheta_1 \leq \vartheta_2,$$

where $\lambda_1 \leq \lambda_2$ are the (real) roots of

$$F_\gamma(\lambda) := |\lambda|^\beta + \frac{\gamma + 1 - \alpha}{\alpha - 1}\lambda + \frac{C_\gamma}{\alpha - 1} = 0. \tag{2.3}$$

If $\eta_2 = 0$ in Theorem 2.4, then we do not need to assume $\gamma + \alpha(\vartheta_2 - 1) + \eta_2 > -1$, since this inequality is satisfied automatically thanks to the properties of the roots, see Lemma 3.6. Observe that under the setting of Theorem 2.4, there are only extremal solutions (when $p < 0$) or intermediate solutions (when $p > 0$). In the case $C_\gamma = K_\gamma$, generally oscillation or nonoscillation of (1.1) can occur. Nonoscillation is guaranteed e.g. by $t^\alpha p(t)/r(t) \leq C_\gamma$ (this follows from the Sturm type theorem, see [6]), or by the conditions of [9, Theorem 2.2, Theorem 3.2], or by some suitable nonoscillation criterion, see, e.g., [6, Chapter 3].

Theorem 2.4. *Let $C_\gamma \in (-\infty, K_\gamma] \setminus \{0\}$ and $r \in \mathcal{NRV}(\gamma) \cap C^1$, $\gamma \in \mathbb{R}$. For the relations involving the classes $\mathcal{L}(\vartheta_i, \eta_i)$, $i = 1, 2$, assume, in addition, $|L(\vartheta_i, \eta_i, \cdot)| \in \mathcal{RV}(\eta_i - 1)$, where $\eta_1, \eta_2 \le 0$, and $\gamma + \alpha(\vartheta_2 - 1) + \eta_2 > -1$. Then $\mathcal{S} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2)$, $\mathcal{S}_{\mathcal{NRV}}(\vartheta_i) \neq \emptyset$, $i = 1, 2$, and the following hold:*

(i) Assume that $C_\gamma < 0$. Then

$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{L}(\vartheta_1, \eta_1) = \mathcal{P}, \quad \vartheta_1 < 0,$$
$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{L}(\vartheta_2, \eta_2), \quad \vartheta_2 > 0.$$

(ii) Assume that $0 < C_\gamma \le K_\gamma$; the strict inequality $C_\gamma < K_\gamma$ is required only when the relations involving the classes $\mathcal{L}(\vartheta_i, \eta_i)$, $i = 1, 2$, are considered. If $C_\gamma = K_\gamma$, we assume, in addition, nonoscillation of (1.1).

(ii-a) If $\gamma < \alpha - 1$, then

$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{S} = \mathcal{IS} = \mathcal{IS}_{\infty 0}, \quad \vartheta_1, \vartheta_2 > 0,$$
$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) = \mathcal{L}(\vartheta_1, \eta_1) = \mathcal{P}, \quad \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{L}(\vartheta_2, \eta_2).$$

(ii-b) If $\gamma > \alpha - 1$, then

$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{S} = \mathcal{DS} = \mathcal{DS}_{0\infty}, \quad \vartheta_1, \vartheta_2 < 0$$
$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) = \mathcal{L}(\vartheta_1, \eta_1) = \mathcal{P}, \quad \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{L}(\vartheta_2, \eta_2).$$

For various examples that illustrate, in particular, the asymptotic formulae in particular settings, see [20,22]. Among others it is shown that the situation where $\int_a^\infty |L(\vartheta_1, \eta_1, s)| \, ds = \infty$ and $\int_a^\infty |L(\vartheta_2, \eta_2, s)| \, ds < \infty$ (or vice versa) can occur even when $\eta_1 = \eta_2 = 0$.

In [21, 22] we explore how some of the above results can be applied to the half-linear equation of the form

$$(\Phi(y'))' + a(t)\Phi(y') + b(t)\Phi(y) = 0$$

to analyze its Poincaré–Perron solutions (that is the solutions y such that $\lim_{t\to\infty} y'(t)/y(t)$ exists as a finite number). The equation can be viewed as a perturbation of the equation with constant coefficients. A key role is played by a suitable transformation, and we believe that the new results of this paper could be extended in this sense. Another direction is an extension to the critical (double-root) case which is roughly explained at the beginning of this section. Since theory of regularly varying sequences is at disposal and difference equations often show their particularities (when compared with their continuous counterparts), a discrete version of our results is also of interest.

3 Auxiliary statements and proofs

We start with selected properties of regularly varying functions.

Proposition 3.1.

(i) If $f \in \mathcal{RV}(\vartheta)$, then $\ln f(t) / \ln t \to \vartheta$ as $t \to \infty$. It then clearly implies that $\lim_{t\to\infty} f(t) = 0$ provided $\vartheta < 0$, and $\lim_{t\to\infty} f(t) = \infty$ provided $\vartheta > 0$.

(ii) If $f \in \mathcal{RV}(\vartheta)$, then $f^\alpha \in \mathcal{RV}(\alpha\vartheta)$ for every $\alpha \in \mathbb{R}$.

(iii) If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, $f_2(t) \to \infty$ as $t \to \infty$, then $f_1 \circ f_2 \in \mathcal{RV}(\vartheta_1 \vartheta_2)$.

(iv) If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1,2$, then $f_1 + f_2 \in \mathcal{RV}(\max\{\vartheta_1, \vartheta_2\})$.

(v) If $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1,2$, then $f_1 f_2 \in \mathcal{RV}(\vartheta_1 + \vartheta_2)$.

(vi) If $f_1, \dots, f_n \in \mathcal{RV}$, $n \in \mathbb{N}$, and $R(x_1, \dots, x_n)$ is a rational function with nonnegative coefficients, then $R(f_1, \dots, f_n) \in \mathcal{RV}$.

(vii) If $L \in \mathcal{SV}$ and $\vartheta > 0$, then $t^\vartheta L(t) \to \infty$, $t^{-\vartheta} L(t) \to 0$ as $t \to \infty$.

(viii) If $f \in \mathcal{RV}(\vartheta)$ and a measurable g is such that $g(t) \sim f(t)$ as $t \to \infty$. Then $g \in \mathcal{RV}(\vartheta)$.

(ix) If $f \in \mathcal{RV}(\vartheta)$, $\vartheta \neq 0$, then there exists $g \in C^1$ with $g(t) \sim f(t)$ as $t \to \infty$ and such that $tg'(t)/g(t) \to \vartheta$, whence $g \in \mathcal{NRV}(\vartheta)$. Moreover, g can be taken such that $|g'| \in \mathcal{NRV}(\vartheta - 1)$.

(x) Let f be eventually positive and differentiable, and let $\lim_{t\to\infty} tf'(t)/f(t) = \vartheta$. Then $f \in \mathcal{NRV}(\vartheta)$.

(xi) If $|f'| \in \mathcal{RV}(\vartheta)$, $\vartheta \neq -1$, with f' being eventually of one sign, then $f \in \mathcal{NRV}(\vartheta + 1)$.

Proof. The proofs of (i)–(x) are either easy or can be found in [1,8]. For (xi) see [19]. □

The following statement (the so-called Karamata integration theorem) is of great importance in our theory.

Theorem 3.2 ([1]). *Let $L \in \mathcal{SV}$.*

(i) If $\vartheta < -1$, then $\int_t^\infty s^\vartheta L(s)\,\mathrm{d}s \sim t^{\vartheta+1} L(t)/(-\vartheta - 1)$ as $t \to \infty$.

(ii) If $\vartheta > -1$, then $\int_a^t s^\vartheta L(s)\,\mathrm{d}s \sim t^{\vartheta+1} L(t)/(\vartheta + 1)$ as $t \to \infty$.

(iii) If $\int_a^\infty L(s)/s\,\mathrm{d}s$ converges, then $\widetilde{L}(t) = \int_t^\infty L(s)/s\,\mathrm{d}s$ is a $\mathcal{SV}$ function; if $\int_a^\infty L(s)/s\,\mathrm{d}s$ diverges, then $\widetilde{L}(t) = \int_a^t L(s)/s\,\mathrm{d}s$ is a $\mathcal{SV}$ function; in both cases, $L(t)/\widetilde{L}(t) \to 0$ as $t \to \infty$.

Finiteness of the limit in (1.2) guarantees (in nonoscillatory case) regular variation of all positive solutions.

Theorem 3.3 ([22]). *Let $r \in \mathcal{RV}(\gamma)$, $\gamma \in \mathbb{R}$, and $C_\gamma \in (-\infty, K_\gamma]$ be defined by (1.2), $K_\gamma = (|\alpha - 1 - \gamma|/\alpha)^\alpha$. We assume, in addition, nonoscillation of (1.1) when $C = K_\gamma$ with $t^\alpha p(t)/r(t) \not\leq K_\gamma$ (in all other cases, nonoscillation is automatically guaranteed). Then $\mathcal{S} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2)$ with $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \neq \emptyset \neq \mathcal{S}_{\mathcal{NRV}}(\vartheta_2)$, where $\lambda_i = \Phi(\vartheta_i)$, $i = 1,2$, are the roots of (2.3).*

Remark 3.4. In the proof of Theorem 3.3 it is actually shown that for any $y \in \mathcal{S}$, we have $\lim_{t\to\infty} ty'(t)/y(t) \in \{\vartheta_1, \vartheta_2\}$. That is why any regularly varying solution is automatically normalized; in other words, (2.1) holds. But even without a-priori assuming (1.2), it can be proved that $\mathcal{S}_{\mathcal{RV}}(\vartheta) \subseteq \mathcal{S}_{\mathcal{NRV}}(\vartheta)$ under the assumption of regular variation of r, by means of Lemma 3.5 and Proposition 3.1. Normality follows also from the asymptotic formulae or from monotonicity of solutions and quasiderivatives with the help of the properties of regularly varying functions.

Under our setting, condition (1.2) is necessary for the existence of a regularly varying solution.

Lemma 3.5 ([22]). *Let (1.11) hold with $\delta \neq -1$ and $\gamma \neq \alpha - 1$. If $\mathcal{S}_{\mathcal{RV}}(\vartheta) \neq \emptyset$, where $\lambda = \Phi(\vartheta)$ is a real root of (2.3), then $\lim_{t\to\infty} t^\alpha p(t)/r(t) = C_\gamma$ and $\delta + \alpha \leq \gamma$.*

Lemma 3.6 ([22]). *Let $\lambda_1^\pm \leq \lambda_2^\pm$ denote the (real) roots of (2.3) when $\operatorname{sgn}(\alpha - 1 - \gamma) = \pm 1$ and let $\lambda_1 \leq \lambda_2$ denote the (real) roots of (2.3) when $\gamma = \alpha - 1$. Set $\vartheta_i^\pm = \Phi(\lambda_i^\pm)$ and $\vartheta_i = \Phi(\lambda_i)$, $i = 1,2$.*

(i) Let $\gamma \neq \alpha - 1$. *If* $C_\gamma < 0$, *then* $\vartheta_1^{\pm}\vartheta_2^{\pm} < 0$, $|\vartheta_1^+| = \vartheta_2^-$, *and* $|\vartheta_1^-| = \vartheta_2^+ > |\varrho|$. *If* $C_\gamma = 0$, *then* $\vartheta_1^+ = \vartheta_2^- = 0$ *and* $-\vartheta_1^- = \vartheta_2^+ = |\varrho|$. *If* $C_\gamma \in (0, K_\gamma)$, *then* $\vartheta_1^{\pm}\vartheta_2^{\pm} > 0$ *and* $\vartheta_1^+ = |\vartheta_2^-| < |\gamma+1-\alpha|/\alpha < \vartheta_2^+ = |\vartheta_1^-| < |\varrho|$. *If* $C_\gamma = K_\gamma$, *then* $-\vartheta_1^- = -\vartheta_2^- = \vartheta_1^+ = \vartheta_2^+ = |\gamma+1-\alpha|/\alpha$.

(ii) Let $\gamma = \alpha - 1$. *Then* $C_\gamma \leq 0$ *with* $\vartheta_1 = \vartheta_2 = 0$ *when* $C_\gamma = 0$ *while* $\vartheta_{1,2} = \pm(|C_\gamma|/(\alpha-1))^{1/\alpha}$ *when* $C_\gamma < 0$.

Denote

$$J_1 = \int_a^\infty V_1(t)\,\mathrm{d}t, \quad J_2 = \int_a^\infty V_2(t)\,\mathrm{d}t, \quad R_1 = \int_a^\infty W_1(t)\,\mathrm{d}t, \quad R_2 = \int_a^\infty W_2(t)\,\mathrm{d}t, \tag{3.1}$$

where

$$V_1(t) = r^{1-\beta}(t)\left(\int_a^t |p(s)|\,\mathrm{d}s\right)^{\beta-1}, \qquad V_2(t) = r^{1-\beta}(t)\left(\int_t^\infty |p(s)|\,\mathrm{d}s\right)^{\beta-1},$$
$$W_1(t) = |p(t)|\left(\int_a^t r^{1-\beta}(s)\,\mathrm{d}s\right)^{\alpha-1}, \qquad W_2(t) = |p(t)|\left(\int_t^\infty r^{1-\beta}(s)\,\mathrm{d}s\right)^{\alpha-1}.$$

These integrals naturally occur when studying (non)emptiness of the classes $\mathcal{IS}_{uv}, \mathcal{DS}_{uv}$ and play an important role also in characterization of principal solutions, see [2–6]. Later, in the proofs we use some of these results.

Since we work in the framework of regular variation, some specific and useful properties of V_1, V_2, W_1, W_2 can be derived.

Lemma 3.7. *Let* (1.11) *hold. Then*

(i) $V_i(t) \sim |G(t)|/|\delta+1|^{\beta-1}$ *as* $t \to \infty$, *where* $i = 1$ *when* $\delta > -1$ *while* $i = 2$ *when* $\delta < -1$.

(ii) $W_i(t) \sim |H(t)|/|\gamma(1-\beta)+1|^{\alpha-1}$ *as* $t \to \infty$, *where* $i = 1$ *when* $\gamma < \alpha - 1$ *while* $i = 2$ *when* $\gamma > \alpha - 1$.

(iii) If $\delta < -1$, *then* $V_1(t) \sim J_p^{\beta-1} r^{1-\beta}(t)$ *as* $t \to \infty$, *where* J_p *is defined in* (1.3).

(iv) If $\gamma > \alpha - 1$, *then* $W_1(t) \sim J_r^{\alpha-1}|p(t)|$ *as* $t \to \infty$, *where* J_r *is defined in* (1.3).

Proof. The asymptotic formulae in (i) and (ii) follow from the Karamata Integration Theorem (Theorem 3.2). The relations in (iii) and (iv) are obvious; convergence of the integrals J_p and J_r, respectively, is a consequence of Theorem 3.2. □

Remark 3.8. Let (1.11) hold. If $\delta > -1$, then $\int_a^\infty |p(s)|\,\mathrm{d}s = \infty$, thus $\int_a^\infty V_2(s)\,\mathrm{d}s$ cannot converge. If $\gamma < \alpha - 1$, then $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty$, thus $\int_a^\infty W_2(s)\,\mathrm{d}s$ cannot converge. Now from Lemma 3.7 it easily follows that:

(i) Let $\delta > -1$. Then a) $J_1 = \infty \Leftrightarrow J = \infty$, b) $J_2 = \infty$.

(ii) Let $\delta < -1$. Then a) $J_1 = \infty \Leftrightarrow J_r = \infty$, b) $J_2 = \infty \Leftrightarrow J = \infty$.

(iii) Let $\gamma < \alpha - 1$. Then a) $R_1 = \infty \Leftrightarrow R = \infty$, b) $R_2 = \infty$.

(iv) Let $\gamma > \alpha - 1$. Then a) $R_1 = \infty \Leftrightarrow J_p = \infty$, b) $R_2 = \infty \Leftrightarrow R = \infty$.

The first statement in the following lemma is sometimes called the *reciprocity principle* and equation (3.2) is called the *reciprocal equation* (to equation (1.1)).

Lemma 3.9. *Let y be a solution of (1.1) with $p \neq 0$. If $u = |y^{[1]}|$, then u is a solution of*

$$(\widehat{r}(t)\Phi^{-1}(u'))' + \widehat{p}(t)\Phi^{-1}(u) = 0, \tag{3.2}$$

where $\widehat{r} = |p|^{1-\beta}$ and $\widehat{p} = r^{1-\beta}\operatorname{sgn} p$. In particular, if $y \in \mathcal{S}$, then

$$u \in \widehat{\mathcal{S}} = \{u : u \text{ is an eventually positive solution of } (3.2)\}.$$

If $\widehat{G}(t) = \Phi(t\widehat{p}(t)/\widehat{r}(t))$ and $\widehat{H}(t) = t^{\beta-1}\widehat{p}(t)/\widehat{r}(t)$, then

$$\widehat{G} = H \quad \text{and} \quad \widehat{H} = G. \tag{3.3}$$

If (1.11) holds, then

$$|\widehat{p}| \in \mathcal{RV}(\widehat{\delta}) \text{ and } \widehat{r} \in \mathcal{RV}(\widehat{\gamma}), \text{ where } \widehat{\delta} = \gamma(1-\beta) \text{ and } \widehat{\gamma} = \delta(1-\beta). \tag{3.4}$$

Proof. Since $u' = -p\Phi(y)$, we get $y = -|p|^{1-\beta}\Phi^{-1}(u')\operatorname{sgn} p$. From $u = r\Phi(y')$, we have $y' = r^{1-\beta}\Phi^{-1}(u)$. Thus we find that u satisfies (3.2). The relations in (3.3) are obvious. The relations in (3.4) follow easily by Proposition 3.1. □

Remark 3.10. For the notation of subclasses of $\widehat{\mathcal{S}}$ we use the "circumflex analog" of the notation of subclasses of $\mathcal{S}$. For instance, $\widehat{\mathcal{DS}}$ and $\widehat{\mathcal{DS}}_{B0}$ mean the set of eventually decreasing solutions of (3.2) and the subset of $\widehat{\mathcal{DS}}$ where $u \in \widehat{\mathcal{DS}}_{B0}$ tends to a positive constant with $\lim_{t\to\infty} \widehat{r}(t)\Phi^{-1}(u(t)) = 0$, respectively. Similarly we approach to the notation of the classes for the solutions satisfying prescribed asymptotic formulae. For example, $\widehat{\mathcal{G}}_2$ is defined as $\widehat{\mathcal{G}}_2 = \{u \in \widehat{\mathcal{S}} : u(t) = M_u\mathfrak{E}(t, \infty, 1/\Phi(\widehat{\delta}+1), \widehat{G})\}$, where $M_u = \lim_{t\to\infty} u(t)$.

Lemma 3.11. *Let (1.11) be satisfied with $\delta \neq -1$ and $\gamma \neq \alpha - 1$. Then the following hold:*

(i) $\mathcal{DS}_{B0} \cup \mathcal{DS}_{B\infty} \cup \mathcal{IS}_{B0} \cup \mathcal{IS}_{B\infty} \subseteq X$, *where $X = \mathcal{B}_5$ when $\delta + \alpha = \gamma$, while $X = \mathcal{B}_3$ when $\delta + \alpha < \gamma$.*

(ii) $\mathcal{DS}_{0B} \cup \mathcal{IS}_{\infty B} \subseteq X$, *where $X = \mathcal{B}_6$ when $\delta + \alpha = \gamma$, while $X = \mathcal{B}_4$ when $\delta + \alpha < \gamma$.*

(iii) $\mathcal{IS}_{BB} \cup \mathcal{DS}_{BB} \subseteq \mathcal{B}_i$, *$i = 1, 2$.*

Proof. (i) Let $y \in \mathcal{DS}_{B0} \cup \mathcal{DS}_{B\infty} \cup \mathcal{IS}_{B0} \cup \mathcal{IS}_{B\infty}$. Then $y \in \mathcal{S}_{\mathcal{SV}}$, and so $y \in \mathcal{S}_{\mathcal{NSV}}$, see Remark 3.4. Integrating (1.1) we get $y^{[1]}(t) \sim P_y(t)$ as $t \to \infty$, where $P_y(t) = \int_t^\infty p(s)\Phi(y(s))\,\mathrm{d}s$ or $P_y(t) = -\int_{t_0}^t p(s)\Phi(y(s))\,\mathrm{d}s$ according to whether $\delta < -1$ or $\delta > -1$, respectively. Applying Theorem 3.2 and using $y(t) \sim M_y$, where $M_y = \lim_{t\to\infty} y(t)$, in both cases we get

$$y^{[1]}(t) \sim \frac{-1}{\delta+1} tp(t)\Phi(y(t)) \sim \frac{-1}{\delta+1} tp(t)\Phi(M_y)$$

as $t \to \infty$, thus $y'(t) \sim -M_y G(t)/\Phi^{-1}(\delta+1)$ as $t \to \infty$. Integrating the last relation from t to ∞, we obtain

$$M_y - y(t) \sim \frac{-M_y}{\Phi^{-1}(\delta+1)} \int_t^\infty G(s)\,\mathrm{d}s \tag{3.5}$$

as $t \to \infty$. Assume that $\delta + \alpha = \gamma$. Since $|G| \in \mathcal{RV}(-1)$, from Theorem 3.2 we get $t|G(t)| = o\left(\int_t^\infty |G(s)|\,\mathrm{d}s\right)$ as $t \to \infty$. Combining the last relation with (3.5), we find that $y \in \mathcal{B}_5$. Assume that $\delta + \alpha < \gamma$. Then, in view of Proposition 3.1, $|G| \in \mathcal{RV}(\zeta - 1)$, where $\zeta =$

$(\beta-1)(\delta+1-\gamma)+1$. From $\delta+\alpha<\gamma$ we have $\zeta<0$. Hence, Theorem 3.2 yields $\int_t^\infty G(s)\,\mathrm{d}s \sim -tG(t)/\zeta$ as $t\to\infty$, thus (3.5) implies $y\in\mathcal{B}_3$.

(ii) Let $y\in\mathcal{DS}_{0B}\cup\mathcal{IS}_{\infty B}\subseteq X$. Set $u=|y^{[1]}|$. We have $u=\pm y^{[1]}$ according to whether $y\in\mathcal{IS}$ or $y\in\mathcal{DS}$, respectively. Then u satisfies (3.2), and $\lim_{t\to\infty}u(t)=M_u$ where $M_u=|N_y|$, $N_y=\lim_{t\to\infty}y^{[1]}(t)$. Since, in addition,

$$u^{[1]}=\widehat{r}\Phi^{-1}(u')=|p|^{1-\beta}\Phi^{-1}(\pm(r\Phi(y'))')=\mp|p|^{1-\beta}\Phi^{-1}(p\Phi(y))=\mp y\,\mathrm{sgn}\,p, \tag{3.6}$$

we get

$$u\in\widehat{\mathcal{DS}}_{B0}\cup\widehat{\mathcal{DS}}_{B\infty}\cup\widehat{\mathcal{IS}}_{B0}\cup\widehat{\mathcal{IS}}_{B\infty}. \tag{3.7}$$

We use the convention introduced in Lemma 3.9 and Remark 3.10. The reciprocal version of $\delta+\alpha\le\gamma$ is $\widehat{\delta}+\beta\le\widehat{\gamma}$; it is easy to see that the inequalities are in fact the same. In view of (3.7), we can apply part (i) of Lemma 3.11 to the reciprocal equation. If $\delta+\alpha<\gamma$, then

$$\begin{aligned}|N_y|-|y^{[1]}(t)|=M_u-u(t)&\sim\frac{M_u(\beta-1)}{\Phi(\widehat{\delta}+1)(\widehat{\delta}+\beta-\widehat{\gamma})}t\Phi\left(\frac{t\widehat{p}(t)}{\widehat{r}(t)}\right)\\&=\frac{|N_y|}{\Phi(\gamma(1-\beta)+1)(-\gamma+\beta/(\beta-1)+\delta)}t^\alpha\frac{p(t)}{r(t)}=\frac{-|N_y|}{\Phi(\varrho)(\delta+\alpha-\gamma)}tH(t)\end{aligned}$$

as $t\to\infty$. Consequently, $y\in\mathcal{B}_4$. Similarly we find that $\mathcal{B}_6$ is reciprocal version of $\mathcal{B}_5$.

(iii) Let $y\in\mathcal{IS}_{BB}\cup\mathcal{DS}_{BB}$. From (1.1), $(y^{[1]}(t))'\sim -M_y^{\alpha-1}p(t)$ as $t\to\infty$, where $M_y=\lim_{t\to\infty}y(t)$. Theorem 3.2 yields

$$N_y-y^{[1]}(t)\sim -M_y^{\alpha-1}\int_t^\infty p(s)\,\mathrm{d}s\sim\frac{-M_y^{\alpha-1}}{-(\delta+1)}tp(t)$$

as $t\to\infty$, where $N_y=\lim_{t\to\infty}y^{[1]}(t)$. This implies $\mathcal{IS}_{BB}\cup\mathcal{DS}_{BB}\subseteq\mathcal{B}_2$. From the relation $y^{[1]}(t)\sim N_y$ as $t\to\infty$, which is equivalent to $y'(t)\sim\Phi^{-1}(N_y/r(t))$, by Theorem 3.2, we obtain

$$M_y-y(t)\sim\Phi^{-1}(N_y)\int_t^\infty r^{1-\beta}(s)\,\mathrm{d}s\sim\frac{\Phi^{-1}(N_y)}{-((1-\beta)\gamma+1)}tr^{1-\beta}(t)=-\frac{\Phi^{-1}(N_y)}{\varrho}tr^{1-\beta}(t)$$

as $t\to\infty$. This implies $\mathcal{IS}_{BB}\cup\mathcal{DS}_{BB}\subseteq\mathcal{B}_1$. □

Lemma 3.12. *Let (1.11) be satisfied with $\delta\neq-1$ and $\gamma\neq\alpha-1$. Then the following hold:*

(i) If $J=\infty$, then $(\mathcal{DS}_{00}\cup\mathcal{IS}_{\infty\infty}\cup\mathcal{IS}_{\infty0}\cup\mathcal{DS}_{0\infty})\cap\mathcal{SV}\subseteq\mathcal{G}_1$.

(ii) If $J<\infty$, then $\mathcal{IS}_{B0}\cup\mathcal{DS}_{B0}\cup\mathcal{IS}_{B\infty}\cup\mathcal{DS}_{B\infty}\subseteq\mathcal{G}_2$.

Proof. Take $y\in\mathcal{S}_{\mathcal{SV}}$. Note that in (i) slow variation is assumed, in (ii) it clearly holds, and $\mathcal{S}_{\mathcal{SV}}=\mathcal{S}_{\mathcal{NSV}}$, see Remark 3.4. We have $|p|\Phi(y)\in\mathcal{RV}(\delta)$ by Proposition 3.1. Let $\delta<-1$. Then $\int_a^\infty|p(s)|\Phi(y(s))\,\mathrm{d}s<\infty$ by Theorem 3.2. Observe that the classes considered in the lemma, which correspond to this setting, are $\mathcal{DS}_{x0},\mathcal{IS}_{x0}$. Indeed, from (1.1) we have

$$\left|y^{[1]}(t)-y^{[1]}(t_0)\right|=\int_{t_0}^t|p(s)|\Phi(y(s))\,\mathrm{d}s \tag{3.8}$$

and because of the convergence of the integral we cannot have $\lim_{t\to\infty}|y^{[1]}(t)|=\infty$. Assume that y belongs to such classes. Integrating (1.1) from t to ∞, Theorem 3.2 yields

$$-y^{[1]}(t)=-\int_t^\infty p(s)\Phi(y(s))\,\mathrm{d}s\sim\frac{1}{\delta+1}tp(t)\Phi(y(t)) \tag{3.9}$$

as $t \to \infty$. Similarly, under the condition $\delta > -1$, which corresponds to the classes $\mathcal{DS}_{x\infty}, \mathcal{IS}_{x\infty}$ (this follows from (3.8) and the divergence of the integral), integration of (1.1) from t_0 to t and Theorem 3.2 lead to

$$y^{[1]}(t) = y^{[1]}(t_0) - \int_{t_0}^{t} p(s)\Phi(y(s))\,\mathrm{d}s \sim -\int_{t_0}^{t} p(s)\Phi(y(s))\,\mathrm{d}s \sim -\frac{1}{\delta+1} tp(t)\Phi(y(t)) \tag{3.10}$$

as $t \to \infty$. Consequently, no matter what $\delta \neq -1$ is, both (3.9) and (3.10) lead to

$$\frac{y'(t)}{y(t)} \sim \Phi^{-1}\left(\frac{-1}{\delta+1}\right)\Phi^{-1}\left(\frac{tp(t)}{r(t)}\right) = \Phi^{-1}\left(\frac{-1}{\delta+1}\right)G(t) \tag{3.11}$$

as $t \to \infty$. The following observation which was established in [22] will be useful in the sequel. Let $A \in \mathbb{R}$, $\varepsilon_1(t) \to 0$ as $t \to \infty$, and f be a positive function such that $\int_a^\infty f(t)\,\mathrm{d}t = \infty$. Then there exists $\varepsilon_2(t) \to 0$ as $t \to \infty$ such that

$$A + \int_a^t (1+\varepsilon_1(s))f(s)\,\mathrm{d}s = \int_a^t (1+\varepsilon_2(s))f(s)\,\mathrm{d}s. \tag{3.12}$$

If $J = \infty$, then integration of (3.11) from t_0 to t yields

$$\begin{aligned}
\ln y(t) &= \ln y(t_0) + \int_{t_0}^t (1+o(1))\Phi^{-1}\left(\frac{-1}{\delta+1}\right)G(s)\,\mathrm{d}s \\
&= \int_{t_0}^t (1+o(1))\Phi^{-1}\left(\frac{-1}{\delta+1}\right)G(s)\,\mathrm{d}s \\
&= \int_{a}^t (1+o(1))\Phi^{-1}\left(\frac{-1}{\delta+1}\right)G(s)\,\mathrm{d}s
\end{aligned}$$

as $t \to \infty$, where we applied (3.12) twice. Taking exponential, we find that $y \in \mathcal{G}_1$. If $J < \infty$, then integration of (3.11) from t to ∞ yields

$$-\ln\frac{y(t)}{M_y} = \int_t^\infty \Phi^{-1}\left(\frac{-(1+o(1))}{\delta+1}\right)G(s)\,\mathrm{d}s$$

as $t \to \infty$, where $M_y = \lim_{t\to\infty} y(t)$, which leads to $y \in \mathcal{G}_2$. □

Remark 3.13. Let (1.11) hold with $\delta \neq -1$ and $\gamma \neq \alpha - 1$. Let $\mathcal{S}_{\mathcal{NSV}} \neq \emptyset$ and recall that it implies (1.2) with $C_\gamma = 0$ by Lemma 3.5. Assume that $J = \infty$ and note that then necessarily $\delta + \alpha = \gamma$. Indeed, $\delta + \alpha < \gamma$ would imply $J < \infty$ while $\delta + \alpha > \gamma$ would imply $\mathcal{S}_{\mathcal{NSV}} = \emptyset$. From [19, Section 6] and [23, Section 4] it follows that if $p < 0$, then

$$\begin{aligned}
\mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{DS}_{00} \quad \text{provided} \quad \delta < -1, \\
\mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{IS}_{\infty\infty} \quad \text{provided} \quad \delta > -1.
\end{aligned}$$

From [22, Section 5] we have, if $p > 0$, then

$$\begin{aligned}
\mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{IS}_{\infty 0} \quad \text{provided} \quad \delta < -1, \\
\mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{DS}_{0\infty} \quad \text{provided} \quad \delta > -1.
\end{aligned}$$

Assume that $J < \infty$. From [19, Section 6], [22, Section 5], and [23, Section 4] we have, if $p < 0$, then

$$\begin{aligned}
\mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{DS}_{B0} \quad \text{provided} \quad \delta < -1, \gamma < \alpha - 1, \\
\mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{IS}_{B\infty} \quad \text{provided} \quad \delta > -1, \gamma > \alpha - 1.
\end{aligned}$$

From [22, Section 5] we have, if $p > 0$, then

$$\begin{aligned}
\mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{IS}_{B0} \quad \text{provided} \quad \delta < -1, \gamma < \alpha - 1, \\
\mathcal{S}_{\mathcal{NSV}} &\subseteq \mathcal{DS}_{B\infty} \quad \text{provided} \quad \delta > -1, \gamma > \alpha - 1.
\end{aligned}$$

Lemma 3.14. *Let* (1.11) *be satisfied with* $\delta \neq -1$ *and* $\gamma \neq \alpha - 1$. *Then the following hold:*

(i) If $R = \infty$, *then* $(\mathcal{DS}_{00} \cup \mathcal{DS}_{0\infty}) \cap \mathcal{RV}(\varrho) \subseteq \mathcal{H}_2 = \mathcal{H}_0$ *and* $(\mathcal{IS}_{\infty\infty} \cup \mathcal{IS}_{\infty 0}) \cap \mathcal{RV}(\varrho) \subseteq \mathcal{H}_1 = \mathcal{H}_0$.

(ii) If $R < \infty$, *then* $\mathcal{DS}_{0B} \subseteq \mathcal{H}_4$ *and* $\mathcal{IS}_{\infty B} \subseteq \mathcal{H}_3$.

(iii) If $R = \infty$, *then* $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_0$.

Proof. We will prove the case when $R < \infty$ for the class $\mathcal{DS}_{0B}$ with details. The other cases in (i) and (ii) can be proved similarly. Let $y \in \mathcal{DS}_{0B}$. Set $u = -y^{[1]}$. Then u satisfies reciprocal equation (3.2) and $u \in \widehat{S}$ by Lemma 3.9. Since $y \in \mathcal{DS}_{0B}$, we get $u(t) \sim M_u$ as $t \to \infty$, where $M_u = -N_y = -\lim_{t\to\infty} y^{[1]}(t)$. As in (3.6), we get $u^{[1]} = y \operatorname{sgn} p$, and therefore $u^{[1]}(t) \to 0$ as $t \to \infty$. Consequently, $u \in \widehat{\mathcal{DS}}_{B0}$ or $u \in \widehat{\mathcal{IS}}_{B0}$ according to whether $p < 0$ or $p > 0$, respectively. In view of Lemma 3.12-(ii), we get $u \in \widehat{\mathcal{G}}_2$, that is

$$u(t) = M_u \exp\left\{ \int_t^\infty \frac{1 + o(1)}{\Phi(\widehat{\delta} + 1)} \Phi\left(\frac{s\widehat{p}(s)}{\widehat{r}(s)} \right) \mathrm{d}s \right\}$$

as $t \to \infty$. We use the convention from Lemma 3.9 and Remark 3.10. Thus we find that

$$-r(t)\Phi(y'(t)) = u(t) = -N_y \exp\left\{ \int_t^\infty (1 + o(1)) \frac{1}{\Phi(\varrho)} H(s)\, \mathrm{d}s \right\},$$

which yields

$$y'(t) = \Phi^{-1}(N_y) r^{1-\beta}(t) \exp\left\{ \int_t^\infty (1 + o(1)) \frac{\beta - 1}{\Phi(\varrho)} H(s)\, \mathrm{d}s \right\},$$

as $t \to \infty$. Since $y \in \mathcal{DS}_0$, integration from t to ∞ leads to $y \in \mathcal{H}_4$.

It remains to prove $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_0$ when $R = \infty$. Take $y \in \mathcal{H}_1$. In view of (1.13) and representation (1.9), we have $\mathfrak{E}(a, \cdot, -(\beta-1)\Phi(\varrho), H) \in \mathcal{SV}$. Therefore, $r^{1-\beta}\mathfrak{E}(a, \cdot, -(\beta - 1)\Phi(\varrho), H) \in \mathcal{RV}(\gamma(1-\beta))$ by Proposition 3.1. Hence, from Theorem 3.2 and thanks to divergence of $\int_a^\infty |H(t)|\, \mathrm{d}t$, utilizing (3.12), we obtain

$$\begin{aligned} y(t) &= (1 + o(1)) \frac{tr^{1-\beta}(t)}{|\varrho|} \mathfrak{E}\left(a, t, -\frac{\beta - 1}{\Phi(\varrho)}, H \right) \\ &= tr^{1-\beta}(t)\, \mathrm{e}^{\ln \frac{1+o(1)}{|\varrho|}}\, \mathfrak{E}\left(a, t, -\frac{\beta - 1}{\Phi(\varrho)}, H \right) = tr^{1-\beta}(t) \mathfrak{E}\left(a, t, -\frac{\beta - 1}{\Phi(\varrho)}, H \right) \end{aligned}$$

as $t \to \infty$. Thus $\mathcal{H}_1 \subseteq \mathcal{H}_0$. Using similar ideas, we obtain the opposite inclusion. The equality $\mathcal{H}_2 = \mathcal{H}_0$ can be proved analogously. □

Remark 3.15. Let (1.11) hold with $\delta \neq -1$ and $\gamma \neq \alpha - 1$. From the reciprocity principle (see Lemma 3.9) combined with the ideas of Remark 3.13, recalling the relations $u = \pm y^{[1]}$, $u^{[1]} = \mp y \operatorname{sgn} p$ (see (3.6)) and $\widehat{G} = H$ (see (3.3)), we obtain the following claims. Assume $R = \infty$ (which implies $\delta + \alpha = \gamma$). Then

$$\begin{aligned} &\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty\infty} \quad \text{provided} \quad \delta < -1, p < 0, \\ &\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{00} \quad \text{provided} \quad \delta > -1, p < 0, \\ &\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty 0} \quad \text{provided} \quad \delta < -1, p > 0, \\ &\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{0\infty} \quad \text{provided} \quad \delta > -1, p > 0. \end{aligned}$$

Assume $R < \infty$. Then

$$\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty B} \quad \text{provided} \quad \delta < -1, \gamma < \alpha - 1,$$
$$\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{0B} \quad \text{provided} \quad \delta > -1, \gamma > \alpha - 1.$$

Lemma 3.16 ([22]). *Let $r \in \mathcal{NRV}(\gamma) \cap C^1$, $\gamma \in \mathbb{R}$, and (1.2) hold with $C_\gamma < K_\gamma$, K_γ being defined by (2.2). Assume that $|L(\vartheta_i, \eta_i, \cdot)| \in \mathcal{RV}(\eta_i - 1)$, $i = 1, 2$, where $\Phi(\vartheta_1) < \Phi(\vartheta_2)$ are the roots of (2.3), $\eta_1, \eta_2 \leq 0$, and $\gamma + \alpha(\vartheta_2 - 1) + \eta_2 > -1$. Then*

$$\mathcal{S}_{\mathcal{NRV}}(\vartheta_i) \subseteq \mathcal{L}_k(\vartheta_i, \eta_i), \qquad i = 1, 2, \tag{3.13}$$

where $k = 1$ when $\int_a^\infty |L(\vartheta_i, \eta_i, s)|\,\mathrm{d}s = \infty$, while $k = 2$ when $\int_a^\infty |L(\vartheta_i, \eta_i, s)|\,\mathrm{d}s < \infty$; if $C_\gamma = 0$, we consider only the nonzero root in (3.13).

Lemma 3.17. *Let (1.11) be satisfied with $p > 0$, $\delta \neq -1$, and $\gamma \neq \alpha - 1$. Then the following hold:*

(i) If $y \in \mathcal{S}_1 \cap \mathcal{RV}(\vartheta)$, $\vartheta \in \mathbb{R}$, where $\mathcal{S}_1 = \mathcal{IS}_{\infty 0} \cup \mathcal{DS}_{0\infty} \cup \mathcal{IS}_{B0} \cup \mathcal{DS}_{B\infty}$, then $|y^{[1]}| \in \mathcal{RV}(\delta + 1 + (\alpha - 1)\vartheta)$ and $|y'| \in \mathcal{RV}((\beta - 1)(\delta + 1 - \gamma) + \vartheta)$. If $y \in \mathcal{S}_1 \cap \mathcal{RV}(\vartheta)$ and $\delta + \alpha = \gamma$, then $|y'| \in \mathcal{RV}(\vartheta - 1)$. If, in addition $\vartheta = \varrho$, then $|y^{[1]}| \in \mathcal{SV}$.

(ii) If $y \in \mathcal{S}_2 \cap \mathcal{RV}(\vartheta)$, where $\mathcal{S}_2 = \mathcal{IS}_{\infty B} \cup \mathcal{DS}_{0B}$, then $\vartheta = \varrho$, $|y^{[1]}| \in \mathcal{SV}$, and $|y'| \in \mathcal{RV}(\varrho - 1)$.

Proof. (i) Let $y \in \mathcal{S}_1 \cap \mathcal{RV}(\vartheta)$. Then $|y^{[1]}|$ tends to 0 or ∞ and $p\Phi(y) \in \mathcal{RV}(\delta + \vartheta(\alpha - 1))$ by Proposition 3.1. Hence, integrating (1.1) from t_0 to t or from t to ∞ (according to whether $\delta + \vartheta(\alpha - 1)$ is positive or negative, respectively), realizing that $y^{[1]}(t) - y^{[1]}(t_0) \sim y^{[1]}(t)$ in the former case, and using Theorem 3.2, we get

$$|y^{[1]}(t)| \sim \frac{1}{|\delta + 1 + \vartheta(\alpha - 1)|} t p(t) \Phi(y(t))$$

as $t \to \infty$, which implies $|y^{[1]}| \in \mathcal{RV}(\delta + 1 + \vartheta(\alpha - 1))$. In view of Proposition 3.1, we get $|y'| \in \mathcal{RV}((\beta - 1)[\delta + 1 + (\alpha - 1)\vartheta - \gamma]) = \mathcal{RV}((\beta - 1)(\delta + 1 - \gamma) + \vartheta)$. If $\delta + \alpha = \gamma$, then the last index reduces to $\vartheta - 1$. If $\vartheta = \varrho$, then for the index associated to $|y^{[1]}|$ we have $\delta + 1 + \vartheta(\alpha - 1) = \delta + \alpha - \gamma = 0$.

(ii) Let $y \in \mathcal{S}_2 \cap \mathcal{RV}(\vartheta)$. Then $y^{[1]}(t) \sim N_y$ as $t \to \infty$, i.e.

$$y'(t) \sim \Phi^{-1}(N_y) r^{1-\beta}(t) \tag{3.14}$$

as $t \to \infty$. Integrating this relation from t_0 to t or from t to ∞ (according to whether $\gamma < \alpha - 1$ or $\gamma > \alpha - 1$, respectively), realizing that $y(t) - y(t_0) \sim y(t)$ in the former case, and using Theorem 3.2, we get

$$y(t) \sim \frac{|\Phi^{-1}(N_y)|}{|(1 - \beta)\gamma + 1|} t r^{1-\beta} \in \mathcal{RV}((1 - \beta)\gamma + 1) = \mathcal{RV}(\varrho),$$

thus $\vartheta = \varrho$. In view of (3.14), we get $|y'| \in \mathcal{RV}(-\gamma(1 - \beta)) = \mathcal{RV}(\varrho - 1)$. □

Lemma 3.18. *Let (1.11) hold with $\gamma \neq \alpha - 1$. If $\mathcal{NSV} \cap (\mathcal{DS}_0 \cup \mathcal{IS}_\infty) \neq \emptyset$, then $\gamma = \delta + \alpha$, $\mathcal{NSV} \cap \mathcal{DS}_0 = \mathcal{DS}_{00} \cup \mathcal{DS}_{0\infty}$, and $\mathcal{NSV} \cap \mathcal{IS}_\infty = \mathcal{IS}_{\infty\infty} \cup \mathcal{IS}_{\infty 0}$.*

Proof. Take $y \in \mathcal{NSV} \cap (\mathcal{DS}_0 \cup \mathcal{IS}_\infty)$. Then, in view of Proposition 3.1, $|(r\Phi(y'))'| = |p|y^{\alpha-1} \in \mathcal{RV}(\delta)$. If $\int_a^\infty |p(s)|y^{\alpha-1}(s)\,\mathrm{d}s$ diverges, then $\lim_{t\to\infty} |y^{[1]}(t)| = \infty$, and the Karamata Integration Theorem (Theorem 3.2) applied to equation (1.1) after integration yields

$$r(t)|y'(t)|^{\alpha-1} \sim |r(t)\Phi(y'(t)) - r(t)\Phi(y'(t_0))| \sim \int_{t_0}^t |p(s)|y^{\alpha-1}(s)\,\mathrm{d}s \in \mathcal{RV}(\delta+1)$$

as $t \to \infty$. Similarly, if $\int_a^\infty |p(s)|y^{\alpha-1}(s)\,\mathrm{d}s$ converges, then

$$r(t)|y'(t)|^{\alpha-1} = \int_t^\infty |p(s)|y^{\alpha-1}(s)\,\mathrm{d}s \in \mathcal{RV}(\delta+1)$$

by Theorem 3.2. Indeed, $\lim_{t\to\infty} y^{[1]}(t) = N_y$ would lead to $y'(t) \sim \Phi^{-1}(N_y)r^{1-\beta}(t)$, so $y \in \mathcal{RV}(\varrho)$, $\varrho \neq 0$, contradiction. Thus in any case, $|y'|^{\alpha-1} \in \mathcal{RV}(\delta+1-\gamma)$, and therefore $|y'| \in \mathcal{RV}((\delta+1-\gamma)/(\alpha-1))$ by Proposition 3.1. Since $y \in \mathcal{DS}_0$ or $y \in \mathcal{IS}_\infty$, in view of the Karamata Theorem, $y \in \mathcal{RV}((\delta+1-\gamma)/(\alpha-1)+1) = \mathcal{RV}((\delta+\alpha-\gamma)(\beta-1))$. But $y \in \mathcal{SV}$, and so it must hold that $\gamma = \delta+\alpha$. ☐

In spite of the fact that many of the claims which are included in the next statement were already proved above (as it was within the more general setting), for completeness and easier reference we prefer to present some conclusions from [19] in the form of individual lemma.

Lemma 3.19 ([19]). *Let $p < 0$, $C_\gamma = 0$, and (1.11) hold, where $\gamma = \delta+\alpha$.*

(i) Let $\delta < -1$. If $J = \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{DS} = \mathcal{DS}_{00} \subseteq \mathcal{G}_1$. If $J < \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{DS} = \mathcal{DS}_{B0} \subseteq \mathcal{G}_2$. If $R = \infty$, then $\mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty\infty} \subseteq \mathcal{H}_1$. If $R < \infty$, then $\mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty B} \subseteq \mathcal{H}_3$.

(ii) Let $\delta > -1$. If $J = \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{IS} = \mathcal{IS}_{\infty\infty} \subseteq \mathcal{G}_1$. If $J < \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{IS} = \mathcal{IS}_{B\infty} \subseteq \mathcal{G}_2$. If $R = \infty$, then $\mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{00} \subseteq \mathcal{H}_2$. If $R < \infty$, then $\mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{DS} = \mathcal{DS}_{0B} \subseteq \mathcal{H}_4$.

Theorem 3.20 ([5]). *Let $p < 0$. Then*

$$\mathcal{P} = \begin{cases} \mathcal{DS}_B & \text{if } J_1 = \infty \text{ and } J_2 < \infty, \\ \mathcal{DS}_0 & \text{otherwise.} \end{cases}$$

The lower limit a in the integrals in Theorems 3.21, 3.24, 3.26 is taken such that $y(t) > 0$ and $y'(t) \neq 0$ for $t \geq a$. In the paper [3], an example is given showing that condition (3.15) cannot be omitted. As we will see, in our proofs, the cases where (3.15) fails to hold can fortunately be treated by Theorem 3.24.

Theorem 3.21 ([3,6]). *Let $p > 0$ and (1.1) be nonoscillatory. Assume that*

$$J_r = \infty \text{ and } \alpha \geq 2 \quad \text{or} \quad J_p = \infty \text{ and } 1 < \alpha \leq 2. \tag{3.15}$$

Then, for $y \in \mathcal{S}$,

$$y \in \mathcal{P} \quad \text{if and only if} \quad \int_a^\infty \mathcal{F}[y](t)\,\mathrm{d}t = \infty,$$

where $\mathcal{F}[y] = y'/(y^2 y^{[1]})$.

Theorem 3.22 ([2]). *Let $p > 0$ and (1.1) be nonoscillatory. Assume that $J_r + J_p = \infty$. Then*

$$y \in \mathcal{P} \quad \text{if and only if} \quad |y^{[1]}| \in \widehat{\mathcal{P}},$$

where $\widehat{\mathcal{P}} = \{u \in \widehat{\mathcal{S}} : u \text{ is principal}\}$.

For $\xi \in (1, \infty)$, define the function $\varphi_\xi : [0, 1] \to \mathbb{R}$ by

$$\varphi_\xi(t) = \begin{cases} \frac{1-t^\xi}{1-t} + (1-t)^{\xi-1} & \text{if } t \in [0, 1), \\ \xi & \text{if } t = 1. \end{cases} \tag{3.16}$$

Denote $m := \min\{\varphi_\beta(t) : t \in [0, 1]\}$, $M := \max\{\varphi_\beta(t) : t \in [0, 1]\}$, where β is the conjugate number of α.

Lemma 3.23. *It holds that $\varphi_\beta(0) = 2$, $\varphi_\beta(1) = \beta$, $\varphi_\beta(1/2) = 2$, and $m > 1$. If $1 < \alpha < 2$ (i.e., $\beta > 2$), then φ_β is strictly convex on $[0, 1]$ and, in particular, $M = \beta$.*

Proof. The equalities $\varphi_\beta(0) = 2$, $\varphi_\beta(1) = \beta$, $\varphi_\beta(1/2) = 2$ are obvious. The convexity of φ_β on $[0, 1]$ when $\alpha \in (1, 2)$ can be demonstrated via standard calculus tools. The equality $M = \beta$ follows from the convexity of φ_β. □

Theorem 3.24 ([13]). *Let $J_r = \infty$ and $y \in \mathcal{S}$. Denote $\mathcal{T}_K[y] = r^{1-\beta} y^{-K}$, $K \in \mathbb{R}$.*

(i) If $y \in \mathcal{P}$, then $\int_a^\infty \mathcal{T}_m[y](s)\, ds = \infty$.

(ii) If $\int_a^\infty \mathcal{T}_M[y](s)\, ds = \infty$, then $y \in \mathcal{P}$.

Remark 3.25. By means of the reciprocity principle (see Lemma 3.9), with help of Theorem 3.22, the condition $J_r = \infty$ in Theorem 3.24 can actually be relaxed to $J_r + J_p = \infty$; $\mathcal{T}_m, \mathcal{T}_M$ are then appropriately modified. For details see the proofs of Theorems 2.1, 2.2, 2.3, and 2.4, where this trick is used.

Theorem 3.26 ([2]). *Let $p > 0$ and $J_r + J_p < \infty$. Then*

$$y \in \mathcal{P} \quad \text{if and only} \quad \int_a^\infty \frac{1}{r^{\beta-1}(t) y^2(t)}\, dt = \infty.$$

In view of their common setting, it is senseful to prove Theorems 2.1 and 2.2 simultaneously.

Proof of Theorems 2.1 and 2.2. Let $p < 0$. If $J = \infty$ and $\delta < -1$, then $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{DS}_{00} \subseteq \mathcal{G}_1$ by Lemma 3.12-(i) and Remark 3.13. Since $G(t) = \Phi^{-1}(L_p(t)/L_r(t))/t$ and $\lim_{t\to\infty} L_p(t)/L_r(t) = 0$, we have $\mathcal{G}_1 \subseteq \mathcal{S}_{\mathcal{SV}} = \mathcal{S}_{\mathcal{NSV}}$ by the Representation Theorem (see (1.9)) and Remark 3.4. From [23] we know that $\mathcal{DS} \subseteq \mathcal{NSV}$, thus $\mathcal{DS}_{00} \subseteq \mathcal{NSV}$. In view of (1.4)

$$\mathcal{S} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho), \qquad \mathcal{S}_{\mathcal{NSV}} \neq \emptyset, \qquad \mathcal{S}_{\mathcal{NRV}}(\varrho) \neq \emptyset \tag{3.17}$$

(which follows from Theorem 3.3), we get $\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{00} = \mathcal{G}_1$. Analogously we obtain $\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{G}_1$ when $J = \infty$ and $\delta > -1$. If $J < \infty$, then in a similar manner as above we use Lemma 3.12-(ii), the obvious fact $\mathcal{G}_2 \subseteq \mathcal{SV}$, (3.17), (1.4), Lemma 3.19, and, in addition, Lemma 3.11-(i), to get $\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{B0} = \mathcal{G}_2 = \mathcal{B}_5$ when $\delta < -1$ and $\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{B\infty} = \mathcal{G}_2 = \mathcal{B}_5$ when $\delta > -1$. Note that Lemma 3.11-(i) yields $\mathcal{DS}_{B0} \subseteq \mathcal{B}_5$ and $\mathcal{IS}_{B\infty} \subseteq \mathcal{B}_5$, respectively. The opposite inclusions are obvious, since y belonging to

$\mathcal{B}_5$ is slowly varying and there are no other slowly varying solutions than $\mathcal{DS}_{B0}$ and $\mathcal{IS}_{B\infty}$, respectively, see Remark 3.13. Let $R = \infty$ and $\delta < -1$. Observe that $\mathcal{H}_1 \subseteq \mathcal{RV}(\varrho)$. Indeed, if $y \in \mathcal{H}_1$, then $y(t) \sim \int_{t_0}^t r^{1-\beta}(s)\mathfrak{E}(a,s,-(\beta-1)/\Phi(\varrho),H)\,\mathrm{d}s \in ([\gamma(1-\beta)+0]+1) = \mathcal{RV}(\varrho)$, where we use (1.9), Proposition 3.1, and Theorem 3.2. From Lemma 3.14-(i) and Remark 3.15, taking into account Lemma 3.19, (3.17), and (1.4), we get $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS} = \mathcal{IS}_{\infty\infty} = \mathcal{H}_1$. Lemma 3.14-(iii) gives $\mathcal{H}_1 = \mathcal{H}_0$. Because $\delta + \alpha = \gamma$ and ϱ is the bigger root of (2.3) when $\delta < -1$, the condition $\gamma + \alpha(\vartheta_2 - 1) + \eta_2 > -1$ from Lemma 3.16 reads as $\delta < -1 + \eta(\alpha - 1)$ which is assumed in Theorems 2.1, 2.2. Since also all other assumptions of Lemma 3.16 are satisfied, we may apply it to obtain $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{L}(\varrho,\eta)$; we use convention (1.15). Since $\lim_{t\to\infty} tL(\vartheta,\eta,t) = 0$, from the Representation Theorem (see (1.9)) it follows that $\mathcal{L}(\varrho,\eta) \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Analogously we proceed when $R = \infty$ and $\delta > -1$. Let us only note that in this case, ϱ is the lesser root of (2.3) (since $\varrho < 0$) and therefore we do not need to verify the condition $\gamma + \alpha(\vartheta_2 - 1) + \eta_2 > -1$ from Lemma 3.16. The case $R < \infty$ can also be treated similarly; we use, in addition, Lemma 3.14-(ii) and Lemma 3.11. Next we derive the relations with $\mathcal{P}$. If $\delta > -1$, then $J_p = \infty$ by Theorem 3.2, thus $J_2 = \infty$. Hence, $\mathcal{P} = \mathcal{DS}_0$ by Theorem 3.20. If $\delta < -1$, then, in view of $\delta + \alpha = \gamma$, we have $\gamma < \alpha - 1$, thus $r^{1-\beta} \in \mathcal{RV}((1-\beta)\gamma)$ with the index greater than -1, and so $J_r = \infty$ (see Theorem 3.2), which implies $J_1 = \infty$. Further, by Lemma 3.7, $V_2(t) \sim |G(t)|/|\delta+1|^{\beta-1} \in \mathcal{RV}(-1)$ as $t \to \infty$. Hence, in general, J_2 can converge or diverge. But we see that $J_2 = \infty$ if and only if $J = \infty$. According to Theorem 3.20, if $J = \infty$, then $\mathcal{P} = \mathcal{DS}_0$, while if $J < \infty$, then $\mathcal{P} = \mathcal{DS}_B$. Adding the relations between $\mathcal{P}$ and $\mathcal{DS}_0$ resp. $\mathcal{P}$ and $\mathcal{DS}_B$ to the other relations we obtain the complete picture in the case $p < 0$.

Let $p > 0$. First of all note that by Theorem 3.3, (3.17) holds. Assume that $\delta < -1$. Then $\gamma < \alpha - 1$, $r^{1-\beta}$ thus has the index of regular variation greater than -1, and so $J_r = \infty$ by Theorem 3.2. Hence, (1.5) holds. Note that ϱ in (3.17) is now positive. If $J = \infty$, then by Lemma 3.12 and Remark 3.13, we get $\mathcal{S}_{\mathcal{NSV}} \cap \mathcal{IS}_{\infty 0} \subseteq \mathcal{G}_1$ and $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{IS}_{\infty 0}$. In view of $\mathcal{G}_1 \subseteq \mathcal{S}_{\mathcal{NSV}}$ (which follows from (1.9)) and (2.1), we have $\mathcal{S}_{\mathcal{NSV}} = \mathcal{G}_1 = \mathcal{IS}_{\infty 0}$. If $R = \infty$, then by Lemma 3.14 and Remark 3.15, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \cap \mathcal{IS}_{\infty 0} \subseteq \mathcal{H}_1$ and $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty 0}$. In view of $\mathcal{H}_1 \subseteq \mathcal{S}_{\mathcal{RV}}(\varrho) = \mathcal{S}_{\mathcal{NRV}}(\varrho)$ (which follows from (1.9)), we have $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{H}_1 = \mathcal{IS}_{\infty 0}$. By Lemma 3.14, $\mathcal{H}_1 = \mathcal{H}_0$. Assume that $J = \infty$ and $R = \infty$. Because of (3.17), (1.5), and the observations from the previous parts, we have $\mathcal{S} = \mathcal{S}_{\mathcal{NSV}} \cup \mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty 0} \subseteq \mathcal{IS} = \mathcal{S}$. If $J < \infty$, then by Lemma 3.12 and Remark 3.13, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{IS}_{B0}$, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{G}_2$. If $y \in \mathcal{IS}_B$, then it is clearly slowly varying and we get $\mathcal{IS}_{B0} = \mathcal{S}_{\mathcal{NSV}}$. Since $\mathcal{G}_2 \subseteq \mathcal{SV}$ and (2.1) holds, we have $\mathcal{S}_{\mathcal{NSV}} = \mathcal{G}_2$. In view of Lemma 3.11, we obtain $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{B}_5$; the opposite inclusion obviously holds as well. If $R < \infty$, then by Lemma 3.14 and Remark 3.15 it follows that $\mathcal{IS}_{\infty B} \subseteq \mathcal{H}_3$ and $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty B}$. From (1.9), Proposition 3.1, and Theorem 3.2, we have $\mathcal{H}_3 \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. If $y \in \mathcal{IS}_{\infty B}$, then $y^{[1]}(t) \sim N_y \in (0,\infty)$ as $t \to \infty$. Expressing y' and integrating, Theorem 3.2 and Proposition 3.1 yield

$$y(t) \sim \left|\frac{\Phi(N_y)}{\gamma(1-\beta)+1}\right| tr^{1-\beta}(t) \in \mathcal{RV}(\gamma(1-\beta)+1) = \mathcal{RV}(\varrho) \tag{3.18}$$

as $t \to \infty$. Hence, $\mathcal{IS}_{\infty B} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Consequently, in view of the fact that regular variation of solutions is normalized, we have $\mathcal{IS}_{\infty B} = \mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{H}_3$. From Lemma 3.11 we get $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{B}_6$. The opposite inclusion is obvious. The settings $J < \infty, R < \infty$, or $J = \infty, R < \infty$, or $J < \infty, R = \infty$, can be treated by suitable combinations of the above presented observations. Similarly as in the case $p < 0$, with the help Lemma 3.16, we show $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{L}(\varrho,\eta)$; we use convention (1.15).

The case $p > 0$ and $\delta > -1$ can be proved analogously to the case $p > 0$ and $\delta < -1$ (applying again (3.17), (1.9), Lemma 3.11, Lemma 3.12, Remark 3.13, Lemma 3.14, Lemma 3.16), and therefore it is omitted.

In the last part of the proof we will show how $\mathcal{P}$ is related to the other classes when $p > 0$. From the above established classification we see that any $y \in \mathcal{S}$ must belong either to $\mathcal{S}_1$ or $\mathcal{S}_2$ under the assumptions of Theorem 2.1 and Theorem 2.2. By Lemma 3.17 we have that $\mathcal{F}[y]$ is regularly varying. Let Ω denote the index of regular variation of $\mathcal{F}[y]$.

Assume first that (3.15) holds. If $y \in \mathcal{S}_{\mathcal{NSV}}$, then $\Omega = -\delta - 2$. If $\delta < -1$, then $\Omega > -1$, and so $\int_a^\infty \mathcal{F}[y](s)\,\mathrm{d}s = \infty$ by Theorem 3.2. This yields $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{P}$ by Theorem 3.21. Similarly we obtain $\mathcal{S}_{\mathcal{NSV}} \cap \mathcal{P} = \emptyset$ when $\delta > -1$. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Then $\Omega = (\beta-1)(\delta+1-\gamma) + \varrho - 2\varrho = -\varrho - 1$, see Lemma 3.17. If $\delta < -1$, then $\varrho > 0$, i.e., $-\varrho - 1 < -1$, which implies $\int_a^\infty \mathcal{F}[y](s)\,\mathrm{d}s < \infty$, and we obtain $\mathcal{S}_{\mathcal{NRV}}(\varrho) \cap \mathcal{P} = \emptyset$ by Theorem 3.21. Similarly we get $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{P}$ when $\delta > -1$. Altogether, in view of (3.17), $\mathcal{P} = \mathcal{S}_{\mathcal{NSV}}$ when $\delta < -1$, while $\mathcal{P} = \mathcal{S}_{\mathcal{NRV}}(\varrho)$ when $\delta > -1$.

Assume now that (3.15) fails to hold. The constants m, M will have the same meaning as in Theorem 3.24. Let $J_r = \infty$ (this means $\gamma < \alpha - 1$, thus, $\delta < -1$ since we assume $\gamma \neq \alpha - 1$ and $\delta + \alpha = \gamma$) and $\alpha < 2$. If $y \in \mathcal{S}_{\mathcal{NSV}}$, then $r^{1-\beta}y^{-M} \in \mathcal{RV}(-\gamma/(\alpha-1))$ by Proposition 3.1. In view of $\gamma < \alpha - 1$, the index is greater -1, and so $\int_a^\infty r^{1-\beta}(s)y^{-M}(s)\,\mathrm{d}s = \infty$ by Theorem 3.2. Hence, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{P}$ by Theorem 3.24. If $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$, then $r^{1-\beta}y^{-m} \in \mathcal{RV}(-\gamma/(\alpha-1) - \varrho m)$. It clearly holds $-\gamma/(\alpha-1) - \varrho m < -1$ if and only if $(\alpha - 1 - \gamma)(1 - m) < 0$. The latter inequality holds since $m > 1$, see Lemma 3.23, and $\alpha - 1 > \gamma$. Consequently, $\int_a^\infty r^{1-\beta}(s)y^{-M}(s)\,\mathrm{d}s < \infty$ by Theorem 3.2, and so Theorem 3.24 yields $\mathcal{S}_{\mathcal{NRV}}(\varrho) \cap \mathcal{P} = \emptyset$. In view of (3.17), we have $\mathcal{S}_{\mathcal{NSV}} = \mathcal{P}$.

Let $J_p = \infty$ (i.e., $\delta > -1$, i.e., $\gamma > \alpha - 1$) and $\alpha > 2$. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$ and note that $\varrho < 0$ and $\mathcal{S} = \mathcal{DS}$. Set $u = -y^{[1]}$. Then u is positive and satisfies (3.2), thus $u \in \widehat{\mathcal{S}}$. By Lemma 3.17, $u \in \widehat{\mathcal{S}}_{\mathcal{NSV}}$. Because of our assumptions we have $\widehat{\delta} < -1$ and $\widehat{\gamma} < \beta - 1$, where $\widehat{\delta}$ and $\widehat{\gamma}$ are defined in (3.4). Thus we can apply Theorem 3.24 to reciprocal equation (3.2). Denote $\widehat{M} = \max\{\varphi_\alpha(t) : t \in [0,1]\}$ and note that φ_α can be understood as a reciprocal counterpart to φ_β. Since $\widehat{r}^{1-\alpha}u^{-\widehat{M}} \in \mathcal{RV}(-\widehat{\gamma}(\alpha-1))$, where $\widehat{\gamma}(\alpha-1) > -1$, we have $\int_a^\infty \widehat{r}^{1-\alpha}(s)u^{-\widehat{M}}(s)\,\mathrm{d}s = \infty$, which implies $u \in \widehat{\mathcal{P}}$. In view of Theorem 3.22, we get $y \in \mathcal{P}$, thus $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{P}$. Now take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$ and $x \in \mathcal{S}_{\mathcal{NSV}}$. Then, since we have $ty'(t)/y(t) \to \varrho < 0$ and $tx'(t)/x(t) \to 0$ with $t \to \infty$, we get $y'(t)/y(t) < x'(t)/x(t)$ for large t, hence $x \notin \mathcal{P}$ by definition. Consequently, $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{P}$. $\square$

Proof of Theorem 2.3. Since $t^\alpha p(t)/r(t) \in \mathcal{RV}(\delta + \alpha - \gamma)$ (by Proposition 3.1) and $\delta + \alpha < \gamma$, we have $C_\gamma = 0$. Consequently (3.17) holds. The following observation will be repeatedly utilized in the sequel. Thanks to (1.11), $|G| \in \mathcal{RV}((\delta + 1 - \gamma)(\beta - 1))$ and $|H| \in \mathcal{RV}(\alpha - 1 + \delta - \gamma)$ by Proposition 3.1. It is easy to see that $\delta + \alpha < \gamma$ is equivalent to $(\delta + 1 - \gamma)(\beta - 1) < -1$. Hence, both the indices of $|G|$ and $|H|$ are less than -1, and so

$$J < \infty \quad \text{and} \quad R < \infty. \tag{3.19}$$

(i-a) Let $\delta < -1$, $\gamma < \alpha - 1$, and $p < 0$. Take $y \in \mathcal{S}_{\mathcal{NSV}}$. Then $y \in \mathcal{DS}$. Indeed, if $y \in \mathcal{IS}$, then $y^{[1]}$ is positive increasing, hence there is $A > 0$ such that $y^{[1]}(t) \geq A$ for large t, say $t \geq t_0$. Consequently, by Theorem 3.2 and Proposition 3.1,

$$y(t) \geq y(t_0) + A^{\beta-1} \int_{t_0}^t r^{1-\beta}(s)\,\mathrm{d}s \in \mathcal{RV}(\gamma(1-\beta)) = \mathcal{RV}(\varrho).$$

Hence, y is greater than or equal to a regularly varying function with a positive index, thus cannot be slowly varying. Using similar arguments we find that for $y \in \mathcal{DS}$, the quasiderivative $y^{[1]}$ (which is negative increasing) must tend to zero. Moreover, $y \in \mathcal{DS}_{B0}$. Indeed, if $y \in \mathcal{DS}_{00}$, then $\gamma = \delta + \alpha$ (see Lemma 3.18), which contradicts to $\delta + \alpha < \gamma$. Hence, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{DS}_{B0} \subseteq \mathcal{DS}$. On the other hand, if $y \in \mathcal{DS}$, then it cannot be in $\mathcal{NRV}(\varrho)$ since $\varrho > 0$ (and the functions with a positive index always tend to infinity, see Proposition 3.1), consequently, in view of (3.17), we get $\mathcal{DS} \subseteq \mathcal{S}_{\mathcal{NSV}}$. Therefore, $\mathcal{S}_{\mathcal{NSV}} = \mathcal{DS}_{B0} = \mathcal{DS}$. Consider the class $\mathcal{S}_{\mathcal{NRV}}(\varrho)$. From the previous part we know that slowly varying solutions cannot be increasing. Recalling (3.17), we get $\mathcal{IS} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Applying Remark 3.8 and (3.19) we obtain $J_2 < \infty$ and $R_1 < \infty$. Condition $\gamma < \alpha - 1$ implies $J_r = \infty$ and that is why $J_1 = \infty$ and $R_2 = \infty$, see Remark 3.8. According to [5, Theorem 1], see also [6, Chapter 4], we get $\mathcal{IS} = \mathcal{IS}_{\infty B}$. Moreover $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$ cannot be decreasing since $\varrho > 0$, thus $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}$. We obtain $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{IS}_{\infty B} = \mathcal{IS}$.

It is not difficult to see that the relations of $\mathcal{S}_{\mathcal{NSV}}$ with $\mathcal{G}_2, \mathcal{B}_3$ and of $\mathcal{S}_{\mathcal{NRV}}(\varrho)$ with $\mathcal{H}_3, \mathcal{B}_4, \mathcal{L}(\varrho, \eta)$ follow similarly as they were established in the proof of Theorems 2.1 and 2.2, with the help of Lemma 3.12, Remark 3.13, Lemma 3.14, Remark 3.15, Lemma 3.11, Lemma 3.16, formula (1.9), and [22, Section 5].

(i-b) Let $\delta < -1$, $\gamma < \alpha - 1$, and $p > 0$. Thanks to $\gamma < \alpha - 1$ and $r^{1-\beta} \in \mathcal{RV}(\gamma(1-\beta))$, we have $J_r = \infty$ (see Theorem 3.2), which implies (1.5). Take $y \in \mathcal{S}_{\mathcal{NSV}}$. Then $\lim_{t\to\infty} y^{[1]}(t) = 0$. Indeed, $y^{[1]}$ is positive decreasing and if $y^{[1]}(t) \sim N_y > 0$ as $t \to \infty$, then as in (3.18), we get $y \in \mathcal{RV}(\varrho)$, contradiction with $y \in \mathcal{S}_{\mathcal{NSV}}$. Moreover, y cannot be in $\mathcal{IS}_{\infty 0}$ otherwise we would get $\gamma = \delta + \alpha$ (see Lemma 3.18), which contradicts to $\delta + \alpha < \gamma$. Consequently, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{IS}_{B0}$. The opposite inclusion is obvious, in view of (2.1). Take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$. From the previous part we get that $y \in \mathcal{IS}_{\infty 0} \cup \mathcal{IS}_{\infty B}$. We claim that $\mathcal{IS}_{\infty 0} = \emptyset$. Indeed, if $y \in \mathcal{IS}_{\infty 0}$, then as in (3.9) we obtain

$$y^{[1]}(t) \sim \frac{-tp(t)\Phi(y(t))}{\delta + 1} \tag{3.20}$$

as $t \to \infty$, which leads to (3.11). Integration of this relation from t to ∞, in view $\lim_{t\to\infty} y(t) = \infty$, would give $J = \infty$. This however contradicts to (3.19). Hence, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{IS}_{\infty B}$. In fact, we have the equality here because of $\mathcal{S}_{\mathcal{NSV}} = \mathcal{IS}_{B0}$ and (3.17). The relations of $\mathcal{S}_{\mathcal{NSV}}$ and $\mathcal{S}_{\mathcal{NRV}}(\varrho)$ with $\mathcal{G}, \mathcal{H}, \mathcal{L}, \mathcal{B}$ type classes can be treated as in the part (i-a).

(ii-a) Let $\delta > -1$, $\gamma > \alpha - 1$, and $p < 0$. Take $y \in \mathcal{S}_{\mathcal{NSV}}$. Then $y \in \mathcal{IS}$. Indeed, if $y \in \mathcal{DS}_{\mathcal{NSV}}$, then $y^{[1]}$ is negative increasing, thus $\lim_{t\to\infty} y^{[1]}(t) \in (-\infty, 0]$. But at the same time, as in (3.10) we get (3.20), where $|tp\Phi(y(t))| \in \mathcal{RV}(\delta+1)$. Hence, $y^{[1]}(t) \in \mathcal{RV}(\delta+1)$, which yields $\lim_{t\to\infty} y^{[1]}(t) = \infty$, contradiction with $y \in \mathcal{DS}$. We have $\mathcal{IS} = \mathcal{IS}_{B\infty} \cup \mathcal{IS}_{\infty\infty}$. But if $y \in \mathcal{IS}_{\infty\infty}$, then $\gamma = \delta + \alpha$ by Lemma 3.18, contradiction with $\gamma > \delta + \alpha$. Thus $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{IS}_{B\infty}$. The opposite inclusion clearly holds as well, in view of (2.1). Consider the class $\mathcal{S}_{\mathcal{NRV}}(\varrho)$. First note that $\mathcal{DS} = \mathcal{DS}_{0B}$. Indeed, similarly as in the proof of the part (i-a), from (3.19), Lemma 3.7, and Remark 3.8, we find that $J_1 < \infty, J_2 = \infty, R_1 = \infty$, and $R_2 < \infty$, and the claim follows by [5, Theorem 1], see also [6, Chapter 4]. Since $\varrho < 0$, $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$ cannot be in $\mathcal{IS}$ (see Proposition 3.1), therefore $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{0B}$. On the other hand, if $y \in \mathcal{DS}_{0B}$, then $y^{[1]}(t) \sim N_y < 0$ as $t \to \infty$ which yields (3.18), and so $\mathcal{DS}_{0B} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$.

(ii-b) Let $\delta > -1$, $\gamma > \alpha - 1$, and $p > 0$. Since $p \in \mathcal{RV}(\delta)$, we have $J_p = \infty$, and so (1.6) holds. Take $y \in \mathcal{S}_{\mathcal{NSV}}$. Then $y^{[1]}$ is negative decreasing and from (3.20), we get $\lim_{t\to\infty} y^{[1]}(t) = -\infty$. Moreover, y cannot be in $\mathcal{DS}_{0\infty}$, otherwise we would get $\gamma = \delta + \alpha$, see Lemma 3.18. Consequently, $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{DS}_{B\infty}$. The opposite inclusion is obvious. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$. We know that $y \in \mathcal{DS}_{0\infty} \cup \mathcal{DS}_{0B}$. We claim that $y \notin \mathcal{DS}_{0\infty}$. Indeed, if

$y \in \mathcal{DS}_{0\infty}$, then from (3.10) we get (3.11). Since $y(t) \to 0$ as $t \to \infty$, integration of (3.11) yields $J = \infty$, contradiction with (3.19). Thus, $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{0B}$ and in view of $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{DS}_{B\infty}$ and (3.17), we get $\mathcal{DS}_{0B} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. The relations of $\mathcal{S}_{\mathcal{NSV}}$ and $\mathcal{S}_{\mathcal{NRV}}(\varrho)$ with $\mathcal{G}, \mathcal{H}, \mathcal{L}, \mathcal{B}$ type classes in the setting of (ii-a) and (ii-b) can be treated as in the part (i).

(iii) Let $\delta < -1$ and $\gamma > \alpha - 1$. Then $J_p < \infty$ and $J_r < \infty$. Hence, clearly $J_i < \infty, R_i < \infty$, $i = 1,2$. Assume that $p < 0$. By [5, Theorem 1], see also [6, Chapter 4], we get $\mathcal{IS} = \mathcal{IS}_B$. Hence, $\mathcal{IS} \subseteq \mathcal{S}_{\mathcal{SV}} = \mathcal{S}_{\mathcal{NSV}}$, in view of (2.1). If $y \in \mathcal{IS}$, then from (1.1), $(y^{[1]}(t))' \sim -M_y^{\alpha-1} p(t)$ as $t \to \infty$, where $M_y = \lim_{t\to\infty} y(t)$, and because of the convergence of J_p, we get $\mathcal{IS} = \mathcal{IS}_{BB}$. Indeed, $y^{[1]}$ is positive increasing and if $\lim_{t\to\infty} y^{[1]}(t) = \infty$, then $J_p = \infty$, contradiction. By [5, Theorem 1], see also [6, Chapter 4], we get $\mathcal{DS} = \mathcal{DS}_{0B} \cup \mathcal{DS}_B$, where both subclasses are nonempty. As in (3.18), we obtain $y \in \mathcal{RV}(\varrho)$ provided $y \in \mathcal{DS}_{0B}$, thus $\mathcal{DS}_{0B} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Since $\varrho < 0$ and except of $\mathcal{DS}_{0B}$ all other possible subclasses ($\mathcal{IS}_B, \mathcal{DS}_B$) are subsets of $\mathcal{SV}$, we get $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{DS}_{0B}$. Further, in view of [5, Theorem 1], $\mathcal{DS}_B = \mathcal{DS}_{B0} \cup \mathcal{DS}_{BB}$, where both subclasses are nonempty. Altogether we get $\mathcal{DS}_{B0} \cup \mathcal{DS}_{BB} \cup \mathcal{IS}_{BB} = \mathcal{S}_{\mathcal{NSV}}$.

From Lemma 3.11 we get $\mathcal{DS}_{0B} \subseteq \mathcal{B}_4$, $\mathcal{DS}_{B0} \subseteq \mathcal{B}_3$, and $\mathcal{DS}_{BB} \cup \mathcal{IS}_{BB} \subseteq \mathcal{B}_j$, $j = 1,2$. Lemma 3.12 yields $\mathcal{DS}_{B0} \subseteq \mathcal{G}_2$. From Lemma 3.14 and Lemma 3.16, we obtain $\mathcal{DS}_{0B} \subseteq \mathcal{H}_4$ and $\mathcal{DS}_{0B} \subseteq \mathcal{L}(\varrho, \eta)$, respectively. By definition, if $y \in \mathcal{B}_4 \cap \mathcal{DS}$, then $y \in \mathcal{DS}_{0B} \cup \mathcal{DS}_{BB}$. Suppose by a contradiction that $y \in \mathcal{DS}_{BB}$. We know that $\mathcal{DS}_{BB} \subseteq \mathcal{B}_2$. Thus, $|N_y - y^{[1]}| \in \mathcal{RV}(\delta + 1)$ by Proposition 3.1. But at the same time we have $y \in \mathcal{B}_4$, which yields $|N_y - y^{[1]}| \in \mathcal{RV}(\alpha + \delta - \gamma)$ by Proposition 3.1. This implies – because of necessary equality of indices of regular variation – that $\gamma = \alpha - 1$, contradiction. Thus $\mathcal{B}_4 \cap \mathcal{DS} \subseteq \mathcal{DS}_{0B}$. By definition and because of the above established classification, if $y \in \mathcal{B}_3 \cap \mathcal{DS}$, then $y \in \mathcal{DS}_{BB} \cup \mathcal{DS}_{B0}$. Let $y \in \mathcal{DS}_{BB}$. We know that $\mathcal{DS}_{BB} \subseteq \mathcal{B}_1$ by Lemma 3.11. Consequently, by Proposition 3.1, $|M_y - y| \in \mathcal{RV}(1 + \gamma(1 - \beta))$. But at the same time we have $y \in \mathcal{B}_3$, and so $|M_y - y| \in \mathcal{RV}((\beta - 1)(\delta + 1 - \gamma) + 1)$. For the indices we then get $(\beta - 1)(\alpha - 1 - \gamma) = (\beta - 1)(\delta + 1 - \gamma + \alpha - 1)$, which gives $\delta = -1$, contradiction. Thus $\mathcal{B}_3 \cap \mathcal{DS} \subseteq \mathcal{DS}_{B0}$. By definition, $\mathcal{B}_j \cap \mathcal{IS} \subseteq \mathcal{IS}_{BB}$ and $\mathcal{B}_j \cap \mathcal{DS} \subseteq \mathcal{DS}_{BB}$, $j = 1,2$. If $y \in \mathcal{G}_2 \cap \mathcal{DS}$, then $y \in \mathcal{DS}_B$. Differentiating the relation which defines $\mathcal{G}_2$, applying Φ to the both sides and multiplying by r, we obtain, as $t \to \infty$, $|y^{[1]}| \sim Kt|p(t)| \in \mathcal{RV}(\delta + 1)$, where K is a positive constant. Consequently, in view of Proposition 3.1, $y \in \mathcal{DS}_{B0}$. If $y \in \mathcal{H}_4$ or $y \in \mathcal{L}(\varrho, \eta)$, then clearly the only class for y among the ones that are allowed in the setting $\delta < -1$, $\gamma > \alpha - 1$, $p < 0$ is $\mathcal{DS}_{0B}$.

Assume that $p > 0$. By [4, Theorems 2 and 4 and their proofs], we have $\mathcal{S} = \mathcal{IS}_{B0} \cup \mathcal{IS}_{BB} \cup \mathcal{DS}_{0B} \cup \mathcal{DS}_{BB}$ with all these subclasses to be nonempty. Hence, $\mathcal{IS} \cup \mathcal{DS}_{BB} \subseteq \mathcal{S}_{\mathcal{NSV}}$. In view of (3.18), $\mathcal{DS}_{0B} \subseteq \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Taking into account (3.17), we get $\mathcal{DS}_{BB} \cup \mathcal{IS}_{B0} \cup \mathcal{IS}_{BB} = \mathcal{S}_{\mathcal{NSV}}$ and $\mathcal{DS}_{0B} = \mathcal{S}_{\mathcal{NRV}}(\varrho)$. The relations with the classes $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{G}_2, \mathcal{H}_3$, and $\mathcal{L}(\varrho, \eta)$ can be shown similarly as in the case $p < 0$

In the last part of this proof we establish the relations with the class $\mathcal{P}$ under the condition $\delta + \alpha < \gamma$. First consider the case $p < 0$. Let $\gamma < \alpha - 1$ and $\delta < -1$. Then, as it was established in the previous parts, $J_1 = \infty$ and $J_2 < \infty$. Theorem 3.20 now yields $\mathcal{P} = \mathcal{DS}_B$. From the previous computations we know that $\mathcal{DS}_B = \mathcal{DS}$. Let $\gamma > \alpha - 1$ and $\delta > -1$. Then, as was established already earlier, we have $J_1 < \infty$. Theorem 3.20 and the equality $\mathcal{DS}_0 = \mathcal{DS}$ (which holds to be true) in this case yield $\mathcal{P} = \mathcal{DS}$. If $\delta < -1$ and $\gamma > \alpha - 1$, then $J_p < \infty$ and $J_r < \infty$. Consequently, $J_1 < \infty$ and thus Theorem 3.20 yields $\mathcal{P} = \mathcal{DS}_0$. The above established classification implies $\mathcal{DS}_0 = \mathcal{DS}_{0B}$, hence $\mathcal{P} = \mathcal{DS}_{0B}$.

Let $p > 0$. If $y \in \mathcal{S}_{\mathcal{NSV}}$, then $r^{1-\beta} y^{-M} \in \mathcal{RV}(-\gamma/(\alpha - 1))$ by Proposition 3.1. If $\gamma < \alpha - 1$ and $\delta < -1$, then $\int_a^\infty \mathcal{T}_m[y](s)\,\mathrm{d}s = \infty$, and hence $\mathcal{S}_{\mathcal{NSV}} \subseteq \mathcal{P}$, in view of Theorem 3.24. Since $\varrho > 0$, $ty'(t)/y(t) \to 0$ and $tx'(t)/x(t) \to \varrho$ as $t \to \infty$ for $x \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$, we get

$\mathcal{S}_{\mathcal{NRV}}(\varrho) \cap \mathcal{P} = \emptyset$ by definition. Consequently, $\mathcal{S}_{\mathcal{NSV}} = \mathcal{P}$. Assume that $\gamma > \alpha - 1$ and $\delta > -1$. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$. Then by the classification made in the previous parts, we obtain $y \in \mathcal{S}_2$, $\mathcal{S}_2$ being defined in Lemma 3.17, and $\mathcal{F}[y] \in \mathcal{RV}(-\varrho - 1)$ (see Lemma 3.17), $\mathcal{F}$ being defined in Theorem 3.21. Since $\varrho < 0$, we have $\int_a^\infty \mathcal{F}[y]\,\mathrm{d}s = \infty$. Assuming (3.15), we get $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{P}$ by Theorem 3.21. Further, $\mathcal{S}_{\mathcal{NSV}} \cap \mathcal{P} = \emptyset$ by definition, since for $x \in \mathcal{S}_{\mathcal{NSV}}$, $tx'(t)/t \to 0$ as $t \to \infty$ and $\varrho < 0$. Thus $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{P}$. If (3.15) fails to hold, then we can proceed similarly as at the end of the proof of Theorems 2.1 and 2.2, since the discussion made there is valid no matter whether $\delta + \alpha = \gamma$ or $\delta + \alpha < \gamma$. We again obtain $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{P}$. It remains to examine principal solutions when $\delta < -1$ and $\gamma > \alpha - 1$, i.e., $J_p + J_r < \infty$ under the condition $p > 0$. We will use Theorem 3.26. If $y \in \mathcal{S}_{\mathcal{NSV}}$, then $r^{1-\beta}y^{-2} \in \mathcal{RV}(\gamma(1-\beta))$. The index is less than -1, thus $\int_a^\infty r^{1-\beta}(s)y^{-2}(s)\,\mathrm{d}s < \infty$ and $\mathcal{S}_{\mathcal{NSV}} \cap \mathcal{P} = \emptyset$ by Theorem 3.26. If $y \in \mathcal{S}_{\mathcal{NRV}}(\varrho)$, then $r^{1-\beta}y^{-2} \in \mathcal{RV}(\gamma(1-\beta) - 2\varrho) = \mathcal{RV}(-1-\varrho)$. In view of $\varrho < 0$, the index is greater than -1, thus $\int_a^\infty r^{1-\beta}(s)y^{-2}(s)\,\mathrm{d}s = \infty$, and $\mathcal{S}_{\mathcal{NRV}}(\varrho) \subseteq \mathcal{P}$ by Theorem 3.26. Hence, in view of (3.17), $\mathcal{S}_{\mathcal{NRV}}(\varrho) = \mathcal{P}$. □

Proof of Theorem 2.4. Let $p < 0$. Since

$$\mathcal{S} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2), \qquad \mathcal{S}_{\mathcal{NRV}}(\vartheta_i) \neq \emptyset, \qquad i = 1, 2, \tag{3.21}$$

$\mathcal{S} = \mathcal{IS} \cup \mathcal{DS}$, and $\vartheta_1 < 0 < \vartheta_2$ (see Lemma 3.6), in view of Proposition 3.1, we get $\mathcal{IS} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_2)$ and $\mathcal{DS} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_1)$. Thanks to the positivity of ϑ_2, we have $\mathcal{IS} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) \subseteq \mathcal{IS}_\infty \subseteq \mathcal{IS}$ by Proposition 3.1. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) = \mathcal{IS} = \mathcal{IS}_\infty$. Since $y^{[1]}$ is positive increasing, we have $\mathcal{IS}_\infty = \mathcal{IS}_{\infty\infty} \cup \mathcal{IS}_{\infty B}$. But if $y \in \mathcal{IS}_{\infty B}$, we get $y \in \mathcal{RV}(\varrho)$ by Lemma 3.17-(ii), contradiction because of $\vartheta_2 \neq \varrho$ (see Lemma 3.6). Therefore $\mathcal{IS} = \mathcal{IS}_{\infty\infty}$. Similarly we find that $\mathcal{DS} \subseteq \mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \subseteq \mathcal{DS}_0 = \mathcal{DS}_{00} \cup \mathcal{DS}_{0B} = \mathcal{DS}_{00} \subseteq \mathcal{DS}$, and the equalities follow. From Lemma 3.16, $\mathcal{S}_{\mathcal{NRV}}(\vartheta_i) \subseteq \mathcal{L}(\vartheta_i, \eta_i)$, $i = 1, 2$. Condition (1.2) and $r \in \mathcal{NRV}(\gamma) \cap C^1$ imply $\lim_{t\to\infty} tL(\vartheta_i, \eta_i, t) = 0$. Hence, by the Representation Theorem (see (1.9)), $\mathcal{L}(\vartheta_i, \eta_i) \subseteq \mathcal{S}_{\mathcal{NRV}}(\vartheta_i)$, $i = 1, 2$. In view of Theorem 3.20, $\mathcal{P} = \mathcal{DS}_B$ or $\mathcal{P} = \mathcal{DS}_0$. But $\mathcal{DS}_B = \emptyset$, thus only the latter possibility occurs. Note that $J_2 = \infty$ by (1.2).

Let $p > 0$. Since we assume that $C_\gamma \in (0, K_\alpha]$, we have $\gamma \neq \alpha - 1$, otherwise K_α would be zero. Let $\gamma < \alpha - 1$. Then $J_r = \infty$ by Theorem 3.2, and so (1.5) holds. The class $\mathcal{IS}_{B0}$ is empty because of (3.21), where ϑ_1, ϑ_2 are positive by Lemma 3.6. The class $\mathcal{IS}_{\infty B}$ is also empty. Indeed, if $y \in \mathcal{IS}_{\infty B}$, then $y \in \mathcal{RV}(\varrho)$ by Lemma 3.17. But according to Lemma 3.6, $0 < \vartheta_1 \leq \vartheta_2 < \varrho$, contradiction. Thus $\mathcal{IS} \subseteq \mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \cup \mathcal{S}_{\mathcal{NRV}}(\vartheta_2) \subseteq \mathcal{IS}_{\infty 0} \subseteq \mathcal{IS}$. Let $\gamma > \alpha - 1$. Then $J_p = \infty$ since $p(t) \sim C_\gamma t^{-\alpha} r(t) \in \mathcal{RV}(\gamma - \alpha)$. Thus (1.6) holds. Similarly as before (using Lemma 3.6 and Lemma 3.17), we get $\mathcal{DS}_{B\infty} = \emptyset = \mathcal{DS}_{0B}$. Consequently, $\mathcal{DS} \subseteq \mathcal{NRV}(\vartheta_1) \cup \mathcal{NRV}(\vartheta_2) \subseteq \mathcal{DS}_{0\infty} \subseteq \mathcal{DS}$. The inclusions $\mathcal{S}_{\mathcal{NRV}}(\vartheta_i) \subseteq \mathcal{L}(\vartheta_i, \eta_i) \subseteq \mathcal{S}_{\mathcal{NRV}}(\vartheta_i)$, $i = 1, 2$, can be proved analogously as in the case $p < 0$.

Finally we show the relations with the class $\mathcal{P}$ when $p > 0$. Take $y \in \mathcal{S}_{\mathcal{NRV}}(\vartheta)$, where $\vartheta = \vartheta_1$ or $\vartheta = \vartheta_2$. From the previous part we know that $y \in \mathcal{IS}_{\infty 0} \cup \mathcal{DS}_{0\infty} \subseteq \mathcal{S}_1$. Recall that $\delta = \gamma - \alpha$ and $\gamma \neq \alpha - 1$. Assume that (3.15) holds. From Lemma 3.17 and Proposition 3.1, we get $\mathcal{F}[y] \in \mathcal{RV}(\Omega)$, $\mathcal{F}$ being defined in Theorem 3.21, where $\Omega = \vartheta - 1 - 2\vartheta - \delta - 1 - (\alpha - 1)\vartheta = \alpha - \gamma - 2 - \alpha\vartheta$. Clearly, $\Omega \gtrless -1$ if and only if $\vartheta \gtrless (\alpha - 1 - \gamma)/\alpha$. Since $C_\gamma \in (0, K_\alpha]$, from Lemma 3.6 we have $\vartheta_1 < (\alpha - 1 - \gamma)/\alpha < \vartheta_2$. Thus $\int_a^\infty \mathcal{F}[y](s)\,\mathrm{d}s = \infty$ when $\vartheta = \vartheta_1$ while $\int_a^\infty \mathcal{F}[y](s)\,\mathrm{d}s < \infty$ when $\vartheta = \vartheta_2$ by Theorem 3.2. Theorem 3.21 yields $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1) \subseteq \mathcal{P}$ and $\mathcal{S}_{\mathcal{NRV}}(\vartheta_2) \cap \mathcal{P} = \emptyset$. In view of (3.21), we get $\mathcal{P} = \mathcal{S}_{\mathcal{NRV}}(\vartheta_1)$. Now assume that (3.15) fails to hold and let $J_r = \infty$ (i.e., in our setting, $\gamma < \alpha - 1$) and $\alpha < 2$. The constant M is defined in Theorem 3.24. If $y \in \mathcal{S}_{\mathcal{NRV}}(\vartheta_1)$, then $r^{1-\beta}y^{-M} \in \mathcal{RV}(-\gamma(\beta - 1) - M\vartheta_1)$ by

Proposition 3.1. For the index we have $-\gamma(\beta-1)-M\vartheta_1>-1$ if and only if $M\vartheta_1(\alpha-1)<\alpha-1-\gamma$. From Lemma 3.23 we know that $M=\beta$; recall we assume $\alpha<2$. Thus the inequality $M\vartheta_1(\alpha-1)<\alpha-1-\gamma$ reads as $\vartheta_1<(\alpha-1-\gamma)/\alpha$ which is true by Lemma 3.6. Consequently, $\int_a^\infty \mathcal{T}_M[y](s)\,\mathrm{d}s=\infty$, and so Theorem 3.24 yields $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1)\subseteq\mathcal{P}$. The class $\mathcal{S}_{\mathcal{NRV}}(\vartheta_2)$ will be treated later. Now assume that (3.15) fails to hold in the sense that $J_p=\infty$ and $\alpha>2$. Note that then $\delta>-1$ and $\gamma>\alpha-1$, and so $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1)\cup\mathcal{S}_{\mathcal{NRV}}(\vartheta_2)=\mathcal{DS}=\mathcal{DS}_{0\infty}\subseteq\mathcal{S}_1$, where ϑ_1,ϑ_2 are negative. Take $y\in\mathcal{S}_{\mathcal{NRV}}(\vartheta_1)$ and set $u=-y^{[1]}$. Then $u\in\widehat{\mathcal{S}}$, see Lemma 3.9. We want to show that $u\in\widehat{\mathcal{P}}$; $\widehat{\mathcal{P}}$ is the set of principal solutions in $\widehat{\mathcal{S}}$. Denote $\widehat{\vartheta}_1=\gamma-\alpha+\vartheta_1(\alpha-1)+1$. Then, owing to Lemma 3.17, $u\in\mathcal{RV}(\widehat{\vartheta}_1)$. Recall that $\widehat{\gamma}$ is the index of regular variation of $\widehat{r}$ and let $\widehat{M}=\max\{\varphi_\alpha(t):t\in[0,1]\}$. Thanks to Proposition 3.1, we have $\widehat{r}^{1-\alpha}u^{-\widehat{M}}\in\mathcal{RV}(\widehat{\Psi})$, where $\widehat{\Psi}=-\widehat{\gamma}(\alpha-1)-\widehat{\vartheta}_1\widehat{M}$. Since we assume $\alpha>2$, we have $\beta<2$, and Lemma 3.23 yields $\widehat{M}=\alpha$. Recalling $\widehat{\gamma}=(\alpha-\gamma)/(\alpha-1)$, for the index $\widehat{\Psi}$ we get $\widehat{\Psi}=-\alpha+\gamma-\alpha(\gamma-\alpha+\vartheta_1(\alpha-1)+1)$. It is now easy to see that $\widehat{\Psi}>-1$ if and only if $\vartheta_1<(\alpha-1-\gamma)/\alpha$ where the last inequality is true by Lemma 3.6. Hence, $\int_a^\infty \widehat{r}^{1-\alpha}(s)u^{-\widehat{M}}(s)\,\mathrm{d}s=\infty$, and noting that $\int_a^\infty \widehat{r}^{1-\alpha}(s)\,\mathrm{d}s=\infty$ (since $\widehat{\gamma}(1-\alpha)>-1$), applying Theorem 3.24 to reciprocal equation (3.2), we get $u\in\widehat{\mathcal{P}}$. According to Theorem 3.22 we have $y\in\mathcal{P}$, and so again $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1)\subseteq\mathcal{P}$. The rest of the observations is made under the general assumption $\gamma\neq\alpha-1$. Take $y_i\in\mathcal{S}_{\mathcal{NRV}}(\vartheta_i)$, $i=1,2$. Then, no matter whether (3.15) holds or does not hold, $\lim_{t\to\infty} ty_i'(t)/y_i(t)=\vartheta_i$, $i=1,2$, and since $\vartheta_1<\vartheta_2$, we get $y_1'(t)/y_1(t)<y_2'(t)/y_2(t)$ for large t, which implies $\mathcal{S}_{\mathcal{NRV}}(\vartheta_2)\cap\mathcal{P}=\emptyset$. Altogether we get $\mathcal{S}_{\mathcal{NRV}}(\vartheta_1)=\mathcal{P}$. □

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Biharmonic system with Hartree-type critical nonlinearity

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Abstract. In this article, we investigate the multiplicity results of the following biharmonic Choquard system involving critical nonlinearities with sign-changing weight function:

$$\begin{cases} \Delta^2 u = \lambda F(x)|u|^{r-2}u + H(x)\left(\int_\Omega \frac{H(y)|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u & \text{in } \Omega,\\ \Delta^2 v = \mu G(x)|v|^{r-2}v + H(x)\left(\int_\Omega \frac{H(y)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|v|^{2^*_\alpha-2}v & \text{in } \Omega,\\ u = v = \nabla u = \nabla v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 5$, $1 < r < 2$, $0 < \alpha < N$, $2^*_\alpha = \frac{2N-\alpha}{N-4}$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality and Δ^2 denotes the biharmonic operator. The functions F, G and $H : \overline{\Omega} \to \mathbb{R}$ are sign-changing weight functions satisfying $F, G \in L^{\frac{2^*}{2^*-r}}(\Omega)$ and $H \in L^\infty(\Omega)$ respectively. By adopting Nehari manifold and fibering map technique, we prove that the system admits at least two nontrivial solutions with respect to parameter $(\lambda,\mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$.

Keywords: biharmonic system, sign-changing weight function, Nehari manifold, Hatree-type critical nonlinearity.

2020 Mathematics Subject Classification: 35A15, 35B33.

1 Introduction

We consider the following biharmonic Choquard system involving concave-convex nonlinearities with critical exponent and sign-changing weight functions

$$\begin{cases} \Delta^2 u = \lambda F(x)|u|^{r-2}u + H(x)\left(\int_\Omega \frac{H(y)|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u & \text{in } \Omega,\\ \Delta^2 v = \mu G(x)|v|^{r-2}v + H(x)\left(\int_\Omega \frac{H(y)|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|v|^{2^*_\alpha-2}v & \text{in } \Omega,\\ u = v = \nabla u = \nabla v = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{D}_{\lambda,\mu}$}$$

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where Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 5$, $0 < \alpha < N$, $1 < r < 2$, $2^*_\alpha = \frac{2N-\alpha}{N-4}$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality, Δ^2 denotes the biharmonic operator and λ, μ are the parameter such that $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$. We assume the following additive assumptions on the weight functions F, G and H:

$(Z1)$ $F, G \in L^\beta(\Omega)$ with $\beta = \frac{2^*}{2^*-r}$ and $2^* = \frac{2N}{N-4}$, $F^\pm = \max\{\pm F, 0\} \not\equiv 0$ in $\overline{\Omega}$ and $G^\pm = \max\{\pm G, 0\} \not\equiv 0$ in $\overline{\Omega}$.

$(Z2)$ $H \in L^\infty(\Omega)$ and $H^+ = \max\{H, 0\} \not\equiv 0$ in Ω.

Over the last many decades, biharmonic equations have been studied by many authors. These equations have wide application in many physical problems such as phase field models of multi-phase systems, in thin film theory, micro electro-mechanical system, nonlinear surface diffusion on solids, interface dynamics, flow in Hele–Shaw cells, incompressible flows, in theory of elasticity and the deformation of a nonlinear elastic beam (see [16,27,28,33,37]).

In recent years, many researchers are highly attracted to the study of nonlinear Choquard equation because of its applications in physical models (see [35,41]). The origin of nonlinear Choquard equation is related to the work of S. Pekar in 1976 [38] and P. Choquard. They used the elliptic equations with Hardy–Littlewood–Sobolev type nonlinearity to describe the model of an electron trapped in its hole in the Hartree–Fock theory of one component plasma and the quantum theory of a polaron at rest respectively.

Here, we are interested to study the biharmonic system with Choquard type nonlinearity because such type of equations occur in many applications. For this, consider the following Schrödinger–Hartree equation

$$i\partial_t u + \alpha(t)\Delta u + \beta(t)\Delta^2 u = \theta(|x|^{-\lambda} * |u|^2)u = 0, \qquad x \in \mathbb{R}^N, \qquad t \in \mathbb{R}$$
$$u(x, t_0) = u_0(x),\ x \in \mathbb{R}^N,$$

where $u(x,t)$ is a complex valued function in space-time $\mathbb{R}^N \times \mathbb{R}$, $N \geq 1$, α, β are real valued functions denoting the variable dispersion, $\theta \neq 0$ represents the focusing or defocus behaviour and λ is a positive parameter. The above model can be used in nonlinear optics for the electromagnetic wave propagation in optical fibers exhibiting particular nonlinearities, where there exists a repulsive (Hartree) force with strength θ, and when α, β experience variations in time due to the need of balance effect of the nonlinearity and the dispersions ([1,2]).

Towards the study of biharmonic equations, Bernis et al. [5] have examined the following critical biharmonic equation with Dirichlet and Navier boundary conditions

$$\begin{aligned} \Delta^2 u &= \lambda |u|^{q-2}u + |u|^{2^*-2}u, \quad \text{in } \Omega, \\ u &= \frac{\partial u}{\partial n} = 0 \quad \text{or} \quad u = \Delta u = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda > 0$, $2^* = \frac{2N}{N-4}$. The authors proved that there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, (1.1) has infinitely many solutions. Moreover, they also showed the existence of at least two positive solutions of (1.1) in the critical case. We suggest some literature ([11,12,15,21,24,32,39]) for reader's convenience and references therein.
Starting with the work of Pekar and Choquard [30,38], there has been a lot of work done involving Laplace, p-Laplace and nonlocal operator with Choquard type nonlinearity (see [9,10,29,36,43]). In [34], Moroz and Schaftingen studied the following Hartree equation (or Choquard equation)

$$-\Delta u + u = (I_\alpha * |u|^p)\,|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \qquad p > 1, \tag{1.2}$$

where I_α denotes the Riesz potential, defined as

$$I_\alpha(x) = \frac{B_\alpha}{|x|^{N-\alpha}}, \quad \text{with} \quad B_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\pi^{\frac{N}{2}}2^\alpha\right)}, \qquad \alpha \in (0, N).$$

and the term $(I_\alpha * |u|^p)\,|u|^{p-2}u$ is also known as Hartree-type nonlinearity. They proved the existence, positivity and radial symmetry of ground state solution. In 2018, Gao and Yang [19] investigated Brézis–Nirenberg type critical Choquard equation regarded as

$$-\Delta u = \left(\int_\Omega \frac{|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u + \lambda u \quad \text{in } \Omega, \tag{1.3}$$

where Ω is an open and bounded subset in $\mathbb{R}^N$ with Lipschitz boundary, $N \geq 3$, $2^*_\alpha = \frac{2N-\alpha}{N-2}$, $\alpha \in (0, N)$ and λ is a parameter. They established the existence and nonexistence of the nontrivial solution for (1.3) using variational methods. For more literature in this direction, we cite [4, 17, 18, 20, 45] and references therein. Recently, there are few works concerning the system involving nonlinear Choquard term. In [49], You and Zhao studied the following system with critical Choquard type nonlinearity

$$\begin{aligned} -\Delta u + \lambda_1 u &= \mu_1\left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu}\right)|u|^{2^*_\mu-1} + \beta\left(\frac{1}{|x|^\mu} * |v|^{2^*_\mu}\right)|u|^{2^*_\mu-1},\ x \in \Omega, \\ -\Delta v + \lambda_2 v &= \mu_1\left(\frac{1}{|x|^\mu} * |v|^{2^*_\mu}\right)|v|^{2^*_\mu-1} + \beta\left(\frac{1}{|x|^\mu} * |u|^{2^*_\mu}\right)|v|^{2^*_\mu-1},\ x \in \Omega, \\ u, v &\geq 0 \quad \text{in } \Omega, \quad u = v = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where $\mu_1, \mu_2 > 0$, $\beta \neq 0$, $-\lambda_1(\Omega) < \lambda_1, \lambda_2 > 0$, $\lambda_1(\Omega)$ is the first eigen value and $2^*_\mu = \frac{2N-\mu}{N-2}$ is the critical exponent in sense of Hardy–Littlewood–Sobolev inequality. The author proved the existence of a positive ground state solution using variational methods. Moreover, for elliptic system involving Laplace and fractional Laplacian with Choquard nonlinearity, we cite [23, 25, 26, 46, 48] and references therein.

Recently, Sang et al. [42] examined the critical Choquard equation with weighted terms and Sobolev-Hardy exponent in the case of Laplacian. They showed the existence of multiple positive solutions corresponding to the problem using variational methods and Lusternik–Schnirelmann category. Afterwards, Rani and Sarika [40] investigated the critical Choquard equation for biharmonic operator involving sign-changing weight functions and proved the multiplicity results analogous to the problem using the method of Nehari manifold and fibering map analysis. Considering all these facts as mentioned above, we have studied the system of critical Choquard equation involving sign-changing weight functions for biharmonic operator and proved the multiplicity results of nontrivial solution related to the system ($\mathcal{D}_{\lambda,\mu}$) with the help of Nehari manifold and fibering map techniques ([7, 8, 13]).

To the best of our knowledge, no work has been done on biharmonic system involving critical Choquard nonlinearity with sign-changing weight function. Apart from that, the minimizers for $S_{H,L}$ demonstrated here are entirely novel in the case of biharmonic system. Moreover, the results obtained in this article are completely fresh and new in the case of Laplacian also however the approach may be familiar.

In this article, we will discuss the existence and multiplicity results of nontrivial solutions for the system ($\mathcal{D}_{\lambda,\mu}$) with respect to parameter λ and μ. Using the Nehari manifold and fibering map analysis [7, 8, 13], we establish the existence of at least two nontrivial solutions for system involving critical Choquard nonlinearities with sign-changing weight functions with respect to the pair of parameters λ, μ belongs to a suitable subset of $\mathbb{R}^2$. The conspicuous aspect of this article is the study of the critical level (c_∞) below which the Palais–Smale condition is satisfied. Altogether, this article amplifies the branch of knowledge and gives a novel addition to the literature of the critical Choquard system.

In order to present our main results, we define the constant Υ_1 as

$$\Upsilon_1 := \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right)^{\frac{2}{2-r}} \left[\frac{2-r}{2(22^*_\alpha - r)} \|H^+\|_\infty^{-2} (\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{1}{2^*_\alpha - 1}} S^{\frac{r}{2-r}},$$

where $\bar{S}_{H,L}$ and S are defined later.

Now we state our following main results.

Theorem 1.1. *If $1 \leq r < 2$, $0 < \alpha < N$ and $\lambda, \mu > 0$ satisfy $0 < (\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, then the system ($\mathcal{D}_{\lambda,\mu}$) has at least one nontrivial solution in $H_0^2(\Omega) \times H_0^2(\Omega)$.*

For multiplicity result, we need the following assumptions on F, G and H respectively:

($Z3$) There exist a_0, b_0 and $r_0 > 0$ such that $B(0, 2r_0) \subset \Omega$ and $F(x) \geq a_0$, $G(x) \geq b_0$ for all $x \in B(0, 2r_0)$.

($Z4$) There exists $\delta_0 > \frac{2N-\alpha}{2}$ such that $\|H^+\|_\infty = H(0) = \max_{x \in \overline{\Omega}} h(x)$, $H(x) > 0$ for all $x \in B(0, 2r_0)$ and
$$H(x) = H(0) + o\left(|x|^{\delta_0}\right) \text{ as } x \to 0.$$

Theorem 1.2. *If $1 \leq r < 2$, $0 < \alpha < N$ and $\lambda, \mu > 0$ satisfy $0 < (\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_2$ (where $\Upsilon_2 \leq \Upsilon_1$) , then the system ($\mathcal{D}_{\lambda,\mu}$) has at least two nontrivial solution in $H_0^2(\Omega) \times H_0^2(\Omega)$. Moreover, the solutions corresponding to the system ($\mathcal{D}_{\lambda,\mu}$) are not semi-trivial.*

Remark 1.3. We note that the multiplicity results for the system ($\mathcal{D}_{\lambda,\mu}$) can be generalized to the following polyharmonic system

$$\begin{cases} (-\Delta)^m u = \lambda F(x)|u|^{r-2}u + H(x)\left(\int_\Omega \frac{H(y)|v(y)|^{2^*_{\alpha,m}}}{|x-y|^\alpha} dy\right)|u|^{2^*_{\alpha,m}-2}u & \text{in } \Omega, \\ (-\Delta)^m v = \mu G(x)|v|^{r-2}v + H(x)\left(\int_\Omega \frac{H(y)|u(y)|^{2^*_{\alpha,m}}}{|x-y|^\alpha} dy\right)|v|^{2^*_{\alpha,m}-2}v & \text{in } \Omega, \\ D^k u = D^k v = 0 \quad \text{for all } |k| \leq m-1 & \text{on } \partial\Omega, \end{cases}$$

where $(-\Delta)^m$ denotes the polyharmonic operators, $m \in \mathbb{N}$, $N \geq 2m+1$, $0 < \alpha < N$, $1 < r < 2$, $2^*_{\alpha,m} = \frac{2N-\alpha}{N-2m}$ is the critical exponent in the sense of Hardy–Littlewood–Sobolev inequality, and λ, μ are the parameter such that $(\lambda, \mu) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$.

Let S be the best Sobolev constant defined as

$$S := \inf_{u \in H_0^m(\Omega) \setminus \{0\}} \frac{\int_\Omega |D^m u|^2 dx}{\left(\int_\Omega |u|^{2^*_m} dx\right)^{\frac{2}{2^*_m}}},$$

where $2^*_m = \frac{2N}{N-2m}$. Then it is well known that S is achieved if and only if $\Omega = \mathbb{R}^N$, by the function

$$U(x) = \frac{C_{N,m}^{\frac{N-2m}{4m}}}{(1+|x|^2)^{\frac{N-2m}{2}}}$$

(see [44]). All the minimizers of S are obtained by

$$U_\epsilon(x) = \epsilon^{\frac{2m-N}{2}} U\left(\frac{x}{\epsilon}\right) = \frac{C_{N,m}^{\frac{N-2m}{4m}} \epsilon^{\frac{N-2m}{2}}}{(\epsilon^2 + |x|^2)^{\frac{N-2m}{2}}},$$

where $\epsilon > 0$ with $C_{N,m} := C(N,m) = \prod_{j=1-m}^{m}(N-2j)$.

Define $S_{H,L}$ to be the best constant as

$$S_{H,L} := \inf_{u \in H_0^m(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |D^m u|^2 dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_{\alpha,m}}|u(y)|^{2^*_{\alpha,m}}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_{\alpha,m}}}}.$$

One can obtain a family of minimizers for $S_{H,L}$ in the similar manner as shown in section 2 for the case $m = 2$ by taking $\widetilde{U_\epsilon}(x) = S^{\frac{(N-\mu)(2m-N))}{4m(N+2m-\mu)}} (C(N,\alpha))^{\frac{2m-N}{2(N+2m-\mu)}} U_\epsilon(x)$, where $\epsilon > 0$ and $\widetilde{U_\epsilon}(x)$ provides a family of minimizers for $S_{H,L}$. Using the same approach, multiplicity results can be established with respect to parameter λ and μ.

Organization of the article is as follows: In Section 2, variational setting for the problem $(\mathcal{D}_{\lambda,\mu})$ and some essential results are proved. Besides this, we show various asymptotic estimates which perform a vital role in the study of a second solution for the critical case. In Section 3, we discuss that the Palais–Smale condition holds for the energy functional associated with $(\mathcal{D}_{\lambda,\mu})$ at energy level in a suitable range related to the best Sobolev constant. Further, Nehari manifold and fibering map analysis are discussed precisely in Section 4. In Section 5, we prove the existence of Palais–Smale sequences and showed the existence of first nontrivial solution by the proof of Theorem 1.1. In Section 6, we give the detail of proof of the Theorem 1.2.

2 Preliminaries and some important results

We are using Sobolev space $\mathcal{H} := H_0^2(\Omega) \times H_0^2(\Omega)$ as a function space with standard norm $\|(u,v)\| = \left(\|u\|^2 + \|v\|^2\right)^{\frac{1}{2}}$, where $\|u\| = \left(\int_\Omega |\Delta u|^2 dx\right)^{\frac{1}{2}}$ and $\|u\|_p = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}$ be the usual $L^p(\Omega)$ norm.

Now, we state the well known Hardy–Littlewood–Sobolev inequality that plays a crucial role in solving the problem involving Choquard type nonlinearity.

Proposition 2.1 (Hardy–Littlewood–Sobolev inequality [31])**.** *Let t, $q > 1$ and $0 < \alpha < N$ with $1/t + \alpha/N + 1/q = 2$, $g \in L^t(\mathbb{R}^N)$ and $h \in L^q(\mathbb{R}^N)$. Then there exists a sharp constant $C(t,N,\alpha,q)$, independent of g, h such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x)h(y)}{|x-y|^\alpha} dxdy \le C(t,N,\alpha,q)\|g\|_{L^t(\mathbb{R}^N)}\|h\|_{L^q(\mathbb{R}^N)}. \tag{2.1}$$

If $t = q = \frac{2N}{2N-\alpha}$ then

$$C(t,N,\alpha,q) = C(N,\alpha) = \pi^{\frac{\alpha}{2}} \frac{\Gamma\left(\frac{N}{2} - \frac{\alpha}{2}\right)}{\Gamma\left(N - \frac{\alpha}{2}\right)} \left\{\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right\}^{-1+\frac{\alpha}{N}}.$$

In this case there is equality in (2.1) if and only if $g \equiv Ch$ and

$$h(x) = A(b^2 + |x-a|^2)^{-\frac{(2N-\alpha)}{2}},$$

for some $A \in \mathbb{C}$, $0 \neq b \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Thus, if $|u|^s \in L^t(\mathbb{R}^N)$ for $t > 1$ such that $\frac{2}{t} + \frac{\alpha}{N} = 2$, then by the Hardy–Littlewood–Sobolev inequality, the integral $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^s|u(y)|^s}{|x-y|^\alpha}dxdy$ is well defined. Hence for $u \in H^2(\mathbb{R}^N)$, by Sobolev embedding theorems, we obtain

$$2_\alpha := \frac{2N-\alpha}{N} \leq s \leq \frac{2N-\alpha}{N-4} =: 2^*_\alpha,$$

where 2_α and 2^*_α are known as lower and upper critical exponent respectively in the sense of Hardy–Littlewood–Sobolev inequality.

Therefore, for all $u \in H^2(\mathbb{R}^N)$, by the Hardy–Littlewood–Sobolev inequality, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy \leq C(N,\alpha)\|u\|_{2^*}^{22^*_\alpha},$$

where $C(N,\alpha)$ is same as defined in Proposition 2.1. One can easily see that in the Hardy–Littlewood–Sobolev inequality, equality takes place if and only if

$$h(x) = C\left(\frac{k}{k^2+|x-a|^2}\right)^{\frac{2N-\alpha}{2}},$$

where $C > 0$ is fixed constant. Thus, $u = C\left(\frac{k}{k^2+|x-a|^2}\right)^{\frac{N-4}{2}}$ if and only if

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}} = (C(N,\alpha))^{\frac{1}{2^*_\alpha}}\left(\int_{\mathbb{R}^N}|u(x)|^{2^*}dx\right)^{\frac{2}{2^*}}. \tag{2.2}$$

Let S be the best Sobolev constant defined as

$$S = \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^2dx}{(\int_\Omega |u(x)|^{2^*}dx)^{\frac{2}{2^*}}}.$$

The best constant S is attained by the function $U(x) = \frac{[N(N+2)(N-2)(N-4)]^{\frac{N-4}{8}}}{(1+|x|^2)^{\frac{N-4}{2}}}$ and all the minimizers of S are obtained by

$$U_\epsilon(x) = \epsilon^{\frac{4-N}{2}}U\left(\frac{x}{\epsilon}\right), \quad \text{where } \epsilon > 0, \tag{2.3}$$

which satisfies the equation $\Delta^2 u = |u|^{2^*-2}u$ in $\mathbb{R}^N$, with

$$\|U_\epsilon(x)\|^2 = \|U_\epsilon(x)\|_{2^*}^{2^*} = S^{\frac{N}{4}}.$$

Further, we define $S_{H,L}$ to be the best constant as

$$S_{H,L} := \inf_{u\in D^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\Delta u|^2dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}}.$$

Next, we show the relation between S and $S_{H,L}$ by the following lemma in which the leading concept is taken from [19].

Theorem 2.2. *The constant $S_{H,L}$ is achieved if and only if*

$$u = C\left(\frac{k}{k^2+|x-a|^2}\right)^{\frac{N-4}{2}},$$

where $C > 0$ is a constant, $a \in \mathbb{R}^N$ and $k \in \mathbb{R}^+$. Furthermore

$$S_{H,L} = \frac{S}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}}}. \tag{2.4}$$

Proof. The Hardy–Littlewood–Sobolev inequality yields that

$$S_{H,L} \geq \frac{1}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}}} \inf_{u\in D^{2,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\Delta u|^2dx}{\left(\int_{\mathbb{R}^N}|u|^{2^*}\right)^{\frac{2}{2^*}}} = \frac{S}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}}}.$$

Further, it follows by the definition of $S_{H,L}$ and (2.2) that

$$S_{H,L} \leq \frac{\int_{\mathbb{R}^N}|\Delta u|^2dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}} \leq \frac{\int_{\mathbb{R}^N}|\Delta u|^2dx}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}}\left(\int_{\mathbb{R}^N}|u(x)|^{2^*}dx\right)^{\frac{2}{2^*}}} \leq \frac{S}{(C(N,\alpha))^{\frac{1}{2^*_\alpha}}}.$$

In conclusion, we obtain the required result. □

Take $\widetilde{U_\epsilon}(x) = S^{\frac{(N-\alpha)(4-N)}{8(N+4-\alpha)}}(C(N,\alpha))^{\frac{4-N}{2(N+4-\alpha)}}U_\epsilon(x)$, then $\widetilde{U_\epsilon}$ gives a family of minimizers for $S_{H,L}$ and satisfies the equation

$$\Delta^2 u = \left(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\alpha}}{|x-y|^\alpha}dy\right)|u|^{2^*_\alpha-2}u \quad \text{in } \mathbb{R}^N.$$

Moreover,

$$\int_{\mathbb{R}^N}|\Delta\widetilde{U_\epsilon}|^2dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\widetilde{U_\epsilon}(x)|^{2^*_\alpha}|\widetilde{U_\epsilon}(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy = (S_{H,L})^{\frac{2N-\alpha}{N+4-\alpha}}.$$

Consider the best constant $\bar{S}_{H,L}$ given as

$$\bar{S}_{H,L} := \inf_{u\in\mathcal{H}\setminus\{(0,0)\}} \frac{\|(u,v)\|^2}{\left(\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}}.$$

Now, we state an important lemma which is used to show the relation between $\bar{S}_{H,L}$ and $S_{H,L}$.

Lemma 2.3. *For $u, v \in L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$, $0 < \alpha < N$ and $s \in [2_\alpha, 2^*_\alpha]$, the following inequality holds true*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^s|v(y)|^s}{|x-y|^\alpha}dxdy \leq \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^s|u(y)|^s}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)|^s|v(y)|^s}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2}}.$$

Proof. The proof is similar as given in [22]. □

Afterwards, we build a relation that connecting $\bar{S}_{H,L}$ and $S_{H,L}$ by adopting an idea from [3].

Lemma 2.4. *The following relation holds*

$$\bar{S}_{H,L} = 2S_{H,L}.$$

Proof. Let $\{k_n\} \subset H_0^2(\Omega)$ be a minimizing sequence for $S_{H,L}$. Choose the sequences $\{u_n = sk_n\}$ and $\{v_n = tk_n\}$ in $H_0^2(\Omega)$, where $s, t > 0$. Then the definition of $\bar{S}_{H,L}$ implies that

$$\bar{S}_{H,L} \leq \frac{\|(u_n, v_n)\|^2}{\left(\int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|v_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}}} = \left(\frac{s}{t} + \frac{t}{s}\right) \frac{\|k_n\|^2}{\left(\int_\Omega \int_\Omega \frac{|k_n(x)|^{2^*_\alpha}|k_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}}}. \tag{2.5}$$

Further, define a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x) = x + \frac{1}{x}$. Then $f(\frac{s}{t}) = \frac{s}{t} + \frac{t}{s}$ and f achieves its minimum at $x_0 = 1$. Thus, we have

$$\min_{x \in \mathbb{R}^+} f(x) = f(x_0) = 2.$$

Now, choose s, t in such a way that $s = t$ and taking $n \to \infty$ in (2.5), we obtain

$$\bar{S}_{H,L} \leq 2S_{H,L}. \tag{2.6}$$

At the same time, let $\{(u_n, v_n)\}$ be a minimizing sequence of $\bar{S}_{H,L}$. Take $a_n = s_n v_n$ for some $s_n > 0$ such that $\int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy = \int_\Omega \int_\Omega \frac{|a_n(x)|^{2^*_\alpha}|a_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy$.

This together with Lemma 2.3 implies that

$$\begin{aligned}
&\int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|a_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy \\
&\quad \leq \left(\int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2}} \left(\int_\Omega \int_\Omega \frac{|a_n(x)|^{2^*_\alpha}|a_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2}} \\
&\quad = \int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{\|(u_n, v_n)\|^2}{\left(\int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|v_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}}} &= s_n \frac{\|(u_n, v_n)\|^2}{\left(\int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|a_n(y)|^{2^*_\alpha}}{|x-y|^\alpha} dxdy\right)^{\frac{1}{2^*_\alpha}}} \\
&\geq s_n \frac{\|u_n\|^2}{\left(\int_\Omega \int_\Omega \frac{|u_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)^{\frac{1}{2^*_\alpha}}} + s_n^{-1} \frac{\|a_n\|^2}{\left(\int_\Omega \int_\Omega \frac{|a_n(x)|^{2^*_\alpha}|a_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)^{\frac{1}{2^*_\alpha}}} \\
&\geq \left(s_n + s_n^{-1}\right) S_{H,L} \\
&\geq \gamma(x_0) S_{H,L}.
\end{aligned}$$

Now passing the limit as $n \to \infty$

$$\bar{S}_{H,L} \geq 2S_{H,L}. \tag{2.7}$$

We desire our result after combining (2.6) and (2.7). □

Now, we prove some estimates, which are useful to obtain the critical level. Without loss of generality, we may assume that $0 \in \Omega$ and $B(0,2\gamma) \subset \Omega$. Let $\phi \in C_c^\infty(\Omega)$ be a fixed cut-off function such that $0 \le \phi \le 1$ in $\mathbb{R}^N$, $\phi(x) = 1$ on $B_\gamma = B(0,\gamma)$ and $\phi(x) = 0$ in $\mathbb{R}^N \setminus B_{2\gamma}$ with $|\nabla\phi| \le C, |\Delta\phi| \le C$. Define

$$\overline{U}_\epsilon(x) = \phi U_\epsilon(x),$$

where $U_\epsilon(x)$ is define in (2.3). Accordingly, we have the following norm estimates (see[12]).

Lemma 2.5. *The following estimates are true for $\epsilon > 0$ small enough.*

$$\|\overline{U}_\epsilon(x)\|^2 = S^{\frac{N}{4}} + o(\epsilon^{N-4}).$$

$$\int_\Omega |\overline{U}_\epsilon(x)|^{2^*} = S^{\frac{N}{4}} + o(\epsilon^N).$$

$$\int_\Omega |\overline{U}_\epsilon(x)|^r dx = \begin{cases} o\left(\epsilon^{\frac{N-4}{2}r}\right), & r < \frac{N}{N-4} \\ o\left(\epsilon^{N-\frac{N-4}{2}r}|\ln\epsilon|\right), & r = \frac{N}{N-4} \\ o\left(\epsilon^{N-\frac{N-4}{2}r}\right), & r > \frac{N}{N-4}. \end{cases} \tag{2.8}$$

Lemma 2.6. *For Choquard term, the following estimate is true:*

$$\begin{aligned} 0 &\le \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N-4}{4}} - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right) \\ &\le \left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}} \\ &\le \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N-4}{4}}. \end{aligned} \tag{2.9}$$

Proof. By assumption $(Z2)$, there exists $0 < \gamma \le r_0$ such that for all $x \in B(0,2\gamma)$ with $\delta_0 > \frac{2N-\alpha}{2}$

$$H(x) = H(0) + o(|x|^{\delta_0}), \quad \text{as } x \to 0. \tag{2.10}$$

Using the Hardy–Littlewood–Sobolev inequality and (2.4), we have

$$\begin{aligned} \left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}} &\le \|H^+\|_\infty^{\frac{2}{2^*_\alpha}} (C(N,\alpha))^{\frac{1}{2^*_\alpha}} \|\overline{U}_\epsilon(x)\|_{2^*}^2 \\ &= \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}} (C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}} S_{H,L}^{\frac{N-4}{4}} + o(\epsilon^{N-4}). \end{aligned}$$

Thus

$$\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy \le \|H^+\|_\infty^2 (C(N,\alpha))^{\frac{N}{4}} S_{H,L}^{\frac{2N-\alpha}{4}} + o(\epsilon^{2N-\alpha}).$$

Consider

$$\begin{aligned} &\epsilon^{\alpha-2N}\|H^+\|_\infty^2 (C(N,\alpha))^{\frac{N}{4}} S_{H,L}^{\frac{2N-\alpha}{4}} - \epsilon^{\alpha-2N}\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy \\ &= A^{2N-\alpha}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(H(0))^2}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} \\ &\quad - \epsilon^{\alpha-2N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha} \end{aligned}$$

$$
\begin{aligned}
&= A^{2N-\alpha}\left[\int_{\mathbb{R}^N\setminus B_\gamma} \frac{H(0)\left(H(0)-H(x)|\phi(x)|^{2^*_\alpha}\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N} \frac{1}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\right.\\
&\quad+\int_{\mathbb{R}^N\setminus B_\gamma} \frac{H(x)|\phi(x)|^{2^*_\alpha}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N} \frac{H(0)-H(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\\
&\quad+\int_{B_\gamma} \frac{H(0)\left(H(0)-H(x)\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N} \frac{1}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\\
&\quad\left.+\int_{B_\gamma} \frac{H(x)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N} \frac{H(0)-H(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\right]\\
&= E_1+E_2+E_3+E_4, \qquad (2.11)
\end{aligned}
$$

where $A=[N(N+2)(N-2)(N-4)]^{\frac{N-4}{8}}$.

On taking E_1, we have

$$
\begin{aligned}
E_1 &= A^{2N-\alpha}\left[\int_{\mathbb{R}^N\setminus B_\gamma} \frac{H(0)\left(H(0)-H(x)|\phi(x)|^{2^*_\alpha}\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N\setminus B_\gamma} \frac{1}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\right.\\
&\quad\left.+\int_{\mathbb{R}^N\setminus B_\gamma} \frac{H(0)\left(H(0)-H(x)|\phi(x)|^{2^*_\alpha}\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{B_\gamma} \frac{1}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\right]\\
&= E_{1,1}+E_{1,2}.
\end{aligned}
$$

Applying the Hardy–Littlewood–Sobolev inequality on $E_{1,1}$ and $E_{1,2}$ respectively, we get

$$
\begin{aligned}
E_{1,1} &\le C_1\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{dx}{(\epsilon^2+|x|^2)^N}\right)^{\frac{2N-\alpha}{2N}}\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{dy}{(\epsilon^2+|y|^2)^N}\right)^{\frac{2N-\alpha}{2N}}\\
&= C_1\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{dt}{(\epsilon^2+|t|^2)^N}\right)^{\frac{2N-\alpha}{N}} \le C_1\left(\int_\gamma^\infty \frac{r^{N-1}}{r^{2N}}dr\right)^{\frac{2N-\alpha}{N}} = C_2.
\end{aligned}
$$

and

$$
\begin{aligned}
E_{1,2} &\le C_3\int_{\mathbb{R}^N\setminus B_\gamma}\int_{B_\gamma}\frac{1}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dxdy\\
&\le C_3\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{dx}{(\epsilon^2+|x|^2)^N}\right)^{\frac{2N-\alpha}{2N}}\left(\int_{B_\gamma}\frac{dy}{(\epsilon^2+|y|^2)^N}\right)^{\frac{2N-\alpha}{2N}}\\
&\le C_4\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{dx}{|x|^{2N}}\right)^{\frac{2N-\alpha}{2N}}\left(\int_0^\gamma\frac{r^{N-1}dr}{(\epsilon^2+r^2)^N}\right)^{\frac{2N-\alpha}{2N}}\\
&\le o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right)\left(\int_0^{\frac{\gamma}{\epsilon}}\frac{t^{N-1}dt}{(1+t^2)^N}\right)^{\frac{2N-\alpha}{2N}}\\
&\le o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right)\left(\int_0^\infty\frac{t^{N-1}dt}{(1+t^2)^N}\right)^{\frac{2N-\alpha}{2N}} = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).
\end{aligned}
$$

Thus

$$
E_1 = C_2 + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).
$$

Further on taking E_2, we obtain

$$\begin{aligned}
E_2 = A^{2N-\alpha} \Bigg[&\int_{\mathbb{R}^N \setminus B_\gamma} \frac{H(x)|\phi(x)|^{2^*_\alpha}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}} \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{H(0)-H(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dy \right) dx \\
&+ \int_{\mathbb{R}^N \setminus B_\gamma} \frac{H(x)|\phi(x)|^{2^*_\alpha}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}} \left(\int_{B_\gamma} \frac{H(0)-H(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dy \right) dx \Bigg] \\
= E_{2,1} + E_{2,2}.
\end{aligned}$$

Now estimating $E_{2,1}$ same as $E_{1,1}$, we have

$$E_{2,1} \le C_5 \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dy}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} = C_6.$$

Using the Hardy–Littlewood–Sobolev inequality $E_{2,2}$ and (2.10), we get

$$\begin{aligned}
E_{2,2} &\le C_7 \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{B_\gamma} \frac{|y|^{\frac{2N\delta_0}{2N-\alpha}}}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\
&\le C_8 \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{B_\gamma} \frac{|y|^{\frac{2N\delta_0}{2N-\alpha}}}{|y|^{2N}} \right)^{\frac{2N-\alpha}{2N}} \\
&= C_9.
\end{aligned}$$

Hence

$$E_2 = C_6 + C_9.$$

For E_3, we use the Hardy–Littlewood–Sobolev inequality with (2.10) which implies that

$$\begin{aligned}
E_3 = A^{2N-\alpha} \Bigg[&\int_{B_\gamma} \int_{\mathbb{R}^N \setminus B_\gamma} \frac{H(0)\,(H(0)-H(x))}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dxdy \\
&+ \int_{B_\gamma} \int_{B_\gamma} \frac{H(0)\,(H(0)-H(x))}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dxdy \Bigg] \\
= E_{3,1} + E_{3,2}.
\end{aligned}$$

$$\begin{aligned}
E_{3,1} &\le A^{2N-\alpha} \int_{B_\gamma} \int_{\mathbb{R}^N \setminus B_\gamma} \frac{H(0)|x|^{\delta_0}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha} dxdy \\
&\le A^{2N-\alpha} \left(\int_{B_\gamma} \frac{|x|^{\frac{2N\delta_0}{2N-\alpha}} dx}{(\epsilon^2+|x|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dy}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\
&\le C_{10} \left(\int_{B_\gamma} \frac{|x|^{\frac{2N\delta_0}{2N-\alpha}} dx}{|x|^{2N}} \right)^{\frac{2N-\alpha}{2N}} \left(\int_{\mathbb{R}^N \setminus B_\gamma} \frac{dy}{(\epsilon^2+|y|^2)^N} \right)^{\frac{2N-\alpha}{2N}} \\
&= C_{11}.
\end{aligned}$$

$$
\begin{aligned}
E_{3,2} &\leq A^{2N-\alpha}\int_{B_\gamma}\int_{B_\gamma}\frac{H(0)|x|^{\delta_0}}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dxdy\\
&\leq A^{2N-\alpha}\left(\int_{B_\gamma}\frac{|x|^{\frac{2N\delta_0}{2N-\alpha}}dx}{(\epsilon^2+|x|^2)^N}\right)^{\frac{2N-\alpha}{2N}}\left(\int_{B_\gamma}\frac{dy}{(\epsilon^2+|y|^2)^N}\right)^{\frac{2N-\alpha}{2N}}\\
&\leq o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right)\left(\int_{B_\gamma}\frac{|x|^{\frac{2N\delta_0}{2N-\alpha}}dx}{|x|^{2N}}\right)^{\frac{2N-\alpha}{2N}}\left(\int_0^{\frac{\gamma}{\epsilon}}\frac{r^{N-1}dr}{(1+r^2)^N}\right)^{\frac{2N-\alpha}{2N}}\\
&\leq o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right)\left(\int_0^{\infty}\frac{r^{N-1}dr}{r^{2N}}\right)^{\frac{2N-\alpha}{2N}} = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).
\end{aligned}
$$

Thus

$$E_3 = C_{11} + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).$$

Similarly on taking E_4, we have

$$
\begin{aligned}
E_4 = A^{2N-\alpha}&\left[\int_{B_\gamma}\frac{H(x)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}}\left(\int_{\mathbb{R}^N\setminus B_\gamma}\frac{H(0)-H(y)|\phi(y)|^{2^*_\alpha}}{(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dy\right)dx\right.\\
&\left.+\int_{B_\gamma}\int_{B_\gamma}\frac{H(x)\left(H(0)-H(y)\right)}{(\epsilon^2+|x|^2)^{\frac{2N-\alpha}{2}}(\epsilon^2+|y|^2)^{\frac{2N-\alpha}{2}}|x-y|^\alpha}dxdy\right]\\
= E_{4,1}&+E_{4,2}.
\end{aligned}
$$

By the same approach used in $E_{1,2}$ and $E_{3,2}$ respectively, we obtain

$$E_{4,1} = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) \text{ and } E_{4,2} = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).$$

Hence

$$E_4 = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right) = o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).$$

Therefore

$$E_1+E_2+E_3+E_4 = \widehat{C} + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right),$$

where $\widehat{C} = C_2 + C_9 + C_{12}$.

Using (2.11), we obtain

$$
\begin{aligned}
0 &\leq \epsilon^{\alpha-2N}\|H^+\|_\infty^2(C(N,\alpha))^{\frac{N}{4}}S_{H,L}^{\frac{2N-\alpha}{4}} - \epsilon^{\alpha-2N}\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\\
&\leq \widehat{C} + o\left(\epsilon^{-\frac{2N-\alpha}{2}}\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
0 &\leq 1 - \|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\\
&\leq \epsilon^{2N-\alpha}\|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C} + o\left(\epsilon^{\frac{2N-\alpha}{2}}\right).
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
0 &\le 1-\epsilon^{2N-\alpha}\|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C}-o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\\
&\le \|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\\
&\le 1.
\end{aligned}
$$

Now, choose $\epsilon>0$ such that $\epsilon^{2N-\alpha}\|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C}<1$. Thus

$$
\begin{aligned}
0 &\le 1-\epsilon^{2N-\alpha}\|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C}-o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\\
&\le \left(1-\epsilon^{2N-\alpha}\|H^+\|_\infty^{-2}(C(N,\alpha))^{-\frac{N}{4}}S_{H,L}^{-\frac{2N-\alpha}{4}}\widehat{C}-o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\right)^{\frac{1}{2^*_\alpha}}\\
&\le \|H^+\|_\infty^{-\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{-\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{-\frac{N-4}{4}}\left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}\\
&\le 1.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
0 &\le \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N-4}{4}}\\
&\quad -\epsilon^{2N-\alpha}\|H^+\|_\infty^{\frac{2(4-N+\alpha)}{2N-\alpha}}(C(N,\alpha))^{\frac{(\alpha-N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{\alpha-N-4}{4}}\widehat{C}-o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\\
&\le \left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}\le \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N-4}{4}}.
\end{aligned}
$$

Thus, we can write

$$
\begin{aligned}
0 &\le \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N-4}{4}}-o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\\
&\le \left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\overline{U}_\epsilon(x)|^{2^*_\alpha}|\overline{U}_\epsilon(y)|^{2^*_\alpha}}{|x-y|^\alpha}dxdy\right)^{\frac{1}{2^*_\alpha}}\\
&\le \|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}(C(N,\alpha))^{\frac{(N-4)N}{(2N-\alpha)4}}S_{H,L}^{\frac{N-4}{4}}.
\end{aligned}
$$

Thus, the proof is complete. □

Definition 2.7. A pair of functions $(u,v)\in\mathcal{H}$ is said to be a weak solution of the system ($\mathcal{D}_{\lambda,\mu}$) if for all $(\phi_1,\phi_2)\in\mathcal{H}$, the following holds

$$
\begin{aligned}
&\int_\Omega \Delta u\Delta\phi_1 dx+\int_\Omega\Delta v\Delta\phi_2 dx-\lambda\int_\Omega F(x)|u|^{r-2}u\phi_1 dx-\mu\int_\Omega G(x)|v|^{r-2}v\phi_2 dx\\
&\quad -\int_\Omega\int_\Omega H(x)H(y)\left(\frac{|v(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha-2}u(y)\phi_1(y)+|u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha-2}v(y)\phi_2(y)}{|x-y|^\alpha}\right)=0.
\end{aligned}
$$

In order to prove the Palais–Smale condition, we need the following lemma which is inspired by the Brézis–Lieb convergence lemma (see [6]).

Lemma 2.8. *Let $N\ge 5$, $0<\alpha<N$ and $\{u_n\}$ be a bounded sequence in $L^{\frac{2N}{N-4}}(\mathbb{R}^N)$. If $u_n\to u$ a.e. in $\mathbb{R}^N$ as $n\to\infty$, then*

$$
\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}(|x|^{-\alpha}*|u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha}-\int_{\mathbb{R}^N}(|x|^{-\alpha}*|u_n-u|^{2^*_\alpha})|u_n-u|^{2^*_\alpha}\right)=\int_{\mathbb{R}^N}(|x|^{-\alpha}*|u|^{2^*_\alpha})|u|^{2^*_\alpha}.
$$

Proof. The proof is similar to the proof of the Brézis–Lieb Lemma (see [6]) or Lemma 2.2 [19]. But for completeness, we give the detail. Consider

$$\begin{aligned}\int_{\mathbb{R}^N}(|x|^{-\alpha}*|u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha} - \int_{\mathbb{R}^N}(|x|^{-\alpha}*|u_n-u|^{2^*_\alpha})|u_n-u|^{2^*_\alpha} \\ = \int_{\mathbb{R}^N}(|x|^{-\alpha}*(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha}))(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha}) \\ +2\int_{\mathbb{R}^N}(|x|^{-\alpha}*(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha}))|u_n-u|^{2^*_\alpha}.\end{aligned} \tag{2.12}$$

Now by using [34, Lemma 2.5], for $q = 2^*_\alpha = \frac{2N-\alpha}{N-4}$ and $r = \frac{2N}{2N-\alpha}2^*_\alpha$, then we obtain

$$|u_n|^{2^*_\alpha} - |u_n-u|^{2^*_\alpha} \to |u|^{2^*_\alpha} \quad \text{in } L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N) \text{ as } n\to\infty. \tag{2.13}$$

Also the Hardy–Littlewood–Sobolev inequality implies that

$$|x|^{-\alpha}*(|u_n|^{2^*_\alpha}-|u_n-u|^{2^*_\alpha}) \to |x|^{-\alpha}*|u|^{2^*_\alpha} \quad \text{in } L^{\frac{2N}{\alpha}}(\mathbb{R}^N) \text{ as } n\to\infty. \tag{2.14}$$

Hence with the help of [47, Proposition 5.4.7], we obtain $|u_n-u|^{2^*_\alpha} \rightharpoonup 0$ weakly in $L^{\frac{2N}{2N-\alpha}}(\mathbb{R}^N)$ as $n\to\infty$. So using this together with (2.13), (2.14), in (2.12), we obtain the required result. □

Now, we define the energy functional $I_{\lambda,\mu} : \mathcal{H} \to \mathbb{R}$ associated with the system $(\mathcal{D}_{\lambda,\mu})$ as

$$\begin{aligned} I_{\lambda,\mu}(u,v) = \frac{1}{2}\|(u,v)\|^2 - \frac{1}{r}\int_\Omega(\lambda F(x)|u|^r + \mu G(x)|v|^r) \\ - \frac{1}{2^*_\alpha}\int_\Omega\int_\Omega H(x)H(y)\left(\frac{|u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right).\end{aligned} \tag{2.15}$$

Then $I_{\lambda,\mu}(u,v)$ is C^1 function on $\mathcal{H}$. Moreover, the critical points of the functional $I_{\lambda,\mu}$ are the solutions of $(\mathcal{D}_{\lambda,\mu})$. For convenience, we define $P_{\lambda,\mu}(u,v)$ and $Q(u,v)$ as

$$\begin{aligned} P_{\lambda,\mu}(u,v) &:= \int_\Omega(\lambda F(x)|u|^r + \mu G(x)|v|^r)dx, \\ Q(u,v) &:= \int_\Omega\int_\Omega H(x)H(y)\left(\frac{|u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)dxdy,\end{aligned}$$

throughout the article. Then we obtain the estimates on $P_{\lambda,\mu}(u,v)$ and $Q(u,v)$ by using Hölder's inequality, Sobolev's embedding theorem and the definition of $\bar{S}_{H,L}$ as follows

$$\begin{aligned} P_{\lambda,\mu}(u,v) &= \int_\Omega(\lambda F(x)|u|^r + \mu G(x)|v|^r)dx \\ &\le S^{-\frac{r}{2}}\left(\lambda\|F\|_\beta\|u\|^r + \mu\|G\|_\beta\|v\|^r\right) \\ &\le S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\|(u,v)\|^r.\end{aligned} \tag{2.16}$$

$$Q(u,v) \le \|H^+\|^2_\infty(\bar{S}_{H,L})^{-2^*_\alpha}\|(u,v)\|^{22^*_\alpha}. \tag{2.17}$$

Definition 2.9. Let $J : X \to \mathbb{R}$ be a C^1 functional on a Banach space X.

1. For $c \in \mathbb{R}$, a sequence $\{u_k\} \subset X$ is a Palais–Smale sequence at level c ($(PS)_c$) in X for J if $J(u_k) = c + o_k(1)$ and $J'(u_k) \to 0$ in X^{-1} as $k \to \infty$.
2. We say J satisfies $(PS)_c$-condition if for any Palais–Smale sequence $\{u_k\}$ in X for J has a convergent subsequence.

3 The Palais–Smale condition

In this section, we show that the energy functional $\mathcal{I}_\lambda$ satisfies the Palais–Smale condition below a certain level i.e. c_∞, which is used to prove the existence of second solution.

Lemma 3.1. *Consider* (Z1) *and* (Z2) *are true. Suppose* $\{(u_n, v_n\} \subset \mathcal{H}$ *is a* $(PS)_c$*-sequence for* $I_{\lambda,\mu}$ *such that* $(u_n, v_n) \rightharpoonup (u, v)$ *weakly in* $\mathcal{H}$*. Then* $I'_{\lambda,\mu}(u, v) = 0$*. Furthermore, there exists a positive constant* K_0 *depending on* $r, \alpha, N, 2^*_\alpha$ *and* S *such that*

$$I_{\lambda,\mu}(u, v) \geq -K_0 \left((\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} \right),$$

where $K_0 = \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right)\left(\frac{2-r}{2}\right)\left[\left(\frac{2N-\alpha}{N+4-\alpha}\right)\left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}}\right]^{\frac{r}{2-r}} S^{-\frac{r}{2}}$.

Proof. If $\{(u_n, v_n)\}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ with $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $\mathcal{H}$, then by using the standard argument, we get $I'_{\lambda,\mu}(u, v) = 0$. i.e.

$$\|(u, v)\|^2 - P_{\lambda,\mu}(u, v) - 2Q(u, v) = 0.$$

Above with Hölder's inequality, Sobolev embedding theorem and Young's inequality in (2.15) implies that

$$\begin{aligned} I_{\lambda,\mu}(u, v) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right)\|(u, v)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right)\int_\Omega (\lambda F(x)|u|^r + \mu G(x)|v|^r)\,dx \\ &\geq \frac{N+4-\alpha}{2(2N-\alpha)}\|(u, v)\|^2 - \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right)\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} S^{-\frac{r}{2}}\|(u, v)\|^r \\ &\geq \frac{N+4-\alpha}{2(2N-\alpha)}\|(u, v)\|^2 \\ &\quad - \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}}\left[\frac{2-r}{2} l^{\frac{2}{2-r}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right) + \frac{r}{2} l^{-\frac{2}{r}}\|(u, v)\|^2\right] \\ &= K_0\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right), \end{aligned}$$

where, $K_0 = \left(\frac{22^*_\alpha - r}{22^*_\alpha}\right)\left(\frac{2-r}{2}\right)\left[\left(\frac{2N-\alpha}{N+4-\alpha}\right)\left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}}\right]^{\frac{r}{2-r}} S^{-\frac{r}{2}}$ and $l = \left[\left(\frac{2N-\alpha}{N+4-\alpha}\right)\left(\frac{22^*_\alpha - r}{22^*_\alpha}\right) S^{-\frac{r}{2}}\right]^{\frac{r}{2}}$. This completes the proof. □

Lemma 3.2. *Assume* $\{(u_n, v_n)\} \subset \mathcal{H}$ *is a* $(PS)_c$*-sequence for* $I_{\lambda,\mu}$*, then* $\{(u_n, v_n)\}$ *is bounded in* $\mathcal{H}$*.*

Proof. Let $\{(u_n, v_n)\}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ in $\mathcal{H}$, then as per the definition of $(PS)_c$-sequence, $I_{\lambda,\mu}(u_n, v_n) \to c$ and $I'_{\lambda,\mu}(u_n, v_n) \to 0$ in $\mathcal{H}^{-1}$ i.e.

$$\frac{1}{2}\|(u_n, v_n)\|^2 - \frac{1}{r}P_{\lambda,\mu}(u_n, v_n) - \frac{1}{2^*_\alpha}Q(u_n, v_n) = c + o_n(1), \tag{3.1}$$

$$\|(u_n, v_n)\|^2 - P_{\lambda,\mu}(u_n, v_n) - Q(u_n, v_n) = o_n(1). \tag{3.2}$$

Now, our aim is to show that $\{(u_n, v_n)\}$ is bounded. On contrary, assume that $\|(u_n, v_n)\| \to \infty$ as $n \to \infty$ and take $(\widehat{u}_n, \widehat{v}_n) := \frac{(u_n, v_n)}{\|(u_n, v_n)\|}$. It follows that $\{(\widehat{u}_n, \widehat{v}_n)\}$ is a bounded sequence. Consequently, up to a subsequence $(\widehat{u}_n, \widehat{v}_n) \rightharpoonup (\widehat{u}, \widehat{v})$ weakly in $\mathcal{H}$, $(\widehat{u}_n, \widehat{v}_n) \to (\widehat{u}, \widehat{v})$ strongly in $L^m(\Omega)$ for all $1 \leq m < 2^*$ and $(\widehat{u}_n(x), \widehat{v}_n(x)) \to (\widehat{u}(x), \widehat{v}(x))$ pointwise a.e. in $\Omega \times \Omega$.
Using (3.1) and (3.2), we have

$$\frac{1}{2}\|(\widehat{u}_n, \widehat{v}_n)\|^2 - \frac{1}{r}\|(u_n, v_n)\|^{r-2}P_{\lambda,\mu}(\widehat{u}_n, \widehat{v}_n) - \frac{1}{2^*_\alpha}\|(u_n, v_n)\|^{22^*_\alpha - 2}Q(\widehat{u}_n, \widehat{v}_n) = o_n(1), \tag{3.3}$$

and

$$\|(\widehat{u}_n,\widehat{v}_n)\|^2-\|(u_n,v_n)\|^{r-2}P_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n)-\|(u_n,v_n)\|^{22^*_\alpha-2}Q(\widehat{u}_n,\widehat{v}_n)=o_n(1). \tag{3.4}$$

From (3.3) and (3.4), we can deduce that

$$\|(\widehat{u}_n,\widehat{v}_n)\|^2=\frac{2(2^*_\alpha-r)}{r(2^*_\alpha-2)}\|(u_n,v_n)\|^{r-2}P_{\lambda,\mu}(\widehat{u}_n,\widehat{v}_n)+o_n(1). \tag{3.5}$$

Since $1\le r<2$ and $\|(u_n,v_n)\|\to\infty$, then (3.5) implies $\|(\widehat{u}_n,\widehat{v}_n)\|^2\to 0$ as $n\to\infty$, which is a contradiction to the fact that $\|(\widehat{u}_n,\widehat{v}_n)\|=1$. Thus, proof is completed. □

Lemma 3.3. *There exists*

$$c_\infty:=\frac{N+4-\alpha}{2(2N-\alpha)}\left(\frac{\|H^+\|_\infty^{-2}}{2}\right)^{\frac{N-4}{N+4-\alpha}}\bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}-K_0\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}}+(\mu\|G\|_\beta)^{\frac{2}{2-r}}\right),$$

such that the energy functional $I_{\lambda,\mu}$ satisfies the $(PS)_c$-condition with $c\in(-\infty,c_\infty)$ and K_0 is defined in Lemma 3.1.

Proof. Let $\{(u_n,v_n)\}\subset\mathcal{H}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ with $0<c<c_\infty$. Then by Lemma 3.2, $\{(u_n,v_n)\}$ is a bounded sequence in $\mathcal{H}$. Thus, up to a subsequence, $(u_n,v_n)\rightharpoonup(u,v)$ weakly in $\mathcal{H}$. So $u_n\rightharpoonup u$ and $v_n\rightharpoonup v$ weakly in $H^2_0(\Omega)$, $u_n\to u$ and $v_n\to v$ strongly in $L^m(\Omega)$ for all $1\le m<2^*$ and $u_n\to u$, $v_n\to v$ pointwise a.e. in Ω. Therefore

$$P_{\lambda,\mu}(u_n,v_n)=P_{\lambda,\mu}(u,v)+o_n(1). \tag{3.6}$$

Also, $I'_{\lambda,\mu}(u,v)=0$, follows from Lemma 3.1. Now, define $(\tilde{u}_n,\tilde{v}_n)$, where $\tilde{u}_n=u_n-u$, $\tilde{v}_n=v_n-v$. Then by the Brézis–Lieb lemma [6] and Lemma 2.8, we have

$$\begin{aligned}\|(\tilde{u}_n,\tilde{v}_n)\|^2&=\|(u_n,v_n)\|^2-\|(u,v)\|^2+o_n(1),\\ Q(u_n,v_n)&=Q(\tilde{u}_n,\tilde{v}_n)+Q(u,v)+o_n(1).\end{aligned} \tag{3.7}$$

Using $I_{\lambda,\mu}(u_n,v_n)=c+o_n(1)$, $I'_{\lambda,\mu}(u_n,v_n)=o_n(1)$ and (3.6)–(3.7), we obtain

$$\frac{1}{2}\|(\tilde{u}_n,\tilde{v}_n)\|^2-\frac{1}{2^*_\alpha}Q(\tilde{u}_n,\tilde{v}_n)=c-I_{\lambda,\mu}(u,v)+o_n(1), \tag{3.8}$$

and

$$\|(\tilde{u}_n,\tilde{v}_n)\|^2-2Q(\tilde{u}_n,\tilde{v}_n)=\langle I'_{\lambda,\mu}(u,v),(u_n-u,v_n-v)\rangle+o_n(1)=o_n(1).$$

Therefore, we may assume that

$$\|(\tilde{u}_n,\tilde{v}_n)\|^2\to d,\quad\text{and}\quad 2\int_\Omega\int_\Omega H(x)H(y)\frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{v}_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}\to d. \tag{3.9}$$

It follows from the definition of $\bar{S}_{H,L}$ that

$$\begin{aligned}\|(\tilde{u}_n,\tilde{v}_n)\|^2&\ge\bar{S}_{H,L}\left(\int_\Omega\int_\Omega\frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{v}_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)^{\frac{1}{2^*_\alpha}}\\ &\ge\bar{S}_{H,L}\|H^+\|_\infty^{-\frac{2}{2^*_\alpha}}\left(\int_\Omega\int_\Omega H(x)H(y)\frac{|\tilde{u}_n(x)|^{2^*_\alpha}|\tilde{v}_n(y)|^{2^*_\alpha}}{|x-y|^\alpha}\right)^{\frac{1}{2^*_\alpha}}.\end{aligned} \tag{3.10}$$

On combining (3.9) and (3.10), we have

$$d \geq \bar{S}_{H,L} \|H^+\|_\infty^{-\frac{2}{2^*_\alpha}} \left(\frac{d}{2}\right)^{\frac{1}{2^*_\alpha}},$$

which gives either

$$d = 0 \quad \text{or} \quad d \geq \left(\frac{\|H^+\|_\infty^{-\frac{2}{2^*_\alpha}}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}.$$

Further, if $d = 0$ then the proof is complete. If

$$d \geq \left(\frac{\|H^+\|_\infty^{-\frac{2}{2^*_\alpha}}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}},$$

then according to (3.8), (3.9) and Lemma 3.1, we get

$$\begin{aligned} c &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) d + I_{\lambda,\mu}(u,v) \\ &\geq \frac{N+4-\alpha}{2(2N-\alpha)} \left(\frac{\|H^+\|_\infty^{-2}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - K_0 \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right) =: c_\infty, \end{aligned}$$

a contradiction to $c < c_\infty$. Hence, $d = 0$ and with this we end the proof. □

4 Nehari manifold and fibering map analysis

In this section, we elaborate some important results for Nehari manifold and analysis of fibering map on $I_{\lambda,\mu}$. Notice that the energy functional $I_{\lambda,\mu}$ is unbounded below on $\mathcal{H}$. So we restrict $I_{\lambda,\mu}$ on an appropriate subset $\mathcal{N}_{\lambda,\mu}$ of $\mathcal{H}$, called Nehari manifold and defined as

$$\mathcal{N}_{\lambda,\mu} := \left\{(u,v) \in \mathcal{H} \setminus \{(0,0)\} : \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\right\}.$$

Thus, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = \|(u,v)\|^2 - P_{\lambda,\mu}(u,v) - 2Q(u,v) = 0. \tag{4.1}$$

Next, we see that $I_{\lambda,\mu}$ is bounded from below on $\mathcal{N}_{\lambda,\mu}$ in the following lemma.

Lemma 4.1. *The energy functional $I_{\lambda,\mu}$ is coercive and bounded below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Let $(u,v) \in \mathcal{N}_{\lambda,\mu}$ for $\lambda, \mu > 0$, then using (4.1) and (2.16), we have

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u,v)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u,v) \\ &\geq \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u,v)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) S^{-\frac{r}{2}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} \|(u,v)\|^r, \end{aligned} \tag{4.2}$$

Since $1 < r < 2$. Therefore, $I_{\lambda,\mu}$ is coercive.

Now, consider the function $\varrho : \mathbb{R} \to \mathbb{R}$ as $\varrho(t) = b_1 t^2 - b_2 t^r$. Then one can easily see that $\varrho'(t) = 0$ if and only if $t = \left(\frac{b_2 r}{2b_1}\right)^{\frac{1}{2-r}} =: t^*$ and $\varrho''(t^*) > 0$. So ϱ attains its minimum at t^*. Moreover,

$$\varrho(t) \geq \varrho(t^*) := -(2-r)\left(\frac{b_2}{2}\right)^{\frac{2}{2-r}}\left(\frac{r}{b_1}\right)^{\frac{r}{2-r}}.$$

Taking $b_1 = \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right)$, $b_2 = \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right)S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}$ and $t = \|(u,v)\|$ in the function ϱ, we obtain

$$I_{\lambda,\mu}(u,v) \geq \varrho(\|(u,v)\|) \geq \varrho(t^*).$$

which yields the required assertion. □

The Nehari manifold is intently related to the behaviour of map $\Psi_{u,v} : t \to I_{\lambda,\mu}(tu,tv)$ for $t > 0$, defined as

$$\Psi_{u,v}(t) := I_{\lambda,\mu}(tu,tv) = \frac{t^2}{2}\|(u,v)\|^2 - \frac{t^r}{r}P_{\lambda,\mu}(u,v) - \frac{t^{22^*_\alpha}}{2^*_\alpha}Q(u,v).$$

These maps are known as fibering maps which were introduced by Drábek and Pohozaev in [13]. Thus, $(tu,tv) \in \mathcal{N}_{\lambda,\mu}$ iff $\Psi'_{u,v}(t) = 0$. Furthermore

$$\Psi'_{u,v}(t) = t\|(u,v)\|^2 - t^{r-1}P_{\lambda,\mu}(u,v) - 2t^{2^*_\alpha - 1}Q(u,v),$$
$$\Psi''_{u,v}(t) = \|(u,v)\|^2 - (r-1)t^{r-2}P_{\lambda,\mu}(u,v) - 2(22^*_\alpha - 1)t^{22^*_\alpha - 2}Q(u,v).$$

In particular, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\Psi'_{u,v}(1) = 0$. Therefore it is obvious to split $\mathcal{N}_{\lambda,\mu}$ into three parts namely $\mathcal{N}^+_{\lambda,\mu}$, $\mathcal{N}^-_{\lambda,\mu}$ and $\mathcal{N}^0_{\lambda,\mu}$ corresponding to local minima, local maxima and point of inflexion respectively as:

$$\mathcal{N}^{\pm}_{\lambda,\mu} := \left\{(u,v) \in \mathcal{N}_{\lambda,\mu} : \Psi''_{u,v}(1) \gtrless 0\right\}, \qquad \mathcal{N}^0_{\lambda,\mu} := \left\{(u,v) \in \mathcal{N}_{\lambda,\mu} : \Psi''_{u,v}(1) = 0\right\}.$$

We note that, for $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\Psi''_{u,v}(1) = \begin{cases} (2 - 22^*_\alpha)\|(u,v)\|^2 - (r - 22^*_\alpha)P_{\lambda,\mu}(u,v) \\ (2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v). \end{cases} \tag{4.3}$$

In next lemma, we will show that the local minimizers of $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ are the critical points of $I_{\lambda,\mu}$.

Lemma 4.2. *If (u,v) is the local minimizer for $I_{\lambda,\mu}$ on subset of $\mathcal{N}_{\lambda,\mu}$, namely $\mathcal{N}^+_{\lambda,\mu}$ or $\mathcal{N}^-_{\lambda,\mu}$ such that $(u,v) \notin \mathcal{N}^0_{\lambda,\mu}$. Then $I'_{\lambda,\mu}(u,v) = 0$ in $\mathcal{H}^{-1}$, where $\mathcal{H}^{-1}$ denotes the dual space of $\mathcal{H}$.*

Proof. Suppose (u,v) is a local minimizer for $I_{\lambda,\mu}$ subject to the constrains $\Phi_{\lambda,\mu}(u,v)$: $\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = 0$. Then by Lagrange multipliers, there exists $\delta \in \mathbb{R}$ such that $I'_{\lambda,\mu}(u,v) = \delta\Phi'_{\lambda,\mu}(u,v)$. This implies that $\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = \delta\langle\Phi'_{\lambda,\mu}(u,v),(u,v)\rangle$. As $(u,v) \in \mathcal{N}_{\lambda,\mu}$, then $\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle =$ and $\langle\Phi'_{\lambda,\mu}(u,v),(u,v)\rangle \neq 0$ because of $(u,v) \notin \mathcal{N}^0_{\lambda,\mu}$. Therefore $\delta = 0$. This completes the proof. □

Lemma 4.3. *The following hold:*

(*i*) *If $(u,v) \in \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$, then $P_{\lambda,\mu}(u,v) > 0$.*

(*ii*) *If $(u,v) \in \mathcal{N}^-_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu}$, then $Q(u,v) > 0$.*

Proof. The proof follows directly from (4.3). □

Before analyzing the fibering map, we define a map $\mathcal{S}_{u,v} : \mathbb{R}^+ \to \mathbb{R}$ such that

$$\mathcal{S}_{u,v}(t) := t^{2-r}\|(u,v)\|^2 - 2t^{22^*_\alpha - r}Q(u,v). \tag{4.4}$$

It is noted that for $t > 0$, $(tu, tv) \in \mathcal{N}_{\lambda,\mu}$ if and only if $\mathcal{S}_{u,v}(t) = P_{\lambda,\mu}(u,v)$. We will check the behaviour of $\mathcal{S}_{u,v}$ near 0 and $+\infty$. Since $1 < r < 2$ and $2 < 22^*_\alpha$, this implies that $\lim_{t\to 0^+} \mathcal{S}_{(u,v)}(t) = 0$ and $\lim_{t\to+\infty} \mathcal{S}_{u,v}(t) = -\infty$. Moreover, for critical points

$$\mathcal{S}'_{u,v}(t) = (2-r)t^{1-r}\|(u,v)\|^2 - 2(22^*_\alpha - r)t^{22^*_\alpha - r - 1}Q(u,v).$$

One can easily see that $\mathcal{S}'_{u,v}(t) = 0$ if and only if $t = t_{\max}$, where

$$t_{\max} = \left(\frac{(2-r)\|(u,v)\|^2}{2(22^*_\alpha - r)Q(u,v)}\right)^{\frac{1}{22^*_\alpha - 2}}.$$

Also, $\mathcal{S}''_{u,v}(t) = (2-r)(1-r)t^{-r}\|(u,v)\|^2 - 2(22^*_\alpha - r)(22^*_\alpha - r - 1)t^{22^*_\alpha - r - 2}Q(u,v)$.

$$\begin{aligned}
\mathcal{S}''_{u,v}(t_{\max}) &= (2-r)(1-r)t_{\max}^{-r}\|(u,v)\|^2 - 2(22^*_\alpha - r)(22^*_\alpha - r - 1)t_{\max}^{22^*_\alpha - r - 2}Q(u,v) \\
&= (2-r)(1-r)\left(\frac{2(22^*_\alpha - r)Q(u,v)}{(2-r)\|(u,v)\|^2}\right)^{\frac{r}{22^*_\alpha - 2}} \|(u,v)\|^2 \\
&\quad - 2(22^*_\alpha - r)(22^*_\alpha - r - 1)\left(\frac{(2-r)\|(u,v)\|^2}{2(22^*_\alpha - r)Q(u,v)}\right)^{\frac{22^*_\alpha - r - 2}{22^*_\alpha - 2}} Q(u,v) \\
&= \frac{\|(u,v)\|^{\frac{2(22^*_\alpha - r - 2)}{22^*_\alpha - 2}}}{(Q(u,v))^{-\frac{r}{22^*_\alpha - 2}}}\left[(2-r)(1-r)\left(\frac{2(22^*_\alpha - r)}{2-r}\right)^{\frac{r}{22^*_\alpha - 2}}\right. \\
&\quad \left. - 2(22^*_\alpha - r)(22^*_\alpha - r - 1)\left(\frac{2-r}{2(22^*_\alpha - r)}\right)\left(\frac{2(22^*_\alpha - r)}{2-r}\right)^{\frac{r}{22^*_\alpha - 2}}\right] \\
&= \frac{(2 - 22^*_\alpha)\left(2(22^*_\alpha - r)\right)^{\frac{r}{22^*_\alpha - 2}}}{(2-r)^{\frac{r+2-22^*_\alpha}{22^*_\alpha - 2}}}\|(u,v)\|^{\frac{2(22^*_\alpha - r - 2)}{22^*_\alpha - 2}}(Q(u,v))^{\frac{r}{22^*_\alpha - 2}} \\
&< 0.
\end{aligned}$$

Thus, $\mathcal{S}_{u,v}(t)$ has maximum value at $t_{\max}$. Moreover, we have relation

$$\Psi'_{u,v}(t) = t^r\left(\mathcal{S}_{u,v}(t) - P_{\lambda,\mu}(u,v)\right). \tag{4.5}$$

Lemma 4.4. *Assume that* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$ *and* $(u,v) \in \mathcal{H}$, *the following results hold:*

(i) If $Q(u,v) < 0$ *and* $P_{\lambda,\mu}(u,v) < 0$, *then there does not exists any critical point.*

(ii) If $Q(u,v) \leq 0$ *and* $P_{\lambda,\mu}(u,v) < 0$, *then there exists a unique* (t^+u, t^+v) *such that* $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$ *and* $I_{\lambda,\mu}(t^+u, t^+v) = \inf_{t\geq 0} I_{\lambda,\mu}(tu, tv)$.

(iii) If $Q(u,v) > 0$ *and* $P_{\lambda,\mu}(u,v) \leq 0$, *then there exists a unique* $t^- > t_{\max}$ *such that* $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$ *and* $I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\geq t_{\max}} I_{\lambda,\mu}(tu, tv)$.

(iv) *If* $Q(u,v) > 0$ *and* $P_{\lambda,\mu}(u,v) > 0$*, then there exists unique* t^+ *and* t^- *satisfying* $0 < t^+ < t_{\max} < t^-$ *such that* $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$ *and* $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$*. Moreover*

$$I_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le t_{\max}} I_{\lambda,\mu}(tu, tv); \qquad I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge t_{\max}} I_{\lambda,\mu}(tu, tv).$$

Proof. Let $(0,0) \neq (u,v) \in \mathcal{H}$, then we have following four possible cases:

(i) If $Q(u,v) < 0$ and $P_{\lambda,\mu}(u,v) < 0$, then $\Psi_{u,v}(t) = 0$ at $t = 0$ and $\Psi'_{u,v}(t) > 0$ for all $t > 0$. This implies that Φ_u is strictly increasing and hence no critical point.

(ii) If $Q(u,v) < 0$, then from (4.4) $\mathcal{S}_{u,v}$ is strictly increasing for $t > 0$. As $P_{\lambda,\mu}(u,v) \geq 0$, this implies that there exists a unique t^+ such that $\mathcal{S}_{u,v}(t^+) = P_{\lambda,\mu}(u,v)$ with $\mathcal{S}_{u,v}(t^+) > 0$. Using (4.5), we conclude that $(t^+u, t^+v) \in \mathcal{N}_{\lambda,\mu}$. Further, $\Psi'_{u,v}(t) > 0$ and $\Psi'_{u,v}(t) < 0$ for $t > t^+$ and and $t < t^+$ respectively. Also $\Psi''_{u,v}(t^+) = (t^+)^{1+r}\mathcal{S}'_{u,v}(t^+) > 0$. Thus, $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$ and $I_{\lambda,\mu}(t^+u, t^+v) = \inf_{t\ge 0} I_{\lambda,\mu}(tu, tv)$.

(iii) If $Q(u,v) > 0$, then $t_{\max}$ is the point at which $\mathcal{S}'_{u,v}(t) > 0$ has maximum. Thus, $\mathcal{S}_{u,v}(t)$ is strictly increasing for $0 \le t < t_{\max}$ and strictly decreasing for $t_{\max} < t < \infty$. As $P_{\lambda,\mu}(u,v) \le 0$, so there is a unique $t^- > t_{\max} > 0$ such that $\mathcal{S}_{u,v}(t^-) = P_{\lambda,\mu}(u,v)$ and $\mathcal{S}_{u,v}(t^-) < 0$. Further, (4.5) gives $\Psi'_{u,v}(t^-) = 0$. Thus $(t^-u, t^-v) \in \mathcal{N}_{\lambda,\mu}$. Also, $\Psi''_{u,v}(t^-) = (t^-)^{1+r}\mathcal{S}'_{u,v}(t^-) < 0$ and $\Psi'_{u,v}(t) < 0$ for $t > t_{\max}$, so $\Psi_{u,v}(t^-) = \sup_{t\ge t_{\max}} \Psi_{u,v}(t)$. Hence, $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$ and $I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge t_{\max}} I_{\lambda,\mu}(tu, tv)$.

(iv) Since $Q(u,v) > 0$, $\mathcal{S}_{u,v}(t)$ achieves its maximum at $t = t_{\max}$. Thus

$$\begin{aligned}
\mathcal{S}_{u,v}(t_{\max}) &= \|(u,v)\|^r \left(\frac{2-r}{2(22^*_\alpha - r)}\right)^{\frac{2-r}{22^*_\alpha-2}} \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right) \left(\frac{\|(u,v)\|^{22^*_\alpha}}{Q(u,v)}\right)^{\frac{2-r}{22^*_\alpha-2}} \\
&\ge \left[\left(\frac{2-r}{2(22^*_\alpha - r)}\right) \|H^+\|_\infty^{-2} (\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{2-r}{22^*_\alpha-2}} \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right) \|(u,v)\|^r.
\end{aligned}$$

As $P_{\lambda,\mu}(u,v) > 0$, so

$$\begin{aligned}
\mathcal{S}_{u,v}(t_{\max}) - P_{\lambda,\mu}(u,v) \ge & \left[\left(\frac{2-r}{2(22^*_\alpha - r)}\right) \|H^+\|_\infty^{-2} (\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{2-r}{22^*_\alpha-2}} \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right) \|(u,v)\|^r \\
& - \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} S^{-\frac{r}{2}} \|(u,v)\|^r \\
> & \, 0,
\end{aligned}$$

for $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. Thus, there exists t^+ and t^- with $0 < t^+ < t_{\max} < t^-$ satisfying

$$\mathcal{S}_{u,v}(t^+) = P_{\lambda,\mu}(u,v) = \mathcal{S}_{u,v}(t^-) \quad \text{and} \quad \mathcal{S}'_{u,v}(t^+) < 0 < \mathcal{S}'_{u,v}(t^-).$$

Therefore, $(t^+u, t^+v) \in \mathcal{N}^+_{\lambda,\mu}$, $(t^-u, t^-v) \in \mathcal{N}^-_{\lambda,\mu}$. Furthermore, $\Psi'_{u,v}(t) < 0$ for $t \in (0, t^+)$, $\Psi'_u(t) > 0$ for $t \in (t^+, t^-)$ and $\Psi'_{u,v}(t) < 0$ for $t \in (t^-, \infty)$.

Hence

$$I_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le t_{\max}} I_{\lambda,\mu}(tu, tv); \; I_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge t_{\max}} I_{\lambda,\mu}(tu, tv).$$

holds true. This completes the proof. ∎

Lemma 4.5. *If* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then* $\mathcal{N}^0_{\lambda,\mu}$ *is a null set.*

Proof. We will prove it by contradiction. Let $(u,v) \in \mathcal{N}^0_{\lambda,\mu}$, then (4.3) implies that,

$$\|(u,v)\|^2 = \frac{22^*_\alpha - r}{22^*_\alpha - 2} P_{\lambda,\mu}(u,v) \tag{4.6}$$

and

$$\|(u,v)\|^2 = \frac{2(22^*_\alpha - r)}{2-r} Q(u,v). \tag{4.7}$$

On using (2.16) in (4.6), it is easy to calculate

$$\|(u,v)\| \leq \left(\frac{22^*_\alpha - r}{22^*_\alpha - 2} S^{-\frac{r}{2}}\right)^{\frac{1}{2-r}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{1}{2}}. \tag{4.8}$$

Now, taking (2.17) in (4.7), we find

$$\|(u,v)\|^2 \leq 2\left(\frac{22^*_\alpha - r}{2-r}\right) \|H^+\|^2_\infty (\bar{S}_{H,L})^{-2^*_\alpha} \|(u,v)\|^{22^*_\alpha},$$

or

$$\|(u,v)\| \geq \left[\left(\frac{2-r}{2(22^*_\alpha - r)}\right) \|H^+\|^{-2}_\infty (\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{1}{22^*_\alpha - 2}}. \tag{4.9}$$

Thus, from (4.8) and (4.9), we get

$$(\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} \geq \left(\frac{22^*_\alpha - 2}{22^*_\alpha - r}\right)^{\frac{2}{2-r}} \left[\left(\frac{2-r}{2(22^*_\alpha - r)}\right) \|H^+\|^{-2}_\infty (\bar{S}_{H,L})^{2^*_\alpha}\right]^{\frac{1}{2^*_\alpha - 1}} S^{\frac{r}{2-r}} =: \Upsilon_1,$$

which contradicts the fact that $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. Hence $\mathcal{N}^0_{\lambda,\mu} = \phi$ which completes the proof. □

Consequently, if $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, then we have

$$\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}.$$

Now, we define

$$k_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(u,v); \qquad k^+_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}} I_{\lambda,\mu}(u,v); \qquad k^-_{\lambda,\mu} = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}} I_{\lambda,\mu}(u,v).$$

Lemma 4.6. *The following facts hold:*

(i) If $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then* $k_{\lambda,\mu} \leq k^+_{\lambda,\mu} < 0$.

(ii) If $0 < \lambda < \frac{r}{2}\Upsilon_1$, *then* $k^-_{\lambda,\mu} > d_0$, *where* d_0 *is a positive constant depending on* $\lambda, \mu, \alpha, r, N$, S, $\|F\|_\alpha$, $\|G\|_\alpha$ *and* $\|H^+\|_\infty$.

Proof. (*i*) Let $(u,v) \in \mathcal{N}^+_{\lambda,\mu}$, then (4.3) gives

$$\frac{2-r}{2(22^*_\alpha - r)}\|(u,v)\|^2 > Q(u,v).$$

This together with (2.15) and (4.1) yield

$$I_{\lambda,\mu}(u,v) = \left(\frac{1}{2}-\frac{1}{r}\right)\|(u,v)\|^2 + 2\left(\frac{1}{r}-\frac{1}{22^*_\alpha}\right)Q(u,v) < -\frac{(2-r)(2^*_\alpha-1)}{r22^*_\alpha}\|(u,v)\|^2 < 0.$$

Thus, by the definition of $k_{\lambda,\mu}$ and $k^+_{\lambda,\mu}$, we conclude that $k_{\lambda,\mu} \le k^+_{\lambda,\mu} < 0$.

(*ii*) Let $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$. Then using (4.3) and (2.17), we have

$$\frac{2-r}{2(22^*_\alpha - r)}\|(u,v)\|^2 < Q(u,v) \le \|H^+\|^2_\infty \left(\bar{S}_{H,L}\right)^{-2^*_\alpha}\|(u,v)\|^{22^*_\alpha}.$$

This implies that

$$\|(u,v)\| > \left(\frac{2-r}{2(22^*_\alpha - r)}\|H^+\|^{-2}_\infty \left(\bar{S}_{H,L}\right)^{2^*_\alpha}\right)^{\frac{1}{22^*_\alpha-2}}. \tag{4.10}$$

On combining (4.2) and (4.10), we obtain

$$\begin{aligned}
&I_{\lambda,\mu}(u,v)\\
&\ge \|(u,v)\|^r\left[\left(\frac{1}{2}-\frac{1}{22^*_\alpha}\right)\|(u,v)\|^{2-r} - \left(\frac{1}{r}-\frac{1}{22^*_\alpha}\right)S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\right]\\
&> \left(\frac{2-r}{2(22^*_\alpha - r)}\|H^+\|^{-2}_\infty \left(\bar{S}_{H,L}\right)^{2^*_\alpha}\right)^{\frac{r}{22^*_\alpha-2}}\left[\left(\frac{1}{2}-\frac{1}{22^*_\alpha}\right)\left(\frac{2-r}{2(22^*_\alpha - r)}\|H^+\|^{-2}_\infty \left(\bar{S}_{H,L}\right)^{2^*_\alpha}\right)^{\frac{2-r}{22^*_\alpha-r}}\right.\\
&\quad\left. - \left(\frac{1}{r}-\frac{1}{22^*_\alpha}\right)S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\right]
\end{aligned}$$

Thus, if $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}}\Upsilon_1$, then $I_{\lambda,\mu}(u,v) > d_0$ for all $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$, where d_0 is a positive constant depending on $\lambda,\mu,\alpha,r,N,S,\|F\|_\alpha,\|G\|_\alpha$ and $\|H^+\|_\infty$. □

5 Existence of solution in $\mathcal{N}^+_{\lambda,\mu}$

In this section, we show the existence of Palais–Smale sequence corresponding to energy functional $I_{\lambda,\mu}$ in $\mathcal{N}^\pm_{\lambda,\mu}$, by using the implicit function theorem.

Lemma 5.1. *Suppose* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. *Then for every* $z=(u,v) \in \mathcal{N}_{\lambda,\mu}$, *there exist* $\epsilon > 0$ *and a differentiable mapping* $\zeta : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ *such that* $\zeta(0) = 1$, $\zeta(w)(z-w) \in \mathcal{N}_{\lambda,\mu}$ *and for all* $w = (w_1,w_2) \in \mathcal{H}$

$$\langle \zeta'(0), w\rangle = \frac{2\mathcal{B}(z,w) - r\mathcal{P}_{\lambda,\mu}(z,w) - 2\mathcal{Q}(z,w)}{(2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v)},$$

where

$$\begin{aligned}
\mathcal{B}(z,w) &= \int_\Omega (\Delta u\Delta w_1 + \Delta v\Delta w_2)\,dx,\\
\mathcal{P}_{\lambda,\mu}(z,w) &= \int_\Omega \left(\lambda F(x)|u|^{r-2}uw_1 + \mu G(x)|v|^{r-2}vw_2\right)dx,\\
\mathcal{Q}(z,w) &= \int_\Omega\int_\Omega H(x)H(y)\left(\frac{|v(x)|^{2^*_\alpha}|u(y)|^{2^*_\alpha-2}u(y)z_1 + |u(x)|^{2^*_\alpha}|v(y)|^{2^*_\alpha-2}v(y)z_2}{|x-y|^\alpha}\right)dxdy
\end{aligned}$$

Proof. For $z = (u,v) \in \mathcal{N}_{\lambda,\mu}$, define a map $\xi_z : \mathbb{R} \times \mathcal{H} \to \mathbb{R}$ such that

$$\xi_z(\zeta, w) = \langle I'_{\lambda,\mu}(\zeta(z-w)), \zeta(z-w)\rangle = \zeta^2 \|(u-w_1, v-w_2)\|^2$$
$$- \zeta^r \int_\Omega (\lambda F(x)|u-w_1|^r + \mu G(x)|v-w_2|^r)dx - 2\zeta^{22^*_\alpha} Q(u-w_1, v-w_2)dx$$

Then $\xi_z(1,(0,0)) = \langle I'_{\lambda,\mu}(z), z\rangle = 0$ and

$$\frac{d}{d\zeta}\xi_z(1,(0,0)) = 2\|(u,v)\|^2 - r\int_\Omega (\lambda F(x)|u|^r + \mu G(x)|v|^r)dx - 2(22^*_\alpha)Q(u,v)$$
$$= (2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v) \neq 0.$$

Thus, by the implicit function theorem, there exist $\epsilon > 0$ and a differentiable mapping $\zeta : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ such that $\zeta(0) = 1$,

$$\langle \zeta'(0), w\rangle = \frac{2\mathcal{B}(z,w) - r\mathcal{P}_{\lambda,\mu}(z,w) - 2\mathcal{Q}(z,w)}{(2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v)},$$

$\xi_z(\zeta(w), w) = 0 \; \forall\, w \in B(0,\epsilon)$. Thus,

$$\langle I'_{\lambda,\mu}(\zeta(w)(z-w)), \zeta(w)(z-w)\rangle = 0 \qquad \forall\, w \in B(0,\epsilon).$$

Therefore, $\zeta(w)(z-w) \in \mathcal{N}_{\lambda,\mu}$. □

The similar result is also true for $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$, which is as follows

Lemma 5.2. *Suppose* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. *Then for every* $z = (u,v) \in \mathcal{N}^-_{\lambda,\mu}$, *there exist* $\epsilon > 0$ *and a differentiable mapping* $\zeta^- : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ *such that* $\zeta^-(0) = 1$, $\zeta^-(w)(z-w) \in \mathcal{N}^-_{\lambda,\mu}$ *and for all* $w = (w_1, w_2) \in \mathcal{H}$

$$\langle (\zeta^-)'(0), w\rangle = \frac{2\mathcal{B}(z,w) - r\mathcal{P}_{\lambda,\mu}(z,w) - 2\mathcal{Q}(z,w)}{(2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v)},$$

where $\mathcal{B}(z,w)$, $\mathcal{P}_{\lambda,\mu}(z,w)$ *and* $\mathcal{Q}(z,w)$ *is same as in Lemma 5.1.*

Proof. By the same argument used in Lemma 5.1, there exists $\epsilon > 0$ and a differentiable function $\zeta^- : B(0,\epsilon) \subset \mathcal{H} \to \mathbb{R}^+$ such that $\zeta^-(0) = 1$ and $\zeta^-(w)(z-w) \in \mathcal{N}^-_{\lambda,\mu}$. Since

$$\Psi''_{(u,v)}(1) = (2-r)\|(u,v)\|^2 - 2(22^*_\alpha - r)Q(u,v) < 0.$$

By the continuity of Ψ'' and ζ^-, we have

$$\Psi''_{\zeta^-(w)(z-w)}(1) = (2-r)\|\zeta^-(w)(z-w)\|^2 - 2(22^*_\alpha - r)Q(\zeta^-(w)(z-w), \zeta^-(w)(z-w) < 0,$$

for $\epsilon > 0$ is sufficiently small. Thus, $\zeta^-(w)(z-w) \in \mathcal{N}^-_{\lambda,\mu}$. □

Lemma 5.3. *The following statements are true:*

- *(i) If* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then there exists a* $(PS)_{k_{\lambda,\mu}}$*-sequence* $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ *in* $\mathcal{H}$ *for* $I_{\lambda,\mu}$.
- *(ii) If* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Upsilon_1$, *then there exists a* $(PS)_{k^-_{\lambda,\mu}}$*-sequence* $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$ *in* $\mathcal{H}$ *for* $I_{\lambda,\mu}$.

Proof. (i) According to Lemma 4.1 and Ekeland Variational Principle [14], there exists a minimizing sequence $\{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n, v_n) < k_{\lambda,\mu} + \frac{1}{n},$$
$$I_{\lambda,\mu}(u_n, v_n) < I_{\lambda,\mu}(u, v) + \frac{1}{n}\|(u,v) - (u_n, v_n)\|, \text{ for each } (u,v) \in \mathcal{N}_{\lambda,\mu}. \tag{5.1}$$

Using Lemma 4.6(i) and taking n large, we get

$$I_{\lambda,\mu}(u_n, v_n) = \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right)\|(u_n,v_n)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right)P_{\lambda,\mu}(u_n,v_n)$$
$$< k_{\lambda,\mu} + \frac{1}{n} < \frac{k_{\lambda,\mu}}{2}. \tag{5.2}$$

This implies that

$$0 < -\frac{r2^*_\alpha k_{\lambda,\mu}}{22^*_\alpha - r} < P_{\lambda,\mu}(u_n,v_n) \le S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\|(u_n,v_n)\|^r. \tag{5.3}$$

Wherefore, $(u_n, v_n) \ne (0,0)$. From (5.2), we have

$$\|(u_n,v_n)\| \le \left[\left(\frac{22^*_\alpha - r}{r(2^*_\alpha - 1)}\right)S^{-\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}}\right]^{\frac{1}{2-r}}. \tag{5.4}$$

Further, (5.3) gives us

$$\|(u_n,v_n)\| \ge \left[-\frac{r2^*_\alpha k_{\lambda,\mu}}{22^*_\alpha - r}S^{\frac{r}{2}}\left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{r-2}{2}}\right]^{\frac{1}{r}}.$$

Now, we will prove that

$$\|I'_{\lambda,\mu}(u_n,v_n)\|_{\mathcal{H}^{-1}} \to 0, \quad \text{as } n \to \infty.$$

Using Lemma 5.1 for each $z_n = (u_n, v_n)$ to obtain the mapping $\zeta_n : B(0,\epsilon_n) \to \mathbb{R}^+$ for some $\epsilon_n > 0$ such that $\zeta_n(w)(z_n - w) \in \mathcal{N}_{\lambda,\mu}$. Choose $0 < \eta < \epsilon_n$. Let $z = (u,v) \in \mathcal{H}$ with $z \ne 0$ and take $w^*_\eta = \frac{\eta z}{\|z\|}$. We set $w_\eta = \zeta_n(w^*_\eta)(z_n - w^*_\eta)$. Since $w_\eta \in \mathcal{N}_{\lambda,\mu}$, from (5.1), we get

$$I_{\lambda,\mu}(w_\eta) - I_{\lambda,\mu}(z_n) \ge -\frac{1}{n}\|w_\eta - z_n\|.$$

Using mean value theorem, we obtain

$$\langle I'_{\lambda,\mu}(z_n), w_\eta - z_n\rangle + o(\|w_\eta - z_n\|) \ge -\frac{1}{n}\|w_\eta - z_n\|.$$

Therefore,

$$\langle I'_{\lambda,\mu}(z_n), -w^*_\eta\rangle + (\zeta_n(w^*_\eta) - 1)\langle I'_{\lambda,\mu}(z_n), z_n - w^*_\eta\rangle \ge -\frac{1}{n}\|w_\eta - z_n\| + o(\|w_\eta - z_n\|). \tag{5.5}$$

Since $\zeta_n(w^*_\eta)(z_n - w^*_\eta) \in \mathcal{N}_{\lambda,\mu}$ and from (5.5), we get

$$-\eta\left\langle I'_{\lambda,\mu}(z_n), \frac{z}{\|z\|}\right\rangle + (\zeta_n(w^*_\eta) - 1)\langle I'_{\lambda,\mu}(z_n - w_\eta), z_n - w^*_\eta\rangle \ge -\frac{1}{n}\|w_\eta - z_n\| + o(\|w_\eta - z_n\|).$$

Thus, we have

$$\left\langle I'_{\lambda,\mu}(z_n), \frac{z}{\|z\|} \right\rangle \leq \frac{1}{n\eta}\|w_\eta - z_n\| + \frac{1}{\eta} o(\|w_\eta - z_n\|) + \frac{(\zeta_n(w_\eta^*) - 1)}{\eta} \langle I'_{\lambda,\mu}(z_n - w_\eta), z_n - w_\eta^* \rangle. \tag{5.6}$$

As $\|w_\eta - z_n\| \leq \eta|\zeta_n(w_\eta^*)| + |\zeta_n(w_\eta^*) - 1|\|z_n\|$ and $\lim_{\eta\to 0} \frac{|\zeta_n(w_\eta^*)-1|}{\eta} \leq \|\zeta_n'(0)\|$, if we take $\eta \to 0$ in (5.6) for a fixed $n \in \mathbb{N}$ and using (5.4) we can find a constant $A_1 > 0$, free from η such that

$$\left\langle I'_{\lambda,\mu}(z_n), \frac{z}{\|z\|} \right\rangle \leq \frac{A_1}{n}(1 + \|\zeta_n'(0)\|).$$

Further, we will show that $\|\zeta_n'(0)\|$ is uniformly bounded. By Hölder's inequality and Sobolev's embedding theorem, we have

$$\begin{aligned}
&\int_\Omega \lambda F(x)|u_n|^{r-1}w_1 + \mu G(x)|v_n|^{r-1}w_2 \\
&\leq \lambda\|F\|_\alpha \left(\int_\Omega \left(|u_n|^{r-1}w_1 \right)^{\frac{2^*}{r}} \right)^{\frac{r}{2^*}} + \mu\|G\|_\alpha \left(\int_\Omega \left(|v_n|^{r-1}w_2 \right)^{\frac{2^*}{r}} \right)^{\frac{r}{2^*}} \\
&\leq \lambda\|F\|_\alpha\|u_n\|_{2^*}^{r-1}\|w_1\|_{2^*} + \mu\|G\|_\alpha\|v_n\|_{2^*}^{r-1}\|w_2\|_{2^*} \\
&\leq S^{-\frac{r}{2}} \left(\lambda\|F\|_\alpha + \mu\|G\|_\alpha \right) \|(u_n, v_n)\|^{r-1}\|(w_1, w_2)\|
\end{aligned} \tag{5.7}$$

Further, using the Hardy–Littlewood–Sobolev inequality, Hölder's inequality and Sobolev's embedding theorem, we obtain

$$\begin{aligned}
&\int_\Omega \int_\Omega \left(\frac{|u_n|^{2^*_\alpha}}{|x-y|^\alpha} \right) |v_n|^{2^*_\alpha - 1} w_1 dxdy \\
&\leq C(N, \alpha) \left(\int_\Omega |u_n|^{\frac{2 2^*_\alpha N}{2N-\alpha}} \right)^{\frac{2N-\alpha}{2N}} \left(\int_\Omega \left(|v_n|^{2^*_\alpha - 1} w_1 \right)^{\frac{2N}{2N-\alpha}} \right)^{\frac{2N-\alpha}{2N}} \\
&= C(N, \alpha) \left(\int_\Omega |u_n|^{2^*} \right)^{\frac{2N-\alpha}{2N}} \left(\int_\Omega \left(|v_n|^{2^*_\alpha - 1} w_1 \right)^{\frac{2N}{2N-\alpha}} \right)^{\frac{2N-\alpha}{2N}} \\
&\leq C(N, \alpha) \left(\int_\Omega |u_n|^{2^*} \right)^{\frac{2N-\alpha}{2N}} \left(\int_\Omega |v_n|^{2^*} \right)^{\frac{N+4-\alpha}{2N-\alpha}} \left(\int_\Omega |w_1|^{2^*} \right)^{\frac{1}{2^*}} \\
&\leq \left[\left(S^{-1} \int_\Omega |\Delta u_n|^2 \right)^{\frac{2^*}{2}} \right]^{\frac{2N-\alpha}{2N}} \left[\left(S^{-1} \int_\Omega |\Delta v_n|^2 \right)^{\frac{2^*}{2}} \right]^{\frac{N+4-\alpha}{2N}} \left[\left(S^{-1} \int_\Omega |\Delta w_1|^2 \right)^{\frac{2^*}{2}} \right]^{\frac{1}{2^*}} \\
&\leq A_2\|u_n\|^{2^*_\alpha}\|v_n\|^{\frac{N+4-\alpha}{N-4}}\|w_1\| \\
&\leq A_2\|(u_n, v_n)\|^{\frac{3N+4-2\alpha}{N-4}}\|(w_1, w_2)\|.
\end{aligned} \tag{5.8}$$

Using the same idea, we can calculate

$$\int_\Omega \int_\Omega \left(\frac{|v_n|^{2^*_\alpha}}{|x-y|^\alpha} \right) |u_n|^{2^*_\alpha - 1} w_2 dxdy \leq A_3\|(u_n, v_n)\|^{\frac{3N+4-2\alpha}{N-4}}\|(w_1, w_2)\|. \tag{5.9}$$

Thus, on combining (5.7)–(5.9) and (5.4), we have

$$\left|(\zeta_n'(0), w)\right| \leq \frac{A_4\|(w_1,w_2)\|}{|(2-r)\|(u_n,v_n)\|^2 - 2(22_\alpha^* - r)Q(u_n,v_n)|},$$

where $A_4 > 0$ is a constant.

Now we are left to show that

$$\left|(2-r)\|(u_n,v_n)\|^2 - 2(22_\alpha^* - r)Q(u_n,v_n)\right| \geq A_5,$$

for some $A_5 > 0$ and n is taking large enough. On contradiction argue, suppose there exists a subsequence $\{(u_n,v_n)\}$ such that

$$\left|(2-r)\|(u_n,v_n)\|^2 - 2(22_\alpha^* - r)Q(u_n,v_n)\right| = o_n(1). \tag{5.10}$$

From (5.10) and using $(u_n,v_n) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\|(u_n,v_n)\|^2 = \frac{22_\alpha^* - r}{22_\alpha^* - 2} P_{\lambda,\mu}(u_n,v_n) + o_n(1),$$
$$\|(u_n,v_n)\|^2 = \frac{2(22_\alpha^* - r)}{2-r} Q(u_n,v_n) + o_n(1).$$

By Hölder's inequality, Sobolev embedding theorem and the definition of $\bar{S}_{H,L}$, we obtain

$$\|(u_n,v_n)\| \leq \left(\frac{22_\alpha^* - r}{22_\alpha^* - 2} S^{-\frac{r}{2}}\right)^{\frac{1}{2-r}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{1}{2}} + o_n(1),$$
$$\|(u_n,v_n)\| \geq \left[\left(\frac{2-r}{2(22_\alpha^* - r)}\right) \|H^+\|_\infty^{-2} (2S_{H,L})^{2_\alpha^*}\right]^{\frac{1}{22_\alpha^*-2}} + o_n(1).$$

This implies that $(\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} \geq \Upsilon_1$, which is a contradiction to the fact that $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$. Hence,

$$\left\langle I'_{\lambda,\mu}(u_n,v_n), \frac{(u,v)}{\|(u,v)\|}\right\rangle \leq \frac{A_1}{n}.$$

Thus, proof of (i) is completed.

(ii) Using Lemma 5.2, one can prove (ii) in a similar manner. □

Lemma 5.4. *Let* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon_1$, *then* $I_{\lambda,\mu}$ *has a minimizer* $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ *in* $\mathcal{N}^+_{\lambda,\mu}$ *which satisfies the following:*

- (i) $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = k_{\lambda,\mu} = k^+_{\lambda,\mu} < 0.$
- (ii) $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ *is a nontrivial solution of the system* $(\mathcal{D}_{\lambda,\mu})$.
- (iii) $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \to (0,0)$ *as* $\lambda \to 0^+$, $\mu \to 0^+$.

Proof. By Lemma 5.3 (i), there exists a minimizing sequence $\{(u_n,v_n)\}$ for $I_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(u_n,v_n) = k_{\lambda,\mu} + o_n(1),\ I'_{\lambda,\mu}(u_n,v_n) = o_n(1) \text{ in } \mathcal{H}^{-1}.$$

Lemma 5.1 gives us that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. So up to subsequence $(u_n, v_n) \rightharpoonup (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ weakly in $(u_n, v_n) \to \mathcal{H}$, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $L^m(\Omega)$ $\forall 1 \leq m < 2^*$ and $(u_n(x), v_n(x)) \to (u^1_{\lambda,\mu}(x), v^1_{\lambda,\mu}(x))$ pointwise a.e. in Ω. Then, it is easy to see that

$$\begin{aligned} |u_n|^{2^*_\alpha} \rightharpoonup |u^1_{\lambda,\mu}|^{2^*_\alpha}, \quad |v_n|^{2^*_\alpha} \rightharpoonup |v^1_{\lambda,\mu}|^{2^*_\alpha} \quad \text{in } L^{\frac{2N}{2N-\alpha}}(\Omega) \quad \text{and} \\ |u_n|^{2^*_\alpha-2}u_n \rightharpoonup |u^1_{\lambda,\mu}|^{2^*_\alpha-2}u^1_{\lambda,\mu}, \quad |v_n|^{2^*_\alpha-2}v_n \rightharpoonup |v^1_{\lambda,\mu}|^{2^*_\alpha-2}v^1_{\lambda,\mu} \quad \text{in } L^{\frac{2N}{N+4-\alpha}}(\Omega), \end{aligned} \tag{5.11}$$

as $n \to \infty$. As we know that the Riesz potential defines a continuous linear map from $L^{\frac{2N}{2N-\alpha}}(\Omega)$ to $L^{\frac{2N}{\alpha}}(\Omega)$ which provides

$$|x|^{-\alpha} * |u_n|^{2^*_\alpha} \rightharpoonup |x|^{-\alpha} * |u^1_{\lambda,\mu}|^{2^*_\alpha} \text{ and } |x|^{-\alpha} * |v_n|^{2^*_\alpha} \rightharpoonup |x|^{-\alpha} * |v^1_{\lambda,\mu}|^{2^*_\alpha} \text{ weakly in } L^{\frac{2N}{\alpha}}(\Omega), \tag{5.12}$$

as $n \to \infty$. Thus, (5.11) and (5.12) gives us

$$\left.\begin{aligned} \left(|x|^{-\alpha} * |v_n|^{2^*_\alpha}\right)|u_n|^{2^*_\alpha-2}u_n &\rightharpoonup \left(|x|^{-\alpha} * |v^1_{\lambda,\mu}|^{2^*_\alpha}\right)|u^1_{\lambda,\mu}|^{2^*_\alpha-2}u^1_{\lambda,\mu}, \\ \left(|x|^{-\alpha} * |u_n|^{2^*_\alpha}\right)|v_n|^{2^*_\alpha-2}v_n &\rightharpoonup \left(|x|^{-\alpha} * |u^1_{\lambda,\mu}|^{2^*_\alpha}\right)|v^1_{\lambda,\mu}|^{2^*_\alpha-2}v^1_{\lambda,\mu}, \end{aligned}\right\} \text{ weakly in } L^{\frac{2N}{N+4}}(\Omega), \tag{5.13}$$

as $n \to \infty$. Therefore, for any $(\phi, \psi) \in \mathcal{H}$, we have

$$\begin{aligned} \lim_{n\to\infty} \Bigg[\int_\Omega (\Delta u_n \Delta\phi + \Delta v_n \Delta\psi)\, dx - \int_\Omega (\lambda F(x)|u_n|^{r-2}u\phi + \mu G(x)|v_n|^{r-2}v\psi)\, dx \\ - \int_\Omega\int_\Omega H(x)H(y) \left(\tfrac{|v_n(x)|^{2^*_\alpha}|u_n(y)|^{2^*_\alpha-2}u_n(y)\phi(y) + |u_n(x)|^{2^*_\alpha}|v_n(y)|^{2^*_\alpha-2}v_n(y)\psi(y)}{|x-y|^\alpha}\right) \Bigg] = 0, \end{aligned}$$

because of $\|I'_{\lambda,\mu}(u_n, v_n)\| \to 0$ as $n \to \infty$. Thus, using (5.13), continuity of H and passing the limit as $n \to \infty$, we have

$$\begin{aligned} \int_\Omega \left(\Delta u^1_{\lambda,\mu}\Delta\phi + \Delta v^1_{\lambda,\mu}\Delta\psi\right) dx - \int_\Omega \left(\lambda F(x)|u^1_{\lambda,\mu}|^{r-2}u^1_{\lambda,\mu}\phi + \mu G(x)|v^1_{\lambda,\mu}|^{r-2}v^1_{\lambda,\mu}\psi\right) dx \\ - \int_\Omega\int_\Omega H(x)H(y) \left(\tfrac{|v^1_{\lambda,\mu}(x)|^{2^*_\alpha}|u^1_{\lambda,\mu}(y)|^{2^*_\alpha-2}u^1_{\lambda,\mu}(y)\phi(y) + |u^1_{\lambda,\mu}(x)|^{2^*_\alpha}|v^1_{\lambda,\mu}(y)|^{2^*_\alpha-2}v^1_{\lambda,\mu}(y)\psi(y)}{|x-y|^\alpha}\right) = 0, \end{aligned}$$

i.e. $\langle I'_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}), (\phi, \psi)\rangle \to 0$. This implies that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a weak solution of $(\mathcal{D}_{\lambda,\mu})$. Since $(u_n, v_n) \in \mathcal{N}_{\lambda,\mu}$. So, we have

$$\|(u_n, v_n)\|^2 = P_{\lambda,\mu}(u_n, v_n) + 2Q(u_n, v_n),$$

which gives

$$\begin{aligned} I_{\lambda,\mu}(u_n, v_n) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right)\|(u_n, v_n)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u_n, v_n) \\ &\geq -\left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u_n, v_n). \end{aligned}$$

Taking $n \to \infty$ together with λ, $\mu < 0$, we obtain

$$P_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \geq -\frac{22^*_\alpha k_{\lambda,\mu}}{(22^*_\alpha - r)} > 0. \tag{5.14}$$

Therefore, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is a nontrivial solution of $(\mathcal{D}_{\lambda,\mu})$. Afterwards, we will show that $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathcal{H}$ and $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = k_{\lambda,\mu}$. Using Fatou's lemma, we obtain

$$\begin{aligned} k_{\lambda,\mu} \le I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) &= \left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \\ &\le \liminf_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{22^*_\alpha}\right) \|(u_n, v_n)\|^2 - \left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) P_{\lambda,\mu}(u_n, v_n)\right] \\ &= \liminf_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) = k_{\lambda,\mu}. \end{aligned}$$

This implies that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = k_{\lambda,\mu}$ and $\lim_{n\to\infty} \|(u_n, v_n)\|^2 = \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^2$. Further, the Brézis–Lieb lemma [6] contributes that $(u_n, v_n) \to (u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ strongly in $\mathcal{H}$.

Now, we are left to show that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$. We prove this by contradiction argument. Suppose $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$ Then, from Lemma 4.3 (ii) and (5.14), we have

$$Q(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > 0 \quad \text{and} \quad P_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) > 0.$$

Thus, from Lemma 4.4, there exist unique t_1^+ and t_1^- such that $(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and $(t_1^- u^1_{\lambda,\mu}, t_1^- v^1_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. In particular, we have $t_1^+ < t_1^- = 1$. Since $\Psi'_{(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})}(t_1^+) = 0$ and $\Psi''_{(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})}(t_1^+) > 0$, there exists $t_1^+ < \bar{t} \le t_1^-$ such that $I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar{t} u^1_{\lambda,\mu}, \bar{t} v^1_{\lambda,\mu})$. On using Lemma 4.4, we obtain

$$I_{\lambda,\mu}(t_1^+ u^1_{\lambda,\mu}, t_1^+ v^1_{\lambda,\mu}) < I_{\lambda,\mu}(\bar{t} u^1_{\lambda,\mu}, \bar{t} v^1_{\lambda,\mu}) \le I_{\lambda,\mu}(t_1^- u^1_{\lambda,\mu}, t_1^- v^1_{\lambda,\mu}) = I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) = k_{\lambda,\mu},$$

which is a contradiction. Therefore, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$.

(iii) Further, from Lemma 4.6 (i) and (4.2), we have

$$\begin{aligned} 0 > k^+_{\lambda,\mu} &\ge k_{\lambda,\mu} = I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \\ &> -\left(\frac{1}{r} - \frac{1}{22^*_\alpha}\right) S^{-\frac{r}{2}} \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right)^{\frac{2-r}{2}} \|(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})\|^r, \end{aligned}$$

which implies that $I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \to (0,0)$ as $\lambda \to 0^+$, $\mu \to 0^+$ which completes the proof. □

Proof of Theorem 1.1. From Lemma 5.4, we conclude that $(\mathcal{D}_{\lambda,\mu})$ has a nontrivial solution $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$. □

6 Existence of solution in $\mathcal{N}^-_{\lambda,\mu}$

In this segment, we first prove the critical level by using few estimates which are already proved in Section 1. Then we show the existence of a second weak solution of problem $(\mathcal{D}_{\lambda,\mu})$ under the assumptions (Z1)–(Z4). At the end of this section, we give the proof of Theorem 1.2.

Lemma 6.1. *Assume that* (Z1)–(Z4) *hold and* $\frac{N}{N-4} \le r < 2$, *then there exist* $(u_{\lambda,\mu}, v_{\lambda,\mu})$ *in* $\mathcal{H} \setminus \{(0,0)\}$ *and* $\Upsilon > 0$ *such that for* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon$,

$$\begin{aligned} &\sup_{t\ge 0} I_{\lambda,\mu}(t u_{\lambda,\mu}, t v_{\lambda,\mu}) \\ &\quad < \frac{N+4-\alpha}{2(2N-\alpha)} \left(\frac{\|H^+\|_\infty^{-2}}{2}\right)^{\frac{N-4}{N+4-\alpha}} \bar{S}_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}} - K_0 \left((\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right) =: c_\infty. \end{aligned}$$

Furthermore, $k^-_{\lambda,\mu} < c_\infty$ *for all* $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon$.

Proof. For this, we first define the functional $\mathcal{E} : \mathcal{H} \to \mathbb{R}$ such that

$$\mathcal{E}(u,v) = \frac{1}{2}\|(u,v)\|^2 - \frac{1}{2^*_\alpha}Q(u,v), \qquad \forall\ (u,v) \in \mathcal{H}.$$

Take $U_0 = V_0 = \overline{U}_\epsilon$ with $(U_0, V_0) \in \mathcal{H}$. We define $\phi(t) = \mathcal{E}(tU_0, tV_0)$. Then $\phi(t)$ satisfies $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$ small and $\phi(t) < 0$ for $t > 0$ large. Further, one can easily verify that $\phi(t)$ attains its maximum at

$$t = \left(\frac{\|(U_0,V_0)\|^2}{2Q(U_0,V_0)}\right)^{\frac{1}{22^*_\alpha-2}} =: t^*.$$

Thus from (2.9), we have

$$\begin{aligned}
\sup_{t\geq 0}\mathcal{E}(tU_0,tV_0) &= \frac{(t^*)^2}{2}\|(U_0,V_0)\|^2 - \frac{(t^*)^{22^*_\alpha}}{2^*_\alpha}Q(U_0,V_0)\\
&= \frac{(N+4-\alpha)}{2N-\alpha}\left[\frac{\|\overline{U}_\epsilon\|^2}{(Q(\overline{U}_\epsilon,\overline{U}_\epsilon))^{\frac{1}{2^*_\alpha}}}\right]^{\frac{2N-\alpha}{N+4-\alpha}}\\
&\leq \frac{(N+4-\alpha)}{2N-\alpha}\left[\frac{(C(N,\alpha))^{\frac{N(N-4)}{4(2N-\alpha}}\, S_{H,L}^{\frac{N}{4}} + o(\epsilon^{N-4})}{\|H^+\|_\infty^{\frac{2(N-4)}{2N-\alpha}}\,(C(N,\alpha))^{\frac{N(N-4)}{4(2N-\alpha}}\, S_{H,L}^{\frac{N-4}{4}} - o(\epsilon^{2N-\alpha}) - o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)}\right]^{\frac{2N-\alpha}{N+4-\alpha}}\\
&\leq \frac{(N+4-\alpha)}{2N-\alpha}\left[\frac{\|H^+\|_\infty^{-\frac{2(N-4)}{2N-\alpha}} S_{H,L} + (\epsilon^{N-4})}{1-o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)}\right]^{\frac{2N-\alpha}{N+4-\alpha}}\\
&\leq \frac{(N+4-\alpha)}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} S_{H,L}^{\frac{2N-\alpha}{N+4-\alpha}}\left[1+o(\epsilon^{N-4}) + o\left(\epsilon^{\frac{2N-\alpha}{2}}\right)\right]^{\frac{2N-\alpha}{N+4-\alpha}}\\
&\leq \frac{(N+4-\alpha)}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{\overline{S}_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} + \begin{cases} o(\epsilon^{N-4}), & \alpha \leq 8\\ o(\epsilon^{\frac{2N-\alpha}{2}}), & \alpha > 8.\end{cases}
\end{aligned} \tag{6.1}$$

Further, $\delta_1 > 0$ is chosen in such a way that $c_\infty > 0$ for all $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \delta_1$. Then, the definition of $I_{\lambda,\mu}$ and $\lambda,\mu > 0$ yield that $I_{\lambda,\mu}(tU_0,tV_0) \leq \frac{t^2}{2}\|(U_0,V_0)\|^2$ for $t \geq 0$. This implies that, there exists $t_0 \in (0,1)$ such that

$$\sup_{t\in[0,t_0]} I_{\lambda,\mu}(tU_0,tV_0) < c_\infty\ \forall\ 0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \delta_1.$$

Moreover,

$$\begin{aligned}
\sup_{t\geq t_0} P_{\lambda,\mu}(tU_0,tV_0) &= \sup_{t\geq t_0}\left(\int_\Omega \lambda F(x)|tU_0|^r + \mu G(x)|tV_0|^r\right)\\
&= \sup_{t\geq t_0}\left(t^r\int_\Omega (\lambda F(x) + \mu G(x))\,|\overline{U}_\epsilon|^r dx\right)\\
&\geq (t_0)^r\,(\lambda a_0 + \mu b_0)\int_{B(0,2r_0)} |\overline{U}_\epsilon|^r dx\\
&\geq \frac{\omega}{r}\,(\lambda+\mu)\begin{cases} o(\epsilon^{N-\frac{N-4}{2}r}|\ln\epsilon|), & r = \frac{N}{N-4}\\ o(\epsilon^{N-\frac{N-4}{2}r}), & r > \frac{N}{N-4},\end{cases}
\end{aligned} \tag{6.2}$$

where $\omega = \min\{a_0, b_0\}$.

Thus, on using (2.8), (6.1) and (6.2), we have

$$\begin{aligned}\sup_{t\geq t_0} I_{\lambda,\mu}(tU_0, tV_0) &= \sup_{t\geq t_0}\left(\mathcal{E}(tU_0, tV_0) - \frac{1}{r}P_{\lambda,\mu}(tU_0, tV_0)\right)\\ &\leq \frac{N+4-\alpha}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{S_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} + \begin{cases} o(\epsilon^{N-4}), & \alpha \leq 8\\ o(\epsilon^{\frac{2N-\alpha}{2}}), & \alpha > 8\end{cases}\\ &\quad - \frac{\omega}{r}(\lambda+\mu)\begin{cases} o(\epsilon^{N-\frac{N-4}{2}r}|\ln\epsilon|), & r = \frac{N}{N-4}\\ o(\epsilon^{N-\frac{N-4}{2}r}), & r > \frac{N}{N-4},\end{cases}\end{aligned}$$

or

$$\begin{aligned}\sup_{t\geq t_0} I_{\lambda,\mu}(tU_0, tV_0) &\leq \frac{N+4-\alpha}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{S_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} + o(\epsilon^\rho)\\ &\quad - \frac{\omega}{r}(\lambda+\mu)\begin{cases} o(\epsilon^{N-\frac{N-4}{2}r}|\ln\epsilon|), & r = \frac{N}{N-4}\\ o(\epsilon^{N-\frac{N-4}{2}r}), & r > \frac{N}{N-4},\end{cases}\end{aligned}$$

where $\rho = \min\left\{N-4, \frac{2N-\alpha}{2}\right\}$.

Choose $\delta_2 > 0$ in this way that $0 \leq \epsilon < \delta_2$ and take $\epsilon = \left[(\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}\right]^{\frac{1}{\rho}}$. Thus, we have

$$\begin{aligned}\sup_{t\geq t_0} I_{\lambda,\mu}(tU_0, tV_0) &\leq \frac{N+4-\alpha}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{S_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} + o(\mathcal{D}(\lambda,\mu))\\ &\quad - \frac{\omega}{r}(\lambda+\mu)\begin{cases} o((\mathcal{D}(\lambda,\mu))^{\frac{N}{2\rho}}|\ln\mathcal{D}(\lambda,\mu)|), & r = \frac{N}{N-4}\\ o((\mathcal{D}(\lambda,\mu))^{\frac{N}{\rho}-\frac{N-4}{2\rho}r}), & r > \frac{N}{N-4},\end{cases}\end{aligned}$$

where $\mathcal{D}(\lambda,\mu) = (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}}$.

Case (i): When $\alpha \leq 8$, then $\rho = N-4$.

For $r = \frac{N}{N-4}$, we can choose $\delta_3 > 0$ with $0 < \mathcal{D}(\lambda,\mu) < \delta_3$ such that

$$o(\mathcal{D}(\lambda,\mu)) - \frac{\omega(\lambda+\mu)}{r}o\left((\mathcal{D}(\lambda,\mu))^{\frac{N}{2(N-4)}}\left|\ln(\mathcal{D}(\lambda,\mu))\right|\right) < -K_0(\mathcal{D}(\lambda,\mu)),$$

as $\lambda,\mu \to 0$ and $|\ln(\mathcal{D}(\lambda,\mu))| \to +\infty$.

For $r > \frac{N}{N-4}$, we choose $\delta_4 > 0$ with $0 < \mathcal{D}(\lambda,\mu) < \delta_4$ such that

$$o(\mathcal{D}(\lambda,\mu)) - \frac{\omega(\lambda+\mu)}{r}o\left((\mathcal{D}(\lambda,\mu))^{\frac{N}{N-4}-\frac{r}{2}}\right) < -K_0(\mathcal{D}(\lambda,\mu)),$$

as $1 + \frac{2}{2-r}\left(\frac{N}{N-4} - \frac{r}{2}\right) < \frac{2}{2-r}$ for $r > \frac{N}{N-4}$. Now, we fix $\Upsilon_* = \min\{\delta_1^{\frac{2-r}{2}}, \delta_2^{\frac{(2-r)(N-4)}{2}}, \delta_3^{\frac{2-r}{2}}, \delta_4^{\frac{2-r}{2}}\} > 0$ such that

$$\sup_{t\geq t_0} I_{\lambda,\mu}(tU_0, tV_0) \leq \frac{N+4-\alpha}{2N-\alpha}\|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}}\left(\frac{S_{H,L}}{2}\right)^{\frac{2N-\alpha}{N+4-\alpha}} - K_0(\mathcal{D}(\lambda,\mu)) \text{ for } 0 < \mathcal{D}(\lambda,\mu) < \Upsilon_*.$$

Case (ii): When $\alpha > 8$, then $\rho = \frac{2N-\alpha}{2}$.

For $r = \frac{N}{N-4}$, we choose $\delta_5 > 0$ with $0 < \mathcal{D}(\lambda, \mu) < \delta_5$ such that

$$o(\mathcal{D}(\lambda, \mu)) - \frac{\omega(\lambda+\mu)}{r} o\left((\mathcal{D}(\lambda, \mu))^{\frac{N}{2N-\alpha}} \left| \ln (\mathcal{D}(\lambda, \mu)) \right| \right) < -K_0 \left(\mathcal{D}(\lambda, \mu) \right),$$

as $\lambda, \mu \to 0$ and $|\ln (\mathcal{D}(\lambda, \mu))| \to +\infty$.

For $r > \frac{N}{N-4}$, we choose $\delta_6 > 0$ with $0 < \mathcal{D}(\lambda, \mu) < \delta_6$ such that

$$o(\mathcal{D}(\lambda, \mu)) - \frac{\omega(\lambda+\mu)}{r} o\left((\mathcal{D}(\lambda, \mu))^{\frac{2N}{2N-\alpha} - \frac{N-4}{2N-\alpha} r} \right) < -K_0 \left(\mathcal{D}(\lambda, \mu) \right),$$

as $1 + \frac{2}{2-r} \left(\frac{2N}{2N-\alpha} - \frac{N-4}{2N-\alpha} r \right) < \frac{2}{2-r}$ for $r > \frac{N}{N-4}$. Fix $\Upsilon_{**} = \min\{\delta_1^{\frac{2-r}{2}}, \delta_2^{\frac{(2-r)(2N-\alpha)}{2}}, \delta_5^{\frac{2-r}{2}}, \delta_6^{\frac{2-r}{2}}\} > 0$ to obtain

$$\begin{aligned} &\sup_{t \geq t_0} I_{\lambda,\mu}(tU_0, tV_0) \\ &\quad \leq \frac{N+4-\alpha}{2N-\alpha} \|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} \left(\frac{\overline{S}_{H,L}}{2} \right)^{\frac{2N-\alpha}{N+4-\alpha}} - K_0(\mathcal{D}(\lambda, \mu)) \quad \text{for } 0 < \mathcal{D}(\lambda, \mu) < \Upsilon_{**}. \end{aligned} \tag{6.3}$$

Thereafter, we fix $\Upsilon = \min\{\Upsilon_*, \Upsilon_{**}\}$. Thus, we have

$$\begin{aligned} &\sup_{t \geq t_0} I_{\lambda,\mu}(tU_0, tV_0) \\ &\quad \leq \frac{N+4-\alpha}{2N-\alpha} \|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} \left(\frac{\overline{S}_{H,L}}{2} \right)^{\frac{2N-\alpha}{N+4-\alpha}} - K_0(\mathcal{D}(\lambda, \mu)) =: c_\infty \quad \text{for } 0 < \mathcal{D}(\lambda, \mu) < \Upsilon. \end{aligned}$$

Later, we show that $k^-_{\lambda,\mu} < c_\infty$ for all $0 < (\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} < \Upsilon$. By using $(Z2)$, $(Z4)$ and the definition of (U_0, V_0), we get

$$P_{\lambda,\mu}(U_0, V_0) > 0 \quad \text{and} \quad Q(U_0, V_0) > 0.$$

Further, by Lemma 4.4, definition of $k^-_{\lambda,\mu}$ and (6.3), there exists $t_2(U_0, V_0) \in \mathcal{N}^-_{\lambda,\mu}$ satisfying

$$\begin{aligned} k^-_{\lambda,\mu} &\leq I_{\lambda,\mu}(t_2 U_0, t_2 V_0)) \leq I_{\lambda,\mu}(tU_0, tV_0) \\ &\leq \frac{N+4-\alpha}{2N-\alpha} \|H^+\|_\infty^{-\frac{2(N-4)}{N+4-\alpha}} \left(\frac{\overline{S}_{H,L}}{2} \right)^{\frac{2N-\alpha}{N+4-\alpha}} - K_0(\mathcal{D}(\lambda, \mu)) =: c_\infty, \end{aligned}$$

for each $0 < \mathcal{D}(\lambda, \mu) < \Upsilon$.

Take $(U_0, V_0) = (u_{\lambda,\mu}, v_{\lambda,\mu})$ and with this we complete the proof. □

Lemma 6.2. *Assume that* $(Z1)$–$(Z4)$ *hold. Then* $I_{\lambda,\mu}$ *satisfies the* $(PS)_{k^-_{\lambda,\mu}}$ *condition for all* $0 < (\lambda \|F\|_\beta)^{\frac{2}{2-r}} + (\mu \|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Upsilon_1$ *and has a minimizer* $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ *in* $\mathcal{N}^-_{\lambda,\mu}$ *and satisfies the following conditions:*

(i) $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = k^-_{\lambda,\mu} > 0$.

(ii) $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ *is a nontrivial solution of the system* $(\mathcal{D}_{\lambda,\mu})$.

Proof. By virtue of Lemma 5.3 (ii), for $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Upsilon_1$, there exists a $(PS)_{k^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$ in $\mathcal{H}$ for $I_{\lambda,\mu}$. Then, from Lemma 3.2, we find that $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$. Now, using Lemma 3.3 and Lemma 6.1, $I_{\lambda,\mu}$ satisfies the $(PS)_{k^-_{\lambda,\mu}}$-condition. Then, there exists $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{H}$ such that up to subsequence $(u_n, v_n) \to (u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ in $\mathcal{H}$. Moreover, $I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) = k^-_{\lambda,\mu} > 0$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. Using the argument as applied in Lemma 5.4, one can easily obtain that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is a nontrivial solution of system $(\mathcal{D}_{\lambda,\mu})$ for $0 < (\lambda\|F\|_\beta)^{\frac{2}{2-r}} + (\mu\|G\|_\beta)^{\frac{2}{2-r}} < \left(\frac{r}{2}\right)^{\frac{2}{2-r}} \Upsilon_1$. □

Proof of Theorem 1.2. By Lemma 5.4 and Lemma 6.2, system $(\mathcal{D}_{\lambda,\mu})$ has one solution $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) \in \mathcal{N}^+_{\lambda,\mu}$ and another solution $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) \in \mathcal{N}^-_{\lambda,\mu}$. Afterwards, we show that the solutions $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ and $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ are not semi-trivial. Using Lemma 5.4 (i) and Lemma 6.2 (i) respectively, we get

$$I_{\lambda,\mu}(u^1_{\lambda,\mu}, v^1_{\lambda,\mu}) < 0 \quad \text{and} \quad I_{\lambda,\mu}(u^2_{\lambda,\mu}, v^2_{\lambda,\mu}) > 0. \tag{6.4}$$

We observe that, if $(u, 0)$ (or $(0, v)$) is a semi-trivial solution of system $(\mathcal{D}_{\lambda,\mu})$, then we have

$$\begin{cases} \Delta^2 u = \lambda F(x)|u|^{r-2}u & \text{in } \Omega, \\ u = \nabla u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Now, the energy functional $I_{\lambda,\mu}(u, 0)$ corresponding to (6.5) is

$$I_{\lambda,\mu}(u, 0) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{r}\int_\Omega F(x)|u|^r dx = -\frac{2-r}{2r}\|u\|^2 < 0. \tag{6.6}$$

Thus (6.4) and (6.6), we conclude that $(u^2_{\lambda,\mu}, v^2_{\lambda,\mu})$ is not a semi-trivial solution. Next, we prove that $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is also not a semi-trivial solution. Without loss of generality, we assume that $v^1_{\lambda,\mu} \equiv 0$. Then $u^1_{\lambda,\mu}$ is a non-trivial solution of (6.5) and

$$\|(u^1_{\lambda,\mu}, 0)\|^2 = \|u^1_{\lambda,\mu}\|^2 = \lambda \int_\Omega F(x)|u^1_{\lambda,\mu}|^r dx \geq 0.$$

Moreover, we choose $w \in H^2_0(\Omega) \setminus \{0\}$ such that

$$\|(0, w)\|^2 = \|w\|^2 = \mu \int_\Omega G(x)|w|^r dx > 0.$$

From Lemma 4.4, there exists a unique $0 < t_1 < t_{\max}(u^1_{\lambda,\mu}, w)$ such that $(t_1 u^1_{\lambda,\mu}, t_1 w) \in \mathcal{N}^+_{\lambda,\mu}$, where

$$t_{\max}(u^1_{\lambda,\mu}, w) = \left(\frac{(22^*_\alpha - r)\int_\Omega(\lambda F(x)|u^1_{\lambda,\mu}|^r + \mu G(x)|w|^r)dx}{(22^*_\alpha - 2)\|(u^1_{\lambda,\mu}, w)\|^2}\right)^{\frac{1}{2-r}} = \left(\frac{22^*_\alpha - r}{22^*_\alpha - 2}\right)^{\frac{1}{2-r}} > 1.$$

Furthermore,

$$I_{\lambda,\mu}(t_1 u^1_{\lambda,\mu}, t_1 w) = \inf_{0 \leq t \leq t_{\max}} I_{\lambda,\mu}(t u^1_{\lambda,\mu}, t w).$$

This together with the fact that $(u^1_{\lambda,\mu}, 0) \in \mathcal{N}^+_{\lambda,\mu}$ imply that

$$\mu^+_{\lambda,\mu} \leq I_{\lambda,\mu}(t_1 u^1_{\lambda,\mu}, t_1 w) \leq I_{\lambda,\mu}(u^1_{\lambda,\mu}, w) < I_{\lambda,\mu}(u^1_{\lambda,\mu}, 0) = \mu^+_{\lambda,\mu},$$

which is a contradiction. Hence, $(u^1_{\lambda,\mu}, v^1_{\lambda,\mu})$ is not a semi-trivial solution. □

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Existence of positive solutions for a class of p-Laplacian type generalized quasilinear Schrödinger equations with critical growth and potential vanishing at infinity

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Abstract. In this paper, we study the existence of positive solutions for the following generalized quasilinear Schrödinger equation

$$\begin{aligned} &-\operatorname{div}(g^p(u)|\nabla u|^{p-2}\nabla u) + g^{p-1}(u)g'(u)|\nabla u|^p + V(x)|u|^{p-2}u \\ &\qquad = K(x)f(u) + Q(x)g(u)|G(u)|^{p^*-2}G(u), \qquad x \in \mathbb{R}^N, \end{aligned}$$

where $N \geq 3$, $1 < p \leq N$, $p^* = \frac{Np}{N-p}$, $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^+)$, $V(x)$ and $K(x)$ are positive continuous functions and $G(u) = \int_0^u g(t)dt$. By using a change of variable, we obtain the existence of positive solutions for this problem by using the Mountain Pass Theorem. Our results generalize some existing results.

Keywords: generalized quasilinear Schrödinger equation, positive solutions, critical growth; p-Laplacian.

2020 Mathematics Subject Classification: 35J60, 35J20.

Introduction

This article is concerned with a class of generalized quasilinear Schrödinger equation

$$\begin{aligned} &-\operatorname{div}(g^p(u)|\nabla u|^{p-2}\nabla u) + g^{p-1}(u)g'(u)|\nabla u|^p + V(x)|u|^{p-2}u \\ &\qquad = K(x)f(u) + Q(x)g(u)|G(u)|^{p^*-2}G(u), \qquad x \in \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where $N \geq 3$, $1 < p \leq N$, $p^* = \frac{pN}{N-p}$, $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^+)$, $V(x)$ and $K(x)$ are positive continuous functions, $Q(x) \geq 0$ is a bounded continuous function and $G(u) = \int_0^u g(t)dt$.

If $p = 2$, then (1.1) will be reduced to the following generalized quasilinear Schrödinger equation

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = K(x)f(u) + Q(x)g(u)|G(u)|^{2^*-2}G(u), \quad x \in \mathbb{R}^N.$$

In nonlinear analysis, the existence of solitary wave solutions for the following quasi-linear Schrödinger equation has been widely considered

$$i\partial_t z = -\Delta z + W(x)z - k(x,|z|) - \Delta l(|z|^2)l'(|z|^2)z \tag{1.2}$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. When l is different, the quasilinear equation of the form (1.2) can express several physical phenomenon. Especially, $l(s) = s$ was used for the superfluid film [26,27] equation in fluid mechanics by Kurihara [26]. For more physical background, we can refer to [5,6,11,25,28,36,38,39] and references therein. In addition, many conclusions about the equation (1.2) with $l(t) = t^\alpha$ for some $\alpha \geq 1$ have been studied, see [33–35,37] and the references therein. However, to our knowledge, only in the recent papers [20] and [40], the equation (1.2) with a general l has been studied.

If we let $z(t,x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function, then (1.2) can be reduce to (see [15]):

$$-\Delta u + V(x)u - \Delta l(u^2)l'(u^2)u = h(x,u), \qquad x \in \mathbb{R}^N. \tag{1.3}$$

If we take

$$g^2(u) = 1 + \frac{[(l^2(u))']^2}{2},$$

then (1.3) turns into quasilinear elliptic equations (see [40])

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u), \qquad x \in \mathbb{R}^N. \tag{1.4}$$

Moreover, if we let

$$g^p(u) = 1 + \frac{[(l^2(u))']^p}{p},$$

the (1.1) turns to the following (see [45])

$$-\Delta_p + V(x)|u|^{p-2}u - \Delta_p(l(u^2))l'(u^2)\frac{2u}{p} = h(x,u), \qquad x \in \mathbb{R}^N.$$

For (1.4), in [20,21], Deng et al. proved the existence of positive solutions with critical exponents. In [20,21], they established the critical exponents, which are 2^* and $\alpha 2^*$, respectively. In [18,19], Deng et al. established the existence of nodal solutions. Especially, in [18], the authors gave some existence results about under critical growth condition. Moreover, in [29], Li et al. proved the existence of ground state solutions and geometrically distinct solutions via Nehari manifold method. In [30], the authors studied the existence of a positive solution, a negative solution and infinitely many solutions via symmetric mountain theorem. In [9], Chen et al. considered the existence and concentration behavior of ground state solutions for (1.4) with subcritical growth. Afterwards, Chen et al. [10] proved the existence and concentration behavior of ground state solutions for (1.4) with critical exponential 22^* growth. For more results, the readers can refer to [13,14,31,40–43]. In 2016, Li et al. [31,46] established the existence of sign-changing solutions and ground state solutions with potential vanishing at infinity as follows:

- (g) $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^+)$ is even with $g'(t) \geq 0$ for all $t \in \mathbb{R}^+$ and $g(0) = 1$.
- (V) The potential function V is positive on $\mathbb{R}^N$ and belongs to $L^\infty(\mathbb{R}^N) \cap \mathcal{C}^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.

(K) $K \in L^{\infty}(\mathbb{R}^N) \cap \mathcal{C}^{\alpha}(\mathbb{R}^N)$ is positive.

(K_1) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \leq M$, for all n and some $M > 0$, then we have

$$\lim_{r \to +\infty} \int_{A_n \cap B_r^c(0)} K(x)dx = 0, \quad \text{uniformly in } n \in \mathbb{N},$$

where $B_r^c(0) = \{x \in \mathbb{R}^N : |x| \geq r\}$

(K_2) The following condition holds:

$$\frac{K(x)}{V(x)} \in L^{\infty}(\mathbb{R}^N). \tag{1.5}$$

Note that conditions (V)–(K_2) are called potential vanishing at infinity. By using potential vanishing at infinity, there are many papers (see [1, 4, 12, 23, 24, 31, 32, 43, 44, 46]) to study the existence of solutions for different equations. Especially in [22], Deng et al. proved the existence of positive solutions with critical growth and potential vanishing at infinity by making the change of variables $v = r^{-1}(u)$, where r is defined by

$$\begin{aligned} r'(t) &= \frac{1}{(1+2r^2(t))^{1/2}} \quad && \text{on } [0, +\infty), \\ r(-t) &= r(t) && \text{on } (-\infty, 0]. \end{aligned}$$

However, conditions (V)–(K_2) are weaker than the following well-known condition:

(VK) $V, K : \mathbb{R}^N \to \mathbb{R}_+$ are smooth and there exist positive numbers α, β, a, b, and c such that

$$\frac{a}{1+|x|^{\alpha}} \leq V(x) \leq b, \qquad 0 < K(x) \leq \frac{c}{1+|x|^{\beta}}, \qquad x \in \mathbb{R}^3,$$

which was firstly introduced in [2].

Before stating our results, let us recall some basic notions. Let

$$D^{1,p}(\mathbb{R}^N) = \left\{ u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N) \right\}$$

with the norm

$$\|u\|_{D^{1,p}} = \left(\int_{\mathbb{R}^N} |\nabla u|^p dx \right)^{\frac{1}{p}}.$$

Since the potential may vanish at infinity, it is natural to use the following working space:

$$E = \left\{ v \in D^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^3} V(x)|v|^p dx < \infty \right\}$$

endowed with the norm

$$\|v\| = \left(\int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|v|^p) dx \right)^{\frac{1}{2}}, \qquad v \in E.$$

Moreover, we define the weighted Lebesgue space

$$L_K^q(\mathbb{R}^N) = \left\{ u : u \text{ is measurable on } \mathbb{R}^N \text{ and } \int_{\mathbb{R}^N} K(x)|u|^q dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^q_K} = \left(\int_{\mathbb{R}^N} (K(x)|u|^q)dx \right)^{\frac{1}{q}},$$

for some $q \in (p, p^*)$.

By the conditions (V)–(K_2), in [17], the authors got the following proposition.

Proposition 1.1 (see [17, Lemma 2.2]). *Suppose that (V)–(K_2) are satisfied. Then E is compactly embedded in $L^q_K(\mathbb{R}^N)$ for all $q \in (p, p^*)$ if (1.5) holds.*

To resolve the equation (1.1), due to the appearance of the nonlocal term $\int_{\mathbb{R}^N} g^p(u)|\nabla u|^p dx$, the right working space seems to be

$$E_0 = \left\{ u \in E : \int_{\mathbb{R}^N} g^p(u)|\nabla u|^p dx < \infty \right\}.$$

But it is easy to see that E_0 is not a linear space under the assumption of (g). To overcome this difficulty, a variable substitution as follows: for any $v \in E$, Shen and Wang [40] make a change of variable as

$$u = G^{-1}(v) \quad \text{and} \quad G(u) = \int_0^u g(t)dt,$$

then

$$\int_{\mathbb{R}^N} g^p(u)|\nabla u|^p dx = \int_{\mathbb{R}^N} g^p(G^{-1}(v))|\nabla G^{-1}(v)|^p dx := |\nabla v|_p^p < +\infty, \qquad v \in E.$$

In such a case, we can deduce formally that the Euler–Lagrange functional associated with the equation (1.1) is

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^N} [g^p(u)|\nabla u|^p + V(x)u^p]dx - \int_{\mathbb{R}^N} K(x)F(u)dx - \frac{1}{p^*} \int_{\mathbb{R}^N} Q(x)|G(u)|^{p^*}dx.$$

Therefore, by this change of variables E can be used as the working space and the equation (1.1) in form can be transformed into

$$\begin{aligned} \mathcal{J}(v) &= \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|G^{-1}(v)|^p)dx \\ &\quad - \int_{\mathbb{R}^N} K(x)F(G^{-1}(v))dx - \frac{1}{p^*} \int_{\mathbb{R}^N} Q(x)|v|^{p^*}dx, \qquad x \in \mathbb{R}^N. \end{aligned} \tag{1.6}$$

By the fact of g is a nondecreasing positive function that $|G^{-1}(v)| \le |v|$. From this and our hypotheses, it is clear that $\mathcal{J}$ is well defined in E and $\mathcal{J} \in \mathcal{C}^1$.

Furthermore, one can easily derive that if $v \in \mathcal{C}^2(\mathbb{R}^N)$ is a critical point of (1.6), then $u = G^{-1}(v) \in \mathcal{C}^2(\mathbb{R}^N)$ is a classical solution to the equation (1.1). To obtain a critical point of (1.6), we only need to seek for the weak solution to the following equation

$$-\Delta_p v + V(x)\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))} = K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v))} + Q(x)|v|^{p^*-2}v, \qquad x \in \mathbb{R}^N. \tag{1.7}$$

Here, we say that $v \in E$ is a weak solution to the equation (1.7) if it holds that

$$\begin{aligned} \langle \mathcal{J}'(v), \varphi \rangle &= \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla \varphi + \int_{\mathbb{R}^N} V(x)\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))}\varphi \\ &\quad - \int_{\mathbb{R}^N} K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v))}\varphi - \int_{\mathbb{R}^N} Q(x)|v|^{p^*-2}v\varphi, \qquad \varphi \in E. \end{aligned}$$

Then it is standard to obtain that $v \in E$ is a weak solution to the equation (1.7) if and only if v is a critical point of the functional $\mathcal{J}$ in E. To sum up, it is sufficient to find a critical point of the functional $\mathcal{J}$ in E to achieve a classical solution to the equation (1.1).

Very recently, Song and Chen [45] studied the existence of weak solutions for (1.1) when V is a positive potential bounded away from zero and $h(x,u) = h(u)$ is a nonlinear term of subcritical type. Now, it is natural to ask whether problem (1.1) has the existence of positive solutions in the case where h satisfies critical growth? To the best of our knowledge, there are few results on such above questions in current literature. Actually, this is one of the motivations for us to study the existence of positive solutions of (1.1) with critical growth. Motivated by the above works, in this paper, our goal is to deal with critical growth case and give the existence of positive solutions of (1.1) with potential vanishing at infinity.

Now, we answer the question in the affirmative, which is given in the front of the article. Before stating our results, we need to give the following assumptions on f:

(f) $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$, $f(t) = 0$ for $t \le 0$ and f has a "quasicritical" growth, namely

$$\lim_{|t|\to\infty} \frac{f(t)}{g(t)|G(t)|^{p^*-1}} = 0.$$

(f_1) $\lim_{t\to 0^+} \frac{f(t)}{g(t)|G(t)|^{p-1}} = 0$ if (1.5) holds.

(f_2) There exists a $\mu \in (p, p^*)$ such that for any $t > 0$

$$0 < \mu g(t)F(t) \le G(t)f(t) \quad \text{for all } s \in \mathbb{R},$$

where $F(u) = \int_0^u f(t)dt$.

In addition, we also assume that

(Q_1) There is a point x_0, such that

$$Q(x_0) = \sup_{x\in\mathbb{R}^N} Q(x).$$

(Q_2) For x close to x_0, we have

$$Q(x) = Q(x_0) + O(|x-x_0|^p) \quad \text{as } x \to x_0.$$

Now, we state our main results by the following theorems.

Theorem 1.2. *Suppose that (g), (V)–(K_2), (Q_1)–(Q_2) and (f)–(f_2) are satisfied. Then problem (1.1) has at least one positive solution if either $N \ge p^2$ or $p < N < p^*$ and $\mu > p^* - \frac{p}{p-1}$.*

Applying Theorem 1.2 to the case when $Q(x) = 1$ and $p = 2$, we can get the following corollary.

Corollary 1.3. *Suppose that (g), (V)–(K_2) and (f)–(f_2) are satisfied. Then the following problem*

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = K(x)f(u) + g(u)|G(u)|^{2^*-2}G(u), \qquad x \in \mathbb{R}^N$$

has at least one positive solution if either $N \ge 4$ or $N = 3$ and $\mu > 2^* - 2$.

The paper is organized as follows. In Section 2, we prove a solution of (1.1) with critical growth and potential vanishing at infinity. In Appendix A, we give some useful lemmas, respectively.

In the following, we denote by $L^p(\mathbb{R}^N)$ the usual Lebesgue space with norms $\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}}$, where $1 \le p < \infty$; for any $z \in \mathbb{R}^2$ and $R > 0$, $B_R(z) := \{x \in \mathbb{R}^2 : |x-z| < R\}$; C possibly denotes the different constants in different place.

Main results

In this section, we present some useful lemmas and corollaries. Now, let us recall the following lemma which has been proved in [30].

Lemma 2.1 ([30]). *For the function g, G, and G^{-1}, the following properties hold:*

(1) the functions $G(\cdot)$ and $G^{-1}(\cdot)$ are strictly increasing and odd;

(2) $G(s) \leq g(s)s$ for all $s \geq 0$; $G(s) \geq g(s)s$ for all $s \leq 0$;

(3) $g(G^{-1}(s)) \geq g(0) = 1$ for all $s \in \mathbb{R}$;

(4) $\frac{G^{-1}(s)}{s}$ is decreasing on $(0, +\infty)$ and increasing on $(-\infty, 0)$;

(5) $|G^{-1}(s)| \leq \frac{1}{g(0)}|s| = |s|$ for all $s \in \mathbb{R}$;

(6) $\frac{|G^{-1}(s)|}{g(G^{-1}(s))} \leq \frac{1}{g^2(0)}|s| = |s|$ for all $s \in \mathbb{R}$;

(7) $\frac{G^{-1}(s)s}{g(G^{-1}(s))} \leq |G^{-1}(s)|^2$ for all $s \in \mathbb{R}$;

(8) $\lim_{|s|\to 0} \frac{G^{-1}(s)}{s} = \frac{1}{g(0)} = 1$ and

$$\lim_{|s|\to+\infty} \frac{G^{-1}(s)}{s} = \begin{cases} \frac{1}{g(\infty)}, & \text{if } g \text{ is bounded,} \\ 0, & \text{if } g \text{ is unbounded.} \end{cases}$$

The next two lemmas show that the functional $\mathcal{J}$ verifies the mountain pass geometry.

Lemma 2.2. *Suppose that (V)–(K_2), (Q_1)–(Q_2), and (f)–$(f_2)$are satisfied. Then there exist $\alpha, \rho > 0$ such that $\mathcal{J}(v) \geq \alpha$ for all $\|v\| = \rho$.*

Proof. It follows from (1.6) that

$$\begin{aligned} \mathcal{J}(v) &= \frac{1}{p}\int_{\mathbb{R}^N}[|\nabla v|^p + V(x)|G^{-1}(v)|^p]dx - \int_{\mathbb{R}^N} K(x)F(G^{-1}(v))dx \\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|v^+|^{p^*}dx \\ &= \frac{1}{p}\|\nabla v\|_p^p - \int_{\mathbb{R}^N}\left(-\frac{1}{p}V(x)|G^{-1}(v)|^p + K(x)F(G^{-1}(v))\right)dx \\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|v^+|^{p^*}dx \\ &\geq \frac{1}{p}\|\nabla v\|_p^p - \int_{\mathbb{R}^N}\left(-\frac{1}{p}V(x)|G^{-1}(v)|^p + K(x)F(G^{-1}(v))\right)dx \\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|v|^{p^*}dx. \end{aligned} \tag{2.1}$$

On the one hand, if (1.5) holds and let $A(x,s) := -\frac{1}{p}|G^{-1}(s)|^p + \frac{K(x)}{V(x)}F(G^{-1}(s))$, then by Lemma 2.1-(8), we have

$$\lim_{s\to 0^+} \frac{A(x,s)}{|s|^p} = \lim_{s\to 0}\left[-\frac{1}{p}\left|\frac{G^{-1}(v)}{s}\right|^p + \frac{K(x)}{V(x)}\frac{F(G^{-1}(s))}{|s|^p}\right] = -\frac{1}{p} \tag{2.2}$$

and

$$\lim_{s\to+\infty}\frac{A(x,s)}{|s|^{p^*}}=\lim_{s\to+\infty}\left[-\frac{1}{p}\left|\frac{G^{-1}(v)}{s}\right|^p\left(\frac{1}{|s|^{p^*-p}}\right)+\frac{K(x)}{V(x)}\frac{F(G^{-1}(s))}{|s|^{p^*}}\right]=0, \tag{2.3}$$

since

$$\lim_{|s|\to+\infty}\frac{G^{-1}(s)}{s}=\begin{cases}\frac{1}{g(\infty)}, & \text{if } g \text{ is bounded,}\\ 0, & \text{if } g \text{ is unbounded.}\end{cases}$$

Thus, by (2.2) and (2.3), for $\varepsilon>0$ sufficiently small, there exists a constant $C_\varepsilon>0$ such that

$$V(x)A(x,s)\le\left(-\frac{1}{p}+\varepsilon\right)V(x)|s|^p+C_\varepsilon V(x)|s|^{p^*}. \tag{2.4}$$

Then by Proposition 1.1, (2.4), (2.1) and (Q_1), we have

$$\begin{aligned}\mathcal{J}(v)&\ge\frac{1}{p}|\nabla v|^p-\left(-\frac{1}{p}+\varepsilon\right)\int_{\mathbb{R}^N}V(x)|v|^pdx-C_\varepsilon\int_{\mathbb{R}^N}V(x)|v|^{p^*}dx-\frac{1}{p^*}Q(x_0)\int_{\mathbb{R}^N}|v|^{p^*}dx\\&\ge\frac{1}{p}\|v\|^p-C\int_{\mathbb{R}^N}|v|^{p^*}dx-\frac{1}{p^*}Q(x_0)\int_{\mathbb{R}^N}|v|^{p^*}dx\\&\ge\left(\frac{1}{p}-\varepsilon C\right)\|v\|^p-C\|v\|^{p^*},\end{aligned}$$

since there exists $C>0$ such that $0<K(x)\le C$ and $0<V(x)\le C$. It follows that

$$\mathcal{J}(v)\ge C\|v\|^p-C\|v\|^{p^*}, \tag{2.5}$$

if we choose sufficiently small $\rho>0$, which implies that

$$\mathcal{J}(v)\ge C\rho^p-C\rho^{p^*}=:\alpha>0.$$

This completes the proof. □

Lemma 2.3. *Suppose that (V)–(K_2), (Q_1)–(Q_2), and (f)–$(f_2)$are satisfied. Then there exists $e\in E$ such that $\mathcal{J}(e)<0$ and $\|e\|>\rho$.*

Proof. For any fixed $v_0\in E$ with $v_0\ge0$ and $v_0\not\equiv0$, by (1.6) and Lemma 2.1-(5) , we have

$$\begin{aligned}\mathcal{J}(tv_0)&=\frac{1}{p}\int_{\mathbb{R}^N}[|t\nabla v_0|^p+V(x)|G^{-1}(tv_0)|^p]dx-\int_{\mathbb{R}^N}K(x)F(G^{-1}(tv_0))dx\\&\quad-\frac{1}{p^*}\int_{\mathbb{R}^N}Q(x)|tv_0|^{p^*}dx\\&\le\frac{t^p}{p}\|v_0\|^p-\frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N}Q(x)|v_0|^{p^*}dx\\&\to-\infty,\quad \text{as } t\to+\infty,\end{aligned}$$

which gives that the results hold if we take $e=tv_0$ with t sufficiently large. This completes the proof. □

As a consequence of Lemma 2.2 and Lemma 2.3, for the constant

$$c_0=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}\mathcal{J}(\gamma(t))>0,$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1],E),\ \gamma(0)=0,\ \gamma(1)\neq 0,\ \mathcal{J}(\gamma(1))<0\}.$$

Note that from Lemma 2.3, $\Gamma \neq \emptyset$. By the Mountain Pass Theorem in [3], then we have the existence of sequence $\{v_n\} \subset E$ satisfying

$$\mathcal{J}(v_n) \to c_0 \quad \text{and} \quad \mathcal{J}'(v_n) \to 0 \qquad n \to +\infty. \tag{2.6}$$

The above sequence is called a $(PS)_{c_0}$ sequence for $\mathcal{J}$.

Lemma 2.4. *The sequence $\{v_n\}$ in (2.6) are satisfied. Then $\{v_n\}$ is bounded in E.*

Proof. Since $\{v_n\} \subset E$ is a $(PS)_{c_0}$ sequence for $\mathcal{J}$, we have

$$\begin{aligned} \mathcal{J}(v_n) &= \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|G^{-1}(v_n)|^p)dx - \int_{\mathbb{R}^N} K(x)F(G^{-1}(v_n))dx \\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|v_n^+|^{p^*}dx \to c_0 \end{aligned} \tag{2.7}$$

and for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} \langle \mathcal{J}'(v_n),\varphi\rangle &= \int_{\mathbb{R}^N}|\nabla v_n|^{p-2}\nabla v_n \nabla\varphi + \int_{\mathbb{R}^N} V(x)\frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}\varphi \\ &\quad - \int_{\mathbb{R}^N} K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v_n))}\varphi - \int_{\mathbb{R}^N} Q(x)|v_n^+|^{p^*-2}v_n^+\varphi = o(1)\|\varphi\|, \end{aligned} \tag{2.8}$$

as $n\to\infty$. Since $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is dense in E, by choosing $\varphi = v_n$ we deduce that

$$\begin{aligned} \langle \mathcal{J}'(v_n),v_n\rangle = &\int_{\mathbb{R}^N}|\nabla v_n|^{p} + \int_{\mathbb{R}^N} V(x)\frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n - \int_{\mathbb{R}^N} K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v_n))}v_n \\ &- \int_{\mathbb{R}^N} Q(x)|v_n^+|^{p^*-2}v_n^+ v_n = o(1)\|v_n\|, \end{aligned}$$

as $n\to\infty$. It follows from (2.7), (2.8) and Lemma 2.1 that

$$\begin{aligned} &\mu c_0 + o(1) - \langle \mathcal{J}'(v_n),v_n\rangle \\ &\quad \geq \mu\mathcal{J}(v_n) - \langle \mathcal{J}'(v_n),v_n\rangle \\ &\quad = \frac{\mu-p}{p}\int_{\mathbb{R}^N}|\nabla v_n|^p dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^{p-2}\left[\frac{1}{p}\mu|G^{-1}(v_n)|^2 - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\right]dx \\ &\qquad - \int_{\mathbb{R}^N} K(x)\left(\mu F(G^{-1}(v_n)) - \frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n\right)dx - \left(\frac{\mu}{p^*}-1\right)\int_{\mathbb{R}^N} Q(x)|v_n^+|^{p^*}dx \\ &\quad \geq \frac{\mu-p}{p}\left[\int_{\mathbb{R}^N}|\nabla v_n|^p dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^p dx\right]. \end{aligned} \tag{2.9}$$

By (f_2), we have $F(s) \geq CG(s)^\mu \geq CG(s)^p$ for all $s \geq 1$. Then

$$\begin{aligned} &\int_{\{x:||G^{-1}(v_n)|>1\}} V(x)|v_n|^p dx \\ &\qquad \leq C\int_{\{x:|G^{-1}(v_n)>1\}} K(x)F(G^{-1}(v_n))dx \\ &\qquad \leq C\int_{\mathbb{R}^N} K(x)F(G^{-1}(v_n))dx + \frac{C}{p^*}\int_{\mathbb{R}^N} Q(x)|v_n^+|^{p^*}dx \\ &\qquad \leq C\left[\frac{1}{p}\left(\int_{\mathbb{R}^N}|\nabla v_n|^p dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^p dx\right) - c_0 + o_n(1)\right]. \end{aligned} \tag{2.10}$$

On the other hand, for the case $x \in \{x : |G^{-1}(v_n)| \le 1\}$ we know that

$$\begin{aligned}\frac{1}{g^p(1)} \int_{\{x:||G^{-1}(v_n)|\le 1\}} V(x)|v_n|^p dx &\le C \int_{\{x:|G^{-1}(v_n)\le 1\}} V(x)|G^{-1}(v_n)|^p dx \\ &\le C \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^p dx.\end{aligned} \tag{2.11}$$

Since $g(s)$ is nondecreasing. Combining (2.9), (2.10) with (2.11), we deduce that $\{v_n\}$ is bounded in E. This completes the proof. □

We are going to verify that the level value c_0 is in an interval where the (PS) condition holds. To this end, by the method developed by [8], we also introduce a well-known fact that the minimization problem

$$S = \inf\{|\nabla v|_p^p : v \in D^{1,p}(\mathbb{R}^N), |v|_{p^*} = 1\}$$

has a solution given by

$$v_\epsilon(x) = \frac{c(N,p)\epsilon^{(N-p)/(p^2-p)}}{(\epsilon^{p/(p-1)} + |x-x_0|^{p/(p-1)})^{(N-p)/p}}$$

and

$$|\nabla v_\epsilon|_p^p = |v_\epsilon|_{p^*}^{p^*} = S^{N/p}.$$

For small enough $R > 0$, define a cut-off function $\psi(x) \in C_0^\infty(\mathbb{R}^N)$ such that $\psi(|x|) = 1$ for $|x - x_0| \le R$, $\psi(|x|) \in (0,1)$ for $R < |x - x_0| < 2R$ and $|\nabla\psi| \le \frac{C}{R}$, and $\psi(|x|) = 0$ for $|x - x_0| \ge 2R$. Define

$$w_\epsilon(x) = \psi(x)v_\epsilon(x) \tag{2.12}$$

and

$$\sigma_\epsilon(x) = w_\epsilon(x)\left[\int_{\mathbb{R}^N} Q(x)w_\epsilon^{p^*}(x)dx\right]^{-\frac{1}{p^*}}. \tag{2.13}$$

Denote

$$V_{max} := \max_{x\in B_{2R}(x_0)} V(x),$$
$$K_{min} := \min_{x\in B_{2R}(x_0)} K(x).$$

Similar to the discussion of [17,22], by $\partial v_\epsilon/\partial \overrightarrow{n} \le 0$, we have that

$$\int_{B_R(x_0)} |\nabla w_\epsilon|^p dx = \int_{B_R(x_0)} |\nabla v_\epsilon|^p dx \le \int_{B_R(x_0)} |v_\epsilon|^{p^*} dx,$$

and by the assumption (Q_2) we also have

$$Q(x_0)\int_{B_R(x_0)} |\nabla v_\epsilon|^{p^*} dx \le Q(x)\int_{B_R(x_0)} |\nabla v_\epsilon|^{p^*} dx + O(\epsilon^p).$$

Simple calculations as [16] gives that

$$\int_{\mathbb{R}^N\setminus B_R(x_0)} |v_\epsilon|^{p^*} dx = O(\epsilon^{N/p-1}),$$

$$A_\epsilon := \int_{\mathbb{R}^N\setminus B_R(x_0)} |\nabla w_\epsilon|^{p^*} dx = O(\epsilon^{(N-p)/(p-1})$$

and

$$\int_{\mathbb{R}^N} |\sigma_\epsilon|^2 dx = \begin{cases} k\epsilon^p + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N > p^2, \\ k\epsilon^p |\ln \epsilon| + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N = p^2, \\ O(\epsilon^{(N-p)/(p-1)}), & \text{if } N < p^2, \end{cases} \tag{2.14}$$

as $\epsilon \to 0$, where k is a positive constant. Therefore, we can get

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_\epsilon|^p dx &= \int_{B_R(x_0)} |\nabla w_\epsilon|^p dx + A_\epsilon \\ &\le \int_{B_R(x_0)} |v_\epsilon|^{p^*} dx + A_\epsilon \\ &\le S\left[\int_{B_R(x_0)} |v_\epsilon|^{p^*} dx\right]^{\frac{p}{p^*}} + A_\epsilon \\ &\le S(\|Q\|_{L^\infty(\mathbb{R}^N)})^{-\frac{p}{p^*}} \left[\int_{B_R(x_0)} Q(x)|v_\epsilon|^{p^*} dx\right]^{\frac{p}{p^*}} + O(\epsilon^p) + O(\epsilon^{(N-p)/(p-1)}). \end{aligned}$$

Set $V_\epsilon \equiv \int_{\mathbb{R}^N} |\nabla \sigma_\epsilon|^p dx$, since for small $\epsilon > 0$, say $\epsilon \le \epsilon_0$, it is easy to see that

$$\int_{B_R(x_0)} Q(x)|w_\epsilon|^{p^*} dx \ge C_{\epsilon_0}$$

for some positive constant C_{ϵ_0}. The definition of V_ϵ and the last two inequalities imply that

$$V_\epsilon \le S(\|Q\|_{L^\infty(\mathbb{R}^N)})^{-\frac{p}{p^*}} + O(\epsilon^p) + O(\epsilon^{(N-p)/(p-1)}). \tag{2.15}$$

Lemma 2.5. *Suppose that (V)–(K_2), (Q_1)–(Q_2), and (f)–$(f_2)$are satisfied. Then there exists $v_0 \in E \setminus \{0\}$ such that*

$$0 < \sup_{t\ge 0} \mathcal{J}(tv_0) < \frac{1}{N} S^{N/p} [\|Q\|_{L^\infty(\mathbb{R}^N)}]^{\frac{p-N}{p}} \tag{2.16}$$

if either $N \ge p^2$ or $p < N < p^2$ and $\mu > p^* - \frac{p}{p-1}$.

Proof. Firstly, we claim that for $\epsilon > 0$ small enough, there exists a constant $t_\epsilon > 0$ such that

$$\mathcal{J}(t_\epsilon \sigma_\epsilon) = \max_{t\ge 0} \mathcal{J}(t\sigma_\epsilon)$$

and

$$0 < A_1 < t_\epsilon < A_2 < +\infty \quad \text{for all } \epsilon > 0 \text{ small enough,}$$

where A_1 and A_2 are positive constants independent of ϵ.

By (f)–(f_1), for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$|f(t)| \le \delta g(t)|G(t)|^{p-1} + C_\delta g(t)|G(t)|^{p^*-1}. \tag{2.17}$$

Now, we consider

$$\begin{aligned} \mathcal{J}(t\sigma_\epsilon) &= \frac{1}{p}\int_{\mathbb{R}^N} [t^p|\nabla\sigma_\epsilon|^p + V(x)|G^{-1}(t\sigma_\epsilon)|^p] dx - \int_{\mathbb{R}^N} K(x)F(G^{-1}(t\sigma_\epsilon)) dx \\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|t\sigma_\epsilon^+|^{p^*} dx \\ &\le \frac{t^p}{p}\|\sigma_\epsilon\|^p - \int_{\mathbb{R}^N} K(x)F(G^{-1}(t\sigma_\epsilon)) dx - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} Q(x)|\sigma_\epsilon|^{p^*} dx \\ &\to -\infty, \quad \text{as } t \to \infty. \end{aligned}$$

Clearly, $\lim_{t\to+\infty}\mathcal{J}(t\sigma_\epsilon) = -\infty$ for all $\epsilon > 0$. Since $\mathcal{J}(0) = 0$ and $\mathcal{J}(t\sigma_\epsilon) = -\infty$, there exists $t_\epsilon > 0$ such that

$$\mathcal{J}(t_\epsilon\sigma_\epsilon) = \max_{t\geq 0}\mathcal{J}(t\sigma_\epsilon) \quad \text{and} \quad \left.\frac{d\mathcal{J}(t\sigma_\epsilon)}{dt}\right|_{t=t_\epsilon} = 0.$$

Thus we have

$$\begin{aligned} t_\epsilon^{p-1}\int_{B_{2R}(x_0)}|\nabla\sigma_\epsilon|^p dx + \int_{B_{2R}(x_0)} V(x)\frac{|G^{-1}(t_\epsilon\sigma_\epsilon)|^{p-2}G^{-1}(t_\epsilon\sigma_\epsilon)}{g(G^{-1}(t_\epsilon\sigma_\epsilon))}\sigma_\epsilon dx \\ = \int_{B_{2R}(x_0)} K(x)\frac{f(G^{-1}(t_\epsilon\sigma_\epsilon))}{g(G^{-1}(t_\epsilon\sigma_\epsilon))}\sigma_\epsilon dx + t_\epsilon^{p^*-1}\int_{B_{2R}(x_0)} Q(x)|\sigma_\epsilon|^{p^*}dx. \end{aligned} \tag{2.18}$$

On the one hand, if there is a sequence $t_{\epsilon_n} \to +\infty$, as $\epsilon_n \to 0^+$, by the above equality, we get

$$\begin{aligned} t_{\epsilon_n}^{p-1}\int_{B_{2R}(x_0)}|\nabla\sigma_{\epsilon_n}|^p dx + \int_{B_{2R}(x_0)} V(x)\frac{|G^{-1}(t_{\epsilon_n}\sigma_{\epsilon_n})|^{p-2}G^{-1}(t_{\epsilon_n}\sigma_{\epsilon_n})}{g(G^{-1}(t_\epsilon\sigma_{\epsilon_n}))}\sigma_{\epsilon_n} dx \\ \geq t_{\epsilon_n}^{p^*-1}\int_{B_{2R}(x_0)} Q(x)|\sigma_{\epsilon_n}|^{p^*}dx. \end{aligned}$$

Hence by Lemma 2.1-(7), we get

$$t_{\epsilon_n}^{p-1}\left[\int_{B_{2R}(x_0)}|\nabla\sigma_{\epsilon_n}|^p dx + \int_{B_{2R}(x_0)} V(x)|\sigma_{\epsilon_n}|^p dx\right] \geq t_{\epsilon_n}^{p^*-1}\int_{B_{2R}(x_0)} Q(x)|\sigma_{\epsilon_n}|^{p^*}dx,$$

which gives a contradiction since $p^* > p$.

On the other hand, we suppose there is a sequence $t'_{\epsilon_n} \to 0$ as $\epsilon_n \to 0^+$. If (1.5) holds, by (2.17), for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N} K(x)\frac{f(G^{-1}(t_{\epsilon_n}\sigma_{\epsilon_n}))}{g(G^{-1}(t_{\epsilon_n}\sigma_{\epsilon_n}))}\sigma_{\epsilon_n}dx &\leq \delta {t'}_{\epsilon_n}^{p-1}\int_{\mathbb{R}^N} K(x)|\sigma_{\epsilon_n}|^p dx + C_\delta(t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N} K(x)|\sigma_{\epsilon_n}|^{p^*}dx \\ &\leq \delta C{t'}_{\epsilon_n}^{p-1}\int_{\mathbb{R}^N}\left(|\nabla\sigma_{\epsilon_n}|^p + V(x)|\sigma_{\epsilon_n}|^p\right)dx \\ &\quad + C_\delta(t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N} K(x)|\sigma_{\epsilon_n}|^{p^*}dx. \end{aligned}$$

By (2.18), we have

$$\begin{aligned} {t'}_{\epsilon_n}^{p-1}\left(\int_{\mathbb{R}^N}|\nabla\sigma_{\epsilon_n}|^p dx\right) + \int_{\mathbb{R}^N} V(x)\frac{|G^{-1}(t'_{\epsilon_n}\sigma_{\epsilon_n})|^{p-2}G^{-1}(t'_{\epsilon_n}\sigma_{\epsilon_n})}{(t'_{\epsilon_n}\sigma_{\epsilon_n})g(G^{-1}(t'_\epsilon\sigma_{\epsilon_n}))}\sigma_{\epsilon_n}^2 dx \\ \leq \delta C{t'}_{\epsilon_n}^{p-1}\int_{\mathbb{R}^N}\left(|\nabla\sigma_{\epsilon_n}|^p + V(x)|\sigma_{\epsilon_n}|^p\right)dx + (t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N} Q(x)|\sigma_{\epsilon_n}|^{p^*}dx \\ + C_\delta(t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N} K(x)|\sigma_{\epsilon_n}|^{p^*}dx. \end{aligned}$$

Thus taking $\delta = \frac{1}{2C}$, we have

$$\begin{aligned} {t'}_{\epsilon_n}^{p-1}\left(\frac{1}{p}\int_{\mathbb{R}^N}|\nabla\sigma_{\epsilon_n}|^p dx + \int_{\mathbb{R}^N} V(x)\left[\frac{|G^{-1}(t'_{\epsilon_n}\sigma_{\epsilon_n})|^{p-2}G^{-1}(t'_{\epsilon_n}\sigma_{\epsilon_n})}{|t'_{\epsilon_n}\sigma_{\epsilon_n}|^{p-1}g(G^{-1}(t'_\epsilon\sigma_{\epsilon_n}))} - \frac{1}{p}\right]|\sigma_{\epsilon_n}|^p dx\right) \\ \leq (t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N} Q(x)|\sigma_{\epsilon_n}|^{p^*}dx + C_\delta(t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N} K(x)|\sigma_{\epsilon_n}|^{p^*}dx. \end{aligned} \tag{2.19}$$

When $t'_{\epsilon_n} \to 0$, we have

$$\frac{G^{-1}(t'_{\epsilon_n}\sigma_{\epsilon_n})}{|t'_{\epsilon_n}\sigma_{\epsilon_n}|^{p-1}g(G^{-1}(t'_{\epsilon}\sigma_{\epsilon_n}))} > \frac{1}{p}.$$

Therefore (2.19) is also impossible because of $p^* > p$. So we complete the proof of our claim.

Since $0 < A_1 < t_\epsilon < A_2 < +\infty$ for ϵ small enough, together with the definition of $V_{\max}$ and $K_{\min}$, we know that

$$\begin{aligned}
\mathcal{J}(t\sigma_\epsilon) &= \frac{1}{p}\int_{\mathbb{R}^N}(t^p|\nabla\sigma_\epsilon|^p + V(x)|G^{-1}(t\sigma_\epsilon)|^p)dx - \int_{\mathbb{R}^N} K(x)F(G^{-1}(t\sigma_\epsilon))dx \\
&\quad - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} Q(x)|\sigma_\epsilon|^{p^*}dx \\
&= \frac{t^p}{p}V_\epsilon + \frac{1}{p}\int_{B_{2R}(x_0)} V(x)|G^{-1}(t\sigma_\epsilon)|^p dx - \int_{B_{2R}(x_0)} K(x)F(G^{-1}(t\sigma_\epsilon))dx - \frac{t^{p^*}}{p^*} \\
&\le \frac{t_\epsilon^p}{p}V_\epsilon + \frac{1}{p}\int_{B_{2R}(x_0)} V(x)|G^{-1}(t_\epsilon\sigma_\epsilon)|^p dx - \int_{B_{2R}(x_0)} K(x)F(G^{-1}(t_\epsilon\sigma_\epsilon))dx - \frac{t_\epsilon^{p^*}}{p^*} \\
&\le \frac{t_\epsilon^p}{p}V_\epsilon + \frac{1}{p}V_{\max}\int_{B_{2R}(x_0)} |G^{-1}(t_\epsilon\sigma_\epsilon)|^p dx - K_{\min}\int_{B_{2R}(x_0)} F(G^{-1}(t_\epsilon\sigma_\epsilon))dx - \frac{t_\epsilon^{p^*}}{p^*} \\
&\le \frac{t_\epsilon^p}{p}V_\epsilon + \frac{t_\epsilon^p}{p}V_{\max}\int_{B_{2R}(x_0)} |\sigma_\epsilon|^p dx - K_{\min}\int_{B_{2R}(x_0)} F(G^{-1}(t_\epsilon\sigma_\epsilon))dx - \frac{t_\epsilon^{p^*}}{p^*}.
\end{aligned}$$

By virtue of $\frac{t^p}{p}V_\epsilon - \frac{t^{p^*}}{p^*} \le \frac{1}{N}V_\epsilon^{N/p}$ for all $t \ge 0$, the estimate (2.15) on V_ϵ and the above inequality imply that

$$\begin{aligned}
\sup_{t\ge 0}\mathcal{J}(t\sigma_\epsilon) &= \mathcal{J}(t_\epsilon\sigma_\epsilon) \\
&\le \frac{1}{N}S^{N/p}\left[\|Q\|_{L^\infty(\mathbb{R}^N)}\right]^{-\frac{N-p}{p}} + O(\epsilon^p) + O(\epsilon^{(N-p)/(p-1)}) \\
&\quad + \frac{t_\epsilon^p}{p}V_{\max}\int_{B_{2R}(x_0)}|\sigma_\epsilon|^p dx - K_{\min}\int_{B_{2R}(x_0)} F(G^{-1}(t_\epsilon\sigma_\epsilon))dx \\
&\le \frac{1}{N}S^{N/p}\left[\|Q\|_{L^\infty(\mathbb{R}^N)}\right]^{-\frac{N-p}{p}} - K_{\min}\int_{B_{2R}(x_0)} F(G^{-1}(t_\epsilon\sigma_\epsilon))dx + O(\epsilon^p) \\
&\quad + O(\epsilon^{(N-p)/(p-1)}) + \begin{cases} k\epsilon^p + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N > p^2, \\ k\epsilon^p|\ln\epsilon| + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N = p^2, \\ O(\epsilon^(N-p)/(p-1)), & \text{if } N < p^2. \end{cases}
\end{aligned} \tag{2.20}$$

By (f_2), we have $F(s) \ge CG(s)^\mu$ for all $s > 0$. Therefore

$$\int_{B_{2R}(x_0)} F(G^{-1}(t_\epsilon\sigma_\epsilon))dx \ge C\int_{B_{2R}(x_0)}(t_\epsilon\sigma_\epsilon)^\mu dx \ge CA_1^\mu\int_{B_R(x_0)}(\sigma_\epsilon)^\mu dx.$$

It follows from (2.20), the above inequality and the definition of σ_ϵ that

$$
\begin{aligned}
\sup_{t\geq 0} \mathcal{J}(t\sigma_\epsilon) \leq{}& \frac{1}{N} S^{N/p} \left[\|Q\|_{L^\infty(\mathbb{R}^N)} \right]^{-\frac{N-p}{p}} - CA_1^\mu \int_{B_R(x_0)} (\sigma_\epsilon)^\mu dx + O(\epsilon^p) \\
& + O(\epsilon^{(N-p)/(p-1)}) + \begin{cases} k\epsilon^p + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N > p^2, \\ k\epsilon^p |\ln \epsilon| + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N = p^2, \\ O(\epsilon^(N-p)/(p-1)), & \text{if } N < p^2, \end{cases} \\
\leq{}& \frac{1}{N} S^{N/p} \left[\|Q\|_{L^\infty(\mathbb{R}^N)} \right]^{-\frac{N-p}{p}} - C\epsilon^{N-\frac{N-p}{p}\mu} \int_0^{\frac{R}{\epsilon}} \frac{r^{N-1}}{(1+r^{p/(p-1)})^{\frac{\mu(N-p)}{p}}} dr \\
& + O(\epsilon^p) + O(\epsilon^{(N-p)/(p-1)}) + \begin{cases} k\epsilon^p + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N > p^2, \\ k\epsilon^p |\ln \epsilon| + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N = p^2, \\ O(\epsilon^(N-p)/(p-1)), & \text{if } N < p^2. \end{cases}
\end{aligned} \tag{2.21}
$$

For $N \geq p^2$ and $\mu \in (p, p^*)$, there exists a constant $C > 0$ such that

$$
\int_0^\infty \frac{r^{N-1}}{(1+r^p/(p-1))^{\frac{\mu(N-p)}{p}}} dr \geq C > 0.
$$

If $N \geq p^2$ and $\mu \in (p, p^*)$, then we have

$$
N - \frac{N-p}{p}\mu < p \leq N - p. \tag{2.22}
$$

Combined (2.21) with (2.22), when $\epsilon \to 0$, we have

$$
\sup_{t\geq 0} \mathcal{J}(t\sigma_\epsilon) < \frac{1}{N} S^{N/p} \left[\|Q\|_{L^\infty(\mathbb{R}^N)} \right]^{-\frac{N-p}{p}}. \tag{2.23}
$$

If $p < N < p^2$ and $\mu \in (p^* - p/(p-1), p^*)$, then we know that (2.23) also holds. Then we can get the following inequality

$$
N - \frac{N-p}{p}\mu < N - p < p.
$$

Hence inequality (2.23) also follows from (2.21) if we choose ϵ small enough. Thus we can imply that the inequality (2.16) holds by taking $u_0 = \sigma_\epsilon$ for sufficiently small ϵ. □

Next, we will prove the main results in this paper.

Proof of Theorem 1.2. By Lemma 2.2 and Lemma 2.3, all conditions of Mountain Pass Lemma in [3] are satisfied. Let $\{v_n\}$ be a $(PS)_{c_0}$ sequence of $\mathcal{J}$. Then

$$
\begin{aligned}
\mathcal{J}(v_n) ={}& \frac{1}{p} \int_{\mathbb{R}^N} [|\nabla v_n|^p + V(x)|G^{-1}(v_n)|^p] dx - \int_{\mathbb{R}^N} K(x) F(G^{-1}(v_n)) dx \\
& - \frac{1}{p^*} \int_{\mathbb{R}^N} Q(x) |v_n^+|^{p^*} dx = c_0 + o_n(1)
\end{aligned} \tag{2.24}
$$

and

$$
\begin{aligned}
\langle \mathcal{J}'(v_n), v_n \rangle ={}& \int_{\mathbb{R}^N} \left[|\nabla v_n|^p + V(x) \frac{|G^{-1}(v_n)|^{p-2} G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n \right] - \int_{\mathbb{R}^N} K(x) \frac{f(G^{-1}(v))}{g(G^{-1}(v_n))} v_n \\
& - \int_{\mathbb{R}^N} Q(x) |v_n^+|^{p^*-2} v_n^+ v_n dx = o_n(1) \|v_n\|.
\end{aligned}
$$

From Lemma 2.4, we know that $\{v_n\}$ is bounded in E. Passing to sequence, there exists a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{in } E \\ &v_n \to v \quad \text{in } L^q_K(\mathbb{R}^N) \text{ for } p<q<p^*, \\ &v_n \to v \quad a.e. \text{ in } \mathbb{R}^N. \end{aligned} \tag{2.25}$$

Let

$$\tilde{f}(x,v) = \frac{f(G^{-1}(v))}{g(G^{-1}(v))} + \frac{V(x)}{K(x)}|v|^{p-2}v - \frac{V(x)}{K(x)}\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))},$$

and

$$\tilde{F}(x,v) = \int_0^v \tilde{f}(x,v)dx = F(G^{-1}(v)) + \frac{1}{p}\frac{V(x)}{K(x)}|v|^p - \frac{1}{p}\frac{V(x)}{K(x)}|G^{-1}(v)|^p,$$

then

$$\mathcal{J}(v) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p + V(x)|v|^p)dx - \int_{\mathbb{R}^N} K(x)\tilde{F}(x,v)dx - \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|v^+|^{p^*}dx.$$

Similar to [43], we can verify that

$$\lim_{s\to 0}\frac{\tilde{F}(x,s)}{|s|^p} = 0, \quad \lim_{s\to\infty}\frac{\tilde{F}(x,s)}{|s|^{p^*}} = 0, \quad \lim_{s\to 0}\frac{\tilde{f}(x,s)}{|s|^{p-1}} = 0, \quad \lim_{s\to\infty}\frac{\tilde{f}(x,s)}{|s|^{p^*-1}} = 0. \tag{2.26}$$

By Corollary A.3, we can get

$$\begin{aligned} \lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\tilde{F}(x,G^{-1}(v_n)) &= \int_{\mathbb{R}^N} K(x)\tilde{F}(x,G^{-1}(v)), \\ \lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\frac{\tilde{f}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n &= \int_{\mathbb{R}^N} K(x)\frac{\tilde{f}(x,G^{-1}(v))}{g(G^{-1}(v))}v. \end{aligned} \tag{2.27}$$

Since $\mathcal{J}'(v_n) \to 0$, by (2.27), we can get

$$\int_{\mathbb{R}^N}(|\nabla v|^p + V(x)|v|^p)dx - \int_{\mathbb{R}^N} K(x)\tilde{f}(x,v)vdx - \int_{\mathbb{R}^N} Q(x)|v^+|^{p^*}dx = 0.$$

Denote $\vartheta_n = v_n - v$, then by (2.6) and the Brézis–Lieb Lemma in [7], we have

$$\mathcal{J}(v) + \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla\vartheta_n|^p + V(x)|\vartheta_n|^p)dx - \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|\vartheta_n|^{p^*}dx = c_0 + o(1) \tag{2.28}$$

and

$$\int_{\mathbb{R}^N}(|\nabla\vartheta_n|^p + V(x)|\vartheta_n|^p)dx - \int_{\mathbb{R}^N} Q(x)|\vartheta_n|^{p^*}dx = o(1).$$

Without loss of generality we can suppose

$$\int_{\mathbb{R}^N}(|\nabla\vartheta_n|^p + V(x)|\vartheta_n|^p)dx \to l \quad \text{as } n\to\infty \tag{2.29}$$

and then we have

$$\int_{\mathbb{R}^N} Q(x)|\vartheta_n|^{p^*}dx \to l, \qquad n\to\infty. \tag{2.30}$$

Moreover, by Sobolev's inequality, we know that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \vartheta_n|^p dx &\geq S \left(\int_{\mathbb{R}^N} |\vartheta_n|^{p^*} dx \right)^{p/p^*} \\ &\geq S \left[\|Q\|_{L^\infty(\mathbb{R}^N)} \right]^{-p/p^*} \left(\int_{\mathbb{R}^N} Q(x) |\vartheta_n|^{p^*} dx \right)^{p/p^*}. \end{aligned} \tag{2.31}$$

Using (2.29), (2.30), (2.31), if $l > 0$, then we have

$$l \geq S^{N/p} \left[\|Q\|_{L^\infty(\mathbb{R}^N)} \right]^{\frac{p-N}{p}}.$$

By (2.28), we have

$$\mathcal{J}(v) = \left(c_0 - \frac{1}{p} - \frac{1}{p^*} \right) l \leq c_0 - \frac{1}{N} S^{N/p} \left[\|Q\|_{L^\infty(\mathbb{R}^N)} \right]^{\frac{p-N}{p}} < 0.$$

On the other hand, by (f_2), we have

$$\begin{aligned} \mathcal{J}(v) &= \frac{1}{p} \int_{\mathbb{R}^N} [|\nabla v|^p + V(x) |G^{-1}(v)|^p] dx - \int_{\mathbb{R}^N} K(x) F(G^{-1}(v)) dx \\ &\quad - \frac{1}{p^*} \int_{\mathbb{R}^N} Q(x) |v|^{p^*} dx \\ &= \frac{1}{p} \int_{\mathbb{R}^N} V(x) |G^{-1}(v)|^{p-2} \left[|G^{-1}(v)|^2 - \frac{G^{-1}(v) v}{g(G^{-1}(v))} \right] dx + \left(\frac{1}{p} - \frac{1}{p^*} \right) \int_{\mathbb{R}^N} Q(x) |v|^{p^*} dx \\ &\quad - \int_{\mathbb{R}^N} K(x) \left[F(G^{-1}(v)) - \frac{f(G^{-1}(v))}{g(G^{-1}(v))} v \right] dx \\ &\geq 0, \end{aligned}$$

which is a contradiction. It shows that $l = 0$. By the definition of ϑ_n we conclude that $\mathcal{J}$ satisfies $(PS)_{c_0}$ condition and thus

$$\mathcal{J}(v) = c_0 > 0 \quad \text{and} \quad \mathcal{J}'(v) = 0.$$

which gives that $u = G^{-1}(v)$ is a positive solution of (1.1). This completes the proof. □

Appendix A

In this part, we want to give some very useful lemmas.

Lemma A.1 ([17, Lemma 2.3]). *Suppose that (V)–(K_2) hold, and $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function, which satisfies the following conditions:*

(h_1) *h has a quasicritical growth, that is,* $\lim_{|s| \to +\infty} \frac{h(x,s)}{|s|^{p^*-1}} = 0$;

(h_2) *if (1.5) holds, then h satisfies* $\lim_{s \to 0} \frac{h(x,s)}{|s|^p} = 0$.

If a sequence $\{v_n\}$ converges weakly to v in E, then

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} KH(x, v_n) = \int_{\mathbb{R}^N} KH(x, v),$$
$$\lim_{n \to \infty} \int_{\mathbb{R}^N} Kh(x, v_n) v_n = \int_{\mathbb{R}^N} Kh(x, v) v,$$

where $H(x,s) = \int_0^s h(x,t) dt$ *for all* $s \in \mathbb{R}$.

Lemma A.2. *Under the assumptions of Lemma A.1, if $v_n \rightharpoonup v$ in E, then for each $\phi \in E$ it holds that*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K\left[h(x,v_n) - h(x,v)\right]\phi dx = 0. \tag{A.1}$$

Proof. Motivated by [1,31,46], since $v_n \rightharpoonup v$ in E and $E \hookrightarrow L^{p^*}(\mathbb{R}^N)$, then there exists $M > 0$ such that

$$\|v_n\|, \|v\| \le M \qquad |v|_{p^*}^{p^*} \le M, \qquad n \in \mathbb{N}.$$

Now, we consider the case that (V)–(K_1), (1.5), (h_1) and (h_2) hold. it follows from (h_1) and (h_2) that for any $\varepsilon > 0$ and $q \in (p, p^*)$ there exists $C_\varepsilon > 0$ such that

$$h(x,s) \le \varepsilon(|s|^{p-1} + |s|^{p^*-1}) + C_\varepsilon |s|^{q-1}, \qquad s \in \mathbb{R}. \tag{A.2}$$

By (1.5), we have that

$$K(x)h(x,s) \le \varepsilon(|K/V|_\infty V(x)|s|^p + |K|_\infty |s|^{p^*}) + C_\varepsilon K(x)|s|^{q-1}, \qquad x \in \mathbb{R}^N \text{ and } s \in \mathbb{R}. \tag{A.3}$$

According to Proposition 1.1, it holds that $\int_{\mathbb{R}^N} K|v_n|^q \to \int_{\mathbb{R}^N} K|v|^q$ as $n \to \infty$. Then there exists $R = R_\varepsilon$ large enough such that

$$\int_{B_R^c} K|v_n|^q, \int_{B_R^c} K|v|^q \le \left(\frac{\varepsilon}{C_\varepsilon}\right)^{q/(q-1)}, \qquad n \in \mathbb{N}. \tag{A.4}$$

where $B_R^c = \{x \in \mathbb{R}^N : |x| \ge R\}$. Hence, we can derive from (A.3), the Hölder inequality, (A.2) and (A.4) that

$$\begin{aligned}\int_{B_R^c} K|h(x,v_n)\phi| &\le \int_{B_R^c} \varepsilon(|K/V|_\infty V(x)|v_n|^{p-1} + |K|_\infty |v_n|^{p^*-1}) + C_\varepsilon \int_{B_R^c} K(x)|v_n|^{q-1}|\phi| \\ &\le \varepsilon\left[|K/V|_\infty \|v_n\|\|\phi\| + |K|_\infty |v_n|_{p^*}^{p^*-1}|\phi|_{p^*}\right] + C_\varepsilon \left(\int_{B_R^c} K(x)|v_n|^q\right)^{(q-1)/q} |\phi|_{L_K^q} \\ &\le C\varepsilon. \end{aligned} \tag{A.5}$$

where C is independent of ε. Similarly, it holds that for some constant C_2 independent of ε,

$$\int_{B_R^c} Kh(x,v)\phi \le C\varepsilon. \tag{A.6}$$

Next, we only need to prove that

$$\lim_{n\to\infty} \int_{B_R} Kh(x,v_n)\phi = \int_{B_R} Kh(x,v)\phi. \tag{A.7}$$

In fact, since $v_n \rightharpoonup v$ in E, then exists a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that $v_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^N$. Thus $h(x,v_n) \to h(x,v)$ for a.e. $x \in \mathbb{R}^N$. Moreover, it follows from (A.3) that $\{h(x,v_n)\}$ is bounded in $L^{p^*/(p^*-p)}(B_R)$. Hence $h(x,v_n) \rightharpoonup h(v)$ in $L^{p^*/(p^*-p)}(B_R)$ as $n \to \infty$, and (A.7) holds as a consequence of the fact that $K\phi \in L^{p^*}(\mathbb{R}^N)$. Thus we can get that

$$\lim_{n\to\infty} \int_{B_R} Kh(x,v_n)\phi = \int_{B_R} Kh(x,v)\phi.$$

Combining (A.5), (A.6) with (A.7), (A.1) holds. This completes the proof. □

Corollary A.3. *Under the assumptions of Lemma A.1, if $v_n \rightharpoonup v$ in E, then it holds that*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K\tilde{F}(x, G^{-1}(v_n)) = \int_{\mathbb{R}^N} K\tilde{F}(x, G^{-1}(v)), \tag{A.8}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K\frac{\tilde{f}(x, G^{-1}(v_n))}{g(G^{-1}(v_n))} v_n = \int_{\mathbb{R}^N} K\frac{\tilde{f}(x, G^{-1}(v))}{g(G^{-1}(v))} v, \tag{A.9}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K\frac{\tilde{f}(x, G^{-1}(v_n))}{g(G^{-1}(v_n))} \phi = \int_{\mathbb{R}^N} K\frac{\tilde{f}(x, G^{-1}(v))}{g(G^{-1}(v))} \phi, \qquad \phi \in E. \tag{A.10}$$

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Sharp results for oscillation of second-order neutral delay differential equations

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Abstract. The aim of the present paper is to continue earlier works by the authors on the oscillation problem of second-order half-linear neutral delay differential equations. By revising the set method, we present new oscillation criteria which essentially improve a number of related ones from the literature. A couple of examples illustrate the value of the results obtained.

Keywords: half-linear neutral differential equation, delay, second-order, oscillation.

2020 Mathematics Subject Classification: 34C10, 34K11.

1 Introduction

In the paper, we consider the second-order half-linear neutral delay differential equation

$$\left(r\left(z'\right)^{\alpha}\right)'(t) + q(t)x^{\alpha}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$. As in [10], we will assume

(H$_1$) $\alpha > 0$ is a quotient of odd positive integers;

(H$_2$) $r \in \mathcal{C}([t_0,\infty),(0,\infty))$ satisfies

$$\pi(t_0) := \int_{t_0}^{\infty} r^{-1/\alpha}(s)\mathrm{d}s < \infty;$$

(H$_3$) σ, $\tau \in \mathcal{C}([t_0,\infty),\mathbb{R})$, $\sigma(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$;

(H$_4$) $p \in \mathcal{C}([t_0,\infty),[0,\infty))$ and $q \in \mathcal{C}([t_0,\infty),(0,\infty))$;

(H$_5$) there exists a constant $p_0 \in [0,1)$ such that

$$p_0 \geq \begin{cases} p(t)\frac{\pi(\tau(t))}{\pi(t)} & \text{for } \tau(t) \leq t \\ p(t) & \text{for } \tau(t) \geq t. \end{cases} \tag{1.2}$$

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Under a solution of (1.1), we mean a function $x \in \mathcal{C}([t_a,\infty),\mathbb{R})$ with $t_a = \min\{\tau(t_b),\sigma(t_b)\}$, for some $t_b \geq t_0$, which has the property $z \in \mathcal{C}^1([t_a,\infty),\mathbb{R})$, $r\,(z')^\alpha \in \mathcal{C}^1([t_a,\infty),\mathbb{R})$ and satisfies (1.1) on $[t_b,\infty)$. We only consider those solutions of (1.1) which exist on some half-line $[t_b,\infty)$ and satisfy the condition $\sup\{|x(t)| : t_c \leq t < \infty\} > 0$ for any $t_c \geq t_b$. Oscillation and nonoscillation of such solutions is defined in the usual way.

Oscillation theory of second-order differential equations has gained much research interest in the past decades, and we refer the reader to the monographs by Agarwal et al. [1,3,4], Berezansky et al. [7], and Saker [33] for recent developments and summaries of known results. Due to the importance of second-order neutral differential equations in the modeling of various phenomena in natural sciences and engineering [12,18,33], the qualitative behavior of solutions such equations has been intensively studied through different techniques.

This paper is the second continuation of our earlier work [9] from 2017, followed by [10] in 2020. To start with, let us summarize briefly the two main ideas employed therein. Let x be a nonoscillatory, say positive solution of (1.1) subject to (H_1)–(H_5). Then z is also positive and either strictly increasing or strictly decreasing. These two possible classes of nonoscillatory solutions were treated independently in the literature, see, e.g., [2,5,19,22–24,26,36–39]. In [9], we pointed out that conditions eliminating positive solutions x with z decreasing are sufficient for the nonexistence of those with z increasing. This observation allowed us to remove a redundant but commonly imposed condition and formulate, in contrast with existing works, single-condition oscillation criteria.

To eliminate the important class of positive solutions with z decreasing, the second main idea in [9] was to sharpen the lower bound 1 of the quantity $z(\sigma(t))/z(t)$ using equation (1.1) itself, which, within the Riccati transformation technique, led to qualitatively stronger results. However, such a lower bound strongly depended on properties of first-order delay differential equations and required σ to be nondecreasing.

The ideas from [9] have been extended and applied in investigation of various classes of equations, e.g., half-linear neutral differential equations with: damping term [28,35], sublinear term [13,15,34], several delay arguments [30]; generalized Emden–Fowler neutral differential equations [25,27,32], half-linear neutral difference equations [8,11,16], neutral dynamic equations on time scales [17,31,40,41], and others.

In [10], we continued our work [9] by removing the restrictions (see [9, (H_3)]) $\tau(t) \leq t$ and $\sigma'(t) \geq 0$. For the reader's convenience, we recall the main results from [10], formulated in terms of the following couple of limit inferiors:

$$\beta_* := \frac{1}{\alpha} \liminf_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t) \quad \text{and} \quad \lambda_* := \liminf_{t\to\infty} \frac{\pi(\sigma(t))}{\pi(t)}. \tag{1.3}$$

Theorem A (See [10, Theorem 1, Theorem 2])**.** *If*

$$\beta_* > \begin{cases} 0 & \text{for } \lambda_* = \infty, \\ \dfrac{\max\{b^\alpha(1-b)\lambda_*^{-\alpha b} : 0 < b < 1\}}{(1-p_0)^\alpha} & \text{for } \lambda_* < \infty, \end{cases}$$

then (1.1) *is oscillatory.*

Although the obtained results can be seen as sharp in the sense that they are unimprovable in a nonneutral case, it is easy to observe that Theorem A does not take the influence of $\tau(t) \geq t$ into account and becomes inefficient as p_0 is close to 1. The aim of this paper is to address these issues and to improve Theorem A when $\lambda_* < \infty$ and $p(t) \not\equiv 0$. As in [10], we employ a recent method of sequentially improved monotonicities of nonoscillatory solutions of binomial differential equations, which has been successfully applied in the investigation of second-order half-linear functional differential equations and as well as linear differential and difference equations of higher order. For a discussion on the results already achieved by the method so far, we refer the reader to [21, Section 4].

For the sake of completeness, let us recall the three main steps of the method we used in [10]: firstly, we showed that the positivity of β_* is sufficient for the nonexistence of positive solutions x with z positive and increasing; secondly, we provided, for x positive with z decreasing, bounds of the ratio x/z, i.e.,

$$1 - p_0 \leq \frac{x}{z} \leq 1. \tag{1.4}$$

The third step was intended to improve the lower bound 1 of the quantity $z(\sigma(t))/z(t)$ so that it was, unlike the one we used in [9], independent of the properties of first-order delay differential equations and the monotone growth of σ. We related this problem to that of finding an optimal value $a > 0$ such that

$$a \leq \frac{-r^{1/\alpha} z' \pi}{z},$$

which corresponds to the monotonicity

$$\left(\frac{z}{\pi^a}\right)' < 0,$$

and tackled it by building an appropriate sequence defined in terms of β_* and λ_*. It turned out that the convergence of the given sequence was necessary for the existence of a nonoscillatory solution of (1.1), and Theorem A emerged as a simple consequence of this fact.

In this work, we revise the set method as follows. Firstly, we provide a sharper lower bound of the quantity x/z than in (1.4). Secondly, we sequentially improve both lower and upper bounds of the ratio $-\pi r^{1/\alpha} z'/z$ up to their limit values by building two iteration processes represented by the sequences $\{\beta_{k,n}\}_{n \in \mathbb{N}_0}$ and $\{\gamma_{k,n}\}_{n \in \mathbb{N}_0}$ (see Section 2) such that

$$\beta_{k,n} \leq \frac{-r^{1/\alpha} z' \pi}{z} \leq 1 - \gamma_{k,n},$$

which correspond to the monotonicities

$$\left(\frac{z}{\pi^{\beta_{k,n}}}\right)' < 0 \quad \text{and} \quad \left(\frac{z}{\pi^{1-\gamma_{k,n}}}\right)' \geq 0,$$

allowing us to improve the lower bound of x/z in each iteration step. Finally, we state the main results – sufficient conditions for (1.1) to be oscillatory – as a direct consequence of these obtained bounds. To illustrate the applicability of the results, two examples are given.

2 Notation and preliminary results

In this section, we list all constants and functions used in the paper. For any $k \in \mathbb{N}_0$, we set

$$\beta_k^* := \frac{1}{\alpha} \liminf_{t \to \infty} r^{1/\alpha}(t) \pi^{\alpha+1}(t) q(t) \left(1 + H_k(\sigma(t))\right)^\alpha, \tag{2.1}$$

where

$$H_k(t) = \begin{cases} 0 & \text{for } k = 0, \\ \sum_{i=1}^{k} \prod_{j=0}^{2i-1} p(\tau^j(t)) & \text{for } \tau(t) \leq t \text{ and } k \in \mathbb{N}, \\ \sum_{i=1}^{k} \frac{\pi(\tau^{2i}(t))}{\pi(t)} \prod_{j=0}^{2i-1} p(\tau^j(t)) & \text{for } \tau(t) \geq t \text{ and } k \in \mathbb{N}, \end{cases}$$

where $\tau^0(t) = t$ and $\tau^j(t) = \tau(\tau^{j-1}(t))$ for all $j \in \mathbb{N}$. As in [10], we set

$$\lambda_* := \liminf_{t\to\infty} \frac{\pi(\sigma(t))}{\pi(t)}$$

and, in addition, we put

$$\psi_* := \liminf_{t\to\infty} \frac{\pi(\tau(t))}{\pi(t)} \quad \text{for } \tau(t) \leq t,$$
$$\omega_* := \liminf_{t\to\infty} \frac{\pi(t)}{\pi(\tau(t))} \quad \text{for } \tau(t) \geq t.$$

By virtue of (H_2) and (H_3), it is immediate to see that $\{\lambda_*, \omega_*, \psi_*\} \in [1, \infty)$. Our reasoning will often rely on the obvious fact that there is a $t_1 \geq t_0$ large enough such that, for arbitrary fixed $\beta_k \in (0, \beta_k^*)$, $\lambda \in [1, \lambda_*)$, $\psi \in [1, \psi_*)$, and $\omega \in [1, \omega_*)$, we have

$$\begin{aligned} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t)\left(1 + H_k(\sigma(t))\right)^\alpha &\geq \alpha\beta_k, \\ \frac{\pi(\sigma(t))}{\pi(t)} &\geq \lambda, \\ \frac{\pi(\tau(t))}{\pi(t)} &\geq \psi \quad \text{for } \tau(t) \leq t, \\ \frac{\pi(t)}{\pi(\tau(t))} &\geq \omega \quad \text{for } \tau(t) \geq t, \end{aligned} \tag{2.2}$$

on $[t_1, \infty)$.

Remark 2.1. In our previous work [10], we formulated the results in terms of $\beta_0^* = \beta_*$ (see (1.3)), which we required to be positive. Clearly, for any $k \in \mathbb{N}$, the positivity of β_0^* is sufficient for that of β_k^*.

Lemma 2.2. *If $\tau(t) \leq t$ and $\psi_* = \infty$, or $\tau(t) \geq t$ and $\omega_* = \infty$, then*

$$\liminf_{t\to\infty} H_k(t) = 0 \quad \textit{for any } k \in \mathbb{N}$$

and so $\beta_k^* = \beta_0^*$ for any $k \in \mathbb{N}$.

Proof. Using (H_2) and (H_5), the proof is obvious and hence omitted. □

The method used in this paper is based on the properties of the sequences $\{\beta_{k,n}\}_{n\in\mathbb{N}_0}$ and $\{\gamma_{k,n}\}_{n\in\mathbb{N}_0}$, which we define (as long as they exist) as follows. For positive and finite β_k^*, λ_*, ψ_*, and ω_*, we set, for any $k \in \mathbb{N}_0$ fixed,

$$\beta_{k,0} := (1 - p_0)\sqrt[\alpha]{\beta_k^*},$$
$$\gamma_{k,0} := (1 - p_0)^\alpha \beta_k^* = \beta_{k,0}^\alpha,$$

and for $n \in \mathbb{N}_0$, we put

1. for $\tau(t) \le t$ and $\psi_* = \infty$ or $\tau(t) \ge t$ and $\omega_* = \infty$:

$$\beta_{k,n+1} := \lambda_*^{\beta_{k,n}} \sqrt[\alpha]{\frac{\beta_k^*}{1-\beta_{k,n}}} = \lambda_*^{\beta_{0,n}} \sqrt[\alpha]{\frac{\beta_0^*}{1-\beta_{0,n}}},$$
$$\gamma_{k,n+1} := \beta_k^* \left(\frac{\lambda_*^{\beta_{k,n}}}{1-\gamma_{k,n}}\right)^{\alpha} = \beta_0^* \left(\frac{\lambda_*^{\beta_{0,n}}}{1-\gamma_{0,n}}\right)^{\alpha}$$

2. for $\tau(t) \le t$ and $\psi_* < \infty$:

$$\beta_{k,n+1} := \frac{\beta_{k,0}\lambda_*^{\beta_{k,n}}}{\sqrt[\alpha]{1-\beta_{k,n}}} \left(\frac{1-p_0\psi_*^{-\gamma_{k,n}}}{1-p_0}\right) = \lambda_*^{\beta_{k,n}} \sqrt[\alpha]{\frac{\beta_k^*}{1-\beta_{k,n}}}(1-p_0\psi_*^{-\gamma_{k,n}}),$$
$$\gamma_{k,n+1} := \frac{\gamma_{k,0}\lambda_*^{\alpha\beta_{k,n}}}{(1-\gamma_{k,n})^{\alpha}} \left(\frac{1-p_0\psi_*^{-\gamma_{k,n}}}{1-p_0}\right)^{\alpha} = \beta_k^* \left(\frac{\lambda_*^{\beta_{k,n}}}{1-\gamma_{k,n}}\right)^{\alpha} (1-p_0\psi_*^{-\gamma_{k,n}})^{\alpha}$$

3. for $\tau(t) \ge t$ and $\omega_* < \infty$:

$$\beta_{k,n+1} := \frac{\beta_{k,0}\lambda_*^{\beta_{k,n}}}{\sqrt[\alpha]{1-\beta_{k,n}}} \left(\frac{1-p_0\omega_*^{-\beta_{k,n}}}{1-p_0}\right) = \lambda_*^{\beta_{k,n}} \sqrt[\alpha]{\frac{\beta_k^*}{1-\beta_{k,n}}}(1-p_0\omega_*^{-\beta_{k,n}}),$$
$$\gamma_{k,n+1} := \frac{\gamma_{k,0}\lambda_*^{\alpha\beta_{k,n}}}{(1-\gamma_{k,n})^{\alpha}} \left(\frac{1-p_0\omega_*^{-\beta_{k,n}}}{1-p_0}\right)^{\alpha} = \beta_k^* \left(\frac{\lambda_*^{\beta_{k,n}}}{1-\gamma_{k,n}}\right)^{\alpha} (1-p_0\omega_*^{-\beta_{k,n}})^{\alpha}.$$

It can be easily verified by induction that if for some $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ fixed, $\beta_{k,i} < 1$ and $\gamma_{k,i} < 1$, $i = 0, 1, \ldots, n$, then $\beta_{k,n+1}$ and $\gamma_{k,n+1}$ exist and

$$\begin{aligned} \beta_{k,n+1} &= \ell_{k,n}\beta_{k,n} > \beta_{k,n}, \\ \gamma_{k,n+1} &= h_{k,n}\gamma_{k,n} > \gamma_{k,n}, \end{aligned} \tag{2.3}$$

where $\ell_{k,n}$ and $h_{k,n}$ are defined as follows:

1. for $\tau(t) \le t$ and $\psi_* = \infty$ or $\tau(t) \ge t$ and $\omega_* = \infty$:

$$\ell_{k,0} := \frac{\lambda_*^{\beta_{k,0}}}{(1-p_0)\sqrt[\alpha]{1-\beta_{k,0}}},$$
$$\ell_{k,n+1} := \lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \sqrt[\alpha]{\frac{1-\beta_{k,n}}{1-\ell_{k,n}\beta_{k,n}}}$$

and

$$h_{k,0} := \left[\frac{\lambda_*^{\beta_{k,0}}}{(1-\gamma_{k,0})(1-p_0)}\right]^{\alpha},$$
$$h_{k,n+1} := \left[\lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \left(\frac{1-\gamma_{k,n}}{1-h_{k,n}\gamma_{k,n}}\right)\right]^{\alpha}$$

2. for $\tau(t) \le t$ and $\psi_* < \infty$:

$$\ell_{k,0} := \frac{\lambda_*^{\beta_{k,0}}}{\sqrt[\alpha]{1-\beta_{k,0}}} \left(\frac{1-p_0\psi_*^{-\gamma_{k,0}}}{1-p_0}\right),$$
$$\ell_{k,n+1} := \lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \sqrt[\alpha]{\frac{1-\beta_{k,n}}{1-\ell_{k,n}\beta_{k,n}}} \left(\frac{1-p_0\psi_*^{-h_{k,n}\gamma_{k,n}}}{1-p_0\psi_*^{-\gamma_{k,n}}}\right)$$

and

$$h_{k,0} := \left[\frac{\lambda_*^{\beta_{k,0}}}{1-\gamma_{k,0}} \left(\frac{1-p_0\psi_*^{-\gamma_{k,0}}}{1-p_0} \right) \right]^{\alpha},$$

$$h_{k,n+1} := \left[\lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \left(\frac{1-\gamma_{k,n}}{1-h_{k,n}\gamma_{k,n}} \right) \left(\frac{1-p_0\psi_*^{-h_{k,n}\gamma_{k,n}}}{1-p_0\psi_*^{-\gamma_{k,n}}} \right) \right]^{\alpha}$$

3. for $\tau(t) \geq t$ and $\omega_* < \infty$:

$$\ell_{k,0} := \frac{\lambda_*^{\beta_{k,0}}}{\sqrt[\alpha]{1-\beta_{k,0}}} \left(\frac{1-p_0\omega_*^{-\beta_{k,0}}}{1-p_0} \right),$$

$$\ell_{k,n+1} := \lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \sqrt[\alpha]{\frac{1-\beta_{k,n}}{1-\ell_{k,n}\beta_{k,n}}} \left(\frac{1-p_0\omega_*^{-\ell_{k,n}\beta_{k,n}}}{1-p_0\omega_*^{-\beta_{k,n}}} \right)$$

and

$$h_{k,0} := \left[\frac{\lambda_*^{\beta_{k,0}}}{1-\gamma_{k,0}} \left(\frac{1-p_0\omega_*^{-\beta_{k,0}}}{1-p_0} \right) \right]^{\alpha},$$

$$h_{k,n+1} := \left[\lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \left(\frac{1-\gamma_{k,n}}{1-h_{k,n}\gamma_{k,n}} \right) \left(\frac{1-p_0\omega_*^{-\ell_{k,n}\beta_{k,n}}}{1-p_0\omega_*^{-\beta_{k,n}}} \right) \right]^{\alpha}.$$

The following simple statement, resulting from the definition of the sequences $\{\beta_{k,n}\}_{n\in\mathbb{N}_0}$ and $\{\gamma_{k,n}\}_{n\in\mathbb{N}_0}$ and (2.3), will play an important role in obtaining our main results. As a matter of fact, we will show (see Corollary 3.8) that all assumptions of Lemma 2.3 are necessary for the existence of a nonoscillatory solution of (1.1), i.e., if (1.1) possesses a nonoscillatory solution, then there exists a solution $\{b, g\} \in (0, 1)$ of a particular limit system.

Lemma 2.3. *Let $\beta_0^* > 0$, $\lambda_* < \infty$, and the sequences $\{\beta_{k,n}\}_{n\in\mathbb{N}_0}$ and $\{\gamma_{k,n}\}_{n\in\mathbb{N}_0}$ be well-defined and bounded from above for some fixed $k \in \mathbb{N}_0$. Then*

$$\lim_{n\to\infty} \beta_{k,n} = b \in (0, 1)$$

and

$$\lim_{n\to\infty} \gamma_{k,n} = g \in (0, 1),$$

where $\{b, g\}$ is a solution of the system

1. *for $\tau(t) \leq t$ and $\psi_* = \infty$ or $\tau(t) \geq t$ and $\omega_* = \infty$:*

$$\begin{cases} \beta_0^* = b^{\alpha}(1-b)\lambda_*^{-\alpha b} \\ \beta_0^* = g(1-g)^{\alpha}\lambda_*^{-\alpha b} \end{cases} \tag{2.4}$$

2. *for $\tau(t) \leq t$ and $\psi_* < \infty$:*

$$\begin{cases} \beta_k^* = \dfrac{b^{\alpha}(1-b)\lambda_*^{-\alpha b}}{\left(1-p_0\psi_*^{-g}\right)^{\alpha}} \\ \beta_k^* = \dfrac{g(1-g)^{\alpha}\lambda_*^{-\alpha b}}{\left(1-p_0\psi_*^{-g}\right)^{\alpha}} \end{cases} \tag{2.5}$$

3. *for* $\tau(t) \geq t$ *and* $\omega_* < \infty$:

$$\begin{cases} \beta_k^* = \dfrac{b^\alpha(1-b)\lambda_*^{-\alpha b}}{\left(1 - p_0\omega_*^{-b}\right)^\alpha} \\ \beta_k^* = \dfrac{g(1-g)^\alpha\lambda_*^{-\alpha b}}{\left(1 - p_0\omega_*^{-b}\right)^\alpha}. \end{cases} \tag{2.6}$$

3 Main results

In the sequel, all occurring functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough. As usual and without loss of generality, in the proofs of the main results, we only need to be concerned with positive solutions of (1.1) since the proofs for eventually negative solutions are similar.

We start by recalling an important result from our previous work.

Lemma 3.1 (See [10, Lemma 2])**.** *Let* $\beta_0^* > 0$. *If* x *is an eventually positive solution of* (1.1), *then* z *eventually satisfies*

(i) $z > 0$, $\left(r\left(z'\right)^\alpha\right)' < 0$, *and* $x(t) \geq z(t) - p(t)z(\tau(t))$;

(ii) $z' < 0$;

(iii) $(z/\pi)' \geq 0$;

(iv) $x \geq (1 - p_0)z$;

(v) $\lim_{t\to\infty} z(t) = 0$.

In order to improve the estimate (iv) between x and z, we need the following auxiliary result.

Lemma 3.2. *If* x *is an eventually positive solution of* (1.1), *then* z *eventually satisfies*

$$\begin{aligned} x(t) &\geq z(t) - p(t)z(\tau(t)) \\ &\quad + \sum_{i=1}^{k}\left(\prod_{j=0}^{2i-1} p(\tau^j(t))\right)\left[z(\tau^{2i}(t)) - p(\tau^{2i}(t))z(\tau^{2i+1}(t))\right], \quad k \in \mathbb{N}. \end{aligned} \tag{3.1}$$

Proof. It follows from the definition of z that

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) \\ &= z(t) - p(t)\left[z(\tau(t)) - p(\tau(t))x(\tau^2(t))\right] \\ &= z(t) - p(t)z(\tau(t)) + p(t)p(\tau(t))x(\tau^2(t)). \end{aligned} \tag{3.2}$$

Evaluating (3.2) in $\tau^2(t)$, we get

$$x(\tau^2(t)) = z(\tau^2(t)) - p(\tau^2(t))z(\tau^3(t)) + p(\tau^2(t))p(\tau^3(t))x(\tau^4(t)). \tag{3.3}$$

Now using (3.3) in (3.2), we have

$$\begin{aligned} x(t) &= z(t) - p(t)z(\tau(t)) \\ &\quad + p(t)p(\tau(t))\left[z(\tau^2(t)) - p(\tau^2(t))z(\tau^3(t))\right] \\ &\quad + p(t)p(\tau(t))p(\tau^2(t))p(\tau^3(t))x(\tau^4(t)). \end{aligned}$$

Repeating the process, it is easy to show via induction that

$$\begin{aligned} x(t) &= z(t) - p(t)z(\tau(t)) \\ &+ \sum_{i=1}^{k} \left(\prod_{j=0}^{2i-1} p(\tau^j(t)) \right) \left[z(\tau^{2i}(t)) - p(\tau^{2i}(t))z(\tau^{2i+1}(t)) \right] \\ &+ \left(\prod_{j=0}^{2k+1} p(\tau^j(t)) \right) x(\tau^{2k+2}(t)), \end{aligned}$$

which implies (3.1). The proof is complete. □

Lemma 3.3. *Let $\beta_0^* > 0$. If x is an eventually positive solution of* (1.1), *then z eventually satisfies*

$$x(t) \geq z(t)(1 - p_0)\left(1 + H_k(t)\right), \quad k \in \mathbb{N}_0. \tag{3.4}$$

Proof. First, let $\tau(t) \leq t$. Using the fact that z/π is nondecreasing (see Lemma 3.1 (iii)) and (H_5), we have

$$z(t) - p(t)z(\tau(t)) \geq z(t) - p(t)\frac{\pi(\tau(t))}{\pi(t)} z(t) \geq z(t)(1 - p_0). \tag{3.5}$$

Evaluating (3.5) in $\tau^{2i}(t)$ and using that z is decreasing (see Lemma 3.1 (ii)), we obtain

$$z(\tau^{2i}(t)) - p(\tau^{2i}(t))z(\tau^{2i+1}(t)) \geq z(\tau^{2i}(t))(1 - p_0) \geq z(t)(1 - p_0). \tag{3.6}$$

Using (3.5) and (3.6) in (3.1), we get

$$x(t) \geq z(t)(1 - p_0)\left[1 + \sum_{i=1}^{k} \prod_{j=0}^{2i-1} p(\tau^j(t))\right], \quad k \in \mathbb{N}.$$

and hence, (3.4) holds. Now, let $\tau(t) \geq t$. Again, by Lemma 3.1 (ii), (iii) and (H_5), we see that

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &\geq z(t) - p(t)z(t) \\ &\geq z(t)(1 - p_0) \end{aligned}$$

and

$$\begin{aligned} z(\tau^{2i}(t)) - p(\tau^{2i}(t))z(\tau^{2i+1}(t)) &\geq z(\tau^{2i}(t))\left(1 - p(\tau^{2i}(t))\right) \\ &\geq z(\tau^{2i}(t))(1 - p_0) \\ &\geq z(t)\frac{\pi(\tau^{2i}(t))}{\pi(t)}(1 - p_0), \end{aligned}$$

which in view of (3.1) yields

$$x(t) \geq z(t)(1 - p_0)\left[1 + \sum_{i=1}^{k} \frac{\pi(\tau^{2i}(t))}{\pi(t)} \prod_{j=0}^{2i-1} p(\tau^j(t))\right], \quad k \in \mathbb{N},$$

and hence, (3.4) holds in this case as well. The proof is complete. □

Remark 3.4. In [20], the authors investigated (1.1) with $p(t) \equiv p > 0$ and $\tau(t) < t$, and required, instead of (H_5), that

$$p_* = \sum_{k=0}^{(n-1)/2} p_0^{2k} \left(1 - p\frac{\pi(\tau^{2k+1})(t)}{\pi(\tau^{2k}(t))}\right) > 0, \quad n \in \mathbb{N}.$$

Then they proved that an eventually positive solution of (1.1) satisfies

$$x \geq (1 - p_*)z. \tag{3.7}$$

Note that (H_5) is sufficient for the positivity of p_* and consequently, (3.7) becomes a particular case of (3.4).

The next step of our approach lies in improving Lemma 3.1 (ii)–(iv) by using the equation (1.1) itself, which can be seen as an improved and extended variant of [10, Lemma 3]. While the improved decreasing monotonicity $(i)_0$ results from minor modification of the original proof, the opposite monotonicity $(ii)_0$, needed to sharpen the relation between x and z in $(iii)_0$, extends the original version of [10, Lemma 3].

Lemma 3.5. *Assume $\beta_0^* > 0$. If x is an eventually positive solution of (1.1), then, for any $\beta_k \in (0, \beta_k^*)$ with $k \in \mathbb{N}_0$ fixed,*

(i)$_0$ $(z/\pi^{\sqrt[\alpha]{\beta_k}(1-p_0)})' < 0$;

(ii)$_0$ $(z/\pi^{1-\beta_k(1-p_0)^\alpha})' \geq 0$;

(iii)$_0$ $x \geq a_k(1 + H_k)z$, *where*

$$a_k = \begin{cases} \varepsilon & \text{for } \tau(t) \leq t,\ \psi_* = \infty \text{ or } \tau(t) \geq t,\ \omega_* = \infty, \text{ and any } \varepsilon \in (0,1); \\ 1 - p_0\psi^{-\beta_k(1-p_0)^\alpha} & \text{for } \tau(t) \leq t, \psi_* < \infty, \text{ and any } \psi \in [1, \psi_*); \\ 1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)} & \text{for } \tau(t) \geq t, \omega_* < \infty, \text{ and any } \omega \in [1, \omega_*), \end{cases}$$

eventually.

Proof. Pick $t_1 \geq t_0$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad \text{and } x(\sigma(t)) > 0,$$

z satisfies Lemma 3.1 with (iv) replaced by (3.4), and (2.2) holds for $t \geq t_1$. Using (3.4) in (1.1), we have

$$\left(r\left(z'\right)^\alpha\right)'(t) + (1 - p_0)^\alpha q(t)(1 + H_k(\sigma(t)))^\alpha z^\alpha(\sigma(t)) \leq 0, \quad t \geq t_1,$$

which in view of (2.2) implies

$$\left(r\left(z'\right)^\alpha\right)'(t) + \frac{\beta_k\alpha(1 - p_0)^\alpha}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} z^\alpha(\sigma(t)) \leq 0. \tag{3.8}$$

Now using that z is decreasing (see Lemma 3.1 (ii)) and (H_3), we find

$$\frac{z(\sigma(t))}{z(t)} \geq 1. \tag{3.9}$$

Hence, (3.8) becomes

$$\left(r\left(z'\right)^\alpha\right)'(t) + \frac{\beta_k\alpha(1 - p_0)^\alpha}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} z^\alpha(t) \leq 0. \tag{3.10}$$

(i)$_0$ Integrating (3.10) from t_1 to t and using again Lemma 3.1 (ii), we find

$$\begin{aligned} -r(t)\left(z'(t)\right)^\alpha &\geq -r(t_1)\left(z'(t_1)\right)^\alpha + \beta_k(1-p_0)^\alpha \int_{t_1}^t \frac{\alpha z^\alpha(s)}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)}\mathrm{d}s \\ &\geq -r(t_1)\left(z'(t_1)\right)^\alpha + \beta_k(1-p_0)^\alpha z^\alpha(t) \int_{t_1}^t \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)}\mathrm{d}s \\ &= -r(t_1)\left(z'(t_1)\right)^\alpha + \beta_k(1-p_0)^\alpha z^\alpha(t)\left(\frac{1}{\pi^\alpha(t)} - \frac{1}{\pi^\alpha(t_1)}\right). \end{aligned} \tag{3.11}$$

Since $\lim_{t\to\infty} z(t) = 0$ (see Lemma 3.1 (v)), there exists $t_2 \geq t_1$ such that

$$-r(t_1)\left(z'(t_1)\right)^\alpha > \frac{\beta_k(1-p_0)^\alpha}{\pi^\alpha(t_1)} z^\alpha(t), \quad t \geq t_2.$$

Using this in (3.11) yields

$$-r^{1/\alpha} z' \pi > \sqrt[\alpha]{\beta_k}(1-p_0) z$$

and so (i)$_0$ holds.

(ii)$_0$ Set

$$Z := z + r^{1/\alpha} z' \pi. \tag{3.12}$$

Since z/π is nondecreasing (see Lemma 3.1 (iii)), Z is clearly nonnegative. Differentiating Z and using the chain rule

$$\left(r(z')^\alpha\right)' = \alpha\left(r^{1/\alpha}z'\right)^{\alpha-1}\left(r^{1/\alpha}z'\right)'$$

along with (3.10), we get

$$\begin{aligned} Z' &= \left(r^{1/\alpha}z'\right)'\pi \\ &= \frac{\pi}{\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha}\left(r\left(z'\right)^\alpha\right)' \\ &\leq -\frac{\pi}{\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha}\frac{\beta_k\alpha(1-p_0)^\alpha}{r^{1/\alpha}\pi^{\alpha+1}} z^\alpha \\ &= -\frac{\beta_k(1-p_0)^\alpha}{r^{1/\alpha}\pi^\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha} z^\alpha < 0. \end{aligned} \tag{3.13}$$

Using again Lemma 3.1 (iii) in (3.13), we obtain

$$Z' \leq -\frac{\beta_k(1-p_0)^\alpha}{r^{1/\alpha}\pi^\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha}(-r^{1/\alpha}z')^\alpha\pi^\alpha = \beta_k(1-p_0)^\alpha z'.$$

Integrating the above inequality from t to ∞ and using that z is decreasing and tending to zero eventually (see Lemma 3.1 (ii) and (v)), we have

$$Z(t) \geq Z(\infty) - \beta_k(1-p_0)^\alpha z(\infty) + \beta_k(1-p_0)^\alpha z(t) \geq \beta_k(1-p_0)^\alpha z(t),$$

which in view of the definition of Z gives

$$(1-\beta_k(1-p_0)^\alpha)z \geq -r^{1/\alpha}z'\pi.$$

Hence, (ii)$_0$ holds.

(iii)$_0$ First, let $\tau(t) \le t$. Using (ii)$_0$ and (H$_5$), we see that

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &\ge z(t) - p(t)\left(\frac{\pi(\tau(t))}{\pi(t)}\right)^{1-\beta_k(1-p_0)^\alpha} z(t) \\ &\ge z(t)\left(1 - p_0\left(\frac{\pi(t)}{\pi(\tau(t))}\right)^{\beta_k(1-p_0)^\alpha}\right) \\ &\ge z(t)\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right). \end{aligned} \tag{3.14}$$

Evaluating (3.14) in $\tau^{2i}(t)$ and using the decreasing nature of z (see Lemma 3.1 (ii)), we get

$$\begin{aligned} z(\tau^{2i}(t)) - p(\tau^{2i}(t))z(\tau^{2i+1}(t)) &\ge z(\tau^{2i}(t))\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right) \\ &\ge z(t)\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right). \end{aligned} \tag{3.15}$$

Using (3.14) and (3.15) in (3.1), we find

$$\begin{aligned} x(t) &\ge z(t)\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right)\left[1 + \sum_{i=1}^{k}\prod_{j=0}^{2i-1} p(\tau^j(t))\right] \\ &= z(t)\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right)(1 + H_k(t)). \end{aligned}$$

If $\tau(t) \ge t$, then similarly as before, we get

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &\ge z(t) - p(t)\left(\frac{\pi(\tau(t))}{\pi(t)}\right)^{\sqrt[\alpha]{\beta_k}(1-p_0)} z(t) \\ &\ge z(t)\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right), \end{aligned}$$

where we used (i)$_0$ and (H$_5$). Evaluating the above inequality in $\tau^{2i}(t)$ and using the nonincreasing nature of z/π (see Lemma 3.1 (iii)), we obtain

$$\begin{aligned} z(\tau^{2i}(t)) - p(\tau^{2i}(t))z(\tau^{2i+1}(t)) &\ge z(\tau^{2i}(t))\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right) \\ &\ge z(t)\frac{\pi(\tau^{2i}(t))}{\pi(t)}\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right). \end{aligned}$$

Then,

$$\begin{aligned} x(t) &\ge z(t)\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right)\left[1 + \sum_{i=1}^{k}\frac{\pi(\tau^{2i}(t))}{\pi(t)}\prod_{j=0}^{2i-1} p(\tau^j(t))\right] \\ &= z(t)\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right)(1 + H_k(t)). \end{aligned}$$

Finally, if $\tau(t) \le t$ and $\psi_* = \infty$ [$\tau(t) \ge t$ and $\omega_* = \infty$], then it follows from Lemma 2.2 that for any $\varepsilon \in (0,1)$, there is t sufficiently large such that

$$\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right)(1 + H_k(t)) < \varepsilon \quad \left[\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right)(1 + H_k(t)) < \varepsilon\right].$$

The proof is complete. □

The following result iteratively improves the previous one.

Lemma 3.6. *Assume $\beta_0^* > 0$. If x is an eventually positive solution of (1.1), then, for any k, $n \in \mathbb{N}_0$,*

(i)$_n$ $(z/\pi^{\beta_{k,n}})' < 0$;

(ii)$_n$ $(z/\pi^{1-\gamma_{k,n}})' > 0$;

(iii)$_n$ $x \geq a_{k,n}(1+H_k)z$, *where*

$$a_k = \begin{cases} \varepsilon & \text{for } \tau(t) \leq t,\ \psi_* = \infty \text{ or } \tau(t) \geq t,\ \omega_* = \infty, \text{ and any } \varepsilon \in (0,1); \\ 1 - p_0\psi^{-\gamma_{k,n}} & \text{for } \tau(t) \leq t,\ \psi_* < \infty, \text{ and any } \psi \in [1,\psi_*); \\ 1 - p_0\omega^{-\beta_{k,n}} & \text{for } \tau(t) \geq t,\ \omega_* < \infty, \text{ and any } \omega \in [1,\omega_*), \end{cases}$$

eventually.

Proof. Pick $t_1 \geq t_0$ large enough such that

$$x(t) > 0, \quad x(\sigma(t)) > 0, \quad \text{and } x(\tau(t)) > 0,$$

z satisfies Lemma 3.1 with (iv) replaced by (3.4), and (2.2) holds for $t \geq t_1$. The proof will proceed in two steps.

1. First, we are going to show via induction on n that for arbitrary ${}_\beta\varepsilon_{k,n} \in (0,1)$ and ${}_\gamma\varepsilon_{k,n} \in (0,1)$ one can set

$$\tilde{\beta}_{k,n} = {}_\beta\varepsilon_{k,n}\beta_{k,n}$$
$$\tilde{\gamma}_{k,n} = {}_\gamma\varepsilon_{k,n}\gamma_{k,n}$$

so that

(I)$_n$

$$\left(\frac{z}{\pi^{\tilde{\beta}_{k,n}}}\right)' < 0,$$

(II)$_n$

$$\left(\frac{z}{\pi^{1-\tilde{\gamma}_{k,n}}}\right)' \geq 0,$$

and

(III)$_n$

$$x \geq \tilde{a}_{k,n}(1+H_k)z,$$

where

$$\tilde{a}_{k,n} = \begin{cases} \varepsilon & \text{for } \tau(t) \leq t,\ \psi_* = \infty \text{ or } \tau(t) \geq t,\ \omega_* = \infty; \\ 1 - p_0\psi^{-\tilde{\gamma}_{k,n}} & \text{for } \tau(t) \leq t,\ \psi_* < \infty; \\ 1 - p_0\omega^{-\tilde{\beta}_{k,n}} & \text{for } \tau(t) \geq t,\ \omega_* < \infty. \end{cases}$$

For $n = 0$, the conclusion apparently follows from (i)$_0$–(iii)$_0$ with

$${}_\beta\varepsilon^{\alpha}_{k,0} = {}_\gamma\varepsilon_{k,0} = \frac{\beta_k}{\beta_k^*}.$$

Clearly,

$$\lim_{\beta_k \to \beta_k^*} {}_\beta\varepsilon_{k,0} = \lim_{\beta_k \to \beta_k^*} {}_\gamma\varepsilon_{k,0} = 1.$$

Now, assume that (I)$_n$–(III)$_n$ hold for some $n \geq 1$ and $t \geq t_n \geq t_1$, and we will show that they hold for $n+1$, with ${}_\beta\varepsilon_{k,n+1}$ and ${}_\gamma\varepsilon_{k,n+1}$ defined by:

(a) for either $\tau(t) \le t$ and $\psi_* = \infty$ or $\tau(t) \ge t$ and $\omega_* = \infty$:

$$
\begin{aligned}
{}_{\beta}\varepsilon_{k,n} &= \sqrt[\alpha]{{}_{\beta}\varepsilon_{k,0}}\,\varepsilon \frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}} \sqrt[\alpha]{\frac{1-\beta_{k,n-1}}{1-\tilde{\beta}_{k,n-1}}},\\
{}_{\gamma}\varepsilon_{k,n} &= {}_{\gamma}\varepsilon_{k,0}\,\varepsilon^{\alpha} \left[\frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}} \left(\frac{1-\gamma_{k,n-1}}{1-\tilde{\gamma}_{k,n-1}}\right)\right]^{\alpha}
\end{aligned}
$$

(b) for $\tau(t) \le t$ and $\psi_* < \infty$:

$$
\begin{aligned}
{}_{\beta}\varepsilon_{k,n} &= \sqrt[\alpha]{{}_{\beta}\varepsilon_{k,0}} \frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}} \sqrt[\alpha]{\frac{1-\beta_{k,n-1}}{1-\tilde{\beta}_{k,n-1}}} \left(\frac{1-p_0\psi^{-\tilde{\gamma}_{k,n-1}}}{1-p_0\psi_*^{-\gamma_{k,n-1}}}\right),\\
{}_{\gamma}\varepsilon_{k,n} &= {}_{\gamma}\varepsilon_{k,0} \left[\frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}} \left(\frac{1-\gamma_{k,n-1}}{1-\tilde{\gamma}_{k,n-1}}\right) \left(\frac{1-p_0\psi^{-\tilde{\gamma}_{k,n-1}}}{1-p_0\psi_*^{-\gamma_{k,n-1}}}\right)\right]^{\alpha}
\end{aligned}
$$

(c) for $\tau(t) \ge t$ and $\omega_* < \infty$:

$$
\begin{aligned}
{}_{\beta}\varepsilon_{k,n} &= \sqrt[\alpha]{{}_{\beta}\varepsilon_{k,0}} \frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}} \sqrt[\alpha]{\frac{1-\beta_{k,n-1}}{1-\tilde{\beta}_{k,n-1}}} \left(\frac{1-p_0\omega^{-\tilde{\beta}_{k,n-1}}}{1-p_0\omega_*^{-\beta_{k,n-1}}}\right),\\
{}_{\gamma}\varepsilon_{k,n} &= {}_{\gamma}\varepsilon_{k,0} \left[\frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}} \left(\frac{1-\gamma_{k,n-1}}{1-\tilde{\gamma}_{k,n-1}}\right) \left(\frac{1-p_0\omega^{-\tilde{\beta}_{k,n-1}}}{1-p_0\omega_*^{-\beta_{k,n-1}}}\right)\right]^{\alpha}
\end{aligned}
$$

for $n \in \mathbb{N}$. Clearly, in all three cases, we have

$$
\begin{aligned}
\lim_{(\beta_k,\lambda,\varepsilon)\to(\beta_k^*,\lambda_*,1)} {}_{\beta}\varepsilon_{k,n} &= \lim_{(\beta_k,\lambda,\varepsilon)\to(\beta_k^*,\lambda_*,1)} {}_{\gamma}\varepsilon_{k,n} = 1,\\
\lim_{(\beta_k,\lambda,\psi)\to(\beta_k^*,\lambda_*,\psi_*)} {}_{\beta}\varepsilon_{k,n} &= \lim_{(\beta,\lambda,\psi)\to(\beta_k^*,\lambda_*,\psi_*)} {}_{\gamma}\varepsilon_{k,n} = 1,
\end{aligned}
$$

and

$$
\lim_{(\beta_k,\lambda,\omega)\to(\beta_k^*,\lambda_*,\omega_*)} {}_{\beta}\varepsilon_{k,n} = \lim_{(\beta_k,\lambda,\omega)\to(\beta_k^*,\lambda_*,\omega_*)} {}_{\gamma}\varepsilon_{k,n} = 1,
$$

respectively.

Using (III)$_n$ in (1.1), we get

$$
\left(r\left(z'\right)^{\alpha}\right)'(t) + q(t)\tilde{a}_{k,n}^{\alpha}(1+H_k(\sigma(t)))^{\alpha} z^{\alpha}(\sigma(t)) \le 0, \quad t \ge t_n,
$$

which in view of (2.2) becomes

$$
\left(r\left(z'\right)^{\alpha}\right)'(t) + \frac{\beta_k \alpha \tilde{a}_{k,n}^{\alpha}}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} z^{\alpha}(\sigma(t)) \le 0. \tag{3.16}
$$

Now using that $z/\pi^{\tilde{\beta}_{k,n}}$ is decreasing (see (I)$_n$), (H$_3$) and (2.2), we find

$$
\frac{z(\sigma(t))}{z(t)} \ge \left(\frac{\pi(\sigma(t))}{\pi(t)}\right)^{\tilde{\beta}_{k,n}} \ge \lambda^{\tilde{\beta}_{k,n}}.
$$

Hence, (3.16) becomes

$$
\left(r\left(z'\right)^{\alpha}\right)'(t) + \frac{\beta_k \alpha \tilde{a}_{k,n}^{\alpha} \lambda^{\alpha\tilde{\beta}_{k,n}}}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} z^{\alpha}(t) \le 0. \tag{3.17}
$$

(I)$_{n+1}$ Integrating (3.17) from t_n to t and using (I)$_n$, we have

$$\begin{aligned}-r(t)\left(z'(t)\right)^{\alpha} &\geq -r(t_n)\left(z'(t_n)\right)^{\alpha}\\ &\quad+\beta_k\tilde{a}_{k,n}^{\alpha}\lambda^{\alpha\tilde{\beta}_{k,n}}\left(\frac{z}{\pi^{\tilde{\beta}_{k,n}}}\right)^{\alpha}(t)\int_{t_n}^{t}\frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha(1-\tilde{\beta}_{k,n})+1}(s)}\mathrm{d}s\\ &= -r(t_n)\left(z'(t_n)\right)^{\alpha}\\ &\quad+\frac{\beta_k\tilde{a}_{k,n}^{\alpha}\lambda^{\alpha\tilde{\beta}_{k,n}}}{1-\tilde{\beta}_{k,n}}\left(\frac{z}{\pi^{\tilde{\beta}_{k,n}}}\right)^{\alpha}(t)\left(\frac{1}{\pi^{\alpha(1-\tilde{\beta}_{k,n})}(t)}-\frac{1}{\pi^{\alpha(1-\tilde{\beta}_{k,n})}(t_n)}\right).\end{aligned}\tag{3.18}$$

Similarly as in the proof of [10, Lemma 4, pp. 8–9], it can be shown that

$$\lim_{t\to\infty}\frac{z(t)}{\pi^{\tilde{\beta}_{k,n}}(t)}=0$$

and so, there exists $t_n'\geq t_n$ such that

$$-r(t_n)\left(z'(t_n)\right)^{\alpha}>\frac{\beta_k\tilde{a}_{k,n}^{\alpha}\lambda^{\alpha\tilde{\beta}_{k,n}}}{1-\tilde{\beta}_{k,n}}\left(\frac{z}{\pi^{\tilde{\beta}_{k,n}}}\right)^{\alpha}(t)\frac{1}{\pi^{\alpha(1-\tilde{\beta}_{k,n})}(t_n)},\qquad t\geq t_n'.\tag{3.19}$$

Using (3.19) in (3.18) implies that

$$-\pi r^{1/\alpha}z'>\tilde{a}_{k,n}\lambda^{\tilde{\beta}_{k,n}}\sqrt[\alpha]{\frac{\beta_k}{1-\tilde{\beta}_{k,n}}}z=\tilde{\beta}_{k,n+1}z\tag{3.20}$$

and

$$\left(\frac{z}{\pi^{\tilde{\beta}_{k,n+1}}}\right)'<0,$$

which completes the induction step.

(II)$_{n+1}$ Differentiating as in (3.13) and using (3.17), we get

$$\begin{aligned}Z' &= \left(r^{1/\alpha}z'\right)'\pi\\ &= \frac{\pi}{\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha}\left(r\left(z'\right)^{\alpha}\right)'\\ &\leq -\frac{\pi}{\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha}\frac{\beta_k\alpha\tilde{a}_{k,n}^{\alpha}\lambda^{\alpha\tilde{\beta}_{k,n}}}{r^{1/\alpha}\pi^{\alpha+1}}z^{\alpha}\\ &= -\frac{\beta_k\tilde{a}_{k,n}^{\alpha}\lambda^{\alpha\tilde{\beta}_{k,n}}}{r^{1/\alpha}\pi^{\alpha}}\left(r^{1/\alpha}z'\right)^{1-\alpha}z^{\alpha}<0.\end{aligned}\tag{3.21}$$

Using (II)$_n$, which corresponds to

$$(1-\tilde{\gamma}_{k,n})z\geq -r^{1/\alpha}z'\pi$$

in (3.21), we obtain

$$Z'\leq -\frac{\beta_k\tilde{a}_{k,n}^{\alpha}\lambda^{\alpha\tilde{\beta}_{k,n}}}{r^{1/\alpha}\pi^{\alpha}}\left(r^{1/\alpha}z'\right)^{1-\alpha}\frac{(-r^{1/\alpha}z'\pi)^{\alpha}}{(1-\tilde{\gamma}_{k,n})^{\alpha}}=\frac{\beta_k\tilde{a}_{k,n}^{\alpha}\lambda^{\alpha\tilde{\beta}_{k,n}}}{(1-\tilde{\gamma}_{k,n})^{\alpha}}z'.$$

Integrating the above inequality from t to ∞ and using that z is decreasing and tending to zero eventually (see Lemma 3.1 (ii) and (v)), we have

$$Z(t) \geq Z(\infty) - \frac{\beta_k \tilde{a}_{k,n}^{\alpha} \lambda^{\alpha \tilde{\beta}_{k,n}}}{(1-\tilde{\gamma}_{k,n})^{\alpha}} z(\infty) + \frac{\beta_k \tilde{a}_{k,n}^{\alpha} \lambda^{\alpha \tilde{\beta}_{k,n}}}{(1-\tilde{\gamma}_{k,n})^{\alpha}} z(t) \geq \frac{\beta_k \tilde{a}_{k,n}^{\alpha} \lambda^{\alpha \tilde{\beta}_{k,n}}}{(1-\tilde{\gamma}_{k,n})^{\alpha}} z(t),$$

which in view of the definition of Z (see (3.12)) gives

$$\left(1 - \frac{\beta_k \tilde{a}_{k,n}^{\alpha} \lambda^{\alpha \tilde{\beta}_{k,n}}}{(1-\tilde{\gamma}_{k,n})^{\alpha}}\right) z \geq -r^{1/\alpha} z' \pi$$

and

$$\left(\frac{z}{\pi^{1-\tilde{\gamma}_{k,n+1}}}\right)' \geq 0,$$

which completes the induction step.

$(\mathrm{III})_{n+1}$ The proof proceeds in the same way as in the case $n = 0$ and hence is omitted.

2. To prove the statement, we claim that $(\mathrm{I})_n$ and $(\mathrm{II})_n$ implies $(\mathrm{i})_{n-1}$ and $(\mathrm{ii})_{n-1}$ for $n \in \mathbb{N}$. Clearly, $(\mathrm{I})_n$ and $(\mathrm{II})_n$ correspond to

$$\tilde{\beta}_{k,n} z < -r^{1/\alpha} z' \pi \tag{3.22}$$

and

$$(1 - \tilde{\gamma}_{k,n}) z \geq -r^{1/\alpha} z' \pi \tag{3.23}$$

respectively. Then, by virtue of Lemma 3.1 (ii) and (iii), it is easy to see that

$$\tilde{\beta}_{k,n} < 1 \quad \text{and} \quad \tilde{\gamma}_{k,n} < 1.$$

Using this and (2.3), we have

$$1 > \tilde{\beta}_{k,n} = {}_{\beta}\varepsilon_{k,n} \ell_{k,n-1} \beta_{k,n-1} > \beta_{k,n-1} \tag{3.24}$$

and

$$1 > \tilde{\gamma}_{k,n} = {}_{\gamma}\varepsilon_{k,n} h_{k,n-1} \gamma_{k,n-1} > \gamma_{k,n-1}, \tag{3.25}$$

where we used that ${}_{\beta}\varepsilon_n \in (0,1)$ and ${}_{\gamma}\varepsilon_n \in (0,1)$ are arbitrary. Therefore, (3.22) and (3.23) become

$$\beta_{k,n-1} z \leq -r^{1/\alpha} z' \pi$$

and

$$(1 - \gamma_{k,n-1}) z > -r^{1/\alpha} z' \pi,$$

for $n \in \mathbb{N}$, which proves our claim. Finally, $(\mathrm{iii})_{n-1}$ is just a consequence of $(\mathrm{i})_{n-1}$ and $(\mathrm{ii})_{n-1}$. □

In view of the newly obtained monotonicities $(\mathrm{i})_n$ and $(\mathrm{ii})_n$, our first main result follows immediately.

Theorem 3.7. *Let $\beta_0^* > 0$, $\lambda_* < \infty$, $\beta_{k,i} < 1$ and $\gamma_{k,i} < 1$ for $i = 0, 1, \ldots, n$ for some $k, n \in \mathbb{N}_0$. If*

$$\beta_{k,n+1} + \gamma_{k,n+1} > 1,$$

then (1.1) is oscillatory.

The second main result of this work results as a simple consequence of Lemma 3.6 (see (3.24) and (3.25)).

Corollary 3.8. *Let $\beta_0^* > 0$. If x is an eventually positive solution of (1.1), then, for some $k \in \mathbb{N}_0$, both sequences $\{\beta_{k,n}\}_{n\in\mathbb{N}_0}$ and $\{\gamma_{k,n}\}_{n\in\mathbb{N}_0}$ are well-defined and bounded from above.*

Now we are prepared to state the second main result of this paper, which is a straightforward consequence of Theorem A (condition (C_1)), Corollary 3.8 and Lemma 2.3 (conditions (C_2)–(C_4)).

Theorem 3.9. *If one of the conditions*

(C_1) *$\beta_0^* > 0$ and $\lambda_* = \infty$;*

(C_2) *$\beta_0^* > 0$, $\lambda_* < \infty$, either $\tau(t) \le t$ and $\psi_* = \infty$ or $\tau(t) \ge t$ and $\omega_* = \infty$, and the system (2.4) does not have a solution $\{b, g\} \in (0, 1)$;*

(C_3) *$\beta_0^* > 0$, $\lambda_* < \infty$, $\tau(t) \le t$, $\psi_* < \infty$, and the system (2.5) does not have a solution $\{b, g\} \in (0, 1)$;*

(C_4) *$\beta_0^* > 0$, $\lambda_* < \infty$, $\tau(t) \ge t$, $\omega_* < \infty$, and the system (2.6) does not have a solution $\{b, g\} \in (0, 1)$*

is satisfied for some $k \in \mathbb{N}_0$, then (1.1) is oscillatory.

By stating explicit conditions for the nonexistence of solutions $\{b, g\} \in (0, 1)$ of the systems (2.4)–(2.6), we get the following results.

Corollary 3.10. *If $\lambda_* < \infty$, either $\tau(t) \le t$ and $\psi_* = \infty$ or $\tau(t) \ge t$ and $\omega_* = \infty$, and*

$$\beta_k^* > \max\left\{ b^\alpha (1-b)\lambda_*^{-\alpha b} : 0 < b < 1 \right\},$$

then (1.1) is oscillatory.

Corollary 3.11. *If $\beta_0^* > 0$, $\lambda_* < \infty$, $\tau(t) \le t$, $\psi_* < \infty$, and*

$$\beta_k^* > \max\left\{ \frac{b^\alpha (1-b)\lambda_*^{-\alpha b}}{(1-p_0\psi_*^{-g})^\alpha} : 0 < g < 1, \text{ where } b = -\frac{\ln \frac{\beta_k^*(1-p_0\psi_*^{-g})^\alpha}{g(1-g)^\alpha}}{\alpha \ln \lambda_*} \right\},$$

then (1.1) is oscillatory.

Corollary 3.12. *If $\lambda_* < \infty$, $\tau(t) \ge t$, $\omega_* < \infty$, and*

$$\beta_k^* > \max\left\{ \frac{b^\alpha (1-b)\lambda_*^{-\alpha b}}{(1-p_0\omega_*^{-b})^\alpha} : 0 < b < 1 \right\},$$

then (1.1) is oscillatory.

The method of iteratively improved monotonicity properties gives us useful information about the asymptotic behavior of solutions in case when (1.1) is nonoscillatory (i.e., it possesses a nonoscillatory solution). The following results, which are a direct consequence of Lemma 3.6, improve and complement our previous statement [10, Corollary 1], and also complement and extend the results from [6, 14] in nonneutral linear and half-linear case, respectively. It is worth to note that in the linear case $\alpha = 1$, we have $\beta_{k,n} = \gamma_{k,n}$, which is stated separately for the sake of future reference.

Theorem 3.13. *Let $\beta_0^* > 0$ and $\lambda_* < \infty$. If x is an eventually positive solution of (1.1), then for any $c \in (0,1)$ and $k \in \mathbb{N}_0$,*

$$\frac{z(\sigma(t))}{z(t)} \geq c\lambda_*^{\beta_{k,n}},$$

eventually.

Theorem 3.14. *Let $\beta_0^* > 0$ and $\lambda_* < \infty$. If x is an eventually positive solution of (1.1), then there exist $c_i > 0$, $i = 1,2$, such that*

$$z \leq c_1 \pi^{\beta_{k,n}} \quad \textit{and} \quad z \geq c_2 \pi^{1-\gamma_{k,n}}, \quad k \in \mathbb{N}_0,$$

eventually.

Corollary 3.15. *Let $\beta_0^* > 0$, $\lambda_* < \infty$, and $\alpha = 1$. If x is an eventually positive solution of (1.1), then there exist $c_i > 0$, $i = 1,2$, such that*

$$z \leq c_1 \pi^{\beta_{k,n}} \quad \textit{and} \quad z \geq c_2 \pi^{1-\beta_{k,n}}, \quad k \in \mathbb{N}_0,$$

eventually.

4 Examples

Finally, we illustrate the importance of our results on two examples. The first one is intended to show the progress attained in case when p_0 from (H$_5$) is close to 1.

Example 4.1. Consider the Euler type differential equation

$$\left(t^{\alpha+1}\left(\left(x(t) + \frac{0.99}{t^{(1-\lambda_1)/\alpha}} x\left(t^{\lambda_1}\right)\right)'\right)^{\alpha}\right)' + q_0 x^{\alpha}(\lambda_2 t) = 0, \quad t \geq t_0 > 0, \tag{4.1}$$

where $\alpha > 0$ is a quotient of odd positive integers, $\lambda_1 \in (0,1)$, $\lambda_2 \in (0,1]$, $q_0 > 0$. Here,

$$\pi(t) = \frac{\alpha}{t^{1/\alpha}}, \quad \lambda_* = \frac{1}{\lambda_2^{1/\alpha}}, \quad \psi_* = \lim_{t\to\infty} t^{(1-\lambda_1)/\alpha} = \infty, \quad p_0 = p(t)\frac{\pi(\tau(t))}{\pi(t)} = 0.99,$$

and

$$\beta_k^* = \beta_0^* = q_0 \alpha^{\alpha}.$$

It follows from [29, Theorem 2.8] that

$$\beta_0^* > \frac{\alpha^{\alpha}}{(1-p_0)^{\alpha}(\alpha+1)^{\alpha+1}} = 100^{\alpha}\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} = 100^{\alpha}\max\{b^{\alpha}(1-b) : 0 < b < 1\} \tag{4.2}$$

is sufficient for (4.1) to be oscillatory. By [9, Theorem 2.4] proved by the present authors, the same conclusion is attained if

$$\rho := q_0^{1/\alpha}(1-p_0)^{\alpha}\ln\frac{1}{\lambda_2} > \frac{1}{\mathrm{e}}$$

or, if $\rho \leq 1/\mathrm{e}$ and

$$\beta_0^* > \frac{1}{(1-p_0)^{\alpha} f(\rho)} \cdot \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} = \frac{100^{\alpha}}{f(\rho)}\max\{b^{\alpha}(1-b) : 0 < b < 1\}, \tag{4.3}$$

where

$$f(\rho) = -\frac{W_0(-\rho)}{\rho}, \quad W_0 \text{ is a principal branch of the Lambert function.}$$

We have also showed in [9] that (4.3) simplifies and improves related results from [5,19,22–24, 26,36–39].

By Theorem A (see also [10, Theorem 2]), (4.1) is oscillatory if

$$\beta_0^* > \frac{\max\{b^\alpha(1-b)\lambda_2^b : 0 < b < 1\}}{(1-p_0)^\alpha} = 100^\alpha \max\{b^\alpha(1-b)\lambda_2^b : 0 < b < 1\}, \tag{4.4}$$

which improves (4.3). Finally, by the newly obtained Theorem 3.9, (4.1) is oscillatory if

$$\beta_0^* > \max\{b^\alpha(1-b)\lambda_2^b : 0 < b < 1\}. \tag{4.5}$$

It is obvious that (4.2) does not take λ_2 into account, which is already included in (4.3)–(4.5). Moreover, in Theorem 3.9, the impact of p_0 was removed by that of λ_1 and so (4.5) gives 100^α-times qualitatively better result than (4.4).

Example 4.2. As in [10, Example 1], we consider

$$\left(t^{\alpha+1}\left(\left(x(t)+p_0x(\lambda_1 t)\right)'\right)^\alpha\right)' + q_0x^\alpha(\lambda_2 t) = 0, \quad t \geq t_0 > 0, \tag{4.6}$$

where $\alpha > 0$ is a quotient of odd positive integers, $\lambda_1 > 0$, $\lambda_2 \in (0,1]$, $q_0 > 0$, and

$$p_0 < \begin{cases} \lambda_1^{1/\alpha} & \text{for } \lambda_1 \leq 1, \\ 1 & \text{for } \lambda_1 > 1. \end{cases}$$

Here,

$$\pi(t) = \frac{\alpha}{t^{1/\alpha}}, \quad \lambda_* = \frac{1}{\lambda_2^{1/\alpha}}, \quad \psi_* = \frac{1}{\lambda_1^{1/\alpha}} \text{ (for } \lambda_1 \leq 1), \quad \omega_* = \lambda_1^{1/\alpha} \text{ (for } \lambda_1 > 1),$$

and

$$\beta_0^* = \alpha^\alpha q_0$$

$$\beta_k^* = \begin{cases} \beta_0^*\left(\sum_{i=0}^k p_0^{2i}\right)^\alpha & \text{for } \lambda_1 \leq 1,\ k \in \mathbb{N}, \\ \beta_0^*\left(\sum_{i=0}^k \left(\frac{p_0}{\lambda_1^{1/\alpha}}\right)^{2i}\right)^\alpha & \text{for } \lambda_1 > 1,\ k \in \mathbb{N}. \end{cases}$$

It is easy to compute the limit

$$\beta^* := \lim_{k\to\infty} \beta_k^* = \begin{cases} \dfrac{\beta_0^*}{(1-p_0^2)^\alpha} & \text{for } \lambda_1 \leq 1, \\ \dfrac{\beta_0^*}{\left(1-\left(p_0\lambda_1^{-1/\alpha}\right)^2\right)^\alpha} & \text{for } \lambda_1 > 1. \end{cases}$$

First, assume $\lambda_1 \leq 1$. By Theorem A, (4.6) is oscillatory if

$$\beta_0^* > \frac{\max\{b^\alpha(1-b)\lambda_2^b : 0<b<1\}}{\left(1-p_0\lambda_1^{-1/\alpha}\right)^\alpha}. \tag{4.7}$$

Let us recall (see [10, Example 1]) that (4.6) has a nonoscillatory solution, if

$$\beta_0^* \leq \max\left\{b^\alpha(1-b)\lambda_2^b\left(1+p_0\lambda_1^{-b/\alpha}\right)^\alpha : 0<b<1\right\}. \tag{4.8}$$

In the nonneutral case $p_0 = 0$, Theorem A is clearly sharp. For, e.g.,

$$\lambda_1 = \lambda_2 = p_0 = 0.5, \quad \alpha = 3, \tag{4.9}$$

we conclude that, by Theorem A, (4.6) is oscillatory if

$$q_0 > 0.0464 \tag{4.10}$$

and, by (4.8), (4.6) has a nonoscillatory solution if

$$q_0 \leq 0.0094,$$

meaning that the behavior of (4.6) subject to (4.9) is unknown for $q_0 \in (0.0094, 0.0464]$.

By Theorem 3.9 (C_3), (4.6) is oscillatory if the system

$$\begin{cases} \dfrac{\beta_0^*}{(1-p_0^2)^\alpha} = \dfrac{b^\alpha(1-b)\lambda_2^b}{\left(1-p_0\lambda_1^{-(1-g)/\alpha}\right)^\alpha} \\ \dfrac{\beta_0^*}{(1-p_0^2)^\alpha} = \dfrac{g(1-g)^\alpha\lambda_2^b}{\left(1-p_0\lambda_1^{-(1-g)/\alpha}\right)^\alpha} \end{cases} \tag{4.11}$$

does not have a solution $\{b, g\}$ on $(0,1)$, what happens if, by Corollary 3.11,

$$\frac{\beta_0^*}{\left(1-p_0^2\right)^\alpha} > \max\left\{\frac{b^\alpha(1-b)\lambda_2^b}{\left(1-p_0\lambda_1^{-(1-g)/\alpha}\right)^\alpha} : 0<g<1, \text{ where } b = \frac{\ln\frac{\beta_0^*(1-p_0\lambda_1^{-(1-g)/\alpha})^\alpha}{(1-p_0^2)^\alpha g(1-g)^\alpha}}{\ln\lambda_2}\right\}. \tag{4.12}$$

To show the improvement over Theorem A, assume (4.9) and

$$q_0 > 0.0158.$$

Although (4.10) fails to apply, it can be verified using numerical software that (4.12) is satisfied and the system (4.11) does not possess a positive solution, i.e., (4.6) is oscillatory. An alternative approach to attain the same conclusion is to use Theorem 3.7 by initiating an iterative process (e.g., 2 iterations are needed for $q_0 = 0.04$, 11 iterations for $q_0 = 0.017$, 63 iterations for $q_0 = 0.0158$). How to fill the gap $q_0 \in (0.0094, 0.0158]$ remains open at the moment.

Now, assume $\lambda_1 > 1$. By Theorem A, (4.6) is oscillatory if

$$\beta_0^* > \frac{\max\{b^\alpha(1-b)\lambda_2^b : 0<b<1\}}{(1-p_0)^\alpha}. \tag{4.13}$$

Here, we would like to point out an oversight we made in [10, Example 1], where we stated that (4.7) (instead of (4.13)) is sufficient for oscillation of (4.6). To look at the improvement, we find that by Corollary 3.12, (4.6) is oscillatory if

$$\beta_0^* > \left(1 - \left(p_0 \lambda_1^{-1/\alpha}\right)^2\right)^\alpha \max\left\{ \frac{b^\alpha (1-b)\lambda_2^b}{\left(1 - p_0 \lambda_1^{-b/\alpha}\right)^\alpha} : 0 < b < 1 \right\}. \tag{4.14}$$

It is obvious to see that, in contrast with (4.14), the criterion (4.13) does not take the influence of λ_1 into account. Clearly, for $p_0 \neq 0$,

$$\max\left\{ \frac{b^\alpha (1-b)\lambda_2^b}{\left(1 - p_0 \lambda_1^{-b/\alpha}\right)^\alpha} : 0 < b < 1 \right\} < \frac{\max\{b^\alpha (1-b)\lambda_2^b : 0 < b < 1\}}{(1-p_0)^\alpha}$$

and

$$\left(1 - \left(p_0 \lambda_1^{-1/\alpha}\right)^2\right)^\alpha < 1,$$

and hence the progress is observable.

Remark 4.3. For $k = 0$, the results established in this paper complement those from [21], where (1.1) subject to

$$\pi(t_0) = \infty$$

was studied. We stress that obtaining a corresponding variant of Lemma 3.3 would immediately improve oscillation criteria from [21]. Another interesting task left for further research is to consider the same problem with $p_0 \geq 1$ or $p_0 < 0$.

5 Summary

The aim of the present paper was to continue studying the oscillation problem of (1.1) under conditions (H_1)–(H_5) and to provide new results which improve Theorem A when $p_0 \neq 0$ and $\lambda_* < \infty$. Our results improve all existing works (i.e., the cited related papers and references therein) on this subject so far.

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On some classes of solvable difference equations related to iteration processes

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Abstract. We present several classes of nonlinear difference equations solvable in closed form, which can be obtained from some known iteration processes, and for some of them we give some generalizations by presenting methods for constructing them. We also conduct several analyses and give many comments related to the difference equations and iteration processes.

Keywords: difference equation, iteration process, equations solvable in a closed form.

2020 Mathematics Subject Classification: 39A45.

1 Introduction

The sets of natural numbers, nonnegative integers, integers, real numbers and complex numbers, we denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, respectively, whereas the notation $l = \overline{s,t}$, when $s, t \in \mathbb{Z}$ and $s \le t$ is used instead of writing $s \le l \le t$, $l \in \mathbb{Z}$. By C_j^n, $n \in \mathbb{N}$ and $j = \overline{0,n}$, we denote the binomial coefficients. Recall that

$$C_j^n = \frac{n!}{j!(n-j)!},$$

where we regard that $0! = 1$ (some information on the coefficients can be found, e.g., in [4,22,32,34,43]).

Difference equations and systems naturally appear in many areas of science and mathematics [9,12–14,18,19,22,26,27,29,34,43,51,59]. The problem of finding formulas for their solutions in closed form appeared long time ago, and was treated by many known mathematicians such as D. Bernoulli, de Moivre, Euler, Lagrange and Laplace (see, e.g., [9,13–15,17,23–28]). Unfortunately, for a great majority of the equations and systems it is impossible to find such

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formulas, especially if they are nonlinear. In [1,10,16,17,19,22,30,31,33–35] can be found some classical solvable nonlinear difference equations, as well as systems of difference equations.

Several classes of solvable nonlinear difference equations can be obtained by using some known iteration processes. Some of them can be found, for example, in [11,12,55,56].

Motivated by some recent investigations on solvability of difference equations and systems of difference equations (see, e.g., [3,40,42,53,54,57,58] and the references therein) and some examples in [12], we have studied recently connections between some difference equations obtained from known iteration processes and their solvability. Related equations and topics such as finding invariants and studying equations obtained from solvable ones can be found in [5–8,21,36–39,46,47,49,50,52].

Here we continue the investigation of solvability of difference equations and their relationships with known iteration processes. We deal with some equations of the form

$$x_{n+1} = \widehat{f}(x_n), \quad n \in \mathbb{N}_0,$$

the autonomous difference equation of first order.

First we show that the Newton–Raphson iteration process for finding roots of quadratic equations produces a solvable nonlinear difference equation extending a known example of such a difference equation. Recall that the Newton–Raphson iteration process is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

(see, e.g., [12,18]), where f is a given function.

Based on it and another known difference equation, we present a related class of solvable nonlinear difference equations. Then we present a solvable class of nonlinear difference equations generalizing two known ones which are obtained by the Newton–Raphson iteration process for calculating reciprocals. We also present an interesting method for constructing a class of solvable nonlinear difference equations generalizing a solvable equation obtained from the Halley iteration process for finding square roots. We also conduct several analyses and give many comments related to solvable nonlinear difference equations and iteration processes.

2 Some analyses and main results

In this section we conduct some analyses related to the relationships between solvable difference equations and some known iteration processes, and state and prove our main results.

2.1 Newton–Raphson iteration process for quadratic equations and solvability

Let

$$f(x) = x^2 + px + q, \tag{2.1}$$

be a quadratic function.

By using function (2.1) in (1.1) we get

$$x_{n+1} = x_n - \frac{x_n^2 + px_n + q}{2x_n + p}, \quad n \in \mathbb{N}_0,$$

that is,

$$x_{n+1} = \frac{x_n^2 - q}{2x_n + p}, \quad n \in \mathbb{N}_0. \tag{2.2}$$

If a solution to equation (2.2) converges to a point x^*, then it is clear that x^* must be equal to one of the zeros of function (2.1).

From the numerical point of view the interesting case is when $q \neq 0$ (if $q = 0$, then the roots of (2.1) are obviously 0 and $-p$). Assume additionally that $p^2 \neq 4q$. Then the function has two different zeros, say, a and b, and equation (2.2) can be rewritten in the form

$$x_{n+1} = \frac{x_n^2 - ab}{2x_n - a - b}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

First, assume that $a \neq b$. We consider the cases $a + b = 0$ and $a + b \neq 0$ separately.

Case $a + b = 0$. In this case we have $b = -a$. Hence, equation (2.3) becomes

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a^2}{x_n}\right), \quad n \in \mathbb{N}_0. \tag{2.4}$$

It is well known that the equation is solvable in closed form [12, 22], and that its general solution is given by

$$x_n = a\frac{1 + (\frac{x_0 - a}{x_0 + a})^{2^n}}{1 - (\frac{x_0 - a}{x_0 + a})^{2^n}}, \quad n \in \mathbb{N}_0. \tag{2.5}$$

Recall that the difference equation in (2.4) serves for calculating a square root of number a^2.

Case $a + b \neq 0$. From (2.3) and by some simple calculations, it follows that

$$x_{n+1} - a = \frac{x_n^2 - 2ax_n + a^2}{2x_n - a - b}, \quad n \in \mathbb{N}_0, \tag{2.6}$$

and

$$x_{n+1} - b = \frac{x_n^2 - 2bx_n + b^2}{2x_n - a - b}, \quad n \in \mathbb{N}_0. \tag{2.7}$$

From (2.6) and (2.7) we have

$$\frac{x_{n+1} - a}{x_{n+1} - b} = \left(\frac{x_n - a}{x_n - b}\right)^2, \quad n \in \mathbb{N}_0,$$

and consequently

$$\frac{x_n - a}{x_n - b} = \left(\frac{x_0 - a}{x_0 - b}\right)^{2^n}, \quad n \in \mathbb{N}_0,$$

from which it easily follows that

$$x_n = \frac{b(\frac{x_0 - a}{x_0 - b})^{2^n} - a}{(\frac{x_0 - a}{x_0 - b})^{2^n} - 1}, \quad n \in \mathbb{N}_0. \tag{2.8}$$

The sequence defined in (2.8) is a solution to equation (2.3). Indeed, let

$$y_n := \left(\frac{x_0 - a}{x_0 - b}\right)^{2^n}, \quad n \in \mathbb{N}_0.$$

Then we have

$$\frac{x_n^2-ab}{2x_n-a-b}=\frac{(\frac{by_n-a}{y_n-1})^2-ab}{2\frac{by_n-a}{y_n-1}-a-b}=\frac{b^2y_n^2-2aby_n+a^2-aby_n^2+2aby_n-ab}{(y_n-1)(2by_n-2a-(a+b)y_n+a+b)}$$
$$=\frac{b(b-a)y_n^2-a(b-a)}{(y_n-1)(b-a)(y_n+1)}=\frac{by_n^2-a}{y_n^2-1}=\frac{b(\frac{x_0-a}{x_0-b})^{2^{n+1}}-a}{(\frac{x_0-a}{x_0-b})^{2^{n+1}}-1}$$
$$=x_{n+1}$$

as claimed.

Remark 2.1. Note that from (2.8) with $b=-a$ is obtained formula (2.5).

From (2.8) we easily obtain the following corollary.

Corollary 2.2. *Consider equation (2.3) where $a\neq b$ and $ab\neq 0$. Then the following statements are true.*

(a) If $|\frac{x_0-a}{x_0-b}|<1$, then $\lim_{n\to+\infty}x_n=a$.

(b) If $|\frac{x_0-a}{x_0-b}|>1$, then $\lim_{n\to+\infty}x_n=b$.

(c) If $\frac{x_0-a}{x_0-b}=-1$, that is, $x_0=\frac{a+b}{2}$, then x_1 is not defined.

Remark 2.3. Note that the case

$$\frac{x_0-a}{x_0-b}=1$$

is excluded, since we assume $a\neq b$.

Case $a=b$. If $a=b$ and $x_0=a$, then since in this case equation (2.3) becomes

$$x_{n+1}=\frac{x_n^2-a^2}{2(x_n-a)},\quad n\in\mathbb{N}_0,\tag{2.9}$$

we have that x_1 is not defined, so that in this case the solution to the equation is not well-defined.

If $x_{n_0}=a$ for some $n_0\in\mathbb{N}$, and

$$x_j\neq a,\quad j=\overline{0,n_0-1},$$

then from (2.9) we have

$$a=x_{n_0}=\frac{x_{n_0-1}^2-a^2}{2(x_{n_0-1}-a)}=\frac{x_{n_0-1}+a}{2},$$

and consequently $x_{n_0-1}=a$, which is a contradiction. Therefore, if $x_0\neq a$ we have that

$$x_n\neq a\quad\text{for}\quad n\in\mathbb{N}_0.\tag{2.10}$$

Hence, if $a=b$ and $x_0\neq a$, then from (2.9) and (2.10) we have that equation (2.3) becomes

$$x_{n+1}=\frac{x_n}{2}+\frac{a}{2},\quad n\in\mathbb{N}_0,$$

from which it follows that

$$x_n=\frac{x_0}{2^n}+a\left(1-\frac{1}{2^n}\right),\quad n\in\mathbb{N}_0,\tag{2.11}$$

(for the original source see [25]; see also [10,19,22,34]).

From (2.11) we easily obtain the following corollary.

Corollary 2.4. *Consider equation (2.3) where $a = b$ and $a \neq 0$. Then every solution to equation (2.3) such that $x_0 \neq a$ converges to a.*

Remark 2.5. Equation (2.3) appeared in [55] but we did not consider it, nor did we formulate any of the above results in the case.

Since we assume that $q = ab \neq 0$, the above analysis excluded the case. However, it is of some interest to consider equation (2.3) also in this case.

Case $ab = 0$. First note that, due to the symmetry of equation (2.3) with respect to parameters a and b, we may assume $b = 0$. In this case the difference equation becomes

$$x_{n+1} = \frac{x_n^2}{2x_n - a}, \tag{2.12}$$

for $n \in \mathbb{N}_0$.

If $a = 0$, then we have

$$x_{n+1} = \frac{x_n^2}{2x_n}, \quad n \in \mathbb{N}_0. \tag{2.13}$$

Hence, if $x_0 \neq 0$ we get

$$x_n = \frac{x_0}{2^n}, \quad n \in \mathbb{N}_0,$$

showing the solvability of equation (2.12) in this case. If $x_0 = 0$, then from (2.13) we see that x_1 is not defined. Therefore, the solution to equation (2.13) is also not well-defined.

Now assume that $a \neq 0$. If $x_0 = 0$, then a simple inductive argument shows that

$$x_n = 0, \quad n \in \mathbb{N}_0. \tag{2.14}$$

If $x_{n_1} = 0$ for some $n_1 \in \mathbb{N}$, and

$$x_j \neq 0, \quad j = \overline{0, n_1 - 1}, \tag{2.15}$$

then from (2.12) we have $x_{n_1-1} = 0$, which is a contradiction. From (2.14) and (2.15) we see that when $x_0 \neq 0$ we have that $x_n \neq 0$ for all $n \in \mathbb{N}_0$ for which x_n is defined. Hence, we can use the change of variables

$$x_n = \frac{1}{y_n}, \quad n \in \mathbb{N}_0, \tag{2.16}$$

and obtain the equation

$$y_{n+1} = y_n(2 - ay_n), \quad n \in \mathbb{N}_0.$$

It is well known that general solution to the equation is given by

$$y_n = \frac{1 - (1 - ay_0)^{2^n}}{a}, \quad n \in \mathbb{N}_0,$$

(see, e.g., [11, 12]).

Hence, we have that the general solution to equation (2.12) in this case is given by the formula

$$x_n = \frac{ax_0^{2^n}}{x_0^{2^n} - (x_0 - a)^{2^n}}, \tag{2.17}$$

for $n \in \mathbb{N}_0$.

Remark 2.6. Formula (2.17) can be also obtained from the formula (2.8) with $b = 0$. Indeed, the above consideration in the case $a + b \neq 0$ also holds in the case when $b = 0$. Note also that if $b = 0$, then $a + b = a \neq 0$. Hence, all the conditions there are satisfied if $a \neq 0$ and $b = 0$.

Remark 2.7. The change of variables (2.16) is a basic one and frequently appears in the literature (see, e.g. [4, 51]). One of the basic examples of difference equations where it is applied is the following

$$x_{n+1} = \frac{a_n x_n}{b_n + c_n x_n}, \quad n \in \mathbb{N}_0,$$

which, by the change of variables, is transformed to a nonhomogeneous linear difference equation of first order, which is theoretically solvable (this was shown first by Lagrange [26], then by another method by Laplace [27]; see, also [10, 16, 19, 34]). For some related changes of variables see, e.g., [40, 53, 57] and the related references therein.

2.2 A relative of equation (2.4)

The difference equation

$$x_{n+1} = \frac{2x_n}{x_n^2 + 1}, \quad n \in \mathbb{N}_0, \tag{2.18}$$

is another known difference equation. The long-term behaviour of its solutions can be studied by using standard methods to the governing function

$$f(t) = \frac{2t}{t^2 + 1}, \quad t \in \mathbb{R},$$

(see, e.g., [4, Problems 9.34, 9.35]).

However, the equation is also solvable. Indeed, first note that if x^* is an equilibrium of equation (2.18), then it is easy to see that

$$x^* \in \{-1, 0, 1\}.$$

Since

$$x_{n+1} - 1 = -\frac{(x_n - 1)^2}{x_n^2 + 1}$$

and

$$x_{n+1} + 1 = \frac{(x_n + 1)^2}{x_n^2 + 1}$$

for $n \in \mathbb{N}_0$, we have

$$\frac{x_{n+1} - 1}{x_{n+1} + 1} = -\left(\frac{x_n - 1}{x_n + 1}\right)^2, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$\frac{x_n - 1}{x_n + 1} = -\left(\frac{x_0 - 1}{x_0 + 1}\right)^{2^n}, \quad n \in \mathbb{N},$$

and finally

$$x_n = \frac{1 - \left(\frac{x_0-1}{x_0+1}\right)^{2^n}}{1 + \left(\frac{x_0-1}{x_0+1}\right)^{2^n}}, \quad n \in \mathbb{N}. \tag{2.19}$$

Remark 2.8. Solvability of equation (2.18) is not so surprising. Namely, note that by using the change of variables (2.16) from equation (2.18) it is obtained equation (2.4) with $a = 1$.

By using the change of variables in (2.16) in equation (2.2) we obtain the equation

$$y_{n+1} = \frac{py_n^2 + 2y_n}{1 - qy_n^2}, \quad n \in \mathbb{N}_0. \tag{2.20}$$

Let $p = -(a+b)$ and $q = ab$, then equation (2.20) becomes

$$y_{n+1} = \frac{-(a+b)y_n^2 + 2y_n}{1 - aby_n^2}, \quad n \in \mathbb{N}_0. \tag{2.21}$$

If $a + b \neq 0$, then from (2.8) we obtain

$$y_n = \frac{(\frac{1-ay_0}{1-by_0})^{2^n} - 1}{b(\frac{1-ay_0}{1-by_0})^{2^n} - a}, \quad n \in \mathbb{N}_0, \tag{2.22}$$

whereas if $a = b$, then from (2.11) we obtain

$$y_n = \frac{y_0 2^n}{1 + ay_0(2^n - 1)}, \quad n \in \mathbb{N}_0. \tag{2.23}$$

From (2.22) we obtain the following corollary.

Corollary 2.9. *Consider equation (2.21) where $a \neq b$ and $ab \neq 0$. Then for well-defined solutions of the equation the following statements are true.*

(a) If $|\frac{1-ay_0}{1-by_0}| < 1$, then $\lim_{n\to+\infty} y_n = \frac{1}{a}$.

(b) If $|\frac{1-ay_0}{1-by_0}| > 1$, then $\lim_{n\to+\infty} y_n = \frac{1}{b}$.

(c) If $\frac{1-ay_0}{1-by_0} = -1$, that is, $y_0 = \frac{2}{a+b}$, then $y_n = 0$, $n \in \mathbb{N}$.

(d) If $y_0 = 0$, then $y_n = 0$, $n \in \mathbb{N}_0$.

Remark 2.10. Note that if $y_0 \neq 0$, the case

$$\frac{1 - ay_0}{1 - by_0} = 1$$

is excluded, since we assume $a \neq b$.

From (2.21) and (2.23) we obtain the following corollary.

Corollary 2.11. *Consider equation (2.21) where $a = b \neq 0$. Then every solution to equation (2.21) such that $y_0 \neq 0$, $y_0 \neq 1/a$, and*

$$y_0 \neq \frac{1}{a(1 - 2^n)}, \quad n \in \mathbb{N},$$

converges to $\frac{1}{a}$.

Remark 2.12. Note that if $a = b$ and $y_0 = 0$, then $y_n = 0$ for every $n \in \mathbb{N}$.

Remark 2.13. Note that if $a = b$ and $y_0 = 1/a$ or

$$y_0 = \frac{1}{a(1 - 2^n)}$$

for some $n \in \mathbb{N}$, then y_n is not defined, and consequently the corresponding solution to equation (2.21).

2.3 Newton–Raphson iteration process for calculating reciprocals

It is well known that if we apply the Newton-Raphson iteration process to

$$f(x) = 1 - \frac{1}{ax} \tag{2.24}$$

where $a \neq 0$, we obtain the equation

$$x_{n+1} = 2x_n - ax_n^2, \quad n \in \mathbb{N}_0. \tag{2.25}$$

Recall that the equation is solvable in closed form [11,12,18,55].

If we apply the iteration process

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)}{2f'(x_n)}\left(\frac{f(x_n)}{f'(x_n)}\right)^2, \quad n \in \mathbb{N}_0$$

to the function in (2.24) we obtain the equation

$$x_{n+1} = 3x_n - 3ax_n^2 + a^2x_n^3, \quad n \in \mathbb{N}_0, \tag{2.26}$$

see, e.g., [18], where it is suggested to show that the relation holds

$$\frac{1}{a} - x_{n+1} = a^2\left(\frac{1}{a} - x_n\right)^3, \quad n \in \mathbb{N}_0. \tag{2.27}$$

From (2.27) we see that the relation (2.26) is also solvable in closed form. Indeed, let

$$y_n := \frac{1}{a} - x_n, \tag{2.28}$$

then from (2.26) we have

$$y_n = a^2y_{n-1}^3, \quad n \in \mathbb{N},$$

which is a simple product-type difference equation (for some examples of such difference equations and systems of equations, see, e.g., [54,58] and the related references therein).

By iterating the last relation we get

$$y_n = a^2(a^2y_{n-2}^3)^3 = a^{2(1+3)}y_{n-2}^{3^2}.$$

By a simple inductive argument we obtain

$$y_n = a^{2\sum_{j=0}^{n-1}3^j}y_0^{3^n} = a^{3^n-1}y_0^{3^n}, \quad n \in \mathbb{N}_0,$$

from which along with (2.28) it follows that

$$x_n = \frac{1}{a} - a^{3^n-1}\left(\frac{1}{a} - x_0\right)^{3^n}, \quad n \in \mathbb{N}_0. \tag{2.29}$$

Remark 2.14. The matrix counterpart of equation (2.25)

$$X_{n+1} = (2I - AX_n)X_n, \quad n \in \mathbb{N}_0,$$

is the Schultz iteration process [48] which has been studied a lot.

2.4 A generalization of equations (2.25) and (2.26)

Equations (2.25) and (2.26) are, among other things, obtained from two known iteration processes by employing them to the function in (2.24). Here we show that a sequence of iteration processes, which can be used for calculating reciprocals and containing relations (2.25) and (2.26), can be obtained in a simple way. Moreover, we show that they all are solvable in closed form.

If in the difference equation

$$y_{n+1} = y_n^k, \quad n \in \mathbb{N}_0, \tag{2.30}$$

where $k \in \mathbb{N} \setminus \{1\}$ is a fixed number, we use the change of variables

$$y_n = 1 - ax_n, \quad n \in \mathbb{N}_0, \tag{2.31}$$

where $a \neq 0$, we have

$$1 - ax_{n+1} = \sum_{j=0}^{k} C_j^k (-a)^j x_n^j, \quad n \in \mathbb{N}_0,$$

so after some simple calculation we obtain

$$x_{n+1} = \sum_{j=1}^{k} C_j^k (-a)^{j-1} x_n^j, \quad n \in \mathbb{N}_0. \tag{2.32}$$

From (2.32) for each $k \in \mathbb{N} \setminus \{1\}$ we obtain a difference equation which can be used for calculating reciprocals.

Now note that from (2.30) we have

$$y_n = y_0^{k^n}, \quad n \in \mathbb{N}_0. \tag{2.33}$$

By using (2.33) in (2.31) we get

$$x_n = \frac{1 - (1 - ax_0)^{k^n}}{a}, \quad n \in \mathbb{N}_0. \tag{2.34}$$

Formula (2.34) shows that equation (2.32) is also solvable in closed form.

Remark 2.15. Note that if in equation (2.32) we take $k = 2$, then we obtain equation (2.25), whereas if we take $k = 3$, then we obtain equation (2.26). This means that the difference equation is a natural generalization of the equations (2.25) and (2.26).

Remark 2.16. The matrix counterparts of equations (2.32) have been also studied considerably. Our literature review shows that the topic has been quite popular among scientists working on numerical mathematics for a long time, and it seems that such iteration processes are rediscovered from time to time. There are also some operator counterparts of equations (2.25), (2.26) and (2.32) (see, for example, [2, 41] and the references therein). So, the facts mentioned in this subsection should be folklore. Nevertheless, the above explanation suggests a natural way for constructing the matrix and operator iteration processes. From (2.30) we also see how is naturally obtained an iteration process whose rate of the convergence has a given order (for the notion see, e.g., [12, 18]).

2.5 A relative to equation (2.32)

By using change of variables (2.16) in equation (2.32) we obtain the equation

$$y_{n+1} = \frac{y_n^k}{\sum_{j=1}^k C_j^k (-a)^{j-1} y_n^{k-j}}, \quad n \in \mathbb{N}_0,$$

that is,

$$y_{n+1} = \frac{a y_n^k}{y_n^k - (y_n - a)^k}, \quad n \in \mathbb{N}_0. \tag{2.35}$$

Hence, from (2.34) we have that the general solution to equation (2.35) is given by

$$y_n = \frac{a y_0^{k^n}}{y_0^{k^n} - (y_0 - a)^{k^n}}, \quad n \in \mathbb{N}_0.$$

For example, if $k = 3$, then equation (2.35) becomes

$$y_{n+1} = \frac{y_n^3}{3y_n^2 - 3a y_n + a^2}, \quad n \in \mathbb{N}_0,$$

and its general solution is

$$y_n = \frac{a y_0^{3^n}}{y_0^{3^n} - (y_0 - a)^{3^n}}, \quad n \in \mathbb{N}_0.$$

2.6 Newton–Raphson method for polynomials of the third degree and solvability

Here we conduct some analyses regarding solvability of difference equations obtained by applying the Newton–Raphson iteration process to polynomials of the third degree, and generalise a class of solvable difference equations by presenting a method for constructing the generalization.

Difference equations can be used for calculating roots of some functions, but it is quite a rare situation that they are solvable in closed form. For example, if we want to calculate a root of the function

$$f(x) = x^3 - x$$

(we can easily find all of them by an elementary method), by using the Newton-Raphson process we get the equation

$$x_{n+1} = x_n - \frac{x_n^3 - x_n}{3x_n^2 - 1} = \frac{2x_n^3}{3x_n^2 - 1}, \quad n \in \mathbb{N}_0. \tag{2.36}$$

The equation frequently appears in the literature (see [20, 44]), and this explains how it can be obtained, which is one of the reasons why we mention the equation. Another reason is connected to the method used in dealing with equation (2.3).

Namely, from (2.36) and some calculations we get

$$x_{n+1} - 1 = \frac{(x_n - 1)^2 (2x_n + 1)}{3x_n^2 - 1}$$

and

$$x_{n+1} + 1 = \frac{(x_n + 1)^2 (2x_n - 1)}{3x_n^2 - 1}$$

from which it follows that

$$\frac{x_{n+1}-1}{x_{n+1}+1} = \left(\frac{x_n-1}{x_n+1}\right)^2 \frac{2x_n+1}{2x_n-1}.$$

However, the natural change of variables

$$y_n = \frac{x_n-1}{x_n+1}$$

cannot show the solvability of relation (2.36).

Let us analyse the general case. If we apply the Newton–Raphson iteration process to an arbitrary polynomial of the third order

$$p_3(t) = t^3 + pt^2 + qt + r \tag{2.37}$$

we get

$$x_{n+1} = x_n - \frac{x_n^3 + px_n^2 + qx_n + r}{3x_n^2 + 2px_n + q} = \frac{2x_n^3 + px_n^2 - r}{3x_n^2 + 2px_n + q}, \tag{2.38}$$

for $n \in \mathbb{N}_0$.

If a, b and c are the roots of (2.37), then (2.38) can be written in the form

$$x_{n+1} = \frac{2x_n^3 - (a+b+c)x_n^2 + abc}{3x_n^2 - 2(a+b+c)x_n + ab + bc + ca}, \quad n \in \mathbb{N}_0, \tag{2.39}$$

and by some calculations we have

$$x_{n+1} - a = \frac{2x_n^3 - (4a+b+c)x_n^2 + 2a(a+b+c)x_n - a^2(b+c)}{3x_n^2 - 2(a+b+c)x_n + ab + bc + ca}, \tag{2.40}$$

for $n \in \mathbb{N}_0$.

Let

$$q_3(t) = 2t^3 - (4a+b+c)t^2 + 2a(a+b+c)t - a^2(b+c).$$

Then, a direct calculation shows that $q_3(a) = 0$, from which it follows that

$$q_3(t) = (t-a)(2t^2 - (2a+b+c)t + a(b+c)) = (t-a)^2(2t - (b+c)).$$

Hence (2.40) can be written as follows

$$x_{n+1} - a = \frac{(x_n-a)^2(2x_n - (b+c))}{3x_n^2 - 2(a+b+c)x_n + ab + bc + ca}, \quad n \in \mathbb{N}_0. \tag{2.41}$$

Since the root of (2.37) we chose was arbitrary, we see that from (2.41) the following relations also hold

$$x_{n+1} - b = \frac{(x_n-b)^2(2x_n - (a+c))}{3x_n^2 - 2(a+b+c)x_n + ab + bc + ca}, \quad n \in \mathbb{N}_0, \tag{2.42}$$

$$x_{n+1} - c = \frac{(x_n-c)^2(2x_n - (a+b))}{3x_n^2 - 2(a+b+c)x_n + ab + bc + ca}, \quad n \in \mathbb{N}_0. \tag{2.43}$$

From (2.41)–(2.43), we have

$$\frac{x_{n+1}-a}{x_{n+1}-b} = \left(\frac{x_n-a}{x_n-b}\right)^2 \frac{2x_n-(b+c)}{2x_n-(a+c)}, \tag{2.44}$$

$$\frac{x_{n+1}-b}{x_{n+1}-c} = \left(\frac{x_n-b}{x_n-c}\right)^2 \frac{2x_n-(a+c)}{2x_n-(a+b)}, \tag{2.45}$$

$$\frac{x_{n+1}-c}{x_{n+1}-a} = \left(\frac{x_n-c}{x_n-a}\right)^2 \frac{2x_n-(a+b)}{2x_n-(b+c)}, \tag{2.46}$$

for $n \in \mathbb{N}_0$.

From (2.44)–(2.46) we see that we can obtain a solvable difference equation if $a+b$, $b+c$ and $c+a$ takes some of the values in the set $\{2a, 2b, 2c\}$. However, it is not difficult to see that in all the cases we get $a = b = c$, so that the equations (2.44)–(2.46) become trivial.

This analysis shows that the method used in solving equation (2.3) cannot be applied to equation (2.39). Nevertheless, there are some equations of the form

$$x_{n+1} = \frac{x_n^3 + px_n^2 + qx_n + r}{sx_n^2 + ux_n + v}, \quad n \in \mathbb{N}_0, \tag{2.47}$$

which are solvable in closed form, but are obtained by using some other iteration processes.

Example 2.17. The difference equation [20,33,45]

$$x_{n+1} = \frac{x_n^3 + 3ax_n}{3x_n^2 + a}, \quad n \in \mathbb{N}_0. \tag{2.48}$$

is used for finding a square root of number a. It is interesting that the difference equation is solvable in closed form. See [56] where a class/sequence of solvable difference equations for finding square roots is presented. Beside this, it is also interesting that the equation can be obtained, for example, from the Halley iteration process [18]

$$x_{n+1} = x_n - \frac{2f'(x_n)f(x_n)}{2f'(x_n)^2 - f''(x_n)f(x_n)}, \quad n \in \mathbb{N}_0,$$

applied to the function

$$f(t) = x^2 - a. \tag{2.49}$$

The fact was not mentioned in [56].

A detailed analysis of the method for solving equation (2.48) given in [56], shows that one of the most important facts used in the method is that the following relations hold

$$t^3 + 3at - \sqrt{a}(3t^2 + a) = (t - \sqrt{a})^3$$

and

$$t^3 + 3at + \sqrt{a}(3t^2 + a) = (t + \sqrt{a})^3.$$

Hence it is of interest to see for which values of parameters p, q, r, s, u and v the following identities hold

$$t^3 + pt^2 + qt + r - a(st^2 + ut + v) = (t - a)^3 \tag{2.50}$$

and

$$t^3 + pt^2 + qt + r - d(st^2 + ut + v) = (t - d)^3 \tag{2.51}$$

for some given numbers a and d such that $a \neq d$.

From (2.50) and (2.51) we obtain the following nonlinear algebraic system of equations

$$p - as = -3a, \qquad p - ds = -3d, \tag{2.52}$$

$$q - au = 3a^2, \qquad q - du = 3d^2, \tag{2.53}$$

$$r - av = -a^3, \qquad r - dv = -d^3. \tag{2.54}$$

From (2.52) we have

$$-s(a - d) = -3(a - d)$$

from which along with the assumption $a \neq d$, it follows that $s = 3$. By using it in (2.52) we get $p = 0$.

From (2.53) we have

$$-u(a-d) = 3(a-d)(a+d) \tag{2.55}$$

and

$$2q = 3(a^2 + d^2) + (a+d)u. \tag{2.56}$$

From (2.55) along with the assumption $a \neq d$, it follows that $u = -3(a+d)$. By using it in (2.56) we get $q = -3ad$.

From (2.54) we have

$$v(a-d) = (a-d)(a^2 + ad + d^2) \tag{2.57}$$

and

$$2r = -(a^3 + d^3) + (a+d)v. \tag{2.58}$$

From (2.57) along with the assumption $a \neq d$, it follows that $v = a^2 + ad + d^2$. By using it in (2.58) we get $r = ad(a+d)$.

This analysis suggests that the following special case of equation (2.47)

$$x_{n+1} = \frac{x_n^3 - 3adx_n + ad(a+d)}{3x_n^2 - 3(a+d)x_n + a^2 + ad + d^2}, \quad n \in \mathbb{N}_0, \tag{2.59}$$

is solvable. Indeed, the following theorem holds.

Theorem 2.18. *The equation (2.59), where $a, d \in \mathbb{C}$ are such that $a \neq d$ is solvable in closed form.*

Proof. From (2.59) and some simple calculation we have

$$x_{n+1} - a = \frac{(x_n - a)^3}{3x_n^2 - 3(a+d)x_n + a^2 + ad + d^2}, \quad n \in \mathbb{N}_0, \tag{2.60}$$

and

$$x_{n+1} - d = \frac{(x_n - d)^3}{3x_n^2 - 3(a+d)x_n + a^2 + ad + d^2}, \quad n \in \mathbb{N}_0. \tag{2.61}$$

From (2.60) and (2.61) we have

$$\frac{x_{n+1} - a}{x_{n+1} - d} = \left(\frac{x_n - a}{x_n - d}\right)^3, \quad n \in \mathbb{N}_0,$$

from which it easily follows that

$$\frac{x_n - a}{x_n - d} = \left(\frac{x_0 - a}{x_0 - d}\right)^{3^n}, \quad n \in \mathbb{N}_0,$$

and consequently

$$x_n = \frac{d\left(\frac{x_0 - a}{x_0 - d}\right)^{3^n} - a}{\left(\frac{x_0 - a}{x_0 - d}\right)^{3^n} - 1}, \quad n \in \mathbb{N}_0. \tag{2.62}$$

By some calculations it is checked that (2.62) is a solution to equation (2.59). □

Remark 2.19. Note that if $d = -a$ equation (2.59) reduces to the equation (2.48) where a is replaced by a^2, from which its solvability again follows.

2.7 Difference equations obtained by polynomials of the fourth degree

In [56] we have also shown that the difference equation

$$x_{n+1} = \frac{x_n^4 + 6ax_n^2 + a^2}{4x_n^3 + 4ax_n}, \quad n \in \mathbb{N}_0, \tag{2.63}$$

is solvable in closed form.

An interesting problem is to try to find a generalization of equation (2.63) by using the method above applied to equation (2.47), which is also solvable in closed form. The following equation

$$x_{n+1} = \frac{x_n^4 + px_n^3 + qx_n^2 + rx_n + s}{\alpha x_n^3 + \beta x_n^2 + \gamma x_n + \delta}, \quad n \in \mathbb{N}_0, \tag{2.64}$$

is a natural generalization of equation (2.63).

Following (2.50) and (2.51), it is of interest to see for which values of parameters p, q, r, s, α, β, γ and δ the following identities hold

$$t^4 + pt^3 + qt^2 + rt + s - a(\alpha t^3 + \beta t^2 + \gamma t + \delta) = (t-a)^4 \tag{2.65}$$

and

$$t^4 + pt^3 + qt^2 + rt + s - d(\alpha t^3 + \beta t^2 + \gamma t + \delta) = (t-d)^4 \tag{2.66}$$

for some given numbers a and d such that $a \neq d$.

From (2.65) and (2.66) we obtain the following nonlinear algebraic system of equations

$$p - \alpha a = -4a, \qquad p - \alpha d = -4d, \tag{2.67}$$

$$q - \beta a = 6a^2, \qquad q - \beta d = 6d^2, \tag{2.68}$$

$$r - \gamma a = -4a^3, \qquad r - \gamma d = -4d^3, \tag{2.69}$$

$$s - \delta a = a^4, \qquad s - \delta d = d^4. \tag{2.70}$$

From (2.67) we have

$$-\alpha(a-d) = -4(a-d)$$

from which along with the assumption $a \neq d$, it follows that $\alpha = 4$. By using it in (2.67) we get $p = 0$.

From (2.68) we have

$$-\beta(a-d) = 6(a-d)(a+d) \tag{2.71}$$

and

$$2q = 6(a^2 + d^2) + (a+d)\beta. \tag{2.72}$$

From (2.71) along with the assumption $a \neq d$, it follows that $\beta = -6(a+d)$. By using it in (2.72) we get $q = -6ad$.

From (2.69) we have

$$-\gamma(a-d) = -4(a-d)(a^2 + ad + d^2) \tag{2.73}$$

and

$$2r = -4(a^3 + d^3) + (a+d)\gamma. \tag{2.74}$$

From (2.73) along with the assumption $a \neq d$, it follows that $\gamma = 4(a^2 + ad + d^2)$. By using it in (2.74) we get $r = 4ad(a+d)$.

From (2.70) we have

$$-\delta(a-d) = (a-d)(a^3 + a^2d + ad^2 + d^3) \tag{2.75}$$

and

$$2s = a^4 + d^4 + (a+d)\delta. \tag{2.76}$$

From (2.75) along with the assumption $a \neq d$, it follows that $\delta = -(a^3 + a^2d + ad^2 + d^3)$. By using it in (2.76) we get $s = -ad(a^2 + ad + d^2)$.

From the analysis we obtain the following result.

Theorem 2.20. *The equation*

$$x_{n+1} = \frac{x_n^4 - 6adx_n^2 + 4ad(a+d)x_n - ad(a^2+ad+d^2)}{4x_n^3 - 6(a+d)x_n^2 + 4(a^2+ad+d^2)x_n - (a^3+a^2d+ad^2+d^3)}, \tag{2.77}$$

for $n \in \mathbb{N}_0$, where $a, d \in \mathbb{C}$ are such that $a \neq d$ is solvable in closed form.

Proof. From (2.77) and some simple calculation we have

$$x_{n+1} - a = \frac{(x_n - a)^4}{4x_n^3 - 6(a+d)x_n^2 + 4(a^2+ad+d^2)x_n - (a^3+a^2d+ad^2+d^3)}, \tag{2.78}$$

for $n \in \mathbb{N}_0$, and

$$x_{n+1} - d = \frac{(x_n - d)^4}{4x_n^3 - 6(a+d)x_n^2 + 4(a^2+ad+d^2)x_n - (a^3+a^2d+ad^2+d^3)}, \tag{2.79}$$

for $n \in \mathbb{N}_0$.

Using the relations in (2.78) and (2.79) we have

$$\frac{x_{n+1} - a}{x_{n+1} - d} = \left(\frac{x_n - a}{x_n - d}\right)^4, \quad n \in \mathbb{N}_0.$$

Hence

$$\frac{x_n - a}{x_n - d} = \left(\frac{x_0 - a}{x_0 - d}\right)^{4^n}, \quad n \in \mathbb{N}_0,$$

and consequently

$$x_n = \frac{d\left(\frac{x_0-a}{x_0-d}\right)^{4^n} - a}{\left(\frac{x_0-a}{x_0-d}\right)^{4^n} - 1}, \quad n \in \mathbb{N}_0, \tag{2.80}$$

which is the general solution to equation (2.77). □

Remark 2.21. Note that if $d = -a$ equation (2.77) reduces to the equation (2.63) where a is replaced by a^2, from which its solvability again follows.

2.8 Difference equations obtained by polynomials of the fifth degree

In [56] we have also shown that the difference equation

$$x_{n+1}=\frac{x_n^5+10ax_n^3+5a^2x_n}{5x_n^4+10ax_n^2+a^2},\quad n\in\mathbb{N}_0, \tag{2.81}$$

is solvable in closed form.

Our aim is to find a generalization of equation (2.81) similar to equation (2.47), which is also solvable in closed form. The following equation

$$x_{n+1}=\frac{x_n^5+px_n^4+qx_n^3+rx_n^2+sx_n+u}{\alpha x_n^4+\beta x_n^3+\gamma x_n^2+\delta x_n+\eta},\quad n\in\mathbb{N}_0, \tag{2.82}$$

is a natural generalization of (2.81).

We find the values of parameters p, q, r, s, u, α, β, γ, δ and η such that the following identities hold

$$t^5+pt^4+qt^3+rt^2+st+u-a(\alpha t^4+\beta t^3+\gamma t^2+\delta t+\eta)=(t-a)^5 \tag{2.83}$$

and

$$t^5+pt^4+qt^3+rt^2+st+u-d(\alpha t^4+\beta t^3+\gamma t^2+\delta t+\eta)=(t-d)^5 \tag{2.84}$$

for some given numbers a and d such that $a\neq d$.

From (2.83) and (2.84) we have

$$p-\alpha a=-5a,\qquad p-\alpha d=-5d, \tag{2.85}$$
$$q-\beta a=10a^2,\qquad q-\beta d=10d^2, \tag{2.86}$$
$$r-\gamma a=-10a^3,\qquad r-\gamma d=-10d^3, \tag{2.87}$$
$$s-\delta a=5a^4,\qquad s-\delta d=5d^4, \tag{2.88}$$
$$u-\eta a=-a^5,\qquad u-\eta d=-d^5. \tag{2.89}$$

From (2.85) it follows that

$$-\alpha(a-d)=-5(a-d)$$

from which along with the assumption $a\neq d$, it follows that $\alpha=5$. From this and (2.85) we get $p=0$.

From (2.86) we have

$$-\beta(a-d)=10(a-d)(a+d) \tag{2.90}$$

and

$$2q=10(a^2+d^2)+(a+d)\beta. \tag{2.91}$$

From (2.90) along with the assumption $a\neq d$, it follows that $\beta=-10(a+d)$. By using it in (2.91) we get $q=-10ad$.

From (2.87) we have

$$-\gamma(a-d)=-10(a-d)(a^2+ad+d^2) \tag{2.92}$$

and

$$2r=-10(a^3+d^3)+(a+d)\gamma. \tag{2.93}$$

From (2.92) along with the assumption $a \neq d$, it follows that $\gamma = 10(a^2 + ad + d^2)$. By using it in (2.93) we get $r = 10ad(a + d)$.

From (2.88) we have

$$-\delta(a-d) = 5(a-d)(a^3 + a^2d + ad^2 + d^3) \tag{2.94}$$

and

$$2s = 5a^4 + 5d^4 + (a+d)\delta. \tag{2.95}$$

From (2.94) along with the assumption $a \neq d$, it follows that $\delta = -5(a^3 + a^2d + ad^2 + d^3)$. By using it in (2.95) we get $s = -5ad(a^2 + ad + d^2)$.

From (2.89) we have

$$-\eta(a-d) = -(a-d)(a^4 + a^3d + a^2d^2 + ad^3 + d^4) \tag{2.96}$$

and

$$2u = -(a^5 + d^5) + (a+d)\delta. \tag{2.97}$$

From (2.96) along with the assumption $a \neq d$, it follows that $\eta = a^4 + a^3d + a^2d^2 + ad^3 + d^4$. By using it in (2.97) we get $u = ad(a^3 + a^2d + ad^2 + d^3)$.

From the analysis we obtain the following result.

Theorem 2.22. *Let*

$$p_4(t) = 5t^4 - 10(a+d)t^3 + 10(a^2+ad+d^2)t^2 - 5(a^3+a^2d+ad^2+d^3)t + a^4+a^3d+a^2d^2+ad^3+d^4.$$

Then the equation

$$x_{n+1} = \frac{x_n^5 - 10adx_n^3 + 10ad(a+d)x_n^2 - 5ad(a^2+ad+d^2)x_n + ad(a^3+a^2d+ad^2+d^3)}{p_4(x_n)}, \tag{2.98}$$

for $n \in \mathbb{N}_0$, where $a, d \in \mathbb{C}$ are such that $a \neq d$, is solvable in closed form.

Proof. From (2.98) we have

$$x_{n+1} - a = \frac{(x_n - a)^5}{p_4(x_n)}, \tag{2.99}$$

for $n \in \mathbb{N}_0$, and

$$x_{n+1} - d = \frac{(x_n - d)^5}{p_4(x_n)}, \tag{2.100}$$

for $n \in \mathbb{N}_0$.

Employing (2.99) and (2.100) it follows that

$$\frac{x_{n+1} - a}{x_{n+1} - d} = \left(\frac{x_n - a}{x_n - d}\right)^5, \quad n \in \mathbb{N}_0.$$

Hence

$$\frac{x_n - a}{x_n - d} = \left(\frac{x_0 - a}{x_0 - d}\right)^{5^n}, \quad n \in \mathbb{N}_0,$$

and consequently

$$x_n = \frac{d\left(\frac{x_0-a}{x_0-d}\right)^{5^n} - a}{\left(\frac{x_0-a}{x_0-d}\right)^{5^n} - 1}, \quad n \in \mathbb{N}_0, \tag{2.101}$$

finishing the proof. □

Remark 2.23. Note that if $d = -a$ equation (2.98) reduces to the equation (2.81) where a is replaced by a^2, implying its solvability.

2.9 A generalization of equations (2.63) and (2.81)

A natural question is if above theorems can be generalized to a more general difference equation. Although, at the first sight, the problem looks technically quite complex, it is interesting that the method used in the proofs of the above theorems can be also employed for finding the corresponding class of difference equations solvable in closed form, which are of the form

$$x_{n+1} = \frac{x_n^k + a_1 x_n^{k-1} + \cdots + a_{k-1}x_n + a_k}{b_0 x_n^{k-1} + b_1 x_n^{k-2} + \cdots + b_{k-2}x_n + b_{k-1}}, \quad n \in \mathbb{N}_0, \tag{2.102}$$

where $k \in \mathbb{N}$ and the coefficients

$$a_j,\ j = \overline{1,k}, \quad \text{and} \quad b_l,\ l = \overline{0,k-1}, \tag{2.103}$$

are complex numbers.

We want to find the values of the coefficients in (2.103) such that the following identities hold

$$t^k + a_1 t^{k-1} + \cdots + a_j t^{k-j} + \cdots + a_{k-1}t + a_k \\ - a(b_0 t^{k-1} + b_1 t^{k-2} + \cdots + b_{j-1}t^{k-j} + \cdots + b_{k-2}t + b_{k-1}) = (t-a)^k \tag{2.104}$$

and

$$t^k + a_1 t^{k-1} + \cdots + a_j t^{k-j} + \cdots + a_{k-1}t + a_k \\ - d(b_0 t^{k-1} + b_1 t^{k-2} + \cdots + b_{j-1}t^{k-j} + \cdots + b_{k-2}t + b_{k-1}) = (t-d)^k \tag{2.105}$$

for some given numbers a and d such that $a \neq d$.

From (2.104) and (2.105) we obtain the following nonlinear algebraic system of equations

$$\begin{array}{ll} a_1 - ab_0 = C_1^k(-a), & a_1 - db_0 = C_1^k(-d) \\ \vdots & \vdots \\ a_j - ab_{j-1} = C_j^k(-a)^j, & a_j - db_{j-1} = C_j^k(-d)^j, \\ \vdots & \vdots \\ a_k - ab_{k-1} = C_k^k(-a)^k, & a_k - db_{k-1} = C_k^k(-d)^k. \end{array} \tag{2.106}$$

From (2.106) we have

$$-(a-d)b_{j-1} = C_j^k((-a)^j - (-d)^j) \tag{2.107}$$

and

$$2a_j = C_j^k(-1)^j(a^j + d^j) + (a+d)b_{j-1}, \tag{2.108}$$

$j = \overline{1,k}$.

From (2.107) and since $a \neq d$ we obtain

$$b_{j-1} = C_j^k(-1)^{j+1}\frac{a^j - d^j}{a-d}, \quad j = \overline{1,k}. \tag{2.109}$$

By using (2.109) in (2.108) we have

$$2a_j = C_j^k(-1)^j(a^j + d^j) + (a+d)C_j^k(-1)^{j+1}\frac{a^j - d^j}{a-d},$$

$j = \overline{1,k}$, from which it follows that

$$a_j = adC_j^k(-1)^{j+1}\frac{a^{j-1} - d^{j-1}}{a-d}, \quad j = \overline{1,k}. \tag{2.110}$$

Remark 2.24. Note that from (2.110) with $j = 1$ it follows that $a_1 = 0$, whereas from (2.109) with $j = 1$ it follows that $b_0 = C_1^k = k$. Further, from (2.109) and (2.110) it follows that

$$a_j = adb_{j-1}\frac{a^{j-1} - d^{j-1}}{a^j - d^j}, \quad j = \overline{2,k}.$$

Now we formulate and prove the general result.

Theorem 2.25. *Let equation (2.102) be such that the coefficients a_j, $j = \overline{1,k}$, and b_l, $l = \overline{0,k-1}$, are given by (2.109) and (2.110), where $a, d \in \mathbb{C}$ are such that $a \neq d$. Then the equation is solvable in closed form.*

Proof. Let

$$p_{k-1}(t) = b_0t^{k-1} + b_1t^{k-2} + \cdots + b_{k-2}t + b_{k-1}.$$

Then from (2.102) and the choice of the coefficients a_j, $j = \overline{1,k}$, and b_l, $l = \overline{0,k-1}$ (see (2.104) and (2.105)), we have

$$x_{n+1} - a = \frac{(x_n - a)^k}{p_{k-1}(x_n)}, \tag{2.111}$$

for $n \in \mathbb{N}_0$, and

$$x_{n+1} - d = \frac{(x_n - d)^k}{p_{k-1}(x_n)}, \tag{2.112}$$

for $n \in \mathbb{N}_0$.

From (2.111) and (2.112) we have

$$\frac{x_{n+1} - a}{x_{n+1} - d} = \left(\frac{x_n - a}{x_n - d}\right)^k, \quad n \in \mathbb{N}_0.$$

Hence

$$\frac{x_n - a}{x_n - d} = \left(\frac{x_0 - a}{x_0 - d}\right)^{k^n}, \quad n \in \mathbb{N}_0,$$

and finally

$$x_n = \frac{d\left(\frac{x_0-a}{x_0-d}\right)^{k^n} - a}{\left(\frac{x_0-a}{x_0-d}\right)^{k^n} - 1}, \quad n \in \mathbb{N}_0,$$

as claimed. □

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Global dynamics of a class of age-infection structured cholera model with immigration

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Abstract. This paper is concerned with a class of age-structured cholera model with general infection rates. We first explore the existence and uniqueness, dissipativeness and persistence of the solutions, and the existence of the global attractor by verifying the asymptotical smoothness of the orbits. We then give mathematical analysis on the existence and local stability of the positive equilibrium. Based on the preparation, we further investigate the global behavior of the cholera infection model. Corresponding numerical simulations have been presented. Our results improve and generalize some known results on cholera models.

Keywords: cholera, age-structured, nonlinear incidence, global dynamics, Lyapunov functional.

2020 Mathematics Subject Classification: 92D30, 34K20, 35B35.

1 Introduction

Cholera is an acute water-borne infectious disease caused by *Vibrio cholerae*, with an estimated disease burden of 1.3 to 4.0 million cases and 21 000 to 143 000 deaths every year worldwide, which still affects at least 47 countries around the globe [2]. At present, there are 139 serogroups of *Vibrio cholerae*, of which O1 and O139 can cause cholera outbreaks. The disease peaks in the summer and it can be transmitted to humans by pathogen in the contaminated water and by person-to-person contact [20, 37]. Clinically, cholera can cause severe diarrhea, and the infected person will die of dehydration within a few days without prompt treatment [12]. In 1855, the British scholar John Snow found that the sewage in the city was the source of the spread of cholera epidemic [36], which was a major historical event in public hygiene. In the history of human epidemiology, cholera broke out many times in different countries and regions. In recent years, cholera outbreaks are mainly concentrated in developing countries with low medical and health level and lack of safe and hygienic drinking water sources. For example, cholera broke out in Haiti in 2010, leading to more than 665000 confirmed cases and 8183 deaths [10], and one of the causal factors for this outbreak is the transmission of local

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water source Artibonite river. The incidence rate of cholera will decrease in the future due to the global economic development and the reduction in global poverty [44], however, may increase in the next few decades due to the climate change and ocean changes caused by the extreme weather [1]. Therefore, it is of great theoretical and practical significance to study the transmission mechanism and development trend of cholera.

Recently, the mathematical model of cholera transmission has attracted widespread attention since the earlier study [9] on cholera modeling for the outbreak in the European Mediterranean region. In the aspect of mathematical modeling, Tien and Earn [37] introduced a water compartment into classical SIR model and established a water-borne infectious disease model with multiple transmission routes described by ordinary differential equations. In [37], the susceptible individual can not only be infected by the infected individual, but also be infected by indirect intake of contaminated water from the environment, which could be used to describe the transmission dynamics of cholera. By constructing an appropriate Lyapunov function, the global asymptotical stability of the equilibria of the system was obtained. Considering the hyperinfectious state of vibrio cholerae, Hartley et al. [20] extended the model proposed in [37] and studied the impact of hyperinfectious state on limiting the spread of cholera. Eisenberg et al. [15] aimed to evaluate the effects of patch structure on cholera spread and the type/target reproduction numbers were derived to quantify the strategies of cholera prevention. Some models involving different factors of cholera can be found in [3,33,38,40,41] and the references therein.

In modeling of epidemics, the age structure of individuals and pathogen is a significant characteristic [4, 8, 13, 27, 42]. In [7], Brauer et al. proposed an age-structured cholera model with multiple transmission pathways, which is

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda_s - \mu S(t) - \beta_i S(t) \displaystyle\int_0^\infty q(b)p(t,b)\mathrm{d}b - \beta_d S(t) \int_0^\infty k(a)i(t,a)\mathrm{d}a, \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\delta(a)i(t,a), \\ \dfrac{\partial p(t,b)}{\partial t} + \dfrac{\partial p(t,b)}{\partial b} = -\gamma(b)p(t,b), \end{cases} \tag{1.1}$$

with initial condition

$$S(0) = S_0, \quad i(0,\cdot) = i_0(\cdot) \in \mathcal{L}^1_+(0,+\infty), \quad p(0,\cdot) = p_0(\cdot) \in \mathcal{L}^1_+(0,+\infty), \tag{1.2}$$

and boundary condition

$$\begin{cases} i(t,0) = \beta_i S(t) \displaystyle\int_0^\infty q(b)p(t,b)\mathrm{d}b + \beta_d S(t) \int_0^\infty k(a)i(t,a)\mathrm{d}a, \\ p(t,0) = \displaystyle\int_0^\infty \xi(a)i(t,a)\mathrm{d}a. \end{cases} \tag{1.3}$$

where $S(t)$ is the density of the susceptible population at time t, $i(t,a)$ and $p(t,b)$ are the densities of the infectious population and the pathogen at time t with age a and b, respectively. The parameters of model (1.1) are explained in Table 1.1. Brauer et al. successfully obtained the global dynamics of model (1.1) by using the method of Lyapunov functional. Moreover, some other results for model (1.1) can be found in the studies [7, 42], such as relative compactness of orbits and uniform persistence.

More and more studies showed that immigration of populations has a significant impact on the spread of cholera. Due to the drought, refugees from Mozambique poured into Zimbabwe at the end of 1992, making Zimbabwe face the first cholera epidemic since 1985, which

Parameter	Interpretation
Λ_s	Constant recruitment of susceptible individuals
μ	Natural death rate of susceptible individuals
β_d	Direct transmission coefficient of cholera
β_i	Indirect transmission coefficient of cholera
$\delta(a)$	Age specific removal rate of the infected individuals
$\gamma(b)$	Age specific removal rate of the pathogen
$\xi(a)$	Age specific shedding rate of an infected individual
$k(a)$	Measure the Infectivity of infected individuals
$q(b)$	Measure the Infectivity of pathogen

Table 1.1: Parameters and their biological meaning in model (1.1).

spread to 7 provinces (Zimbabwe has 8 provinces in total) within five months [5]. Research shows that cholera has a history of outbreak through the immigration caused by international flights [14] and international conferences [32]. By analyzing 26 strains isolated from the cholera outbreak in Haiti in 2010, Frerichs et al. [17] believes that this wave of cholera outbreak was caused by the spread of Vibrio cholerae to the local drinking water source by the UN peacekeeping force dispatched by Nepal to Haiti. In fact, for developed countries with safe and hygienic water resources, cholera can also enter through immigration. According to the Centers for Disease Control and Prevention of USA, there was an increase in cholera cases reported in the United States during cholera outbreaks in Latin America in the 1990s and countries close to the United States such as Haiti in 2010 [11]. Five European Union countries reported 26 confirmed cholera cases in 2018, of which 22 were immigrated from India, Pakistan, Thailand, Bangladesh, Myanmar and Tunisia [16]. Therefore, it is urgent to explore the impact of immigration on the development and evolution of cholera infectious disease, which is also one of the important topics in the study of infectious disease dynamics.

From the view of mathematical modeling of infectious disease, immigration of population was always supposed to be of constant recruitment rate in each compartment. Brauer and van den Driessche [6] studied the threshold-like results for disease transmission model with immigration of the infective. By using Lyapunov function, Sigdel and McCluskey [34] investigated the global stability for an SEI model with immigration. More specifically, the endemic equilibrium for the model proposed in [34] is globally asymptotically stable. Considering the vaccination effect in the modeling of infectious diseases, Henshaw and McCluskey [21] presented the results on the global stability of a vaccination model with immigration, by virtue of the key method of constructing appropriate Lyapunov function. Meanwhile, age-dependent immigration rate seems more realistic in the real world and it is meaningful to investigate the age-structured models with immigration. In [30], McCluskey introduced an age-structured epidemic model with immigration. With an ingenious Lyapunov functional, the stability of endemic equilibrium for the SEI model with immigration was proved successfully. Zhang and Liu [46] further extended the study in [30] by introducing general nonlinear incidence. More recently, McCluskey [31] proved a general result for a Lyapunov calculation for the model with immigration and applied the results to a multi-group SIR model.

In (1.1), the incidence rates are assumed to be bilinear. Actually, nonlinear incidence rates are critical for accounting for a variety of nonlinear features of the corresponding biological phenomena. For example, Beddington–DeAngelis [23], Holling type II [24], Crowley–Martin

[45] and general incidence [18,26]. Motivated by the above studies, in this paper, we shall consider a generalization of model (1.1) by taking general incidence rates into account. However, to our best knowledge, there is no study on the age-structured cholera model with immigration. Based on model (1.1), we further introduce the immigration of infectious individuals and pathogen into the cholera model. Let $\Lambda_i(a)$ and $\Lambda_p(b)$ represent the recruitment through immigration into the infectious group and the pathogen group. Let

$$Q(t) = \int_0^\infty q(b)p(t,b)\mathrm{d}b \quad \text{and} \quad J(t) = \int_0^\infty k(a)i(t,a)\mathrm{d}a$$

represent the infectivity of infected individuals with infection age a and the total infectivity of pathogen with pathogen age b. In the current paper, we focus on the following age-structured cholera model with immigration

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda_s - \mu S(t) - S(t)f(J(t)) - S(t)g(Q(t)), \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = \Lambda_i(a) - \delta(a)i(t,a), \\ \dfrac{\partial p(t,b)}{\partial t} + \dfrac{\partial p(t,b)}{\partial b} = \Lambda_p(b) - \gamma(b)p(t,b), \end{cases} \tag{1.4}$$

with boundary condition

$$\begin{cases} i(t,0) = S(t)f(J(t)) + S(t)g(Q(t)), \;\; t > 0, \\ p(t,0) = P(t) := \displaystyle\int_0^\infty \xi(a)i(t,a)\mathrm{d}a, \;\; t > 0, \end{cases} \tag{1.5}$$

and initial condition

$$X_0 := (S(0), i(0,\cdot), p(0,\cdot)) = (S_0, i_0(\cdot), p_0(\cdot)) \in \mathfrak{y}_+, \tag{1.6}$$

where $\mathfrak{y} := \mathbb{R} \times \mathcal{L}^1(0,\infty) \times \mathcal{L}^1(0,\infty)$ with norm

$$\|(\phi,\psi,\varphi)\|_{\mathfrak{y}} := |\phi| + \int_0^\infty |\psi(a)|\mathrm{d}a + \int_0^\infty |\varphi(b)|\mathrm{d}b, \quad \phi \in \mathbb{R}, \quad \psi,\varphi \in \mathcal{L}^1(0,\infty)$$

and $\mathfrak{y}_+ := \mathbb{R}_+ \times \mathcal{L}^1_+(0,\infty) \times \mathcal{L}^1_+(0,\infty)$ is the positive cone of $\mathfrak{y}$. Here $\mathcal{L}^1(0,\infty)$ denotes the space of $\mathcal{L}^1$-integrable functions from the interval $(0,\infty)$ to itself. All the other coefficients in system (1.4)–(1.6) and the corresponding biological interpretation are the same as those in (1.1)–(1.3).

In the current paper, we study the global asymptotical stability of the unique positive equilibrium, which need to construct suitable Lyapunov functional. Mathematically, the age-based immigration rate and the indirect/direct transmission route of cholera generate a huge difficulty in constructing the proper Lyapunov functional. Moreover, the general incidence will bring great trouble to the calculation of the derivative of Lyapunov functional. For the well-posedness of the Lyapunov functional, we also need verify the uniform persistence of the system. The theoretical analysis shows that there exists a unique globally asymptotically stable endemic equilibrium, and the disease persists at the endemic level. The results in present paper not only serve as a supplement and generalization of the works in F. Brauer et al. [7], but also deal with some other new epidemic models with multiple transmission routes and immigration.

The plan of this article is as follows. In Section 2, we make some preliminaries for the system. In Section 3, we explore the asymptotical smoothness and global attractor. In Section 4, we explore the existence and local stability of the positive equilibrium. In Section 5, we construct a Lyapunov functional to discuss the global stability of the equilibrium. Numerical simulation and a brief conclusion will be given in section 6.

2 Preliminaries

Firstly, for system (1.4)–(1.6), we give the following assumptions.

Assumption 1. Constants $\Lambda_s,\ \mu \in \mathbb{R}_+$. For functions $\xi(\cdot),\ k(\cdot),\ q(\cdot),\ \delta(\cdot),\ \gamma(\cdot) \in \mathcal{L}^\infty_+(0,+\infty)$, $\Lambda_i(\cdot),\ \Lambda_p(\cdot) \in \mathcal{L}^1_+(0,+\infty)$, we make the following assumptions.

(I) For $\delta(\cdot)$, denote $\tilde{\delta} := \int_0^\infty \delta(\tau)\mathrm{d}\tau$, and denote $\bar{\delta}$ and $\underline{\delta}$ as the essential supremum and essential infimum of $\delta(\cdot)$, so do $\xi(\cdot),\ k(\cdot),\ q(\cdot),\ \gamma(\cdot),\ \Lambda_i(\cdot)$ and $\Lambda_p(\cdot)$;

(II) $\xi(\cdot),\ k(\cdot),\ q(\cdot),\ \delta(\cdot)$ and $\gamma(\cdot)$ are Lipschitz continuous.

Assumption 2. For functions $f(\cdot),\ g(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, we introduce the following assumptions.

(I) $f(\cdot)$ and $g(\cdot)$ are Lipschitz continuous on $\mathbb{R}_+$ with $f(0) = g(0) = 0$;

(II) $\frac{f(z)}{z} \geqslant f'(z) \geqslant 0,\ \frac{g(z)}{z} \geqslant g'(z) \geq 0$ and $f''(z) \leqslant 0, g''(z) \leqslant 0$, for $z \in \mathbb{R}_+$.

2.1 Existence of unique solution

Denote the following spaces

$$\begin{aligned}
\mathcal{X} &= \mathbb{R} \times \mathbb{R} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}),\\
\mathcal{X}_0 &= \mathbb{R} \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}) \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}),\\
\mathcal{X}_+ &= \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R}_+ \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}),\\
\mathcal{X}_{0+} &= \mathcal{X}_+ \cap \mathcal{X}_0 = \mathbb{R}_+ \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}) \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}),
\end{aligned}$$

One defines a linear operator $\mathbf{A} : \mathrm{Dom}(\mathbf{A}) \subset \mathcal{X} \to \mathcal{X}$ as follows,

$$\mathbf{A}\begin{pmatrix}\phi_1\\ \begin{pmatrix}0\\ \varphi_1\end{pmatrix}\\ \begin{pmatrix}0\\ \varphi_2\end{pmatrix}\end{pmatrix} = \begin{pmatrix}-\mu\phi_1\\ \begin{pmatrix}-\varphi_1(0)\\ -\delta(\cdot)\varphi_1 - \varphi_1'\end{pmatrix}\\ \begin{pmatrix}-\varphi_2(0)\\ -\gamma(\cdot)\varphi_2 - \varphi_2'\end{pmatrix}\end{pmatrix}$$

with $\mathrm{Dom}(\mathbf{A}) = \mathbb{R} \times \{0\} \times \mathcal{W}^{1,1}(0,\infty) \times \{0\} \times \mathcal{W}^{1,1}(0,\infty)$, where $\mathcal{W}^{1,1}(0,\infty)$ denotes the Sobolev space of locally summable functions $y : \mathbb{R}^+ \to \mathbb{R}$ such that for every multi-index α with $\|\alpha\| \leqslant 1$, the weak derivative $D^\alpha y \in \mathcal{L}^1(0,\infty)$ exists. Moreover, define a nonlinear operator $\mathbf{F} : \overline{\mathrm{Dom}(\mathbf{A})} \subset \mathcal{X} \to \mathcal{X}$ as

$$\mathbf{F}\begin{pmatrix}\phi_1\\ \begin{pmatrix}0\\ \varphi_1\end{pmatrix}\\ \begin{pmatrix}0\\ \varphi_2\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\Lambda - \phi_1 f\left(\int_0^\infty k(a)\varphi_1(a)da\right) - \phi_1 g\left(\int_0^\infty q(b)\varphi_2(b)db\right)\\ \begin{pmatrix}\phi_1 f\left(\int_0^\infty k(a)\varphi_1(a)da\right) + \phi_1 g\left(\int_0^\infty q(b)\varphi_2(b)db\right)\\ \Lambda_i(a)\end{pmatrix}\\ \begin{pmatrix}\int_0^\infty \xi(a)\varphi_1(a)da\\ \Lambda_p(b)\end{pmatrix}\end{pmatrix}.$$

Let $u(t) = (S(t), (0, i(t,\cdot))^{\mathrm{T}}, (0, p(t,\cdot))^{\mathrm{T}})^{\mathrm{T}} \in \mathcal{X}_{0+}$, then we can write system (1.4) as the following abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = \mathbf{A}u(t) + \mathbf{F}(u(t)), \quad \forall t \geqslant 0, \\ u(0) = u_0 \in \mathcal{X}_0 \cap \mathcal{X}_{0+}. \end{cases} \tag{2.1}$$

To show the existence of unique solutions for system (1.4), we need to prove the operator $\mathbf{A}$ as a Hille–Yosida operator.

Theorem 2.1. *The operator* $\mathbf{A}$ *is a Hille–Yosida operator.*

Proof. In order to apply Hille–Yosida theorem [29], we need find $\zeta = (\hat{\phi}_1, \hat{\varphi}_{10}, \hat{\varphi}_1, \hat{\varphi}_{20}, \hat{\varphi}_2) \in \mathcal{X}$, such that for $(\phi_1, 0, \varphi_1, 0, \varphi_2) \in \mathrm{Dom}(\mathbf{A})$, there holds

$$(\lambda\mathbf{I} - \mathbf{A})^{-1} \begin{pmatrix} \hat{\phi}_1 \\ \begin{pmatrix} \hat{\varphi}_{10} \\ \hat{\varphi}_1 \end{pmatrix} \\ \begin{pmatrix} \hat{\varphi}_{20} \\ \hat{\varphi}_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix} \end{pmatrix}.$$

From above equation, we then yield

$$\hat{\phi}_1 = (\lambda\mathbf{I} - \mathbf{A})\phi_1 = \lambda\phi_1 - \mathbf{A}\phi_1 = (\lambda + \mu)\phi_1.$$

Thus, $\phi_1 = \frac{\hat{\phi}_1}{\lambda+\mu}$. Besides, we obtain

$$\hat{\varphi}_1 = (\lambda\mathbf{I} - \mathbf{A})\varphi_1 = \lambda\varphi_1 - \mathbf{A}\varphi_1 = \lambda\varphi_1 + \delta(\cdot)\varphi_1 + \varphi_1' = (\lambda + \delta(\cdot))\varphi_1 + \varphi_1'.$$

Thus, there holds

$$\varphi_1' = \hat{\varphi}_1 - (\lambda + \delta(\cdot))\varphi_1,$$

and we have

$$\varphi_1(a) = \hat{\varphi}_{10} e^{-\int_0^a (\lambda+\delta(s))\mathrm{d}s} + \int_0^a \hat{\varphi}_1(\tau) e^{-\int_\tau^a (\lambda+\delta(s))\mathrm{d}s} \mathrm{d}\tau.$$

Similarly, we get that

$$\varphi_2(b) = \hat{\varphi}_{20} e^{-\int_0^b (\lambda+\gamma(s))\mathrm{d}s} + \int_0^b \hat{\varphi}_2(\tau) e^{-\int_\tau^b (\lambda+\gamma(s))\mathrm{d}s} \mathrm{d}\tau.$$

Further, there holds

$$\begin{aligned}
\|\varphi_1\|_{\mathcal{L}_1} &= \int_0^\infty |\varphi_1(a)| \mathrm{d}a \\
&= \int_0^\infty \left| \hat{\varphi}_{10} e^{-\int_0^a (\lambda+\delta(s))\mathrm{d}s} + \int_0^a \hat{\varphi}_1(\tau) e^{-\int_\tau^a (\lambda+\delta(s))\mathrm{d}s} d\tau \right| \mathrm{d}a \\
&\leqslant |\hat{\varphi}_{10}| \int_0^\infty e^{-\int_0^a (\lambda+\delta(s))\mathrm{d}s} \mathrm{d}a + \int_0^\infty \int_0^a |\hat{\varphi}_1(\tau)| e^{-\int_\tau^a (\lambda+\delta(s))\mathrm{d}s} \mathrm{d}\tau \mathrm{d}a \\
&\leqslant |\hat{\varphi}_{10}| \frac{1}{\lambda + \underline{\delta}} + \int_0^\infty \int_\tau^\infty |\hat{\varphi}_1(\tau)| e^{-(a-\tau)(\lambda+\underline{\delta})} \mathrm{d}a \mathrm{d}\tau \\
&= |\hat{\varphi}_{10}| \frac{1}{\lambda + \underline{\delta}} + \int_0^\infty |\hat{\varphi}_1(\tau)| e^{\tau(\lambda+\underline{\delta})} \int_\tau^\infty e^{-a(\lambda+\underline{\delta})} \mathrm{d}a \mathrm{d}\tau \\
&= |\hat{\varphi}_{10}| \frac{1}{\lambda + \underline{\delta}} + \int_0^\infty \frac{|\hat{\varphi}_1(\tau)|}{\lambda + \underline{\delta}} \mathrm{d}\tau \\
&= \frac{|\hat{\varphi}_{10}|}{\lambda + \underline{\delta}} + \frac{\|\hat{\varphi}_1\|_{\mathcal{L}_1}}{\lambda + \underline{\delta}}.
\end{aligned}$$

Similarly, we can obtain that

$$\|\varphi_2\|_{\mathcal{L}_1} \leqslant \frac{|\hat{\varphi}_{20}|}{\lambda+\underline{\gamma}} + \frac{\|\hat{\varphi}_2\|_{\mathcal{L}_1}}{\lambda+\underline{\gamma}}.$$

Let $\eta = \min\{\underline{\delta}, \underline{\gamma}, \mu\}$. For any $\zeta = (\hat{\phi}_1, \hat{\varphi}_{10}, \hat{\varphi}_1, \hat{\varphi}_{20}, \hat{\varphi}_2) \in \mathcal{X}$, we have

$$\begin{aligned}\|(\lambda \mathbf{I} - \mathbf{A})^{-1}\zeta\| &= |\phi_1| + |0| + \|\varphi_1\|_{\mathcal{L}_1} + |0| + \|\varphi_2\|_{\mathcal{L}_1} \\ &\leqslant \frac{|\hat{\phi}_1|}{\lambda+\mu} + \frac{|\hat{\varphi}_{10}|}{\lambda+\underline{\delta}} + \frac{\|\hat{\varphi}_1\|_{\mathcal{L}_1}}{\lambda+\underline{\delta}} + \frac{|\hat{\varphi}_{20}|}{\lambda+\underline{\gamma}} + \frac{\|\hat{\varphi}_2\|_{\mathcal{L}_1}}{\lambda+\underline{\gamma}} \\ &\leqslant \frac{\|\zeta\|}{\lambda+\eta}.\end{aligned}$$

Thus, the linear operator $\mathbf{A}$ is a Hille–Yosida operator due to [29]. □

Let $X_0 = (S(t), (0, i(t,\cdot))^{\mathrm{T}}, (0, p(t,\cdot))^{\mathrm{T}})^{\mathrm{T}} \in \mathcal{X}_{0+}$, thanks to [29, Theorem 5.2.7], we have the following theorem.

Theorem 2.2. *There exists a unique determined semiflow $\{\mathfrak{U}(t)\}_{t\geqslant 0}$ on $\mathcal{X}_{0+}$ such that for any X_0, a unique continuous map $\mathfrak{U} \in C([0,\infty], \mathcal{X}_{0+})$ exists as an integrated solution of the Cauchy problem (2.1), that is,*

$$\begin{cases} \int_0^t \mathfrak{U}(s)X_0 \mathrm{d}s \in \mathrm{Dom}(\mathbf{A}), \\ \mathfrak{U}(t)X_0 = X_0 + \mathbf{A}\int_0^t \mathfrak{U}(s)X_0\mathrm{d}s + \int_0^\infty \mathbf{F}(\mathfrak{U}(s)X_0)\mathrm{d}s, \end{cases}$$

for all $t \geqslant 0$.

2.2 Dissipativeness and persistence

Combining equations (1.5) and (1.6), integrating the last two equations of (1.4) along the characteristic lines yields

$$i(t,a) = \begin{cases} i(t-a,0)\sigma_1(a) + \int_0^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant a \leqslant t, \\ i(0,a-t)\frac{\sigma_1(a)}{\sigma_1(a-t)} + \int_{a-t}^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant t \leqslant a, \end{cases} \tag{2.2}$$

$$p(t,b) = \begin{cases} p(t-b,0)\sigma_2(b) + \int_0^b \Lambda_p(\epsilon)\frac{\sigma_2(b)}{\sigma_2(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant b \leqslant t, \\ p(0,b-t)\frac{\sigma_2(b)}{\sigma_2(b-t)} + \int_{b-t}^b \Lambda_p(\epsilon)\frac{\sigma_2(b)}{\sigma_2(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant t \leqslant b, \end{cases} \tag{2.3}$$

where

$$\sigma_1(a) = e^{-\int_0^a \delta(\tau)\mathrm{d}\tau} \quad \text{and} \quad \sigma_2(b) = e^{-\int_0^b \gamma(\tau)\mathrm{d}\tau}. \tag{2.4}$$

Now, we are concerned with the boundedness of solutions. Let $Y_1 := \frac{\Lambda_s + \tilde{\Lambda}_i}{\eta}$, $Y_2 := \frac{\bar{\xi}Y_1 + \tilde{\Lambda}_p}{\eta}$ and

$$\Pi := \left\{ (S(t), i(t,a), p(t,b)) \in \mathcal{X}_{0+} \;\middle|\; S(t) + \int_0^\infty i(t,a)\mathrm{d}a + \int_0^\infty p(t,b)\mathrm{d}b \leqslant Y_1 + Y_2 \right\}.$$

We arrive at the following theorem.

Theorem 2.3. *For* (1.4), $\mathfrak{U}$ *is point dissipative, which means there is a bounded set* Π *that attracts all points in* $\mathcal{X}_+$.

Proof. Note that

$$\begin{aligned}\int_0^\infty i(t,a)\mathrm{d}a &= \int_0^t i(t,a)\mathrm{d}a + \int_t^\infty i(t,a)\mathrm{d}a\\ &= \int_0^t i(t-a,0)\sigma_1(a)\mathrm{d}a + \int_0^t\int_0^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a\\ &\quad + \int_t^\infty i(0,a-t)\frac{\sigma_1(a)}{\sigma_1(a-t)}\mathrm{d}a + \int_t^\infty\int_{a-t}^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a.\end{aligned}$$

Interchanging the order of integration for the double integrals and making change of integration variable for the two single integrals gives

$$\int_0^\infty i(t,a)\mathrm{d}a = \int_0^t i(\tau,0)\sigma_1(t-\tau)\mathrm{d}\tau + \int_0^\infty i(0,\tau)\frac{\sigma_1(t+\tau)}{\sigma_1(\tau)}\mathrm{d}\tau + \int_0^\infty\int_\epsilon^{\epsilon+t} \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}a\mathrm{d}\epsilon.$$

Thus, there holds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\mathrm{d}a &= \sigma_1(0)i(t,0) + \int_0^t i(\tau,0)\frac{\mathrm{d}}{\mathrm{d}t}\sigma_1(t-\tau)\mathrm{d}\tau\\ &\quad + \int_0^\infty i(0,\tau)\frac{\frac{\mathrm{d}}{\mathrm{d}t}\sigma_1(t+\tau)}{\sigma_1(\tau)}\mathrm{d}\tau + \int_0^\infty \Lambda_i(\epsilon)\frac{\sigma_1(\epsilon+t)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\\ &= i(t,0) - \int_0^t i(\tau,0)\delta(t-\tau)\sigma_1(t-\tau)\mathrm{d}\tau\\ &\quad - \int_0^\infty i(0,\tau)\frac{\delta(t+\tau)\sigma_1(t+\tau)}{\sigma_1(\tau)}\mathrm{d}\tau + \int_0^\infty \Lambda_i(\epsilon)\frac{\sigma_1(\epsilon+t)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\\ &= i(t,0) - \int_0^t i(t-a,0)\delta(a)\sigma_1(a)\mathrm{d}a\\ &\quad - \int_t^\infty i(0,a-t)\frac{\delta(a)\sigma_1(a)}{\sigma_1(a-t)}\mathrm{d}a + \int_0^\infty \Lambda_i(\epsilon)\frac{\sigma_1(\epsilon+t)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon.\end{aligned}$$

Note that

$$\begin{aligned}\int_0^t\int_0^a \Lambda_i(\epsilon)\frac{\sigma_1'(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a + \int_t^\infty\int_{a-t}^a \Lambda_i(\epsilon)\frac{\sigma_1'(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a &= \int_0^\infty\int_\epsilon^{\epsilon+t} \Lambda_i(\epsilon)\frac{\sigma_1'(a)}{\sigma_1(\epsilon)}\mathrm{d}a\mathrm{d}\epsilon\\ &= \int_0^\infty \Lambda_i(\epsilon)\frac{\sigma_1(\epsilon+t)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon - \int_0^\infty \Lambda_i(\epsilon)\mathrm{d}\epsilon,\end{aligned}$$

we thus have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\mathrm{d}a &= i(t,0) - \int_0^t i(t-a,0)\delta(a)\sigma_1(a)\mathrm{d}a - \int_t^\infty i(0,a-t)\frac{\delta(a)\sigma_1(a)}{\sigma_1(a-t)}\mathrm{d}a\\ &\quad + \int_0^t\int_0^a \Lambda_i(\epsilon)\frac{\sigma_1'(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a + \int_t^\infty\int_{a-t}^a \Lambda_i(\epsilon)\frac{\sigma_1'(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a + \int_0^\infty \Lambda_i(\epsilon)\mathrm{d}\epsilon.\\ &= i(t,0) - \int_0^\infty \delta(a)i(t,a)\mathrm{d}a + \tilde{\Lambda}_i.\end{aligned}$$

Together with the first equation of (1.4), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(S(t) + \int_0^\infty i(t,a)\mathrm{d}a\right) \leqslant (\Lambda_s + \tilde{\Lambda}_i) - \eta\left(S(t) + \int_0^\infty i(t,a)\mathrm{d}a\right).$$

Hence,

$$S(t)+\int_0^\infty i(t,a)\mathrm{d}a \leqslant \Upsilon_1 - e^{-\eta t}\left\{\Upsilon_1-\left(S(0)+\int_0^\infty i(0,a)\mathrm{d}a\right)\right\}, \tag{2.5}$$

for any $X_0 \in \Pi$. Similarly, we can derive that

$$\int_0^\infty p(t,b)\mathrm{d}b \leqslant \Upsilon_2 - e^{-\eta t}\left\{\Upsilon_2-\int_0^\infty p(0,b)\mathrm{d}b\right\}, \tag{2.6}$$

for any $X_0 \in \Pi$. Hence, combining (2.5) and (2.6) yields

$$\|\mathfrak{U}(t,X_0)\|_{\mathcal{X}} \leqslant \Upsilon_1+\Upsilon_2 - e^{-\eta t}\left\{\Upsilon_1+\Upsilon_2-\left(S(0)+\int_0^\infty i(0,a)\mathrm{d}a+\int_0^\infty p(0,b)\mathrm{d}b\right)\right\}.$$

This implies that $\|\mathfrak{U}(t,X_0)\|_{\mathcal{X}} \leqslant \Upsilon_1+\Upsilon_2$ for $X_0 \in \Pi$, and the proof is complete. □

From Theorem 2.3, we obtain the following result.

Corollary 2.4. *If $X_0 \in \mathcal{X}_+$ and $\|X_0\|_{\mathcal{X}} \leqslant B$ with some constant $B \geqslant \Upsilon_1+\Upsilon_2$, then for $t \in \mathbb{R}_+$, we have the following statements*

(i) $0 \leqslant S(t)$, $\int_0^\infty i(t,a)\mathrm{d}a \leqslant B$ *and* $\int_0^\infty p(t,b)\mathrm{d}b \leqslant B$;

(ii) $i(t,0) \leqslant (\bar{k}f'(0)+\bar{q}g'(0))B^2$ *and* $p(t,0) \leqslant \bar{\xi}B$.

The following corollary generates a positive asymptotical lower bound of $S(t)$.

Corollary 2.5. *If $X_0 \in \mathcal{X}_+$, then*

$$\liminf_{t\to\infty} S(t) \geqslant \frac{\Lambda_s}{\mu+f'(0)\bar{k}B+g'(0)\bar{q}B}.$$

Proof. For any $\epsilon > 0$, there exists a $t_0 \in \mathbb{R}_+$ such that

$$\int_0^\infty i(t,a)\mathrm{d}a \leqslant B+\varepsilon \quad \text{and} \quad \int_0^\infty p(t,b)\mathrm{d}b \leqslant B+\varepsilon$$

for $t \geqslant t_0$. Then, for $t \geqslant t_0$,

$$\begin{aligned}\frac{\mathrm{dS(t)}}{\mathrm{dt}} &= \Lambda_s - S(t)(\mu+f(J(t))+g(Q(t)))\\ &\geqslant \Lambda_s - S(t)(\mu+f'(0)\bar{k}(B+\varepsilon)+g'(0)\bar{q}(B+\varepsilon)).\end{aligned}$$

This implies that

$$\liminf_{t\to\infty} S(t) \geqslant \frac{\Lambda_s}{\mu+f'(0)\bar{k}(B+\varepsilon)+g'(0)\bar{q}(B+\varepsilon)}.$$

Letting ϵ tend to 0 gives the required result. □

Then by similar verification in [30], we obtain the following proposition.

Theorem 2.6. *There exist $\tilde{t} > 0$ and $\epsilon > 0$ such that $i(t,0) > \epsilon$ and $p(t,0) > \epsilon$ for all $t \geqslant \tilde{t}$.*

3 Asymptotical smoothness and global attractor

For the existence of an attractor, the asymptotical smoothness of the semiflow $\mathfrak{U}$ is necessary. For this, by the similar argument in [30, Proposition 6], we claim that $J(t)$, $Q(t)$ and $P(t)$ are Lipschitz continuous with Lipschitz coefficients L_J, L_Q and L_P. Then we introduce the following lemma for the asymptotical smoothness of the semiflow.

Lemma 3.1 ([35]). *The semiflow $\mathfrak{U} : \mathbb{R}_+ \times \mathcal{X}_+ \to \mathcal{X}_+$ is asymptotically smooth if there exist maps $\mathfrak{U}_1, \mathfrak{U}_2 : \mathbb{R}_+ \times \mathcal{X}_+ \to \mathcal{X}_+$ satisfying $\mathfrak{U}(t,x) = \mathfrak{U}_1(t,x) + \mathfrak{U}_2(t,x)$, and for any bounded closed set $\mathbb{B} \subset \mathcal{X}_+$, which is forward invariant under $\mathfrak{U}$, there holds:*

(i) $\lim_{t\to\infty} \operatorname{diam} \mathfrak{U}_2(t,\mathbb{B}) = 0$;

(ii) *There exists $t_{\mathbb{B}} \geqslant 0$ such that $\mathfrak{U}_1(t,\mathbb{B})$ has compact closure for $t \geqslant t_{\mathbb{B}}$.*

For Lemma 3.1 (ii), we utilize the following lemma.

Lemma 3.2 ([35]). *A set $\mathbb{B} \in \mathcal{L}^1_+(0,\infty)$ has compact closure iff the following conditions hold:*

(i) $\sup_{f\in\mathbb{B}} \int_0^\infty f(z)\mathrm{d}z < \infty$;

(ii) $\lim_{r\to\infty} \int_r^\infty f(z)\mathrm{d}z \to 0$ *uniformly in* $f \in \mathbb{B}$;

(iii) $\lim_{h\to 0^+} \int_0^\infty |f(z+h) - f(z)|\mathrm{d}z \to 0$ *uniformly in* $f \in \mathbb{B}$;

(iv) $\lim_{h\to 0^+} \int_0^h f(z)\mathrm{d}z \to 0$ *uniformly in* $f \in \mathbb{B}$.

Based on above lemmas, we can obtain the following result.

Theorem 3.3. *The semiflow $\mathfrak{U}$ generated by (1.4) is asymptotically smooth.*

Proof. Define maps $\mathfrak{U}_1$ and $\mathfrak{U}_2$ such that $\mathfrak{U} = \mathfrak{U}_1 + \mathfrak{U}_2$, satisfying

$$\begin{cases} \mathfrak{U}_1(t,x_0) = (S(t), \grave{i}(t,\cdot), \grave{p}(t,\cdot)), \\ \mathfrak{U}_2(t,x_0) = (0, \grave{\varphi}_i(t,\cdot), \grave{\varphi}_p(t,\cdot)), \end{cases}$$

where

$$\grave{i}(t,a) = \begin{cases} i(t-a,0)\sigma_1(a), & 0 \leqslant a \leqslant t, \\ 0, & 0 \leqslant t \leqslant a, \end{cases}$$

$$\grave{p}(t,b) = \begin{cases} p(t-b,0)\sigma_2(b), & 0 \leqslant b \leqslant t, \\ 0, & 0 \leqslant t \leqslant b, \end{cases}$$

$$\grave{\varphi}_i(t,a) = \begin{cases} \displaystyle\int_0^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant a \leqslant t, \\ \displaystyle i(0,a-t)\frac{\sigma_1(a)}{\sigma_1(a-t)} + \int_{a-t}^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant t \leqslant a, \end{cases}$$

and

$$\grave{\varphi}_p(t,b) = \begin{cases} \displaystyle\int_0^b \Lambda_p(\epsilon)\frac{\sigma_2(b)}{\sigma_2(\epsilon)}\mathrm{d}\varepsilon, & 0 \leqslant b \leqslant t, \\ \displaystyle p(0,b-t)\frac{\sigma_2(b)}{\sigma_2(b-t)} + \int_{b-t}^b \Lambda_p(\epsilon)\frac{\sigma_2(b)}{\sigma_2(\epsilon)}\mathrm{d}\varepsilon, & 0 \leqslant t \leqslant b. \end{cases}$$

Firstly, we show that $\mathfrak{U}_2$ satisfies Lemma 3.1(i). For $X_0^1, X_0^2 \in \Pi$, letting $\varepsilon_1 = a - t$, we obtain

$$\begin{aligned}\|\grave{\varphi}_i^1(t,\cdot) - \grave{\varphi}_i^2(t,\cdot)\|_{\mathcal{L}_1} &= \int_t^\infty |i^1(0,a-t) - i^2(0,a-t)|\frac{\sigma_1(a)}{\sigma_1(a-t)}da \\ &= \int_0^\infty |i^1(0,\varepsilon_1) - i^2(0,\varepsilon_1)|\frac{\sigma_1(t+\varepsilon_1)}{\sigma_1(\varepsilon_1)}d\varepsilon_1 \\ &\leqslant \int_0^\infty e^{-\underline{\delta}t}|i^1(0,\varepsilon_1) - i^2(0,\varepsilon_1)|d\varepsilon_1 \\ &\leqslant 2Be^{-\underline{\delta}t}.\end{aligned}$$

Similarly, we have $\|\grave{\varphi}_p^1(t,\cdot) - \grave{\varphi}_p^2(t,\cdot)\|_{\mathcal{L}_1} \leqslant 2Be^{-\underline{\Upsilon}t}$. And thus, we have

$$\left\|\mathfrak{U}_2(t,X_0^1) - \mathfrak{U}_2(t,X_0^2)\right\|_{\mathcal{L}_1} \leqslant 2B(e^{-\underline{\delta}t} + e^{-\underline{\Upsilon}t}).$$

Hence, as $t \to \infty$, we have that $\mathrm{diam}\|\mathfrak{U}_2(t,X_0)\|_{\mathcal{X}} \to 0$. This accomplishes the verification of Lemma 3.1(i). Subsequently, we focus on the proof of Lemma 3.1(ii) by virtue of Lemma 3.2. By Proposition 2.4, we claim that conditions (i), (ii) and (iv) of Lemma 3.2 are satisfied since $0 \leqslant \grave{i}(t,a) = i(t-a,0)\sigma_1(a) \leqslant [f'(0)\bar{k} + g'(0)\bar{q}]B^2e^{-\underline{\delta}a}$. It suffices to verify the condition of Lemma 3.2 (iii). Choosing $h \in (0,t)$ small enough, one has

$$\begin{aligned}&\int_0^\infty |\grave{i}(a,t) - \grave{i}(a+h,t)|\mathrm{d}a \\ &\quad\leqslant \int_0^{t-h} |S(t-a-h)(f(J(t-a-h)) + g(Q(t-a-h)))(\sigma_1(a+h) - \sigma_1(a))|\mathrm{d}a \\ &\qquad + \int_0^{t-h} S(t-a-h)(|f(J(t-a-h)) - f(J(t-a))| \\ &\qquad\qquad + |g(Q(t-a-h)) - g(Q(t-a))|)\sigma_1(a)\mathrm{d}a \\ &\qquad + \int_0^{t-h} |S(t-a-h) - S(t-a)|(f(J(t-a)) + g(Q(t-a)))\sigma_1(a)\mathrm{d}a \\ &\qquad + f'(0)\int_{t-h}^{t} |S(t-a)J(t-a)\sigma_1(a)|\mathrm{d}a \\ &\qquad + g'(0)\int_{t-h}^{t} |S(t-a)Q(t-a)\sigma_1(a)|\mathrm{d}a \\ &\quad\leqslant (f'(0)\bar{k} + g'(0)\bar{q})B^2\int_0^{t-h} |\sigma_1(a+h) - \sigma_1(a)|\mathrm{d}a \\ &\qquad + f'(0)\int_0^{t-h} S(t-a-h)|J(t-a-h) - J(t-a)|\sigma_1(a)\mathrm{d}a \\ &\qquad + g'(0)\int_0^{t-h} S(t-a-h)|Q(t-a-h) - Q(t-a)|\sigma_1(a)\mathrm{d}a \\ &\qquad + \int_0^{t-h} (f'(0)J(t-a) + g'(0)Q(t-a))|S(t-a-h) - S(t-a)|\sigma_1(a)\mathrm{d}a \\ &\qquad + (f'(0)\bar{k} + g'(0)\bar{q})B^2h.\end{aligned} \tag{3.1}$$

From (2.4), we have

$$0 \leqslant \int_0^{t-h} |\sigma_1(a+h) - \sigma_1(a)|\mathrm{d}a = \int_0^h \sigma_1(a)\mathrm{d}a - \int_{t-h}^t \sigma_1(a)\mathrm{d}a \leqslant h. \tag{3.2}$$

Note that

$$\left|\frac{\mathrm{d}S(t)}{\mathrm{d}t}\right| \leqslant \Lambda_S + \mu B + (\bar{k}f'(0) + \bar{q}g'(0))B^2,$$

which means that $S(t)$ is Lipschitz continuous. Then

$$\int_0^{t-h} S(t-a-h)|J(t-a-h) - J(t-a)|\sigma_1(a)\mathrm{d}a \leqslant \frac{1}{\underline{\delta}}BL_J h, \tag{3.3}$$

and

$$\int_0^{t-h} S(t-a-h)|Q(t-a-h) - Q(t-a)|\sigma_1(a)\mathrm{d}a \leqslant \frac{1}{\underline{\delta}}BL_Q h. \tag{3.4}$$

Moreover, one has that

$$\begin{aligned}\int_0^{t-h} & (f'(0)J(t-a) + g'(0)Q(t-a))|S(t-a-h) - S(t-a)|\sigma_1(a)\mathrm{d}a \\ & \leqslant \frac{1}{\underline{\delta}}(f'(0)\bar{k} + g'(0)\bar{q})BL_s h,\end{aligned} \tag{3.5}$$

where $L_S := \Lambda_S + \mu B + (\bar{k}f'(0) + \bar{q}g'(0))B^2$. Substituting equations (3.2)–(3.5) into (3.1), one has that

$$\begin{aligned}\int_0^\infty & |\grave{i}(a,t) - \grave{i}(a+h,t)|\mathrm{d}a \\ & \leqslant 2(f'(0)\bar{k} + g'(0)\bar{q})B^2 h + \frac{1}{\underline{\delta}}(f'(0)L_J + g'(0)L_Q)Bh + \frac{1}{\underline{\delta}}(f'(0)\bar{k} + g'(0)\bar{q})BL_s h.\end{aligned} \tag{3.6}$$

Thus, Lemma 3.2 holds. Hence, $\grave{i}(t,a)$ remains in a pre-compact subset in $\mathcal{L}^1_+(0,\infty)$. The same arguments can be derived on $\grave{p}(t,b)$ and this completes the proof. □

According to [19], a global attractor exists since the semiflow $\mathfrak{U}$ is asymptotically smooth.

Theorem 3.4. *The semi-flow $\mathfrak{U}(t)$ has a global attractor in $\mathcal{X}_+$.*

4 Local stability of the infection equilibrium

Because the model introduces immigration terms, there exists no infection-free equilibrium for system (1.4). Assume $E^* = (S^*, i^*(a), p^*(b))$ be an equilibrium for system (1.4), then it satisfies the following equations.

$$\begin{cases} \Lambda_s = \mu S^* + S^* f(J^*) + S^* g(Q^*), \\ \dfrac{\mathrm{d}i^*(a)}{\mathrm{d}a} = \Lambda_i(a) - \delta(a)i^*(a), \\ \dfrac{\mathrm{d}p^*(b)}{\mathrm{d}b} = \Lambda_p(b) - \gamma(b)p^*(b), \\ i^*(0) = S^* f(J^*) + S^* g(Q^*), \\ p^*(0) = \displaystyle\int_0^\infty \xi(a)i^*(a)\mathrm{d}a, \end{cases} \tag{4.1}$$

where $Q^* = \int_0^\infty q(b)p^*(b)\mathrm{d}b$ and $J^* = \int_0^\infty k(a)i^*(a)\mathrm{d}a$. Denote

$$\begin{aligned}
\Xi_1 &= \int_0^\infty k(a)\sigma_1(a)\mathrm{d}a, & \Xi_4 &= \int_0^\infty k(a)\sigma_1(a)\int_0^a \frac{\Lambda_i(\tau)}{\sigma_1(\tau)}\mathrm{d}\tau\mathrm{d}a,\\
\Xi_2 &= \int_0^\infty q(b)\sigma_2(b)\mathrm{d}b, & \Xi_5 &= \int_0^\infty q(b)\sigma_2(b)\int_0^b \frac{\Lambda_p(\tau)}{\sigma_2(\tau)}\mathrm{d}\tau\mathrm{d}b,\\
\Xi_3 &= \int_0^\infty \xi(a)\sigma_1(a)\mathrm{d}a, & \Xi_6 &= \int_0^\infty \xi(a)\sigma_1(a)\int_0^a \frac{\Lambda_i(\tau)}{\sigma_1(\tau)}\mathrm{d}\tau\mathrm{d}a.
\end{aligned} \tag{4.2}$$

Owing to equations (4.1), we derive

$$i^*(0) = \Lambda_s - \mu S^* \tag{4.3}$$

and

$$\begin{aligned}
i^*(a) &= i^*(0)\sigma_1(a) + \int_0^a \Lambda_i(\tau)\frac{\sigma_1(a)}{\sigma_1(\tau)}\mathrm{d}\tau,\\
p^*(b) &= p^*(0)\sigma_2(b) + \int_0^b \Lambda_p(\tau)\frac{\sigma_2(b)}{\sigma_2(\tau)}\mathrm{d}\tau.
\end{aligned} \tag{4.4}$$

Then substituting (4.3) into the first equation of (4.4) yields

$$i^*(a) = (\Lambda_s - \mu S^*)\sigma_1(a) + \int_0^a \Lambda_i(\tau)\frac{\sigma_1(a)}{\sigma_1(\tau)}\mathrm{d}\tau. \tag{4.5}$$

Combining (4.5) and the last equation of (4.1) yields

$$p^*(0) = (\Lambda_s - \mu S^*)\Xi_3 + \Xi_6. \tag{4.6}$$

Further, substituting (4.6) into the second equation of (4.4), one has that

$$p^*(b) = (\Lambda_s - \mu S^*)\Xi_3\sigma_2(b) + \Xi_6\sigma_2(b) + \int_0^b \Lambda_p(\tau)\frac{\sigma_2(b)}{\sigma_2(\tau)}\mathrm{d}\tau. \tag{4.7}$$

Thus, in order to find E^*, inspired from the first equation of (4.1), we need to search for the zero of the following formula

$$h(S) = \Lambda_s - \mu S - Sf((\Lambda_s - \mu S)\Xi_1 + \Xi_4) - Sg((\Lambda_s - \mu S)\Xi_2\Xi_3 + \Xi_2\Xi_6 + \Xi_5).$$

Since $h(0) = \Lambda_s > 0$ and $h(\frac{\Lambda_s}{\mu}) < 0$, by the Intermediate Vale Theorem, $h(S)$ has one zero in $(0, \frac{\Lambda_s}{\mu})$. Thus, there exists at least one $S^* \in \left(0, \frac{\Lambda_s}{\mu}\right)$ and thus at least one positive equilibrium E^* exists.

In the following, we first focus on the local stability.

Theorem 4.1. *System (1.4) has one infection equilibrium E^*, which is locally asymptotically stable.*

Proof. The linearization of system (1.4)–(1.5) on $(S^*, i^*(a), p^*(b))$ is

$$\begin{cases}
\dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = (-\mu - f(J^*) - g(Q^*))S(t) - S^*f'(J^*)J - S^*(t)g'(Q^*)Q,\\
\dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\delta(a)i(t,a),\\
\dfrac{\partial p(t,b)}{\partial t} + \dfrac{\partial p(t,b)}{\partial b} = -\gamma(b)p(t,b),\\
i(t,0) = (f(J^*) + g(Q^*))S + S^*f'(J^*)J + S^*g'(Q^*)Q,\\
p(t,0) = \displaystyle\int_0^\infty \xi(a)i(t,a)\mathrm{d}a.
\end{cases} \tag{4.8}$$

Substituting $(S(t), i(t,a), p(t,b)) = (\hat{S}, \hat{i}(a), \hat{p}(b))e^{\lambda t}$ into (4.8) and dropping the hat, we can obtain

$$\begin{cases} \lambda S = -(\mu + f(J^*) + g(Q^*))S - S^* f'(J^*) \int_0^\infty k(a)i(a)\mathrm{d}a - S^* g'(Q^*) \int_0^\infty q(b)p(b)\mathrm{d}b, \\ i(a) = i(0)e^{-\lambda a}\sigma_1(a), \\ p(b) = p(0)e^{-\lambda b}\sigma_2(b), \\ i(0) = (f(J^*) + g(Q^*))S + S^* g'(Q^*) \int_0^\infty q(b)p(b)\mathrm{d}b + S^* f'(J^*) \int_0^\infty k(a)i(a)\mathrm{d}a, \\ p(0) = \int_0^\infty \xi(a)i(a)\mathrm{d}a. \end{cases} \tag{4.9}$$

Denote

$$\Gamma_1(\lambda) = \int_0^\infty k(a)e^{-\lambda a}\sigma_1(a)\mathrm{d}a, \qquad \Gamma_2(\lambda) = \int_0^\infty q(b)e^{-\lambda b}\sigma_2(b)\mathrm{d}b,$$

and

$$\Gamma_3(\lambda) = \int_0^\infty \xi(a)e^{-\lambda a}\sigma_1(a)\mathrm{d}a.$$

It follows from (4.9) that

$$\begin{cases} (\lambda + \mu + f(J^*) + g(Q^*))S + [S^* f'(J^*)\Gamma_1(\lambda)]i(0) + [S^* g'(Q^*)\Gamma_2(\lambda)]p(0) = 0, \\ (f(J^*) + g(Q^*))S - [1 - S^* f'(J^*)\Gamma_1(\lambda)]i(0) + [S^* g'(Q^*)\Gamma_2(\lambda)]p(0) = 0, \\ \Gamma_3(\lambda)i(0) - p(0) = 0. \end{cases}$$

Thus, the corresponding characteristic equation of the linearization for system (1.4) at infection equilibrium $(S^*, i^*(a), p^*(b))$ is

$$\begin{vmatrix} \lambda + \mu + f(J^*) + g(Q^*) & S^* f'(J^*)\Gamma_1(\lambda) & S^* g'(Q^*)\Gamma_2(\lambda) \\ f(J^*) + g(Q^*) & S^* f'(J^*)\Gamma_1(\lambda) - 1 & S^* g'(Q^*)\Gamma_2(\lambda) \\ 0 & \Gamma_3(\lambda) & -1 \end{vmatrix} = 0.$$

Clearly, $\lambda = -\mu$ is not a root of the above equation, then

$$(\lambda + \mu + f(J^*) + g(Q^*))/(\lambda + \mu) = S^* f'(J^*)\Gamma_1(\lambda) + S^* g'(Q^*)\Gamma_2(\lambda)\Gamma_3(\lambda). \tag{4.10}$$

Assume that equation (4.10) has one root with positive real part. The module of the left side of the equation (4.10) is more than one. The module of the right side is

$$|S^* f'(J^*)\Gamma_1(\lambda) + S^* g'(Q^*)\Gamma_2(\lambda)\Gamma_3(\lambda)| \leqslant \left| S^* \frac{f(J^*)}{J^*}\Gamma_1(\lambda) + S^* \frac{g(Q^*)}{Q^*}\Gamma_2(\lambda)\Gamma_3(\lambda) \right|.$$

Since

$$Q^* = \int_0^\infty q(b)p^*(b)\mathrm{d}b = p^*(0)\Xi_2 + \Xi_5, \qquad J^* = \int_0^\infty k(a)i^*(a)\mathrm{d}a = i^*(0)\Xi_1 + \Xi_4,$$

and

$$\Xi_3 = \int_0^\infty \xi(a)\sigma_1(a)\mathrm{d}a \leqslant \int_0^\infty \xi(a)\frac{i^*(a)}{i^*(0)}\mathrm{d}a = \frac{p^*(0)}{i^*(0)},$$

we have

$$|S^* f'(J^*)\Gamma_1(\lambda) + S^* g'(Q^*)\Gamma_2(\lambda)\Gamma_3(\lambda)| \leqslant \left| \frac{S^* f(J^*)}{i^*(0)} + \frac{S^* g(Q^*)}{p^*(0)}\Xi_3 \right| = 1.$$

This is a contradiction and we finish the proof. □

5 Global asymptotic stability of the positive equilibrium

For the global asymptotic stability of the positive equilibrium, we apply Lyapunov functional method. For this, we introduce a function

$$\hbar(z) = z - 1 - \ln z, \qquad z \in \mathbb{R}_+. \tag{5.1}$$

In order to ensure $\hbar\left(\frac{i(t,a)}{i^*(a)}\right)$ and $\hbar\left(\frac{p(t,b)}{p^*(b)}\right)$ well-defined, we need to show that $\frac{i(t,a)}{i^*(a)}$ and $\frac{p(t,b)}{p^*(b)}$ are bounded by some positive constants through dissipativeness and persistence analysis in Section 2.

For the verification of Lyapunov functional, we need the following lemmas.

Lemma 5.1. $\frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) i^*(a)\xi(a)\left[1 - \frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right]\mathrm{d}a = 0.$

Proof. Since $p(t,0) = \int_0^\infty \xi(a) i(t,a)\mathrm{d}a$ and $p^*(0) = \int_0^\infty \xi(a) i^*(a)\mathrm{d}a$, we have

$$\begin{aligned}
&\frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) i^*(a)\xi(a)\left[1 - \frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right]\mathrm{d}a \\
&= \frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) i^*(a)\xi(a)\mathrm{d}a - \frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*)\xi(a)\frac{i(t,a)p^*(0)}{p(t,0)}\mathrm{d}a \\
&= \frac{1}{\Xi_3} S^* g(Q^*)\int_0^\infty i^*(a)\xi(a)\mathrm{d}a - \frac{1}{\Xi_3} S^* g(Q^*) p^*(0)\frac{1}{p(t,0)}\int_0^\infty \xi(a) i(t,a)\mathrm{d}a \\
&= 0.
\end{aligned}$$

The proof is completed. □

Lemma 5.2. *Define a function h not depending on a and b. Then we have*

$$\frac{1}{\Xi_2}\int_0^\infty S^* g(Q^*) p^*(0)\sigma_2(b) q(b) h \mathrm{d}b = \int_0^\infty S^* g(Q^*) i^*(a)\xi(a) h \mathrm{d}a.$$

Proof. Since $\Xi_2 = \int_0^\infty q(b)\sigma_2(b)\mathrm{d}b$ and $p^*(0) = \int_0^\infty i^*(a)\xi(a)\mathrm{d}a$, we have

$$\frac{1}{\Xi_2}\int_0^\infty S^* g(Q^*) p^*(0)\sigma_2(b) q(b) h \mathrm{d}b = S^* g(Q^*) p^*(0) h = \int_0^\infty S^* g(Q^*) i^*(a)\xi(a) h \mathrm{d}a.$$

This completes the proof. □

Theorem 5.3. *The infection equilibrium E^* of system (1.4) is globally asymptotically stable.*

Proof. We define the Lyapunov function $\ell(t) = \ell_1(t) + \ell_2(t) + \ell_3(t)$ with

$$\ell_1(t) = S^*\hbar\left(\frac{S(t)}{S^*}\right) i^*(0), \qquad \ell_2(t) = \int_0^\infty \Phi(a) i^*(a)\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a,$$

and

$$\ell_3(t) = \frac{1}{\Xi_3}\int_0^\infty \Psi(b) p^*(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b,$$

where

$$\Phi(a) = \frac{1}{\Xi_1}\int_a^\infty S^* f(J^*) k(u) e^{-\int_a^u \delta(\tau)\mathrm{d}\tau}\mathrm{d}u + \frac{1}{\Xi_3}\int_a^\infty \Psi(0)\xi(u) e^{-\int_a^u \delta(\tau)\mathrm{d}\tau}\mathrm{d}u,$$

and

$$\Psi(b) = \frac{1}{\Xi_2}\int_b^\infty S^* g(Q^*)q(v)e^{-\int_b^v \gamma(\tau)\mathrm{d}\tau}\mathrm{d}v.$$

Then, calculating the derivative of $\ell_1(t)$ along (1.4) yields

$$\frac{\mathrm{d}\ell_1(t)}{\mathrm{d}t} = \left(1-\frac{S^*}{S}\right)[\Lambda-\mu S - Sf(J) - Sg(Q)]i^*(0).$$

Using the fact that $\Lambda = \mu S^* + S^* f(J^*) + S^* g(Q^*)$, one has that

$$\begin{aligned}\frac{\mathrm{d}\ell_1(t)}{\mathrm{d}t} &= \left(1-\frac{S^*}{S}\right)[\mu S^* + S^* f(J^*) + S^* g(Q^*) - \mu S - Sf(J) - Sg(Q)]i^*(0)\\ &= \Big[-\frac{\mu}{S}(S-S^*)^2 + S^* f(J^*) + S^* g(Q^*) - Sf(J) - Sg(Q)\\ &\quad - \frac{S^*}{S}S^* f(J^*) - \frac{S^*}{S}S^* g(Q^*) + S^* f(J) + S^* g(Q)\Big] i^*(0).\end{aligned} \tag{5.2}$$

Define $H(a) = \int_0^a \frac{\Lambda_i(\epsilon)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon$ and $K(b) = \int_0^b \frac{\Lambda_p(\epsilon)}{\sigma_2(\epsilon)}\mathrm{d}\epsilon$. In what follows, calculating the derivative of $\ell_2(t)$ along (1.4) and then letting $\tau = t-a$ gives

$$\begin{aligned}\frac{\mathrm{d}\ell_2(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \Phi(a)i^*(a)\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t \Phi(t-\tau)i^*(t-\tau)\hbar\left(\frac{i(\tau,0)+H(t-\tau)}{i^*(0)+H(t-\tau)}\right)\mathrm{d}\tau\\ &= \Phi(0)i^*(0)\hbar\left(\frac{i(t,0)+H(0)}{i^*(0)+H(0)}\right)\\ &\quad + \int_{-\infty}^t \frac{\mathrm{d}}{\mathrm{d}t}[\Phi(t-\tau)i^*(t-\tau)]\hbar\left(\frac{i(\tau,0)+H(t-\tau)}{i^*(0)+H(t-\tau)}\right)\mathrm{d}\tau\\ &\quad + \int_{-\infty}^t \Phi(t-\tau)i^*(t-\tau)\frac{\mathrm{d}}{\mathrm{d}t}\left[\hbar\left(\frac{i(\tau,0)+H(t-\tau)}{i^*(0)+H(t-\tau)}\right)\right]\mathrm{d}\tau.\end{aligned}$$

Letting $a = t-\tau$, we have

$$\begin{aligned}\frac{\mathrm{d}\ell_2(t)}{\mathrm{d}t} &= \Phi(0)i^*(0)\hbar\left(\frac{i(t,0)}{i^*(0)}\right) + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}a}[\Phi(a)i^*(a)]\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a\\ &\quad + \int_0^\infty \Phi(a)i^*(a)\left(1-\frac{i^*(a)}{i(t,a)}\right)\frac{H'(a)}{i^*(0)+H(a)}\left(1-\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a\\ &= \Phi(0)i^*(0)\hbar\left(\frac{i(t,0)}{i^*(0)}\right) + \int_0^\infty [\Phi'(a)i^*(a) + \Phi(a)i^*_a(a)]\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a\\ &\quad + \int_0^\infty \Phi(a)i^*(a)\left(1-\frac{i^*(a)}{i(t,a)}\right)\left(1-\frac{i(t,a)}{i^*(a)}\right)\frac{\Lambda_i(a)}{\sigma_1(a)(i^*(0)+H(a))}\mathrm{d}a\\ &= \Phi(0)i^*(0)\hbar\left(\frac{i(t,0)}{i^*(0)}\right) + \int_0^\infty [\Phi'(a)i^*(a) + \Phi(a)i^*_a(a)]\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a\\ &\quad + \int_0^\infty \Phi(a)\Lambda_i(a)\left(1-\frac{i^*(a)}{i(t,a)}\right)\left(1-\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a.\end{aligned}$$

Since

$$\Phi'(a) = -\frac{1}{\Xi_1}S^* f(J^*)k(a) - \frac{1}{\Xi_3}\Psi(0)\xi(a) + \delta(a)\Phi(a) \quad \text{and} \quad i^*_a(a) = -i^*(a)\delta(a) + \Lambda_i(a),$$

we further derive

$$\frac{\mathrm{d}\ell_2(t)}{\mathrm{d}t} = \Phi(0)i^*(0)\hbar\left(\frac{i(t,0)}{i^*(0)}\right) - \int_0^\infty \Phi(a)\Lambda_i(a)\hbar\left(\frac{i^*(a)}{i(t,a)}\right)\mathrm{d}a$$
$$- \int_0^\infty i^*(a)\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\left[\frac{1}{\Xi_1}S^*f(J^*)k(a) + \frac{1}{\Xi_3}S^*g(Q^*)\xi(a)\right]\mathrm{d}a.$$

Since

$$\Phi(0)i^*(0) = \frac{1}{\Xi_1}\int_0^\infty S^*f(J^*)k(a)i^*(0)\sigma_1(a)\mathrm{d}a + \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)\xi(a)i^*(0)\sigma_1(a)\mathrm{d}a,$$

we subsequently obtain

$$\begin{aligned}\frac{\mathrm{d}\ell_2(t)}{\mathrm{d}t} \leqslant{}& \frac{1}{\Xi_1}\int_0^\infty S^*f(J^*)i^*(0)\sigma_1(a)k(a)\left[\hbar\left(\frac{i(t,0)}{i^*(0)}\right) - \hbar\left(\frac{i(t,a)}{i^*(a)}\right)\right]\mathrm{d}a \\ &+ \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[\hbar\left(\frac{i(t,0)}{i^*(0)}\right) - \hbar\left(\frac{i(t,a)}{i^*(a)}\right)\right]\mathrm{d}a \\ &- \int_0^\infty \Phi(a)\Lambda_i(a)\hbar\left(\frac{i^*(a)}{i(t,a)}\right)\mathrm{d}a \\ &- \int_0^\infty H(a)\sigma_1(a)\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\left[\frac{1}{\Xi_1}S^*f(J^*)k(a) + \frac{1}{\Xi_3}S^*g(Q^*)\xi(a)\right]\mathrm{d}a.\end{aligned} \tag{5.3}$$

Similarly, we have

$$\begin{aligned}\frac{\mathrm{d}\ell_3(t)}{\mathrm{d}t} \leqslant{}& \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)p^*(0)\sigma_2(b)q(b)\left[\hbar\left(\frac{p(t,0)}{p^*(0)}\right) - \hbar\left(\frac{p(t,b)}{p^*(b)}\right)\right]\mathrm{d}b \\ &- \frac{1}{\Xi_3}\int_0^\infty \Psi(b)\Lambda_p(b)\hbar\left(\frac{p^*(b)}{p(t,b)}\right)\mathrm{d}b \\ &- \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)q(b)\sigma_2(b)K(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b.\end{aligned} \tag{5.4}$$

From equations (5.2), (5.3) and (5.4), we yield

$$\begin{aligned}\frac{\mathrm{d}\ell(t)}{\mathrm{d}t} \leqslant{}& \Big[-\frac{\mu}{S}(S-S^*)^2 + S^*f(J^*) + S^*g(Q^*) - Sf(J) - Sg(Q) \\ &- \frac{S^*}{S}S^*f(J^*) - \frac{S^*}{S}S^*g(Q^*) + S^*f(J) + S^*g(Q)\Big]\, i^*(0) \\ &+ \frac{1}{\Xi_1}\int_0^\infty S^*f(J^*)i^*(0)\sigma_1(a)k(a)\left[\frac{i(t,0)}{i^*(0)} - \frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)i^*(a)}{i^*(0)i(t,a)}\right]\mathrm{d}a \\ &+ \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[\frac{i(t,0)}{i^*(0)} - \frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)i^*(a)}{i^*(0)i(t,a)}\right]\mathrm{d}a \\ &+ \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)p^*(0)\sigma_2(b)q(b)\left[\frac{p(t,0)}{p^*(0)} - \frac{p(t,b)}{p^*(b)} - \ln\frac{p(t,0)p^*(b)}{p^*(0)p(t,b)}\right]\mathrm{d}b \\ &- \int_0^\infty \Phi(a)\Lambda_i(a)\hbar\left(\frac{i^*(a)}{i(t,a)}\right)\mathrm{d}a - \frac{1}{\Xi_3}\int_0^\infty \Psi(b)\Lambda_p(b)\hbar\left(\frac{p^*(b)}{p(t,b)}\right)\mathrm{d}b \\ &- \int_0^\infty H(a)\sigma_1(a)\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\left[\frac{1}{\Xi_1}S^*f(J^*)k(a) + \frac{1}{\Xi_3}S^*g(Q^*)\xi(a)\right]\mathrm{d}a \\ &- \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)q(b)\sigma_2(b)K(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b.\end{aligned}$$

Since $\int_0^\infty k(a)\sigma_1(a)\mathrm{d}a = \Xi_1$ and $\int_0^\infty \xi(a)\sigma_1(a)\mathrm{d}a = \Xi_3$, we have

$$\begin{aligned}
&\frac{1}{\Xi_1}\int_0^\infty S^* f(J^*)k(a)i^*(0)\sigma_1(a)\left[1-\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}\right]\mathrm{d}a \\
&\qquad + \frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*)\xi(a)i^*(0)\sigma_1(a)\left[1-\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right]\mathrm{d}a \\
&\quad = S^* f(J^*)\frac{1}{\Xi_1}\int_0^\infty k(a)\sigma_1(a)i^*(0)\mathrm{d}a\left[1-\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}\right] \\
&\qquad + S^* g(Q^*)\frac{1}{\Xi_3}\int_0^\infty \xi(a)\sigma_1(a)i^*(0)\mathrm{d}a\left[1-\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right] \\
&\quad = \left(S^* f(J^*)\left[1-\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}\right] + S^* g(Q^*)\left[1-\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right]\right) i^*(0) \\
&\quad = \left(S^* f(J^*) - \frac{i^*(0)Sf(J)}{i(t,0)} + S^* g(Q^*) - \frac{i^*(0)Sg(Q)}{i(t,0)}\right) i^*(0) \\
&\quad = \left(i^*(0) - i(t,0)\frac{i^*(0)}{i(t,0)}\right) i^*(0) \\
&\quad = 0.
\end{aligned}$$

Then $\frac{\mathrm{d}\ell(t)}{\mathrm{d}t} \leqslant \sum_{i=1}^{6}\Theta_i$, where

$$\begin{aligned}
\Theta_1 :=\ & \left[-\frac{\mu}{S}(S-S^*)^2 - Sf(J) - Sg(Q)\right] i^*(0) + \frac{1}{\Xi_1}\int_0^\infty S^* f(J^*)i^*(0)\sigma_1(a)k(a)\frac{i(t,0)}{i^*(0)}\mathrm{d}a \\
& + \frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*)i^*(0)\sigma_1(a)\xi(a)\frac{i(t,0)}{i^*(0)}\mathrm{d}a, \\
\Theta_2 :=\ & -\frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*)i^*(0)\sigma_1(a)\xi(a)\frac{i(t,a)}{i^*(a)}\mathrm{d}a + \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^* g(Q^*)p^*(0)\sigma_2(b)q(b)\frac{p(t,0)}{p^*(0)}\mathrm{d}b, \\
\Theta_3 :=\ & \left[S^* f(J^*) - \frac{S^*}{S}S^* f(J^*) + S^* f(J)\right] i^*(0) \\
& + \frac{1}{\Xi_1}\int_0^\infty S^* f(J^*)i^*(0)\sigma_1(a)k(a)\left[-\frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)}{i^*(0)} + \ln\frac{i(t,a)}{i^*(a)}\right]\mathrm{d}a \\
& + \frac{1}{\Xi_1}\int_0^\infty S^* f(J^*)i^*(0)\sigma_1(a)k(a)\left[1-\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}\right]\mathrm{d}a, \\
\Theta_4 :=\ & \left[S^* g(Q^*) - \frac{S^*}{S}S^* g(Q^*)\right] i^*(0) + \frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*)i^*(0)\sigma_1(a)\xi(a)\ln\frac{i(t,a)}{i^*(a)}\mathrm{d}a \\
& - \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^* g(Q^*)p^*(0)\sigma_2(b)q(b)\ln\frac{p(t,0)}{p^*(0)}\mathrm{d}b, \\
\Theta_5 :=\ & S^* g(Q)i^*(0) - \frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*)i^*(0)\sigma_1(a)\xi(a)\ln\frac{i(t,0)}{i^*(0)}\mathrm{d}a \\
& + \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^* g(Q^*)p^*(0)\sigma_2(b)q(b)[-\frac{p(t,b)}{p^*(b)} + \ln\frac{p(t,b)}{p^*(b)}]\mathrm{d}b \\
& + \frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[1-\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right]\mathrm{d}a, \\
\Theta_6 :=\ & -\int_0^\infty \Phi(a)\Lambda_i(a)\hbar\left(\frac{i^*(a)}{i(t,a)}\right)\mathrm{d}a - \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^* g(Q^*)q(b)\sigma_2(b)K(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b \\
& - \int_0^\infty H(a)\sigma_1(a)\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\left[\frac{1}{\Xi_1}S^* f(J^*)k(a) + \frac{1}{\Xi_3}S^* g(Q^*)\xi(a)\right]\mathrm{d}a.
\end{aligned}$$

Thanks to $Sf(J)+Sg(Q)=i(t,0)$, one has

$$\begin{aligned}\Theta_1 =& -\frac{\mu}{S}(S-S^*)^2 i^*(0) - i(t,0)i^*(0) + S^* f(J^*) i(t,0) + S^* g(Q^*) i(t,0)\\ =& -\frac{\mu}{S}(S-S^*)^2 i^*(0).\end{aligned} \tag{5.5}$$

By virtue of $\int_0^\infty \xi(a)i(t,a)\mathrm{d}a = p(t,0)$, $\int_0^\infty \sigma_2(b)q(b)\mathrm{d}b = \Xi_2$ and the first equation of (4.4), we obtain that

$$\begin{aligned}\Theta_2 =& \frac{1}{\Xi_3} S^* g(Q^*) \left(\frac{1}{\Xi_2} p(t,0) \int_0^\infty \sigma_2(b) q(b) \mathrm{d}b - \int_0^\infty \xi(a) i(t,a) \mathrm{d}a \right)\\ &+ \frac{1}{\Xi_3} S^* g(Q^*) \int_0^\infty \xi(a) H(a) \sigma_1(a) \frac{i(t,a)}{i^*(a)} \mathrm{d}a\\ =& \frac{1}{\Xi_3} S^* g(Q^*) \int_0^\infty \xi(a) H(a) \sigma_1(a) \frac{i(t,a)}{i^*(a)} \mathrm{d}a.\end{aligned} \tag{5.6}$$

It follows from $\Xi_1 = \int_0^\infty k(a)\sigma_1(a)\mathrm{d}a$ that

$$\begin{aligned}\Theta_3 =& S^* f(J^*) \frac{1}{\Xi_1} \int_0^\infty k(a) i^*(0) \sigma_1(a) \left[1 - \frac{S^*}{S} + \frac{f(J)}{f(J^*)} \right] \mathrm{d}a\\ &+ \frac{1}{\Xi_1} \int_0^\infty S^* f(J^*) k(a) i^*(0) \sigma_1(a) \left[-\frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)} + \ln \frac{i(t,a)}{i^*(a)} \right] \mathrm{d}a\\ &+ \frac{1}{\Xi_1} \int_0^\infty S^* f(J^*) k(a) i^*(0) \sigma_1(a) \left[1 - \frac{i^*(0) S f(J)}{i(t,0) S^* f(J^*)} \right] \mathrm{d}a\\ =& \frac{1}{\Xi_1} \int_0^\infty S^* f(J^*) k(a) i^*(0) \sigma_1(a) \left[-\hbar\left(\frac{S^*}{S}\right) - \hbar\left(\frac{i^*(0) S f(J)}{i(t,0) S^* f(J^*)}\right)\right.\\ &\left. - \hbar\left(\frac{i(t,a)}{i^*(a)}\right) + \hbar\left(\frac{f(J)}{f(J^*)}\right) \right] \mathrm{d}a.\end{aligned} \tag{5.7}$$

Due to

$$\Xi_1 i^*(0) = \int_0^\infty k(a)\sigma_1(a) i^*(0) \mathrm{d}a \leqslant \int_0^\infty k(a) i^*(a) \mathrm{d}a = J^*$$

and Jensen's inequality, we have

$$\begin{aligned}\frac{1}{\Xi_1} \int_0^\infty k(a) i^*(a) \hbar\left(\frac{i(t,a)}{i^*(a)}\right) \mathrm{d}a \geqslant& i^*(0) \int_0^\infty \frac{k(a) i^*(a)}{J^*} \hbar\left(\frac{i(t,a)}{i^*(a)}\right) \mathrm{d}a\\ \geqslant& i^*(0) \hbar\left(\int_0^\infty \frac{k(a) i^*(a)}{J^*} \frac{i(t,a)}{i^*(a)} \mathrm{d}a \right)\\ =& \frac{1}{\Xi_1} \int_0^\infty k(a) i^*(0) \sigma_1(a) \hbar\left(\frac{J}{J^*}\right) \mathrm{d}a\\ \geqslant& \frac{1}{\Xi_1} \int_0^\infty k(a) i^*(0) \sigma_1(a) \hbar\left(\frac{f(J)}{f(J^*)}\right) \mathrm{d}a.\end{aligned} \tag{5.8}$$

Then, combining (5.7) and (5.8), we have

$$\begin{aligned}\Theta_3 \leqslant& \frac{1}{\Xi_1} \int_0^\infty S^* f(J^*) k(a) i^*(0) \sigma_1(a) \left[-\hbar\left(\frac{S^*}{S}\right) - \hbar\left(\frac{i^*(0) S f(J)}{i(t,0) S^* f(J^*)}\right) \right] \mathrm{d}a\\ &+ \frac{1}{\Xi_1} \int_0^\infty S^* f(J^*) k(a) H(a) \sigma_1(a) \hbar\left(\frac{i(t,a)}{i^*(a)}\right) \mathrm{d}a.\end{aligned} \tag{5.9}$$

Recall that $\Xi_2 = \int_0^\infty \sigma_2(b)q(b)\mathrm{d}b$, $p^*(0) = \int_0^\infty \xi(a)i^*(a)\mathrm{d}a$ and Lemmas 5.1 and 5.2, we derive that

$$\begin{aligned}\Theta_4 &= \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[2-\frac{S^*}{S}+\ln\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right]\mathrm{d}a\\ &\quad+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[1-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\ln\frac{p(t,0)}{p^*(0)}\right]\\ &= \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{S^*}{S}\right)-\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)+\ln\frac{S}{S^*}\right]\mathrm{d}a\\ &\quad+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[1-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\ln\frac{p(t,0)}{p^*(0)}\right].\end{aligned}$$

Thus, combining Θ_4 and Θ_5 gives

$$\begin{aligned}\Theta_4+\Theta_5 \leqslant &\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right)\right]\mathrm{d}a\\ &+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{S^*}{S}\right)-\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)\right]\mathrm{d}a\\ &+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[\frac{g(Q)}{g(Q^*)}-\ln\frac{g(Q)}{g(Q^*)}\right]\mathrm{d}a\\ &+\frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)p^*(0)\sigma_2(b)q(b)\left[\ln\frac{p(t,b)}{p^*(b)}-\frac{p(t,b)}{p^*(b)}\right]\mathrm{d}b\\ &+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[1-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\ln\frac{p(t,0)}{p^*(0)}\right].\end{aligned}$$

Using Lemma 5.2, we further have

$$\begin{aligned}\Theta_4+\Theta_5 \leqslant &\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right)\right]\mathrm{d}a\\ &+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{S^*}{S}\right)-\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)\right]\mathrm{d}a\\ &+\frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)p^*(0)\sigma_2(b)q(b)\left[\hbar\left(\frac{g(Q)}{g(Q^*)}\right)-\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\right]\mathrm{d}b\\ &-\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[\frac{g(Q)}{g(Q^*)}-\ln\frac{g(Q)}{g(Q^*)}\right]\mathrm{d}a\\ &+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[1-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\ln\frac{p(t,0)}{p^*(0)}\right].\end{aligned}$$

Since

$$\Xi_2p^*(0)=\int_0^\infty q(b)\sigma_2(b)p^*(0)\mathrm{d}b\leqslant\int_0^\infty q(b)p^*(b)\mathrm{d}b=Q^*$$

and Jensen's inequality, we have

$$\begin{aligned}\frac{1}{\Xi_2}\int_0^\infty q(b)p^*(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b &\geqslant p^*(0)\int_0^\infty\frac{q(b)p^*(b)}{Q^*}\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b\\ &\geqslant p^*(0)\hbar\left(\int_0^\infty\frac{q(b)p^*(b)}{Q^*}\frac{p(t,b)}{p^*(b)}\mathrm{d}b\right)\\ &=\frac{1}{\Xi_2}\int_0^\infty q(b)p^*(0)\sigma_2(b)\hbar\left(\frac{Q}{Q^*}\right)\mathrm{d}b\\ &\geqslant\frac{1}{\Xi_2}\int_0^\infty q(b)p^*(0)\sigma_2(b)\hbar\left(\frac{g(Q)}{g(Q)^*}\right)\mathrm{d}b.\end{aligned}$$

Thus, we finally have

$$
\begin{aligned}
\Theta_4+\Theta_5 \leqslant\ & \frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right)\right]\mathrm{d}a\\
&+\frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{S^*}{S}\right)-\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)\right]\mathrm{d}a\\
&+\frac{1}{\Xi_2\Xi_3}\int_0^\infty S^* g(Q^*) K(b)\sigma_2(b)q(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b\\
&-\frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) H(a)\sigma_1(a)\xi(a)\left[\frac{g(Q)}{g(Q^*)}-\ln\frac{g(Q)}{g(Q^*)}\right]\mathrm{d}a\\
&+\frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) H(a)\sigma_1(a)\xi(a)\left[1-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\ln\frac{p(t,0)}{p^*(0)}\right].
\end{aligned}
\tag{5.10}
$$

Hence, combining (5.6), (5.9) and (5.10), we yield

$$
\begin{aligned}
\frac{\mathrm{d}\ell(t)}{\mathrm{d}t}\leqslant &-\frac{\mu}{S}(S-S^*)^2 i^*(0)\\
&-\int_0^\infty \Phi(a)\Lambda_i(a)\hbar\left(\frac{i^*(a)}{i(t,a)}\right)\mathrm{d}a-\frac{1}{\Xi_3}\int_0^\infty \Psi(b)\Lambda_p(b)\hbar\left(\frac{p^*(b)}{p(t,b)}\right)\mathrm{d}b\\
&-\frac{1}{\Xi_1}\int_0^\infty S^* f(J^*) i^*(0)\sigma_1(a)k(a)\left[\hbar\left(\frac{S^*}{S}\right)+\hbar\left(\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}\right)\right]\mathrm{d}a\\
&-\frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) i^*(0)\sigma_1(a)\xi(a)\left[\hbar\left(\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right)\right]\mathrm{d}a\\
&-\frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) i^*(0)\sigma_1(a)\xi(a)\left[\hbar\left(\frac{S^*}{S}\right)+\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)\right]\mathrm{d}a\\
&-\frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) H(a)\sigma_1(a)\xi(a)\left[\hbar\left(\frac{g(Q)}{g(Q^*)}\right)+\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)\right]\mathrm{d}a.
\end{aligned}
$$

Consequently, from above discussion, we assert that $\frac{\mathrm{d}\ell(t)}{\mathrm{d}t}\leqslant 0$ and the largest invariant subset of set $\left\{\frac{\mathrm{d}\ell(t)}{\mathrm{d}t}=0\right\}$ is E^*. Due to the invariance principle [39, Theorem 4.2], E^* is globally asymptotically stable. □

6 Numerical simulation and conclusion

In this section, we consider a special model with nonlinear functional responses:

$$
\begin{cases}
\dfrac{\mathrm{d}S(t)}{\mathrm{d}t}=\Lambda_s-\mu S(t)-\dfrac{S(t)\int_0^\infty k(a)i(t,a)\mathrm{d}a}{A\int_0^\infty k(a)i(t,a)\mathrm{d}a+1}-\dfrac{S(t)\int_0^\infty q(b)p(t,b)\mathrm{d}b}{A\int_0^\infty q(b)p(t,b)\mathrm{d}b+1},\\
\dfrac{\partial i(t,a)}{\partial t}+\dfrac{\partial i(t,a)}{\partial a}=\Lambda_i(a)-\delta(a)i(t,a),\\
\dfrac{\partial p(t,b)}{\partial t}+\dfrac{\partial p(t,b)}{\partial b}=\Lambda_p(b)-\gamma(b)p(t,b),
\end{cases}
\tag{6.1}
$$

with initial condition (1.6) and boundary condition

$$
\begin{aligned}
i(t,0)&=\frac{S(t)\int_0^\infty k(a)i(t,a)\mathrm{d}a}{A\int_0^\infty k(a)i(t,a)\mathrm{d}a+1}+\frac{S(t)\int_0^\infty q(b)p(t,b)\mathrm{d}b}{A\int_0^\infty q(b)p(t,b)\mathrm{d}b+1},\quad t>0,\\
p(t,0)&=\int_0^\infty \xi(a)i(t,a)\mathrm{d}a,\quad t>0.
\end{aligned}
\tag{6.2}
$$

Then, from Theorem 5.3, we obtain the following corollary:

Corollary 6.1. *The infection equilibrium of system* (6.1) *is globally asymptotically stable.*

To verify the validity of the result, we perform numerical simulations. Let $\Lambda_s = 2000$, $\mu = \frac{1}{70}$ and

$$\begin{aligned} k(a) &= 0.003\left(1+\sin\frac{(a-5)\,\Xi}{10}\right), & q(b) &= 0.01\left(1+\sin\frac{(b-5)\,\Xi}{10}\right) \\ \Lambda_i(a) &= 5\left(1+\sin\frac{(a-5)\,\Xi}{10}\right), & \Lambda_p(b) &= 80\left(1+\sin\frac{(b-5)\,\Xi}{10}\right) \\ \delta(a) &= 0.18\left(1+\sin\frac{(a-5)\,\Xi}{10}\right), & \gamma(b) &= 2\left(1+\sin\frac{(b-5)\,\Xi}{10}\right), \\ \xi(a) &= \left(1+\sin\frac{(a-5)\,\Xi}{10}\right), \end{aligned}$$

for $0 \leqslant a, b \leqslant 10$. Clearly, as in Figure 6.1, all the solutions converge to the positive steady state. In Figure 6.2, we further show the distribution of $i(t,a)$ and $p(t,b)$ at age $a = b = 5$.

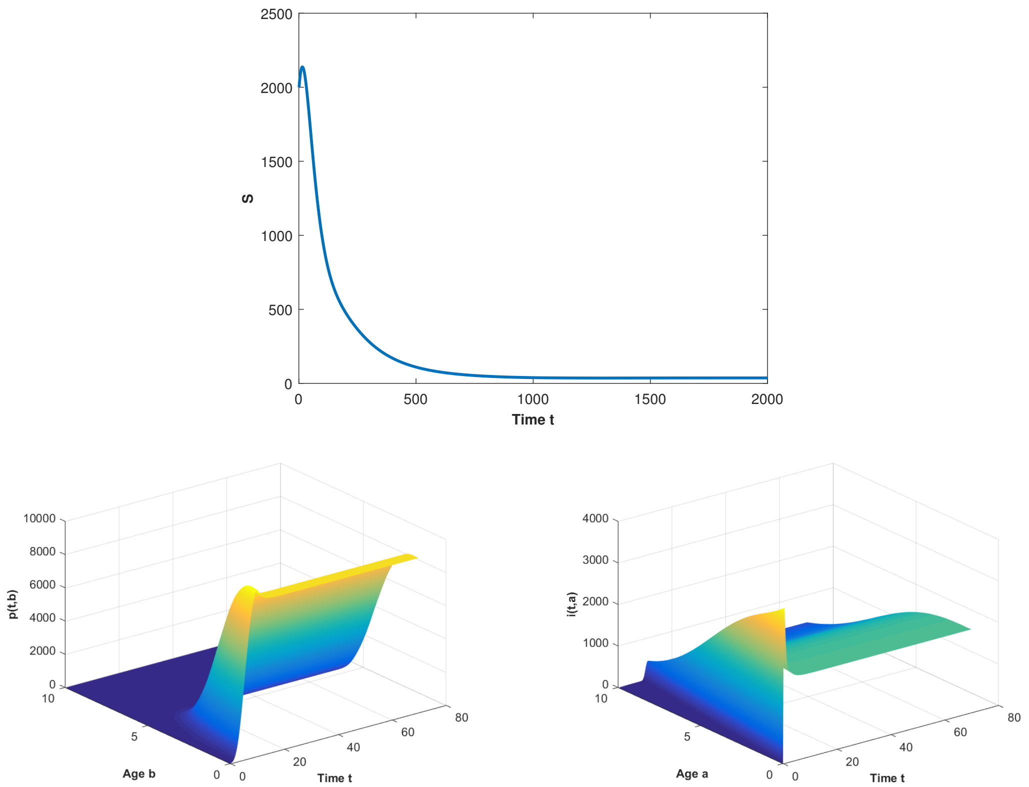

Figure 6.1: Long-time dynamical behavior of system (6.1)–(6.2).

Now, we finish this paper with a conclusion. In this paper, we considered an age-infection model of cholera with general infection rates. We focused on the global asymptotical stability of the unique positive equilibrium under some assumptions. For this, we directly used the Lyapunov functional method. It is necessarily pointed here that the uniform persistence and asymptotical smoothness play the key role for the construction of Lyapunov functional.

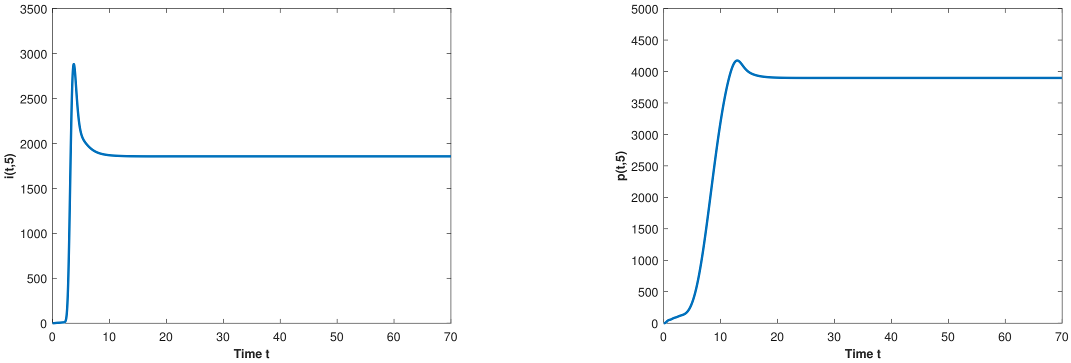

Figure 6.2: Long-time dynamical behavior of $i(t,a)$ and $p(t,b)$ for $a = b = 5$.

Finally, we performed numerical simulations. On account of the waterborne disease, we incorporated indirect pathogen-to-person transmission and direct person-to-person transmission. By taking general infection rates into account, we gain a unified theoretical framework to describe the cholera propagation process. In a recent paper [25], Liu et al. proposed an age-space structured cholera model, and studied the local stability of equilibria, disease persistence and global attractivity of equilibria for their model. How about introducing immigration into the age-space structured cholera model, which will be an interesting problem and we will leave it for the future work.

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An exact bifurcation diagram for a p–q Laplacian boundary value problem

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Abstract. We study positive solutions to the p–q Laplacian two-point boundary value problem:

$$\begin{cases} -\mu[(u')^{p-1}]' - [(u')^{q-1}]' = \lambda u(1-u) & \text{on } (0,1) \\ u(0) = 0 = u(1) \end{cases}$$

when $p = 4$ and $q = 2$. Here $\lambda > 0$ is a parameter and $\mu \geq 0$ is a weight parameter influencing the higher-order diffusion term. When $\mu = 0$ (the Laplacian case) the exact bifurcation diagram for a positive solution is well-known, namely, when $\lambda \leq \pi^2$ there are no positive solutions, and for $\lambda > \pi^2$ there exists a unique positive solution $u_{\lambda,\mu}$ such that $\|u_{\lambda,\mu}\|_\infty \to 0$ as $\lambda \to \pi^2$ and $\|u_{\lambda,\mu}\|_\infty \to 1$ as $\lambda \to \infty$. Here, we will prove that for all $\mu > 0$ similar bifurcation diagrams preserve, and they all bifurcate from $(\lambda, u) = (\pi^2, 0)$. Our results are established via the method of sub-super solutions and a quadrature method. We also present computational evaluations of these bifurcation diagrams for various values of μ and illustrate how they evolve when μ varies.

Keywords: positive solutions, p–q Laplacian, Dirichlet boundary conditions, exact bifurcation diagram.

2020 Mathematics Subject Classification: 34B08, 34B18.

1 Introduction

We analyze positive solutions to the boundary value problem:

$$\begin{cases} -\mu[(u')^{p-1}]' - [(u')^{q-1}]' = \lambda f(u) & \text{on } (0,1), \\ u(0) = 0 = u(1) \end{cases} \tag{1.1}$$

when $p = 4$ and $q = 2$. Here we will choose f to be a smooth function such that $f(0) = 0$, and $\lambda > 0$, $\mu \geq 0$ are parameters, with μ influencing the higher-order diffusion term. Study of p–q

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Laplacian problems have been of interest in the literature (see [1–4, 6]) as they arise as steady states of reaction-diffusion processes when the diffusion involved is of a certain nonlinear class. See [6] in particular, where they note that equations of this type arise in biophysics, plasma physics, and chemical reactor design. However, our motivation of this study is purely mathematical. We will begin with the case $\mu = 0$ when the exact bifurcation diagram for positive solutions is known, and then prove that for all $\mu > 0$ similar bifurcation diagrams preserve, and that they all bifurcate from the branch of trivial solutions at the same point where the bifurcation occurs in the case $\mu = 0$. In particular, in this study we choose

$$f(s) = s(1-s); \qquad s \in [0,1],$$

for which when $\mu = 0$ it is well-known that the bifurcation diagram of positive solutions is exact (see [5, 7]) of the form:

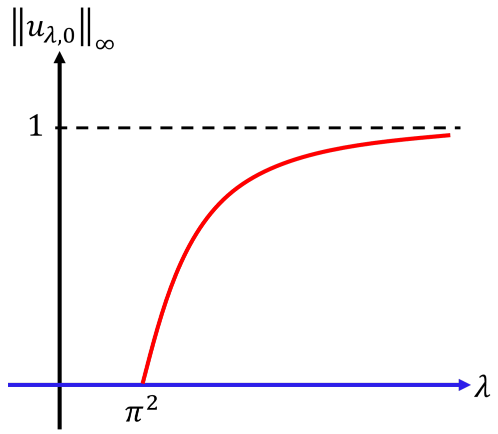

Figure 1.1: A prototypical bifurcation diagram of positive solutions for (1.1) when $f(s) = s(1-s)$ and $\mu = 0$.

Namely, for $\lambda \leq \pi^2$ there are no positive solutions, and for $\lambda > \pi^2$, there is a unique positive solution $u_{\lambda,0}$ such that $\|u_{\lambda,0}\|_\infty \to 0$ as $\lambda \to \pi^2$ and $\|u_{\lambda,0}\|_\infty \to 1$ as $\lambda \to \infty$. Here, we extend the study for the case $\mu > 0$. In particular, we prove:

Theorem 1.1. *Let $\mu > 0$ be fixed. Then for $\lambda \leq \pi^2$, (1.1) has no positive solution, and for $\lambda > \pi^2$, (1.1) has a unique positive solution $u_{\lambda,\mu}$ such that $\|u_{\lambda,\mu}\|_\infty \to 0$ as $\lambda \to \pi^2$ and $\|u_{\lambda,\mu}\|_\infty \to 1$ as $\lambda \to \infty$. Further, for $\lambda > \pi^2$, if $\mu_2 > \mu_1$ then $u_{\lambda,\mu_1}(x) \geq u_{\lambda,\mu_2}(x)$ for all $x \in [0,1]$.*

Remark 1.2. Theorem 1.1 establishes that for each $\mu > 0$, a similar exact bifurcation diagram for positive solutions to the case when $\mu = 0$ preserves and each bifurcates from $(\lambda, u) = (\pi^2, 0)$ (see Figure 1.2).

Remark 1.3. Our analysis uses the relationship (2.3), which determines the bifurcation diagram. The derivation of (2.3) uses $p = 4$ and $q = 2$ (see the proof of Lemma 2.2). Establishing such a result for any $p > q > 1$ is an open problem. Further, our analysis is restricted to the specific f we chose.

We prove our results by the method of sub-super solutions (see [4]) and via using the quadrature method discussed in [2] (an extension of the quadrature method first introduced for the case $\mu = 0$ in [5]). In Section 2 we present preliminaries, in Section 3 we prove Theorem 1.1, and in Section 4 we compute the bifurcation diagrams numerically for several values of μ and demonstrate their evolution as μ varies.

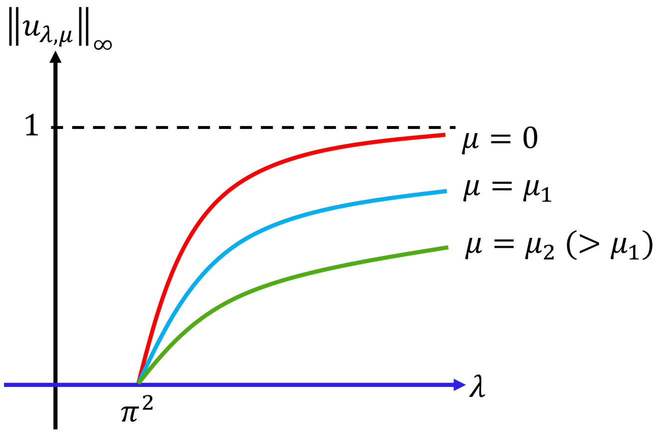

Figure 1.2: Prototypical bifurcation diagrams of positive solutions for (1.1) when $\mu \geq 0$.

2 Preliminaries

In this section, we introduce definitions of a subsolution and a supersolution of (1.1) and state a sub-supersolution theorem that will be used to prove our existence result for positive solutions. We also state a result via a quadrature method which we will use in our analysis (combined with an existence result obtained via sub-supersolutions) to establish exact details on the bifurcation diagram for positive solutions.

By a subsolution of (1.1) we mean $\psi \in C^2((0,1)) \cap C([0,1])$ that satisfies

$$\begin{cases} -\mu[(\psi')^3]' - \psi'' \leq \lambda f(\psi) & \text{on } (0,1), \\ \psi(0) \leq 0, \psi(1) \leq 0. \end{cases} \tag{2.1}$$

By a supersolution of (1.1) we mean $Z \in C^2((0,1)) \cap C([0,1])$ that satisfies

$$\begin{cases} -\mu[(Z')^3]' - Z'' \geq \lambda f(Z) & \text{on } (0,1), \\ Z(0) \geq 0, Z(1) \geq 0. \end{cases} \tag{2.2}$$

Then the following result holds:

Lemma 2.1. *Let ψ and Z be a subsolution and a supersolution of (1.1) respectively such that $\psi \leq Z$. Then (1.1) has a solution $u \in C^2((0,1)) \cap C([0,1])$ such that $u \in [\psi, Z]$.*

Proof. See [4]. □

Lemma 2.2. *Let $\lambda, \mu > 0$ be fixed and $\rho \in (0,1)$. Then (1.1) has a positive solution with $\|u_{\lambda,\mu}\|_\infty = \rho$ if and only if λ and ρ satisfy*

$$G(\lambda,\rho) = \int_0^\rho \frac{ds}{\sqrt{\sqrt{12\mu\lambda[F(\rho) - F(s)] + 1} - 1}} = \frac{1}{2\sqrt{3\mu}}, \tag{2.3}$$

where $F(s) = \int_0^s f(z)dz$.

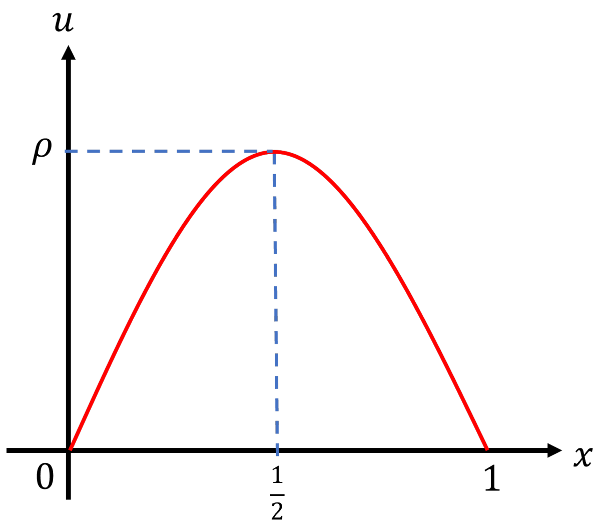

Figure 2.1: A prototypical shape of a positive solution to (1.1).

Proof. (See also [3].) Suppose $u_{\lambda,\mu}$ is a positive solution of (1.1) with $\|u_{\lambda,\mu}\|_\infty = \rho$. Since (1.1) is autonomous, $u_{\lambda,\mu}$ must be symmetric about $x = \frac{1}{2}$, increasing on $(0, \frac{1}{2})$, and decreasing on $(\frac{1}{2}, 1)$. See Figure 2.1.

Multiplying the differential equation in (1.1) by $u'_{\lambda,\mu}(x)$ for $x \in [0, \frac{1}{2}]$, we get

$$-\mu u'_{\lambda,\mu}(x)[(u'_{\lambda,\mu}(x))^3]' - u'_{\lambda,\mu}(x)[u'_{\lambda,\mu}(x)]' = u'_{\lambda,\mu}(x)\lambda f(u_{\lambda,\mu}(x)), \tag{2.4}$$

which can be written as

$$\frac{-3\mu}{4}[(u'_{\lambda,\mu}(x))^4]' - \frac{1}{2}[(u'_{\lambda,\mu}(x))^2]' = \lambda[F(u_{\lambda,\mu}(x))]'; \qquad x \in \left[0, \frac{1}{2}\right]. \tag{2.5}$$

Integrating (2.5) with respect to x over $\left[0, \frac{1}{2}\right]$, we obtain

$$3\mu[u'_{\lambda,\mu}(x)]^4 + 2[u'_{\lambda,\mu}(x)]^2 = 4\lambda[F(\rho) - F(u_{\lambda,\mu}(x))]; \qquad x \in \left[0, \frac{1}{2}\right]. \tag{2.6}$$

Solving (2.6) for $[u'_{\lambda,\mu}(x)]^2$, we obtain

$$[u'_{\lambda,\mu}(x)]^2 = \frac{\sqrt{12\mu\lambda[F(\rho) - F(u_{\lambda,\mu}(x))] + 1} - 1}{3\mu}; \qquad x \in \left[0, \frac{1}{2}\right].$$

Since $u'_{\lambda,\mu}(x) > 0$ for $x \in \left[0, \frac{1}{2}\right]$, it follows that

$$u'_{\lambda,\mu}(x) = \frac{\sqrt{\sqrt{12\mu\lambda[F(\rho) - F(u_{\lambda,\mu}(x))] + 1} - 1}}{\sqrt{3\mu}}; \qquad x \in \left[0, \frac{1}{2}\right]. \tag{2.7}$$

Integrating (2.7) with respect to x over $\left[0, \frac{1}{2}\right)$, we obtain

$$\frac{x}{\sqrt{3\mu}} = \int_0^{u_{\lambda,\mu}(x)} \frac{ds}{\sqrt{\sqrt{12\mu\lambda[F(\rho) - F(s)] + 1} - 1}}; \qquad x \in \left[0, \frac{1}{2}\right), \tag{2.8}$$

and letting $x \to \left(\frac{1}{2}\right)^-$, we obtain (2.3):

$$G(\lambda,\rho) = \int_0^\rho \frac{ds}{\sqrt{\sqrt{12\mu\lambda[F(\rho)-F(s)]+1}-1}} = \frac{1}{2\sqrt{3\mu}}.$$

Conversely, suppose λ and $\rho \in (0,1)$ are such that (2.3) is satisfied. Then for each $x \in [0,\frac{1}{2})$, we can find a unique $u_{\lambda,\mu}(x)$ satisfying (2.8). We can now extend this $u_{\lambda,\mu}$ on $[0,1]$ such that $u_{\lambda,\mu}\left(\frac{1}{2}\right) = \rho$ and $u_{\lambda,\mu}(x) = u_{\lambda,\mu}(1-x)$ for $x \in \left(\frac{1}{2},1\right]$. With the aid of the Implicit Function Theorem, we can show that $u_{\lambda,\mu} \in C^2((0,1)) \cap C([0,1])$ and then it is easy to show it satisfies (1.1). Hence, (2.3) determines the bifurcation diagram of positive solutions $u_{\lambda,\mu}$ for (1.1) with $\|u_{\lambda,\mu}\|_\infty = \rho \in (0,1)$. □

Remark 2.3. If $\mu = 0$, (1.1) becomes the boundary value problem:

$$\begin{cases} -u'' = \lambda f(u) & \text{on } (0,1), \\ u(0) = 0 = u(1) \end{cases} \tag{2.9}$$

and by the quadrature method described in [5], the bifurcation diagram for positive solutions of (2.9) is determined by

$$\lambda = 2\left\{\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}\right\}^2; \qquad \rho \in (0,1). \tag{2.10}$$

3 Proof of Theorem 1.1

Claim: Nonexistence of positive solutions for $\lambda \le \pi^2$.

Suppose $u_{\lambda,\mu} > 0; (0,1)$ is a solution to (1.1) for $\lambda \le \pi^2$. Multiplying each term of the differential equation by $\sin(\pi x)$ and integrating on $(0,1)$, we have

$$-\mu\int_0^1 [(u'_{\lambda,\mu}(x))^3]'\sin(\pi x)dx - \int_0^1 u''_{\lambda,\mu}(x)\sin(\pi x)dx = \lambda\int_0^1 u_{\lambda,\mu}(x)[1-u_{\lambda,\mu}(x)]\sin(\pi x)dx. \tag{3.1}$$

Equivalently, we have

$$-\mu\int_0^1 [(u'_{\lambda,\mu}(x))^3]'\sin(\pi x)dx + \lambda\int_0^1 [u_{\lambda,\mu}(x)]^2\sin(\pi x)dx = (\lambda-\pi^2)\int_0^1 u_{\lambda,\mu}(x)\sin(\pi x)dx. \tag{3.2}$$

Since $\lambda \le \pi^2$, we have

$$(\lambda-\pi^2)\int_0^1 u_{\lambda,\mu}(x)\sin(\pi x)dx \le 0. \tag{3.3}$$

However,

$$
\begin{aligned}
&-\mu\int_0^1[(u'_{\lambda,\mu}(x))^3]'\sin(\pi x)dx+\lambda\int_0^1[u_{\lambda,\mu}(x)]^2\sin(\pi x)dx\\
&\quad=-\mu\left[\underbrace{\sin(\pi x)[u'_{\lambda,\mu}(x)]^3\Big|_0^1}_{=0}-\pi\int_0^1\cos(\pi x)[u'_{\lambda,\mu}(x)]^3dx\right]+\lambda\int_0^1[u_{\lambda,\mu}(x)]^2\sin(\pi x)dx\\
&\quad=\mu\pi\left[\int_0^{1/2}\underbrace{\cos(\pi x)[u'_{\lambda,\mu}(x)]^3}_{>0}dx+\int_{1/2}^1\underbrace{\cos(\pi x)[u'_{\lambda,\mu}(x)]^3}_{>0}dx\right]+\underbrace{\lambda\int_0^1[u_{\lambda,\mu}(x)]^2\sin(\pi x)dx}_{>0}\\
&\quad>0.
\end{aligned}
$$

This contradicts (3.3). Hence (1.1) has no positive solution for $\lambda \le \pi^2$.

Claim: Existence of a positive solution $u_{\lambda,\mu}$ for $\lambda > \pi^2$.

Consider $\psi(x) = \varepsilon\sin(\pi x)$ with $\varepsilon > 0$. Then $\psi''(x) = -\varepsilon\pi^2\sin(\pi x)$ and $(\psi'(x))^3 = \varepsilon^3\pi^3[\cos(\pi x)]^3$. Hence

$$
\begin{aligned}
&-\mu[(\psi'(x))^3]'-\psi''(x)-\lambda\psi(x)(1-\psi(x))\\
&\quad=-\mu\Big(-3\varepsilon^3\pi^4\cos^2(\pi x)\sin(\pi x)\Big)-\Big(-\varepsilon\pi^2\sin(\pi x)\Big)-\lambda\varepsilon\sin(\pi x)[1-\varepsilon\sin(\pi x)]\\
&\quad=\varepsilon\sin(\pi x)\Big(3\mu\varepsilon^2\pi^4\cos^2(\pi x)+\pi^2-\lambda+\lambda\varepsilon\sin(\pi x)\Big)\\
&\quad<0;\qquad x\in(0,1)
\end{aligned}
$$

for $\varepsilon \approx 0$ when $\lambda > \pi^2$. Clearly the boundary conditions are satisfied by ψ. Thus, ψ is a subsolution of (1.1) for $\lambda > \pi^2$. Now $Z \equiv 1$ is a supersolution of (1.1) and $\psi < Z$ for $\varepsilon \approx 0$. Hence by Lemma 2.1, (1.1) has a positive solution $u_{\lambda,\mu} \in [\psi, Z]$ for all $\lambda > \pi^2$.

Claim: Existence of a unique positive solution $u_{\lambda,\mu}$ such that $\|u_{\lambda,\mu}\|_\infty \to 0$ as $\lambda \to \pi^2$ and $\|u_{\lambda,\mu}\|_\infty \to 1$ as $\lambda \to \infty$.

Recall $G(\lambda,\rho)$ from (2.3). Note that

$$
G(\lambda,\rho)=\int_0^\rho\frac{ds}{\sqrt{\sqrt{12\mu\lambda[F(\rho)-F(s)]+1}-1}}=\int_0^1\frac{\rho}{\sqrt{\sqrt{12\mu\lambda[F(\rho)-F(\rho v)]+1}-1}}dv.\quad(3.4)
$$

Now, using (3.4) we have

$$
G_\rho(\lambda,\rho)=\int_0^1\frac{N(v)}{\sqrt{2\lambda\mu\rho^2(2\rho(v^3-1)-3v^2+3)+1}\Big(\sqrt{2\lambda\mu\rho^2(2\rho(v^3-1)-3v^2+3)+1}-1\Big)^{3/2}}dv,\quad(3.5)
$$

where

$$
N(v)=\lambda\mu\rho^2\Big(\rho(v^3-1)-3v^2+3\Big)-\sqrt{2\lambda\mu\rho^2\Big(2\rho(v^3-1)-3v^2+3\Big)+1}+1.
$$

Clearly the denominator of (3.5) is positive. Further, $N(1) = 0$. Hence, if we prove that $N'(v) < 0$, then $N(v)$ has to be positive on $[0,1)$. Now,

$$N'(v) = 3\lambda\mu\rho^2 v(\rho v - 2) - \frac{6\lambda\mu\rho^2 v(\rho v - 1)}{\sqrt{2\lambda\mu\rho^2\left(2\rho(v^3-1) - 3v^2 + 3\right) + 1}},$$

so $N'(v) < 0$ provided that $2 - \rho v > \frac{2(1-\rho v)}{\sqrt{2\lambda\mu\rho^2\sigma(v)+1}}$, where $\sigma(v) = \left(2\rho(v^3-1) - 3v^2 + 3\right)$. But since $\sigma'(v) = 6v(\rho v - 1) < 0$ and $\sigma(1) = 0$, we must have $\sigma(v) \geq 0$; $v \in (0,1)$. Hence, $N'(v) < 0$ provided $2 - \rho v > 2(1 - \rho v)$, which is clearly true. So $G_\rho(\lambda, \rho) > 0$ for $\lambda > 0$ and $\rho \in (0,1)$. Now combining this with our existence of a positive solution for $\lambda > \pi^2$, we see that there exists a unique $\rho \in (0,1)$ such that $G(\lambda, \rho) = \frac{1}{2\sqrt{3\mu}}$. Further, from (2.3) it is easy to see that $G_\lambda(\lambda, \rho) < 0$ for $\lambda > 0$ and $\rho \in (0,1)$ (See Figure 3.1). Thus, by the Implicit Function Theorem, there exists a unique function $\lambda : (0,1) \to (\pi^2, \infty)$ satisfying $G(\lambda(\rho), \rho) = \frac{1}{2\sqrt{3\mu}}$ and

$$\frac{d\lambda}{d\rho} = -\frac{G_\rho(\lambda, \rho)}{G_\lambda(\lambda, \rho)} > 0. \tag{3.6}$$

Recall that we already established a positive solution for $\lambda > \pi^2$. Combining this result with (3.6) we now have a unique positive solution $u_{\lambda,\mu}$ for $\lambda > \pi^2$. Further, combining with our nonexistence result for $\lambda \leq \pi^2$, we have the following:

$$\lim_{\rho \to 0} \lambda(\rho) = \pi^2 \qquad \left(\lim_{\lambda \to \pi^2} \|u_{\lambda,\mu}\|_\infty = 0\right)$$
$$\lim_{\rho \to 1} \lambda(\rho) = \infty \qquad \left(\lim_{\lambda \to \infty} \|u_{\lambda,\mu}\|_\infty = 1\right).$$

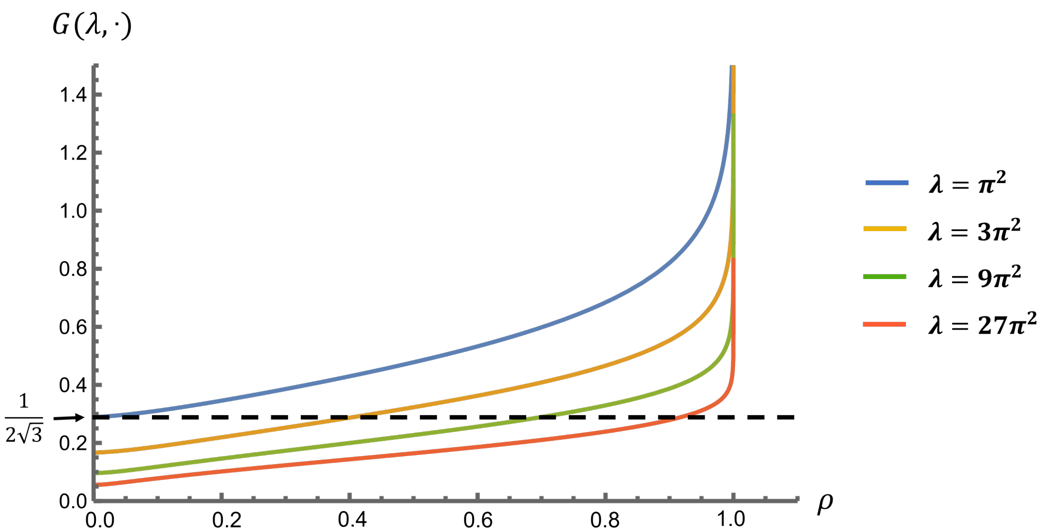

Figure 3.1: Plots of $G(\lambda, \cdot)$ for various λ. Observe their intersections with the level $\frac{1}{2\sqrt{3\mu}}$ when $\mu = 1$.

Claim: For $\mu_2 > \mu_1$, $u_{\lambda,\mu_1}(x) \geq u_{\lambda,\mu_2}(x)$ for all $x \in [0,1]$.

Let $\mu_2 > \mu_1$ and $\lambda > \pi^2$ be fixed. Now, let u_{λ,μ_1} be a positive solution to (1.1) with $\mu = \mu_1$. Then u_{λ,μ_1} satisfies $-\mu_1[(u'_{\lambda,\mu_1}(x))^3]' - u''_{\lambda,\mu_1}(x) = \lambda f(u_{\lambda,\mu_1}(x))$; $x \in (0,1)$. We proceed by showing that u_{λ,μ_1} is a supersolution to (1.1) with $\mu = \mu_2$. Observe that

$$\begin{aligned}
-\mu_2[(u'_{\lambda,\mu_1}(x))^3]' - u''_{\lambda,\mu_1}(x) &= -\mu_2\left(-\frac{1}{\mu_1}\Big(\lambda f(u_{\lambda,\mu_1}(x)) + u''_{\lambda,\mu_1}(x)\Big)\right) - u''_{\lambda,\mu_1}(x)\\
&= \frac{\mu_2}{\mu_1}\Big(\lambda f(u_{\lambda,\mu_1}(x)) + u''_{\lambda,\mu_1}(x)\Big) - u''_{\lambda,\mu_1}(x)\\
&= \frac{\mu_2}{\mu_1}\lambda f(u_{\lambda,\mu_1}(x)) + \Big(\frac{\mu_2}{\mu_1} - 1\Big)u''_{\lambda,\mu_1}(x)\\
&\geq \lambda f(u_{\lambda,\mu_1}(x)); \qquad x \in (0,1)
\end{aligned}$$

provided that

$$\Big(\frac{\mu_2}{\mu_1} - 1\Big)\Big(\lambda f(u_{\lambda,\mu_1}(x)) + u''_{\lambda,\mu_1}(x)\Big) \geq 0; \qquad x \in (0,1). \tag{3.7}$$

Given our assumption that $\mu_2 > \mu_1$, we have $\frac{\mu_2}{\mu_1} - 1 > 0$. By (1.1) with $p = 4$ and $q = 2$, it is easy to see that $u''_{\lambda,\mu_1}(x) = \frac{-\lambda f(u_{\lambda,\mu_1}(x))}{1+3\mu_1(u'_{\lambda,\mu_1}(x))^2}$; $x \in (0,1)$. Hence

$$\lambda f(u_{\lambda,\mu_1}(x)) + u''_{\lambda,\mu_1}(x) = \lambda f(u_{\lambda,\mu_1}(x))\left(1 - \frac{1}{1+3\mu_1(u'_{\lambda,\mu_1}(x))^2}\right) \geq 0; \qquad x \in (0,1).$$

So (3.7) is satisfied and u_{λ,μ_1} is a supersolution to (1.1) with $\mu = \mu_2$. Recall that $\psi(x) = \varepsilon\sin(\pi x)$ with $\varepsilon > 0$ and $\varepsilon \approx 0$ is a subsolution to (1.1) for any $\mu > 0$ when $\lambda > \pi^2$ and clearly $\psi \leq u_{\lambda,\mu_1}$ when $\varepsilon \approx 0$. Thus, the unique positive solution u_{λ,μ_2} to (1.1) with μ_2 when $\lambda > \pi^2$ must be such that $u_{\lambda,\mu_2} \in [\psi, u_{\lambda,\mu_1}]$. Hence, $u_{\lambda,\mu_1}(x) \geq u_{\lambda,\mu_2}(x)$ for all $x \in [0,1]$. □

4 Computation of bifurcation diagrams as μ varies

The bifurcation diagrams for $\mu > 0$ in Figure 4.1 are computed using (2.3). In particular, for a sequence of values $\rho \in (0,1)$, we determine the corresponding sequence of $\lambda > 0$ such that (2.3) is satisfied using the *FindRoot* function in *Mathematica*. The bifurcation curves are generated using linear interpolation of the points $\{(\lambda,\rho)\}$. Similarly, for the $\mu = 0$ case, we apply (2.10).

In Figure 4.2, we generate profiles of positive solutions for $\lambda = 50$, $\mu_1 = 5$, and $\mu_2 = 30$ using (2.8) for $x \in [0, \frac{1}{2})$ and appealing to the symmetry established in Lemma 2.2. This illustrates that $u_{50,5}(x) \geq u_{50,30}(x)$ for all $x \in [0,1]$ as described in Theorem 1.1 for particular choices of μ_1 and μ_2. By considering a uniform sequence of x-values lying in $[0,1]$ and solving (2.8) with corresponding λ, ρ, μ values within a specified tolerance using *FindRoot*, then linearly interpolating the points $\{(x, u_{\lambda,\mu}(x))\}$, we obtain the solution profiles.

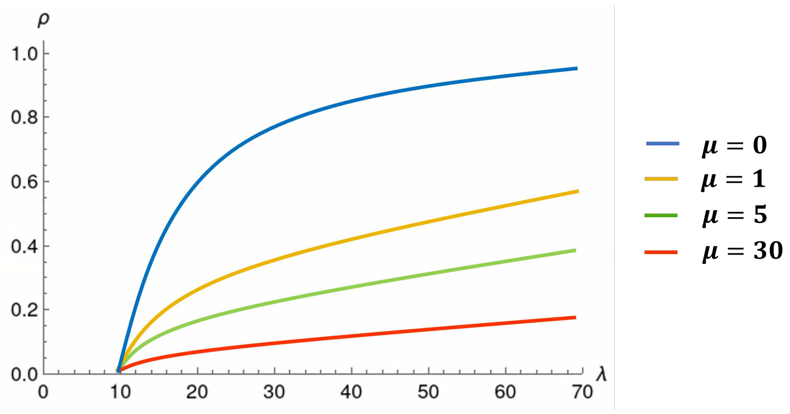

Figure 4.1: Evolution of exact bifurcation diagrams of positive solutions to (1.1) as $\mu \geq 0$ varies.

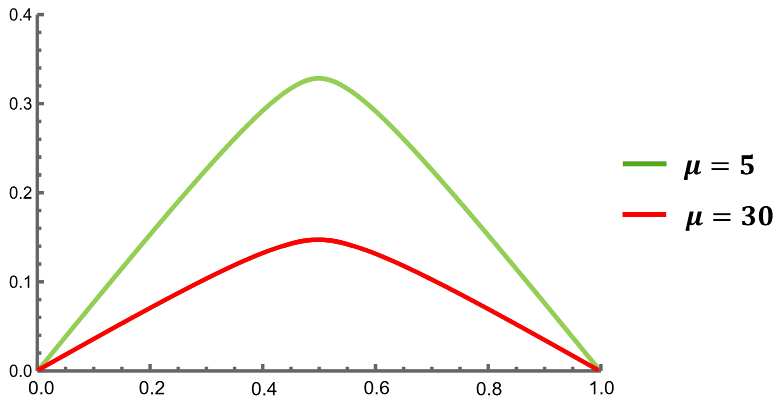

Figure 4.2: Profiles of positive solutions u_{λ,μ_1} and u_{λ,μ_2} to (1.1) for $\lambda = 50$, $\mu_1 = 5$, and $\mu_2 = 30$.

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Ulam–Hyers stability and exponentially dichotomic evolution equations in Banach spaces

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For finite-dimensional linear differential systems with bounded coefficients we prove that their exponential dichotomy on $\mathbb{R}$ is equivalent to their Ulam–Hyers stability on $\mathbb{R}$ with uniqueness. We also consider abstract non-autonomous evolution equations which are exponentially bounded and exponentially dichotomic and prove that Ulam–Hyers stability with uniqueness is maintained when perturbing them with a nonlinear term having a sufficiently small Lipschitz constant.

Keywords: Ulam–Hyers stability, evolution family, nonautonomous, mild solution, exponential dichotomy, small Lipschitz constant.

2020 Mathematics Subject Classification: 34G10, 34G20, 34D09.

1 Introduction

Ulam–Hyers stability of different types of equations is intensively studied in the literature, especially in the last years. The idea of this notion was given by Ulam in 1940. Note that there exists generalizations of the initial notion (see [15]). As far as we know, the first studies on the Ulam–Hyers stability of differential equations were presented by Obłoza [12, 13] in 1993 and 1997, and by Alsina–Ger [1] in 1998.

The special case of finite dimensional linear differential systems with constant and, respectively, continuous periodic coefficients, was considered by Jung [10] in 2006, Bușe–Salieri–Tabassum [5] in 2014, Barbu–Bușe–Tabassum [4] in 2015, and, respectively, by Buică–Tőtős [3] in 2022. These papers emphasized the relation of Ulam–Hyers stability on unbounded intervals of finite dimensional linear differential systems, with their exponential dichotomy.

Ulam–Hyers stability of some nonlinear differential equations were also studied, especially on a compact interval of time. Anyway, it seems that Ulam–Hyers stability on a compact interval is a property of any linear differential system and of the most of the nonlinear ones. I. A. Rus proved this using the Gronwall Lemma technique and other techniques in [14]. In [2] we showed that exponentially stable abstract linear evolution equations are Ulam–Hyers stable on the interval $[0, \infty)$. We also proved that this property is maintained when perturbing this type of equations with a nonlinear term having a sufficiently small Lipschitz constant.

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In this work we show that exponentially dichotomic on $\mathbb{R}$ abstract linear evolution equations are Ulam–Hyers stable on $\mathbb{R}$ with uniqueness. We study the special case of finite dimensional linear differential systems with bounded coefficients and prove that their exponential dichotomy is equivalent to their Ulam–Hyers stability with uniqueness (Theorem 3.5 in Section 3). We also prove that Ulam–Hyers stability with uniqueness is maintained when perturbing this type of linear abstract evolution equations with a nonlinear term having a sufficiently small Lipschitz constant (Theorem 4.2, Theorem 4.4 and Theorem 4.6 in Section 4).

2 Exponential dichotomy of an evolution family. Definition and equivalent condition

Let $(X, |\cdot|)$ be a real or complex Banach space. The zero vector in X will be denoted by 0. $\mathcal{L}(X)$ will stand for the space of bounded linear operators from X into itself. The corresponding norm in $\mathcal{L}(X)$ will also be denoted by $|\cdot|$. The identity operator on X is $I \in \mathcal{L}(X)$. For notations, notions and results presented in this section we used [6, 11].

Definition 2.1 ([6, Definition 3.1])**.** A family of operators $\{U(\theta,\tau)\}_{\theta\geq\tau} \subset \mathcal{L}(X)$, with $\theta, \tau \in \mathbb{R}$, is called an evolution family if

(i) $U(\theta,s)U(s,\tau) = U(\theta,\tau)$ and $U(\theta,\theta) = I$ for all $\theta \geq s \geq \tau$; and

(ii) for each $x \in X$, the function $(\theta,\tau) \mapsto U(\theta,\tau)x$ is continuous for $\theta \geq \tau$.

An evolution family $\{U(\theta,\tau)\}_{\theta\geq\tau}$ is said to be exponentially bounded if, in addition,

(iii) there exist real constants $C \geq 1$ and $\gamma > 0$ such that

$$|U(\theta,\tau)| \leq Ce^{\gamma(\theta-\tau)}, \quad \theta \geq \tau.$$

We now give the definition of exponential dichotomy for an evolution family. Let $P : \mathbb{R} \to \mathcal{L}(X)$ be a projection-valued function (i.e. $P(\theta)P(\theta) = P(\theta)$ for each $\theta \in \mathbb{R}$). The function whose values are the complementary projections is denoted by $Q(\theta) = I - P(\theta)$ for each $\theta \in \mathbb{R}$. If, for all $\theta \geq \tau$, we have

$$P(\theta)U(\theta,\tau) = U(\theta,\tau)P(\tau),$$

then we denote by

$$U_P(\theta,\tau) := P(\theta)U(\theta,\tau)P(\tau), \quad U_Q(\theta,\tau) := Q(\theta)U(\theta,\tau)Q(\tau),$$

the restrictions of the operator $U(\theta,\tau)$ on $\operatorname{Im} P(\tau)$ and $\operatorname{Im} Q(\tau)$, respectively. We stress that $U_P(\theta,\tau)$ is an operator from $\operatorname{Im} P(\tau)$ to $\operatorname{Im}(\theta)$ while $U_Q(\theta,\tau)$ is an operator from $\operatorname{Im} Q(\tau)$ to $\operatorname{Im} Q(\theta)$.

Definition 2.2 ([6, Definition 3.6])**.** An evolution family $\{U(\theta,\tau)\}_{\theta\geq\tau}$ is said to have an exponential dichotomy (with constants $M > 0$ and $\omega > 0$ if there exists a projection-valued function $P : \mathbb{R} \to \mathcal{L}(X)$ such that, for each $x \in X$, the function $\theta \mapsto P(\theta)x$ is continuous and bounded, and, for all $\theta \geq \tau$, the following conditions hold.

(i) $P(\theta)U(\theta,\tau) = U(\theta,\tau)P(\tau)$.

(ii) $U_Q(\theta,\tau)$ is invertible as an operator from $\operatorname{Im} Q(\tau)$ to $\operatorname{Im} Q(\theta)$.

(iii) $|U_P(\theta,\tau)| \le Me^{-\omega(\theta-\tau)}$.

(iv) $|[U_Q(\theta,\tau)]^{-1}| \le Me^{-\omega(\theta-\tau)}$.

Denote by $C_b(\mathbb{R},X) = \{g : \mathbb{R} \to X \text{ continuous and bounded}\}$. It is known that $C_b(\mathbb{R},X)$ with the norm $\|u\| = \max_{t\in\mathbb{R}} |u(t)|$ is a Banach space.

Condition (M). For every $g \in C_b(\mathbb{R},X)$, there exists a unique function $u \in C_b(\mathbb{R},X)$ such that

$$u(\theta) = U(\theta,\tau)u(\tau) + \int_\tau^\theta U(\theta,s)g(s)ds, \quad \theta \ge \tau. \tag{2.1}$$

Theorem 2.3 (Theorem 4.28 in [6]). *An exponentially bounded evolution family has an exponential dichotomy if and only if Condition (M) is satisfied. Moreover, if this is the case, for each $g \in C_b(\mathbb{R},X)$ the solution $u^* \in C_b(\mathbb{R},X)$ of the integral equation (2.1) is given by*

$$u^*(\theta) = \int_{-\infty}^\theta U_P(\theta,\tau)g(\tau)d\tau - \int_\theta^\infty [U_Q(\tau,\theta)]^{-1}g(\tau)d\tau, \quad \theta \in \mathbb{R}. \tag{2.2}$$

Proposition 2.4. *In the hypotheses of Theorem 2.3, the function given by (2.2) satisfies*

$$\|u^*\| \le \frac{2M}{\omega}\|g\|. \tag{2.3}$$

When either $P(t) = I$ for all $t \in \mathbb{R}$, or $Q(t) = I$ for all $t \in \mathbb{R}$, the estimation can be improved as

$$\|u^*\| \le \frac{M}{\omega}\|g\|. \tag{2.4}$$

Proof. For any $t \in \mathbb{R}$ we have

$$\begin{aligned}
|u^*(t)| &\le \left|\int_{-\infty}^t U_P(t,s)g(s)ds\right| + \left|\int_t^\infty [U_Q(s,t)]^{-1}g(s)ds\right| \\
&\le \left|\int_{-\infty}^t |U_P(t,s)|\cdot|g(s)|ds\right| + \left|\int_t^\infty |[U_Q(s,t)]^{-1}|\cdot|g(s)|ds\right| \\
&\le M\|g\|\left[\int_{-\infty}^t e^{-\omega(t-s)}ds + \int_t^\infty e^{-\omega(s-t)}ds\right] = \frac{2M}{\omega}\|g\|.
\end{aligned}$$

In each of the particular cases $P = I$ or $Q = I$, only one of the two integrals appear in the expression (2.2) of u^*. Thus, also in the last line of the display above appears only one of the two integrals, each of them being equal to $1/\omega$. □

3 Exponential dichotomy and Ulam–Hyers stability of finite dimensional linear differential systems

Let $A \in C(\mathbb{R},\mathcal{L}(\mathbb{C}^n))$. We consider the differential system in $X = \mathbb{C}^n$

$$x' = A(t)x. \tag{3.1}$$

We present now the notion of Ulam–Hyers stability on the time interval $\mathbb{R}$ of the finite dimensional linear differential system (3.1).

Definition 3.1. We say that the equation (3.1) is Ulam–Hyers stable when there exists a constant $m > 0$ such that, for any $\varepsilon > 0$ and any $\varphi \in C^1(\mathbb{R}, \mathbb{C}^n)$ with

$$|\varphi'(t) - A(t)\varphi(t)| \leq \varepsilon, \quad t \in \mathbb{R},$$

there exists $\psi \in C^1(\mathbb{R}, X^n)$ a solution of (3.1), such that $(\varphi - \psi) \in C_b(\mathbb{R}, \mathbb{C}^n)$ and

$$\|\varphi - \psi\| \leq m\varepsilon.$$

We say that the equation (3.1) is Ulam–Hyers stable with uniqueness when, for a given φ as above, there exists a unique ψ.

Remark 3.2. Assume, in addition, that there exists $T > 0$ such that $A(T + t) = A(t)$ for all $t \in \mathbb{R}$. It is known that, in this particular case, if equation (3.1) is Ulam–Hyers stable then it is Ulam–Hyers stable with uniqueness. One can see, for example [3].

An important result proved in [3] is the following.

Lemma 3.3 ([3])**.** *The equation $x' = A(t)x$ is Ulam–Hyers stable if and only if for any $g \in C_b(\mathbb{R}, \mathbb{C}^n)$ there is a solution in $C_b(\mathbb{R}, \mathbb{C}^n) \cap C^1(\mathbb{R}, \mathbb{C}^n)$ of $x' = A(t)x + g$.*

Let $Y(t) \in \mathcal{L}(\mathbb{C}^n)$ be the fundamental matrix solution of (3.1) such that $Y(0)$ is the identity matrix, and define

$$U(\theta, \tau) = Y(\theta)Y^{-1}(\tau), \quad \theta, \tau \in \mathbb{R}.$$

It is known (or it can be easily checked) that $\{U(\theta, \tau)\}_{\theta \geq \tau}$ is an evolution family and we have $[U(\theta, \tau)]^{-1} = U(\tau, \theta)$ for all $\theta, \tau \in \mathbb{R}$.

We say that *the equation $x' = A(t)x$ has an exponential dichotomy* whenever $\{U(\theta, \tau)\}_{\theta \geq \tau}$ defined above has an exponential dichotomy (as in Definition 2.2).

In addition, we have the following.

Lemma 3.4 ([7])**.** *If A is a bounded function then $\{U(\theta, \tau)\}_{\theta \geq \tau}$ is exponentially bounded.*

Proof. Fix $\tau \in \mathbb{R}$. Then $U(\cdot, \tau)$ is a matrix solution of the initial value problem $x' = A(t)x$, $x(\tau) = I_n$ (the identity matrix). Then

$$U(\theta, \tau) = I_n + \int_\tau^\theta A(s)U(s, \tau)ds, \quad \theta \geq \tau.$$

Applying the Gronwall inequality we immediately obtain $|U(\theta, \tau)| \leq e^{\gamma(\theta - \tau)}$, $\theta \geq \tau$, where $\gamma > 0$ is such that $|A(t)| \leq \gamma$ for all $t \in \mathbb{R}$. □

As a consequence of Lemma 3.3, Lemma 3.4 and Theorem 2.3 we obtain the following characterizations, which is the main result of this section.

Theorem 3.5. *Let $A \in C(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ be a bounded function. The following conditions are equivalent.*

(i) The equation (3.1) is Ulam–Hyers stable with uniqueness.

(ii) Condition (M) is satisfied for the equation (3.1).

(iii) The equation (3.1) has an exponential dichotomy.

Using Remark 3.2, Theorem 3.5, and a result from [7] we obtain the following corollary. In the statement appears the fundamental matrix solution $Y(t)$ defined before.

Corollary 3.6. *Let $A \in C(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ be a T-periodic function. The following conditions are equivalent.*

(i) The equation (3.1) is Ulam–Hyers stable with uniqueness.

(ii) Condition (M) is satisfied for the equation (3.1).

(iii) The equation (3.1) has an exponential dichotomy.

(iv) No eigenvalue of $Y(T)$ lies on the unit circle.

In the case when $A \in \mathcal{L}(\mathbb{C}^n)$ (is constant) Corollary 3.6 holds true with condition (iv) replaced by "*No eigenvalue of A has zero real part.*". These two corollaries are known, but they were justified using other tools. One can see [3, 4].

4 Main abstract result and applications

The main result of this section concludes the Ulam–Hyers stability of mild solutions of some nonlinear abstract nonautonomous evolution equations. We start by proving a lemma which is essential in the proof of the main result. We present with details two applications of the main abstract result for finite dimensional nonautonomous differential systems and for an abstract autonomous evolution equation whose linear part is the generator of a C_0-semigroup.

Lemma 4.1. *Let $\{U(\theta, \tau)\}_{\theta \geq \tau}$ be an exponentially bounded evolution family on X. In addition, assume that it has an exponential dichotomy and let the constants $M > 0$ and $\omega > 0$ be like in Definition 2.2.*

Let $L > 0$, $g \in C_b(\mathbb{R}, X)$ and $F \in C(\mathbb{R} \times X, X)$ with $F(s, 0) = 0$ for any $s \in \mathbb{R}$. Assume that

(i) $|F(s, u_1) - F(s, u_2)| \leq L|u_1 - u_2|$, $s \in \mathbb{R}$, $u_1, u_2 \in X$,

(ii) $2L < \omega/M$.

Then there exists a unique solution $u^* \in C_b(\mathbb{R}, X)$ of the following integral equation.

$$u(t) = \int_{-\infty}^{t} U_P(t, s)[F(s, u(s)) + g(s)]ds - \int_{t}^{\infty} [U_Q(s, t)]^{-1}[F(s, u(s)) + g(s)]ds. \tag{4.1}$$

Moreover, we have

$$\|u^*\| \leq \frac{M}{\omega/2 - LM}\|g\|. \tag{4.2}$$

When either $P(t) = I$ for all $t \in \mathbb{R}$, or $Q(t) = I$ for all $t \in \mathbb{R}$, condition (ii) *can be replaced by* (ii)′ *$L < \omega/M$ and the estimation (4.2) can be improved as*

$$\|u^*\| \leq \frac{M}{\omega - LM}\|g\|. \tag{4.3}$$

Proof. Consider the operator

$$B : C_b(\mathbb{R}, X) \to C(\mathbb{R}, X)$$

defined for any $u \in C_b(\mathbb{R}, X)$ and for any $t \in \mathbb{R}$ by

$$B(u)(t) = \int_{-\infty}^{t} U_P(t,s)[F(s,u(s)) + g(s)]ds - \int_{t}^{\infty} [U_Q(s,t)]^{-1}[F(s,u(s)) + g(s)]ds.$$

We claim that B is a contraction with the Lipschitz constant $2LM/\omega$. For any $u_1, u_2 \in C_b(\mathbb{R}, X)$ and $t \in \mathbb{R}$ we have

$$\begin{aligned} |B(u_1)(t) - B(u_2)(t)| &\leq \left| \int_{-\infty}^{t} U_P(t,s)[F(s,u_1(s)) - F(s,u_2(s))]ds \right| \\ &\quad + \left| \int_{t}^{\infty} [U_Q(s,t)]^{-1}[F(s,u_1(s)) - F(s,u_2(s))]ds \right| \\ &\leq L \left| \int_{-\infty}^{t} |U_P(t,s)| \cdot |u_1(s) - u_2(s)| ds \right| \\ &\quad + L \left| \int_{t}^{\infty} |[U_Q(s,t)]^{-1}| \cdot |u_1(s) - u_2(s)| ds \right| \\ &\leq LM \|u_1 - u_2\| \left[\int_{-\infty}^{t} e^{-\omega(t-s)} ds + \int_{t}^{\infty} e^{-\omega(s-t)} ds \right] \\ &\leq \frac{2LM}{\omega} \|u_1 - u_2\|. \end{aligned}$$

Then

$$\|B(u_1) - B(u_2)\| \leq \frac{2LM}{\omega} \|u_1 - u_2\|, \quad u_1, u_2 \in C_b(\mathbb{R}, X). \tag{4.4}$$

Thus, the claim is proved.

By Theorem 2.3 we have $B(0) \in C_b(\mathbb{R}, X)$ since its expression is given by (2.2). Then using (2.3) from Proposition 2.4 we have

$$\|B(0)\| \leq \frac{2M}{\omega} \|g\|. \tag{4.5}$$

Relation (4.4) implies that

$$\|B(u)\| \leq \frac{2LM}{\omega} \|u\| + \|B(0)\|, \quad u \in C_b(\mathbb{R}, X). \tag{4.6}$$

Then

$$Bu \in C_b(\mathbb{R}, X), \quad u \in C_b(\mathbb{R}, X),$$

meaning that $C_b(\mathbb{R}, X)$ is invariant for B. The Contraction Mapping Principle assures the existence of a unique fixed point, denoted u^*, of B in $C_b(\mathbb{R}, X)$. Moreover, from (4.6) we deduce that

$$\|u^*\| \leq \frac{2LM}{\omega} \|u^*\| + \|B(0)\|,$$

which, together with (4.5) implies (4.2).

For the last part one needs to use (2.4) instead of (2.3). □

Theorem 4.2. *Let $\{U(\theta,\tau)\}_{\theta\geq\tau}$ be an exponentially bounded evolution family on X. In addition, assume that it has an exponential dichotomy and let the constants $M > 0$ and $\omega > 0$ be like in Definition 2.2.*

Let $f \in C(\mathbb{R}\times X, X)$, $L > 0$ be such that

(i) $|f(s,u_1) - f(s,u_2)| \leq L|u_1 - u_2|$, $s \in \mathbb{R}$, $u_1, u_2 \in X$,

(ii) $2L < \omega/M$.

Let $g \in C_b(\mathbb{R}, X)$. If $\varphi \in C(\mathbb{R}, X)$ is a solution of

$$y(\theta) = U(\theta,\tau)y(\tau) + \int_\tau^\theta U(\theta,s)[f(s,y(s)+g(s)]ds, \quad \theta \geq \tau, \tag{4.7}$$

then there exists a unique solution $\psi \in C(\mathbb{R}, X)$ of

$$x(\theta) = U(\theta,\tau)x(\tau) + \int_\tau^\theta U(\theta,s)f(s,x(s)ds, \quad \theta \geq \tau, \tag{4.8}$$

such that $(\varphi - \psi) \in C_b(\mathbb{R}, X)$ and

$$\|\varphi - \psi\| \leq \frac{M}{\omega/2 - LM}\|g\|. \tag{4.9}$$

When either $P(t) = I$ for all $t \in \mathbb{R}$, or $Q(t) = I$ for all $t \in \mathbb{R}$, condition (ii) *can be replaced by* (ii)′ *$L < \omega/M$ and the estimation (4.9) can be improved as*

$$\|\varphi - \psi\| \leq \frac{M}{\omega - LM}\|g\|. \tag{4.10}$$

Proof. Consider the function $F : \mathbb{R}\times X \to X$ defined by

$$F(s,u) = f(s,\varphi(s)) - f(s,\varphi(s) - u), \quad (s,u) \in \mathbb{R}\times X.$$

It is not difficult to see that F satisfies the hypotheses of Lemma 4.1. In fact, all the hypotheses of this theorem are fulfilled. Then let $u^* \in C_b(\mathbb{R}, X)$ be the unique bounded solution of equation (4.1). Consider the function $g^*(s) = F(s,u^*(s)) + g(s)$, $s \in \mathbb{R}$ which satisfies $g^* \in C_b(\mathbb{R}, X)$. Then, from (4.1) we have that

$$u^*(\theta) = \int_{-\infty}^\theta U_P(\theta,\tau)g^*(\tau)d\tau - \int_\theta^\infty [U_Q(\tau,\theta)]^{-1}g^*(\tau)d\tau, \quad \theta \in \mathbb{R}. \tag{4.11}$$

By Theorem 2.3, the above relation implies that u^* is the unique bounded solution of

$$u(\theta) = U(\theta,\tau)u(\tau) + \int_\tau^\theta U(\theta,s)g^*(s)ds, \quad \theta \geq \tau. \tag{4.12}$$

Now define

$$\psi = \varphi - u^*$$

and note that $\psi \in C(\mathbb{R}, X)$ is a solution of (4.7) which, in addition, by Lemma 4.1, satisfies (4.9). The uniqueness of ψ with mentioned properties follows by the uniqueness of u^* as in Theorem 2.3. □

4.1 Application. Finite dimensional differential systems

Let $A \in C(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ and $f \in C(\mathbb{R} \times \mathbb{C}^n, \mathbb{C}^n)$. We consider the nonlinear differential system in $X = \mathbb{C}^n$

$$x' = A(t)x + f(t,x). \tag{4.13}$$

Recall that we refer to the linear system $x' = A(t)x$ in Section 3.

Definition 4.3. We say that the equation (4.13) is Ulam–Hyers stable when there exists a constant $m > 0$ such that, for any $\varepsilon > 0$ and any $\varphi \in C^1(\mathbb{R}, \mathbb{C}^n)$ with

$$|\varphi'(t) - A(t)\varphi(t) - f(t, \varphi(t))| \leq \varepsilon, \quad t \in \mathbb{R},$$

there exists $\psi \in C^1(\mathbb{R}, \mathbb{C}^n)$ a solution of (4.13), such that $(\varphi - \psi) \in C_b(\mathbb{R}, \mathbb{C}^n)$ and

$$\|\varphi - \psi\| \leq m\varepsilon.$$

We say that the equation (4.13) is Ulam–Hyers stable with uniqueness when, for a given φ as above, there exists a unique ψ.

As a consequence of Theorem 4.2, using also Lemma 3.4, we obtain the following result.

Theorem 4.4. *Assume that A is a bounded function and that the system $x' = A(t)x$ has an exponential dichotomy. Let $M > 0$ and $\omega > 0$ be like in Definition 2.2. Assume that there exists $L > 0$ with $2L < \omega/M$ and such that*

$$|f(s,y) - f(s,x)| \leq L|x - y|, \quad \text{for all } s \in \mathbb{R}, \quad x, y \in \mathbb{C}^n.$$

Then system (4.13) is Ulam–Hyers stable with uniqueness and with constant

$$m = M/(\omega/2 - LM).$$

4.2 Application. Semigroups

For the definition of a C_0-semigroup and other useful results we used [8,9].

Definition 4.5. If the evolution family $\{U(\theta, \tau)\}_{\theta \geq \tau}$ on the Banach space X satisfies in addition

$$U(\theta, \tau)x = U(\theta - \tau, 0)x, \quad \theta \geq \tau, \quad x \in X,$$

then it is called a C_0-semigroup.

Assume from now that $\{U(\theta, \tau)\}_{\theta \geq \tau}$ is a C_0-semigroup. An important remark is that there exists a dense set $D \subset X$ and a linear operator $A : D \to X$ such that if $x \in D$,

$$\lim_{\theta \downarrow 0} \frac{U(\theta, 0)x - x}{\theta} = Ax.$$

The mapping A is in general unbounded and is called the infinitesimal generator of the semigroup. Sometimes the following notation is used

$$e^{tA} := U(t, 0), \quad t \geq 0$$

and it is said that $\{e^{tA}\}_{t\geq 0}$ is a one-parameter C_0-semigroup.

Let $f \in C(\mathbb{R} \times X, X)$ and consider the abstract evolution equation

$$x' + Ax = f(t,x), \tag{4.14}$$

and the abstract evolution inequation

$$|x' + Ax - f(t,x)| \leq \varepsilon. \tag{4.15}$$

We say that $\psi \in C(\mathbb{R}, X)$ is a *mild solution* of equation (4.14) if ψ is a solution of the integral equation (4.8).

We say that $\varphi \in C(\mathbb{R}, X)$ is a *mild solution* of inequation (4.15) if there exists $g \in C(\mathbb{R}, X)$ with $|g(s)| \leq \varepsilon$, $s \in \mathbb{R}$ such that φ is a solution of the integral equation (4.7).

Let $m > 0$. We say that the evolution equation (4.14) is *Ulam–Hyers stable with constant m* if for any $\varepsilon > 0$ and for any mild solution $\varphi \in C(\mathbb{R}, X)$ of inequation (4.15) there exists a mild solution $\psi \in C(\mathbb{R}, X)$ of (4.14) such that $(\varphi - \psi) \in C_b(\mathbb{R}, X)$ and

$$\|\varphi - \psi\| \leq m\varepsilon.$$

We say that the equation (4.14) is Ulam–Hyers stable with uniqueness when, for a given φ as above, there exists a unique ψ.

As a consequence of Theorem 4.2 we obtain the following result.

Theorem 4.6. *Let $A : D \subset X \to X$ be the infinitesimal generator of an exponentially bounded and exponentially dichotomic C_0-semigroup $\{U(\theta,\tau)\}_{\theta\geq\tau}$. Let M and ω be like in Definition 2.2. Assume that there exists $L > 0$ with $2L < \omega/M$ such that*

$$|f(s,y) - f(s,x)| \leq L|x-y|, \quad \text{for all } s \in \mathbb{R}, \quad x,y \in X.$$

Then the abstract evolution equation (4.14) is Ulam–Hyers stable with uniqueness and with constant $m = M/(\omega/2 - LM)$.

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Carleman inequality for a class of super strong degenerate parabolic operators and applications

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Abstract. In this paper, we present a new Carleman estimate for the adjoint equations associated to a class of super strong degenerate parabolic linear problems. Our approach considers a standard geometric imposition on the control domain, which can not be removed in general. Additionally, we also apply the aforementioned main inequality in order to investigate the null controllability of two nonlinear parabolic systems. The first application is concerned a global null controllability result obtained for some semilinear equations, relying on a fixed point argument. In the second one, a local null controllability for some equations with nonlocal terms is also achieved, by using an inverse function theorem.

Keywords: degenerate parabolic equations, Carleman estimates, linear systems in control theory, nonlinear systems in control theory.

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1 Introduction

In this work we derive a new Carleman estimate for the linear super strong degenerate problem

$$\begin{cases} u_t - (x^\alpha u_x)_x + x^{\alpha/2} b_1(x,t) u_x + b_0(x,t) u = f 1_\omega & \text{in } Q, \\ u(1,t) = 0 \text{ and } (x^\alpha u_x)(0,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,1), \end{cases} \tag{1.1}$$

where $Q = (0,1) \times (0,T)$, $\omega \subset (0,1)$ is a non-empty open interval and 1_ω is its associated characteristic function, and $\alpha \geq 2$. Also, we take $b_0 \in L^\infty(Q)$, $h \in L^2(\omega \times (0,T))$, $u_0 \in L^2(0,1)$, and $b_1 \in L^\infty(Q)$ satisfying

$$(x^{\alpha/2} b_1(x,t))_x \in L^\infty(Q). \tag{1.2}$$

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We also consider a geometrical condition on the control domain

$$\exists d > 0; \quad (0,d) \subset \omega. \tag{1.3}$$

As we will see further, (1.1) is controllable at any time $T > 0$, according to the following specification:

Definition 1.1. We say that (1.1) is *null controllable* at $T > 0$ if, for any $u_0 \in L^2(0,1)$, there exists $h \in L^2(\omega \times (0,T))$ such that the solution u of (1.1) satisfies

$$u(x,T) = 0 \quad \text{in } (0,1). \tag{1.4}$$

The null controllability of (1.1) is well understood for $\alpha \in (0,2)$, see [1,9] and references therein. Following the terminology adopted in these works, we say that (1.1) is *weakly degenerate* if $\alpha \in (0,1)$ and *strongly degenerate* if $\alpha \in (1,2)$. Despite there are many works for the case $\alpha \in (0,2)$, little has been done for the *super strong degenerate case*, i.e. when $\alpha \geq 2$, although this is a very relevant case of the degenerate problem. Indeed, when $\alpha = 2$, the Black-Scholes equation can be obtained from (1.1) and this equation has a key role in several financial applications.

Regarding the global null controllability of (1.1), the fact is that this problem is not null controllable for $\alpha \geq 2$, in general. As pointed out in [9], a suitable change of variables transforms (1.1) into a non-degenerate problem in an unbounded domain, which fails to be null controllable in general, as proved in [14]. However, if the new control domain $\tilde{\omega}$ has bounded complement, it can be controlled, as proved in [4,7].

Because of that, in [8], it was introduced a weaker kind of null controllability for this problem, called *regional null controllability*. It means that for any $u_0 \in L^2(0,T)$, $\omega = (a,b) \subset (0,1)$ and $\delta \in (0, b-a)$, there exists a control $f \in L^2(Q)$ such that the solution u of (1.1) satisfies

$$u(x,T) = 0 \quad \forall x \in (a+\delta, 1). \tag{1.5}$$

They established *regional null controllability* for a linear problem like (1.1), but with $b_1 = 0$. In [6], this result was extended for a system like (1.1) with the first order term and a semilinar case with a nonlinearity independent of it, i.e., *regional null controllability* was achieved for (1.1) and for the following system

$$\begin{cases} u_t - (x^\alpha u_x)_x + g(x,t,u) = f 1_\omega & \text{in } Q, \\ u(1,t) = 0 \text{ and } (x^\alpha u_x)(0,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,1). \end{cases} \tag{1.6}$$

Finally, in [5], those results were extended considering a nonlinearity of the type $g(x,t,u,u_x)$, but the restriction $\alpha \in (0,2)$ was made. These works were concerned with *regional null controllability*, more recently, in [3], the authors came up with the new geometrical condition (1.3), which allows to prove a global null controllability result for (1.1), when $\alpha = 2$. In this work, under the same geometrical condition, we will extend that result for $\alpha > 2$.

A significant number of papers on null controllability of parabolic degenerate equations follows a standard approach based on the Hilbert Uniqueness Method (HUM). It goes through obtaining a Carleman estimate that leads to an observability inequality. This way, the null controllability property can be deduced from the observability inequality. The particularity of [3] and [8] is that the authors applied a change of variables to transform the system (1.1) into a non-degenerate problem in unbounded domains. There, a Carleman estimate is obtained for this non-degenerate system.

Although the approach of transforming the degenerate problem into a non-degenerate one, in an unbounded domain, works fine for linear problems, this procedure can meet difficulties to deal with some related problems. Indeed, when we work with some autonomous semilinear problems, for example, this change of variable leads it to a nonautonomous semilinear problem. And, if we work with a certain nonlocal problems, it is lead to an even more complicated one. In this work we present a Carleman estimate for (1.1), without passing by this change of variables method. To our best knowledge, this estimate and some consequences presented in the sequel mean some novelties for the super strong degenerate case.

The second part of the introduction is all about the presentation of our main results.

Statement of the results

First of all, let us consider the adjoint system associated to (1.1):

$$\begin{cases} v_t + (x^\alpha v_x)_x + (x^{\alpha/2} b_1 v)_x - b_0(x,t)v = h & \text{in } Q, \\ v(1,t) = 0 \quad \text{and} \quad (x^\alpha v_x)(0,t) = 0 & \text{in } (0,T), \\ v(x,T) = v_T(x) & \text{in } (0,1), \end{cases} \tag{1.7}$$

where $h \in L^2(Q)$ and $v_T \in L^2(0,1)$.

Now, for $\lambda > 0$, let us introduce some weight functions given by θ, p_0 and σ_0 with

$$\theta(t) := \frac{1}{(t(T-t))^4}, \quad \eta(x) := -x^2/2, \quad \xi(x,t) = \theta(t) e^{\lambda(2|\eta|_\infty + \eta(x))}$$
$$\text{and} \quad \sigma(x,t) := \theta(t) e^{4\lambda|\eta|_\infty} - \xi(x,t). \tag{1.8}$$

The assumption (1.3) and the weight function η are the key points that allow us to build the following Carleman estimate:

Theorem 1.2. *Assume* (1.2) *and* (1.3). *There exists positive constants C, s_0 and λ_0, depending only on ω, $\|b_0\|_\infty$, T, d and α such that, for any $s \geq s_0$, any $\lambda \geq \lambda_0$ and any solution v to* (1.7), *one has:*

$$\iint_Q e^{-2s\sigma} \left[s^{-1}\lambda^{-1}\xi^{-1}(|v_t|^2 + |(x^\alpha v_x)_x|^2) + s\lambda^2 \xi x^\alpha |v_x|^2 + s^3\lambda^4\xi^3|v|^2 \right] dx\, dt$$
$$\leq C \left[\|e^{-s\sigma} h\|^2 + s^3\lambda^4 \iint_{\omega_T} e^{-2s\sigma}\xi^3 |v|^2\, dx\, dt \right], \tag{1.9}$$

where $\omega_T := \omega \times (0,T)$.

The proof of Theorem 1.2 will be given in section 3.

As a consequence of Theorem 1.2 we have the following null controllability result:

Theorem 1.3. *Assume* (1.2) *and* (1.3). *Then the system* (1.1) *is null controllable.*

Next, the same Carleman estimate allows us to prove a null controllability result for the following semilinear problem

$$\begin{cases} u_t - (x^\alpha u_x)_x + g(x,t,u,u_x) = f 1_\omega & \text{in } Q, \\ u(1,t) = 0 \text{ and } (x^\alpha u_x)(0,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,1), \end{cases} \tag{1.10}$$

where $\alpha \geq 2$ and $g : Q \times \mathbb{R}^2 \to \mathbb{R}$ must satisfies the following assumptions:

$$\begin{cases} g \quad \text{is Lebesgue measurable;} \\ g(x,t,\cdot,\cdot) \in C^1(\mathbb{R}^2) \text{ uniformly in } (x,t) \in Q; \\ g(x,t,0,0) = 0 \quad \forall (x,t) \in Q; \\ \exists K > 0 \quad \text{such that} \quad |g_r(x,t,r,s)| + x^{-\alpha/2}|g_s(x,t,r,s)| \leq K \quad \forall (x,t,r,s) \in Q \times \mathbb{R}^2. \end{cases} \tag{1.11}$$

Theorem 1.4. *Assume* (1.3) *and* (1.11)*. Then the system* (1.10) *is null controllable.*

In [15], a null controllability result is obtained for (1.10), when $\alpha \in (0,2)$. In this current work, we extend this fact for the super strong degenerate case applying a similar technique. At this point, we recall that the classical null controllability for (1.10) does not hold in general, but the geometrical assumption (1.3) provided the inequality (1.9), which can be applied to prove Theorem 1.4. In other words, the obtainment of Theorem 1.4 is possible because the degeneracy point $x = 0$ belongs to the boundary of the control domain ω. It is worth observe that, Cannarsa and Fragnelli proved, in [5], *regional null controllability* results for (1.10), when $\alpha \in (0,2)$. Summarizing, we emphasize that the investigation of [5] does not rely on the localization of ω near $x = 0$, as in this paper, but it only allows to find a control which drives the state to zero in a portion of ω far from the degeneracy point $x = 0$.

As a second application of our Carleman estimate (1.9), we will also obtain the local null controllability for the following degenerate nonlocal problem

$$\begin{cases} u_t - \ell\left(\int_0^1 u\,dx\right)(x^\alpha u_x)_x = f1_\omega & \text{in } Q, \\ u(1,t) = 0 \text{ and } (x^\alpha u_x)(0,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,1), \end{cases} \tag{1.12}$$

where $\ell : \mathbb{R} \to \mathbb{R}$ is a C^1 function with bounded derivative, with $\ell(0) = 1$. At this point, we should observe that our results remain the same if we just consider $\ell(0) > 0$. The null controllability for this problem is studied in [11], when $\alpha \in (0,1)$, and in [10] when $\alpha \in [1,2)$. Under the hypotheses (1.2) and (1.3), we extend this investigation for $\alpha \in [2,+\infty)$, as described below:

Theorem 1.5. *Assume* (1.3)*. The nonlinear system* (1.12) *is locally null-controllable at any time* $T > 0$*, i.e, there exists* $\varepsilon > 0$ *such that, whenever* $u_0 \in H^1_\alpha$ *and* $|u_0|_{H^1_\alpha} \leq \varepsilon$*, there exists a control* $f \in L^2(\omega \times (0,T))$*, associated to a state* u*, satisfying*

$$u(x,T) = 0, \text{ for every } x \in [0,1].$$

The rest of this paper is organized as follows. In Section 2, we state some classical well-posedness results related to the systems (1.1) and (1.10). In Section 3, we present an α-independent Carleman inequality for solutions of (1.7) (see Theorem 1.2), which provides an observability estimate and, consequently, the null controllability of (1.1). Sections 4 and 5 are devoted to some applications of Theorem 1.2. More precisely, in Section 4, we use a fixed point argument to obtain a null controllability result to the degenerate semilinear system (1.10) (see Theorem 1.4); in Section 5, an inverse function argument allows us to prove a local null controllability result for the degenerate nonlocal system (1.12) (see Theorem 1.5).

2 Well-posedness results

The usual norms in $L^2(0,1)$ and $L^2(Q)$ will be denoted by $|\cdot|_2$ and $\|\cdot\|_2$, respectively, related to the usual inner products $(\cdot,\cdot)$ and $((\cdot,\cdot))$. Moreover, the norms in $L^\infty(0,1)$ and in $L^\infty(Q)$ will be denoted respectively by $|\cdot|_\infty$ and $\|\cdot\|_\infty$.

Let us consider the functional spaces

$$H^1_\alpha := \Big\{u \in L^2(0,1); u \text{ is locally absolutely continuous in } (0,1],\ x^{\alpha/2}u_x \in L^2(0,1),\ u(1) = 0\Big\}.$$

and

$$H^2_\alpha := \Big\{u \in H^1_\alpha;\ x^\alpha u_x \in H^1(0,1)\Big\},$$

with the norms

$$|u|_{H^1_\alpha} := \left[\int_0^1 (u^2 + x^\alpha |u|^2)\, dx\right]^{1/2}, \quad \text{if } u \in H^1_\alpha$$

and

$$|u|_{H^2_\alpha} := \left[\int_0^1 (u^2 + x^\alpha |u|^2 + |(x^\alpha u_x)_x|^2)\, dx\right]^{1/2}, \quad \text{if } u \in H^2_\alpha.$$

With these norms, we observe that H^1_α and H^2_α are two Hilbert spaces. In [8, Proposition 2.1], the authors provided the following characterization:

$$H^2_\alpha = \Big\{u \in L^2(0,1);\ u \text{ is locally absolutely continuous in } (0,1],$$
$$x^\alpha u \in H^1_0(0,1),\ x^\alpha u_x \in H^1(0,1),\ (x^\alpha u_x)(0) = 0\Big\}.$$

Now, for the reader's convenience, let us introduce the notations

$$\mathcal{M} = C(0,T;L^2(0,1)) \cap L^2(0,T;H^1_\alpha) \quad \text{and} \quad \mathcal{N} = H^1(0,T;L^2(0,T)) \cap L^2(0,T;H^2_\alpha).$$

In [15], the authors proved that the embedding $\mathcal{M} \hookrightarrow \mathcal{N}$ is compact (in fact, their result was proved for $\alpha \in (0,2)$, but the proof does not depend on α).

The next result, proved in [8], establishes the well-posedness of system (1.1).

Proposition 2.1. *Assume $b_0, b_1 \in L^\infty(Q)$. For any $f \in L^2(Q)$ and any $u_0 \in L^2(0,1)$, there exists exactly one solution $u \in \mathcal{M}$ to (1.1). Furthermore, there exists a constant $C > 0$ only depending on T, α, b_1 and b_0, such that*

$$\sup_{t\in[0,T]} |u(\cdot,t)|_2^2 + \|x^{\alpha/2}u_x\|_2^2 \le C(\|f1_\omega\|_2^2 + |u_0|_2^2).$$

Furthermore, if $u_0 \in H^1_\alpha$, then $u \in \mathcal{N} \cap C^0([0,T];H^1_\alpha)$ and we have the following estimate:

$$\sup_{t\in[0,T]} |u(\cdot,t)|^2_{H^1_\alpha} + \|u_t\|_2^2 + \|(x^\alpha u_x)_x\|_2^2 \le C\left(\|f1_\omega\|_2^2 + |u_0|^2_{H^1_\alpha}\right).$$

We also state the well-posedness of (1.10), whose proof can be seen in [15, Theorem 2.1].

Proposition 2.2. *Assume g satisfies (1.11). For any $f \in L^2(Q)$ and any $u_0 \in L^2(0,1)$, there exists exactly one solution $u \in \mathcal{M}$ to the system (1.10).*

3 Carleman and observability inequalities

The aim of this section is to prove the Carleman estimate (1.9) and, as a consequence, an observability inequality, which yields the null controllability of the linear system (1.1).

It suffices to prove Theorem 1.2 for $b_1 = b_0 = 0$, since the general case follows taking $\tilde{h} = h - b_0 v - (x^{\alpha/2} b_1 v)_x$.

Let us take $\delta \in (0, d)$ and let v be the solution to (1.7) (where $v_T \in L^2(0,1)$ and $h \in L^2(Q)$). For any $s \geq s_0 > 0$, we set $z = e^{-s\sigma} v$. By a density argument we can assume without loss of generality that v is regular enough. A simple computation gives us

$$v_t = e^{s\sigma}[s\sigma_t z + z_t] \quad \text{and} \quad (x^\alpha v_x)_x = e^{s\sigma}[s^2 \sigma_x^2 x^\alpha z + 2s\sigma_x x^\alpha z_x + s(\sigma_x x^\alpha)_x z + (x^\alpha z_x)_x].$$

Consequently,

$$P^+ z + P^- z = G, \tag{3.1}$$

where

$$\begin{aligned} P^- z &:= -2s\lambda^2 \xi x^{\alpha+2} z + 2s\lambda\xi x^{\alpha+1} z_x + z_t := I_{11} + I_{12} + I_{13}, \\ P^+ z &:= s^2\lambda^2\xi^2 x^{\alpha+2} z + (x^\alpha z_x)_x + s\sigma_t z := I_{21} + I_{22} + I_{23} \end{aligned}$$

and

$$G = e^{-s\sigma} h - s\lambda^2 \xi x^{\alpha+2} z - (\alpha+1)s\lambda\xi x^\alpha z.$$

From (3.1) one has

$$\|P^- z\|_2^2 + \|P^+ z\|_2^2 + 2((P^- z, P^+ z)) = \|G\|_2^2. \tag{3.2}$$

Now let us estimate $((P^- z, P^+ z))$. We have that

$$((I_{11}, I_{21})) = -2s^3\lambda^4 \iint_Q \xi^3 x^{2\alpha+4} |z|^2 \, dx\, dt,$$

$$\begin{aligned} ((I_{12}, I_{21})) &= s^3\lambda^3 \iint_Q \xi^3 x^{2\alpha+3} (|z|^2)_x \, dx\, dt \\ &= 3s^3\lambda^4 \iint_Q \xi^3 x^{2\alpha+4} |z|^2 \, dx\, dt - (2\alpha+3)s^3\lambda^3 \iint_Q \xi^3 x^{2\alpha+2} |z|^2 \, dx\, dt \end{aligned}$$

and

$$((I_{13}, I_{21})) = \frac{1}{2} s^2\lambda^2 \iint_Q \xi^2 x^{\alpha+2} (|z|^2)_t \, dx\, dt = -s^2\lambda^2 \iint_Q \xi\xi_t x^{\alpha+2} |z|^2 \, dx\, dt.$$

Thus

$$\begin{aligned} ((P^- z, I_{21})) &= s^3\lambda^4 \iint_Q \xi^3 x^{2\alpha+4} |z|^2 \, dx\, dt - (2\alpha+3)s^3\lambda^3 \iint_Q \xi^3 x^{2\alpha+2} |z|^2 \, dx\, dt \\ &\quad - s^2\lambda^2 \iint_Q \xi\xi_t x^{\alpha+2} |z|^2 \, dx\, dt. \end{aligned}$$

Since $|\xi\xi_t| \leq C\xi^3$, for λ_0 and s_0 large enough, we can deduce that

$$
\begin{aligned}
((P^-z, I_{21})) &\geq s^3\lambda^4 \int_0^T \left[\int_0^\delta \xi^3 x^{2\alpha+4}|z|^2\,dx + \int_\delta^1 \xi^3 x^{2\alpha+4}|z|^2\,dx\right]\,dt \\
&\quad -Cs^3\lambda^3\left((2\alpha+3)+\frac{C}{\lambda_0 s_0}\right)\iint_Q \xi^3|z|^2\,dx\,dt \\
&\geq s^3\lambda^4\int_0^T\int_\delta^1 \xi^3 x^{2\alpha+4}|z|^2\,dx\,dt - Cs^3\lambda^3\iint_Q \xi^3|z|^2\,dx\,dt \\
&\geq \delta^{2\alpha+4}s^3\lambda^4\int_0^T\int_\delta^1 \xi^3|z|^2\,dx\,dt - Cs^3\lambda^3\iint_Q \xi^3|z|^2\,dx\,dt \\
&\geq Cs^3\lambda^4\int_0^T\int_\delta^1 \xi^3|z|^2\,dx\,dt - Cs^3\lambda^3\iint_Q \xi^3|z|^2\,dx\,dt \qquad (3.3)\\
&= Cs^3\lambda^4\iint_Q \xi^3|z|^2\,dx\,dt - Cs^3\lambda^4\int_0^T\int_0^\delta \xi^3|z|^2\,dx\,dt \\
&\quad -Cs^3\lambda^3\iint_Q \xi^3|z|^2\,dx\,dt \\
&\geq Cs^3\lambda^4\left(1-\frac{1}{\lambda_0}\right)\iint_Q \xi^3|z|^2\,dx\,dt - Cs^3\lambda^4\int_0^T\int_0^\delta \xi^3|z|^2\,dx\,dt \\
&\geq Cs^3\lambda^4\iint_Q \xi^3|z|^2\,dx\,dt - Cs^3\lambda^4\int_0^T\int_0^\delta \xi^3|z|^2\,dx\,dt.
\end{aligned}
$$

Note that C depends on δ and α, where $\delta \in (0,d)$ is a fixed number as before.

Furthermore,

$$((I_{11}, I_{23})) = -2s^2\lambda^2\iint_Q \xi\sigma_t x^{\alpha+2}|z|^2\,dx\,dt,$$

$$
\begin{aligned}
((I_{12}, I_{23})) &= s^2\lambda\iint_Q \xi\sigma_t x^{\alpha+1}(|z|^2)_x\,dx\,dt \\
&= s^2\lambda^2\iint_Q \xi(\sigma_t+\xi_t)x^{\alpha+2}|z|^2\,dx\,dt - (\alpha+1)s^2\lambda\iint_Q \xi\sigma_t x^\alpha|z|^2\,dx\,dt
\end{aligned}
$$

and

$$((I_{13}, I_{23})) = \frac{s}{2}\iint_Q \sigma_t(|z|^2)_t\,dx\,dt = -\frac{s}{2}\iint_Q \sigma_{tt}|z|^2\,dx\,dt.$$

Thus

$$
\begin{aligned}
((P^-z, I_{23})) = &-s^2\lambda^2\iint_Q \xi(\xi_t+\sigma_t)x^{\alpha+2}|z|^2\,dx\,dt - (\alpha+1)s^2\lambda\iint_Q \xi\sigma_t x^\alpha|z|^2\,dx\,dt \\
&-\frac{s}{2}\iint_Q \sigma_{tt}|z|^2\,dx\,dt.
\end{aligned}
$$

We can see that $|\xi_t|, |\sigma_t| \leq C\xi^2$ and $|\sigma_{tt}| \leq C\xi^3$. Hence, from (3.3), we have

$$
\begin{aligned}
((P^-z, I_{21}+I_{23})) &\geq Cs^3\lambda^4\iint_Q \xi^3|z|^2\,dx\,dt - Cs^3\lambda^4\int_0^T\int_0^\delta \xi^3|z|^2\,dx\,dt - Cs^2\lambda^2\iint_Q \xi^3|z|^2\,dx\,dt \\
&\quad -C(\alpha+1)s^2\lambda\iint_Q \xi^3|z|^2\,dx\,dt - C\frac{s}{2}\iint_Q \xi^3|z|^2\,dx\,dt \\
&\geq Cs^3\lambda^4\left(1-\frac{1}{s_0\lambda_0^2}-\frac{1}{s_0\lambda_0^3}-\frac{1}{s_0^2\lambda_0^4}\right)\iint_Q \xi^3|z|^2\,dx\,dt \\
&\quad -Cs^3\lambda^4\int_0^T\int_0^\delta \xi^3|z|^2\,dx\,dt.
\end{aligned}
$$

Therefore, for λ_0 and s_0 large enough, we have

$$((P^- z, I_{21} + I_{23})) \geq Cs^3\lambda^4 \iint_Q \xi^3 |z|^2 \, dx \, dt - Cs^3\lambda^4 \int_0^T \int_0^\delta \xi^3 |z|^2 \, dx \, dt. \tag{3.4}$$

Moreover, we have that

$$\begin{aligned}
((I_{11}, I_{22})) &= -2s\lambda^2 \iint_Q \xi x^{\alpha+2} z (x^\alpha z_x)_x \, dx \, dt \\
&= 2s\lambda^2 \iint_Q [-\lambda \xi x^{2\alpha+3} z z_x + (\alpha+2)\xi x^{2\alpha+1} z z_x + \xi x^{2\alpha+2} |z_x|^2] \, dx \, dt \\
&= s\lambda^3 \iint_Q \xi[-\lambda x^{2\alpha+4} + (2\alpha+3) x^{2\alpha+2}] |z|^2 \, dx \, dt \\
&\quad -(\alpha+2) s\lambda^2 \iint_Q \xi[-\lambda x^{2\alpha+2} + (2\alpha+1) x^{2\alpha}] |z|^2 \, dx \, dt + 2s\lambda^2 \iint_Q \xi x^{2\alpha+2} |z_x|^2 \, dx \, dt
\end{aligned}$$

and

$$\begin{aligned}
((I_{13}, I_{22})) &= \iint_Q z_t (x^\alpha z_x)_x \, dx \, dt = -\iint_Q z (x^\alpha z_{tx})_x \, dx \, dt = \iint_Q x^\alpha z_x z_{xt} \, dx \, dt \\
&= \frac{1}{2} \iint_Q (x^\alpha |z_x|)_t \, dx \, dt = 0.
\end{aligned}$$

Thus

$$((I_{11} + I_{13}, I_{22})) \geq -Cs\lambda^4 \iint_Q \xi^3 |z|^2 \, dx \, dt + 2s\lambda^2 \iint_Q \xi x^{2\alpha+2} |z_x|^2 \, dx \, dt. \tag{3.5}$$

On the other hand

$$\begin{aligned}
2s\lambda^2 \iint_Q \xi x^{2\alpha+2} |z_x|^2 \, dx \, dt &= 2s\lambda \int_0^T \left[\int_0^\delta \xi x^{2\alpha+2} |z_x|^2 \, dx + \int_\delta^1 \xi x^{2\alpha+2} |z_x|^2 \, dx \right] dt \\
&\geq 2s\lambda \delta^{\alpha+2} \int_0^T \int_\delta^T \xi x^\alpha |z_x|^2 \, dx \, dt \\
&= Cs\lambda^2 \iint_Q \xi x^\alpha |z_x|^2 \, dx \, dt - Cs\lambda^2 \int_0^T \int_0^\delta \xi x^\alpha |z_x|^2 \, dx \, dt.
\end{aligned}$$

Hence, from (3.5) we deduce that

$$((I_{11} + I_{13}, I_{22})) \geq Cs\lambda^2 \iint_Q \xi x^\alpha |z_x|^2 \, dx \, dt - Cs\lambda^2 \int_0^T \int_0^\delta \xi x^\alpha |z_x|^2 \, dx \, dt - Cs\lambda^4 \iint_Q \xi^3 |z|^2 \, dx \, dt. \tag{3.6}$$

Finally, working as before we obtain

$$\begin{aligned}
((I_{12}, I_{22})) &= 2s\lambda \iint_Q \xi x x^\alpha z_x (x^\alpha z_x)_x \, dx \, dt = s\lambda \iint_Q \xi x (|x^\alpha z_x|^2)_x \, dx \, dt \\
&= s\lambda^2 \iint_Q \xi x^{2\alpha+2} |z_x|^2 \, dx \, dt - s\lambda \iint_Q \xi x^{2\alpha} |z_x|^2 \, dx \, dt + s\lambda \int_0^T \xi |z_x(1,t)|^2 \, dt \\
&\geq Cs\lambda^2 \iint_Q \xi x^\alpha |z_x|^2 \, dx \, dt - Cs\lambda^2 \int_0^T \int_0^\delta \xi x^\alpha |z_x|^2 \, dx \, dt.
\end{aligned}$$

Thus, from (3.6) we get

$$((P^- z, I_{22})) \geq Cs\lambda^2 \iint_Q \xi x^\alpha |z_x|^2 \, dx \, dt - Cs\lambda^2 \int_0^T \int_0^\delta \xi x^\alpha |z_x|^2 \, dx \, dt - Cs\lambda^4 \iint_Q \xi^3 |z|^2 \, dx \, dt. \tag{3.7}$$

Combining (3.4) and (3.7) we obtain that

$$((P^-z, P^+z)) \geq C \iint_Q [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt - C\int_0^T \int_0^\delta [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt.$$

Whence,

$$C \iint_Q [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt \leq 2((P^-z, P^+z)) + C\int_0^T \int_0^\delta [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt. \quad (3.8)$$

From (3.2) and (3.8) we obtain

$$\begin{aligned} &\|P^-z\|_2^2 + \|P^+z\|_2^2 + C\iint_Q [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt \\ &\leq \|P^-z\|_2^2 + \|P^+z\|_2^2 + 2((P^-z, P^+z)) + \int_0^T \int_0^\delta [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt \\ &\leq \|G\|_2^2 + \int_0^T \int_0^\delta [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt. \end{aligned}$$

Hence, if we set $C_0 = 1/\min\{1, C\}$, we have that

$$\frac{1}{C_0}\left(\|P^-z\|_2^2 + \|P^+z\|_2^2 + \iint_Q [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt\right) \leq \|G\|_2^2 + \int_0^T \int_0^\delta [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt,$$

whence

$$\|P^-z\|_2^2 + \|P^+z\|_2^2 + \iint_Q [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt \leq C_0\left(\|G\|_2^2 + \int_0^T \int_0^\delta [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt\right). \quad (3.9)$$

Using (3.9) and the definitions of P^-z and P^+z one has

$$\begin{aligned} s^{-1}\iint_Q \xi^{-1}|z_t|^2\, dx\, dt &\leq s^{-1}\iint_Q \xi^{-1}[|P^-z|^2 + 4s^2\lambda^4\xi^2 x^{2\alpha+4}|z|^2 + 4s^2\lambda^2\xi^2 x^{2\alpha+2}|z_x|^2]\, dx\, dt \\ &\leq s^{-1}\|P^-z\|_2^2 + Cs\lambda^4 \iint_Q \xi^2|z|^2\, dx\, dt + Cs\lambda^2 \iint_Q \xi x^\alpha |z_x|^2\, dx\, dt \\ &\leq C\left(\|G\|_2^2 + \int_0^T \int_0^\delta [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt\right) \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} s^{-1}\iint_Q \xi^{-1}|(x^\alpha z_x)_x|^2\, dx\, dt &\leq s^{-1}\iint_Q \xi^{-1}[|P^+z|^2 + s^4\lambda^4\xi^4 x^{2\alpha+4}|z|^2 + s^2\xi^3|z|^2]\, dx\, dt \\ &\leq s^{-1}\|P^+z\|_2^2 + Cs^3\lambda^4 \iint_Q \xi^3|z|^2\, dx\, dt + s\iint_Q \xi^2|z|^2\, dx\, dt \\ &\leq C\left(\|G\|_2^2 + \int_0^T \int_0^\delta [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\, dx\, dt\right). \end{aligned} \quad (3.11)$$

Combining (3.9)–(3.11) we conclude that

$$\iint_Q \left[s^{-1}\xi^{-1}(|z_t|^2 + |(x^\alpha z_x)_x|^2) + s\lambda^2\xi x^\alpha |z_x|^2 + s^3\lambda^4\xi^3|z|^2\right] dx\,dt$$
$$\leq C\left(\|G\|_2^2 + \int_0^T \int_0^\delta [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\,dx\,dt\right). \quad (3.12)$$

On the other hand, from the definition of g one has

$$\|G\|_2^2 \leq \|e^{-s\sigma}h\|_2^2 + Cs^2\lambda^4 \iint_Q \xi^2|z|^2\,dx\,dt.$$

Hence, for s_0 large enough, (3.12) gives

$$\iint_Q \left[s^{-1}\xi^{-1}(|z_t|^2 + |(x^\alpha z_x)_x|^2) + s\lambda^2\xi x^\alpha |z_x|^2 + s^3\lambda^4\xi^3|z|^2\right] dx\,dt$$
$$\leq C\left(\|e^{-s\sigma}h\|_2^2 + \int_0^T \int_0^\delta [s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi x^\alpha |z_x|^2]\,dx\,dt\right). \quad (3.13)$$

Now let us consider $\delta_1 \in (\delta, d)$ and take a cut off function $\psi \in C^\infty([0,1])$ such that $0 \leq \phi \leq 1$, $\psi = 1$ in $[0,\delta]$ and $\psi = 0$ in $[\delta_1, 1]$. For any $\epsilon > 0$ we have that

$$\begin{aligned}
s\lambda^2 \int_0^T \int_0^\delta \xi x^\alpha |z_x|^2\,dx\,dt &\leq s\lambda^2 \int_0^T \int_0^{\delta_1} \xi\psi x^\alpha |z_x|^2\,dx\,dt \\
&= \int_0^T \int_0^{\delta_1} \left[s\lambda^3\xi\psi x^{\alpha+1} z_x z - s\lambda^2\xi\psi' x^\alpha z_x z - s\lambda^2\xi\psi(x^\alpha z_x)_x z\right] dx\,dt \\
&\leq C\epsilon^{-1}s^3\lambda^4 \int_0^T \int_0^{\delta_1} \xi^3|z|^2\,dx\,dt + \iint_Q [s^2\lambda^4\xi^2|z|^2 + \lambda^2 x^\alpha |z_x|^2]\,dx\,dt \\
&\quad + \epsilon s^{-1} \iint_Q \xi^{-1}|(x^\alpha z_x)_x|^2\,dx\,dt.
\end{aligned}$$

Hence, taking ϵ small enough and s_0 large enough, from (3.13) we conclude that

$$\iint_Q \left[s^{-1}\xi^{-1}(|z_t|^2 + |(x^\alpha z_x)_x|^2) + s\lambda^2\xi x^\alpha |z_x|^2 + s^3\lambda^4\xi^3|z|^2\right] dx\,dt$$
$$\leq C\left(\|e^{-s\sigma}h\|_2^2 + s^3\lambda^4 \int_0^T \int_0^{\delta_1} \xi^3|z|^2\,dx\,dt\right).$$

Using classical and well known arguments, we can coming back to the original variable v and finish the proof. □

It is well known that a observability inequality for solutions of (1.7) leads to Theorem 1.3. So, it is sufficient to prove the following inequality:

Proposition 3.1 (Observability inequality)**.** *Assume* (1.2) *and* (1.3)*. There exists a constant $C > 0$ such that, for any $v_T \in L^2(0,1)$ and v solution of* (1.7) *with $h = 0$, one has*

$$|v(\cdot,0)|_2^2 \leq C \iint_{\omega_T} e^{-2s\sigma}\xi^3|v|^2\,dx\,dt, \quad (3.14)$$

where we recall that $\omega_T = \omega \times (0,T)$.

Proof. From Theorem 1.2 we have that

$$s^3\lambda^4 \iint_Q e^{-2s\sigma}\xi^3|v|^2\,dx\,dt \le Cs^3\lambda^4 \int_0^T\int_\omega e^{-2s\sigma}\xi^3|v|^2\,dx\,dt. \tag{3.15}$$

Multiplying the equation in (1.7) by v and integrating on $(0,1)$ we obtain that

$$-\frac{1}{2}\frac{d}{dt}|v(\cdot,t)|_2^2 + \int_0^1 x^\alpha|v_x|^2\,dx = \int_0^1 b_1x^{\alpha/2}v_xv\,dx - \int_0^1 b_0|v|^2\,dx.$$

Hence

$$-\frac{1}{2}\frac{d}{dt}|v(\cdot,t)|^2 + \frac{1}{2}\int_0^1 x^\alpha|v_x|^2\,dx \le C|v(\cdot,t)|^2.$$

Thus

$$|v(\cdot,0)|_2^2 \le e^{2Ct}|v(\cdot,t)|^2 \quad \forall t\in(0,T). \tag{3.16}$$

Integrating (3.16) on $(T/4,3T/4)$ and using (3.15) we deduce that

$$\begin{aligned}|v(\cdot,0)|_2^2 &= \frac{2}{T}\int_{T/4}^{3T/4}|v(\cdot,0)|^2\,dt \le C\int_{T/4}^{3T/4}\int_0^1|v|^2\,dx\,dt\\ &\le C\int_{T/4}^{3T/4}\int_0^1 s^3\lambda^4e^{-2s\sigma}\xi^3|v|^2\,dx\,dt \le C\int_0^T\int_\omega e^{-2s\sigma}\xi^3|v|^2\,dx\,dt.\end{aligned}$$

□

4 The degenerate semilinear problem

As we have pointed out in the introduction, in [15] the authors proved a null controllability result for (1.10) with $\alpha\in(0,2)$. However, most of the arguments in that work does not depend on α. Indeed, the only result in that paper that only works for $\alpha\in(0,2)$ is an observability estimate for system (1.1) of [6]. In (3.14), we give such an estimate that works for $\alpha\ge 2$. So, the majority of the arguments of [15] can now be adapted to deal with (1.10) with $\alpha\ge 2$. For readers convenience, we will reproduce their main guideline, but we will not present the proof of the results.

Firstly, for each $w\in L^2(0,T;H^1_\alpha)$, let us set the following notations

$$b_0[w](x,t) = \int_0^1 g_s(x,t,\lambda w(x,t),\lambda w_x(x,t))\,d\lambda$$

and

$$b_1[w](x,t) = x^{-\alpha/2}\int_0^1 g_p(x,t,\lambda w(x,t),\lambda w_x(x,t))\,d\lambda.$$

From (1.11) we have

$$\|b_0[w]\|_\infty + \|b_1[w]\|_\infty \le 2K \quad \forall w\in L^2(0,T;H^1_\alpha). \tag{4.1}$$

Furthermore,

$$g(x,t,u,u_x) = b_0[u](x,t)u(x,t) + x^{\alpha/2}b_1[u](x,t)u_x(x,t) \quad \forall u\in L^2(0,T;H^1_\alpha) \text{ and } a.e. \text{ in } Q. \tag{4.2}$$

As we will see, from (4.2) we can develop a fixed point argument to prove Theorem 1.4.

For now, let us assume that $u_0\in H^1_\alpha$ and for each $\varepsilon>0$ consider the functional $J_\varepsilon\colon L^2(Q)\to\mathbb{R}$ given by

$$J_\varepsilon(h) = \frac{1}{2}\int_0^T\int_\omega |h|^2\,dx\,dt + \frac{1}{2\varepsilon}\int_0^1|u(x,T)|^2\,dx,$$

where u is the solution of (1.1) with $f=h$. The first step is to establishes an approximate null controllability result for the linear system:

Proposition 4.1. *Assume that $u_0 \in H^1_\alpha$ and (1.3). Then, there exists $C > 0$ (that does not depend on ε) and $h_\varepsilon \in L^2(Q)$ such that*

1. $J_\varepsilon(h_\varepsilon) \leq J_\varepsilon(h) \quad \forall h \in L^2(Q)$;
2. $\int_0^T \int_\omega |h_\varepsilon|^2 \, dx \, dt \leq C|u_0|^2$;
3. *if u_ε is the solution of (1.1) with $f = h_\varepsilon$, then $|u_\varepsilon(\cdot, T)| \leq \varepsilon$.*

The idea of the proof of Proposition 4.1 is to verify that the minimum point of J_ε is precisely $h_\varepsilon = -\varphi_\varepsilon 1_\omega$, where φ_ε is the solution of the adjoint system of (1.1), with final datum $\varphi_\varepsilon(x, T) = \frac{1}{\varepsilon} u_\varepsilon(x, T)$. Then, it is possible to work with the adjoint equation to obtain the estimates given in the items 2 and 3.

Now, a standard argument based on the Schauder's fixed point theorem can be applied to obtain an approximate null controllability result for the semilinear system (1.10).

Proposition 4.2. *Assume that $u_0 \in H^1_\alpha$ and (1.3). Then, for each $\varepsilon > 0$ there exists $h_\varepsilon \in L^2(Q)$ and $C > 0$ (that does not depends on ε) such that:*

1. $\int_0^T \int_\omega |h_\varepsilon|^2 \, dx \, dt \leq C|u_0|^2$;
2. *if u_ε is the solution of (1.10) with $f = h_\varepsilon$, then $|u_\varepsilon(\cdot, T)| \leq \varepsilon$.*

As we have said at the beginning of this section, the detailed proofs of Propositions 4.1 and 4.2 can be found in [15]. Proposition 4.2 allows us to prove a null controllability result for the semilinear system (1.10), with the initial data in H^1_α.

Proposition 4.3. *Assume that $u_0 \in H^1_\alpha$ and (1.3). Then the system (1.10) is null controllable.*

Proof. Given $\varepsilon > 0$, let us take the control h_ε and the solution u_ε given by Proposition 4.2. From Proposition 4.2-1, there exists $\bar{h} \in L^2(Q)$ such that $h_\varepsilon \rightharpoonup \bar{h}$ in $L^2(Q)$. Furthermore, using Proposition 4.2-1 and the energy estimates given in Theorem 1.2, we can deduce that $|u_\varepsilon|^2_{\mathcal{N}} \leq C|u_0|^2$. Thus, there also exists $\bar{u} \in \mathcal{N}$ such that $u_\varepsilon \rightharpoonup \bar{u}$ in $\mathcal{N}$. From the compact embedding $\mathcal{N} \hookrightarrow \mathcal{M}$, we conclude that $u_\varepsilon \to \bar{u}$ in $\mathcal{M}$. Then $\bar{u}$ is the solution of (1.10) with $f = \bar{h}$ and, from Proposition 4.2-2, $\bar{u}(\cdot, T) = 0$. □

Finally, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let u_1 be the weak solution of the following system

$$\begin{cases} u_t - (x^\alpha u_x)_x + g(x,t,u,u_x) = 0 & \text{in } (0,1) \times (0, T_0), \\ u(1,t) = 0 \text{ and } (x^\alpha u_x)(0,t) = 0 & \text{in } (0, T_0), \\ u(x,0) = u_0(x) & \text{in } (0,1), \end{cases} \tag{4.3}$$

where $T_0 \in (0, T)$.

Now, let us consider the following system

$$\begin{cases} u_t - (x^\alpha u_x)_x + g(x,t,u,u_x) = h 1_\omega & \text{in } (0,1) \times (T_0/2, T) \\ u(1,t) = 0 \text{ and } (x^\alpha u_x)(0,t) = 0 & \text{in } (T_0/2, T), \\ u(x, T_0/2) = u_1(x, T_0/2) & \text{in } (0,1). \end{cases} \tag{4.4}$$

From Theorem 1.2, $u_1(\cdot, T_0/2) \in H^1_\alpha$. Hence, from Proposition 4.3, there exists a control $h_1 \in L^2((0,1)\times(T_0/2,T))$ such that the associated weak solution u^2 of (4.4) satisfies $u_2(\cdot,T) = 0$ in $(0,1)$. Now we can take $u \in C([0,T];L^2(Q))$ and $h \in L^2(Q)$ given by

$$u(x,t) = \begin{cases} u_1(x,t), & \text{if } t \in [0,T_0/2], \\ u_2(x,t), & \text{if } t \in [T_0/2,T], \end{cases} \quad \text{and} \quad h(x,t) = \begin{cases} 0, & \text{if } t \in [0,T_0/2], \\ h_1(x,t), & \text{if } t \in [T_0/2,T]. \end{cases}$$

It is easy to see that $u \in \mathcal{M}$ is the solution of (1.10), with $f = h$, satisfying $u(\cdot,T) = 0$. □

5 The degenerate nonlocal problem

In this section, we will obtain the local null controllability for the problem (1.12). The proof is based on a meticulous inverse function argument, as specified later on.

5.1 Functional spaces

The remainder of this section is devoted to a brief explanation about the most important strategies to prove Theorem 1.5. At this point, *Lyusternik's inverse mapping theorem* (see [2,13], for instance) is our main tool. Let us recall its statement:

Theorem 5.1 (Lyusternik). *Let E and F be two Banach spaces, consider $H \in C^1(E,F)$ and put $\eta_0 = H(0)$. If $H'(0) \in \mathcal{L}(E,F)$ is onto, then there exist $r > 0$ and $\tilde{H} : B_r(\eta_0) \subset F \to E$ such that*

$$H(\tilde{H}(\xi)) = \xi, \quad \forall \xi \in B_r(\eta_0),$$

which means that $\tilde{H}$ is a right inverse of H in $B_r(\eta_0)$. In addition, there exists $K > 0$ such that

$$\|\tilde{H}(\xi)\|_E \le K\|\xi - \eta_0\|_F, \quad \forall \xi \in B_r(\eta_0).$$

To be more precise, let us indicate how the proof of Theorem 1.5 can be seen as an application of Theorem 5.1. Even though we have not set the desired Hilbert spaces E and F yet, let us put

$$H(u,h) = (H_1(u,h), H_2(u,h)), \tag{5.1}$$

where

$$H_1(u,h) := u_t - \ell\left(\int_0^1 u\right)(au_x)_x - f\chi_\omega \quad \text{and} \quad H_2(u,h) := u(0,\cdot).$$

We should notice that, for $u_0 \in H^1_\alpha$, the first and the last relations in (1.12) are satisfied if, and only if, there exists $(u,h) \in E$ solving

$$H(u,h) = (0,u_0).$$

From this point, we realize that, among other properties, E and F must be built:

- considering the boundary conditions mentioned in (1.12);
- having some imposition on its elements assuring that $u(\cdot,T) \equiv 0$. It is done having in mind some modified weights which come from (5.5). We remark that these new weights exponentially explode at $t = T$;

- having in mind that we want $H'(0,0) \in \mathcal{L}(E,F)$ to be onto.

  In fact, we can see that

$$H'(0,0)(u,h) = (u_t - \ell(0)(au_x)_x - f\chi_\omega, u(0)).$$

  Recalling we have assumed that $\ell(0) = 1$, $H'(0,0) \in \mathcal{L}(E,F)$ is onto if, and only if, given any $(g, u_0) \in F$, the linear system

$$\begin{cases} u_t - (x^\alpha u_x)_x = f\chi_\omega + g, & (x,t) \in Q; \\ u(1,t) = (x^\alpha u_x)(0,t) = 0, & \text{in } (0,T), \\ u(x,0) = u_0(x), & x \in (0,1), \end{cases} \tag{5.2}$$

  is globally null-controllable at $T > 0$, where $f \in L^2(\omega \times (0,T))$ is the control function. Hence, it seems that E should contain some information involving the well-posedness (and additional regularity) of the linear system (5.2).

From now on, we will be focused on explicitly describing the spaces E and F, as well as, their Hilbertian norms. To do so, we consider the useful notation below.

Definition 5.2. Let $\delta = \delta(x,t)$ and $f = f(x,t)$ be two real-valued measurable functions defined in Q, where δ is non-negative. We say that f *belongs to* $L^2(Q;\delta)$ *if* $\sqrt{\delta} f \in L^2(Q)$. *Moreover, the natural norm of* $L^2(Q;\delta)$ *will be denoted by* $\|\cdot\|_\delta$, *that is,*

$$\|f\|_\delta = \left(\int_0^T \int_0^1 \delta f^2 \, dx \, dt \right)^{1/2}$$

for each $f \in L^2(Q;\delta)$.

In order to prove the global null-controllability for the linearized system (5.2), we first need to establish a Carleman estimate with new weight functions that do not vanish at $t = 0$. Namely, consider a function $m \in C^\infty([0,T])$ satisfying

$$\begin{cases} m(t) \geq t^4(T-t)^4, & t \in (0, T/2]; \\ m(t) = t^4(T-t)^4, & t \in [T/2, T]; \\ m(0) > 0, \end{cases}$$

and define

$$\tau(t) := \frac{1}{m(t)}, \quad \zeta(x,t) := \tau(t) e^{\lambda(1+\eta(x))} \quad \text{and} \quad A(x,t) := \tau(t)\left(e^{2\lambda} - e^{\lambda(1+\eta(x))}\right), \tag{5.3}$$

where $(t,x) \in [0,T) \times [0,1]$ (see Remark 5.5).

Let us note that the adjoint system associated to (5.2) is

$$\begin{cases} -v_t - (x^\alpha v_x)_x = h & \text{in } Q, \\ v(1,t) = (x^\alpha v_x)(0,t) = 0 & \text{in } (0,T), \\ v(x,T) = v_T(x) & \text{in } (0,1), \end{cases} \tag{5.4}$$

where $h \in L^2(Q)$ and $v_T \in L^2(0,1)$. Next, we state a very convenient Carleman estimate verified by any solution of (5.4).

Proposition 5.3. *Assuming* (1.3), *there exist* $C > 0$, $\lambda_0 > 0$ *and* $s_0 > 0$ *such that, for any* $s \geq s_0$, $\lambda \geq \lambda_0$ *and* $v_T \in L^2(Q)$, *the corresponding solution* v *to* (5.4) *satisfies*

$$\int_0^T \int_0^1 e^{-2sA} \left(s\lambda^2 \zeta x^\alpha |v_x|^2 + s^3\lambda^4\zeta^2|v|^2 \right) dx\,dt \leq C \left(\int_0^T \int_0^1 e^{-2sA}|h|^2\,dx\,dt + s^3\lambda^4 \int_0^T \int_\omega e^{-2sA}\zeta^6|v|^2\,dx\,dt \right). \quad (5.5)$$

The obtainment of (5.5) is a consequence of (1.9), by following the same steps detailed in [12, Proposition 4].

The factors multiplying v in (5.5) inspire the definition of the new weight functions

$$\rho_i = e^{sA}\zeta^{-i}, \quad \textit{where } i = 0,1,2,3. \quad (5.6)$$

As a matter of fact, ρ_1^{-2} and ρ_3^{-2} appears in the two integrals involving v, while ρ_2 was chosen in order to satisfies $\rho_2^2 = \rho_1\rho_3$. Besides, we have $\rho_3 \leq C\rho_2 \leq C\rho_1 \leq C\rho_0$ and, since $\rho_i \geq C_T > 0$ for all $i = 1,2,3$, we also know that $L^2(Q;\rho_i^2) \hookrightarrow L^2(Q)$. Here, for completeness, let us state the expected observability inequality which can be derived from (5.5).

Corollary 5.4. *Assuming* (1.3), *there exist* $C > 0$, $\lambda_0 > 0$ *and* $s_0 > 0$ *with the following property: given* $s \geq s_0$, $\lambda \geq \lambda_0$ *and* $v_T \in L^2(Q)$, *then the corresponding solution* v *to* (5.4), *with* $h \equiv 0$, *satisfies*

$$|v(\cdot,0)|_2^2 \leq Cs^3\lambda^4 \int_0^T \int_\omega \rho_3^{-2}|v|^2\,dx\,dt. \quad (5.7)$$

Remark 5.5. In (5.3), we have redefined the functions given in (1.8), replacing $\theta = \theta(t)$, which satisfies $\lim_{t\to 0^+} \theta(t) = +\infty$, by $\tau = \tau(t)$ fulfilling $\lim_{t\to 0^+} \tau(t) = \tau(0) > 0$. That is a crucial point in order to guarantee that (1.12) is locally null-controllable at $T > 0$, as stated in Theorem 1.5. Let us clarify this point: in fact, the definition of each ρ_i, with $i \in \{1,2,3\}$, is based on those weights which appear in (5.5), however, it comes from (5.3) that $\rho_1(t) \to +\infty$, as $t \to T^-$, and $\rho_1(0) > 0$ (since $m(0) > 0$). Because of that, $u(x,T) = 0$ for any $u \in L^2(Q;\rho_1^2)$. Hence, it seems reasonable to require that, if $(u,h) \in E$, then u belongs to $L^2(Q;\rho_1^2)$.

Finally, we are ready to define E and F. Let us consider

$$\mathcal{U} := H^1(0,T;L^2(0,1)) \cap L^2(0,T;H_\alpha^2) \cap C^0([0,T];H_\alpha^1)$$

and put $\mathcal{L}u := u_t - (x^\alpha u_x)_x$ for each $u \in \mathcal{U}$. Under all these previous notations, we set the Hilbert spaces

$$E := \left\{ (u,h) \in \mathcal{U} \times L^2(\omega_T;\rho_3^2) : u, (\mathcal{L}u - f\chi_\omega) \in L^2(Q;\rho_1^2) \right\},$$

and

$$F := L^2(Q;\rho_1^2) \times H_\alpha^1,$$

equipped with the norms

$$\|(u,h)\|_E := \left(\|u\|_{\rho_1^2}^2 + \|h\|_{\rho_3^2}^2 + \|\mathcal{L}u - f\chi_\omega\|_{\rho_1^2}^2 + \|u(0,\cdot)\|_{H_\alpha^1}^2 \right)^{1/2},$$

and

$$\|(g,v)\|_F := \left(\|g\|_{\rho_1^2}^2 + \|v\|_{H_\alpha^1}^2 \right)^{1/2},$$

respectively. The remainder of this work is devoted to check that the mapping $H : E \to F$ accomplishes everything which is required in order to apply Theorem 5.1.

5.2 Global null-controllability for the linearized system

The goal of this section is to prove a global null-controllability result for the linear system (5.2) and establish some important additional estimates. As previously discussed, the global null-controllability will guarantee that $H'(0,0)$ is surjective, which is required by *Lyusternik's theorem*, and the additional estimates will allow us to prove that H is well defined and of class C^1. As the first step here, let us define what we mean by a solution to the problem (5.2).

Definition 5.6. Given $u_0 \in H^1_\alpha$, $f \in L^2(\omega_T)$ and $g \in L^2(Q)$, we say that $u \in L^2(Q)$ is a *solution by transposition* of (5.2) if, for each $(h, v_T) \in L^2(Q) \times L^2(0,1)$, we have

$$\int_0^T \int_0^1 uh\,dx\,dt = \int_0^1 u_0 v(x,0)\,dx + \int_0^T \int_0^1 (f1_\omega + g)v\,dx\,dt,$$

for any v solution to (5.4).

The main result of this section is the following:

Proposition 5.7. *Assume* (1.3). *If* $u_0 \in H^1_\alpha$ *and* $g \in L^2(Q;\rho_1^2)$, *then there exists a control* $f \in L^2(\omega_T;\rho_3^2)$ *to* (5.2), *with associated state* $u \in L^2(Q;\rho_1^2)$, *such that*

$$\|u\|^2_{\rho_1^2} + \|f\|^2_{\rho_3^2} \le C\left(\|u_0\|^2_{H^1_\alpha} + \|g\|^2_{\rho_1^2}\right).$$

In particular, it guarantees that (5.2) *is globally null-controllable. Furthermore, we have*

$$x^{\alpha/2}u_x \in L^2(Q;\rho_2^2),\ u_t, (x^\alpha u_x)_x \in L^2(Q;\rho_3^2)$$

and there exists $C > 0$ *such that*

$$\|x^{\alpha/2}u_x\|^2_{\rho_2^2} + \|u_t\|^2_{\rho_3^2} + \|(x^\alpha u_x)_x\|^2_{\rho_3^2} \le C\left(\|u\|^2_{\rho_1^2} + \|h\chi_\omega\|^2_{\rho_3^2} + \|g\|^2_{\rho_1^2} + \|u_0\|^2_{H^1_\alpha}\right). \tag{5.8}$$

Proof. Let us define the set

$$P_{0\alpha} = \{w \in C^2(\bar{Q});\ w(1,t) = x^\alpha w_x(0,t) = 0,\ t \in (0,T)\}.$$

Recalling the definition of $\mathcal{L}$, we can see that its formal adjoint is given by $\mathcal{L}^* v = -v_t - (x^\alpha v_x)_x$. Hence, analyzing the right-hand side of (5.5), we can define the following symmetric, positive defined bilinear form

$$a(w_1, w_2) = \int_0^T \int_0^1 \rho_0^{-2}\mathcal{L}^* w_1 \mathcal{L}^* w_2\,dx\,dt + \int_0^T \int_0^1 \rho_3^{-2} w_1 w_2 1_\omega\,dx\,dt,\ \forall w_1, w_2 \in P_{0\alpha}.$$

Thus, let us denote by P_α the completion of $P_{0\alpha}$ with respect to the inner product defined by a. Hence, P_α is a Hilbert space with norm given by $\|v\|_{P_\alpha} = a(v,v)^{1/2}$.

Now, let us define the continuous linear functional $L : L^2(Q) \to \mathbb{R}$ given by

$$Lv = \int_0^1 u_0 v(x,0)\,dx + \int_0^T \int_0^1 gv\,dx\,dt.$$

In this case, Lax–Milgram theorem yields $\hat{v} \in P_\alpha$ such that

$$a(\hat{v}, v) = Lv,\ \forall v \in P_\alpha,$$

that is,

$$\int_0^T\int_0^1 \rho_0^{-2}\mathcal{L}^*\hat{v}\mathcal{L}^*v_2\,dx\,dt+\int_0^T\int_0^1 \rho_3^{-2}\hat{v}v1_\omega\,dx\,dt=\int_0^1 u_0v(x,0)\,dx+\int_0^T\int_0^1 gv\,dx\,dt,\ \forall v\in P_\alpha.$$

According to Definition 5.6, it means that $f:=-\rho_3^{-2}\hat{v}1_\omega$ is a control and $u:=\rho_0^{-2}\mathcal{L}^*\hat{v}$ the associated state to the problem (5.2). Indeed, for any $(h,v_T)\in L^2(Q)\times L^2(0,1)$, if v is a solution to (5.4), then

$$\int_0^T\int_0^1 uh\,dx\,dt=\int_0^1 u_0v(x,0)\,dx+\int_0^T\int_0^1(f1_\omega+g)v\,dx\,dt.$$

Furthermore, from Carleman and observability inequalities, given in (5.5) and (5.7) respectively, we have

$$\begin{aligned}\|\hat{v}\|^2_{P_\alpha}&=L\hat{v}\le\|u_0\|\|\hat{v}(\cdot,0)\|+\|g\|_{\rho_1^2}\left(\int_0^T\int_0^1\rho_1^{-2}\hat{v}^2\,dx\,dt\right)^{1/2}\\&\le\left(\|u_0\|^2+\|g\|^2_{\rho_1^2}\right)^{1/2}\left(\|\hat{v}(\cdot,0)\|^2+\int_0^T\int_0^1\rho_1^{-2}\hat{v}^2\,dx\,dt\right)^{1/2}\\&\le C\left(\|u_0\|^2+\|g\|^2_{\rho_1^2}\right)^{1/2}a(\hat{v},\hat{v})^{1/2}\\&=C\left(\|u_0\|^2+\|g\|^2_{\rho_1^2}\right)^{1/2}\|\hat{v}\|_{P_\alpha},\end{aligned}$$

whence

$$\|\hat{v}\|_{P_\alpha}\le C\left(\|u_0\|^2+\|g\|^2_{\rho_1^2}\right)^{1/2}.$$

Using the explicit expressions $f=-\rho_3^{-2}\hat{v}1_\omega$ and $u=\rho_0^{-2}\mathcal{L}^*\hat{v}$, as well as, recalling the norm $\|\cdot\|_{P_\alpha}$, we easily get

$$\begin{aligned}\|u\|^2_{\rho_1^2}+\|f\|^2_{\rho_3^2}&\le C\int_0^T\int_0^1\rho_0^2u^2\,dx\,dt+\int_0^T\int_0^1\rho_3^2f^2\,dx\,dt\\&=\int_0^T\int_0^1\rho_0^{-2}|\mathcal{L}^*\hat{v}|^2\,dx\,dt+\int_0^T\int_0^1\rho_3^{-2}\hat{v}^21_\omega\,dx\,dt\\&\le C\left(\|u_0\|^2+\|g\|^2_{\rho_1^2}\right),\end{aligned}$$

as desired.

At this moment, we would like to say that the obtainment of (5.8) will be left for the two subsequent lemmas. □

Lemma 5.8. *Assume* (1.3). *Given $u_0\in H^1_\alpha$ and $g\in L^2(Q;\rho_1^2)$, if $(u,h)\in\mathcal{U}\times L^2(Q_\omega;\rho_3^2)$ is a solution to* (5.2), *then $x^{\alpha/2}u_x\in L^2(Q;\rho_2)$ and there exists $C>0$ such that*

$$\|x^{\alpha/2}u_x\|^2_{\rho_2^2}\le C\left(\|u\|^2_{\rho_1^2}+\|h\chi_\omega\|^2_{\rho_3^2}+\|g\|^2_{\rho_1^2}+\|u_0\|^2_{H^1_\alpha}\right).$$

Proof. Multiplying the equation in (5.2) by ρ_2^2u, integrating in $[0,1]$ and using the two relations

$$\frac{1}{2}\frac{d}{dt}\int_0^1\rho_2^2u^2\,dx=\int_0^1\rho_2^2u_tu\,dx+\int_0^1\rho_2(\rho_2)_tu^2\,dx$$

and

$$\int_0^1 \rho_2^2 (x^{\alpha/2} u_x)_x u\, dx \ = -2 \int_0^1 \rho_2 (\rho_2)_x x^\alpha u u_x \, dx - \int_0^1 \rho_2^2 x^\alpha u_x^2 \, dx,$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 \rho_2^2 u^2\, dx \ + \int_0^1 \rho_2^2 x^\alpha u_x^2\, dx = & -\int_0^1 \rho_2^2 c u^2\, dx + \int_0^1 \rho_2^2 u h \chi_\omega \, dx + \int_0^1 \rho_2^2 g u\, dx \\ & + \int_0^1 \rho_2 (\rho_2)_t u^2 \, dx - 2\int_0^1 \rho_2 (\rho_2)_x x^\alpha u u_x \, dx \\ = & \, I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{5.9}$$

Now, using $\rho_i \leq C\rho_j$, for $i \geq j$, and $\rho_1\rho_3 = \rho_2^2$, we obtain

$$\begin{aligned} I_1 &\leq C \int_0^1 \rho_1^2 |u|^2 \, dx, \\ I_2 &\leq C \left(\frac{1}{2}\int_0^1 \rho_3^2 |h\chi_\omega|^2\, dx \ + \frac{1}{2}\int_0^1 \rho_1^2 |u|^2\, dx \right) \end{aligned}$$

and

$$I_3 \leq C \left(\frac{1}{2}\int_0^1 \rho_1^2 |g|^2\, dx \ + \frac{1}{2}\int_0^1 \rho_1^2 |u|^2\, dx \right).$$

Let us estimate I_4. Firstly, we will rewrite A as $A(t,x) = \varsigma(t,x)\bar{\mu}(x)$, where

$$\bar{\mu}(x) := (e^{M\lambda} - e^{\lambda(1+\eta(x))})/\mu(x).$$

Secondly, note that

$$\rho_2(\rho_2)_t = e^{sA}\varsigma^{-2}(s e^{sA}\varsigma_t \bar{\mu}\varsigma^{-2} - 2e^{sA}\varsigma^{-3}\varsigma_t) = e^{sA}\varsigma^{-2}(s\varsigma^{-2}\bar{\mu} - 2\varsigma^{-3})\varsigma_t$$

Then, for all $t \in [0,T]$,

$$|\rho_2(\rho_2)_t| \leq C\rho_1^2 \varsigma^{-2}|\varsigma_t| \leq C\rho_1^2,$$

whence

$$I_4 \leq C\int_0^1 \rho_1^2 |u|^2\, dx.$$

Now, using

$$|(\rho_2)_x|^2 x^\alpha u^2 \leq C e^{-2sA}\varsigma^{-2}\left|\varsigma^{-2} + \varsigma^{-4}\right| |\varsigma_x^2| x^\alpha u^2 \leq C\rho_1^2 u^2,$$

we obtain

$$\begin{aligned} I_5 \leq 2\int_0^1 |\rho_2 x^{\alpha/2} u_x||(\rho_2)_x x^{\alpha/2} u|\, dx &\leq \frac{1}{2}\int_0^1 \rho_2^2 x^\alpha u_x^2\, dx + 2\int_0^1 |(\rho_2)_x|^2 x^\alpha u^2\, dx \\ &\leq \frac{1}{2}\int_0^1 \rho_2^2 x^\alpha u_x^2 \, dx + C\int_0^1 \rho_1^2 u^2\, dx. \end{aligned}$$

Hence, (5.9) gives us

$$\frac{d}{dt}\int_0^1 \rho_2^2 |u|^2\, dx + \int_0^1 \rho_2^2 x^\alpha |u_x|^2 \, dx \leq C\left(\int_0^1 \rho_1^2 |u|^2\, dx + \int_0^1 \rho_3^2 |h\chi_\omega|^2\, dx + \int_0^1 \rho_1^2 |g|^2\, dx \right).$$

Integrating in time, the desired result follows. □

Lemma 5.9. *Assume* (1.3)*. Given* $u_0 \in H^1_\alpha$ *and* $g \in L^2(Q;\rho_1^2)$*, if* $(u,h) \in \mathcal{U} \times L^2(Q_\omega;\rho_3^2)$ *is a solution to* (5.2)*, then* $u_t, (au_x)_x \in L^2(Q;\rho_3^2)$ *and there exists* $C > 0$ *such that*

$$\|u_t\|^2_{\rho_3^2} + \|(x^\alpha u_x)_x\|^2_{\rho_3^2} \le C\left(\|u\|^2_{\rho_1^2} + \|h\chi_\omega\|^2_{\rho_3^2} + \|g\|^2_{\rho_1^2} + \|u_0\|^2_{H^1_\alpha}\right).$$

Proof. In the first step, we will estimate the first term of left side of the inequality. Multiplying equation in (5.2) by $\rho_3^2 u_t$ and integrating in $[0,1]$, we have

$$\begin{aligned}\int_0^1 \rho_3^2 u_t^2\,dx &= \int_0^1 \rho_3^2 u_t h\chi_\omega\,dx + \int_0^1 \rho_3^2 g u_t\,dx - \int_0^1 c(x,t)\rho_3^2 u u_t\,dx + \int_0^1 \rho_3^2 (x^\alpha u_x)_x u_t\,dx \\ &=: I_1 + I_2 - I_3 + I_4.\end{aligned} \tag{5.10}$$

Using Young's inequality with ε and $\rho_i \le C\rho_j$, for $i \ge j$, we obtain

$$\begin{aligned}I_1 &\le \int_0^1 \rho_3^2 |h\chi_\omega||u_t|\,dx \ \le \varepsilon\int_0^1 \rho_3^2|u_t|^2\,dx + \frac{1}{4\varepsilon}\int_0^1 \rho_3^2|h\chi_\omega|^2\,dx,\\ I_2 &\le \int_0^1 \rho_3^2|g u_t|\,dx \le \varepsilon\int_0^1 \rho_3^2|u_t|^2\,dx + \frac{1}{4\varepsilon}\int_0^1 \rho_3^2|g|^2\,dx \le \varepsilon\int_0^1 \rho_3^2|u_t|^2\,dx + C\int_0^1 \rho_1^2|g|^2\,dx\end{aligned}$$

and

$$-I_3 \le \int_0^1 |c(t,x)|\rho_3^2|u u_t|\,dx \le \varepsilon\int_0^1 \rho_3^2 u_t^2\,dx + C\int_0^1 \rho_1^2 u^2\,dx.$$

Now, integrating I_4 by parts, we can see that

$$\begin{aligned}I_4 &= \rho_3^2 x^\alpha u_x u_t\big|_{x=0}^{x=1} - \int_0^1 (\rho_3^2 u_t)_x x^\alpha u_x\,dx\\ &= -2\int_0^1 \rho_3(\rho_3)_x x^\alpha u_t u_x\,dx - \frac{1}{2}\frac{d}{dt}\int_0^1 \rho_3^2 x^\alpha u_x^2\,dx + \frac{1}{2}\int_0^1 (\rho_3^2)_t x^\alpha u_x^2\,dx.\end{aligned} \tag{5.11}$$

If we set

$$I_{41} := \int_0^1 \rho_3(\rho_3)_x x^\alpha u_t u_x\,dx \text{ and } I_{42} := \int_0^1 (\rho_3^2)_t x^\alpha u_x^2\,dx,$$

we have,

$$\int_0^1 \rho_3^2|u_t|^2\,dx \ + \frac{1}{2}\frac{d}{dt}\int_0^1 \rho_3^2 x^\alpha |u_x|^2\,dx \ = I_1 + I_2 - I_3 - 2I_{41} + \frac{1}{2}I_{42}. \tag{5.12}$$

Since $|(\rho_3)_x| \le C\rho_2$ and $|(\rho_3^2)_t| \le C\rho_2^2$, observe that

$$|\rho_3(\rho_3)_x x^\alpha u_x u_t| \le C|\rho_3 u_t||\rho_2 x^{\alpha/2} u_x|$$

and

$$(\rho_3^2)_t| = 2|\rho_3(\rho_3)_t| \le C\rho_2^2.$$

So that,

$$I_{41} \le \frac{1}{4}\int_0^1 \rho_3^2 u_t^2\,dx + C\int_0^1 \rho_2^2 x^\alpha u_x^2\,dx$$

and

$$I_{42} \le C\int_0^1 \rho_2^2 x^\alpha u_x^2\,dx.$$

Using the estimates obtained for I_1, I_2, I_3, I_{41} and I_{42}, the relation (5.12) provides

$$\int_0^1 \rho_3^2 u_t^2\,dx + \frac{1}{2}\frac{d}{dt}\int_0^1 \rho_3^2 x^\alpha u_x^2\,dx \\ \leq C\left(\int_0^1 \rho_3^2|h\chi_\omega|^2\,dx + \int_0^1 \rho_1^2 g^2\,dx + \int_0^1 \rho_1^2 u^2\,dx + \int_0^1 \rho_2^2 x^\alpha u_x^2\,dx\right),$$

and, consequently,

$$\int_0^T\int_0^1 \rho_3^2 u_t^2\,dx \leq C\left(\int_0^T\int_0^1 \rho_1^2 u^2\,dx + \int_0^T\int_\omega \rho_3^2 h^2\,dx + \int_0^T\int_0^1 \rho_1^2 g^2 + \|u_0\|^2_{H^1_\alpha}\,dx\right). \quad (5.13)$$

In the second part, we must estimate the term $\int_0^T\int_0^1 \rho_3^2|(x^\alpha u_x)_x|^2$. Multiplying the equation in (5.2) by $-\rho_3^2(x^\alpha u_x)_x$ and integrating in $[0,1]$, we take

$$\begin{aligned}\int_0^1 \rho_3^2|(x^\alpha u_x)_x|^2\,dx &= -\int_0^1 \rho_3^2 h\chi_\omega (x^\alpha u_x)_x\,dx - \int_0^1 \rho_3^2 g (x^\alpha u_x)_x\,dx \\ &\quad + \int_0^1 c(x,t)\rho_3^2 u (x^\alpha u_x)_x\,dx + \int_0^1 \rho_3^2 u_t (x^\alpha u_x)_x\,dx \\ &= -J_1 - J_2 + J_3 + I_4.\end{aligned}$$

As earlier in this proof, applying Young's inequality with ε, we obtain

$$\begin{aligned} J_1 &\leq \int_0^1 \rho_3^2|h\chi_\omega||(x^\alpha u_x)_x|\,dx \leq \varepsilon\int_0^1 \rho_3^2|(x^\alpha u_x)_x|^2\,dx + \frac{1}{4\varepsilon}\int_0^1 \rho_3^2|h\chi_\omega|^2\,dx,\\ J_2 &\leq \int_0^1 \rho_3^2|g||(x^\alpha u_x)_x|\,dx \leq \varepsilon\int_0^1 \rho_3^2|(x^\alpha u_x)_x|^2\,dx + \frac{1}{4\varepsilon}\int_0^1 \rho_1^2 g^2\,dx,\\ J_3 &\leq C\left(\varepsilon\int_0^1 \rho_3^2|(x^\alpha u_x)_x|^2\,dx + \frac{1}{4\varepsilon}\int_0^1 \rho_1^2 u^2\,dx\right).\end{aligned}$$

From (5.11) and (5.13), we achieve

$$\int_0^1 \rho_3^2|(x^\alpha u_x)_x|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_0^1 \rho_3^2 x^\alpha|u_x|^2\,dx \\ \leq C\left(\int_0^1 \rho_3^2|h\chi_\omega|^2\,dx + \int_0^1 \rho_1^2|g|^2\,dx + \int_0^1 \rho_1^2|u|^2\,dx + \int_0^1 \rho_2^2 x^\alpha|u_x|^2\,dx\right)$$

Integrating in time, we conclude the proof. □

5.3 Local null-controllability for the nonlinear system

In this section, our goal is to prove Theorem 1.5, which is based on Theorem 5.1. Indeed, it will allow us to conclude that $H : E \to F$, given in (5.1), has a right inverse mapping defined in a small ball $B \subset F = L^2(Q;\rho_1^2) \times H^1_a$. Since Theorem 5.7 already guarantees that $H'(0,0) \in \mathcal{L}(E,F)$ is onto, it remains to verify that

- H is well-defined;
- $H \in C^1(E,F)$.

We will clarify that in Propositions 5.10 and 5.12.

Proposition 5.10. *The mapping $H : E \to F$, given in* (5.1)*, is well defined.*

Proof. Given $(u,h) \in E$, we intend to prove that $H(u,h)$ belongs to $L^2(Q;\rho_1^2) \times H^1_\alpha$. From definition of E, it is clear that $H_2(u,h) = u(0,\cdot) \in H^1_\alpha$. Let us see that $H_1(u,h) \in L^2(Q;\rho_1^2)$.

In fact, since $\ell(0) = 1$ and ℓ is Lipschitz continuous, we have

$$\begin{aligned}
\int_0^T \int_0^1 \rho_1^2 |H_1(u,h)|^2 \,dx\,dt &= \int_0^T \int_0^1 \rho_1^2 \left| u_t - \ell\left(\int_0^1 u\,dx\right)(x^\alpha u_x)_x - h\chi_\omega \right|^2 dxdt \\
&\le 4\int_0^T \int_0^1 \rho_1^2 \,|\mathcal{L}(u) - h\chi_\omega|^2 \,dx\,dt + 4\int_0^T \int_0^1 \rho_1^2 \left| \left[\ell\left(\int_0^1 u\,dx\right) - \ell(0)\right](x^\alpha u_x)_x \right|^2 dx\,dt \\
&\le 4\|(u,h)\|_E^2 + 4\int_0^T \int_0^1 \rho_1^2 \left(\int_0^1 u\,dx\right)^2 |(x^\alpha u_x)_x|^2 \,dx\,dt.
\end{aligned}$$

Hence, we just need to prove that the last integral is bounded from above by $\|(u,h)\|_E^2$. Indeed, note that

$$\begin{aligned}
\int_0^T \int_0^1 \rho_1^2 \left(\int_0^1 u\,dx\right)^2 |(x^\alpha u_x)_x|^2 \,dx\,dt &= \int_0^T \int_0^1 \rho_1^2 \rho_3^{-2} \left(\int_0^1 u\,dx\right)^2 \rho_3^2 |(x^\alpha u_x)_x|^2 \,dx\,dt \\
&\le C \sup_{[0,T]} \left(\tau^4 \left(\int_0^1 u\,dx\right)^2\right) \int_0^T \int_0^1 \rho_3^2 |(x^\alpha u_x)_x|^2 \,dx\,dt \\
&\le C \sup_{[0,T]} \left(\tau^4 \left(\int_0^1 u\,dx\right)^2\right) \|(u,h)\|_E^2 \\
&\le C\|(u,h)\|_E^4,
\end{aligned}$$

where the last inequality is a consequence of Lemma 5.11, since $\tau^4 \le Ce^{M_s/m(t)}$. □

Lemma 5.11. *Given $s > 0$, there exists $M_s > 0$ such that*

$$\sup_{t\in[0,T]} \left\{ e^{\frac{M_s}{m(t)}} \left(\int_0^1 u\,dx\right)^2 \right\} \le C\|(u,h)\|_E^2,$$

for all $(u,h) \in E$, where $m = m(t)$ is the the function defined in (5.3)*.*

Proof. Firstly, for $s > 0$, let us consider $(u,h) \in E$ and the function $q : [0,T] \to \mathbb{R}$

$$q(t) := e^{\frac{M_s}{m(t)}} \left(\int_0^1 u(x,t)dx\right)^2,$$

for all $t \in [0,T]$, where $M_s > 0$ will be specified later.

<u>Claim 1</u>: Given $s > 0$, there exist $M_s > 0$ and $C > 0$ such that

$$e^{\frac{M_s}{m(t)}} \le C\rho_1^2.$$

Indeed, for any $K > 0$, we know that

$$e^{\frac{-k}{m}} \le \frac{2}{k^2}[m(t)]^2 \quad \text{for all } t \in [0,T].$$

In particular, taking $k = s\beta_\lambda$ and $M_s = \frac{s\beta_\lambda}{2}$, we obtain

$$\rho_1^2 = e^{2sA}\varsigma^{-2} \geq e^{-2\lambda} m^2 e^{2sA} \geq \frac{e^{-2\lambda}k^2}{2} e^{2sA - \frac{k}{m}} = C_{\lambda,s} e^{\frac{2s\beta_\lambda - k}{m}} = C_{\lambda,s} e^{\frac{2M_s}{m}}, \tag{5.14}$$

where $C_{\lambda,s} = \frac{e^{-2\lambda}s^2\beta_\lambda^2}{2}$.

<u>Claim 2</u>: There exist $K_1 = K_1(\lambda, s) > 0$ and $K_2 = K_2(\lambda, s) > 0$, such that

$$\frac{\rho_3^2}{m^2} \leq K_1 \rho_1^2 \quad \textit{and} \quad e^{\frac{2M_s}{m}} \leq K_2 \rho_3^2. \tag{5.15}$$

As a consequence, $q \in H^1(0, T) \hookrightarrow C^0([0, T])$.

In fact, arguing as in Claim 1, we can get

$$\frac{\rho_3^2}{m^4} = \frac{e^{2sA}\tau^{-2}}{\mu^2} \leq \frac{\rho_1^2}{\mu^6} \leq K_1 \rho_1^2$$

and

$$\rho_3^2 = \frac{e^{2sA}m^6}{\mu^6} \geq e^{-6\lambda} e^{2sA} \frac{k^6}{6!} e^{\frac{-k}{m}} = \frac{e^{-6\lambda}k^6}{6!} e^{\frac{2s\beta_\lambda - k}{m}} = \frac{1}{K_2} e^{\frac{2M_s}{m}},$$

where we have taken $k = s\beta_\lambda$, $M_s = \frac{s\beta_\lambda}{2}$ and $K_2 = \frac{6!}{e^{-6\lambda}(s\beta_\lambda)^6}$. In this case,

$$\int_0^T |q|^2\, dt \leq \int_0^T \int_0^1 e^{\frac{2M_s}{m}} |u|^2\, dxdt \leq \frac{1}{C_{\lambda,s}} \int_0^T \int_0^1 \rho_1^2 |u|^2\, dxdt \leq C \|(u, h)\|_E^2$$

and

$$\begin{aligned}
\int_0^T |q'|^2\, dt &\leq C \left(\int_0^T \int_0^1 \frac{M_s^2 (m')^2}{m^4} e^{\frac{2M_s}{m}} |u|^2\, dxdt + \int_0^T \int_0^1 e^{\frac{2M_s}{m}} |u_t|^2\, dxdt \right) \\
&\leq C \left(\int_0^T \int_0^1 \frac{\rho_3^2}{m^4} |u|^2\, dxdt + \int_0^T \int_0^1 \rho_3^2 |u_t|^2\, dxdt \right) \\
&\leq C \left(\int_0^T \int_0^1 \rho_1^2 |u|^2\, dxdt + \int_0^T \int_0^1 \rho_3^2 |u_t|^2\, dxdt \right) \\
&\leq C \|(u, h)\|_E^2,
\end{aligned}$$

following the desired result. □

Proposition 5.12. *The mapping H belongs to $C^1(E, F)$.*

Proof. It is clear that $H_2 \in C^1$. We will prove that H_1 has a continuous Gateaux derivative on E. In fact, some well-known calculation allows us to see that the Gateaux derivative of H_1 at $(u, h) \in E$ is given by

$$H_1'(u, h)(\bar{u}, \bar{h}) := \bar{u}_t - \ell' \left(\int_0^1 u\, dx \right) \int_0^1 \bar{u}\, dx\, (x^\alpha u_x)_x - \ell \left(\int_0^1 u\, dx \right) (x^\alpha \bar{u}_x)_x - \bar{h}\chi_\omega,$$

for each $(\bar{u}, \bar{h}) \in E$. We just need to prove that the Gateaux derivative $H_1' : E \to \mathcal{L}(E; L^2(Q; \rho_1^2))$ is continuous. On this purpose, given $(u, h) \in E$, let $((u^n, h^n))_{n=1}^\infty$ be a sequence in E such that

$\|(u^n,h^n)-(u,h)\|_E \to 0$. We must prove that $\|H_1'(u^n,h^n)-H_1'(u,h)\|_{\mathcal{L}(E;L^2(Q;\rho_1^2))} \to 0$. In fact, taking $(\bar{u},\bar{h})$ on the unit sphere of E, we can see that

$$
\begin{aligned}
&\|(H_1'(u^n,h^n)-H_1'(u,h))(\bar{u},\bar{h})\|^2_{\rho_1^2} \\
&\quad = \int_0^T\int_0^1 \rho_1^2 \bigg| -\ell'\left(\int_0^1 u^n\,dx\right)\int_0^1 \bar{u}\,dx\,(x^\alpha u^n_x)_x - \ell\left(\int_0^1 u^n\,dx\right)(x^\alpha \bar{u}_x)_x\,dx\,dt \\
&\qquad\qquad + \ell'\left(\int_0^1 u\,dx\right)\int_0^1 \bar{u}\,dx\,(x^\alpha u_x)_x + \ell\left(\int_0^1 u\,dx\right)(x^\alpha \bar{u}_x)_x\,dx\,dt\bigg|^2 \\
&\quad \le C\int_0^T\int_0^1 \rho_1^2\left(\int_0^1 \bar{u}\,dx\right)^2\left(\ell'\left(\int_0^1 u^n\,dx\right)\right)^2 |(x^\alpha(u^n_x-u_x))_x|^2\,dx\,dt \\
&\qquad + C\int_0^T\int_0^1 \rho_1^2\left(\int_0^1 \bar{u}\,dx\right)^2\left(\ell'\left(\int_0^1 u^n\,dx\right)-\ell'\left(\int_0^1 u\,dx\right)\right)^2 |(x^\alpha u^n_x)_x|^2\,dx\,dt \\
&\qquad + C\int_0^T\int_0^1 \rho_1^2\left(\int_0^1 \bar{u}\,dx\right)^2\left(\ell\left(\int_0^1 u^n\,dx\right)-\ell\left(\int_0^1 u\,dx\right)\right)^2 |(x^\alpha \bar{u}_x)_x|^2\,dx\,dt.
\end{aligned}
$$

Proceeding as in [12], using that $\ell \in C^1(\mathbb{R},\mathbb{R})$ has bounded derivatives and applying Lebesgue's dominated convergence theorem, we can prove that each of these three last integral converges to zero, as $n \to +\infty$. Hence, H_1' is continuous, as desired. □

Proof of Theorem 1.4. We already know that the mapping $H : E \to F$ is well defined and belongs to $C^1(E,F)$ (Propositions 5.10 and 5.12). We state that $H'(0,0) \in \mathcal{L}(E,F)$ is onto. In fact, given $(g,u_0) \in F = L^2(Q;\rho_1^2)\times H^1_\alpha$, we apply Proposition 5.7 in order to obtain $(u,h) \in L^2(Q;\rho_1^2)\times L^2(\omega_T;\rho_3^2)$ which solves (5.2) and satisfies (5.8). It means that $(u,h)\in E$ and $H'(0,0)(u,h) = (g,u_0)$, as desired.

Hence, by *Lyusternik's inverse mapping theorem* (Theorem 5.1) , there exist $\varepsilon > 0$ and a mapping $\tilde{H} : B_\varepsilon(0) \subset L^2(Q;\rho_1^2)\times H^1_\alpha \to E$ such that

$$
H(\tilde{H}(y)) = y \quad \textit{for each } y \in B_\varepsilon(0) \subset L^2(Q;\rho_1^2)\times H^1_\alpha.
$$

In particular, if $\bar{u}_0 \in H^1_\alpha$ and $\|\bar{u}_0\|_{H^1_\alpha} < \varepsilon$, we conclude that $(\bar{u},\bar{h}) = \tilde{H}(0,\bar{u}_0) \in E$ solves $H(\bar{u},\bar{h}) = (0,\bar{u}_0)$. Finally, since $\bar{u} \in L^2(Q;\rho_1^2)$, we get $\bar{u}(x,T) = 0$ almost everywhere in $[0,1]$ (see Remark 5.5). It completes the proof. □

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The logistic equation in the context of Stieltjes differential and integral equations

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Abstract. In this paper, we introduce logistic equations with Stieltjes derivatives and provide explicit solution formulas. As an application, we present a population model which involves intraspecific competition, periods of hibernation, as well as seasonal reproductive cycles. We also deal with various forms of Stieltjes integral equations, and find the corresponding logistic equations. We show that our work extends earlier results for dynamic equations on time scales, which served as an inspiration for this paper.

Keywords: logistic equation, Stieltjes differential equation, Stieltjes integral equation, dynamic equation, population dynamics.

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1 Introduction

The logistic equation is ubiquitous in population dynamics. The simplest version of this equation, which was proposed by Pierre-François Verhulst in 1838 (see [2]), has the form

$$x'(t) = rx(t)\left(1 - \frac{x(t)}{K}\right),$$

where $x(t)$ represents the size of a population at time t, r is the population growth rate, and K is the carrying capacity of the environment, i.e., the maximum population size that can be sustained by the environment. More realistic models assume that r and K are no longer constants and are, in fact, functions of time, which leads to the equation

$$x'(t) = r(t)x(t)\left(1 - \frac{x(t)}{K(t)}\right). \tag{1.1}$$

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Observe that the logistic equation above is nonlinear; however, dividing Eq. (1.1) by $-x(t)^2$ and substituting $y(t) = 1/x(t)$, we obtain the nonhomogeneous linear equation

$$y'(t) = -r(t)y(t) + \frac{r(t)}{K(t)},$$

whose solution can be obtained using the variation of constants formula. Conversely, starting with the general first-order nonhomogenous linear equation

$$y'(t) = p(t)y(t) + f(t)$$

and performing the change of variables $x(t) = 1/y(t)$, we get the logistic equation

$$x'(t) = -p(t)x(t) - f(t)x(t)^2.$$

Thus, the logistic equation can be regarded as an equation for $x = 1/y$, where y is a (nonzero) solution of a first-order nonhomogenous linear equation. This idea has been employed in [3], which deals with dynamic equations on time scales. Beginning with the first-order nonhomogeneous linear Δ-dynamic equation

$$u^\Delta(t) = p(t)u(t) + f(t), \tag{1.2}$$

the authors found that $y = 1/u$ satisfies

$$y^\Delta(t) = -(p(t) + f(t)y(t))y(\sigma(t)), \tag{1.3}$$

where σ is the forward jump operator. Similarly, starting with the adjoint equation of Eq. (1.2), namely,

$$v^\Delta(t) = -p(t)v(\sigma(t)) + f(t)$$

they found that $x = 1/v$ satisfies

$$x^\Delta(t) = (p(t) - f(t)x(\sigma(t))x(t). \tag{1.4}$$

Hence, it is reasonable to refer to Eq. (1.3) and Eq. (1.4) as to logistic dynamic equations.

In the present paper, we deal with two classes of equations that are more general than dynamic equations, and whose solutions need not be continuous. First, we focus on Stieltjes differential equations, which were introduced and studied e.g. in [7–12, 14]. The concept of the Stieltjes derivative of a function with respect to a left-continuous nondecreasing function g is recalled in Section 2, where we also recall some basic facts on linear Stieltjes differential equations. In Section 3, we show that if u is a (nonzero) solution of the Stieltjes differential equation

$$u'_g(t) = p(t)u(t) + f(t), \tag{1.5}$$

then $y = 1/u$ is a solution of

$$y'_g(t)(1 + (p(t) + f(t)y(t))\Delta^+ g(t)) + p(t)y(t) + f(t)y(t)^2 = 0 \tag{1.6}$$

(where $\Delta^+ g(t) = g(t+) - g(t)$), or equivalently

$$y'_g(t) = -(p(t) + f(t)y(t))y(t+).$$

Similarly, if v is a solution of the adjoint linear equation to Eq. (1.5), i.e.,

$$v'_g(t) = -\frac{p(t)}{1+p(t)\Delta^+ g(t)}v(t) + \frac{f(t)}{1+p(t)\Delta^+ g(t)}, \tag{1.7}$$

then $x = 1/v$ satisfies

$$x'_g(t)(1+\Delta^+ g(t)f(t)x(t)) - p(t)x(t) + f(t)x(t)^2 = 0, \tag{1.8}$$

or equivalently

$$x'_g(t) = (p(t) - f(t)x(t+))x(t).$$

In view of these facts, we refer to Eq. (1.6) and Eq. (1.8) as to logistic equations with Stieltjes derivatives. We provide explicit solution formulas for both equations.

In Section 4, we show that the logistic Δ-dynamic equations (1.3) and (1.4) represent special cases of the Stieltjes differential equations (1.6) and (1.8) corresponding to a suitable function g.

Theoretical results on logistic differential equations with Stieltjes derivatives are illustrated on an example in Section 5. It describes a simple model of grizzly bears, whose population dynamics involves competition between individuals, periods of hibernation, as well as a seasonal reproductive cycle.

The second part of the paper deals with Stieltjes integral equations. In Section 6, we recall some basic properties of Stieltjes integrals, and present a generalization of the quotient rule; as far as we are aware, this is the first appearance of the quotient rule for Stieltjes integrals in the literature.

In Section 7, we consider three types of linear nonhomogeneous Stieltjes integral equations, namely

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s) + f(s))\,\mathrm{d}g(s),$$

as well as the pair of dual equations

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s-) + f(s))\,\mathrm{d}g(s),$$
$$x(t) = x(t_0) + \int_{t_0}^{t} (-p(s)x(s+) + f(s))\,\mathrm{d}g(s),$$

which were recently studied in [13, 20]. For each of the three equations, we find the corresponding logistic equation satisfied by the function $y = 1/x$. In comparison with earlier sections, we only assume that g has bounded variation, and do not require left-continuity. Because of this, the corresponding theory covers not only Δ-dynamic equations on time scales (where the corresponding g is left-continuous), but also ∇-dynamic equations (where g is right-continuous). These facts are utilized in Section 8, where we explore the relations between Stieltjes integral versions of the logistic equation and both types of dynamic equations.

2 Preliminaries on Stieltjes derivatives

Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function. We shall denote by μ_g the Lebesgue–Stieltjes measure associated to g given by

$$\mu_g([c,d)) = g(d) - g(c), \quad c, d \in \mathbb{R},\ c < d,$$

see [6,17,18]. We will use the term "g-measurable" for a set or function to refer to μ_g-measurability in the corresponding sense, and we denote by $\mathcal{L}^1_g(X,\mathbb{R})$ the set of Lebesgue–Stieltjes μ_g-integrable functions on a g-measurable set X with values in $\mathbb{R}$, whose integral we denote by

$$\int_X f(s)\,\mathrm{d}\mu_g(s), \quad f \in \mathcal{L}^1_g(X,\mathbb{R}).$$

Similarly, we will talk about properties holding g-almost everywhere in a set X (shortened to g-a.e. in X), or holding for g-almost all (or simply, g-a.a.) $x \in X$, as a simplified way to express that they hold μ_g-almost everywhere in X or for μ_g-almost all $x \in X$, respectively.

Define

$$C_g = \{t \in \mathbb{R} : g \text{ is constant on } (t-\varepsilon, t+\varepsilon) \text{ for some } \varepsilon > 0\},$$
$$D_g = \{t \in \mathbb{R} : \Delta^+ g(t) > 0\}.$$

Observe that, as pointed out in [9], the set C_g has null g-measure and it is open in the usual topology, so it can be uniquely expressed as the countable union of open disjoint intervals, say

$$C_g = \bigcup_{n\in\mathbb{N}} (a_n, b_n),$$

for some $a_n, b_n \in [-\infty, +\infty]$, $n \in \mathbb{N}$. With this notation, we also define

$$N_g^- = \{a_n \in \mathbb{R} : n \in \mathbb{N}\} \setminus D_g, \quad N_g^+ = \{b_n \in \mathbb{R} : n \in \mathbb{N}\} \setminus D_g, \quad N_g = N_g^- \cup N_g^+.$$

We are now in position to introduce the definition of the Stieltjes derivative of a real-valued function as in [9,11].

Definition 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R} \setminus C_g$. We define the *Stieltjes derivative*, or *g-derivative*, of f at t as follows, provided the corresponding limit exists:

$$f_g'(t) = \begin{cases} \lim\limits_{s\to t} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \notin D_g \cup N_g, \\ \lim\limits_{s\to t^-} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \in N_g^-, \\ \lim\limits_{s\to t^+} \dfrac{f(s)-f(t)}{g(s)-g(t)}, & t \in D_g \cup N_g^+, \end{cases}$$

In that case, we say that f is *g-differentiable at t*.

Remark 2.2. It is important to note that, as explained in [11, Remark 2.2], for $t \in N_g$ we have

$$f_g'(t) = \lim_{s\to t} \frac{f(s)-f(t)}{g(s)-g(t)},$$

as the domain of the quotient function gives the corresponding one-sided limit. Furthermore, since g is a regulated function, it follows that the g-derivative of f at a point $t \in D_g$ exists if and only if $f(t+)$ exists and, in that case,

$$f_g'(t) = \frac{\Delta^+ f(t)}{\Delta^+ g(t)}.$$

The following result, [11, Proposition 2.5], contains some basic properties of Stieltjes derivatives such as linearity and the product and quotient rules.

Proposition 2.3. *Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be two g-differentiable functions at $t \in \mathbb{R} \setminus C_g$. Then:*

- *The function $\lambda_1 f_1 + \lambda_2 f_2$ is g-differentiable at t for any $\lambda_1, \lambda_2 \in \mathbb{R}$ and*

$$(\lambda_1 f_1 + \lambda_2 f_2)'_g(t) = \lambda_1 (f_1)'_g(t) + \lambda_2 (f_2)'_g(t).$$

- *The product $f_1 f_2$ is g-differentiable at t and*

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t) + (f_2)'_g(t) f_1(t) + (f_1)'_g(t)\,(f_2)'_g(t)\Delta^+ g(t).$$

- *If $f_2(t)\,(f_2(t) + (f_2)'_g(t)\,\Delta^+ g(t)) \neq 0$, the quotient f_1/f_2 is g-differentiable at t and*

$$\left(\frac{f_1}{f_2}\right)'_g(t) = \frac{(f_1)'_g(t) f_2(t) - (f_2)'_g(t) f_1(t)}{f_2(t)\,(f_2(t) + (f_2)'_g(t)\,\Delta^+ g(t))}. \tag{2.1}$$

Next, we present the concept of g-absolute continuity introduced in [9], as well as some of its properties. For simplicity, we introduce such concept as part of the following result from [9, Proposition 5.4].

Theorem 2.4. *Let $F : [a,b] \to \mathbb{R}$. The following conditions are equivalent:*

1. *The function F is* g-absolutely continuous on $[a,b]$ *according to the following definition: for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every open pairwise disjoint family of subintervals $\{(a_n, b_n)\}_{n=1}^m$ satisfying*

$$\sum_{n=1}^{m} (g(b_n) - g(a_n)) < \delta,$$

we have

$$\sum_{n=1}^{m} |F(b_n) - F(a_n)| < \varepsilon.$$

2. *The function F satisfies the following conditions:*

   (i) *there exists $F'_g(t)$ for g-a.a. $t \in [a,b)$;*

   (ii) *$F'_g \in \mathcal{L}^1_g([a,b), \mathbb{R})$;*

   (iii) *for each $t \in [a,b]$,*

$$F(t) = F(a) + \int_{[a,t)} F'_g(s)\, \mathrm{d}\mu_g(s). \tag{2.2}$$

Remark 2.5. Observe that the equality in Eq. (2.2) is, indeed, true for $t = a$ as we are considering the integral over the empty set, which makes the integral null.

We denote by $\mathcal{AC}_g([a,b], \mathbb{R})$ the set of g-absolutely continuous functions in $[a,b]$ with values on $\mathbb{R}$. With this notation, we present [9, Proposition 2.4], a result that, in a way, is the converse of Theorem 2.4.

Theorem 2.6. *Let $f \in \mathcal{L}^1_g([a,b), \mathbb{R})$. Then, the function $F : [a,b] \to \mathbb{R}$ defined as*

$$F(t) = \int_{[a,t)} f(s)\, \mathrm{d}\mu_g(s),$$

is an element of $\mathcal{AC}_g([a,b], \mathbb{R})$ and $F'_g(t) = f(t)$ for g-a.a. $t \in [a,b)$.

We include the following lemma that, to the best of our knowledge, is not available in the literature. (Only the fact that if f has bounded variation and is bounded away from zero, then $1/f$ has bounded variation, is known; see for example [1, Exercise 1.1].) This result shows that, under certain conditions, the multiplicative inverse of a g-absolutely continuous function is also g-absolutely continuous.

Lemma 2.7. *Let $f : [a,b] \to \mathbb{R}$ be a regulated function such that*

$$f(t) \neq 0,\ t \in [a,b]; \quad f(t+) \neq 0,\ t \in [a,b); \quad f(t-) \neq 0,\ t \in (a,b].$$

Then, there exists $M > 0$ such that $|f(t)| \geq M$ for all $t \in [a,b]$. Furthermore:

(i) *If f has bounded variation on $[a,b]$, then so does $1/f$.*

(ii) *If f is g-absolutely continuous on $[a,b]$, then so is $1/f$.*

Proof. First, note that for each $t \in (a,b)$, $f(t-), f(t+) \neq 0$ so we can find $\delta_t > 0$ such that

$$|f(s)| > \frac{|f(t-)|}{2},\ s \in (t-\delta_t, t) \quad \text{and} \quad |f(s)| > \frac{|f(t+)|}{2},\ s \in (t, t+\delta_t).$$

Consequently, $|f(s)| \geq M_t := \min\{|f(t-)|/2, |f(t+)|/2, |f(t)|\} > 0$ for all $s \in (t-\delta_t, t+\delta_t)$. A similar reasoning shows that there exist $\delta_a, \delta_b > 0$ such that

$$|f(s)| \geq M_a := \min\left\{\frac{|f(a+)|}{2}, |f(a)|\right\}, \quad s \in [a, a+\delta_a),$$
$$|f(s)| \geq M_b := \min\left\{\frac{|f(b-)|}{2}, |f(b)|\right\}, \quad s \in (b-\delta_b, b].$$

Note that the family $\{(t-\delta_t, t+\delta_t)\}_{t\in[a,b]}$ is an open cover of $[a,b]$, which is compact, so there must be a finite subcover, i.e., there exist $t_1, t_2, \ldots, t_N \in [a,b]$ such that $\{(t_k - \delta_{t_k}, t_k + \delta_{t_k})\}_{k=1}^N$ covers $[a,b]$. Now, it is enough to take $M = \min\{M_{t_1}, M_{t_2}, \ldots, M_{t_N}\}$ to obtain the first part of the result.

Now, in order to prove (i)–(ii), note that given $c, d \in [a,b]$, $c < d$, we have

$$\left|\frac{1}{f(d)} - \frac{1}{f(c)}\right| = \left|\frac{f(c)-f(d)}{f(d)f(c)}\right| \leq \frac{|f(c)-f(d)|}{M^2}. \tag{2.3}$$

Assume that f has bounded variation on $[a,b]$. Let $a = t_0 < t_1 < \cdots < t_m = b$ be a partition of $[a,b]$. Then, (2.3) yields

$$\sum_{i=1}^{m}\left|\frac{1}{f(t_i)} - \frac{1}{f(t_{i-1})}\right| \leq \frac{1}{M^2}\sum_{i=1}^{m}|f(t_{i-1}) - f(t_i)| \leq \frac{1}{M^2}\operatorname{var}(f,[a,b]),$$

which shows that $1/f$ has bounded variation on $[a,b]$.

Finally, assume that f is g-absolutely continuous on $[a,b]$ and let $\varepsilon > 0$. In that case, there exists $\delta > 0$ such that if $\{(a_n, b_n)\}_{n=1}^m$ is a family of open pairwise disjoint subintervals of $[a,b]$ satisfying that $\sum_{n=1}^m (g(b_n) - g(a_n)) < \delta$, then

$$\sum_{n=1}^{m}|f(b_n) - f(a_n)| < M^2\varepsilon.$$

Consequently, if $\{(a_n, b_n)\}_{n=1}^m$ is a family of open pairwise disjoint subintervals satisfying $\sum_{n=1}^m (g(b_n) - g(a_n)) < \delta$, using (2.3) we have

$$\sum_{n=1}^{m} \left| \frac{1}{f(b_n)} - \frac{1}{f(a_n)} \right| \leq \frac{1}{M^2} \sum_{n=1}^{m} |f(b_n) - f(a_n)| < \varepsilon,$$

which proves that $1/f \in \mathcal{AC}_g([a,b], \mathbb{R})$. □

As shown in [8, Proposition 5.5], every g-absolutely continuous function is g-continuous according to the following definition from [8].

Definition 2.8. A function $f : [a,b] \to \mathbb{R}$ is *g-continuous* at a point $t \in [a,b]$, or *continuous with respect to g* at t, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(s)| < \varepsilon, \quad \text{for all } s \in [a,b], \ |g(t) - g(s)| < \delta.$$

If f is g-continuous at every point $t \in A \subset [a,b]$, we say that f is g-continuous on A.

The following result, [8, Proposition 3.2], describes some properties of g-continuous functions, and thus, of g-absolutely continuous functions.

Proposition 2.9. *If $f : [a,b] \to \mathbb{R}$ is g-continuous on $[a,b]$, then:*

- *f is continuous from the left at every $t \in (a,b]$;*
- *if g is continuous at $t \in [a,b)$, then so is f;*
- *if g is constant on some $[\alpha, \beta] \subset [a,b]$, then so is f.*

In particular, g-continuous functions on $[a,b]$ are continuous on $[a,b]$ when g is continuous on $[a,b)$.

Finally, we provide some context and information on differential problems with Stieltjes derivatives of the form

$$u_g'(t) = F(t, u(t)), \qquad u(t_0) = u_0, \tag{2.4}$$

with $t_0, T, u_0 \in \mathbb{R}$, $T > 0$, and $F : [t_0, t_0 + T] \times \mathbb{R} \to \mathbb{R}$. Let us start by clarifying the concept of solution for this type of equations.

Definition 2.10. Given $\tau \in (0, T]$, a *solution* of Eq. (2.4) on $[t_0, t_0 + \tau]$ is a function $u \in \mathcal{AC}_g([t_0, t_0 + \tau], \mathbb{R})$ such that $u(t_0) = u_0$ and

$$u_g'(t) = F(t, u(t)), \quad g\text{-a.a. } t \in [t_0, t_0 + \tau).$$

As usual, one of the most important equations in the context of Stieltjes derivatives is the linear differential equation, which has been deeply studied in [7,8,11]. In the following result, which can be found in [11, Theorem 3.2] or, more generally, in [7, Theorem 4.3], we introduce the g-exponential map, the unique solution of the homogeneous linear problem.

Theorem 2.11. *Let $p \in \mathcal{L}_g^1([t_0, t_0 + T), \mathbb{R})$ be such that*

$$1 + p(t)\Delta^+ g(t) \neq 0, \quad \textit{for all } t \in [t_0, t_0 + T) \cap D_g. \tag{2.5}$$

Then, the set $T_p^- = \{t \in [t_0, t_0+T) \cap D_g : 1 + p(t)\Delta^+ g(t) < 0\}$ *has finite cardinality. Furthermore, if* $T_p^- = \{t_1, \dots, t_k\}$, $t_0 \le t_1 < t_2 < \dots < t_k < t_{k+1} = t_0 + T$, *then the map* $\widehat{p} : [t_0, t_0+T) \to \mathbb{R}$ *defined as*

$$\widehat{p}(t) = \begin{cases} p(t), & \text{if } t \in [t_0, t_0+T) \setminus D_g, \\ \dfrac{\log(1 + p(t)\Delta^+ g(t))}{\Delta^+ g(t)}, & \text{if } t \in [t_0, t_0+T) \cap D_g, \end{cases}$$

belongs to $\mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$*; the map* $\exp_g(p, \cdot) : [t_0, t_0+T] \to (0, +\infty)$ *given by*

$$\exp_g(p,t) = \begin{cases} \exp\left(\displaystyle\int_{[t_0,t)} \widehat{p}(s)\, d\mu_g(s)\right), & \text{if } t_0 \le t \le t_1, \\ (-1)^j \exp\left(\displaystyle\int_{[t_0,t)} \widehat{p}(s)\, d\mu_g(s)\right), & \text{if } t_j < t \le t_{j+1},\ j = 1, \dots, k, \end{cases}$$

is g-absolutely continuous on $[t_0, t_0+T]$*; and the function* $u(t) = u_0 \exp_g(p,t)$, $t \in [t_0, t_0+T]$, *is the unique solution of*

$$u'_g(t) = p(t)u(t), \quad u(t_0) = u_0.$$

Finally, in [11, Theorem 3.5] and [7, Proposition 4.12], using the method of variation of constants, the authors obtained the explicit expression of the unique solution of the nonhomogeneous linear equation, which we present in the next theorem.

Theorem 2.12. *Let* $p, f \in \mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$ *and suppose that* (2.5) *holds. Then, the function* $u : [t_0, t_0+T] \to \mathbb{R}$ *defined as*

$$u(t) = x_0 \exp_g(p,t) + \exp_g(p,t) \int_{[t_0,t)} \frac{f(s)}{1 + p(s)\Delta^+ g(s)} \exp_g(p,s)^{-1} d\mu_g(s), \quad t \in [a,b], \tag{2.6}$$

is the unique solution of

$$u'_g(t) = p(t)u(t) + f(t), \quad u(t_0) = u_0. \tag{2.7}$$

3 The logistic equation in the context of Stieltjes derivatives

In the setting of ordinary differential equations and dynamic equations on time scales, one way of defining the logistic equation is to consider it as the equation for which a change of variables of the form $u(t) = (x(t))^{-1}$ yields a linear equation in the corresponding setting. Hence, following the reasonings in [5, Section 2.4], we will obtain the form of the logistic equation in the context of Stieltjes derivatives through the mentioned change of variables.

In what follows we assume that $x_0, t_0, T \in \mathbb{R}$, $T > 0$. Let us start by looking at the change of variables above. Suppose u is a function which is a solution of Eq. (2.7). If $u(t) = (x(t))^{-1}$, provided that the corresponding hypotheses are satisfied, we can compute the g-derivative of x using Proposition 2.3. Indeed, clearly, the function 1 is g-differentiable everywhere (except on C_g) and has null g-derivative so, under suitable conditions, (2.1) ensures that

$$\begin{aligned} x'_g(t) &= -\frac{u'_g(t)}{u(t)(u(t) + u'_g(t)\Delta^+ g(t))} = -\frac{p(t)u(t) + f(t)}{u(t)(u(t) + (p(t)u(t) + f(t))\Delta^+ g(t))} \\ &= -\frac{p(t) + f(t)(u(t))^{-1}}{1 + (p(t) + f(t)(u(t))^{-1})\Delta^+ g(t)} \frac{1}{u(t)} = -\frac{p(t) + f(t)x(t)}{1 + \Delta^+ g(t)(p(t) + f(t)x(t))} x(t). \end{aligned} \tag{3.1}$$

At this point, one might be inclined to define the logistic equation with Stieltjes derivatives on $[t_0, t_0 + T]$ as Eq. (3.1) as it is in the form of Eq. (2.4). However, in doing so, one needs to require that any solution on $[t_0, t_0 + \tau)$ in the sense of Definition 2.10 must also satisfy that $1 + \Delta^+ g(t)(p(t) + f(t)x(t)) \neq 0$ for every $t \in [t_0, t_0 + \tau) \cap D_g$. Alternatively, instead of Eq. (3.1), we can consider the more general equation

$$x'_g(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \tag{3.2}$$

which no longer requires such consideration at the cost of moving away from problems of the form (2.4). Observe that when the Stieltjes derivative coincides with the usual derivative (namely, when $g = \mathrm{Id}$), Eq. (3.2) yields the usual logistic equation.

After these considerations, we define the *logistic equation with Stieltjes derivatives* as the initial value problem

$$x'_g(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad x(t_0) = x_0, \tag{3.3}$$

with $p, f \in \mathcal{L}^1_g([t_0, t_0 + T), \mathbb{R})$. Naturally, since Eq. (3.3) is no longer in the framework of Eq. (2.4), we need to define the concept of solution for this problem in a similar manner.

Definition 3.1. Given $\tau \in (0, T]$, a *solution* of Eq. (3.3) on $[t_0, t_0 + \tau]$ is a function $x \in \mathcal{AC}_g([t_0, t_0 + \tau], \mathbb{R})$ such that $x(t_0) = x_0$ and

$$x'_g(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad g\text{-a.a. } t \in [t_0, t_0 + \tau).$$

Remark 3.2. Observe that, if $x_0 = 0$, the map $x(t) = 0$, $t \in [t_0, t_0 + T]$, is a solution of Eq. (3.3) so, without loss of generality, we shall assume that $x_0 \neq 0$ for the remaining of the section.

Remark 3.3. Remark 2.2 and Proposition 2.9 imply that, for any $x \in \mathcal{AC}_g([t_0, t_0 + \tau], \mathbb{R})$,

$$x'_g(t)\Delta^+ g(t) = x(t+) - x(t), \quad t \in [t_0, t_0 + \tau].$$

Hence, it is clear that x is a solution of Eq. (3.3) if and only if it is a solution of

$$x'_g(t) = -(p(t) + f(t)x(t))x(t+), \quad x(t_0) = x_0. \tag{3.4}$$

The following result provides an explicit expression for a solution of Eq. (3.3), which is obtained through the solution of the nonhomogeneous linear equation, Eq. (2.6).

Theorem 3.4. *Let $p, f \in \mathcal{L}^1_g([t_0, t_0 + T), \mathbb{R})$ be such that (2.5) holds and define*

$$\phi(t) = \frac{1}{x_0} + \int_{[t_0, t)} \frac{f(s)}{1 + p(s)\Delta^+ g(s)} \exp_g(p, s)^{-1} \mathrm{d}\mu_g(s), \quad t \in [t_0, t_0 + T).$$

If there exists $\tau \in (0, T]$ such that $\phi(t) \neq 0$ for $t \in [t_0, t_0 + \tau]$ and

$$\phi(t) \neq -\frac{f(t)\Delta^+ g(t)}{1 + p(t)\Delta^+ g(t)} \exp_g(p, t)^{-1}, \quad t \in [t_0, t_0 + \tau] \cap D_g, \tag{3.5}$$

then, the map $x : [t_0, t_0 + \tau] \to \mathbb{R}$ defined as

$$x(t) = \frac{1}{\exp_g(p, t)\phi(t)}, \quad t \in [t_0, t_0 + \tau] \tag{3.6}$$

is a solution of Eq. (3.3) on $[t_0, t_0 + \tau]$.

Proof. Let us denote
$$\Phi(t) = \exp_g(p,t)\phi(t), \quad t \in [t_0, t_0 + T).$$
Observe that Theorem 2.12 ensures that $\Phi \in \mathcal{AC}_g([t_0, t_0 + T], \mathbb{R})$. Also, there exists $N \subset [t_0, t_0 + \tau)$ such that $\mu_g(N) = 0$ and
$$\Phi_g'(t) = p(t)\Phi(t) + f(t), \quad t \in [t_0, t_0 + \tau) \setminus N.$$
Furthermore, for $t \in [t_0, t_0 + \tau)$, since $\phi(t) \neq 0$ by hypothesis and $\exp_g(p,t) \neq 0$ by definition, we have that $\Phi(t) \neq 0$, which ensures that x is well-defined. Hence, in order to prove that $x \in \mathcal{AC}_g([t_0, t_0 + T], \mathbb{R})$ it is enough to show that
$$\exists \lim_{s \to t^-} \Phi(s) \neq 0, \quad t \in (t_0, t_0 + \tau] \tag{3.7}$$
$$\exists \lim_{s \to t^+} \Phi(s) \neq 0, \quad t \in [t_0, t_0 + \tau) \tag{3.8}$$
as in that case, Lemma 2.7 ensures the g-absolute continuity.

Since $\Phi \in \mathcal{AC}_g([t_0, t_0 + T], \mathbb{R})$, Φ is left-continuous at every $t \in (t_0, t_0 + \tau]$ (see Proposition 2.9), so for each $t \in (t_0, t_0 + \tau]$, $\Phi(t-) = \Phi(t) \neq 0$, which proves (3.7). Similarly, if $t \in [t_0, t_0 + \tau) \setminus D_g$, Proposition 2.9 ensures that Φ is continuous at t, so $\Phi(t+) = \Phi(t) \neq 0$. Finally, if $t \in [t_0, t_0 + \tau) \cap D_g$, then $t \notin N$, so it follows from Remark 2.2 and (3.5) that
$$\begin{aligned}
\Phi(t+) &= \Phi(t) + \Phi_g'(t)\Delta^+ g(t) \\
&= \Phi(t) + (p(t)\Phi(t) + f(t))\Delta^+ g(t) \\
&= (1 + p(t)\Delta^+ g(t))\Phi(t) + f(t)\Delta^+ g(t) \\
&= (1 + p(t)\Delta^+ g(t)) \exp_g(p,t)\phi(t) + f(t)\Delta^+ g(t) \neq 0,
\end{aligned}$$
which shows that (3.8) holds.

Finally, we prove that x satisfies the equation g-a.e. in $[t_0, t_0 + \tau]$. Note that the reasoning above and the fact that $\Phi \neq 0$ ensure that $\Phi(t) + \Phi_g'(t)\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0 + \tau) \setminus N$. Hence, given that the map $h(t) = 1$, $t \in [t_0, t_0 + \tau)$, is g-differentiable on $[t_0, t_0 + \tau)$ with null g-derivative, Proposition 2.3 guarantees that x is g-differentiable for each $t \in [t_0, t_0 + \tau) \setminus N$ and
$$\begin{aligned}
x_g'(t) &= -\frac{\Phi_g'(t)}{\Phi(t)(\Phi(t) + \Phi_g'(t)\Delta^+ g(t))} \\
&= -\frac{p(t)\Phi(t) + f(t)}{\Phi(t)(1 + (p(t)\Phi(t) + f(t))\Delta^+ g(t))} = -\frac{p(t) + f(t)x(t)}{1 + (p(t) + f(t)x(t))\Delta^+ g(t)} x(t),
\end{aligned} \tag{3.9}$$
so we have that, for $t \in [t_0, t_0 + \tau) \setminus N$,
$$\begin{aligned}
&x_g'(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 \\
&\quad = -(p(t) + f(t)x(t))x(t) + p(t)x(t) + f(t)x(t)^2 = 0,
\end{aligned}$$
which finishes the proof. □

Remark 3.5. Let us briefly reflect on the conditions that we are requiring on the map ϕ in the hypotheses of Theorem 3.4. When we ask for ϕ to not vanish on the interval, we are essentially asking for the solution of the nonhomogeneous linear equation to be different from zero on the whole interval, which allows us to properly define the map x on that set. Observe that this condition is also necessary in the ODE setting. Condition (3.5), on the other hand, is a condition that is only relevant in this context (as $D_g = \emptyset$ when $g = \mathrm{Id}$) and it is equivalent to the requirements for the quotient rule in Proposition 2.3, guaranteeing that the derivative of x exists wherever the derivative of the solution of the nonhomogeneous linear equation exists.

A careful reader might have noticed that in the proof of Theorem 3.4, we obtained (3.9). In other words, we showed that the map x in (3.6) satisfies Eq. (3.1). This might be a bit surprising since Eq. (3.2) is the more general equation. However, as we show in the next result, under the assumption that (2.5) holds, Eq. (3.1) and Eq. (3.2) are equivalent problems in the sense that a solution of one of the problems is a solution of the other one.

Proposition 3.6. *Let $\tau \in (0, T]$ and assume that $1 + p(t)\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0 + \tau) \cap D_g$.*
If $x : [t_0, t_0 + \tau] \to \mathbb{R}$ is such that

$$x'_g(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad g\text{-a.a. } t \in [t_0, t_0 + \tau), \tag{3.10}$$

then, $1 + (p(t) + f(t)x(t))\Delta^+ g(t) \neq 0$ for g-a.a. $t \in [t_0, t_0 + \tau)$ and

$$x'_g(t) = -\frac{p(t) + f(t)x(t)}{1 + \Delta^+ g(t)(p(t) + f(t)x(t))} x(t), \quad g\text{-a.a. } t \in [t_0, t_0 + \tau). \tag{3.11}$$

Conversely, if $x : [t_0, t_0 + \tau] \to \mathbb{R}$ is such that (3.11) holds (in which case, we are implicitly assuming that $1 + (p(t) + f(t)x(t))\Delta^+ g(t) \neq 0$ for g-a.a. $t \in [t_0, t_0 + \tau)$), then x satisfies (3.10).

Proof. First, let $x : [t_0, t_0 + \tau] \to \mathbb{R}$ be such that (3.10) holds. In that case, there exists $N \subset [t_0, t_0 + \tau)$ such that $\mu_g(N) = 0$ and

$$x'_g(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad t \in [t_0, t_0 + \tau) \setminus N. \tag{3.12}$$

Let us first show that

$$1 + (p(t) + f(t)x(t))\Delta^+ g(t) \neq 0, \quad t \in [t_0, t_0 + \tau) \setminus N. \tag{3.13}$$

Observe that this is clear for $t \in [t_0, t_0 + \tau) \setminus (N \cup D_g)$ as $\Delta^+ g(t) = 0$ in that case. Thus, in order to prove (3.13) we need to show that $1 + (p(t) + f(t)x(t))\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0 + \tau) \cap D_g$.

Choose an arbitrary $t \in [t_0, t_0 + \tau) \cap D_g$ and suppose for the sake of contradiction that $1 + (p(t) + f(t)x(t))\Delta^+ g(t) = 0$. Then, since $t \in D_g$, we have $\Delta^+ g(t) > 0$, so we can write $p(t) + f(t)x(t) = -1/\Delta^+ g(t)$. In that case, (3.12) yields

$$0 = p(t)x(t) + f(t)x(t)^2 = (p(t) + f(t)x(t))x(t) = -\frac{x(t)}{\Delta^+ g(t)},$$

which means that $x(t) = 0$. Thus $0 = 1 + (p(t) + f(t)x(t))\Delta^+ g(t) = 1 + p(t)\Delta^+ g(t)$, which contradicts the assumption of the result. Thus, (3.13) must hold.

Now, (3.11) is a direct consequence of (3.12) and (3.13), which finishes the proof of the first part of the result. The second part of the result is trivial since we are implicitly assuming that $1 + (p(t) + f(t)x(t))\Delta^+ g(t) \neq 0$ for g-a.a. $t \in [t_0, t_0 + \tau)$. □

In [13, Section 3], the authors introduced the adjoint linear equation of Eq. (2.7) as the equation

$$y'_g(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)} y(t) + \frac{f(t)}{1 + p(t)\Delta^+ g(t)}, \qquad y(t_0) = y_0, \tag{3.14}$$

with $y_0 \in \mathbb{R}$ and $p, f \in \mathcal{L}^1_g([t_0, t_0+T], \mathbb{R})$ such that (2.5) holds. Observe that if we define

$$P(t) = -\frac{p(t)}{1+p(t)\Delta^+ g(t)}, \quad F(t) = \frac{f(t)}{1+p(t)\Delta^+ g(t)}, \quad t \in [t_0, t_0+T], \tag{3.15}$$

then Eq. (3.14) can be rewritten as

$$y'_g(t) = P(t)y(t) + F(t), \qquad y(t_0) = y_0,$$

i.e., it can be regarded as a particular case of Eq. (2.7) since [13, Lemma 3.4, statement (iii)] ensures that $P, F \in \mathcal{L}^1_g([t_0, t_0+T], \mathbb{R})$ and, furthermore,

$$1 + P(t)\Delta^+ g(t) = 1 - \frac{p(t)}{1+p(t)\Delta^+ g(t)}\Delta^+ g(t) = \frac{1}{1+p(t)\Delta^+ g(t)} \neq 0, \quad t \in [t_0, t_0+T) \cap D_g.$$

Hence, we have a logistic equation associated with Eq. (3.14), which is determined by

$$\begin{aligned}
0 &= y'_g(t)(1 + (P(t) + F(t)y(t))\Delta^+ g(t)) + P(t)y(t) + F(t)y(t)^2 \\
&= y'_g(t)\left(1 - \frac{p(t)\Delta^+ g(t)}{1+p(t)\Delta^+ g(t)} + \frac{f(t)\Delta^+ g(t)}{1+p(t)\Delta^+ g(t)}y(t)\right) - \frac{p(t)}{1+p(t)\Delta^+ g(t)}y(t) \\
&\quad + \frac{f(t)}{1+p(t)\Delta^+ g(t)}y(t)^2 \\
&= \frac{1}{1+p(t)\Delta^+ g(t)}(y'_g(t)(1 + \Delta^+ g(t) f(t) y(t)) - p(t)y(t) + f(t)y(t)^2).
\end{aligned}$$

Therefore, we define the *adjoint logistic equation with Stieltjes derivatives* – that is, the logistic equation associated with the adjoint equation (3.14) – as the initial value problem

$$y'_g(t)(1 + \Delta^+ g(t) f(t) y(t)) - p(t)y(t) + f(t)y(t)^2 = 0, \quad y(t_0) = y_0, \tag{3.16}$$

with $p, f \in \mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$ such that (3.13) holds. This equation turns out to be a much simpler version of Eq. (3.3).

Remark 3.7. In a similar fashion to Remark 3.3, we can see that Eq. (3.3) is equivalent to

$$y'_g(t) = (p(t) - f(t)y(t+))y(t), \quad y(t_0) = y_0. \tag{3.17}$$

As a direct consequence of Theorem 3.4, we have the following result providing an explicit solution for (3.16).

Theorem 3.8. *Let $p, f \in \mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$ be such that (2.5) holds and define*

$$\varphi(t) = \frac{1}{y_0} + \int_{[t_0,t)} f(s) \exp_g(p, s)\, d\mu_g(s), \quad t \in [t_0, t_0+T).$$

If there exists $\tau \in (0, T]$ such that $\varphi(t) \neq 0$ for $t \in [t_0, t_0+\tau]$ and

$$\varphi(t) \neq -f(t)\exp_g(p, t), \quad t \in [t_0, t_0+\tau] \cap D_g, \tag{3.18}$$

then, the map $y : [t_0, t_0+\tau] \to \mathbb{R}$ defined as

$$y(t) = \frac{\exp_g(p, t)}{\varphi(t)}, \quad t \in [t_0, t_0+\tau] \tag{3.19}$$

is a solution of Eq. (3.16) on $[t_0, t_0+\tau]$.

Proof. First, observe that, given that (3.13) holds, y is a solution of Eq. (3.16) if and only if x solves

$$y_g'(t)(1+(P(t)+F(t)y(t))\Delta^+g(t))+P(t)y(t)+F(t)y(t)^2=0, \quad y(t_0)=y_0, \tag{3.20}$$

for P, F as in (3.15). Let us check that P, F satisfy the conditions of Theorem 3.4. Since we have already shown that $P, F \in \mathcal{L}_g^1([t_0, t_0+T], \mathbb{R})$ and $1+P(t)\Delta^+g(t) \neq 0$, $t \in [t_0, t_0+T) \cap D_g$, all that is left to do is check that the map ϕ in Theorem 3.4 satisfies the corresponding conditions under our hypotheses.

First, observe that

$$\frac{F(t)}{1+P(t)\Delta^+g(t)} = \frac{\dfrac{f(t)}{1+p(t)\Delta^+g(t)}}{1-\dfrac{p(t)}{1+p(t)\Delta^+g(t)}\Delta^+g(t)} = \frac{\dfrac{f(t)}{1+p(t)\Delta^+g(t)}}{\dfrac{1}{1+p(t)\Delta^+g(t)}\Delta^+g(t)} = f(t)$$

for all $t \in [t_0, t_0+T]$. Now, by definition,

$$\begin{aligned}\phi(t) &= \frac{1}{y_0} + \int_{[t_0,t)} \frac{F(s)}{1+P(s)\Delta^+g(s)} \exp_g(P,s)^{-1} \mathrm{d}\mu_g(s)\\ &= \frac{1}{y_0} + \int_{[t_0,t)} f(s)\exp_g(p,s)\mathrm{d}\mu_g(s) = \varphi(t),\end{aligned}$$

where we have used the identity $\exp_g(P,\cdot)^{-1} = \exp_g(p,\cdot)$, see [7, Proposition 4.6]. Therefore, $\Phi(t) = \varphi(t) \neq 0$ for $t \in [t_0, t_0+\tau]$ and, using the identity $\exp_g(P,\cdot)^{-1} = \exp_g(p,\cdot)$ once again,

$$\phi(t) = \varphi(t) \neq -f(t)\exp_g(p,t) = -\frac{F(t)\Delta^+g(t)}{1+P(t)\Delta^+g(t)}\exp_g(P,t)^{-1}, \quad t \in [t_0, t_0+\tau] \cap D_g.$$

Therefore, ϕ satisfies the conditions in Theorem 3.4 so the map

$$y(t) = \frac{1}{\exp_g(P,t)\phi(t)} = \frac{\exp_g(p,t)}{\varphi(t)}, \quad t \in [t_0, t_0+\tau],$$

is a solution of Eq. (3.20) and, thus, a solution of Eq. (3.16) as we wanted to show. □

Finally, note that it is possible to adapt Proposition 3.6 for (3.16) in a similar way to Theorem 3.8, which yields the following result. We leave the proof to the reader.

Proposition 3.9. *Let $\tau \in (0, T]$ and assume that $1+p(t)\Delta^+g(t) \neq 0$ for all $t \in [t_0, t_0+\tau) \cap D_g$.*
If $y : [t_0, t_0+\tau] \to \mathbb{R}$ is such that

$$y_g'(t)(1+\Delta^+g(t)f(t)y(t)) - p(t)y(t) + f(t)y(t)^2 = 0, \quad g\text{-a.a. } t \in [t_0, t_0+\tau), \tag{3.21}$$

then, $1+\Delta^+g(t)f(t)y(t) \neq 0$ for g-a.a. $t \in [t_0, t_0+\tau)$ and

$$y_g'(t) = \frac{p(t)-f(t)y(t)}{1+\Delta^+g(t)f(t)y(t)}y(t), \quad g\text{-a.a. } t \in [t_0, t_0+\tau). \tag{3.22}$$

Conversely, if $y : [t_0, t_0+\tau] \to \mathbb{R}$ is such that (3.22) holds (in which case, we are implicitly assuming that $1+\Delta^+g(t)f(t)y(t) \neq 0$ for g-a.a. $t \in [t_0, t_0+\tau)$), then y satisfies (3.21).

4 Relations between Stieltjes differential equations and dynamic equations

Throughout this section, we assume that the reader is familiar with time scale calculus and dynamic equations. For more information on these topics, see [4,5].

Let $\mathbb{T}$ be a time scale, $t_0, t_0+T \in \mathbb{T}$, $T>0$, and denote $[t_0, t_0+T)_{\mathbb{T}} = [t_0, t_0+T) \cap \mathbb{T}$. The aim of this section is the study of possible relations between the logistic equation with Stieltjes derivatives, Eq. (3.3), and its corresponding counterpart in the context of dynamic equations as described in [5],

$$x^{\Delta}(t) = -(p(t) + f(t)x(t))x(\sigma(t)), \quad t \in [t_0, t_0+T)_{\mathbb{T}}, \tag{4.1}$$

where x^{Δ} denotes the Δ-derivative of x and $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator. Here, we assume that p and f are defined on the whole $[t_0, t_0+T)$ despite the fact that we only need them to be defined on $[t_0, t_0+T)_{\mathbb{T}}$ for Eq. (4.1). We do this so that we can easily compare Eq. (3.3) and Eq. (4.1). Similarly, we also want to consider the relations that might take place between the adjoint logistic equation, Eq. (3.16), and the corresponding logistic equation that can be deduced from the adjoint linear equation in [5], namely

$$y^{\Delta}(t) = (p(t) - f(t)y(\sigma(t)))y(t), \quad t \in [t_0, t_0+T)_{\mathbb{T}}. \tag{4.2}$$

In order to discuss the possible relations between the different logistic equations, we need to consider a context in which we can compare the two types of differential problems. In [8, Section 8.3] and [12, Section 3.3.3], it is shown that equations on time scales can be regarded as a particular case of Stieltjes differential equations when we consider the nondecreasing and left-continuous map $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(t) = \begin{cases} t_0, & t \le t_0, \\ \inf\{s \in \mathbb{T} : s \ge t\}, & t_0 < t \le t_0+T, \\ t_0+T, & t > t_0+T. \end{cases} \tag{4.3}$$

As pointed out in [8, Section 8.3], $g(t) = t$ for all $t \in [t_0, t_0+T)_{\mathbb{T}}$, from which it follows that

$$\Delta^+ g(t) = g(t+) - g(t) = \inf\{s \in \mathbb{T} : s > t\} - t = \sigma(t) - t = \mu(t), \quad t \in [t_0, t_0+T)_{\mathbb{T}}, \tag{4.4}$$

where $\mu : \mathbb{T} \to \mathbb{T}$ denotes the graininess function.

Theorems 3.49 and 3.51 in [12] establish the mentioned relation between Stieltjes differential problems and dynamic equations on time scales. Furthermore, a closer look at the proofs of these results shows that, in fact, the equivalence is between the Stieltjes derivative and the Δ-derivative. We gathered this information in the following result. Observe that, unlike [12, Theorem 3.49] we do not require continuity from the left at right-scattered points as such condition is always satisfied for Δ-differentiable maps, see [4, Theorem 1.16 (i)].

Theorem 4.1. *If $u : [t_0, t_0+T)_{\mathbb{T}} \to \mathbb{R}$ is Δ-differentiable for each $t \in [t_0, t_0+T)_{\mathbb{T}}$, then the map $\widetilde{u} = u \circ g$ for g as in (4.3) is g-differentiable for g-a.a. $t \in [t_0, t_0+T)$ and, furthermore,*

$$\widetilde{u}(t) = u(t), \quad \widetilde{u}'_g(t) = u^{\Delta}(t), \quad g\text{-a.a. } t \in [t_0, t_0+T).$$

Conversely, if $\widetilde{u} : [t_0, t_0+T] \to \mathbb{R}$ is a g-continuous function which is g-differentiable for each $t \in [t_0, t_0+T)_{\mathbb{T}}$, then $u = \widetilde{u}|_{[t_0, t_0+T]_{\mathbb{T}}}$ is Δ-differentiable on $[t_0, t_0+T)_{\mathbb{T}}$ and, furthermore,

$$u^{\Delta}(t) = \widetilde{u}'_g(t), \quad t \in [t_0, t_0+T)_{\mathbb{T}}.$$

Now, given Theorem 4.1, the equivalence between Eq. (3.3) and Eq. (4.1) should be clear. Indeed, if x satisfies Eq. (4.1), [4, Theorem 1.16 (iv)] ensures that

$$x^{\Delta}(t) = -(p(t) + f(t)x(t))(x(t) + \mu(t)x^{\Delta}(t)), \quad t \in [t_0, t_0 + T)_{\mathbb{T}},$$

or, equivalently,

$$x^{\Delta}(t)(1 + (p(t) + f(t)x(t))\mu(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad t \in [t_0, t_0 + T)_{\mathbb{T}}. \tag{4.5}$$

Hence, Theorem 4.1 ensures that if $\widetilde{x} = x \circ g$ with g as in (4.3), then for g-a.a. $t \in [t_0, t_0 + T)$,

$$\begin{aligned} 0 &= \widetilde{x}'_g(t)(1 + (p(t) + f(t)\widetilde{x}(t))\mu(t)) + p(t)\widetilde{x}(t) + f(t)\widetilde{x}(t)^2 \\ &= \widetilde{x}'_g(t)(1 + (p(t) + f(t)\widetilde{x}(t))\Delta^+ g(t)) + p(t)\widetilde{x}(t) + f(t)\widetilde{x}(t)^2, \end{aligned}$$

where the last equality follows from (4.4). Hence, $\widetilde{x}$ satisfies Eq. (3.3).

Conversely, if $\widetilde{x}$ is a g-continuous function satisfying Eq. (3.3), then $x = \widetilde{x}|_{[t_0, t_0+T]_{\mathbb{T}}}$ is such that

$$x^{\Delta}(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad t \in [t_0, t_0 + T)_{\mathbb{T}},$$

so, once again, given (4.4), we see that (4.5) holds. Now [4, Theorem 1.16 (iv)] is enough to guarantee that x satisfies Eq. (4.1).

The equivalence between (3.16) and (4.2) is done in an analogous manner and we leave it to the reader.

5 Applications to population models

Impulsive differential equations and equations on time scales can be regarded as particular cases of differential equations with Stieltjes derivatives, see [8, Section 8]. This fact was taken into account in [8, Section 9], where the authors showed that some real-life phenomena can be modelled in the context of Stieltjes calculus. Similarly, in [10, Sections 5 and 6], the authors used these relations to show that Stieltjes differential equations can be a better tool than ODEs for population models of species that exhibit very short periods of reproductions or are subject to dormant states in which the population size is unlikely to change in a noticeable manner. With these ideas in mind, and bearing the applications of the usual logistic equation for population models, we want to show that the logistic equations with Stieltjes derivative introduced above can be an adequate tool to describe the behavior of certain species.

During the winter and early spring months, the grizzly bears, like many other bears, enter a stupor stage, during which they reduce their activity as much as possible in order to survive that time of the year. This is possible because, in the months prior to the hibernation stage, they build a layer of fat that they will use to sustain themself during this dormant state. Naturally, this might cause a population of grizzly bears to compete for resources during the months leading to winter. Interestingly, the mating of the grizzly bear occurs during this period of time when the grizzly bear is preparing itself for the winter. However, the development of the embryos goes on hold until the hibernation stage, which eventually leads to the introduction of newborn cubs towards the end of the stupor stage.

We claim that a logistic equation with Stieltjes derivatives can be used to represent the evolution of a population of grizzly bears. To that end, we shall divide years into the four different seasons and we shall assume that one unit of time, denoted by t, represents a full season, which leads to the following classification of time intervals:

Season	Time intervals
Winter	$(4k, 4k+1]$, $k = 0, 1, 2, \dots$
Spring	$(4k+1, 4k+2]$, $k = 0, 1, 2, \dots$
Summer	$(4k+2, 4k+3]$, $k = 0, 1, 2, \dots$
Fall	$(4k+3, 4k+4]$, $k = 0, 1, 2, \dots$

With this notation, the intervals $(4k, 4k+\frac{3}{2}]$, $k = 0, 1, 2, \dots$, represent the hibernation periods of the population and, for simplicity, we shall assume that the remaining times of the year, namely, $(4k+\frac{3}{2}, 4k+4]$, $k = 0, 1, 2, \dots$, represent the period of time when the bears prepare for the next winter.

The next step is to select an adequate nondecreasing and left-continuous map $g : \mathbb{R} \to \mathbb{R}$ which reflects the behavior explained above, keeping in mind the information in [10, Section 5]: "[the map g] can be regarded as a time modulator. Discontinuities correspond to sudden changes ... while constancy intervals correspond to dormant states ... The greater the slope, the more influence the corresponding times have in the process". Hence, we would like the map g to exhibit the following properties:

(a) On intervals of the form $(4k, 4k+\frac{3}{2}]$, $k = 0, 1, 2, \dots$, the map g should remain constant as during these times, the population is hibernating and, thus, very unlikely to change drastically.

(b) At times of the form $4k+\frac{3}{2}$, $k = 0, 1, 2, \dots$, the map g should possess a jump discontinuity, representing the introduction of newborns into the population, which we shall assume to happen simultaneously so that they can be represented by impulses. The map g must be continuous everywhere else as there are no other sudden changes in the population.

(c) In the months directly after new individuals are born, g must have a greater slope as newborns are weaker and, therefore, the population size is more volatile. As time progresses, the slope of the function should flatten as new individuals get stronger. In the times immediately prior to the hibernation periods we would want g to have a less steep slope, representing the slowing down of the population as they approach their dormant state.

Since we will be assuming that the evolution of the population starts at $t = 0$, for simplicity, we shall assume that g is constant on $(-\infty, 0]$. Furthermore, given the cyclical nature of the previously described annual phenomena, we will assume that there exists $c \in \mathbb{R}$ such that

$$g(t) - g(t-4) = c, \quad t \geq 4. \tag{5.1}$$

Observe that, in particular, this implies that $\Delta^+ g(t) = \Delta^+ g(\frac{3}{2})$ for all $t \in D_g$.

An example of a map $g : \mathbb{R} \to \mathbb{R}$ satisfying conditions (a)–(c) and the extra assumptions is

$$g(t) = \begin{cases} 0, & t \in \left(-\infty, \dfrac{3}{2}\right], \\ 1 + 5\sin\left(\dfrac{\pi}{5}\left(t - \dfrac{3}{2}\right)\right), & t \in \left(\dfrac{3}{2}, 4\right], \end{cases} \tag{5.2}$$

and $g(t) = g(4) + g(t-4)$ for $t > 4$, see Figure 5.1.

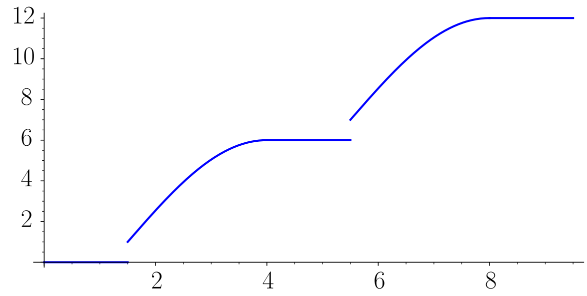

Figure 5.1: Graph of the map g in (5.2).

We now consider the initial value problem

$$x_g'(t) = F(t, x(t)), \quad x(0) = x_0, \tag{5.3}$$

where $x_0 > 0$ and $F : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is defined as

$$F(t,x) = \begin{cases} -\beta x, & \text{if } t \in \bigcup_{k=0}^{\infty} \left[4k, 4k + \dfrac{3}{2}\right), \ x \in \mathbb{R}, \\ \alpha x, & \text{if } t = 4k + \dfrac{3}{2}, k = 0, 1, 2, \ldots, \ x \in \mathbb{R}, \\ -\beta x\,(1 + \gamma x), & \text{if } t \in \bigcup_{k=0}^{\infty} \left(4k + \dfrac{3}{2}, 4k + 4\right), \ x \in \mathbb{R}, \end{cases}$$

where $\beta > 0$ represents the death rate of the population; $\alpha > 0$, the reproduction rate; and $\gamma > 0$ represents the competition strength. Naturally, (5.3) only represents the evolution of a population as long as $x(t) \geq 0$, which will be the case for our solution as we will show later. Furthermore, observe that for $t \neq 4k + \frac{3}{2}$, $k = 0, 1, 2, \ldots$, and $x(t) > 0$, we have $x_g'(t) \leq 0$, which shows that the population is bound to decay over time; while $x_g'(t) \geq 0$ for $t = 4k + \frac{3}{2}$, $k = 0, 1, 2, \ldots$ and $x(t) > 0$, which is consistent with the fact that only new members of the population are introduced at such times. Furthermore, during intervals of the form $(4k + 3, 4k + 4]$, $k = 0, 1, 2, \ldots$, the population decays faster as the population increases. The competition term is not present in the equation on the intervals $(4k, 4k + \frac{3}{2}]$, $k = 0, 1, 2, \ldots$, as during hibernation, there is no competition for resources. Of course, given our choice of g, this is not relevant for $(4k, 4k + \frac{3}{2})$, $k = 0, 1, 2, \ldots$, as they belong to C_g and, thus, have measure zero. Nevertheless, for other choices of g this might be relevant.

Consider the maps $p, f : [0, +\infty) \to \mathbb{R}$ defined as

$$p(t) = \begin{cases} -\beta, & \text{if } t \neq 4k + \dfrac{3}{2}, \ k = 0, 1, 2, \ldots, \\ \alpha, & \text{if } t = 4k + \dfrac{3}{2}, \ k = 0, 1, 2, \ldots, \end{cases} \tag{5.4}$$

$$f(t) = \begin{cases} \beta\gamma, & \text{if } t \in \bigcup_{k=0}^{\infty} \left(4k + \dfrac{3}{2}, 4k + 4\right), \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

We claim that (5.3) can be rewritten as

$$x_g'(t)(1 + \Delta^+ g(t) f(t) x(t)) - p(t)x(t) + f(t)x(t)^2 = 0, \quad x(0) = x_0,$$

that is, it is an adjoint logistic equation with Stieltjes derivatives of the form (3.16). Indeed, given that $f(t)=0$ for $t\notin\bigcup_{k=0}^{\infty}(4k+\frac{3}{2},4k+4)$, it follows that

$$\begin{aligned}x_g'(t)(1+\Delta^+g(t)f(t)x(t))-p(t)x(t)+f(t)x(t)^2\\ =F(t,x(t))-p(t)x(t),\quad t\notin\bigcup_{k=0}^{\infty}\left(4k+\frac{3}{2},4k+4\right).\end{aligned}$$

Observe that if $t=4k+\frac{3}{2}, k=0,1,2,\dots$, then

$$F(t,x(t))-p(t)x(t)=\alpha x(t)-\alpha x(t)=0;$$

while for $t\in\bigcup_{k=0}^{\infty}\left[4k,4k+\frac{3}{2}\right)$,

$$F(t,x(t))-p(t)x(t)=-\beta x(t)-(-\beta)x(t)=0.$$

Now, if $t\in\bigcup_{k=0}^{\infty}(4k+\frac{3}{2},4k+4)$, then $t\notin D_g=\{4k+\frac{3}{2}:k=0,1,2,\dots\}$, so $\Delta^+g(t)=0$. Thus, for $t\in\bigcup_{k=0}^{\infty}(4k+\frac{3}{2},4k+4)$,

$$\begin{aligned}x_g'(t)(1+\Delta^+g(t)f(t)x(t))-p(t)x(t)+f(t)x(t)^2&=F(t,x(t))-p(t)x(t)+f(t)x(t)^2\\ &=-\beta x(t)\left(1+\gamma x(t)\right)-(-\beta x(t))+\beta\gamma x(t)^2=0.\end{aligned}$$

Thus, we can apply Theorem 3.8 on an interval $[0,T]$, $T>0$, to obtain a solution of (5.3). To that end, we need to check that p and f in (5.4) and (5.5) satisfy the corresponding hypotheses.

Let $T>0$. First, observe that p and f are Borel-measurable maps which guarantees that they are g-measurable. Hence, since they are bounded, it follows that $p,f\in\mathcal{L}_g^1([0,T),\mathbb{R})$. Furthermore, observe that (2.5) holds since

$$1+p(t)\Delta^+g(t)=1+\alpha\Delta^+g(t)>0,\quad t\in[0,T]\cap D_g.$$

Observe that, in particular, this implies that $\exp_g(p,t)>0$ for all $t\in[0,T]$, see Theorem 2.11.

Consider

$$\varphi(t)=\frac{1}{x_0}+\int_{[0,t)}f(s)\exp_g(p,s)\,\mathrm{d}\mu_g(s),\quad t\in[0,T).$$

Given that $f(t)\geq 0$ for all $t\in[0,T)$, it follows that φ is nondecreasing. Therefore, $\varphi(t)\geq\varphi(0)=x_0^{-1}>0$ for all $t\in[0,T]$. In particular, this proves that $\varphi(t)\neq 0$ on $[0,T]$, which also shows that (3.18) holds since $f(t)=0$ for $t\in D_g$. Therefore, since the conditions of Theorem 3.8 are satisfied on the whole $[0,T]$, we know that the map

$$x(t)=\frac{\exp_g(p,t)}{\varphi(t)},\quad t\in[0,T],$$

is a solution of (5.3). Since $\exp_g(p,t),\varphi(t)>0$ for $t\in[0,T]$, it follows that $x(t)>0$ for all $t\in[0,T]$ as we claimed before. Given that Theorem 3.8 can be applied for each $T>0$, we can obtain a solution on $[0,+\infty)$. The following result provides a recursive expression for such map.

Theorem 5.1. *The solution of* (5.3) *on* $[0,+\infty)$ *given by* Theorem 3.8 *is the map* $x:[0,+\infty)\to\mathbb{R}$ *defined as* $x(0)=x_0$ *and, for* $k=0,1,2,\dots$,

$$x(t)=\begin{cases}x(4k), & 4k<t\leq 4k+\dfrac{3}{2}\\ \dfrac{x(4k)\left(1+\widetilde{\alpha}\right)}{e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)}+x(4k)\gamma\left(1+\widetilde{\alpha}\right)\left(e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)}-1\right)}, & 4k+\dfrac{3}{2}<t\leq 4(k+1),\end{cases}$$

with $\widetilde{\alpha}=\alpha\Delta^+g(\frac{3}{2})$.

Proof. First, observe that by definition, $\exp_g(p,0) = 1$ and $\varphi(0) = x_0^{-1}$, and so $x(0) = x_0$. Next, note that since g is constant on each interval of the form $[4k, 4k+\frac{3}{2}]$, $k = 0,1,2,\dots$, and $\exp_g(p,\cdot), \varphi$ are g-absolutely continuous maps, they are also constant on the same interval, see Proposition 2.9. Therefore,

$$x(t) = \frac{\exp_g(p,t)}{\varphi(t)} = \frac{\exp_g(p,4k)}{\varphi(4k)} = x(4k), \quad t \in \left[4k, 4k+\frac{3}{2}\right],\ k = 0,1,2,\dots.$$

Hence, all that is left to do is to show that, for $k = 0,1,2,\dots$,

$$x(t) = \frac{x(4k)\,(1+\alpha_k)}{e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)} + x(4k)\gamma\,(1+\alpha_k)\left(e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)}-1\right)}, \quad t \in \left[4k+\frac{3}{2}, 4(k+1)\right].$$

Let $k \in \{0,1,2,\dots\}$. Observe that, by definition, for $t \in \left(4k+\frac{3}{2}, 4(k+1)\right]$,

$$\begin{aligned}
\exp_g(p,t) &= \exp\left(\int_{[0,t)} \widehat{p}(s)\mathrm{d}\mu_g(s)\right) \\
&= \exp_g\left(p, 4k+\frac{3}{2}\right)\exp\left(\int_{\left[4k+\frac{3}{2},t\right)} \widehat{p}(s)\mathrm{d}\mu_g(s)\right) \\
&= \exp_g\left(p,4k\right)\exp\left(\int_{\left[4k+\frac{3}{2},t\right)} \widehat{p}(s)\mathrm{d}\mu_g(s)\right), \\
\varphi(t) &= \frac{1}{x_0} + \int_{[0,t)} f(s)\exp_g(p,s)\mathrm{d}\mu_g(s) \\
&= \varphi\left(4k+\frac{3}{2}\right) + \int_{\left[4k+\frac{3}{2},t\right)} f(s)\exp_g(p,s)\mathrm{d}\mu_g(s) \\
&= \varphi\left(4k\right) + \int_{\left[4k+\frac{3}{2},t\right)} f(s)\exp_g(p,s)\mathrm{d}\mu_g(s).
\end{aligned}$$

Now, for $t \in (4k+\frac{3}{2}, 4(k+1)]$,

$$\begin{aligned}
\int_{\left[4k+\frac{3}{2},t\right)} \widehat{p}(s)\mathrm{d}\mu_g(s) &= \int_{\left\{4k+\frac{3}{2}\right\}} \widehat{p}(s)\mathrm{d}\mu_g(s) + \int_{\left(4k+\frac{3}{2},t\right)} \widehat{p}(s)\mathrm{d}\mu_g(s) \\
&= \widehat{p}\left(4k+\frac{3}{2}\right)\Delta^+ g\left(4k+\frac{3}{2}\right) + \int_{\left(4k+\frac{3}{2},t\right)} p(s)\mathrm{d}\mu_g(s) \\
&= \log\left(1+p\left(4k+\frac{3}{2}\right)\Delta^+ g\left(4k+\frac{3}{2}\right)\right) - \int_{\left(4k+\frac{3}{2},t\right)} \beta\mathrm{d}\mu_g(s) \\
&= \log\left(1+\alpha\Delta^+ g\left(\frac{3}{2}\right)\right) + \beta\left(g\left(4k+\frac{3}{2}+\right) - g(t)\right) \\
&= \log\left(1+\widetilde{\alpha}\right) + \beta\left(g\left(4k+\frac{3}{2}+\right) - g(t)\right).
\end{aligned}$$

Hence, for $t \in (4k+\frac{3}{2}, 4(k+1)]$, we have

$$\exp\left(\int_{\left[4k+\frac{3}{2},t\right)} \widehat{p}(s)\mathrm{d}\mu_g(s)\right) = (1+\widetilde{\alpha})\,e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}.$$

On the other hand, for $t \in (4k+\frac{3}{2}, 4(k+1)]$, since $f(4k+\frac{3}{2}) = 0$, we have

$$\begin{aligned}\int_{[4k+\frac{3}{2},t)} f(s)\exp_g(p,s)\mathrm{d}\mu_g(s) &= \int_{(4k+\frac{3}{2},t)} f(s)\exp_g(p,s)\mathrm{d}\mu_g(s)\\ &= \int_{(4k+\frac{3}{2},t)} \beta\gamma\exp_g(p,s)\mathrm{d}\mu_g(s) = -\gamma\int_{(4k+\frac{3}{2},t)} -\beta\exp_g(p,s)\mathrm{d}\mu_g(s)\\ &= -\gamma\int_{(4k+\frac{3}{2},t)} p(s)\exp_g(p,s)\mathrm{d}\mu_g(s) = -\gamma\int_{(4k+\frac{3}{2},t)} (\exp_g(p,\cdot))_g'(s)\mathrm{d}\mu_g(s).\end{aligned}$$

Now, using the Fundamental Theorem of Calculus, Theorem 2.6, it follows that

$$\begin{aligned}\int_{[4k+\frac{3}{2},t)} f(s)\exp_g(p,s)\mathrm{d}\mu_g(s) &= -\gamma\left(\exp_g(p,t) - \exp_g\left(p, 4k+\frac{3}{2}+\right)\right)\\ &= -\gamma\exp_g(p,4k)\,(1+\widetilde{\alpha})\left(e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}-1\right)\\ &= \gamma\exp_g(p,4k)\,(1+\widetilde{\alpha})\left(1-e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}\right).\end{aligned}$$

Therefore, for $t \in (4k+\frac{3}{2}, 4(k+1)]$,

$$\begin{aligned}x(t) &= \frac{\exp_g(p,4k)\exp\left(\int_{[4k+\frac{3}{2},t)} \widehat{p}(s)\mathrm{d}\mu_g(s)\right)}{\varphi(4k) + \int_{[4k+\frac{3}{2},t)} f(s)\exp_g(p,s)\mathrm{d}\mu_g(s)}\\ &= \frac{\exp_g(p,4k)\,(1+\widetilde{\alpha})\,e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}}{\varphi(4k)+\gamma\exp_g(p,4k)\,(1+\widetilde{\alpha})\left(1-e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}\right)}\\ &= \frac{\dfrac{\exp_g(p,4k)}{\varphi(4k)}\,(1+\widetilde{\alpha})\,e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}}{1+\dfrac{\exp_g(p,4k)}{\varphi(4k)}\gamma\,(1+\widetilde{\alpha})\left(1-e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}\right)}\\ &= \frac{x(4k)\,(1+\widetilde{\alpha})}{e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)}+x(4k)\gamma\,(1+\widetilde{\alpha})\left(e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)}-1\right)},\end{aligned}$$

as we needed to show. □

In Figure 5.2 we have plotted the solution above for different values of γ. Observe that the population presents the behavior we expected. Indeed, first note that the population remains constant during the hibernation periods. Furthermore, the population decays between generations, and the rate of this decay depends on the competition strength, γ. This can be easily observed by noting that $x(\frac{3}{2}+) = (1+\alpha)x_0 = \frac{17}{10}$ in all the graphs in Figure 5.2, however, the population levels at $t = 4$ are lower for higher values of γ.

In order to study the asymptotic behavior of the solution of (5.3), we will look at the sequences $\{P_k\}_{k=0}^{\infty} = \{x(4k+\frac{3}{2})\}_{k=0}^{\infty}$ and $\{\widetilde{P}_k\}_{k=0}^{\infty} = \{x(4k+\frac{3}{2}+)\}_{k=0}^{\infty}$ representing the population at the end of the hibernation period and the population after newborns are introduced, respectively. Using the expression for x obtained in Theorem 5.1, we see that $\{P_k\}_{k=0}^{\infty}$ satisfies

$$P_0 = x_0, \quad P_{k+1} = \frac{P_k(1+\widetilde{\alpha})}{e^{\beta(g(4k)-g(4(k-1)+\frac{3}{2}+))}+P_k\gamma(1+\widetilde{\alpha})(e^{\beta(g(4k)-g(4(k-1)+\frac{3}{2}+)}-1)}, \quad k = 0, 1, \ldots,$$

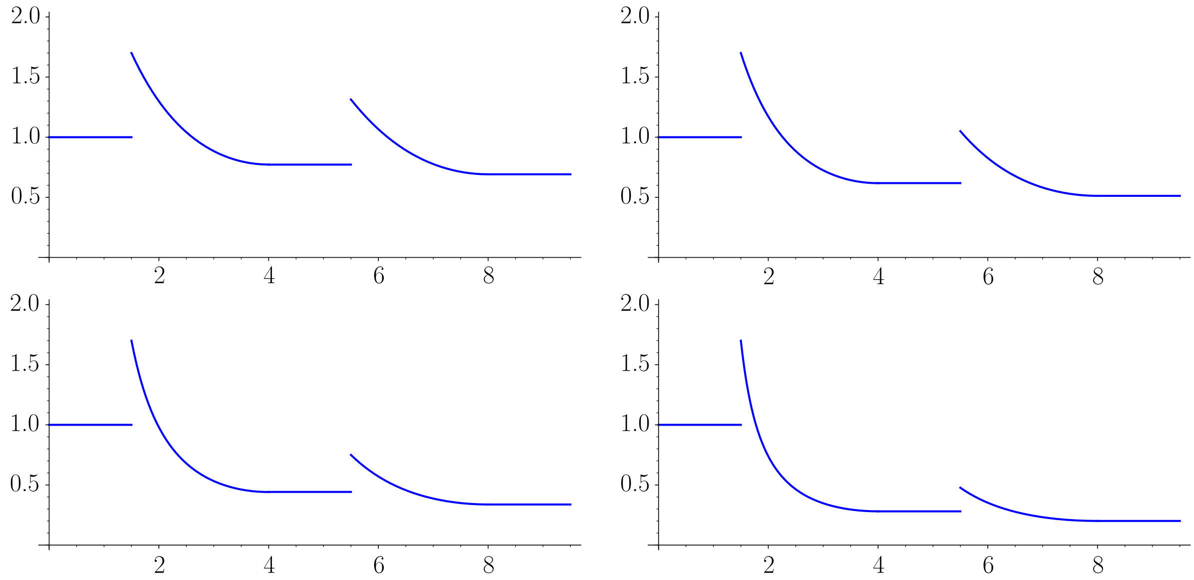

Figure 5.2: Graphs of the solution of (5.3) for g as in (5.2), $x_0 = 1$, $\alpha = \frac{7}{10}$, $\beta = \frac{1}{10}$ and different values of γ. In order, $\gamma = \frac{1}{2}$, $\gamma = 1$, $\gamma = 2$ and $\gamma = 4$.

which, thanks to (5.1), simplifies to

$$P_{k+1} = \frac{P_k(1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)}, \quad k = 0, 1, \ldots, \tag{5.6}$$

with $\widetilde{\beta} = \beta(g(4) - g(\frac{3}{2}+))$. Furthermore, $\widetilde{P}_k = (1+\widetilde{\alpha})P_k$ for $k = 0, 1, 2, \ldots$. Let us rewrite Eq. (5.6) in the form

$$P_{k+1} = H(P_k), \quad k = 0, 1, \ldots,$$

where

$$H(t) = \frac{t(1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + t\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)}, \quad t \in [0, \infty).$$

A simple calculation shows that the map H has, in general, two fixed points, namely zero and

$$L = \frac{1+\widetilde{\alpha} - e^{\widetilde{\beta}}}{\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)}.$$

The next result shows that the asymptotic behavior of the sequences $\{P_k\}_{k=0}^{\infty}$ and $\{\widetilde{P}_k\}_{k=0}^{\infty}$ (and therefore of the whole solution x) depends on whether L is positive (i.e., $e^{\widetilde{\beta}} < 1+\widetilde{\alpha}$) or nonpositive (i.e., $e^{\widetilde{\beta}} \geq 1+\widetilde{\alpha}$).

Theorem 5.2. *Denote* $\widetilde{\alpha} = \alpha\Delta^+ g(\frac{3}{2}) > 0$, $\widetilde{\beta} = \beta(g(4) - g(\frac{3}{2}+)) > 0$.

(a) *If* $e^{\widetilde{\beta}} \geq 1+\widetilde{\alpha}$, *the sequence* $\{P_k\}_{k=0}^{\infty}$ *is nonincreasing and converges to* 0. *As a consequence,* $\{\widetilde{P}_k\}_{k=0}^{\infty}$ *has the same behavior, and* $\lim_{t\to\infty} x(t) = 0$.

(b) *If* $e^{\widetilde{\beta}} < 1+\widetilde{\alpha}$, *we distinguish two cases:*

   (i) *If* $x_0 \geq L$, *the sequence* $\{P_k\}_{k=0}^{\infty}$ *is nonincreasing and converges to* L. *As a consequence,* $\{\widetilde{P}_k\}_{k=0}^{\infty}$ *is also nonincreasing and converges to* $(1+\widetilde{\alpha})L$.

(ii) *If* $x_0 \leq L$*, the sequence* $\{P_k\}_{k=0}^{\infty}$ *is nondecreasing and converges to* L*. As a consequence,* $\{\widetilde{P}_k\}_{k=0}^{\infty}$ *is also nondecreasing and converges to* $(1+\widetilde{\alpha})L$*.*

Proof. We shall only prove the result for $\{P_k\}_{k=0}^{\infty}$ as the properties for $\{\widetilde{P}_k\}_{k=0}^{\infty}$ follow from the relation $\widetilde{P}_k = (1+\widetilde{\alpha})P_k$ for $k = 0,1,2,\dots$.

First, assume that $e^{\widetilde{\beta}} \geq 1+\widetilde{\alpha}$. Observe that, for $k = 0,1,2,\dots,$

$$P_{k+1} = H(P_k) = \frac{P_k(1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} \leq P_k\frac{1+\widetilde{\alpha}}{e^{\widetilde{\beta}}} \leq P_k,$$

which proves that the sequence is nonincreasing. Furthermore, by definition, we have that $P_k > 0$ for $k = 0,1,2,\dots$. Hence, the sequence $\{P_k\}_{k=0}^{\infty}$ is nonincreasing and bounded from below, so it is convergent. Since the only nonnegative fixed point of H is zero, it follows that $\{P_k\}_{k=0}^{\infty}$ converges to 0.

Next, we assume that $e^{\widetilde{\beta}} < 1+\widetilde{\alpha}$. Standard computations show that

$$H'(t) = \frac{e^{\widetilde{\beta}}(1+\widetilde{\alpha})}{(e^{\widetilde{\beta}} + t\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1))^2}, \quad t \geq 0,$$

so it follows that H is nondecreasing on $[0,+\infty)$. Recalling that $H(L) = L$ and $P_{k+1} = H(P_k)$, it follows that if $x_0 \geq L$, then $P_k \geq L$, $k = 0,1,2,\dots$; and if $x_0 \leq L$, then $P_k \leq L$, $k = 0,1,2,\dots$.

Now, suppose that $x_0 \geq L$. In that case, for $k = 0,1,2,\dots$

$$\begin{aligned} P_k - P_{k+1} &= P_k - \frac{P_k(1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} = P_k\frac{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1) - (1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} \\ &\geq P_k\frac{e^{\widetilde{\beta}} + L\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1) - (1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} = P_k\frac{e^{\widetilde{\beta}} + 1 + \widetilde{\alpha} - e^{\widetilde{\beta}} - (1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} = 0. \end{aligned}$$

Hence, the sequence $\{P_k\}_{k=0}^{\infty}$ is nonincreasing and bounded from below by the unique positive fixed point L, so it is convergent to L.

On the other hand, if $x_0 \leq L$ then, for $k = 0,1,2,\dots,$

$$P_k - P_{k+1} = P_k\frac{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1) - (1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} \leq P_k\frac{e^{\widetilde{\beta}} + L\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1) - (1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} = 0$$

In this case, the sequence $\{P_k\}_{k=0}^{\infty}$ is nondecreasing and bounded from above by the unique positive fixed point L, so it is convergent to L. □

Remark 5.3. Observe that, if $e^{\widetilde{\beta}} < 1+\widetilde{\alpha}$ and $x_0 = L$, then the sequences $\{P_k\}_{k=0}^{\infty}$ and $\{\widetilde{P}_k\}_{k=0}^{\infty}$ are constant and equal to x_0 and $(1+\widetilde{\alpha})x_0$, respectively. Hence, it follows from Theorem 5.1 that the solution is 4-periodic in this case.

In Figure 5.3 we can observe the different asymptotic behaviors that we can expect from the solution of Eq. (5.3) as described by Theorem 5.2. In particular, we can see that when $e^{\widetilde{\beta}} \geq 1+\widetilde{\alpha}$ (i.e., when the death rate is high enough) the population is bound to extinction as presented in the first of the graphs. On the other hand, the second and third plot show that if $e^{\widetilde{\beta}} < 1+\widetilde{\alpha}$ (i.e., when the reproduction rate is high enough), we can expect the population to approach an equilibrium state corresponding to a 4-periodic solution shown in the fourth plot.

As a final note, observe that the example here provided is relatively simple. More complicated models can be obtained if we consider the parameters α, β and K to be functions instead, or if we relax the condition (5.1).

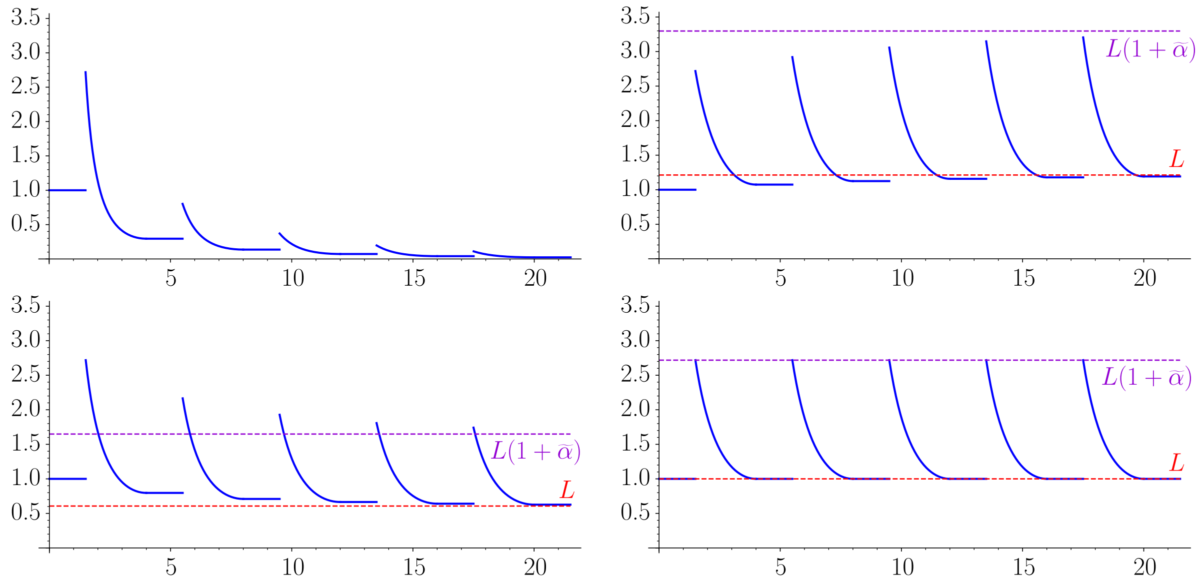

Figure 5.3: Graphs of the solution of (5.3) for g as in (5.2) showing the different asymptotic behaviors for $x_0 = 1$, $\alpha = e - 1$ and different values of the parameters (β, γ). In order, $(\frac{3}{10}, \frac{1}{2})$, $(\frac{1}{10}, \frac{1}{2})$, $(\frac{1}{10}, 1)$ and $(\frac{1}{10}, \frac{1}{\sqrt{e}})$.

6 Preliminaries on Stieltjes integrals

In the rest of the paper, we focus on the logistic equation in the context of Stieltjes integral equations. We will work with Kurzweil–Stieltjes integrals (also known as Perron–Stieltjes integrals), but we only need some basic properties of these integrals, which are summarized in the present section. A much more comprehensive treatment is available in [15]. Alternatively, it would be possible to work with the Young integral, which coincides with the Kurzweil–Stieltjes integral if the integrand and integrator are regulated and one of them has bounded variation (cf. [15, Theorem 6.13.1]).

We need the substitution theorem for the Kurzweil–Stieltjes integral (see [15, Theorem 6.6.1]).

Theorem 6.1. *Assume that $h : [a, b] \to \mathbb{R}$ is bounded and $f, g : [a, b] \to \mathbb{R}$ are such that $\int_a^b f \, \mathrm{d}g$ exists. Then*

$$\int_a^b h(t) \, \mathrm{d}\left(\int_a^t f(s) \, \mathrm{d}g(s)\right) = \int_a^b h(t) f(t) \, \mathrm{d}g(t),$$

whenever either side of the equation exists.

The next result describes the properties of indefinite Kurzweil–Stieltjes integrals (see [15, Corollary 6.5.5]).

Theorem 6.2. *Let $f, g : [a, b] \to \mathbb{R}$ be such that g is regulated and $\int_a^b f \, \mathrm{d}g$ exists. Then, for every $t_0 \in [a, b]$, the function*

$$h(t) = \int_{t_0}^t f \, \mathrm{d}g, \quad t \in [a, b]$$

is regulated and satisfies

$$h(t+) = h(t) + f(t)\Delta^+ g(t), \quad t \in [a, b),$$
$$h(t-) = h(t) - f(t)\Delta^- g(t), \quad t \in (a, b].$$

Moreover, if f is regulated and g has bounded variation, then h has bounded variation.

For the next result, see [15, Exercise 6.3.5].

Lemma 6.3. *If $f : [\alpha, \beta] \to \mathbb{R}$ is an arbitrary function and $g : [\alpha, \beta] \to \mathbb{R}$ is such that $g(t) = c$ for each $t \in (\alpha, \beta)$, then*

$$\int_{\alpha}^{\beta} f(t)\,\mathrm{d}g(t) = f(\beta)g(\beta) - f(\alpha)g(\alpha) - c(f(\beta) - f(\alpha)).$$

Our next goal is to obtain an integral version of the formula

$$\left(\frac{1}{g(t)}\right)' = -\frac{g'(t)}{g(t)^2}.$$

We begin with the case when g is a step function.

Lemma 6.4. *If $g : [a, b] \to \mathbb{R}$ is a step function, which is nonzero on $[a, b]$, then*

$$\int_a^b \mathrm{d}\left(\frac{1}{g(t)}\right) = \frac{1}{g(b)} - \frac{1}{g(a)} = -\int_a^b \frac{1}{g(t-)g(t+)}\,\mathrm{d}g(t),$$

with the convention that $g(a-) = g(a)$ and $g(b+) = g(b)$.

Proof. The first equality is obvious from the definition of the integral; let us verify the second one.

Since g is a step function, there exists a partition $a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b$ and constants $c_1, \ldots, c_m \in \mathbb{R}$ such that $g(t) = c_j$ for each $t \in (\alpha_{j-1}, \alpha_j)$. Let us also denote $c_0 = g(a)$, $c_{m+1} = g(b)$. Then $g(\alpha_{j-1}-) = c_{j-1}$ and $g(\alpha_{j-1}+) = c_j$ for each $j \in \{1, \ldots, m+1\}$. Applying Lemma 6.3 to each interval $[\alpha_{j-1}, \alpha_j]$, $j \in \{1, \ldots, m\}$, we calculate

$$\begin{aligned}
\int_a^b \frac{1}{g(t-)g(t+)}\,\mathrm{d}g(t) &= \sum_{j=1}^m c_j \left(\frac{1}{g(\alpha_{j-1}-)g(\alpha_{j-1}+)} - \frac{1}{g(\alpha_j-)g(\alpha_j+)}\right) \\
&\quad + \sum_{j=1}^m \left(\frac{g(\alpha_j)}{g(\alpha_j-)g(\alpha_j+)} - \frac{g(\alpha_{j-1})}{g(\alpha_{j-1}-)g(\alpha_{j-1}+)}\right) \\
&= \sum_{j=1}^m c_j \left(\frac{1}{c_{j-1}c_j} - \frac{1}{c_j c_{j+1}}\right) + \frac{g(b)}{g(b-)g(b+)} - \frac{g(a)}{g(a-)g(a+)} \\
&= \sum_{j=1}^m \frac{1}{c_{j-1}} - \sum_{j=1}^m \frac{1}{c_{j+1}} + \frac{1}{g(b-)} - \frac{1}{g(a+)} \\
&= \sum_{j=1}^m \frac{1}{c_{j-1}} - \sum_{j=1}^m \frac{1}{c_{j+1}} + \frac{1}{c_m} - \frac{1}{c_1} \\
&= \frac{1}{c_0} - \frac{1}{c_{m+1}} = \frac{1}{g(a)} - \frac{1}{g(b)}.
\end{aligned}$$

□

We now generalize Lemma 6.4 to functions of bounded variation.

Theorem 6.5. *If $g : [a, b] \to \mathbb{R}$ has bounded variation and for each $t \in [a, b]$, we have $g(t) \neq 0$, $g(t-) \neq 0$, and $g(t+) \neq 0$, then*

$$\int_a^b \mathrm{d}\left(\frac{1}{g(t)}\right) = \frac{1}{g(b)} - \frac{1}{g(a)} = -\int_a^b \frac{1}{g(t-)g(t+)}\,\mathrm{d}g(t),$$

with the convention that $g(a-) = g(a)$ and $g(b+) = g(b)$.

Proof. It suffices to prove the second equality. Since g has bounded variation, there exist nondecreasing functions $g^1, g^2 : [a,b] \to \mathbb{R}$ such that $g = g^1 - g^2$. Also, for each $i \in \{1,2\}$, there exists a sequence of nondecreasing step functions $\{g_n^i\}_{n=1}^{\infty}$ which is uniformly convergent to g^i. Without loss of generality, we can assume that these sequences are such that

$$g^i(a) \le g_n^i(a) \le g_n^i(b) \le g^i(b)$$

for all $n \in \mathbb{N}$ and $i \in \{1,2\}$. Therefore,

$$\operatorname{var}(g_n^i, [a,b]) = g_n^i(b) - g_n^i(a) \le g^i(b) - g^i(a), \quad n \in \mathbb{N}, \quad i \in \{1,2\}.$$

Consequently, by letting $g_n = g_n^1 - g_n^2$ for all $n \in \mathbb{N}$, we obtain a sequence of finite step functions $\{g_n\}_{n=1}^{\infty}$, which is uniformly convergent to g, and its members have uniformly bounded variation.

Let us again use the convention that $g_n(a-) = g_n(a)$ and $g_n(b+) = g_n(b)$ for each $n \in \mathbb{N}$. Note that $g_n(t-) \rightrightarrows g(t-)$ and $g_n(t+) \rightrightarrows g(t+)$ with respect to $t \in [a,b]$ (see [15, Lemma 4.2.3]).

Also, there exists an $M > 0$ such that

$$|g(t-)| \ge M, \quad t \in [a,b]$$

(apply Lemma 2.7 to $f(t) = g(t-)$). Hence, for sufficiently large $n \in \mathbb{N}$, we have

$$|g_n(t-)| \ge M/2, \quad t \in [a,b],$$

and therefore

$$\left| \frac{1}{g_n(t-)} - \frac{1}{g(t-)} \right| = \left| \frac{g(t-) - g_n(t-)}{g_n(t-)g(t-)} \right| \le \frac{2}{M^2} |g(t-) - g_n(t-)|,$$

which shows that $1/g_n(t-) \rightrightarrows 1/g(t-)$ with respect to $t \in [a,b]$. In a similar way, one can show that $1/g_n(t+) \rightrightarrows 1/g(t+)$ with respect to $t \in [a,b]$. Consequently,

$$\frac{1}{g_n(t-)g_n(t+)} \rightrightarrows \frac{1}{g(t-)g(t+)}$$

with respect to $t \in [a,b]$. Thus, we conclude that

$$\begin{aligned} \frac{1}{g(b)} - \frac{1}{g(a)} = \lim_{n\to\infty} \left(\frac{1}{g_n(b)} - \frac{1}{g_n(a)} \right) &= -\lim_{n\to\infty} \int_a^b \frac{1}{g_n(t-)g_n(t+)} \, \mathrm{d}g_n(t) \\ &= -\int_a^b \frac{1}{g(t-)g(t+)} \, \mathrm{d}g(t), \end{aligned}$$

where the second equality follows from Lemma 6.3 and the third from the uniform convergence theorem for integrals whose integrators have uniformly bounded variation (see [15, Theorem 6.8.8]). □

Once we have Theorem 6.5, it is not difficult to obtain the following integral version of the quotient rule, i.e., of the classical formula

$$\left(\frac{f(t)}{g(t)} \right)' = \frac{f'(t)}{g(t)} - \frac{f(t)g'(t)}{g(t)^2}.$$

Theorem 6.6. *If $f, g : [a,b] \to \mathbb{R}$ have bounded variation and for each $t \in [a,b]$, we have $g(t) \neq 0$, $g(t-) \neq 0$, and $g(t+) \neq 0$, then*

$$\int_a^b \mathrm{d}\left(\frac{f(t)}{g(t)}\right) = \frac{f(b)}{g(b)} - \frac{f(a)}{g(a)} = \int_a^b \frac{\mathrm{d}f(t)}{g(t+)} - \int_a^b \frac{f(t-)\,\mathrm{d}g(t)}{g(t-)g(t+)},$$

with the convention that $g(a-) = g(a)$ and $g(b+) = g(b)$.

Proof. It suffices to prove the second equality. Lemma 2.7 implies that $1/g$ has bounded variation. Using the integration by parts formula in the form presented in [13, Theorem B.6], we get

$$\int_a^b \frac{\mathrm{d}f(t)}{g(t+)} = \frac{f(b)}{g(b)} - \frac{f(a)}{g(a)} - \int_a^b f(t-)\,\mathrm{d}\left(\frac{1}{g(t)}\right).$$

The definition of the integral, Theorem 6.5 and Theorem 6.1 imply

$$\int_a^b f(t-)\,\mathrm{d}\left(\frac{1}{g(t)}\right) = \int_a^b f(t-)\,\mathrm{d}\left(\frac{1}{g(t)} - \frac{1}{g(a)}\right)$$
$$= -\int_a^b f(t-)\,\mathrm{d}\left(\int_a^t \frac{\mathrm{d}g(s)}{g(s-)g(s+)}\right) = -\int_a^b \frac{f(t-)\,\mathrm{d}g(t)}{g(t-)g(t+)},$$

which completes the proof. □

Theorem 6.6 is not needed in the rest of this paper, but we hope it might be useful for subsequent research.

7 Stieltjes-integral versions of the logistic equation

We are now ready to deal with Stieltjes integral equations. In this section, we always assume that $g : [a,b] \to \mathbb{R}$ has bounded variation (left-continuity is no longer required). We begin with the linear nonhomogeneous equation

$$x(t) = x(t_0) + \int_{t_0}^t (p(s)x(s) + f(s))\,\mathrm{d}g(s), \quad t \in [a,b], \tag{7.1}$$

and try to obtain the corresponding logistic equation as an integral equation whose solution is the function $y(t) = x(t)^{-1}$.

Theorem 7.1. *Suppose that $g : [a,b] \to \mathbb{R}$ has bounded variation, $p : [a,b] \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ are regulated, and $x : [a,b] \to \mathbb{R}$ satisfies Eq. (7.1). If $x(t) \neq 0$, $x(t-) \neq 0$, and $x(t+) \neq 0$ for all $t \in [a,b]$, then the function $y(t) = x(t)^{-1}$ satisfies*

$$y(t) = y(t_0) - \int_{t_0}^t \frac{(p(s) + f(s)y(s))y(s)}{(1 - (p(s) + f(s)y(s))\Delta^- g(s))(1 + (p(s) + f(s)y(s))\Delta^+ g(s))}\,\mathrm{d}g(s) \tag{7.2}$$

for all $t \in [a,b]$, with the convention that $\Delta^+ g(s) = 0$ if $s = \max(t, t_0)$, and $\Delta^- g(s) = 0$ if $s = \min(t, t_0)$.

Proof. According to Theorem 6.5, we have

$$y(t) - y(t_0) = \frac{1}{x(t)} - \frac{1}{x(t_0)} = -\int_{t_0}^t \frac{1}{x(s-)x(s+)}\,\mathrm{d}x(s),$$

with the convention that $x(s-) = x(s)$ if $s = \min(t, t_0)$, and $x(s+) = x(s)$ if $s = \max(t, t_0)$. Using Eq. (7.1) and Theorem 6.1, we get

$$y(t) - y(t_0) = -\int_{t_0}^{t} \frac{p(s)x(s) + f(s)}{x(s-)x(s+)} \, \mathrm{d}g(s).$$

Theorem 6.2 yields

$$x(s+) = x(s) + (p(s)x(s) + f(s))\Delta^+ g(s) = x(s)(1 + (p(s) + f(s)y(s))\Delta^+ g(s)), \tag{7.3}$$
$$x(s-) = x(s) - (p(s)x(s) + f(s))\Delta^- g(s) = x(s)(1 - (p(s) + f(s)y(s))\Delta^- g(s)). \tag{7.4}$$

Therefore,

$$y(t) - y(t_0) = -\int_{t_0}^{t} \frac{(p(s) + f(s)y(s))y(s)}{(1 - (p(s) + f(s)y(s))\Delta^- g(s))(1 + (p(s) + f(s)y(s))\Delta^+ g(s))} \, \mathrm{d}g(s),$$

with the convention that $\Delta^+ g(s) = 0$ if $s = \max(t, t_0)$, and $\Delta^- g(s) = 0$ if $s = \min(t, t_0)$. □

Remark 7.2. Theorem 7.1 requires that $x(t) \neq 0$, $x(t-) \neq 0$, and $x(t+) \neq 0$ for all $t \in [a, b]$. The first condition is obviously necessary, for otherwise the definition of y would not make sense. If this condition is satisfied, then Eq. (7.3) and (7.4) show that the latter two conditions are equivalent to

$$1 + (p(t) + f(t)y(t))\Delta^+ g(t) \neq 0, \tag{7.5}$$
$$1 - (p(t) + f(t)y(t))\Delta^- g(t) \neq 0 \tag{7.6}$$

for all $t \in [a, b]$. Since these terms appear in the denominator on the right-hand side of the logistic equation, it is clear that the two conditions are necessary as well. Recalling that $y(t) = 1/x(t)$, we can rewrite the conditions (7.5) and (7.6) as

$$x(t) \neq -\frac{f(t)\Delta^+ g(t)}{1 + p(t)\Delta^+ g(t)}, \tag{7.7}$$
$$x(t) \neq \frac{f(t)\Delta^- g(t)}{1 - p(t)\Delta^- g(t)} \tag{7.8}$$

whenever the denominators are nonzero.

Remark 7.3. In the theory of Stieltjes differential equations, it is always assumed that g is a left-continuous nondecreasing function. In this case, Eq. (7.1) is the integral version of the Stieltjes differential equation $x'_g(t) = p(t)x(t) + f(t)$, and Eq. (7.2) simplifies to

$$y(t) = y(t_0) - \int_{t_0}^{t} \frac{(p(s) + f(s)y(s))y(s)}{1 + (p(s) + f(s)y(s))\Delta^+ g(s)} \, \mathrm{d}g(s), \quad t \in [a, b],$$

which is the integral version of the Stieltjes differential equation (3.1). Thus, we see that the form of the logistic equation (7.2) is consistent with the form obtained in Section 3. Condition (7.7) corresponds to the earlier condition (3.5), and condition (7.8) reduces to $x(t) \neq 0$.

Note that Eq. (7.1) is a special case of a generalized linear differential equation, whose solution can be explicitly expressed using the variation of constants formula (see e.g. [15, Theorems 7.8.4 and 7.8.5]). Thus, the reciprocal of this solution is a solution of the logistic equation given in Theorem 7.1.

Besides Eq. (7.1), one can also investigate the linear nonhomogeneous Stieltjes equations

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s-) + f(s))\,\mathrm{d}g(s), \quad t \in [a,b], \tag{7.9}$$

$$x(t) = x(t_0) + \int_{t_0}^{t} (-p(s)x(s+) + f(s))\,\mathrm{d}g(s), \quad t \in [a,b], \tag{7.10}$$

which were studied in [13, 20], and which are dual to each other. Note that the one-sided limits $x(s-)$ and $x(s+)$ in the integrands have to be interpreted as $x(s)$ when s coincides with the lower or upper limit of the integral, respectively.

Starting with a solution x of Eq. (7.9) or Eq. (7.10), let us find the corresponding integral equation for the function $y(t) = x(t)^{-1}$. Interestingly, we will see that the resulting logistic equations are simpler than the logistic equation obtained in Theorem 7.1. We need the following modification of Theorem 6.1.

Lemma 7.4. *Assume that $g, h : [a,b] \to \mathbb{R}$ have bounded variation and $k, x : [a,b] \to \mathbb{R}$ are regulated.*

1. *If*

$$y(t) = \int_{t_0}^{t} k(s)x(s+)\,\mathrm{d}g(s), \quad t \in [a,b],$$

with the convention that $x(s+)$ means $x(s)$ if $s = \max(t, t_0)$, then for each $t \in [a,b]$, we have

$$\int_{t_0}^{t} h(s)\,\mathrm{d}y(s) = \int_{t_0}^{t} h(s)k(s)x(s+)\,\mathrm{d}g(s)$$

with the convention that $x(s+)$ means $x(s)$ if $s = \max(t, t_0)$.

2. *If*

$$y(t) = \int_{t_0}^{t} k(s)x(s-)\,\mathrm{d}g(s), \quad t \in [a,b],$$

with the convention that $x(s-)$ means $x(s)$ if $s = \min(t, t_0)$, then for each $t \in [a,b]$, we have

$$\int_{t_0}^{t} h(s)\,\mathrm{d}y(s) = \int_{t_0}^{t} h(s)k(s)x(s-)\,\mathrm{d}g(s)$$

with the convention that $x(s-)$ means $x(s)$ if $s = \min(t, t_0)$.

Proof. Let us prove the first statement. We will use the symbol χ_A to denote the characteristic (indicator) function of a set $A \subset \mathbb{R}$. Suppose first that $t > t_0$. Using Theorem 6.1 and the formula $\int_{t_0}^{t} p(s)\chi_{\{t\}}(s)\,\mathrm{d}q(s) = p(t)\Delta^- q(t)$, which holds for each $t > t_0$ and all functions $p, q : [a,b] \to \mathbb{R}$, we get

$$\begin{aligned}
\int_{t_0}^{t} h(s)\,\mathrm{d}y(s) &= \int_{t_0}^{t} h(s)\,\mathrm{d}\left(\int_{t_0}^{s} k(\tau)(x(\tau+)\chi_{[t_0,s)}(\tau) + x(\tau)\chi_{\{s\}}(\tau))\,\mathrm{d}g(\tau)\right) \\
&= \int_{t_0}^{t} h(s)\,\mathrm{d}\left(\int_{t_0}^{s} k(\tau)x(\tau+)\,\mathrm{d}g(\tau)\right) - \int_{t_0}^{t} h(s)\,\mathrm{d}\left(\int_{t_0}^{s} k(\tau)\chi_{\{s\}}(\tau)\Delta^+ x(\tau)\,\mathrm{d}g(\tau)\right) \\
&= \int_{t_0}^{t} h(s)k(s)x(s+)\,\mathrm{d}g(s) - \int_{t_0}^{t} h(s)\,\mathrm{d}\left(\chi_{(t_0,t]}(s)k(s)\Delta^+ x(s)\Delta^- g(s)\right) \\
&= \int_{t_0}^{t} h(s)k(s)(x(s+)\chi_{[t_0,t)}(s) + x(s)\chi_{\{t\}}(s))\,\mathrm{d}g(s) \\
&\quad + \int_{t_0}^{t} h(s)k(s)\Delta^+ x(s)\chi_{\{t\}}(s)\,\mathrm{d}g(s) - \int_{t_0}^{t} h(s)\,\mathrm{d}\left(\chi_{(t_0,t]}(s)k(s)\Delta^+ x(s)\Delta^- g(s)\right).
\end{aligned}$$

The last two integrals cancel each other out, since both have the value $h(t)k(t)\Delta^+x(t)\Delta^-g(t)$; for the latter integral, this follows from [15, Lemma 6.3.16] (note that the integrand has bounded variation, the integrator is regulated and vanishes in all points with at most countably many exceptions). This settles the case $t > t_0$. Similarly, if $t < t_0$, we have

$$\begin{aligned}\int_{t_0}^{t} h(s)\,\mathrm{d}y(s) &= \int_{t_0}^{t} h(s)\,\mathrm{d}\left(\int_{t_0}^{s} k(\tau)(x(\tau+)\chi_{[s,t_0)}(\tau) + x(\tau)\chi_{\{t_0\}}(\tau))\,\mathrm{d}g(\tau)\right)\\ &= \int_{t_0}^{t} h(s)\,\mathrm{d}\left(\int_{t_0}^{s} k(\tau)x(\tau+)\,\mathrm{d}g(\tau)\right) + \int_{t_0}^{t} h(s)\,\mathrm{d}\left(\int_{s}^{t_0} k(\tau)\chi_{\{t_0\}}(\tau)\Delta^+x(\tau)\,\mathrm{d}g(\tau)\right)\\ &= \int_{t_0}^{t} h(s)k(s)x(s+)\,\mathrm{d}g(s) + \int_{t_0}^{t} h(s)\,\mathrm{d}\left(\chi_{[t,t_0)}(s)k(t_0)\Delta^+x(t_0)\Delta^-g(t_0)\right)\\ &= \int_{t_0}^{t} h(s)k(s)(x(s+)\chi_{[t,t_0)}(s) + x(s)\chi_{\{t_0\}}(s))\,\mathrm{d}g(s)\\ &\quad - \int_{t}^{t_0} h(s)k(s)\Delta^+x(s)\chi_{\{t_0\}}(s)\,\mathrm{d}g(s) - \int_{t}^{t_0} h(s)\,\mathrm{d}\left(\chi_{[t,t_0)}(s)k(t_0)\Delta^+x(t_0)\Delta^-g(t_0)\right).\end{aligned}$$

The last two integrals cancel each other out, since the former equals $h(t_0)k(t_0)\Delta^+x(t_0)\Delta^-g(t_0)$, while the latter has the opposite value. This completes the proof of the first statement.

The second statement can be proved in a similar way. □

We can now obtain the logistic equations corresponding to Eq. (7.9) and Eq. (7.10).

Theorem 7.5. *Suppose that $g : [a,b] \to \mathbb{R}$ has bounded variation, $p : [a,b] \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ are regulated.*

1. *Suppose that $x : [a,b] \to \mathbb{R}$ satisfies Eq. (7.9). If $x(t) \neq 0$, $x(t-) \neq 0$, and $x(t+) \neq 0$ for all $t \in [a,b]$, then the function $y(t) = x(t)^{-1}$ satisfies*

$$y(t) = y(t_0) - \int_{t_0}^{t} (p(s) + f(s)y(s-))y(s+)\,\mathrm{d}g(s), \quad t \in [a,b]. \tag{7.11}$$

2. *Suppose that $x : [a,b] \to \mathbb{R}$ satisfies Eq. (7.10). If $x(t) \neq 0$, $x(t-) \neq 0$, and $x(t+) \neq 0$ for all $t \in [a,b]$, then the function $y(t) = x(t)^{-1}$ satisfies*

$$y(t) = y(t_0) - \int_{t_0}^{t} (-p(s) + f(s)y(s+))y(s-)\,\mathrm{d}g(s), \quad t \in [a,b]. \tag{7.12}$$

In both cases, $y(s-)$ or $y(s+)$ in the integrands should be understood as $y(s)$ when s coincides with the lower or upper limit of the integral, respectively.

Proof. Let us prove the first statement. According to Theorem 6.5 and Eq. (7.9), we have

$$\begin{aligned}y(t) - y(t_0) &= \frac{1}{x(t)} - \frac{1}{x(t_0)} = -\int_{t_0}^{t} \frac{\mathrm{d}x(s)}{x(s-)x(s+)}\\ &= -\int_{t_0}^{t} \frac{1}{x(s-)x(s+)}\,\mathrm{d}\left(\int_{t_0}^{t} (p(s)x(s-) + f(s))\,\mathrm{d}g(s)\right).\end{aligned}$$

Note that Eq. (7.9) implies that x has bounded variation, and by Lemma 2.7, the function $1/x$ has the same property. Hence, the functions $s \mapsto 1/x(s-)$ and $s \mapsto 1/x(s+)$ as well as their product have bounded variation. Using Lemma 7.4 and Theorem 6.1, we get

$$y(t) - y(t_0) = -\int_{t_0}^{t} \frac{p(s)x(s-) + f(s)}{x(s-)x(s+)}\,\mathrm{d}g(s) = -\int_{t_0}^{t} (p(s) + f(s)y(s-))y(s+)\,\mathrm{d}g(s),$$

where the second equality follows from the fact that $x(s+)^{-1} = y(s+)$ and $x(s-)^{-1} = y(s-)$.

The proof of the second statement is similar. □

Remark 7.6. If g is left-continuous, then a function x satisfying (7.9) or (7.10) is also left-continuous, i.e., $x(t-) = x(t)$ for all t. In this case, Eq. (7.9) coincides with Eq. (7.1), i.e., we have the following pair of equations:

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s) + f(s))\,\mathrm{d}g(s), \quad t \in [a,b], \tag{7.13}$$

$$x(t) = x(t_0) + \int_{t_0}^{t} (-p(s)x(s+) + f(s))\,\mathrm{d}g(s), \quad t \in [a,b]. \tag{7.14}$$

According to Theorem 7.5, if $x(t) \neq 0$ and $x(t+) \neq 0$ for all t, then $y = 1/x$ satisfies one of the following equations:

$$y(t) = y(t_0) - \int_{t_0}^{t} (p(s) + f(s)y(s))y(s+)\,\mathrm{d}g(s), \quad t \in [a,b], \tag{7.15}$$

$$y(t) = y(t_0) - \int_{t_0}^{t} (-p(s) + f(s)y(s+))y(s)\,\mathrm{d}g(s), \quad t \in [a,b]. \tag{7.16}$$

These are integral versions of the Stieltjes differential equations of Eq. (3.4) and Eq. (3.17), respectively, which are equivalent to the two logistic equations presented in Section 3.

Remark 7.7. General solution formulas for Eq. (7.9) and (7.10) were recently published in [20]. They resemble the well-known variation of constants formula, and involve solutions of the corresponding homogenenous Stieltjes integral equations. According to Theorem 7.5, explicit solutions of Eq. (7.9) and (7.10) immediately give rise to explicit solution formulas for the two versions of the logistic equation.

Remark 7.8. Theorem 7.5 again requires that $x(t) \neq 0$, $x(t-) \neq 0$, and $x(t+) \neq 0$ for all $t \in [a,b]$. The first condition is obviously necessary, for otherwise the definition of y would not make sense. Let us have a closer look on the latter two conditions, trying to avoid $x(t-)$ and $x(t+)$, and express both conditions in terms of $x(t)$.

Suppose first that $x : [a,b] \to \mathbb{R}$ satisfies Eq. (7.9). Using the properties of the Kurzweil–Stieltjes integral and performing similar calculations as in the proof of [13, Lemma 6.5] (which corresponds to the homogenous case $f = 0$), we find that

$$x(t-)(1 + p(t)\Delta g(t)) = x(t)(1 + p(t)\Delta^+ g(t)) - f(t)\Delta^- g(t), \qquad t \in (a, t_0), \tag{7.17}$$

$$x(t-)(1 + p(t)\Delta^- g(t)) = x(t) - f(t)\Delta^- g(t), \qquad t \in [t_0, b], \tag{7.18}$$

$$x(t+) = x(t)(1 + p(t)\Delta^+ g(t)) + f(t)\Delta^+ g(t), \qquad t \in [a, t_0], \tag{7.19}$$

$$x(t+) = x(t) + x(t-)p(t)\Delta^+ g(t) + f(t)\Delta^+ g(t), \quad t \in (t_0, b). \tag{7.20}$$

First, we deal with $x(t+)$. Taking $t \in [a, t_0]$, Eq. (7.19) implies that $x(t+) \neq 0$ if and only if $x(t)(1 + p(t)\Delta^+ g(t)) + f(t)\Delta^+ g(t) \neq 0$; assuming that $1 + p(t)\Delta^+ g(t) \neq 0$, this is equivalent to

$$x(t) \neq -\frac{f(t)\Delta^+ g(t)}{1 + p(t)\Delta^+ g(t)}, \quad t \in [a, t_0]. \tag{7.21}$$

For $t \in (t_0, b)$, if $1 + p(t)\Delta^- g(t) \neq 0$, we can express $x(t-)$ from Eq. (7.18) and substitute to Eq. (7.20) to obtain

$$x(t+) = x(t) + \frac{x(t) - f(t)\Delta^- g(t)}{1 + p(t)\Delta^- g(t)} p(t)\Delta^+ g(t) + f(t)\Delta^+ g(t), \quad t \in [a, t_0].$$

Hence, to ensure that $x(t+) \neq 0$, we need

$$x(t)(1+p(t)\Delta^{-}g(t)) + (x(t) - f(t)\Delta^{-}g(t))p(t)\Delta^{+}g(t) + f(t)\Delta^{+}g(t)(1+p(t)\Delta^{-}g(t)) \neq 0,$$

which simplifies to

$$x(t)(1+p(t)\Delta g(t)) + f(t)\Delta^{+}g(t) \neq 0,$$

and if $1+p(t)\Delta g(t) \neq 0$, this is equivalent to

$$x(t) \neq -\frac{f(t)\Delta^{+}g(t)}{1+p(t)\Delta g(t)}, \quad t \in (t_0, b). \tag{7.22}$$

Next, we focus on $x(t-)$. If $t \in (a, t_0)$ and $1+p(t)\Delta g(t) \neq 0$, then Eq. (7.17) implies that $x(t-) \neq 0$ if and only if

$$x(t)(1+p(t)\Delta^{+}g(t)) - f(t)\Delta^{-}g(t) \neq 0,$$

and if $1+p(t)\Delta^{+}g(t) \neq 0$, this is equivalent to

$$x(t) \neq \frac{f(t)\Delta^{-}g(t)}{1+p(t)\Delta^{+}g(t)}, \quad t \in (a, t_0). \tag{7.23}$$

Similarly, if $t \in [t_0, b]$ and $1+p(t)\Delta^{-}g(t) \neq 0$, then Eq. (7.18) implies that $x(t-) \neq 0$ if and only if

$$x(t) - f(t)\Delta^{-}g(t) \neq 0,$$

or equivalently

$$x(t) \neq f(t)\Delta^{-}g(t), \quad t \in [t_0, b]. \tag{7.24}$$

Thus, we have shown how to reformulate the conditions $x(t+) \neq 0$ and $x(t-) \neq 0$ in terms of $x(t)$. Note that if g is left-continuous, then the conditions in (7.22) and (7.21) coincide, and the conditions in (7.23) and (7.24) reduce to $x(t) \neq 0$.

A similar analysis can be performed for Eq. (7.10). However, it is easier to observe that $x : [a, b] \to \mathbb{R}$ satisfies

$$x(t) = x(t_0) + \int_{t_0}^{t} (-p(s)x(s+) + f(s))\, \mathrm{d}g(s), \quad t \in [a, b],$$

if and only if the function $y : [-b, -a] \to \mathbb{R}$ given by $y(t) = x(-t)$ satisfies

$$y(t) = y(-t_0) + \int_{-t_0}^{t} (\tilde{p}(s)y(s-) + \tilde{f}(s))\, \mathrm{d}\tilde{g}(s), \quad t \in [-b, -a],$$

where $\tilde{p}(s) = p(-s)$, $\tilde{f}(s) = -f(-s)$, $\tilde{g}(s) = -g(-s)$. The proof of the fact is similar to the proof in [13, Remark 6.4] (which corresponds to the case $f = 0$). Notice that we have $x(t+) = y(-t-)$, $x(t-) = y(-t+)$, $\Delta^{+}g(t) = \Delta^{-}g(-t)$, and $\Delta^{-}g(t) = \Delta^{+}g(-t)$. Using these relations, it is clear that the conditions guaranteeing that $x(t+) \neq 0$ and $x(t-) \neq 0$ for Eq. (7.10) can obtained from the conditions derived earlier for Eq. (7.9) by interchanging $\Delta^{+}g$ and $\Delta^{-}g$, f and $-f$, and a and b. In this way, we obtain the following counterparts to conditions (7.21)–(7.24):

$$x(t) \neq \frac{f(t)\Delta^{-}g(t)}{1+p(t)\Delta^{-}g(t)}, \quad t \in [t_0, b], \tag{7.25}$$

$$x(t) \neq \frac{f(t)\Delta^{-}g(t)}{1+p(t)\Delta g(t)}, \quad t \in (a, t_0), \tag{7.26}$$

$$x(t) \neq -\frac{f(t)\Delta^{+}g(t)}{1+p(t)\Delta^{-}g(t)}, \quad t \in (t_0, b), \tag{7.27}$$

$$x(t) \neq -f(t)\Delta^{+}g(t), \quad t \in [a, t_0]. \tag{7.28}$$

8 Relations between Stieltjes integral equations and dynamic equations

It has been known for a long time that dynamic equations on time scales represent a special case of Stieltjes integral equations (also known as measure differential equations), see [19]. Hence, it is interesting to check whether the logistic equations obtained in the previous section are consistent with logistic dynamic equations on time scales. In comparison with Section 4, we will discuss both Δ- and ∇-dynamic equations.

Let $\mathbb{T}$ be a time scale. It is convenient to work with a fixed time scale interval $[a,b]_{\mathbb{T}} = [a,b]\cap\mathbb{T}$, where $a,b\in\mathbb{T}$, $a<b$. We need the functions

$$g(t) = \inf\{s\in\mathbb{T} : s\geq t\}, \quad t\in[a,b], \tag{8.1}$$

$$h(t) = \sup\{s\in\mathbb{T} : s\leq t\}, \quad t\in[a,b]. \tag{8.2}$$

The function g is left-continuous, and h is right-continuous.

The relations between Stieltjes integral equations and dynamic equations are described in [15, Section 8.7]. They are based on the following relation (see [15, Corollary 8.6.9]) between Kurzweil–Stieltjes integrals and Henstock–Kurzweil Δ- and ∇-integrals, which were introduced in [16].

Theorem 8.1. *Consider a function $f:[a,b]\to\mathbb{R}$. Then the following statements hold:*

1. *The Henstock–Kurzweil Δ-integral $\int_a^b f(t)\,\Delta t$ exists if and only if the Kurzweil–Stieltjes integral $\int_a^b f(t)\,\mathrm{d}g(t)$ exists; in this case, both integrals have the same value.*
2. *The Henstock-Kurzweil ∇-integral $\int_a^b f(t)\,\nabla t$ exists if and only if the Kurzweil-Stieltjes integral $\int_a^b f(t)\,\mathrm{d}h(t)$ exists; in this case, both integrals have the same value.*

Hence, Δ-dynamic equations on time scales are special cases of Stieltjes integral equations with the integrator g given by Eq. (8.1). In particular, the Δ-dynamic equation

$$x(t) = x(t_0) + \int_{t_0}^t (p(s)x(s) + f(s))\Delta s$$

is a special case of Eq. (7.9); note that a solution x of Eq. (7.9) satisfies $x(s-) = x(s)$ for all s, because g is left-continuous, and therefore x has the same property. The corresponding logistic equation (7.11) given by Theorem 7.5 is then equivalent to the Δ-dynamic equation

$$y(t) = y(t_0) - \int_{t_0}^t (p(s) + f(s)y(s))y(\sigma(s))\Delta s, \tag{8.3}$$

where σ is the forward jump operator. Indeed, if y is a solution of Eq. (7.11), then $y(s-) = y(s)$ (because g is left-continuous). Moreover, g is constant on each interval $(\alpha,\beta]\subset[a,b]$ such that $(\alpha,\beta)\cap\mathbb{T}=\emptyset$. Thus, y has the same property, and $y(s+) = y(\sigma(s))$ for each $s\in[a,b)_{\mathbb{T}}$.

Similarly, the Δ-dynamic equation

$$x(t) = x(t_0) + \int_{t_0}^t (-p(s)x(\sigma(s)) + f(s))\Delta s$$

is a special case of Eq. (7.10). The corresponding logistic equation (7.12) given by Theorem 7.5 is then equivalent to the Δ-dynamic equation

$$y(t) = y(t_0) - \int_{t_0}^t (-p(s) + f(s)y(\sigma(s)))y(s)\Delta s. \tag{8.4}$$

Equations (8.3) and (8.4) are integral forms of the two versions of Δ-dynamic logistic equation described in [3] and mentioned in the introduction of the present paper.

To deal with ∇-dynamic equations, we replace g by the integrator h given by Eq. (8.2). The ∇-dynamic equation

$$x(t) = x(t_0) + \int_{t_0}^{t} (-p(s)x(s) + f(s))\nabla s$$

is then a special case of Eq. (7.10) with g replaced by h; note that a solution x of Eq. (7.10) satisfies $x(s+) = x(s)$ for all s, because h is right-continuous, and therefore x has the same property. The corresponding logistic equation (7.12) given by Theorem 7.5 is then equivalent to the ∇-dynamic equation

$$y(t) = y(t_0) - \int_{t_0}^{t} (-p(s) + f(s)y(s))y(\rho(s))\nabla s, \tag{8.5}$$

where ρ is the backward jump operator. Indeed, if y is a solution of Eq. (7.12), then $y(s+) = y(s)$ (because h is left-continuous). Moreover, h is constant on each interval $[\alpha, \beta) \subset [a, b]$ such that $(\alpha, \beta) \cap \mathbb{T} = \emptyset$. Thus, y has the same property, and $y(s-) = y(\rho(s))$ for each $s \in (a, b]_{\mathbb{T}}$.

Similarly, the ∇-dynamic equation

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(\rho(s)) + f(s))\nabla s$$

is a special case of Eq. (7.9). The corresponding logistic equation (7.11) given by Theorem 7.5 is then equivalent to the ∇-dynamic equation

$$y(t) = y(t_0) - \int_{t_0}^{t} (p(s) + f(s)y(\rho(s)))y(s)\nabla s. \tag{8.6}$$

As far as we are aware, the ∇-dynamic logistic equations (8.5) and (8.6) did not appear in the literature yet.

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Long time behavior of the solution to a chemotaxis system with nonlinear indirect signal production and logistic source

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Abstract. This paper is devoted to studying the following quasilinear parabolic-elliptic-elliptic chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + au - bu^{\gamma}, & x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + w^{\gamma_1}, & x \in \Omega,\ t > 0, \\ 0 = \Delta w - w + u^{\gamma_2}, & x \in \Omega,\ t > 0, \end{cases}$$

with homogeneous Neumann boundary conditions in a bounded and smooth domain $\Omega \subset \mathbb{R}^n (n \geq 1)$, where $a, b, \gamma_2 > 0, \gamma_1 \geq 1, \gamma > 1$ and the functions $\varphi, \psi \in C^2([0, \infty)$ satisfy $\varphi(s) \geq a_0(s+1)^{\alpha}$ and $|\psi(s)| \leq b_0 s(1+s)^{\beta-1}$ for all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. It is proved that if $\gamma - \beta \geq \gamma_1\gamma_2$, the classical solution of system would be globally bounded. Furthermore, a specific model for $\gamma_1 = 1, \gamma_2 = \kappa$ and $\gamma = \kappa + 1$ with $\kappa > 0$ is considered. If $\beta \leq 1$ and $b > 0$ is large enough, there exist $C_\kappa, \mu_1, \mu_2 > 0$ such that the solution(u, v, w) satisfies

$$\left\| u(\cdot,t) - \left(\frac{b}{a}\right)^{\frac{1}{\kappa}} \right\|_{L^\infty(\Omega)} + \left\| v(\cdot,t) - \frac{b}{a} \right\|_{L^\infty(\Omega)} + \left\| w(\cdot,t) - \frac{b}{a} \right\|_{L^\infty(\Omega)} \leq \begin{cases} C_\kappa e^{-\mu_1 t}, & \text{if } \kappa \in (0,1], \\ C_\kappa e^{-\mu_2 t}, & \text{if } \kappa \in (1,\infty), \end{cases}$$

for all $t \geq 0$. The above results generalize some existing results.

Keywords: chemotaxis system, nonlinear indirect secretion, global boundedness, long time behavior.

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1 Introduction

Chemotaxis is one of the basic physiological reactions of cells or organisms, which refers to the directional movement of biological cells or organisms along the concentration gradient of stimulants under the stimulation of some chemicals in the environment. The establishment of chemotactic mathematical model can be traced back to the pioneering work proposed by Keller and Segel [16] to describe the aggregation of cellular slime molds, which is given by

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $\tau \in \{0,1\}$, ν denotes the outward unit normal vector on $\partial\Omega$, $u(x,t)$ denotes the cell density and $v(x,t)$ represents the concentration of the chemical signal. Here, $f(u)$ describes cell proliferation and death, $\nabla \cdot (\varphi(u)\nabla u)$ and $-\nabla \cdot (\psi(u)\nabla v)$ represent self-diffusion and cross-diffusion, respectively. It is well known that chemotaxis research has many important applications in both biology and medicine so that it has been one of the hottest research focuses in applied mathematics nowadays. In the past few decades, a large number of valuable theoretical results have been established. Among them, one of the main issues related to (1.1) is to study whether there is a global in-time bounded solution or when blow-up occurs. For $\tau = 1, \varphi(u) = 1, \psi(u) = \chi u$ and $f(u) = 0$ with $\chi > 0$, it has been shown that the system (1.1) has globally bounded classical solution when $n = 1$ [24] or $n = 2$ and $\int_\Omega u_0 dx < \frac{4\pi}{\chi}$ [5, 23], whereas the system (1.1) has finite time blow-up solution in the case of $n = 2$ and $\int_\Omega u_0 dx > \frac{4\pi}{\chi}$ [9, 26] or in the case of $n \geq 3$ [36, 39]. Inter alia, when $f(u) = u - \mu u^2$ with $\mu > 0$, under the restrictions that $\tau = 1$ and Ω is convex, Winkler [40] proved that if the ratio $\frac{\mu}{\chi}$ is sufficiently large, then the unique nontrivial spatially homogeneous equilibrium given by $u = v \equiv \frac{1}{\mu}$ is globally asymptotically stable. Later on, Cao [2] used an approach based on maximal Sobolev regularity and improved Winkler's results without the restrictions $\tau = 1$ and the convexity of Ω. When the chemical substance diffuses much faster than the diffusion of cells, the system (1.1) can be reduced to the simplified parabolic-elliptic model, i.e. $\tau = 0$. Such model was first studied for $\varphi(u) = 1, \psi(u) = \chi u$ and $f(u) = 0$ in [14]. Recently, when $f(u) = Au - bu^\alpha$ with $\alpha > 1, A \geq 0$ and $b > 0$, in [35], a concept of very weak solutions was introduced, and global existence of such solutions for any nonnegative initial data $u_0 \in L^1(\Omega)$ was proved under the assumption that $\alpha > 2 - \frac{1}{n}$, moreover, boundedness properties of the constructed solutions were studied by Winkler. Thereafter various variants of (1.1) have been considered by many other scholars [6, 11, 31, 34]. In general, diffusion functions $\varphi(u)$ and $\psi(u)$ may not be linear forms, such as diffusion in porous media and volume filling effect. When $\varphi(u), \psi(u)$ are nonlinear and $f(u) = 0$ or $f(u) \neq 0$, a lot of scholars have studied the finite time blow-up of solution and the existence of globally bounded classical solution to system (1.1). We refer the readers to [8, 12, 13, 37, 38] for more details.

With regard to the system (1.1), the term of chemotaxis signal production v is produced directly by the cell density u. However, the mechanism of signal production might be very complex in realistic biological processes. On the one hand, the signal generation usually undergoes intermediate stages, i.e. signal v is not produced directly by cells u, but is governed by some other signal substances w. The related models can be described as

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + w, & x \in \Omega,\ t > 0, \\ \tau w_t = \Delta w - w + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where u, v, w represent the density of cells, the density of chemical substances and the concentration of indirect signal, respectively. Such problem has been widely studied in recent years. For $\tau = 1, \varphi(u) = 1, \psi(u) = u$ and $f(u) = \mu(u - u^\gamma)$, the authors in [46] proved that if $\gamma > \frac{n}{4} + \frac{1}{2}$, then the system possesses globally bounded classical solution. Moreover, if μ is large enough, the solution (u, v, w) converges to $(1, 1, 1)$ in L^∞-norm as $t \to \infty$. When φ and ψ satisfy some nonlinear conditions and smoothness conditions, it also has been showed that the solution to system (1.2) is globally bounded in [30]. Recently, the authors in [18] have studied the system (1.2) for $\tau = 0$, where $\varphi(s) \geq a_0(s+1)^\alpha$ and $|\psi(s)| \leq b_0 s(1+s)^{\beta-1}$ for all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. They have proved that the nonnegative classical solution to (1.2) is global in time and bounded. In addition, if μ satisfy some suitable conditions, the solution (u, v, w) converges to $(1, 1, 1)$ in L^∞-norm as $t \to \infty$. More relevant results on the system with indirect signal production can refer to [10, 19].

One the other hand, the signal generation may be in a nonlinear form, which is given by

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + g(u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded, smooth domain. When $\tau = 0, \varphi(u) = 1, \psi(u) = \chi u, f(u) = au - bu^\theta$ and $g(u) = u^\kappa$ with $\chi, b, \kappa > 0, a \in \mathbb{R}$ and $\theta > 1$, Xiang [44] obtained the global existence and boundedness of solution for (1.3) under either $\kappa + 1 < \max\{\theta, 1 + \frac{2}{n}\}$ or $\theta = \kappa + 1, b \geq \frac{(\kappa n - 2)}{kn}\chi$. Besides, they studied the dynamical behavior of the solution on the interactions among nonlinear cross-diffusion, generalized logistic source and signal production. In addition, When $\tau = 1, \varphi(u) = 1, \psi(u) = \chi u, f(u) = 0$ and $g(u) \in C^1([0, \infty))$ satisfying $0 \leq g(u) \leq Ku^\alpha$ with some constants $K, \alpha > 0$, Liu and Tao [21] proved that the classical solution of the system (1.3) is globally bounded if $0 < \alpha < \frac{2}{n}$. When the second equation degenerates into an elliptic equation (i.e. $\tau = 0$), $\varphi(u) = 1, \psi(u) = \chi u, f(u) = 0$ and v is replaced by $\mu(t) = \frac{1}{|\Omega|}\int_\Omega g(u), g(u) \geq ku^k$ for all $u \geq 0$ with some $k > 0$, Winkler [43] derived a blow-up critical exponent $k = \frac{2}{n}$, which asserted that the radially symmetric solution blows up in finite time if the parameter k satisfies $k > \frac{2}{n}$. Conversely, when $k < \frac{2}{n}$, they proved that there exists suitable initial value such that the system has globally bounded classical solution. Later on, the authors in [45] considered the case $f(u) = \lambda u - \mu u^\alpha$ with $\lambda, \mu > 0$ and $\alpha > 1$, and they generalized the blow-up results developed in [43] with $k + 1 > \alpha\left(\frac{2}{n} + 1\right)$. Intuitively, the existing literatures show that the logistics source (i.e. $f(u) = \lambda u - \mu u^\alpha$ with $\lambda, \mu > 0$ and $\alpha > 1$) and its possibly damping behavior have important influences on the behavior of the solution. For instance, the strong logistic damping (i.e. μ is suitably large) may ensure the system has globally bounded classical solution, especially in higher-dimensional case. More precisely, when $\alpha = 2$, Tello and Winkler [29] proved that for all suitably regular initial data, the system had a unique globally bounded classical solution if $\mu > \max\{0, \frac{n-2}{n}\chi\}$. Afterwards, Cao and Zheng [3] proved that such global solution to a quasilinear system (1.3) is also known to exist for all nonnegative and smooth initial data if μ is suitably large. However, "logistic source" does not always prevent chemotactic collapse. When $\alpha = 2$, such assertion was verified in [41] for one-dimensional case by Winkler, and also could be found in [15] for higher-dimensional setting. Recently, Winkler [42] obtained a condition on initial data to ensure the occurrence of finite-time blow-up to system (1.3) for

$$\alpha < \begin{cases} \frac{7}{6}, & \text{if } n \in \{3,4\}, \\ 1 + \frac{1}{2(n-1)}, & \text{if } n \geq 5. \end{cases}$$

Some boundedness or blow-up results to variants of system (1.3) can also be found in [20, 22, 25, 32, 33, 47].

Among the existing literatures, it is not difficult to find that there are very few papers to study the chemotaxis system, where chemical signal production is not only indirect but also nonlinear. Based on the complexity of biological process, such signal production mechanism could be more in line with the actual situation. Inspired by the above works, in this paper, we are concerned with the following system

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + au - bu^\gamma, & x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + w^{\gamma_1}, & x \in \Omega,\ t > 0, \\ 0 = \Delta w - w + u^{\gamma_2}, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary, ν denotes the outward unit normal vector on $\partial\Omega$, the parameters satisfy $a, b, \gamma_2 > 0, \gamma_1 \geq 1$ and $\gamma > 1$, and $\varphi(u), \psi(u)$ are self-diffusion and cross-diffusion functions, respectively. Since from a physical point of view, the equation modeling the migration of cells should rather be regarded as nonlinear diffusion [27]. Thus, here we assume that the diffusion functions $\varphi, \psi \in C^2[0, \infty)$ fulfill

$$\varphi(s) \geq a_0(s+1)^\alpha \tag{1.5}$$

and

$$|\psi(s)| \leq b_0 s(s+1)^{\beta-1}, \tag{1.6}$$

for all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$.

The main purpose of the present paper is to explore the interplay of nonlinear diffusion functions φ, ψ and logistic source term $au - bu^\gamma$ as well as nonlinear indirect signal production mechanism for system (1.4). To the best of our knowledge, studying the fully parabolic chemotaxis system need to use the method of variation-of-constants formula and heat semigroup, which can not be applied to the system (1.4). In this paper, we shall use a different method to reveal the influence of nonlinear diffusion functions φ, ψ and logistic source term $au - bu^\gamma$ as well as nonlinear indirect signal production mechanism on the dynamical behavior of the solution to system (1.4).

Firstly, we state our boundedness result to system (1.4) as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded and smooth domain. Assume that $a, b, \gamma_2 > 0, \gamma > 1, \gamma_1 \geq 1$ and functions $\varphi, \psi \in C^2[0, \infty)$ with $\varphi(s) \geq a_0(s+1)^\alpha$ and $|\psi(s)| \leq b_0 s(s+1)^{\beta-1}$ for* *all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. If $\gamma - \beta \geq \gamma_1\gamma_2$, then for any nonnegative initial data $0 \not\equiv u_0 \in C(\bar{\Omega})$, the system (1.4) admits a unique nonnegative classical solution (u, v, w) belonging to $C[(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))]$. Moreover, the solution of system (1.4) is bounded in $\Omega \times (0,\infty)$, namely, there exists a constant $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \tag{1.7}$$

for all $t > 0$.

In contrast to the boundedness criterion obtained in [18], the boundedness condition in Theorem 1.1 is more generalized involving nonlinear diffusion and logistic source term as well as nonlinear indirect signal production mechanism.

From the viewpoint of biological evolution, it has profound theoretical and practical significance to study the long time behavior of chemotaxis system. Based on [7,18,44], we have also studied the long time behavior of solution to a special case (see system (3.1) in Section 3) of system 1.1 (i.e. $\gamma_1 = 1, \gamma_2 = \kappa$ and $\gamma = \kappa + 1$ with $\kappa > 0$). Here, it should be pointed out that from the above Theorem 1.1 if $\beta \leq 1$, the corresponding system has globally bounded classical solution for this case. Thus, from Theorem 1.1, there exists $R > 0$ independent of $a, b, \alpha, \beta, a_0, b_0$ and κ such that

$$u(x,t) \leq R \tag{1.8}$$

holds on $\bar{\Omega} \times [0,\infty)$. Moreover, we can also find $\lambda > 0$ independent of a, b, a_0, b_0 and κ such that

$$(u+1)^{2\beta-\alpha-2} \leq \lambda \tag{1.9}$$

holds on $\bar{\Omega} \times [0,\infty)$.

Therefore, the second conclusion of this paper can be stated as

Theorem 1.2. *Let $0 \leq u_0 \in C(\bar{\Omega})$ and $a, b, \kappa > 0$. Assume that functions $\varphi, \psi \in C^2[0,\infty)$ with $\varphi(s) \geq a_0(s+1)^\alpha$ and $|\psi(s)| \leq b_0 s(s+1)^{\beta-1}$ for all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. If $\beta \leq 1$ and*

$$\begin{cases} b > \frac{b_0}{4}\sqrt{\frac{\lambda a}{a_0}}, & \kappa \in (0,1], \\ b > \dfrac{\lambda b_0^2\left[(\kappa-1)R^\kappa + \sqrt{(\kappa-1)^2R^{2\kappa} + \frac{64aa_0}{\lambda b_0^2}}\right]}{32a_0}, & \kappa \in (1,\infty), \end{cases} \tag{1.10}$$

then there exists $C_\kappa > 0$ large enough such that the classical solution (u, v, w) to system (3.1) satisfies

$$\left\|u(\cdot,t) - \left(\frac{b}{a}\right)^{\frac{1}{\kappa}}\right\|_{L^\infty(\Omega)} + \left\|v(\cdot,t) - \frac{b}{a}\right\|_{L^\infty(\Omega)} + \left\|w(\cdot,t) - \frac{b}{a}\right\|_{L^\infty(\Omega)} \leq \begin{cases} C_\kappa e^{-\mu_1 t}, & \kappa \in (0,1], \\ C_\kappa e^{-\mu_2 t}, & \kappa \in (1,\infty), \end{cases}$$

for all $t \geq 0$, where

$$\mu_1 = \frac{\kappa a}{(n+2)b^2}\left(b^2 - \frac{\lambda a b_0^2}{16a_0}\right) \tag{1.11}$$

and

$$\mu_2 = \frac{\kappa\left(\frac{a}{b}\right)^{\frac{2-\kappa}{\kappa}}}{(n+2)}\left\{b - \frac{\lambda b_0^2}{16a_0}\left[\frac{a}{b} + (\kappa-1)R^\kappa\right]\right\}, \tag{1.12}$$

with $R > 0$ and $\lambda > 0$ defined in (1.8) and (1.9), respectively.

The results in Theorem 1.2 are similar to those in [44, Theorem 5.1(i)], but more general, since self-diffusion, cross-diffusion and indirect secretion mechanism are involved. We need to modify the method in [44] to overcome the difficulties from these terms (see (3.10) and (3.25) in the proof of Lemma 3.2). In addition, our conclusion in Theorem 1.2 can also be seen as an extension of [7] or [18]. Comparing with [7], in Theorem 1.2, we calculate the exponential convergence rate explicitly in terms of the model parameters with diffusion functions, generalized logistic source and nonlinear indirect secretion. But in [7], the convergence rate estimates were derived but not stated explicitly (see [7, Theorem 1]) for special logistic source and linear secretion. Comparing with [18], since our model is nonlinear indirect production, we have to divide the range of κ into $(0,1]$ and $(1,+\infty)$ to construct different functionals $A(t)$ and $H(t)$ (see Lemma 3.2) to prove Theorem 1.2.

Remark 1.3. It is relevant to point out that by the limitation of the method, we also have no idea the long time behavior of solution to system (1.4) for generalized parameters γ_1, γ_2 and γ satisfying the condition in Theorem 1.1.

The outline of this paper is as follows. In Section 2, the global existence and boundedness of classical solution to (1.4) is proved. In Section 3, by applying the method of energy functional, we obtain that the solution to system (3.1) exponentially converges to the point $((\frac{a}{b})^{\frac{1}{\kappa}}, \frac{a}{b}, \frac{a}{b})$ as $t \to \infty$.

2 Global existence and boundedness

In this section, we will obtain the existence and boundedness of globally classical solution to system (1.4). At the beginning, we give a statement on the local existence of classical solutions. The proof depends on the Schauder fixed theorem. We omit it for brevity and refer the readers to [30] for more details.

Lemma 2.1. *Let $a, b, \gamma_2 > 0, \gamma_1 \geq 1, \gamma > 1$ and $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded and smooth domain. Assume that $\varphi, \psi \in C^2[0,\infty)$ satisfy (1.5) and (1.6), respectively. For any nonnegative initial data $0 \not\equiv u_0 \in C(\bar{\Omega})$, there exists $T_{\max} \in (0,\infty]$ such that the system (1.4) admits a unique nonnegative classical solution (u,v,w) belonging to $C[(\bar{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\max}))]$ in $\Omega \times (0,T_{\max})$ with*

$$u, v, w \geq 0 \quad \text{in } \bar{\Omega} \times (0, T_{\max}). \tag{2.1}$$

Furthermore,

$$\text{if} \quad T_{\max} < \infty, \quad \text{then} \quad \lim_{t \nearrow T_{\max}} \sup \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.2}$$

Lemma 2.2. *Let $a, b, \gamma_2 > 0, \gamma_1 \geq 1, \gamma > 1$ and (u,v,w) be a solution of system (1.4). Assume that $\varphi, \psi \in C^2[0,\infty)$ satisfy (1.5) and (1.6). Then for any $\eta_1, \eta_2 > 0$ and $\theta > 1$, there exist $c_0, c_1 > 0$ depending only on $\gamma_1, \gamma_2, \eta_1, \eta_2, \theta$ such that*

$$\int_\Omega w^\theta \leq \eta_2 \int_\Omega (u+1)^{\gamma_2 \theta} + c_0 \tag{2.3}$$

and

$$\int_\Omega v^\theta \leq \eta_1 \eta_2 \int_\Omega (u+1)^{\gamma_1 \gamma_2 \theta} + c_1, \tag{2.4}$$

for all $t \in (0, T_{\max})$.

Proof. Integrating the first equation of system (1.4) over Ω, we find

$$\frac{d}{dt}\int_\Omega udx = \int_\Omega au - bu^\gamma \le a\int_\Omega u - \frac{b}{|\Omega|^{\gamma-1}}\left(\int_\Omega u\right)^\gamma \quad \text{for all } t \in (0, T_{\max}), \tag{2.5}$$

where we have used Hölder's inequality. Thus, using a standard ODE comparison theory, it shows that

$$\int_\Omega u \le \max\left\{\int_\Omega u_0, \left(\frac{a}{b}\right)^{\frac{1}{\gamma-1}}|\Omega|\right\} \quad \text{for all } t \in (0, T_{\max}). \tag{2.6}$$

Moreover, we can derive directly by integrating the third equation over Ω,

$$\|w\|_{L^1(\Omega)} = \|u^{\gamma_2}\|_{L^1(\Omega)} \le \|(u+1)^{\gamma_2}\|_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\max}). \tag{2.7}$$

Multiplying the third equation of system(1.4) with $w^{\theta-1}$ and integrating by parts over Ω, we can get

$$\frac{4(\theta-1)}{\theta^2}\int_\Omega |\nabla w^{\frac{\theta}{2}}|^2 + \int_\Omega w^\theta = \int_\Omega u^{\gamma_2}w^{\theta-1} \le \frac{\theta-1}{\theta}\int_\Omega w^\theta + \frac{1}{\theta}\int_\Omega u^{\gamma_2\theta} \tag{2.8}$$

by Young's inequality. Hence

$$\|w\|_{L^\theta(\Omega)} \le \|u^{\gamma_2}\|_{L^\theta(\Omega)} \le \|(u+1)^{\gamma_2}\|_{L^\theta(\Omega)} \quad \text{for all } t \in (0, T_{\max}) \tag{2.9}$$

and

$$\frac{4(\theta-1)}{\theta}\int_\Omega |\nabla w^{\frac{\theta}{2}}|^2 \le \int_\Omega u^{\gamma_2\theta} \le \int_\Omega (u+1)^{\gamma_2\theta} \quad \text{for all } t \in (0, T_{\max}). \tag{2.10}$$

By Ehrling's lemma, for any $\eta_2 > 0, \theta > 1$ and function $\phi \in W^{1,2}(\Omega)$, there exists $C_0 = C_0(\eta_2, \theta) > 0$ such that

$$\|\phi\|^2_{L^2(\Omega)} \le \eta_2\|\phi\|^2_{W^{1,2}(\Omega)} + C_0\|\phi\|^\theta_{L^{\frac{2}{\theta}}(\Omega)}. \tag{2.11}$$

Let $\phi = w^{\frac{\theta}{2}}$, from (2.7),(2.9) and (2.10), there exists $C_1 = C_1(\eta_2, \theta) > 0$ such that

$$\int_\Omega w^\theta \le \eta_2\int_\Omega (u+1)^{\gamma_2\theta} + C_1\|(u+1)^{\gamma_2}\|^\theta_{L^1(\Omega)}. \tag{2.12}$$

For $\gamma_2 \in (0,1]$, using Hölder's inequality, one may obtain from (2.6)

$$\|(u+1)^{\gamma_2}\|^\theta_{L^1(\Omega)} \le C_2 \tag{2.13}$$

with $C_2 = C_2(\eta_2, \theta, \gamma_2) > 0$. For $\gamma_2 \in (1, \infty)$, using interpolation inequality and Young's inequality, from (2.6) we deduce

$$\|(u+1)^{\gamma_2}\|^\theta_{L^1(\Omega)} \le \|(u+1)^{\gamma_2}\|^{\theta\tau}_{L^\theta(\Omega)}\|(u+1)^{\gamma_2}\|^{\theta(1-\tau)}_{L^{\frac{1}{\gamma_2}}(\Omega)} \le \eta_2\int_\Omega (u+1)^{\gamma_2\theta} + C_3 \tag{2.14}$$

where $\tau = \frac{\gamma_2 - 1}{\gamma_2 - \frac{1}{\theta}} \in (0,1)$ and $C_3 = C_3(\eta_2, \theta, \gamma_2) > 0$. Thus (2.3) is the direct result of combining (2.12)–(2.14). Similarly, multiplying the second equation of system(1.4) with $v^{\theta-1}$, by the same procedure as above, we can obtain for any $\eta_1 > 0$ and $\theta > 1$

$$\int_\Omega v^\theta \le \eta_1\int_\Omega w^{\gamma_1\theta} + C_4 \quad \text{for all } t \in (0, T_{\max}) \tag{2.15}$$

with $C_4 = C_4(\eta_1, \theta, \gamma_1) > 0$. Since $\gamma_1 \ge 1$, we can obtain from (2.3)

$$\int_\Omega w^{\gamma_1\theta} \le \eta_2\int_\Omega (u+1)^{\gamma_1\gamma_2\theta} + C_5 \quad \text{for all } t \in (0, T_{\max}) \tag{2.16}$$

with $C_5 > 0$. Combining (2.15)–(2.16) yields (2.4). This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $a, b, \gamma_2 > 0, \gamma_1 \geq 1, \gamma > 1$ and (u, v, w) be a solution of system (1.4). Assume that functions $\varphi, \psi \in C^2[0,\infty)$ satisfying (1.5) and (1.6) for all $s \geq 0$ with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. If $\gamma - \beta \geq \gamma_1\gamma_2$, then for any $p > \max\{1, 1-\beta\}$, there exists a constant $C > 0$ such that*

$$\int_\Omega (u+1)^p \leq C \tag{2.17}$$

for all $t \in (0, T_{\max})$.

Proof. Multiplying the first equation of system (1.4) by $(u+1)^{p-1}$ and integrating by parts over Ω, we derive

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p = & -(p-1)\int_\Omega (u+1)^{p-2}\varphi(u)|\nabla u|^2 + (p-1)\int_\Omega (u+1)^{p-2}\psi(u)\nabla u\cdot\nabla v \\ & + a\int_\Omega u(u+1)^{p-1} - b\int_\Omega u^\gamma (u+1)^{p-1}\end{aligned} \tag{2.18}$$

for all $t \in (0, T_{\max})$. Since φ satisfies (1.5), we can estimate the first term on the right-hand side of (2.18) as

$$\begin{aligned}-(p-1)\int_\Omega (u+1)^{p-2}\varphi(u)|\nabla u|^2 \leq & -(p-1)\int_\Omega a_0(1+u)^\alpha (1+u)^{p-2}|\nabla u|^2 \\ \leq & -\frac{4a_0(p-1)}{(p+\alpha)^2}\int_\Omega |\nabla (u+1)^{\frac{p+\alpha}{2}}|^2\end{aligned} \tag{2.19}$$

for all $t \in (0, T_{\max})$. Let $\Psi(u) = \int_0^u (\xi+1)^{p-2}\psi(\xi)d\xi$, thus

$$\nabla\Psi(u) = (u+1)^{p-2}\psi(u)\nabla u \tag{2.20}$$

and

$$|\Psi(u)| \leq \frac{b_0}{\beta+p-1}(u+1)^{\beta+p-1} \tag{2.21}$$

for all $t \in (0, T_{\max})$. From (2.20) and (2.21), we can get

$$\begin{aligned}(p-1)\int_\Omega (u+1)^{p-2}\psi(u)\nabla u\cdot\nabla v &= (p-1)\int_\Omega \nabla\Psi(u)\cdot\nabla v = -(p-1)\int_\Omega \Psi(u)\Delta v \\ &\leq (p-1)\int_\Omega \Psi(u)|\Delta v| \leq \frac{b_0(p-1)}{\beta+p-1}\int_\Omega (u+1)^{\beta+p-1}|\Delta v|\end{aligned} \tag{2.22}$$

for all $t \in (0, T_{\max})$. By the basic inequality $(u+1)^\gamma < 2^\gamma(u^\gamma+1)$ with $\gamma > 1$, we have

$$-b\int_\Omega u^\gamma (u+1)^{p-1} \leq -\frac{b}{2^\gamma}\int_\Omega (u+1)^{p+\gamma-1} + b\int_\Omega (u+1)^{p-1} \tag{2.23}$$

for all $t \in (0, T_{\max})$. Denoting $m_0 = \max\{a, b\}$, from (2.18)–(2.19) and (2.22)–(2.23), we can get

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p \leq & -\frac{4a_0(p-1)}{(p+\alpha)^2}\int_\Omega |\nabla (u+1)^{\frac{p+\alpha}{2}}|^2 + \frac{b_0(p-1)}{\beta+p-1}\int_\Omega (u+1)^{\beta+p-1}|\Delta v| \\ & + m_0\int_\Omega u(u+1)^{p-1} - \frac{b}{2^\gamma}\int_\Omega (u+1)^{p+\gamma-1} + m_0\int_\Omega (u+1)^{p-1} \\ \leq & \frac{b_0(p-1)}{\beta+p-1}\int_\Omega (u+1)^{\beta+p-1}|v - w^{\gamma_1}| + m_0\int_\Omega (u+1)^p - \frac{b}{2^\gamma}\int_\Omega (u+1)^{p+\gamma-1} \\ \leq & \frac{b_0(p-1)}{\beta+p-1}\int_\Omega (u+1)^{\beta+p-1}v + \frac{b_0(p-1)}{\beta+p-1}\int_\Omega (u+1)^{\beta+p-1}w^{\gamma_1} \\ & + m_0\int_\Omega (u+1)^p - \frac{b}{2^\gamma}\int_\Omega (u+1)^{p+\gamma-1}\end{aligned} \tag{2.24}$$

for all $t \in (0, T_{\max})$, where we have made use of the second identity $0 = \Delta v - v + w^{\gamma_1}$ in system (1.4). In the sequel, we estimate (2.24) in two different cases.

Case 1 ($\gamma - \beta > \gamma_1\gamma_2$). In this case, using Young's inequality, we can derive

$$\int_\Omega (u+1)^{\beta+p-1} w^{\gamma_1} \le \frac{b(\beta+p-1)}{2^{\gamma+3} b_0(p-1)} \int_\Omega (u+1)^{p+\gamma-1} + C_6 \int_\Omega w^{\frac{(p+\gamma-1)\gamma_1}{\gamma-\beta}} \tag{2.25}$$

with $C_6 = \left(\frac{2^{\gamma+3} b_0(p-1)}{b(\gamma+p-1)}\right)^{\frac{\gamma+p-1}{\gamma-\beta}}$. Since $\gamma - \beta > \gamma_1\gamma_2$, with applications of Young's inequality, we get from Lemma 2.2 with $\theta = \frac{p+\gamma-1}{\gamma_2} > 1$

$$\begin{aligned} \int_\Omega w^{\frac{(p+\gamma-1)\gamma_1}{\gamma-\beta}} &\le \frac{b(\beta+p-1)}{2^{\gamma+3} C_6 b_0 \eta_2 (p-1)} \int_\Omega w^{\frac{p+\gamma-1}{\gamma_2}} + C_7 \\ &\le \frac{b(\beta+p-1)}{2^{\gamma+3} C_6 b_0 (p-1)} \int_\Omega (u+1)^{p+\gamma-1} + C_8, \end{aligned} \tag{2.26}$$

where $C_8 = C_7 + c_0$ with $C_7 = \left(\frac{2^{\gamma+3} C_6 b_0 \eta_2 (p-1)}{b(\beta+p-1)}\right)^{\frac{\gamma_1\gamma_2}{\gamma-\beta-\gamma_1\gamma_2}} |\Omega|$. Similarly, we have

$$\int_\Omega (u+1)^{\beta+p-1} v \le \frac{b(\beta+p-1)}{2^{\gamma+3} b_0(p-1)} \int_\Omega (u+1)^{p+\gamma-1} + C_9 \int_\Omega v^{\frac{p+\gamma-1}{\gamma-\beta}}. \tag{2.27}$$

Since $\gamma_1 \ge 1$, in view of Young's inequality, we can obtain from Lemma 2.2 with $\theta = \frac{p+\gamma-1}{\gamma-\beta} > 1$

$$\begin{aligned} \int_\Omega v^{\frac{p+\gamma-1}{\gamma-\beta}} &\le \eta_1\eta_2 \int_\Omega (u+1)^{\frac{(p+\gamma-1)\gamma_2}{\gamma-\beta}} + c_1 \\ &\le \eta_1\eta_2 \left(\frac{1}{\eta_1\eta_2} \int_\Omega (u+1)^{\frac{p+\gamma-1}{\gamma_1}} + (\eta_1\eta_2)^{\frac{\gamma-\beta-\gamma_1\gamma_2}{\gamma_1\gamma_2}} |\Omega| \right) + c_1 \\ &\le \int_\Omega (u+1)^{\frac{p+\gamma-1}{\gamma_1}} + (\eta_1\eta_2)^{1+\frac{\gamma-\beta-\gamma_1\gamma_2}{\gamma_1\gamma_2}} |\Omega| + c_1 \\ &\le \frac{b(\beta+p-1)}{2^{\gamma+3} b_0(p-1) C_9} \int_\Omega (u+1)^{p+\gamma-1} + C_{10} \end{aligned} \tag{2.28}$$

with $C_{10} = \left(\frac{2^{\gamma+3} b_0(p-1) C_9}{b(\gamma+p-1)}\right)^{\frac{1}{\gamma_1-1}} + (\eta_1\eta_2)^{1+\frac{\gamma-\beta-\gamma_1\gamma_2}{\gamma_1\gamma_2}} |\Omega| + c_1$. Since $\gamma > 1$, using Young's inequality, there exists $C_{11} = \frac{b}{2^{\gamma+2}(m_0+1)}$ such that

$$\int_\Omega (u+1)^p \le C_{11} \int_\Omega (u+1)^{p+\gamma-1} + C_{12} \tag{2.29}$$

with $C_{12} = \left(\frac{2^{\gamma+2}(m_0+1)}{b}\right)^{\frac{p}{\gamma-1}} |\Omega|$. Using (2.24)–(2.29), we can obtain

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega (u+1)^p + \int_\Omega (u+1)^p \\ &\quad \le \frac{b_0(p-1)}{\beta+p-1} \int_\Omega (u+1)^{\beta+p-1} v + \frac{b_0(p-1)}{\beta+p-1} \int_\Omega (u+1)^{\beta+p-1} w^{\gamma_1} + (m_0+1) \int_\Omega (u+1)^p \\ &\quad\quad - \frac{b}{2^\gamma} \int_\Omega (u+1)^{p+\gamma-1} \\ &\quad \le \frac{b_0(p-1)}{\beta+p-1} \left[\frac{b(\beta+p-1)}{2^{\gamma+2} b_0(p-1)} \int_\Omega (u+1)^{p+\gamma-1} + C_9 \int_\Omega v^{\frac{p+\gamma-1}{\gamma-\beta}} + C_6 \int_\Omega w^{\frac{(p+\gamma-1)\gamma_1}{\gamma-\beta}} \right] \\ &\quad\quad + (m_0+1)\left(C_{11} \int_\Omega (u+1)^{p+\gamma-1} + C_{12} \right) - \frac{b}{2^\gamma} \int_\Omega (u+1)^{p+\gamma-1} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{b_0(p-1)}{\beta+p-1}\left[\frac{b(\beta+p-1)}{2^{\gamma+1}b_0(p-1)}\int_\Omega (u+1)^{p+\gamma-1}+C_8+C_{10}\right] \\
&\quad -\frac{3b}{2^{\gamma+2}}\int_\Omega (u+1)^{p+\gamma-1}+C_{12}(m_0+1) \\
&\leq -\frac{b}{2^{\gamma+2}}\int_\Omega (u+1)^{p+\gamma-1}+(C_8+C_{10})\frac{b_0(p-1)}{\beta+p-1}+C_{12}(m_0+1) \qquad (2.30)
\end{aligned}$$

for all $t \in (0, T_{\max})$, which means that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u+1)^p + p\int_\Omega (u+1)^p \\
&\qquad \leq -\frac{bp}{2^{\gamma+2}}\int_\Omega (u+1)^{p+\gamma-1}+(C_8+C_{10})\frac{b_0p(p-1)}{\beta+p-1}+C_{12}p(m_0+1).
\end{aligned} \qquad (2.31)$$

Thus we can get the conclusion immediately by the ODE comparison principle.

Case 2 ($\gamma-\beta=\gamma_1\gamma_2$). Recalling (2.25) and (2.27), we know

$$\int_\Omega (u+1)^{\beta+p-1}w^{\gamma_1} \leq \frac{b(\beta+p-1)}{2^{\gamma+3}b_0(p-1)}\int_\Omega (u+1)^{p+\gamma-1}+C_6\int_\Omega w^{\frac{(p+\gamma-1)\gamma_1}{\gamma-\beta}} \qquad (2.32)$$

and

$$\int_\Omega (u+1)^{\beta+p-1}v \leq \frac{b(\beta+p-1)}{2^{\gamma+3}b_0(p-1)}\int_\Omega (u+1)^{p+\gamma-1}+C_9\int_\Omega v^{\frac{p+\gamma-1}{\gamma-\beta}}. \qquad (2.33)$$

Since $\gamma-\beta=\gamma_1\gamma_2$, for any $\eta_1,\eta_2>0$, we can obtain from Lemma 2.2

$$\int_\Omega w^{\frac{(p+\gamma-1)\gamma_1}{\gamma-\beta}} = \int_\Omega w^{\frac{(p+\gamma-1)}{\gamma_2}} \leq \eta_2\int_\Omega (u+1)^{p+\gamma-1}+c_0 \qquad (2.34)$$

and

$$\int_\Omega v^{\frac{p+\gamma-1}{\gamma-\beta}} = \int_\Omega v^{\frac{p+\gamma-1}{\gamma_1\gamma_2}} \leq \eta_1\eta_2\int_\Omega (u+1)^{p+\gamma-1}+c_1 \qquad (2.35)$$

for all $t \in (0, T_{\max})$. Because of the arbitrariness of η_1 and η_2, we choose $\eta_2 = \frac{b(\beta+p-1)}{2^{\gamma+3}C_6b_0(p-1)}$ and $\eta_1\eta_2 = \frac{b(\beta+p-1)}{2^{\gamma+3}C_9b_0(p-1)}$ in (2.34) and (2.35), respectively. From (2.24), (2.29) and (2.32)–(2.35), we can obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u+1)^p + p\int_\Omega (u+1)^p \\
&\qquad \leq -\frac{bp}{2^{\gamma+2}}\int_\Omega (u+1)^{p+\gamma-1}+(c_0C_6+c_1C_9)\frac{b_0p(p-1)}{\beta+p-1}+C_{12}p(m_0+1), \qquad (2.36)
\end{aligned}$$

for all $t \in (0, T_{\max})$. Using the ODE comparison principle, we can prove the conclusion. The proof of Lemma 2.3 is completed. □

Proof of Theorem 1.1. Let $a, b, \gamma_2 > 0, \gamma_1 \geq 1, \gamma > 1$ and (u, v, w) be a solution of system (1.4). From Lemma 2.3, for any $p > \max\{1, 1-\beta\}$, there exists $C_{13} > 0$ such that $\|u\|_{L^p(\Omega)} \leq C_{13}$ for all $t \in (0, T_{\max})$. By the elliptic L^p-estimate applied to the second and third equations in system (1.4), we have

$$\|w(\cdot,t)\|_{W^{2,p/\gamma_2}(\Omega)} + \|v(\cdot,t)\|_{W^{2,p/\gamma_1\gamma_2}(\Omega)} \leq C_{14} \qquad (2.37)$$

for all $t \in (0, T_{\max})$, with some $C_{14} > 0$. Using the Sobolev imbedding theorem, we can get

$$\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C_{15} \tag{2.38}$$

for all $t \in (0, T_{\max})$, with some $C_{15} > 0$. Thus by standard Alikakos–Moser iteration ([28, Lemma A.1]), we can find a constant $C_{16} > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{16}$$

for all $t \in (0, T_{\max})$, which together with Lemma 2.1 implies that $T_{\max} = \infty$. Hence, by standard elliptic regularity theory, we know that (u, v, w) is a globally bounded classical solution of system (1.4). The proof of Theorem 1.1 is completed. □

3 Long time behavior of the solution for a specific model

In this section, we shall study the long time behavior of the solution for a specific model (i.e. $\gamma_1 = 1, \gamma_2 = \kappa$ and $\gamma = \kappa + 1$ with $\kappa > 0$) with nonlinear indirect signal production and logistic source as follows

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + u(a - bu^\kappa), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + w, & x \in \Omega,\ t > 0, \\ 0 = \Delta w - w + u^\kappa, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded and smooth domain, the parameters $a, b, \kappa > 0$ and functions $\varphi, \psi \in C^2[0, \infty)$ satisfy conditions (1.5) and (1.6), respectively.

Based on Theorem 1.1, it is easy to check that if $\beta \leq 1$, then the system (3.1) admits a unique globally bounded classical solution (u, v, w). Furthermore, such classical solution (u, v, w) may be strictly positive which can be ensured by choosing some suitable $0 \leq u_0 \in C(\bar{\Omega})$ from Theorem 1.1. Thus let us assume that the classical solution (u, v, w) to system (3.1) is strictly positive throughout the proof of Theorem 1.2. For the convenience, we repeat the description stated in (1.8) and (1.9), i.e. there exists $R > 0$ which does not depend on $a, b, \alpha, \beta, a_0, b_0$ and κ such that

$$0 < u(x,t) \leq R \tag{3.2}$$

holds on $\bar{\Omega} \times [0, \infty)$. Moreover, we can also find $\lambda > 0$ independent of a, b, a_0, b_0 and κ such that

$$(u+1)^{2\beta - \alpha - 2} \leq \lambda \tag{3.3}$$

holds on $\bar{\Omega} \times [0, \infty)$.

In order to prove Theorem 1.2, we introduce a useful lemma.

Lemma 3.1 (cf. [1, Lemma 3.1.]). *Let $g : (t_0, \infty) \to [0, \infty)$ be uniformly continuous such that $\int_{t_0}^\infty g(t)dt < \infty$ with $t_0 > 0$. Then*

$$g(t) \to 0 \quad \text{as } t \to \infty. \tag{3.4}$$

The key to prove Theorem 1.2 relies on seeking so-called Lyapunov functional inspired from [1, 7]. In the following, we need to construct appropriate energy functionals to system (3.1), which is prepared for the proof of Theorem 1.2.

Lemma 3.2. *Let $0 \le u_0 \in C(\bar{\Omega})$ and $a, b, \kappa > 0$. Assume that $\varphi, \psi \in C^2[0,\infty)$ satisfy (1.5) and (1.6) with $a_0, b_0 > 0$ and $\alpha, \beta \in \mathbb{R}$. If $\beta \le 1$ and the condition (1.10) in Theorem 1.2 holds, then the solution (u, v, w) has the following L^2−convergence*

$$\int_\Omega \left(u - (\frac{b}{a})^{\frac{1}{\kappa}}\right)^2 + \int_\Omega \left(v - \frac{b}{a}\right)^2 + \int_\Omega \left(w - \frac{b}{a}\right)^2 \to 0 \quad as\ t \to \infty. \tag{3.5}$$

Proof. For $\kappa \in (0, 1]$, we define the functional

$$A(t) = \int_\Omega u - c - c\ln\left(\frac{u}{c}\right), \quad t > 0, \tag{3.6}$$

for $u > 0$, with $c = (\frac{a}{b})^{\frac{1}{\kappa}}$. By taking derivative, we can easily obtain that $a(s) = s - c - c\ln(\frac{s}{c})$ with $s > 0$ has global minimum zero at $s = c$. Hence, $A(t) \ge 0$ for all $t \ge 0$.

Using Young's inequality and the fact (3.3), we deduce from the first equation of system (3.1)

$$\begin{aligned}
\frac{d}{dt}A(t) =& \int_\Omega \frac{u-c}{u} u_t \\
=& \int_\Omega \frac{u-c}{u} [\nabla \cdot (\varphi(u)\nabla u - \psi(u)\nabla v) + u(a - bu^\kappa)] \\
=& -c\int_\Omega \varphi(u)\frac{|\nabla u|^2}{u^2} + c\int_\Omega \psi(u)\frac{\nabla u \cdot \nabla v}{u^2} - b\int_\Omega (u-c)(u^\kappa - c^\kappa) \\
\le& -a_0 c\int_\Omega (u+1)^\alpha \frac{|\nabla u|^2}{u^2} + \frac{a_0 c}{\lambda}\int_\Omega (u+1)^{2\beta}\frac{|\nabla u|^2}{u^2} + \frac{\lambda b_0^2 c}{4a_0}\int_\Omega |\nabla v|^2 \\
&- b\int_\Omega (u-c)(u^\kappa - c^\kappa) \\
\le& \frac{\lambda b_0^2 c}{4a_0}\int_\Omega |\nabla v|^2 - b\int_\Omega (u-c)(u^\kappa - c^\kappa).
\end{aligned} \tag{3.7}$$

Multiplying the third equation in system (3.1) by $w - c^\kappa$, we get

$$\int_\Omega |\nabla w|^2 = -\int_\Omega (w - c^\kappa)^2 + \int_\Omega (w - c^\kappa)(u^\kappa - c^\kappa). \tag{3.8}$$

Similarly, multiplying the second equation in system (3.1) by $v - c^\kappa$, we derive

$$\int_\Omega |\nabla v|^2 = -\int_\Omega (v - c^\kappa)^2 + \int_\Omega (v - c^\kappa)(w - c^\kappa). \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7), by Young's inequality we see

$$\begin{aligned}
\frac{d}{dt}A(t) \le& -b\int_\Omega (u-c)(u^\kappa - c^\kappa) + \frac{\lambda b_0^2 c}{4a_0}\int_\Omega |\nabla v|^2 - \frac{\lambda b_0^2 c}{4a_0}\int_\Omega |\nabla v|^2 \\
&- \frac{\lambda b_0^2 c}{4a_0}\int_\Omega (v - c^\kappa)^2 + \frac{\lambda b_0^2 c}{4a_0}\int_\Omega (v - c^\kappa)(w - c^\kappa) - \frac{\lambda b_0^2 c}{8a_0}\int_\Omega |\nabla w|^2 \\
&- \frac{\lambda b_0^2 c}{8a_0}\int_\Omega (w - c^\kappa)^2 + \frac{\lambda b_0^2 c}{8a_0}\int_\Omega (w - c^\kappa)(u^\kappa - c^\kappa) \\
\le& -b\int_\Omega (u-c)(u^\kappa - c^\kappa) + \frac{\lambda b_0^2 c}{16a_0}\int_\Omega (w - c^\kappa)^2 - \frac{\lambda b_0^2 c}{8a_0}\int_\Omega (w - c^\kappa)^2 \\
&+ \frac{\lambda b_0^2 c}{8a_0}\int_\Omega (w - c^\kappa)(u^\kappa - c^\kappa) \\
\le& -b\int_\Omega (u-c)(u^\kappa - c^\kappa) + \frac{\lambda b_0^2 c}{16a_0}\int_\Omega (u^\kappa - c^\kappa)^2.
\end{aligned} \tag{3.10}$$

For $\kappa \in (0,1]$, we have the following basic inequality

$$(u^\kappa - c^\kappa)^2 \le c^{\kappa-1}(u-c)(u^\kappa - c^\kappa). \tag{3.11}$$

Thus, from (3.10) and (3.11), we derive

$$\begin{aligned}\frac{d}{dt}A(t) &\le -(b - \frac{\lambda b_0^2 c^\kappa}{16a_0}) \int_\Omega (u-c)(u^\kappa - c^\kappa) \\ &= -\delta \int_\Omega (u-c)(u^\kappa - c^\kappa),\end{aligned} \tag{3.12}$$

where $\delta = b - \frac{\lambda b_0^2 c^\kappa}{16a_0}$. For any $t_0 \ge 0$, integrating both sides of (3.12) on $[t_0, t]$, one can obtain

$$A(t) - A(t_0) \le -\delta \int_{t_0}^t \int_\Omega (u-c)(u^\kappa - c^\kappa). \tag{3.13}$$

Since $A(t) \ge 0$ and δ is nonnegative ensured by $b > \frac{b_0}{4}\sqrt{\frac{\lambda a}{a_0}}$. Thus

$$\int_{t_0}^t \int_\Omega (u-c)(u^\kappa - c^\kappa) \le \frac{A(t_0)}{\delta} < \infty. \tag{3.14}$$

From Theorem 1.1, we know that (u, v, w) is a globally bounded classical solution. Hence, by standard parabolic regularity for parabolic equations [17], we can find $\sigma \in (0,1)$ and $C > 0$ such that

$$\|u\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])} + \|v\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])} + \|w\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])} \le C, \quad \forall t \ge 1. \tag{3.15}$$

This clearly implies that $\int_\Omega (u-c)(u^\kappa - c^\kappa)$ is globally bounded and uniformly continuous with respect to t. Using (3.11) once again, we can obtain from Lemma 3.1

$$\frac{1}{c^{\kappa-1}} \int_\Omega (u^\kappa - c^\kappa)^2 \le \int_\Omega (u-c)(u^\kappa - c^\kappa) \to 0 \quad \text{as } t \to \infty. \tag{3.16}$$

On the other hand, using Young's inequality to (3.8), we get

$$\int_\Omega |\nabla w|^2 = -\frac{1}{2}\int_\Omega (w - c^\kappa)^2 + \frac{1}{2}\int_\Omega (u^\kappa - c^\kappa)^2 \tag{3.17}$$

and so

$$\int_\Omega (w - c^\kappa)^2 \le \int_\Omega (u^\kappa - c^\kappa)^2 \to 0 \quad \text{as } t \to \infty. \tag{3.18}$$

Similarly,

$$\int_\Omega |\nabla v|^2 = -\frac{1}{2}\int_\Omega (v - c^\kappa)^2 + \frac{1}{2}\int_\Omega (w - c^\kappa)^2 \tag{3.19}$$

and so

$$\int_\Omega (v - c^\kappa)^2 \le \int_\Omega (w - c^\kappa)^2 \to 0 \quad \text{as } t \to \infty. \tag{3.20}$$

Define $z(s) = s^{\frac{1}{\kappa}}$. By mean value theorem and (3.2), one may obtain

$$u - c = z(u^\kappa) - z(c^\kappa) = \frac{1}{\kappa}\xi^{\frac{1-\kappa}{\kappa}}(u^\kappa - c^\kappa) \tag{3.21}$$

for some ξ between R^κ and c^κ. Thus

$$\int_\Omega (u-c)^2 \leq \frac{1}{\kappa^2} R^{\frac{2(1-\kappa)}{\kappa}} \int_\Omega (u^\kappa - c^\kappa)^2 \to 0 \quad \text{as } t \to \infty. \tag{3.22}$$

Therefore, from (3.18), (3.20) and (3.22), we can get (3.5) for $\kappa \in (0,1]$.

For $\kappa \in (1,+\infty)$, we define the following functional

$$H(t) = \frac{1}{\kappa} \int_\Omega \left(u^\kappa - \frac{a}{b} - \frac{a}{b} \ln\left(\frac{bu^\kappa}{a}\right)\right), \quad t > 0, \tag{3.23}$$

for $u > 0$. We can easily obtain the function $h(s) = s - \frac{a}{b} - \frac{a}{b}\ln(\frac{bs}{a})$ has global minimum zero over $(0,\infty)$ at $s = \frac{a}{b}$. Thus

$$H(t) = \frac{1}{\kappa} \int_\Omega h(u^\kappa) \geq 0 \quad \text{for all } t \geq 0. \tag{3.24}$$

By Young's inequality, we can obtain from (1.5)–(1.6) and (3.2)–(3.3) that

$$\begin{aligned}
\frac{d}{dt}H(t) &= \int_\Omega \frac{u^\kappa - \frac{a}{b}}{u} u_t \\
&= \int_\Omega \frac{u^\kappa - \frac{a}{b}}{u} \left[\nabla\cdot(\varphi(u)\nabla u - \psi(u)\nabla v) + u(a - bu^\kappa)\right] \\
&= -\frac{a}{b}\int_\Omega \varphi(u)\frac{|\nabla u|^2}{u^2} + \frac{a}{b}\int_\Omega \psi(u)\frac{\nabla u\cdot\nabla v}{u^2} - (\kappa-1)\int_\Omega u^{\kappa-2}\varphi(u)|\nabla u|^2 \\
&\quad + (\kappa-1)\int_\Omega u^{\kappa-2}\psi(u)\nabla u\cdot\nabla v - b\int_\Omega \left(u^\kappa - \frac{a}{b}\right)^2 \\
&\leq -\frac{aa_0}{b}\int_\Omega (u+1)^\alpha \frac{|\nabla u|^2}{u^2} + \frac{ab_0}{b}\int_\Omega (u+1)^{\beta-1}\frac{\nabla u\cdot\nabla v}{u} - (\kappa-1)\int_\Omega u^{\kappa-2}\varphi(u)|\nabla u|^2 \\
&\quad + (\kappa-1)\int_\Omega u^{\kappa-2}\psi(u)\nabla u\cdot\nabla v - b\int_\Omega \left(u^\kappa - \frac{a}{b}\right)^2 \\
&\leq \frac{\lambda a b_0^2}{4ba_0}\int_\Omega |\nabla v|^2 - (\kappa-1)\int_\Omega \left(\sqrt{\varphi(u)}u^{\frac{\kappa}{2}-1}\nabla u - \frac{\psi(u)}{2\sqrt{\varphi(u)}}u^{\frac{\kappa}{2}-1}\nabla v\right)^2 \\
&\quad + \frac{\kappa-1}{4}\int_\Omega \frac{\psi^2(u)}{\varphi(u)}u^{\kappa-2}|\nabla v|^2 - b\int_\Omega \left(u^\kappa - \frac{a}{b}\right)^2 \\
&\leq \frac{\lambda b_0^2}{4a_0}\left[\frac{a}{b} + (\kappa-1)R^\kappa\right]\int_\Omega |\nabla v|^2 - b\int_\Omega \left(u^\kappa - \frac{a}{b}\right)^2 \\
&= \vartheta \int_\Omega |\nabla v|^2 - b\int_\Omega \left(u^\kappa - \frac{a}{b}\right)^2
\end{aligned} \tag{3.25}$$

where $\vartheta = \frac{\lambda b_0^2}{4a_0}\left[\frac{a}{b} + (\kappa-1)R^\kappa\right]$. Multiplying the second equation in system (3.1) by $(v - \frac{a}{b})$, we have

$$\int_\Omega |\nabla v|^2 = -\int_\Omega \left(v - \frac{a}{b}\right)^2 + \int_\Omega \left(v - \frac{a}{b}\right)\left(w - \frac{a}{b}\right). \tag{3.26}$$

Similarly, for the third equation, we get

$$\int_\Omega |\nabla w|^2 = -\int_\Omega \left(w - \frac{a}{b}\right)^2 + \int_\Omega \left(w - \frac{a}{b}\right)\left(u^\kappa - \frac{a}{b}\right). \tag{3.27}$$

Combining (3.26), (3.27) with (3.25) and using Young's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}H(t)\leq & -\vartheta\int_\Omega\left(v-\frac{a}{b}\right)^2+\vartheta\int_\Omega\left(v-\frac{a}{b}\right)\left(w-\frac{a}{b}\right)-b\int_\Omega(u^\kappa-\frac{a}{b})^2\\ & -\frac{\vartheta}{2}\int_\Omega|\nabla w|^2-\frac{\vartheta}{2}\int_\Omega\left(w-\frac{a}{b}\right)^2+\frac{\vartheta}{2}\int_\Omega\left(w-\frac{a}{b}\right)\left(u^\kappa-\frac{a}{b}\right)\\ \leq & \frac{\vartheta}{4}\int_\Omega\left(w-\frac{a}{b}\right)^2-b\int_\Omega\left(u^\kappa-\frac{a}{b}\right)^2-\frac{\vartheta}{2}\int_\Omega\left(w-\frac{a}{b}\right)^2\\ & +\frac{\vartheta}{4}\int_\Omega\left(w-\frac{a}{b}\right)^2+\frac{\vartheta}{4}\int_\Omega\left(u^\kappa-\frac{a}{b}\right)^2\\ = & -\epsilon\int_\Omega(u^\kappa-\frac{a}{b})^2\end{aligned}\tag{3.28}$$

where $\epsilon=b-\frac{\vartheta}{4}$. By the assumption (1.10) in Theorem 1.2, we know that $\epsilon>0$. Then for any $t_0\geq 0$, an integration of the inequality (3.28) from t_0 to t entails

$$H(t)-H(t_0)\leq-\epsilon\int_{t_0}^t\int_\Omega\left(u^\kappa-\frac{a}{b}\right)^2.\tag{3.29}$$

Thus the nonnegativity of H yields

$$\int_{t_0}^\infty\int_\Omega\left(u^\kappa-\frac{a}{b}\right)^2\leq\frac{H(t_0)}{\epsilon}<\infty.\tag{3.30}$$

From Lemma 3.1, the global boundedness and uniform continuity of $\int_\Omega(u^\kappa-\frac{a}{b})^2$ in t entails

$$\int_\Omega\left(u^\kappa-\frac{a}{b}\right)^2\to 0\quad\text{as } t\to\infty.\tag{3.31}$$

A simple use of Young's inequality to (3.27) immediately shows

$$\int_\Omega|\nabla w|^2\leq-\frac{1}{2}\int_\Omega\left(w-\frac{a}{b}\right)^2+\frac{1}{2}\int_\Omega\left(u^\kappa-\frac{a}{b}\right)^2\tag{3.32}$$

and so

$$\int_\Omega(w-\frac{a}{b})^2\leq\int_\Omega\left(u^\kappa-\frac{a}{b}\right)^2\to 0\quad\text{as } t\to\infty.\tag{3.33}$$

Similarly, we have

$$\int_\Omega|\nabla v|^2\leq-\frac{1}{2}\int_\Omega\left(v-\frac{a}{b}\right)^2+\frac{1}{2}\int_\Omega\left(w-\frac{a}{b}\right)^2.\tag{3.34}$$

Thus

$$\int_\Omega\left(v-\frac{a}{b}\right)^2\leq\int_\Omega\left(w-\frac{a}{b}\right)^2\to 0\quad\text{as } t\to\infty.\tag{3.35}$$

Since $\kappa\in(1,\infty)$, then there exists a constant $M>0$ such that

$$M=\sup_{z\in(0,\infty)}\frac{\left(z-\left(\frac{a}{b}\right)^{\frac{1}{\kappa}}\right)^2}{(z^\kappa-\frac{a}{b})^2}<\infty.\tag{3.36}$$

Therefore

$$\int_\Omega \left(u-\left(\frac{a}{b}\right)^{\frac{1}{\kappa}}\right)^2 \le M\int_\Omega \left(u^\kappa-\frac{a}{b}\right)^2 \to 0 \quad \text{as } t\to\infty. \tag{3.37}$$

This completes the proof of L^2-convergence of the solution to system (3.1). □

Proof of Theorem 1.2. In view of the Gagliardo–Nirenberg inequality [4], we conclude from (3.5), (3.15), (3.22) and (3.37) that

$$\begin{aligned}\left\|u(\cdot,t)-\left(\frac{a}{b}\right)^{\frac{1}{\kappa}}\right\|_{L^\infty(\Omega)} &\le C_{GN}\left\|u(\cdot,t)-\left(\frac{a}{b}\right)^{\frac{1}{\kappa}}\right\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\left\|u(\cdot,t)-\left(\frac{a}{b}\right)^{\frac{1}{\kappa}}\right\|_{L^2}^{\frac{2}{n+2}}\\ &\le C\left\|u(\cdot,t)-\left(\frac{a}{b}\right)^{\frac{1}{\kappa}}\right\|_{L^2}^{\frac{2}{n+2}}\\ &\le C_\kappa\left\|u^\kappa(\cdot,t)-\frac{a}{b}\right\|_{L^2}^{\frac{2}{n+2}} \to 0 \quad \text{as } t\to\infty. \end{aligned}\tag{3.38}$$

For $\kappa\in(0,1]$, by the L'Hospital rule, we get

$$\lim_{u\to c}\frac{a(u)}{(u-c)(u^\kappa-c^\kappa)} = \lim_{u\to c}\frac{u-c-c\ln(\frac{u}{c})}{(u-c)(u^\kappa-c^\kappa)} = \frac{1}{2\kappa c^\kappa}, \qquad c=\left(\frac{a}{b}\right)^\kappa. \tag{3.39}$$

Based on (3.38) and(3.39), we choose $t_1>0$ such that

$$\frac{1}{4\kappa c^\kappa}(u-c)(u^\kappa-c^\kappa)\le a(u)\le \frac{1}{\kappa c^\kappa}(u-c)(u^\kappa-c^\kappa), \quad t\ge t_1, \tag{3.40}$$

and so

$$\frac{1}{4\kappa c^\kappa}\int_\Omega (u-c)(u^\kappa-c^\kappa)\le A(t)\le \frac{1}{\kappa c^\kappa}\int_\Omega (u-c)(u^\kappa-c^\kappa), \quad t\ge t_1. \tag{3.41}$$

Using (3.12) and (3.41), we get

$$\frac{d}{dt}A(t)\le -\delta\kappa c^\kappa A(t), \quad t\ge t_1, \tag{3.42}$$

thus

$$A(t)\le A(t_1)\mathrm{e}^{-\delta\kappa c^\kappa(t-t_1)}, \quad t\ge t_1. \tag{3.43}$$

From (3.11), (3.38), (3.41) and (3.42), we can deduce

$$\begin{aligned}\|u(\cdot,t)-c\|_{L^\infty(\Omega)} &\le C_\kappa\|u^\kappa(\cdot,t)-c^\kappa\|_{L^2}^{\frac{2}{n+2}}\\ &\le C_\kappa\left[\int_\Omega (u^\kappa-c^\kappa)^2\right]^{\frac{1}{n+2}}\\ &\le C_\kappa\left[\int_\Omega c^{\kappa-1}(u-c)(u^\kappa-c^\kappa)\right]^{\frac{1}{n+2}}\\ &\le C_\kappa\left[4\kappa c^{2\kappa-1}A(u)\right]^{\frac{1}{n+2}}\\ &\le C_\kappa(4\kappa c^{2\kappa-1}A(t_1))^{\frac{1}{n+2}}\mathrm{e}^{-\frac{\kappa\delta c^\kappa(t-t_1)}{n+2}}, \quad t\ge t_1. \end{aligned}\tag{3.44}$$

Repeating the similar steps for w and v, we can obtain from (3.18), (3.19) and (3.44)

$$\|w(\cdot,t)-c^\kappa\|_{L^\infty(\Omega)} \le C_\kappa \left(4\kappa c^{2\kappa-1}A(t_1)\right)^{\frac{1}{n+2}} \mathrm{e}^{-\frac{\kappa\delta c^\kappa(t-t_1)}{n+2}}, \quad t \ge t_1 \tag{3.45}$$

and

$$\|v(\cdot,t)-c^\kappa\|_{L^\infty(\Omega)} \le C_\kappa \left(4\kappa c^{2\kappa-1}A(t_1)\right)^{\frac{1}{n+2}} \mathrm{e}^{-\frac{\kappa\delta c^\kappa(t-t_1)}{n+2}}, \quad t \ge t_1. \tag{3.46}$$

For $\kappa \in (1,\infty)$, using the the L'Hospital rule, we deduce

$$\lim_{u\to c} \frac{h(u^\kappa)}{(u^\kappa-c^\kappa)^2\kappa} = \lim_{z\to c^\kappa} \frac{z-c^\kappa-c^\kappa\ln(\frac{z}{c^\kappa})}{(z-c^\kappa)^2\kappa} = \frac{c^{\kappa-2}}{2\kappa}. \tag{3.47}$$

From (3.38) and (3.47), we pick $t_2 \ge 0$ such that

$$\frac{c^{\kappa-2}}{4\kappa}\int_\Omega (u^\kappa-c^\kappa)^2 \le H(t) \le \frac{c^{\kappa-2}}{\kappa}\int_\Omega (u^\kappa-c^\kappa)^2, \quad t \ge t_2. \tag{3.48}$$

Using (3.28) and (3.48), we get

$$\frac{d}{dt}H(t) \le -\epsilon\kappa c^{2-\kappa}H(t), \quad t \ge t_2, \tag{3.49}$$

which implies

$$H(t) \le H(t_2)\mathrm{e}^{-\epsilon\kappa c^{2-\kappa}(t-t_2)}, \quad t \ge t_2. \tag{3.50}$$

From (3.38), (3.48) and (3.50), we infer that

$$\begin{aligned}\|u(\cdot,t)-c\|_{L^\infty(\Omega)} &\le C_\kappa \|u^\kappa(\cdot,t)-c^\kappa\|_{L^2}^{\frac{2}{n+2}} \\ &\le C_\kappa(4\kappa c^{2\kappa-1}H(t_2))^{\frac{1}{n+2}}\mathrm{e}^{-\frac{\epsilon\kappa c^{2-\kappa}(t-t_2)}{n+2}}, \quad t \ge t_2.\end{aligned} \tag{3.51}$$

Analogously, taking (3.33), (3.35) and (3.51) into account, we can obtain

$$\|w(\cdot,t)-c^\kappa\|_{L^\infty(\Omega)} \le C_\kappa(4\kappa c^{2\kappa-1}H(t_2))^{\frac{1}{n+2}}\mathrm{e}^{-\frac{\epsilon\kappa c^{2-\kappa}(t-t_2)}{n+2}}, \quad t \ge t_2 \tag{3.52}$$

and

$$\|v(\cdot,t)-c^\kappa\|_{L^\infty(\Omega)} \le C_\kappa(4\kappa c^{2\kappa-1}H(t_2))^{\frac{1}{n+2}}\mathrm{e}^{-\frac{\epsilon\kappa c^{2-\kappa}(t-t_2)}{n+2}}, \quad t \ge t_2. \tag{3.53}$$

Finally, plugging δ and ϵ into (3.44)–(3.46) and (3.51)–(3.53), we take C_κ large enough and then complete the proof of Theorem 1.2. □

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Existence of nontrivial weak solutions for nonuniformly elliptic equation with mixed boundary condition in a variable exponent Sobolev space

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Abstract. In this paper, we consider a mixed boundary value problem for nonuniformly elliptic equation in a variable exponent Sobolev space containing $p(\cdot)$-Laplacian and mean curvature operator. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of a nontrivial weak solution and at least two nontrivial weak solutions according to some hypotheses on given functions.

Keywords: $p(\cdot)$-Laplacian type equation, mean curvature operator, mixed boundary value problem, Ekeland variational principle.

2020 Mathematics Subject Classification: 35H30, 35D05, 35J60, 35J70.

1 Introduction

In this paper, we consider the following problem

$$\begin{cases} -\mathrm{div}\,[\boldsymbol{a}(x,\nabla u(x))] = f(x,u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma_1, \\ \boldsymbol{n}(x)\cdot \boldsymbol{a}(x,\nabla u(x)) = g(x,u(x)) & \text{on } \Gamma_2. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a Lipschitz-continuous ($C^{0,1}$ for short) boundary Γ satisfying that

$$\Gamma_1 \text{ and } \Gamma_2 \text{ are disjoint open subsets of } \Gamma \text{ such that } \overline{\Gamma_1}\cup\overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset, \tag{1.2}$$

and the vector field $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. The function $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega\times\mathbb{R}^d$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$. Then the operator $u \mapsto \mathrm{div}\,[\boldsymbol{a}(x,\nabla u(x))]$ is more general than the $p(\cdot)$-Laplacian $\Delta_{p(x)}u(x) = \mathrm{div}\,[|\nabla u(x)|^{p(x)-2}\nabla u(x)]$ and the mean curvature

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operator div $[(1+|\nabla u(x)|^2)^{(p(x)-2)/2}\nabla u(x)]$. These generalities bring about difficulties and requires some conditions.

We impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and $g:\Gamma_2\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying some conditions.

The study of differential equations with $p(\cdot)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [28]), in electrorheological fluids (Diening [7], Halsey [15], Mihăilescu and Rădulescu [18], Růžička [20]).

Over the last two decades, there are many articles on the existence of weak solutions for the Dirichlet boundary condition, that is, in the case $\Gamma_2=\emptyset$ in (1.1), (for example, see Mashiyev et al. [17], Duc and Vu [10], Wei and Chen [22], Yücedağ [25], Nápoli and Mariani [19]).

However, since we can only find a few of papers associate with the problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1). See Aramaki [1–3]. We are convinced of the reason for existence of this paper.

In particular, the authors in [10] considered the problem (1.1) when $p(x)=p=$ const. and $\Gamma_2=\emptyset$, and derived the existence of a nontrivial weak solution to (1.1). This paper is an extension of the article [10] to the case of variable exponent and mixed boundary value problem. In the paper [10], the authors derived the weakly continuous differentiability of the corresponding energy functional and then applied a version of the Mountain-pass lemma introduced in Duc [9]. However, in this paper we show that the corresponding energy functional is of class C^1, and so it suffices to apply the standard Mountain-pass lemma.

The paper is organized as follows. Section 2 consists of two subsections. In Subsection 2.1, we recall some results on variable exponent Lebesgue-Sobolev spaces. In Subsection 2.2, we give the assumptions to the main theorems. In Section 3, we state the main theorems (Theorem 3.3 and Theorem 3.5) on the existence of at least one and two nontrivial weak solutions. The proofs of the main theorems are given in Section 4.

2 Preliminaries and the main theorems

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d\geq 2)$ with a $C^{0,1}$-boundary Γ. Moreover, we assume that Γ satisfies (1.2).

Throughout this paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^d by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a}=(a_1,\dots,a_d)$ and $\boldsymbol{b}=(b_1,\dots,b_d)$ in $\mathbb{R}^d$ by $\boldsymbol{a}\cdot\boldsymbol{b}=\sum_{i=1}^d a_ib_i$ and $|\boldsymbol{a}|=(\boldsymbol{a}\cdot\boldsymbol{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle\cdot,\cdot\rangle_{B^*,B}$.

2.1 Variable exponent Lebesgue and Sobolev spaces

In this subsection, we recall some well-known results on variable exponent Lebesgue–Sobolev spaces. See Diening et al. [8], Fan and Zhang [12], Kováčik and Rákosník [16] and references therein for more detail. Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^d$ with a $C^{0,1}$-boundary Γ and Ω is locally on the same side of Γ. Define $\mathcal{P}(\Omega)=\{p:\Omega\to[1,\infty); p \text{ is a measurable function}\}$, and for any $p\in\mathcal{P}(\Omega)$, put

$$p^+=\operatorname*{ess\,sup}_{x\in\Omega}p(x)\quad\text{and}\quad p^-=\operatorname*{ess\,inf}_{x\in\Omega}p(x).$$

For any measurable function u on Ω, a modular $\rho_{p(\cdot)} = \rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\}$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0; \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define, for any integer $m \geq 0$,

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); \partial^\alpha u \in L^{p(\cdot)}(\Omega) \text{ for } |\alpha| \leq m\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, $|\alpha| = \sum_{i=1}^d \alpha_i$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ and $\partial_i = \partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{m,p(\cdot)}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

Of course, $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$. Define

$$W_0^{m,p(\cdot)}(\Omega) = \text{ the closure of the set of } W^{m,p(\cdot)}(\Omega)\text{-functions with compact supports in } \Omega.$$

The following three propositions are well known (see Fan et al. [14,22], Fan and Zhao [13], Zhao et al. [27], and [25]).

Proposition 2.1. *Let $p \in \mathcal{P}(\Omega)$ and let $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, \ldots)$ Then we have*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1, > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{p(\cdot)}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let $p \in \mathcal{P}_+(\Omega)$, where*

$$\mathcal{P}_+(\Omega) = \{p \in \mathcal{P}(\Omega); 1 < p^- \leq p^+ < \infty\}.$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have

$$\int_\Omega |u(x)v(x)| dx \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \leq 2\|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}.$$

Here and from now on, $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$, that is, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

For $p \in \mathcal{P}(\Omega)$, define

$$p^*(x) = \begin{cases} \frac{dp(x)}{d-p(x)} & \text{if } p(x) < d, \\ \infty & \text{if } p(x) \geq d. \end{cases}$$

Proposition 2.3. *Let Ω be a bounded domain with $C^{0,1}$-boundary and let $p \in \mathcal{P}_+(\Omega)$ and $m \geq 0$ be an integer. Then we have the following:*

(i) *The spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.*

(i) *If $q(\cdot) \in \mathcal{P}_+(\Omega)$ and satisfies $q(x) \leq p(x)$ for all $x \in \Omega$, then $W^{m,p(\cdot)}(\Omega) \hookrightarrow W^{m,q(\cdot)}(\Omega)$, where $\hookrightarrow$ means that the embedding is continuous.*

(i) *If $q(x) \in \mathcal{P}_+(\Omega)$ satisfies that $q(x) \leq p^*(x)$ for all $x \in \Omega$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous. Moreover, if $q(x) < p^*(x)$ for all $x \in \Omega$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.*

We say that $p \in \mathcal{P}(\Omega)$ belongs to $\mathcal{P}^{\log}(\Omega)$ if p has the log-Hölder continuity in Ω, that is, $p : \Omega \to \mathbb{R}$ satisfies that there exists a constant $C_{\log}(p) > 0$ such that

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/|x-y|)} \quad \text{for all } x, y \in \Omega.$$

We also write $\mathcal{P}_+^{\log}(\Omega) = \{p \in \mathcal{P}^{\log}(\Omega); 1 < p^- \leq p^+ < \infty\}$.

Proposition 2.4. *If $p \in \mathcal{P}_+^{\log}(\Omega)$ and $m \geq 0$ is an integer, then $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ is dense in $W_0^{m,p(\cdot)}(\Omega)$.*

For the proof, see [8, Corollary 11.2.4].

Next we consider the notion of trace. Let Ω be a domain of $\mathbb{R}^d$ with a $C^{0,1}$-boundary Γ and $p \in \mathcal{P}_+(\overline{\Omega})$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}_{\text{loc}}(\Omega)$, the trace $\gamma(u) = u|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1_{\text{loc}}(\Gamma)$. We define

$$\text{Tr}(W^{1,p(\cdot)}(\Omega)) = (\text{Tr}\, W^{1,p(\cdot)})(\Gamma) = \{f; f \text{ is the trace to } \Gamma \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|f\|_{(\text{Tr}\, W^{1,p(\cdot)})(\Gamma)} = \inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)}; F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F|_\Gamma = f\}$$

for $f \in (\text{Tr}\, W^{1,p(\cdot)})(\Gamma)$, where the infimum can be achieved. Then $(\text{Tr}\, W^{1,p(\cdot)})(\Gamma)$ is a Banach space. More precisely, see [8, Chapter 12]. In the later we also write $F|_\Gamma = f$ by $F = f$ on Γ. Moreover, we denote

$$(\text{Tr}\, W^{1,p(\cdot)})(\Gamma_i) = \{f|_{\Gamma_i}; f \in (\text{Tr}\, W^{1,p(\cdot)})(\Gamma)\} \quad \text{for } i = 1, 2$$

equipped with the norm

$$\|g\|_{(\text{Tr}\, W^{1,p(\cdot)})(\Gamma_i)} = \inf\{\|f\|_{(\text{Tr}\, W^{1,p(\cdot)})(\Gamma)}; f \in (\text{Tr}\, W^{1,p(\cdot)})(\Gamma) \text{ satisfying } f|_{\Gamma_i} = g\},$$

where the infimum can also be achieved, so for any $g \in (\text{Tr}\, W^{1,p(\cdot)})(\Gamma_i)$, there exists $F \in W^{1,p(\cdot)}(\Omega)$ such that $F|_{\Gamma_i} = g$ and $\|F\|_{W^{1,p(\cdot)}(\Omega)} = \|g\|_{(\text{Tr}\, W^{1,p(\cdot)})(\Gamma_i)}$.

Let $q \in \mathcal{P}_+(\Gamma) := \{q \in \mathcal{P}(\Gamma); q^- > 1\}$ and denote the surface measure on Γ induced from the Lebesgue measure dx on Ω by $d\sigma$. We define

$$L^{q(\cdot)}(\Gamma) = \Big\{u; u : \Gamma \to \mathbb{R} \text{ is a measurable function with respect to } d\sigma \quad \text{satisfying } \int_\Gamma |u(x)|^{q(x)} d\sigma < \infty\Big\}$$

equipped with the norm

$$\|u\|_{L^{q(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0; \int_\Gamma \left|\frac{u(x)}{\lambda}\right|^{q(x)} d\sigma \le 1\right\},$$

and we also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u) = \int_\Gamma |u(x)|^{q(x)} d\sigma.$$

Proposition 2.5. *We have the following properties.*

(i) $\|u\|_{L^{q(\cdot)}(\Gamma)} \ge 1 \Longrightarrow \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.

(ii) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 \Longrightarrow \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.

Proposition 2.6. *Let Ω be a bounded domain with a $C^{0,1}$-boundary Γ and let $p \in \mathcal{P}^{\log}_+(\overline{\Omega})$. If $f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma)$ and there exists a constant $C > 0$ such that*

$$\|f\|_{L^{p(\cdot)}(\Gamma)} \le C\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}.$$

In particular, if $f \in (\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma_i)$ and $\|f\|_{L^{p(\cdot)}(\Gamma_i)} \le C\|f\|_{(\operatorname{Tr} W^{1,p(\cdot)})(\Gamma)}$.

For $p \in \mathcal{P}_+(\overline{\Omega})$, define

$$p^\partial(x) = \begin{cases} \frac{(d-1)p(x)}{d-p(x)} & \text{if } p(x) < d, \\ \infty & \text{if } p(x) \ge d. \end{cases}$$

Proposition 2.7. *Let $p \in \mathcal{P}_+(\overline{\Omega})$. Then if $q(x) \in \mathcal{P}_+(\Gamma)$ satisfies $q(x) < p^\partial(x)$ for all $x \in \Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is well defined and compact. In particular, the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Gamma)$ is compact and there exists a constant $C > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Gamma)} \le C\|u\|_{W^{1,p(\cdot)}(\Omega)} \quad \textit{for } u \in W^{1,p(\cdot)}(\Omega).$$

For the proof, see Yao [24, Proposition 2.6].

Define a space by

$$X = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma_1\}. \tag{2.1}$$

Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable Banach space. We show the following Poincaré type inequality (cf. Ciarlet and Dinca [6]).

Lemma 2.8. *Let $p \in \mathcal{P}^{\log}_+(\Omega)$. Then there exists a constant $C = C(\Omega, d, p) > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{\mathbf{L}^{p(\cdot)}(\Omega)} \quad \textit{for all } u \in X,$$

where $\|\nabla u\|_{\mathbf{L}^{p(\cdot)}(\Omega)} := \||\nabla u|\|_{L^{p(\cdot)}(\Omega)}$.

In particular, the norm $\|\nabla u\|_{\mathbf{L}^{p(\cdot)}(\Omega)}$ is equivalent to $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ for $u \in X$.

For the direct proof, see Aramaki [4, Lemma 2.5].

Thus we can define the norm on the space X defined by (2.1) so that

$$\|v\|_X = \|\nabla v\|_{L^{p(\cdot)}(\Omega)} \quad \text{for } v \in X, \tag{2.2}$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$ from Lemma 2.8.

2.2 Assumptions to the main theorems

In this subsection, we state the assumptions to the main theorems. Let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$ be fixed.

Let $A : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a function satisfying that for a.e. $x \in \Omega$, the function $A(x, \cdot) : \mathbb{R}^d \ni \boldsymbol{\xi} \mapsto A(x, \boldsymbol{\xi})$ is of C^1-class, and for all $\boldsymbol{\xi} \in \mathbb{R}^d$, the function $A(\cdot, \boldsymbol{\xi}) : \Omega \ni x \mapsto A(x, \boldsymbol{\xi})$ is measurable. Moreover, suppose that $A(x, \mathbf{0}) = 0$ and put $\boldsymbol{a}(x, \boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} A(x, \boldsymbol{\xi})$. Then $\boldsymbol{a}(x, \boldsymbol{\xi})$ is a Carathéodory function. Assume that there exist constants $c_0, k_0, k_1 > 0$ and nonnegative functions $h_0 \in L^{p'(\cdot)}(\Omega)$ and $h_1 \in L^1_{\text{loc}}(\Omega)$ with $h_1(x) \geq 1$ a.e. $x \in \Omega$ such that the following conditions hold.

(A1) $|\boldsymbol{a}(x, \boldsymbol{\xi})| \leq c_0(h_0(x) + h_1(x)|\boldsymbol{\xi}|^{p(x)-1})$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $x \in \Omega$.

(A2) A is $p(\cdot)$-uniformly convex, that is,

$$A\left(x, \frac{\boldsymbol{\xi} + \boldsymbol{\eta}}{2}\right) + k_1 h_1(x)|\boldsymbol{\xi} - \boldsymbol{\eta}|^{p(x)} \leq \frac{1}{2}A(x, \boldsymbol{\xi}) + \frac{1}{2}A(x, \boldsymbol{\eta})$$
$$\text{for all } \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d \text{ and a.e. } x \in \Omega.$$

(A3) A is $p(\cdot)$-subhomogeneous, that is,

$$0 \leq \boldsymbol{a}(x, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \leq p(x)A(x, \boldsymbol{\xi}) \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d \text{ and a.e. } x \in \Omega.$$

(A4) $A(x, \boldsymbol{\xi}) \geq k_0 h_1(x)|\boldsymbol{\xi}|^{p(x)}$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and a.e. $x \in \Omega$.

Example 2.9.

(i) $A(x, \boldsymbol{\xi}) = \frac{h(x)}{p(x)}|\boldsymbol{\xi}|^{p(x)}$ with $p^- \geq 2$, $h \in L^1_{\text{loc}}(\Omega)$ satisfying $h(x) \geq 1$.

(ii) $A(x, \boldsymbol{\xi}) = \frac{h(x)}{p(x)}((1 + |\boldsymbol{\xi}|^2)^{p(x)/2} - 1)$ with $p^- \geq 2$, $h \in L^{p'(\cdot)}(\Omega)$ satisfying $h(x) \geq 1$ a.e. $x \in \Omega$.

Then $A(x, \boldsymbol{\xi})$ and $\boldsymbol{a}(x, \boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} A(x, \boldsymbol{\xi})$ satisfy (A1)–(A4).

Remark 2.10. When $h(x) \equiv 1$, (i) corresponds to the $p(\cdot)$-Laplacian and (ii) corresponds to the prescribed mean curvature operator for nonparametric surface.

For the function $h_1 \in L^1_{\text{loc}}(\Omega)$ with $h_1(x) \geq 1$ a.e. $x \in \Omega$, we define a modular

$$\rho_{p(\cdot),h_1(\cdot)}(\nabla v) = \int_\Omega h_1(x)|\nabla v(x)|^{p(x)}dx \quad \text{for } v \in W^{1,p(\cdot)}(\Omega).$$

Define our basic space

$$Y = \{v \in X; \rho_{p(\cdot),h_1(\cdot)}(\nabla v) < \infty\} \tag{2.3}$$

equipped with the norm

$$\|v\|_Y = \inf\left\{\lambda > 0; \rho_{p(\cdot),h_1(\cdot)}\left(\frac{\nabla v}{\lambda}\right) \leq 1\right\},$$

then Y is a Banach space (see Lemma 2.12 below). We note that $C_0^\infty(\Omega) \subset Y$. Since

$$\rho_{p(\cdot),h_1(\cdot)}(\nabla v) = \rho_{p(\cdot)}(h_1^{1/p(\cdot)}\nabla v),$$

we have

$$\|v\|_Y = \|h_1^{1/p(\cdot)}\nabla v\|_{L^{p(\cdot)}(\Omega)}. \tag{2.4}$$

Then we have the following lemma.

Lemma 2.11.

(i) *$Y \hookrightarrow X$ and $\|v\|_X \leq \|v\|_Y$ for all $v \in Y$.*

(ii) *Let $v \in Y$. Then $\|v\|_Y > 1 (= 1, < 1) \Longleftrightarrow \rho_{p(\cdot),h_1(\cdot)}(\nabla v) > 1 (= 1, < 1)$.*

(iii) *Let $v \in Y$. Then $\|v\|_Y > 1 \Longrightarrow \|v\|_Y^{p^-} \leq \rho_{p(\cdot),h_1(\cdot)}(\nabla v) \leq \|v\|_Y^{p^+}$.*

(iv) *Let $v \in Y$. Then $\|v\|_Y < 1 \Longrightarrow \|v\|_Y^{p^+} \leq \rho_{p(\cdot),h_1(\cdot)}(\nabla v) \leq \|v\|_Y^{p^-}$.*

(v) *Let $u_n, u \in Y$. Then $\lim_{n\to\infty} \|u_n - u\|_Y = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot),h_1(\cdot)}(\nabla u_n - \nabla u) = 0$.*

(vi) *Let $u_n \in Y$. Then $\|u_n\|_Y \to \infty$ as $n \to \infty \Longleftrightarrow \rho_{p(\cdot),h_1(\cdot)}(\nabla u_n) \to \infty$ as $n \to \infty$.*

When $q \in \mathcal{P}_+^{\log}(\overline{\Omega})$ satisfies $q(x) \leq p^*(x)$ for all $x \in \Omega$, define

$$\lambda_q = \inf\left\{\frac{\|v\|_Y}{\|v\|_{L^{q(\cdot)}(\Omega)}}; v \in Y \setminus \{0\}\right\}. \tag{2.5}$$

By Proposition 2.3 and Lemma 2.11, there exists a constant $c > 0$ such that $\|v\|_{L^{q(\cdot)}(\Omega)} \leq c\|v\|_X \leq c\|v\|_Y$ for all $v \in Y$, so we can see that $\lambda_q > 0$.

When $q \in \mathcal{P}_+^{\log}(\overline{\Omega})$ satisfies $q(x) \leq p^\partial(x)$ for all $x \in \Gamma_2$, define

$$\mu_q = \inf\left\{\frac{\|v\|_Y}{\|v\|_{L^{q(\cdot)}(\Gamma_2)}}; v \in Y \text{ with } v \neq 0 \text{ on } \Gamma_2\right\}. \tag{2.6}$$

By Proposition 2.7 and Lemma 2.11, there exists a constant $c > 0$ such that $\|v\|_{L^{q(\cdot)}(\Gamma_2)} \leq c\|v\|_X \leq c\|v\|_Y$ for all $v \in Y$, so we can see that $\mu_q > 0$.

Lemma 2.12. *The space $(Y, \|\cdot\|_Y)$ is a reflexive Banach space.*

Proof. Since $\|v\|_Y = \|h_1^{1/p(\cdot)}\nabla v\|_{L^{p(\cdot)}(\Omega)}$ for $v \in Y (\subset X)$, it is clear that Y is a normed linear space. Let $\{v_n\}$ be a Cauchy sequence in Y. Then $\{\|v_n\|_Y\}$ is bounded, so $\{\rho_{p(\cdot),h_1(\cdot)}(\nabla v_n)\}$ is bounded from Lemma 2.11 (vi) and we have

$$\lim_{n\to\infty} \liminf_{j\to\infty} \int_\Omega h_1(x)|\nabla u_j(x) - \nabla u_n(x)|^{p(x)} dx = 0.$$

Since $\|v\|_X \le \|v\|_Y$ for all $v \in Y$, $\{v_n\}$ is also a Cauchy sequence in X. Hence there exists $v \in X$ such that $v_n \to v$ in X, that is, $\nabla v_n \to \nabla v$ in $L^{p(\cdot)}(\Omega)$. So there exists a subsequence $\{v_{n'}\}$ of $\{v_n\}$ such that $\nabla v_{n'}(x) \to \nabla v(x)$ a.e. in Ω. By the Fatou lemma,

$$\int_\Omega h_1(x)|\nabla v(x)|^{p(x)}dx \le \liminf_{n'\to\infty} \int_\Omega h_1(x)|\nabla v_{n'}(x)|^{p(x)}dx < \infty.$$

Thereby $v \in Y$. Applying again the Fatou lemma,

$$\lim_{n'\to\infty} \int_\Omega h_1(x)|\nabla v(x) - \nabla v_{n'}(x)|^{p(x)}dx \le \lim_{n'\to\infty} \liminf_{j'\to\infty} \int_\Omega h_1(x)|\nabla v_{j'}(x) - \nabla v_{n'}(x)|^{p(x)}dx = 0.$$

This implies $v_{n'} \to v$ in Y. Since $\{v_n\}$ is a Cauchy sequence in Y, we see that $v_n \to v$ in Y, so $(Y, \|\cdot\|_Y)$ is a Banach space.

We claim that $(Y, \|\cdot\|_Y)$ is a uniformly convex Banach space. Since $L^{p(\cdot)}(\Omega)$ is uniformly convex, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $u, v \in L^{p(\cdot)}(\Omega)$ satisfy $\|u\|_{L^{p(\cdot)}(\Omega)} \le 1$, $\|v\|_{L^{p(\cdot)}(\Omega)} \le 1$ and $\|u - v\|_{L^{p(\cdot)}(\Omega)} > \varepsilon$, then $\|(u+v)/2\|_{L^{p(\cdot)}(\Omega)} < 1 - \delta$. Thus if $u, v \in Y$ satisfy $\|u\|_Y \le 1$, $\|v\|_Y \le 1$ and $\|u - v\|_Y > \varepsilon$, then $\|h_1^{1/p(\cdot)}\nabla u\|_{L^{p(\cdot)}(\Omega)} \le 1$, $\|h_1^{1/p(\cdot)}\nabla v\|_{L^{p(\cdot)}(\Omega)} \le 1$ and $\|h_1^{1/p(\cdot)}\nabla u - h_1^{1/p(\cdot)}\nabla v\|_{L^{p(\cdot)}(\Omega)} > \varepsilon$ from (2.4). Hence we have

$$\|(h_1^{1/p(\cdot)}\nabla u + h_1^{1/p(\cdot)}\nabla v)/2\|_{L^{p(\cdot)}(\Omega)} \le 1 - \delta.$$

Therefore we get $\|(u+v)/2\|_Y \le 1 - \delta$. This implies the uniform convexity of Y. So it follows from the Milman theorem (cf. Brezis [5, Theorem III.29]) that Y is reflexive. □

We continue to state the assumptions of f and g in (1.1).

Let f is a real Carathéodory function on $\Omega \times \mathbb{R}$ having the following properties.

(F1) $|f(x,t)| \le c_1(1 + |t|^{q(x)-1})$ for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$, where c_1 is a positive constant and $q \in \mathcal{P}_+^{\log}(\overline{\Omega})$ such that $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ and $p^+ < q^-$.

(F2) There exist $\theta > p^+$ and $t_0 > 0$ such that

$$0 < \theta F(x,t) \le f(x,t)t \quad \text{for all } t \in \mathbb{R} \setminus (-t_0, t_0) \text{ and a.e. } x \in \Omega,$$

where

$$F(x,t) = \int_0^t f(x,s)ds. \tag{2.7}$$

(F3) Let λ_{p^+} be defined by (2.5). There exist $\lambda \in (0, k_0 p^+ (\lambda_{p^+})^{p^+}/4)$ and $0 < \delta < 1$ such that

$$\frac{f(x,t)}{|t|^{p^+-2}t} \le \lambda \quad \text{for all } t \in (-\delta, \delta) \setminus \{0\} \text{ and a.e. } x \in \Omega.$$

Let g be a real Carathéodory function on $\Gamma_2 \times \mathbb{R}$ having the following properties.

(G1) $|g(x,t)| \le c_2(1 + |t|^{r(x)-1})$ for all $t \in \mathbb{R}$ and a.e. $x \in \Gamma_2$, where c_2 is a positive constant and $r \in \mathcal{P}_+^{\log}(\overline{\Omega})$ such that $r(x) < p^\partial(x)$ for all $x \in \overline{\Gamma_2}$ and $p^+ < r^-$.

(G2) Let θ and t_0 be as in (F2). That is, there exist $\theta > p^+$ and $t_0 > 0$ such that

$$0 < \theta G(x,t) \le g(x,t)t \quad \text{for all } t \in \mathbb{R} \setminus (-t_0, t_0) \text{ and a.e. } x \in \Gamma_2,$$

where

$$G(x,t) = \int_0^t g(x,s)ds. \tag{2.8}$$

(G3) Let μ_{p^+} be defined by (2.6). There exist $\mu \in (0, k_0 p^+ (\mu_{p^+})^{p^+}/4)$ and $0 < \delta < 1$ such that

$$\frac{g(x,t)}{|t|^{p^+-2}t} \leq \mu \quad \text{for all } t \in (-\delta, \delta) \setminus \{0\} \text{ and a.e. } x \in \Gamma_2.$$

3 Main theorems

In this section, we state the main theorems.

Definition 3.1. We say $u \in Y$ is a weak solution of (1.1) if u satisfies that

$$\int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla v(x) dx = \int_\Omega f(x, u(x)) v(x) dx + \int_{\Gamma_2} g(x, u(x)) v(x) d\sigma \quad \text{for all } v \in Y. \tag{3.1}$$

Remark 3.2. Since $\{\varphi \in C^\infty(\overline{\Omega}); \varphi = 0 \text{ on } \Gamma_1\} \subset Y$, if $u \in Y$ satisfies (3.1), then the equation (1.1) holds in the distribution sense.

Then we obtain the following two theorems.

Theorem 3.3. *Let Ω be a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a $C^{0,1}$-boundary Γ satisfying* (1.2)*. Under the hypotheses (A1)-(A4), (F1)-(F3) and (G1)-(G3), the problem* (1.1) *has a nontrivial weak solution.*

Remark 3.4. This theorem extends the result of [10] in which the authors considered the case where $p(x) = p = \text{const.}$ and $\Gamma_2 = \emptyset$.

We impose one more assumption.

(F4) There exist a constant $c > 0$ and $0 < m < 1$ such that $f(x,t) \geq ct^{m-1}$ for $0 < t \leq \delta$ and a.e. $x \in \Omega$, where $\delta > 0$ is as in (F3).

Theorem 3.5. *Addition to the hypotheses of Theorem 3.3, assume that (F4) also holds. Then the problem* (1.1) *has at least two nontrivial weak solutions.*

Remark 3.6. The authors in [17] considered the equation

$$-\mathrm{div}\,[\boldsymbol{a}(x, \nabla u(x))] = m(x)|u(x)|^{r(x)-2}u(x) + n(x)|u(x)|^{s(x)-2}u(x)$$

and $\Gamma_2 = \emptyset$. The authors got the same result of Theorem 3.5 under stronger hypotheses than (A1) and (A4), that is, $h_1(x) \equiv 1$. However, they use an inequality $A(x, t\boldsymbol{\xi}) \leq t^{p(x)} A(x, \boldsymbol{\xi})$ for small $t > 0$ which does not hold for the function in Example 2.9 (ii). To overcome their mistake, we assume a stronger condition (F4).

4 Proofs of Theorem 3.3 and Theorem 3.5

In this section, we give proofs of Theorem 3.3 and Theorem 3.5. In order to do so, we use the variational method. Define a functional on Y

$$I(u) = E(u) - J(u) - K(u) \tag{4.1}$$

where

$$E(u) = \int_\Omega A(x, \nabla u(x))dx, \tag{4.2}$$

$$J(u) = \int_\Omega F(x, u(x))dx, \qquad F \text{ is defined by (2.7)}, \tag{4.3}$$

$$K(u) = \int_{\Gamma_2} G(x, u(x))d\sigma, \qquad G \text{ is defined by (2.8)}. \tag{4.4}$$

The proof of Theorem 3.3 consists of several lemmas and propositions.

Lemma 4.1.

(i) $|A(x,\boldsymbol{\xi})| \le c_0(h_0(x)|\boldsymbol{\xi}| + h_1(x)|\boldsymbol{\xi}|^{p(x)})$ *for all* $\boldsymbol{\xi} \in \mathbb{R}^d$ *and a.e.* $x \in \Omega$.

(ii)
$$E\left(\frac{u+v}{2}\right) + k_1\rho_{p(\cdot),h_1(\cdot)}(\nabla u - \nabla v) \le \frac{1}{2}E(u) + \frac{1}{2}E(v) \quad \text{for all } u, v \in Y$$

and

$$E((1-\tau)u + \tau v) \le (1-\tau)E(u) + \tau E(v) \quad \text{for all } u, v \in Y \text{ and } \tau \in [0,1].$$

(iii) *There exists a constant* $c_3 > 0$ *such that* $|F(x,t)| \le c_3(1 + |t|^{q(x)})$ *for all* $t \in \mathbb{R}$ *and a.e.* $x \in \Omega$.

(iv) *There exists* $\gamma \in L^\infty(\Omega)$ *such that* $\gamma(x) > 0$ *a.e.* $x \in \Omega$ *and* $F(x,t) \ge \gamma(x)t^\theta$ *for all* $t \in [t_0, \infty)$ *and a.e.* $x \in \Omega$.

(v) *There exists a constant* $c_4 > 0$ *such that* $|G(x,t)| \le c_4(1 + |t|^{r(x)})$ *for all* $t \in \mathbb{R}$ *and a.e.* $x \in \Gamma_2$.

(vi) *There exists* $\delta \in L^\infty(\Gamma_2)$ *such that* $\delta(x) > 0$ *a.e.* $x \in \Gamma_2$ *and* $G(x,t) \ge \delta(x)t^\theta$ *for all* $t \in [t_0, \infty)$ *and a.e.* $x \in \Gamma_2$.

Proof. (i) Using (A1), we have

$$\begin{aligned}
|A(x,\boldsymbol{\xi})| &= |A(x,\boldsymbol{\xi}) - A(x,\mathbf{0})| \\
&= \left|\int_0^1 \frac{d}{dt}A(x,t\boldsymbol{\xi})dt\right| \\
&= \left|\int_0^1 \boldsymbol{a}(x,t\boldsymbol{\xi})\cdot\boldsymbol{\xi}dt\right| \\
&\le c_0\int_0^1 (h_0(x) + h_1(x)t^{p(x)-1}|\boldsymbol{\xi}|^{p(x)-1})|\boldsymbol{\xi}|dt \\
&\le c_0(h_0(x)|\boldsymbol{\xi}| + h_1(x)|\boldsymbol{\xi}|^{p(x)}).
\end{aligned}$$

(ii) The first inequality easily follows from (A2). Since $A(x,\boldsymbol{\xi})$ is continuous with respect to $\boldsymbol{\xi}$, it follows from (A2) that $A(x,(1-\tau)\boldsymbol{\xi} + \tau\boldsymbol{\eta}) \le (1-\tau)A(x,\boldsymbol{\xi}) + \tau A(x,\boldsymbol{\eta})$ for all $\boldsymbol{\xi},\boldsymbol{\eta} \in \mathbb{R}^d$ and $\tau \in [0,1]$, so the second inequality follows from this inequality.

(iii) From (F1),

$$|F(x,t)| = \left|\int_0^t f(x,\tau)d\tau\right| \le c_1\left|\int_0^t (1 + |\tau|^{q(x)-1})d\tau\right| \le c_1\left(|t| + \frac{1}{q(x)}|t|^{q(x)}\right).$$

Since $q(x) > 1$, we have $|t| \le 1 + |t|^{q(x)}$, so (iii) follows.

(iv) From (F2), for $t \ge t_0$,

$$0 < \theta F(x,t) \le f(x,t)t. \tag{4.5}$$

Put $\gamma(x) = F(x, t_0)t_0^{-\theta}$. Then $\gamma(x) > 0$ a.e. $x \in \Omega$ and it follows from (iii) that

$$\gamma(x) \le c_3(1 + t_0^{q(x)})t_0^{-\theta} \le c_3(1 + \max\{t_0^{q^+}, t_0^{q^-}\})t_0^{-\theta} < \infty.$$

So $\gamma \in L^\infty(\Omega)$. From (4.5),

$$\frac{\theta}{\tau} \le \frac{f(x,\tau)}{F(x,\tau)} = \frac{\frac{\partial F}{\partial \tau}(x,\tau)}{F(x,\tau)}.$$

Integrating this inequality over (t_0, t), we have

$$\theta \log \frac{t}{t_0} \le \log \frac{F(x,t)}{F(x,t_0)} \quad \text{for all } t \ge t_0.$$

This implies $F(x,t) \ge \gamma(x)t^\theta$ for all $t \ge t_0$.

(v) and (vi) follow from the same arguments as (iii) and (iv), respectively. □

Proposition 4.2. *The functionals $E, J, K \in C^1(Y, \mathbb{R})$ and the Fréchet derivatives E', J' and K' satisfy the following equalities.*

$$\langle E'(u), v\rangle_{Y^*,Y} = \int_\Omega \boldsymbol{a}(x, \nabla u(x)) \cdot \nabla v(x)dx, \tag{4.6}$$

$$\langle J'(u), v\rangle_{Y^*,Y} = \int_\Omega f(x, u(x))v(x)dx, \tag{4.7}$$

$$\langle K'(u), v\rangle_{Y^*,Y} = \int_{\Gamma_2} g(x, u(x))v(x)d\sigma \tag{4.8}$$

for all $u, v \in Y$.

Proof. Step 1. We show that E is continuous on Y. Let $u_n \to u$ in Y as $n \to \infty$. Then from (2.4),

$$\|h_1^{1/p(\cdot)}\nabla u_n - h_1^{1/p(\cdot)}\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \to 0 \quad \text{as } n \to \infty. \tag{4.9}$$

From [2, Proposition A.1], there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $k \in L^{p(\cdot)}(\Omega)$ such that $h_1(x)^{1/p(x)}\nabla u_{n'}(x) \to h_1(x)^{1/p(x)}\nabla u(x)$ a.e. $x \in \Omega$, and since $h_1(x) \ge 1$ a.e. $x \in \Omega$,

$$|\nabla u_{n'}(x)| \le |h_1(x)^{1/p(x)}\nabla u_{n'}(x)| \le k(x) \quad \text{a.e. } x \in \Omega.$$

In particular, $\nabla u_{n'}(x) \to \nabla u(x)$ a.e. $x \in \Omega$. Since $A(x, \boldsymbol{\xi})$ is a Carathéodory function, $A(x, \nabla u_{n'}(x)) \to A(x, \nabla u(x))$ a.e. $x \in \Omega$ as $n' \to \infty$. By Lemma 4.1 (i),

$$|A(x, \nabla u_{n'}(x))| \le c_0(h_0(x)|\nabla u_{n'}(x)| + h_1(x)|\nabla u_{n'}(x)|^{p(x)}) \le c_0(h_0(x)k(x) + k(x)^{p(x)}).$$

Since $h_0 \in L^{p'(\cdot)}(\Omega)$ and $k \in L^{p(\cdot)}(\Omega)$, taking the Hölder inequality (Proposition 2.2) into consideration, we see that the last term is an integrable function independent of n'. By the Lebesgue dominated convergence theorem, we have

$$\lim_{n' \to \infty} \int_\Omega A(x, \nabla u_{n'}(x))dx = \int_\Omega A(x, \nabla u(x))dx.$$

By the convergent principle (cf. Zeidler [26, Proposition 10.13 (i)], for the full sequence $\{u_n\}$,

$$\lim_{n \to \infty} \int_\Omega A(x, \nabla u_n(x))dx = \int_\Omega A(x, \nabla u(x))dx.$$

This means that $E(u_n) \to E(u)$ as $n \to \infty$, so the functional E is continuous in Y.

Step 2. We derive that E is Gateaux differentiable in Y. Let $u, v \in Y$ and $0 < |t| \le 1$. By the mean value theorem,

$$\begin{aligned}\frac{E(u+tv)-E(u)}{t} &= \int_\Omega \frac{A(x,\nabla u(x)+t\nabla v(x))-A(x,\nabla u(x)}{t}dx \\ &= \int_\Omega \int_0^1 \boldsymbol{a}(x,\nabla u(x)+\tau t\nabla v(x))\cdot\nabla v(x)d\tau dx.\end{aligned}$$

From (A1), we have

$$\begin{aligned}&|\boldsymbol{a}(x,\nabla u(x)+\tau t\nabla v(x))\cdot\nabla v(x)| \\ &\quad = c_0(h_0(x)+h_1(x)|\nabla u(x)+\tau t\nabla v(x)|^{p(x)-1})|\nabla v(x)| \\ &\quad \le c_0(h_0(x)|\nabla v(x)|+h_1(x)^{1/p(x)}|\nabla v(x)|h_1(x)^{(p(x)-1)/p(x)}(|\nabla u(x)|+|\nabla v(x)|)^{p(x)-1}) \\ &\quad \le c_0(h_0(x)|\nabla v(x)|+h_1(x)^{1/p(x)}|\nabla v(x)|\big((h_1(x)^{1/p(x)}(|\nabla u(x)|+|\nabla v(x)|)\big)^{p(x)-1}.\end{aligned}$$

Here since $u, v \in Y, h_0 \in L^{p'(\cdot)}(\Omega), h_1^{1/p(\cdot)}|\nabla v| \in L^{p(\cdot)}(\Omega)$ and

$$\big((h_1(\cdot)^{1/p(\cdot)}|\nabla u(\cdot)|+h_1(\cdot)^{1/p(\cdot)}|\nabla v(x)|)\big)^{p(\cdot)-1} \in L^{p'(\cdot)}(\Omega),$$

it follows from the Hölder inequality (Proposition 2.2), the last term of the above inequality is an integrable function independent of t. On the other hand, $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function, we have

$$\boldsymbol{a}(x,\nabla u(x)+\tau t\nabla v(x))\cdot\nabla v(x) \to \boldsymbol{a}(x,\nabla u(x))\cdot\nabla v(x)$$

as $t \to 0$. Using again the Lebesgue dominated convergence theorem, we have

$$\frac{E(u+tv)-E(u)}{t} \to \int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla v(x)dx \quad \text{as } t \to 0.$$

Thus E is Gateaux differentiable at u and the Gateaux derivative DE satisfies

$$DE(u)(v) = \int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla v(x)dx.$$

Clearly $DE(u)$ is linear in Y.

Step 3. We show that for every $u \in Y$, we have $DE(u) \in Y^*$. For any $v \in Y$,

$$\begin{aligned}DE(u)(v) &= \int_\Omega \boldsymbol{a}(x,\nabla u(x))\cdot\nabla v(x)dx \\ &= \int_\Omega h_1(x)^{-1/p(x)}\boldsymbol{a}(x,\nabla u(x))\cdot h_1(x)^{1/p(x)}\nabla v(x)dx.\end{aligned}$$

We note that $\|v\|_Y = \|h_1^{1/p(\cdot)}\nabla v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$ from (2.4). On the other hand, from (A1),

$$\begin{aligned}&\rho_{p'(\cdot)}(h_1^{-1/p(\cdot)}\boldsymbol{a}(\cdot,\boldsymbol{u}(\cdot))) \\ &\quad = \int_\Omega h_1(x)^{-p'(x)/p(x)}|\boldsymbol{a}(x,\nabla u(x))|^{p'(x)}dx \\ &\quad \le \int_\Omega h_1(x)^{-p'(x)/p(x)}\big(c_0(h_0(x)+h_1(x)|\nabla u(x)|^{p(x)-1}\big)^{p'(x)}dx \\ &\quad \le \max\left\{c_0^{(p')^+}, c_0^{(p')^-}\right\}2^{(p')^+-1}\int_\Omega (h_0(x)^{p'(x)}+h_1(x)|\nabla u(x)|^{p(x)})dx < \infty.\end{aligned}$$

Hence $h_1^{-1/p(\cdot)}\boldsymbol{a}(\cdot,\nabla u) \in \boldsymbol{L}^{p'(\cdot)}(\Omega)$. By the Hölder inequality (Proposition 2.2), we have

$$|DE(u)(v)| \le 2\|h_1^{-1/p(\cdot)}\boldsymbol{a}(\cdot,\nabla u(\cdot))\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}\|v\|_Y \quad \text{for all } v \in Y.$$

Hence we see that $DE(u) \in Y^*$ and

$$\|DE(u)\|_{Y^*} \le 2\|h_1^{-1/p(\cdot)}\boldsymbol{a}(\cdot,\nabla u(\cdot))\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}. \tag{4.10}$$

Step 4. We derive that the map $Y \ni u \mapsto DE(u) \in Y^*$ is continuous. Let $u_n \to u$ in Y as $n \to \infty$. Then (4.9) holds. So there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $\widetilde{k} \in L^{p(\cdot)}(\Omega)$ such that $\nabla u_{n'}(x) \to \nabla u(x)$ a.e. $x \in \Omega$ and $h_1(x)^{1/p(x)}|\nabla u_{n'}(x)| \le \widetilde{k}(x)$ a.e. $x \in \Omega$ and all n'. By (4.10),

$$\|DE(u_{n'}) - DE(u)\|_{Y^*} \le 2\|h_1(\cdot)^{-1/p(\cdot)}\big(\boldsymbol{a}(\cdot,\nabla u_{n'}(\cdot)) - \boldsymbol{a}(\cdot,\nabla u(\cdot))\big)\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}.$$

In order to show that the right-hand side converges to zero, taking Proposition 2.1 into consideration, it suffices to derive that

$$\rho_{p'(\cdot)}\big(h_1(\cdot)^{-1/p(\cdot)}(\boldsymbol{a}(\cdot,\nabla u_{n'}(\cdot)) - \boldsymbol{a}(\cdot,\nabla u(\cdot)))\big) \to 0 \quad \text{as } n' \to \infty,$$

that is,

$$\int_\Omega h_1(x)^{-p'(x)/p(x)}|\boldsymbol{a}(x,\nabla u_{n'}(x)) - \boldsymbol{a}(x,\nabla u(x))|^{p'(x)}dx \to 0 \quad \text{as } n' \to \infty. \tag{4.11}$$

Since $\boldsymbol{a}(x,\boldsymbol{\xi})$ is a Carathéodory function, and $\nabla u_{n'}(x) \to \nabla u(x)$ a.e. $x \in \Omega$, we have

$$h_1(x)^{-p'(x)/p(x)}|\boldsymbol{a}(x,\nabla u_{n'}(x)) - \boldsymbol{a}(x,\nabla u(x))|^{p'(x)} \to 0 \quad \text{a.e. } x \in \Omega.$$

As in the argument in Step 3, we have

$$h_1(x)^{-p'(x)/p(x)}|\boldsymbol{a}(x,\nabla u_{n'}(x))|^{p'(x)} \le \max\left\{c_0^{(p')^+}, c_0^{(p')^-}\right\} 2^{(p')^+-1}(h_0(x)^{p'(x)} + \widetilde{k}(x)^{p(x)}).$$

The right-hand side is an integrable function in Ω independent of n'. By the Lebesgue dominated convergence theorem, (4.11) holds. Thus $\|DE(u_{n'}) - DE(u)\|_{Y^*} \to 0$ as $n' \to \infty$. By the convergent principle (cf. [26, Proposition 10.13 (i)], for full sequence $\{u_n\}$ we have $\|DE(u_n) - DE(u)\|_{Y^*} \to 0$ as $n \to \infty$. Therefore, since the Gateaux differential DE is continuous in Y, we see that E is Fréchet differentiable and the Fréchet derivative E' is equal to the Gateaux derivative DE. Hence $E \in C^1(Y,\mathbb{R})$ and (4.6) holds.

Step 5. We show that J and K belong to $C^1(Y,\mathbb{R})$ and (4.7) and (4.8) hold. By Lemma 4.1 (iii) and [2, Proposition 2.12], the Nemytskii operator $N_F : L^{q(\cdot)}(\Omega) \ni u \mapsto F(\cdot,u(\cdot)) \in L^1(\Omega)$ is continuous. From (F1), we have $Y \hookrightarrow X \hookrightarrow L^{q(\cdot)}(\Omega)$, so N_F is continuous in Y, so we see that J is continuous in Y. Since $F(x,t)$ is a C^1-function with respect to t, clearly J is Gateaux differentiable in Y and

$$DJ(u)(v) = \int_\Omega f(x,u(x))v(x)dx \quad \text{for all } u,v \in Y.$$

By the Hölder inequality (Proposition 2.2),

$$|DJ(u)(v)| \le 2\|f(\cdot,u(\cdot))\|_{L^{q'(\cdot)}(\Omega)}\|v\|_{L^{q(\cdot)}(\Omega)} \le C\|f(\cdot,u(\cdot))\|_{L^{q'(\cdot)}(\Omega)}\|v\|_Y \quad \text{for all } v \in Y.$$

Hence $DJ(u) \in Y^*$ and $\|DJ(u)\|_{Y^*} \le C\|f(\cdot,u(\cdot))\|_{L^{q'(\cdot)}(\Omega)}$. Since $|f(x,t)| \le c_1(1+|t|^{q(x)-1}) = c_1(1+|t|^{q(x)/q'(x)})$ from (F1), Nemytskii operator $N_f : u \mapsto f(\cdot,u(\cdot))$ is continuous from $L^{q(\cdot)}(\Omega)$ to $L^{q'(\cdot)}(\Omega)$ (cf. [1, Proposition 2.9]). Thus if $u_n \to u$ in $L^{q(\cdot)}(\Omega)$, then

$$\|f(\cdot,u_n(\cdot)) - f(\cdot,u(\cdot))\|_{L^{q'(\cdot)}(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Since $Y \hookrightarrow X \hookrightarrow L^{q(\cdot)}(\Omega)$, we can see that $J \in C^1(Y,\mathbb{R})$ and (4.7) holds. Similarly, we can prove that $K \in C^1(Y,\mathbb{R})$ and (4.8) holds. □

Remark 4.3. When $p(\cdot) = p =$ const. and $\Gamma_2 = \emptyset$, the authors of [10] only prove the weakly continuously differentiable on Y, and so they must use a version of the Mountain-pass lemma introduced in [9]. However, since we derived that E belongs to $C^1(Y,\mathbb{R})$, it suffices to use the standard Mountain-pass lemma later.

Proposition 4.4.

(i) *The functionals J and K are weakly continuous in Y, that is, if $u_n \to u$ weakly in Y as $n \to \infty$, then $J(u_n) \to J(u)$ and $K(u_n) \to K(u)$ as $n \to \infty$.*

(ii) *The functional E is weakly lower semi-continuous in Y, that is, if $u_n \to u$ weakly in Y as $n \to \infty$, then $E(u) \le \liminf_{n\to\infty} E(u_n)$.*

(iii) *$E(u) - E(v) \ge \langle E'(v), u - v\rangle_{Y^*,Y}$ for all $u, v \in Y$.*

Proof. (i) Let $u_n \to u$ weakly in Y as $n \to \infty$. Since the embedding $Y \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact, we see that $u_n \to u$ strongly in $L^{q(\cdot)}(\Omega)$. Since J and K are continuous on $L^{q(\cdot)}(\Omega)$, we see that $J(u_n) \to J(u)$ and $K(u_n) \to K(u)$ as $n \to \infty$.

(ii) $A(x,\boldsymbol{\xi})$ is a Carathéodory function on $\Omega \times \mathbb{R}^d$ and $A(x,\boldsymbol{\xi}) \ge 0$ by (A4). Moreover, from (A2), $A(x,\boldsymbol{\xi})$ is convex with respect to $\boldsymbol{\xi}$ for a.e. $x \in \Omega$. If $u_n \to u$ weakly in Y, then $u_n, u \in W^{1,1}(\Omega)$ and $u_n \to u$ strongly in $L^1(\Omega)$ and $\nabla u_n \to \nabla u$ weakly in $\boldsymbol{L}^1(\Omega)$. Hence it follows from Struwe [21, Theorem 1.6, p. 9] that $E(u) \le \liminf_{n\to\infty} E(u_n)$.

(iii) Since E is convex function in Y, for $u, v \in Y$ and $0 < \tau < 1$,

$$\begin{aligned}\frac{E(v+\tau(u-v)) - E(v)}{\tau} &= \frac{E((1-\tau)v + \tau u) - E(v)}{\tau} \\ &\le \frac{(1-\tau)E(v) + \tau E(u) - E(v)}{\tau} \\ &= E(u) - E(v).\end{aligned}$$

Letting $\tau \to +0$, we get $\langle E'(v), u - v\rangle_{Y^*,Y} \le E(u) - E(v)$, so (iii) holds. □

Lemma 4.5.

(i) *There exist constants $k_3 > 0$ and $c_3 > 0$ such that*

$$I(u) \ge \|u\|_Y^{p^+}\left(k_3 - c_3\left(\|u\|_Y^{q^- - p^+} + \|u\|_Y^{r^- - p^+}\right)\right) \text{ for all } u \in Y \text{ with } \|u\|_Y < 1.$$

(ii) *There exist constants $c_3 > 0$ and $k_4 \in \mathbb{R}$ such that*

$$I(u) \ge \|u\|_Y\left(c_4 \min\left\{\|u\|_Y^{p^+-1}, \|u\|_Y^{p^--1}\right\} - \frac{1}{\theta}\|I'(u)\|_{Y^*}\right) + k_4 \quad \text{for all } u \in Y.$$

Proof. (i) From (F3), for a.e. $x \in \Omega$,

$$F(x,t) = \int_0^t f(x,s)dx \leq \frac{\lambda}{p^+}|t|^{p^+} \quad \text{for all } t \in (-\delta,\delta).$$

On the other hand, by Lemma 4.1 (iii), there exists $c_3' > 0$ such that $|F(x,t)| \leq c_3'|t|^{q(x)}$ for all $t \in \mathbb{R} \setminus (-\delta,\delta)$. Hence

$$F(x,t) \leq \frac{\lambda}{p^+}|t|^{p^+} + c_3'|t|^{q(x)} \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Therefore, we have

$$\begin{aligned} J(u) = \int_\Omega F(x,u(x))dx &\leq \frac{\lambda}{p^+}\int_\Omega |u(x)|^{p^+}dx + c_3'\int_\Omega |u(x)|^{q(x)}dx \\ &\leq \frac{\lambda}{p^+}\|u\|^{p^+}_{L^{p^+}(\Omega)} + c_3' \max\left\{\|u\|^{q^+}_{L^{q(\cdot)}(\Omega)}, \|u\|^{q^-}_{L^{q(\cdot)}(\Omega)}\right\}. \end{aligned}$$

Similarly, there exists $c_4' > 0$ such that

$$K(u) \leq \frac{\mu}{p^+}\|u\|^{p^+}_{L^{p^+}(\Gamma_2)} + c_4' \max\left\{\|u\|^{p^+}_{L^{r(\cdot)}(\Gamma_2)}, \|u\|^{r^-}_{L^{r(\cdot)}(\Gamma_2)}\right\}.$$

Since $p^+ < q^- \leq q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ from (F1), we have $Y \hookrightarrow X \hookrightarrow L^{p^+}(\Omega)$, $L^{q(\cdot)}(\Omega)$. By (2.5), $\|u\|_{L^{p^+}(\Omega)} \leq \frac{1}{\lambda_{p^+}}\|u\|_Y$ and $\|u\|_{L^{q(\cdot)}(\Omega)} \leq \frac{1}{\lambda_q}\|u\|_Y$ for all $u \in Y$. Since we have $p^+ < r^- \leq r(x) < p^\partial(x)$ for all $x \in \overline{\Gamma_2}$ from (G2), it follows from (2.6) that we can see that $Y \hookrightarrow X \hookrightarrow L^{p^+}(\Gamma_2), L^{r(\cdot)}(\Gamma_2)$. Thus we have $\|u\|_{L^{p^+}(\Gamma_2)} \leq \frac{1}{\mu_{p^+}}\|u\|_Y$ and $\|u\|_{L^{r(\cdot)}(\Gamma_2)} \leq \frac{1}{\mu_r}\|u\|_Y$ for all $u \in Y$. When $\|u\|_Y < 1$, there exist positive constants c_5 and c_6 such that

$$\begin{aligned} J(u) &\leq \frac{\lambda}{p^+}\frac{1}{(\lambda_{p^+})^{p^+}}\|u\|_Y^{p^+} + c_5\|u\|_Y^{q^-}, \\ K(u) &\leq \frac{\mu}{p^+}\frac{1}{(\mu_{p^+})^{p^+}}\|u\|_Y^{p^+} + c_6\|u\|_Y^{r^-}. \end{aligned}$$

On the other hand, from (A4),

$$E(u) = \int_\Omega A(x,\nabla u(x))dx \geq k_0\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}dx \geq k_0\|u\|_Y^{p^+}.$$

Thus we have

$$\begin{aligned} I(u) = E(u) - J(u) - K(u) &\geq \frac{k_0}{2}\|u\|_Y^{p^+} - c_5\|u\|_Y^{q^-} - c_6\|u\|_Y^{r^-} \\ &= \|u\|_Y^{p^+}\left(k_3 - c_3(\|u\|_Y^{q^- - p^+} + \|u\|_Y^{r^- - p^+})\right), \end{aligned}$$

where $k_3 = k_0/2$ and $c_3 = \max\{c_5, c_6\}$ for all $u \in Y$ with $\|u\|_Y < 1$.

(ii) From (A3) and (A4), for any $u \in Y$,

$$\begin{aligned} E(u) - \frac{1}{\theta}\langle E'(u), u\rangle_{Y^*,Y} &= \int_\Omega A(x,\nabla u(x))dx - \frac{1}{\theta}\int_\Omega \boldsymbol{a}(x,\nabla u(x)) \cdot \nabla u(x)dx \\ &\geq \int_\Omega A(x,\nabla u(x))dx - \frac{1}{\theta}\int_\Omega p(x)A(x,\nabla u(x))dx \\ &\geq \left(1 - \frac{p^+}{\theta}\right)\int_\Omega A(x,\nabla u(x))dx \\ &\geq \left(1 - \frac{p^+}{\theta}\right)k_0\int_\Omega h_1(x)|\nabla u(x)|^{p(x)}dx \\ &\geq c_4 \min\left\{\|u\|_Y^{p^+}, \|u\|_Y^{p^-}\right\}, \end{aligned}$$

where $c_4 = k_0(1 - p^+/\theta) > 0$. Put $\Omega_u = \{x \in \Omega; |u(x)| > t_0\}$ and $\Gamma_u = \{x \in \Gamma_2; |u(x)| > t_0\}$. From (F2) and (G2),

$$\frac{1}{\theta}f(x,u(x))u(x) - F(x,u(x)) \geq 0 \quad \text{for a.e. } x \in \Omega_u,$$
$$\frac{1}{\theta}g(x,u(x))u(x) - G(x,u(x)) \geq 0 \quad \text{for a.e. } x \in \Gamma_u,$$

and there exists a constant $M > 0$ such that

$$\left|\frac{1}{\theta}f(x,u(x))u(x) - F(x,u(x))\right| \leq M \quad \text{for a.e. } x \in \Omega \setminus \Omega_u,$$
$$\left|\frac{1}{\theta}g(x,u(x))u(x) - G(x,u(x))\right| \leq M \quad \text{for a.e. } x \in \Gamma_2 \setminus \Gamma_u.$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{\theta}\langle J'(u), u\rangle_{Y^*,Y} - J(u) \\
&\quad = \int_{\Omega_u}\left(\frac{1}{\theta}f(x,u(x))u(x) - F(x,u(x))\right)dx + \int_{\Omega\setminus\Omega_u}\left(\frac{1}{\theta}f(x,u(x))u(x) - F(x,u(x))\right)dx \\
&\quad \geq -M|\Omega \setminus \Omega_u| \geq -M|\Omega|
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\theta}\langle K'(u), u\rangle_{Y^*,Y} - K(u) \\
&\quad = \int_{\Gamma_u}\left(\frac{1}{\theta}g(x,u(x))u(x) - G(x,u(x))\right)d\sigma + \int_{\Gamma_2\setminus\Gamma_u}\left(\frac{1}{\theta}g(x,u(x))u(x) - G(x,u(x))\right)d\sigma \\
&\quad \geq -M|\Gamma_2 \setminus \Gamma_u| \geq -M|\Gamma_2|.
\end{aligned}$$

Put $k_4 = -M|\Omega| - M|\Gamma_2|$. Summing up, we have

$$\begin{aligned}
&I(u) - \frac{1}{\theta}\langle I'(u), u\rangle_{Y^*,Y} \\
&\quad = E(u) - \frac{1}{\theta}\langle E'(u), u\rangle_{Y^*,Y} - J(u) + \frac{1}{\theta}\langle J'(u), u\rangle_{Y^*,Y} - K(u) + \frac{1}{\theta}\langle K'(u), u\rangle_{Y^*,Y} \\
&\quad \geq c_4 \min\left\{\|u\|_Y^{p^+}, \|u\|_Y^{p^-}\right\} + k_4.
\end{aligned}$$

Hence

$$\begin{aligned}
I(u) &\geq c_4 \min\left\{\|u\|_Y^{p^+}, \|u\|_Y^{p^-}\right\} + \frac{1}{\theta}\langle I'(u), u\rangle_{Y^*,Y} + k_4 \\
&\geq c_4 \min\left\{\|u\|_Y^{p^+}, \|u\|_Y^{p^-}\right\} - \frac{1}{\theta}\|I'(u)\|_{Y^*}\|u\|_Y + k_4.
\end{aligned}$$ □

For a proof of Theorem 3.3, we apply the following standard Mountain-pass lemma (cf. Willem [23]).

Proposition 4.6. *Let $(V, \|\cdot\|_V)$ be a Banach space and $I \in C^1(V, \mathbb{R})$ be a functional satisfying the Palais–Smale condition, that is, if a sequence $\{u_n\} \subset V$ satisfies that $\lim_{n\to\infty} I(u_n) = \gamma$ exists and $\lim_{n\to\infty}\|I'(u_n)\|_{V^*} = 0$, then $\{u_n\}$ has a convergent subsequence. Assume that $I(0) = 0$, and there exist $\rho > 0$ and $z_0 \in V$ such that $\|z_0\|_V > \rho$, $I(z_0) \leq I(0) = 0$ and*

$$\alpha := \inf\{I(u); u \in V \text{ with } \|u\|_V = \rho\} > 0.$$

Put $G = \{\varphi \in C([0,1], V); \varphi(0) = 0, \varphi(1) = z_0\}$ and $\beta = \inf\{\max I(\varphi([0,1])); \varphi \in G\}$. Then $\beta \geq \alpha$ and β is a critical value of I.

We apply Proposition 4.6 with $(V, \|\cdot\|_V) = (Y, \|\cdot\|_Y)$. In order to do so, we must show the following proposition.

Proposition 4.7.

(i) *The functional I satisfies the Palais–Smale condition.*

(ii) $I(0) = 0$.

(iii) *There exists $\rho > 0$ such that* $\inf\{I(u); u \in Y \text{ with } \|u\|_Y = \rho\} > 0$.

(iv) *There exists $z_0 \in Y$ such that $\|z_0\|_Y > \rho$ and $I(z_0) \le 0$.*

(v) $G \neq \emptyset$.

Proof. (i) Assume that a sequence $\{u_n\} \subset Y$ satisfies that $\lim_{n\to\infty} I(u_n) = \gamma$ exists and $\lim_{n\to\infty} \|I'(u_n)\|_{Y^*} = 0$.

Step 1. The sequence $\{u_n\}$ is bounded in Y. Indeed, if $\{u_n\}$ is unbounded, there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\|u_{n'}\| \ge n'$ for any $n' \in \mathbb{N}$. By Lemma 4.5 (ii),

$$I(u_{n'}) \ge \|u_{n'}\|_Y \left(c_4 \|u_{n'}\|_Y^{p^- -1} - \frac{1}{\theta} \|I'(u_{n'})\|_{Y^*} \right) + k_4 \to \infty \quad \text{as } n' \to \infty.$$

This contradicts $\lim_{n'\to\infty} I(u_{n'}) = \gamma$.

Step 2. Since $\{u_n\}$ is bounded in Y and Y is a reflexive Banach space, passing to a subsequence, we may assume that $u_n \to u$ weakly in Y. By Proposition 4.4 (ii) and (iii),

$$E(u) \le \liminf_{n\to\infty} E(u_n) = \lim_{n\to\infty} (J(u_n) + K(u_n) + I(u_n)) = J(u) + K(u) + \gamma.$$

Since $\{\|u_n - u\|_Y\}$ is a bounded sequence and $\lim_{n\to\infty} \|I'(u_n)\|_{Y^*} = 0$, we see that $\langle I'(u_n), u - u_n\rangle_{Y^*,Y} \to 0$ as $n \to \infty$. By the Rellich–Kondrachov theorem, $u_n \to u$ strongly in $L^{q(\cdot)}(\Omega)$ and $u_n \to u$ strongly in $L^{r(\cdot)}(\Gamma_2)$. By (F1) and (G1), $|f(\cdot, u_n(\cdot))|$ is bounded in $L^{q'(\cdot)}(\Omega)$ and $|g(\cdot, u_n(\cdot))|$ is also bounded in $L^{r'(\cdot)}(\Gamma_2)$. Hence

$$\lim_{n\to\infty} \langle J'(u_n), u - u_n\rangle_{Y^*,Y} = \lim_{n\to\infty} \int_\Omega f(x, u_n(x))(u(x) - u_n(x))dx = 0$$

and

$$\lim_{n\to\infty} \langle K'(u_n), u - u_n\rangle_{Y^*,Y} = \lim_{n\to\infty} \int_{\Gamma_2} g(x, u_n(x))(u(x) - u_n(x))d\sigma = 0.$$

Therefore.

$$\begin{aligned} &\lim_{n\to\infty} \langle E'(u_n), u - u_n\rangle_{Y^*,Y} \\ &\qquad = \lim_{n\to\infty} \big(\langle I'(u_n), u - u_n\rangle_{Y^*,Y} + \langle J'(u_n), u - u_n\rangle_{Y^*,Y} + \langle K'(u_n), u - u_n\rangle_{Y^*,Y} \big) = 0. \end{aligned}$$

On the other hand, by Proposition 4.4 (iii) and the above equality,

$$E(u) - \limsup_{n\to\infty} E(u_n) = \liminf_{n\to\infty} (E(u) - E(u_n)) \ge \lim_{n\to\infty} \langle E'(u_n), u - u_n\rangle_{Y^*,Y} = 0.$$

Thus by Proposition 4.4 (ii), we have $\lim_{n\to\infty} E(u_n) = E(u)$.

Step 3. We show that $u_n \to u$ strongly in Y, that is, $\rho_{p(\cdot),h_1(\cdot)}(\nabla u_n - \nabla u) \to 0$ as $n \to \infty$. If this is not satisfied, there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $\varepsilon_0 > 0$ such that

$$\rho_{p(\cdot),h_1(\cdot)}(\nabla u_{n'} - \nabla u) \geq \varepsilon_0 \quad \text{for all } n' \in \mathbb{N}.$$

By Lemma 4.1,

$$\frac{1}{2}E(u_{n'}) + \frac{1}{2}E(u) - E\left(\frac{u_{n'}+u}{2}\right) \geq k_1\rho_{p(\cdot),h_1(\cdot)}(\nabla u_{n'} - \nabla u) \geq k_1\varepsilon_0.$$

Letting $n' \to \infty$ and using Step 2, we have

$$E(u) - \liminf_{n'\to\infty} E\left(\frac{u_{n'}+u}{2}\right) \geq k_1\varepsilon_0. \tag{4.12}$$

On the other hand, since $\frac{u_{n'}+u}{2} \to u$ weakly in Y, it follows from Proposition 4.4 (ii) that

$$E(u) \leq \liminf_{n'\to\infty} E\left(\frac{u_{n'}+u}{2}\right).$$

This contradicts (4.12).

(ii) Since $E(0) = J(0) = K(0) = 0$, we have $I(0) = 0$.

(iii) Since $k_3 > 0$, $q^- > p^+$ and $r^- > p^+$, there exists $0 < \rho < 1$ such that

$$\rho^{p^+}(k_3 - c_3\rho^{q^- - p^+} - c_3\rho^{r^- - p^+}) > 0.$$

By Lemma 4.5 (i),

$$\begin{aligned} I(u) &\geq \|u\|_Y^{p^+}(k_3 - c_3\|u\|_Y^{q^- - p^+} - c_3\|u\|_Y^{r^- - p^+}) \\ &= \rho^{p^+}(k_3 - c_3\rho^{q^- - p^+} - c_3\rho^{r^- - p^+}) > 0 \end{aligned}$$

for all $u \in Y$ with $\|u\|_Y = \rho$.

(iv) Let $t > 1$ and choose $v_0 \in C_0^\infty(\Omega)(\subset Y)$ such that $v_0(x) \geq 0$ and $W = \{x \in \Omega; v_0(x) \geq t_0\}$ has a positive measure. By (F2), $F(x, v_0(x)) > 0$ for a.e. $x \in \Omega$. If we put $W_t = \{x \in \Omega; tv_0(x) \geq t_0\}$, then $W \subset W_t$. By Lemma 4.1 (iv),

$$\int_{W_t} F(x, tv_0(x))dx \geq \int_{W_t} \gamma(x)t^\theta v_0(x)^\theta dx \geq t^\theta L(v_0),$$

where $L(v_0) = \int_W \gamma(x)v_0(x)^\theta dx > 0$. By Lemma 4.1 (iii), there exists a constant $M > 0$ such that $|F(x,t)| \leq M$ for $t \in [0, t_0]$ and a.e. $x \in \Omega$. We note that (F2) implies that

$$F(x, st) \geq F(x,t)s^\theta \quad \text{for all } t \in \mathbb{R} \setminus (-t_0, t_0), s > 1 \text{ and a.e. } x \in \Omega. \tag{4.13}$$

Indeed, if we define $g(s) = F(x, st)$, then

$$g'(s) = f(x, st)t = \frac{1}{s}f(x, st)st \geq \frac{\theta}{s}F(x, st) = \frac{\theta}{s}g(s).$$

Thus $g'(s)/g(s) \geq \theta/s$, so $\log g(s)/g(1) \geq \theta \log s$. This implies $g(s) \geq g(1)s^\theta$.

On the other hand, (A3) implies that

$$A(x, s\boldsymbol{\xi}) \leq A(x, \boldsymbol{\xi})s^{p(x)} \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d, \text{ a.e. } x \in \Omega \text{ and } s > 1. \tag{4.14}$$

In fact, if we define $g(s) = A(x, s\boldsymbol{\xi})$, then

$$g'(s) = \boldsymbol{a}(x, s\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \le \frac{p(x)}{s} A(x, s\boldsymbol{\xi}) = \frac{p(x)}{s} g(s).$$

Hence $g'(s)/g(s) \le p(x)/s$. We also get $g(s) \ge g(1)s^{p(x)}$. From (4.14), we have

$$\begin{aligned} E(tv_0) &= \int_\Omega A(x, t\nabla v_0(x))dx \le \int_\Omega A(x, \nabla v_0(x))t^{p(x)}dx \\ &\le t^{p^+} \int_\Omega A(x, \nabla v_0(x))dx = t^{p^+} E(v_0). \end{aligned}$$

Since $v_0 \in C_0^\infty(\Omega)$, we have $K(tv_0) = 0$. Therefore,

$$\begin{aligned} I(tv_0) &= E(tv_0) - J(tv_0) \\ &= E(tv_0) - \int_{W_t} F(x, tv_0(x))dx - \int_{\Omega \setminus W_t} F(x, tv_0(x))dx \\ &\le t^{p^+} E(v_0) - t^\theta L(v_0) + M|\Omega|. \end{aligned}$$

Since $\theta > p^+$ and $L(v_0) > 0$, $I(tv_0) \to -\infty$ as $t \to \infty$. Hence there exists $t_1 > 1$ such that $\|t_1 v_0\|_Y > \rho$ and $I(t_1 v_0) \le 0$. If we put $z_0 = t_1 v_0$, then the conclusion of (iv) holds.

(v) If we define $\varphi(t) = tz_0$, then $\varphi \in G$, so $G \ne \emptyset$. □

Proof of Theorem 3.3. By Propositions 4.2 and 4.7, we see that all the hypotheses in Proposition 4.6 hold. Hence there exists $u_0 \in Y$ such that $0 < \alpha \le I(u_0) = \beta$ and $I'(u_0) = 0$, so

$$\begin{aligned} \langle I'(u_0), v\rangle_{Y^*,Y} &= \int_\Omega \boldsymbol{a}(x, \nabla u_0(x)) \cdot \nabla v(x)dx \\ &= \int_\Omega f(x, u_0(x))v(x)dx + \int_{\Gamma_2} g(x, u_0(x))v(x)d\sigma \text{ for all } v \in Y. \end{aligned}$$

Thus u_0 is a weak solution of (1.1). Since $I(u_0) = \beta > I(0) = 0$, u_0 is a nontrivial weak solution of (1.1). □

Proof of Theorem 3.5. By (F4),

$$F(x,t) = \int_0^t f(x,\tau)d\tau \ge \frac{c}{m}t^m \quad \text{for } 0 \le t \le \delta \text{ and a.e. } x \in \Omega.$$

Choose $\varphi \in C_0^\infty(\Omega)$ so that $0 \le \varphi \le 1$ and $\varphi \not\equiv 0$. Let $0 < t < \delta(< 1)$. Since $A(x,\boldsymbol{\xi})$ is convex with respect to $\boldsymbol{\xi}$ and $A(x,\mathbf{0}) = 0$, we have $A(x, t\boldsymbol{\xi}) = A(x, t\boldsymbol{\xi} + (1-t)\mathbf{0}) \le tA(x,\boldsymbol{\xi})$. Thus

$$\begin{aligned} I(t\varphi) &= E(t\varphi) - J(t\varphi) \\ &= \int_\Omega A(x, t\nabla\varphi(x))dx - \int_\Omega F(x, t\varphi(x))dx \\ &\le t\int_\Omega A(x, \nabla\varphi(x))dx - \frac{c}{m}t^m \int_\Omega \varphi(x)^m dx. \end{aligned}$$

Since $m < 1$ and $\frac{c}{m}\int_\Omega \varphi(x)^m dx > 0$, we see that $I(t\varphi) < 0$ for small $t > 0$. By Lemma 4.5 (i), I is bounded from below on $\overline{B_\rho(0)}$, where $B_\rho(0) = \{v \in Y; \|v\|_Y < \rho\}$. Hence

$$-\infty < \underline{c} := \inf_{v \in \overline{B_\rho(0)}} I(v) < 0.$$

Let $0 < \varepsilon < \inf_{v \in \partial B_\rho(0)} I(v) - \inf_{v \in \overline{B_\rho(0)}} I(v)$. Then there exists $u \in \overline{B_\rho(0)}$ such that

$$\inf_{v \in \overline{B_\rho(0)}} I(v) \le I(u) \le \inf_{v \in \overline{B_\rho(0)}} I(v) + \varepsilon^2.$$

Since $\inf_{v \in \overline{B_\rho(0)}} I(v) < 0$, we can choose $u \in \overline{B_\rho(0)}$ so that $I(u) < 0$. Applying the Ekeland variational principle [11, Theorem 1.1] to the complete metric space $\overline{B_\rho(0)}$, there exists $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$I(u_\varepsilon) \le I(u), \tag{4.15}$$

$$I(u_\varepsilon) \le I(v) + \varepsilon \|v - u_\varepsilon\|_Y \text{ for all } v \in \overline{B_\rho(0)}, \tag{4.16}$$

$$\|u - u_\varepsilon\|_Y \le \varepsilon. \tag{4.17}$$

Define $\Phi : \overline{B_\rho(0)} \to \mathbb{R}$ by $\Phi(v) = I(v) + \varepsilon \|v - u_\varepsilon\|_Y$ for $v \in \overline{B_\rho(0)}$. Since $I(u_\varepsilon) \le I(u) < 0$ and $I(v) > 0$ for all $v \in \partial B_\rho(0)$, we have $u_\varepsilon \in B_\rho(0)$. Choose $\rho' > 0$ small enough, so that if $w \in \overline{B_{\rho'}(0)}$, then $u_\varepsilon + w \in \overline{B_\rho(0)}$. From (4.16), since $\Phi(u_\varepsilon) \le \Phi(u_\varepsilon + w)$ for all $w \in \overline{B_{\rho'}(0)}$. We have

$$\begin{aligned} &\frac{\langle I'(u_\varepsilon), w\rangle_{Y^*,Y} + \varepsilon \|w\|_Y}{\|w\|_Y} \\ &\quad = \frac{\langle I'(u_\varepsilon), tw\rangle_{Y^*,Y} + \varepsilon t \|w\|_Y - (\Phi(u_\varepsilon + tw) - \Phi(u_\varepsilon))}{t\|w\|_Y} + \frac{\Phi(u_\varepsilon + tw) - \Phi(u_\varepsilon)}{t\|w\|_Y} \\ &\quad \ge \frac{\langle I'(u_\varepsilon), tw\rangle_{Y^*,Y} - (I(u_\varepsilon + tw) - I(u_\varepsilon))}{t\|w\|_Y} \to 0 \quad \text{as } t \to +0. \end{aligned}$$

Hence $\langle I'(u_\varepsilon), w\rangle_{Y^*,Y} + \varepsilon \|w\|_Y \ge 0$ for all $w \in \overline{B_\rho(0)}$, so $\langle I'(u_\varepsilon), w\rangle_{Y^*,Y} \ge -\varepsilon \|w\|_Y$. Replacing w with $-w$, we have $|\langle I'(u_\varepsilon), w\rangle_{Y^*,Y}| \le \varepsilon \|w\|_Y$ for all $w \in \overline{B_\rho(0)}$. Thus $\|I'(u_\varepsilon)\|_{Y^*} \le \varepsilon$. Letting $\varepsilon \to 0$, we see that $I(u_\varepsilon) \to \underline{c}$ and $I'(u_\varepsilon) \to 0$ in Y^*. Since I satisfies the Palais–Smale condition in Y and $I \in C^1(Y, \mathbb{R})$, there exist a subsequence $\{u_n\}$ of $\{u_\varepsilon\}$ and $u_2 \in \overline{B_\rho(0)}$ such that $u_n \to u_2$ in Y and $I'(u_2) = 0$. Therefore, u_2 is a weak solution of (1.1). Since $I(u_2) = \underline{c} < 0 = I(0)$, u_2 is a nontrivial weak solution of (1.1). Since $I(u_2) = \underline{c} < 0 < I(u_1)$, we have $u_1 \ne u_2$. This completes the proof of Theorem 3.5. □

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Further study on second order nonlocal problems monitored by an operator: an approach without compactness.

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Abstract. In this note we prove the existence of mild solutions for nonlocal problems governed by semilinear second order differential inclusions which involves a nonlinear term driven by an operator. A first result is obtained in suitable Banach spaces in the lack of compactness both on the fundamental operator, generated by the linear part, and on the nonlinear multivalued term. This purpose is achieved by combining a fixed point theorem, a selection theorem and a containment theorem. Further we provide another existence result in reflexive spaces by using the classical Hahn–Banach theorem and a new selection proposition, proved here, for a multimap guided by an operator. This setting allows us to remove some assumptions required in the previous existence theorem. As a consequence of this last result we obtain the controllability of a problem driven by a wave equation on which an appropriate perturbation acts.

Keywords: semilinear second order differential inclusion, perturbation effect, fundamental system, De Blasi measure of noncompactness, controllability problem, wave equation.

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1 Introduction

This research is a continuation of the recent papers [8] and [9]. In this paper we proceed the study, started in [8], concerning the existence of mild solutions for the following problem driven by a non-autonomous semilinear second order differential inclusion with nonlocal conditions

$$\begin{cases} x''(t) \in A(t)x(t) + F(t, N(t)x), \ t \in J = [0, a] \\ x(0) = g(x) \\ x'(0) = h(x) \end{cases} \qquad (\mathbf{P})_{\mathbf{N}}$$

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where $\{A(t)\}_{t\in J}$ is a family of linear operators generating a fundamental system and $F : J \times X \to \mathcal{P}(X)$, $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J;X);X)$, $g,h : \mathcal{C}(J;X) \to X$ are suitable maps, with X a appropriate Banach space and

$$\hat{\mathcal{C}}_w(\mathcal{C}(J;X);X) = \{f : \mathcal{C}(J;X) \to X : \ f \text{ is } (w\text{-}w)\text{sequentially continuous}\}. \tag{1.1}$$

We remember that in infinite dimensional spaces the nonlocal problems are investigated with different kinds of approach. By using topological methods the existence of mild solutions for these problems is studied with fixed point theorems applied to a suitable solution operator. This often requires strong compactness conditions, which are usually not satisfied in an infinite dimensional framework (see [4,5,8,10,21]).

The purpose of this work is twofold:

(1) to obtain existence results of mild solutions for the abstract problem $(\mathbf{P})_\mathbf{N}$ in the lack of compactness,

in order to establish

(2) controllability of the problem driven by the following non-autonomous wave equation

$$\frac{\partial^2 w}{\partial t^2}(t,\xi) = \frac{\partial^2 w}{\partial \xi^2}(t,\xi) + b(t)\frac{\partial w}{\partial \xi}(t,\xi) + f(t,\xi,\frac{\mathfrak{Im}\, e(\hat{w})}{2e(\hat{w})+1}\int_0^t p(s)\,ds) + u(t,\xi), \tag{E}$$

subjected to the "mixed conditions": the trajectory $w : [0,a] \times \mathbb{R} \to \mathbb{C}$ satisfies, with respect to the first variable, a "periodicity condition" and it has a fixed initial velocity. The control $u(t,\xi)$ belongs to a set of admissible controls. Moreover, $b \in \mathcal{C}^1(J)$, $p \in L^1(J)$, $f : J \times \mathbb{R} \times \mathbb{C} \to \mathbb{C}$, $e : \mathcal{C}(J, L^2(\mathbb{T},\mathbb{C})) \to \mathbb{C}$ are suitable maps, while $\hat{w} : J \to L^2(\mathbb{T},\mathbb{C})$, is defined as $\hat{w}(t) = w(t,\cdot)$ and $\mathfrak{Im}\, e(\hat{w})$ denotes the imaginary part of the complex number $e(\hat{w})$.

The study of aim (1) in *absence* of the operator N is already addressed in [9] by using a combination of two techniques: one technique is based on the concept of measure of noncompactness, while another makes use of the weak topology. This method is also used in [3] in order to prove the existence of mild solutions for problems monitored by semilinear first order differential inclusions.

On the other hand the controllability of the mentioned problem governed by (**E**) is brought back by using classical arguments (see, for example [42]) to purpose (1).

About the study referred in this note, let us recall that the theory of semilinear differential inclusions is well documented in literature. Various aspects of this field catch the attention of many researchers and are widely employed in the study of several dynamical problems arising from physics, economics, biology, social sciences. Several authors have studied abstract semilinear second order equations/inclusions in the autonomous case starting from the initial researches of Kato [25–27] (see, e.g. [15,28,32,36]). On the other hand, the theory dealing with non-autonomous second order abstract inclusions is only recently investigated, starting from Kozak's pioneering work [30]. On this subject we recall [8–10,13,21].

Moreover, with regard to nonlocal conditions we mention the reference [7] of Byszewski. In many cases it is advantageous to treat the nonlocal conditions since they are more appropriate then the classical initial conditions to describe natural phenomena (see [16,41] and the reference therein).

Finally, as is known, the controllability problems appear as a natural description of observed evolution phenomena of the real world. The attention of the researchers to such problems is increasing in literature. For example, for the notions and facts of controllability for first order differential equations/inclusions, the reader is referred to [1, 2, 14, 33], while we recall [15, 34, 35, 39] for nonlinear second order differential cases.

In this paper our main contributions are the followings:

(I) a new sufficient condition for the existence of mild solutions for the nonlocal problem $(\mathbf{P})_\mathbf{N}$ in weakly compactly generated Banach spaces (see Theorem 4.7, obtained as a consequence of our Propositions 4.3, 4.4, 4.5, 4.6);

(II) a new selection theorem for a multimap guided by an operator in reflexive Banach spaces (see Theorem 4.10);

(III) a version of the existence result for $(\mathbf{P})_\mathbf{N}$ in reflexive Banach spaces (see Theorem 4.13, proved by using our Propositions 4.4, 4.11, 4.12);

(IV) the existence of a mild solutions for an abstract problem satisfying a "periodicity condition" and having a fixed initial velocity (see Corollary 4.16);

(V) an application of Corollary 4.16 to the study of the controllability for the perturbed problem driven by (**E**) (see Theorem 5.1).

Regarding the proof of our first existence result for $(\mathbf{P})_\mathbf{N}$, in the setting of weakly compactly generated Banach spaces, we apply a fixed point theorem for multimaps, recently proved in [9]. This fixed point result allows us to work with weak topology and De Blasi measure of weak noncompactness. So we can avoid requests of compactness on the family generated by the linear part and on the multivalued term. The existence of mild solutions for the nonlocal problem $(\mathbf{P})_\mathbf{N}$ is also obtained as a consequence of a selection theorem and a containment theorem.

Then, to study the case of reflexive Banach spaces, in addition to use the fixed point theorem of [9], we need to achieve a new selection theorem for multimaps driven by a suitable operator. Combining this result with the classical Hahn–Banach Theorem and the weak upper semicontinuity property we are able to remove some assumptions required in the previous existence Theorem 4.7.

Finally we are in a position to study the purpose (2), thanks to the definition of a suitable operator N.

Let us note that, since we have not used the strong compactness property, our existence results extend in a broad sense those presented in [8]. On the other hand, although it is possible to reduce problem $(\mathbf{P})_\mathbf{N}$ to the one studied in [9] by considering an appropriate operator N (see Remark 4.15), the presence of the required boundedness property on N in our existence theorems makes us that problem $(\mathbf{P})_\mathbf{N}$ is not reduced to that examined in [9].

The paper is organized as follows. Section 2 is devoted to the collection of all notions, propositions and theorems known in literature and used in the sequel: so that the paper is self contained. The problem setting is presented in Section 3. Section 4 is divided into two subsections. The first one presents an existence theorem in weakly compactly generated Banach spaces, obtained by proving some preliminary propositions. The second one is aimed at examining the existence of mild solutions in the setting of reflexive Banach spaces. Finally, in Section 5 the controllability of the problem governed by (**E**) is given as a consequence of the last result presented in the previous section.

2 Preliminaries

In this section, we recall a few results, notations and definitions needed to establish our theorems. We introduce certain notations which are used throughout the article without any further mention. Let $(X, \|\cdot\|_X)$ be a Banach space, X^* be the dual space of X and τ_w be the weak topology on X. In this paper $\overline{B}_X(0,r)$ denotes the closed ball centered at the origin and of radius $r > 0$ of X. Moreover, we recall that a Banach space X is said to be *weakly compactly generated* (WCG, for short) if there exists a weakly compact subset H of X such that $X = \overline{\text{span}}\{H\}$ (see [20]). Let us note that every separable space is weakly compactly generated as well as the reflexive ones (see [20]).

Now, put $J = [0, a]$ an interval of the real line endowed with the usual Lebesgue measure μ, we denote by $\mathcal{M}(J)$ the family of all Lebesgue measurable sets, by $\mathcal{C}(J;X)$ the space of all continuous functions from J to X provided with the norm $\|\cdot\|_\infty$ of uniform convergence. We precise that to define the set $\hat{\mathcal{C}}_w(\mathcal{C}(J;X);X)$, presented in (1.1), we say that $f : \mathcal{C}(J;X) \to X$ is *(w-w)sequentially continuous* if for every sequence $(x_n)_n$, $x_n \in \mathcal{C}(J;X)$, $x_n \rightharpoonup x$, then $f(x_n) \rightharpoonup f(x)$.

We recall the following version of Theorem 4 of [29], which characterizes the weak convergence in the space $\mathcal{C}(J;X)$.

Proposition 2.1. *Let X be a normed space, $(g_n)_n$ be a sequence in $\mathcal{C}(J;X)$ and $g \in \mathcal{C}(J;X)$. Then $g_n \rightharpoonup g$ if and only if $(g_n - g)_n$ is uniformly bounded and $g_n(t) \rightharpoonup g(t)$, for every $t \in J$.*

Next, if (Ω, Σ) is a measurable space, a function $u : \Omega \to X$ is said to be $\Sigma \otimes \mathcal{B}(X)$-*measurable* if, for all $A \in \mathcal{B}(X)$, $u^{-1}(A) \in \Sigma$, where $\mathcal{B}(X)$ denotes the Borel σ-field of X (see [18, Definition 2.1.48]). In the case $(\Omega, \Sigma) = (J, \mathcal{M}(J))$, $u : J \to X$ is said to be *Bochner-measurable* (B-measurable, for short) if there is a sequence of simple functions which converges to u almost everywhere in J (see [18, Definition 3.10.1 (a)]) and $u : J \to X$ is said to be *weakly measurable* if for each $l \in X^*$, the real valued function $l(u)$ is measurable (see [18, Definition 3.10.1 (b)]).

Proposition 2.2 ([18, Corollary 3.10.5])**.** *If X is a separable Banach space and $u : J \to X$, then the following conditions are equivalent:*

(a) u is B-measurable;

(b) u is weakly measurable;

(c) u is $\mathcal{M}(J) \otimes \mathcal{B}(X)$-measurable.

Moreover, $L^1(J;X)$ is the space of all X-valued Bochner integrable functions on J with norm $\|u\|_{L^1(J;X)} = \int_0^a \|u(t)\|_X \, dt$ and $L^1_+(J) = \{f \in L^1(J;\mathbb{R}) : \ f(t) \geq 0, \ a.e. \ t \in J\}$. If $X = \mathbb{R}$ we put $\|\cdot\|_1 = \|\cdot\|_{L^1(J;\mathbb{R})}$. For L^1_+-functions the following result holds.

Proposition 2.3 ([12, Lemma 3.1])**.** *For every $k > 0$, $\nu \in L^1_+(J)$, there exists $n := n(k,\nu) \in \mathbb{N}$ such that*

$$\sup_{t \in J} \int_0^t k\nu(\xi) e^{-n(t-\xi)} \, d\xi < 1.$$

A set $A \subset L^1(J;X)$ has the property of *equi-absolute continuity of the integral* if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, for every $E \in \mathcal{M}(J)$, $\mu(E) < \delta_\varepsilon$, we have

$$\int_E \|f(t)\|_X \, dt < \varepsilon$$

whenever $f \in A$, while $A \subset L^1(J;X)$ is *integrably bounded* if there exists $\nu \in L^1_+(J)$ such that

$$\|f(t)\|_X \leq \nu(t), \quad a.e.\ t \in J, \text{ for every } f \in A.$$

Clearly every integrably bounded set has the property of equi-absolute continuity of the integral. We recall that the equi-absolute continuity of the integral is fundamental to characterize the relative weak compactness of a bounded set in $L^1(J;X)$.

Proposition 2.4 ([38, Corollary 9]). *Let A be a bounded subset of $L^1(J;X)$ such that it has the property of equi-absolute continuity of the integral and, for a.e. $t \in J$, the set $A(t) = \{f(t) : f \in A\}$ is relatively weakly compact. Then A is relatively weakly compact.*

Then, if H is a subset of the Banach space X, we denote by the symbol $\overline{H}^w$ the weak closure of H. As is well known, a bounded subset H of a reflexive space X is relatively weakly compact. Moreover, we recall that a subset H of a Banach space X is called *relatively weakly sequentially compact* if any sequence of points in H has a subsequence weakly convergent to a point in X (see [31]). Now we recall the classical Eberlein–Šmulian result.

Proposition 2.5 ([18, Theorem 3.5.3]). *A subset of a Banach space is relatively weakly compact if and only if it is relatively weakly sequentially compact. In particular, a subset of a Banach space is weakly compact if and only if it is weakly sequentially compact.*

In the sequel we use the following version of Theorem 3 obtained by H. Vogt in [40].

Proposition 2.6. *Let H be a relatively weakly compact subset of a Banach space X. Then H is weakly closed if and only if H is weakly sequentially closed.*

Further, if $\mathcal{P}(X)$ is the family of all nonempty subsets of X, we use the following notations:

$$\begin{aligned}
\mathcal{P}_b(X) &= \{H \in \mathcal{P}(X) : \ H \text{ bounded}\}, \\
\mathcal{P}_f(X) &= \{H \in \mathcal{P}(X) : \ H \text{ closed}\}, \\
\mathcal{P}_k(X) &= \{H \in \mathcal{P}(X) : \ H \text{ compact}\}, \\
\mathcal{P}_{wk}(X) &= \{H \in \mathcal{P}(X) : \ H \text{ weakly compact}\}.
\end{aligned}$$

Now, let $(A_n)_n$ be a sequence, $A_n \in \mathcal{P}(X)$, we consider the *"Kuratowski limit superior"* (see [23, Definition 7.1.3])

$$w - \limsup_{n \to +\infty} A_n = \{x \in X : \ \exists (x_{n_k})_k, \ x_{n_k} \in A_{n_k}, \ n_k < n_{k+1}, \ x_{n_k} \rightharpoonup x\}$$

Proposition 2.7 ([23, Proposition 7.3.9]). *Let X be a Banach space, $1 \leq p < \infty$, $G : J \to \mathcal{P}_{wk}(X)$ and $(f_n)_n$, $f_n \in L^p(J;X)$, be a sequence such that*

i) there exists $f \in L^p(J;X)$ such that $f_n \rightharpoonup f$;

ii) $f_n(t) \in G(t)$ a.e. $t \in J$, $n \in \mathbb{N}$.

Then

$$f(t) \in \overline{\text{co}}\ w - \limsup_{n \to \infty} \{f_n(t)\}, \quad a.e.\ t \in J,$$

where $\overline{\text{co}}$ denotes the closure of the convex hull of a set.

Furthermore a multimap $F : \Omega \to \mathcal{P}(Y)$, where Y is a topological space, is said to be *measurable* if for every open set $V \subset Y$ one has $F^{-}(V) = \{x \in \Omega\ F(x) \cap V \neq \emptyset\} \in \Sigma$ (see [24, Definition 1.3.1]).

Proposition 2.8 ([18, Proposition 4.2.4]). *Let (Ω, Σ) be a measurable space and Y be a separable metric space. A multimap $F : \Omega \to \mathcal{P}(Y)$ is measurable if and only if for every $y \in Y$ the function $x \mapsto dist(y, F(x))$ is $\Sigma \otimes \mathcal{B}(Y)$-measurable.*

Proposition 2.9 ([18, Theorem 4.3.1]). *If (Ω, Σ) is a measurable space, Y is a Polish space and $F : \Omega \to \mathcal{P}_f(Y)$ is measurable, then F has a $\Sigma \otimes \mathcal{B}(Y)$-measurable selection.*

If T is a topological space, a multimap $F : T \to \mathcal{P}(Y)$ is said to be *upper semicontinuous* in T if, for every $\overline{x} \in T$, it is upper semicontinuous at $\overline{x}$, i.e. for every open $W \subset Y$ such that $F(\overline{x}) \subset W$, there exists a neighborhood $V(\overline{x})$ of $\overline{x}$ with the property $F(V(\overline{x})) \subset W$.

A multimap $F : T \to \mathcal{P}(Y)$ has *closed graph* if the set graph $F = \{(x, y) \in T \times Y :\ y \in F(x)\}$ is closed in $T \times Y$.

Moreover, F is said to be *compact* if $\overline{F(T)}$ is compact in Y, while F is said to be *locally compact* if every $x \in T$ there exists a neighborhood $V(x)$ such that the restriction $F_{|V(x)}$ is compact.

Proposition 2.10 ([24, Theorem 1.1.5]). *Let T, Y be topological spaces and $F : T \to \mathcal{P}_k(Y)$ be a closed and locally compact multimap. Then F is upper semicontinuos in T.*

If Y is a linear topological space, $F : X \to \mathcal{P}(Y)$ has *(s-w)sequentially closed graph* [*(w-w)sequentially closed graph*] if for every $(x_n)_n$, $x_n \in X$, $x_n \to x$ [$x_n \rightharpoonup x$] and for every $(y_n)_n$, $y_n \in F(x_n)$, $y_n \rightharpoonup y$, we have $y \in F(x)$. In the sequel, the "$(w$-$w)$sequential continuity" and the "$(w$-$w)$ sequentially closed graph property" are named "weak sequential continuity" and "weakly sequentially closed graph property" respectively.

A multimap $F : J \to \mathcal{P}(X)$ is said to have a *B-measurable selection* if there exists a B-measurable function $f : J \to X$ such that $f(t) \in F(t)$, a.e. $t \in J$.

Now, we recall the following results that ensures the existence of a B-selection for a multimap.

Theorem 2.11 ([24, Theorem 1.3.5]). *Let X, Y be Banach spaces and $F : J \times X \to \mathcal{P}_k(Y)$ be a multimap such that*

i) for every $x \in X$, $F(\cdot, x) : J \to \mathcal{P}_k(Y)$ has a B-measurable selection;

ii) for a.e. $t \in J$, $F(t, \cdot) : X \to \mathcal{P}_k(Y)$ is upper semicontinuous in X.

Then for every B-measurable function $q : J \to Y$, there exists a B-measurable selection $f : J \to X$ of the multimap $F(\cdot, q(\cdot))$.

Let us recall that for $v : J \to M$, where M is a metric space, the B-measurability is generalized in [22] by using again the simple functions. Thanks to this definition, the following result holds.

Theorem 2.12 ([9, Theorem 4.2] (Selection Theorem)). *Let M be a metric space, X be a Banach space and $F : J \times M \to \mathcal{P}(X)$ be a multimap such that*

f1) for a.e. $t \in J$, for every $x \in M$, the set $F(t, x)$ is convex ;

f2) for every $x \in M$, $F(\cdot, x) : J \to \mathcal{P}(X)$ has a B-measurable selection;

f3) for a.e. $t \in J$, $F(t,\cdot) : M \to \mathcal{P}(X)$ *has a (s-w)sequentially closed graph in* $M \times X$;

f4) there exists $\varphi : J \to [0,\infty)$, $\varphi \in L^1_+(J)$, *such that*

$$\sup_{z \in F(t,M)} \| z \| \leq \varphi(t),\ a.e.\ t \in J;$$

f5) for almost all $t \in J$ *and every convergent sequence* $(x_n)_n$ *in* M, *the set* $\bigcup_n F(t,x_n)$ *is relatively weakly compact.*

Then, for every B-measurable $v : J \to M$, *there is a B-measurable* $y : J \to X$ *such that* $y(t) \in F(t,v(t))$ *for a.e.* $t \in J$.

Next, we recall that, if H is a subset of X, $F : H \to \mathcal{P}(X)$ is a multimap and $x_0 \in H$, a closed convex set $M_0 \subset H$ is said to be (x_0,F)*-fundamental*, if $x_0 \in M_0$ and $F(M_0) \subset M_0$ (see [3, p. 620]). In this setting we recall the following result which allows to characterize the *smallest* (x_0,F)*-fundamental* set

Theorem 2.13 ([3, Theorem 3.1])**.** *Let* X *be a locally convex Hausdorff space,* $H \subset X$ *and* $x_0 \in H$. *Let* $F : H \to \mathcal{P}(X)$ *be a multimap such that* $\overline{\text{co}}(F(H) \cup \{x_0\}) \subset H$. *Then*

1) $\mathcal{F} = \{H : H \text{ is a } (x_0,F)\text{-fundamental set}\} \neq \varnothing$;

2) put $M_0 = \bigcap_{H \in \mathcal{F}} H$, *we have* $M_0 \in \mathcal{F}$ *and* $M_0 = \overline{\text{co}}(F(M_0) \cup \{x_0\})$.

Now we present a fixed point result and a "Containment Theorem", which play a key role in our existence results.

Theorem 2.14 ([9, Corollary 4.4])**.** *Let* X *be a Banach space,* $H \subset X$, $x_0 \in H$ *and* $F : H \to \mathcal{P}(X)$ *be a multimap such that*

i) $F(x)$ *convex, for every* $x \in H$;

ii) $\overline{\text{co}}(F(H) \cup \{x_0\}) \subset H$;

iii) M_0 *is weakly compact;*

iv) $F_{|M_0}$ *has weakly sequentially closed graph,*

where M_0 *is the smallest* (x_0,F)*-fundamental set.*

Then there exists at least one point $\overline{x} \in M_0$ *such that* $\overline{x} \in F(\overline{x})$.

Theorem 2.15 ([3, Theorem 4.4] (Containment Theorem))**.** *Let* X *be a Banach space and* $G_n, G : J \to \mathcal{P}(X)$ *be such that*

α) a.e. $t \in J$, *for every* $(u_n)_n$, $u_n \in G_n(t)$, *there exists a subsequence* $(u_{n_k})_k$ *of* $(u_n)_n$ *and* $u \in G(t)$ *such that* $u_{n_k} \rightharpoonup u$;

αα) there exists a sequence $(y_n)_n$, $y_n : J \to X$, *having the property of equi-absolute continuity of the integral, such that* $y_n(t) \in G_n(t)$, *a.e.* $t \in J$, *for all* $n \in \mathbb{N}$.

Then there exists a subsequence $(y_{n_k})_k$ *of* $(y_n)_n$ *such that* $y_{n_k} \rightharpoonup y$ *in* $L^1(J;X)$ *and, moreover,* $y(t) \in \overline{\text{co}}G(t)$, *a.e.* $t \in J$.

Now, a function $\omega : \mathcal{P}_b(X) \to \mathbb{R}^+_0$ is said to be a *measure of weak noncompactness* (MwNC, for short) if the following properties are satisfied (see [11, Definition 4.1]):

ω_1) ω is a Sadowskii functional, i.e. $\omega(\overline{\text{co}}(H)) = \omega(H)$, for every $H \in \mathcal{P}_b(X)$;

ω_2) ω is regular, i.e. $\omega(H) = 0$ if and only if $\overline{H}^w$ is weakly compact.

Further, a measure of weak noncompactness $\omega : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ is said to be:
monotone if $H_1, H_2 \in \mathcal{P}_b(X) :\ H_1 \subset H_2$ implies $\omega(H_1) \le \omega(H_2)$;
nonsingular if $\omega(\{x\} \cup H) = \omega(H)$, $x \in X$, $H \in \mathcal{P}_b(X)$;
x_0-stable if, fixed $x_0 \in X$, $\omega(\{x_0\} \cup H) = \omega(H)$, $H \in \mathcal{P}_b(X)$;
invariant under closure if $\omega(\overline{H}) = \omega(H)$, $H \in \mathcal{P}_b(X)$;
invariant with respect to the union with compact set if $\omega(H \cup C) = \omega(H)$, for every relatively compact set $C \subset X$ and $H \in \mathcal{P}_b(X)$.

In particular in [17] De Blasi introduces the MwNC function $\beta : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ so defined

$$\beta(H) = \inf\{\varepsilon \in [0, \infty[: \text{there exists } C \subset X \text{ weakly compact} :\ H \subseteq C + B_X(0, \varepsilon)\},$$

(named in literature *De Blasi*–MwNC) and he proves that β has all the properties mentioned before and it is also *algebrically subadditive*, i.e. $\beta\left(\sum_{k=1}^n H_k\right) \le \sum_{k=1}^n \beta(H_k)$, where $H_k \in \mathcal{P}_b(X)$, $k = 1, \dots, n$.

Moreover, for every bounded linear operator $L : X \to X$ the following property holds (see [19, Lemma 1])

$$\beta(L(H)) \le \|L\|\, \beta(H),\ H \in \mathcal{P}_b(X), \tag{2.1}$$

where $\|L\|$ denotes the norm of the operator L.

We recall the following interesting result for the De Blasi–MwNC $\beta : \mathcal{P}_b(X) \to \mathbb{R}_0^+$.

Proposition 2.16 ([3, Theorem 2.7])**.** *Let (Ω, Σ, μ) be a finite positive measure space and X be a weakly compactly generated Banach space. Then for every countable bounded set $C \subset L^1(J; X)$ having the property of equi-absolute continuity of the integral, the function $\beta(C(\cdot))$ is $\mathcal{M}(J) \otimes \mathcal{M}(\mathbb{R})$ measurable and*

$$\beta\left(\left\{\int_\Omega x(s)\, ds :\ x \in C\right\}\right) \le \int_\Omega \beta(C(s))\, ds.$$

In the sequel, fixed $\alpha \in \mathbb{R}$, we use the following Sadowskii functional $\beta_\alpha : \mathcal{P}_b(\mathcal{C}(J; X)) \to \mathbb{R}_0^+$, so defined (see [3, Definition 3.9])

$$\beta_\alpha(M) = \sup_{\substack{C \subset M \\ \text{countable}}} \sup_{t \in J} \beta(C(t)) e^{-\alpha t}, \quad M \in \mathcal{P}_b(\mathcal{C}(J; X)), \tag{2.2}$$

where β is the De Blasi MwNC and, for every $t \in J$, $C(t) = \{x(t) :\ x \in C\}$. We recall that the Sadowskii functional β_α is *x_0-stable* and *monotone* (see [3, Proposition 3.10]) and β_α has the two following properties (see [9, Remark 2.11])

(I) β_α is algebraically subadditive;

(II) $M \subset \mathcal{C}(J; X)$ is relatively weakly compact $\Rightarrow \beta_\alpha(M) = 0$.

3 Problem setting

First of all, on the linear part of the second order differential inclusion, presented in the nonlocal problem $(\mathbf{P})_\mathbf{N}$, we assume the following property:

(A) $\{A(t)\}_{t\in J}$ is a family of linear operators $A(t) : D(A) \to X$, where $D(A)$, independent on $t \in J$, is a subset dense in X such that, for each $x \in D(A)$, the function $t \mapsto A(t)x$ is continuous on J and generating a fundamental system $\{S(t,s)\}_{t,s\in J}$.

The notion of fundamental system is introduced by Kozak in [30] and it is recently used in [8–10, 21]. In some works, for every $t \in J$, the linear operator $A(t) : D(A) \to X$ is also closed (see [21, 30]) and bounded (see [8–10]), but we leave out these properties on $A(t)$ since they are not necessary in order to prove the existence of mild solutions (see [37]).

Definition 3.1. A family $\{S(t,s)\}_{t,s\in J}$ of bounded linear operators $S(t,s) : X \to X$ is called a *fundamental system* generated by the family $\{A(t)\}_{t\in J}$ if

S1. for each $x \in X$, the function $S(\cdot,\cdot)x : J \times J \to X$ is of class $\mathcal{C}^1$ and

- a. for every $t \in J$, $S(t,t)x = 0,\ x \in X$;
- b. for every $t, s \in J$ and every $x \in X$, $\frac{\partial}{\partial t} S(t,s)\big|_{t=s} x = x$ and $\frac{\partial}{\partial s} S(t,s)\big|_{t=s} x = -x$;

S2. for all $t, s \in J$, $x \in D(A)$, then $S(t,s)x \in D(A)$, the map $S(\cdot,\cdot)x : J \times J \to X$ is of class $\mathcal{C}^2$ and

- a′. $\frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x$;
- b′. $\frac{\partial^2}{\partial s^2} S(t,s)x = S(t,s)A(s)x$;
- c′. $\frac{\partial^2}{\partial t \partial s} S(t,s)\big|_{t=s} x = 0$;

S3. for all $t, s \in J$, $x \in D(A)$, then $\frac{\partial}{\partial s} S(t,s)x \in D(A)$. Moreover, there exist $\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x$ such that

- a″. $\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x = A(t)\frac{\partial}{\partial s} S(t,s)x$;
- b″. $\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x = \frac{\partial}{\partial t} S(t,s)A(s)x$;

and, for all $x \in D(A)$, the function $(t,s) \mapsto A(t)\frac{\partial}{\partial s} S(t,s)x$ is continuous in $J \times J$.

As in [21], a map $S : J \times J \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the space of all bounded linear operators in X with the norm $\|\cdot\|_{\mathcal{L}(X)}$, is said to be a *fundamental operator* if the family $\{S(t,s)\}_{t,s\in J}$ is a fundamental system.

Moreover, for every $(t,s) \in J \times J$, we consider the linear operator, named *"cosine operator"*, $C(t,s) = -\frac{\partial}{\partial s} S(t,s) : X \to X$.

In [10] it is pointed out that the Banach–Steinhaus Theorem allows to establish the existence of two constant $K, K^* > 0$ such that

p1. $\|C(t,s)\|_{\mathcal{L}(X)} \le K$, $(t,s) \in J \times J$;

p2. $\|S(t,s)\|_{\mathcal{L}(X)} \le K|t-s|$, $(t,s) \in J \times J$;

p3. $\|S(t,s)\|_{\mathcal{L}(X)} \le Ka$, $(t,s) \in J \times J$;

p4. $\|S(t_2,s) - S(t_1,s)\|_{\mathcal{L}(X)} \le K^*|t_2 - t_1|,\ t_1, t_2, s \in J$.

Further, as in [10], we denote with $G_S : L^1(J;X) \to \mathcal{C}(J;X)$ the *fundamental Cauchy operator* defined as

$$G_S f(t) = \int_0^t S(t,s)f(s)\,ds, \quad t \in J, \ f \in L^1(J;X). \tag{3.1}$$

Let us note that, fixed $t \in J$, the map $p_t : [0,t] \times X \to X$ such that $p_t(\xi, x) = S(t,\xi)x$, $(\xi, x) \in [0,t] \times X$, satisfies all the assumptions of Theorem 2.11 (see S1.). Hence, for every $f \in L^1(J;X)$, by p3. it easy to deduce that $p_t(\cdot, f(\cdot))$ is B-integrable in $[0,t]$. Then, by using p4. we also have $G_S f \in \mathcal{C}(J;X)$. So G_S is well posed. Moreover, the fundamental Cauchy operator has the properties declared in the following

Proposition 3.2 ([9, Proposition 4.1]). *If $\{S(t,s)\}_{(t,s)\in J\times J}$ is a fundamental system, then the fundamental Cauchy operator $G_S : L^1(J;X) \to \mathcal{C}(J;X)$ is linear, bounded and weakly continuous (so it is also weakly sequentially continuous).*

We investigate the existence of mild solutions for the nonlocal problem $(\mathbf{P})_\mathbf{N}$ (see [8, Definition 2.2])

Definition 3.3. A continuous function $u : J \to X$ is a *mild solution* for $(\mathbf{P})_\mathbf{N}$ if

$$u(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,\xi)f(\xi)\,d\xi, \quad t \in J,$$

where $f \in S^1_{F(\cdot,N(\cdot)u)} = \{f \in L^1(J;X) : \ f(t) \in F(t, N(t)u), \ a.e. \ t \in J\}$.

4 Existence results for the nonlocal problem $(\mathbf{P})_\mathbf{N}$

In this section, put X a Banach space, we consider the following properties on the multimap $F : J \times X \to \mathcal{P}(X)$ and on the map $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J;X);X)$

F1 for every $(t,x) \in J \times X$, the set $F(t,x)$ is convex;

F2 for every $x \in X$, $F(\cdot, x) : J \to \mathcal{P}(X)$ admits a B-measurable selection;

F3 for a.e. $t \in J$, $F(t,\cdot) : X \to \mathcal{P}(X)$ has a weakly sequentially closed graph;

F4 there exists $(\varphi_n)_n$, $\varphi_n \in L^1_+(J)$ such that

$$\limsup_{n\to\infty} \frac{Ka\int_0^a \varphi_n(t)\,dt}{n} < 1 \tag{4.1}$$

and

$$\|F(t, \overline{B}_X(0,n))\| \le \varphi_n(t), \quad a.e. \ t \in J, \ n \in \mathbb{N}; \tag{4.2}$$

where K is the constant presented in p1. of Section 3;

FN there exists $A \subset J$, $\mu(A) = 0$: for all $n \in \mathbb{N}$ there exists $\nu_n \in L^1_+(J)$ such that, for every $t \in J \setminus A$

$$\beta(C_1) \le \nu_n(t)\beta(C_0(t)), \tag{4.3}$$

for all countable C_0, C_1 with

$$C_0 \subseteq \overline{B}_{\mathcal{C}(J;X)}(0,n), \quad C_1 \subseteq F(t, C_0(t) \cup N(t)C_0),$$

where β is the De Blasi measure of weak noncompactness;

N1 for every $u \in \mathcal{C}(J;X)$, $N(\cdot)u$ is B-measurable;

N2 there exists $\bar{c} \in \mathbb{N}$ such that $\|N(t)u\|_X \leq \bar{c}$, for all $t \in J$, $u \in \mathcal{C}(J;X)$.

Moreover, we consider the following properties on the functions $g, h : \mathcal{C}(J;X) \to X$

gh1 g, h are weakly sequentially continuous;

gh2 for every countable $H \subset \mathcal{C}(J;X)$, the sets $g(H)$ and $h(H)$ are relatively weakly compact;

gh3 for every bounded and closed subset M of $\mathcal{C}(J;X)$, the sets

$$C(\cdot,0)g(M) \quad \text{and} \quad S(\cdot,0)h(M)$$

are relatively weakly compact in $\mathcal{C}(J;X)$.

Remark 4.1. First of all we note that, under assumptions F1, F3 and FN,

$$F(t, N(t)x) \text{ is closed in } X, \text{ for a.e. } t \in J \text{ and for every } x \in \mathcal{C}(J;X).$$

Denoted by H^* a null measure set such that F3 and FN hold in $J \setminus H^*$, we fix $t \in J \setminus H^*$ and $x \in \mathcal{C}(J;X)$. Now we prove that the set $F(t, N(t)x)$ is relatively w-compact. To this aim we consider $C_0 = \{x\}$ and $C_1 = \{y_n : n \in \mathbb{N}\}$, where $y_n \in F(t, N(t)x)$, $n \in \mathbb{N}$. Note that $C_0 \subset \overline{B}_{\mathcal{C}(J;X)}(0,p)$, for a suitable $p \in \mathbb{N}$, and $C_1 \subset F(t, C_0(t) \cup N(t)C_0)$, so by (4.3) we have $\beta(C_1) \leq \nu_p(t)\beta(C_0(t)) = 0$. By the regularity of β the set C_1 is relatively w-compact and so there exist $(y_{n_k})_k \subset (y_n)_n$ and $y \in X$ such that $y_{n_k} \rightharpoonup y$. Then by the arbitrariness of the sequence $(y_n)_n$, by using the Eberlein–Šmulian Theorem we have that the set $F(t, N(t)x)$ is relatively weakly compact too. By virtue of F1 and F3 we also know that the set $F(t, N(t)x)$ is convex and weakly sequentially closed. So, by using Proposition 2.6 we have that the set $F(t, N(t)x)$ is closed in X.

Remark 4.2. We note that, in the setting of reflexive Banach spaces and under assumptions F1, F3 and F4, by using again Proposition 2.6 we have

$$F(t,x) \text{ is closed, for a.e. } t \in J \text{ and for every } x \in X, \tag{4.4}$$

(see the beginning of the proof of Theorem 5.3 of [9].)

4.1 Existence of mild solutions in WCG Banach spaces

In this subsection, by combining the Containment Theorem 2.15 and a selection result, which is a consequence of Theorem 2.12, we obtain the existence of mild solutions to the nonlocal problem $(\mathbf{P})_{\mathbf{N}}$, assuming that X is a WCG Banach space. Note that our technique allows us to avoid hypotheses of compactness both on the family generated by the linear part and on the nonlinear multivalued term. We achive our goal by applying the fixed point Theorem 2.14 to the following multioperator $T : \mathcal{C}(J;X) \to \mathcal{P}(\mathcal{C}(J;X))$ defined as (see (3.1))

$$Tu = \{y \in \mathcal{C}(J;X) : y(t) = C(t,0)g(u) + S(t,0)h(u) + G_S f(t),\ t \in J,\ f \in S^1_{F(\cdot,N(\cdot)u)}\}, \tag{4.5}$$

where

$$S^1_{F(\cdot,N(\cdot)u)} = \{f \in L^1(J;X) : f(t) \in F(t, N(t)u) \ a.e.\ t \in J\}. \tag{4.6}$$

To make the propositions that we will present below of greater applicability, allow us to request, at first, that the following property holds

(T) $S^1_{F(\cdot,N(\cdot)u)} \neq \emptyset,\ u \in \mathcal{C}(J;X)$,

so we have the multioperator T is well posed.

Obviously the fixed points of the integral multioperator T are mild solutions for the problem $(\mathbf{P})_\mathbf{N}$.

At first, thanks to the Containment Theorem 2.15, we establish the following property on T.

Proposition 4.3. *Let X be a Banach space, under assumptions* (A), (T), F1, F3, F4, FN, N2 *and* gh1, *the multioperator T has a weakly sequentially closed graph.*

Proof. Let $(q_n)_n$ and $(x_n)_n$ be two sequences of $\mathcal{C}(J;X)$ such that

$$x_n \in Tq_n, \quad n \in \mathbb{N} \tag{4.7}$$

and

$$q_n \rightharpoonup q, \quad x_n \rightharpoonup x, \tag{4.8}$$

where $q, x \in \mathcal{C}(J;X)$. We have to show that $x \in Tq$.

First of all, by Proposition 2.1 the weak convergence of $(q_n)_n$ implies that

$$q_n(t) \rightharpoonup q(t), \quad t \in J \tag{4.9}$$

and the existence of $\overline{n} \in \mathbb{N}$ such that

$$\|q_n\|_{\mathcal{C}(J;X)} \leq \overline{n}, \quad n \in \mathbb{N}. \tag{4.10}$$

Now we prove that Containment Theorem 2.15 can be applied to the multimaps $G_n : J \to \mathcal{P}(X)$, $n \in \mathbb{N}$ and $G : J \to \mathcal{P}(X)$ respectively so defined

$$G_n(t) = F(t, N(t)q_n), \quad t \in J, \tag{4.11}$$

$$G(t) = F(t, N(t)q), \quad t \in J. \tag{4.12}$$

First we establish α) of Theorem 2.15. To this aim we consider the null measure set H^* for which F3 and FN hold. Fixed $t \in J \setminus H^*$, we consider a sequence $(u_n)_n$ such that

$$u_n \in G_n(t), \quad n \in \mathbb{N}. \tag{4.13}$$

Now we fix the countable subset of $\overline{B}_{\mathcal{C}(J;X)}(0,\overline{n})$, where $\overline{n}$ is presented in (4.10), so defined

$$C_0 = \{q_n : n \in \mathbb{N}\} \tag{4.14}$$

and the countable set of X

$$C_1 = \{u_n : \ n \in \mathbb{N}\}$$

satisfying (see (4.13), (4.11) and (4.14))

$$C_1 \subset F(t, \{N(t)q_n\}_n) \subset F(t, C_0(t) \cup N(t)C_0).$$

So, by FN there exists $\nu_{\overline{n}} \in L^1_+(J)$ such that

$$\beta(C_1) \leq \nu_{\overline{n}}(t)\beta(C_0(t)). \tag{4.15}$$

Now, since the set $C_0(t)$ is relatively weakly sequentially compact (see (4.14) and (4.9)), by the regularity of β we have $\beta(C_0(t)) = 0$. By the virtue of (4.15) and of the Eberlein–Šmulian Theorem, we deduce that C_1 is relatively weakly sequentially compact too, i.e. there exist $(u_{n_k})_k \subset (u_n)_n$ and $u \in X$ such that

$$u_{n_k} \rightharpoonup u. \tag{4.16}$$

Therefore, taking into account the weak sequential continuity of $N(t)$ and F3, from (4.8), (4.11), (4.12), (4.13) and (4.16) we have $u \in G(t)$. So $\alpha)$ of Theorem 2.15 holds.

Next, we prove $\alpha\alpha)$ of Theorem 2.15. By (4.7), (4.5) and (T), for every $n \in \mathbb{N}$, there exists $f_n \in S^1_{F(\cdot,N(\cdot)q_n)}$ such that

$$x_n(t) = C(t,0)g(q_n) + S(t,0)h(q_n) + G_S f_n(t), \quad t \in J.$$

First we observe that the sequence $(f_n)_n$, $f_n : J \to X$, is such that

$$f_n(t) \in G_n(t) = F(t, N(t)q_n), \quad \text{a.e. } t \in J,\ n \in \mathbb{N}. \tag{4.17}$$

Moreover, thanks to (4.17) and to hypotheses N2 and F4, the sequence $(f_n)_n$ is integrably bounded. So $(f_n)_n$ has the property of equi-absolute continuity of the integral, i.e. $\alpha\alpha)$ holds.

Now we are in a position to apply the mentioned Containment Theorem, so there exists a subsequence $(f_{n_k})_k \subset (f_n)_n$ such that

$$f_{n_k} \rightharpoonup f \quad \text{in } L^1(J;X),$$

where (see (4.12), F1 and, taking into account F3 and FN, Remark 4.1)

$$f(t) \in \overline{co}G(t) = \overline{co}F(t, N(t)q) = F(t, N(t)q), \quad a.e.\ t \in J.$$

Hence we can conclude that

$$f \in S^1_{F(\cdot,N(\cdot)q)}. \tag{4.18}$$

Moreover, the weak continuity of the fundamental Cauchy operator G_S (see Proposition 3.2) implies that $G_S f_{n_k} \rightharpoonup G_S f$. Then, by using again Proposition 2.1, hypothesis gh1, continuity and linearity of $S(t,0)$ and $C(t,0)$, $t \in J$, we have

$$x_{n_k}(t) \rightharpoonup C(t,0)g(q) + S(t,0)h(q) + G_S f(t) =: \tilde{x}(t), \quad t \in J. \tag{4.19}$$

On the other hand by (4.8) we deduce $x_{n_k}(t) \rightharpoonup x(t)$, for all $t \in J$ and then the uniqueness of the weak limit implies

$$x(t) = \tilde{x}(t),\ t \in J. \tag{4.20}$$

Finally, from (4.20), (4.19), (4.18) and (4.5) we have that $x \in Tq$. Therefore we can conclude that T has a weakly sequentially closed graph. □

Proposition 4.4. *Let X be a Banach space, under assumptions* (A), (T), F4, N2 *and* gh2, *there exists $\bar{r} \in \mathbb{N}$, $\bar{r} > \bar{c}$, such that the operator T maps the closed ball $K_{\bar{r}} = \overline{B}_{C(J;X)}(0,\bar{r})$ into itself, where* 0 *denotes the null function of $C(J;X)$.*

Proof. First of all, from N2, we know that there exists a constant $\bar{c} \in \mathbb{N}$ such that $\|N(t)x\|_X \leq \bar{c}$, for every $t \in J$, $x \in C(J;X)$.

We show that there exists $\bar{r} \in \mathbb{N}$, $\bar{r} > \bar{c}$, such that

$$T(K_{\bar{r}}) \subset K_{\bar{r}}. \tag{4.21}$$

Let assume by contradiction that, for every $n \in \mathbb{N}$ such that $n > \bar{c}$ we have

$$T(K_n) \not\subseteq K_n.$$

Then, there exist $q_n \in \mathcal{C}(J;X)$ with $\|q_n\|_{\mathcal{C}(J;X)} \le n$ and $x_{q_n} \in Tq_n$ such that $\|x_{q_n}\|_{\mathcal{C}(J;X)} > n$.

Being $\|x_{q_n}\|_{\mathcal{C}(J;X)} > n$, there exists $t_n \in J = [0,a]$: $\|x_{q_n}(t_n)\|_X > n$. By gh2 we have that $g(\{q_n,\ n \in \mathbb{N} :\ n > \bar{c}\})$ and $h(\{q_n,\ n \in \mathbb{N} :\ n > \bar{c}\})$ are relatively weakly compact. Hence there exists a subsequence $(q_{n_k})_k$ such that $(g(q_{n_k}))_k$ and $(h(q_{n_k}))_k$ are weakly convergent, so there exists $Q > 0$ such that (see [6], Proposition 3.5 (iii)) $\|g(q_{n_k})\|_X \le Q$, $\|h(q_{n_k})\|_X \le Q$, for every $n_k \in \mathbb{N}$, $n_k > \bar{c}$. Now, being $f_{q_{n_k}} \in S^1_{F(\cdot,N(\cdot)q_{n_k})}$, taking into account p1. and p3. we can write

$$\begin{aligned} n_k < \|x_{q_{n_k}}(t_{n_k})\|_X &\le \|C(t_{n_k},0)\|_{\mathcal{L}(X)}\|g(q_{n_k})\|_X + \|S(t_{n_k},0)\|_{\mathcal{L}(X)}\|h(q_{n_k})\|_X \\ &+ \int_0^{t_{n_k}} \|S(t_{n_k},\xi)\|_{\mathcal{L}(X)}\|f_{q_{n_k}}(\xi)\|_X\,d\xi \le KQ + KaQ + Ka\int_0^a \|f_{q_{n_k}}(\xi)\|_X\,d\xi, \end{aligned} \tag{4.22}$$

Next, from N2 we have $\|N(t)q_{n_k}\|_X \le \bar{c} < n_k$, $t \in J$. So $f_{q_{n_k}}(t) \in F(t, N(t)q_{n_k}) \subset F(t, \overline{B}_X(0,n_k))$, a.e. $t \in J$. Now by (4.2) of F4 there exists $\varphi_{n_k} \in L^1_+(J)$ such that

$$\|f_{q_{n_k}}(t)\|_X \le \varphi_{n_k}(t), \quad a.e.\ t \in J,$$

then by (4.22) the following inequality holds

$$n_k < \|x_{q_{n_k}}(t_{n_k})\|_X \le KQ + KaQ + Ka\int_0^a \varphi_{n_k}(\xi)\,d\xi. \tag{4.23}$$

Therefore, since (4.23) is true for every natural number $n_k > \bar{c}$, we have

$$1 \le \frac{KQ+KaQ}{n_k} + \frac{Ka\int_0^a \varphi_{n_k}(\xi)\,d\xi}{n_k}, \quad n_k \in \mathbb{N},\ n_k > \bar{c}.$$

Hence, passing to the superior limit, by (4.1) we deduce the following contradiction

$$1 \le \limsup_{k\to\infty}\left(\frac{KQ+KaQ}{n_k} + \frac{Ka\int_0^a \varphi_{n_k}(\xi)\,d\xi}{n_k}\right) \le \limsup_{n\to\infty}\frac{Ka\int_0^a \varphi_n(\xi)\,d\xi}{n} < 1.$$

Therefore we can conclude that (4.21) is true, i.e. there exists $\bar{r} \in \mathbb{N}$ with $\bar{r} > \bar{c}$ such that

$$K_{\bar{r}} = \overline{B}_{\mathcal{C}(J;X)}(0,\bar{r}) \tag{4.24}$$

is invariant under the action of the operator T. □

If the Banach space X is also WCG, we have the following result for the multimap $T_{\bar{r}} = T_{|K_{\bar{r}}} : K_{\bar{r}} \to \mathcal{P}(\mathcal{C}(J;X))$, which is the restriction of the multimap T on the set $K_{\bar{r}}$ defined in (4.24).

Proposition 4.5. *If X is a weakly compactly generated Banach space, under assumptions* (A), (T), F4, FN, N2, gh2 *and* gh3 *there exists M_0 the smallest $(0, T_{\bar{r}})$-fundamental set which is weakly compact, with $\bar{r} > \bar{c}$ such that $T(K_{\bar{r}}) \subset K_{\bar{r}}$.*

Proof. First of all, we consider $x_0 = 0 \in \mathcal{C}(J;X)$ and the set $K_{\bar r}$ in the locally convex Hausdorff space $\mathcal{C}(J;X)$ equipped with the weak topology. Since $T_{\bar r}(K_{\bar r}) \subset K_{\bar r}$, clearly we have $\overline{\text{co}}(T_{\bar r}(K_{\bar r}) \cup \{0\}) \subset K_{\bar r}$. Hence, being true the assumptions of Theorem 2.13, there exists the smallest $(0, T_{\bar r})$-fundamental set $M_0 \subset \mathcal{C}(J;X)$ such that

$$M_0 \subset K_{\bar r} = \overline{B}_{\mathcal{C}(J;X)}(0, \bar r) \tag{4.25}$$

and

$$M_0 = \overline{\text{co}}(T_{\bar r}(M_0) \cup \{0\}). \tag{4.26}$$

Now, we prove that M_0 is weakly compact.

We consider the Sadovskij functional β_α, where $\alpha \in \mathbb{R}^+$, defined in (2.2). Being β_α 0-stable we can write (see (4.26))

$$\beta_\alpha(T_{\bar r}(M_0)) = \beta_\alpha(M_0). \tag{4.27}$$

Hence, since β_α satisfies (I) and (II), by (4.27), (4.5) and gh3 we have (see (2.2) and (3.1))

$$\begin{aligned}
\beta_\alpha(M_0) &= \beta_\alpha\left(\{C(\cdot,0)g(u) + S(\cdot,0)h(u) + G_S f : \ f \in S^1_{F(\cdot,N(\cdot)u)},\ u \in M_0\}\right) \\
&\leq \beta_\alpha(C(\cdot,0)g(M_0)) + \beta_\alpha(S(\cdot,0)h(M_0)) + \beta_\alpha(\{G_S f : \ f \in S^1_{F(\cdot,N(\cdot)u)},\ u \in M_0\}) \\
&= \beta_\alpha(\{G_S f : \ f \in S^1_{F(\cdot,N(\cdot)u)},\ u \in M_0\}) \\
&= \sup_{\substack{C \subset S^1_{F(\cdot,N(\cdot)M_0)} \\ C \text{ countable}}} \ \sup_{t \in J} \beta\left(\left\{\int_0^t S(t,\xi)f(\xi)\,d\xi : \ f \in C\right\}\right) e^{-\alpha t}.
\end{aligned} \tag{4.28}$$

Now, fixed $t \in J$ and a countable set $C \subset S^1_{F(\cdot,N(\cdot)M_0)}$ we define

$$C_t^* = \{S(t,\cdot)f(\cdot) : \ f \in C\}.$$

Recalling that $\bar r > \bar c$, by using p3. and F4, we can say that the set C_t^* is integrably bounded and so it is bounded in $L^1(J;X)$ and it has the property of equi-absolute continuity of the integral. Then, by recalling that X is a WCG Banach space, we are in the position to apply Proposition 2.16 to the countable set C_t^*, so we have

$$\begin{aligned}
\beta\left(\left\{\int_0^t S(t,\xi)f(\xi)\,d\xi : \ f \in C\right\}\right) &\leq \int_0^t \beta(C_t^*(\xi))\,d\xi \\
&= \int_0^t \beta(\{S(t,\xi)f(\xi) : \ f \in C\})\,d\xi.
\end{aligned} \tag{4.29}$$

Further let us note that for every $f \in C$ we can consider, by the Axiom of Choice, a continuous map $q_f \in M_0$ such that $f(\xi) \in F(\xi, N(\xi)q_f)$, a.e. $\xi \in J$. So the set $C_0^C = \{q_f \in M_0 : \ f \in C\}$ is countable too. Now, taking into account the numerability of C, there exists a null measure set $I \subset J$ containing the set A defined in FN, such that

$$f(\xi) \in F(\xi, N(\xi)q_f),\ \xi \in J \setminus I,\ f \in C,$$

where $q_f \in C_0^C$.

Hence, fixed $\xi \in J \setminus I$, we observe that $C(\xi) \subset F(\xi, C_0^C(\xi) \cup N(\xi)C_0^C)$. Now, since the countable set $C_0^C \subset M_0 \subset K_{\bar r}$, by FN there exists $\nu_{\bar r} \in L^1_+(J)$ such that

$$\beta(C(\xi)) \leq \nu_{\bar r}(\xi)\beta(C_0^C(\xi)).$$

The above considerations allow us to claim that for every countable set $C \subset S^1_{F(\cdot,N(\cdot)M_0)}$ there exists a countable subset $C_0^C \subset M_0$ such that

$$\int_0^t \beta(C(\xi))\,d\xi \leq \int_0^t \nu_{\bar{r}}(\xi)\beta(C_0^C(\xi))\,d\xi \leq \int_0^t \nu_{\bar{r}}(\xi) \sup_{\substack{C_0\subset M_0\\ C_0 \text{ countable}}} \beta(C_0(\xi))\,d\xi. \tag{4.30}$$

Now, by using (4.28), (4.29), (2.1), p3. and (4.30) we can write (see (2.2))

$$\begin{aligned}
\beta_\alpha(M_0) &\leq \sup_{\substack{C\subset S^1_{F(\cdot,N(\cdot)M_0)}\\ C \text{ countable}}} \sup_{t\in J} \left(\int_0^t \|S(t,\xi)\|_{\mathcal{L}(X)}\beta(C(\xi))\,d\xi\right) e^{-\alpha t}\\
&\leq \sup_{\substack{C\subset S^1_{F(\cdot,N(\cdot)M_0)}\\ C \text{ countable}}} \sup_{t\in J} \left(Ka\int_0^t \beta(C(\xi))\,d\xi\right) e^{-\alpha t}\\
&\leq \sup_{\substack{C\subset S^1_{F(\cdot,N(\cdot)M_0)}\\ C \text{ countable}}} \sup_{t\in J} \left(Ka\int_0^t \nu_{\bar{r}}(\xi) \sup_{\substack{C_0\subset M_0\\ C_0 \text{ countable}}} \beta(C_0(\xi))\,d\xi\right) e^{-\alpha t}\\
&\leq \sup_{t\in J} \left(Ka\int_0^t e^{-\alpha(t-\xi)}\nu_{\bar{r}}(\xi) \sup_{\substack{C_0\subset M_0\\ C_0 \text{ countable}}} \sup_{\xi\in J} e^{-\alpha\xi}\beta(C_0(\xi))\,d\xi\right)\\
&= \beta_\alpha(M_0) \sup_{t\in J} \int_0^t Ka e^{-\alpha(t-\xi)}\nu_{\bar{r}}(\xi)\,d\xi.
\end{aligned} \tag{4.31}$$

By virtue of Proposition 2.3 we can say that there exists $m \in \mathbb{N}$ such that

$$\sup_{t\in J} \int_0^t Ka e^{-m(t-\xi)}\nu_{\bar{r}}(\xi)\,d\xi < 1. \tag{4.32}$$

Now, if we assume that $\beta_m(M_0) > 0$, where m is the constant characterized in (4.32), from (4.31) we have the contradiction

$$\beta_m(M_0) \leq \beta_m(M_0) \sup_{t\in J} \int_0^t Ka e^{-m(t-\xi)}\nu_{\overline{p}}(\xi)\,d\xi < \beta_m(M_0).$$

So we can claim

$$\beta_m(M_0) = 0. \tag{4.33}$$

By definition of $\beta_m(M_0)$, we have that, for every $t \in J$, the set $M_0(t)$ is relatively weakly sequentially compact. Indeed, fixed $t \in J$ and a sequence $(q_n(t))_n$, $q_n(t) \in M_0(t)$, $n \in \mathbb{N}$, we consider the countable set

$$\tilde{C}(t) = \{q_n(t) :\ n \in \mathbb{N}\}.$$

By (4.33) we have $\beta(\tilde{C}(t)) = 0$ and, since β is regular the set $\tilde{C}(t)$ is relatively weakly compact. So, by the Eberlein–Šmulian Theorem $\tilde{C}(t)$ is relatively weakly sequentially compact too. Hence there exists a subsequence $(q_{n_k}(t))_k$ of $(q_n(t))_n$ such that $q_{n_k}(t) \rightharpoonup q(t) \in X$. Therefore, by the arbitrariness of the sequence $(q_n(t))_n$ we can conclude that $M_0(t)$ is relatively weakly sequentially compact, and, by using again Proposition 2.5, $M_0(t)$ is relatively weakly compact too.

Now, we use Proposition 2.4 to prove that the set $S^1_{F(\cdot,N(\cdot)M_0)}$ is relatively weakly compact in $L^1(J;X)$.

First of all, since $N(t)M_0 \subset \overline{B}_X(0,\overline{r})$ for every $t \in J$ (see N2 and recalling that $\overline{r} > \overline{c}$), we prove that $S^1_{F(\cdot,N(\cdot)M_0)}$ is integrably bounded. By (4.2) of F4 we have that there exists $\varphi_{\overline{r}} \in L^1_+(J)$ such that

$$\|f(t)\|_X \le \varphi_{\overline{r}}(t), \quad \text{a.e. } t \in J,\ f \in S^1_{F(\cdot,N(\cdot)M_0)}.$$

Therefore we can deduce that $S^1_{F(\cdot,N(\cdot)M_0)}$ is bounded in $L^1(J;X)$ and it has the property of equi-absolute continuity of the integral.

Finally we show that $S^1_{F(t,N(t)M_0)}$ is relatively weakly compact in X, for a.e. $t \in J$.

Let us fix $t \in J \setminus H^*$, where H^* is the null measure set containing the set A presented in FN and such that $\|F(t, N(t)M_0\|_X \le \varphi_{\overline{r}}(t)$, for every $t \in J \setminus H^*$. First of all, we note that $S^1_{F(t,N(t)M_0)}$ is norm bounded in X by the constant $\varphi_{\overline{r}}(t)$.

Next, we consider a sequence $(y_n)_n \subset S^1_{F(t,N(t)M_0)}$. Obviously there exists a sequence $(f_n)_n$ such that $f_n \in S^1_{F(\cdot,N(\cdot)M_0)}$ and $f_n(t) = y_n$, $n \in \mathbb{N}$. So we have

$$y_n \in F(t, N(t)M_0), \quad n \in \mathbb{N}. \tag{4.34}$$

Let us note that, for every $n \in \mathbb{N}$, from (4.34), there exists $q_n \in M_0$ such that

$$y_n \in F(t, N(t)q_n). \tag{4.35}$$

Now, by considering the two countable sets $C_0 = \{q_n : n \in \mathbb{N}\} \subset K_{\overline{r}}$ (see (4.25)) and $C_1 = \{y_n : n \in \mathbb{N}\}$ we have (see (4.35))

$$C_1 \subset F(t, N(t)C_0) \subset F(t, C_0(t) \cup N(t)C_0).$$

So, by FN and recalling that $M_0(t)$ is relatively weakly compact, we write

$$0 \le \beta(C_1) \le \nu_{\overline{r}}(t)\beta(C_0(t)) \le \nu_{\overline{r}}(t)\beta(M_0(t)) = 0,$$

so $\beta(C_1) = 0$, i.e. C_1 is relatively weakly compact. Hence there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ weakly convergent.

By the arbitrariness of $(y_n)_n$ in $S^1_{F(t,N(t)M_0)}$, we can conclude that $S^1_{F(t,N(t)M_0)}$ is relatively weakly sequentially compact. By using again the Eberlein–Šmulian Theorem the set $S^1_{F(t,N(t)M_0)}$ is relatively weakly compact.

Therefore, we are in the position to apply Proposition 2.4, hence $S^1_{F(\cdot,N(\cdot)M_0)}$ is relatively weakly compact in $L^1(J;X)$.

In order to prove the weak compactness of M_0, by (4.26) it is sufficient to show that $T(M_0)$ is relatively weakly compact.

To this aim we fix a sequence $(x_n)_n$, $x_n \in T(M_0)$. Then there exists $(q_n)_n$, $q_n \in M_0$, such that $x_n \in Tq_n$, $n \in \mathbb{N}$. Hence

$$x_n(t) = C(t,0)g(q_n) + S(t,0)h(q_n) + \int_0^t S(t,\xi)f_n(\xi)\,d\xi, \quad t \in J,$$

where $f_n \in S^1_{F(\cdot,N(\cdot)q_n)} \subset S^1_{F(\cdot,N(\cdot)M_0)}$.

By the relative weak sequential compactness of $S^1_{F(\cdot,N(\cdot)M_0)}$ we can find a subsequence $(f_{n_k})_k$ of $(f_n)_n$, such that $f_{n_k} \rightharpoonup f$ in $L^1(J;X)$ and by using the mentioned Proposition 3.2, we have

$$G_S f_{n_k} \rightharpoonup G_S f. \tag{4.36}$$

Moreover, if we consider the countable set $\{q_{n_k} : n_k \in \mathbb{N}\}$, thanks to gh2, there exist a subsequence of $(q_{n_k})_k$, w.l.o.g we name also $(q_{n_k})_k$, and $x, y \in X$ such that

$$g(q_{n_k}) \rightharpoonup x \quad \text{and} \quad h(q_{n_k}) \rightharpoonup y. \tag{4.37}$$

Now, let us consider the subsequence $(x_{n_k})_k$ of $(x_n)_n$. First of all, for every $t \in J$, since $S(t,0)$ and $C(t,0)$ are linear and bounded, from (4.36) and (4.37) we deduce

$$x_{n_k}(t) \rightharpoonup C(t,0)x + S(t,0)y + G_S f(t) := \overline{x}(t),$$

where $\overline{x} : J \to X$ is a continuous function.

Then, since $x_{n_k} \in T(M_0)$, $k \in \mathbb{N}$, by (4.25) we can write

$$\|x_{n_k} - \overline{x}\|_{C(J;X)} \leq \overline{r} + \|\overline{x}\|_{C(J;X)},$$

i.e. the sequence $(x_{n_k} - \overline{x})_k$ is uniformly bounded. So, thanks to Proposition 2.1 $x_{n_k} \rightharpoonup \overline{x}$. Then we deduce that $T(M_0)$ is relatively weakly sequentially compact and so $T(M_0)$ is relatively weakly compact.

Finally, recalling (4.26) we can conclude that M_0 is weakly compact. □

Now we state some conditions on F and N in order to have property (T) true in a Banach space X, i.e. the following selection proposition.

Proposition 4.6. *Let X be a Banach space, $F : J \times X \to \mathcal{P}(X)$ be a multimap having the properties* F1, F2, F3, F4 *and $N : J \to \hat{\mathcal{C}}_w(C(J;X);X)$ be a map satisfying* N1, N2. *If* FN *holds, then for every $u \in C(J;X)$ the set $S^1_{F(\cdot,N(\cdot)u)}$ is nonempty.*

Proof. First of all, fixed $u \in C(J;X)$, we define the following function $q_u : J \to X$

$$q_u(t) = N(t)u, \quad t \in J \tag{4.38}$$

and by N1 we say that q_u is B-measurable. Moreover, in correspondence of the costant $\overline{c} \in \mathbb{N}$ of N2, there exists $r_u \in \mathbb{N}$, $r_u > \overline{c}$ such that $\|u(t)\|_X \leq r_u$, $t \in J$. Now, we consider $F_{|J\times M_u} : J \times M_u \to \mathcal{P}(X)$, where

$$M_u = \overline{B}_X(0, r_u). \tag{4.39}$$

and we consider on M_u the metric d induced by that on the Banach space X.

We will show that this multimap satisfies all the hypotheses of Theorem 2.12.

First of all *f1)*, *f2)* and *f3)* of Theorem 2.12 are true for the restriction $F_{|J\times M_u}$ (see F1, F2 and F3 respectively).

Now, for the fixed r_u in (4.39), from F4 there exists $\varphi_{r_u} \in L^1_+(J)$ such that (see (4.2))

$$\sup_{z \in F(t,M_u)} \|z\| \leq \varphi_{r_u}(t), \quad a.e.\ t \in J,$$

i.e. *f4)* of Theorem 2.12 is true for $F_{|J\times M_u}$.

Finally, fixed a sequence $(u_n)_n$, $u_n \in M_u$, such that $u_n \to v$ in M_u, we consider, for every $n \in \mathbb{N}$, $g_n : J \to X$ so defined

$$g_n(t) = u_n, \quad t \in J. \tag{4.40}$$

Clearly $g_n \in \overline{B}_{C(J;X)}(0, r_u)$ (see (4.39)). Then, fixed $t \in J \setminus A$, where A is defined in FN, we show that $\bigcup_n F(t, u_n)$ is relatively weakly compact. Indeed, considering a sequence $(x_p)_p$, $x_p \in$

$\bigcup_n F(t,u_n)$, we fix the following countable sets $C_0 = \{g_n : n \in \mathbb{N}\}$ and $C_1 = \{x_p : p \in \mathbb{N}\}$. Now $C_0 \subset \overline{B}_{\mathcal{C}(J;X)}(0,r_u)$ and C_1 has the property (see (4.40))

$$C_1 \subset \bigcup_n F(t,u_n) = \bigcup_n F(t,g_n(t)) \subset F(t,C_0(t) \cup N(t)C_0).$$

By hypothesis FN there exists $\nu_{r_u} \in L^1_+(J)$ such that

$$\beta(C_1) \le \nu_{r_u}(t)\beta(C_0(t)).$$

Recalling the convergence $u_n \to v$ we have $\beta(C_0(t)) = 0$ and so $\beta(C_1) = 0$, i.e. C_1 is relatively weakly compact. Then there exists a subsequence of $(x_p)_p$ weakly convergent in X. By the arbitrariness of $(x_p)_p$, the set $\bigcup_n F(t,u_n)$ is relatively sequentially weakly compact and so it is also relatively weakly compact. Therefore property *f5)* of Theorem 2.12 holds for $F_{|J\times M_u}$.

Hence, in correspondence of $q_u : J \to X$ defined in (4.38), there exists a B-selection $g_{q_u} : J \to X$, for the multimap $F_{|J\times M_u}(\cdot, q_u(\cdot))$. Now, recalling that $r_u > \bar{c}$ for every $t \in J$, $N(t)u \in M_u$ (see (4.39) and N2), and by (4.38) we have $F_{|J\times M_u}(t,q_u(t)) = F(t,N(t)u)$. So

$$g_{q_u}(t) \in F(t,N(t)u), \quad \text{a.e. } t \in J. \tag{4.41}$$

Moreover, by using F4 there exists $\varphi_{r_u} \in L^1_+(J)$ such that (see (4.39))

$$\|g_{q_u}(t)\|_X \le \varphi_{r_u}(t), \quad \text{a.e. } t \in J.$$

Hence $g_{q_u} \in L^1(J;X)$ and, by (4.41), we have $g_{q_u} \in S^1_{F(\cdot,N(\cdot)u)}$, i.e. the set $S^1_{F(\cdot,N(\cdot)u)}$ is nonempty. □

Finally we are in a position to establish the following existence result of mild solutions for **(P)**$_{\mathbf{N}}$ in weakly compactly generated Banach spaces.

Theorem 4.7. *Let X be a WCG Banach space and $\{A(t)\}_{t\in J}$ a family of operators which satisfies the property* (A).

Let $F : J \times X \to \mathcal{P}(X)$ be a multimap and $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J;X);X)$ be a map satisfying F1, F2, F3, F4 *and* N1, N2 *respectively, and* FN. *Let $g, h : \mathcal{C}(J;X) \to X$ be two functions satisfying* gh1, gh2 *and* gh3.

Then there exists at least one mild solution for the nonlocal problem **(P)**$_{\mathbf{N}}$.

Proof. First of all by using Proposition 4.4 and Proposition 4.6 we can say that there exists $\bar{r} > \bar{c}$, $T(K_{\bar{r}}) \subset K_{\bar{r}}$, such that the map $T_{\bar{r}} = T_{|K_{\bar{r}}} : K_{\bar{r}} \to \mathcal{P}(\mathcal{C}(J;X))$ defined as in (4.5) is well posed, where $K_{\bar{r}}$ is presented in (4.24).

In order to obtain the thesis we want to apply the fixed point Theorem 2.14 to $T_{\bar{r}}$. At first, by F1 we deduce that $T_{\bar{r}}$ takes convex values, i.e. *i)* of Theorem 2.14 is satisfied.

Moreover, since $T_{\bar{r}}(K_{\bar{r}}) \subset K_{\bar{r}}$, we have $\overline{\text{co}}(T_{\bar{r}}(K_{\bar{r}} \cup \{0\})) \subset K_{\bar{r}}$, i.e. hypothesis *ii)* of Theorem 2.14 holds.

Next, Proposition 4.5 ensures the existence and the weakly compactness of the smallest $(0, T_{\bar{r}})$-fundamental set M_0, i.e. *iii)* of Theorem 2.14 is true.

Finally, thanks to Proposition 4.3, the restriction of $T_{\bar{r}}$ to the set weakly compact set M_0 has weakly sequentially closed graph, i.e. *iv)* of Theorem 2.14 holds.

In conclusion we can apply Theorem 2.14 to $T_{\bar{r}}$. Hence the multioperator T has a fixed point in M_0, i.e. there exists $x \in M_0$ such that

$$x(t) = C(t,0)g(x) + S(t,0)h(x) + \int_0^t S(t,\xi)f(\xi)\,d\xi, \quad t \in J$$

where $f \in S^1_{F(\cdot,N(\cdot)x)}$. Of course, x is a mild solution for **(P)**$_{\mathbf{N}}$. □

Clearly, an immediate consequence of Theorem 4.7 is the following existence result for Cauchy problems.

Corollary 4.8. *Let X be a WCG Banach space and $x_0, x_1 \in X$. Under the assumptions (A),* F1, F2, F3, F4, FN *and* N1, N2, *there exists at least one mild solution for the Cauchy problem*

$$\begin{cases} x''(t) \in A(t)x(t) + F(t, N(t)x), \ t \in J \\ x(0) = x_0 \\ x'(0) = x_1. \end{cases}$$

Remark 4.9. Let us note that Theorem 4.7 extends in broad sense Proposition 4.3 of [8]. In particular we remove the hypothesis of compactness on the operators $S(t,s)$, $(t,s) \in J \times J$, and we use the weak topology instead of the strong one on the maps involved in the nonlocal problem.

4.2 Existence of mild solutions in reflexive spaces

In this subsection we discuss the existence of mild solutions to the problem $(\mathbf{P})_\mathbf{N}$ in the particular case of reflexive Banach spaces. In this case we are able to omit assumptions FN and gh3 of Theorem 4.7. Let us note that the lack of these hypotheses implies that this result is new with respect to Theorem 4.7 since the reflexivity does not imply that gh3 holds. For this reason it is necessary to modify in some points the proof of the previous existence result and also to prove a variant of Theorem 2.12. Let us note that the reflexivity of the space allows us to remove hypothesis *f5)* of Theorem 2.12 in order to establish the existence of a selection for multimaps perturbed by an operator.

This new proposition plays a key role in order to show the good position of the solution multioperator in this new setting. Moreover, in order to prove the existence of at least a mild solution for the nonlocal problem $(\mathbf{P})_\mathbf{N}$, we use again the fixed point Theorem 2.14, but instead of the Containment Theorem, we work with the property of weak upper semicontinuity and with the classical Hahn–Banach Theorem.

Theorem 4.10. *Let $J = [0, a]$, M be a metric space, X a reflexive Banach space, $F : J \times M \to \mathcal{P}(X)$ a multimap having the properties f1), f2), f3), f4) of Theorem 2.12 and $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J; X); M)$ be a map satifying* N1.

Then, for every $u \in \mathcal{C}(J; X)$, there exists $y \in L^1(J; X)$ with $y(t) \in F(t, N(t)u)$ for a.e. $t \in J$.

Proof. First of all, by *f2)* we easily can deduce

f2)$_w$ for every simple function $s : J \to M$, the multimap $F(\cdot, s(\cdot))$ has a B-measurable selection.

Now, fixed $u \in \mathcal{C}(J; X)$, we define $q^u : J \to M$ as

$$q^u(t) = N(t)u, \quad t \in J \tag{4.42}$$

Clearly, by N1 the map q^u is B-measurable and so there exists a sequence of simple functions $(s_n^u)_n$, $s_n^u : J \to M$, such that

$$s_n^u(t) \to q^u(t), \quad \text{a.e. } t \in J. \tag{4.43}$$

Hence, fixed $n \in \mathbb{N}$, in correspondence of the simple function s_n^u, by *f2)$_w$* there exists a B-measurable function $y_n^u : J \to X$ such that

$$y_n^u(t) \in F(t, s_n^u(t)) \quad \text{a.e. } t \in J. \tag{4.44}$$

Now, let us consider $A = \{y_n^u,\ n \in \mathbb{N}\} \subset L^1(J;X)$. By *f4)* and (4.44) we have that

$$\|y_n^u(t)\|_X \le \varphi(t), \quad \text{a.e. } t \in J, n \in \mathbb{N}, \tag{4.45}$$

i.e. $A(t)$ is bounded in X, a.e. $t \in J$. So, since X is a reflexive Banach space, $A(t)$ is relatively weakly compact, for a.e. $t \in J$. Moreover, (4.45) implies that A is bounded in $L^1(J;X)$ and it has the property of equi-absolute continuity of the integral. Therefore, since the set A satisfies all the hypotheses of Proposition 2.4, we can conclude that A is relatively weakly compact. Hence there exist $(y_{n_k}^u)_k \subset (y_n^u)_n$ and $y \in L^1(J;X)$ such that $y_{n_k}^u \rightharpoonup y$.

Now, we can apply Proposition 2.7 to the multimap $G : J \to \mathcal{P}_{wk}(X)$, defined by $G(s) = \overline{A(s)}^w$, for every $s \in J$, and to the sequence $(y_{n_k}^u)_k$ of $L^1(J;X)$, so we deduce

$$y(t) \in \overline{\text{co}}\; w - \limsup_{k\to\infty}\{y_{n_k}^u(t)\}, \quad \text{a.e. } t \in J. \tag{4.46}$$

Next we fix H the null measure subset of J such that (4.43), (4.44), (4.46), *f1)* and *f3)* hold for every $t \in J \setminus H$. By (4.44) we can claim

$$\overline{\text{co}}\; w - \limsup_{k\to\infty}\{y_{n_k}(t)\} \subset \overline{\text{co}}\; w - \limsup_{k\to\infty} F(t, s_{n_k}^u(t)), \quad t \in J \setminus H. \tag{4.47}$$

Moreover, we are able to prove

$$w - \limsup_{k\to\infty} F(t, s_{n_k}^u(t)) \subset F(t, q^u(t)), \quad t \in J \setminus H. \tag{4.48}$$

To this aim, let us fix $t \in J \setminus H$ and $z \in w - \limsup_{k\to\infty} F(t, s_{n_k}^u(t))$. Then there exists $z_{n_{k_p}} \in F(t, s_{n_{k_p}}^u(t))$ such that $z_{n_{k_p}} \rightharpoonup z$, where $(n_{k_p})_p$ is an increasing sequence. Now, by (4.43) we know that

$$s_{n_{k_p}}^u(t) \to q^u(t),$$

therefore, since $t \notin H$, hypothesis *f3)* implies $z \in F(t, q^u(t))$. For the arbitrariness of z we can conclude that (4.48) is true.

In virtue of *f3)* the convex set $F(t, q^u(t))$ is closed in X so, by (4.48) and (4.42) we can write

$$\overline{\text{co}}\; w - \limsup_{k\to\infty} F(t, s_{n_k}^u(t)) \subset F(t, N(t)u). \tag{4.49}$$

Finally, thanks to (4.46), (4.47), (4.49), we can conclude that the map $y \in L^1(J;X)$ satisfies $y(t) \in F(t, N(t)u)$ a.e. $t \in J$, so the thesis holds. □

Now we show that in reflexive Banach spaces we can omit some assumptions on the multimap F and on the map N required in Proposition 4.3 and Proposition 4.5.

Proposition 4.11. *Let X be a reflexive Banach space. Under assumptions* (A), (T), F1, F3, F4, N2 *and* gh1, *the multioperator T has a weakly sequentially closed graph.*

Proof. As in Proposition 4.3 we fix two sequences $(q_n)_n \subset \mathcal{C}(J;X)$ and $(x_n)_n \subset \mathcal{C}(J;X)$, weakly convergent to $q, x \in \mathcal{C}(J;X)$ respectively, with $x_n \in Tq_n$, $n \in \mathbb{N}$.

By (T) we can say that, for every $n \in \mathbb{N}$, there exists (see (4.6))

$$f_n \in S^1_{F(\cdot, N(\cdot)q_n)} \tag{4.50}$$

such that (see (4.5))

$$x_n(t) = C(t,0)g(q_n) + S(t,0)h(q_n) + \int_0^t S(t,\xi)f_n(\xi)\,d\xi, \quad t \in J.$$

Now we want to prove that the set $A = \{f_n : n \in \mathbb{N}\}$ satisfies all the hypotheses of Proposition 2.4. Obviously, by (4.50), A is a subset of $L^1(J;X)$ and we have

$$f_n(t) \in F(t, N(t)q_n) \subset F(t, \overline{B}_X(0,\bar{c})), \quad a.e.\ t \in J,\ n \in \mathbb{N}, \tag{4.51}$$

where $\bar{c}$ is the constant presented in N2.

Now, put H the null measure set for which F4 and (4.51) hold, there exists $\varphi_{\bar{c}} \in L^1_+(J)$ such that (see (4.2))

$$\|f_n(t)\|_X \le \varphi_{\bar{c}}(t), \quad t \in J \setminus H,\ n \in \mathbb{N}.$$

So A is bounded in $L^1(J;X)$ and A has also the property of equi-absolute continuity of the integral. Moreover, for every $t \in J \setminus H$, $A(t)$ being bounded on the reflexive space X, $A(t)$ is relatively weakly compact.

Hence, applying Proposition 2.4, the set A is relatively weakly compact in $L^1(J;X)$. So there exist a subsequence $(f_{n_k})_k$ of $(f_n)_n$ and $f \in L^1(J;X)$ such that

$$f_{n_k} \rightharpoonup f. \tag{4.52}$$

Now, from (4.52) and by using the classical Mazur Theorem there exists a sequence $(\tilde{f}_{n_k})_k$ made up of convex combinations of f_{n_k}'s, such that $\tilde{f}_{n_k} \to f$ in $L^1(J;X)$. So, up to a subsequence, we have

$$\tilde{f}_{n_{k_p}}(t) \to f(t), \quad a.e.\ t \in J. \tag{4.53}$$

Let H^* be the null measure set for which (4.53), (4.4), (4.51), F3 and F4 hold. In order to show that $f \in S^1_{F(\cdot,N(\cdot)q)}$ we prove that

$$f(t) \in F(t, N(t)q), \quad t \in J \setminus H^*. \tag{4.54}$$

If we assume that (4.54) is false, there exists $\bar{t} \in J \setminus H^*$ such that $f(\bar{t}) \notin F(\bar{t}, N(\bar{t})q)$.

First, we want to establish that the multimap $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},\cdot)$ is weakly upper semicontinuous and, to this aim, we show that all the hypotheses of Proposition 2.10 are satisfied.

For every $x \in \overline{B}_X(0,\bar{c})$ from F4 we can write $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},x) \subset \overline{B}_X(0,\varphi_{\bar{c}}(\bar{t}))$. Therefore, since X is reflexive, from (4.4) (see F1, F3 and F4) we can say that the bounded set $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},x)$ is weakly compact. Moreover, the weak compactness of $\overline{B}_X(0,\varphi_{\bar{c}}(\bar{t}))$ obviously implies that the multimap $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},\cdot)$ is weakly compact. Further, recalling hypothesis F3, from Proposition 2.6 we deduce that $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},\cdot)$ is a weakly closed multimap.

Hence, since all the hypotheses of Proposition 2.10 are satisfied, the multimap $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},\cdot)$ is weakly upper semicontinuous.

Now, let us consider the convex, closed set $F_{|\overline{B}_X(0,\bar{c})}(\bar{t}, N(\bar{t})q)$ and the compact set $\{f(\bar{t})\}$. Since we have assumed that $f(\bar{t}) \notin F_{|\overline{B}_X(0,\bar{c})}(\bar{t}, N(\bar{t})q)$, by the classical Hahn–Banach Theorem there exists a weakly open convex set $V \supset F_{|\overline{B}_X(0,\bar{c})}(\bar{t}, N(\bar{t})q)$ satisfying

$$f(\bar{t}) \notin \overline{V} = \overline{V}^w \tag{4.55}$$

Next, taking into account the weak upper semicontinuity of $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},\cdot)$, there exists a weak neighborhood $W_{N(\bar{t})q}$ of the point $N(\bar{t})q$ such that

$$F_{|\overline{B}_X(0,\bar{c})}(\bar{t},x) \subset V, \quad x \in W_{N(\bar{t})q} \cap \overline{B}_X(0,\bar{c}). \tag{4.56}$$

Moreover, by the weak convergence of $(q_n)_n$ to q and the weak sequential continuity of $N(\bar{t})$, the subsequence $(N(\bar{t})q_{n_{k_p}})_p$, indexed as in (4.53), weakly converges to $N(\bar{t})q$. So, there exists $\overline{N} \in \mathbb{N}$ such that, for every $n_{k_p} \geq \overline{N}$, $N(\bar{t})q_{n_{k_p}} \in W_{N(\bar{t})q}$. Since from N2 $N(\bar{t})q_{n_{k_p}} \in \overline{B}_X(0, \bar{c})$, $n_{k_p} \geq \overline{N}$, we deduce that (see (4.56))

$$f_{n_{k_p}}(\bar{t}) \in F_{|\overline{B}_X(0,\bar{c})}(\bar{t}, N(\bar{t})q_{n_{k_p}})) \subset V, \; n_{k_p} \geq \overline{N}.$$

Now, thanks to the convexity of V, we can claim that the convex combinations $\tilde{f}_{n_{k_p}}$, satisfying (4.53), have the following property

$$\tilde{f}_{n_{k_p}}(\bar{t}) \in V, \quad n_{k_p} \geq \overline{N}$$

and so

$$f(\bar{t}) \in \overline{V} = \overline{V}^{w},$$

which contradicts (4.55). Therefore (4.54) is true. By recalling that $f \in L^1(J; X)$ we obtain $f \in S^1_{F(\cdot, N(\cdot)q)}$. Now, by using gh1 and the same technique of the final part of Proposition 4.3 we obtain $x \in Tq$.

Therefore we can conclude that T has a weakly sequentially closed graph. □

Proposition 4.12. *Let X be a reflexive Banach space and assumptions* (A), (T), F4, N2 *and* gh2 *hold. If there exists $\bar{r} > \bar{c}$ such that $T(K_{\bar{r}}) \subset K_{\bar{r}}$, where $K_{\bar{r}} = \overline{B}_{C(J;X)}(0, \bar{r})$, then there exists M_0 the smallest $(0, T_{\bar{r}})$-fundamental set which is weakly compact, where $T_{\bar{r}}$ is the restriction of T to the set $K_{\bar{r}}$.*

Proof. First of all, by using Theorem 2.13 and reasoning as at the beginning of Proposition 4.5, there exists

$$M_0 \subset \overline{B}_{C(J;X)}(0, \bar{r}) = K_{\bar{r}}$$

such that

$$M_0 = \overline{\text{co}}(T_{\bar{r}}(M_0) \cup \{0\}). \tag{4.57}$$

Now, we prove that M_0 is weakly compact. To this end we establish that the set $T_{\bar{r}}(M_0)$ is relatively weakly compact.

Let $(q_n)_n$ be a sequence in M_0 and $(x_n)_n$ be a sequence in $C(J; X)$ such that $x_n \in T_{\bar{r}}q_n$, $n \in \mathbb{N}$. Now, by (T), there exists a sequence $(f_n)_n$, $f_n \in S^1_{F(\cdot, N(\cdot)q_n)}$, such that (see (4.5))

$$x_n(t) = C(t, 0)g(q_n) + S(t, 0)h(q_n) + \int_0^t S(t, \xi)f_n(\xi)\, d\xi, \quad t \in J.$$

Next, put $A = \{f_n : \; n \in \mathbb{N}\}$, reasoning as in Proposition 4.11, we can apply Proposition 2.4 (see N2 and F4). Therefore we have that the subset A of $L^1(J; X)$ is relatively weakly compact. So there exist $(f_{n_k})_k$ subsequence of $(f_n)_n$ and $f \in L^1(J; X)$ such that $f_{n_k} \rightharpoonup f$.

Now, by the weak sequential continuity of G_S (see Proposition 3.2), we can write

$$G_S f_{n_k} \rightharpoonup G_S f.$$

Next, thanks to hypothesis gh2, reasoning as in the final part of the proof of Proposition 4.5, the subsequence $(x_{n_k})_k$ of $(x_n)_n$ weakly converges to a continuous function. Therefore $T(M_0)$ is relatively weakly sequentially compact and so $T(M_0)$ is also relatively weakly compact. Recalling (4.57) we deduce that the subset M_0 of $C(J; X)$ is weakly compact. □

Theorem 4.13. *Let X be a reflexive Banach space, $J = [0, a]$ and $\{A(t)\}_{t\in J}$ a family which satisfies the property* (A).

Let $F : J \times X \to \mathcal{P}(X)$ be a multimap and $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J; X); X)$ be a map satisfying F1, F2, F3, F4 *and* N1, N2 *respectively. Let $g, h : \mathcal{C}(J; X) \to X$ be two functions having the properties* gh1 *and* gh2.

Then there exists at least one mild solution for the nonlocal problem $(\mathbf{P})_{\mathbf{N}}$.

Proof. First of all, in the setting of theorem we have that property (T) is true. Indeed, fixed $u \in \mathcal{C}(J; X)$, we define a function $q_u : J \to X$ as in (4.38). Moreover, in correspondence of $\bar{c} \in \mathbb{N}$, presented in N2, there exists $r_u \in \mathbb{N}$, $r_u > \bar{c}$, such that $\|u(t)\|_X \leq r_u$, for every $t \in J$. Now we consider $F_{|J\times M_u} : J \times M_u \to \mathcal{P}(X)$, where $M_u = \overline{B}_X(0, r_u)$, and we note that hypotheses F1, F2, F3, F4 imply *f1), f2), f3), f4)* of Theorem 4.10 respectively. Moreover, the map N satisfies N1 of Theorem 4.10. So, by considering on M_u the metric d induced by that on X, we can conclude that there exists $g_u \in S^1_{F(\cdot,N(\cdot)u)}$ (see (4.6)). Therefore, since (T) holds, the map $T_{\bar{r}} = T_{|K_{\bar{r}}} : K_{\bar{r}} \to \mathcal{P}(\mathcal{C}(J; X))$, where $K_{\bar{r}}$ is defined in (4.24), is well posed.

Thanks to Propositions 4.4, 4.11, 4.12 and analogous arguments used in the proof of Theorem 4.7 we are in a position to apply Theorem 2.14.

So there exists at least one one mild solution for $(\mathbf{P})_{\mathbf{N}}$. □

Remark 4.14. Let us note that in reflexive Banach spaces we can omit hypothesis FN in Corollary 4.8.

Remark 4.15. We observe that $(\mathbf{P})_{\mathbf{N}}$ can be rewritten as the problem studied in [9]

$$\begin{cases} x''(t) \in A(t)x(t) + F(t, x(t)), \ t \in J \\ x(0) = g(x) \\ x'(0) = h(x) \end{cases} \tag{P}$$

by considering the map $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J; X); X)$ so defined

$$N(t)u = u(t), \quad t \in J, \ u \in \mathcal{C}(J; X).$$

Let us note that N is well posed by using Proposition 2.1.

Unfortunately Theorems 4.7 and 4.13 does not allow us to prove the existence of mild solutions for (**P**) because the map N has not the property N2.

Finally we deduce as a consequence of Theorem 4.13 the existence of mild solutions satisfying a "periodicity condition" and having a fixed initial velocity.

Corollary 4.16. *Let X be a reflexive Banach space and $\overline{x} \in X$. Under the assumptions* (A), F1, F2, F3, F4 *and* N1, N2 *there exists at least one mild solution for the problem*

$$\begin{cases} x''(t) \in A(t)x(t) + F(t, N(t)x), \ t \in J = [0, a] \\ x(0) = x(a) \\ x'(0) = \overline{x}. \end{cases} \tag{$\mathbf{P}$}'_{\mathbf{N}}$$

Proof. By considering the maps $g, h : \mathcal{C}(J; X) \to X$ so defined

$$g(x) = x(a) \qquad \text{and} \quad h(x) = \overline{x}, \quad x \in \mathcal{C}(J; X)$$

it is easy to see by using Proposition 2.1 that gh1 is true. Moreover, the reflexivity of X allow us to say that gh2 holds too. Since all the hypotheses of Theorem 4.13 are satisfied, we conclude that there exists at least one mild solution of the problem $(\mathbf{P})'_{\mathbf{N}}$. □

5 Controllability for a problem driven by (E)

Now, we are in a position to study the controllability for the problem mentioned in the introduction, subjected to mixed conditions and governed by the wave equation (E) under the action of a suitable operator.

First of all we use, as in [21], the identification between functions defined on the quotient group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with values in $\mathbb{C}$ and 2π-periodic functions from $\mathbb{R}$ to $\mathbb{C}$. Moreover, we consider the space $L^2(\mathbb{T}, \mathbb{C})$, i.e. the space of all functions defined in $\mathbb{R}$ and assuming values in $\mathbb{C}$, 2-integrable in $[0, 2\pi]$ and 2π-periodic, with the usual norm $\|\cdot\|_{L^2(\mathbb{T},\mathbb{C})}$.

In particular, we want to study the following problem

$$\begin{cases} \frac{\partial^2 w}{\partial t^2}(t,\xi) = \frac{\partial^2 w}{\partial \xi^2}(t,\xi) + b(t)\frac{\partial w}{\partial \xi}(t,\xi) + f(t,\xi,\frac{\Im m\ e(\hat{w})}{2e(\hat{w})+1}\int_0^t p(s)\,ds) + u(t,\xi) \\ w(t,0) = w(t,2\pi),\ t \in J \\ \frac{\partial w}{\partial \xi}(t,0) = \frac{\partial w}{\partial \xi}(t,2\pi),\ t \in J \\ w(0,\xi) = w(a,\xi),\ \text{a.e. } \xi \in \mathbb{R} \\ \frac{\partial w}{\partial t}(0,\xi) = x_0,\ \text{a.e. } \xi \in \mathbb{R} \\ u(t,\xi) \in U(t),\ \text{a.e. } t \in J,\ \xi \in \mathbb{R} \end{cases} \tag{C}$$

where $x_0 \in \mathbb{C}$, $J = [0, a]$, $a > 0$, $b \in \mathcal{C}^1(J)$, $p \in L^1(J)$, $f : J \times \mathbb{R} \times \mathbb{C} \to \mathbb{C}$, $e : \mathcal{C}(J, L^2(\mathbb{T}, \mathbb{C})) \to \mathbb{C}$ are suitable maps, $\hat{w} : J \to L^2(\mathbb{T}, \mathbb{C})$, where $\hat{w}(t) = w(t, \cdot)$, and $U : J \to \mathcal{P}(\mathbb{C})$.

In order to rewrite problem (C) into the abstract form $(\mathbf{P})'_{\mathbf{N}}$, it is necessary to define the Banach space X, the family $\{A(t) : t \in J\}$ and the nonlinear term F.

First of all we assume the Banach space $X = L^2(\mathbb{T}, \mathbb{C})$. Moreover, we denote by $H^1(\mathbb{T}, \mathbb{C})$ and by $H^2(\mathbb{T}, \mathbb{C})$ respectively the following Sobolev spaces

$$H^1(\mathbb{T},\mathbb{C}) = \left\{ x \in L^2(\mathbb{T},\mathbb{C}) :\ \text{there exists } \frac{dx}{d\xi} \in L^2(\mathbb{T},\mathbb{C}) \right\}$$

$$H^2(\mathbb{T},\mathbb{C}) = \left\{ x \in L^2(\mathbb{T},\mathbb{C}) :\ \text{there exist } \frac{dx}{d\xi}, \frac{d^2x}{d\xi^2} \in L^2(\mathbb{T},\mathbb{C}) \right\}.$$

Further we consider the operator $A_0 : H^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$ so defined

$$A_0 x = \frac{d^2x}{d\xi^2}, \quad x \in H^2(\mathbb{T},\mathbb{C}) \tag{5.1}$$

and we recall that the A_0 is the infinitesimal generator of a strongly continuous cosine family $\{C_0(t)\}_{t\in\mathbb{R}}$, where $C_0(t) : L^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$, for every $t \in \mathbb{R}$ (see [21]).

Then we fixed the function $P : J \to \mathcal{L}(H^1(\mathbb{T}, \mathbb{C}), L^2(\mathbb{T}, \mathbb{C}))$ defined as

$$P(t)x = b(t)\frac{dx}{d\xi}, \quad t \in J,\ x \in H^1(\mathbb{T},\mathbb{C}). \tag{5.2}$$

Now we can introduce the family $\{A(t) : t \in J\}$ where, for every $t \in J$, $A(t) : D(A) = H^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$ is the following operator

$$A(t) := A_0 + P(t). \tag{5.3}$$

Let us note that the family $\{A(t) : t \in J\}$ generates a fundamental system $\{S(t,s)\}_{(t,s)\in J\times J}$ (see [21, Lemma 4.1]) and, for every $x \in D(A)$, the map $t \mapsto A(t)x$ is continuous.

Moreover, let us consider $e : \mathcal{C}(J, L^2(\mathbb{T}, \mathbb{C})) \to \mathbb{C}$ an operator such that e)$_1$ e linear, bounded such that $e(h) \neq -\frac{1}{2}$, for all $h \in \mathcal{C}(J, L^2(\mathbb{T}, \mathbb{C}))$,

the map $f : J \times \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ having the following properties

f)$_1$ $f(t, \xi + 2\pi, z) = f(t, \xi, z)$, for all $(t, \xi, z) \in J \times \mathbb{R} \times \mathbb{C}$;

f)$_2$ $f(t, \cdot, y(\cdot)) \in L^2(\mathbb{T}, \mathbb{C})$, for every $t \in J$, $y \in L^2(\mathbb{T}, \mathbb{C})$;

f)$_3$ for every $y \in L^2(\mathbb{T}, \mathbb{C})$, the map $t \mapsto f(t, \cdot, y(\cdot))$ is weakly measurable;

f)$_4$ there exists $\alpha \in L^1_+(J)$ such that

$$Ka \int_0^a \alpha(\theta)\, d\theta < 1$$

where K is presented in Section 3 and

$$\|f(t, \xi, z) - f(t, \xi, w)\|_{\mathbb{C}} \leq \alpha(t)\|z - w\|_{\mathbb{C}}, \quad z, w \in \mathbb{C},\ \xi \in \mathbb{R},\ \text{a.e. } t \in J; \tag{5.4}$$

f)$_5$ for a.e. $t \in J$ and every $(y_n)_n$, $y_n \in L^2(\mathbb{T}, \mathbb{C})$ such that $y_n \rightharpoonup y$, $y \in L^2(\mathbb{T}, \mathbb{C})$, the sequence $(f(t, \cdot, y_n(\cdot)))_n$ uniformly converges to $f(t, \cdot, y(\cdot))$ in $\mathbb{R}$;

f)$_6$ for every $t \in J$, $f(t, \cdot, 0) \in L^2(\mathbb{T}, \mathbb{C})$ and the map $t \mapsto \|f(t, \cdot, 0)\|_{L^2(\mathbb{T}, \mathbb{C})}$ is in $L^1(J)$;

and the multimap $U : J \to \mathcal{P}(\mathbb{C})$ having the following properties:

U)$_1$ for every $t \in J$, $U(t)$ is closed and convex;

U)$_2$ for every $y \in L^2(\mathbb{T}, \mathbb{C})$, the map $t \mapsto \inf_{z \in U(t)} \left(\int_0^{2\pi} \|y(\xi) - z\|^2_{\mathbb{C}}\, d\xi \right)^{\frac{1}{2}}$ is $\mathcal{M}(J) \otimes \mathcal{B}(\mathbb{R})$-measurable.

Let us note that, since our goal is to prove the existence of a mild solution, the existence of derivatives is not necessary. So it is sufficient to consider that $w(t, \cdot) \in L^2(\mathbb{T}, \mathbb{C})$, instead of $w(t, \cdot) \in H^2(\mathbb{T}, \mathbb{C})$. In that follows, we revise functions $w, u : J \times \mathbb{R} \to \mathbb{C}$ such that $w(t, \cdot), u(t, \cdot) \in L^2(\mathbb{T}, \mathbb{C})$, for every $t \in J$, as two maps $x, v : J \to L^2(\mathbb{T}, \mathbb{C})$ respectively so defined

$$x(t)(\xi) = w(t, \xi), \quad t \in J,\ \xi \in \mathbb{R}$$

$$v(t)(\xi) = u(t, \xi), \quad t \in J,\ \xi \in \mathbb{R}.$$

Now by using f)$_2$ we can define the function $\tilde{f} : J \times L^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$ such that

$$\tilde{f}(t, y)(\xi) = f(t, \xi, y(\xi)), \quad t \in J,\ \xi \in \mathbb{R},\ y \in L^2(\mathbb{T}, \mathbb{C}). \tag{5.5}$$

Next we consider the multimap $\tilde{U} : J \to \mathcal{P}(L^2(\mathbb{T}, \mathbb{C}))$

$$\tilde{U}(t) = \{v \in L^2(\mathbb{T}, \mathbb{C}) :\ \exists z \in U(t) \text{ such that } v(\xi) = z, \text{ a.e. } \xi \in \mathbb{R}\}, \quad t \in J, \tag{5.6}$$

which is obviously well defined. Thanks to hypothesis U)$_1$ we deduce that $\tilde{U}(t)$ is closed and convex, for every $t \in J$. Then, taking into account U)$_2$ and the separability of $L^2(\mathbb{T}, \mathbb{C})$, Proposition 2.8 implies the measurability of $\tilde{U}$.

Moreover, we define the multimap $F : J \times L^2(\mathbb{T}, \mathbb{C}) \to \mathcal{P}(L^2(\mathbb{T}, \mathbb{C}))$ in the following way

$$F(t, y) = \{\tilde{f}(t, y) + v :\ v \in \tilde{U}(t)\}, \quad t \in J,\ y \in L^2(\mathbb{T}, \mathbb{C}). \tag{5.7}$$

Finally, by using the linear and bounded operator $e : \mathcal{C}(J, L^2(\mathbb{T}, \mathbb{C})) \to \mathbb{C}$ and the L^1-map $p : J \to \mathbb{R}$, we construct $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J; L^2(\mathbb{T}, \mathbb{C})); L^2(\mathbb{T}, \mathbb{C}))$ such that, for every $t \in J$ and every $h \in \mathcal{C}(J; L^2(\mathbb{T}, \mathbb{C}))$, the map $N(t)h$ is so defined (see e)$_1$)

$$[N(t)h](\xi) = \frac{\mathfrak{Im}\, e(h)}{2e(h)+1} \int_0^t p(s)\, ds, \quad \xi \in \mathbb{R}. \tag{5.8}$$

Clearly, being the map $N(t)h$ constant on $\mathbb{R}$, we have that $N(t)h \in L^2(\mathbb{T}, \mathbb{C})$ and, for every $t \in J$, $N(t)$ is weakly sequentially continuous. Therefore N is correctly defined too.

So by recalling (5.1), (5.2), (5.3), (5.5), (5.6), (5.7) and (5.8), problem **(C)** can be rewritten in the abstract form

$$\begin{cases} x''(t) \in A_0 x(t) + P(t)x(t) + F(t, N(t)x) = A(t)x(t) + F(t, N(t)x), \ t \in J \\ x(0) = x(a) \\ x'(0) = \hat{x_0} \end{cases}$$

where $\hat{x_0} : \mathbb{R} \to \mathbb{C}$ is the function of $L^2(\mathbb{T}, \mathbb{C})$ such that

$$\hat{x_0}(\xi) = x_0, \quad \xi \in \mathbb{R}. \tag{5.9}$$

At this point let us show that we can apply Corollary 4.16.

First of all, we note that the Banach space $L^2(\mathbb{T}, \mathbb{C})$ is obviously reflexive. Moreover, hypothesis (A) is clearly true thanks to the construction of the family $\{A(t) : \ t \in J\}$.

Now, let us show that hypotheses N1 and N2 are satisfied.

First of all, fixed $h \in \mathcal{C}(J; L^2(\mathbb{T}, \mathbb{C}))$ we note that $N(\cdot)h$ is a continuous map on J. Indeed, fixed $\bar{t} \in J$, we write (see (5.8))

$$\begin{aligned} \|N(t)h - N(\bar{t})h\|_{L^2(\mathbb{T},\mathbb{C})} &= \left\{ \int_0^{2\pi} \|[N(t)h](\xi) - [N(\bar{t})h](\xi)\|^2_{\mathbb{C}}\, d\xi \right\}^{\frac{1}{2}} \\ &= \left| \int_{\bar{t}}^t p(s)\, ds \right| \left\| \frac{\mathfrak{Im}\, e(h)}{2e(h)+1} \right\|_{\mathbb{C}} \sqrt{2\pi} \end{aligned}$$

and, by the absolute continuity of the integral we have the continuity of $N(\cdot)h$ in $\bar{t}$. Now, being $N(\cdot)h \in \mathcal{C}(J; L^2(\mathbb{T}, \mathbb{C}))$, obviously N satisfies hypothesis N1.

Moreover, for every $t \in J$ and every $h \in \mathcal{C}(J; L^2(\mathbb{T}, \mathbb{C}))$, since $\|\mathfrak{Im}\, e(h)\|_{\mathbb{C}} \leq \|2e(h)+1\|_{\mathbb{C}}$, we have

$$\|N(t)h\|_{L^2(\mathbb{T},\mathbb{C})} \leq \sqrt{2\pi}\|p\|_1 := \bar{c},$$

i.e. hypothesis N2 holds.

Now we show that the multimap F satisfies hypotheses F1–F4.

First of all, since $\tilde{U}$ has convex values, we can say that F takes convex values too, i.e. F1 holds.

Next we prove that, fixed $y \in L^2(\mathbb{T}, \mathbb{C})$, the multimap $F(\cdot, y)$ has a B-selection.

Since $L^2(\mathbb{T}, \mathbb{C})$ is a separable Banach space and taking into account f)$_3$, Proposition 2.2 allows us to say that $\tilde{f}(\cdot, y)$ is B-measurable.

Moreover, by recalling that $\tilde{U}$ is measurable and takes closed values, by using Proposition 2.9 there exists a $\mathcal{M}(J) \otimes \mathcal{B}(L^2(\mathbb{T}, \mathbb{C}))$-measurable $\tilde{u} : J \to L^2(\mathbb{T}, \mathbb{C})$ such that $\tilde{u}(t) \in \tilde{U}(t)$, for every $t \in J$. Then, by using the separability of the space $L^2(\mathbb{T}, \mathbb{C})$, $\tilde{u}$ is also B-measurable (see again Proposition 2.2).

So, $q_y : J \to L^2(\mathbb{T},\mathbb{C})$ so defined $q_y(t) := \tilde{f}(t,y) + \tilde{u}(t)$, $t \in J$, is B-measurable and such that $q_y(t) \in F(t,y)$, for every $t \in J$ (see (5.7)). Therefore, hypothesis F2 holds.

Now, we show that also hypothesis F3 is true.

Put H the null measure set for which f)$_5$ holds, we fix $t \in J \setminus H$ and $(y_n)_n$, $(q_n)_n$ two sequences in $L^2(\mathbb{T},\mathbb{C})$ such that $y_n \rightharpoonup y$, $q_n \rightharpoonup q$, where $y, q \in L^2(\mathbb{T},\mathbb{C})$, and $q_n \in F(t,y_n)$, for every $n \in \mathbb{N}$. Now, from (5.7), there exists $v_n \in \tilde{U}(t)$ such that

$$q_n = \tilde{f}(t,y_n) + v_n, \quad n \in \mathbb{N}. \tag{5.10}$$

Recalling f)$_5$ we have that $(f(t,\cdot,y_n(\cdot)))_n$ uniformly converges to $f(t,\cdot,y(\cdot))$ hence, by (5.5), we can write for every $\varepsilon > 0$ there exists $\overline{n}_t = \overline{n}_t(\frac{\varepsilon}{\sqrt{2\pi}}) \in \mathbb{N}$ such that, for every $n \geq \overline{n}_t$, $\|\tilde{f}(t,y_n)(\xi) - \tilde{f}(t,y)(\xi)\|_{\mathbb{C}} < \frac{\varepsilon}{\sqrt{2\pi}}$, $\xi \in \mathbb{T}$, and so we have $\|\tilde{f}(t,y_n) - \tilde{f}(t,y)\|_{L^2(\mathbb{T},\mathbb{C})} < \varepsilon$. Then we can say

$$\tilde{f}(t,y_n) \to \tilde{f}(t,y) \quad \text{in } L^2(\mathbb{T},\mathbb{C}). \tag{5.11}$$

Now, by (5.10) we have

$$v_n = q_n - \tilde{f}(t,y_n), \quad n \in \mathbb{N},$$

so (5.11) and the weak convergence of $(q_n)_n$ imply

$$v_n \rightharpoonup q - \tilde{f}(t,y) =: v,$$

where $v \in L^2(\mathbb{T},\mathbb{C})$.

Further, since $(v_n)_n$ is a sequence in the weakly closed set $\tilde{U}(t)$, the weak limit $v \in \tilde{U}(t)$ hence, by (5.7), we deduce $q = \tilde{f}(t,y) + v \in F(t,y)$. Therefore F3 holds.

Finally we prove that F4 is also true.

First of all, let $\tilde{H}$ the null measure set for which (5.4) of f)$_4$ holds. For every $n \in \mathbb{N}$, let us fix $y \in \overline{B}_{L^2(\mathbb{T},\mathbb{C})}(0,n)$ and $t \in J \setminus \tilde{H}$. Now, fixed $q \in F(t,y)$, by (5.7) there exists $v \in \tilde{U}(t)$ such that $q = \tilde{f}(t,y) + v$ and, named $z \in \mathbb{C}$ such that $v(\xi) = z$ a.e. $\xi \in \mathbb{T}$ (see (5.6)), we have (see (5.5) and f)$_4$)

$$\begin{aligned}
\|q\|_{L^2(\mathbb{T},\mathbb{C})} &\leq \|\tilde{f}(t,y)\|_{L^2(\mathbb{T},\mathbb{C})} + \|v\|_{L^2(\mathbb{T},\mathbb{C})} \\
&\leq \left\{ \int_0^{2\pi} [\alpha(t)\|y(\xi)\|_{\mathbb{C}} + \|f(t,\xi,0)\|_{\mathbb{C}}]^2 \, d\xi \right\}^{\frac{1}{2}} + \left\{ \int_0^{2\pi} \|z\|^2_{\mathbb{C}} \, d\xi \right\}^{\frac{1}{2}} \\
&\leq \left\{ \int_0^{2\pi} \alpha^2(t)\|y(\xi)\|^2_{\mathbb{C}} \, d\xi \right\}^{\frac{1}{2}} + \left\{ \int_0^{2\pi} \|f(t,\xi,0)\|^2_{\mathbb{C}} \, d\xi \right\}^{\frac{1}{2}} \\
&\quad + \left\{ \int_0^{2\pi} 2\alpha(t)\|f(t,\xi,0)\|_{\mathbb{C}}\|y(\xi)\|_{\mathbb{C}} \, d\xi \right\}^{\frac{1}{2}} + \sqrt{2\pi}\|z\|_{\mathbb{C}} \\
&\leq \alpha(t)\|y\|_{L^2(\mathbb{T},\mathbb{C})} + \|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})} \\
&\quad + \sqrt{2\alpha(t)}\sqrt{\|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})}}\sqrt{\|y\|_{L^2(\mathbb{T},\mathbb{C})}} + \sqrt{2\pi}\|z\|_{\mathbb{C}}.
\end{aligned}$$

So, recalling that $y \in \overline{B}_{L^2(\mathbb{T},\mathbb{C})}(0,n)$, we have

$$\|q\|_{L^2(\mathbb{T},\mathbb{C})} \leq \alpha(t)n + \|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})} + \sqrt{2n\alpha(t)}\sqrt{\|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})}} + \sqrt{2\pi}\|z\|_{\mathbb{C}}.$$

Therefore, put $\varphi_n : J \to \mathbb{R}^+_0$ so defined

$$\varphi_n(t) := \alpha(t)n + \|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})} + \sqrt{2n}\sqrt{\alpha(t)}\sqrt{\|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})}} + \sqrt{2\pi}\|z\|_{\mathbb{C}},$$

by the arbitrariness of $q \in F(t,y)$ and $y \in \overline{B}_{L^2(\mathbb{T},\mathbb{C})}(0,n)$ we deduce

$$\|F(t,\overline{B}_{L^2(\mathbb{T},\mathbb{C})}(0,n))\| \le \varphi_n(t) \quad \text{a.e. } t \in J. \tag{5.12}$$

As a consequence of f)$_4$, f)$_6$ and by using Hölder inequality it is easy to see that $\varphi_n \in L^1_+(J)$.

Moreover, by f)$_4$ we also have

$$\limsup_{n\to\infty} \frac{Ka\int_0^a \varphi_n(t)\,dt}{n} = Ka\int_0^a \alpha(t)\,dt < 1. \tag{5.13}$$

So (5.12) and (5.13) establish F4.

By virtue of arguments above presented, we are in the position to apply Corollary 4.16. Then there exists a continuous function $\hat{x} : J \to L^2(\mathbb{T},\mathbb{C})$ such that

$$\hat{x}(t) = C(t,0)\hat{x}(a) + S(t,0)\hat{x}_0 + \int_0^t S(t,s)\hat{q}(s)\,ds, \quad t \in J \tag{5.14}$$

$$\hat{x}(0) = \hat{x}(a) \qquad \hat{x}'(0) = \hat{x}_0, \tag{5.15}$$

where $\hat{q} \in S^1_{F(\cdot,N(\cdot)\hat{x})} = \{q \in L^1(J; L^2(\mathbb{T},\mathbb{C})) : \; q(t) \in F(t,N(t)\hat{x}) \text{ a.e. } t \in J\}$.

Now we consider $v_{\hat{x}} : J \to L^2(\mathbb{T},\mathbb{C})$ as

$$v_{\hat{x}}(t) = \hat{q}(t) - \tilde{f}(t,N(t)\hat{x}), \; t \in J. \tag{5.16}$$

In order to prove that $v_{\hat{x}}$ is B-measurable, we begin by showing that $\tilde{f}(\cdot,N(\cdot)\hat{x})$ is B-measurable. To this aim we consider the multimap $G : J \times L^2(\mathbb{T},\mathbb{C}) \to \mathcal{P}_k(L^2(\mathbb{T},\mathbb{C}))$ so defined (see (5.5))

$$G(t,y) = \{\tilde{f}(t,y)\}, \quad t \in J, \; y \in L^2(\mathbb{T},\mathbb{C})$$

and we establish that G satisfies all the hypotheses of Theorem 2.11.

First of all, fixed $y \in L^2(\mathbb{T},\mathbb{C})$, thanks to hypothesis f)$_3$ and to the separability of $L^2(\mathbb{T},\mathbb{C})$, $G(\cdot,y) = \{\tilde{f}(\cdot,y)\}$ has obviously a B-measurable selection, so hypothesis *i)* of Theorem 2.11 holds. Next, let us fix $t \in J \setminus H$, where H is the null measure set for which (5.4) is true, and $\overline{y} \in L^2(\mathbb{T},\mathbb{C})$. Then by (5.5) we have

$$\|\tilde{f}(t,\overline{y}) - \tilde{f}(t,y)\|_{L^2(\mathbb{T},\mathbb{C})} = \alpha(t)\|\overline{y} - y\|_{L^2(\mathbb{T},\mathbb{C})}, \quad y \in L^2(\mathbb{T},\mathbb{C}).$$

So, passing to the limit for $y \to \overline{y}$ we obtain that $\tilde{f}(t,\cdot)$ is continuous in $\overline{y}$. Obviously, for a.e. $t \in J$, $G(t,\cdot)$ is upper semicontinuous in $L^2(\mathbb{T},\mathbb{C})$, i.e. hypothesis *ii)* of Theorem 2.11.

Finally, since we have already proved N1, we know that $N(\cdot)\hat{x}$ is B-measurable. By using Theorem 2.11, $\tilde{f}(\cdot,N(\cdot)\hat{x})$ is B-measurable too. Therefore, being $\hat{q}$ B-measurable, also $v_{\hat{x}}$ is B-measurable.

At this point, put $w : J \times \mathbb{R} \to \mathbb{C}$ and $u : J \times \mathbb{R} \to \mathbb{C}$ respectively so defined

$$w(t,\xi) = \hat{x}(t)(\xi), \; u(t,\xi) = v_{\hat{x}}(t)(\xi), \quad t \in J, \; \xi \in \mathbb{R}, \tag{5.17}$$

we show that $\{w,u\}$ is an admissible mild-pair for **(C)**. By (5.14) and (5.15) we immediately have that, for every $\xi \in \mathbb{R}$, $w(\cdot,\xi)$ is continuous on J, $w(\xi,0) = w(\xi,a)$ and, for every $t \in J$, $w(t,\cdot)$ is 2-integrable on $[0,2\pi]$ and 2π-periodic. Let us note that, for every $\xi \in \mathbb{R}$ for which $w(\cdot,\xi)$ is derivable at 0, we have $\frac{\partial w}{\partial t}(0,\xi) = x_0$ (see (5.15) and (5.9)).

Then, fixed $t \in J \setminus H$, where H is the null measure set such that $\hat{q}(\cdot) \in F(\cdot,N(\cdot)\hat{x})$ in $J \setminus H$, by (5.7) there exists $v_t \in \tilde{U}(t)$ such that

$$\hat{q}(t) = \tilde{f}(t,N(t)\hat{x}) + v_t.$$

On the other hand, from (5.16) we have

$$\hat{q}(t) = \tilde{f}(t, N(t)\hat{x}) + v_{\hat{x}}(t).$$

Therefore, $v_{\hat{x}}(t) = v_t$. Hence $v_{\hat{x}}(t) \in \tilde{U}(t)$, a.e. $t \in J$. Since by (5.6) we have $v_{\hat{x}}(t)(\xi) \in U(t)$, a.e. $t \in J$, $\xi \in \mathbb{R}$, by (5.17) we can write $u(t,\xi) \in U(t)$, a.e. $t \in J$, $\xi \in \mathbb{R}$. Then, by the B-measurability of $v_{\hat{x}}$, for a.e. $\xi \in \mathbb{R}$, the map $u(\cdot,\xi)$ is B-measurable too. Clearly for every $t \in J$, $u(t,\cdot)$ is 2-integrable on $[0,2\pi]$ and 2π-periodic.

Hence, we can conclude that $\{w,u\}$ is an admissible mild-pair for **(C)**.

Finally we are able to enunciate the following result.

Theorem 5.1. *In the framework above described, there exist $w, u : J \times \mathbb{R} \to \mathbb{C}$ satisfying the following properties*

(w1) for every $t \in J$, $w(t,\cdot)$ is 2-integrable on $[0,2\pi]$ and 2π-periodic;

(w2) for every $\xi \in \mathbb{R}$, $w(\cdot,\xi)$ is continuous on J;

(w3) $w(0,\xi) = w(a,\xi)$, for every $\xi \in \mathbb{R}$;

(w4) for every $\xi \in \mathbb{R}$ such that $w(\cdot,\xi)$ is derivable at 0, we have $\frac{\partial w}{\partial t}(0,\xi) = x_0$;

(u1) for every $t \in J$, $u(t,\cdot)$ is 2-integrable on $[0,2\pi]$ and 2π-periodic;

(u2) for every $\xi \in \mathbb{R}$, $u(\cdot,\xi)$ is B-measurable and such that $u(t,\xi) \in U(t)$, for a.e. $t \in J$, $\xi \in \mathbb{R}$,

*i.e. $\{w,u\}$ is an admissible pair for **(C)** such that*

$$w(t,\xi) = [C(t,0)w(a,\cdot)](\xi) + [S(t,0)\hat{x}_0](\xi) + \int_0^t [S(t,s)q(s,\cdot)](\xi)\, ds, \quad t \in J,\ \xi \in \mathbb{R}$$

where $\hat{x}_0(\xi) = x_0$ for every $\xi \in \mathbb{R}$ and $q : J \times \mathbb{R} \to \mathbb{C}$ is so defined

$$q(t,\xi) = f\left(t, \xi, \frac{\mathfrak{Im}\, e(\hat{w})}{2e(\hat{w}) + 1} \int_0^t p(s)\, ds\right) + u(t,\xi), \quad t \in J,\ \xi \in \mathbb{R},$$

being $\hat{w} : J \to L^2(\mathbb{T},\mathbb{C})$, a map such that $\hat{w}(t) = w(t,\cdot)$, for every $t \in J$.

6 Conclusions and future studies

In this paper, the existence of mild solutions to a nonlocal problem governed by a semilinear second order differential inclusion in Banach spaces is investigated. The novelty with respect to the known results of [9] is the presence of an operator which intervenes on the dynamics described through a second order differential inclusion. Our first result is obtained with a fixed point approach, by applying ideas about measures of weak noncompactness, a selection theorem and a containment theorem. Further, in order to analyze the case of reflexive spaces a new selection theorem is proved and a combination of this result with the classical Hahn–Banach Theorem and the weak upper semicontinuity property is used. The applied method enables us obtaining the existence results without any compactness requirement both on the family generated by the linear part and on the nonlinear multivalued term. Finally our theoretical theorems are applied to study the controllability of a problem driven by a wave equation.

A possible future direction of research related to this topic could be to broaden the class of models to which it can be applied. For example could be interesting to remove the boundedness type property on the perturbation operator N, perhaps using a different fixed point theorem. As we noted in Remark 4.15, this assumption does not allow to see a non-perturbed problem as a particular case of a perturbed one. Moreover, in a contest of lack of compactness, this boundedness property on N does not make it possible to investigate problems involving operators N having a stabilization effect on the solution, like those studied in [8] under strong compactness assumptions.

Author contributions

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Concentration of solutions for an (N, q)-Laplacian equation with Trudinger–Moser nonlinearity

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Abstract. In this article, we consider the concentration of positive solutions for the following equation with Trudinger–Moser nonlinearity:

$$\begin{cases} -\Delta_N u - \Delta_q u + V(\varepsilon x)(|u|^{N-2}u + |u|^{q-2}u) = f(u), & x \in \mathbb{R}^N, \\ u \in W^{1,N}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), & x \in \mathbb{R}^N, \end{cases}$$

where V is a positive continuous function and has a local minimum, $\varepsilon > 0$ is a small parameter, $2 \leq N < q < +\infty$, f is C^1 with subcritical growth. When V and f satisfy some appropriate assumptions, we construct the solution u_ε that concentrates around any given isolated local minimum of V by applying the penalization method for the above equation.

Keywords: (N, q)-Laplacian equation, penalization method, variational methods.

2020 Mathematics Subject Classification: 35A15, 35B38, 35J60.

1 Introduction and main result

In this article, we consider the concentration of positive solutions for an (N, q)-Laplacian equation with Trudinger–Moser nonlinearity:

$$\begin{cases} -\Delta_N u - \Delta_q u + V(\varepsilon x)(|u|^{N-2}u + |u|^{q-2}u) = f(u), & x \in \mathbb{R}^N, \\ u \in W^{1,N}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $V : \mathbb{R}^N \mapsto \mathbb{R}$ is a function that satisfies continuity and has a local minimum, $\varepsilon > 0$ is a small parameter, $2 \leq N < q < +\infty$, $f \in C^1$ is subcritical.

We first introduce some background about (p, q)-Laplacian equation. As described in [14], problem (1.1) originates from the following reaction-diffusion equation:

$$u_t = C(x, u) + \operatorname{div}(D(u)\nabla u), \quad D(u) = |\nabla u|^{q-2} + |\nabla u|^{p-2}.$$

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It is widely used in physics or chemistry, such as solid state physics, chemical reaction design, biophysics and plasma physics. Note that, in general reaction-diffusion equation, the physical meaning of u is concentration, and the physical meaning of $\operatorname{div}(D(u)\nabla u)$ is the diffusion generated by $D(u)$. $C(x,u)$ is related to the source and loss process. Generally, $C(x,u)$ is a polynomial with variable coefficients related to u in chemical and biological applications.

When $p < q < N$, Zhang et al. in [36] studied the following double phase problem

$$\begin{cases} (-\Delta)_q^m u + (-\Delta)_p^m u + V(\varepsilon x)\left(|u|^{q-2}u + |u|^{p-2}u\right) = \lambda f(u) + |u|^{r-2}u, & x \in \mathbb{R}^N, \\ u \in W^{m,p}\left(\mathbb{R}^N\right) \cap W^{m,q}\left(\mathbb{R}^N\right), u > 0, & x \in \mathbb{R}^N, \end{cases}$$

where ε is a parameter small enough but λ is required to be large enough, $0 < m < 1$, $r = q_m^* = Nq/(N - mq)$, $2 \leqslant p < q < N/m$, $(-\Delta)_t^m$ is the fractional t-Laplace operator and the potential $V : \mathbb{R}^N \mapsto \mathbb{R}$ is a continuous function. The authors obtained the existence and concentration properties of multiple positive solutions to the above problem. Note that, [36] assumed that the nonlinearity satisfies the Ambrosetti–Rabinowitz condition, that is, for all $t > 0$, there is $\theta \in (q, q_m^*)$ that satisfies $0 < \theta F(t) := \theta \int_0^t f(\tau)\mathrm{d}\tau \leqslant f(t)t$. So the authors can get the existence and concentration properties of multiple positive solutions by using Nehari manifold.

When $1 < q < N = p$, the authors in [12] investigated the existence of solutions for the (N,q)-Laplacian equation:

$$-\Delta_q u - \Delta_N u = f(u) \text{ in } \mathbb{R}^N, \tag{1.2}$$

where the nonlinear term $f(u)$ satisfies exponential critical growth in the sense of Trudinger–Moser. In order to detect the solution, they used a variational method related to the new Trudinger–Moser type inequality. Figueiredo and Nunes in [19] used Nehari manifold method to studied the existence of positive solutions for the following class of quasilinear problems

$$\begin{cases} -\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

It is worth pointing out that Theorems 1.1 and 1.2 in [19] are valid for the problem (1.2) if $\mathbb{R}^N$ is replaced by Ω which is a smooth bounded domain. In [15], Costa and Figueiredo studied a class of quasilinear equation with exponential critical growth. They used variational methods and del Pino and Felmer's technique (del Pino and Felmer 1996) in order to overcome the lack of compactness, and got the existence of a family nodal solutions, which concentrate on the minimum points set of the potential function, changes sign exactly once in $\mathbb{R}^N$.

When $p = N/m < q$, Nguyen in [29] studied the following Schrödinger equation involving the fractional (N,q)-Laplace operator and Trudinger–Moser nonlinear term

$$(-\Delta)_{N/m}^m u + (-\Delta)_q^m u + V(\varepsilon x)\left(|u|^{\frac{N}{m}-2}u + |u|^{q-2}u\right) = f(u) \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon > 0$ is a parameter small enough, $m \in (0,1)$, $N = pm$, $2 \leq p = N/m < q$, the potential $V : \mathbb{R}^N \mapsto \mathbb{R}$ is a continuous function that satisfies some suitable conditions. The nonlinear term $f(u)$ satisfies exponential growth. In order to obtain existence and concentration properties of nontrivial nonnegative solutions, the author in [29] used the Ljusternik–Schnirelmann theory and Nehari manifold.

It is worth mentioning that both the nonlinearities of [12] and [29] satisfy the Ambrosetti–Rabinowitz condition. Inspired by the above works, it seems quite natural to ask if $f(u)$ does not satisfy the Ambrosetti–Rabinowitz condition but satisfies Beresticky–Lions type assumptions, do the same results hold for (N, q)-Laplacian problem? In this paper, we give a positive answer.

In the present paper, we assume that the potential $V : \mathbb{R}^N \mapsto \mathbb{R}$ is a continuous function satisfying the following conditions which are always called del Pino–Felmer type conditions (cf. [16]).

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ such that $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0$.

(V_2) There exists a bounded domain $\Lambda \subset \mathbb{R}^N$ satisfies

$$m := \inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

Moreover, we can assume $0 \in \mathcal{M} := \{x \in \Lambda : V(x) = m\}$.

The nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, for $t \leq 0$, we assume that $f(t) = 0$. Furthermore, $f(t)$ satisfies the following hypotheses:

(f_1) $\lim_{t\to 0} \frac{f(t)}{t^{q-1}} = 0$;

(f_2) $\forall \alpha > 0$, for $t \geq 0$, there is a $C_\alpha > 0$ satisfies $|f(t)| \leq C_\alpha e^{\alpha t^{\frac{N}{N-1}}}$;

(f_3) there is $T > 0$ satisfies $F(T) > \frac{m}{N} T^N + \frac{m}{q} T^q$.

Next, we state the main conclusion as follows:

Theorem 1.1. *If (V_1)–(V_2) and (f_1)–(f_3) are true, for small $\varepsilon > 0$, equation (1.1) has a positive solution u_ε which has a maximum point x_ε satisfying*

$$\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathcal{M}) = 0.$$

Moreover, for any x_ε, as $\varepsilon \to 0$ (up to a subsequence), $v_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ converges uniformly to a least energy solution of the following equation:

$$\begin{cases} -\Delta_q u - \Delta_N u + m(|u|^{q-2}u + |u|^{N-2}u) = f(u), & x \in \mathbb{R}^N, \\ u \in W^{1,q}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N), & x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

Furthermore, we have

$$u_\varepsilon(x) \leq C_1 e^{-C_2|x-x_\varepsilon|}, \quad \forall x \in \mathbb{R}^N,\ C_1,\ C_2 > 0.$$

Remark 1.2. Without loss of generality, it can be assumed that $V_0 = 1$.

As far as we know, there is no result on the concentration of positive solutions for (N, q)-Laplacian problems with Berestycki–Lions nonlinearity.

Finally, we point out that Theorem 1.1 is proved by variational method, and there are four main difficulties we encounter during the preparation of manuscript:

(1) The nonlinear term $f(u)$ does not satisfy the Ambrosetti–Rabinowitz condition, and for $u > 0$, the function $\frac{f(u)}{u^{q-1}}$ is not increasing. They both prevent us from getting the boundedness of Palais–Smale sequence and using the Nehari manifold. Moreover, we can not apply the method in [16].

(2) Since $\mathbb{R}^N$ is unbounded, it will lead to the loss of compactness. In the later proof, we will find that this difficulty will prevent us from directly using the variational method.

(3) When $N > 2$, the working space X_ε is no longer a Hilbert space. This makes it more complicated to prove the following formula in Lemma 3.11:

$$J_\varepsilon(u_\varepsilon) \geq J_\varepsilon\left(u_\varepsilon^1\right) + J_\varepsilon\left(u_\varepsilon^2\right) + o(1)$$

as $\varepsilon \to 0$.

(4) Due to $N = p < q$, we can not use the method of [2] to obtain that $b_m \geq c_m$ in Lemma 3.6.

In order to overcome the above difficulties, inspired by [8, 18, 22, 25], we recover the compactness by penalization method described in [10].

The plan of this paper is as follows. In Section 2, we give some definitions of function spaces and lemmas to be used later. In Section 3, we give the proof of Theorem 1.1.

2 Preliminary

In this section, we will give some definitions of symbols, and review some existing results that need to be used in the future.

Let $u : \mathbb{R}^N \mapsto \mathbb{R}$. For $2 \leq N < q < +\infty$, let us define $D^{1,N}(\mathbb{R}^N) = \overline{C^\infty(\mathbb{R}^N)}^{|\nabla\cdot|_N}$. We denote the following fractional Sobolev space

$$W^{1,N}(\mathbb{R}^N) = \{u : |\nabla u|_N < +\infty,\ |u|_N < +\infty\}$$

equipped with the natural norm

$$\|u\|_{W^{1,N}(\mathbb{R}^N)} = \left(\ |\nabla u|_N^N + |u|_N^N\right)^{1/N},$$

where $|\cdot|_N^N := \int_{\mathbb{R}^N} |\cdot|^N \mathrm{d}x$.

For all $u, v \in W^{1,N}(\mathbb{R}^N)$, we define

$$\langle u, v\rangle_{W^{1,N}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\nabla u|^{N-2}\nabla u \nabla v + |u|^{N-2}uv)\mathrm{d}x.$$

In this article, we need to introduce a work space

$$X = W^{1,N}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$$

whose norm is defined as

$$\|u\|_X := \|u\|_{W^{1,q}(\mathbb{R}^N)} + \|u\|_{W^{1,N}(\mathbb{R}^N)}.$$

When $V(x) = V_0$, we define space

$$X_0 := \left\{u \in X : \int_{\mathbb{R}^N} V_0(|u|^q + |u|^N)\mathrm{d}x < +\infty\right\}$$

equipped with the norm as

$$\|u\|_{X_0} = \|u\|_{V_0,q} + \|u\|_{V_0,N},$$

where $\|u\|_{V_0,r}^r = \int_{\mathbb{R}^N} (|\nabla u|^r + V_0|u|^r)\, dx$, $\forall r \in \{N, q\}$. It should be noted that X_0 is a separable reflexive Banach space. Due to the Theorem 6.9 in [28], for any $\nu \in [N, +\infty)$, it is easy to see that the embedding from X_0 into $L^\nu(\mathbb{R}^N)$ is continuous. Then for all $\nu \in [N, +\infty)$, there exists $A_{\nu,m} > 0$ satisfies

$$A_{\nu,m} = \inf_{u\neq 0, u\in X_0} \frac{\|u\|_{X_0}}{\|u\|_{L^\nu(\mathbb{R}^N)}}.$$

This implies

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq A_{\nu,m}^{-1}\|u\|_{X_0} \quad \text{for all } \ u \in X_0. \tag{2.1}$$

Fix $\varepsilon \geq 0$, we also need to introduce the following space

$$X_\varepsilon := \left\{ u \in X : \int_{\mathbb{R}^N} V(\varepsilon x)(|u|^q + |u|^N) dx < +\infty \right\}$$

whose norm is defined as

$$\|u\|_{X_\varepsilon} := \|u\|_{V_\varepsilon,q} + \|u\|_{V_\varepsilon,N},$$

where $\|u\|_{V_\varepsilon,r}^{1,r} = \int_{\mathbb{R}^N} (|\nabla u|^r + V(\varepsilon x)|u|^r)\, dx$, $\forall r \in \{N, q\}$. According to Lemma 10 in [31], we obtain that X_ε is uniformly convex Banach space. Moreover, for any $\nu \in [N, +\infty)$, the embedding

$$X_\varepsilon \hookrightarrow L^\nu(\mathbb{R}^N)$$

is continuous. Then for all $\nu \in [N, +\infty)$, there is $S_{\nu,\varepsilon} > 0$ satisfies:

$$S_{\nu,\varepsilon} = \inf_{u\neq 0, u\in X_\varepsilon} \frac{\|u\|_{X_\varepsilon}}{\|u\|_{L^\nu(\mathbb{R}^N)}}.$$

It can be seen that

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq S_{\nu,\varepsilon}^{-1}\|u\|_{X_\varepsilon}, \quad \forall u \in X_\varepsilon. \tag{2.2}$$

Finally, we consider

$$X_{\text{rad}} := \{u \in X : u(x) = u(|x|)\}.$$

Lemma 2.1 (see [34, Theorem 2.8]). *Assume that X is a Banach space, M_0 is a closed subspace of the metric space M, $\Gamma_0 \subset \mathcal{C}(M_0, X)$. Consider*

$$\Gamma := \{\gamma \in \mathcal{C}(M, X) : \gamma|_{M_0} \in \Gamma_0\}.$$

Assume $\varphi \in C^1(X, \mathbb{R})$ satisfies

$$\infty > c := \inf_{\gamma\in\Gamma} \sup_{u\in M} \varphi(\gamma(u)) > a := \sup_{\gamma_0\in\Gamma_0} \sup_{u\in M_0} \varphi(\gamma_0(u)).$$

For any $\varepsilon \in (0, (c-a)/2), \delta > 0$ and $\gamma \in \Gamma$ such that $\sup_M \varphi \circ \gamma \leq c + \varepsilon$, there is $u \in X$ satisfies

(a) $c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon$;

(b) $\text{dist}(u, \gamma(M)) \leq 2\delta$;

(c) $\|\varphi'(u)\| \leq \frac{8\varepsilon}{\delta}$.

Now, we recall follow Lemma 2.2 from J. M. do Ó [17] (or see [11]). The Lemma 2.3 follows from Adachi and Tanaka [1].

Lemma 2.2 (see [17]). *Assume $N \geq 2$, $u \in W^{1,N}\left(\mathbb{R}^N\right)$ and $\alpha > 0$, we have*

$$\int_{\mathbb{R}^N} \left(\exp\left(\alpha|u|^{N/(N-1)}\right) - S_{N-2}(\alpha, u)\right) \mathrm{d}x < \infty,$$

where

$$S_{N-2}(\alpha, u) = \sum_{k=0}^{N-2} \frac{\alpha^k}{k!} |u|^{\frac{kN}{(N-1)}}.$$

In addition, when $\alpha < \alpha_N$, for $\forall M > 0$, there is $C = C(\alpha, N, M)$ satisfies

$$\int_{\mathbb{R}^N} \left(\exp\left(\alpha|u|^{N/(N-1)}\right) - S_{N-2}(\alpha, u)\right) \mathrm{d}x \leq C, \quad \forall u \in W^{1,N}(\mathbb{R}^N).$$

We also have $\|u\|_N \leq M$ and $\|\nabla u\|_N \leq 1$.

Lemma 2.3 (see [1]). *Assume $N \geq 2$, $\alpha \in (0, \alpha_N)$, there is a constant $C_\alpha > 0$ that satisfies*

$$\|\nabla u\|_N^N \int_{\mathbb{R}^N} \Psi_N\left(\frac{u}{\|\nabla u\|_N}\right) \mathrm{d}x \leq C_\alpha \|u\|_N^N, \quad \forall u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}.$$

Here $\Psi_N(t) = e^{\alpha|t|^{N/(N-1)}} - S_{N-2}(\alpha, t)$.

3 Proof of Theorem 1.1

For $\forall B \subset \mathbb{R}^N$, $\varepsilon > 0$, B_ε can be define as $B_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in B\}$. Next, we will use the method in [16,21] to modify f. According to (f_1), there exists $a > 0$ such that

$$f(t) \leq \frac{t^{N-1}}{2}, \ \forall t \in (0, a).$$

For $t \in \mathbb{R}, x \in \mathbb{R}^N$, assume that

$$g(x, t) = (1 - \chi_\Lambda(x))\,\widetilde{f}(t) + \chi_\Lambda(x) f(t),$$

where

$$\widetilde{f}(t) = \begin{cases} f(t), & t \leq a, \\ \min\left\{f(t), \frac{1}{2}t^{N-1}\right\}, & t > a \end{cases}$$

and

$$\chi_\Lambda(x) = \begin{cases} 1, & x \in \Lambda, \\ 0, & x \notin \Lambda. \end{cases}$$

Obviously, $\forall x \in \mathbb{R}^N, t \in [0, a]$, we have $g(x, t) = f(t)$. Moreover, for $\forall x \in \mathbb{R}^N, t \geq 0$, we also obtain that $g(x, t) \leq f(t)$. Now, considering the modified problem

$$\begin{cases} -\Delta_N u - \Delta_q u + V_\varepsilon(|u|^{N-2}u + |u|^{q-2}u) = g(\varepsilon x, u), & x \in \mathbb{R}^N, \\ u \in X_\varepsilon, \ u > 0, & x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

where $g(\varepsilon x, t) = (1 - \chi_{\Lambda_\varepsilon}(x))\,\widetilde{f}(t) + \chi_{\Lambda_\varepsilon}(x) f(t)$. Clearly, for $x \in \mathbb{R}^N \backslash \Lambda_\varepsilon$, if u_ε satisfies $u_\varepsilon(x) \leq a$ and it is a solution of (3.1), we know that u_ε is the solution of the original problem (1.1).

As to $u \in X_\varepsilon$, we assume that

$$I_\varepsilon(u) = \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + V_\varepsilon |u|^q)\,\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N} \left(|\nabla u|^N + V_\varepsilon |u|^N\right)\mathrm{d}x - \int_{\mathbb{R}^N} G(\varepsilon x, u)\mathrm{d}x,$$

where $G(x,t) = \int_0^t g(x,\varrho)\mathrm{d}\varrho$. For $\forall \mu > 0$, define

$$\chi_\varepsilon(x) = \begin{cases} \varepsilon^{-\mu}, & x \in \mathbb{R}^N \setminus \Lambda_\varepsilon, \\ 0, & x \in \Lambda_\varepsilon, \end{cases}$$

$$Q_\varepsilon(u) = \left(\int_{\mathbb{R}^N} \chi_\varepsilon |u|^N \mathrm{d}x - 1\right)_+^2.$$

This penalization first appeared in [10] (or see [8]). It has the advantage that it can make the concentration phenomena to occur in Λ. Now, we define $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ as follows:

$$J_\varepsilon(u) = Q_\varepsilon(u) + I_\varepsilon(u).$$

Clearly, $J_\varepsilon \in C^1(X_\varepsilon)$. Next, to find the solutions of equation (3.1) concentrated around the local minimum of potential function as $\varepsilon \to 0$, we will find the critical points of J_ε which make Q_ε zero.

3.1 Limit problem

First, considering the limit problem, i.e.

$$\begin{cases} -\Delta_q u - \Delta_N u + m(|u|^{q-2}u + |u|^{N-2}u) = f(u), & x \in \mathbb{R}^N, \\ u \in X, & x \in \mathbb{R}^N. \end{cases} \tag{3.2}$$

The energy functional corresponding to (3.2) is defined as follows

$$I_m(u) = \frac{1}{N}\int_{\mathbb{R}^N} \left(|\nabla u|^N + m|u|^N\right)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + m|u|^q)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

In view of [30], assuming that $u \in X_0$ is the weak solution of problem (3.2), it is easy to get the Pohozǎev identity:

$$P_m(u) = \frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x + m\int_{\mathbb{R}^N} |u|^N \mathrm{d}x + \frac{Nm}{q}\int_{\mathbb{R}^N} |u|^q \mathrm{d}x - N\int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

Lemma 3.1. *I_m has the Mountain-Pass geometry.*

Proof. According to (f_1), $\forall |t| \le \delta$, $\exists \varepsilon > 0$ and $\delta > 0$ such that

$$|f(t)| \le \varepsilon |t|^{q-1}.$$

In addition, by using the condition (f_1) and f is a function that satisfies continuity, $\forall \tau > q$, $\forall |t| \ge \delta$, it is easy to find a constant $C = C(\tau,\delta) > 0$ satisfies

$$|f(t)| \le C|t|^{\tau-1}\Psi_N(t).$$

Combining the above two formulas, we get

$$|f(t)| \le \varepsilon|t|^{q-1} + C|t|^{\tau-1}\Psi_N(t), \quad \forall t \ge 0.$$

Then

$$|F(t)| \leq \varepsilon|t|^q + C|t|^\tau \Psi_N(t).$$

So, for $2 \leq N < q < q^*$,

$$\begin{aligned}
I_m(u) &= \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + m|u|^q)\,\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N}\left(|\nabla u|^N + m|u|^N\right)\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x \\
&\geq \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla u|^N + m|u|^N)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla u|^q + m|u|^q)\mathrm{d}x - \varepsilon|u|_q^q \\
&\quad - C\int_{\mathbb{R}^N}|t|^\tau \Psi_N(u)\mathrm{d}x.
\end{aligned}$$

Using Hölder's inequality, we have

$$\int_{\mathbb{R}^N}\Psi_N(u)\,|u|^\tau \mathrm{d}x \leq \|u\|^\tau_{L^{\tau t'}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}(\Phi_N(u))^t\,\mathrm{d}x\right)^{\frac{1}{t}},$$

where $\frac{1}{t}+\frac{1}{t'} = 1(t' > 1,\ t > 1)$. Due to Lemma 2.3, we may find a constant $D > 0$ satisfies

$$\left(\int_{\mathbb{R}^N}(\Phi_N(u))^t\,\mathrm{d}x\right)^{\frac{1}{t}} \leq D.$$

By using (2.1), we obtain that

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq A^{-1}_{\nu,m}\|u\|_{X_0} \quad \text{for all} \ \ u \in X_0.$$

Hence, when $\|u\|_{X_0}$ is small enough, we obtain that

$$\begin{aligned}
I_m(u) &\geq \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla u|^q + m|u|^q)\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla u|^N + m|u|^N)\mathrm{d}x \\
&\quad - C\int_{\mathbb{R}^N}|t|^\tau\Psi_N(u)\mathrm{d}x - \varepsilon|u|_q^q \\
&\geq \frac{1}{q\cdot 2^{q-1}}\|u\|^q_{X_0} - \varepsilon A^{-q}_{q,m}\|u\|^q_{X_0} - CDA^{-\tau}_{\tau t',m}\|u\|^\tau_{X_0} \\
&= \|u\|^q_{X_0}\left(\frac{1}{q\cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m} - CDA^{-\tau}_{\tau t',m}\|u\|^{\tau-q}_{X_0}\right).
\end{aligned}$$

From which we deduce that $\frac{1}{q\cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m} > 0$ for ε small enough. Let

$$h(t) = \frac{1}{q\cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m} - CDA^{-\tau}_{\tau t',m}t^{\tau-q}, \quad t \geq 0.$$

Next, we will prove there is $t_0 > 0$ small enough such that $\frac{1}{2}\big(\frac{1}{q\cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m}\big) \leq h(t_0)$. Obviously, if $t \in [0,+\infty)$, h is a continuous function. Note that $\lim_{t\to 0^+} h(t) = \frac{1}{q\cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m}$, then we can find t_0 that satisfies $h(t) \geq \frac{1}{q\cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m} - \varepsilon_1$, $\forall t \in (0,t_0)$, t_0 is small enough. Choosing $\varepsilon_1 = \frac{1}{2}\big(\frac{1}{q\cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m}\big)$, we have

$$h(t) \geq \frac{1}{2}\left(\frac{1}{q\cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m}\right)$$

for all $0 \leq t \leq t_0$. In particularly,

$$h(t_0) \geq \frac{1}{2}\left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon A_{q,m}^{-q}\right).$$

So, for $\|u\|_{X_0} = t_0$, we get

$$I_m(u) \geq \frac{t_0^q}{2} \cdot \left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon A_{q,m}^{-q}\right) = \rho_0 > 0.$$

Now, $\forall R > 0$, define $w_R(x, y)$ as follows:

$$w_R(x,y) := \begin{cases} T, & x \in B_R^+(0), \\ 0, & x \in \mathbb{R}_+^N \backslash B_{R+1}^+(0), \\ T\left(R + 1 - \sqrt{|x|}\right), & x \in B_{R+1}^+(0) \backslash B_R^+(0). \end{cases}$$

It is easy to get that $w_R \in X_{\text{rad}}\left(\mathbb{R}^N\right)$. It is worth noting that, for $R > 0$ large enough, according to (f_3), we have that

$$\int_{\mathbb{R}^N}\left[F\left(w_R(x)\right) - \frac{m}{N} w_R^N(x) - \frac{m}{q} w_R^q(x)\right] \mathrm{d}x \geq 0.$$

Next, consider $w_{R,\theta}(x) := w_R\left(\frac{x}{e^\theta}\right)$. Fix $R > 0$, then we have

$$\begin{aligned} I_m\left(w_{R,\theta}\right) = &\frac{1}{q} e^{(N-q)\theta} \int_{\mathbb{R}_+^N} |\nabla u|^q \mathrm{d}x - e^{N\theta} \int_{\mathbb{R}^N}\left[F\left(w_R(x)\right) - \frac{m}{N} w_R^N(x) - \frac{m}{q} w_R^q(x)\right] \mathrm{d}x \\ &\rightarrow -\infty \quad \text{as } \theta \rightarrow \infty. \end{aligned}$$

This ends the proof. □

Therefore, according to Lemma 3.1, we may define c_m as follows:

$$c_m := \inf_{\gamma \in \Gamma_m} \sup_{t \in [0,1]} I_m(\gamma(t)). \tag{3.3}$$

Here Γ_m is defined by

$$\Gamma_m := \{\gamma \in C([0,1], X_0) : \gamma(0) = 0 \text{ and } I_m(\gamma(1)) < 0\}. \tag{3.4}$$

Clearly, $c_m > 0$. Moreover, similar to [2], we note that

$$c_m = c_{m,\text{rad}},$$

where

$$c_{m,\text{rad}} := \inf_{\gamma \in \Gamma_{m,\text{rad}}} \max_{t \in [0,1]} I_m(\gamma(t))$$

and

$$\Gamma_{m,\text{rad}} := \left\{\gamma \in C\left([0,1], X_{\text{rad}}(\mathbb{R}^N)\right) : I_m(\gamma(1)) < 0, \gamma(0) = 0\right\}.$$

Next, we will construct a (PS) sequence $\{w_n\}_{n=1}^{\infty}$ for I_m at the level c_m that satisfies $I_m'(w_n) \rightarrow 0$ as $n \rightarrow \infty$, that is

Proposition 3.2. *There exists a sequence* $\{w_n\}_{n=1}^{\infty}$ *in* X_0 *that satisfies, as* $n \to \infty$,

$$I_m(w_n) \to c_m, \quad I_m'(w_n) \to 0, \quad P_m(w_n) \to 0. \tag{3.5}$$

Proof. For $(\theta, u) \in \mathbb{R} \times X_{\text{rad}}(\mathbb{R}^N)$, define $\widetilde{I}_m(\theta, u) := (I_m \circ \Phi)(\theta, u)$, where $\Phi(\theta, u) := u(\frac{x}{e^\theta})$. The standard norm of $\mathbb{R} \times X_{\text{rad}}(\mathbb{R}^N)$ is defined as

$$\|(\theta, u)\|_{\mathbb{R}\times X_0} = \left(\|u\|_{X_0}^2 + |\theta|^2\right)^{\frac{1}{2}}.$$

According to Lemma 3.1, $\widetilde{I}_m$ has a mountain pass geometry, so we can define $\tilde{c}_m$ as follows:

$$\tilde{c}_m = \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}_m} \max_{t\in[0,1]} \widetilde{I}_m(\widetilde{\gamma}(t)),$$

where

$$\widetilde{\Gamma}_m = \left\{\widetilde{\gamma} \in C\left([0,1], \mathbb{R} \times X_{\text{rad}}(\mathbb{R}^N)\right) : \widetilde{I}_m(\widetilde{\gamma}(1)) < 0, \widetilde{\gamma}(0) = (0)\right\}.$$

It is easy to prove that $\widetilde{c}_m = c_m$(see [3,23]). Then according to Lemma 2.1, we obtain that there exists a sequence $(\theta_n, u_n) \subset \mathbb{R} \times X_{\text{rad}}(\mathbb{R}^N)$ such that, as $n \to \infty$,

(i) $(I_m \circ \Phi)(\theta_n, u_n) \to c_m$,

(ii) $(I_m \circ \Phi)'(\theta_n, u_n) \to 0$,

(iii) $\theta_n \to 0$.

In fact, let $\delta = \delta_n = \frac{1}{n}, \varepsilon = \varepsilon_n = \frac{1}{n^2}$ in Lemma 2.1, by using (a) and (c) in Lemma 2.1, we can obtain (i) and (ii). Due to (3.3) and (3.4), for $\varepsilon = \varepsilon_n = \frac{1}{n^2}$, it is easy to find that $\gamma_n \in \Gamma_m$ such that $\sup_{t\in[0,1]} I_m(\gamma_n(t)) \le c_m + \frac{1}{n^2}$. Now define $\widetilde{\gamma}_n(t) = (0, \gamma_n(t))$, we obtain

$$\sup_{t\in[0,1]} (I_m \circ \Phi)(\widetilde{\gamma}_n(t)) = \sup_{t\in[0,1]} I_m(\gamma_n(t)) \le c_m + \frac{1}{n^2}.$$

According to (b) in Lemma 2.1, then there is $(\theta_n, u_n) \in \mathbb{R} \times X_0$ such that

$$\operatorname*{dist}_{\mathbb{R}\times X_0} \left((0, \gamma_n(t)), (\theta_n, u_n)\right) \le \frac{2}{n},$$

so (iii) holds. Now, for $A \subset \mathbb{R} \times X_0$, define

$$\operatorname*{dist}_{\mathbb{R}\times X_0}((\theta, u), A) = \inf_{(\tau,v)\in\mathbb{R}\times X_0} \left(\|u - v\|_{X_0}^2 + |\theta - \tau|^2\right)^{\frac{1}{2}}.$$

So, for $(h, w) \in \mathbb{R} \times X_0$, we have

$$\left\langle (I_m \circ \Phi)'(\theta_n, u_n), (h, w)\right\rangle = P_m(\Phi(\theta_n, u_n))\, h + \left\langle I_m'(\Phi(\theta_n, u_n)), \Phi'(\theta_n, w)\right\rangle. \tag{3.6}$$

Now, put $w = 0$ and $h = 1$, it is easy to get

$$P_m(\Phi(\theta_n, u_n)) \to 0.$$

Moreover, for all $v \in X_0$, we only take $h = 0$ and $w(x) = v\left(e^{\theta_n} x\right)$ in (3.6), by using (ii), (iii), we get

$$o(1)\|v\|_{X_0} = o(1)\left\|v\left(e^{\theta_n} x\right)\right\|_{X_0} = \left\langle I_m'(\Phi(\theta_n, u_n)), v\right\rangle.$$

Hence, $w_n = \Phi(\theta_n, u_n)$ is just the sequence we need. □

Lemma 3.3. *The sequence (w_n) that satisfies (3.5) is bounded in X_0.*

Proof. According to (3.5), we have

$$\begin{aligned}
c_m + o_n(1) &= I_m(w_n) - \frac{1}{N}P_m(w_n)\\
&= \frac{1}{N}\int_{\mathbb{R}^N}|\nabla w_n|^N \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}|\nabla w_n|^q \mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N} m|w_n|^N \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} m|w_n|^q \mathrm{d}x\\
&\quad - \int_{\mathbb{R}^N} F(w_n)\mathrm{d}x - \frac{1}{N}\left(\frac{N-q}{q}\int_{\mathbb{R}^N}|\nabla w_n|^q \mathrm{d}x + m\int_{\mathbb{R}^N}|w_n|^p \mathrm{d}x\right.\\
&\quad \left. + \frac{N}{q}\int_{\mathbb{R}^N} m|w_n|^q \mathrm{d}x - N\int_{\mathbb{R}^N} F(w_n)\mathrm{d}x\right)\\
&= \frac{1}{N}\left(\int_{\mathbb{R}^N}|\nabla w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N}|\nabla w_n|^q \mathrm{d}x\right).
\end{aligned}$$

Hence, we get that $\int_{\mathbb{R}^N}|\nabla w_n|^N \mathrm{d}x$ and $\int_{\mathbb{R}^N}|\nabla w_n|^q \mathrm{d}x$ are bounded in $\mathbb{R}$. Moreover, $P_m(w_n) = o_n(1)$ and (f_1)–(f_2) show that

$$\begin{aligned}
&\frac{N-q}{q}\int_{\mathbb{R}^N}|\nabla w_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|w_n|^N \mathrm{d}x + \frac{N}{q}\int_{\mathbb{R}^N} m|w_n|^q \mathrm{d}x\\
&\quad = o_n(1) + N\int_{\mathbb{R}^N} F(w_n)\mathrm{d}x\\
&\quad \le o_n(1) + \varepsilon N|w_n|_q^q + NC\int_{\mathbb{R}^N}|w_n|^\tau \Psi_N(w_n)\mathrm{d}x.
\end{aligned}$$

According to the boundedness of $\int_{\mathbb{R}^N}|w_n|^\tau \Psi_N(w_n)\mathrm{d}x$ and choosing $\varepsilon > 0$ small enough, we can deduce that $(|w_n|_N)$ and $(|w_n|_q)$ are bounded in $\mathbb{R}$. Therefore, (w_n) is bounded in X_0. □

According to the method in [33], we have:

Lemma 3.4 (see [33]). *Assume that (u_n) is a bounded sequence in X_0, if there exist for some $R > 0, t \ge N$ such that*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|u_n(x)|^t \mathrm{d}x = 0,$$

then for all $\xi \in (t, +\infty)$, $u_n \to 0$ in $L^\xi(\mathbb{R}^N)$.

Lemma 3.5. *Assume (w_n) satisfies Proposition 3.2, then there exist a sequence $(x_n) \subset \mathbb{R}^N$ and constants $R > 0, \beta > 0$ satisfy*

$$\int_{B_R(x_n)} w_n^q(x)\mathrm{d}x \ge \beta.$$

Proof. In fact, we assume that the conclusion is not true. According to Lemma 3.4, it is easy to get

$$w_n(\cdot) \to 0 \text{ in } L^\xi\left(\mathbb{R}^N\right), \quad \forall \xi \in (t, +\infty). \tag{3.7}$$

Therefore, due to (f_1) and (f_2), we obtain that

$$\int_{\mathbb{R}^N} f(w_n(x))\, w_n(x)\mathrm{d}x = o_n(1).$$

According to $\langle I_m'(w_n), w_n\rangle = o_n(1)$, we can obtain that

$$\int_{\mathbb{R}^N}|\nabla w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N}|\nabla w_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} m|w_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} f(w_n)w_n \mathrm{d}x = o_n(1),$$

and so we deduce that $\|w_n\|_{X_0} \to 0$. Therefore, $I_m(w_n) \to 0$ and then we get contradiction since $c_m > 0$. □

Next, define

$$\mathcal{T}_m := \left\{ u \in X(\mathbb{R}^N)\backslash\{0\} : \max_{x\in\mathbb{R}^N} u(x) = u(0), I'_m(u) = 0 \right\},$$

$$b_m := \inf_{u\in\mathcal{T}_m} I_m(u),$$

and

$$\mathcal{S}_m := \{u \in \mathcal{T}_{V_0} : I_m(u) = b_m\}.$$

Lemma 3.6. *There exists $u \in \mathcal{S}_m$.*

Proof. Assume (w_n) satisfies Proposition 3.2. Let $\widetilde{w}_n(x) := w_n(x_n + x)$, here x_n comes from Lemma 3.5. According to Lemma 3.4, we can see that (w_n) is bounded in $X_{\text{rad}}(\mathbb{R}^N)$, that is, for all $n \in \mathbb{N}$, we have $\|w_n\|_{X_{\text{rad}}(\mathbb{R}^N)} \le C$. Going if necessary to a subsequence, for some $\widetilde{w} \in X_{\text{rad}}(\mathbb{R}^N) \setminus \{0\}$, we assume that $\widetilde{w}_n \rightharpoonup \widetilde{w}$ in $X_{\text{rad}}(\mathbb{R}^N)$, then

$$\widetilde{w}_n(x) \to \widetilde{w}(x) \quad \text{in } L^{\xi}(\mathbb{R}^N), \quad \forall \xi \in (N, +\infty).$$

So

$$\int_{\mathbb{R}^N} f(\widetilde{w}_n)\widetilde{w_n} \to \int_{\mathbb{R}^N} f(\widetilde{w})\widetilde{w}. \tag{3.8}$$

Moreover, $\widetilde{w}$ satisfies

$$(-\Delta)_N\widetilde{w} + (-\Delta)_q\widetilde{w} + m(|\widetilde{w}|^{N-2}\widetilde{w} + |\widetilde{w}|^{q-2}\widetilde{w}) = f(\widetilde{w}) \quad \text{in } \mathbb{R}^N. \tag{3.9}$$

From (3.8) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla\widetilde{w}|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla\widetilde{w}|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}|^N \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}|^q \mathrm{d}x \\
&\le \liminf_{n\to\infty} \left[\int_{\mathbb{R}^N} |\nabla\widetilde{w}_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla\widetilde{w}_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}_n|^q \mathrm{d}x \right] \\
&\le \limsup_{n\to\infty} \left[\int_{\mathbb{R}^N} |\nabla\widetilde{w}_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla\widetilde{w}_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}_n|^q \mathrm{d}x \right] \\
&= \limsup_{n\to\infty} \left[\int_{\mathbb{R}^N} |\nabla w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} m|w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|w_n|^q \mathrm{d}x \right] \\
&= \limsup_{n\to\infty} \int_{\mathbb{R}^N} f(w_n)w_n \mathrm{d}x \\
&= \limsup_{n\to\infty} \int_{\mathbb{R}^N} f(\widetilde{w}_n)\widetilde{w}_n \mathrm{d}x \\
&= \int_{\mathbb{R}^N} f(\widetilde{w})\widetilde{w} \mathrm{d}x \\
&= \int_{\mathbb{R}^N} |\nabla\widetilde{w}|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla\widetilde{w}|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}|^p \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}|^q \mathrm{d}x,
\end{aligned}$$

which implies that $\|\widetilde{w}_n\|_{X_0} \to \|\widetilde{w}\|_{X_0}$ and thus $\widetilde{w}_n \to \widetilde{w}$ in X_0. Therefore, by $I_m(w_n) = I_m(\widetilde{w}_n) \to c_m$ and $I'_m(w_n) = I'_m(\widetilde{w}_n) \to 0$, we obtain that $I_m(\widetilde{w}) = c_m$ and $I'_m(\widetilde{w}) = 0$. Due to $\widetilde{w} \neq 0$, we get that $c_m \ge b_m$.

Now, let $w \in X_0\backslash\{0\}$ be an arbitrary solution of (3.2). We define

$$w_t(x) := \begin{cases} w\left(\frac{x}{t}\right) & \text{for } t > 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Next, choosing the real number $\theta_1 > t_1 > 1 > t_0 > 0$, we denote the curve γ consisting of three parts as follows:

$$\gamma(\theta) = \begin{cases} \theta w_{t_0}, & \theta \in [0, t_0], \\ \theta w_\theta, & \theta \in [t_0, t_1], \\ \theta w_{t_1}, & \theta \in [t_1, \theta_1]. \end{cases}$$

Due to w is a weak solution, then

$$\int_{\mathbb{R}^N} f(w) w \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla w|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|w|^N \mathrm{d}x + \int_{\mathbb{R}^N} m|w|^q \mathrm{d}x > 0.$$

Hence, we can find $\theta_1 > 1$ such that

$$\int_{\mathbb{R}^N} f(\theta w) w \mathrm{d}x > 0, \quad \forall \theta \in [1, \theta_1].$$

Let $\varphi(s) = \frac{f(s)}{s^{q-1}}$. Due to (f_1), we know that $\varphi \in C(\mathbb{R}, \mathbb{R})$. Hence, we have

$$\int_{\mathbb{R}^N} \varphi(\theta w) w^q \mathrm{d}x > 0, \quad \forall \theta \in [1, \theta_1]. \tag{3.10}$$

Moreover,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta} I_m(\theta w_t) &= \langle I_m'(\theta w_t), w_t \rangle \\ &= \theta^{N-1} \int_{\mathbb{R}^N} |\nabla w_t|^N \mathrm{d}x + \theta^{q-1} \int_{\mathbb{R}^N} |\nabla w_t|^q \mathrm{d}x + \theta^{N-1} \int_{\mathbb{R}^N} m|w_t|^N \mathrm{d}x \\ &\quad + \theta^{q-1} \int_{\mathbb{R}^N} m|w_t|^q \mathrm{d}x - \theta^{q-1} \int_{\mathbb{R}^N} \varphi(\theta w_t) w_t^q \mathrm{d}x \\ &= \theta^{N-1} \int_{\mathbb{R}^N} |\nabla w_t|^N \mathrm{d}x + \theta^{q-1} \int_{\mathbb{R}^N} |\nabla w_t|^q \mathrm{d}x + \theta^{N-1} \int_{\mathbb{R}^N} m|w_t|^N \mathrm{d}x \\ &\quad + \theta^{q-1} \int_{\mathbb{R}^N} m|w_t|^q \mathrm{d}x - \frac{\theta^{q-1}}{2} \int_{\mathbb{R}^N} \varphi(\theta w_t) w_t^q \mathrm{d}x - \frac{\theta^{q-1}}{2} \int_{\mathbb{R}^N} \varphi(\theta w_t) w_t^q \mathrm{d}x \\ &= \theta^{N-1} \left(\int_{\mathbb{R}^N} |\nabla w|^N \mathrm{d}x + t^N \int_{\mathbb{R}^N} m|w|^N \mathrm{d}x - \frac{\theta^{q-N} t^N}{2} \int_{\mathbb{R}^N} \varphi(\theta w) w^q \mathrm{d}x \right) \\ &\quad + \theta^{N-1} \cdot t^{N-q} \left(\int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x + t^q \int_{\mathbb{R}^N} m|w|^q \mathrm{d}x - \frac{t^q}{2} \int_{\mathbb{R}^N} \varphi(\theta w) w^q \mathrm{d}x \right). \end{aligned}$$

Selecting $t_0 \in (0, 1)$ small enough, we obtain

$$\int_{\mathbb{R}^N} |\nabla w|^N \mathrm{d}x + t_0^N \int_{\mathbb{R}^N} m|w|^N \mathrm{d}x - \frac{\theta^{q-N} t_0^N}{2} \int_{\mathbb{R}^N} \varphi(\theta w) w^q \mathrm{d}x > 0 \quad \text{for all } \theta \in [1, \theta_1] \tag{3.11}$$

and

$$\int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x + t_0^q \int_{\mathbb{R}^N} m|w|^q \mathrm{d}x - \frac{t_0^q}{2} \int_{\mathbb{R}^N} \varphi(\theta w) w^q \mathrm{d}x > 0 \quad \text{for all } \theta \in [1, \theta_1]. \tag{3.12}$$

According to (3.10), for all $\theta \in [1, \theta_1]$, we select $t_1 > 1$ such that

$$\int_{\mathbb{R}^N} |\nabla w|^N \mathrm{d}x + t_1^N \int_{\mathbb{R}^N} m|w|^N \mathrm{d}x - \frac{\theta^{q-N} t_1^N}{2} \int_{\mathbb{R}^N} \varphi(\theta w) w^q \mathrm{d}x \leq -\frac{N}{\theta_1^N - 1} \int_{\mathbb{R}^N} |\nabla w|^N \mathrm{d}x, \tag{3.13}$$

and

$$\int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x + t_1^q \int_{\mathbb{R}^N} m|w|^q \mathrm{d}x - \frac{t_1^q}{2} \int_{\mathbb{R}^N} \varphi(\theta w) w^q \mathrm{d}x \leq -\frac{N t_1^{q-N}}{(\theta_1^N - 1)} \int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x. \tag{3.14}$$

Therefore, according to (3.11) and (3.12), we know $I(\gamma(\theta))$ increases at the interval $[0, t_0]$, then takes its maximum value at $\theta = 1$. According to the Pohozǎev identity:

$$P_m(u) = \frac{N-q}{q} \int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x + m \int_{\mathbb{R}^N} |u|^N \mathrm{d}x + \frac{Nm}{q} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x - N \int_{\mathbb{R}^N} F(u) \mathrm{d}x.$$

Consequently,

$$\begin{aligned}
I_m(w_{t_1}(x)) &\leq I_m(w(x)) \\
&= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla w|^N \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x + \frac{m}{N} \int_{\mathbb{R}^N} |w|^N \mathrm{d}x + \frac{m}{q} \int_{\mathbb{R}^N} |w|^q \mathrm{d}x \\
&\quad - \frac{1}{N} \left(\frac{N-q}{q} \int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x + m \int_{\mathbb{R}^N} |w|^N \mathrm{d}x + \frac{N}{q} \int_{\mathbb{R}^N} m|w|^q \mathrm{d}x \right) \\
&= \frac{1}{N} \left(\int_{\mathbb{R}^N} |\nabla w|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x \right).
\end{aligned}$$

Now by using (3.13) and (3.14), we have

$$\begin{aligned}
I_m(\theta_1 w_{t_1}) &= I_m(w_{t_1}) + \int_1^{\theta_1} \frac{\mathrm{d}}{\mathrm{d}\theta} I(\theta w_{t_1}) \mathrm{d}\theta \\
&\leq \frac{1}{N} \left(\int_{\mathbb{R}^N} |\nabla w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x \right) - \frac{N}{\theta_1^N - 1} \int_{\mathbb{R}^N} |\nabla w|^N \mathrm{d}x \int_1^{\theta_1} \theta^{N-1} \mathrm{d}\theta \\
&\quad - \frac{N t_1^{q-N}}{(\theta_1^N - 1)} \int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x \cdot t_1^{N-q} \int_1^{\theta_1} \theta^{N-1} \mathrm{d}\theta \\
&= \left(\frac{1}{N} - 1 \right) \int_{\mathbb{R}^N} |\nabla w_n|^N \mathrm{d}x + \left(\frac{1}{N} - 1 \right) \int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x < 0.
\end{aligned}$$

So we know $\gamma(\theta) \in \Gamma_m$. According to the definition of c_m, we have $I_m(\gamma(\theta)) \geq c_m$. Due to w is arbitrary, we obtain that $b_m \geq c_m$ and this means $b_m = c_m$.

Selecting $w^- = \min\{w, 0\}$ as a test function of (3.2), we infer that $w \geq 0$ in $\mathbb{R}^N$. Using (f_1)–(f_2) and according to the Moser iteration (see [3, 13]), it is easy to obtain that $w \in L^\infty(\mathbb{R}^N)$. By means of Corollary 2.1 in [4], we can see that $w \in C^\sigma(\mathbb{R}^N)$ for some $\sigma \in (0,1)$. Similar to the proof of Theorem 1.1-(ii) in [24], we obtain that $w > 0$ in $\mathbb{R}^N$. □

Remark 3.7. As to $m > 0$, we define

$$I_{m'}(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x + \frac{m'}{p} \int_{\mathbb{R}^N} |u|^p \mathrm{d}x + \frac{m'}{q} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x - \int_{\mathbb{R}^N} F(u) \mathrm{d}x,$$

the mountain pass level is $c_{m'}$. By using standard method, we can prove that $c_{m'_1} > c_{m'_2}$ when $m'_1 > m'_2$.

In the following, we will prove that $\mathcal{S}_{V_0}$ is compact in X_0.

Lemma 3.8. *$\mathcal{S}_{V_0}$ is compact in X_0.*

Proof. For any $U \in \mathcal{S}_{V_0}$, we have that

$$\begin{aligned} c_m + o_n(1) &= I_m(U) - \frac{1}{N} P_m(U) \\ &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla U|^N \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla U|^q \mathrm{d}x + \frac{m}{N}\int_{\mathbb{R}^N} |U|^N \mathrm{d}x + \frac{m}{q}\int_{\mathbb{R}^N} |U|^q \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} F(U)\mathrm{d}x - \frac{1}{N}\left(\frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla U|^q \mathrm{d}x + m\int_{\mathbb{R}^N} |U|^p \mathrm{d}x \right. \\ &\quad \left. + \frac{Nm}{q}\int_{\mathbb{R}^N} |U|^q \mathrm{d}x - N\int_{\mathbb{R}^N} F(U)\mathrm{d}x\right) \\ &= \frac{1}{N}\left(\int_{\mathbb{R}^N} |\nabla U|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla U|^q \mathrm{d}x\right). \end{aligned}$$

So $\mathcal{S}_m$ is bounded in X_0.

For any sequence $\{U_k\} \subset \mathcal{S}_{V_0}$, up to a subsequence, we can find a $U_0 \in X_0$ satisfies

$$U_k \rightharpoonup U_0 \quad \text{in } X_0 \tag{3.15}$$

and U_0 satisfies

$$-\Delta_N U_0 - \Delta_q U_0 + m(|U_0|^{N-2}U_0 + |U_0|^{q-2}U_0) = f(U_0), \quad \text{in } \mathbb{R}^N,\ U_0 \geq 0.$$

Next, we will prove that U_0 is nontrivial. Note that, up to a subsequence, we have

$$U_k \to U_0 \text{ in } L^t_{\mathrm{loc}}(\mathbb{R}^N), \quad t \in (N, +\infty). \tag{3.16}$$

By using (3.16), any bounded region in $\mathbb{R}^N$, (U_k^t) is uniformly integrable. According to Lemma 2.2 (i) in [22], $\|U_k\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}^N)} \leq C$. In view of [26], there exists $\alpha \in (0,1)$ such that $\|U_k\|_{C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)} \leq C$. Due to $(U_k) \subset \mathcal{S}_{V_0}$, by Lemma 3.6, we have that $U_k > 0$. We can prove that $\liminf_{k\to\infty} \|U_k\|_\infty > 0$ because of $\lim_{t\to 0} \frac{f(t)}{t^{q-1}} = 0$. In fact, since U_k satisfies (3.1), we have that

$$-\Delta_N U_k - \Delta_q U_k + m(|U_k|^{N-2}U_k + |U_k|^{q-2}U_k) = f(U_k),$$

that is

$$\int_{\mathbb{R}^N} |\nabla U_k|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla U_k|^q \mathrm{d}x + m\int_{\mathbb{R}^N} |U_k|^N \mathrm{d}x + m\int_{\mathbb{R}^N} |U_k|^q \mathrm{d}x = \int_{\mathbb{R}^N} f(U_k)U_k \mathrm{d}x.$$

According to $\lim_{t\to 0} \frac{f(t)}{t^{q-1}} = 0$, $\forall \varepsilon > 0$, we can find $\delta > 0$ satisfies

$$f(t) < \varepsilon t^{q-1}, \quad |t| < \delta,$$

then $f(U_k)U_k < \varepsilon |U_k|^q$. Assume by contradiction, we have $\liminf_{k\to\infty} \|U_k\|_\infty = 0$, then for δ given above, we have $|U_k| < \delta$. Therefore,

$$\int_{\mathbb{R}^N} |\nabla U_k|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla U_k|^q \mathrm{d}x = \int_{\mathbb{R}^N} f(U_k)U_k \mathrm{d}x - m\int_{\mathbb{R}^N} |U_k|^N \mathrm{d}x - m\int_{\mathbb{R}^N} |U_k|^q \mathrm{d}x < 0,$$

which leads to a contradiction. Noting that $U_k(0) = \|U_k\|_\infty$, we get that $U_0 \not\equiv 0$. Therefore, there exists $\exists C_0 > 0$ such that $U_k(0) \geq C_0 > 0$, then $U_0(0) \geq C_0 > 0$, this means that U_0 is nontrivial. Using the same method as Lemma 3.6, we get $I_m(U_0) = c_m$ and $U_k \to U_0$ in X_0. Therefore, $\mathcal{S}_m$ is compact in X_0. □

3.2 Proof of Theorem 1.1

This section will prove Theorem 1.1. For $U \in \mathcal{S}_m$, set $c_m = I_m(U)$ and $10\delta = \operatorname{dist}\{\mathcal{M}, \mathbb{R}^N \backslash \Lambda\}$. Now, fix a $\beta \in (0,\delta)$ and a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^N)$ satisfies

$$\varphi := \begin{cases} 1, & |x| \le \beta, \\ 0, & |x| \ge 2\beta \end{cases}$$

and $|\nabla \varphi| \le C/\beta$. Moreover, let $y \in \mathbb{R}^N$, $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$. For $\varepsilon > 0$ small enough, we will look for solutions of (1.1) near the set

$$Y_\varepsilon := \left\{ \varphi(\varepsilon y - x)\, U\left(y - \frac{x}{\varepsilon}\right) : x \in \mathcal{M}^\beta, U \in \mathcal{S}_m \right\},$$

where $\mathcal{M}^\beta := \{ y \in \mathbb{R}^N : \inf_{z\in\mathcal{M}} |z-y| \le \beta \}$. Moreover, as to $A \subset X_\varepsilon$, define

$$A^a := \left\{ u \in X_\varepsilon : \inf_{v\in A} \|u - v\|_{X_\varepsilon} \le a \right\}.$$

For any $U \in \mathcal{S}_m$, define $W_{\varepsilon,t}(x) := \varphi(\varepsilon x) U\left(\frac{x}{t}\right)$.

Next, we show that J_ε has the Mountain-Pass geometry. Let $U_t(x) := U(\frac{x}{t})$, by using the same proof as in Lemma 3.1, we have

$$\begin{aligned} I_m(U_t) &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla U|^N \mathrm{d}x + \frac{t^N}{N}\int_{\mathbb{R}^N} m|U|^N \mathrm{d}x + \frac{t^{N-q}}{q}\int_{\mathbb{R}^N} |\nabla U|^q \mathrm{d}x \\ &\quad + \frac{t^N}{q}\int_{\mathbb{R}^N} m|U|^q \mathrm{d}x - t^N \int_{\mathbb{R}^N} F(U)\mathrm{d}x \\ &\to -\infty \quad \text{as } t \to \infty. \end{aligned}$$

So there exists $t_0 > 0$ such that $I_m(U_{t_0}) < -3$.

Clearly, $Q_\varepsilon(W_{\varepsilon,t_0}) = 0$. As to $\varepsilon > 0$ sufficiently small, by using the Dominated Convergence Theorem, one has

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,t_0}) &= I_\varepsilon(W_{\varepsilon,t_0}) \\ &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla W_{\varepsilon,t_0}|^N \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla W_{\varepsilon,t_0}|^q \mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N} V(\varepsilon x)|W_{\varepsilon,t_0}|^p \mathrm{d}x \\ &\quad + \frac{1}{q}\int_{\mathbb{R}^N} V(\varepsilon x)|W_{\varepsilon,t_0}|^q \mathrm{d}x - \int_{\mathbb{R}^N} F(W_{\varepsilon,t_0})\mathrm{d}x \\ &\overset{\tilde{x}=\frac{x}{t_0}}{=} \frac{1}{N}\int_{\mathbb{R}^N} \left|\varepsilon t_0^2 \nabla\varphi(\varepsilon t_0 \tilde{x}) U(\tilde{x}) + \varphi(\varepsilon\tilde{x})\nabla U(\tilde{x})\right|^N \mathrm{d}\tilde{x} \\ &\quad + \frac{t_0^{N-q}}{q}\int_{\mathbb{R}^N} \left|\varepsilon t_0^2 \nabla\varphi(\varepsilon t_0 \tilde{x}) U(\tilde{x}) + \varphi(\varepsilon t_0\tilde{x})\nabla U(\tilde{x})\right|^q \mathrm{d}\tilde{x} \\ &\quad + \frac{t_0^N}{N}\int_{\mathbb{R}^N} V(\varepsilon t_0\tilde{x})|\varphi(\varepsilon t_0 \tilde{x}) U(\tilde{x})|^N \mathrm{d}\tilde{x} \\ &\quad + \frac{t_0^N}{q}\int_{\mathbb{R}^N} V(\varepsilon t_0\tilde{x})|\varphi(\varepsilon t_0 \tilde{x}) U(\tilde{x})|^q \mathrm{d}\tilde{x} \\ &\quad - t^N \int_{\mathbb{R}^N} F(\varphi(\varepsilon t_0\tilde{x}) U(\tilde{x})\mathrm{d}\tilde{x} \\ &= I_m(U_{t_0}) + o(1) < -2. \end{aligned} \tag{3.17}$$

According to (f_1) and (f_2), it is easy to see that

$$|F(t)| \leq \varepsilon|t|^q + C|t|^\tau \Psi_N(t).$$

So, for $2 \leq N < q < q^*$, we get

$$\begin{aligned}
J_\varepsilon(u) &\geq I_\varepsilon(u) \\
&= \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + V_\varepsilon |u|^q)\, dx + \frac{1}{N}\int_{\mathbb{R}^N} \left(|\nabla u|^N + V_\varepsilon |u|^N\right) dx - \int_{\mathbb{R}^N} F(u) dx \\
&\geq \frac{1}{N}\int_{\mathbb{R}^N} (|\nabla u|^N + V_\varepsilon|u|^N) dx + \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + V_\varepsilon |u|^q) dx - \varepsilon|u|_q^q - C\int_{\mathbb{R}^N} |t|^\tau \Psi_N(u) dx.
\end{aligned}$$

Using Hölder's inequality, it is easy to get

$$\int_{\mathbb{R}^N} |u|^\tau \Psi_N(u)\, dx \leq \|u\|^\tau_{L^{\tau t'}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} (\Phi_N(u))^t\, dx\right)^{\frac{1}{t}},$$

where $\frac{1}{t} + \frac{1}{t'} = 1(t' > 1,\ t > 1)$. Due to Lemma 2.3, we can find a constant $D > 0$ satisfies

$$\left(\int_{\mathbb{R}^N} (\Phi_N(u))^t\, dx\right)^{\frac{1}{t}} \leq D.$$

From (2.2), we have

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq S^{-1}_{\nu,\varepsilon}\|u\|_{X_\varepsilon}, \quad \forall u \in X_\varepsilon.$$

Hence, when $\|u\|_{X_\varepsilon}$ is small, we get

$$\begin{aligned}
J_\varepsilon(u) \geq{} & \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + V_\varepsilon|u|^q) dx + \frac{1}{N}\int_{\mathbb{R}^N} (|\nabla u|^N + V_\varepsilon |u|^N) dx \\
& - \varepsilon|u|_q^q - C\int_{\mathbb{R}^N} |t|^\tau \Psi_N(u) dx \\
\geq{} & \frac{1}{q\cdot 2^{q-1}}\|u\|^q_{X_\varepsilon} - \varepsilon S^{-q}_{q,\varepsilon}\|u\|^q_{X_\varepsilon} - CDS^{-\tau}_{\tau t',\varepsilon}\|u\|^\tau_{X_\varepsilon} \\
={} & \|u\|^q_{X_\varepsilon}\left(\frac{1}{q\cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon} - CDS^{-\tau}_{\tau t',\varepsilon}\|u\|^{\tau-q}_{X_\varepsilon}\right).
\end{aligned}$$

We see $\frac{1}{q\cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon} > 0$ for ε small enough. Let

$$h(t) = \frac{1}{q\cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon} - CDS^{-\tau}_{\tau t',\varepsilon} t^{\tau-q}, \quad t \geq 0.$$

Next, we will find $t_0 > 0$ small that satisfies $h(t_0) \geq \frac{1}{2}\left(\frac{1}{q\cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon}\right)$. Clearly, $\lim_{t\to 0^+} h(t) = \frac{1}{q\cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon}$ and h is continuous function on $[0,+\infty)$, so there exists t_0 satisfies $h(t) \geq \frac{1}{q\cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon} - \varepsilon_1$, $\forall t \in (0, t_0)$, t_0 is small enough. Choosing $\varepsilon_1 = \frac{1}{2}\left(\frac{1}{q\cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon}\right)$, we get that

$$h(t) \geq \frac{1}{2}\left(\frac{1}{q\cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon}\right)$$

for all $0 \leq t \leq t_0$. In particularly,

$$h(t_0) \geq \frac{1}{2}\left(\frac{1}{q\cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon}\right).$$

So, for $\|u\|_{X_\varepsilon} = t_0$, we have

$$J_\varepsilon(u) \geq \frac{t_0^q}{2} \cdot \left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon S_{q,\varepsilon}^{-q} \right) = \rho_0 > 0.$$

Therefore, we can define c_ε as follows:

$$c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{s \in [0,1]} J_\varepsilon(\gamma(s)).$$

Here Γ_ε is defined by

$$\Gamma_\varepsilon := \left\{ \gamma \in C\left([0,1], X_\varepsilon\right) \mid \gamma(1) = W_{\varepsilon,t_0}, \gamma(0) = 0 \right\}.$$

Lemma 3.9. *There holds*

$$\varlimsup_{\varepsilon \to 0} c_\varepsilon \leq c_m.$$

Proof. Denote $W_{\varepsilon,0} = \lim_{t \to 0} W_{\varepsilon,t}$ in X_ε sense, then it is easy to get $W_{\varepsilon,0} = 0$. Consequently, let $\gamma(s) := W_{\varepsilon,st_0} (0 \leq s \leq 1)$, then $\gamma(s) \in \Gamma_\varepsilon$, so

$$c_\varepsilon \leq \max_{s \in [0,1]} J_\varepsilon(\gamma(s)) = \max_{t \in [0,t_0]} J_\varepsilon\left(W_{\varepsilon,t}\right).$$

Now, we only need to prove

$$\varlimsup_{\varepsilon \to 0} \max_{t \in [0,t_0]} J_\varepsilon\left(W_{\varepsilon,t}\right) \leq c_m.$$

In fact, similar to (3.17), we obtain that

$$\begin{aligned} \max_{t \in [0,t_0]} J_\varepsilon\left(W_{\varepsilon,t}\right) &= \max_{t \in [0,t_0]} I_m\left(U_t\right) + o(1) \\ &\leq o(1) + \max_{t \in [0,\infty)} I_m\left(U_t\right) \\ &= I_m(U) + o(1) = o(1) + c_m. \end{aligned}$$

This finishes the proof. □

Lemma 3.10. *There holds*

$$\varliminf_{\varepsilon \to 0} c_\varepsilon \geq c_m.$$

Proof. Assuming $\lim_{\varepsilon \to 0} c_\varepsilon < c_m$, we can find $\delta_0 > 0$, $\gamma_n \in \Gamma_{\varepsilon_n}$ and $\varepsilon_n \to 0$ satisfy, for $s \in [0,1]$, $J_{\varepsilon_n}\left(\gamma_n(s)\right) < c_m - \delta_0$. Now, fixed an $\varepsilon_n > 0$, we have

$$\frac{1}{N} m \varepsilon_n \left(1 + \left(1 + c_m\right)^{1/2}\right) < \min\left\{\delta_0, 1\right\}. \tag{3.18}$$

Due to $I_{\varepsilon_n}\left(\gamma_n(0)\right) = 0$ and $I_{\varepsilon_n}\left(\gamma_n(1)\right) \leq J_{\varepsilon_n}\left(\gamma_n(1)\right) = J_{\varepsilon_n}\left(W_{\varepsilon_n,t_0}\right) < -2$, we can look for an $s_n \in (0,1)$ such that $I_{\varepsilon_n}\left(\gamma_n(s)\right) \geq -1$ for $s \in [0, s_n]$ and $I_{\varepsilon_n}\left(\gamma_n\left(s_n\right)\right) = -1$. Moreover, for any $s \in [0, s_n]$, we have that

$$Q_{\varepsilon_n}\left(\gamma_n(s)\right) = J_{\varepsilon_n}\left(\gamma_n(s)\right) - I_{\varepsilon_n}\left(\gamma_n(s)\right) \leq 1 + c_m - \delta_0,$$

which implies that

$$\int_{\mathbb{R}^N \setminus (\Lambda/\varepsilon_n)} \gamma_n^N(s) \mathrm{d}x \leq \varepsilon_n \left(1 + \left(1 + c_m\right)^{1/2}\right) \quad \text{for } s \in [0, s_n].$$

So for $s\in[0,s_n]$, we have

$$
\begin{aligned}
& I_{\varepsilon_n}(\gamma_n(s)) \\
&= I_m(\gamma_n(s)) + \frac{1}{N}\int_{\mathbb{R}^N} (V(\varepsilon_n x)-m)\,\gamma_n^N(s)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} (V(\varepsilon_n x)-m)\,\gamma_n^q(s)\mathrm{d}x \\
&\geq I_m(\gamma_n(s)) + \frac{1}{N}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_n)} (V(\varepsilon_n x)-m)\,\gamma_n^N(s)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_n)} (V(\varepsilon_n x)-m)\,\gamma_n^q(s)\mathrm{d}x \\
&\geq I_m(\gamma_n(s)) + \frac{1}{N}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_n)} (V(\varepsilon_n x)-m)\,\gamma_n^N(s)\mathrm{d}x \\
&\geq I_m(\gamma_n(s)) - \frac{1}{N}m\varepsilon_n\left(1+(1+c_m)^{1/2}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
I_m(\gamma_n(s_n)) &\leq I_{\varepsilon_n}(\gamma_n(s_n)) + \frac{1}{N}m\varepsilon_n\left(1+(1+c_m)^{1/2}\right) \\
&= -1 + \frac{1}{N}m\varepsilon_n\left(1+(1+c_m)^{1/2}\right) < 0,
\end{aligned}
$$

and recalling (3.3), we obtain that

$$
\max_{s\in[0,s_n]} I_m(\gamma_n(s)) \geq c_m.
$$

Therefore, we get that

$$
\begin{aligned}
c_m-\delta_0 &\geq \max_{s\in[0,1]} J_{\varepsilon_n}(\gamma_n(s)) \geq \max_{s\in[0,1]} I_{\varepsilon_n}(\gamma_n(s)) \geq \max_{s\in[0,s_n]} I_{\varepsilon_n}(\gamma_n(s)) \\
&\geq -\frac{1}{N}m\varepsilon_n\left(1+(1+c_m)^{1/2}\right) + \max_{s\in[0,s_n]} I_m(\gamma_n(s)),
\end{aligned}
$$

that is $0<\delta_0\leq \frac{1}{N}m\varepsilon_n\big(1+(1+c_m)^{1/2}\big)$, which contradicts (3.18). As desired. □

By using Lemmas 3.9 and 3.10, it follows

$$
0=\lim_{\varepsilon\to 0}\left(\max_{s\in[0,1]} J_\varepsilon(\gamma_\varepsilon(s)) - c_\varepsilon\right).
$$

Here $\forall s\in[0,1]$, $\gamma_\varepsilon(s)=W_{\varepsilon,st_0}$. Denote

$$
\tilde{c}_\varepsilon := \max_{s\in[0,1]} J_\varepsilon(\gamma_\varepsilon(s)).
$$

Clearly, $c_\varepsilon\leq\tilde{c}_\varepsilon$,

$$
c_m=\lim_{\varepsilon\to 0}\tilde{c}_\varepsilon=\lim_{\varepsilon\to 0}c_\varepsilon.
$$

Now define

$$
J_\varepsilon^\alpha=\{u\in X_\varepsilon \mid J_\varepsilon(u)\leq\alpha\}.
$$

For $\alpha>0$ and $\forall A\subset X_\varepsilon$, set $A^\alpha=\left\{u\in X_\varepsilon \mid \inf_{v\in A}\|u-v\|_{X_\varepsilon}\leq\alpha\right\}$.

Lemma 3.11. *Assume $\{\varepsilon_i\}_{i=1}^\infty$ satisfies $\lim_{i\to\infty}\varepsilon_i=0$, $\{u_{\varepsilon_i}(\cdot)\}\subset Y_{\varepsilon_i}^d$ and*

$$
\lim_{i\to\infty} J'_{\varepsilon_i}(u_{\varepsilon_i}(\cdot))=0,\quad \lim_{i\to\infty} J_{\varepsilon_i}(u_{\varepsilon_i}(\cdot))\leq c_m.
$$

Then, $\forall d>0$ small enough, up to a subsequence, there exist $x\in\mathcal{M}$, $\{y_i\}_{i=1}^\infty\subset\mathbb{R}^N$, $U\in\mathcal{S}_m$ satisfy

$$
\lim_{i\to\infty}\|\varphi_{\varepsilon_i}(\cdot-y_i)\,U(\cdot-y_i)-u_{\varepsilon_i}(\cdot)\|_{X_{\varepsilon_i}}=0 \quad \textit{and} \quad \lim_{i\to\infty}|x-\varepsilon_i y_i|=0.
$$

Proof. Now, write ε_i as ε. According to

$$Y_\varepsilon := \left\{\varphi\left(\varepsilon y - x\right) U\left(y - \frac{x}{\varepsilon}\right) : x \in \mathcal{M}^\beta, U \in \mathcal{S}_m\right\},$$

we can find $\{U_\varepsilon\} \subset \mathcal{S}_m$ and $\{x_\varepsilon\} \subset \mathcal{M}^\beta$ satisfy

$$\left\|\varphi_\varepsilon\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) U_\varepsilon\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) - u_\varepsilon(\cdot)\right\|_{X_\varepsilon} \le d.$$

Due to $\mathcal{S}_m$, $\mathcal{M}^\beta$ are compact, there exist $Z \in \mathcal{S}_m, x \in \mathcal{M}^\beta$ satisfy $U_\varepsilon \to Z$ in X_ε and $x_\varepsilon \to x$. Hence, for $\varepsilon > 0$ small enough,

$$\left\|\varphi_\varepsilon\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) Z\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) - u_\varepsilon(\cdot)\right\|_{X_\varepsilon} \le 2d. \tag{3.19}$$

In addition, according to (f_2), we can suppose that $\sup \|u_\varepsilon\|_{X_\varepsilon} \le 1$.

Step 1. First we will prove

$$0 = \liminf_{\varepsilon \to 0} \sup_{y \in A_\varepsilon} \int_{B(y,1)} |u_\varepsilon|^N \, \mathrm{d}x, \tag{3.20}$$

here $A_\varepsilon = B\left(\frac{x_\varepsilon}{\varepsilon}, \frac{3\beta}{\varepsilon}\right) \backslash B\left(\frac{x_\varepsilon}{\varepsilon}, \frac{\beta}{2\varepsilon}\right)$.

Assume the formula (3.20) is true, according to Lions' lemma, for any $\xi > N$, we have that $u_\varepsilon \to 0$ in $L^\xi\left(B_\varepsilon\right)$, where $B_\varepsilon = B\left(\frac{x_\varepsilon}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \backslash B\left(\frac{x_\varepsilon}{\varepsilon}, \frac{\beta}{\varepsilon}\right)$.

Now, we assume the formula (3.20) is not true, then we can find $r > 0$ that satisfies

$$\liminf_{\varepsilon \to 0} \sup_{y \in A_\varepsilon} \int_{B(y,1)} |u_\varepsilon|^N \, \mathrm{d}x = 2r > 0.$$

So, for $\varepsilon > 0$ small enough, we also can find that $y_\varepsilon \in A_\varepsilon$ satisfies $\int_{B(y_\varepsilon,1)} |u_\varepsilon|^N \, \mathrm{d}x \ge r$. It is necessary to mention that, there is $x_0 \in \mathcal{M}^{4\beta} \subset \Lambda$ satisfying $\varepsilon y_\varepsilon \to x_0$. Assume $v_\varepsilon(y) = u_\varepsilon\left(y + y_\varepsilon\right)$, it is easy to obtain that

$$\begin{aligned} &-\Delta_N v_\varepsilon - \Delta_q v_\varepsilon + V_\varepsilon\left(y + y_\varepsilon\right) |v_\varepsilon|^{N-2} v_\varepsilon - g\left(\varepsilon y + \varepsilon y_\varepsilon, v_\varepsilon\right) + V_\varepsilon\left(y + y_\varepsilon\right) |v_\varepsilon|^{q-2} v_\varepsilon \\ &\quad = h_\varepsilon - 2NQ_\varepsilon^{1/2}\left(u_\varepsilon\right) \chi_\varepsilon\left(y + y_\varepsilon\right) |v_\varepsilon|^{N-2} v_\varepsilon. \end{aligned} \tag{3.21}$$

Taking ε adequately small, we have

$$\int_{B(0,1)} |v_\varepsilon|^N \, \mathrm{d}y \ge r. \tag{3.22}$$

Going if necessary to a subsequence, then we get $v_\varepsilon \rightharpoonup v$ in X_ε, and almost everywhere in $\mathbb{R}^N$. Note that the embedding $X_\varepsilon \hookrightarrow L^N(B(0,1))$ is compact, by using (3.22), we get $v \not\equiv 0$. Next, we will prove v satisfies

$$-\Delta_q v - \Delta_N v + V\left(x_0\right) |v|^{q-2} v + V\left(x_0\right) |v|^{N-2} v = f(v) \quad \text{in } \mathbb{R}^N. \tag{3.23}$$

Indeed, for any $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$, in (3.21), we use $\left(v_\varepsilon - v\right) \varphi$ as a test function. For ε small enough, according to χ and g, we have that

$$\chi_\varepsilon\left(y + y_\varepsilon\right) |v_\varepsilon|^{N-2} v_\varepsilon\left(v_\varepsilon - v\right) \varphi = 0, \quad \forall y \in \mathbb{R}^N,$$

$$g\left(\varepsilon y+\varepsilon y_{\varepsilon}, v_{\varepsilon}\right)\left(v_{\varepsilon}-v\right) \varphi=f\left(v_{\varepsilon}\right)\left(v_{\varepsilon}-v\right) \varphi, \quad \forall y \in \mathbb{R}^{N},$$
$$\chi_{\varepsilon}\left(y+y_{\varepsilon}\right)\left|v_{\varepsilon}\right|^{q-2} v_{\varepsilon}\left(v_{\varepsilon}-v\right) \varphi=0, \quad \forall y \in \mathbb{R}^{N}.$$

$\forall \xi \geq N$, we know that the embedding $X_{\varepsilon} \hookrightarrow L^{\xi}\left(\mathbb{R}^{N}\right)$ is local compact. Hence,

$$\int_{\mathbb{R}^{N}} V_{\varepsilon}\left(y+y_{\varepsilon}\right)\left|v_{\varepsilon}\right|^{N-2} v_{\varepsilon} \varphi \mathrm{d} y \rightarrow \int_{\mathbb{R}^{N}} V\left(x_{0}\right)|v|^{N-2} v \varphi \mathrm{d} y$$

and

$$\int_{\mathbb{R}^{N}} V_{\varepsilon}\left(y+y_{\varepsilon}\right)\left|v_{\varepsilon}\right|^{q-2} v_{\varepsilon} \varphi \mathrm{d} y \rightarrow \int_{\mathbb{R}^{N}} V\left(x_{0}\right)|v|^{q-2} v \varphi \mathrm{d} y.$$

By Lemma 2.2, (f_1), and $\left\|f\left(v_{\varepsilon}\right)\right\|_{N}<\infty$, we obtain that

$$\int_{\mathbb{R}^{N}} f\left(v_{\varepsilon}\right)\left(v_{\varepsilon}-v\right) \varphi \mathrm{d} y=\int_{\mathbb{R}^{N}} g\left(\varepsilon y+\varepsilon y_{\varepsilon}, v_{\varepsilon}\right)\left(v_{\varepsilon}-v\right) \varphi \mathrm{d} y \rightarrow 0.$$

Therefore, similar to the proof of Lemma 3 in [6], we have that

$$\int_{\mathbb{R}^{N}}\left|\nabla v_{\varepsilon}\right|^{N-2} \nabla v_{\varepsilon} \nabla \varphi \mathrm{d} y \rightarrow \int_{\mathbb{R}^{N}}|\nabla v|^{N-2} \nabla v \nabla \varphi \mathrm{d} y$$

and

$$\int_{\mathbb{R}^{N}}\left|\nabla v_{\varepsilon}\right|^{q-2} \nabla v_{\varepsilon} \nabla \varphi \mathrm{d} y \rightarrow \int_{\mathbb{R}^{N}}|\nabla v|^{q-2} \nabla v \nabla \varphi \mathrm{d} y.$$

According to (f_1), (f_2), the compactness lemma of Strauss [32] and Lemma 2.2, we get that

$$\int_{\mathbb{R}^{N}} g\left(\varepsilon y+\varepsilon y_{\varepsilon}, v_{\varepsilon}\right) \varphi \mathrm{d} y \rightarrow \int_{\mathbb{R}^{N}} f(v) \varphi \mathrm{d} y.$$

Therefore, (3.23) has a nontrivial solution v. According to definition, $I_{V(x_0)}(v) \geq c_{V(x_0)}$. For $R>0$ large enough, because of Fatou's lemma, it is easy to get

$$\liminf_{\varepsilon \rightarrow 0} \int_{B\left(x_{\varepsilon}, R\right)}\left|\nabla u_{\varepsilon}\right|^{N} \mathrm{d} y \geq \frac{1}{2} \int_{\mathbb{R}^{N}}|\nabla v|^{N} \mathrm{d} y, \tag{3.24}$$

and

$$\liminf_{\varepsilon \rightarrow 0} \int_{B\left(x_{\varepsilon}, R\right)}\left|\nabla u_{\varepsilon}\right|^{q} \mathrm{d} y \geq \frac{1}{2} \int_{\mathbb{R}^{N}}|\nabla v|^{q} \mathrm{d} y. \tag{3.25}$$

Now, recalling from Remark 3.7 that $c_a>c_b$ when $a>b$, it is easy to see that $c_{V(x_0)} \geq c_m$ because of $V\left(x_{0}\right) \geq m$. According to Pohozǎev identity, for any $U \in \mathcal{S}_m$,

$$\frac{1}{N}\left(\int_{\mathbb{R}^{N}}|\nabla U|^{N} \mathrm{d} x+\int_{\mathbb{R}^{N}}|\nabla U|^{q} \mathrm{d} x\right)=I_{m}(U). \tag{3.26}$$

Thus,it follows from (3.24), (3.25) and (3.26) that

$$\liminf_{\varepsilon \rightarrow 0} \int_{B\left(y_{\varepsilon}, R\right)}\left|\nabla u_{\varepsilon}\right|^{N} \mathrm{d} y+\liminf_{\varepsilon \rightarrow 0} \int_{B\left(y_{\varepsilon}, R\right)}\left|\nabla u_{\varepsilon}\right|^{q} \mathrm{d} y \geq \frac{N}{2} I_{V(x_0)}(v) \geq \frac{N}{2} c_{m}>0.$$

When d is small enough, this is a contradiction with (3.19) .

Step 2. Define $u_{\varepsilon}^{2}=u_{\varepsilon}-u_{\varepsilon}^{1}$, where $u_{\varepsilon}^{1}(y)=\varphi_{\varepsilon}\left(y-x_{\varepsilon} / \varepsilon\right) u_{\varepsilon}(y)$. For $d>0$ small enough, we will prove, $J_{\varepsilon}\left(u_{\varepsilon}^{2}\right) \geq 0$ and

$$J_{\varepsilon}\left(u_{\varepsilon}\right) \geq o(1)+J_{\varepsilon}\left(u_{\varepsilon}^{1}\right)+J_{\varepsilon}\left(u_{\varepsilon}^{2}\right) \quad \text{as } \varepsilon \rightarrow 0. \tag{3.27}$$

Clearly, for small enough $\varepsilon > 0$, we have $Q_\varepsilon\left(u_\varepsilon^1\right) = 0$ and $Q_\varepsilon\left(u_\varepsilon\right) = Q_\varepsilon\left(u_\varepsilon^2\right)$. Moreover, $\forall y \in \mathbb{R}^N$, $u_\varepsilon^1(y)u_\varepsilon^2(y) \geq 0$, we get

$$\begin{aligned} |u_\varepsilon(y)|^q &= \left(\left|u_\varepsilon^1(y)\right|^2 + \left|u_\varepsilon^2(y)\right|^2 + 2u_\varepsilon^1(y)u_\varepsilon^2(y)\right)^{q/2} \\ &\geq \left(\left|u_\varepsilon^1(y)\right|^2 + \left|u_\varepsilon^2(y)\right|^2\right)^{q/2} \\ &\geq \left|u_\varepsilon^1(y)\right|^q + \left|u_\varepsilon^2(y)\right|^q \end{aligned}$$

and

$$\begin{aligned} |u_\varepsilon(y)|^N &= \left(\left|u_\varepsilon^1(y)\right|^2 + \left|u_\varepsilon^2(y)\right|^2 + 2u_\varepsilon^1(y)u_\varepsilon^2(y)\right)^{N/2} \\ &\geq \left(\left|u_\varepsilon^1(y)\right|^2 + \left|u_\varepsilon^2(y)\right|^2\right)^{N/2} \\ &\geq \left|u_\varepsilon^1(y)\right|^N + \left|u_\varepsilon^2(y)\right|^N. \end{aligned}$$

So

$$\int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon^1\right|^N \mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon^2\right|^N \mathrm{d}y \leq \int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon\right|^N \mathrm{d}y$$

and

$$\int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon\right|^q \mathrm{d}y \geq \int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon^1\right|^q \mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon^2\right|^q \mathrm{d}y.$$

Moreover, it is easy to verify that

$$\begin{aligned} \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^1\right|^N \mathrm{d}y &= \int_{\mathbb{R}^N} \varphi_\varepsilon^N\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) |\nabla u_\varepsilon|^N \mathrm{d}y + o(1), \\ \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^2\right|^N \mathrm{d}y &= \int_{\mathbb{R}^N} \left(1 - \varphi_\varepsilon\left(-\frac{x_\varepsilon}{\varepsilon}\right)\right)^N |\nabla u_\varepsilon|^N \mathrm{d}y + o(1), \\ \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^2\right|^q \mathrm{d}y &= \int_{\mathbb{R}^N} \left(1 - \varphi_\varepsilon\left(-\frac{x_\varepsilon}{\varepsilon}\right)\right)^q |\nabla u_\varepsilon|^N \mathrm{d}y + o(1), \\ \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^1\right|^q \mathrm{d}y &= \int_{\mathbb{R}^N} \varphi_\varepsilon^N\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) |\nabla u_\varepsilon|^q \mathrm{d}y + o(1). \end{aligned}$$

Obviously, for any $y \in \mathbb{R}^N$, we have

$$\varphi_\varepsilon^2\left(y - x_\varepsilon/\varepsilon\right) |\nabla u_\varepsilon(y)|^2 + \left(1 - \varphi_\varepsilon\left(y - x_\varepsilon/\varepsilon\right)\right)^2 |\nabla u_\varepsilon(y)|^2 \leq |\nabla u_\varepsilon(y)|^2.$$

Therefore, we have

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^N \mathrm{d}y \geq \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^1\right|^N \mathrm{d}y + \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^2\right|^N \mathrm{d}y + o(1)$$

and

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^q \mathrm{d}y \geq \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^1\right|^q \mathrm{d}y + \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^2\right|^q \mathrm{d}y + o(1).$$

Hence, we have that

$$J_\varepsilon\left(u_\varepsilon\right) \geq o(1) - \int_{B_\varepsilon} \left(G\left(\varepsilon y, u_\varepsilon\right) - G(\varepsilon y, u_\varepsilon^1) - G\left(\varepsilon y, u_\varepsilon^2\right)\right) \mathrm{d}y + J_\varepsilon(u_\varepsilon^1) + J_\varepsilon(u_\varepsilon^2).$$

According to (f_1) and (f_2), then we obtain

$$\varepsilon|t|^q + C|t|^\tau \Psi_N(t) \geq |F(t)|. \tag{3.28}$$

Using the same proof as that in Lemma 3.1, we get

$$\int_{\mathbb{R}^N} |u|^\tau \Psi_N(u)\,\mathrm{d}x \leq \|u\|^\tau_{L^{\tau t'}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} (\Phi_N(u))^t\,\mathrm{d}x\right)^{\frac{1}{t}}.$$

By using Step 1, we know that $u_\varepsilon \to 0$ in $L^q(B_\varepsilon)$, so

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_{B_\varepsilon} \left(G(\varepsilon y, u_\varepsilon) - G(\varepsilon y, u_\varepsilon^2) - G\left(\varepsilon y, u_\varepsilon^1\right)\right)\mathrm{d}y \\
&= \limsup_{\varepsilon\to 0} \left|\int_{B_\varepsilon} \left(F(u_\varepsilon) - F\left(u_\varepsilon^1\right) - F\left(u_\varepsilon^2\right)\right)\mathrm{d}y\right| \\
&\leq \limsup_{\varepsilon\to 0} \int_{B_\varepsilon} \left(C|u_\varepsilon|^\tau \Psi_N(u_\varepsilon) + \varepsilon|u_\varepsilon|^q\right)\mathrm{d}y \\
&\leq c\varepsilon.
\end{aligned}$$

Due to ε being arbitrary, as $\varepsilon \to 0$ we get $\int_{B_\varepsilon}\left(F(u_\varepsilon) - F\left(u_\varepsilon^1\right) - F\left(u_\varepsilon^2\right)\right)\mathrm{d}y = o(1)$. So there is $C > 0$ satisfies

$$\begin{aligned}
J_\varepsilon\left(u_\varepsilon^2\right) \geq I\left(u_\varepsilon^2\right) &\geq \frac{1}{N}\|u_\varepsilon^2\|^N_{X_\varepsilon} + \frac{1}{q}\|u_\varepsilon^2\|^q_{X_\varepsilon} - C\int_{\mathbb{R}^N} |u_\varepsilon|^\tau \Psi_N\left(u_\varepsilon^2\right)\mathrm{d}y - \varepsilon\left\|u_\varepsilon^2\right\|^q_{X_\varepsilon} \\
&\geq \frac{1}{q\cdot 2^{q-1}}\|u_\varepsilon^2\|^q_{X_\varepsilon} - C\left\|u_\varepsilon^2\right\|^\tau_{X_\varepsilon}.
\end{aligned}$$

Hence, by using $\tau > q$, we get that $J_\varepsilon\left(u_\varepsilon^2\right) \geq 0$ for $d > 0$ small.

Step 3. Now, assume $w_\varepsilon(y) := u_\varepsilon^1\left(y + \frac{x_\varepsilon}{\varepsilon}\right) = \varphi_\varepsilon(y)u_\varepsilon\left(y + \frac{x_\varepsilon}{\varepsilon}\right)$. Up to a subsequence, we have $w_\varepsilon \rightharpoonup w$ in $X_\varepsilon, w_\varepsilon \to w$ almost everywhere in $\mathbb{R}^N$. Next, we will prove that

$$w_\varepsilon \to w \quad \text{in } L^\tau(\mathbb{R}^N),$$

where τ is given in (3.28). By contradiction, if there is $r > 0$ that satisfies

$$0 < 2r = \liminf_{\varepsilon\to 0} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |w_\varepsilon - w|^\tau\,\mathrm{d}y.$$

So there is $z_\varepsilon \in \mathbb{R}^N$ that satisfies

$$\liminf_{\varepsilon\to 0} \int_{B(z_\varepsilon,1)} |w_\varepsilon - w|^\tau > r.$$

It is easy to see that (z_ε) is unbounded. We may assume that $|z_\varepsilon| = \infty$ as $\varepsilon \to 0$, then,

$$r \leq \liminf_{\varepsilon\to 0} \int_{B(z_\varepsilon,1)} |w_\varepsilon|^\tau\,\mathrm{d}y,$$

i.e.

$$\liminf_{\varepsilon\to 0} \int_{B(z_\varepsilon,1)} \left|\varphi_\varepsilon(y)u_\varepsilon\left(y + \frac{x_\varepsilon}{\varepsilon}\right)\right|^\tau \mathrm{d}y \geq r.$$

Using the same proof method as [9], for ε small enough, we have that $|z_\varepsilon| \leq \frac{\beta}{2\varepsilon}$. Assume that

$$\varepsilon z_\varepsilon \to z_0 \in \overline{B(0,\beta/2)},$$

$$\widetilde{w}_\varepsilon = w_\varepsilon\left(y+z_\varepsilon\right) \rightharpoonup \widetilde{w} \quad \text{in } X_\varepsilon,$$
$$\widetilde{w}_\varepsilon \to \widetilde{w} \quad \text{a.e. in } \mathbb{R}^N.$$

So $\widetilde{w} \not\equiv 0$ and according to Step 1, $\widetilde{w}$ satisfies

$$-\Delta_q \widetilde{w}(y) - \Delta_N \widetilde{w}(y) + V\left(x+z_0\right)|\widetilde{w}(y)|^{q-2}\widetilde{w}(y) + V\left(x+z_0\right)|\widetilde{w}(y)|^{N-2}\widetilde{w}(y) = f(\widetilde{w}(y)), \quad y \in \mathbb{R}^N.$$

Using the same approach as Step 1, we obtain a contradiction for $d>0$ small enough. Therefore, $w_\varepsilon \to w$ in $L^\tau\left(\mathbb{R}^N\right)$.

Step 4. According to Step 3, it follows that

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} G\left(\varepsilon x, u_\varepsilon^1\right) dx &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} G\left(\varepsilon x + x_\varepsilon, w_\varepsilon\right) dx \\ &= \lim_{\varepsilon\to 0}\int_{\Lambda_\varepsilon - x_\varepsilon/\varepsilon} F\left(w_\varepsilon\right) dx = \int_{\mathbb{R}^N} F(w) dx. \end{aligned} \tag{3.29}$$

By using $w_\varepsilon \rightharpoonup w$ in X_ε, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} J_\varepsilon\left(u_\varepsilon^1\right) \\ &\geq \liminf_{\varepsilon\to 0} I_\varepsilon\left(u_\varepsilon^1\right) \\ &= \liminf_{\varepsilon\to 0} \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla w_\varepsilon(y)|^N + V_\varepsilon|w_\varepsilon(y)|^N)dy + \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla w_\varepsilon(y)|^q + V_\varepsilon|w_\varepsilon(y)|^q)dy \\ &\quad - \int_{\mathbb{R}^N} F\left(w_\varepsilon(y)\right) dy \\ &\geq \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla w|^N + m|w|^N)dy - \int_{\mathbb{R}^N} F(w)\, dy + \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla w|^q + m|w|^q)dy \\ &\geq c_m. \end{aligned} \tag{3.30}$$

On the other hand, since $\lim_{\varepsilon\to 0} J_\varepsilon\left(u_\varepsilon\right) \leq c_m$, $J_\varepsilon\left(u_\varepsilon^2\right) \geq 0$ and (3.27), we have

$$\limsup_{\varepsilon\to 0} J_\varepsilon\left(u_\varepsilon^1\right) \leq c_m. \tag{3.31}$$

Combining (3.30) and (3.31), we obtain that $J_\varepsilon(w) = c_m$. Similar to [25], we can obtain that $x \in \mathcal{M}$. So it is easy to see that $w(y) = U(y-z)$, $U \in \mathcal{S}_m$, $z \in \mathbb{R}^N$.

Lastly, due to (3.29) and (3.31) and $V(y) \geq m$ on Λ, by using (3.30), we have

$$\begin{aligned} \int_{\mathbb{R}^N}\left(|\nabla w|^N + m|w|^N\right) dy &\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\left|\nabla u_\varepsilon^1(y)\right|^N + V(\varepsilon y)|u_\varepsilon^1(y)|^N\right) dy \\ &\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\left|\nabla u_\varepsilon^1(y)\right|^N + m|u_\varepsilon^1(y)|^N\right) dy \\ &\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(|\nabla w_\varepsilon(y)|^N + m|w_\varepsilon(y)|^N\right) dy \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N}(|\nabla w|^q + m|w|^q)\, dy &\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\left|\nabla u_\varepsilon^1(y)\right|^q + V(\varepsilon y)|u_\varepsilon^1(y)|^q\right) dy \\ &\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}\left(\left|\nabla u_\varepsilon^1(y)\right|^q + m|u_\varepsilon^1(y)|^q\right) dy \\ &\geq \limsup_{\varepsilon\to 0}\int_{\mathbb{R}^N}(|\nabla w_\varepsilon(y)|^q + m|w_\varepsilon(y)|^q)\, dy. \end{aligned}$$

Moreover, by using weak lower semi-continuity, we prove $u_\varepsilon^1 \to w$ in X_ε. Especially, let $y_\varepsilon = z + \frac{x}{\varepsilon}$, then $u_\varepsilon^1 \to U(\cdot - y_\varepsilon)\varphi_\varepsilon(\cdot - y_\varepsilon)$ in X_ε. So we get $u_\varepsilon^1 \to U(\cdot - y_\varepsilon)\varphi_\varepsilon(\cdot - y_\varepsilon)$ in X_ε.

In order to prove the desired conclusion, we only prove that $u_\varepsilon^2 \to 0$ in X_ε. Since $\{u_\varepsilon\}_\varepsilon$ is bounded, for small $\varepsilon > 0$, it is easy to see from (3.19) that $\|u_\varepsilon^2\|_\varepsilon \le 4d$. Now, using (3.27), $\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon^1) = c_m$ and the estimation of $J_\varepsilon(u_\varepsilon^2)$, we have that for some $C > 0$,

$$c_m \ge \lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) \ge c_m + \|u_\varepsilon^2\|_{X_\varepsilon}^q \left(\frac{1}{q\cdot 2^{q-1}} - C(4d)^{\tau-q}\right) + o(\varepsilon).$$

This proves that $u_\varepsilon^2 \to 0$ in X_ε, which completes the proof. □

Lemma 3.12. *For $0 < d_2 < d_1$ small enough, there exist $\omega > 0$ and $\varepsilon_0 > 0$ that satisfy $|J_\varepsilon'(u)| \ge \omega$, where $\varepsilon \in (0, \varepsilon_0)$, $u \in J_\varepsilon^{\tilde{c}_\varepsilon} \cap (Y_\varepsilon^{d_1} \backslash Y_\varepsilon^{d_2})$.*

Proof. By contradiction, we can suppose $0 < d_2 < d_1$ small enough, there are $\{\varepsilon_i\}_{i=1}^\infty$ with $\lim_{i\to\infty}\varepsilon_i = 0$ and $u_{\varepsilon_i} \in Y_{\varepsilon_i}^{d_1} \backslash Y_{\varepsilon_i}^{d_2}$ satisfying $\lim_{i\to\infty} J_{\varepsilon_i}(u_{\varepsilon_i}) \le c_m$ and $\lim_{i\to\infty}\left|J'_{\varepsilon_i}(u_{\varepsilon_i})\right| = 0$. For the convenience of description, we write ε for ε_i. Due to Lemma 3.11, for some $U \in \mathcal{S}_m$ and $x \in \mathcal{M}$, there is $\{y_\varepsilon\}_\varepsilon \subset \mathbb{R}^N$ such that

$$\lim_{\varepsilon\to 0}\|\varphi_\varepsilon(\cdot - y_\varepsilon)U(\cdot - y_\varepsilon) - u_\varepsilon\|_{X_\varepsilon} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}|x - \varepsilon y_\varepsilon| = 0.$$

It follows from Y_ε that $\lim_{\varepsilon\to 0}\operatorname{dist}(Y_\varepsilon, u_\varepsilon) = 0$. Obviously contradictory because of $u_\varepsilon \notin Y_\varepsilon^{d_2}$. □

According to Lemma 3.12, fix a $d > 0$ small enough, there exist $\omega > 0$ and $\varepsilon_0 > 0$ that satisfy $|J_\varepsilon'(u)| \ge \omega$, where $\varepsilon \in (0, \varepsilon_0)$, $u \in J_\varepsilon^{\tilde{c}_\varepsilon} \cap (Y_\varepsilon^{d_1} \setminus Y_\varepsilon^{d_2})$. So we have

Lemma 3.13. *For $\varepsilon > 0$ small enough, we can find $\alpha > 0$ satisfies $J_\varepsilon(\gamma_\varepsilon(s)) \ge c_\varepsilon - \alpha$, then $\gamma_\varepsilon(s) \in Y_\varepsilon^{d/2}$ where $\gamma_\varepsilon(s) = W_{\varepsilon, st_0}(s)$.*

Proof. For each $s \in [0, 1]$, due to $\mathcal{M}_\varepsilon^{2\beta} \supset \operatorname{supp}(\gamma_\varepsilon(s))$, we have $I_\varepsilon(\gamma_\varepsilon(s)) = J_\varepsilon(\gamma_\varepsilon(s))$. In addition, it is easy to see that

$$\begin{aligned}
I_\varepsilon(\gamma_\varepsilon(s)) &= \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla\gamma_\varepsilon(s)|^q + V_\varepsilon|\gamma_\varepsilon(s)|^q)\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla\gamma_\varepsilon(s)|^N + V_\varepsilon|\gamma_\varepsilon(s)|^N)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} F(\gamma_\varepsilon(s))\mathrm{d}x \\
&= \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla\gamma_\varepsilon(s)|^q + m|\gamma_\varepsilon(s)|^q)\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla\gamma_\varepsilon(s)|^N + m|\gamma_\varepsilon(s)|^N)\mathrm{d}x \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N}(V_\varepsilon(x) - m)|\gamma_\varepsilon(s)|^q)\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N}(V_\varepsilon(x) - m)|\gamma_\varepsilon(s)|^N)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} F(\gamma_\varepsilon(s))\mathrm{d}x \\
&= \frac{1}{N}\int_{\mathbb{R}^N}|\nabla U|^N\mathrm{d}x + \frac{(st_0)^{N-q}}{q}\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x + \frac{(st_0)^N}{N}\int_{\mathbb{R}^N} m|U|^N\mathrm{d}x \\
&\quad + \frac{(st_0)^N}{q}\int_{\mathbb{R}^N} m|U|^q\mathrm{d}x - (st_0)^N\int_{\mathbb{R}^N} F(U)\mathrm{d}x + O(\varepsilon).
\end{aligned}$$

Using the Pohozăev identity, we have

$$\begin{aligned} J_\varepsilon(\gamma_\varepsilon(s)) &= I_\varepsilon(\gamma_\varepsilon(s)) \\ &= \frac{1}{N}\int_{\mathbb{R}^N}|\nabla U|^N\mathrm{d}x + \frac{(st_0)^{N-q}}{q}\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x - \frac{N-q}{Nq}(st_0)^N\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x + O(\varepsilon) \\ &= \frac{1}{N}\int_{\mathbb{R}^N}|\nabla U|^N\mathrm{d}x + \left(\frac{(st_0)^{N-q}}{q} - \frac{N-q}{Nq}(st_0)^N\right)\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x + O(\varepsilon). \end{aligned}$$

Note that

$$c_m = \left(\frac{t^{N-q}}{q} - \frac{N-q}{Nq}t^N\right)\int_{\mathbb{R}^N}|\nabla U|^q\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N}|\nabla U|^N\mathrm{d}x$$

and $\lim_{\varepsilon\to 0} c_\varepsilon = c_m$. Denote $g_1(t) = -\frac{N-q}{Nq}t^N + \frac{t^{N-q}}{q}$, then

$$g_1'(t)\begin{cases} < 0, & t > 1, \\ = 0, & t = 1, \\ > 0, & t \in (0,1). \end{cases}$$

So we have $g_1''(1) = q - N < 0$, the conclusion follows. □

Lemma 3.14. *For $\varepsilon > 0$ small enough, we can find $\{u_n\}_{n=1}^\infty \subset Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$ satisfies as $n \to \infty$, $J_\varepsilon'(u_n) \to 0$.*

Proof. According to Lemma 3.13, for $\varepsilon > 0$ small enough, due to $\exists \alpha > 0$ satisfies $J_\varepsilon(\gamma_\varepsilon(s)) \geq c_\varepsilon - \alpha$. So $\gamma_\varepsilon(s) \in Y_\varepsilon^{d/2}$. Now, we assume that Lemma 3.14 is not true, then for $\varepsilon > 0$ small enough, we can find $a(\varepsilon) > 0$ satisfies $|J_\varepsilon'(u)| \geq a(\varepsilon)$ on $Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$. Moreover, by using Lemma 3.12, we also can find $\omega > 0$, independent of $\varepsilon > 0$, satisfies for $u \in J_\varepsilon^{\tilde{c}_\varepsilon} \cap \left(Y_\varepsilon^d \backslash Y_\varepsilon^{d/2}\right)$, $|J_\varepsilon'(u)| \geq \omega$. Therefore, recalling that $\lim_{\varepsilon\to 0}(c_\varepsilon - \tilde{c}_\varepsilon) = 0$, according to a deformation lemma, for $\varepsilon > 0$ small enough, we can construct a path $\gamma \in \Gamma_\varepsilon$ satisfying $J_\varepsilon(\gamma(s)) < c_\varepsilon, s \in [0,1]$. Obviously contradictory. □

Lemma 3.15. *For $\varepsilon > 0$ sufficiently small, $u_\varepsilon \in Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$ is a critical point of J_ε.*

Proof. For $\varepsilon > 0$ sufficiently small. According to Lemma 3.14, there exists a sequence $\{u_{n,\varepsilon}\}_{n=1}^\infty \subset Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$ that satisfies, as $n \to \infty$, $|J_\varepsilon'(u_{n,\varepsilon})| \to 0$. Due to Y_ε^d is bounded, so as $n \to \infty$, $u_{n,\varepsilon} \rightharpoonup u_\varepsilon$ in X_ε. Using the same proof as [10, Proposition 3], we obtain that

$$0 = \lim_{R\to\infty}\sup_{n\geq 1}\int_{|x|\geq R}\left(V_\varepsilon|u_{n,\varepsilon}|^N + |\nabla u_{n,\varepsilon}|^N\right)\mathrm{d}x \tag{3.32}$$

and

$$0 = \lim_{R\to\infty}\sup_{n\geq 1}\int_{|x|\geq R}\left(V_\varepsilon|u_{n,\varepsilon}|^q + |\nabla u_{n,\varepsilon}|^q\right)\mathrm{d}x, \tag{3.33}$$

so as $n \to \infty$, $u_{n,\varepsilon} \to u_\varepsilon$ in $L^r(\mathbb{R}^N)$ $(N \leq r < +\infty)$. In addition, using (f_1)–(f_2), we have $\sup\|f(u_{n,\varepsilon})\| < \infty$. Now, $\forall\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f(u_{n,\varepsilon})(u_{n,\varepsilon} - u_\varepsilon)\varphi\mathrm{d}x \to 0, \quad n \to \infty.$$

Using the same argument as in [21, Proposition 5.3], we have $u_{n,\varepsilon} \to u_\varepsilon$ in X_ε as $n \to \infty$. Hence, $u_\varepsilon \in Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$ and $J_\varepsilon'(u_\varepsilon) = 0$ in X_ε. This completes the proof. □

Next, we will use Moser iteration in [27] to obtain L^{∞}-estimate.

Lemma 3.16. *Let (u_n) is the sequence in Lemma 3.11. Then, $J_{\varepsilon_n}(u_n) \to c_m$ in $\mathbb{R}$ as $n \to \infty$, and there is some sequence $(\hat{y}_n) \subset \mathbb{R}^N$ that satisfies $v_n(\cdot) := u_n(\cdot + \hat{y}_n) \in L^{\infty}(\mathbb{R}^N)$ and $|v_n|_{L^{\infty}(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$.*

Proof. Proceeding as in the proof of Lemmas 3.9 and 3.10, as $n \to \infty$, we know that $J_{\varepsilon_n}(u_n) \to c_m$ in $\mathbb{R}$. According to Lemma 3.11, as $n \to \infty$, we can find $(\hat{y}_n) \subset \mathbb{R}^N$ satisfies $v_n(\cdot) := u_n(\cdot + \hat{y}_n) \to v(\cdot) \in X_{\varepsilon}$ and $y_n := \varepsilon_n \hat{y}_n \to y_0 \in \mathcal{M}$.

For all $L > 0$ and $\beta > 1$, consider

$$\phi(v_n) = \phi_{L,\beta}(v_n) = v_n v_{L,n}^{N(\beta-1)} \in X_{\varepsilon}, v_{L,n} = \min\{v_n, L\}.$$

Set

$$\Phi(t) = \int_0^t (\phi'(t))^{\frac{1}{N}} \, d\tau, \quad \Upsilon(t) = \frac{|t|^N}{N}.$$

According to [5], we have

$$|\Phi(a) - \Phi(b)|^N \leq \Upsilon'(a-b)(\phi(a) - \phi(b)), \quad \forall a \in \mathbb{R}, \ b \in \mathbb{R}. \tag{3.34}$$

According to (3.34), we have

$$\begin{aligned} &|\Phi(v_n(x)) - \Phi(v_n(y))|^N \\ &\quad \leq (v_n(x) - v_n(y)) \left(\left(v_n v_{L,n}^{N(\beta-1)} \right)(x) - \left(v_n v_{L,n}^{N(\beta-1)} \right)(y) \right) |v_n(x) - v_n(y)|^{N-2}. \end{aligned} \tag{3.35}$$

Therefore, taking $\phi(v_n) = v_n v_{L,n}^{N(\beta-1)}$ as a test function, we obtain that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla v_n|^{N-1} \phi(v_n) \, dx + \int_{\mathbb{R}^N} |\nabla v_n|^{q-1} \phi(v_n) \, dx \\ &\qquad + \int_{\mathbb{R}^N} V(y_n + \varepsilon_n x) |v_n|^{N-2} v_n \phi(v_n) \, dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |v_n|^{q-2} v_n \phi(v_n) \, dx \\ &\quad = \int_{\mathbb{R}^N} f(\varepsilon_n x + y_n, v_n) \phi(v_n) \, dx. \end{aligned}$$

Due to (f_1) and (f_2), $\forall \varepsilon > 0$, we can find $C(\varepsilon) > 0$ satisfies

$$|f(t)| \leq \varepsilon |t|^{q-1} + C(\varepsilon)|t|^{N-1} \Psi_N(t), \quad \forall t \in \mathbb{R}.$$

According to method of [5], it is easy to get

$$\int_{\mathbb{R}^N} |\nabla v_n|^N v_{L,n}^{p(\beta-1)} dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |v_n|^N v_{L,n}^{p(\beta-1)} dx \leq \int_{\mathbb{R}^N} f(v_n) v_n v_{L,n}^{N(\beta-1)} dx.$$

Since $\Phi(v_n) \geq \frac{1}{\beta} v_n v_{L,n}^{\beta-1}, v_n v_{L,n}^{\beta-1} \geq \Phi(v_n)$ and the embedding from $X_{\varepsilon} \to L^{N^*}(\mathbb{R}^N)$ $(N^* > N)$ is continuous, so we can find $S_* > 0$ that satifies

$$\frac{1}{\beta^N} S_* \left\| v_n v_{L,n}^{\beta-1} \right\|_{L^{N^*}(\mathbb{R}^N)}^N \leq S_* \|\Phi(v_n)\|_{L^{N^*}(\mathbb{R}^N)}^N \leq \|\Phi(v_n)\|_{X_{\varepsilon}}^N. \tag{3.36}$$

Since $X_{\varepsilon} \to L^{\nu}(\mathbb{R}^N)$ $(\nu \geq N)$ is continuous, there exists $\mathcal{S}_{\nu}$ satisfying

$$\mathcal{S}_{\nu} = \inf_{u \neq 0, u \in X_{\varepsilon}} \frac{\|u\|_{X_{\varepsilon}}}{\|u\|_{L^{\nu}(\mathbb{R}^N)}}, \quad \nu \geq N.$$

This implies

$$\|u\|_{L^N(\mathbb{R}^N)} \le \mathcal{S}_N^{-1}\|u\|_{X_\varepsilon}, \quad \forall u \in X_\varepsilon. \tag{3.37}$$

Then we obtain

$$\begin{aligned}
\|\Phi(v_n)\|^N_{m,X(\mathbb{R}^N)} &\le \varepsilon \int_{\mathbb{R}^N} \left|v_n v_{L,n}^{\beta-1}\right|^N \mathrm{d}x + C(\varepsilon)\int_{\mathbb{R}^N} \Psi_N(v_n)\left|v_n v_{L,n}^{\beta-1}\right|^p \mathrm{d}x \\
&\le \varepsilon\beta^N \int_{\mathbb{R}^N} |\Phi(v_n)|^N \mathrm{d}x + C(\varepsilon)\int_{\mathbb{R}^N} \Psi_N(v_n)\left|v_n v_{L,n}^{\beta-1}\right|^N \mathrm{d}x \\
&\le \varepsilon\beta^N \mathcal{S}_N^{-N}\|\Phi(v_n)\|^N_{m,X(\mathbb{R}^N)} + C(\varepsilon)\int_{\mathbb{R}^N} \Psi_N(v_n)\left|v_n v_{L,n}^{\beta-1}\right|^N \mathrm{d}x.
\end{aligned} \tag{3.38}$$

Choose $0 < \varepsilon < \beta^{-N}\mathcal{S}_N^N$, then (3.38) implies

$$\begin{aligned}
&\frac{1}{\beta^N} S_*\left(1-\varepsilon\beta^N\mathcal{S}_N^{-N}\right)\left\|v_n v_{L,n}^{\beta-1}\right\|^N_{L^{N^*}(\mathbb{R}^N)} \\
&\le C(\varepsilon)\left(\int_{\mathbb{R}^N}(\Psi_N(v_n))^{q'}\mathrm{d}x\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^N}\left|v_n v_{L,n}^{\beta-1}\right|^{qN}\mathrm{d}x\right)^{\frac{1}{q}}.
\end{aligned}$$

Now, by the Trudinger–Moser inequality with $N << q$ such that $N^* > qN = N^{**}$. Note that, q' near 1 but $q' > 1$. So we can find $D > 0$ satisfies

$$\left\|v_n v_{L,n}^{\beta-1}\right\|^N_{L^{N^*}(\mathbb{R}^N)} \le D\beta^N\left\|v_n v_{L,n}^{\beta-1}\right\|^N_{L^{qN}(\mathbb{R}^N)}.$$

Let $L \to +\infty$, we obtain

$$\|v_n\|_{L^{N^*\beta}} \le D^{\frac{1}{N\beta}}\beta^{\frac{1}{\beta}}\|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)}. \tag{3.39}$$

Let $\beta = \frac{N^*}{N^{**}} > 1$. Then $\beta^2 N^{**} = \beta N^*$. Replace β with β^2, (3.39) holds. Hence,

$$\begin{aligned}
\|v_n\|_{L^{N^*\beta^2}} &\le D^{\frac{1}{N\beta^2}}\beta^{\frac{2}{\beta^2}}\|v_n\|_{L^{N^{**}\beta^2}(\mathbb{R}^N)} \\
&= D^{\frac{1}{N\beta^2}}\beta^{\frac{2}{\beta^2}}\|v_n\|_{L^{N^*\beta}(\mathbb{R}^N)} \\
&\le D^{\frac{1}{N}\left(\frac{1}{\beta}+\frac{1}{\beta^2}\right)}\beta^{\frac{1}{\beta}+\frac{2}{\beta^2}}\|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)}.
\end{aligned} \tag{3.40}$$

Now iterating the process, as shown in (3.40), for any positive integer m, we get that

$$\|v_n\|_{L^{N^*\beta^\sigma}} \le D^{\sum\limits_{j=1}^{\sigma}\frac{1}{N\beta^j}}\beta^{\sum\limits_{j=1}^{\sigma}j\beta^{-j}}\|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)}. \tag{3.41}$$

Taking the limit in (3.41) as $\sigma \to \infty$, we have

$$\|v_n\|_{L^\infty(\mathbb{R}^N)} \le C$$

for all n, where $C = D^{\sum\limits_{j=1}^{\infty}\frac{1}{N\beta^j}}\beta^{\sum_{j=1}^{\infty}j\beta^{-j}}\sup\limits_n\|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)} < +\infty.$ □

Proof of Theorem 1.1. For $\varepsilon \in (0,\varepsilon_0)$, according to Lemma 3.15, there are $d,\ \varepsilon_0 > 0$ that satisfy J_ε has a critical point $u_\varepsilon \in Y_\varepsilon^d \cap \Gamma_\varepsilon^{\tilde{c}_\varepsilon}$. Since u_ε satisfies

$$-\Delta_N u_\varepsilon - \Delta_q u_\varepsilon + V(\varepsilon x)(|u_\varepsilon|^{N-2}u_\varepsilon + |u_\varepsilon|^{q-2}u_\varepsilon) = f(u_\varepsilon) + 4\left(\int_{\mathbb{R}^N}\chi_\varepsilon u_\varepsilon^p \mathrm{d}x - 1\right)_+ \chi_\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}^N.$$

When $t \leq 0$, we know $f(t) = 0$. So $u_\varepsilon > 0$ in $\mathbb{R}^N$. In addition, by using Lemma 3.16, it is easy to get $\{\|u_\varepsilon\|_{L^\infty}\}_\varepsilon$ is bounded. Now by using Lemma 3.11, we have

$$\lim_{\varepsilon \to 0} \left[\frac{1}{N} \left(\int_{\mathbb{R}^N \setminus \mathcal{M}_\varepsilon^{2\delta}} |\nabla u_\varepsilon|^N + V_\varepsilon (u_\varepsilon)^N \, dx \right) + \frac{1}{q} \left(\int_{\mathbb{R}^N \setminus \mathcal{M}_\varepsilon^{2\delta}} |\nabla u_\varepsilon|^q + V_\varepsilon (u_\varepsilon)^q \, dx \right) \right] = 0.$$

According to elliptic estimates in [20], we know

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N \setminus \mathcal{M}_\varepsilon^{2\delta})} = 0.$$

Similar to [35], there are $C > 0$, $c > 0$ that satisfy

$$u(x) \leq Ce^{-c|x|}.$$

In fact, by using the Radial Lemma in [7], one has

$$u(x) \leq C \frac{\|u\|_{L^N}}{|x|}, \quad \forall x \neq 0,$$

here C is related to N, p. Therefore, for $u \in \mathcal{S}_m$, we have $\lim_{|x| \to \infty} u(x) = 0$ uniformly. According to the comparison principle, we have that $C > 0$, $c > 0$ satisfy

$$u(x) \leq Ce^{-c|x|}, \quad \forall x \in \mathbb{R}^N.$$

According to a comparison principle, for some $C, c > 0$, we obtain that

$$u_\varepsilon(x) \leq C \exp\left(-c \operatorname{dist}\left(x, \mathcal{M}_\varepsilon^{2\delta}\right)\right).$$

So $Q_\varepsilon(u_\varepsilon) = 0$, then u_ε satisfies (1.1). Lastly, assume u_ε has a maximum point x_ε. According to Lemma 3.8 and Lemma 3.11, for some $x \in \mathcal{M}$, we get that $\varepsilon x_\varepsilon \to x$ as $\varepsilon \to 0$. Moreover, as to $C > 0$, $c > 0$,

$$u_\varepsilon(x) \leq Ce^{-c|x - x_\varepsilon|}.$$

This completes the final proof. □

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Existence and exponential stability of periodic solutions of Nicholson-type systems with nonlinear density-dependent mortality and linear harvesting

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Abstract. In this work we study a Nicholson-type periodic system with variable delay, density-dependent mortality and linear harvesting rate. Using the topological degree and Lyapunov stability theories, we obtain sufficient conditions that allow us to demonstrate the existence of periodic solutions for the Nicholson-type system and, under suitable conditions, the uniqueness and local exponential stability of the periodic solution is established. We illustrate our results with an example and numerical simulations.

Keywords: Nicholson type systems, delay differential systems, periodic solutions, exponential stability.

2020 Mathematics Subject Classification: 34K13, 34K60, 92D25.

1 Introduction

In recent years, the question of the existence of periodic solutions for Nicholson-type systems with periodic coefficients has received the attention of many researchers. This class of systems of differential equations with delays was introduced as a coupled patch population model for marine protected areas and B-cell chronic lymphocytic leukemia [7]. However, it has been pointed out that the new models applied to the fishery must consider nonlinear density-dependent mortality rates [6]. Consequently, research on Nicholson-type equations and systems with density-dependent mortality has developed rapidly. But despite that, few studies have considered periodic Nicholson models with density-dependent mortality and harvesting. The goal of this article is to investigate the existence and stability of positive periodic solutions for a m-dimensional Nicholson-type system with periodic coefficients, nonlinear mortality rates, and linear harvesting.

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1.1 The Nicholson models

In [16] Gurney, Blythe and Nisbet proposed a model to describe the behavior of a population of flies that had been studied in the 1950s by Nicholson [27]. The model corresponds to the following delayed differential equation

$$\dot{x}(t) = -mx(t) + bx(t-\tau)\exp\left\{-\gamma^{-1}x(t-\tau)\right\}, \tag{1.1}$$

where x is the density of the adult population, m is the per capita mortality rate, b the maximum birth rate, τ is the time to maturity and γ indicates where the unimodal function reaches its maximum. Equation (1.1) is known as the Nicholson model.

In [7] Berezansky, Idels and Troib studied the dynamics of metapopulation models with migration between two patches. Within the models studied, the authors considered a model of a marine population, with an age structure that inhabits two areas, one protected and the other for extraction. From this model, they obtained the system of differential equations with delay:

$$\begin{aligned} \dot{x_1}(t) &= -(m_1+d_1)x_1(t) + b_1x_1(t-\tau)\exp\left\{-\gamma_1^{-1}x_1(t-\tau)\right\} + d_2x_2(t) \\ \dot{x_2}(t) &= -(m_2+d_2+h)x_2(t) + b_2x_2(t-\tau)\exp\left\{-\gamma_2^{-1}x_2(t-\tau)\right\} + d_1x_1(t), \end{aligned} \tag{1.2}$$

where x_i corresponds to the densities of adult populations, m_i are the per capita mortality rates, d_i are the diffusion rates between patches, b_i are the maximum birth rates, γ_i indicates where the unimodal functions reaches its maximum, τ is the time to maturity, and h is the harvesting rate. Due to the presence of a nonlinear birth rate that considers delay, models similar to (1.2) are known as Nicholson-type systems.

The model (1.2) has been extended to the non-autonomous case to consider variations due to the passage of time, such as the seasons of the year, which has led to the study of periodic and almost periodic solutions, see [14,15,22,28,29,35].

Since the model (1.2) allows predicting the dynamics of an adult population, it is relevant to include some types of harvesting in them so that they can be applied in models of fishery or agricultural livestock production. Different authors have considered Nicholson-type equations and systems with linear harvesting [13,24,38] and nonlinear harvesting [1,4,5] among others.

Berezansky, Braverman, and Idels in [6] mention that for marine populations at low densities it is appropriate a linear model of density-dependent mortality and that new fishery models must consider nonlinear density-dependent mortality rates. Afterward, research on Nicholson-type equations and systems with density-dependent mortality has been developing rapidly, see [3,8,9,19,23,25,30,33]. However, the study of periodic Nicholson models with density-dependent nonlinear mortality and harvesting terms have not yet been sufficiently explored and this work aims to contribute in this direction.

1.2 Novelty of this work

We consider a Nicholson-type system with nonlinear density-dependent mortality, linear harvesting terms, and several concentrated delays of the form

$$x_i'(t) = -\frac{\delta_{ii}(t)x_i(t)}{c_{ii}(t)+x_i(t)} + \sum_{j=1}^{n} b_{ij}(t)r(x_i(t-\tau_{ij}(t))) + \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(t)x_j(t)}{c_{ij}(t)+x_j(t)} - h_i(t)x_i(t) \tag{1.3}$$

where $r(x) = x\exp(-x)$, and δ_{ij} , c_{ij}, b_{ij} , $\tau_{ij}, h_i : \mathbb{R} \to (0, +\infty)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, are bounded, continuous and ω–periodic functions.

Note that the above system includes the case where each patch considers a different Ricker-type function, namely $r_i(y_i) = y_i e^{-\gamma_i^{-1} y_i}$. In fact, in this case the system (1.3) is obtained by making the change of variable $y_i = \gamma_i x_i$.

Our objective is to apply topological degree and Lyapunov stability theory to the system (1.3) to determine the conditions that guarantee the existence and exponential stability of periodic solutions of the system.

1.3 Outline

Section 2 deals with fundamental preliminary aspects of this work, particularly the theory of differential equations with delay and a theorem of continuation of the topological degree; In addition, a result of the existence of solutions and a priori estimates are obtained. Section 3 establishes the main results of this work: Theorem 3.1 provides sufficient conditions for the existence of positive periodic solutions, while Theorems 3.3 and 3.5 prove the local asymptotic and exponential stability, respectively. Section 4 focuses on an example and its numerical simulations. Section 5 is dedicated to the conclusions and discussion of the results, particularly the possible extension of the present study to one involving nonlinear harvesting terms previously considered in population models, see [18, 34].

2 Preliminaries

2.1 Delay differential equations

Time delays occur naturally in many population dynamical models and their presence is due, among others, to factors like sexual maturity or gestation. Mathematical models with time-delays has a significant role in population dynamics, we refer the reader to [12, 26, 32, 36]. Delayed differential equations may exhibit more complex dynamics than ODE's because of the presence of delay may induce a Hopf bifurcation, periodic and oscillatory solutions or chaos, see [17, 21, 36].

We introduce some definitions and notation for delay differential equations. For $\overline{\tau} \geq 0$, we consider $\mathcal{C} = C([-\overline{\tau}, 0], \mathbb{R}^m)$ the Banach space with the norm $\|\varphi\|_{\overline{\tau}} = \sup_{-\overline{\tau} \leq \theta \leq 0} \|\varphi(\theta)\|$, where $\|\cdot\|$ is the maximum norm in $\mathbb{R}^m$. Any vector $\mathbf{v} \in \mathbb{R}^m$ is identified in $\mathcal{C}$ with the constant function $v(\theta) = \mathbf{v}$ for $\theta \in [-\overline{\tau}, 0]$. A general system of functional differential equations take the form

$$\dot{x}(t) = f(t, x_t), \tag{2.1}$$

where $f : \mathbb{R} \times \mathcal{C} \supset D \mapsto \mathbb{R}^m$ and x_t corresponds to the translation of a function $x(t)$ on the interval $[t - \overline{\tau}, t]$ to the interval $[-\overline{\tau}, 0]$, more precisely $x_t \in \mathcal{C}$ is given by $x_t(\theta) = x(t + \theta)$, $\theta \in [-\overline{\tau}, 0]$.

A function x is said to be a *solution* of system (2.1) on $[-\overline{\tau}, A)$ if there is $A > 0$ such that $x \in C([-\overline{\tau}, A), \mathbb{R}^m), (t, x_t) \in D$ and $x(t)$ satisfies (2.1) for $t \in [0, A)$. For given $\phi \in \mathcal{C}$, we say $x(t; 0, \phi)$ is a solution of system (2.1) *with initial value* ϕ at 0 if there is an $A > 0$ such that $x(t; 0, \phi)$ is a solution of equation (2.1) on $[-\overline{\tau}, A)$ and $x_0(t; 0, \phi) = \phi$. In addition, for a given continuous and bounded function $f \in C(\mathbb{R}, \mathbb{R})$ we will denote by f^+ and f^- respectively, the supremum and infimum of f over $\mathbb{R}$. Now, for system (1.3) we consider $\overline{\tau} := \max\{\tau_{ij}^+, 1 \leq i \leq m, 1 \leq j \leq n\}$.

Since nonnegative solutions are significant for population models, the following subsets of $\mathcal{C}$ are often introduced :

$$\mathcal{C}^+ := C([-\overline{\tau},0],\mathbb{R}^m_+), \quad \mathcal{C}_0 := \{\phi \in \mathcal{C}^+ \,:\, \phi_i(0) > 0,\, 1 \le i \le m\}.$$

Theorem 2.1. *The system* (1.3) *has a unique nonnegative solution defined over* $[-\overline{\tau},+\infty)$ *for each initial condition* $\phi \in \mathcal{C}^+$.

Proof. We will denote by $F_i(t,x(t),x(t-\tau_{i1}(t)),\dots,x(t-\tau_{ij}(t)))$ the right hand side of system (1.3) and $x(t) = (x_1(t),\dots,x_m(t))^T$, then (1.3) can be written as,

$$\dot{x}(t) = F(t,x(t),x(t-\tau_{11}(t)),\dots,x(t-\tau_{mn}(t))), \tag{2.2}$$

where $F : \mathbb{R}_+ \times (\mathbb{R}^m_+)^{mn+1} \to \mathbb{R}^m$. We denote F_x to the derivative of F respect to the state $x(t)$, consequently the map $F_x : \mathbb{R}_+ \times (\mathbb{R}^m_+)^{mn+1} \to M(\mathbb{R})_{m\times m}$ defined by

$$F_x = \begin{pmatrix} F_1/\partial x_1 & F_1/\partial x_2 & \dots & F_1/\partial x_m \\ F_2/\partial x_1 & F_2/\partial x_2 & \dots & F_2/\partial x_m \\ \vdots & \vdots & \dots & \vdots \\ F_m/\partial x_1 & F_m/\partial x_2 & \dots & F_m/\partial x_m \end{pmatrix}$$

is continuous over $\mathbb{R}_+ \times (\mathbb{R}^m_+)^{mn+1}$. Now, applying Theorems 3.1 and 3.2 of [36], it follows that the system (1.3) has a unique solution defined over a maximal interval, for each initial condition $\phi \in \mathcal{C}^+$. In order to show that $x(t;0,\phi)$ takes nonnegative values, we fix $i \in \{1,\dots,m\}$ and t in the maximal interval, in addition we assume that entries of the function F are nonnegative vectors while $x \in \mathbb{R}^m_+$ is such that $x_i = 0$, then

$$\begin{aligned} F_i(t,x,\cdot) &= -\frac{\delta_{ii}(t)x_i}{c_{ii}(t)+x_i} + \sum_{j=1}^{n} b_{ij}(t)r(\cdot) + \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(t)x_j}{c_{ij}(t)+x_j} - h_i(t)x_i \\ &= \sum_{j=1}^{n} b_{ij}(t)r(\cdot) + \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(t)x_j}{c_{ij}(t)+x_j} \ge 0. \end{aligned}$$

Consequently, each nonnegative initial condition ϕ has a corresponding solution $x(t;0,\phi)$ that takes nonnegative values for t in the maximal interval. Now we will prove that the solutions of (1.3), corresponding to nonnegative initial conditions, are defined for all $t \ge 0$. Otherwise, they would be defined over an interval $[-\overline{\tau},A)$, where $0 < A < \infty$. Since $x(t)$ is a solution of (1.3), it follows that $x_i(t)$ satisfies

$$\begin{aligned} x_i'(t) &= -\frac{\delta_{ii}(t)x_i(t)}{c_{ii}(t)+x_i(t)} + \sum_{j=1}^{n} b_{ij}(t)r(x_i(t-\tau_{ij}(t))) + \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(t)x_j(t)}{c_{ij}(t)+x_j(t)} - h_i(t)x_i(t) \\ &\le \sum_{j=1}^{n} b_{ij}(t)r(x_i(t-\tau_{ij}(t))) + \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(t)x_j(t)}{c_{ij}(t)+x_j(t)} \\ &\le \sum_{j=1}^{n} b^+_{ij}e^{-1} + \sum_{j=1,j\neq i}^{m} \delta^+_{ij}. \end{aligned}$$

Whence, integrating the above estimation we obtain

$$x_i(t) \le x_i(0) + \left(\sum_{j=1}^{n} b^+_{ij}e^{-1} + \sum_{j=1,j\neq i}^{m} \delta^+_{ij} \right) t, \quad 0 \le t < A.$$

This estimates ensure that $A = +\infty$, because if $A < +\infty$ then $|x(t)| \to \infty$ as $t \to A$, contradicting the estimates. □

2.2 Topological degree and periodic functions

We begin this subsection by recalling some definitions and notations that will be used in this work. The closure and the boundary of a subset A of a topological space will be denoted respectively by $\overline{A}$ and ∂A. Let

$$C_\omega := \{x(t) = (x_i(t)) \in C(\mathbb{R}, \mathbb{R}^m) : x(t+\omega) = x(t) \text{ for all } t \in \mathbb{R}\}$$

the Banach space of the continuous vector functions ω periodic with the norm

$$\|x\| = \max_{1\le i\le m} \left\{ \sup_{t\in[0,\omega]} \|x_i(t)\| \right\}.$$

It is useful consider the usual notation for the natural embedding $\mathbb{R}^m \to C_\omega$ given by $\mathbf{y} \to y$, where $y(t) = \mathbf{y}$ for $t \in \mathbb{R}$. Given a continuous function and ω periodic $f \in C(\mathbb{R}, \mathbb{R})$ notice that f^+ and f^- coincide, respectively, with the maximum and the minimum value of f over the interval $[0, \omega]$.

The existence of periodic solutions of the system (1.3) will be proved as a consequence of a general continuation theorem, see [2, Theorem 6.3], in our case we consider:

Lemma 2.2. *Assume there exists an open bounded $\Omega \subset C_\omega$ such that:*

i) The system

$$x'(t) = \lambda F(t, x(t), x(t-\tau_{11}(t)), \dots, x(t-\tau_{mn}(t))) \tag{2.3}$$

has no solutions on $\partial\Omega$ for $\lambda \in (0,1)$.

ii) $g(\mathbf{x}) \neq 0$ for $\mathbf{x} \in \partial\Omega \cap \mathbb{R}^m$, where $g = (g_i) : \mathbb{R}^m \to \mathbb{R}^m$ is given by

$$g_i(\mathbf{x}) = \frac{1}{\omega}\int_0^\omega \left(\frac{\delta_{ii}(t)x_i}{c_{ii}(t)+x_i} - \sum_{j=1}^{n} b_{ij}(t) r(x_i) - \sum_{j=1, j\neq i}^{m} \frac{\delta_{ij}(t)x_j}{c_{ij}(t)+x_j} + h_i(t)x_i \right) dt.$$

iii) $\deg_B(g, \Omega \cap \mathbb{R}^m, 0) \neq 0$.

Then there exist at least one solution of (1.3) in $\overline{\Omega}$.

To study conditions ii) and iii) is useful introduce additional notation, let $I^m = \Pi_{i=1}^m [a_i, b_i]$ be a bounded and closed subset of $\mathbb{R}^m$ and $\mathbf{x} = (x_i) \in \mathbb{R}^m$, for each $1 \le i \le m$ let us denote

$$I_i^- := \{\mathbf{x} \in I^m : x_i = a_i\}, \quad I_i^+ := \{\mathbf{x} \in I^m : x_i = b_i\},$$

the i-th opposite faces. Condition iii) of the lemma 2.2 will be obtained by the construction of an affine isomorphism homotopic to g combined with the homotopy invariance property of the Brouwer degree.

2.3 A priori bounds

To prove the existence of a periodic solution of (1.3) by using the theory of topological degree we need to find some a priori bounds for any ω-periodic solution of the system (2.3). Next, we will state some propositions related to upper and lower a priori bounds that will be useful when proving the existence of positive periodic solutions of (1.3). To obtain the existence of upper bounds for the solutions of the system (2.3) we consider the following assumption:

(H1) The coefficients of the system satisfy:

$$\min_{\xi\in[0,\omega]}\left(\delta_{ii}(\xi)-\frac{1}{e}\sum_{j=1}^{n}b_{ij}(\xi)-\sum_{j=1,j\neq i}^{m}\delta_{ij}(\xi)\right)>0,\quad i=1,\ldots,m.$$

Proposition 2.3. *If **(H1)** holds, then every non-negative ω-periodic solution of (2.3) is bounded above for any $\lambda\in(0,1)$.*

Proof. Let $(x_i(t))$ an ω-periodic solution of (2.3) and $x_i^+=R_i\geq x_j^+$, for $i\neq j$ let $\xi\in[0,\omega]$ such that $x_i^+=x_i(\xi)$, since $x_i'(\xi)=0$ it follows that

$$0=\lambda\left[-\frac{\delta_{ii}(\xi)x_i(\xi)}{c_{ii}(\xi)+x_i(\xi)}+\sum_{j=1}^{n}b_{ij}(\xi)r(x_i(\xi-\tau_{ij}(\xi)))+\sum_{j=1,j\neq i}^{m}\frac{\delta_{ij}(\xi)x_j(\xi)}{c_{ij}(\xi)+x_j(\xi)}-h_i(\xi)x_i(\xi)\right].$$

Now, combining the monotonicity of the map $u\mapsto\frac{\delta u}{c+u}$, the assumptions over the functions $b_{ij}(\cdot),\delta_{ij}(\cdot),c_{ij}(\cdot),h_i(\cdot)$ and, the fact that $r(u)\leq\frac{1}{e}$ for $u\in\mathbb{R}^+$ we obtain

$$0\geq\frac{\delta_{ii}(\xi)R}{c_{ii}(\xi)+R}-\frac{1}{e}\sum_{j=1}^{n}b_{ij}(\xi)-\sum_{j=1,j\neq i}^{m}\frac{\delta_{ij}(\xi)R}{c_{ij}(\xi)+R}.$$

Next, adding and subtracting the terms $\delta_{ii}(\xi)+\sum_{j=1,j\neq i}^{m}\delta_{ij}(\xi)$, we can assert that

$$0\geq\left(\delta_{ii}(\xi)-\frac{1}{e}\sum_{j=1}^{n}b_{ij}(\xi)-\sum_{j=1,j\neq i}^{m}\delta_{ij}(\xi)\right)-\delta_{ii}^{+}\left(1-\frac{R}{c_{ii}(\xi)+R}\right)+\sum_{j=1,j\neq i}^{m}\delta_{ij}^{-}\left(1-\frac{R}{c_{ij}(\xi)+R}\right).$$

The above inequality implies

$$0\geq\left(\delta_{ii}(\xi)-\frac{1}{e}\sum_{j=1}^{n}b_{ij}(\xi)-\sum_{j=1,j\neq i}^{m}\delta_{ij}(\xi)\right)-\delta_{ii}^{+}\left(1-\frac{R}{c_{ii}(\xi)+R}\right).\tag{2.4}$$

On the other hand, (**H1**) and the continuity of the coefficients imply that there is $\zeta>0$ such that

$$\min_{\xi\in[0,\omega]}\left(\delta_{ii}(\xi)-\frac{1}{e}\sum_{j=1}^{n}b_{ij}(\xi)-\sum_{j=1,j\neq i}^{m}\delta_{ij}(\xi)-\zeta\right)>0.\tag{2.5}$$

Note that $\lim_{R\to\infty}\left(1-\frac{R}{c_{ii}(\xi)+R}\right)=0$ uniformly on $\xi\in[0,\omega]$, so there exists $R\gg0$ such that

$$-\zeta\leq-\delta_{ii}^{+}\left(1-\frac{R}{c_{ii}(\xi)+R}\right)<0,\quad\xi\in[0,\omega].\tag{2.6}$$

Now, for $R\gg0$ taking the minimum in (2.4), by using the estimations (2.5) and (2.6) we obtain the contradiction

$$0\geq\min_{\xi\in[0,\omega]}\left[\delta_{ii}(\xi)-\frac{1}{e}\sum_{j=1}^{n}b_{ij}(\xi)-\sum_{j=1,j\neq i}^{m}\delta_{ij}(\xi)-\delta_{ii}^{+}\left(1-\frac{R}{c_{ii}(\xi)+R}\right)\right]>0.$$

Consequently there is a positive number R_0 such that

$$x_i(t)<R_0,\quad\text{for }t\in\mathbb{R}\text{ and }i=1,2,\ldots,m.\tag{2.7}$$

□

To study the a priori lower bounds for the solutions of the system (2.3) we will proceed in a similar way to the proof of the proposition 2.3, but this time the key hypothesis is:

(H2) For $i = 1, 2, \dots, m$ we have:

$$\max_{\eta\in[0,\omega]} \left(\frac{\delta_{ii}(\eta)}{c_{ii}(\eta)} - \sum_{j=1}^{n} b_{ij}(\eta) - \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(\eta)}{c_{ij}(\eta)} + h_i(\eta) \right) < 0.$$

Proposition 2.4. *If **(H1)** and **(H2)** hold, then every positive ω-periodic solution of* (2.3) *is bounded below by a positive constant for any $\lambda \in (0,1)$.*

Proof. Consider $\varepsilon = \min\{x_1^-, x_2^-, \dots, x_m^-\}$ and, without loss of generality, we suppose that $x_i(\eta) = \varepsilon$ for some $\eta \in [0,\omega]$, then we obtain $x_i'(\eta) = 0$ whence

$$0 = \frac{\delta_{ii}(\eta)x_i(\eta)}{c_{ii}(\eta)+x_i(\eta)} - \sum_{j=1}^{n} b_{ij}(\eta) r(x_i(\eta-\tau_{ij}(\eta))) - \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(\eta)x_j(\eta)}{c_{ij}(\eta)+x_j(\eta)} + h_i(\eta)x_i(\eta). \tag{2.8}$$

Since **(H1)** holds, proposition 2.3 implies that the periodic solutions of (2.3) are bounded from above by R_0.

We assume that $R_0 \geq 1$ and consider ρ_0 as the unique value in $(0,1]$ such that $r(\rho_0) = r(R_0)$. We may suppose that $\varepsilon \leq \rho_0$ since otherwise, we have trivially a lower bounds for the solutions of (2.3), from $\rho_0 < x_i(t)$, for $t \in \mathbb{R}$. Now, since $\varepsilon \leq \rho_0$, it follows

$$\varepsilon \leq x_i(\eta - \tau_{ij}(\eta)) \leq R_0, \quad \text{and} \quad r(x_i(\eta-\tau_{ij}(\eta))) \geq r(\varepsilon), \quad 1 \leq j \leq n.$$

By adding and subtracting the terms $\frac{\delta_{ii}(\eta)\varepsilon}{c_{ii}(\eta)}$, $\sum_{j=1}^{n} b_{ij}(\eta)\varepsilon$, and $\varepsilon \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(\eta)}{c_{ij}(\eta)}$ to equation (2.8), we obtain

$$\begin{aligned}
0 &= \frac{\delta_{ii}(\eta)\varepsilon}{c_{ii}(\eta)+\varepsilon} - \sum_{j=1}^{n} b_{ij}(\eta) r(x_i(\eta-\tau_{ij}(\eta))) - \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(\eta)x_j(\eta)}{c_{ij}(\eta)+x_j(\eta)} + h_i(\eta)\varepsilon \\
&\leq \frac{\delta_{ii}(\eta)\varepsilon}{c_{ii}(\eta)+\varepsilon} - \sum_{j=1}^{n} b_{ij}(\eta)\varepsilon e^{-\varepsilon} - \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(\eta)\varepsilon}{c_{ij}(\eta)+\varepsilon} + h_i(\eta)\varepsilon \\
&= \frac{\delta_{ii}(\eta)\varepsilon}{c_{ii}(\eta)} - \sum_{j=1}^{n} b_{ij}(\eta)\varepsilon - \varepsilon \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(\eta)}{c_{ij}(\eta)} + h_i(\eta)\varepsilon \\
&\quad - \delta_{ii}(\eta)\varepsilon \left(\frac{1}{c_{ii}(\eta)} - \frac{1}{c_{ii}(\eta)+\varepsilon} \right) + \sum_{j=1}^{n} b_{ij}(\eta)\varepsilon(1-e^{-\varepsilon}) \\
&\quad + \sum_{j=1,j\neq i}^{m} \delta_{ij}(\eta)\varepsilon \left(\frac{1}{c_{ij}(\eta)} - \frac{1}{c_{ij}(\eta)+\varepsilon} \right) \\
&\leq \frac{\delta_{ii}(\eta)\varepsilon}{c_{ii}(\eta)} - \sum_{j=1}^{n} b_{ij}(\eta)\varepsilon - \varepsilon \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(\eta)}{c_{ij}(\eta)} + h_i(\eta)\varepsilon \\
&\quad + \sum_{j=1}^{n} b_{ij}^{+}\varepsilon(1-e^{-\varepsilon}) + \sum_{j=1,j\neq i}^{m} \delta_{ij}^{+}\varepsilon \left(\frac{1}{c_{ij}(\eta)} - \frac{1}{c_{ij}(\eta)+\varepsilon} \right).
\end{aligned}$$

Since $\varepsilon > 0$, the above inequality is equivalent to

$$\begin{aligned}
0 &\leq \frac{\delta_{ii}(\eta)}{c_{ii}(\eta)} - \sum_{j=1}^{n} b_{ij}(\eta) - \sum_{j=1,j\neq i}^{m} \frac{\delta_{ij}(\eta)}{c_{ij}(\eta)} + h_i(\eta) \\
&\quad + \sum_{j=1}^{n} b_{ij}^{+}(1-e^{-\varepsilon}) + \sum_{j=1,j\neq i}^{m} \delta_{ij}^{+} \left(\frac{1}{c_{ij}(\eta)} - \frac{1}{c_{ij}(\eta)+\varepsilon} \right).
\end{aligned} \tag{2.9}$$

On the other hand, (**H2**) and the continuity of the coefficients imply that there is $\zeta > 0$ such that

$$\max_{\eta\in[0,\omega]}\left(\frac{\delta_{ii}(\eta)}{c_{ii}(\eta)}-\sum_{j=1}^{n}b_{ij}(\eta)-\sum_{j=1,j\neq i}^{m}\frac{\delta_{ij}(\eta)}{c_{ij}(\eta)}+h_i(\eta)+\zeta\right)<0.$$

Note that there exists $0<\varepsilon\ll 1$ such that

$$0<\sum_{j=1}^{n}b_{ij}^{+}(1-e^{-\varepsilon})+\sum_{j=1,j\neq i}^{m}\delta_{ij}^{+}\left(\frac{1}{c_{ij}(\eta)}-\frac{1}{c_{ij}(\eta)+\varepsilon}\right)\leq\zeta,\quad \eta\in[0,\omega].$$

Therefore, for $\varepsilon>0$ arbitrarily small values we obtain

$$\begin{aligned}0\leq \max_{\eta\in[0,\omega]}\Bigg[&\frac{\delta_{ii}(\eta)}{c_{ii}(\eta)}-\sum_{j=1}^{n}b_{ij}(\eta)-\sum_{j=1,j\neq i}^{m}\frac{\delta_{ij}(\eta)}{c_{ij}(\eta)}+h_i(\eta)\\&+\sum_{j=1}^{n}b_{ij}^{+}(1-e^{-\varepsilon})+\sum_{j=1,j\neq i}^{m}\delta_{ij}^{+}\left(\frac{1}{c_{ij}(\eta)}-\frac{1}{c_{ij}(\eta)+\varepsilon}\right)\Bigg]<0,\end{aligned}$$

a contradiction. Consequently there is a positive number ε_0 such that

$$\varepsilon_0<x_i(t)<R_0,\quad \text{for } t\in\mathbb{R} \text{ and } i=1,2,\dots,m.$$ □

3 Results

In this section, we address the problem of the existence and local stability of positive periodic solution for (1.3). We prove the existence of at least one periodic solution of the system (1.3) under assumptions **(H1)** and **(H2)** by using the degree topological theory.

Theorem 3.1. *Assume that* **(H1)** *and* **(H2)** *hold. Then system* (1.3) *has at least one ω-periodic positive solution.*

Proof. The proof of this result is supported by lemma 2.2. Since **(H1)** and **(H2)** hold, we apply propositions 2.3 and 2.4 to obtain lower and upper bounds for the periodic solutions of (2.3) for all $\lambda\in(0,1)$. Next define the set $\Omega\subset C_\omega$ as

$$\Omega:=\{(x_i(t))\in C_\omega:\varepsilon_0<x_i(t)<R_0,\ t\in[0,\omega],\ i=1,2,\dots,m\},\tag{3.1}$$

where the positive constants R_0 and ε_0 are, respectively, the upper and lower bounds given by propositions 2.3 and 2.4, we note that $\Omega\cap\mathbb{R}^m=(\varepsilon_0,R_0)^m$. As a consequence of these propositions, it follows that the system (2.3) has no solution in $\partial\Omega$ for any $\lambda\in(0,1)$. We will prove that there are positive constants ε and R such that $g(\mathbf{x})\neq 0$ for $\mathbf{x}\in\partial I$, where $I=[\varepsilon,R]^m$.

We recall that, for $i=1,2,\dots,m$ and $\mathbf{x}=(x_i)\in\mathbb{R}^m$, we have

$$g_i(\mathbf{x})=\frac{1}{\omega}\int_0^{\omega}\left(\frac{\delta_{ii}(t)x_i}{c_{ii}(t)+x_i}-\sum_{j=1}^{n}b_{ij}(t)r(x_i)-\sum_{j=1,j\neq i}^{m}\frac{\delta_{ij}(t)x_j}{c_{ij}(t)+x_j}+h_i(t)x_i\right)dt.\tag{3.2}$$

From the definition of $g_i(\mathbf{x})$, considering the notation $\mathbf{1}=(1,1,\dots,1)$, it follows that for $\mathbf{z}\in I_i^-$ we obtain

$$\begin{aligned}g_i(\mathbf{z})&=\frac{1}{\omega}\int_0^{\omega}\left(\frac{\delta_{ii}(t)\varepsilon}{c_{ii}(t)+\varepsilon}-\sum_{j=1}^{n}b_{ij}(t)r(\varepsilon)-\sum_{j=1,j\neq i}^{m}\frac{\delta_{ij}(t)z_j}{c_{ij}(t)+z_j}+h_i(t)\varepsilon\right)dt\\&\leq\frac{\varepsilon}{\omega}\int_0^{\omega}\left(\frac{\delta_{ii}(t)}{c_{ii}(t)+\varepsilon}-\sum_{j=1}^{n}b_{ij}(t)e^{-\varepsilon}-\sum_{j=1,j\neq i}^{m}\frac{\delta_{ij}(t)}{c_{ij}(t)+\varepsilon}+h_i(t)\right)dt\\&=g_i(\varepsilon\mathbf{1}).\end{aligned}$$

Analogously to the estimates made in the proof of proposition 2.4, we deduce that

$$g_i(\varepsilon\mathbf{1}) \leq \max_{\eta\in[0,\omega]}\left[\frac{\delta_{ii}(\eta)}{c_{ii}(\eta)} - \sum_{j=1}^{n} b_{ij}(\eta) - \sum_{j=1,j\neq i}^{m}\frac{\delta_{ij}(\eta)}{c_{ij}(\eta)} + h_i(\eta) + \sum_{j=1}^{n} b_{ij}^{+}(1-e^{-\varepsilon}) + \sum_{j=1,j\neq i}^{m}\delta_{ij}^{+}\left(\frac{1}{c_{ij}(\eta)} - \frac{1}{c_{ij}(\eta)+\varepsilon}\right)\right].$$

From **(H2)**, it follows that there exists some $0 < \varepsilon \ll 1$ such that

$$\max_{\eta\in[0,\omega]}\left[\frac{\delta_{ii}(\eta)}{c_{ii}(\eta)} - \sum_{j=1}^{n} b_{ij}(\eta) - \sum_{j=1,j\neq i}^{m}\frac{\delta_{ij}(\eta)}{c_{ij}(\eta)} + h_i(\eta) + \sum_{j=1}^{n} b_{ij}^{+}(1-e^{-\varepsilon}) + \sum_{j=1,j\neq i}^{m}\delta_{ij}^{+}\left(\frac{1}{c_{ij}(\eta)} - \frac{1}{c_{ij}(\eta)+\varepsilon}\right)\right] < 0.$$

Therefore, there exists a positive number ε_1 such that if $\varepsilon \leq \varepsilon_1$ we have

$$g_i(\mathbf{z}) \leq g_i(\varepsilon\mathbf{1}) < 0 \text{ for } \mathbf{z} \in I_i^{-}. \tag{3.3}$$

On the other hand, if $\mathbf{z} \in I_i^{+}$ then

$$\begin{aligned} g_i(\mathbf{z}) &= \frac{1}{\omega}\int_0^{\omega}\left(\frac{\delta_{ii}(t)R}{c_{ii}(t)+R} - \sum_{j=1,j\neq i}^{m}\frac{\delta_{ij}(t)z_j}{c_{ij}(t)+z_j} - \sum_{j=1}^{n} b_{ij}(t)r(R) + h_i(t)R\right)dt \\ &\geq \frac{1}{\omega}\int_0^{\omega}\left(\frac{\delta_{ii}(t)R}{c_{ii}(t)+R} - \sum_{j=1,j\neq i}^{m}\frac{\delta_{ij}(t)R}{c_{ij}(t)+R} - \sum_{j=1}^{n} b_{ij}(t)Re^{-R} + h_i(t)R\right)dt \\ &= g_i(R\mathbf{1}). \end{aligned}$$

Since $r(R) \leq \frac{1}{e}$ for $R \in \mathbb{R}^{+}$ and analogously to the estimates made in the proof of proposition 2.3, for $\mathbf{z} \in I_i^{+}$ we obtain

$$g_i(R\mathbf{1}) > \min_{\xi\in[0,\omega]}\left[\delta_{ii}(\xi) - \frac{1}{e}\sum_{j=1}^{n} b_{ij}(\xi) - \sum_{j=1,j\neq i}^{m}\delta_{ij}(\xi) - \delta_{ii}^{+}\left(1 - \frac{R}{c_{ii}(\xi)+R}\right)\right].$$

From **(H1)**, it follows that there exists some $R > R_0$ such that

$$\min_{\xi\in[0,\omega]}\left[\delta_{ii}(\xi) - \frac{1}{e}\sum_{j=1}^{n} b_{ij}(\xi) - \sum_{j=1,j\neq i}^{m}\delta_{ij}(\xi) - \delta_{ii}^{+}\left(1 - \frac{R}{c_{ii}(\xi)+R}\right)\right] > 0.$$

Hence there is $R_1 > 0$ such that if $R \geq R_1$, then

$$g_i(\mathbf{z}) \geq g_i(R\mathbf{1}) > 0 \quad \text{for } \mathbf{z} \in I_i^{+}. \tag{3.4}$$

We have proved that if $\varepsilon < \varepsilon_1$ and $R > R_1$, then $g(\mathbf{x}) \neq 0$ for $\mathbf{x} \in \partial I$, where $I = [\varepsilon, R]^m$. We claim that g is homotopic to an affine isomorphism. In fact we consider $\mathcal{A} : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\mathcal{A}(\mathbf{x}) = \mathbf{b} + M\mathbf{x},$$

where $\mathbf{b} \in \mathbb{R}^m$ and the diagonal matrix $M \in \mathbb{M}_{m\times m}$ are completely defined by the systems of linear equation

$$\begin{aligned} b_i + m_{ii}\varepsilon &= g_i(\varepsilon\mathbf{1}), \\ b_i + m_{ii}R &= g_i(R\mathbf{1}). \end{aligned}$$

It follows immediately that $m_{ii} = (g_i(R\mathbf{1}) - g_i(\varepsilon\mathbf{1}))/(R-\varepsilon) > 0$, and $b_i = g_i(\varepsilon\mathbf{1}) - m_{ii} < 0$. Furthermore, there is a unique vector $\bar{\mathbf{x}} = (\bar{x}_i)$ with $\bar{x}_i \in (\varepsilon, R)$ satisfying $b_i + m_{ii}\bar{x}_i = 0$, hence $\bar{\mathbf{x}}$ is the unique vector in the interior of I such that $\mathcal{A}(\bar{\mathbf{x}}) = \mathbf{0}$. Next we define the map $H : \mathbb{R}^m \times [0,1] \to \mathbb{R}^m$ given by

$$H(\mathbf{x},\sigma) = \sigma g(\mathbf{x}) + (1-\sigma)\mathcal{A}(\mathbf{x}),$$

which is a homotopy between $\mathcal{A}$ and g. Since $\operatorname{sign} g(I_i^+) = \operatorname{sign} \mathcal{A}(I_i^+)$ and $\operatorname{sign} g(I_i^-) = \operatorname{sign} \mathcal{A}(I_i^-)$ it follows that $H(\cdot,\sigma)$ does not vanish on ∂I for any $\sigma \in [0,1]$, and we conclude that g is homotopic to the affine isomorphism $\mathcal{A}$. The homotopy invariance property of Brouwer degree implies that

$$\deg_B(g, \Omega \cap \mathbb{R}^m, 0) = \deg_B(\mathcal{A}, \Omega \cap \mathbb{R}^m, 0),$$

and by the definition of Brouwer degree it follows that

$$\deg_B(\mathcal{A}, \Omega \cap \mathbb{R}^m, 0) = \operatorname{sign}\left(\det(D\mathcal{A}(\bar{\mathbf{x}}))\right) = \operatorname{sign}\left(\prod_{i=1}^m m_{ii}\right) = 1.$$

Finally we apply Lemma 2.2 to conclude that the system (1.3) has at least one solution $x(t) \in \overline{\Omega}$. □

Remark 3.2. Several types of delayed harvesting terms have been considered for the Nicholson scalar equation. If we modify the harvesting terms $h_i(t)x_i(t)$ in our model to delayed terms similar to those used in the work of Qiyuan Zhou in [38], then we obtain the system

$$\begin{aligned} x_i'(t) = &-\frac{\delta_{ii}(t)x_i(t)}{c_{ii}(t)+x_i(t)} + \sum_{j=1}^n b_{ij}(t) r(x_i(t-\tau_{ij}(t))) \\ &+ \sum_{j=1, j\neq i}^m \frac{\delta_{ij}(t)x_j(t)}{c_{ij}(t)+x_j(t)} - \sum_{j=1}^n h_{ij}(t)x_i(t-\tau_{ij}(t)). \end{aligned} \tag{3.5}$$

Then it is possible to obtain a result analogous to proposition 2.4 and theorem 3.1 considering **(H1)** and changing **(H2)** by:

(H2') There exists a positive upper bound R_0 for the solutions of system (3.5), such that for $i = 1,2,\ldots,m$ we have:

$$\max_{\eta\in[0,\omega]} \left(\frac{\delta_{ii}(\eta)}{c_{ii}(\eta)} - \sum_{j=1, j\neq i}^m \frac{\delta_{ij}(\eta)}{c_{ij}(\eta)} + R_0 \sum_{j=1}^n \left[h_{ij}(\eta) - b_{ij}(\eta) e^{-R_0} \right] \right) < 0.$$

Next, we will address the asymptotic and exponential stability of the system (1.3). As is common in the literature on Nicholson-type models, our results are obtained by constructing appropriate Lyapunov functions. We define the region of stability of the solutions of our system as the set

$$B = \{(x_i(t)) \in C(\mathbb{R}, \mathbb{R}^m) : 0 < x_i(t) < K_i,\ i = 1,2,\ldots,m\}. \tag{3.6}$$

To achieve our stability results, we assume the following:

(H3) The delays involve in the model (1.3) are continuously differentiable and satisfy:

$$\tau'_{ij}(t) \le \tau^*_{ij} < 1, \quad (i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}.$$

(H4) For $i = 1,2,\dots,m$ we have

$$\frac{\delta^-_{ii} c^-_{ii}}{(c^+_{ii} + K_i)^2} > \sum_{j=1, j\ne i}^{m} \frac{\delta^+_{ij} c^+_{ij}}{(c^-_{ij})^2} - h^-_i + \sum_{j=1}^{n} \frac{b^+_{ij}}{1-\tau^*_{ij}}.$$

Now we state and prove our first stability theorem.

Theorem 3.3. *If assumptions* **(H1)–(H4)** *hold, then there is a unique asymptotically stable ω-periodic solution of system* (1.3) *in B.*

Proof. Let $x(t) = (x_i(t))$ and $y(t) = (y_i(t))$ two solutions in B of system (1.3). We consider the functions:

$$V_i(t) = |y_i(t) - x_i(t)| + \sum_{j=1}^{n} \frac{b^+_{ij}}{1-\tau^*_{ij}} \int_{t-\tau_{ij}(t)}^{t} |y_i(s) - x_i(s)| ds, \quad i = 1,2,\dots,m.$$

Calculating the upper right Dini derivative of $V_i(t)$ along the solutions of (1.3), since $0 \le x_i(t), y_i(t) \le K_i$ and $|r'(x)| \le 1$ for $x \in [0,+\infty)$, then proceeding similarly to theorem 2 in [31] we have

$$\begin{aligned} D^+V_i(t) \le & -\frac{\delta_{ii}(t) c_{ii}(t) |y_i(t) - x_i(t)|}{(c_{ii}(t) + y_i(t))(c_{ii}(t) + x_i(t))} + \sum_{j=1, j\ne i}^{m} \frac{\delta_{ij}(t) c_{ij}(t) |y_j(t) - x_j(t)|}{(c_{ij}(t) + y_i(t))(c_{ij}(t) + x_i(t))} \\ & + \sum_{j=1}^{n} b_{ij}(t) |r(y_i(t - \tau_{ij}(t))) - r(x_i(t - \tau_{ij}(t)))| - h_i(t) |y_i(t) - x_i(t)| \\ & + \sum_{j=1}^{n} \frac{b^+_{ij}}{1-\tau^*_{ij}} |y_i(t) - x_i(t)| - \sum_{j=1}^{n} \frac{b^+_{ij}}{1-\tau^*_{ij}} |y_1(t - \tau_{ij}(t)) - x_i(t - \tau_{ij}(t))| (1 - \tau'_{ij}(t)). \end{aligned}$$

Notice that assumption (**H3**) implies that

$$1 < \frac{1 - \tau'_{ij}(t)}{1 - \tau^*_{ij}},$$

hence we obtain the following estimate

$$\begin{aligned} D^+V_i(t) \le & -\frac{\delta^-_{ii} c^-_{ii} |y_i(t) - x_i(t)|}{(c^+_{ii} + K_i)^2} + \sum_{j=1, j\ne i}^{m} \frac{\delta^+_{ij} c^+_{ij} |y_j(t) - x_j(t)|}{(c^-_{ij})^2} \\ & + \sum_{j=1}^{n} b^+_{ij} |y_1(t - \tau_{ij}(t)) - x_i(t - \tau_{ij}(t))| - h^-_i |y_i(t) - x_i(t)| \\ & + \sum_{j=1}^{n} b^+_{ij} \frac{|y_i(t) - x_i(t)|}{1-\tau^*_{ij}} - \sum_{j=1}^{n} b^+_{ij} |y_j(t - \tau_{ij}(t)) - x_i(t - \tau_{ij}(t))| \\ \le & \left(-\frac{\delta^-_{ii} c^-_{ii}}{(c^+_{ii} + K_i)^2} - h^-_i + \sum_{j=1}^{n} \frac{b^+_{ij}}{1-\tau^*_{ij}} \right) |y_i(t) - x_i(t)| \\ & + \sum_{j=1, j\ne i}^{m} \frac{\delta^+_{ij} c^+_{ij}}{(c^-_{ij})^2} |y_j(t) - x_j(t)|. \end{aligned}$$

Now, we define the Lyapunov functional $V(t) := \sum_{i=1}^{m} V_i(t)$, and by a straightforward computation of the corresponding sums it follows

$$D^+V(t) \leq \sum_{i=1}^{m} \left(-\frac{\delta_{ii}^- c_{ii}^-}{(c_{ii}^+ + K_i)^2} - h_i^- + \sum_{j=1}^{n} \frac{b_{ij}^+}{1-\tau_{ij}^*} + \sum_{j=1, j\neq i}^{m} \frac{\delta_{ij}^+ c_{ij}^+}{(c_{ij}^-)^2} \right) |y_i(t) - x_i(t)|.$$

Hypothesis **(H4)** ensure the existence of a positive constant μ such that

$$D^+V(t) \leq -\mu \sum_{i=1}^{m} |y_i(t) - x_i(t)|, \quad t \geq 0,$$

then we get

$$V(t) + \mu \int_0^t \sum_{i=1}^{m} |y_i(s) - x_i(s)| ds \leq V(0) < +\infty, \quad t \geq 0,$$

and

$$\int_0^t \sum_{i=1}^{m} |y_i(s) - x_i(s)| ds \leq \frac{V(0)}{\mu} < +\infty, \quad t \geq 0. \tag{3.7}$$

It follows that $H_i(s) := |y_i(s) - x_i(s)| \in L^1([0, +\infty]), 1 \leq i \leq m$ and, since $H_i(t)$ are uniformly continuous in $[0, +\infty)$, we can apply the Barbalat's Lemma [20, Lemma 8.2] to conclude:

$$\lim_{t\to+\infty} \sum_{i=1}^{m} |y_i(t) - x_i(t)| = 0.$$

Therefore, all solution of the system (1.3) in B converge to an ω-periodic solution, hence there is a unique periodic solution of (1.3) in B. □

Remark 3.4. Note that in the proof of theorem (3.3), we use arguments similar to those presented in the proof of theorem (4.5) of [37]. Both results are supported by considering the derivative of Dini and the definition of an adequate Lyapunov functional, in addition to the uniform continuity of the integrands of (3.7) of our proof, equivalent to the integrand given in (4.13) of the proof used in [37]. These are key aspects in the literature on stability in Nicholson-type models, see for instance [13] and references therein.

In order to state and prove our second stability theorem we define, for $i = 1, \ldots, m$, the continuous functions $G_i : \mathbb{R} \to \mathbb{R}$ given by

$$G_i(\varepsilon) = \frac{\delta_{ii}^- c_{ii}^-}{(c_{ii}^+ + K_i)^2} - \varepsilon - \sum_{j=1, j\neq i}^{m} \frac{\delta_{ij}^+ c_{ij}^+}{(c_{ij}^-)^2} + h_i^- - \sum_{j=1}^{n} \frac{b_{ij}^+}{1-\tau_{ij}^*} e^{\varepsilon \tau_{ij}^+}. \tag{3.8}$$

Notice that hypothesis **(H4)** ensures that $G_i(0) > 0$ for each $i = 1, \ldots, m$, furthermore, the continuity of G_i guarantees the existence of positive constants r_i, such that

$$G_i(\varepsilon) > 0, \quad \text{for } 0 \leq \varepsilon \leq r_i, \tag{3.9}$$

and we define $\lambda_0 := \min_{1\leq i\leq m}\{r_i\}$, so $G_i(\lambda_0) > 0$ for $i = 1, \ldots, m$.

Theorem 3.5. *If the hypotheses **(H1)–(H4)** hold, then all solution of system (1.3) in B converge exponentially to the ω-periodic solution.*

Proof. We consider $x(t) = (x_i(t))$ and $y(t) = (y_i(t))$ two arbitrary solutions in B of system (1.3) and we define the functions:

$$W_i(t) \quad = \quad |y_i(t) - x_i(t)|e^{\lambda t} + \sum_{j=1}^{n} b_{ij}^{+} \frac{1}{1-\tau_{ij}^{*}} \int_{t-\tau_{ij}(t)}^{t} |y_i(s) - x_i(s)|e^{\lambda(s+\tau_{ij}^{+})} ds\,.$$

Calculating the upper right Dini derivative of $W_i(t)$ along the solutions of model (1.3) we have

$$\begin{aligned} D^{+}W_i(t) = {} & |y_i(t) - x_i(t)|\lambda e^{\lambda t} + [y_i'(t) - x_i'(t)] \times \operatorname{sgn}\{y_i(t) - x_i(t)\} \times e^{\lambda t} \\ & + \sum_{j=1}^{n} b_{ij}^{+} \frac{1}{1-\tau_{ij}^{*}} |y_i(t) - x_i(t)|e^{\lambda(t+\tau_{ij}^{+})} \\ & - \sum_{j=1}^{n} b_{ij}^{+} \frac{1}{1-\tau_{ij}^{*}} |y_i(t-\tau_{ij}(t)) - x_i(t-\tau_{ij}(t))|(1-\tau_{ij}'(t))e^{\lambda(t-\tau_{ij}(t)+\tau_{ij}^{+})}\,. \end{aligned}$$

Replacing x_i and y_i given in the system, applying triangular inequality, considering (H3), $0 \le x_i(t), y_i(t) \le K_i$, $|r'(x)| \le 1$ for $x \in [0, +\infty)$ and grouping we obtain

$$\begin{aligned} D^{+}W_i(t) \le e^{\lambda t} \Bigg[& |y_i(t) - x_i(t)|\lambda - \frac{\delta_{ii}^{-} c_{ii}^{-} |y_i(t) - x_i(t)|}{(c_{ii}^{+} + K_i)^2} + \sum_{j=1, j\neq i}^{m} \frac{\delta_{ij}^{+} c_{ij}^{+} |y_j(t) - x_j(t)|}{(c_{ij}^{-})^2} \\ & + \sum_{j=1}^{n} b_{ij}^{+} |r(y_i(t-\tau_{ij}(t))) - r(x_i(t-\tau_{ij}(t)))| - h_i^{-} |y_i(t) - x_i(t)| \\ & + \sum_{j=1}^{n} b_{ij}^{+} \frac{|y_i(t) - x_i(t)|}{1-\tau_{ij}^{*}} e^{\lambda \tau_{ij}^{+}} - \sum_{j=1}^{n} b_{ij}^{+} |y_i(t-\tau_{ij}(t)) - x_i(t-\tau_{ij}(t)) \Bigg] \end{aligned}$$

$$\begin{aligned} D^{+}W_i(t) \le e^{\lambda t} \Bigg[& |y_i(t) - x_i(t)|\lambda - \frac{\delta_{ii}^{-} c_{ii}^{-} |y_i(t) - x_i(t)|}{(c_{ii}^{+} + K_i)^2} + \sum_{j=1, j\neq i}^{m} \frac{\delta_{ij}^{+} c_{ij}^{+} |y_j(t) - x_j(t)|}{(c_{ij}^{-})^2} \\ & + \sum_{j=1}^{n} b_{ij}^{+} |y_i(t-\tau_{ij}(t)) - x_i(t-\tau_{ij}(t))| - h_i^{-} |y_i(t) - x_i(t)| \\ & + \sum_{j=1}^{n} b_{ij}^{+} \frac{|y_i(t) - x_1(t)|}{1-\tau_{ij}^{*}} e^{\lambda \tau_{ij}^{+}} - \sum_{j=1}^{n} b_{ij}^{+} |y_i(t-\tau_{ij}(t)) - x_i(t-\tau_{ij}(t))| \Bigg] \\ \le {} & -e^{\lambda t} \left(-\lambda + \frac{\delta_{ii}^{-} c_{ii}^{-}}{(c_{ii}^{+} + K_i)^2} + h_i^{-} - \sum_{j=1}^{n} \frac{b_{ij}^{+} e^{\lambda \tau_{ij}^{+}}}{1-\tau_{ij}^{*}} \right) |y_i(t) - x_i(t)| \\ & + e^{\lambda t} \sum_{j=1, j\neq i}^{m} \frac{\delta_{ij}^{+} c_{ij}^{+}}{(c_{ij}^{-})^2} |y_j(t) - x_j(t)| \\ = {} & -e^{\lambda t} \left(\left[-\lambda + \frac{\delta_{ii}^{-} c_{ii}^{-}}{(c_{ii}^{+} + K_i)^2} + h_i^{-} - \sum_{j=1}^{n} \frac{b_{ij}^{+} e^{\lambda \tau_{ij}^{+}}}{1-\tau_{ij}^{*}} \right] |y_i(t) - x_i(t)| \right. \\ & \left. - \sum_{j=1, j\neq i}^{m} \frac{\delta_{ij}^{+} c_{ij}^{+}}{(c_{ij}^{-})^2} |y_j(t) - x_j(t)| \right). \end{aligned}$$

Extending the sum for $i = 1$ to m and grouping terms we obtain that the Lyapunov functional $W(t) = \sum_{i=1}^m W_i(t)$ satisfies

$$D^+W(t) \le -e^{-\lambda t} \sum_{i=1}^m G_i(\lambda)|y_i(t) - x_i(t)|.$$

We fix $\lambda = \lambda_0 = \min_{1\le i\le m}\{r_i\}$, since (3.8) and (3.9) hold we deduce that

$$D^+W(t) \le -e^{-\lambda_0 t} \sum_{i=1}^m G_i(\lambda_0)|y_i(t) - x_i(t)| < 0, \quad \forall t \in (0,\infty).$$

It follows that $W(t)$ is decreasing for all $t > 0$ along the solutions of system (1.3), consequently we have

$$\sum_{i=1}^m |y_i(t) - x_i(t)|e^{\lambda_0 t} \le W(t) \le W(0),$$

whence

$$\sum_{i=1}^m |y_i(t) - x_i(t)| \le W(t)e^{-\lambda_0 t} < W(0)e^{-\lambda_0 t},$$

and the exponential convergence it is obtained for solutions of (1.3) in B. □

4 Examples

In this section we show an example of the asymptotic stability of the solution and include numerical simulations performed in R software using the library *PBSddesolve*, see for instance [11]. In this example x_i is the density of biomass in patch i, $s(t) = \sin(2\pi t/365)$, $c(t) = \cos(2\pi t/365)$, and $i \in \{1,2,3\}$.

Example 4.1. We consider the system of differential equations with delay,

$$\begin{aligned}
x_1'(t) = &-\frac{(6+0.5c(t))x_1(t)}{2+x_1(t)} + 3(1+0.5s(t))r(x_1(t-60)) \\
&+ \left(\frac{(1+0.125c(t))x_2(t)}{5+x_2(t)} + \frac{1+0.125c(t))x_3(t)}{5+x_3(t)}\right) - 0.1x_1(t), \\
x_2'(t) = &-\frac{(4+0.5c(t))x_2(t)}{1.5+x_2(t)} + 3(1+0.5s(t))r(x_2(t-60)) \\
&+ \left(\frac{(1.5+0.125c(t))x_1(t)}{35+x_1(t)} + \frac{0.75+0.0625c(t))x_2(t)}{35+x_2(t)}\right), \\
x_3'(t) = &-\frac{(5+0.5c(t))x_3(t)}{1+x_3(t)} + 3(1+0.5s(t))r(x_3(t-60)) \\
&+ \left(\frac{(1.5+0.125c(t))x_1(t)}{12+x_1(t)} + \frac{(0.75+0.0625c(t))x_2(t)}{12+x_2(t)}\right) - 0.2x_3(t).
\end{aligned} \tag{4.1}$$

Hypotheses **(H1)–(H4)** are verified where $K_1 < 1.087$, $K_2 < 1.2814$, $K_3 < 1.1086$. The numerical simulations are presented in Figure 4.1.

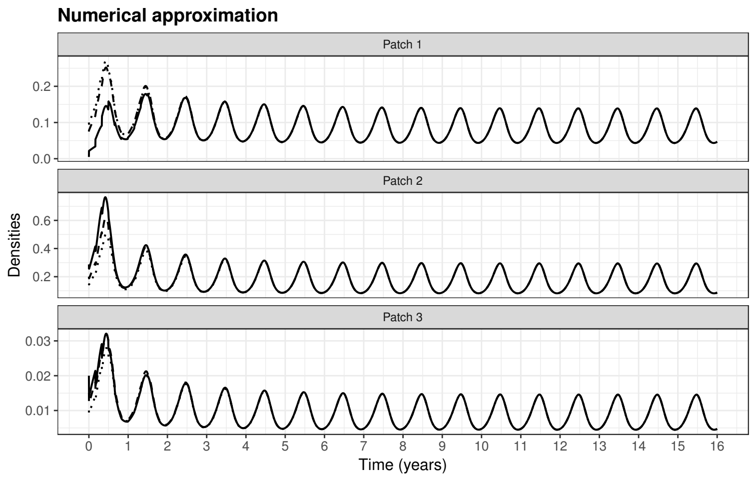

Figure 4.1: Numerical simulation of (4.1) for sixteen years. Initial conditions: $(x_1(\theta), x_2(\theta), x_3(\theta)) \equiv (0.05, 0.287, 0.02), \theta \in [-60, 0]$ (solid curve), $(x_1(\theta), x_2(\theta), x_3(\theta)) \equiv (0.075, 0.2, 0.015), \theta \in [-60, 0]$ (dashed curve), $(x_1(\theta), x_2(\theta), x_3(\theta)) \equiv (0.1, 0.15, 0.01), \theta \in [-60, 0]$ (dotted curve).

5 Conclusion and further work

A Nicholson-type system with nonlinear density-dependent mortality and linear harvesting has been studied in this paper. Based on the theory of topological degree, has been obtained sufficient conditions for the existence of a positive periodic solution of the model. In addition, by using the Lyapunov–Krasovskii functional method, the uniqueness, stability, and exponential stability of the Nicholson-type system were addressed. Numerical simulations were performed based on an example to illustrate the results obtained.

Among the projections of this work, we will focus on the possible extension of the present study to one involving nonlinear harvesting terms. We recall that in the works [1,4,5] advances in this direction have been developed. However, from the point of view of applications, it seems more realistic to consider the harvesting terms, proposed by Clark and Mangel in [10], of the form

$$h(E, x) = \frac{qEx}{cE + \ell x},$$

where q is the catch coefficient, E is the external effort dedicated to the harvest, c and ℓ are constants. Population models with terms of this type have been studied in [18,34]. Thus, a new version of the system (1.3) naturally arises, this time with these nonlinear harvesting terms as a new research goal. We anticipate that the main aspects to take into account when applying the methods presented in this work to these nonlinear terms is to search for alternative hypotheses to **(H2)** and **(H4)**, which can be deduced after a careful reading of this work.

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Qualitative analysis of a mechanical system of coupled nonlinear oscillators

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Abstract. In this paper we investigate nonlinear systems of second order ODEs describing the dynamics of two coupled nonlinear oscillators of a mechanical system. We obtain, under certain assumptions, some stability results for the null solution. Also, we show that in the presence of a time-dependent external force, every solution starting from sufficiently small initial data and its derivative are bounded or go to zero as the time tends to $+\infty$, provided that suitable conditions are satisfied. Our theoretical results are illustrated with numerical simulations.

Keywords: coupled oscillators, uniform stability, asymptotic stability, uniform asymptotic stability.

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1. Introduction

Consider a mechanical system of coupled nonlinear oscillators, as shown in Figure 1.1. Specifically, the block of mass m_1 is anchored to a fixed horizontal wall and the block of mass m_2 by springs and dampers, and the block of mass m_2 is also attached to the wall by a pair of springs and dampers. Suppose that the stiffnesses and the dampings are represented by the functions $k_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $d_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \{1, 2, 3\}$, and $\widehat{g}_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, 2\}$, denote external forces acting on the blocks. One may also consider an external force $\widehat{f}(t)$ acting on the block of mass m_1, but for the moment, we restrict our attention to the case $\widehat{f} \equiv 0$. We assume that when the two blocks are in their equilibrium positions, the springs and the dampers are also in their equilibrium positions. Let $x(t)$ and $y(t)$ be the vertical displacements of the blocks from their equilibrium positions.

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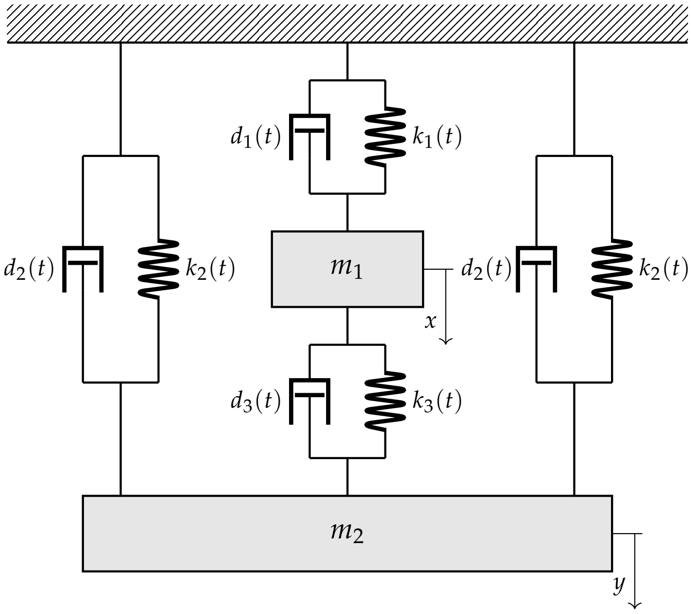


Figure 1.1: A mechanical system of coupled nonlinear oscillators

Then the system of ODEs describing the motion is (see, e.g., [27])

$$\begin{cases} m_1\ddot{x} + k_1(t)x + 2d_1(t)\dot{x} - k_3(t)(y-x) - 2d_3(t)(\dot{y}-\dot{x}) = \widehat{g}_1(t,x,y), \\ m_2\ddot{y} + 2k_2(t)y + 4d_2(t)\dot{y} + k_3(t)(y-x) + 2d_3(t)(\dot{y}-\dot{x}) = \widehat{g}_2(t,x,y), \end{cases}$$

or

$$\begin{cases} \ddot{x} + 2f_1(t)\dot{x} - f_3(t)\dot{y} + \beta(t)x - \gamma_1(t)y + g_1(t,x,y) = 0, \\ \ddot{y} + 2f_2(t)\dot{y} - f_4(t)\dot{x} - \gamma_2(t)x + \delta(t)y + g_2(t,x,y) = 0, \end{cases} \tag{1.1}$$

where

$$\begin{aligned} f_1(t) &:= \frac{1}{m_1}(d_1(t)+d_3(t)), & f_2(t) &:= \frac{1}{m_2}(2d_2(t)+d_3(t)), \\ f_3(t) &:= \frac{2}{m_1}d_3(t), & f_4(t) &:= \frac{2}{m_2}d_3(t), \\ \beta(t) &:= \frac{1}{m_1}(k_1(t)+k_3(t)), & \delta(t) &:= \frac{1}{m_2}(2k_2(t)+k_3(t)), \\ \gamma_1(t) &:= \frac{1}{m_1}k_3(t), & \gamma_2(t) &:= \frac{1}{m_2}k_3(t), \\ g_1(t,x,y) &:= -\frac{1}{m_1}\widehat{g}_1(t,x,y), & g_2(t,x,y) &:= -\frac{1}{m_2}\widehat{g}_2(t,x,y). \end{aligned}$$

The general case of a single 1-D damped nonlinear oscillator is described by the following equation which is well-known in the literature

$$\ddot{x} + 2f^*(t)\dot{x} + \beta^*(t)x + g^*(t,x) = 0, \quad t \in \mathbb{R}_+. \tag{1.2}$$

T. A. Burton and T. Furumochi [2] introduced a new method, based on the Schauder fixed point theorem, to study the stability of the null solution of Eq. (1.2) in the case $\beta^*(t) = 1$. In [14] we reported new stability results for the same equation. Our approach was based on elementary arguments only, involving in particular some Bernoulli type differential inequalities. In [15] we considered Eq. (1.2) under more general assumptions, which required more sophisticated arguments. For other investigations regarding the asymptotic stability of the equilibrium of a single damped nonlinear oscillator, we refer the reader to [7,8,10,11,24], and the references therein.

In the present paper, in Section 2 we will study the stability of the null solution of system (1.1), by two approaches, based on classical differential inequalities and on Lyapunov's method. For other results regarding the asymptotic stability of the equilibria of coupled damped nonlinear oscillators, we refer the reader to [9, 16, 17, 20–23, 25], and the references therein. For fundamental concepts and results in stability theory we refer the reader to [1,3,5,6,13,19].

In Section 3 we will consider that the block of mass m_1 is subject to the action of a time dependent external force $\widehat{f} : \mathbb{R}_+ \to \mathbb{R}$. In this case, the system of ODEs describing the dynamics of the mechanical system is

$$\begin{cases} \ddot{x} + 2f_1(t)\dot{x} - f_3(t)\dot{y} + \beta(t)x - \gamma_1(t)y - f(t) + g_1(t,x,y) = 0, \\ \ddot{y} + 2f_2(t)\dot{y} - f_4(t)\dot{x} - \gamma_2(t)x + \delta(t)y + g_2(t,x,y) = 0, \end{cases} \tag{1.3}$$

with the same functions as before, and $f(t) := \frac{1}{m_1}\widehat{f}(t)$, and we will derive certain qualitative properties of the solutions of system (1.3) with initial data small enough.

The model in Figure 1.1 could be used, e.g., to describe the dynamics in vertical direction of vibration reduction systems for horizontal cranes with loadings suspended in two sides [12, 28]. For other models of coupled oscillators or for models from electric circuit theory, structural dynamics, described by systems of type (1.1) or (1.3), we refer the reader to the monographs [4,18,26].

2. A stability result for the system (1.1)

In this section we shall use the following hypotheses.

(H1) $f_i \in C^1(\mathbb{R}_+), f_j \in C(\mathbb{R}_+)$, $f_i(t) \geq 0$, $f_j(t) \geq 0$, $\forall t \in \mathbb{R}_+$, and $\int_0^{+\infty} f_j(t)\mathrm{d}t < +\infty$, $\forall i \in \{1,2\}$, $\forall j \in \{3,4\}$;

(H2) there exist constants h, K_1, $K_2 \geq 0$ such that

$$\left|\dot{f}_i(t) + f_i^2(t)\right| \leq K_i\widetilde{f}(t), \ \forall t \in [h,+\infty), \ \forall i \in \{1,2\},$$

where $\widetilde{f}(t) := \min\{f_1(t), f_2(t)\}$, $\forall t \in \mathbb{R}_+$;

(H3) $\int_0^{+\infty} \widetilde{f}(t)\mathrm{d}t = +\infty$.

(H4) β, $\delta \in C^1(\mathbb{R}_+)$, β, δ are decreasing and

$$\beta(t) \geq \beta_0 > 0, \ \delta(t) \geq \delta_0 > 0, \quad \forall t \in \mathbb{R}_+,$$

where β_0, δ_0 are constants such that

$$\frac{K_1}{\sqrt{\beta_0}} + \frac{K_2}{\sqrt{\delta_0}} < 1;$$

(H5) $\gamma_i \in C(\mathbb{R}_+)$, $\gamma_i(t) \geq 0$, $\forall t \in \mathbb{R}_+$, and $\int_0^{+\infty} \gamma_i(t)\mathrm{d}t < +\infty$, $\forall i \in \{1,2\}$;

(H6) $g_i = g_i(t,x,y) \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$, g_i are locally Lipschitzian with respect to x, y, $i \in \{1,2\}$, and fulfill the relations

$$|g_1(t,x,y)| \leq r_1(t)\mathrm{O}(|x|), \quad \forall t \in \mathbb{R}_+, \ \forall y \in \mathbb{R}, \tag{2.1}$$

$$|g_2(t,x,y)| \leq r_2(t)\mathrm{O}(|y|), \quad \forall t \in \mathbb{R}_+, \ \forall x \in \mathbb{R}, \tag{2.2}$$

where $r_i \in C(\mathbb{R}_+)$, $r_i(t) \geq 0$, $\forall t \in \mathbb{R}_+$, $\int_0^{+\infty} r_i(t)\mathrm{d}t < +\infty$, $\forall i \in \{1,2\}$, and $\mathrm{O}(|x|)$ denotes the big-O Landau symbol as $x \to 0$ (similarly for $\mathrm{O}(|y|)$);

(H7) There is a $p > 0$, such that $f_i(t) \geq p$, $\forall t \geq 0$, $\forall i \in \{1,2\}$.

Remark 2.1. If (H1) and (H2) hold, then f_i, $\dot{f}_i$ are bounded, $i \in \{1,2\}$. Indeed, by (H2) we see that

$$(t \geq h, \ f_i(t) > K_i) \Longrightarrow \dot{f}_i(t) < 0.$$

This, combined with (H1), implies

$$f_i(t) \leq M_i := \max\{f_i(h), K_i\}, \quad \forall t \geq h.$$

So, using again (H2), we obtain

$$\left|\dot{f}_i(t)\right| \leq 2M_i^2, \quad \forall t \geq h.$$

This concludes the proof, since, by (H1), f_i, $\dot{f}_i \in C[0,h]$, $i \in \{1,2\}$.

Remark 2.2. Since we are going to discuss the stability of the null solution of system (1.1) and the large-time behavior of the solutions to (1.3) starting from small initial data, we can replace the inequalities (2.1) and (2.2) by

$$|g_1(t,x,y)| \leq r_1(t)|x|, \quad |g_2(t,x,y)| \leq r_2(t)|y|, \quad \forall t \in \mathbb{R}_+, \ \forall x, \ y \in \mathbb{R}, \tag{2.3}$$

possibly with $M_i r_i(t)$ instead of $r_i(t)$, where $M_i > 0$, and some functions $\widetilde{g}_i$ instead of g_i, $\forall i \in \{1,2\}$.

Indeed, from (2.1) there exist M_1, $a_1 > 0$, such that

$$|g_1(t,x,y)| \leq r_1(t)M_1|x|, \quad \text{if } |x| < a_1.$$

If we define the function $\widetilde{g}_1 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$\widetilde{g}_1(t,x,y) := \begin{cases} g_1(t,a_1,y), \text{ if } x \geq a_1, \\ g_1(t,x,y), \text{ if } |x| < a_1, \\ g_1(t,-a_1,y), \text{ if } x \leq -a_1, \end{cases}$$

for all $t \geq 0$, $y \in \mathbb{R}$, then

$$|\widetilde{g}_1(t,x,y)| \leq r_1(t)M_1|x|, \quad \forall (t,x,y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R},$$

$\widetilde{g}_1 \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$, and $\widetilde{g}_1$ is locally Lipschitzian in x, y. Similar reasonings work for the functions g_2 and r_2, possibly with another constant a_2.

2.1. A stability result via differential inequalities

We can state and prove the following stability result.

Theorem 2.3.

a) Suppose that the hypotheses (H1), (H2), (H4)–(H6) are satisfied. Then the null solution of the system (1.1) *is uniformly stable.*

b) If the hypotheses (H1)–(H6) are fulfilled, then the null solution of (1.1) *is asymptotically stable.*

c) If the hypotheses (H1), (H2), (H4)–(H7) are fulfilled, then the null solution of (1.1) *is uniformly asymptotically stable.*

Proof. By using the following transformation (inspired from [2])

$$\begin{cases} \dot{x} = u - f_1(t)x \\ \dot{u} = \left[\dot{f}_1(t) + f_1^2(t) - \beta(t)\right]x - f_1(t)u + [\gamma_1(t) - f_2(t)f_3(t)]y + f_3(t)v - g_1(t,x,y) \\ \dot{y} = v - f_2(t)y \\ \dot{v} = [\gamma_2(t) - f_1(t)f_4(t)]x + f_4(t)u + \left[\dot{f}_2(t) + f_2^2(t) - \delta(t)\right]y - f_2(t)v - g_2(t,x,y) \end{cases} \tag{2.4}$$

the system (1.1) becomes

$$\dot{z} = A(t)z + B(t)z + F(t,z), \tag{2.5}$$

where

$$z = \begin{pmatrix} x \\ u \\ y \\ v \end{pmatrix}, \quad A(t) = \begin{pmatrix} -f_1(t) & 1 & 0 & 0 \\ -\beta(t) & -f_1(t) & \gamma_1(t) & 0 \\ 0 & 0 & -f_2(t) & 1 \\ \gamma_2(t) & 0 & -\delta(t) & -f_2(t) \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dot{f}_1(t) + f_1^2(t) & 0 & -f_2(t)f_3(t) & f_3(t) \\ 0 & 0 & 0 & 0 \\ -f_1(t)f_4(t) & f_4(t) & \dot{f}_2(t) + f_2^2(t) & 0 \end{pmatrix}, \quad F(t,z) = \begin{pmatrix} 0 \\ -g_1(t,x,y) \\ 0 \\ -g_2(t,x,y) \end{pmatrix}.$$

Using the boundedness of the functions f_i, $\dot{f}_i$, f_j, β, γ_i, δ, r_i, $\forall i \in \{1,2\}$, $\forall j \in \{3,4\}$, we easily deduce that our stability question of the null solution of the system (1.1) reduces to the stability of the null solution $z(t) = 0$ of the system (2.5) .

Let $t_0 \geq 0$ and

$$Z(t,t_0) = \left(a_{ij}(t,t_0)\right)_{i,j\in\overline{1,4}}, \ t \geq t_0,$$

be the fundamental matrix of the system

$$\dot{z} = A(t)z, \tag{2.6}$$

which equals the identity matrix for $t = t_0$. Then we deduce

$$\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0) + \delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0) \leq \beta(t_0)e^{\int_{t_0}^t \left[-2\widetilde{f}(u) + \frac{\gamma(u)}{\sqrt{\zeta(u)}}\right]du}, \tag{2.7}$$

$$\beta(t)a_{12}^2(t,t_0) + a_{22}^2(t,t_0) + \delta(t)a_{32}^2(t,t_0) + a_{42}^2(t,t_0) \leq e^{\int_{t_0}^t \left[-2\widetilde{f}(u) + \frac{\gamma(u)}{\sqrt{\zeta(u)}}\right]du}, \tag{2.8}$$

$$\beta(t)a_{13}^2(t,t_0) + a_{23}^2(t,t_0) + \delta(t)a_{33}^2(t,t_0) + a_{43}^2(t,t_0) \leq \delta(t_0)e^{\int_{t_0}^t \left[-2\widetilde{f}(u) + \frac{\gamma(u)}{\sqrt{\zeta(u)}}\right]du}, \tag{2.9}$$

$$\beta(t)a_{14}^2(t,t_0) + a_{24}^2(t,t_0) + \delta(t)a_{34}^2(t,t_0) + a_{44}^2(t,t_0) \leq e^{\int_{t_0}^t \left[-2\widetilde{f}(u) + \frac{\gamma(u)}{\sqrt{\zeta(u)}}\right]du}, \tag{2.10}$$

for all $t \geq t_0$, where $\gamma(t) := \max\{\gamma_1(t), \gamma_2(t)\}$, $\zeta(t) := \min\{\beta(t), \delta(t)\}$, $\forall t \in \mathbb{R}_+$.

Indeed, from (2.6) we get the following system

$$\begin{cases} \dot{a}_{11}(t,t_0) = -f_1(t)a_{11}(t,t_0) + a_{21}(t,t_0) \\ \dot{a}_{21}(t,t_0) = -\beta(t)a_{11}(t,t_0) - f_1(t)a_{21}(t,t_0) + \gamma_1(t)a_{31}(t,t_0) \\ \dot{a}_{31}(t,t_0) = -f_2(t)a_{31}(t,t_0) + a_{41}(t,t_0) \\ \dot{a}_{41}(t,t_0) = \gamma_2(t)a_{11}(t,t_0) - \delta(t)a_{31}(t,t_0) - f_2(t)a_{41}(t,t_0). \end{cases} \tag{2.11}$$

From the first two equations of (2.11) and hypothesis (H4) we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0)\right] \\ &\qquad \leq -f_1(t)\left[\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0)\right] + \gamma_1(t)a_{21}(t,t_0)a_{31}(t,t_0) \end{aligned} \tag{2.12}$$

and, similarly,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0)\right] \\ &\qquad \leq -f_2(t)\left[\delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0)\right] + \gamma_2(t)a_{11}(t,t_0)a_{41}(t,t_0). \end{aligned} \tag{2.13}$$

By relations (2.12) and (2.13) we obtain successively

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0) + \delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0)\right] \\ &\quad \leq -f_1(t)\left[\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0)\right] - f_2(t)\left[\delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0)\right] \\ &\qquad + \gamma_1(t)a_{21}(t,t_0)a_{31}(t,t_0) + \gamma_2(t)a_{11}(t,t_0)a_{41}(t,t_0) \\ &\quad \leq \left[-\tilde{f}(t) + \frac{\gamma(t)}{2\sqrt{\zeta(t)}}\right]\left[\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0) + \delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0)\right], \end{aligned}$$

for all $t \geq t_0$, and (2.7) follows immediately. The inequalities (2.8)–(2.10) can be derived in the same way.

Let $\|\cdot\|_0$ be the norm in $\mathbb{R}^4$ defined by

$$\|z\|_0 = \left(\beta_0 x^2 + u^2 + \delta_0 y^2 + v^2\right)^{1/2}, \quad \text{for } z = (x,u,y,v)^\top, \tag{2.14}$$

which is equivalent to the Euclidean norm.

For $z_0 = (x_0, u_0, y_0, v_0)^\top \in \mathbb{R}^4$, from (2.7)–(2.10) and (H4), we deduce

$$\|Z(t,t_0)z_0\|_0 \leq \lambda\|z_0\|_0\sqrt{\beta(t_0) + \delta(t_0) + 2}\,\mathrm{e}^{\int_{t_0}^t\left[-\tilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}, \quad \forall t \geq t_0, \tag{2.15}$$

where $\lambda := \max\{1, 1/\sqrt{\beta_0}, 1/\sqrt{\delta_0}\}$,

$$\begin{aligned} \left\|Z(t,t_0)Z(s,t_0)^{-1}e_2\right\|_0 &\leq \mathrm{e}^{\int_s^t\left[-\tilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}, \\ \left\|Z(t,t_0)Z(s,t_0)^{-1}e_4\right\|_0 &\leq \mathrm{e}^{\int_s^t\left[-\tilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}, \end{aligned} \tag{2.16}$$

for all $t \geq s \geq t_0 \geq 0$, where $e_2 = (0,1,0,0)^\top$, $e_4 = (0,0,0,1)^\top$.

Proof of a). Let $z_0 \neq 0$ with $\|z_0\|_0$ small enough, $t_0 \geq 0$, and $z(t,t_0,z_0) = (x(t,t_0,z_0), u(t,t_0,z_0), y(t,t_0,z_0), \; v(t,t_0,z_0))^\top$ be the unique solution of (2.5) which equals z_0 for $t = t_0$.

From the continuity and the boundedness of the functions f_i, $\dot{f}_i$, f_j, β, γ_i, δ, r_i, $\forall i \in \{1,2\}$, $\forall j \in \{3,4\}$, there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous and bounded function, such that

$$\|A(t)z + B(t)z + F(t,z)\|_0 \leq \psi(t)\|z\|_0, \quad \forall (t,z) \in \mathbb{R}_+ \times \mathbb{R}^4.$$

By applying a classical result of global existence in the future to system (2.5) (see, e.g., [3, Corollary, p. 53]) it follows that $z(t,t_0,z_0)$ exists on the whole interval $[t_0, +\infty)$.

We have

$$z(t,t_0,z_0) = Z(t,t_0)z_0 + \int_{t_0}^{t} Z(t,t_0)Z(s,t_0)^{-1}[B(s)z(s,t_0,z_0) + F(s,z(s,t_0,z_0))]\mathrm{d}s, \tag{2.17}$$

for all $t \geq t_0$.

From the relations (2.15)–(2.17) we get

$$\begin{aligned}\|z(t,t_0,z_0)\|_0 \leq \lambda\|z_0\|_0\sqrt{\beta(t_0)+\delta(t_0)+2}\,\mathrm{e}^{\int_{t_0}^{t}\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u} + \int_{t_0}^{t}\mathrm{e}^{\int_{s}^{t}\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}\\ \times\Big[|\dot{f}_1(s)+f_1^2(s)||x(s,t_0,z_0)| + |\dot{f}_2(s)+f_2^2(s)||y(s,t_0,z_0)|\\ + f_1(s)f_4(s)|x(s,t_0,z_0)| + f_2(s)f_3(s)|y(s,t_0,z_0)|\\ + f_3(s)|v(s,t_0,z_0)| + f_4(s)|u(s,t_0,z_0)|\\ + |g_1(s,x(s,t_0,z_0),y(s,t_0,z_0))|\\ + |g_2(s,x(s,t_0,z_0),y(s,t_0,z_0))|\Big]\mathrm{d}s,\end{aligned} \tag{2.18}$$

for all $t \geq t_0$.

In what follows we consider two cases.

Case 1: $0 \leq t_0 < h$. Since $f_i \in C^1[t_0,h]$, $f_j, \beta, \gamma_i, \delta, \in C[t_0,h]$, $g_i \in C([t_0,h] \times \mathbb{R} \times \mathbb{R})$, $\forall i \in \{1,2\}$, $\forall j \in \{3,4\}$, from (2.18) it results that

$$\|z(t,t_0,z_0)\|_0 \leq \lambda D_1\sqrt{\beta(t_0)+\delta(t_0)+2}\|z_0\|_0 + D\int_{t_0}^{t}\|z(s,t_0,z_0)\|_0\mathrm{d}s, \qquad \forall t \in [t_0,h],$$

with D, D_1 positive constants. Using the Gronwall lemma we get

$$\|z(t,t_0,z_0)\|_0 \leq \lambda D_1\sqrt{\beta(t_0)+\delta(t_0)+2}\|z_0\|_0\mathrm{e}^{Dh}, \qquad \forall t \in [t_0,h]. \tag{2.19}$$

For all $t \geq h$, from the relation (2.18) and the hypothesis (H2) we deduce

$$\begin{aligned}\|z(t,t_0,z_0)\|_0 \leq \lambda\sqrt{\beta(h)+\delta(h)+2}\|z(h,t_0,z_0)\|_0\mathrm{e}^{\int_{h}^{t}\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}\\ + \int_{h}^{t}\mathrm{e}^{\int_{s}^{t}\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}\Big[K_1\widetilde{f}(s)|x(s,t_0,z_0)| + K_2\widetilde{f}(s)|y(s,t_0,z_0)|\\ + f_1(s)f_4(s)|x(s,t_0,z_0)| + f_2(s)f_3(s)|y(s,t_0,z_0)|\\ + f_3(s)|v(s,t_0,z_0)| + f_4(s)|u(s,t_0,z_0)|\\ + |g_1(s,x(s,t_0,z_0),y(s,t_0,z_0))|\\ + |g_2(s,x(s,t_0,z_0),y(s,t_0,z_0))|\Big]\mathrm{d}s.\end{aligned} \tag{2.20}$$

By (2.3) and (2.20) we obtain

$$\begin{aligned}\|z(t,t_0,z_0)\|_0 &\le \lambda\sqrt{\beta(h)+\delta(h)+2}\|z(h,t_0,z_0)\|_0 e^{\int_h^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]du}\\&\quad+\int_h^t e^{\int_s^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]du}\left[\left(\frac{K_1}{\sqrt{\beta_0}}+\frac{K_2}{\sqrt{\delta_0}}\right)\widetilde{f}(s)\right.\\&\quad+\frac{f_1(s)f_4(s)}{\sqrt{\beta_0}}+\frac{f_2(s)f_3(s)}{\sqrt{\delta_0}}+f_3(s)+f_4(s)\\&\quad\left.+\frac{r_1(s)}{\sqrt{\beta_0}}+\frac{r_2(s)}{\sqrt{\delta_0}}\right]\|z(s,t_0,z_0)\|_0 ds\\&=: \sigma(t), \qquad \forall t\ge h. \end{aligned} \tag{2.21}$$

Straightforward calculations lead us to

$$\dot{\sigma}(t)\le\omega(t)\sigma(t), \qquad \forall t\ge h, \tag{2.22}$$

$$\sigma(h)=\lambda\sqrt{\beta(h)+\delta(h)+2}\|z(h,t_0,z_0)\|_0.$$

where

$$\omega(t):=-K\widetilde{f}(t)+\varphi(t), \qquad \forall t\ge 0,\ K=1-\frac{K_1}{\sqrt{\beta_0}}-\frac{K_2}{\sqrt{\delta_0}},$$

$$\varphi(t):=\frac{\gamma(t)}{2\sqrt{\zeta(t)}}+\frac{f_1(t)f_4(t)}{\sqrt{\beta_0}}+\frac{f_2(t)f_3(t)}{\sqrt{\delta_0}}+f_3(t)+f_4(t)+\frac{r_1(t)}{\sqrt{\beta_0}}+\frac{r_2(t)}{\sqrt{\delta_0}}, \qquad \forall t\ge 0.$$

From (2.21) and (2.22) using classical differential inequalities, we obtain

$$\|z(t,t_0,z_0)\|_0\le\lambda\sqrt{\beta(h)+\delta(h)+2}\|z(h,t_0,z_0)\|_0 e^{-K\int_h^t\widetilde{f}(s)ds}e^{\int_h^t\varphi(s)ds}, \qquad \forall t\ge h. \tag{2.23}$$

It is readily seen from the hypotheses (H1), (H5), (H6), and Remark 2.1, that

$$\int_h^{+\infty}\varphi(s)ds<+\infty.$$

Let $\varepsilon>0$ be arbitrary and

$$\eta=\eta(\varepsilon):=\frac{\varepsilon e^{-\int_h^{+\infty}\varphi(s)ds}e^{-Dh}}{\lambda^2 D_1\sqrt{\beta(0)+\delta(0)+2}\sqrt{\beta(h)+\delta(h)+2}}.$$

Then, if $\|z_0\|_0<\eta$, by (2.19) and the hypothesis (H4) it results

$$\|z(t)\|_0\le\frac{\varepsilon e^{-\int_h^{+\infty}\varphi(s)ds}}{\lambda\sqrt{\beta(h)+\delta(h)+2}}, \qquad \forall t\in[t_0,h]. \tag{2.24}$$

From the relations (2.23), (2.24), and the hypothesis (H4), it follows that $\|z(t,t_0,z_0)\|_0<\varepsilon$, $\forall t\ge h$.

Case 2: $t_0\ge h$. We similarly get

$$\|z(t,t_0,z_0)\|_0\le\lambda\sqrt{\beta(t_0)+\delta(t_0)+2}\|z_0\|_0 e^{-K\int_{t_0}^t\widetilde{f}(s)ds}e^{\int_{t_0}^t\varphi(s)ds}, \tag{2.25}$$

for all $t \geq t_0$. With the same η as before, if $\|z_0\|_0 < \eta$, then $\|z(t, t_0, z_0)\|_0 < \varepsilon$, $\forall t \geq t_0$.

Therefore, the null solution of (1.1) is uniformly stable.

Proof of b). If, in addition (H3) holds, then from (2.25) we can easily obtain that the null solution of (1.1) is asymptotically stable.

Proof of c). We know from a) that the null solution of (1.1) is uniformly stable. It remains to prove that there exists $\xi > 0$, such that for every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$, such that $\|z_0\|_0 < \xi$ implies $\|z(t, t_0, z_0)\|_0 < \varepsilon$, for all $t_0 \geq 0$ and $t \geq t_0 + T$.

Indeed, if (H7) also holds, then $\int_{t_0}^{t} \widetilde{f}(s)\mathrm{d}s \geq p(t - t_0)$, $\forall t \geq t_0 \geq 0$. From (2.25) we obtain for all $t \geq t_0 \geq 0$, that

$$\|z(t, t_0, z_0)\|_0 \leq \lambda\sqrt{\beta(0) + \delta(0) + 2}\|z_0\|_0 \mathrm{e}^{-Kp(t-t_0)}N, \tag{2.26}$$

where $N := \mathrm{e}^{\int_0^{+\infty} \varphi(s)\mathrm{d}s}$. Let $\xi := \frac{1}{\lambda\sqrt{\beta(0)+\delta(0)+2}}$, $\varepsilon > 0$, and

$$T = T(\varepsilon) := \begin{cases} \frac{1}{Kp} \ln \frac{N}{\varepsilon}, \text{ if } \varepsilon < N, \\ 0, \text{ if } \varepsilon \geq N. \end{cases}$$

Consider $z_0 \in \mathbb{R}^4$, $z_0 \neq 0$, with $\|z_0\|_0 < \xi$ and let $t_0 \geq 0$. Then for all $t \geq t_0 + T$, by (2.26) we successively deduce

$$\|z(t, t_0, z_0)\|_0 < \lambda\sqrt{\beta(0) + \delta(0) + 2}\xi \mathrm{e}^{-Kp(t-t_0)}N = N\mathrm{e}^{-Kp(t-t_0)} \leq \varepsilon.$$

Therefore the null solution of (1.1) is uniformly asymptotically stable. □

Example 2.4. An example of functions f_i, f_j, β, δ, γ_i, g_i, $i \in \{1, 2\}$, $j \in \{3, 4\}$, is

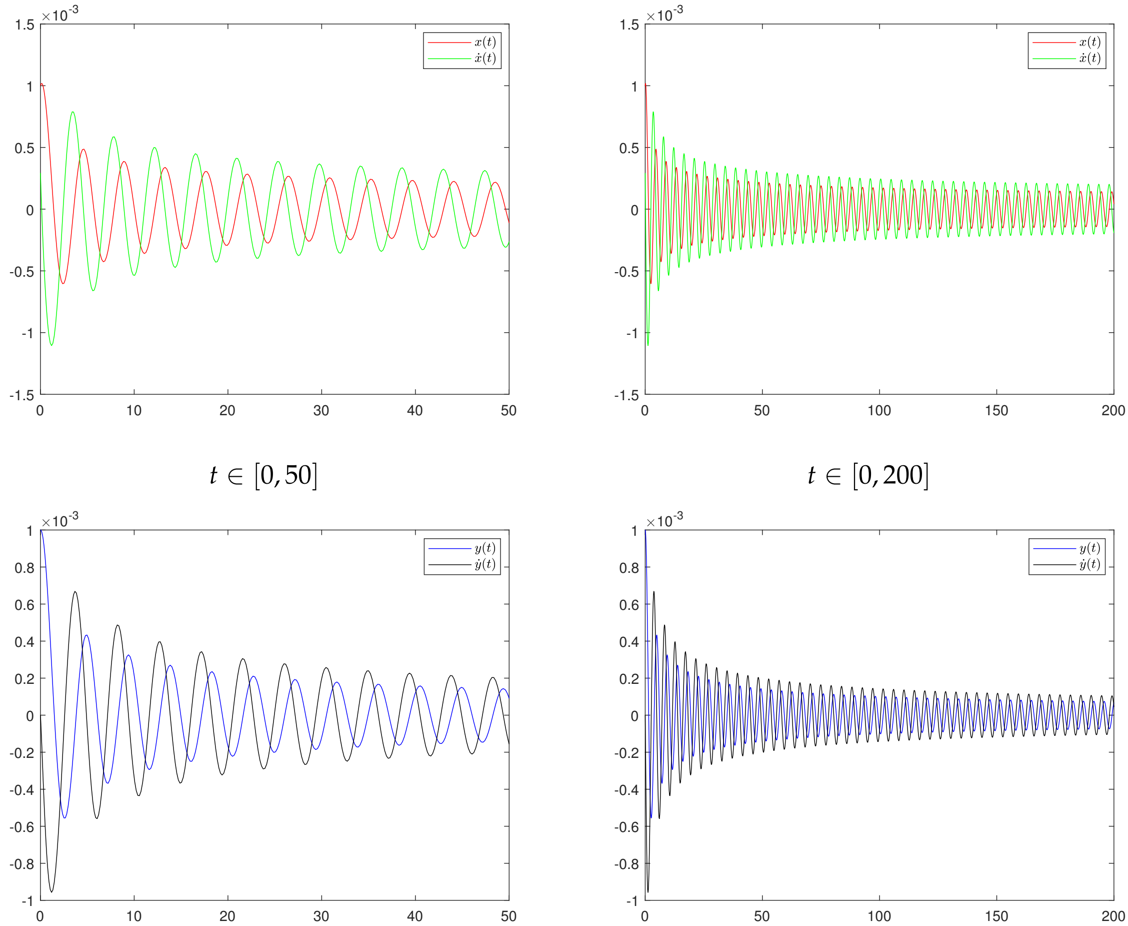
$$f_1(t) = \frac{1}{2t + \sqrt{t^2 + 2}}, \quad f_2(t) = \frac{1}{t + \sqrt{t^2 + 1}}, \quad f_3(t) = \frac{1}{(t+1)^4}, \quad f_4(t) = \frac{2}{(t+1)^3}, \quad \forall t \geq 0,$$

$$\beta(t) = \frac{2t + 3}{t + 1}, \quad \delta(t) = \frac{2t^3 + 5}{t^3 + 2}, \quad \gamma_1(t) = \frac{1}{t\sqrt{t^2 + 1} + 1}, \quad \gamma_2(t) = \mathrm{e}^{-t/2}, \quad \forall t \geq 0,$$

$$g_1(t, x, y) = \mathrm{e}^{-t^2/2}x^3, \quad g_2(t, x, y) = \frac{3}{t^2\sqrt{t + 1}}y^4, \quad \forall t \geq 0, \ \forall x, \ y \in \mathbb{R}.$$

These functions satisfy the hypotheses (H1)–(H6), with $\beta_0 = 2$, $\delta_0 = 2$, $K_1 = 1/\sqrt{2}$, $K_2 = (2 + \sqrt{3}) \times (3 - 2\sqrt{2})$, $h = 1$, $r_1(t) = \mathrm{e}^{-t^2/2}$, $r_2(t) = \frac{3}{t^2\sqrt{t+1}}$, $\forall t \geq 0$. In Figure 2.1 the solution of (1.1) and its derivative are plotted on two time intervals, for small initial data. The solution in the planes $(x, \dot{x})$ and $(y, \dot{y})$ on the same time intervals can be observed in Figure 2.2.

Example 2.5. If in Example 2.4 one changes only f_1, f_2 to $f_1(t) = \frac{1}{10} + \frac{1}{t+1}$, respectively $f_2(t) = \frac{1}{5} + \frac{2}{t+1}$, $\forall t \geq 0$, then the hypotheses (H1), (H2), (H4)–(H7) are verified with $K_1 = 1/5$, $K_2 = 4/5$, $h = 7$, $p = \frac{1}{10}$, and the same β_0, δ_0, $r_1(t)$, $r_2(t)$ and we obtain the solution of (1.1) and its derivative plotted in Figure 2.3 on the same time intervals and for the same initial data. In Figure 2.4 the solution is generated in the planes $(x, \dot{x})$ and $(y, \dot{y})$.



$t \in [0, 50]$ $t \in [0, 200]$

$t \in [0, 50]$ $t \in [0, 200]$

Figure 2.1: The solution of system (1.1) and its derivative, with the initial data $z_0 = [0.01, 0.01, 0.01, 0.01]$ and the functions f_1, f_2, f_3, f_4, β, δ, γ_1, γ_2, g_1, g_2 given in Example 2.4.

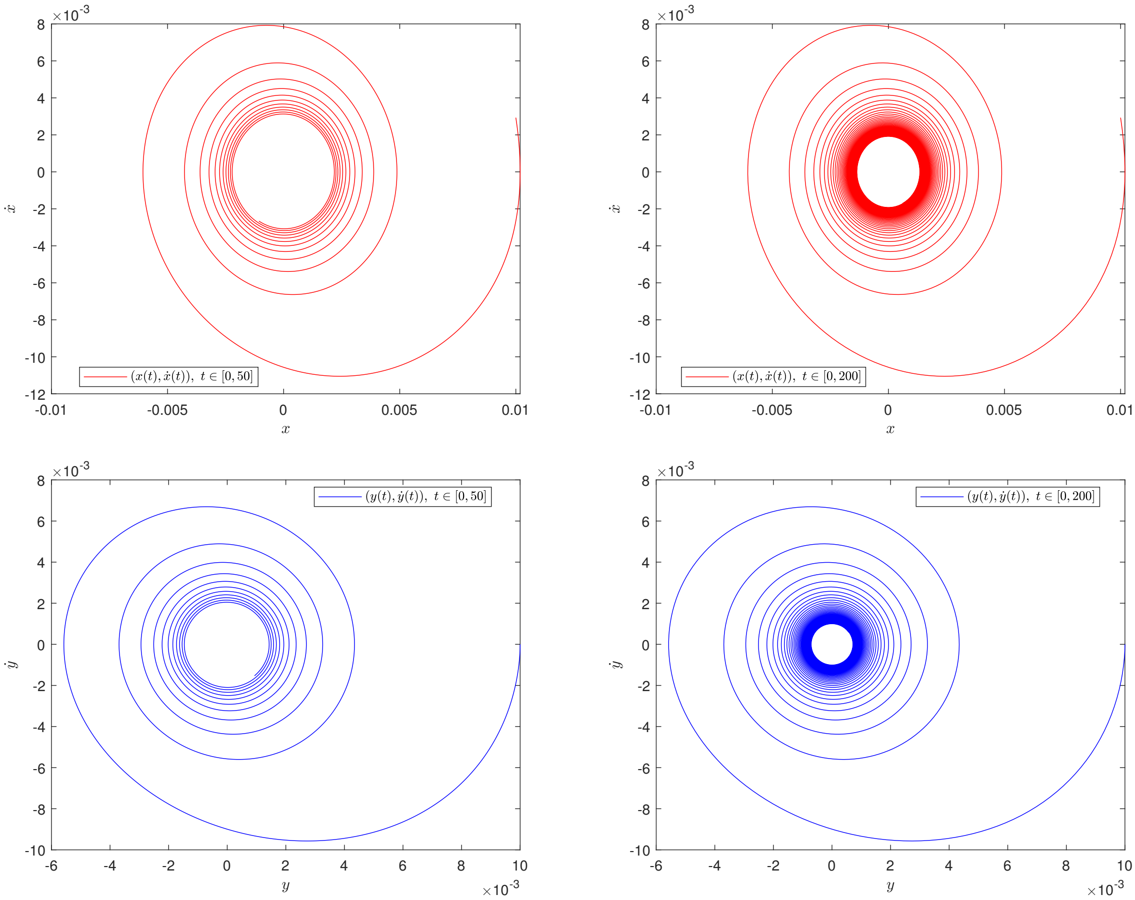
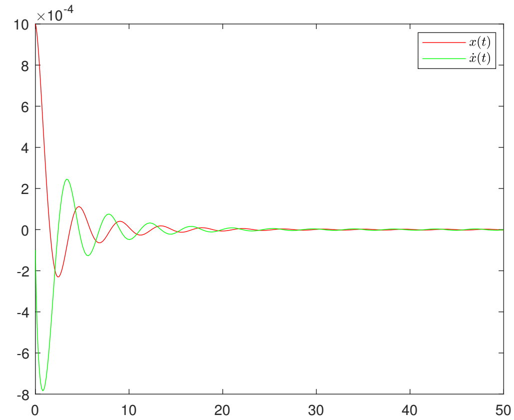
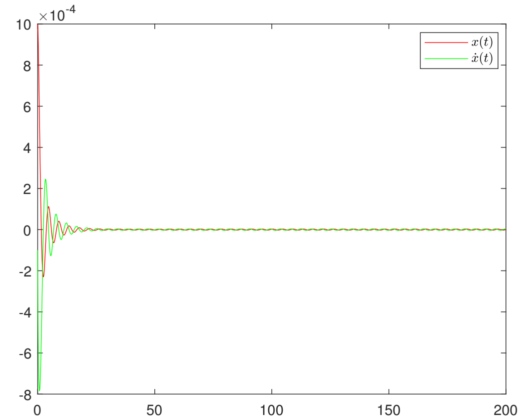


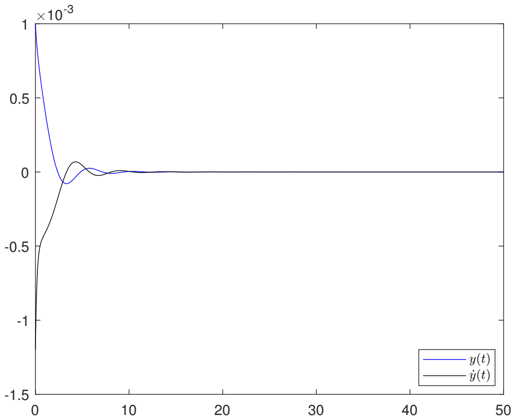
Figure 2.2: The solution of (1.1) in the planes $(x, \dot{x})$ and $(y, \dot{y})$, with the data from Example 2.4.



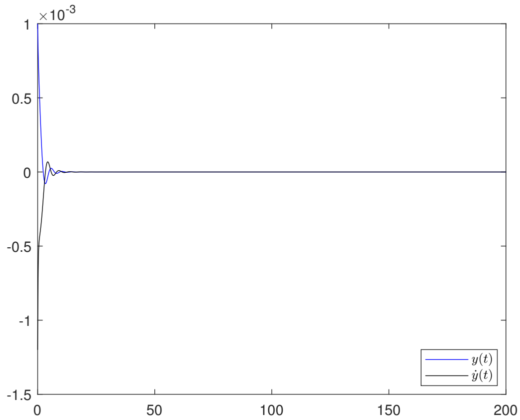
$t \in [0, 50]$



$t \in [0, 200]$



$t \in [0, 50]$



$t \in [0, 200]$

Figure 2.3: The solution of system (1.1) and its derivative, with the initial data $z_0 = [0.01, 0.01, 0.01, 0.01]$ and the functions f_1, f_2, f_3, f_4, β, δ, γ_1, γ_2, g_1, g_2 given in Example 2.5.

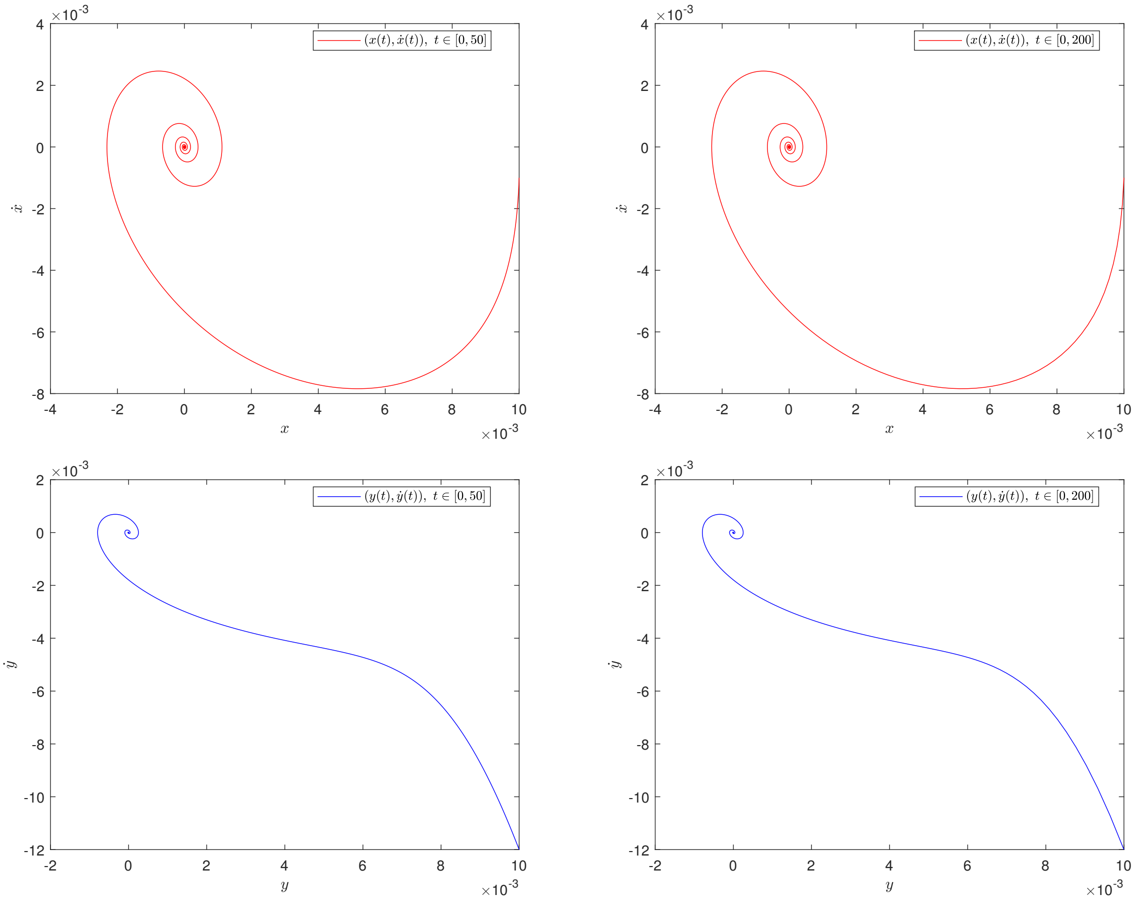


Figure 2.4: The solution of (1.1) in the planes $(x, \dot{x})$ and $(y, \dot{y})$, with the data from Example 2.5.

2.2. A stability result via Lyapunov's method

We are going to use the following additional assumptions.

(H1*) $f_i \in C(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, $f_j \in C(\mathbb{R}_+)$, $f_i(t) \geq 0$, $f_j(t) \geq 0$, $\forall t \in \mathbb{R}_+$, and $\int_0^{+\infty} f_j(t)\mathrm{d}t < +\infty$, $\forall i \in \{1,2\}$, $\forall j \in \{3,4\}$;

(H3*) $\int_0^{+\infty} \widetilde{f}(t)\mathrm{d}t < +\infty$;

(H4*) β, $\delta \in C^1(\mathbb{R}_+)$, β, δ are decreasing and

$$\beta(t) \geq \beta_0 > 0,\ \delta(t) \geq \delta_0 > 0, \quad \forall t \in \mathbb{R}_+.$$

Let us state and prove the following result.

Theorem 2.6. *Suppose that the hypotheses (H1*), (H3*), (H4*), (H5), (H6) are fulfilled. Then the null solution of the system* (1.1) *is uniformly stable.*

Proof. Let us remark that using the classical change of variables $x = x$, $u = \dot{x}$, $y = y$, $v = \dot{y}$, the system (1.1) becomes

$$\dot{z} = F(t, z), \tag{2.27}$$

where

$$z = \begin{pmatrix} x \\ u \\ y \\ v \end{pmatrix}, \qquad F(t,z) = \begin{pmatrix} u \\ -\beta(t)x - 2f_1(t)u + \gamma_1(t)y + f_3(t)v - g_1(t,x,y) \\ v \\ \gamma_2(t)x + f_4(t)u - \delta(t)y - 2f_2(t)v - g_2(t,x,y) \end{pmatrix}$$

and our stability question reduces to the stability of the null solution $z(t) = 0$ of the system (2.27). Let us remark that the global existence in the future of the solutions of (2.27) follows as in the proof of Theorem 2.3, this time the boundedness of the functions f_1, f_2 being ensured by the hypothesis (H1*).

We are going to use again the norm $\|\cdot\|_0$ defined by (2.14). Consider the function $V : \mathbb{R}_+ \times \Delta \to \mathbb{R}$,

$$V(t,z) = \frac{1}{2}\left[\beta(t)x^2 + u^2 + \delta(t)y^2 + v^2\right] \mathrm{e}^{-\int_0^t \left[\tilde{f}(s) + f_3(s) + f_4(s) + \frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s},$$

for $z = (x,u,y,v)^\top \in \Delta$, where $\Delta \subset \mathbb{R}^4$ is a neighborhood of the origin of $\mathbb{R}^4$,

$$\Delta = \left\{ z \in \mathbb{R}^4,\ \|z\|_0 < a \right\},$$

where $a = \min\{a_1\sqrt{\beta_0},\ a_2\sqrt{\delta_0}\}$, $a_1 > 0$, $a_2 > 0$ are as in Remark 2.2, $\gamma(t) := \max\{\gamma_1(t), \gamma_2(t)\}$, $\zeta(t) := \min\{\beta(t), \delta(t)\}$, $\forall t \in \mathbb{R}_+$, and $r(t) := \max\{r_1(t), r_2(t)\}$, $\forall t \geq 0$.

Obviously,

$$\begin{aligned} V(t,z) &\geq \frac{1}{2}\left(\beta_0 x^2 + u^2 + \delta_0 y^2 + v^2\right) \mathrm{e}^{-\int_0^t \left[\tilde{f}(s) + f_3(s) + f_4(s) + \frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ &= \frac{1}{2}\|z\|_0^2 \mathrm{e}^{-\int_0^t \left[\tilde{f}(s) + f_3(s) + f_4(s) + \frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s}, \end{aligned}$$

for all $(t,z) \in \mathbb{R}_+ \times \Delta$.

By using hypotheses (H1*), (H3*), (H4*), (H5), (H6), we deduce

$$V(t,z) \geq \frac{1}{2}\|z\|_0^2 \mathrm{e}^{-\left[\int_0^{+\infty} \tilde{f}(s)\mathrm{d}s + \int_0^{+\infty} f_3(s)\mathrm{d}s + \int_0^{+\infty} f_4(s)\mathrm{d}s + \int_0^{+\infty} \frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\mathrm{d}s\right]}, \quad \forall (t,z) \in \mathbb{R}_+ \times \Delta$$

and so the function V is positive definite.

The function V is also decrescent. Indeed,

$$\begin{aligned} V(t,z) &\leq \frac{1}{2}\left[\beta(0)x^2 + u^2 + \delta(0)y^2 + v^2\right] \mathrm{e}^{-\int_0^t \left[\tilde{f}(s) + f_3(s) + f_4(s) + \frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ &\leq \frac{1}{2}\max\left\{\frac{\beta(0)}{\beta_0}, \frac{\delta(0)}{\delta_0}\right\}\|z\|_0^2, \quad \forall (t,z) \in \mathbb{R}_+ \times \Delta. \end{aligned}$$

We prove that the time derivative of V along the solutions of the system (2.27) is less than or equal to 0. Indeed, for every $(t,z) \in \mathbb{R}_+ \times \Delta$,

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) &= \frac{1}{2}\big[\dot{\beta}(t)x^2 + 2\beta(t)x\dot{x} + 2u\dot{u} + \dot{\delta}(t)y^2 + 2\delta(t)y\dot{y} + 2v\dot{v}\big] \\
&\quad \times \mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\
&\quad - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right] V(t,z) \\
&\le \{\gamma(t)(|u||y| + |x||v|) + [f_3(t)+f_4(t)]|u||v| + |u||g_1(t,x,y)| + |v||g_2(t,x,y)|\} \\
&\quad \times \mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\
&\quad - 2\big[f_1(t)u^2 + f_2(t)v^2\big]\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\
&\quad - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right] V(t,z).
\end{aligned} \tag{2.28}$$

From (2.28) and (2.3) for all $(t,z) \in \mathbb{R}_+ \times \Delta$ we successively obtain

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) &\le \Big\{\gamma(t)(|u||y| + |x||v|) + [f_3(t)+f_4(t)]|u||v| + [r_1(t)|x||u| + r_2(t)|y||v|] \\
&\quad - 2\big[f_1(t)u^2 + f_2(t)v^2\big]\Big\}\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\
&\quad - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + 2\frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right] V(t,z) \\
&\le \left[f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right] V - 2\big[f_1(t)u^2 + f_2(t)v^2\big] \\
&\quad \times \mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\
&\quad - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right] V(t,z) \\
&= -\widetilde{f}(t)V - 2\big[f_1(t)u^2 + f_2(t)v^2\big]\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s}.
\end{aligned} \tag{2.29}$$

Then, from (2.29) we easily get

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) \le 0, \quad \forall (t,z) \in \mathbb{R}_+ \times \Delta.$$

From Persidski's Theorem (see, e.g., [3, second Corollary, p. 101], [17, Theorem 2.1]), it follows that the null solution of (1.1) is uniformly stable. □

Remark 2.7. Let us remark that by using the transformation (2.4) we obtained the uniform, the asymptotic, and the uniform asymptotic stability, while by using the classical transformation ($x = x$, $u = \dot{x}$, $y = y$, $v = \dot{y}$) and the Lyapunov's method we were only able to achieve the uniform stability of the null solution of (1.1). Hence the first method, based on the transformation (2.4), is more effective.

Remark 2.8. Note that the null solution of the system (1.1) can be uniformly stable and not asymptotically stable. Indeed, this can be seen by considering the following functions

$$f_1(t) = \frac{e^{-t}}{t+1}, \quad f_2(t) = \frac{|\cos^3 t|}{t^2+4}, \quad \forall t \geq 0, \quad f_3(t) = \frac{|\sin t^2|}{t+2}, \quad f_4(t) = \frac{e^{-t^2}}{t+1}, \quad \forall t \geq 0,$$

$$\beta(t) = 0.3 + \frac{1}{t^2+1}, \quad \delta(t) = 0.2 + \frac{1}{\sqrt{t^2+2}}, \quad \gamma_1(t) = \frac{t}{t+2} e^{-t^2}, \quad \gamma_2(t) = \frac{3|\cos t|}{(t+1)^2}, \quad \forall t \geq 0,$$

$$g_1(t,x,y) = \frac{3x^3}{(t^2+2)^2}, \quad g_2(t,x,y) = \frac{2y^2}{(t+1)^3}, \quad \forall t \geq 0, \quad \forall x, y \in \mathbb{R}.$$

These functions satisfy the hypotheses (H1*), (H3*), (H4*), (H5), (H6), with $\beta_0 = 0.3$, $\delta_0 = 0.2$, $r_1(t) = \frac{3}{(t^2+2)^2}$, $r_2(t) = \frac{2}{(t+1)^3}$, $\forall t \geq 0$. For small initial data, the solution of (1.1) and its derivative can be observed in Figure 2.5 on some time intervals. The plottings of the solution in the planes $(x, \dot{x})$, $(y, \dot{y})$ are given in Figure 2.6.

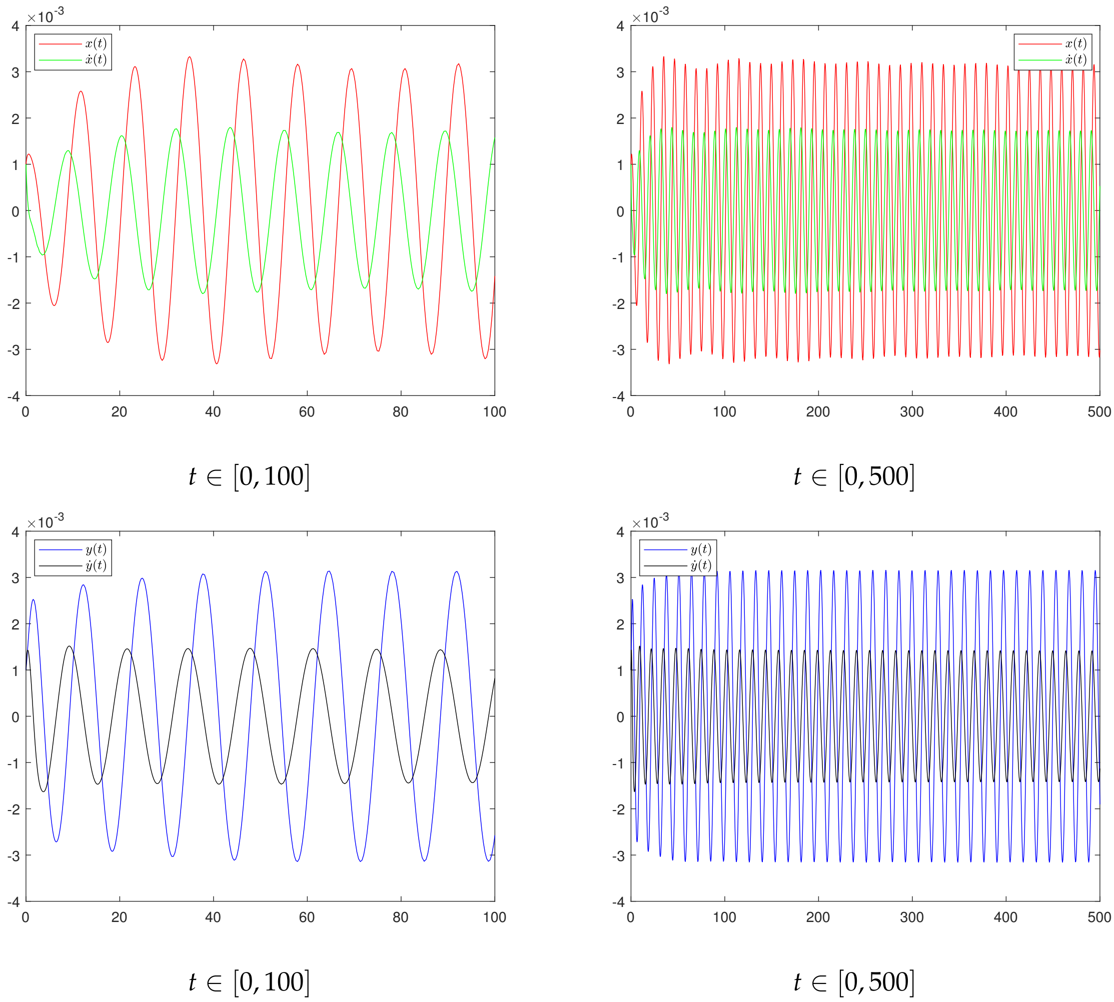


Figure 2.5: The solution of (1.1) and its derivative, with the initial data $z_0 = [0.001, 0.001, 0.001, 0.001]$ and the functions f_1, f_2, f_3, f_4, β, δ, γ_1, γ_2, g_1, g_2 given in Remark 2.8.

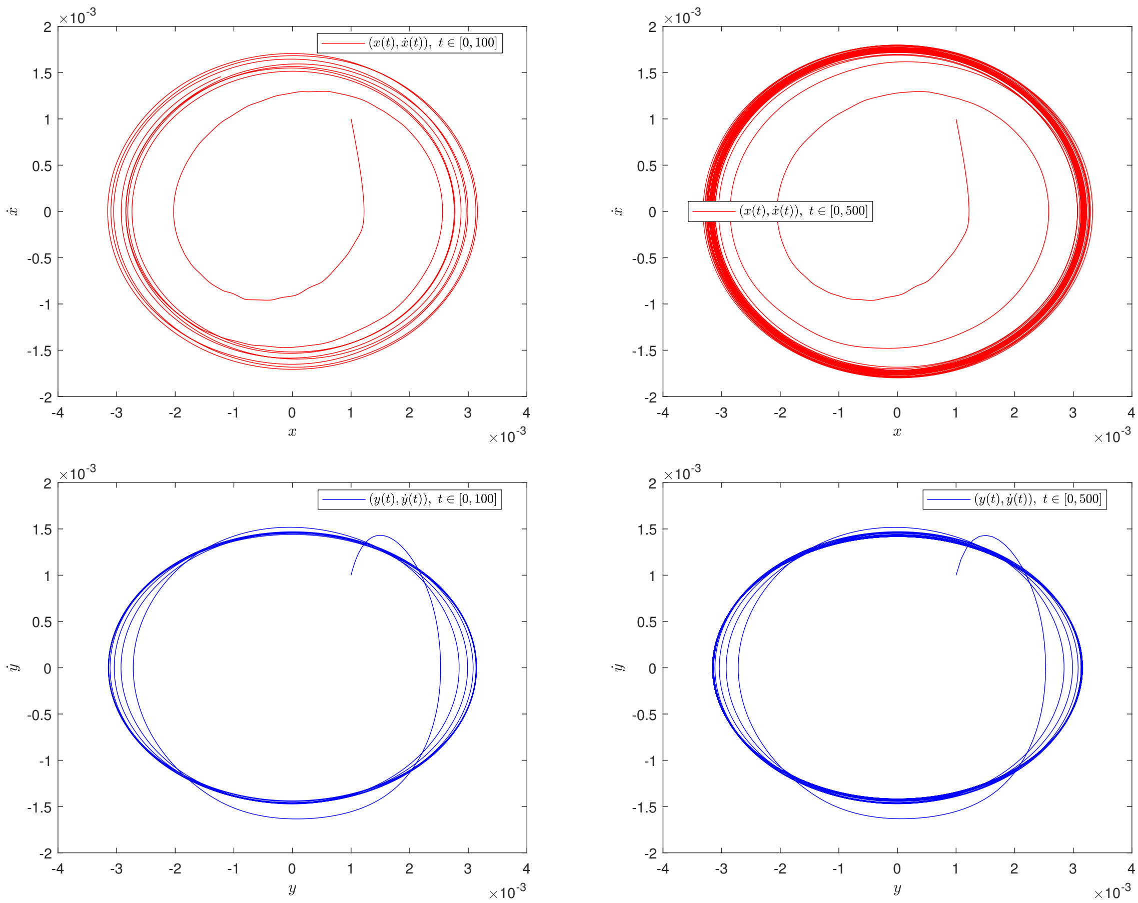


Figure 2.6: The solution of (1.1) in the planes $(x,\dot{x})$ and $(y,\dot{y})$, with the data from Remark 2.8.

3. Analysis of the inhomogeneous system (1.3)

Suppose that the block of mass m_1 is subject to the action of a time-dependent external force $\widehat{f} : \mathbb{R}_+ \to \mathbb{R}$. In this case, we obtain the inhomogeneous system (1.3).

We are going to use the following hypotheses.

(H8) $f \in C(\mathbb{R}_+)$ and $f \in L^1(\mathbb{R}_+)$;

(H9) $f \in C(\mathbb{R}_+)$ and $\lim_{t\to+\infty} f(t) = 0$.

3.1. Qualitative properties of solutions via differential inequalities

Theorem 3.1.

a) Suppose that the hypotheses (H1), (H2), (H4)–(H6), (H8) are fulfilled. Then every solution of the system (1.3) starting from sufficiently small initial data and its derivative are bounded.

b) If the hypotheses (H1), (H2), (H4)–(H6), (H7) with p big enough, and (H9) are satisfied, then for every solution (x, y) of (1.3) starting from small initial data, we have $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \dot{y}(t) = 0$.

Proof. This time we use the following transformation (of the same type as the one from [2])

$$\begin{cases} \dot{x} = u - f_1(t)x \\ \dot{u} = [\dot{f}_1(t) + f_1^2(t) - \beta(t)]x - f_1(t)u + [\gamma_1(t) - f_2(t)f_3(t)]y + f_3(t)v + f(t) - g_1(t,x,y) \\ \dot{y} = v - f_2(t)y \\ \dot{v} = [\gamma_2(t) - f_1(t)f_4(t)]x + f_4(t)u + \left[\dot{f}_2(t) + f_2^2(t) - \delta(t)\right]y - f_2(t)v - g_2(t,x,y) \end{cases} \tag{3.1}$$

and the system (1.3) becomes

$$\dot{z} = A(t)z + B(t)z + G(t,z), \tag{3.2}$$

where

$$G(t,z) = \begin{pmatrix} 0 \\ f(t) - g_1(t,x,y) \\ 0 \\ -g_2(t,x,y) \end{pmatrix}$$

and $A(t)$ and $B(t)$ are the same as in the proof of Theorem 2.3.

Let $z_0 \in \mathbb{R}^4 \setminus \{0\}$ with $\|z_0\|_0$ small enough, $t_0 \geq 0$, and

$$z(t,t_0,z_0) = (x(t,t_0,z_0), u(t,t_0,z_0), y(t,t_0,z_0), v(t,t_0,z_0))^\top$$

be the unique solution of (3.2) which is equal to z_0 for $t = t_0$.

Similarly (by applying, e.g., [3, Corollary, p. 53]) we conclude that $z(t,t_0,z_0)$ exists on $[t_0, +\infty)$, this time having

$$\|A(t)z + B(t)z + G(t,z)\|_0 \leq \psi(t)\|z\|_0 + |f(t)|, \quad \forall (t,z) \in \mathbb{R}_+ \times \mathbb{R}^4.$$

As before we deduce

$$\begin{aligned}\|z(t,t_0,z_0)\|_0 &\le \lambda\|z_0\|_0\sqrt{\beta(t_0)+\delta(t_0)+2}\,\mathrm{e}^{\int_{t_0}^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}+\int_{t_0}^t \mathrm{e}^{\int_s^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}\\
&\times\Big[\big|\dot f_1(s)+f_1^2(s)\big||x(s,t_0,z_0)|+\big|\dot f_2(s)+f_2^2(s)\big||y(s,t_0,z_0)|\\
&+f_1(s)f_4(s)|x(s,t_0,z_0)|+f_2(s)f_3(s)|y(s,t_0,z_0)|\\
&+f_3(s)|v(s,t_0,z_0)|+f_4(s)|u(s,t_0,z_0)|+|f(s)|\\
&+|g_1(s,x(s,t_0,z_0),y(s,t_0,z_0))|\\
&+|g_2(s,x(s,t_0,z_0),y(s,t_0,z_0))|\Big]\mathrm{d}s, \qquad (3.3)\end{aligned}$$

for all $t \ge t_0$.

We distinguish two cases again.

Case 1: $0 \le t_0 < h$. As in the proof of Theorem 2.3, we obtain the relation (2.19), with D, $D_1 > 0$.

From (3.3) and using Remark 2.2, we deduce for all $t \ge h$

$$\begin{aligned}\|z(t,t_0,z_0)\|_0 &\le \lambda\sqrt{\beta(h)+\delta(h)+2}\|z(h,t_0,z_0)\|_0\mathrm{e}^{\int_h^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}\\
&+\int_h^t \mathrm{e}^{\int_s^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}\Bigg\{\Bigg[\left(\frac{K_1}{\sqrt{\beta_0}}+\frac{K_2}{\sqrt{\delta_0}}\right)\widetilde{f}(s)\\
&+\frac{f_1(s)f_4(s)}{\sqrt{\beta_0}}+\frac{f_2(s)f_3(s)}{\sqrt{\delta_0}}+f_3(s)+f_4(s)\\
&+\frac{r_1(s)}{\sqrt{\beta_0}}+\frac{r_2(s)}{\sqrt{\delta_0}}\Bigg]\|z(s,t_0,z_0)\|_0+|f(s)|\Bigg\}\mathrm{d}s\\
&=: \rho(t), \qquad \forall t\ge h.\end{aligned}$$

Straightforward calculations lead us to

$$\begin{cases}\dot\rho(t)\le \omega(t)\rho(t)+|f(t)|, & \forall t\ge h,\\ \rho(h)=\lambda\sqrt{\beta(h)+\delta(h)+2}\|z(h,t_0,z_0)\|_0,\end{cases}$$

with $\omega(t)$, $t \ge 0$, as in the proof of Theorem 2.3.

We easily deduce

$$\begin{aligned}\|z(t,t_0,z_0)\|_0 &\le \left(\rho(h)+\int_h^t \mathrm{e}^{-\int_h^s[-K\widetilde{f}(u)+\varphi(u)]\mathrm{d}u}|f(s)|\mathrm{d}s\right)\mathrm{e}^{\int_h^t[-K\widetilde{f}(s)+\varphi(s)]\mathrm{d}s}\\
&=: \mu(t), \quad \forall t\ge h. \qquad (3.4)\end{aligned}$$

Proof of a). By using the hypotheses (H1), (H5), (H6), and Remark 2.1, it is readily seen that $\overline{\varphi} := \int_0^{+\infty}\varphi(t)\mathrm{d}t < +\infty$. From (3.4) and the hypothesis (H8) we derive that

$$\begin{aligned}\|z(t,t_0,z_0)\|_0 &\le \rho(h)\mathrm{e}^{\int_h^t\varphi(u)\mathrm{d}u}+\int_h^t \mathrm{e}^{\int_s^t\varphi(u)\mathrm{d}u}|f(s)|\mathrm{d}s\\
&\le \mathrm{e}^{\overline{\varphi}}\left(\rho(h)+\int_h^t|f(s)|\mathrm{d}s\right)\\
&\le \mathrm{e}^{\overline{\varphi}}\left(\rho(h)+\|f\|_{L^1[0,+\infty)}\right)<+\infty, \quad \forall t\ge h\end{aligned}$$

and so every solution of (1.3) with initial data small enough is bounded. The boundedness of $\dot{z}(t,t_0,z_0)$ follows immediately.

Proof of b). Let us estimate the limit of μ at $+\infty$. We have

$$\lim_{t\to+\infty}\mu(t)=\lim_{t\to+\infty}\frac{\rho(h)+\int_h^t e^{-\int_h^s[-K\widetilde{f}(u)+\varphi(u)]\mathrm{d}u}|f(s)|\mathrm{d}s}{e^{-\int_h^t[-K\widetilde{f}(s)+\varphi(s)]\mathrm{d}s}}. \tag{3.5}$$

If $\int_h^{+\infty} e^{-\int_h^s[-K\widetilde{f}(u)+\varphi(u)]\mathrm{d}u}|f(s)|\mathrm{d}s<+\infty$, then, from (3.5) and the hypothesis (H7), we easily obtain

$$\lim_{t\to+\infty}\mu(t)=0.$$

If $\int_h^{+\infty} e^{-\int_h^s[-K\widetilde{f}(u)+\varphi(u)]\mathrm{d}u}|f(s)|\mathrm{d}s=+\infty$, then we estimate

$$\lim_{t\to+\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho(h)+\int_h^t e^{-\int_h^s[-K\widetilde{f}(u)+\varphi(u)]\mathrm{d}u}|f(s)|\mathrm{d}s\right)}{\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\int_h^t[-K\widetilde{f}(s)+\varphi(s)]\mathrm{d}s}\right)}=\lim_{t\to+\infty}\frac{|f(t)|}{K\widetilde{f}(t)-\varphi(t)}. \tag{3.6}$$

Using the hypotheses (H1), (H5)–(H7), and Remark 2.1,

$$K\widetilde{f}(t)-\varphi(t)\geq Kp-\varphi_0,\quad \forall t\geq 0,$$

where $\varphi_0=\sup_{t\geq 0}\{\varphi(t)\}$. Hence, if $p>\frac{\varphi_0}{K}$, then $K\widetilde{f}(t)-\varphi(t)>0$, $\forall t\geq 0$, and, from (3.6), the hypothesis (H9), and L'Hospital's rule, we obtain $\lim_{t\to+\infty}\mu(t)=0$. Hence, by (3.4) it follows that $\lim_{t\to+\infty}\|z(t,t_0,z_0)\|_0=0$ and we also infer $\lim_{t\to+\infty}\|\dot{z}(t,t_0,z_0)\|_0=0$.

Case 2: $t_0\geq h$. The proofs of a) and b) follow as in *Case 1*, this time by using the inequality

$$\|z(t,t_0,z_0)\|_0\leq\left(\lambda\sqrt{\beta(t_0)+\delta(t_0)+2}\|z_0\|_0+\int_{t_0}^t e^{-\int_{t_0}^s[-K\widetilde{f}(u)+\varphi(u)]\mathrm{d}u}|f(s)|\mathrm{d}s\right)\times e^{\int_{t_0}^t[-K\widetilde{f}(s)+\varphi(s)]\mathrm{d}s},\qquad \forall t\geq t_0.$$

□

Example 3.2. If we consider the functions

$$f_1(t)=\begin{cases}\frac{\ln t}{t}, & t\geq e\\ \frac{t}{e^3}(2e-t), & t\in[0,e)\end{cases},\quad f_2(t)=\begin{cases}\frac{\ln t}{t-1}, & t\geq e\\ \frac{t}{e(e-1)^2}(2e-1-t), & t\in[0,e)\end{cases},$$

$$f_3(t)=\frac{\arctan t}{(t+1)^2},\quad f_4(t)=\frac{\sqrt{t}}{(t+2)^2},\quad f(t)=\frac{2t+3}{t+2}e^{-t},\quad \forall t\geq 0,$$

$$\beta(t)=\frac{9}{e^2}+\frac{1}{\sqrt{t+2}},\ \delta(t)=\frac{49}{4(e-1)^2}+e^{-2t},\ \gamma_1(t)=\frac{e^{-3t}}{t^2+1},\ \gamma_2(t)=\frac{\sin^2 t}{(t+1)^3},\ \forall t\geq 0,$$

$$g_1(t,x,y)=\frac{2|\sin t|x^3}{t\sqrt{t+1}},\quad g_2(t,x,y)=\frac{3|\cos t|y^2}{(t+1)\sqrt{t+1}},\quad \forall t\geq 0,\ \forall x,y\in\mathbb{R},$$

then the hypotheses (H1), (H2), (H4)–(H6), (H8) are fulfilled with $\beta_0=\frac{9}{e^2}$, $\delta_0=\frac{49}{4(e-1)^2}$, $K_1=2/e$, $K_2=1/(e-1)$, $h=e$, $r_1(t)=\frac{2|\sin t|}{t\sqrt{t+1}}$, $r_2(t)=\frac{3|\cos t|}{(t+1)\sqrt{t+1}}$, $\forall t\geq 0$. In Figure 3.1 one can observe the solution of (1.3) and its derivative, for small initial data on two time intervals and in Figure 3.2 the solution is plotted in the planes $(x,\dot{x})$, $(y,\dot{y})$ on the same time intervals.

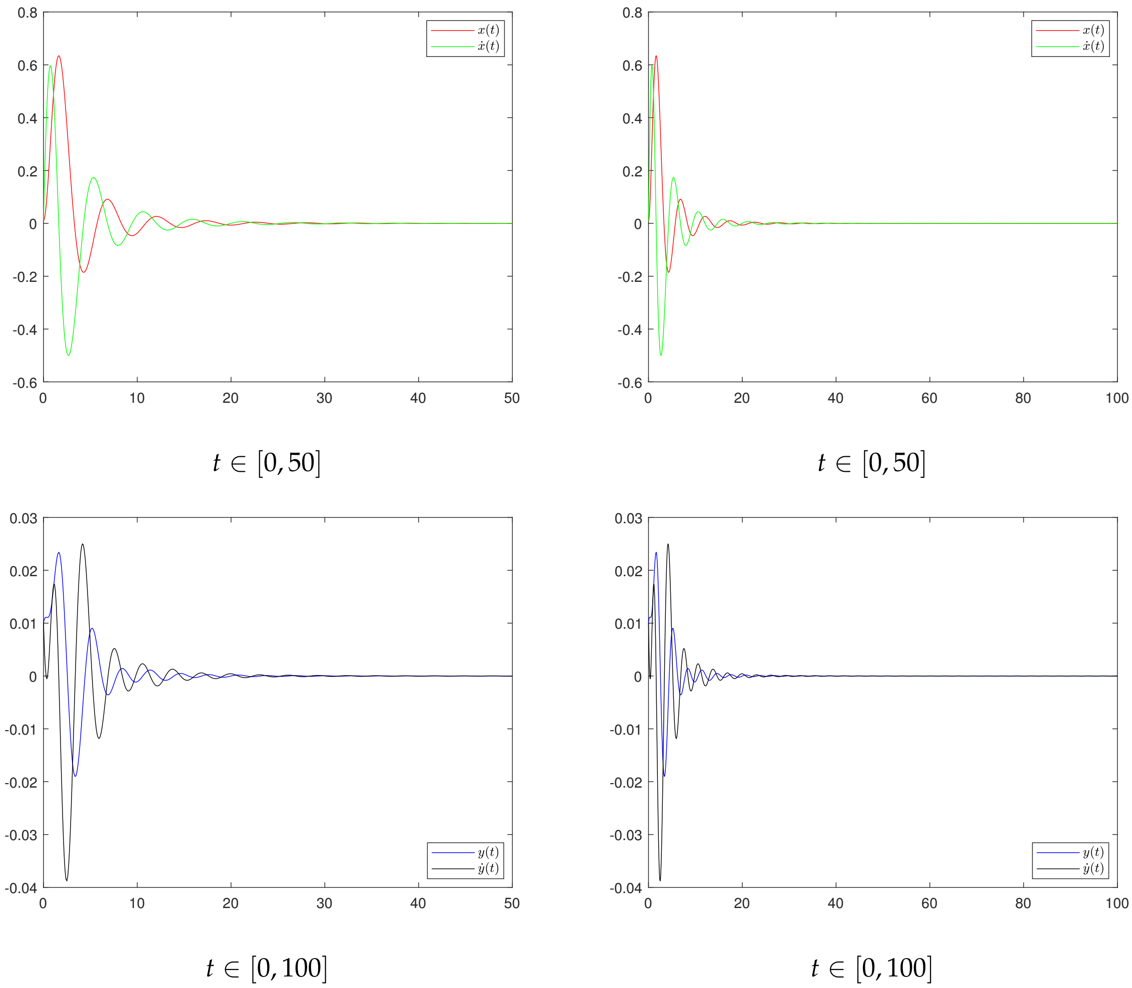


Figure 3.1: The solution of (1.3) and its derivative, with the initial data $z_0 = [0.01, 0.01, 0.01, 0.01]$ and the functions f_1, f_2, f_3, f_4, f, β, δ, γ_1, γ_2, g_1, g_2 given in Example 3.2.

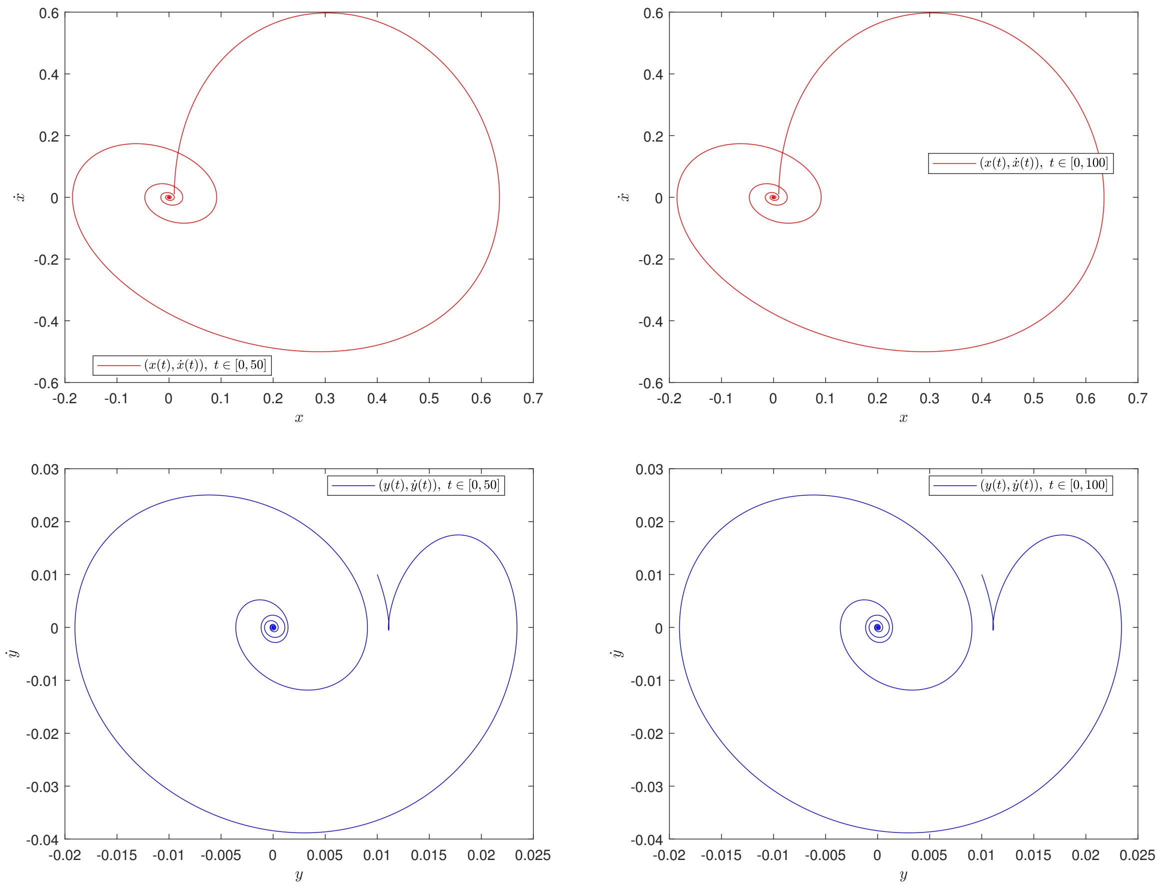


Figure 3.2: The solution of (1.3) in the planes $(x, \dot{x})$ and $(y, \dot{y})$, with the data from Example 3.2.

Remark 3.3. Let us remark the difference between the graphs of the first and second components of the solution near the origin. Due to the action of the external force $\widehat{f}(t)$ on the first block m_1, at least near the origin, the absolute values of $x = x(t)$ are much bigger than the ones of $y = y(t)$.

3.2. Boundedness of solutions

Theorem 3.4. *Suppose that the hypotheses (H1*), (H4*), (H5), (H6), (H8) are fulfilled. Then every solution of the system* (1.3) *with sufficiently small initial data is bounded.*

Proof. Let us remark that using the classical change of variables $x = x$, $u = \dot{x}$, $y = y$, $v = \dot{y}$, the system (1.3) becomes

$$\dot{z} = F(t, z), \tag{3.7}$$

where

$$z = \begin{pmatrix} x \\ u \\ y \\ v \end{pmatrix} \qquad F(t,z) = \begin{pmatrix} u \\ -\beta(t)x - 2f_1(t)u + \gamma_1(t)y + f_3(t)v + f(t) - g_1(t,x,y) \\ v \\ \gamma_2(t)x + f_4(t)u - \delta(t)y - 2f_2(t)v - g_2(t,x,y) \end{pmatrix}.$$

We will use again the norm $\|\cdot\|_0$ defined by (2.14) and the function $V : \mathbb{R}_+ \times \Delta \to \mathbb{R}$,

$$V(t,z) = \frac{1}{2}\left[\beta(t)x^2 + u^2 + \delta(t)y^2 + v^2\right]\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s},$$

for $z = (x,u,y,v)^\top \in \Delta$, where $\Delta \subset \mathbb{R}^4$ is as in the proof of Theorem 2.6, $\gamma(t) := \max\{\gamma_1(t), \gamma_2(t)\}$, $\zeta(t) := \min\{\beta(t), \delta(t)\}$, $\forall t \in \mathbb{R}_+$, and $r(t) := \max\{r_1(t), r_2(t)\}$, $\forall t \geq 0$.

Let us calculate the time derivative of V along the solutions of the system (3.7), whose global existence in the future is deduced as in the proof of Theorem 2.6. For every $(t,z) \in \mathbb{R} \times \Delta$ we have

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) &= \frac{1}{2}\left[\dot\beta(t)x^2 + 2\beta(t)x\dot x + 2u\dot u + \dot\delta(t)y^2 + 2\delta(t)y\dot y + 2v\dot v\right] \\ &\quad\times \mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ &\quad - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right]V(t,z).\end{aligned}$$

By hypothesis (H4*) we get for every $(t,z) \in \mathbb{R}_+ \times \Delta$,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) &\leq \{\gamma(t)(|u||y| + |x||v|) + [f_3(t) + f_4(t)]|u||v| + |u||g_1(t,x,y)| + |v||g_2(t,x,y)| \\ &\quad + |f(t)||u|\} \times \mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ &\quad - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right]V(t,z) \\ &\quad - 2\left[f_1(t)u^2 + f_2(t)v^2\right]\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s}.\end{aligned} \tag{3.8}$$

From relations (3.8) and Remark 2.2, we successively deduce

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) &\leq \Big\{\gamma(t)(|u||y| + |x||v|) + [f_3(t) + f_4(t)]|u||v| + [r_1(t)|x||u| + r_2(t)|y||v| \\ &\quad + |f(t)||u|] - 2\left[f_1(t)u^2 + f_2(t)v^2\right]\Big\}\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ &\quad - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right]V(t,z) \\ &\leq \left[f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right]V(t,z) + |f(t)||u|\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ &\quad - 2\left[f_1(t)u^2 + f_2(t)v^2\right]\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ &\quad - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right]V(t,z) \\ &\leq -\widetilde{f}(t)V(t,z) + |f(t)||u|\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s},\end{aligned} \tag{3.9}$$

for all $(t,z) \in \mathbb{R}_+ \times \Delta$. Then, from (3.9) we easily obtain $\forall (t,z) \in \mathbb{R}_+ \times \Delta$

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) &\le -\widetilde{f}(t)V(t,z) + |f(t)|\sqrt{\beta(t)x^2+u^2+\delta(t)y+v^2} \\ &\quad \times \mathrm{e}^{-\int_0^t\left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ &\le -\widetilde{f}(t)V(t,z)+|f(t)|\sqrt{2V(t,z)}\mathrm{e}^{-\frac{1}{2}\int_0^t\widetilde{f}(s)\mathrm{d}s},\end{aligned}$$

which actually represents an inequality of Bernoulli type.

Let $z_0 \in \Delta$, $t_0 \ge 0$, and $z(t,t_0,z_0)$ be the unique solution of (3.7) which is equal to z_0 for $t = t_0$. Using classical differential estimates, we find

$$V(t,z(t,t_0,z_0)) \le \mathrm{e}^{-\int_{t_0}^t\widetilde{f}(s)\mathrm{d}s}\left[\sqrt{V(t_0,z_0)}+\frac{\sqrt{2}}{2}\int_{t_0}^t|f(s)|\mathrm{e}^{-\frac{1}{2}\int_0^{t_0}\widetilde{f}(u)\mathrm{d}u}\mathrm{d}s\right]^2, \quad \forall t \ge t_0.$$

Therefore, by using the hypotheses (H1*), (H5), (H6), it follows that

$$\|z(t,t_0,z_0)\|_0 \le M\left[\sqrt{V(t_0,z_0)}+\frac{\sqrt{2}}{2}\int_{t_0}^t|f(s)|\mathrm{e}^{-\frac{1}{2}\int_0^{t_0}\widetilde{f}(u)\mathrm{d}u}\mathrm{d}s\right], \quad \forall t \ge t_0,$$

where $M := \sqrt{2}\mathrm{e}^{\frac{1}{2}\int_0^{t_0}\widetilde{f}(s)\mathrm{d}s+\frac{1}{2}\int_0^{+\infty}\left[f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s}$. If the hypothesis (H8) comes into play, then

$$\|z(t,t_0,z_0)\|_0 \le M\left[\sqrt{V(t_0,z_0)}+\frac{\sqrt{2}}{2}\|f\|_{L^1[0,+\infty)}\mathrm{e}^{-\frac{1}{2}\int_0^{t_0}\widetilde{f}(s)\mathrm{d}s}\right], \quad \forall t \ge t_0. \qquad \square$$

Remark 3.5. Note that by using the classical transformation $(x = x,\ u = \dot{x},\ y = y,\ v = \dot{y})$, we could only deduce the boundedness of the solutions of (1.3) for initial data small enough. In contrast, the transformation (3.1) allowed us to obtain in addition that the solutions of (1.3), starting from sufficiently small initial data, have the limit zero at $+\infty$.

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Existence and asymptotic behavior of nontrivial solutions for the Klein–Gordon–Maxwell system with steep potential well

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Abstract. In this paper, we consider the following nonlinear Klein–Gordon–Maxwell system with a steep potential well

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u - \mu(2\omega + \phi)\phi u = f(x, u), & \text{in } \mathbb{R}^3, \\ \Delta\phi = \mu(\omega + \phi)u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $\omega > 0$ is a constant, μ and λ are positive parameters, $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and the nonlinearity f satisfies the Ambrosetti–Rabinowitz condition. We use parameter-dependent compactness lemma to prove the existence of nontrivial solution for μ small and λ large enough, then explore the asymptotic behavior as $\mu \to 0$ and $\lambda \to \infty$. Moreover, we also use truncation technique to study the existence and asymptotic behavior of positive solutions of the Klein–Gordon–Maxwell system when $f(u) := |u|^{q-2}u$ where $2 < q < 4$.

Keywords: Klein–Gordon–Maxwell system, asymptotic behavior, variational method.

2020 Mathematics Subject Classification: 35J60, 35J20.

1 Introduction

In recent years, the Klein–Gordon–Maxwell system has been widely studied. It is well known that this type of system has a strong physical meaning, and it arises in a very interesting physical context: as a model describing the nonlinear Klein–Gordon field interacting with the electromagnetic field. More specifically, the model represents standing waves $\psi = u(x)e^{i\omega t}$ in equilibrium with a purely electrostatic field $E = -\nabla\phi(x)$, where ϕ is the gauge potential. Using the variational method, Benci and Fortunato [4, 5] first introduced the Klein–Gordon–Maxwell equations. In addition, they first studied the following special Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + [m_0^2 - (\omega + \phi)^2]u = |u|^{q-2}\, u, & \text{in } \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

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where $q \in (4,6)$, $m_0 > 0$ and $\omega > 0$ are constants, and establish the existence of infinitely many solitary wave solutions when $0 < \omega < m_0$ and $4 < q < 6$. D'Aprile and Mugnai in [12] also obtained the same conclusion for system (1.1) if one of the following assumptions holds:

(i) $0 < \omega < \sqrt{(q-2)/2}m_0$ and $q \in (2,4)$;

(ii) $q \in [4,6)$ and $0 < \omega < m_0$.

By a Pohozaev-type argument, D'Aprile and Mugnai in [13] showed that (1.1) only has a trivial solution when $0 < q \le 2$ or $q \ge 6$. Inspired by [5, 12], Azzollini and Pomponio [1] proved that (1.1) admits a ground state solution if one of the following conditions holds:

(i) $4 \le q < 6$ and $0 < \omega < m_0$;

(ii) $2 < q < 4$ and $0 < \omega < \sqrt{(q-2)/(6-q)}m_0$.

This range has been improved by authors in references [2] and [25]. We point out that the approaches used in [1, 2, 25] are heavily dependent on the form $f(u) := |u|^{q-2}u$. After that, many mathematicians focused on the more general system. For instance, Chen and Tang in[10] generalized the above results to the nonlinear term $f(u)$. They obtained a ground state solution with positive energy under some parameter limitations and f satisfied a superlinear condition.

It can be seen that many early articles are about Klein–Gordon–Maxwell with constant potential, and later more and more researchers concentrated on the non-constant potential. In recent years, there are a large number of articles concerning the existence, nonexistence and multiplicity of nontrivial solutions for the following problem (1.2).

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

In [16], He obtained infinitely many solutions of (1.2). Later, Li and Tang [17] improved the results of [16]. From these two references, we can see that $V(x)$ satisfies the following condition:

$(\widehat{V})$ $V(x) \in C(\mathbb{R}^3, \mathbb{R})$, $\inf_{\mathbb{R}^3} V(x) > 0$ and there exists $a_0 > 0$ such that

$$\lim_{|y|\to\infty} \operatorname{meas}\left\{x \in \mathbb{R}^3 : |x - y| \le a_0, V(x) \le M\right\} = 0, \quad \forall M > 0.$$

Condition $(\widehat{V})$ plays a crucial role in guaranteeing the compactness of embedding of the weighted Sobolev space. If $V(x)$ is radially symmetric, we recall (see [6] or [23]) that, for $2 < s < 6$, the embedding $H^1_r(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is compact. Without this conditions we can see that the compactness is lost. It will make it more difficult for us to deal with the Klein–Gordon–Maxwell system. In this paper we consider the potential satisfied (a_1)–(a_3) below. The conditions (a_1)–(a_3) were first introduced in [3] and $\lambda a(x) + 1$ was called a steep potential well when λ was large. In [20], Liu, Kang and Tang studied the existence of positive solution for the Klein–Gordon–Maxwell system with steep potential well where $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ satisfied the following conditions:

(f_1') There exists a $\bar{C} > 0$ such that $|f(x,t)| \le \bar{C}|t|$ for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$.

(f_2') There exists a $k \in \{1, 2, \dots\}$ such that uniformly in $x \in \mathbb{R}^3$,

$$v_k < \liminf_{|t|\to 0} \frac{f(x,t)}{t} \le \limsup_{|t|\to 0} \frac{f(x,t)}{t} < v_{k+1}.$$

(f_3') $\limsup\limits_{|t|\to\infty} \frac{f(x,t)}{t} < v_1$.

Where $0 < v_1 < v_2 < v_3 < \cdots$ were the eigenvalues of the following eigenvalue problem (1.3) and can be written as $0 < \mu_1 < \mu_2 \le \mu_3 \le \cdots$ counting their multiplicity.

$$\begin{cases} -\Delta u + u = \mu u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

For more articles about steep potential well, readers can refer to [7–9, 11, 17, 19, 21, 22, 24] and references therein for relevant conclusions.

In [27], Zhang and Du studied the existence and asymptotic behavior of positive solutions for Kirchhoff type problems with steep potential well by combining the truncation technique and the parameter-dependent compactness lemma. Motivated by the above works, one of the purposes of this paper is to investigate the existence and asymptotic behavior of nontrivial solution for the following Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u - \mu(2\omega + \phi)\phi u = f(x, u), & \text{in } \mathbb{R}^3, \\ \Delta\phi = \mu(\omega + \phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.4}$$

where $\omega > 0$ is a constant, μ and λ are positive parameters, $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $a(x)$ satisfy the following conditions:

(a_1) $a(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $a(x) \ge 0$ on $\mathbb{R}^3$;

(a_2) there exists $c > 0$ such that $A_c := \{x \in \mathbb{R}^3 : a(x) < c\}$ is nonempty and bounded;

(a_3) $\Omega = \operatorname{int} a^{-1}(0)$ is non-empty and has smooth boundary with $\overline{\Omega} = a^{-1}(0)$;

(f_1) $\lim\limits_{|s|\to 0} \frac{f(x,s)}{s} = 0$ uniformly for $x \in \mathbb{R}^3$;

(f_2) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist $c_1 > 0$, and $p \in (4, 6)$ such that

$$|f(x,s)| \le c_1(1 + |s|^{p-1});$$

(f_3) there exist $\alpha > 4$ such that $0 < \alpha F(x,s) \le f(x,s)s$ uniformly $x \in \mathbb{R}^3$.

Remark 1.1. In [16], in order to show that the associated functional has a mountain pass geometry and obtain the boundedness of Cerami sequence, the authors used a global Ambrosetti–Rabinowitz condition (f_3). To the best of our knowledge, there are only a few articles about the asymptotic behavior of the solution of the Klein–Gordon–Maxwell system with a steep potential well and f satisfies the super-quartic condition.

The following results holds:

Theorem 1.2. *Suppose that* (a_1)–(a_3) *and* (f_1)–(f_3) *are satisfied. Then there exist* λ_1^* *and* $\mu_0 > 0$ *such that for* $\lambda > \lambda_1^*$ *and* $\mu \in (0, \mu_0)$*, problem* (1.4) *has at least a nontrivial solution* $u_{\lambda,\mu} \in E_\lambda$*. Moreover, exist constants* $\tau_0, M > 0$ *(independent of* λ *and* μ*) such that*

$$\tau_0 \le \|u_{\lambda,\mu}\|_\lambda \le 2\sqrt{M} \quad \text{for all } \lambda \text{ and } \mu. \tag{1.5}$$

Then we show the asymptotic behavior of the nontrivial solution for system (1.4) as $\mu \to 0$ and $\lambda \to \infty$. By means of Theorem 1.2, we have the following results.

Theorem 1.3. *Let* $u_{\lambda,\mu}$ *be the nontrivial solution of* (1.4) *obtained by Theorem 1.2. Then for each* $\mu \in (0, \mu_0)$ *be fixed,* $u_{\lambda,\mu} \to u_\mu$ *in* $H^1(\Omega)$ *as* $\lambda \to \infty$*, where* u_μ *is a nontrivial solution of*

$$\begin{cases} -\Delta u + u - \mu(2\omega + \phi)\phi u = f(x,u), & \text{in } \Omega, \\ \Delta \phi = \mu(\omega + \phi)u^2, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{1.6}$$

Theorem 1.4. *Let* $u_{\lambda,\mu}$ *be the nontrivial solution of* (1.4) *obtained by Theorem 1.2. Then for each* $\lambda \in (\lambda_1^*, \infty)$ *be fixed,* $u_{\lambda,\mu} \to u_\lambda$ *in* E_λ *as* $\mu \to 0$*, where* u_λ *is a nontrivial solution of*

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u = f(x,u), & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.7}$$

Theorem 1.5. *Let* $u_{\lambda,\mu}$ *be the nontrivial solution of* (1.4) *obtained by Theorem 1.2. Then* $u_{\lambda,\mu} \to u_0$ *in* $H^1(\Omega)$ *as* $\mu \to 0$ *and* $\lambda \to \infty$*, where* u_0 *is a nontrivial solution of*

$$\begin{cases} -\Delta u + u = f(x,u), & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Remark 1.6. For Theorem 1.2, applying the Mountain Pass Theorem directly to the associated functional $I_{\lambda,\mu}$, we can get a Cerami sequence for $\mu > 0$ small enough. Then we will obtain the boundedness of this Cerami sequence.

Next, we consider the following Klein–Gordon–Maxwell system where $f(u) := |u|^{q-2}u$,

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u - \mu(2\omega + \phi)\phi u = |u|^{q-2}u, & \text{in } \mathbb{R}^3, \\ \Delta \phi = \mu(\omega + \phi)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.9}$$

where $\omega > 0$ is a constant, μ and λ are positive parameters, $2 < q < 4$ and $a(x)$ also satisfies $(a_1) - (a_3)$.

Remark 1.7. In particular, we note that the nonlinearity $u \mapsto f(u) := |u|^{q-2}u$ with $2 < q < 4$ does not satisfy the Ambrosetti–Rabinowitz type condition which would readily obtain a bounded Palais–Smale sequence or Cerami sequence.

Then we have the following results.

Theorem 1.8. *Suppose that* $(a_1) - (a_3)$ *and* $2 < q < 4$ *are satisfied. Then there exist* λ_2^* *and* $\mu_1, \mu_2 > 0$ *such that for* $\lambda > \lambda_2^*$ *and* $\mu \in (0, \min\{\mu_1, \mu_2\})$*, problem* (1.9) *has at least a positive solution* $\hat{u}_{\lambda,\mu} \in E_\lambda$*. Moreover, exist constants* $\tau_1, T > 0$ *(independent of* λ *and* μ*) such that*

$$\tau_1 \le \|\hat{u}_{\lambda,\mu}\|_\lambda \le T \quad \text{for all } \lambda \text{ and } \mu. \tag{1.10}$$

Theorem 1.9. *Let $\hat{u}_{\lambda,\mu}$ be the positive solution of* (1.9) *obtained by Theorem 1.8. Then for each $\mu \in (0, \min\{\mu_1, \mu_2\})$ be fixed, $\hat{u}_{\lambda,\mu} \to \hat{u}_\mu$ in $H^1(\Omega)$ as $\lambda \to \infty$, where $\hat{u}_\mu$ is a positive solution of*

$$\begin{cases} -\Delta u + u - \mu(2\omega + \phi)\phi u = |u|^{p-2}u, & \text{in } \Omega, \\ \Delta \phi = \mu(\omega + \phi)u^2, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases} \tag{1.11}$$

Theorem 1.10. *Let $\hat{u}_{\lambda,\mu}$ be the positive solution of* (1.9) *obtained by Theorem 1.8. Then for each $\lambda \in (\lambda_2^*, \infty)$ be fixed, $\hat{u}_{\lambda,\mu} \to \hat{u}_\lambda$ in E_λ as $\mu \to 0$, where $\hat{u}_\lambda$ is a positive solution of*

$$\begin{cases} -\Delta u + (\lambda a(x) + 1)u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.12}$$

Theorem 1.11. *Let $\hat{u}_{\lambda,\mu}$ be the positive solution of* (1.9) *obtained by Theorem 1.8. Then $\hat{u}_{\lambda,\mu} \to \hat{u}_0$ in $H^1(\Omega)$ as $\mu \to 0$ and $\lambda \to \infty$, where $\hat{u}_0$ is a positive solution of*

$$\begin{cases} -\Delta u + u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.13}$$

Remark 1.12. For Theorem 1.8, if we apply the Mountain Pass Theorem directly to the associated functional $\hat{I}_{\lambda,\mu}$, we also can get a Cerami sequence for $\mu > 0$ small enough. But it is difficult to obtain the boundedness of this Cerami sequence. Then we will use a new method (refer to [27]) called truncation technique to get over this difficulty.

The remainder of this paper is organized as follows. Next Section 2 we derive a variational setting for problems and give some preliminary lemmas. In Section 3 we will prove Theorem 1.2 to Theorem 1.5. Section 4 is devoted to the proof of Theorem 1.8 to Theorem 1.11.

2 Variational setting and preliminaries

Throughout this paper, we use the standard notations. We denote by $C, c_i, C_i, i = 1, 2, \ldots$ for various positive constants whose exact value may change from lines to lines but are not essential to the analysis of problem. We use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space respectively. We will write $o(1)$ to denote quantity that tends to 0 as $n \to \infty$. X' denotes the dual space of X.

$|\cdot|_q$ denotes the usual Lebesgue space with the norm $L^q(\mathbb{R}^3)$ for any $q \in [1, \infty]$. $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space with the standard scalar product and norm $\|\cdot\|_{H^1}$. $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{D^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}}$.

In the paper, we work in the following Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} a(x)u^2 dx < \infty \right\}$$

with the inner product and norm

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + (a(x) + 1)u^2 dx) \right)^{\frac{1}{2}}, \quad \|u\| = \langle u, u \rangle^{\frac{1}{2}}.$$

For $\lambda > 0$, we also need the following inner product and norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^3}(|\nabla u|^2 + (\lambda a(x) + 1)u^2)dx\right)^{\frac{1}{2}}, \quad \|u\|_\lambda = \langle u, u\rangle_\lambda^{\frac{1}{2}}.$$

It is clear that $\|u\| \leq \|u\|_\lambda$ for $\lambda \geq 1$. Set $E_\lambda = (E, \|\cdot\|_\lambda)$.

Referring to [28], it is well known that $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $s \in [2,6]$. Thus, combining Sobolev embedding, for each $s \in [2,6]$, there exists $d_s > 0$ (independent of $\lambda \geq 1$) such that

$$|u|_s \leq d_s \|u\| \leq d_s \|u\|_\lambda \quad \text{for } u \in E. \tag{2.1}$$

S is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$ and

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{|\nabla u|_2^2}{|u|_6^2}.$$

It is easy to see that the weak solutions $(u,\phi) \in E_\lambda \times D^{1,2}(\mathbb{R}^3)$ of system (1.4) are critical points of the functional given by

$$G_{\lambda,\mu}(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + (\lambda a(x)+1)u^2 - |\nabla\phi|^2 - \mu(2\omega+\phi)\phi u^2\right)dx - \int_{\mathbb{R}^3} F(x,u)dx. \tag{2.2}$$

The functional $G_{\lambda,\mu}(u,\phi)$ is strongly indefinite, i.e., unbounded from below and from above on infinite dimensional spaces. We need the following technical results to study of the functional in the only variable u.

Lemma 2.1 ([4, 12]). *For any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ which solves equation*

$$-\Delta\phi + u^2\phi = -\omega u^2. \tag{2.3}$$

Moreover, the map $\Phi : u \in H^1(\mathbb{R}^3) \mapsto \Phi[u] := \phi_u \in D^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and

(i) $-\omega \leq \phi_u \leq 0$ on the set $\{x \in \mathbb{R}^3 | u(x) \neq 0\}$;

(ii) $\|\phi_u\|_{D^{1,2}} \leq C\|u\|^2$ and $\int_{\mathbb{R}^3}|\phi_u|u^2dx \leq C\|u\|_{12/5}^4 \leq C\|u\|^4$.

Lemma 2.2 ([1, Lemma 2.7]). *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$. As a consequence $I'(u_n) \to I'(u)$ in the sense of distributions.*

The following lemma is a stronger version of the Mountain Pass Theorem, so we can find a Cerami sequence.

Proposition 2.3 ([15]). *Let X be a real Banach space with its dual space X', and suppose that $J \in C^1(X,\mathbb{R})$ satisfies*

$$\max\{J(0), J(e)\} \leq \mu < \eta \leq \inf_{\|u\|_X = \rho} J(u)$$

for some $\mu < \eta$, $\rho > 0$ and $e \in X$ with $\|e\|_X > \rho$. Let $c \geq \eta$ be characterized by

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e. Then there exists a sequence $\{u_n\} \subset X$ such that

$$J(u_n) \to c \geq \eta \quad \text{and} \quad (1 + \|u_n\|_X)\left\|J'(u_n)\right\|_{X'} \to 0 \quad \text{as } n \to \infty.$$

3 The existence and concentration phenomenon of solution of (1.4)

We proof Theorem 1.2 to Theorem 1.5 in this section. By (2.3), multiplying both sides by ϕ_u and integrating we obtain

$$\int_{\mathbb{R}^3} |\nabla\phi|^2 dx = -\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 dx. \tag{3.1}$$

Using (3.1), we can rewrite $G_{\lambda,\mu}$ as a C^1 functional $I_{\lambda,\mu} : E_\lambda \to \mathbb{R}$ given by

$$I_{\lambda,\mu}(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + (\lambda a(x)+1)u^2)dx - \frac{\mu}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx. \tag{3.2}$$

Moreover, for any $u, v \in E_\lambda$, we have

$$\langle I'_{\lambda,\mu}(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u\cdot\nabla v + (\lambda a(x)+1)uv)\, dx - \mu\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u uv dx - \int_{\mathbb{R}^3} f(x,u)v dx. \tag{3.3}$$

To begin with, we show that the $I_{\lambda,\mu}$ has the mountain pass geometry.

Lemma 3.1. *Suppose (a_1)–(a_3) and (f_1)–(f_3) are satisfied. There exists $\mu_0 > 0$ for each $\mu \in (0,\mu_0)$ and $\lambda \geq 1$. Then there exist $\rho, \beta > 0$ and $e_0 \in E_\lambda$, $\|e_0\|_\lambda > \rho$, such that*

$$\inf_{\|u\|=\rho} I_{\lambda,\mu}(u) \geq \beta > 0 \geq \max\{I_{\lambda,\mu}(0), I_{\lambda,\mu}(e_0)\}.$$

Proof. From (f_1) and (f_2), for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $(x,s) \in \mathbb{R}^3\times\mathbb{R}$

$$|f(x,s)| \leq \varepsilon|s| + C_\varepsilon|s|^{p-1} \tag{3.4}$$

and

$$|F(x,s)| \leq \frac{\varepsilon}{2}|s|^2 + \frac{C_\varepsilon}{p}|s|^p. \tag{3.5}$$

We choose $\varepsilon = \frac{1}{2d_2^2}$, where $d_2 > 0$ is from (2.1). For each $u \in E_\lambda$, by Lemma 2.1, (2.1), (3.2) and (3.5) we have

$$\begin{aligned} I_{\lambda,\mu}(u) &\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{\varepsilon}{2}|u|_2^2 - \frac{C_\varepsilon}{p}|u|_p^p \\ &\geq \left(\frac{1}{2} - \frac{\varepsilon d_2^2}{2}\right)\|u\|_\lambda^2 - \frac{C_\varepsilon d_p^p}{p}\|u\|_\lambda^p \\ &= \left(\frac{1}{4} - \frac{C_\varepsilon d_p^p}{p}\|u\|_\lambda^{p-2}\right)\|u\|_\lambda^2, \end{aligned}$$

where the constants $d_p > 0$ and $C_\varepsilon > 0$ are independent of μ and λ. Then there exist $\rho > 0$ small enough and $\beta > 0$, such that $\inf\limits_{\|u\|=\rho} I_{\lambda,\mu}(u) \geq \beta > 0$.

Then, we define the functional $J_\lambda : E_\lambda \to \mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + (\lambda a(x)+1)u^2)dx - \int_{\mathbb{R}^3} F(x,u)dx.$$

By (f_1) and (f_3), there exist $c_3, c_4 > 0$ such that

$$F(x,s) \geq c_3|s|^\alpha - c_4 s^2. \tag{3.6}$$

Let $e \in C_0^\infty(\Omega)$ be a positive smooth function, since $\alpha > 4$ then we have

$$\begin{aligned} J_\lambda(te) &= \frac{t^2}{2}\int_\Omega (|\nabla e|^2 + e^2)dx - \int_\Omega F(x,te)dx \\ &\leq \frac{t^2}{2}\int_\Omega (|\nabla e|^2 + e^2)dx + c_4 t^2 \int_\Omega e^2 dx - c_3 t^\alpha \int_\Omega |e|^\alpha dx \\ &\to -\infty, \end{aligned}$$

as $t \to \infty$. Therefore, there exist $t_0 > 0$ large enough and let $e_0 := t_0 e$ such that $J_\lambda(e_0) \leq -1$ with $\|e_0\|_\lambda > \rho$. From Lemma 2.1(i), then

$$\begin{aligned} I_{\lambda,\mu}(e_0) &= J_\lambda(e_0) - \frac{\mu}{2}\int_{\mathbb{R}^3} \omega \phi_{e_0} e_0^2 dx \\ &\leq -1 + \frac{\mu\omega^2}{2}|e_0|_2^2, \end{aligned}$$

there exists $\mu_0 := \frac{2}{\omega^2 |e_0|_2^2} > 0$ (independent of λ) such that $I_{\lambda,\mu}(e_0) < 0$ for each $\lambda \geq 1$ and $\mu \in (0, \mu_0)$. The proof is completed. □

Then we consider the mountain pass value

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_{\lambda,\mu}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e_0\}$. From Proposition 2.3 and Lemma 3.1, we can obtain that for each $\lambda \geq 1$ and $\mu \in (0, \mu_0)$, there exists a Cerami sequence $\{u_n\} \subset E_\lambda$ such that

$$I_{\lambda,\mu}(u_n) \to c_{\lambda,\mu} > 0 \quad \text{and} \quad (1 + \|u_n\|_\lambda)\left\|I'_{\lambda,\mu}(u_n)\right\|_{E'_\lambda} \to 0 \quad \text{as } n \to \infty. \tag{3.7}$$

Next we prove that $c_{\lambda,\mu}$ has an upper bound.

Lemma 3.2. *Suppose (a_1)–(a_3) and $(f_1), (f_3)$ hold. Then for each $\lambda \geq 1$ and $\mu \in (0, \mu_0)$, there exists $M > 0$ (independent of μ and λ) such that $c_{\lambda,\mu} \leq M$.*

Proof. By $(f_1), (f_3)$, we have (3.6). Since $e_0 \in C_0^\infty(\Omega)$ and Lemma 3.1, we obtain that

$$\begin{aligned} I_{\lambda,\mu}(te_0) &= \frac{t^2}{2}\int_\Omega (|\nabla e_0|^2 + e_0^2)dx - \frac{\mu}{2}\int_\Omega \omega \phi_{te_0}(te_0)^2 dx - \int_\Omega F(x,te_0)dx \\ &\leq \frac{t^2}{2}\int_\Omega (|\nabla e_0|^2 + e_0^2)dx + \frac{\mu_0\omega^2}{2}t^2\int_\Omega e_0^2 dx + c_4 t^2 \int_\Omega e_0^2 dx - c_3 t^\alpha \int_\Omega |e_0|^\alpha dx, \end{aligned}$$

where $\alpha > 4$. Therefore, there exists $M > 0$ (independent of μ and λ) such that

$$c_{\lambda,\mu} \leq \max_{t\in[0,1]} I_{\lambda,\mu}(te_0) \leq M.$$

This completes the proof. □

Lemma 3.3. *Assume (a_1)–(a_3) and (f_1)–(f_3) hold, for each $\lambda > 1$, $\mu \in (0, \mu_0)$, if $\{u_n\} \subset E_\lambda$ is a sequence satisfying (3.7), then we have, up to a subsequence, $\{u_n\}$ is bounded in E_λ.*

Proof. From (3.2), (3.3), (f_3), Lemma 2.1(i) and Lemma 3.2, $\alpha > 4$, for $n \to \infty$ we have

$$\begin{aligned}
M + o(1) \geq c_{\lambda,\mu} + o(1) =& I_{\lambda,\mu}(u_n) - \frac{1}{\alpha}\langle I'_{\lambda,\mu}(u_n), u_n\rangle \\
=& \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|_\lambda^2 + \left(\frac{2}{\alpha} - \frac{1}{2}\right)\mu \int_{\mathbb{R}^3} \omega\phi_{u_n} u_n^2 dx \\
&+ \frac{\mu}{\alpha}\int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx + \int_{\mathbb{R}^3}\left(\frac{1}{\alpha} f(x,u_n)u_n - F(x,u_n)\right)dx \\
\geq& \frac{1}{4}\|u_n\|_\lambda^2,
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in E_λ and $\|u_n\|_\lambda \leq 2\sqrt{M}$ as $n \to \infty$, where M is given by Lemma 3.2. □

Then we will give the compactness conditions for $I_{\lambda,\mu}$. Before that, we introduce a lemma to deal with nonlinear term.

Lemma 3.4 ([14])**.** *Assume that (f_1) and (f_2) hold. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then along a subsequence of $\{u_n\}$,*

$$\lim_{n\to\infty} \sup_{\varphi \in H^1(\mathbb{R}^3), \|\varphi\|_{H^1} \leq 1} \left| \int_{\mathbb{R}^3} [f(x,u_n) - f(x,u_n - u) - f(x,u)]\varphi dx \right| = 0.$$

Lemma 3.5. *Suppose that (a_1)–(a_3) and (f_1)–(f_3) hold. If $\{u_n\} \subset E_\lambda$ is a sequence satisfying (3.7), up to a subsequence, there exists $\lambda_1^* \geq 1$ such that for each $\mu \in (0,\mu_0)$ and $\lambda \in (\lambda_1^*, \infty)$, $\{u_n\} \subset E_\lambda$ contains a convergent subsequence.*

Proof. By Lemma 3.3, we know that $\{u_n\}$ is bounded. We may assume that there exists $u \in E_\lambda$ such that

$$\begin{aligned}
u_n &\rightharpoonup u \quad \text{in } E_\lambda, \\
u_n &\to u \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^3), 2 \leq s < 6, \\
u_n &\to u \quad \text{a.e. in } \mathbb{R}^3.
\end{aligned} \tag{3.8}$$

From Lemma 2.2 and (3.7), we have $\langle I'_{\lambda,\mu}(u), u\rangle = 0$, i.e.,

$$\|u\|_\lambda^2 - 2\mu \int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \mu \int_{\mathbb{R}^3} \phi_u^2 u^2 dx - \int_{\mathbb{R}^3} f(x,u)u dx = 0. \tag{3.9}$$

Next, we prove that $u_n \to u$ in E_λ. Let $v_n = u_n - u$, by (a_2), then

$$|v_n|_2^2 = \int_{\mathbb{R}^3\setminus A_c} v_n^2 dx + \int_{A_c} v_n^2 dx \leq \frac{1}{c\lambda + 1}\|v_n\|_\lambda^2 + o(1) \leq \frac{1}{c\lambda}\|v_n\|_\lambda^2 + o(1). \tag{3.10}$$

It follows from Brézis–Lieb Lemma the ([26, Lemma 1.32]) that

$$\|u_n\|_\lambda^2 - \|u\|_\lambda^2 = \|v_n\|_\lambda^2 + o(1). \tag{3.11}$$

Then, by (3.10), Hölder and Sobolev inequalities, we have

$$|v_n|_p \leq |v_n|_2^\theta |v_n|_6^{1-\theta} \leq S^{\frac{\theta-1}{2}} |v_n|_2^\theta |\nabla v_n|_2^{1-\theta} \leq S^{\frac{\theta-1}{2}} (c\lambda)^{-\frac{\theta}{2}} \|v_n\|_\lambda + o(1), \tag{3.12}$$

where $\theta = \frac{6-p}{2p} > 0$. Employing Lemma 3.4, we have

$$\left| \frac{1}{\|u_n\|_\lambda} \int_{\mathbb{R}^3} [f(x,u_n) - f(x,v_n) - f(x,u)]u_n dx \right| = o(1).$$

From (3.8), $v_n \rightharpoonup 0$ in E_λ, $v_n \to 0$ in $L^s_{loc}(\mathbb{R}^3)$ for $2 \le s < 6$ and $v_n \to 0$ a.e. on $\mathbb{R}^3$, there have

$$\begin{aligned}\frac{1}{\|u_n\|_\lambda}\int_{\mathbb{R}^3} f(x,u_n)u_n dx \le & \frac{1}{\|u_n\|_\lambda}\left[\int_{\mathbb{R}^3} f(x,u)u + f(x,v_n)v_n + f(x,v_n)u + f(x,u)v_n dx\right] \\ & + \left|\frac{1}{\|u_n\|_\lambda}\int_{\mathbb{R}^3}[f(x,u_n) - f(x,v_n) - f(x,u)]u_n dx\right| \\ = & \frac{1}{\|u_n\|_\lambda}\left[\int_{\mathbb{R}^3} f(x,u)udx + \int_{\mathbb{R}^3} f(x,v_n)v_n dx\right] + o(1).\end{aligned} \tag{3.13}$$

From (3.5), (3.10)–(3.13), Lemma 3.3 and Fatou's Lemma, choose $\varepsilon = \frac{1}{2d_2^2}$, then

$$\begin{aligned} o(1) &= \langle I'_{\lambda,\mu}(u_n), u_n\rangle - \langle I'_{\lambda,\mu}(u), u\rangle \\ &= \|u_n\|^2_\lambda - \mu\int_{\mathbb{R}^3}(2\omega + \phi_{u_n})\phi_{u_n}u_n^2 dx - \int_{\mathbb{R}^3} f(x,u_n)u_n dx \\ &\quad -\|u\|^2_\lambda + \mu\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u u^2 dx + \int_{\mathbb{R}^3} f(x,u)udx \\ &\ge \|v_n\|^2_\lambda - \int_{\mathbb{R}^3} f(x,v_n)v_n dx + o(1) \\ &\ge \|v_n\|^2_\lambda - \varepsilon|v_n|_2^2 - C_\varepsilon|v_n|_p^p + o(1) \\ &\ge [\frac{1}{2} - C_\varepsilon(4d_p\sqrt{M})^{p-2}S^{\theta-1}(c\lambda)^{-\theta}]\|v_n\|^2_\lambda + o(1).\end{aligned}$$

Hence, there exists $\lambda_1 = [2C_\varepsilon(4d_p\sqrt{M})^{p-2}S^{\theta-1}c^{-\theta}]^{\frac{1}{\theta}}$ such that the previous coefficient of $\|v_n\|^2_\lambda$ is greater than 0 when $\lambda > \lambda_1$, where M is given by Lemma 3.2. Then choose $\lambda_1^* = \max\{\lambda_1, 1\}$ such that $v_n \to 0$ in E_λ for all $\lambda > \lambda_1^*$. □

Proof of Theorem 1.2. Assume (a_1)–(a_3) and (f_1)–(f_3) are satisfied. By Lemma 3.1, there exists $\mu_0 > 0$ such that for every $\lambda \ge 1$ and $\mu \in (0,\mu_0)$, $I_{\lambda,\mu}$ possesses a Cerami sequence $\{u_n\}$ at the mountain pass level $c_{\lambda,\mu}$ and satisfied

$$I_{\lambda,\mu}(u_n) \to c_{\lambda,\mu} > 0 \quad \text{and} \quad (1 + \|u_n\|_\lambda)\|I'_{\lambda,\mu}(u_n)\|_{E'_\lambda} \to 0, \text{ as } n \to \infty.$$

From Lemmas 3.2 and 3.3, we thus deduce that for every $\lambda \ge 1$ and $\mu \in (0,\mu_0)$, after passing to a subsequence, $\{u_n\}$ is bounded in E_λ and $\|u_n\|_\lambda \le 2\sqrt{M}$ as $n \to \infty$. It follows from Lemma 3.5 that exists $\lambda_1^* \ge 1$ such that for each $\mu \in (0,\mu_0)$ and $\lambda \in (\lambda_1^*, \infty)$, the sequence $\{u_n\}$ has a convergent subsequence in E_λ. Then there exists $u_{\lambda,\mu} \in E_\lambda$, such that $u_n \to u_{\lambda,\mu}$ as $n \to \infty$, and thus

$$\|u_{\lambda,\mu}\|_\lambda \le 2\sqrt{M}, \quad I_{\lambda,\mu}(u_{\lambda,\mu}) = c_{\lambda,\mu} \quad \text{and} \quad I'_{\lambda,\mu}(u_{\lambda,\mu}) = 0.$$

Now we claim that $u_{\lambda,\mu} \ne 0$. Otherwise, $I_{\lambda,\mu}(u_{\lambda,\mu}) = 0 = c_{\lambda,\mu}$, which is a contradiction to $c_{\lambda,\mu} > 0$. Moreover, by the Hölder inequality, Lemma 2.1(ii) and Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^3}\phi^2_{u_{\lambda,\mu}}u^2_{\lambda,\mu}dx &\le \left(\int_{\mathbb{R}^3}(\phi^2_{u_{\lambda,\mu}})^3 dx\right)^{\frac{1}{3}}\left(\int_{\mathbb{R}^3}(u^2_{\lambda,\mu})^{\frac{3}{2}}dx\right)^{\frac{2}{3}} \\ &\le Cd_3^2\|\phi_{u_{\lambda,\mu}}\|^2_{D^{1,2}}\|u_{\lambda,\mu}\|^2_\lambda \\ &\le \widetilde{C}\|u_{\lambda,\mu}\|^4_\lambda.\end{aligned}$$

Then, since $\langle I'_{\lambda,\mu}(u_{\lambda,\mu}), u_{\lambda,\mu}\rangle = 0$, Lemma 2.1(i) and (3.4), there exists $\varepsilon = \frac{1}{2d_2^2}$, we have

$$\begin{aligned}\|u_{\lambda,\mu}\|_\lambda^2 &= 2\mu \int_{\mathbb{R}^3} \omega\phi_{u_{\lambda,\mu}} u_{\lambda,\mu}^2 dx + \mu \int_{\mathbb{R}^3} \phi_{u_{\lambda,\mu}}^2 u_{\lambda,\mu}^2 dx + \int_{\mathbb{R}^3} f(x, u_{\lambda,\mu}) u_{\lambda,\mu} dx \\ &\le \mu_0 \widetilde{C}_1 \|u_{\lambda,\mu}\|_\lambda^4 + \varepsilon d_2^2 \|u_{\lambda,\mu}\|_\lambda^2 + C_\varepsilon d_p^p \|u_{\lambda,\mu}\|_\lambda^p.\end{aligned}$$

Hence, there exists $\tau_0 > 0$ (independent of μ and λ) such that $\|u_{\lambda,\mu}\|_\lambda \ge \tau_0$ for all $\mu \in (0, \mu_0)$ and $\lambda \in (\lambda_1^*, \infty)$. This finishes the proof. □

Proof of Theorem 1.3. Let $\mu \in (0, \mu_0)$ be fixed, then for any sequence $\lambda_n \to +\infty$. Let $u_n := u_{\lambda_n,\mu}$ be the nontrivial solution of (1.4) obtained by Theorem 1.2. From Theorem 1.2 we have

$$0 < \tau_0 \le \|u_n\|_{\lambda_n} \le 2\sqrt{M} \quad \text{for } n \to \infty. \tag{3.14}$$

Thus, up to a subsequence, we may assume that

$$\begin{aligned} u_n &\rightharpoonup u_\mu \quad ; \text{in } E, \\ u_n &\to u_\mu \quad \text{in } L^s_{\text{loc}}(\mathbb{R}^3),\ 2 \le s < 6, \\ u_n &\to u_\mu \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

It follows from (3.14), Fatou's Lemma and (a_1) that

$$0 \le \int_{\mathbb{R}^3} (a(x)+1) u_\mu^2 dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^3} (a(x)+1) u_n^2 dx \le \liminf_{n\to\infty} \frac{\|u_n\|_{\lambda_n}^2}{\lambda_n} = 0.$$

Hence, $u_\mu = 0$ a.e. in $\mathbb{R}^3 \setminus a^{-1}(0)$, and $u_\mu \in H^1(\Omega)$ by the condition (a_3).

Now we show that $u_n \to u_\mu$ in $L^s(\mathbb{R}^3)$ for all $s \in (2,6)$. Otherwise, by Lions' vanishing Lemma ([18,26]) there exist $\delta, r > 0$ and $x_n \in \mathbb{R}^3$ such that

$$\int_{B_r(x_n)} (u_n - u_\mu)^2 dx \ge \delta.$$

This implies that $|x_n| \to \infty$ as $n \to \infty$, and so $|B_r(x_n) \cap A_c| \to 0$. By the Hölder inequality, we then conclude that

$$\int_{B_r(x_n)\cap A_c} (u_n - u_\mu)^2 dx \to 0 \quad \text{as } n \to \infty.$$

Consequently, we get

$$\begin{aligned}\|u_n\|_{\lambda_n}^2 &\ge (c\lambda_n + 1) \int_{B_r(x_n)\cap\{a(x)\ge c\}} u_n^2 dx = (c\lambda_n+1) \int_{B_r(x_n)\cap\{a(x)\ge c\}} (u_n - u_\mu)^2 dx \\ &= (c\lambda_n+1)\left(\int_{B_r(x_n)} (u_n-u_\mu)^2 dx - \int_{B_r(x_n)\cap A_c} (u_n - u_\mu)^2 dx \right) \to \infty,\end{aligned}$$

as $n \to \infty$, which contradicts (3.14).

We then prove that $u_n \to u_\mu$ in E. Since

$$\langle I'_{\lambda_n,\mu}(u_n), u_n\rangle = \langle I'_{\lambda_n,\mu}(u_n), u_\mu\rangle = 0,$$

we have

$$\|u_n\|_{\lambda_n}^2 - 2\mu \int_{\mathbb{R}^3} \omega\phi_{u_n} u_n^2 dx - \mu \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx - \int_{\mathbb{R}^3} f(x,u_n) u_n dx = 0,$$

$$\int_{\mathbb{R}^3} (\nabla u_n \nabla u_\mu + (\lambda_n a(x)+1)u_n u_\mu)\,dx - 2\mu \int_{\mathbb{R}^3} \omega\phi_{u_n} u_n u_\mu dx - \mu \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n u_\mu dx - \int_{\mathbb{R}^3} f(x,u_n)u_\mu dx = 0.$$

Since $u_\mu = 0$ a.e. in $\mathbb{R}^3 \setminus a^{-1}(0)$ and by Lemma 2.1(ii), (3.14), we deduce that

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n(u_n - u_\mu)dx \le |\phi_{u_n}|_6 |u_n|_2 |u_n - u_\mu|_3 \le C\|\phi_{u_n}\|_{D^{1,2}} \|u_n\|_\lambda |u_n - u_\mu|_3 \to 0,$$

$$\int_{\mathbb{R}^3} \phi_{u_n}^2 u_n(u_n - u_\mu)dx \le |\phi_{u_n}|_6^2 |u_n|_3 |u_n - u_\mu|_3 \le C\|\phi_{u_n}\|_{D^{1,2}}^2 \|u_n\|_\lambda |u_n - u_\mu|_3 \to 0,$$

$$\int_{\mathbb{R}^3} f(x,u_n)(u_n - u_\mu)dx \to 0.$$

Thus, $\lim_{n\to\infty} \|u_n\|_{\lambda_n}^2 = \|u_\mu\|^2$. Then from the weakly lower semi-continuity of norm, we have

$$\|u_\mu\|^2 \le \liminf_{n\to\infty} \|u_n\|^2 \le \limsup_{n\to\infty} \|u_n\|^2 \le \limsup_{n\to\infty} \|u_n\|_{\lambda_n}^2 = \|u_\mu\|^2. \tag{3.15}$$

Consequently, we yield that $u_n \to u_\mu$ in E.

Finally, we only need to show that u_μ is a weak solution of (1.6). Now for any $v \in C_0^\infty(\Omega)$, since $\langle I'_{\lambda_n,\mu}(u_n), v\rangle = 0$, it is easy to check that

$$\int_\Omega (\nabla u_\mu \nabla v + u_\mu v)dx - \mu \int_\Omega (2\omega + \phi_{u_\mu})\phi_{u_\mu} u_\mu v dx - \int_\Omega f(x,u_\mu)v dx = 0.$$

i.e., u_μ is a weak solution of (1.6) by the density of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. From (3.14) and (3.15), we can see that

$$\|u_\mu\| = \lim_{n\to\infty} \|u_n\|_{\lambda_n} \ge \tau_0 > 0,$$

so $u_\mu \ne 0$. Thus, u_μ is a nontrivial weak solution of (1.6). □

Proof of Theorem 1.4. Let $\lambda \in (\lambda_1^*, \infty)$ be fixed, then for any sequence $\mu_n \to 0$. Let $u_n := u_{\lambda,\mu_n}$ be the nontrivial solution of (1.4) obtained by Theorem 1.2. From Theorem 1.2, we have

$$0 < \tau_0 \le \|u_n\|_{\lambda_n} \le 2\sqrt{M} \quad \text{for } n \to \infty. \tag{3.16}$$

Passing to a subsequence , we may assume that $u_n \rightharpoonup u_\lambda$ in E_λ. Note that $I'_{\lambda,\mu_n}(u_n) = 0$, we can obtain that $u_n \to u_\lambda$ in E_λ as the proof of Lemma 3.5.

To complete the proof, we will show that u_λ is a nontrivial solution of (1.7). Now for any $v \in E_\lambda$, since $\langle I'_{\lambda,\mu_n}(u_n), v\rangle = 0$, it is easy to check that

$$\int_{\mathbb{R}^3} \nabla u_\lambda \nabla v + (\lambda a(x)+1)u_\lambda v dx = \int_{\mathbb{R}^3} f(x,u_\lambda)v dx,$$

i.e., u_λ is a weak solution of (1.7). Then, by (3.16) we see that $u_\lambda \ne 0$. Therefore, u_λ is a nontrivial weak solution of (1.7). This completes the proof. □

Proof of Theorem 1.5. Following the same argument as in the proof of Theorems 1.3 and 1.4, we get the conclusion. □

4 The existence and concentration phenomenon of solution of (1.9)

In this section, we will give the asymptotic behavior of positive solution of (1.9), and devote to prove Theorem 1.8 to Theorem 1.11. We will use truncation technique to obtain the boundedness of Cerami sequence. Before that, we write the functional corresponding to (1.9). $\hat{I}_{\lambda,\mu} : E_\lambda \to \mathbb{R}$ given by

$$\hat{I}_{\lambda,\mu}(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + (\lambda a(x)+1)u^2)dx - \frac{\mu}{2}\int_{\mathbb{R}^3}\omega\phi_u u^2 dx - \frac{1}{q}\int_{\mathbb{R}^3}|u^+|^q dx \tag{4.1}$$

and $\hat{I}_{\lambda,\mu} \in C^1$. Moreover, for any $u, v \in E_\lambda$, we have

$$\langle \hat{I}'_{\lambda,\mu}(u), v\rangle = \int_{\mathbb{R}^3}(\nabla u\cdot\nabla v + (\lambda a(x)+1)uv)\,dx - \mu\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u uv dx - \int_{\mathbb{R}^3}|u^+|^{q-2}u^+ v dx. \tag{4.2}$$

Then we define a cut-off function $\eta \in C^1([0,\infty),\mathbb{R})$ satisfying $0 \le \eta \le 1$, $\eta(t) = 1$ if $0 \le t \le 1$, $\eta(t) = 0$ if $t \ge 2$, $\max_{t>0}|\eta'(t)| \le 2$ and $\eta'(t) \le 0$ for each $t > 0$. Using η, for every $T > 0$ we then consider the truncated functional $\hat{I}^T_{\lambda,\mu} : E_\lambda \to \mathbb{R}$ defined by

$$\hat{I}^T_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2_\lambda - \frac{\mu}{2}\eta\left(\frac{\|u\|^2_\lambda}{T^2}\right)\int_{\mathbb{R}^3}\omega\phi_u u^2 dx - \frac{1}{q}\int_{\mathbb{R}^3}|u^+|^q dx. \tag{4.3}$$

We can see that $\hat{I}^T_{\lambda,\mu}$ is of class C^1, then for each $u, v \in E_\lambda$,

$$\begin{aligned}\langle(\hat{I}^T_{\lambda,\mu})'(u), v\rangle =& \langle u,v\rangle_\lambda - \frac{\mu}{T^2}\eta'\left(\frac{\|u\|^2_\lambda}{T^2}4.1\right)\langle u,v\rangle_\lambda\int_{\mathbb{R}^3}\omega\phi_u u^2 dx\\ &- \mu\eta\left(\frac{\|u\|^2_\lambda}{T^2}\right)\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u uv dx - \int_{\mathbb{R}^3}|u^+|^{q-2}u^+ v dx.\end{aligned} \tag{4.4}$$

It is easy to see that every nontrivial critical point of $\hat{I}_{\lambda,\mu}$ is a positive solution of (1.9), and we will prove it in the following lemma.

Lemma 4.1. *Suppose that $2 < q < 4$ and (a_1)–(a_3) are satisfied. Then every nontrivial critical point of $\hat{I}_{\lambda,\mu}$ is a positive solution of (1.9).*

Proof. Let $u \in E_\lambda$ be a nontrivial critical point of $\hat{I}_{\lambda,\mu}$, then $\langle \hat{I}'_{\lambda,\mu}(u), v\rangle = 0$ for all $v \in E_\lambda$. We have

$$\int_{\mathbb{R}^3}(\nabla u\cdot\nabla v + (\lambda a(x)+1)uv)\,dx - \mu\int_{\mathbb{R}^3}(2\omega+\phi_u)\phi_u uv dx - \int_{\mathbb{R}^3}|u^+|^{p-2}u^+ v dx = 0. \tag{4.5}$$

Taking $v = u^- = -\min\{u, 0\}$ in (4.5), by Lemma 2.1(i) we obtain that

$$\|u^-\|^2_\lambda \le \|u^-\|^2_\lambda - \mu\int_{\mathbb{R}^3}(2\omega+\phi_{u^-})\phi_{u^-}|u^-|^2 dx = 0$$

which is a contradiction. Then we can see $u \ge 0$ in $\mathbb{R}^3$. Hence, the strong maximum principle and the fact $u \ne 0$ imply that $u > 0$ in $\mathbb{R}^3$, and the proof is ready. □

Lemma 4.2. *Suppose $2 < q < 4$ and (a_1)–(a_3) are satisfied. There exists $\mu_1 > 0$ for each $T > 0$, $\mu \in (0, \mu_1)$ and $\lambda \ge 1$. Then there exist $\bar{\rho}, \bar{\beta} > 0$ and $\bar{e}_0 \in E_\lambda$, $\|\bar{e}_0\|_\lambda > \bar{\rho}$, such that*

$$\inf_{\|u\|=\bar{\rho}} \hat{I}^T_{\lambda,\mu}(u) \ge \bar{\beta} > 0 \ge \max\left\{\hat{I}^T_{\lambda,\mu}(0), \hat{I}^T_{\lambda,\mu}(\bar{e}_0)\right\}.$$

Proof. From Lemma 2.1, (2.1) and (4.3), for each $u \in E_\lambda$, we have

$$\hat{I}^T_{\lambda,\mu}(u) \geq \frac{1}{2}\|u\|^2_\lambda - \frac{1}{q}d_q^q\|u\|^q_\lambda = \|u\|^2_\lambda\left(\frac{1}{2} - \frac{1}{q}d_q^q\|u\|^{q-2}_\lambda\right),$$

where the constant $d_q > 0$ is independent of T, μ and λ. Since $q > 2$, there exist $\bar{\rho} > 0$ small enough and $\bar{\beta} > 0$, such that $\inf_{\|u\|=\bar{\rho}} \hat{I}^T_{\lambda,\mu}(u) \geq \bar{\beta} > 0$.

Then, we define the functional $\hat{J}_\lambda : E_\lambda \to \mathbb{R}$ by

$$\hat{J}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + (\lambda a(x) + 1)u^2)dx - \frac{1}{q}\int_{\mathbb{R}^3}|u^+|^q dx.$$

Since $2 < q < 4$, similar to Lemma 3.1, there also exist $\bar{t}_0 > 0$ large enough and let $\bar{e}_0 := \bar{t}_0 e$ such that $\hat{J}_\lambda(\bar{e}_0) \leq -1$ with $\|\bar{e}_0\|_\lambda > \bar{\rho}$. From Lemma 2.1(i), then

$$\begin{aligned}\hat{I}^T_{\lambda,\mu}(\bar{e}_0) &= \hat{J}_\lambda(\bar{e}_0) - \frac{\mu}{2}\eta\left(\frac{\|\bar{e}_0\|^2_\lambda}{T^2}\right)\int_{\mathbb{R}^3}\omega\phi_{\bar{e}_0}\bar{e}_0^2 dx \\ &\leq -1 + \frac{\mu\omega^2}{2}|\bar{e}_0|^2_2,\end{aligned}$$

there exists $\mu_1 := \frac{2}{\omega^2|\bar{e}_0|^2_2} > 0$ (independent of λ and T) such that $\hat{I}^T_{\lambda,\mu}(\bar{e}_0) < 0$ for each $T, \lambda \geq 1$ and $\mu \in (0, \mu_1)$. The proof is completed. □

Then we also can consider the mountain pass value

$$\hat{c}^T_{\lambda,\mu} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\hat{I}^T_{\lambda,\mu}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = \bar{e}_0\}$. From Proposition 2.3 and Lemma 4.2, we obtain that for each $T > 0$, $\lambda \geq 1$ and $\mu \in (0, \mu_1)$, there exists a Cerami sequence $\{\hat{u}_n\} \subset E_\lambda$ such that

$$\hat{I}^T_{\lambda,\mu}(\hat{u}_n) \to \hat{c}^T_{\lambda,\mu} > 0 \quad \text{and} \quad (1 + \|\hat{u}_n\|_\lambda)\left\|(\hat{I}^T_{\lambda,\mu})'(\hat{u}_n)\right\|_{E'_\lambda} \to 0 \quad \text{as } n \to \infty. \tag{4.6}$$

Since $2 < q < 4$ and the definition of $\eta(t)$, similar proof to Lemma 3.2, we can find a $\bar{M} > 0$ such that $\hat{c}^T_{\lambda,\mu}$ has an upper bound, i.e.,

$$\hat{c}^T_{\lambda,\mu} \leq \bar{M}. \tag{4.7}$$

Lemma 4.3. *Assume $2 < q < 4$ and (a_1)–(a_3) hold, let $T = \sqrt{\frac{2q(\bar{M}+1)}{q-2}}$. Then there exists $\mu_2 > 0$ small enough, for each $\lambda \geq 1$, $\mu \in (0, \min\{\mu_1, \mu_2\})$, if $\{\hat{u}_n\} \subset E_\lambda$ is a sequence satisfying (4.6), then we have, up to a subsequence,*

$$\sup_{n\in\mathbb{N}}\|\hat{u}_n\|_\lambda \leq T.$$

Proof. Otherwise, there exists a subsequence of $\{\hat{u}_n\}$, still denoted by $\{\hat{u}_n\}$ such that $\|\hat{u}_n\|_\lambda > T$. It can be divided into two situations:

(i) $T < \|\hat{u}_n\|_\lambda < \sqrt{2}T$; (ii) $\|\hat{u}_n\|_\lambda \geq \sqrt{2}T$.

Firstly, for the case (i), due to (4.3), (4.4) and Lemma 2.1 we have

$$\begin{aligned}\bar{M}+o(1)\geq \hat{c}^T_{\lambda,\mu}+o(1) &= \hat{I}^T_{\lambda,\mu}(\hat{u}_n)-\frac{1}{q}\langle(\hat{I}^T_{\lambda,\mu})'(\hat{u}_n),\hat{u}_n\rangle\\ &=\left(\frac{1}{2}-\frac{1}{q}\right)\|\hat{u}_n\|^2_\lambda+\frac{\mu}{qT^2}\eta'\left(\frac{\|\hat{u}_n\|^2_\lambda}{T^2}\right)\|\hat{u}_n\|^2_\lambda\int_{\mathbb{R}^3}\omega\phi_{\hat{u}_n}\hat{u}_n^2dx\\ &\quad+\left(\frac{2}{q}-\frac{1}{2}\right)\mu\eta\left(\frac{\|\hat{u}_n\|^2_\lambda}{T^2}\right)\int_{\mathbb{R}^3}\omega\phi_{\hat{u}_n}\hat{u}_n^2dx+\frac{\mu}{q}\eta\left(\frac{\|\hat{u}_n\|^2_\lambda}{T^2}\right)\int_{\mathbb{R}^3}\phi^2_{\hat{u}_n}\hat{u}_n^2dx\\ &\geq\left(\frac{1}{2}-\frac{1}{q}\right)\|\hat{u}_n\|^2_\lambda-\left(\frac{2}{q}-\frac{1}{2}\right)\mu\omega^2d_2^2\|\hat{u}_n\|^2_\lambda\\ &\geq(\bar{M}+1)-\frac{2(4-q)\mu\omega^2d_2^2}{q-2}(\bar{M}+1),\end{aligned}$$

which is a contradiction when we choose $\mu_2 := \frac{q-2}{2(4-q)\omega^2 d_2^2(\bar{M}+1)} > 0$ such that $\mu \in (0,\min\{\mu_1,\mu_2\})$. Then we deduce that $\|\hat{u}_n\|_\lambda \geq \sqrt{2}T$ for n large enough. With the definition of $\eta(t)$, we conclude that

$$\begin{aligned}\bar{M}+o(1)\geq \hat{c}^T_{\lambda,\mu}+o(1) =&\hat{I}^T_{\lambda,\mu}(\hat{u}_n)-\frac{1}{q}\langle(\hat{I}^T_{\lambda,\mu})'(\hat{u}_n),\hat{u}_n\rangle\\ =&\left(\frac{1}{2}-\frac{1}{q}\right)\|\hat{u}_n\|^2_\lambda\\ \geq&2(\bar{M}+1),\end{aligned}$$

this is obviously a contradiction. The proof of this lemma ends. □

Up to now, we have proved that the sequence $\{\hat{u}_n\}$ given by (4.6) satisfies $\|\hat{u}_n\|_\lambda \leq T$. In particular, this sequence $\{\hat{u}_n\}$ is also a Cerami sequence at level $\hat{c}^T_{\lambda,\mu}$ for $\hat{I}_{\lambda,\mu}$, i.e.,

$$\hat{I}_{\lambda,\mu}(\hat{u}_n)\to\hat{c}^T_{\lambda,\mu}>0\quad\text{and}\quad(1+\|\hat{u}_n\|_\lambda)\left\|\hat{I}'_{\lambda,\mu}(\hat{u}_n)\right\|_{E'_\lambda}\to 0\quad\text{as } n\to\infty.$$

Then we will give the compactness conditions for $\hat{I}_{\lambda,\mu}$.

Lemma 4.4. *Suppose that $2 < q < 4$ and (a_1)–(a_3) hold. If $\{\hat{u}_n\} \subset E_\lambda$ is a sequence satisfying (4.6), up to a subsequence, there exists $\lambda_2^* \geq 1$ such that for each $\mu \in (0,\min\{\mu_1,\mu_2\})$ and $\lambda \in (\lambda_2^*,\infty)$, $\{\hat{u}_n\} \subset E_\lambda$ contains a convergent subsequence.*

Proof. Proof is similar to Lemma 3.5, there exists $\lambda_2 = [(2d_qT)^{q-2}S^{\theta-1}c^{-\theta}]^{\frac{1}{\theta}}$ where $\theta = \frac{6-q}{2q} > 0$ and choose $\lambda_2^* = \max\{\lambda_2, 1\}$ such that $(\hat{u}_n - \hat{u}) \to 0$ in E_λ for all $\lambda > \lambda_2^*$. □

Proof of Theorem 1.8. Assume $2 < q < 4$ and (a_1)–(a_3) are satisfied. By Lemma 4.2, there exists $\mu_1 > 0$ such that for every $\lambda \geq 1$ and $\mu \in (0,\mu_1)$, $\hat{I}^T_{\lambda,\mu}$ possesses a Cerami sequence $\{\hat{u}_n\}$ at the mountain pass level $\hat{c}^T_{\lambda,\mu}$. From (4.7) and Lemma 4.3, we thus deduce that there exist $\mu_2 > 0$ such that for every $\lambda \geq 1$ and $\mu \in (0,\min\{\mu_1,\mu_2\})$, after passing to a subsequence, $\{\hat{u}_n\}$ is a Cerami sequence of $\hat{I}_{\lambda,\mu}$ satisfying $\|\hat{u}_n\|_\lambda \leq T$, i.e.,

$$\sup_{n\in\mathbb{N}}\|\hat{u}_n\|_\lambda\leq T,\quad\hat{I}_{\lambda,\mu}(\hat{u}_n)\to\hat{c}^T_{\lambda,\mu}\quad\text{and}\quad(1+\|\hat{u}_n\|_\lambda)\|\hat{I}'_{\lambda,\mu}(\hat{u}_n)\|_{E'_\lambda}\to 0,\text{ as } n\to\infty.$$

It follows from Lemma 4.4 that exists $\lambda_2^* \geq 1$ such that for each $\mu \in (0, \min\{\mu_1, \mu_2\})$ and $\lambda \in (\lambda_2^*, \infty)$, the sequence $\{\hat{u}_n\}$ has a convergent subsequence in E_λ. Then there exists $\hat{u}_{\lambda,\mu} \in E_\lambda$, such that $\hat{u}_n \to \hat{u}_{\lambda,\mu}$ as $n \to \infty$, and thus

$$\|\hat{u}_{\lambda,\mu}\|_\lambda \leq T, \quad \hat{I}_{\lambda,\mu}(\hat{u}_{\lambda,\mu}) = \hat{c}^T_{\lambda,\mu} \quad \text{and} \quad \hat{I}'_{\lambda,\mu}(\hat{u}_{\lambda,\mu}) = 0.$$

Similarly, we can prove that $\hat{u}_{\lambda,\mu} \neq 0$ and there exists $\tau_1 > 0$ (independent of μ and λ) such that $\|\hat{u}_{\lambda,\mu}\|_\lambda \geq \tau_1$ for all $\mu \in (0, \min\{\mu_1, \mu_2\})$ and $\lambda \in (\lambda_2^*, \infty)$. □

Proof of Theorem 1.9 to Theorem 1.11. Please refer to the proofs of Theorems 1.3 to 1.5. The detailed proofs are omitted here.

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Periodic perturbations of reducible scalar second order functional differential equations

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Abstract. Using a topological approach we investigate the structure of the set of forced oscillations of a class of reducible second order functional retarded differential equations subject to periodic forcing. More precisely, we consider a delay-type functional dependence involving a gamma probability distribution and, using a linear chain trick, we formulate a first order system of ODEs whose T-periodic solutions correspond to those of the functional equation.

Keywords: functional differential equations, branches of periodic solutions, linear chain trick, Brouwer degree.

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1 Introduction and setting of the problem

Integro-differential and functional-delayed equations are often used to describe phenomena whose dynamics depends on the past states of the system of concern. We mention, in particular, scalar second order functional differential equation of the following form:

$$\ddot{x}(t) = g\left(x(t), \dot{x}(t), \int_{-\infty}^{t} \mathcal{K}(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s\right), \tag{1.1}$$

where $g\colon \mathbb{R}^3 \to \mathbb{R}$ is a continuous map, $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\mathcal{K}$ is some integral kernel. Such equations, often studied in the first order case, have been considered in many contexts. A comprehensive bibliography is beyond the scope of this paper, we only mention a few relevant papers and books, e.g., [14, 22, 32, 34, 35, 40, 41].

In this paper we focus on the case when the integral kernel in (1.1) is the gamma probability distribution γ_a^b, for $a > 0$ and $b \in \mathbb{N} \setminus \{0\}$ given by

$$\gamma_a^b(s) = \frac{a^b s^{b-1} \mathrm{e}^{-as}}{(b-1)!} \ \text{ for } s \geq 0, \qquad \gamma_a^b(s) = 0 \ \text{ for } s < 0, \tag{1.2}$$

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with mean b/a and variance b/a^2. Namely, we study the following equation:

$$\ddot{x}(t) = g\left(x(t), \dot{x}(t), \int_{-\infty}^{t} \gamma_a^b(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s\right). \tag{1.3}$$

An equation as (1.3) is often called *reducible* because, unlike equation (1.1) whose dynamics is, in general, infinite dimensional (see, e.g., [31]), equation (1.3) can be essentially reduced to a system of ordinary differential equations by the so-called *linear chain trick* (see Section 2). Indeed, backtracking the linear chain trick procedure, one can see how equations of the form (1.3) may arise from the coupling of a nonlinear equation with a linear one (see, e.g., [35, Ch. 10]). We will briefly come back on this last topic in Section 8 on perspectives and further developments. Another of the reasons that motivates the interest in this particular type of kernels is that, at least for some linear functional equations containing a convolution-like term as in equation (1.1) with general kernel, it is possible to construct gamma-like kernels such that the solutions of the corresponding equations approximate those of the equation we started with (see [6, 9, 10]).

The particular dependence on the past of the solution that is considered in equation (1.3) naturally arises in many contexts (not necessarily for second order equations) see, e.g., [5, 10, 21, 22, 34, 35, 40, 41]. In the equation we examine, the delay is spread along the whole history but more concentrated at a given distance in the past. We can interpret this, from a probabilistic point of view, as a delay that follows a gamma-type distribution with a given mean (and variance). This approach seems to be reasonable in contexts when the delay cannot be measured with precision, but its mean value and variance are known, or it is genuinely spread in time as, e.g., in [18, 19]. Notice, in particular, that letting a and b tend to infinity in such a way that the quotient $r := a/b$ remains constant (for instance, put $a = nr$ and $b = n$ and let $n \to \infty$), one concentrates the memory effect close to the delay r. Indeed, at least in the sense of distributions, (1.3) approximates the second order delay differential equation

$$\ddot{x}(t) = g\left(x(t), \dot{x}(t), \varphi\big(x(t-r), \dot{x}(t-r)\big)\right).$$

Observe that the function γ_a^b defined in (1.2) is continuous for $b = 2, 3, \dots$ but not for $b = 1$. However, also in the latter case, assuming x in C^1, the function

$$t \mapsto \int_{-\infty}^{t} \gamma_a^1(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s$$

is continuous. Then the right hand side of (1.3) is continuous as well; consequently, any C^1 solution of (1.3) is actually of class C^2. It is also worth noticing (and we will use this fact later) that, since in the integral one has $t - s \geq 0$, then

$$\int_{-\infty}^{t} \gamma_a^1(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s = \int_{-\infty}^{t} a\mathrm{e}^{-a(t-s)}\varphi(x(s), \dot{x}(s))\,\mathrm{d}s,$$

so that the function $t \mapsto \int_{-\infty}^{t} \gamma_a^1(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s$ is actually in C^1.

Our main concern will be the structure of periodic solutions of (1.3) when subject to a periodic forcing. Given $T > 0$, we consider the following T-periodic perturbation of (1.3):

$$\ddot{x}(t) = g\left(x(t), \dot{x}(t), \int_{-\infty}^{t} \gamma_a^b(t-s)\varphi(x(s), \dot{x}(s))\,\mathrm{d}s\right) + \lambda f\big(t, x(t), \dot{x}(t)\big), \tag{1.4}$$

in which we assume that the map $f\colon \mathbb{R}^3 \to \mathbb{R}$ is continuous and T-periodic in the first variable, and λ is a nonnegative real parameter. Our main purpose is to investigate the set of T-periodic solutions of (1.4). Here, given $\lambda \geq 0$, by a *T-periodic solution* (on $\mathbb{R}$) of (1.4) we mean a C^2 function $x\colon \mathbb{R} \to \mathbb{R}$ of period T that satisfies identically equality (1.4). In that case, we will call (λ, x) a *T-forced pair* of (1.4). The T-forced pairs of the type $(0, x)$ with x constant will be called *trivial*.

Roughly speaking, we explicitly construct a scalar function Φ whose change of sign implies the existence of a connected set, called a *"branch"*, of nontrivial pairs (λ, x) that emanates out of the set of zeros of Φ and whose closure – in a suitable Banach space – is not compact.

We also give sufficient conditions yielding the multiplicity of T-periodic solutions of (1.4) for $\lambda > 0$ small. Such conditions are essentially based upon the notion of *ejecting set* (see, e.g., [26]).

The methods that we employ are topological in nature and are based on the Brouwer degree. Nevertheless, the use of our main results is accessible even without a familiarity with this notion since the application of the degree theory is restricted to the proofs and some preliminary results proved elsewhere. This is possible since we deal with scalar equations for which degree-theoretic assumptions can be replaced with more elementary ones. It would be not so if the vector case was considered, see Section 8 for a brief discussion. However, for completeness, we provide a short summary of degree theory in Section 4.

Our results are so-to-speak dual to those of [12], where periodic perturbations containing delay terms are applied to scalar, second order ODEs. The study, by means of topological methods, of the branching and multiplicity of periodic solutions of periodically perturbed equations, is now a well-investigated subject in the case of ODEs both in Euclidean spaces and on manifolds (see, e.g., [25, 28]). The case of ODEs perturbed with delayed forcing terms is also studied in the literature, although not so broadly (see, e.g., [12, 13]). However, the presence of delay terms in the *unperturbed* equation (1.3) is peculiar of the problem addressed here. In spite of the apparent similarities, a different approach is called upon in order to manage the unperturbed equation.

In the undelayed case, that is for periodically forced second-order scalar autonomous ODEs, both the problems of existence and multiplicity of periodic solutions are quite classical. However, this is still the subject of active research by the mathematical community. There are many approaches that have been successfully pursued to get multiplicity results: among the others let us mention here, e.g., the recent contributions [3, 4, 11, 23], the survey papers [37, 38] as well as the monograph [24] and the references therein.

The strategy adopted in this paper is inspired to [42] and can be summarized as follows. First we construct a system of $b+2$, T-periodically perturbed first-order ordinary differential equations whose T-periodic solutions correspond to that of (1.4). We then use known results about these perturbed systems to ensure the existence of a branch as sought when the topological degree of the unperturbed field G has nonzero degree and finally, we show, by homotopy techniques, that the change of sign of Φ implies that G has nonzero degree.

It should be noted that the result just described does not guarantee the existence of forced oscillations even for very small values of λ. For this reason, we also provide a nonlocal condition, based on an inequality of J. A. Yorke [43], implying that the branch projects nontrivially onto $[0, \infty)$.

In order to develop a better understanding of the nature of the branch of T-periodic solutions and its relation with multiplicity results we discuss, following [2] and [42], a method to visualize, in finite dimension, a homemorphic set that retains all the relevant properties. We illustrate the procedure with a numeric example.

2 The linear chain trick

As pointed out in the introduction, the purpose of this paper is to investigate the structure of the set of the T-periodic solutions of the parametric ODE

$$\ddot{x}(t) = g\left(x(t), \dot{x}(t), \int_{-\infty}^{t} \gamma_a^b(t-s)\varphi(x(s), \dot{x}(s))\, \mathrm{d}s\right) + \lambda f\big(t, x(t), \dot{x}(t)\big), \tag{2.1}$$

where we make the following set of assumptions:

$$\begin{aligned} &i. \quad g\colon \mathbb{R}^3 \to \mathbb{R} \text{ is a continuous map;}\\ &ii. \quad \varphi\colon \mathbb{R}^2 \to \mathbb{R} \text{ is a continuous map;}\\ &iii. \quad \gamma_a^b, \text{ for } a > 0 \text{ and } b \in \mathbb{N}\setminus\{0\}, \text{ represents the gamma probability distribution (1.2);}\\ &iv. \quad f\colon \mathbb{R}^3 \to \mathbb{R} \text{ is continuous and } T\text{-periodic in the first variable;}\\ &v. \quad \lambda \geq 0 \text{ is a real parameter.} \end{aligned} \tag{2.2}$$

We start with a crucial remark; namely, the fact that the T-periodic solutions of (2.1) and those of the following system of $b+2$ ordinary differential equations correspond in some sense:

$$\dot{\xi} = G(\xi) + \lambda F(t, \xi), \tag{2.3}$$

where $\xi = (u, v_0, v_1, \dots, v_b) \in \mathbb{R}^{b+2}$, and the maps $G\colon \mathbb{R}^{b+2} \to \mathbb{R}^{b+2}$ and $F\colon \mathbb{R}\times\mathbb{R}^{b+2} \to \mathbb{R}^{b+2}$ are respectively defined as

$$G(u, v_0, v_1, \dots, v_b) = \Big(v_0, g(u, v_0, v_b), a\big(\varphi(u, v_0) - v_1\big), a(v_1 - v_2)\dots, a(v_{b-1} - v_b)\Big) \tag{2.4}$$

and

$$F(t, u, v_0, v_1, \dots, v_b) = \Big(0, f(t, u, v_0), \underbrace{0, \dots, 0}_{b \text{ times}}\Big),$$

with g, φ and f as in equation (2.1). Clearly, G and F are continuous maps. By a T-periodic solution (on $\mathbb{R}$) of system (2.3) we mean a C^1 function $\xi\colon \mathbb{R} \to \mathbb{R}^{b+2}$ of period T that satisfies (2.3) identically.

Let us show now how a "linear chain trick" (see, e.g., [41, 42]) can be used to prove the correspondence between T-periodic solutions of the second order equation (2.1) and of the first order system (2.3).

Let us introduce some notation. By $C_T^n(\mathbb{R}^s)$, $n = 0, 1, 2$, we will denote the Banach space of the T-periodic C^n maps $x\colon \mathbb{R} \to \mathbb{R}^s$ with the the standard norm

$$\|x\|_{C^n} = \sum_{i=0}^{n} \max_{t\in\mathbb{R}} |x^{(i)}(t)|.$$

Here, $x^{(i)}$ denotes the i-th derivative of x, in particular $x^{(0)}$ coincides with x.

Theorem 2.1. *Assume* (2.2) *and suppose x_0 is a T-periodic solution of* (2.1), *and let*

$$\begin{cases} y_0(t) := \dot{x}_0(t), \\ y_i(t) := \int_{-\infty}^{t} \gamma_a^i(t-s)\varphi(x_0(s), y_0(s))\, \mathrm{d}s, \quad i = 1, \dots, b \end{cases}$$

for $t \in \mathbb{R}$. Then, $(x_0, y_0, y_1 \dots, y_b)$ is a T-periodic solution of (2.3).

Proof. First notice that, since $x_0 \in C^2_T(\mathbb{R})$ and $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, it is not difficult to prove that the functions y_i, $i = 0, \dots, b$, are in $C^1_T(\mathbb{R})$.

Moreover, we claim that, for any $t \in \mathbb{R}$,

$$\begin{cases} \dot{x}_0(t) = y_0(t), \\ \dot{y}_0(t) = g\big(x_0(t), y_0(t), y_b(t)\big) + \lambda f\big(t, x_0(t), y_0(t)\big), \quad \lambda \geq 0, \\ \dot{y}_1(t) = a\big(\varphi(x_0(t), y_0(t)) - y_1(t)\big), \\ \dot{y}_i(t) = a\big(y_{i-1}(t) - y_i(t)\big), \quad i = 2, \dots, b. \end{cases}$$

Indeed, the first equality follows by definition. Since x_0 is a solution of (2.1), for $\lambda \geq 0$ we have

$$\begin{aligned} \dot{y}_0(t) = \ddot{x}_0(t) &= g\left(x_0(t), \dot{x}_0(t), \int_{-\infty}^{t} \gamma_a^b(t-s)\varphi(x_0(s), \dot{x}_0(s))\, \mathrm{d}s\right) + \lambda f\big(t, x_0(t), \dot{x}_0(t)\big) \\ &= g\big(x_0(t), y_0(t), y_b(t)\big) + \lambda f\big(t, x_0(t) y_0(t)\big), \quad \forall t \in \mathbb{R}. \end{aligned}$$

Now observe that, when $i = 1$, we have for any $t \in \mathbb{R}$,

$$y_1(t) := \int_{-\infty}^{t} \gamma_a^1(t-s)\varphi(x_0(s), y_0(s))\, \mathrm{d}s = \int_{-\infty}^{t} a\mathrm{e}^{-a(t-s)}\varphi(x_0(s), y_0(s))\, \mathrm{d}s.$$

Thus, taking the derivative under the integral sign,

$$\frac{\mathrm{d}}{\mathrm{d}t} y_1(t) = a\varphi(x_0(t), y_0(t)) - a^2 \int_{-\infty}^{t} \mathrm{e}^{-a(t-s)}\, \varphi(x_0(s), y_0(s))\, \mathrm{d}s$$

so that

$$\dot{y}_1(t) = a\big(\varphi(x_0(t), y_0(t)) - y_1(t)\big).$$

Also, for $i = 2, \dots, b$, we get for any $t \in \mathbb{R}$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} y_i(t) &= \gamma_a^i(0)\, \varphi(x_0(t), y_0(t)) + \int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}t} \gamma_a^i(t-s)\, \varphi(x_0(s), y_0(s))\, \mathrm{d}s \\ &= \int_{-\infty}^{t} a\left(\gamma_a^{i-1}(t-s) - \gamma_a^i(t-s)\right)\, \varphi(x_0(s), y_0(s))\, \mathrm{d}s \\ &= a\left(\int_{-\infty}^{t} \gamma_a^{i-1}(t-s)\, \varphi(x_0(s), y_0(s))\, \mathrm{d}s - \int_{-\infty}^{t} \gamma_a^i(t-s)\, \varphi(x_0(s), y_0(s))\, \mathrm{d}s\right) \\ &= a\,(y_{i-1}(t) - y_i(t)) \end{aligned}$$

proving our claim.

Finally, to see that $(x_0, y_0, y_1 \dots, y_b)$ is T-periodic, observe that so are x_0 and y_0, and with the change of variable $\sigma = s - T$ we get

$$\begin{aligned} y_i(t+T) &= \int_{-\infty}^{t+T} \gamma_a^i(T+t-s)\varphi(x_0(s), y_0(s))\, \mathrm{d}s \\ &= \int_{-\infty}^{t} \gamma_a^i(t-\sigma)\varphi(x_0(\sigma+T), y_0(\sigma+T))\, \mathrm{d}\sigma \\ &= \int_{-\infty}^{t} \gamma_a^i(t-\sigma)\varphi(x_0(\sigma), y_0(\sigma))\, \mathrm{d}\sigma = y_i(t), \end{aligned}$$

for $i = 1, \dots, b$ and any $t \in \mathbb{R}$, and this completes the proof. □

Conversely, we have the following:

Theorem 2.2. *Assume* (2.2) *and suppose that* $(x_0, y_0, y_1 \dots, y_b)$ *is a* T*-periodic solution of* (2.3)*, then* x_0 *is a* T*-periodic solution of* (2.1)*.*

The proof of Theorem 2.2 is based on the following technical lemma on linear systems of ODEs, see [42, Lemma 3.3], cf. also [41, Prop. 7.3]. The proof is omitted.

Lemma 2.3 ([42], Lemma 3.3)**.** *Given any continuous and bounded function* $z_0 \colon \mathbb{R} \to \mathbb{R}$ *and any* $a > 0$*, there exists a unique* C^1 *solution* $z = (z_1, \dots, z_b)$*,* $z_i \colon \mathbb{R} \to \mathbb{R}$*,* $i = 1, \dots, b$*, of the system in* $\mathbb{R}^b$

$$\dot{z}_i(t) = a\big(z_{i-1}(t) - z_i(t)\big),$$

which is bounded in the C^1 *norm. This solution is given by*

$$z_i(t) = \int_{-\infty}^{t} \gamma_a^i(t-s) z_0(s)\, \mathrm{d}s, \quad i = 1, \dots, b.$$

Remark 2.4. Observe in particular that, for $i = 1$ in the previous lemma, one has

$$z_1(t) = \int_{-\infty}^{t} \gamma_a^1(t-s) z_0(s)\, \mathrm{d}s = a \int_{-\infty}^{t} \mathrm{e}^{-a(t-s)} z_0(s)\, \mathrm{d}s,$$

so that z_1 is actually a C^1 function, and thus so are all the z_i's for $i = 2, \dots, b$.

Proof of Theorem 2.2. Let $(x_0, y_0, y_1 \dots, y_b)$ be a T-periodic solution of (2.3), and define $z_0(t) = \varphi(x_0(t), y_0(t))$ for $t \in \mathbb{R}$. Observe that z_0 is bounded and continuous. Thus, by Lemma 2.3,

$$y_i(t) = \int_{-\infty}^{t} \gamma_a^i(t-s) \varphi(x_0(s), y_0(s))\, \mathrm{d}s, \quad i = 1, \dots, b,$$

is the unique solution of class C^1 of

$$\begin{cases} \dot{y}_1(t) = a\big(\varphi(x_0(t), y_0(t)) - y_1(t)\big), \\ \dot{y}_i(t) = a\big(y_{i-1}(t) - y_i(t)\big), \quad i = 2, \dots, b. \end{cases}$$

In particular, we have

$$y_b(t) = \int_{-\infty}^{t} \gamma_a^b(t-s) \varphi(x_0(s), y_0(s))\, \mathrm{d}s.$$

Thus, from (2.3),

$$\begin{aligned} \ddot{x}_0(t) = \dot{y}_0(t) &= g\big(x_0(t), y_0(t), y_b(t)\big) + \lambda f\big(t, x_0(t) y_0(t)\big) \\ &= g\left(x_0(t), \dot{x}_0(t), \int_{-\infty}^{t} \gamma_a^b(t-s) \varphi(x_0(s), \dot{x}_0(s))\, \mathrm{d}s\right) + \lambda f\big(t, x_0(t), \dot{x}_0(t)\big), \quad \lambda \geq 0, \end{aligned}$$

for all $t \in \mathbb{R}$, whence the assertion. □

3 Branches of T-pairs

This section investigates the structure of the set of T-periodic solutions of (2.1). We begin by recalling some notation and basic facts.

Consider the following first-order parameterized ODE on $\mathbb{R}^k$:

$$\dot{x}(t) = \mathfrak{g}\big(x(t)\big) + \lambda \mathfrak{f}\big(t, x(t)\big), \tag{3.1}$$

where $\lambda \geq 0$, the maps $\mathfrak{g}\colon \mathbb{R}^k \to \mathbb{R}^k$ and $\mathfrak{f}\colon \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ are continuous and $\mathfrak{f}$ is T-periodic in the first variable.

We say that a pair $(\lambda, x) \in [0, \infty) \times C_T(\mathbb{R}^k)$ is a *T-pair* (for (3.1)) if x is a T-periodic solution of (3.1) corresponding to λ. If $\lambda = 0$ and x is constant then the T-pair is said *trivial*. It is not hard to see that the trivial T-periodic pairs of (3.1) correspond to the zeros of $\mathfrak{g}$. From now on, given any $p \in \mathbb{R}^k$, we will denote by $\overline{p}$ the constant map $t \mapsto p$, $t \in \mathbb{R}$. Given an open subset $\mathcal{O}$ of $[0, \infty) \times C_T(\mathbb{R}^k)$, we will denote by $\widetilde{\mathcal{O}} \subseteq \mathbb{R}^k$ the open set $\widetilde{\mathcal{O}} := \big\{p : (0, \overline{p}) \in \mathcal{O}\big\}$.

We have the following fact concerning the T-pairs of (3.1), see [27, Theorem 3.3]. The statement of Theorem 3.1 involves the Brouwer degree of the map $\mathfrak{g}$, see Section 4 for a definition and related notions.

Theorem 3.1 ([27]). *Let $\mathcal{O}$ be open in $[0, \infty) \times C_T(\mathbb{R}^k)$, and assume that $\deg(\mathfrak{g}, \widetilde{\mathcal{O}})$ is well defined and nonzero. Then there exists a connected set $\Gamma \subseteq \mathcal{O}$ of nontrivial T-pairs whose closure in $\mathcal{O}$ is not compact and meets the set of trivial T-pairs contained in $\mathcal{O}$, namely the set: $\big\{(0, \overline{p}) \in \mathcal{O} : \mathfrak{g}(p) = 0\big\}$.*

Let us now go back to the second-order equation (2.1). Roughly speaking we will state a global bifurcation result, analogous to Theorem 3.1, whose assumptions do not involve the topological degree, but rely only on sign-changing properties of the real-valued function $\Phi\colon \mathbb{R} \to \mathbb{R}$, given by

$$\Phi(u) = g(u, 0, \varphi(u, 0)). \tag{3.2}$$

To make a precise statement we need some further notation. The pairs $(\lambda, x) \in [0, +\infty) \times C^1_T(\mathbb{R})$, with $x\colon \mathbb{R} \to \mathbb{R}$ a T-periodic solution of (2.1) corresponding to λ, will be called *T-forced pairs* (of (2.1)), and we denote by X the set of these pairs. Among the T-forced pairs we shall consider as *trivial* those of the type $(0, x)$ with x constant. Given an open subset Ω of $[0, \infty) \times C^1_T(\mathbb{R})$, we will denote by $\widetilde{\Omega} := \big\{u \in \mathbb{R} : (0, \overline{u}) \in \Omega\big\}$, where by $\overline{u}$ we mean the constant map $t \mapsto u$, $t \in \mathbb{R}$.

As pointed out in Section 2, there is a correspondence between T-periodic solutions of the second order equation (2.1) and of the first order system in (2.3). A similar correspondence holds between their "T-periodic pairs" in a sense that we are going to specify. We point out that, since equation (2.3) can be seen as a special case of (3.1), in accordance with the notation introduced above, a pair $(\lambda, \xi) \in [0, \infty) \times C_T(\mathbb{R}^{b+2})$ is called a *T-pair* if ξ is a T-periodic solution of (2.3) corresponding to λ. We also recall that the set of T-forced pairs of (2.1) is regarded as a subset of $[0, +\infty) \times C^1_T(\mathbb{R})$.

As a first remark we relate the corresponding "trivial T-periodic pairs". This observation will be deduced from the following link between the zeros of the map G, defined in (2.4), and those of Φ, introduced in (3.2).

Remark 3.2. Observe that if $(\bar{u}, v_0, v_1, \ldots, v_b) \in G^{-1}(0)$, then $\Phi(\bar{u}) = 0$, $v_0 = 0$ and $v_1 = v_2 = \ldots = v_b = \varphi(\bar{u}, 0)$. Conversely, for any $\bar{u} \in \Phi^{-1}(0)$, then $G(\bar{u}, 0, \varphi(\bar{u}, 0) \ldots, \varphi(\bar{u}, 0)) = 0$.

Remark 3.3. Observe that if $(0, \overline{q})$, with $q \in \mathbb{R}$, is a trivial T-forced pair of (2.1), then $(0, \overline{\xi_0})$ is a trivial T-pair for equation (2.3), where $\xi_0 \in \mathbb{R}^{b+2}$ is given by

$$\xi_0 := (q, 0, \varphi(q, 0) \ldots, \varphi(q, 0)),$$

that is, by Remark 3.2, $\xi_0 \in G^{-1}(0)$. Conversely, any trivial T-pair of (2.3) must be of the form $(0, \overline{\xi_0})$, where $\xi_0 \in \mathbb{R}^{b+2}$ is such that $G(\xi_0) = 0$. Thus, again by Remark 3.2, we have $\xi_0 = (q, 0, \varphi(q,0) \ldots, \varphi(q,0))$ for some $q \in \mathbb{R}$, and consequently $(0, \overline{q})$ is a trivial T-forced pair for (2.3).

Let us now establish a general correspondence between the sets of T-forced pairs and of T-pairs, that preserves the notion of triviality. Let $\mathcal{J} \colon [0, +\infty) \times C^1_T(\mathbb{R}) \to [0, \infty) \times C_T(\mathbb{R}^{b+2})$ be the map defined as follows:

$$\mathcal{J} : (\lambda, x_0) \mapsto (\lambda, \xi)$$

where $\xi := (x_0, y_0, y_1 \ldots, y_b)$ is given by

$$\begin{cases} y_0(t) := \dot{x}_0(t), \\ y_i(t) := \int_{-\infty}^{t} \gamma_a^i(t-s)\varphi(x_0(s), y_0(s))\, \mathrm{d}s, \quad i = 1, \ldots, b. \end{cases}$$

As above, denote by X the set of the T-forced pairs of (2.1), and let $Y \subseteq [0, \infty) \times C_T(\mathbb{R}^{b+2})$ be the set of the T-pairs of (2.3).

Lemma 3.4. *Let $\mathcal{J}|_X : X \to Y$ be the restriction to X of $\mathcal{J}$. Then $\mathcal{J}|_X$ is a homeomorphism of X onto Y, which establishes a bijective correspondence between the trivial T-forced pairs in X and the trivial T-pairs in Y.*

Proof. Since φ is continuous, the continuity of $\mathcal{J}$ is obtained by construction. Whence we get the continuity of the restriction $\mathcal{J}|_X$. Injectivity and surjectivity of $\mathcal{J}|_X : X \to Y$ follow from Theorems 2.1 and 2.2. The inverse map $(\mathcal{J}|_X)^{-1}$ can be seen merely as the projection onto the first two components, so it is continuous. Finally the last part of the assertion follows by Remark 3.3. □

The following fact will be crucial for the proof of our main result. Its proof heavily relies on the properties of the topological degree, it is therefore postponed to Section 4 where this concept and its features are discussed. Given a bounded open interval $(\alpha, \beta) \subseteq \mathbb{R}$, define the open subset $W^* = (\alpha, \beta) \times \mathbb{R}^{b+1}$ of $\mathbb{R}^{b+2}$.

Theorem 3.5. *Assume (2.2), let G be as in (2.4) and let Φ be the real-valued function defined in (3.2). Suppose that $\Phi(\alpha) \cdot \Phi(\beta) < 0$. Then G is admissible for the degree in W^* and we have $\deg(G, W^*) \neq 0$.*

We are now in a position to state and prove our main result concerning the set of T-forced pairs of (2.1).

Theorem 3.6. *Consider equation (2.1) and assume that (2.2) hold. Denote by $X \subseteq [0, \infty) \times C^1_T(\mathbb{R})$ the set of its T-forced pairs. Let Φ be the real-valued function defined in (3.2), and suppose that $(\alpha, \beta) \subseteq \mathbb{R}$ is such that $\Phi(\alpha) \cdot \Phi(\beta) < 0$. Let $\Omega \subseteq [0, \infty) \times C^1_T(\mathbb{R})$ be open and such that $\widetilde{\Omega} = (\alpha, \beta)$. Then, in $X \cap \Omega$ there is a connected subset Γ of nontrivial T-forced pairs of (2.1) whose closure relative to Ω is not compact and intersects the set*

$$\big\{(0, \overline{u}) \in \Omega : u \in (\alpha, \beta) \cap \Phi^{-1}(0)\big\}. \tag{3.3}$$

Proof. Let $\mathcal{O} = \Omega \times C_T(\mathbb{R} \times \mathbb{R}^b)$. Clearly, $\widetilde{\mathcal{O}} = \widetilde{\Omega} \times \mathbb{R} \times \mathbb{R}^b = (\alpha, \beta) \times \mathbb{R} \times \mathbb{R}^b$. Thus, Theorem 3.5 implies that $\deg(G, \widetilde{\mathcal{O}})$ is well-defined and nonzero. So, there exists a connected set Y of nontrivial T-pairs for (2.3) as in Theorem 3.1. Let $\Gamma := \mathcal{J}^{-1}(\mathrm{Y})$; by Lemma 3.4, this is a connected set, it is made up of nontrivial T-forced pairs of (2.1), and its closure relative to Ω must intersect the set (3.3) of the trivial T-forced pairs contained in Ω. We claim that Γ is not contained in any compact subset of Ω. By contradiction, assume that there exists a compact set $K \subseteq \Omega$ containing Γ. Since X is closed, $K \cap X$ is compact. Thus, its image $\mathcal{J}(K \cap X)$ is a compact subset of $\mathcal{O}$ containing Υ, which is a contradiction. The assertion follows. $\square$

We remark that, as in the well-known case of the resonant harmonic oscillator

$$\ddot{x} = -x + \lambda \sin t,$$

this unbounded branch is possibly contained in the slice $\{0\} \times C^1_T(\mathbb{R})$. We will discuss some conditions preventing this "pathological" situation in Section 5.

4 Computation of the degree

4.1 Brouwer degree in Euclidean spaces

We will make use of the Brouwer degree in $\mathbb{R}^k$ in a slightly extended version (see e.g. [1,20,39]). Let U be an open subset of $\mathbb{R}^k$, f a continuous $\mathbb{R}^k$-valued map whose domain contains the closure $\overline{U}$ of U, and $q \in \mathbb{R}^k$. We say that the triple (f, U, q) is *admissible (for the Brouwer degree)* if $f^{-1}(q) \cap U$ is compact.

The Brouwer degree is a function that to any admissible triple (f, U, q) assigns an integer, denoted by $\deg(f, U, q)$ and called the *Brouwer degree of f in U with target q*. Roughly speaking, $\deg(f, U, q)$ is an algebraic count of the solutions in U of the equation $f(p) = q$. In fact, one of the properties of this integer-valued function is given by the following *computation formula.* Recall that, if $f\colon U \to \mathbb{R}^k$ is a C^1 map, an element $p \in U$ is said to be a *regular point* (of f) if the differential of f at p, df_p, is surjective. Non-regular points are called *critical (points)*. The *critical values* of f are those points of $\mathbb{R}^k$ which lie in the image $f(C)$ of the set C of critical points. Any $q \in \mathbb{R}^k$ which is not in $f(C)$ is a *regular value*. Therefore, q is a regular value for f in U if and only if $\det(df_p) \neq 0$, $\forall p \in f^{-1}(q) \cap U$. Observe, in particular, that any element of $\mathbb{R}^k$ which is not in the image of f is a regular value.

Computation formula. If (f, U, q) is admissible, f is smooth, and q is a regular value for f in U, then

$$\deg(f, U, q) = \sum_{p \in f^{-1}(q) \cap U} \operatorname{sign} \det(df_p). \tag{4.1}$$

This formula is actually the basic definition of the Brouwer degree, and the integer associated to any admissible triple (g, U, r) is defined by

$$\deg(g, U, r) := \deg(f, U, q),$$

where f and q satisfy the assumptions of the Computation Formula and are, respectively, "sufficiently close" to g and r. It is known that this is a well-posed definition.

The more classical and well-known definition of Brouwer degree is usually given in the subclass of triples (f, U, q) such that $f\colon \overline{U} \to \mathbb{R}^k$ is continuous, U is bounded and $q \notin f(\partial U)$. However, all the standard properties of the degree, such as homotopy invariance, excision, additivity, existence, are still valid in this more general context. For a detailed list of such properties we refer, e.g., to [30,33,39].

Since in this paper the target point q will always be the origin, for the sake of simplicity, we will simply write $\deg(f, U)$ instead of $\deg(f, U, 0)$. In this context, we will say that an element $p \in f^{-1}(0)$ is a *nondegenerate zero* (of f) if $\det(df_p) \neq 0$; this means, equivalently, that p is a regular point. Accordingly, we will also say that (f, U) is an *admissible pair* (or that the map f is admissible in U) if so is the triple $(f, U, 0)$. Observe that $\deg(f, U)$ can be regarded also as the degree (or characteristic, or rotation) of the map f seen as a tangent vector field on $\mathbb{R}^k$.

In what follows we will make use of the following elementary fact whose proof we include for the sake of completeness.

Lemma 4.1. *Let $\psi\colon [a,b] \to \mathbb{R}$ be a continuous function with $\psi(a)\psi(b) < 0$. Then the pair $\big(\psi, (a,b)\big)$ is admissible and*

$$\deg\big(\psi, (a,b)\big) = \operatorname{sign}\psi(b).$$

In particular $\deg\big(\psi, (a,b)\big) \neq 0$.

Proof. It is sufficient to observe that if $\psi(b) > 0$ the map $H\colon [0,1] \times [a,b] \to \mathbb{R}$ given by $H(\lambda, s) = \lambda\big(s - \frac{b+a}{2}\big) + (1-\lambda)\psi(s)$ is an admissible homotopy with the identity map translated by $\frac{b+a}{2}$. Hence

$$\deg\big(\psi, (a,b)\big) = \deg\big(H(0,\cdot), (a,b)\big) = \deg\big(H(1,\cdot), (a,b)\big) = +1.$$

If, otherwise, one has $\psi(b) < 0$, it is possible to define an admissible homotopy H by setting $H(\lambda, s) = -\lambda\left(s - \frac{b+a}{2}\right) + (1-\lambda)\psi(s)$. Thus, in this case, $\deg\big(\psi, (a,b)\big) = -1$. □

4.2 The degree of the map G

The last part of this section is devoted to the proof of Theorem 3.5. Roughly speaking we will relate the degree of the map G, defined in (2.4) by

$$G(u, v_0, v_1, \dots, v_b) = \big(v_0, g(u, v_0, v_b), a(\varphi(u, v_0) - v_1), a(v_1 - v_2) \dots, a(v_{b-1} - v_b)\big),$$

with that of the function $\Phi\colon \mathbb{R} \to \mathbb{R}$ in (3.2), given by

$$\Phi(u) = g(u, 0, \varphi(u, 0)).$$

To simplify the computation of the degree, it is convenient to introduce the next map on $\mathbb{R}^{b+2}$:

$$\mathcal{G}(u, v_0, v_1, \dots, v_b) := \big(v_0, g(u, v_0, v_b), a(\varphi(u, v_0) - v_b), a(v_1 - v_2) \dots, a(v_{b-1} - v_b)\big).$$

Remark 4.2. As in Remark 3.2, we have that the zeros of $\mathcal{G}$ and those of Φ correspond. In fact, if $\mathcal{G}(\bar{u}, v_0, v_1, \dots, v_b) = 0$ then $\Phi(\bar{u}) = 0$, $v_0 = 0$ and $v_1 = v_2 = \ldots = v_b = \varphi(\bar{u}, 0) =: \bar{w}$. Conversely, for any $\bar{u} \in \Phi^{-1}(0)$, then $\mathcal{G}(\bar{u}, 0, \bar{w} \dots, \bar{w}) = 0$. Indeed, $G^{-1}(0) = \mathcal{G}^{-1}(0)$.

Lemma 4.3. *Let G and $\mathcal{G}$ be as above, and let $V \subseteq \mathbb{R}^{b+2}$ be open. Suppose that one of the maps G or $\mathcal{G}$ is admissible for the Brouwer degree in V; then, so is the other and they are admissibly homotopic in V.*

Proof. For $(\lambda, u, v_0, v_1, \dots, v_b) \in [0,1] \times V$, consider the map

$$\begin{aligned}\mathcal{H}&(\lambda, u, v_0, v_1, \dots, v_b)\\ &= \big(v_0, g(u, v_0, v_b), a(\varphi(u, v_0) - [\lambda v_b + (1-\lambda)v_1]), a(v_1 - v_2) \dots, a(v_{b-1} - v_b)\big),\end{aligned}$$

and observe that $\mathcal{H}(\lambda, u, v_0, v_1, \dots, v_b) = 0$ if and only if $(u, v_0, v_1, \dots, v_b) \in V \cap G^{-1}(0) = V \cap \mathcal{G}^{-1}(0)$. Hence $\mathcal{H}$ is an admissible homotopy. □

Now we prove that if Φ is smooth and the zeros of Φ are nondegenerate, so are those of $\mathcal{G}$.

Lemma 4.4. *Assume that g and φ are of class C^1. Then, all zeros of Φ are nondegenerate if and only if all zeros of $\mathcal{G}$ are nondegenerate.*

Proof. First notice that, by Remark 4.2 the zeros of $\mathcal{G}$ and those of Φ correspond. Assume now that $\bar{\xi} := (\bar{u}, 0, \bar{w} \ldots, \bar{w})$ is a nondegenerate zero of $\mathcal{G}$. We represent the differential $d\mathcal{G}_{\bar{\xi}}$ of $\mathcal{G}$ at $\bar{\xi}$ as the following Jacobian matrix:

$$\left(\begin{array}{cc|ccccc|c}
0 & 1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\
\partial_1 g(\bar{u},0,\bar{w}) & \partial_2 g(\bar{u},0,\bar{w}) & 0 & 0 & \ldots & \ldots & 0 & \partial_3 g(\bar{u},0,\bar{w}) \\
a\,\partial_1 \varphi(\bar{u},0) & a\,\partial_2 \varphi(\bar{u},0) & 0 & 0 & \ldots & \ldots & 0 & -a \\
\hline
0 & 0 & a & -a & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & a & -a & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & a & -a
\end{array}\right),$$

where "∂_i", $i = 1,2,3$, represents the partial derivation with respect to the i-th variable. By direct computation based on Laplace expansion we obtain

$$\begin{aligned}
\det d\mathcal{G}_{\bar{\xi}} &= (-a)^{b-1} \det \begin{pmatrix} 0 & 1 & 0 \\ \partial_1 g(\bar{u},0,\bar{w}) & \partial_2 g(\bar{u},0,\bar{w}) & \partial_3 g(\bar{u},0,\bar{w}) \\ a\partial_1 \varphi(\bar{u},0) & a\partial_2 \varphi(\bar{u},0) & -a \end{pmatrix} \\
&= -(-a)^{b-1} \det \begin{pmatrix} \partial_1 g(\bar{u},0,\bar{w}) & \partial_3 g(\bar{u},0,\bar{w}) \\ a\partial_1 \varphi(\bar{u},0) & -a \end{pmatrix} \\
&= (-1)^{b-1} a^b \cdot [\partial_1 g(\bar{u},0,\bar{w}) + \partial_3 g(\bar{u},0,\bar{w})\, \partial_1 \varphi(\bar{u},0)] = (-1)^{b-1} a^b \cdot \Phi'(\bar{u}).
\end{aligned}$$

Thus $\bar{u}$ is a nondegenerate zero of the function Φ. The proof of the converse implication is analogous. □

Let us now compute the degree of the map $\mathcal{G}$ in the smooth case.

Lemma 4.5. *Assume that g and φ are of class C^1. Assume further that Φ is admissible on an open set $U \subseteq \mathbb{R}$ and all its zeros are nondegenerate. Then, $\mathcal{G}$ is admissible in $U^* = U \times \mathbb{R}^{b+1}$ and*

$$\deg(\mathcal{G}, U^*) = (-1)^{b-1} \deg(\Phi, U).$$

Proof. By Remark 4.2 we have that all zeros of $\mathcal{G}$ are of the form $\bar{\xi} := (\bar{u}, 0, \bar{w} \ldots, \bar{w})$ with $\Phi(\bar{u}) = 0$. Moreover, they are all nondegenerate. In particular, if Φ is admissible in U then $\Phi^{-1}(0) \cap U$ is compact and so is $\mathcal{G}^{-1}(0) \cap U^*$, whence the admissibility of $\mathcal{G}$ in U^*. Let now $\bar{u} \in \Phi^{-1}(0) \cap U$ be a nondegenerate zero of Φ. As in the proof of Lemma 4.4 we have

$$\det d\mathcal{G}_{\bar{\xi}} = (-1)^{b-1} a^b \cdot \Phi'(\bar{u})$$

and, consequently,

$$\operatorname{sign} \det \left(d\mathcal{G}_{\bar{\xi}} \right) = (-1)^{b-1} \operatorname{sign} \det \Phi'(\bar{u})$$

Thus, by formula (4.1),

$$\begin{aligned}\deg(\mathcal{G}, U^*) &= \sum_{\bar{\xi} \in \mathcal{G}^{-1}(0) \cap U^*} \operatorname{sign}\det\left(d\mathcal{G}_{\bar{\xi}}\right) \\ &= \sum_{\bar{u} \in \Phi^{-1}(0) \cap U} (-1)^{b-1} \operatorname{sign}\det \Phi'(\bar{u}) = (-1)^{b-1} \deg(\Phi, U),\end{aligned}$$

whence the assertion. □

The above lemmas imply now the assertion of Theorem 3.5.

Proof of Theorem 3.5. First notice that, by Lemma 4.1, assumption $\Phi(\alpha) \cdot \Phi(\beta) < 0$ implies that Φ is admissible in (α, β) and $\deg(\Phi, (\alpha, \beta)) \neq 0$. Thus, if Φ is of class C^1 and all zeros of Φ in (α, β) are nondegenerate, the assertion follows from Lemmas 4.3 and 4.5.

Consider now the case in which Φ is merely continuous, or the zeros of Φ in (α, β) may not be nondegenerate. Clearly the above Lemmas 4.4 and 4.5 are not directly applicable, nevertheless it is possible to prove the assertion by applying some well-known approximation theorems and the properties of the Brouwer degree. More precisely, using the so-called smooth Urysohn Theorem and Sard's Theorem, we will construct a real-valued function $\widetilde{\Phi}_r \colon \mathbb{R} \to \mathbb{R}$, depending on a suitable small parameter r, satisfying the assumptions of Lemma 4.5 in (α, β).

For this purpose, observe first that, as a consequence of Remark 4.2, the map $\mathcal{G}$ is admissible in W^* so that $\deg(\mathcal{G}, W^*)$ is well-defined. Let $\mathcal{W} \subseteq W^*$ be an open relatively compact neighborhood of the compact set $\mathcal{G}^{-1}(0) \cap W^*$. Without loss of generality, one can choose $\mathcal{W}$ with the following properties:

$$\begin{aligned}&i. \quad \mathcal{W} = (\alpha, \beta) \times \mathcal{U} \text{ where } \mathcal{U} \subseteq \mathbb{R}^{b+1} \text{ is open and bounded;} \\ &ii. \quad (u, 0, \varphi(u,0) \ldots, \varphi(u,0)) \in \overline{\mathcal{W}} \text{ for all } u \in [\alpha, \beta].\end{aligned} \tag{4.2}$$

Let $\alpha' < \alpha$ and $\beta' > \beta$ be such that Φ does not change sign in the intervals $[\alpha', \alpha]$ and $[\beta, \beta']$. Let also $\mathcal{U}' \subseteq \mathbb{R}^{b+1}$ be an open and bounded set such that $\overline{\mathcal{U}} \subseteq \mathcal{U}'$. Consider the open set

$$\mathcal{V} = (\alpha', \beta') \times \mathcal{U}'.$$

Note that $\mathcal{V}$ is bounded and contains $\overline{\mathcal{W}}$; in particular, $\mathcal{G}^{-1}(0) \cap W^* \subseteq \mathcal{W} \subseteq \mathcal{V}$.

By Lemma 4.3 and the excision property of the degree,

$$\deg(G, W^*) = \deg(\mathcal{G}, W^*) = \deg(\mathcal{G}, \mathcal{W}). \tag{4.3}$$

Now, take $\delta > 0$ such that

$$\begin{aligned}&i. \quad \delta < \min\{|\Phi(\alpha)|, |\Phi(\beta)|\}; \\ &ii. \quad \delta < \min_{(u,v_0,\ldots,v_b) \in \overline{\mathcal{V}} \setminus \mathcal{W}} \|\mathcal{G}(u, v_0, \ldots, v_b)\|.\end{aligned} \tag{4.4}$$

By a smooth approximation argument similar to the one used in the definition of the degree (cfr. [30, Ch. 5, §1], see in particular [30, Theorem. 2.6]), we can choose smooth approximations $\widetilde{g}$ and $\widetilde{\varphi}$ of g and φ, respectively, such that, setting

$$\widetilde{\mathcal{G}}(u, v_0, v_1, \ldots, v_b) = \Big(v_0, \widetilde{g}(u, v_0, v_b), a\big(\widetilde{\varphi}(u, v_0) - v_b\big), a(v_1 - v_2) \ldots, a(v_{b-1} - v_b)\Big),$$

the map $\widetilde{\mathcal{G}}$ has the following properties: $\big(\widetilde{\mathcal{G}}\big)^{-1}(0) \cap W^* \subseteq \mathcal{W}$, all the zeros of $\widetilde{\mathcal{G}}$ are nondegenerate, we have $\deg(\mathcal{G}, W^*) = \deg(\widetilde{\mathcal{G}}, W^*)$ and

$$\max_{(u,v_0,\dots,v_b)\in\overline{\mathcal{V}}} \|\mathcal{G}(u, v_0, v_1, \dots, v_b) - \widetilde{\mathcal{G}}(u, v_0, v_1, \dots, v_b)\| < \delta/4. \tag{4.5}$$

Let now $\widetilde{\Phi}(u) = \widetilde{g}\big(u, 0, \widetilde{\varphi}(u, 0)\big)$. By (4.2)–*(ii)* and (4.5) it follows $|\widetilde{\Phi}(\alpha) - \Phi(\alpha)| < \delta/4$ and $|\widetilde{\Phi}(\beta) - \Phi(\beta)| < \delta/4$, so that from $\Phi(\alpha)\Phi(\beta) < 0$ we get

$$\widetilde{\Phi}(\alpha)\widetilde{\Phi}(\beta) < 0. \tag{4.6}$$

Now, recall that by Sard's Theorem (see, e.g., [30, Ch. 3, §1] or [29, Ch. 1, §7]) the set of the critical values of a smooth function has Lebesgue measure zero. Thus, one can choose some $r \in (-\delta/4, \delta/4)$ such that 0 is a regular value of the map

$$u \mapsto \widetilde{g}\big(u, 0, \widetilde{\varphi}(u, 0)\big) - r. \tag{4.7}$$

In particular, for such an r, all the zeros of this map are nondegenerate. Let $\sigma\colon \mathbb{R}^{b+2} \to [0, 1]$ be a smooth function that vanishes identically on $\mathbb{R}^{b+2} \setminus \mathcal{V}$ and is identically equal to 1 in $\overline{\mathcal{W}}$. The existence of such a function follows from the so-called smooth Urysohn Theorem; see, e.g., [29, Ch. 1, §8, Exercise 15]. Define

$$\widetilde{\Phi}_r(u) = \widetilde{g}\big(u, 0, \widetilde{\varphi}(u, 0)\big) - \sigma\big(u, 0, \widetilde{\varphi}(u, 0), \dots, \widetilde{\varphi}(u, 0)\big)r.$$

Observe that (4.2)-*(ii)* implies that $\widetilde{\Phi}_r$ coincides with the map in (4.7) in $[\alpha, \beta]$; consequently, all of its zeros in this interval are nondegenerate. From (4.6) it follows $\widetilde{\Phi}_r(\alpha)\widetilde{\Phi}_r(\beta) < 0$, which implies that $\widetilde{\Phi}_r$ is admissible in (α, β). Let now

$$\begin{aligned}\widetilde{\mathcal{G}}_r&(u, v_0, v_1, \dots, v_b)\\ &= \Big(v_0, \widetilde{g}(u, v_0, v_b) - \sigma(u, v_0, \dots, v_b)r, a\big(\widetilde{\varphi}(u, v_0) - v_b\big), a(v_1 - v_2) \dots, a(v_{b-1} - v_b)\Big),\end{aligned}$$

where σ is the scalar function introduced above. Applying Lemma 4.1, Lemma 4.4 and Lemma 4.5 to $\widetilde{\Phi}_r$ and $\widetilde{\mathcal{G}}_r$ we obtain that $\widetilde{\mathcal{G}}_r$ is admissible in $W^* = (\alpha, \beta) \times \mathbb{R}^{b+1}$, meaning that $\big(\widetilde{\mathcal{G}}_r\big)^{-1}(0) \cap W^*$ is compact, and

$$\deg(\widetilde{\mathcal{G}}_r, W^*) = \deg\Big(\widetilde{\Phi}_r, (\alpha, \beta)\Big) \neq 0. \tag{4.8}$$

Furthermore, by construction, the map $\widetilde{\mathcal{G}}_r$ coincides with $\widetilde{\mathcal{G}}$ on $\mathbb{R}^{b+2} \setminus \mathcal{V}$, hence the compact set $\big(\widetilde{\mathcal{G}}_r\big)^{-1}(0) \cap W^*$ is contained in $\mathcal{V} \cap W^*$. Actually, as a consequence of (4.4)-*(ii)*, (4.5) and the choice of r, $\big(\widetilde{\mathcal{G}}_r\big)^{-1}(0) \cap W^* \subseteq \mathcal{W}$. Thus, from (4.8) and the excision property of the degree we get

$$\deg(\widetilde{\mathcal{G}}_r, \mathcal{W}) \neq 0. \tag{4.9}$$

Moreover, note that

$$\max_{(u,v_0,\dots,v_b)\in\overline{\mathcal{W}}} \|\mathcal{G}(u, v_0, v_1, \dots, v_b) - \widetilde{\mathcal{G}}_r(u, v_0, v_1, \dots, v_b)\| < \delta/2,$$

and consider the homotopy $H\colon [0, 1] \times \overline{\mathcal{W}} \to \mathbb{R}$ defined by

$$H(\lambda, u, v_0, v_1, \dots, v_b) = \lambda\mathcal{G}(u, v_0, v_1, \dots, v_b) + (1 - \lambda)\widetilde{\mathcal{G}}_r(u, v_0, v_1, \dots, v_b).$$

Observe that, by construction,

$$\min\{\|H(\lambda,u,v_0,v_1,\dots,v_b)\| : \lambda \in [0,1], (u,v_0,\dots,v_b) \in \partial\mathcal{W}\} > \delta/2 > 0;$$

hence, the compact set

$$\{(\lambda,u,v_0,\dots,v_b) \in [0,1]\times\overline{\mathcal{W}} : H((\lambda,u,v_0,\dots,v_b) = 0\},$$

is contained in $[0,1]\times\mathcal{W}$. Thus, H is an admissible homotopy, so that

$$\deg(\mathcal{G},\mathcal{W}) = \deg(\widetilde{\mathcal{G}}_r,\mathcal{W}). \tag{4.10}$$

Finally, from (4.3) and (4.10) we get

$$\deg(G,W^*) = \deg(\mathcal{G},\mathcal{W}) = \deg(\widetilde{\mathcal{G}}_r,\mathcal{W})$$

and the assertion follows from inequality (4.9). □

5 Ejecting points and small perturbations

As we have seen by the simple example at the end of Section 3, it may happen that the branch Γ of T-forced pairs of equation (2.1), as in the assertion of Theorem 3.6, is completely contained in the slice $\{0\}\times C^1_T(\mathbb{R})$. In this section we show simple conditions ensuring that this is not the case. Such conditions are based on the key notion of *ejecting set* (see, e.g., [26]).

Let us first introduce the following notation: let Y be a subset of $[0,\infty)\times C^1_T(\mathbb{R})$. Given $\lambda \ge 0$, let Y_λ be the slice $\{x \in C^1_T(\mathbb{R}) : (\lambda,x) \in Y\}$. Below, we adapt to our context the definition of ejecting set of [26]:

Definition 5.1. Let $X \subseteq [0,\infty)\times C^1_T(\mathbb{R})$ be the set of T-forced pairs of (2.1), and let X_0 be the slice of X at $\lambda = 0$. We say that $A \subset X_0$ is an *ejecting set* (for X) if it is relatively open in X_0 and there exists a connected subset of X which meets A and is not contained in X_0. In particular, when $A = \{p_0\}$ is a singleton we say that p_0 is an *ejecting point*.

We now discuss a sufficient condition for an isolated point of Φ to be ejecting for the set X of T-forced pairs of (2.1). This condition is based on a result by J. Yorke ([43], see also [7]) concerning the period of solutions of an autonomous ODE with Lipschitz continuous right-hand side.

We point out that an analysis of (2.3), for $\lambda = 0$, linearized at its zeros leads to a different, not entirely comparable, approach. A discussion of the latter technique, which is based on the notion of T-resonance and requires the knowledge of the spectrum of the linearized equation is outside the scope of the present paper (see, e.g., [17, Ch. 7] and [16, Ch. 2], a similar idea can be traced back to Poincaré see, e.g., [38]; see also [2] for an application of this idea to multiplicity results).

The aforementioned result of Yorke is the following:

Theorem 5.2 ([43])**.** *Let ξ be a nonconstant τ-periodic solution of*

$$\dot{x} = \mathcal{F}(x)$$

where $\mathcal{F}\colon W \subseteq \mathbb{R}^k \to \mathbb{R}^k$ is Lipschitz of constant L. Then the period τ of ξ satisfies $\tau \ge 2\pi/L$.

Inequalities on the period like that of Theorem 5.2 have been object of study in different contexts, see e.g., the introduction of [8] for an interesting discussion.

Since we are going to use Theorem 5.2, besides our standard assumptions (2.2), we need to assume the following fact on the maps g and φ throughout this and the following sections:

$$g \text{ and } \varphi \text{ are such that } G\text{, defined in (2.4), is (locally) Lipschitz.} \tag{5.1}$$

Observe that taking g and φ locally Lipschitz allows us to apply Theorem 5.2 in a suitable set $W \subseteq \mathbb{R}^{b+2}$. In particular, since we are going to apply such a result only to unperturbed part of (2.3), no further hypothesis on f is necessary.

From Theorem 5.2, we immediately deduce the following facts concerning the branch of T-forced pairs of (2.1) given by Theorem 3.6:

Corollary 5.3. *Consider equation (2.1), assume that (2.2) hold and let (α, β), Ω and Φ be as in Theorem 3.6. Suppose further that g and φ are such that G is Lipschitz with constant L_G. If the period T of the forcing term f satisfies $T < 2\pi/L_G$, then the set Γ of Theorem 3.6 cannot intersect the slice $\{0\} \times C^1_T(\mathbb{R})$.*

Proof. Assume by contradiction that there exists a nontrivial T-forced pair $(0, x_0) \in \Gamma$. Then x_0 corresponds (in the sense of Theorem 2.1) to a T-periodic solution of $\dot{\xi} = G(\xi)$ with $T < 2\pi/L_G$, violating Theorem 5.2. □

A similar argument yields the following result:

Corollary 5.4. *Consider equation (2.1), suppose that (2.2) hold and let (α, β), Ω and Φ be as in Theorem 3.6. Let $p \in (\alpha, \beta)$ be an isolated zero of Φ and assume that g and φ are such that G is locally Lipschitz in a neighborhood W of the corresponding zero $P = \big(p, 0, \varphi(p, 0), \dots, \varphi(p, 0)\big)$ of G. Suppose that a set Υ of T-pairs for (2.3) contains $\big(0, \overline{P}\big)$. Let $\Lambda = \mathcal{J}^{-1}(\Upsilon)$ be the corresponding set of T-forced pairs of (2.1). As above, call L_G the Lipschitz constant of G in W. Then, the trivial T-forced pair $(0, \bar{p})$ is isolated in the slice Λ_0. In particular, if $T < 2\pi/L_G$, the set Λ cannot be contained in the slice $\{0\} \times C^1_T(\mathbb{R})$.*

Observe that, unlike Γ in Corollary 5.3, the set Υ in Corollary 5.4 does not necessarily consist of nontrivial T-pairs. Consequently, Λ does not necessarily consist of nontrivial T-forced pairs as well. Notice also that since p is an isolated zero of Φ then $\big(0, \overline{P}\big)$ is isolated in the set of trivial T-pairs of (2.3) and, similarly, $(0, \bar{p})$ is isolated in the set of trivial T-forced pairs of (2.1).

Proof of Corollary 5.4. By construction of the map $\mathcal{J}$, the T-forced pairs of (2.1) that lie in the slice $\{0\} \times C^1_T(\mathbb{R})$ corresponds bijectively to the T-pairs of (2.3) contained in $\{0\} \times C_T(\mathbb{R}^{b+2})$. Theorem 5.2 implies that there are not nonconstant T-periodic solutions of (2.3) contained in W. Thus Υ cannot be contained in the slice $\{0\} \times C_T(\mathbb{R}^{b+2})$. Hence, the same is true for $\Lambda = \mathcal{J}^{-1}(\Upsilon)$. □

Remark 5.5. Suppose that g and φ are such that G is C^1 in a neighborhood of a nondegenerate zero, then it is locally Lipschitz in this neighborhood. Then by Corollary 5.4 we have that *if the frequency of the forcing term is sufficiently high, then a nondegenerate zero of the unperturbed vector field is necessarily an ejecting point of nontrivial T-forced pairs of (2.1).*

6 Ejecting sets and multiplicity

This section is devoted to the illustration of sufficient conditions on f, g, and φ yielding the multiplicity of T-periodic solutions of (2.1) for $\lambda > 0$ small.

As above, let G be as in (2.4), and let $X \subseteq [0,\infty) \times C^1_T(\mathbb{R})$ be the set of nontrivial T-forced pairs of (2.1).

For brevity, we will say that a continuous function $\psi : (\alpha,\beta) \to \mathbb{R}$ *changes sign at the isolated zero* $p \in (\alpha,\beta)$ if there exists $\delta > 0$ such that $\forall x \in (p-\delta, p)$, $\forall y \in (p, p+\delta)$ we have $\psi(x) \cdot \psi(y) < 0$.

Corollary 5.4 and Theorem 3.6 yield the following result about multiple periodic solutions of (2.1).

Theorem 6.1. *Consider equation* (2.1) *and suppose that* (2.2) *hold. Let* $(\alpha,\beta) \subseteq \mathbb{R}$ *be an open interval and assume that* Φ *changes sign at the isolated zeros* $p_1,\dots,p_n \in (\alpha,\beta)$*. For* $i = 1,\dots,n$*, let* $P_i = \big(p_i, 0, \varphi(p_i,0)\dots,\varphi(p_i,0)\big)$*. Assume that* g *and* φ *are such that* G *is Lipschitz with constant* L *on a neighborhood of* $\{P_1,\dots,P_n\}$ *and the period* T *of* f *satisfies* $T < 2\pi/L$*. Then:*

1. *for* $i = 1,\dots,n$*, there exist open subsets* Ω_i *of* $[0,\infty) \times C^1_T(\mathbb{R})$ *and connected sets* $\Gamma_i \subseteq \Omega_i$ *of nontrivial* T*-pairs of* (2.1) *whose closure* Ξ_i *in* Ω_i *contains* $(0,\bar{p}_i)$ *and is not compact;*
2. *there exists* $\lambda_* > 0$ *such that the projection of each* Ξ_i*,* $i = 1,\dots,n$*, on the first component contains* $[0,\lambda_*]$*. Thus,* (2.1) *has at least* n *solutions* $x_1^\lambda,\dots,x_n^\lambda$ *of period* T *for* $\lambda \in [0,\lambda_*]$*;*
3. *the images of* $x_1^\lambda,\dots,x_n^\lambda$ *are pairwise disjoint.*

Proof. Let I_i, $i = 1,\dots,n$, be open intervals with pairwise disjoint closures each isolating the zero p_i. Let $U_i = C^1_T(\mathbb{R}, I_i)$ and $\Omega_i = [0,\infty) \times U_i$. Then by Theorem 3.6, for each $i \in \{1,\dots,n\}$, there is a connected set Γ_i in $X \cap \Omega_i$ whose closure, $\Xi_i := \mathrm{cl}_{\Omega_i}(\Gamma_i)$, in Ω_i is not compact and intersect the set

$$\big\{(0,\bar{u}) \in \Omega_i : u \in I_i \cap \Phi^{-1}(0)\big\} = \{(0,\bar{p}_i)\}.$$

This proves assertion (1).

To prove assertion (2) let, for $i = 1,\dots,n$, $K_i = \{(0,\bar{p}_i)\}$. By Corollary 5.4, K_i is isolated in the slice $\{0\} \times U_i$. Since Ξ_i is not compact, it does not consist only of the singleton $\{(0,\overline{p}_i)\}$. Also, since Ξ_i is connected it cannot be completely contained in $\{0\} \times U_i$ (otherwise $(0,\overline{p}_i)$ would not be an isolated point). So, the projection π_1 of Ξ_i onto the first factor of $[0,\infty) \times C^1_T(\mathbb{R})$ cannot reduce to $\{0\}$. Thus, π_1 being continuous, $\pi_1\big(\Xi_i\big)$ is a nontrivial interval J_i with $0 \in J_i$. Let $\delta_i > 0$ be such that $[0,\delta_i] \subseteq J_i$. The proof is completed by taking $\lambda_* = \min\{\delta_1,\dots,\delta_n\}$.

To prove the last assertion, observe that by part (2) there exists $\lambda_* > 0$ such that each $\{\lambda\} \times U_i$, contains at least one T-forced pair, say (λ, x_i^λ) for any $\lambda \in [0,\lambda_*]$. Hence, for $j,k = 1,\dots,n$ and $j \neq k$, the images of x_j^λ and x_k^λ are confined to the disjoint sets I_i and I_k. □

Restricting to a neighborhood of the set of zeros of G corresponding to $p_1,\dots,p_n$, we can give a somewhat less technical and perhaps more elegant formulation of our multiplicity result.

Corollary 6.2. *Consider equation* (2.1) *and suppose that* (2.2) *and* (5.1) *hold. Assume that* Φ *changes sign at the isolated zeros* $p_1,\dots,p_n \in (\alpha,\beta)$*. Then, for sufficiently high frequency of the perturbing term* f *and sufficiently small* $\lambda > 0$*, equation* (2.1) *has at least* n *solutions of period* T *whose images are pairwise disjoint.*

7 Visual representation of branches

We briefly discuss a method allowing to represent graphically the infinite dimensional set of T-forced pairs of (2.1). In other words, as in [2], we create a homeomorphic finite dimensional image of the set Γ yielded by Theorem 3.6 and show a graph of some relevant functions of the point of Γ as, for instance, the sup-norm or the diameter of the orbit of the solution x in any T-forced pair (λ, x).

In this section we assume g and φ as well as the perturbing term f to be at least Lipschitz continuous, so that continuous dependence on initial data of (2.3) holds.

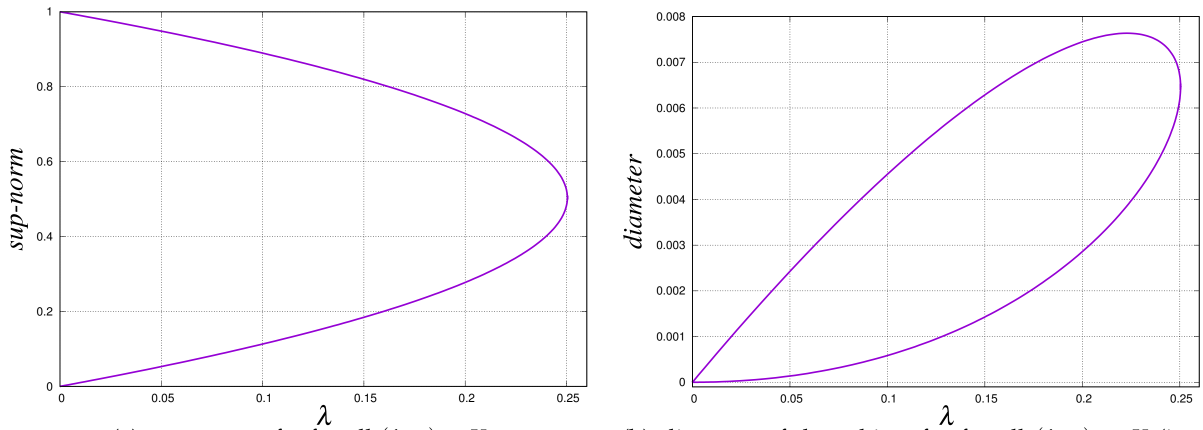
Let us consider the set

$$S = \left\{ (\lambda, q, p_0, \dots, p_b) \in [0,\infty) \times \mathbb{R}^{b+2} \;\middle|\; \begin{array}{l} (q, p_0, \dots, p_b) \text{ is an initial condition at } t = 0 \\ \text{for a } T\text{-periodic solution of (2.3)} \end{array} \right\}.$$

The elements of S are called *starting points* for (2.3). A starting point $(\lambda, q, p_0, \dots, p_b)$ is *trivial* when $\lambda = 0$ and the solution of (2.3) starting a time $t = 0$ from $(q, p_0, \dots, p_b)$ is constant. By uniqueness and continuous dependence on initial data the map $\mathfrak{p}\colon Y \to S$ given by

$$(\lambda, x_0, y_0, \dots . y_b) \mapsto (\lambda, x_0(0), y_0(0), \dots . y_b(0))$$

is a homeomorphism that establishes a correspondence between trivial T-pairs and trivial starting points. Thus, the composition $\mathfrak{h} = \mathfrak{p} \circ \mathcal{J}^{-1}\colon X \to S$ is as well a homeomorphism that establishes a correspondence between trivial T-forced pairs and trivial starting points. In other words, $\Sigma := \mathfrak{h}(\Gamma) \subseteq [0,\infty) \times \mathbb{R}^{b+2}$ is the desired homemorphic image of Γ.



(a) sup-norm of x for all $(\lambda, x) \in X$

(b) diameter of the orbits of x for all $(\lambda, x) \in X$ (i.e., $\max_{t\in[0,T]} x(t) - \min_{t\in[0,T]} x(t)$)

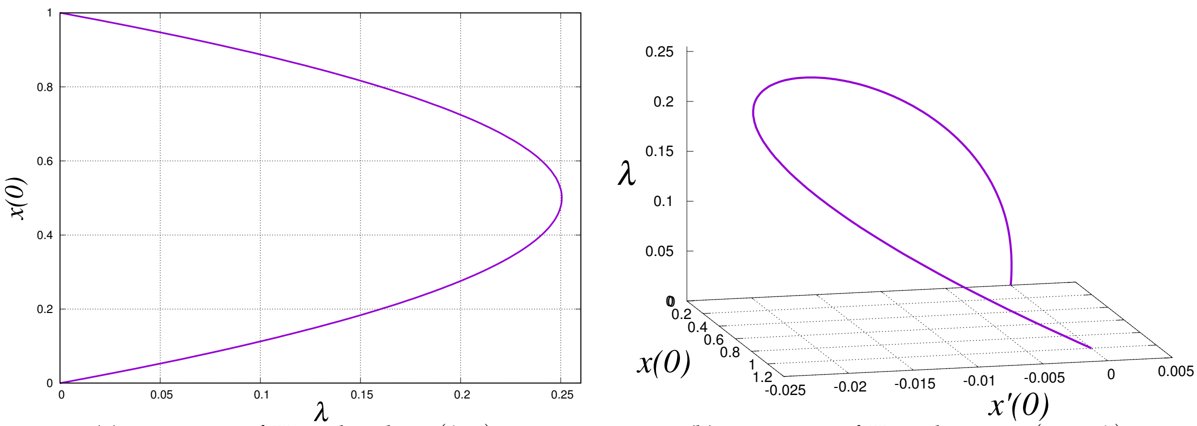
Figure 7.1: Sup-norm and diameter of points in Γ

Example 7.1. Consider the following scalar equation:

$$\ddot{x}(t) = -x(t)\left(1 + \int_{-\infty}^{t} \gamma_2^2(t-s)\big(\dot{x}(s) - x(s)\big)\,\mathrm{d}s\right) + \lambda\big(1 + x(t)\sin(2\pi t)\big), \tag{7.1}$$

with $\lambda \geq 0$. Here, $T = 1$, $g(\xi, \eta, \zeta) = -\xi(1+\zeta)$, $\varphi(p, q) = q - p$, so that

$$\Phi(u) = g\big(u, 0, \varphi(u, 0)\big) = -u(1-u).$$



(a) projection of X on the plane (λ, q) (b) projection of X on the space (q, p_0, λ)

Figure 7.2: Projections of Σ: Initial values and speed of x for $(\lambda, x) \in \Gamma$

One has

$$G(u, v_0, v_1, v_2) = \big(v_0, -u(1 + v_2), 2(v_0 - u - v_1), 2(v_1 - v_2)\big).$$

Note that Φ changes sign at the zeros 0 and 1, the corresponding zeros of G being $P_0 := (0, 0, 0, 0)$ and $P_1 := (1, 0, -1, -1)$. Observe that G is locally Lipschitz. In particular, sufficiently small neighborhoods of P_0 and P_1 can be chosen where the Lipschitz constant L of G is smaller than 2π. The validity of this statement can be checked by computing the operator norms of the Jacobian matrices of G at P_0 and P_1 that actually turn out to be strictly smaller than 4. Thus, by Theorem 6.1 there are connected sets Γ_0 and Γ_1 of nontrivial T-forced pairs of (7.1) emanating, respectively, from $(0, \bar{0})$ and $(0, \bar{1})$. In particular, by Theorem 6.1 it follows that there exists $\lambda_* > 0$ such that for $\lambda \in [0, \lambda_*)$ there are two periodic solutions of (7.1) with disjoint images. Corollary 6.2 shows that the same is true (with possibly different values of λ_*) for all perturbations with sufficiently high frequency.

Figure 7.1 shows sup-norm and diameter of the solutions of the T-forced pairs of (7.1). Figure 7.2, instead, shows the projections of Σ on the plane (λ, q) and on the 3-dimensional space (q, p_0, λ). Indeed, Figures 7.1 and 7.2 suggest that, for Γ_0 and Γ_1 maximal, $\Gamma_1 = \Gamma_0$. The figures suggest that the value of λ_* in Theorem 6.1 is about 0.25.

8 Perspectives and future developments

The results of this paper can be naturally extended along two lines. The first and more direct one is to consider systems of equations, say in $\mathbb{R}^n$, allowing φ to take vector values, say in $\mathbb{R}^k$. In this case one considers (1.3) with $g\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ and $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$. This generalization that, for example, finds application to coupled nonlinear oscillators with memory, has been considered in [36]. Here we only mention that the analogous of the function Φ that has to be constructed in this more general environment is a map of an open set of $\mathbb{R}^n$ in $\mathbb{R}^n$, hence the sign change hypothesis has to be replaced with an assumption about the degree.

A second extension that can be considered is the case when more than one distributed delay is allowed. This situation is considered in some models (see, e.g., [35, 41]), indeed it arises naturally in some situations where a nonlinear equation is coupled with a linear "subsystem". To illustrate this point we consider the following example:

Example 8.1. Let $f\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\varphi\colon\mathbb{R}\to\mathbb{R}$ be given continuous functions. We are interested in the bounded (in C^0) solutions, if any, of the following system of coupled scalar differential equations

$$\begin{cases} \ddot{x}(t) = f\big(t, x(t), y(t)\big), \\ \ddot{y}(t) + (a+b)\,\dot{y}(t) + ab\,y(t) = \varphi\big(x(t)\big), \end{cases} \tag{8.1}$$

with $a\neq b$ positive constants. It is well known that if x is a bounded function, the second equation in (8.1) admits a unique bounded solution which, as it is easy to check, is given by (see, e.g., [15, Lemma 4.1])

$$\begin{aligned} y(t) &= \frac{1}{b-a}\left(\int_{-\infty}^{t} \mathrm{e}^{-a(t-s)}\varphi\big(x(s)\big)\,\mathrm{d}s - \int_{-\infty}^{t} \mathrm{e}^{-b(t-s)}\varphi\big(x(s)\big)\,\mathrm{d}s\right) \\ &= \frac{1}{b-a}\left(\frac{1}{a}\int_{-\infty}^{t}\gamma_a^1(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s - \frac{1}{b}\int_{-\infty}^{t}\gamma_b^1(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s\right). \end{aligned}$$

Using this expression for y, one sees that the bounded solutions of (8.1) are determined by the bounded solutions of the following equation with two distributed delays:

$$\ddot{x}(t) = F\left(t, x(t), \frac{1}{a}\int_{-\infty}^{t}\gamma_a^1(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s - \frac{1}{b}\int_{-\infty}^{t}\gamma_b^1(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s\right),$$

with $F(t,\xi,\eta) = f\big(t,\xi,\eta/(b-a)\big)$. Notice that when $a=b$ one obtains the unique bounded solution of the second equation in (8.1) as

$$y(t) = \int_{-\infty}^{t}(t-s)\mathrm{e}^{-a(t-s)}\varphi\big(x(s)\big)\,\mathrm{d}s = \frac{1}{a^2}\int_{-\infty}^{t}\gamma_a^2(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s.$$

Hence, in this case, the bounded solutions of (8.1) are determined by the following differential equation with a single distributed delay:

$$\ddot{x}(t) = f\left(t, x(t), \frac{1}{a^2}\int_{-\infty}^{t}\gamma_a^2(t-s)\varphi\big(x(s)\big)\,\mathrm{d}s\right).$$

We observe that the above perspective extensions can be combined and further expanded to the case where the ambient space is a differentiable manifold. Such generalization will be investigated elsewhere.

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On the existence and multiplicity of solutions for nonlinear Klein–Gordon–Maxwell systems

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Abstract. In this paper, we study the existence and multiplicity solutions for the following Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\omega > 0$ is a constant and the nonlinearity $f(x,u)$ is either asymptotically linear in u at infinity or the primitive of $f(x,u)$ is of 4-superlinear growth in u at infinity. Under some suitable assumptions, the existence and multiplicity of solutions are proved by using the Mountain Pass theorem and the fountain theorem, respectively.

Keywords: Klein–Gordon–Maxwell system, sign-changing potential, 4-superlinear, asymptotically linear.

2020 Mathematics Subject Classification: 35B33, 35J65, 35Q55.

1 Introduction and main results

In this paper we consider the following nonlinear Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{KGM}$$

where $\omega > 0$ is a constant. We are interested in the existence and multiplicity solutions of system (*KGM*) when the nonlinearity $f(x,u)$ is either asymptotically linear in u at infinity or the primitive of $f(x,u)$ is of 4-superlinear growth at infinity.

Such system has been firstly studied by Benci and Fortunato [6] as a model which describes nonlinear Klein–Gordon fields in three dimensional space interacting with the electrostatic field. For more details on the physical aspects of the problem we refer the readers to see [7] and the references therein.

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In 2002, Benci and Fortunato [7] first studied the following Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + [m^2 - (\omega+\phi)^2]\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi + \phi u^2 = -\omega u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

with the pure power type nonlinearity, i.e. $f(x,u) = |u|^{q-2}u$, where ω and m are constants. By using a version of the mountain pass theorem, they proved that system (1.1) has infinitely many radially symmetric solutions under $|m| > |\omega|$ and $4 < q < 6$. It was complemented and improved by [3] and [19]. Azzollini and Pomponio [2] obtained the existence of a ground state solution for (1.1) under one of the conditions

(i) $4 \le q < 6$ and $m > \omega$;

(ii) $2 < q < 4$ and $m\sqrt{q-2} > \omega\sqrt{6-q}$.

Soon afterwards, it is improved by Wang [33]. Motivated by the methods of [7], Cassani [9] considered (1.1) for the critical case by adding a lower order perturbation:

$$\begin{cases} -\Delta u + [m^2 - (\omega+\phi)^2]\phi u = \mu|u|^{q-2}u + |u|^{2^*-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\mu > 0$ and $2^* = 6$. He showed that (1.2) has at least a radially symmetric solution under one of the following conditions:

(i) $4 < q < 6, |m| > |\omega|$ and $\mu > 0$;

(ii) $q = 4, |m| > |\omega|$ and μ is sufficiently large.

It is improved and generalized by the results in [10] and [32]. Recently, the authors in [11,17,37] proved the existence of positive ground state solutions for the problem (1.2) with a periodic potential V or V is a constant:

$$\begin{cases} -\Delta u + V(x)u - (2\omega+\phi)\phi u = \mu|u|^{q-2}u + |u|^{2^*-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2, & x \in \mathbb{R}^3. \end{cases}$$

In [23], Georgiev and Visciglia introduced a system like (1.1) with potentials, however they considered a small external Coulomb potential in the corresponding Lagrangian density. Inspired by these works, He [24] first considered the existence of infinitely many solutions for system (*KGM*). The nonlinearity f satisfied (AR) condition:

(AR) There exists $\theta > 4$ such that $\theta F(x,t) \le tf(x,t)$, for all $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$, where $F(x,t) = \int_0^t f(x,s)ds$.

Very recently, Ding and Li [21] obtained the existence of infinitely many solutions for (*KGM*) under the following condition:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ is bounded below and, for every $C > 0$, $\mathrm{meas}\{x \in \mathbb{R}^3 : V(x) \le C\} < +\infty$, where meas denotes the Lebesgue measures;

(F_1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $|f(x,t)| \le C_1|t| + C_2|t|^{p-1}$ for $4 \le p < 2^*$, where C_1, C_2 are positive constants, $f(x,t)t \ge 0$ for $t \ge 0$;

(F_2) $\frac{F(x,t)}{t^4} \to +\infty$ as $|t| \to +\infty$;

(F_3) Let $\mathcal{F}(x,t) := \frac{1}{4}f(x,t)t - F(x,t)$, there exists $r_0 > 0$ such that if $|t| \geq r_0$, then $\mathcal{F}(x,t) \geq 0$ uniformly for $x \in \mathbb{R}^3$;

(F_4) $f(x,-t) = -f(x,t)$ for any $x \in \mathbb{R}^3, t \in \mathbb{R}$.

Cunha [18] considered the existence of positive and ground state solutions for (KGM) with periodic potential $V(x)$. By the Ekeland variational principle and the Mountain Pass Theorem, Li, A. Boucherif and N. D. Merzagui [27] obtained the existence of two different solutions for (KGM). Other related results about Klein–Gordon–Maxwell system on $\mathbb{R}^3$ can be found in [16, 20, 26, 28, 35]. By the way, we recall that Klein–Gordon–Maxwell system with nonhomogeneous nonlinearity is studied in [14, 22, 36, 39] and the existence of infinitely many radial solitary waves solutions are studied in [12].

Before giving our main results, we give some notations. Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the standard scalar and norm

$$(u,v)_H = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)dx; \qquad \|u\|_H^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)dx.$$

$D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_D^2 := \|u\|_{D^{1,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

The norm on $L^s = L^s(\mathbb{R}^3)$ with $1 < s < \infty$ is given by $|u|_s^s = \int_{\mathbb{R}^3} |u|^s dx$.

System (KGM) has a variational structure. Indeed, we consider the functional $\mathcal{J} : H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{J}(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx - \frac{1}{2}\int_{\mathbb{R}^3} (2\omega + \phi)\phi u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx.$$

The solutions $(u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ of system (KGM) are critical points of $\mathcal{J}$. However, the functional $\mathcal{J}$ is strongly indefinite and is difficult to investigate. Fortunately, this indefiniteness can be removed by using the reduction method described in [8]. Then we are led to the study of a new functional $I(u)$ which does not present such strongly indefinite nature.

Motivated by the above works, in the present paper we first consider system (KGM) with the superlinear case, and hence make the following assumptions:

(f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist $C > 0$ and $p \in (4,6)$ such that

$$|f(x,t)| \leq C(1 + |t|^{p-1});$$

(f_2) $f(x,t) = o(t)$ uniformly in x as $t \to 0$;

(f_3) $\frac{F(x,t)}{t^4} \to +\infty$ uniformly in x as $|t| \to +\infty$;

(f_4) There exists a positive constant b such that $\mathcal{F}(x,t) := \frac{1}{4}f(x,t)t - F(x,t) \geq -bt^2$.

Remark 1.1. We emphasize that unlike all previous results about system (KGM), see e.g. [18, 24, 26], we have not assume that the potential V is positive. This means that we allow the potential V be sign changing.

Remark 1.2. It is well known that the condition (AR) is widely used in the studies of elliptic problem by variational methods. The condition (AR) is used not only to prove that the Euler-Lagrange function associated has a mountain pass geometry, but also to guarantee that the Palais–Smale sequences, or Cerami sequences are bounded. Obviously, we can observe that the condition (AR) implies the following condition:

(A_1) There exist $\theta > 4$ and $C_1, C_2 > 0$ such that $F(x,t) \geq C_1|t|^\theta - C_2$, for every t sufficiently large.

Moreover, the condition (A_1) implies our condition (f_3).

Another widely employed condition is the following condition, which is first introduced by Jeanjean [25].

(Je) There exist $\theta \geq 1$ such that $\theta\mathcal{F}(x,t) \geq \mathcal{F}(x,st)$ for all $s \in [0,1]$ and $t \in \mathbb{R}$, where $\mathcal{F}(x,t)$ is given in (f_4).

We can observe that when $s = 0$, then $\mathcal{F}(x,t) \geq 0$, but for our condition (f_4), $\mathcal{F}(x,t)$ may assume negative values. Therefore, it is interesting to consider 4-superlinear problems under the conditions (f_3) and (f_4).

The condition (f_4) is motivated by Alves, Soares and Souto [1]. Supposing in addition

$$\alpha = \inf_{x\in\mathbb{R}^3} V(x) > 0 \tag{1.3}$$

and $b \in [0,\alpha)$, they proved that all Cerami sequences are bounded. In 2015, Chen and Liu [13] also used conditions (f_3) and (f_4) to show the existence of infinitely many solutions for Schrödinger–Maxwell systems. In our case, however, many technical difficulties arise to the presence of a non-local term ϕ, which is not homogeneous as it is in the Schrödinger–Maxwell systems. Hence, a more careful analysis of the interaction between the couple (u,ϕ) is required.

By (V), we know that V is bounded from below, hence we may choose $V_0 > 0$ such that

$$\tilde{V}(x) := V(x) + V_0 > 1, \quad \forall x \in \mathbb{R}^3$$

and define a new Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < \infty \right\}$$

with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla v + \tilde{V}(x)uv\right) dx$$

and the norm $\|u\| = \langle u,u\rangle^{1/2}$. Obviously, the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous, for any $s \in [2,2^*]$. The norm on $L^s = L^s(\mathbb{R}^3)$ with $1 < s < \infty$ is given by $|u|_s^s = \int_{\mathbb{R}^3} |u|^s dx$. Consequently, for each $s \in [2,6]$, there exists a constant $d_s > 0$ such that

$$|u|_s \leq d_s\|u\|, \quad \forall u \in E. \tag{1.4}$$

Furthermore, we have that under the condition (V), the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $s \in [2,6)$ (see [4]). By the compact embedding $E \hookrightarrow L^2(\mathbb{R}^3)$ and the standard elliptic theory [40], it is easy to see that the eigenvalue problem

$$-\Delta u + V(x)u = \lambda u, \quad u \in E \tag{1.5}$$

possesses a complete sequence of eigenvalues

$$-\infty < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots, \quad \lambda_j \to +\infty.$$

Each λ_j has finite multiplicity and $|\lambda_j|_2 = 1$. Denote e_j be the eigenfunction of λ_j. E^- is spanned by the eigenfunctions corresponding to negative eigenvalues. Note that the negative space E^- of the quadratic part of I is nontrivial if and only if some λ_j is negative.

Now we can state our first result.

Theorem 1.3. *Suppose (V), (f_1)–(f_4) are satisfied, and f is odd in t. If 0 is not an eigenvalue of (1.5), then (KGM) has a sequence of solutions $(u_n, \phi_n) \in E \times D^{1,2}(\mathbb{R}^3)$ such that the energy $\mathcal{J}(u_n, \phi_n) \to +\infty$.*

Remark 1.4. If u is a critical point of I, then $I(u) = \mathcal{J}(u, \phi_u)$ (see (2.1)). Therefore, in order to prove Theorem 1.1, we only need to find a sequence of critical points $\{u_n\}$ of I such that $I(u_n) \to +\infty$.

Remark 1.5. Theorem 1.3 improves the recent results in [24]. In that paper, the author assumed in addition (1.3), and (AR) or (Je). When V is positive, the quadratic part of the functional I (see (2.1)) is positively definite, and I has a mountain pass geometry. Therefore, the mountain pass lemma [30] can be applied. In our case, the quadratic part may possesses a nontrivial negative space E^-, so I no longer possesses the mountain pass geometry. Therefore the methods in [21, 24] cannot be applied. To obtain our result, we adopt a technique developed in [13].

In the second part of this paper, we deal with the system (KGM) when the nonlinearity $f(x,t)$ is asymptotically linear at infinity in the second variable t. Set

$$\Omega = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx}{\int_{\mathbb{R}^3} u^2 dx},$$

i.e. Ω is the infimum of the spectrum of the Schrödinger operator $-\Delta + V$.

We make the following assumptions:

(H_1) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $V(x) \geq D_0 > 0$ for all $x \in \mathbb{R}^3$;

(H_2) $\lim_{|x| \to +\infty} V(x) = V_\infty \in (0, +\infty)$;

(H_3) $f(x,t) \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $\lim_{t \to 0} \frac{f(x,t)}{t} = 0$ uniformly in x;

(H_4) There exists $A \in (\Omega, V_\infty)$ such that $\lim_{t \to +\infty} \frac{f(x,t)}{t} = A$ uniformly in x and $0 \leq \frac{f(x,t)}{t} \leq A$ for all $t \neq 0$.

Theorem 1.6. *Assume (H_1)–(H_4) hold, then there exists a constant $\omega^* > 0$ such that (KGM) has a positive solution for any $\omega \in (0, \omega^*)$.*

Theorem 1.7. *Assume (H_1)–(H_4) hold, then there exists a constant $\omega^\sharp > 0$ such that (KGM) has no nontrival solution for any $\omega > \omega^\sharp$.*

Remark 1.8.

(a) It follows from the condition $\Omega < A < V_\infty$ that $V(x)$ is not a constant.

(b) By Theorem 1.7, it is easy to know that ω^* is finite.

Remark 1.9. To our best knowledge, it seems that there are few results for system (*KGM*) in this case: the nonlinear term $f(x,t)$ in t is asymptotically linear at infinity. In order to get our results, we have to solve some difficulties. The first difficult is how to prove the variational function satisfies the assumptions of the Mountain Pass Theorem. The second difficult is how to check the (PS) condition, i.e., how to verify the boundedness and compactness of a (PS) sequence. To overcome these difficulties we use some techniques used in [29], [31] and [34]. However, it seems difficult to use this method to the case $f(x,t)$ is superlinear in t at infinity.

We denote by $'' \rightharpoonup ''$ weak convergence and by $'' \to ''$ strong convergence. Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again $\{u_n\}$.

The paper is organized as follows. In Section 2, we will introduce the variational setting for the problem, give some related preliminaries and prove Theorem 1.3. We give the proofs of Theorem 1.6 and Theorem 1.7 in Section 3.

2 Proof of Theorem 1.3

By [3], we know that the signs of ω is not relevant for the existence of solutions, so we can assume that $\omega > 0$. Evidently, the properties of ϕ_u plays an important role in the study of $\mathcal{J}$. So we need the following technical results.

Proposition 2.1. *For any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ which satisfies*

$$\Delta\phi = (\phi + \omega)u^2 \quad \text{in } \mathbb{R}^3.$$

Moreover, the map $\Phi : u \in H^1(\mathbb{R}^3) \mapsto \phi_u \in D^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and

(i) $-\omega \le \phi_u \le 0$ *on the set* $\{x \in \mathbb{R}^3 | u(x) \neq 0\}$;

(ii) $\|\phi_u\|_D^2 \le C\|u\|^2$ *and* $\int_{\mathbb{R}^3} \phi_u u^2 dx \le C|u|_{12/5}^4 \le C\|u\|^4$.

The proof is similar to Proposition 2.1 in [24] by using the fact $E \hookrightarrow L^s(\mathbb{R}^3)$, for any $s \in [2,6]$ is continuous.

By Proposition 2.1, we can consider the functional $I : H^1(\mathbb{R}^3) \mapsto \mathbb{R}$ defined by $I(u) = \mathcal{J}(u, \phi_u)$.

Multiplying $-\Delta\phi_u + \phi_u u^2 = -\omega u^2$ by ϕ_u and integration by parts, we obtain

$$\int_{\mathbb{R}^3} (|\nabla\phi_u|^2 + \phi_u^2 u^2)dx = -\int_{\mathbb{R}^3} \omega\phi_u u^2 dx.$$

By the above equality and the definition of $\mathcal{J}$, we obtain a C^1 functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx - \frac{1}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx \tag{2.1}$$

and its Gateaux derivative is

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)dx - \int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u uv dx - \int_{\mathbb{R}^3} f(x,u)v dx$$

for all $v \in H^1(\mathbb{R}^3)$. Here we use the fact that $(\Delta - u^2)^{-1}[\omega u^2] = \phi_u$.

If $\lambda_1 > 0$, we can easy to prove that I has the mountain pass geometry, so we omit this case. Since 0 is not an eigenvalue of (1.5), we assume that there exists $l \geq 1$ such that $0 \in (\lambda_l, \lambda_{l+1})$. Set

$$E^- = \text{span}\{e_1, \dots, e_l\}, \quad E^+ = (E^-)^\perp. \tag{2.2}$$

Then E^- and E^+ are the negative space and positive space of the quadratic form

$$N(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx$$

respectively, and $\dim E^- < \infty$. Moreover, there is a positive constant B such that

$$\pm N(u) \geq B\|u\|^2, \quad u \in E^\pm. \tag{2.3}$$

In order to prove Theorem 1.3, we shall use the fountain theorem of Bartsch [5], see also Theorem 3.6 in [38]. For $k = 1, 2, \dots,$ set

$$Y_k = \text{span}\{e_1, \dots, e_k\}, \quad Z_k = \overline{\text{span}\{e_{k+1}, \dots, \}}. \tag{2.4}$$

Proposition 2.2 (Fountain Theorem). *Assume that the even functional $I \in C^1(E, \mathbb{R})$ satisfies the (PS) condition. If there is a positive constant K such that for any $k \geq K$ there exist $\rho_k > r_k > 0$ such that*

(i) $a_k = \max_{u \in Y_k, \|u\| = \rho_k} I(u) \leq 0,$

(ii) $b_k = \inf_{u \in Z_k, \|u\| = r_k} I(u) \to +\infty$ *as* $k \to +\infty,$

then I has a sequence of critical points $\{u_k\}$ such that $I(u_k) \to +\infty$.

In order to study the functional I, we will write the functional I in a form in which the quadratic part is $\|u\|^2$. Let $g(x,t) = f(x,t) + V_0 t$. Then, by an computation, we obtain that

$$G(x,t) := \int_0^t g(x,s)ds \leq \frac{t}{4}g(x,t) + \frac{\tilde{V_0}}{4}t^2, \quad \tilde{V_0} := 4b + V_0 > 0. \tag{2.5}$$

By (f_3) we have

$$\lim_{|t| \to \infty} \frac{g(x,t)t}{t^4} = +\infty. \tag{2.6}$$

Furthermore, by (f_2) we obtain

$$\lim_{|t| \to 0} \frac{g(x,t)t}{t^4} = \lim_{|t| \to 0} \left(\frac{t^2}{t^4} \cdot \frac{f(x,t)t + V_0 t^2}{t^2} \right) = +\infty.$$

Hence there exists $M > 0$ such that

$$g(x,t)t \geq -Mt^4, \quad \forall t \in \mathbb{R}. \tag{2.7}$$

With the modified nonlinearity g, the functional $I : E \to \mathbb{R}$ can be rewritten in the following

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\omega}{2}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} G(x,u)dx \tag{2.8}$$

with the derivative

$$\langle I'(u), v\rangle = \langle u, v\rangle - \int_{\mathbb{R}^3}(2\omega + \phi_u)\phi_u uv dx - \int_{\mathbb{R}^3} g(x,u)v dx.$$

Lemma 2.3. *Suppose (V), (f_1)–(f_4) are satisfied, then the function I satisfies the (PS) condition.*

Proof. It follows from $\frac{1}{4}tf(x,t) - F(x,t) \geq -bt^2$ that the condition (f_3) is equivalent to

$$\lim_{|t|\to+\infty} \frac{G(x,t)}{t^4} = +\infty.$$

Let $\{u_n\}$ be a (PS) sequence, i.e.,

$$\sup_n |I(u_n)| < \infty, \quad I'(u_n) \to 0.$$

We first prove that $\{u_n\}$ is bounded in E. Arguing by contradiction, suppose that $\{u_n\}$ is unbounded, passing to a subsequence, by (2.5), we obtian

$$\begin{aligned} 4\sup_n I(u_n) + \|u_n\| &\geq 4I(u_n) - \langle I'(u_n), u_n\rangle \\ &= \|u_n\|^2 + \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx + \int_{\mathbb{R}^3} (g(x,u_n)u_n - 4G(x,u_n))dx \\ &\geq \|u_n\|^2 - \tilde{V}_0 \int_{\mathbb{R}^3} u_n^2 dx. \end{aligned} \tag{2.9}$$

Let $v_n = \frac{u_n}{\|u_n\|}$. Then, going if necessary to a subsequence, by the compact embedding $E \hookrightarrow L^2(\mathbb{R}^3)$ we may assume that

$$\begin{aligned} v_n &\rightharpoonup v_0 && \text{in } E; \\ v_n &\to v_0 && \text{in } L^2(\mathbb{R}^3); \\ v_n(x) &\to v_0(x) && \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Dividing both sides of (2.9) by $\|u_n\|^2$, we have

$$\tilde{V}_0 \int_{\mathbb{R}^3} v_0^2 dx \geq 1 \quad \text{as } n \to \infty.$$

Consequently, we have that $v_0 \neq 0$.

By (1.4) and (2.7), we have

$$\begin{aligned} \int_{v_0=0} \frac{g(x,u_n)u_n}{\|u_n\|^4} dx &= \int_{v_0=0} \frac{g(x,u_n)u_n}{u_n^4} v_n^4 dx \\ &\geq -M \int_{v_0=0} v_n^4 dx \geq -M \int_{\mathbb{R}^3} v_n^4 dx \\ &= -M|v_n|_4^4 \geq -Md_4^4 > -\infty. \end{aligned} \tag{2.10}$$

For $x \in \{x \in \mathbb{R}^3 | v_0 \neq 0\}$, we have $|u_n(x)| \to +\infty$ as $n \to \infty$. By (2.6) we have

$$\frac{g(x,u_n(x))u_n(x)}{\|u_n\|^4} = \frac{g(x,u_n(x))u_n(x)}{u_n^4(x)} v_n^4(x) \to +\infty. \tag{2.11}$$

Hence, by (2.10) and (2.11) and Fatou's lemma we obtain

$$\int_{\mathbb{R}^3} \frac{g(x,u_n)u_n}{\|u_n\|^4} dx \geq \int_{v_0\neq 0} \frac{g(x,u_n)u_n}{u_n^4} v_n^4(x) dx - Md_4^4 \to +\infty. \tag{2.12}$$

Hence we obtain that

$$\int_{\mathbb{R}^3} \frac{G(x,u_n)}{\|u_n\|^4} dx \to +\infty. \tag{2.13}$$

Since $\{u_n\}$ is a (PS) sequence, using Proposition 2.1 and (2.12), for n large enough, we obtain

$$\begin{aligned} c\omega + 1 &\geq \frac{1}{\|u_n\|^4}\left(\frac{1}{2}\|u_n\|^2 - \frac{\omega}{2}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - I(u_n)\right) \\ &= \int_{\mathbb{R}^3} \frac{G(x,u_n)}{\|u_n\|^4} dx \to +\infty, \end{aligned} \tag{2.14}$$

which is a contradiction.

Now we have proved that $\{u_n\}$ is bounded in E. By a similar argument in [15], the compact embedding $E \hookrightarrow L^2(\mathbb{R}^3)$ and

$$E = \overline{\bigcup_{n\in\mathbb{N}} E_n},$$

we can show that $\{u_n\}$ has a subsequence converging to a critical point of I. □

Lemma 2.4. *Let X be a finite dimensional subspace of E, then I is anti-coercive on X, i.e.*

$$I(u) \to -\infty, \quad \text{as } \|u\| \to \infty,\ u \in X.$$

Proof. If it is not true, we can choose a sequence $\{u_n\} \subset X$ and ξ is a real number such that

$$\|u_n\| \to \infty, \quad I(u_n) \geq \xi. \tag{2.15}$$

Let $v_n = \frac{u_n}{\|u_n\|}$. Since $\dim X < \infty$, going if necessary to a subsequence we have

$$\|v_n - v_0\| \to 0, \quad v_n(x) \to v_0(x) \quad \text{a.e. in } \mathbb{R}^3$$

for every $v_0 \in X$, with $\|v_0\| = 1$. Since $v_0 \neq 0$, similar to (2.13) we obtain that

$$\int_{\mathbb{R}^3} \frac{G(x,u_n)}{\|u_n\|^4} dx \to +\infty.$$

And arguing similar to (2.14), it follows from $\sup_n |I(u_n)| < \infty$ that

$$I(u_n) = \|u_n\|^4 \left(\frac{1}{2\|u_n\|^2} - \frac{\omega}{2\|u_n\|^4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} \frac{G(x,u_n)}{\|u_n\|^4} dx\right) \to -\infty,$$

which is contradict with $I(u_n) \geq \xi$. The proof is complete. □

Now, we are ready to prove our main result.

Proof of Theorem 1.3. We will find a sequence of critical points $\{u_n\}$ of I such that $I(u_n) \to +\infty$.

Since $f(x,t)$ is odd in t, I is an even function. It follows from Lemma 2.3 that I satisfies (PS) condition. Therefore, it suffices to verify (i) and (ii) of Proposition 2.2.

(i) Since $\dim Y_k < \infty$, by Lemma 2.4, we get the conclusion of (i).

(ii) By $(f_1), (f_2)$, we have

$$|f(x,t)| \leq \epsilon|t| + C_\epsilon |t|^{p-1}, \quad |F(x,t)| \leq \frac{\epsilon}{2}t^2 + \frac{C_\epsilon}{p}|t|^p,$$

where $\epsilon > 0$ is very small. Then we have

$$|F(x,t)| \leq \frac{B}{2d_2^2}t^2 + \frac{CB}{p}|t|^p, \tag{2.16}$$

where B is defined in (2.3). We assume that $0 \in [\lambda_l, \lambda_{l+1})$. Then if $k > l$, we have that $Z_k \subset E^+$, where E^+ is defined in (2.2). Now we have

$$N(u) \geq B\|u\|^2, \quad u \in Z_k \tag{2.17}$$

and, as proof of Lemma 3.8 in [38],

$$\beta_p(k) = \sup_{u \in Z_k, \|u\|=1} |u|_p \to 0, \quad \text{as } k \to \infty.$$

Let $r_k = (Cp\beta_p(k))^{1/(2-p)}$, where C is chosen as in (2.16). For $u \in Z_k \subset E^+$ with $\|u\| = r_k$, $\phi_u \leq 0$, by (2.17) we deduce that

$$\begin{aligned} I(u) &= N(u) - \frac{1}{2}\omega \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx \\ &\geq B\|u\|^2 - \frac{B}{2d_2^2}|u|_2^2 - CB|u|_p^p \\ &\geq B\left(\frac{1}{2}\|u\|^2 - C\beta_p^p\|u\|^p\right) \\ &= B\left(\frac{1}{2} - \frac{1}{p}\right)(Cp\beta_p^p)^{2/(2-p)}, \end{aligned}$$

where $\beta_p^p := (\beta_p(k))^p$. Since $\beta_p(k) \to 0$ and $p > 2$, it follows that

$$b_k = \inf_{u \in Z_k, \|u\|=r_k} I(u) \to +\infty.$$

We get the conclusion of (ii). The proof is complete. □

3 Proofs of Theorem 1.6 and Theorem 1.7

Under the condition (H_1), we define a new Hilbert space

$$F := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx < \infty \right\}.$$

with the inner product

$$(u,v)_F = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)\, dx$$

and the norm $\|u\|_F = (u,u)_F^{1/2}$, which is equivalent to the usual Sobolev norm on $H^1(\mathbb{R}^3)$. Obviously, the embedding $F \hookrightarrow L^s(\mathbb{R}^3)$ is continuous, for any $s \in [2, 2^*]$. Consequently, for each $s \in [2,6]$, there exists a constant $v_s > 0$ such that

$$|u|_s \leq v_s\|u\|_F, \quad \forall u \in F. \tag{3.1}$$

Furthermore, we know that under assumption (H_1), the embedding $F \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $s \in [2, 2^*)$ (see [40]).

By Proposition 2.1, we can consider the functional I_ω on $(F, \|\cdot\|_F)$:

$$I_\omega(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2)dx - \int_{\mathbb{R}^3} F(x,u)dx,$$

with its Gateaux derivative is

$$\langle I'_\omega(u), v\rangle = \int_{\mathbb{R}^3}[\nabla u \nabla v + V(x)uv - (2\omega + \phi_u)\phi_u uv - f(x,u)v]dx.$$

Lemma 3.1. *Suppose (H_1)–(H_4) hold. Then there exist some positive constants ρ_0, α_0 such that $I_\omega(u)|_{\|u\|_F=\rho_0} \geq \alpha_0$ for all $u \in F$. Moreover, there exists a function $u_0 \in F$ with $\|u_0\|_F > \rho_0$ and $\omega^* > 0$ such that $I_\omega(u_0) < 0$ for $0 < \omega < \omega^*$.*

Proof. By (H_3), (H_4), for any $\varepsilon > 0$, there exists q with $1 < q < 5$ and $M_1 = M_1(\varepsilon, p) > 0$ such that

$$|F(x,t)| \leq \frac{\varepsilon}{2}t^2 + M_1 t^{q+1}, \quad \text{for all } t > 0. \tag{3.2}$$

By $\phi_u \leq 0$ and the Sobolev inequality, we get that

$$\begin{aligned} I_\omega(u) &= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2)dx - \int_{\mathbb{R}^3} F(x,u)dx \\ &\geq \frac{1}{2}\|u\|_F^2 - \frac{\varepsilon}{2}v_2^2\|u\|_F^2 - M_1 v_{q+1}^{q+1}\|u\|_F^{q+1} \\ &= \left(\frac{1}{2} - \frac{\varepsilon}{2}v_2^2\right)\|u\|_F^2 - M_1 v_{q+1}^{q+1}\|u\|_F^{q+1}. \end{aligned}$$

Since $1 < q < 5$, let $\varepsilon = \frac{1}{2v_2^2}$ and $\|u\|_F = \rho_0 > 0$ small enough, then we can obtain $I_\omega(u)|_{\|u\|_F=\rho} \geq \alpha_0$ for all $u \in F$.

By (H_4), we have $A > \Omega$. From the definition of Ω, there exists a nonnegative function $u_1 \in H^1(\mathbb{R}^3)$ such that

$$\|u_1\|_F^2 = \int_{\mathbb{R}^3}(|\nabla u_1|^2 + V(x)u_1^2)dx < A\int_{\mathbb{R}^3} u_1^2 dx = A|u_1|_2^2.$$

Hence, by (H_4) and Fatou's lemma we obtain that

$$\begin{aligned} \lim_{t\to+\infty}\frac{I_0(tu_1)}{t^2} &= \frac{1}{2}\|u_1\|_F^2 - \lim_{t\to+\infty}\int_{\mathbb{R}^3}\frac{F(x,tu_1)}{t^2u_1^2}u_1^2 dx \\ &\leq \frac{1}{2}\|u_1\|_F^2 - \frac{A}{2}\int_{\mathbb{R}^3}u_1^2 dx \\ &= \frac{1}{2}(\|u_1\|_F^2 - A|u_1|_2^2) < 0. \end{aligned}$$

If $I_0(tu_1) \to -\infty$ as $t \to +\infty$, then there is $u_0 \in F$ with $\|u_0\|_F > \rho_0$ such that $I_0(u_0) < 0$. Since $I_\omega(u_0) \to I_0(u_0)$ as $\omega \to 0^+$. We have that there is a positive constant $\omega^* > 0$ such that $I_\omega(u_0) < 0$ for all $0 < \omega < \omega^*$. The proof is complete. □

Lemma 3.2. *Suppose (H_1)–(H_4) hold. Then any sequence $\{u_n\} \subset F$ satisfying*

$$I_\omega(u_n) \to c > 0, \quad \langle I'_\omega(u_n), u_n\rangle \to 0$$

is bounded in F. Moreover, $\{u_n\}$ has a strongly convergent subsequence in F.

Proof. (i) We first to prove that $\{u_n\}$ is bounded. For any fixed $L > 0$, let $\eta_L \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be a cut-off function such that

$$\eta_L = \begin{cases} 0, & \text{for } |x| \le L/2, \\ 1, & \text{for } |x| \ge L, \end{cases}$$

and $|\nabla \eta_L| \le \frac{C}{L}$ for all $x \in \mathbb{R}^3$ and C is a positive constant. For any $u \in F$ and all $L \ge 1$, there exists a constant $C_0 > 0$, which is independent of L, such that $\|\eta_L u\|_F \le C_0 \|u\|_F$.

Since $I'_\omega(u_n) \to 0$ as $n \to +\infty$ in $H^{-1}(\mathbb{R}^3)$, for n large enough we have that

$$\langle I'_\omega(u_n), \eta_L u_n \rangle \le \|I'_\omega(u_n)\|_{F^{-1}} \|\eta_L u_n\|_F \le \|u_n\|_F, \tag{3.3}$$

and

$$\begin{aligned} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x) u_n^2) \eta_L dx + \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \eta_L dx - \int_{\mathbb{R}^3} (2\omega + \phi_{u_n}) \phi_{u_n} \eta_L u_n^2 dx \\ \le \int_{\mathbb{R}^3} f(x, u_n) u_n \eta_L dx + \|u_n\|_F, \end{aligned} \tag{3.4}$$

where F^{-1} is the dual space of F.

By assumptions (H_2) and (H_4), there exist $\gamma > 0$ and $L_1 > 0$ such that $V(x) \ge A + \gamma$ for all $|x| \ge L_1$. Choosing $L > 2L_1$, since $|\nabla \eta_L(x)| \le \frac{C}{L}$ for all $x \in \mathbb{R}^3$, $2\omega + \phi_{u_n} \ge 0$ and $f(x, u_n(x)) u_n(x) \le A u_n^2(x)$ for all $x \in \mathbb{R}^3$ by (H_4). Following from (3.4) we get that

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + \gamma u_n^2) \eta_L dx \le \frac{C}{L} \left(\int_{\mathbb{R}^3} u_n^2 dx + \int_{\mathbb{R}^3} \nabla u_n^2 dx \right) + \|u_n\|_F. \tag{3.5}$$

Similar to (3.3), we have that $\langle I'_\omega(u_n), u_n \rangle \le \|u_n\|_F$, that is

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x) u_n^2 - 2\omega \phi_{u_n} u_n^2 - \phi_{u_n}^2 u_n^2 - f(x, u_n) u_n) dx \le \|u_n\|_F. \tag{3.6}$$

Motivated by [31] (see also [39]), we give an inequality by using the second equality of system (*KGM*). Multiplying both sides of $\Delta \phi_{u_n} = (\omega + \phi_{u_n}) u_n^2$ by $|u_n|$,integrating by parts and using the Young's inequality, we have

$$\sqrt{\frac{3}{4}} \int_{\mathbb{R}^3} (\omega + \phi_{u_n}) |u_n|^3 dx \le \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{3}{4} \int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2 dx. \tag{3.7}$$

Then by Proposition 2.1, one has

$$\begin{aligned} \sqrt{3} \int_{\mathbb{R}^3} (\omega + \phi_{u_n}) |u_n|^3 dx &\le \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2 dx \\ &\le \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \frac{3}{2} \int_{\mathbb{R}^3} \omega \phi_{u_n} u_n^2 dx - \frac{3}{2} \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx \\ &\le \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \frac{3}{2} \int_{\mathbb{R}^3} \omega \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx. \end{aligned} \tag{3.8}$$

By (3.6), (3.7), (3.8) and $V(x) > 0, \phi_{u_n} \le 0$, $f(x, u_n(x)) u_n(x) \le A u_n^2(x)$ for all $x \in \mathbb{R}^3$ we have that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2+V(x)u_n^2)dx+\int_{\mathbb{R}^3}(\sqrt{3}(\omega+\phi_{u_n})|u_n|^3-Au_n^2)dx\\
&\quad\le \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2+V(x)u_n^2)dx+\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2dx-\frac{3}{2}\int_{\mathbb{R}^3}\omega\phi_{u_n}u_n^2dx\\
&\quad\quad-\int_{\mathbb{R}^3}\phi_{u_n}^2u_n^2dx-\int_{\mathbb{R}^3}f(x,u_n)u_ndx\\
&\quad=\int_{\mathbb{R}^3}(|\nabla u_n|^2+V(x)u_n^2-2\omega\phi_{u_n}u_n^2-\phi_{u_n}^2u_n^2-f(x,u_n)u_n)dx\\
&\quad\quad-\frac{1}{2}\int_{\mathbb{R}^3}V(x)u_n^2dx+\frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_{u_n}u_n^2dx\\
&\quad\le \|u_n\|_F,
\end{aligned}$$

that is

$$\frac{1}{2}\|u_n\|_F^2+\int_{\mathbb{R}^3}h(u_n)dx\le\|u_n\|_F, \tag{3.9}$$

where $h(u_n)=\sqrt{3}(\omega+\phi_{u_n})|u_n|^3-Au_n^2$.

By (3.5), there is a positive constant $C_1>0$ (independent of L) such that

$$\int_{|x|\ge L}u_n^2dx\le\frac{C_1}{L}\|u_n\|_F^2+C_1\|u_n\|_F.$$

Let $\delta=\inf_{t\in R}h(t)$. Then $\delta\in(-\infty,0)$ and by above inequality we have

$$\begin{aligned}
\int_{\mathbb{R}^3}h(u_n)dx&\ge\int_{|x|\le L}\delta dx+\int_{|x|\ge L}(-Au_n^2)dx\\
&\ge\delta|B_L(0)|-\frac{AC_1}{L}\|u_n\|_F^2-AC_1\|u_n\|_F,
\end{aligned}\tag{3.10}$$

where $|B_L(0)|$ denotes the volume of $B_L(0)$. It follows from (3.9) and (3.10) that

$$\frac{1}{2}\|u_n\|_F^2\le|\delta||B_L(0)|+\frac{AC_1}{L}\|u_n\|_F^2+AC_1\|u_n\|_F+\|u_n\|_F.$$

Since C_1 is a constant independent of L, we can choose L large enough such that $\frac{AC_1}{L}<\frac{1}{2}$. Then we obtain that $\{u_n\}$ is bounded in F by above inequality.

(ii) Now we shall show that $\{u_n\}$ has a strongly convergent subsequence in F. From case (i), $\{u_n\}$ is bounded in F. Then (3.3) and (3.5) become

$$\langle I'_\omega(u_n),\eta_L u_n\rangle=\circ(1)$$

and

$$\int_{|x|\ge L}(|\nabla u_n|^2+u_n^2)dx\le\frac{C}{L}\|u_n\|_F^2+\circ(1), \tag{3.11}$$

respectively. Therefore, for any $\varepsilon>0$, there exists $L>0$ such that for n large enough,

$$\int_{|x|\ge L}(|\nabla u_n|^2+u_n^2)dx\le\varepsilon. \tag{3.12}$$

Since $\{u_n\}$ is bounded in F, passing to a subsequence if necessary, there exists $u\in F$ such that $u_n\rightharpoonup u$ in F. In view of the embedding $F\hookrightarrow L^s(\mathbb{R}^3)$ are compact for any $s\in[2,6)$, $u_n\to u$

in $L^s(\mathbb{R}^3)$ for $1 < s < 6$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^3$. Hence it follows from assumptions of Lemma 3.2 and the derivative of I_ω, we easily obtain

$$\|u_n - u\|_F^2 = \langle I'_\omega(u_n) - I'_\omega(u), u_n - u\rangle + \int_{\mathbb{R}^3}(f(x,u_n) - f(x,u))(u_n - u)dx \\ +2\omega \int_{\mathbb{R}^3}(\phi_{u_n}u_n - \phi_u u)(u_n - u)dx + \int_{\mathbb{R}^3}(\phi_{u_n}^2 u_n - \phi_u^2 u)(u_n - u)dx.$$

It is clear that

$$\langle I'_\omega(u_n) - I'_\omega(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

By Proposition 2.1, the Hölder inequality and the Sobolev inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\phi_{u_n}u_n(u_n - u)dx\right| &\le |\phi_{u_n}|_6|u_n|_{12/5}|u_n - u|_{12/5} \\ &\le C_1\|\phi_{u_n}\|_D|u_n|_{12/5}|u_n - u|_{12/5} \\ &\le C_2|u_n|_{12/5}^3|u_n - u|_{12/5} \to 0.\end{aligned}$$

Since $u_n \to u$ in $L^s(\mathbb{R}^3)$ for any $s \in [2, 2^*)$. We obtain

$$\int_{\mathbb{R}^3}(\phi_{u_n} - \phi_u)u_n(u_n - u)dx \to 0 \quad \text{as } n \to \infty$$

and

$$\int_{\mathbb{R}^3}\phi_u(u_n - u)^2dx \le |\phi_u|_6|u_n - u|_3|u_n - u|_2 \to 0 \quad \text{as } n \to \infty$$

Thus, we get

$$\begin{aligned}\int_{\mathbb{R}^3}(\phi_{u_n}u_n - \phi_u u)(u_n - u)dx &= \int_{\mathbb{R}^3}(\phi_{u_n} - \phi_u)u_n(u_n - u)dx + \int_{\mathbb{R}^3}\phi_u(u_n - u)^2dx \\ &\to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Now, we shall prove

$$\int_{\mathbb{R}^3}f(x,u_n)(u_n - u)dx = \circ(1) \quad \text{and} \quad \int_{\mathbb{R}^3}f(x,u)(u_n - u) = \circ(1). \tag{3.13}$$

We only to prove the first one and the second one is similar. Since $|f(x,u_n)| \le A|u_n|$ and $\|u_n\|_F$ is bounded, by (3.12), the Hölder inequality and the Sobolev inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3}f(x,u_n)(u_n - u)dx\right| &\le \left|\int_{|x|\le L}f(x,u_n)(u_n - u)dx\right| + \left|\int_{|x|\ge L}f(x,u_n)(u_n - u)dx\right| \\ &\le C|u_n - u|_{L^2(B_L(0))} + C\left(\int_{|x|\ge L}u_n^2dx\right)^{1/2} \\ &\to 0 \quad \text{as } n \to \infty \text{ and } L \to +\infty.\end{aligned}$$

So (3.13) hold. Therefore, $\|u_n - u\|_F \to 0$ as $n \to \infty$. The proof is complete. □

Now we can prove our main results Theorem 1.6 and Theorem 1.7.

Proof of Theorem 1.6. By Lemma 3.1 and Lemma 3.2 we can obtain that u is a solution of system (*KGM*). And by using bootstrap arguments and the maximum principle, we can conclude that u is positive. The proof is complete. □

Proof of Theorem 1.7. Let $(u, \phi_u) \in F \times D^{1,2}(\mathbb{R}^3)$ be a solution of (*KGM*). Then $\langle I'_\omega(u), u\rangle = 0$, i.e.

$$\langle I'_\omega(u), u\rangle = \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2 - (2\omega + \phi_u)\phi_u u^2 - f(x,u)u]dx = 0. \tag{3.14}$$

Similar to (3.8), we deduce that

$$\sqrt{3}\int_{\mathbb{R}^3} (\omega + \phi_u)|u|^3 dx \leq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} \omega\phi_{u_n} u^2 dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 dx. \tag{3.15}$$

By (H_3) and (H_4), there exists $C = C(D_0)$ such that

$$f(x,u)u \leq D_0 u^2 + C|u|^3. \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14), we obtain that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2 - (2\omega + \phi_u)\phi_u u^2 - f(x,u)u]dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 dx \\ &\quad + \int_{\mathbb{R}^3} (V(x) - D_0)u^2 dx - \int_{\mathbb{R}^3} C|u|^3 dx \\ &\geq \int_{\mathbb{R}^3} [\sqrt{3}(\omega + \phi_u) - C]|u|^3 dx + \int_{\mathbb{R}^3} (V(x) - D_0)u^2 dx \\ &\geq \int_{\mathbb{R}^3} [\sqrt{3}(\omega + \phi_u) - C]|u|^3 dx. \end{aligned}$$

Therefore, if ω is large enough such that $\omega + \phi_u > \frac{\sqrt{3}}{3}C$, system (*KGM*) only has the trivial solution $u \equiv 0$. The proof is complete. □

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Positive solutions for a class of concave-convex semilinear elliptic systems with double critical exponents

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Abstract. In this paper, we consider the following concave-convex semilinear elliptic system with double critical exponents:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \frac{\alpha}{2^*}|u|^{\alpha-2}|v|^{\beta}u + \lambda|u|^{q-2}u, & \text{in } \Omega, \\ -\Delta v = |v|^{2^*-2}v + \frac{\beta}{2^*}|u|^{\alpha}|v|^{\beta-2}v + \mu|v|^{q-2}v, & \text{in } \Omega, \\ u,\ v > 0, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $\lambda,\ \mu > 0,\ 1 < q < 2,\ \alpha > 1,\ \beta > 1,\ \alpha + \beta = 2^* = \frac{2N}{N-2}$. By the Nehari manifold method and variational method, we obtain two positive solutions which improves the recent results in the literature.

Keywords: semilinear elliptic system, double critical exponents, positive solutions, Nehari manifold, variational method.

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1 Introduction and main result

In this paper, we mainly study the following concave-convex semilinear elliptic system with double critical exponents

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \frac{\alpha}{2^*}|u|^{\alpha-2}|v|^{\beta}u + \lambda|u|^{q-2}u, & \text{in } \Omega, \\ -\Delta v = |v|^{2^*-2}v + \frac{\beta}{2^*}|u|^{\alpha}|v|^{\beta-2}v + \mu|v|^{q-2}v, & \text{in } \Omega, \\ u,\ v > 0, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $\lambda,\ \mu > 0$, $1 < q < 2$, $\alpha > 1$, $\beta > 1$, $\alpha + \beta = 2^* = \frac{2N}{N-2}$. System (1.1) is abstracted from some physical phenomenon, especially some description in nonlinear optics. As we all known, it is also a model in Hartree–Fock theory for a double condensate, i.e., a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$, which was first discovered and proposed by B.D. Esry et al in 1997. There is a lot of literatures about the origin and physical background of system (1.1), and we refer the readers to see [5,7,11,26].

It is well known that many results have been obtained in these years on critical semilinear elliptic equation and system. For example, in 1983, Brézis and Nirenberg in reference [3] studied the case of positive solutions of semilinear elliptic equations with critical exponent in different dimensions and got many important results. In 1994, Ambrosetti et al in [2] showed that some problems of critical elliptic equation with concave-convex nonlinearities. With the development of variational methods, people gradually shifted their focus from equation to system. In 2000, Alves et al first studied elliptic system involving subcritical or critical Sobolev exponent in [1] as following

$$\begin{cases} -\Delta u = au + bv + \frac{2\alpha}{\alpha+\beta} u|u|^{\alpha-2}|v|^{\beta}, & \text{in } \Omega, \\ -\Delta v = bu + cv + \frac{2\beta}{\alpha+\beta} |u|^{\alpha} v|v|^{\beta-2}, & \text{in } \Omega, \\ u, v > 0, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ with smooth boundary, and $a,\ b,\ c \in \mathbb{R}$, $\alpha,\ \beta > 1$, $\alpha + \beta = 2^*$. They obtained some existence results and nonexistence results for the corresponding elliptic system with different dimensions and in different domain's shapes. In 2009, Hsu and Lin in [19] studied the following critical elliptic system

$$\begin{cases} -\Delta u = \lambda |u|^{q-2} u + \frac{2\alpha}{\alpha+\beta} |u|^{\alpha-2} u|v|^{\beta}, & \text{in } \Omega, \\ -\Delta v = \mu |v|^{q-2} v + \frac{2\beta}{\alpha+\beta} |u|^{\alpha} |v|^{\beta-2} v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$ with $N \geq 3$, and $\lambda,\ \mu > 0$, $\alpha,\ \beta > 1$, $\alpha + \beta = 2^*$. For $1 < q < 2$, they got two positive solutions when $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \Lambda$, where Λ is a positive constant. What's more, there are some other references on semilinear elliptic system with critical exponent, such as [5,6,8,9,12,13,16–18,20–22,24–26]. However, among the references mentioned above, the elliptic system involving double critical exponential terms with one strongly coupled and the other weakly coupled was studied only in [9] and [8]. Recently, Duan, Wei and Yang in [9], on an incompressible bounded domain, studied the following nonhomogeneous semilinear elliptic system with double critical exponents

$$\begin{cases} -\Delta u = |u|^{2^*-2} u + \frac{\alpha}{2^*} |u|^{\alpha-2} u|v|^{\beta} + \varepsilon f, & \text{in } \Omega, \\ -\Delta v = |v|^{2^*-2} v + \frac{\beta}{2^*} |u|^{\alpha} |v|^{\beta-2} v + \varepsilon g, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\alpha, \beta > 1$, $\alpha + \beta = 2^*$, $\varepsilon > 0$, for non-homogeneous terms $f,\ g$, which satisfy $0 \leq f(x),\ g(x) \in L^{\infty}(\Omega)$, $f,\ g \not\equiv 0$, and for the incompressible bounded domains with smooth boundary Ω satisfies the following condition:

(V) $\Omega \subset \mathbb{R}^N (N \geq 3)$, and there exist two positive constants $0 < R_1 < R_2 < \infty$ such that

$$\{x \in \mathbb{R}^N : R_1 < |x| < R_2\} \subset \Omega, \qquad \{x \in \mathbb{R}^N : |x| < R_1\} \not\subset \bar{\Omega}.$$

If condition (V) holds, by splitting Nehari manifold and the knowledge of topology, they got that there is a $\varepsilon' > 0$, for any $0 < \varepsilon < \varepsilon'$, such that the above system has at least three solutions in the incompressible domain Ω, one of which is a positive ground state solution. Furthermore, if R_1 in condition (V) is small enough, then there is a ε'' such that for any $0 < \varepsilon < \varepsilon''$ there are at least four solutions on the incompressible domain Ω.

Inspired by [9], we replace the abstract inhomogeneous terms with the concave-convex terms. In order to get a more general result, we extend the constraints of the "incompressible" domain to the general bounded domain. So, we study system (1.1).

We denote the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}}$ of $H_0^1(\Omega)$; and $E = H_0^1(\Omega) \times H_0^1(\Omega)$ with the norm:

$$\|(u,v)\|_E = \left[\int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx\right]^{\frac{1}{2}}.$$

Then, we use $|\cdot|_p$ to denote the $L^p(\Omega)$-norm, and denote S as the Sobolev optimal embedding constant, where S is defined as follows:

$$S = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}}} = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{2^*} dx\right)^{\frac{2}{2^*}}} > 0. \tag{1.2}$$

From reference [27], we know that S is achieved by the function:

$$U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}, \qquad (x \in \mathbb{R}^N) \tag{1.3}$$

which is also a solution of the following equation:

$$\begin{cases} -\Delta u = u^{2^*-1}, & x \in \mathbb{R}^N, \\ u > 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

with $|\Delta U|_2^2 = |U|_{2^*}^{2^*} = S^{\frac{N}{2}}$. Let

$$S_{\alpha,\beta} = \inf_{(u,v) \in E\setminus\{(0,0)\}} \frac{\int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx}{\left[\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx\right]^{\frac{2}{2^*}}} = f(\tau_{min}) S. \tag{1.5}$$

According to [9, Lemma 1], we know that $S_{\alpha,\beta} = f(\tau_{min})S$, where

$$f(\tau) = \frac{1+\tau^2}{(1+\tau^{2^*}+\tau^\beta)^{\frac{2}{2^*}}}$$

and $f(\tau_{min}) \in [2^{-\frac{2}{2^*}}, 1]$ for any $\tau \geq 0$.

Based on (1.1), we know that the corresponding energy functional as follows:

$$I_{\lambda,\mu}(u,v) = \frac{1}{2}\|(u,v)\|_E^2 - \frac{1}{2^*}\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx - \frac{1}{q}\int_\Omega (\lambda |u|^q + \mu |v|^q) dx \tag{1.6}$$

and (u,v) is a weak solution of system (1.1) if for any $(\xi_1,\xi_2)\in E$ it satisfies

$$\begin{aligned}
&\langle I'_{\lambda,\mu}(u,v),(\xi_1,\xi_2)\rangle\\
&\quad=\int_\Omega(\nabla u\nabla\xi_1+\nabla v\nabla\xi_2)dx-\int_\Omega\left(|u|^{2^*-2}u\xi_1+|v|^{2^*-2}v\xi_2\right)dx\\
&\quad\quad-\int_\Omega\left(\frac{\alpha}{2^*}|u|^{\alpha-2}|v|^\beta u\xi_1+\frac{\beta}{2^*}|u|^\alpha|v|^{\beta-2}v\xi_2\right)dx-\int_\Omega\left(\lambda|u|^{q-2}u\xi_1+\mu|v|^{q-2}v\xi_2\right)dx=0.
\end{aligned}$$

When $(\xi_1,\xi_2)=(u,v)$, we can get:

$$\|(u,v)\|_E^2-\int_\Omega(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta)dx-\int_\Omega(\lambda|u|^q+\mu|v|^q)dx=0. \tag{1.7}$$

Define Nehari manifold as follows:

$$\mathcal{N}_{\lambda,\mu}=\{(u,v)\in E:\langle I'_{\lambda,\mu}(u,v),(u,v)\rangle=0\}. \tag{1.8}$$

Set $z=(u,v)$, $\|z\|_E=\|(u,v)\|_E=\left(\|u\|^2+\|v\|^2\right)^{\frac{1}{2}}$. Define the function $\Psi(z)=\langle I'_{\lambda,\mu}(z),z\rangle$, such that

$$\begin{aligned}
\langle\Psi'(z),z\rangle&=2\|z\|_E^2-2^*\int_\Omega(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta)dx-q\int_\Omega(\lambda|u|^q+\mu|v|^q)dx\\
&=(2-q)\|z\|_E^2-(2^*-q)\int_\Omega(|u|^{2^*}+|v|^{2^*}+|u|^\alpha|v|^\beta)dx\\
&=(2-2^*)\|z\|_E^2-(q-2^*)\int_\Omega(\lambda|u|^q+\mu|v|^q)dx
\end{aligned}\tag{1.9}$$

for any $z=(u,v)\in\mathcal{N}_{\lambda,\mu}$. To obtain the two positive solutions, we now split $\mathcal{N}_{\lambda,\mu}$ into three parts as follows:

$$\begin{aligned}
\mathcal{N}^+_{\lambda,\mu}&=\{z\in\mathcal{N}_{\lambda,\mu}:\langle\Psi'(z),z\rangle>0\},\\
\mathcal{N}^0_{\lambda,\mu}&=\{z\in\mathcal{N}_{\lambda,\mu}:\langle\Psi'(z),z\rangle=0\},\\
\mathcal{N}^-_{\lambda,\mu}&=\{z\in\mathcal{N}_{\lambda,\mu}:\langle\Psi'(z),z\rangle<0\},
\end{aligned}\tag{1.10}$$

where $\mathcal{N}_{\lambda,\mu}=\mathcal{N}^+_{\lambda,\mu}\cup\mathcal{N}^0_{\lambda,\mu}\cup\mathcal{N}^-_{\lambda,\mu}$. In addition, we will prove $\mathcal{N}^\pm_{\lambda,\mu}\neq\emptyset$ and $\mathcal{N}^0_{\lambda,\mu}=\{(0,0)\}$ for $0<\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}<T$, where

$$T=\left(\frac{2-q}{2^*-q}\right)^{\frac{2}{2^*-2}}\left(\frac{2^*-2}{2^*-q}\right)^{\frac{2}{2-q}}(S_{\alpha,\beta})^{\frac{2^*}{2^*-2}}S^{\frac{q}{2-q}}|\Omega|^{-\frac{2(2^*-q)}{2^*(2-q)}} \tag{1.11}$$

in Section 2.

Here is our main result.

Theorem 1.1. *Assume that $1<q<2$ and $\Omega\subset\mathbb{R}^N(N\geq3)$ is a bounded domain with smooth boundary, $\lambda,\ \mu>0,\ \alpha>1,\ \beta>1,\ \alpha+\beta=\frac{2N}{N-2}$. Then,*

(i) for any $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,T)$, system (1.1) has a positive ground state solution;

(ii) for any $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,(\frac{q}{2})^{\frac{2}{2-q}}T)$, system (1.1) has two positive solutions, one of which is the positive ground state solution.

Remark 1.2. To the best of our knowledge, our result is up to date. On the one hand, we generalize [9] to system (1.1) on general bound domain and obtain two positive solutions. On the other hand, noting that [9, Claim 2], that is,

$$\int_\Omega (u_1 + tu_\delta^{\sigma,\rho})^{2^*} - u_1^{2^*} - (tu_\delta^{\sigma,\rho})^{2^*} - 2^* u_1^{2^*-1} tu_\delta^{\sigma,\rho} dx \geq O(\delta^{\frac{N-2}{2}})$$

and

$$\int_\Omega (v_1 + tv_\delta^{\sigma,\rho})^{2^*} - v_1^{2^*} - (tv_\delta^{\sigma,\rho})^{2^*} - 2^* v_1^{2^*-1} tv_\delta^{\sigma,\rho} dx \geq O(\delta^{\frac{N-2}{2}}).$$

From [9] we know that [15, (4.7)] is used in the proof of Claim 2. However, [15, (4.7)] has a restriction of $q \geq 3$ on the exponential q. For $2^* = \frac{2N}{N-2} \geq 3$, it implies that $N \leq 6$. Thus, when $N > 6$ the inequality in [9, Claim 2] may not hold, which may have some influence on the estimation of corresponding energy functional. So, for $N \geq 3$, we revalued [9, Claim 2], which is important for estimating the value of corresponding energy functional $I_{\lambda,\mu}$.

The content structure of this paper is organized as the following way. In Section 2, we will give some important lemmas for preparation to prove our main result. In Section 3, we will give the proof of the existence of positive ground state solutions for system (1.1). Finally, we will prove the existence of two positive solutions in Section 4.

2 Some preliminary results

In this section, we first give some important lemmas which are valuable preparation for the proof of our main result.

Lemma 2.1. *Assume that $z = (u, v) \in E \backslash \{(0, 0)\}$ with $\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx > 0$, then:*

(i) there exist unique t^+, t^- with $0 < t^+ < t_{max} < t^-$ when $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$, such that $t^+ z \in \mathcal{N}^+_{\lambda,\mu}$, $t^- z \in \mathcal{N}^-_{\lambda,\mu}$ and

$$I_{\lambda,\mu}(t^+ z) = \inf_{0 \leq t \leq t_{max}} I_{\lambda,\mu}(tz), \qquad I_{\lambda,\mu}(t^- z) = \sup_{t_{max} \leq t} I_{\lambda,\mu}(tz); \tag{2.1}$$

(ii) for $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$, $\mathcal{N}^0_{\lambda,\mu} = \{(0, 0)\}$ and $\mathcal{N}^-_{\lambda,\mu}$ is a closed set.

Proof. (i) For $t \geq 0$, $z = (u, v) \in E$ such that $\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx > 0$, we have

$$\langle I'_{\lambda,\mu}(tz), tz \rangle = t^2 \|z\|_E^2 - t^{2^*} \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx - t^q \int_\Omega (\lambda |u|^q + \mu |v|^q) dx.$$

Then, set $y_1, y_2 : \mathbb{R}^+ \to \mathbb{R}$,

$$y_1(t) = t^{2-q} \|z\|_E^2 - t^{2^*-q} \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx - \int_\Omega (\lambda |u|^q + \mu |v|^q) dx, \tag{2.2}$$

$$y_2(t) = t^{2-q} \|z\|_E^2 - t^{2^*-q} \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx. \tag{2.3}$$

Obviously, $y_1(t) = y_2(t) - \int_\Omega (\lambda |u|^q + \mu |v|^q)\, dx$. We now proceed with the analysis of $y_2(t)$,

$$\begin{aligned} y_2'(t) &= (2-q) t^{1-q} \|z\|_E^2 - (2^* - q) t^{2^*-1-q} \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx \\ &= t^{1-q} \left[(2-q) \|z\|_E^2 - (2^* - q) t^{2^*-2} \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx \right]. \end{aligned}$$

It is easy to figure out that $y_2'(t_{max}) = 0$ with

$$t_{max} = \left[\frac{(2-q)\|z\|_E^2}{(2^*-q)\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right]^{\frac{1}{2^*-2}} > 0.$$

Moreover, $y_2'(t) > 0$ for all $0 < t < t_{max}$, and $y_2'(t) < 0$ for all $t > t_{max}$. Through a simple analysis, we can get that

$$y_2(t_{max}) = \max y_2(t) = \left[\frac{(2-q)\|z\|_E^2}{(2^*-q)\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right]^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q} \|z\|_E^2.$$

According to the definition of $S_{\alpha,\beta}$, Hölder's inequality and (1.2), one has

$$\begin{aligned}
y_1(t_{max}) &= y_2(t_{max}) - \int_\Omega (\lambda |u|^q + \mu |v|^q)\, dx \\
&= \left[\frac{(2-q)\|z\|_E^2}{(2^*-q)\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right]^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q} \|z\|_E^2 \\
&\quad - \int_\Omega (\lambda |u|^q + \mu |v|^q)\, dx \\
&\geq \left(\frac{2-q}{2^*-q} \right)^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q} \|z\|_E^2 \left(\frac{\|z\|_E^2}{\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right)^{\frac{2-q}{2^*-2}} \\
&\quad - (\lambda \|u\|^q + \mu \|v\|^q)\, |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\
&\geq \left(\frac{2-q}{2^*-q} \right)^{\frac{2-q}{2^*-q}} \frac{2^*-2}{2^*-q} \|z\|_E^2 \left(\frac{\|z\|_E^2}{\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right)^{\frac{2-q}{2^*-2}} \\
&\quad - \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \right)^{\frac{2-q}{2}} (\|u\|^2 + \|v\|^2)^{\frac{q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\
&= \left(\frac{2-q}{2^*-q} \right)^{\frac{2-q}{2^*-q}} \frac{2^*-2}{2^*-q} \|z\|_E^2 \left(\frac{\|z\|_E^2}{\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx} \right)^{\frac{2-q}{2^*-2}} \\
&\quad - \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \right)^{\frac{2-q}{2}} \|z\|_E^q |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\
&\geq \left(\frac{2-q}{2^*-q} \right)^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q} \|z\|_E^2 \left(\frac{\|z\|_E^2}{(S_{\alpha,\beta})^{-\frac{2^*}{2}} \|z\|_E^{2^*}} \right)^{\frac{2-q}{2^*-2}} \\
&\quad - \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \right)^{\frac{2-q}{2}} \|z\|_E^q |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\
&= \|z\|_E^q \left[\left(\frac{2-q}{2^*-q} \right)^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q} (S_{\alpha,\beta})^{\frac{2^*(2-q)}{2(2^*-2)}} - \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \right)^{\frac{2-q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \right] \\
&> 0,
\end{aligned} \tag{2.4}$$

for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$, where T is defined by (1.11). Because $y_1(t)$ is a continuous function, according to inequality preserving for continuous functions and (2.4), there exist unique t^+, t^- with $0 < t^+ < t_{max} < t^-$, which makes

$$y_1(t^+) = y_1(t^-) = 0.$$

So, for $t^+ < t_{max} < t^-$, since $y_2'(t^+) > 0$ and $y_2'(t^-) < 0$, we have $t^+z \in \mathcal{N}^+_{\lambda,\mu}$, $t^-z \in \mathcal{N}^-_{\lambda,\mu}$. Moreover, one has

$$I_{\lambda,\mu}(t^-z) \geq I_{\lambda,\mu}(tz) \geq I_{\lambda,\mu}(t^+z),$$

for each $t \in [t^+, t^-]$, and $I_{\lambda,\mu}(t^+z) < I_{\lambda,\mu}(tz)$ for each $t \in [0, t^+)$. Thus, one obtains

$$I_{\lambda,\mu}(t^+z) = \inf_{0\leq t\leq t_{max}} I_{\lambda,\mu}(tz), \qquad I_{\lambda,\mu}(t^-z) = \sup_{t_{max}\leq t} I_{\lambda,\mu}(tz).$$

(ii) Set $z_0 = (u_0, v_0) \neq (0,0) \in \mathcal{N}^0_{\lambda,\mu}$, from (1.9) and (1.10), we know

$$\frac{2^*-2}{2^*-q}\|z_0\|_E^2 = \int_\Omega (\lambda|u_0|^q + \mu|v_0|^q), \tag{2.5}$$

$$\frac{2-q}{2^*-q}\|z_0\|_E^2 = \int_\Omega |u_0|^{2^*} + |v_0|^{2^*} + |u_0|^\alpha|v_0|^\beta dx. \tag{2.6}$$

We can deduce from (2.4), (2.5) and (2.6) that

$$\begin{aligned} 0 &< \left[\frac{(2-q)\|z_0\|_E^2}{(2^*-q)\int_\Omega |u_0|^{2^*} + |v_0|^{2^*} + |u_0|^\alpha|v_0|^\beta dx}\right]^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q}\|z_0\|_E^2 \\ &\quad - \int_\Omega (\lambda|u_0|^q + \mu|v_0|^q)\,dx \\ &= \left[\frac{(2-q)\|z_0\|_E^2}{(2^*-q)\frac{2-q}{2^*-q}\|z_0\|_E^2}\right]^{\frac{2-q}{2^*-2}} \frac{2^*-2}{2^*-q}\|z_0\|_E^2 - \frac{2^*-2}{2^*-q}\|z_0\|_E^2 \\ &= \frac{2^*-2}{2^*-q}\|z_0\|_E^2 - \frac{2^*-2}{2^*-q}\|z_0\|_E^2 \\ &= 0, \end{aligned}$$

which is a contradiction for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0,T)$. So, for $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0,T)$, we obtain $\mathcal{N}^0_{\lambda,\mu} = \{(0,0)\}$. Then, we will prove $\mathcal{N}^-_{\lambda,\mu}$ is a closed set when $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0,T)$. Assume $\{z_n\} \subset \mathcal{N}^-_{\lambda,\mu}$, $z_n \to z$, $z \in E$, and now we prove $z \in \mathcal{N}^-_{\lambda,\mu}$. From (1.10), we have

$$(2-q)\|z_n\|_E^2 - (2^*-q)\int_\Omega |u_n|^{2^*} + |v_n|^{2^*} + |u_n|^\alpha|v_n|^\beta dx < 0. \tag{2.7}$$

According to $z_n \to z$, $z \in E$ and (2.7), one has

$$(2-q)\|z\|_E^2 - (2^*-q)\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha|v|^\beta)dx \leq 0. \tag{2.8}$$

From (2.8), we can get $z \in \mathcal{N}^0_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$. We already know from the above proof that $\mathcal{N}^0_{\lambda,\mu} = \{(0,0)\}$ when $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0,T)$. So, if $z \in \mathcal{N}^0_{\lambda,\mu}$, then $z = (0,0)$. According to (1.5) and (2.7), we obtain

$$\|z_n\|_E \geq \left[\frac{(2-q)}{(2^*-q)}(S_{\alpha,\beta})^{\frac{2^*}{2}}\right]^{\frac{1}{2^*-2}} > 0,$$

which implies a contradiction with $z = (0,0)$. Thus, $z \in \mathcal{N}^-_{\lambda,\mu}$ for $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0,T)$. So, we can prove that $\mathcal{N}^-_{\lambda,\mu}$ is a closed set. The proof of Lemma 2.1 is complete. □

Lemma 2.2. *The energy functional $I_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Assume that $z = (u, v) \in \mathcal{N}_{\lambda,\mu}$. By the Hölder inequality, (1.2) and (1.7), one has

$$\begin{aligned} I_{\lambda,\mu}(z) &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\|z\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\int_\Omega (\lambda|u|^q + \mu|v|^q)dx \\ &= \frac{1}{N}\|z\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\int_\Omega (\lambda|u|^q + \mu|v|^q)dx \\ &\geq \frac{1}{N}\|z\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)(\lambda\|u\|^q + \mu\|v\|^q)|\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\ &\geq \frac{1}{N}\|z\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} (\|u\|^2 + \|v\|^2)^{\frac{q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\ &= \frac{1}{N}\|z\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|z\|_E^q |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}. \end{aligned} \tag{2.9}$$

Because $1 < q < 2 < 2^*$, from (2.9) we know that $I_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$. The proof of Lemma 2.2 is completed. □

According to Lemma 2.1 and Lemma 2.2, we set $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^0_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$. And we define

$$m = \inf_{z\in\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(z), \qquad m^+ = \inf_{z\in\mathcal{N}^+_{\lambda,\mu}} I_{\lambda,\mu}(z), \qquad m^- = \inf_{z\in\mathcal{N}^-_{\lambda,\mu}} I_{\lambda,\mu}(z). \tag{2.10}$$

Lemma 2.3.

(*i*) *We have $m \leq m^+ < 0$, for $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$;*

(*ii*) *there exists a positive constant m_0 depending on λ, μ, S, N, such that $m^- \geq m_0 > 0$ for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in \left(0, (\frac{q}{2})^{\frac{2}{2-q}} T\right)$.*

Proof. (*i*) Assume that $z = (u, v) \in \mathcal{N}^+_{\lambda,\mu}$, by (1.7), (1.9) and (1.10), we can get

$$\frac{2-q}{2^*-q}\|z\|_E^2 > \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha|v|^\beta)dx. \tag{2.11}$$

Then, by (2.11) we have

$$\begin{aligned} I_{\lambda,\mu}(z) &= \left(\frac{1}{2} - \frac{1}{q}\right)\|z\|_E^2 + \left(\frac{1}{q} - \frac{1}{2^*}\right)\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha|v|^\beta)dx \\ &< \left(\frac{1}{2} - \frac{1}{q}\right)\|z\|_E^2 + \left(\frac{1}{q} - \frac{1}{2^*}\right)\left(\frac{2-q}{2^*-q}\right)\|z\|_E^2 \\ &= \frac{q-2}{Nq}\|z\|_E^2 \\ &< 0. \end{aligned}$$

So, $m = \inf_{z\in\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(z) \leq m^+ = \inf_{z\in\mathcal{N}^+_{\lambda,\mu}} I_{\lambda,\mu}(z) \leq I_{\lambda,\mu}(z) < 0$.

(*ii*) For $z = (u, v) \in \mathcal{N}^-_{\lambda,\mu}$, we can deduce that

$$\frac{2-q}{2^*-q}\|z\|_E^2 < \int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha|v|^\beta)dx \leq S_{\alpha,\beta}^{-\frac{2^*}{2}}\|z\|_E^{2^*}.$$

Consequently, from (1.5), (1.9) and (1.10), one has

$$\|z\|_E > \left(\frac{2-q}{2^*-q}\right)^{\frac{1}{2^*-2}} S_{\alpha,\beta}^{\frac{2^*}{2(2^*-2)}}. \tag{2.12}$$

By (2.9) and (2.12), for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in \big(0, (\frac{q}{2})^{\frac{2}{2-q}} T\big)$, we will get

$$\begin{aligned}
I_{\lambda,\mu}(z) &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right) \|z\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right) \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|z\|_E^q |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \\
&= \|z\|_E^q \Bigg[\left(\frac{1}{2} - \frac{1}{2^*}\right) \|z\|_E^{2-q} \\
&\quad - \left(\frac{1}{q} - \frac{1}{2^*}\right) \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \Bigg] \\
&> \|z\|_E^q \Bigg[\left(\frac{1}{2} - \frac{1}{2^*}\right) \left(\frac{2-q}{2^*-q}\right)^{\frac{2-q}{2^*-2}} S_{\alpha,\beta}^{\frac{2^*(2-q)}{2(2^*-2)}} \\
&\quad - \left(\frac{1}{q} - \frac{1}{2^*}\right) \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}} \Bigg] \\
&\geq m_0 \\
&> 0,
\end{aligned} \tag{2.13}$$

where m_0 is a positive constant. So, $m^- = \inf_{z \in \mathcal{N}_{\lambda,\mu}^-} I_{\lambda,\mu}(z) \geq m_0 > 0$. Then, the proof of Lemma 2.3 is complete. □

Lemma 2.4. *Suppose $z_0 \in E$ is a local minimizer of $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, then we have $I'_{\lambda,\mu}(z_0) = 0$ in E^{-1}.*

Proof. Set $z_0 = (u_0, v_0) \in E$ is a local minimizer of $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$. Then, $I_{\lambda,\mu}(z_0) = \min_{z \in \mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(z)$. According to the Lagrange multiplier theorem, there is a $\theta \in \mathbb{R}$ such that $I'_{\lambda,\mu}(z_0) = \theta \Psi'(z_0)$, where $\Psi(z) = \langle I'_{\lambda,\mu}(z), z\rangle$. Due to $z_0 \in \mathcal{N}_{\lambda,\mu}$, we have

$$0 = \langle I'_{\lambda,\mu}(z_0), z_0\rangle = \theta \langle \Psi'(z_0), z_0\rangle.$$

By Lemma 2.3, if $z_0 \notin \mathcal{N}_{\lambda,\mu}^0$, we can get $\langle \Psi'(z_0), z_0\rangle \neq 0$ for $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$. Thus, $\theta = 0$, $I'_{\lambda,\mu}(z_0) = 0$. The Lemma 2.4 is proved. □

3 The positive ground state solution

Lemma 3.1. *For any $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$, then there exists a $(PS)_m$-sequence $\{z_n\} = \{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ for $I_{\lambda,\mu}$ and T is defined as (1.11).*

Proof. The proof process is the same as [28, Proposition 9], which is omitted here. □

Lemma 3.2. *The energy functional $I_{\lambda,\mu}$ has a minimizer $z_* = (u_*, v_*) \in \mathcal{N}_{\lambda,\mu}^+$, for $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, T)$. What's more z_* is a positive ground state solution of system (1.1), which makes $I_{\lambda,\mu}(z_*) = m = m^+ < 0$.*

Proof. According to Lemma 3.1, there is a $(PS)_m$-sequence, which is recorded as $\{z_n\} = \{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$. Then, we have

$$I_{\lambda,\mu}(z_n) = m + o_n(1), \qquad I'_{\lambda,\mu}(z_n) = o_n(1), \tag{3.1}$$

where $o_n(1) \to 0$ as $n \to \infty$. Combining with (2.9) and (3.1), we can get

$$\begin{aligned} m + o_n(1) &= I_{\lambda,\mu}(z_n) \\ &\geq \frac{1}{N}\|z_n\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|z_n\|_E^q |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}. \end{aligned}$$

Thus, $\{z_n\}$ is bounded in E. Then, $\{z_n\}$ has a subsequence (still denoted by $\{z_n\}$) which weakly converges to $z_* = (u_*, v_*) \in E$, and

$$\begin{cases} u_n \rightharpoonup u_*,\ v_n \rightharpoonup v_*, & \text{in } H_0^1(\Omega), \\ u_n \to u_*,\ v_n \to v_*, & \text{in } L^s(\Omega)(1 \leq s < 2^*), \\ u_n(x) \to u_*(x),\ v_n(x) \to v_*(x), & \text{a.e. in } \Omega. \end{cases} \tag{3.2}$$

According to (3.1), we have $\langle I'_{\lambda,\mu}(z_n), \xi\rangle \to 0$ as $n \to \infty$ for any $\xi \in E$. What's more, combining with (3.2), we have

$$\langle I'_{\lambda,\mu}(z_*), \xi\rangle = 0, \quad \text{for all } \xi \in E,$$

which implies that z_* is a solution of system (1.1) and $z_* \in \mathcal{N}_{\lambda,\mu}$.

Then, we will prove $z_n \to z_*$. By using the Lebesgue dominated convergence theorem, we can get

$$\lim_{n\to\infty} \int_\Omega (\lambda|u_n|^q + \mu|v_n|^q)dx = \int_\Omega (\lambda|u_*|^q + \mu|v_*|^q)dx. \tag{3.3}$$

Since $z_* \in \mathcal{N}_{\lambda,\mu}$, by Fatou's Lemma and (3.3), one has

$$\begin{aligned} m &\leq I_{\lambda,\mu}(z_*) \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\|z_*\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\int_\Omega (\lambda|u_*|^q + \mu|v_*|^q)dx \\ &= \frac{1}{N}\|z_*\|_E^2 - \frac{2^*-q}{2^*q}\int_\Omega (\lambda|u_*|^q + \mu|v_*|^q)dx \\ &\leq \liminf_{n\to\infty}\left(\frac{1}{N}\|z_n\|_E^2 - \frac{2^*-q}{2^*q}\int_\Omega (\lambda|u_n|^q + \mu|v_n|^q)dx\right) \\ &= \liminf_{n\to\infty} I_{\lambda,\mu}(z_n) \\ &= m, \end{aligned}$$

which implies $I_{\lambda,\mu}(z_*) = m$, $\|z_n\|_E^2 \to \|z_*\|_E^2$. By combining with (3.2), we can derive $z_n \to z_*$ in E. Thus, z_* is a solution of system (1.1) that means $z_* \in \mathcal{N}_{\lambda,\mu}$. Moreover, we are going to prove $z_* \in \mathcal{N}^+_{\lambda,\mu}$. Since $z_* \in \mathcal{N}_{\lambda,\mu}$, from (1.6) and (1.7), we have

$$\begin{aligned} \int_\Omega (\lambda|u_*|^q + \mu|v_*|^q)dx &= \frac{q(2^*-2)}{2(2^*-q)}\|z_*\|_E^2 - \frac{2^*q}{2^*-q}m \\ &\geq -\frac{2^*q}{2^*-q}m \\ &> 0. \end{aligned} \tag{3.4}$$

Then, $z_* \neq (0,0)$, which implies $z_* \in \mathcal{N}_{\lambda,\mu}^+$ or $z_* \in \mathcal{N}_{\lambda,\mu}^-$. If $z_* \in \mathcal{N}_{\lambda,\mu}^-$, by Lemma 2.1 there are unique t^+, t^- with $t^+ < t^- = 1$ such that $t^+ z_* \in \mathcal{N}_{\lambda,\mu}^+$, $t^- z_* \in \mathcal{N}_{\lambda,\mu}^-$. From (1.10) we know that

$$\frac{d}{dt} I_{\lambda,\mu}(t^+ z_*) = 0, \qquad \frac{d^2}{dt^2} I_{\lambda,\mu}(t^+ z_*) > 0.$$

Moreover, according to Lemma 2.1 , for any t with $t^+ < t < t^- = 1$, one gets

$$m^+ \leq I_{\lambda,\mu}(t^+ z_*) < I_{\lambda,\mu}(t z_*) \leq I_{\lambda,\mu}(t^- z_*) = I_{\lambda,\mu}(z_*) = m,$$

which implies a contradiction. Thus, $z_* \in \mathcal{N}_{\lambda,\mu}^+$ and $m = m^+$, and according to Lemma 2.3 (i), we have $m^+ = I_{\lambda,\mu}(z_*) < 0$.

Finally, we are going to prove z_* is a positive solution. We have $z_* \neq (0,0)$ from (3.4). Then, the main purpose now is to exclude semi-trivial solutions. Assume that $u_* \not\equiv 0, v_* \equiv 0$, then u_* is a nontrivial solution to the following equation:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda |u|^{q-2}u, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0. & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Because $(u_*, 0)$ is a solution of equation (3.5), we have

$$\|(u_*, 0)\|_E^2 = W_*(u_*, 0) > 0,$$

where $W_*(u_*, 0) = \int_\Omega |u_*|^{2^*} dx + \int_\Omega \lambda u_*^q dx$. And similarly, we could take $\phi \in H_0^1(\Omega) \backslash \{0\}$ such that

$$\|(0, \phi)\|_E^2 = W_*(0, \phi) > 0.$$

Now,

$$W_*(u_*, \phi) = \|(u_*, \phi)\|_E^2 = W_*(u_*, 0) + W_*(0, \phi).$$

According to Lemma 2.1, there exists a unique $0 < t^+ < t_{max}$ such that $(t^+ u_*, t^+ \phi) \in \mathcal{N}_{\lambda,\mu}^+$, where

$$t_{max} = \left[\frac{(2^* - q) W_*(u_*, \phi)}{(2^* - 2)\|(u_*, \phi)\|_E^2} \right]^{\frac{1}{2-q}} = \left(\frac{2^* - q}{2^* - 2} \right)^{\frac{1}{2-q}} > 1$$

and

$$I_{\lambda,\mu}(t^+ u_*, t^+ \phi) = \inf_{0 \leq t \leq t_{max}} I_{\lambda,\mu}(t u_*, t\phi).$$

Then, we can deduce the following result:

$$m^+ \leq I_{\lambda,\mu}(t^+ u_*, t^+ \phi) \leq I_{\lambda,\mu}(u_*, \phi) < I_{\lambda,\mu}(u_*, 0) = m^+.$$

It is impossible. Finally, we can know that $u_*, v_* > 0$ in Ω by using the strong maximum principle, and $z_* = (u_*, v_*)$ is a positive solution of system (1.1). The proof of Theorem 1.1 (i) is complete. □

4 Proof of Theorem 1.1

In this part, we will prove Theorem 1.1 (ii), and obtain the second positive solution of system (1.1). Before that, due to lacking of compactness condition for $I_{\lambda,\mu}$, we first give the local $(PS)_c$ condition which is satisfied for the corresponding energy function.

Lemma 4.1. *Let $\{z_n = (u_n, v_n)\}$ be a $(PS)_c$ sequence of $I_{\lambda,\mu}$ with*

$$c < m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}},$$

we can get that $I_{\lambda,\mu}$ satisfies the $(PS)_c$ condition in E.

Proof. Let $\{z_n\} = \{(u_n, v_n)\}$ be a $(PS)_c$-sequence for $I_{\lambda,\mu}$ such that

$$I_{\lambda,\mu}(z_n) = c + o_n(1), \qquad I'_{\lambda,\mu}(z_n) = o_n(1). \tag{4.1}$$

Combining with (2.9), we have

$$\begin{aligned} c + o_n(1) &= I_{\lambda,\mu}(z_n) \\ &\geq \frac{1}{N}\|z_n\|_E^2 - \left(\frac{1}{q} - \frac{1}{2^*}\right)\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|z_n\|_E^q |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{q}{2}}. \end{aligned}$$

Since $1 < q < 2$, we know that $\{z_n\}$ is bounded in E. Passing to a subsequence (still denoted by $\{z_n\}$), there exists $z = (u, v) \in E$ such that $z_n \rightharpoonup z$ in E, and we have

$$\begin{cases} u_n \rightharpoonup u,\ v_n \rightharpoonup v, & \text{in } H_0^1(\Omega), \\ u_n \to u,\ v_n \to v, & \text{in } L^s(\Omega)(1 \leq s < 2^*), \\ u_n(x) \to u(x),\ v_n(x) \to v(x), & \text{a.e. in } \Omega. \end{cases} \tag{4.2}$$

Similar to [9, Proposition 1], as $n \to \infty$, from (4.1) and (4.2), one has

$$\begin{aligned} \lim_{n\to\infty} \langle I'_{\lambda,\mu}(z_n), \xi\rangle &= \langle I'_{\lambda,\mu}(z), \xi\rangle \\ &= \int_\Omega (\nabla u \nabla \xi_1 + \nabla v \nabla \xi_2) dx \\ &\quad - \int_\Omega \left(|u|^{2^*-2} u\xi_1 + |v|^{2^*-2} v\xi_2\right) dx \\ &\quad - \int_\Omega \left(\frac{\alpha}{2^*}|u|^{\alpha-2}|v|^\beta u\xi_1 + \frac{\beta}{2^*}|u|^\alpha |v|^{\beta-2} v\xi_2\right) dx \\ &\quad - \int_\Omega \left(\lambda|u|^{q-2} u\xi_1 + \mu|v|^{q-2} v\xi_2\right) dx \\ &= 0, \end{aligned}$$

for any $\xi = (\xi_1, \xi_2) \in E$. Particularly, choosing $\xi = z$, one obtains $\langle I'_{\lambda,\mu}(z), z\rangle$=0 and $z = (u, v) \in \mathcal{N}_{\lambda,\mu}$.

Set $\{(\eta_n, \mu_n)\} = \{(u_n - u, v_n - v)\}$ in E, then, $(\eta_n, \mu_n) \rightharpoonup (0, 0)$ in E. And next, we give the following version of Brézis–Lieb Lemma from [14, Lemma 3.4]

$$\int_\Omega |u_n|^\alpha |v_n|^\beta dx = \int_\Omega (|\eta_n|^\alpha |\mu_n|^\beta + |u|^\alpha |v|^\beta) dx + o_n(1), \tag{4.3}$$

and the Brézis–Lieb Lemma for the other terms,

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega (|\nabla \eta_n|^2 + |\nabla u|^2) dx + o_n(1), \tag{4.4}$$

$$\int_\Omega |u_n|^{2^*} dx = \int_\Omega (|\eta_n|^{2^*} + |u|^{2^*}) dx + o_n(1), \tag{4.5}$$

where (4.4) and (4.5) are equally applicable to v_n. Moreover, according to the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_\Omega (\lambda |u_n|^q + \mu |v_n|^q) dx = \int_\Omega (\lambda |u|^q + \mu |v|^q) dx. \tag{4.6}$$

Then from (4.3)–(4.5), we have

$$\begin{aligned} o_n(1) &= \langle I'_{\lambda,\mu}(u_n, v_n), (u_n, v_n)\rangle \\ &= \|(\eta_n, \mu_n)\|_E^2 - \int_\Omega (|\eta_n|^{2^*} + |\mu_n|^{2^*} + |\eta_n|^\alpha |\mu_n|^\beta) dx + o_n(1). \end{aligned} \tag{4.7}$$

Assume that there exists a constant l, which makes $\|(\eta_n, \mu_n)\|_E^2 \to l$. Then, from (4.7) we can get $\int_\Omega (|\eta_n|^{2^*} + |\mu_n|^{2^*} + |\eta_n|^\alpha |\mu_n|^\beta) dx \to l$. According to (1.5), one obtains

$$S_{\alpha,\beta} \left[\int_\Omega (|u|^{2^*} + |v|^{2^*} + |u|^\alpha |v|^\beta) dx \right]^{\frac{2}{2^*}} \le \int_\Omega (|\nabla u|^2 + |\nabla v|^2) dx.$$

Then, $l \ge S_{\alpha,\beta} l^{\frac{2}{2^*}}$, which implies that $l = 0$ or $l \ge S_{\alpha,\beta}^{\frac{N}{2}}$. On the one hand, if $l = 0$, the proof is complete. On the other hand, if $l \ge S_{\alpha,\beta}^{\frac{N}{2}}$, according to the definition of m and $(u, v) \in \mathcal{N}$, it follows from (4.3)–(4.7) that

$$\begin{aligned} c &= I_{\lambda,\mu}(u, v) + \frac{1}{2}\|(\eta_n, \mu_n)\|^2 - \frac{1}{2^*} \int_\Omega (|\eta_n|^{2^*} + |\mu_n|^{2^*} + |\eta_n|^\alpha |u_n|^\beta) dx + o_n(1) \\ &= m + \left(\frac{1}{2} - \frac{1}{2^*}\right) l \\ &\ge m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}, \end{aligned}$$

which is contrary to the given condition of $c < m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}$. So, $l = 0$, i.e. $(u_n, v_n) \to (u, v)$ in E. The proof of Lemma 4.1 is complete. □

Set $\psi \in C_0^\infty$ and satisfies $0 \le \psi \le 1$, $|\nabla \psi| \le C$. The definition of ψ as follows:

$$\psi(x) = \begin{cases} 1, & |x| \le \frac{\rho_0}{2} \\ 0, & |x| \ge \rho_0, \end{cases}$$

where $\varepsilon \in (0, 1)$. Moreover, setting

$$u_\varepsilon(x) = \psi(x) U_\varepsilon(x) \in H_0^1(\Omega), \qquad v_\varepsilon(x) = \tau_{min} \psi(x) U_\varepsilon(x) \in H_0^1(\Omega). \tag{4.8}$$

Then, we will have the following estimates.

Lemma 4.2. *Under the assumptions of Theorem 1.1 (ii), for any* $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, (\frac{q}{2})^{\frac{2}{2-q}} T)$, *there exist* $\varepsilon_0 > 0$ *for any* $\varepsilon \in (0, \varepsilon_0)$ *such that*

$$\sup_{t \geq 0} I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) < m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}.$$

Proof. From [3], we can obtain the following classical conclusion:

$$|u_\varepsilon|_{2^*}^2 = |U_\varepsilon|_{2^*}^2 + O(\varepsilon^N); \tag{4.9}$$

$$\|u_\varepsilon\|^2 = \|U_\varepsilon\|^2 + O(\varepsilon^{N-2}). \tag{4.10}$$

Then, we have

$$\begin{aligned}
I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) &= \frac{1}{2}\|(u_* + tu_\varepsilon, v_* + tv_\varepsilon)\|_E^2 \\
&\quad - \frac{1}{2^*}\int_\Omega (u_* + tu_\varepsilon)^{2^*} + (v_* + tv_\varepsilon)^{2^*} + (u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta dx \\
&\quad - \frac{1}{q}\int_\Omega \lambda(u_* + tu_\varepsilon)^q + \mu(v_* + tv_\varepsilon)^q dx \\
&= \frac{1}{2}\|(u_*, v_*)\|_E^2 + \frac{1}{2}\|(tu_\varepsilon, tv_\varepsilon)\|_E^2 \\
&\quad + t\int_\Omega u_*^{2^*-1}u_\varepsilon + \frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta u_\varepsilon + \lambda u_*^{q-1}u_\varepsilon dx \\
&\quad + t\int_\Omega v_*^{2^*-1}v_\varepsilon + \frac{\beta}{2^*}u_*^\alpha v_*^{\beta-1}v_\varepsilon + \mu v_*^{q-1}v_\varepsilon dx \\
&\quad - \frac{1}{2^*}\int_\Omega (u_* + tu_\varepsilon)^{2^*} + (v_* + tv_\varepsilon)^{2^*} + (u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta dx \\
&\quad - \frac{1}{q}\int_\Omega \lambda(u_* + tu_\varepsilon)^q + \mu(v_* + tv_\varepsilon)^q dx.
\end{aligned}$$

According to [23, (4.11)]:

$$(a+b)^q \geq a^q + qa^{q-1}b,\ a, b > 0,\ 1 < q < 2. \tag{4.11}$$

Then, we have

$$\begin{aligned}
&t\int_\Omega u_*^{2^*-1}u_\varepsilon + \frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta u_\varepsilon + \lambda u_*^{q-1}u_\varepsilon dx \\
&\qquad + t\int_\Omega v_*^{2^*-1}v_\varepsilon + \frac{\beta}{2^*}u_*^\alpha v_*^{\beta-1}v_\varepsilon + \mu v_*^{q-1}v_\varepsilon dx - \frac{1}{q}\int_\Omega \lambda(u_* + tu_\varepsilon)^q + \mu(v_* + tv_\varepsilon)^q dx \\
&\quad \leq t\int_\Omega u_*^{2^*-1}u_\varepsilon + v_*^{2^*-1}v_\varepsilon + \frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta u_\varepsilon + \frac{\beta}{2^*}u_*^\alpha v_*^{\beta-1}v_\varepsilon dx - \frac{1}{q}\int_\Omega (\lambda u_*^q + \mu v_*^q) dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) &= \frac{1}{2}\|(u_*, v_*)\|_E^2 + \frac{1}{2}\|(tu_\varepsilon, tv_\varepsilon)\|_E^2 \\
&\quad + t\int_\Omega u_*^{2^*-1}u_\varepsilon + \frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta u_\varepsilon + \lambda u_*^{q-1}u_\varepsilon dx \\
&\quad + t\int_\Omega v_*^{2^*-1}v_\varepsilon + \frac{\beta}{2^*}u_*^\alpha v_*^{\beta-1}v_\varepsilon + \mu v_*^{q-1}v_\varepsilon dx
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2^*}\int_\Omega (u_* + tu_\varepsilon)^{2^*} + (v_* + tv_\varepsilon)^{2^*} + (u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta dx \\
&- \frac{1}{q}\int_\Omega \lambda (u_* + tu_\varepsilon)^q + \mu (v_* + tv_\varepsilon)^q dx \\
\le\ & \frac{1}{2}\|(u_*, v_*)\|_E^2 + \frac{1}{2}\|(tu_\varepsilon, tv_\varepsilon)\|_E^2 \\
&- \frac{1}{2^*}\int_\Omega (u_* + tu_\varepsilon)^{2^*} + (v_* + tv_\varepsilon)^{2^*} + (u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta dx \\
&+ \int_\Omega (u_*^{2^*-1} tu_\varepsilon + v_*^{2^*-1} tv_\varepsilon + \frac{\alpha}{2^*} u_*^{\alpha-1} v_*^\beta tu_\varepsilon + \frac{\beta}{2^*} u_*^\alpha v_*^{\beta-1} tv_\varepsilon) dx \\
&- \frac{1}{q}\int_\Omega (\lambda u_*^q + \mu v_*^q) dx \\
=\ & I_{\lambda,\mu}(u_*, v_*) + \frac{1}{2}\|(tu_\varepsilon, tv_\varepsilon)\|_E^2 \\
&- \frac{1}{2^*}\int_\Omega (tu_\varepsilon)^{2^*} + (tv_\varepsilon)^{2^*} + (tu_\varepsilon)^\alpha (tv_\varepsilon)^\beta dx \\
&- \frac{1}{2^*}\int_\Omega (u_* + tu_\varepsilon)^{2^*} - u_*^{2^*} - (tu_\varepsilon)^{2^*} - 2^* u_*^{2^*-1} tu_\varepsilon dx \\
&- \frac{1}{2^*}\int_\Omega (v_* + tv_\varepsilon)^{2^*} - v_*^{2^*} - (tv_\varepsilon)^{2^*} - 2^* v_*^{2^*-1} tv_\varepsilon dx \\
&- \frac{1}{2^*}\int_\Omega (u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta - u_*^\alpha v_*^\beta - (tu_\varepsilon)^\alpha (tv_\varepsilon)^\beta \\
&- \alpha u_*^{\alpha-1} v_*^\beta tu_\varepsilon - \beta u_*^\alpha v_*^{\beta-1} tv_\varepsilon dx. \qquad (4.12)
\end{aligned}
$$

Let $\Phi_\varepsilon(t) = \Phi_{\varepsilon,1}(t) + \Phi_{\varepsilon,2}(t) + \Phi_{\varepsilon,3}(t) + \Phi_{\varepsilon,4}(t)$, where

$$\Phi_{\varepsilon,1}(t) = \frac{1}{2}\|(tu_\varepsilon, tv_\varepsilon)\|_E^2 - \frac{1}{2^*}\int_\Omega (tu_\varepsilon)^{2^*} + (tv_\varepsilon)^{2^*} + (tu_\varepsilon)^\alpha (tv_\varepsilon)^\beta dx, \qquad (4.13)$$

$$\Phi_{\varepsilon,2}(t) = \frac{1}{2^*}\int_\Omega (u_* + tu_\varepsilon)^{2^*} - u_*^{2^*} - (tu_\varepsilon)^{2^*} - 2^* u_*^{2^*-1} tu_\varepsilon dx, \qquad (4.14)$$

$$\Phi_{\varepsilon,3}(t) = \frac{1}{2^*}\int_\Omega (v_* + tv_\varepsilon)^{2^*} - v_*^{2^*} - (tv_\varepsilon)^{2^*} - 2^* v_*^{2^*-1} tv_\varepsilon dx, \qquad (4.15)$$

$$
\begin{aligned}
\Phi_{\varepsilon,4}(t) = & \frac{1}{2^*}\int_\Omega (u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta - u_*^\alpha v_*^\beta - (tu_\varepsilon)^\alpha (tv_\varepsilon)^\beta \\
& - \alpha u_*^{\alpha-1} v_*^\beta tu_\varepsilon - \beta u_*^\alpha v_*^{\beta-1} tv_\varepsilon dx. \qquad (4.16)
\end{aligned}
$$

Notice that $\Phi_\varepsilon(0) = 0$, $\lim_{t\to+\infty} \Phi_\varepsilon(t) = -\infty$, and $\lim_{t\to 0^+} \Phi_\varepsilon(t) = 0$ uniformly for all ε. On the one hand, when $\inf\sup_{t\ge 0} \Phi_\varepsilon(t) \le 0$, one has $I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) \le I_{\lambda,\mu}(u_*, v_*) = m < m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}$. Conclusion naturally holds in this case. On the other hand, when $\inf\sup_{t\ge 0} \Phi_\varepsilon(t) > 0$, then, $\sup_{t\ge 0} \Phi_\varepsilon(t) > 0$ and it attains for some $t_\varepsilon > 0$, that is, $\sup_{t\ge 0} \Phi_\varepsilon(t) = \Phi_\varepsilon(t_\varepsilon)$. According to the monotonicity of Φ_ε near $t = 0$, we can find two positive constants $\overline{T_0}, \underline{T_0}$, such that

$$|\Phi_\varepsilon(\underline{T_0})| = |\Phi_\varepsilon(\underline{T_0}) - \Phi_\varepsilon(0)| < \xi = \frac{\Phi_\varepsilon(t_\varepsilon)}{4}.$$

Similarly, we can obtain $t_\varepsilon < \overline{T_0}$. So, $\underline{T_0} < t_\varepsilon < \overline{T_0}$ is bounded. Now, we evaluate the four parts separately. Let us evaluate (4.13) first, from (4.8) we can get:

$$\begin{aligned}\Phi_{\varepsilon,1}(t_\varepsilon) &= \frac{t_\varepsilon^2}{2}\|(u_\varepsilon, v_\varepsilon)\|_E^2 - \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega (|u_\varepsilon|^{2^*} + |v_\varepsilon|^{2^*} + |u_\varepsilon|^\alpha |v_\varepsilon|^\beta)dx \\ &= \frac{t_\varepsilon^2}{2}\int_\Omega (|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2)dx - \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega (|u_\varepsilon|^{2^*} + |v_\varepsilon|^{2^*} + |u_\varepsilon|^\alpha |v_\varepsilon|^\beta)dx \\ &= \frac{t_\varepsilon^2}{2}(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx - \frac{t_\varepsilon^{2^*}}{2^*}(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx.\end{aligned}$$

Then, define

$$J(t) = \frac{t^2}{2}\left[(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx\right] - \frac{t^{2^*}}{2^*}\left[(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx\right].$$

Obviously, $J'(t_{max}^\varepsilon) = 0$ with

$$t_{max}^\varepsilon = \left[\frac{(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx}{(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon^{2^*}|dx}\right]^{\frac{1}{2^*-2}} > 0.$$

By simple analysis, we get that $J(t)$ attains maximum at t_{max}^ε. Next, by using (1.5), (4.9), (4.10), we have the following result:

$$\begin{aligned}J(t_{max}^\varepsilon) &= \frac{1}{2}\left[\frac{(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx}{(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx}\right]^{\frac{2}{2^*-2}}\left[(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx\right] \\ &\quad - \frac{1}{2^*}\left[\frac{(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx}{(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx}\right]^{\frac{2^*}{2^*-2}}\left[(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx\right] \\ &= \frac{1}{N}\left\{\frac{(1+\tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx}{\left[(1+\tau_{min}^{2^*}+\tau_{min}^\beta)\int_\Omega |u_\varepsilon|^{2^*}dx\right]^{\frac{2}{2^*}}}\right\}^{\frac{N}{2}} \\ &\le \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} + O(\varepsilon^{N-2}).\end{aligned}$$

Then, we can get

$$\Phi_{\varepsilon,1}(t_\varepsilon) \le J(t_{max}^\varepsilon) \le \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}} + O(\varepsilon^{N-2}). \tag{4.17}$$

Next, let us analyze (4.14). According to [23, (4.12)]:

$$(a+b)^\gamma \ge a^\gamma + b^\gamma + \gamma a^{\gamma-1}b + C_1 a b^{\gamma-1}, \qquad 0 \le a \le M,\ b \ge 1,\ M > 0,\ \gamma > 2. \tag{4.18}$$

Then, according to (1.5) we can find a positive constant $C_2 > 1$, where C_2 satisfies:

$$\begin{aligned}S^{\frac{N}{2}} &= \left(\frac{1}{f(\tau_{min})}S_{\alpha,\beta}\right)^{\frac{N}{2}} \\ &\le C_2 S_{\alpha,\beta}^{\frac{N}{2}}.\end{aligned} \tag{4.19}$$

Moreover, by the standard elliptic estimates, we know that $u_*, v_* \in C(\overline{\Omega})$. Assume that $t_\varepsilon u_\varepsilon \geq 1$ for all $t_\varepsilon \geq \frac{1}{\sqrt{C_2 N(1+\tau_{min}^2)}}$, by using (4.18), we get

$$\begin{aligned}
\Phi_{\varepsilon,2}(t_\varepsilon) &= \frac{1}{2^*}\int_\Omega [(u_* + t_\varepsilon u_\varepsilon)^{2^*} - u_*^{2^*} - (t_\varepsilon u_\varepsilon)^{2^*} - 2^* u_*^{2^*-1} t_\varepsilon u_\varepsilon] dx \\
&\geq \frac{1}{2^*}\int_\Omega [u_*^{2^*} + (t_\varepsilon u_\varepsilon)^{2^*} + 2^* u_*^{2^*-1} t_\varepsilon u_\varepsilon \\
&\quad - u_*^{2^*} - (t_\varepsilon u_\varepsilon)^{2^*} - 2^* u_*^{2^*-1} t_\varepsilon u_\varepsilon + C_1 u_* t_\varepsilon^{2^*-1} u_\varepsilon^{2^*-1}] dx \\
&= \frac{t_\varepsilon^{2^*-1}}{2^*}\int_\Omega C_1 u_* u_\varepsilon^{2^*-1} dx \\
&\geq O(\varepsilon^{\frac{N-2}{2}}).
\end{aligned} \tag{4.20}$$

By using the same method as (4.20) to (4.15), we can get

$$\Phi_{\varepsilon,3}(t_\varepsilon) \geq O(\varepsilon^{\frac{N-2}{2}}). \tag{4.21}$$

At last, let's evaluate (4.16). First of all, we define a new function $f(x,y) : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$, and

$$f(x,y) = (1+x)^\alpha (1+y)^\beta - x^\alpha y^\beta - \alpha x - \beta y - 1.$$

Since

$$\begin{aligned}
\frac{\partial f(x,y)}{\partial x} &= \alpha(1+x)^{\alpha-1}(1+y)^\beta - \alpha x^{\alpha-1} y^\beta - \alpha \\
&\geq \alpha(1+x)^{\alpha-1}(1+y^\beta) - \alpha x^{\alpha-1} y^\beta - \alpha \\
&= \alpha(1+x)^{\alpha-1} - \alpha + \alpha(1+x)^{\alpha-1} y^\beta - \alpha x^{\alpha-1} y^\beta \\
&\geq 0.
\end{aligned}$$

We can also get $\dfrac{\partial f(x,y)}{\partial y} \geq 0$ in the same way. Obviously, $f(0,0) = 0$, so for any $x \geq 0, y \geq 0$, we have $f(x,y) \geq 0$. Because of $u_*, v_* > 0$, we have

$$\begin{aligned}
\Phi_{\varepsilon,4}(t_\varepsilon) &= \frac{1}{2^*}\int_\Omega (u_* + t_\varepsilon u_\varepsilon)^\alpha (v_* + t_\varepsilon v_\varepsilon)^\beta - u_*^\alpha v_*^\beta - (t_\varepsilon u_\varepsilon)^\alpha (t_\varepsilon v_\varepsilon)^\beta \\
&\quad - \alpha u_*^{\alpha-1} v_*^\beta t_\varepsilon u_\varepsilon - \beta u_*^\alpha v_*^{\beta-1} t_\varepsilon v_\varepsilon dx \\
&\geq 0.
\end{aligned} \tag{4.22}$$

Therefore, for $t = t_\varepsilon \geq \frac{1}{\sqrt{C_2 N(1+\tau_{min}^2)}}$, we know that there exists a $\varepsilon_1 > 0$ such for any $\varepsilon \in (0, \varepsilon_1)$ that

$$\begin{aligned}
I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) &\leq m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}} + O(\varepsilon^{N-2}) - O(\varepsilon^{\frac{N-2}{2}}) \\
&< m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}
\end{aligned} \tag{4.23}$$

by (4.17), (4.20), (4.21) and (4.22).

When $0 < t < \frac{1}{\sqrt{C_2 N(1+\tau_{min}^2)}}$, according to (4.8)–(4.11) and (4.19), there is a $\varepsilon_2 > 0$, when $\varepsilon \in (0, \varepsilon_2)$ we have the following estimates:

$$\begin{aligned}
I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) &= \frac{1}{2}\|(u_*, v_*)\|_E^2 + \frac{1}{2}\|(tu_\varepsilon, tv_\varepsilon)\|_E^2 \\
&\quad + t\int_\Omega (u_*^{2^*-1}u_\varepsilon + \frac{\alpha}{2^*}u_*^{\alpha-1}v_*^\beta u_\varepsilon + \lambda u_*^{q-1}u_\varepsilon)dx \\
&\quad + t\int_\Omega (v_*^{2^*-1}v_\varepsilon + \frac{\beta}{2^*}u_*^{\alpha}v_*^{\beta-1} v_\varepsilon + \mu v_*^{q-1}v_\varepsilon)dx \\
&\quad - \frac{1}{2^*}\int_\Omega [(u_* + tu_\varepsilon)^{2^*} + (v_* + tv_\varepsilon)^{2^*} + (u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta]dx \\
&\quad - \frac{1}{q}\int_\Omega [\lambda(u_* + tu_\varepsilon)^q + \mu(v_* + tv_\varepsilon)^q]dx \\
&= I_{\lambda,\mu}(u_*, v_*) + \frac{t^2}{2}\|(u_\varepsilon, v_\varepsilon)\|_E^2 \\
&\quad + \frac{1}{2^*}\int_\Omega (u_*^{2^*} + v_*^{2^*} + 2^* u_*^{2^*-1}tu_\varepsilon + 2^* v_*^{2^*-1}tv_\varepsilon)dx \\
&\quad - \frac{1}{2^*}\int_\Omega [(u_* + tu_\varepsilon)^{2^*} + (v_* + tv_\varepsilon)^{2^*}]dx \\
&\quad + \frac{1}{2^*}\int_\Omega (u_*^\alpha v_*^\beta + \alpha u_*^{\alpha-1}v_*^\beta tu_\varepsilon + \beta u_*^\alpha v_*^{\beta-1}tv_\varepsilon)dx \\
&\quad - \frac{1}{2^*}\int_\Omega [(u_* + tu_\varepsilon)^\alpha (v_* + tv_\varepsilon)^\beta]dx \\
&\quad + \frac{1}{q}\int_\Omega (\lambda u_*^q + \mu v_*^q)dx \\
&\quad - \frac{1}{q}\int_\Omega [\lambda(u_* + tu_\varepsilon)^q + \mu(v_* + tv_\varepsilon)^q]dx \\
&\le m + \frac{t^2}{2}\|(u_\varepsilon, v_\varepsilon)\|_E^2 \\
&= m + \frac{t^2}{2}\int_\Omega (|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2)dx \\
&= m + \frac{t^2}{2}(1 + \tau_{min}^2)\int_\Omega |\nabla u_\varepsilon|^2 dx \\
&= m + \frac{t^2}{2}(1 + \tau_{min}^2)\left[\|U_\varepsilon\|_E^2 + O(\varepsilon^{N-2})\right] \\
&\le m + t^2(1 + \tau_{min}^2)S^{\frac{N}{2}} \\
&\le m + t^2(1 + \tau_{min}^2)C_2 S_{\alpha,\beta}^{\frac{N}{2}} \\
&< m + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}.
\end{aligned} \tag{4.24}$$

Therefore, choosing $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$, for any $0 < \varepsilon < \varepsilon_0$, we can draw a conclusion

$$\sup_{t\ge 0} I_{\lambda,\mu}(u_* + tu_\varepsilon, v_* + tv_\varepsilon) < m + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}$$

from (4.23) and (4.24). The proof of Lemma 4.2 is finished. □

Lemma 4.3. *There is a $t_\varepsilon^-(u_\varepsilon) > 0$ such that $(u_* + t_\varepsilon^- u_\varepsilon, v_* + t_\varepsilon^- v_\varepsilon) \in \mathcal{N}_{\lambda,\mu}^-$, when $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, (\frac{q}{2})^{\frac{2}{2-q}}T)$. What is more, $0 < m^- < m + \frac{1}{N}S_{\alpha,\beta}^{\frac{N}{2}}$.*

Proof. According to Lemma 2.1, there is a $t^-(z) > 0$ for any $z = (u, v) \in E\backslash\{(0, 0)\}$ such that $t^-(z)z \in \mathcal{N}^-_{\lambda,\mu}$. Let

$$E_1 = \left\{ z \in E : u = 0 \text{ or } \|z\| < t^- \left(\frac{z}{\|z\|_E} \right) \right\},$$
$$E_2 = \left\{ z \in E : \|z\| > t^- \left(\frac{z}{\|z\|_E} \right) \right\}.$$

Then, we have $\mathcal{N}^-_{\lambda,\mu} = \left\{ z \in E : \|z\| = t^-\left(\frac{z}{\|z\|_E}\right)\right\}$. So, $E = E_1 \cup E_2 \cup \mathcal{N}^-_{\lambda,\mu}$. We have $\mathcal{N}^+_{\lambda,\mu} \subset E_1$, since $t^+ < t^-$. Now, there is a positive constant M_1 such that $0 < t^-(z) < M_1$ for $\|z\|_E = 1$. When $t_0 = \frac{\left|M_1 - \|(u_*,v_*)\|_E^2\right|^{\frac{1}{2}}}{\|(u_\varepsilon,v_\varepsilon)\|_E^2} + 1$, we claim that

$$\omega_\varepsilon = (u_* + t_0 u_\varepsilon, v_* + t_0 v_\varepsilon) \in E_2,$$

for $\varepsilon > 0$ small enough. By (4.10), we can deduce that

$$\begin{aligned}
\|(u_* + t_0 u_\varepsilon, v_* + t_0 v_\varepsilon)\|_E^2 &= \|(u_*, v_*)\|_E^2 + \|(t_0 u_\varepsilon, t_0 v_\varepsilon)\|_E^2 \\
&\quad + 2t_0 \int_\Omega (u_*^{2^*-1} u_\varepsilon + \frac{\alpha}{2^*} u_*^{\alpha-1} v_*^\beta u_\varepsilon + \lambda u_*^{q-1} u_\varepsilon) dx \\
&\quad + 2t_0 \int_\Omega (v_*^{2^*-1} v_\varepsilon + \frac{\beta}{2^*} u_*^{\alpha} v_*^{\beta-1} v_\varepsilon + \mu v_*^{q-1} v_\varepsilon) dx \\
&\geq \|(u_*, v_*)\|_E^2 + t_0^2 \|(u_\varepsilon, v_\varepsilon)\|_E^2 + o_n(1) \\
&> M_1^2 \\
&\geq \left[t^- \left(\frac{\omega_\varepsilon}{\|\omega_\varepsilon\|_E} \right) \right]^2.
\end{aligned}$$

We denote $h : [0, 1] \to E$ by $h(t) = u_* + t t_0 v_\varepsilon$, then there exists $0 < (t_\varepsilon)^- < t_0$, which makes $(u_* + (t_\varepsilon)^- u_\varepsilon, v_* + (t_\varepsilon)^- v_\varepsilon) \in \mathcal{N}^-_{\lambda,\mu}$. Moreover, from Lemma 4.2 and Lemma 2.3 (*ii*), one has

$$0 < m^- \leq I_{\lambda,\mu}(u_* + (t_\varepsilon)^- u_\varepsilon, v_* + (t_\varepsilon)^- v_\varepsilon) \leq \sup_{t \geq 0} I_{\lambda,\mu}(u_* + t u_\varepsilon, v_* + t v_\varepsilon) < m + \frac{1}{N} S_{\alpha,\beta}^{\frac{N}{2}}.$$

Thus, the proof of Lemma 4.3 is complete. □

Next, for $z = (u, v)$, $\varphi = (\varphi_1, \varphi_2) \in E$, we define

$$\begin{aligned}
z - \varphi &= (u - \varphi_1, v - \varphi_2), \\
\langle z, \varphi \rangle &= \int_\Omega \nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 dx, \\
G_{\lambda,\mu}(z, \varphi) &= \int_\Omega \left(\lambda |u|^{q-2} u \varphi_1 + \mu |v|^{q-2} v \varphi_2 \right) dx, \\
H(z, \varphi) &= \int_\Omega \left(|u|^{2^*-2} u \varphi_1 + |v|^{2^*-2} v \varphi_2 \right) dx + \int_\Omega \left(\frac{\alpha}{2^*} |u|^{\alpha-2} |v|^\beta u \varphi_1 + \frac{\beta}{2^*} |u|^\alpha |v|^{\beta-2} v \varphi_2 \right) dx.
\end{aligned}$$

Then, we have the following conclusion.

Lemma 4.4. *When $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, (\frac{q}{2})^{\frac{2}{2-q}} T)$, there exist $\eta > 0$ and a differentiable function $\zeta : B_\eta(0) \subset E \to \mathbb{R}^+$, for $z = (u, v) \in \mathcal{N}^-_{\lambda,\mu}$ such that $\zeta(0) = 1$, $\zeta(\varphi)(z - \varphi) \in \mathcal{N}^-_{\lambda,\mu}$ for any $\varphi = (\varphi_1, \varphi_2) \in B_\eta(0)$, and*

$$\langle \zeta'(0), \phi \rangle = \frac{2\langle z, \phi \rangle - 2^* H(z, \phi) - q G_{\lambda,\mu}(z, \phi)}{(2 - q)\|z\|_E^2 - (2^* - q) H(z, z)}$$

for all $\phi = (\phi_1, \phi_2) \in E$.

Proof. For $z \in \mathcal{N}^{-}_{\lambda,\mu}$, we define a function $F_z : \mathbb{R} \times E \to \mathbb{R}$ and

$$\begin{aligned} F_z(\zeta,\phi) &= \left\langle I'_{\lambda,\mu}(\zeta(z-\phi)), \zeta(z-\phi) \right\rangle \\ &= \zeta^2\|z-\phi\|_E^2 - \zeta^{2^*}H(z-\phi,z-\phi) - \zeta^q G_{\lambda,\mu}(z-\phi,z-\phi). \end{aligned}$$

Then, we have $F_z(1,0) = \langle I'_{\lambda,\mu}(z), z\rangle = 0$, moreover, from (1.8) and (1.9) we have

$$\begin{aligned} \frac{d}{d\zeta}F_z(1,0) &= 2\|z\|_E^2 - 2^*H(z,z) - qG_{\lambda,\mu}(z,z) \\ &= (2-q)\|z\|_E^2 - (2^*-q)H(z,z) \\ &< 0. \end{aligned}$$

According to the implicit function theorem, there is a $\eta > 0$ and a differential function $\zeta : B_\eta(0) \subset E \to \mathbb{R}$, which makes $\zeta(0) = 1$, then, $F_z(\zeta(0),0) = F_z(1,0) = 0$, one has

$$\langle \zeta'(0), \phi \rangle = \frac{2\langle z,\phi\rangle - 2^*H(z,\phi) - qG_{\lambda,\mu}(z,\phi)}{(2-q)\|z\|_E^2 - (2^*-q)H(z,z)}$$

and

$$F_z(\zeta(\varphi),\varphi) = 0, \quad \text{for all } \varphi \in B_\eta(0)$$

which is equivalent to

$$\left\langle I'_{\lambda,\mu}(\zeta(\varphi)(z-\varphi)), \zeta(\varphi)(z-\varphi) \right\rangle = 0, \quad \text{for all } \varphi \in B_\eta(0).$$

This means that for all $\varphi \in B_\eta(0)$, we have $\zeta(\varphi)(z-\varphi) \in \mathcal{N}_{\lambda,\mu}$. The proof of Lemma 4.4 is complete. □

4.1 The proof of Theorem 1.1

Proof. For $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, (\frac{q}{2})^{\frac{2}{2-q}}T)$, there exists a $z \in \mathcal{N}^{-}_{\lambda,\mu}$ such that $m^- = \inf I_{\lambda,\mu}(z) > 0$, by Lemma 2.3. Setting $\{(u_n,v_n)\} \subset E$, which is a minimizing sequence of $I_{\lambda,\mu}$ at m^-. Now, we are going to prove that $\{(u_n,v_n)\}$ is a $(PS)_{m^-}$-sequence of $I_{\lambda,\mu}$. According to Ekeland's variational principle (see [10]), there exists a sequence (we still denote it as $\{(u_n,v_n)\}$) that satisfies

(i) $I_{\lambda,\mu}(u_n,v_n) < m^- + \dfrac{1}{n}$;

(ii) $I_{\lambda,\mu}(u_n,v_n) \leq I_{\lambda,\mu}(w_1,w_2) + \dfrac{\|(w_1,w_2)-(u_n,v_n)\|_E}{n}, \qquad (w_1,w_2) \in \mathcal{N}^{-}_{\lambda,\mu}.$

So, we only need to prove $I'_{\lambda,\mu}(u_n,v_n) \to 0$ as $n \to \infty$ in E^{-1} to get that $\{(u_n,v_n)\}$ is a $(PS)_{m^-}$-sequence of $I_{\lambda,\mu}$. According to Lemma 4.4, there exist a $\eta_n > 0$ and differentiable function $\zeta_n : B((0,0);\eta_n) \subset E \to \mathbb{R}^+$ such that $\zeta_n(0,0) = 1$, $\zeta_n(w_1,w_2)((u_n,v_n)-(w_1,w_2)) \in \mathcal{N}^{-}_{\lambda,\mu}$, for any $(w_1,w_2) \in B((0,0);\eta_n)$. Let $(\varphi_1,\varphi_2) \in E$, $\|(\varphi_1,\varphi_2)\|_E = 1$, and $0 < \sigma < \eta_n$. Then we choose $(w_1,w_2) = \sigma(\varphi_1,\varphi_2)$, which makes $(w_1,w_2) = \sigma(\varphi_1,\varphi_2) \in B((0,0);\eta_n)$ and $\omega_{\sigma,n} = \zeta_n(\sigma(\varphi_1,\varphi_2))((u_n,v_n) - \sigma(\phi_1,\phi_2)) \in \mathcal{N}^{-}_{\lambda,\mu}$, for $0 < \sigma < \eta_n$. From (ii) and the mean value theorem, let $\sigma \to 0^+$, we have

$$
\begin{aligned}
\frac{\|\omega_{\sigma,n} - (u_n, v_n)\|_E}{n} &\geq I_{\lambda,\mu}(u_n, v_n) - I_{\lambda,\mu}(\omega_{\sigma,n}) \\
&= \left\langle I'_{\lambda,\mu}(t_0(u_n, v_n) + (1-t_0)\omega_{\sigma,n}), (u_n, v_n) - \omega_{\sigma,n} \right\rangle \\
&= \left\langle I'_{\lambda,\mu}(u_n, v_n), (u_n, v_n) - \omega_{\sigma,n} \right\rangle + o(\|(u_n, v_n) - \omega_{\sigma,n}\|_E) \\
&= \sigma\zeta_n(\sigma(\varphi_1, \varphi_2)) \left\langle I'_{\lambda,\mu}(u_n, v_n), (\varphi_1, \varphi_2) \right\rangle \\
&\quad + (1 - \zeta_n(\sigma(\varphi_1, \varphi_2))) \left\langle I'_{\lambda,\mu}(u_n, v_n), (u_n, v_n) \right\rangle + o(\|(u_n, v_n) - \omega_{\sigma,n}\|_E) \\
&= \sigma\zeta_n(\sigma(\varphi_1, \varphi_2)) \left\langle I'_{\lambda,\mu}(u_n, v_n), (\varphi_1, \varphi_2) \right\rangle + o(\|(u_n, v_n) - \omega_{\sigma,n}\|_E),
\end{aligned}
$$

where $t_0 \in (0, 1)$. Next, let $\sigma \to 0^+$, we have

$$
\begin{aligned}
&\left\langle I'_{\lambda,\mu}(u_n, v_n), (\varphi_1, \varphi_2) \right\rangle \\
&\leq \frac{\|\omega_{\sigma,n} - (u_n, v_n)\|_E(\frac{1}{n} + o(1))}{\sigma|\zeta_n(\sigma(\varphi_1, \varphi_2))|} \\
&\leq \frac{\|(u_n, v_n)(\zeta_n(\sigma(\varphi_1, \varphi_2)) - \zeta_n(0,0)) - \sigma(\varphi_1, \varphi_2)\zeta_n(\sigma(\varphi_1, \varphi_2))\|_E(\frac{1}{n} + |o(1)|)}{\sigma|\zeta_n(\sigma(\varphi_1, \varphi_2))|} \\
&\leq \frac{\|(u_n, v_n)\|_E|\zeta_n(\sigma(\varphi_1, \varphi_2)) - \zeta_n(0,0)| + \sigma\|(\varphi_1, \varphi_2)\|_E|\zeta_n(\sigma(\varphi_1, \varphi_2))|}{\sigma|\zeta_n(\sigma(\varphi_1, \varphi_2))|} \left(\frac{1}{n} + |o(1)|\right) \\
&\leq C(1 + \|\zeta'_n(0,0)\|) \left(\frac{1}{n} + |o(1)|\right).
\end{aligned}
$$

Due to $\{(u_n, v_n)\}$ and $\zeta'_n(0,0)$ are bounded, we could learn that $I'_{\lambda,\mu}(u_n, v_n) \to 0$ in E^{-1} as $n \to \infty$. Thus, $\{(u_n, v_n)\}$ is a $(PS)_{m^-}$-sequence of $I_{\lambda,\mu}$.

In accordance with Lemma 4.1, Lemma 4.2 and Lemma 4.3, there is a list of convergent subsequences $\{(u_n, v_n)\}$, such that $(u_n, v_n) \to (u_{**}, v_{**})$, where $(u_{**}, v_{**}) \in \mathcal{N}^-_{\lambda,\mu}$. What's more, when $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in \left(0, (\frac{q}{2})^{\frac{2}{2-q}}T\right)$, we can get $I_{\lambda,\mu}(u_{**}, v_{**}) = m^- > 0$. Since $I_{\lambda,\mu}(u_{**}, v_{**}) = I_{\lambda,\mu}(|u_{**}|, |v_{**}|)$ and $(u_{**}, v_{**}) \in \mathcal{N}^-_{\lambda,\mu}$, we can deduce that

$$
\int_\Omega |u_{**}|^{2^*} + |v_{**}|^{2^*} + |u_{**}|^\alpha |v_{**}|^\beta dx > \frac{2-q}{2^*-q}\|(u_{**}, v_{**})\|_E^2 > 0 \tag{4.25}
$$

from (1.9) and (1.10). So, $(u_{**}, v_{**}) \neq 0$. Applying the strong maximum principle, we could get that (u_{**}, v_{**}) is a positive solution of system (1.1). Finally, due to $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$, which implies that (u_*, v_*) and (u_{**}, v_{**}) are entirely different. The proof of Theorem 1.1 is complete. □

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Convergence of weak solutions of elliptic problems with datum in L^1

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Abstract. Motivated by the Q-condition result proven by Arcoya and Boccardo in [*J. Funct. Anal.* **268**(2015), No. 5, 1153–1166], we analyze the behaviour of the weak solutions $\{u_\varepsilon\}$ of the problems

$$\begin{cases} -\Delta_p u_\varepsilon + \varepsilon |f(x)| u_\varepsilon = f(x) & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

when ε tends to 0. Here, Ω denotes a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplacian operator $(1 < p < \infty)$ and $f(x)$ is an $L^1(\Omega)$ function.

We show that this sequence converges in some sense to u, the entropy solution of the problem

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In the semilinear case, we prove stronger results provided the weak solution of that problem exists.

Keywords: nonlinear elliptic equations, entropy solution, Q-condition.

2020 Mathematics Subject Classification: 35D30, 35J25, 35J60.

1 Introduction

In this paper we develop a new method to approach solutions (in a broad sense that we will discuss later) of a problem with data only in L^1 that does not require the approximation of such data by more regular functions. Specifically, we consider the following boundary value problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + b(x)g(u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

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where Ω is a bounded open set of $\mathbb{R}^N$ ($N \geq 2$) and $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a nonlinear Leray–Lions operator, i.e., it is a Carathéodory function such that for every $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$ ($\xi \neq \eta$), and for almost every $x \in \Omega$ satisfies

$$a(x,s,\xi)\xi \geq \alpha|\xi|^p, \tag{1.1}$$

$$|a(x,s,\xi)| \leq h(x) + \beta|\xi|^{p-1}, \tag{1.2}$$

$$(a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) > 0, \tag{1.3}$$

where $1 < p < \infty$, $h(x) \in L^{p'}(\Omega)$ and $\alpha, \beta > 0$. With respect to the coefficient $b(x)$ of the lower order term and to the datum $f(x)$, it assumed that

$$0 \leq b(x) \in L^1(\Omega), \qquad f(x) \in L^1(\Omega), \tag{1.4}$$

and $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying that

$$g \text{ is increasing, odd and } \lim_{s \to +\infty} g(s) = +\infty. \tag{1.5}$$

A simple model of function g for the reader may be $g(s) = |s|^{\gamma-1}s$ for $\gamma > 0$.

We remark that the problem (P) under the previous hypotheses (1.1), (1.2), (1.3), (1.4) and (1.5) does not always have a solution in the usual sense when f belongs to $L^1(\Omega)$. Moreover, in the case in which the solution of the problem (P) exists with a right-hand side in $L^1(\Omega)$ it is not necessarily bounded; in fact, it may not even be in the $W^{1,1}_{\mathrm{loc}}(\Omega)$ space when $p \leq 2 - \frac{1}{N}$. Motivated by this, many authors started to study if there was a more general concept of solution in which existence and uniqueness were guaranteed; see, for example, the paper [6], where they use the concept of renormalized solution, or [5], where the concept of entropy solution is introduced.

Nevertheless, under some extra conditions the existence of a weak solution of (P) can be ensured. In [2] (see also [1]), the authors proved that if there exists certain relation between the coefficient $b(x)$ of the lower order term and the datum $f(x)$, then the existence of a bounded weak solution is granted even if $f(x)$ only belongs to $L^1(\Omega)$. Concretely, they showed that if the so-called Q-condition is satisfied, i.e., if there exists some $Q > 0$ such that

$$|f(x)| \leq Qb(x), \tag{1.6}$$

then the problem (P) has a unique weak solution $u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$. Moreover, they also gave an $L^\infty(\Omega)$-estimate for u, namely

$$\|u\|_\infty \leq g^{-1}(Q).$$

Therefore, they put in evidence that this interplay between the coefficients provides a regularizing effect on the problem (P). After the publication of these works, several number of papers studying this kind of regularizing effects given by the interplay between coefficients in other types of problems were published, such as [3,4], giving rise to a prolific and original line of modern research.

Motivated by this result, in this paper we approach the problem (P) in such a way that the resulting approximated problems satisfy the relation (1.6) and we study the convergence of the sequence of solutions. Concretely, we consider the following approximated elliptic problems

$$\begin{cases} -\mathrm{div}(a(x,u_n,\nabla u_n)) + \left[b(x) + \frac{1}{n}|f(x)|\right]g(u_n) = f(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_n}$$

Observe that the coefficients of these problems satisfy the relation (1.6) since

$$|f(x)| \leq n\big[b(x) + \tfrac{1}{n}|f(x)|\big],$$

so, if we assume the hypotheses (1.1), (1.2), (1.3), (1.4) and (1.5), the results of [2] provide for each $n \in \mathbb{N}$ the existence of a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of (P_n) which also satisfies that

$$\|u_n\|_\infty \leq g^{-1}(n). \tag{1.7}$$

The purpose of this paper will be to study the behaviour of the sequence $\{u_n\}$ when n goes to ∞. We stress that similar studies can be done on other problems for which existence or regularity results have been proven thanks to some Q-condition type hypothesis. Therefore, this paper can be the beginning of a productive line of research.

The main result, stated below, is related with the entropy solution of (P), whose existence and uniqueness is guaranteed thanks to the results of [5]. We also point out that the proof of our theorem is, in fact, an alternative existence proof to the one given in [5], where the major difference between both are the approximate problems considered.

Theorem 1.1. *Suppose that $a(x,s,\xi)$ satisfies (1.1), (1.2) and (1.3), that $b(x)$ and $f(x)$ verify (1.4) and that g satisfies (1.5). Then the solution of (P) in the sense of Definition 2.5 exists and the sequence $\{u_n\}$ of weak solutions of (P_n) converges in measure to that solution.*

Note that the sequence of weak solutions $\{u_n\}$ of (P_n), in general, cannot converge weakly in $W_0^{1,p}(\Omega)$ because, in this case, that would imply the existence of a weak solution of (P). Recall that this type of solution (see Definition 2.4) does not always exist for problem (P).

In the semilinear case, i.e., when $p = 2$, we study if this stronger convergence can be proved as long as the weak solution of (P) exists. For this purpose, we consider the linear operator $a(x,s,\xi) = M(x)\xi$, where $M(x)$ is a symmetric bounded elliptic matrix, i.e., there exist $\alpha, \beta > 0$ such that

$$\alpha|\xi|^2 \leq M(x)\xi\xi, \tag{1.8}$$

$$|M(x)| \leq \beta \tag{1.9}$$

for every $\xi \in \mathbb{R}^N$ and for almost every x in Ω.

The mentioned result for the semilinear case is the following one.

Theorem 1.2. *Suppose that $a(x,s,\xi) = M(x)\xi$ with $M(x)$ a symmetric matrix satisfying (1.8) and (1.9). Assume also that $b(x)$ and $f(x)$ verify (1.4) and that g satisfies (1.5). If the weak solution $u \in H_0^1(\Omega)$ of (P) exists and it is in $L^\infty(\Omega)$, then $\{u_n\}$, the sequence of weak solutions of (P_n), verifies that*

$$u_n \rightharpoonup u \quad \textit{in } H_0^1(\Omega).$$

We stress that, unlike Theorem 1.1, this theorem is not an existence result since we are assuming that the weak solution of (P) exists.

In order to prove these results we will follow the next structure in the work. In Section 2 we state the theorem of [2] in which our study is motivated, we take a brief review of the Marcinkiewicz spaces, we remind the concept of entropy solution of (P) and we give other preliminary results. In Section 3 we prove Theorem 1.1, the main result of this paper. Finally, in Section 4, we deal with the semilinear case and we give the proof of Theorem 1.2.

2 Preliminaries

First of all, we state here the result of [2] that has motivated this research and then we indicate the key of the proof. As we will see, the tools which are used in the proof are not excessively sophisticated, so the approach we adopt in this paper is elemental.

Theorem 2.1 ([2]). *Suppose that $a(x,s,\xi)$ satisfies (1.1), (1.2) and (1.3), that $b(x)$ and $f(x)$ verify (1.4) and that g satisfies (1.5). If the relation (1.6) between $b(x)$ and $f(x)$ is verified, then there exists a unique $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ weak solution of (P) which also satisfies that*

$$\|u\|_\infty \leq g^{-1}(Q).$$

Remark 2.2. The key of the proof is to obtain an a priori $L^\infty(\Omega)$-estimate. The idea is to approximate (*P*) in such a way that its coefficients still satisfy the Q-condition (1.6), and this condition allows us to prove the uniform boundedness in $L^\infty(\Omega)$ of the sequence of approximated solutions.

Formally speaking, this $L^\infty(\Omega)$-estimate is obtained by taking as test function in (*P*) the mapping $G_k(u) := \max\{\min\{u+k,0\}, u-k\}$ with $k > 0$. In particular, using (1.1) and (1.6) we get that

$$\alpha \int_\Omega |\nabla G_k(u)|^p + \int_\Omega b(x)g(u)G_k(u) \leq \int_\Omega f(x)G_k(u) \leq \int_\Omega Qb(x)|G_k(u)|,$$

i.e., that

$$\alpha \int_\Omega |\nabla G_k(u)|^p + \int_\Omega b(x)\left[|g(u)| - Q\right]|G_k(u)| \leq 0.$$

Observe that g^{-1} exists thanks to (1.5) and that we can choose $k = g^{-1}(Q)$ in the above inequality to get that the second integral is nonnegative and, as a consequence, it is deduced that $g^{-1}(Q)$ is an a priori bound in $L^\infty(\Omega)$. □

In several parts of this paper we work with the Marcinkiewicz spaces. For the convenience of the reader, we recall here their definition and some of their properties. For $0 < q < \infty$, we denote by $\mathcal{M}^q(\Omega)$ the set of measurable functions $v\colon \Omega \to \mathbb{R}$ such that there exists $C > 0$ satisfying that

$$\text{meas}\{|v| > k\} \leq \frac{C}{k^q}, \qquad \forall k > 0. \tag{2.1}$$

This space is a complete quasi-normed space with the quasi-norm

$$\|v\|^q_{\mathcal{M}^q(\Omega)} = \inf\{C > 0 : (2.1) \text{ holds}\}.$$

We also recall that, since Ω is bounded, then

$$\mathcal{M}^{q_2}(\Omega) \hookrightarrow L^{q_1}(\Omega) \hookrightarrow \mathcal{M}^{q_1}(\Omega)$$

for $0 < q_1 < q_2 < \infty$.

Related with these spaces we state the following lemma whose proof can be found in [5, Lemma 4.1]. For any $k > 0$ we set $T_k(s) = \min\{k, \max\{s, -k\}\}$.

Lemma 2.3 ([5]). *Let $u\colon \Omega \to \mathbb{R}$ be a function such that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$ and*

$$\frac{1}{k}\int_{\{|u|<k\}} |\nabla u|^p \leq M$$

for some constant $M > 0$ and for every $k > 0$. Then $u \in \mathcal{M}^{p_1}(\Omega)$ for $p_1 = \frac{N(p-1)}{N-p}$ if $1 < p < N$ and for every $p_1 > 1$ if $p \geq N$. More precisely, there exists $C = C(M, N, p) > 0$ such that

$$\text{meas}\{|u| > k\} \leq \frac{C}{k^{p_1}}, \qquad \forall k > 0.$$

We also recall here the concepts of weak solution and entropy solution of (P).

Definition 2.4. A function $u\colon \Omega \to \mathbb{R}$ is a *weak solution* of the problem (P) if $u \in W_0^{1,p}(\Omega)$, $b(x)g(u) \in L^1(\Omega)$ and

$$\int_\Omega a(x, u, \nabla u)\nabla\varphi + \int_\Omega b(x)g(u)\varphi = \int_\Omega f(x)\varphi$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Definition 2.5. A function $u\colon \Omega \to \mathbb{R}$ is an *entropy solution* of (P) if $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$, $b(x)g(u) \in L^1(\Omega)$ and

$$\int_\Omega a(x, u, \nabla u)\nabla T_k(u - \varphi) + \int_\Omega b(x)g(u)T_k(u - \varphi) = \int_\Omega f(x)T_k(u - \varphi)$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and every $k > 0$.

Observe that the concept of entropy solution is more general than the concept of weak solution, i.e., every weak solution is an entropy solution. Although the reciprocal is not true in general, if an entropy solution of (P) is in $W_0^{1,p}(\Omega)$, then is also a weak solution of (P) (see [5, Corollary 4.3]).

Regarding the uniqueness, both types of solutions are unique (see [5, Theorem 5.1]). However, unlike the weak solution, which may not exist when $p \leq 2 - \frac{1}{N}$, it was proved in [5, Theorem 6.1] that the entropy solution of (P) always exists.

Finally, we end this section with a convergence lemma that we will use throughout this paper.

Lemma 2.6. *Suppose that $a(x, s, \xi)$ satisfies (1.1), (1.2) and (1.3), that $b(x)$ and $f(x)$ verify (1.4) and that g satisfies (1.5). If the sequence $\{u_n\}$ of weak solutions of (P_n) is bounded in $\mathcal{M}^q(\Omega)$ for some $q > 0$ and satisfies that $u_n \to u$* a.e. *in Ω for some function u, then*

$$b(x)g(u_n) \to b(x)g(u) \quad \textit{in } L^1(\Omega).$$

Proof. Let $\psi_{k,\delta}\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$\psi_{k,\delta}(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq k, \\ \frac{1}{\delta}(s - k) & \text{if } k < s < k + \delta, \\ 1 & \text{if } s \geq k + \delta, \\ -\psi_{k,\delta}(-s) & \text{if } s < 0. \end{cases}$$

Taking $\psi_{k,\delta}(u_n) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in (P_n) and dropping two nonnegative terms we obtain that

$$\int_\Omega b(x)g(u_n)\psi_{k,\delta}(u_n) \leq \int_\Omega |f(x)|\,|\psi_{k,\delta}(u_n)|,$$

what implies that

$$\int_{\{k+\delta\le|u_n|\}} b(x)|g(u_n)| \le \int_{\{k\le|u_n|\}} |f(x)|.$$

If $\delta \to 0$, Fatou Lemma gives

$$\int_{\{k\le|u_n|\}} b(x)|g(u_n)| \le \int_{\{k\le|u_n|\}} |f(x)|.$$

We claim that $\{b(x)g(u_n)\}$ is uniformly integrable. Fix $\varepsilon > 0$. Since $b(x)$ is nonnegative by (1.4) and g is increasing and odd by (1.5), we deduce from the above inequality that for every measurable set $E \subset \Omega$ we have

$$\begin{aligned}\int_E b(x)|g(u_n)| &= \int_{E\cap\{|u_n|\le k\}} b(x)|g(u_n)| + \int_{E\cap\{k\le|u_n|\}} b(x)|g(u_n)| \\ &\le g(k)\int_E b(x) + \int_{\{k\le|u_n|\}} |f(x)|.\end{aligned}$$

On the one hand, since $f(x) \in L^1(\Omega)$, thanks to the absolute continuity of the integral there exists some $\delta' > 0$ such that if $E \subset \Omega$ is a measurable set with $\text{meas}(E) < \delta'$ then $\int_E |f(x)| < \frac{\varepsilon}{2}$. As $\{u_n\}$ is bounded in $\mathcal{M}^q(\Omega)$, we can fix $k > 0$ large enough such that $\text{meas}\{|u_n| \ge k\} \le \delta'$ for every $n \in \mathbb{N}$. Thus,

$$\int_{\{k\le|u_n|\}} |f(x)| \le \frac{\varepsilon}{2}, \qquad \forall n \in \mathbb{N}.$$

On the other hand, since $b(x) \in L^1(\Omega)$, again by the absolute continuity of the integral there exists some $\delta > 0$ such that $E \subset \Omega$ is a measurable set with $\text{meas}(E) < \delta$ then

$$\int_E b(x) < \frac{\varepsilon}{2g(k)}.$$

In this way, we have that if $E \subset \Omega$ is a measurable set with $\text{meas}(E) < \delta$ then

$$\int_E b(x)|g(u_n)| \le g(k)\int_E b(x) + \int_{\{k\le|u_n|\}} |f(x)| < \varepsilon, \qquad \forall n \in \mathbb{N}.$$

Therefore, the sequence $\{b(x)g(u_n)\}$ is uniformly integrable. As we also have that this sequence $b(x)g(u_n) \to b(x)g(u)$ a.e. in Ω, we can apply Vitali's Theorem (since $\text{meas}(\Omega) < \infty$) to conclude that $b(x)g(u) \in L^1(\Omega)$ and that

$$b(x)g(u_n) \to b(x)g(u) \quad \text{in } L^1(\Omega).$$ □

3 Convergence to the entropy solution

In this section we give the proof of Theorem 1.1.

Proof of Theorem 1.1. First, let us remember that as u_n are weak solutions of (P_n), then for every $n \in \mathbb{N}$ and for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ we have that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\varphi + \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right]g(u_n)\varphi = \int_\Omega f(x)\varphi. \tag{3.1}$$

Now we begin with the proof.

Step 1. $\{u_n\}$ is bounded on some Marcinkiewicz space.

Taking $T_k(u_n) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ as test function in (3.1) we obtain for every $n \in \mathbb{N}$ and for every $k > 0$ that

$$\int_{\{|u_n| \le k\}} a(x, u_n, \nabla u_n) \nabla T_k(u_n) + \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right] g(u_n) T_k(u_n) = \int_\Omega f(x) T_k(u_n).$$

Observe that the second integral is nonnegative since $g(s)s \ge 0$ for every $s \in \mathbb{R}$ by (1.5) and that we can apply (1.1) on the first integral since $a(x, u_n, \nabla u_n) = a(x, T_k(u_n), \nabla T_k(u_n))$ on the set $\{|u_n| < k\}$. So, from the above equality we deduce that

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p = \alpha \int_{\{|u_n|<k\}} |\nabla u_n|^p \le \int_\Omega f(x) T_k(u_n) \le k \|f\|_1, \qquad \forall n \in \mathbb{N},\ \forall k > 0. \tag{3.2}$$

Thus, we can apply Lemma 2.3 to assure that there exists a constant $C > 0$ depending only of N, p, α and f such that

$$\text{meas}\{|u_n| > k\} \le C k^{-\frac{N(p-1)}{N-p}}, \tag{3.3}$$

for every $n \in \mathbb{N}$ and every $k > 0$. As a consequence, we deduce that $\{u_n\}$ is bounded on the space $\mathcal{M}^{p_1}(\Omega)$ with $p_1 = \frac{N(p-1)}{N-p}$.

Step 2. $\{u_n\}$ converges in measure to some function u.

To show that $\{u_n\}$ converges in measure it suffices to show that it is Cauchy in measure. Let $\varepsilon > 0$ and let $t > 0$. As

$$\{|u_n - u_m| > t\} \subseteq \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > t\},$$

then

$$\text{meas}\{|u_n - u_m| > t\} \le \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > t\}.$$

Thanks to (3.3), we can fix $k_0 > 0$ large enough to obtain that

$$\text{meas}\{|u_n| > k_0\} < \frac{\varepsilon}{3}, \qquad \forall n \in \mathbb{N}.$$

By (3.2), we deduce that $\{T_k(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$ for every $k > 0$. Thus, for every fixed $k > 0$ there exists a subsequence $\{u_{\sigma_k(n)}\}$ of $\{u_n\}$ such that $\{T_k(u_{\sigma_k(n)})\}$ is Cauchy in $L^p(\Omega)$. Using the Cantor's diagonal argument, we can build a subsequence $\{u_{\sigma(n)}\}$ of $\{u_n\}$ such that $\{T_k(u_{\sigma(n)})\}$ is Cauchy in $L^p(\Omega)$ for every $k > 0$. For the sake of simplicity, we still denote $\{u_{\sigma(n)}\}$ by $\{u_n\}$.

So, since $\{T_{k_0}(u_n)\}$ is a Cauchy sequence in $L^p(\Omega)$, there exists $n_0 \in \mathbb{N}$ such that

$$\text{meas}\{|T_{k_0}(u_n) - T_{k_0}(u_m)| > t\} \le t^{-p} \int_\Omega |T_{k_0}(u_n) - T_{k_0}(u_m)|^p < \frac{\varepsilon}{3}, \qquad \forall m, n \ge n_0.$$

Thus, it is proven that $\{u_n\}$ is Cauchy in measure and hence there exists some measurable function u such that $u_n \to u$ in measure. As a consequence, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$u_n \to u \quad \text{a.e. in } \Omega.$$

Now, since for $k>0$ fixed the sequence $\{T_k(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$ by (3.2) and $T_k(u)$ is its only possible almost everywhere limit because of the continuity of T_k, we can conclude that

$$\begin{aligned} T_k(u_n) &\rightharpoonup T_k(u) \quad \text{in } W_0^{1,p}(\Omega),\\ T_k(u_n) &\to T_k(u) \quad \text{in } L^p(\Omega),\\ T_k(u_n) &\to T_k(u) \quad \text{a.e. in } \Omega. \end{aligned}$$

Observe that this implies that $T_k(u)\in W_0^{1,p}(\Omega)$ for every $k>0$.

Step 3. $T_k(u_n)$ strongly converges to $T_k(u)$ in $W_0^{1,p}(\Omega)$ for every $k>0$.

Following the ideas of [8], in order to obtain the strong convergence of the truncations in the $W_0^{1,p}(\Omega)$ space we choose

$$w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$$

with $h>k>0$ as test function in (3.1). See that if we set $M=4k+h$ then we have that $\nabla w_n = 0$ on the set $\{|u_n|>M\}$. Thus, we can write

$$\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\nabla w_n + \int_\Omega \left[b(x)+\tfrac{1}{n}|f(x)|\right]g(u_n)w_n = \int_\Omega f(x)w_n. \tag{3.4}$$

Now, we split the first integral on the sets $\{|u_n|<k\}$ and $\{|u_n|\ge k\}$. On the one hand, observing that $\{|T_k(u_n)-T_k(u)|\le 2k\}=\Omega$, that $\nabla T_k(u_n)=0$ on the set $\{|u_n|\ge k\}$ and that $a(x,s,0)=0$ by (1.1), we obtain that

$$\begin{aligned} &\int_{\{|u_n|<k\}} a(x,T_M(u_n),\nabla T_M(u_n))\nabla w_n\\ &\quad= \int_{\{|u_n|<k\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_{2k}(T_k(u_n)-T_k(u))\\ &\quad= \int_{\{|u_n|<k\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla (T_k(u_n)-T_k(u))\\ &\quad= \int_{\Omega} a(x,T_k(u_n),\nabla T_k(u_n))\nabla (T_k(u_n)-T_k(u)). \end{aligned} \tag{3.5}$$

On the other hand, using (1.1) we deduce that

$$\begin{aligned} &a(x,T_M(u_n),\nabla T_M(u_n))\nabla(G_h(u_n)-T_k(u))\\ &\quad= a(x,T_M(u_n),\nabla T_M(u_n))\nabla G_h(u_n) - a(x,T_M(u_n),\nabla T_M(u_n))\nabla T_k(u)\\ &\quad\ge -a(x,T_M(u_n),\nabla T_M(u_n))\nabla T_k(u) \end{aligned}$$

and, thus, we have

$$\begin{aligned} &\int_{\{|u_n|\ge k\}} a(x,T_M(u_n),\nabla T_M(u_n))\nabla w_n\\ &\quad= \int_{\{|u_n|\ge k\}\cap\{|G_h(u_n)+k+T_k(u)|\le 2k\}} a(x,T_M(u_n),\nabla T_M(u_n))\nabla(G_h(u_n)-T_k(u))\\ &\quad\ge -\int_{\{|u_n|\ge k\}} |a(x,T_M(u_n),\nabla T_M(u_n))|\,|\nabla T_k(u)|. \end{aligned} \tag{3.6}$$

From equations (3.5) and (3.6) we deduce that

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla(T_k(u_n) - T_k(u)) \\ \leq \int_{\{|u_n|\geq k\}} |a(x, T_M(u_n), \nabla T_M(u_n))|\,|\nabla T_k(u)| + \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))\nabla w_n.$$

Adding $-\int_\Omega a(x, T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u))$ to both sides of the previous inequality and using (3.4) we obtain that

$$\begin{aligned} &\int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))]\nabla(T_k(u_n) - T_k(u)) \\ &\quad\leq \int_{\{|u_n|\geq k\}} |a(x, T_M(u_n), \nabla T_M(u_n))|\,|\nabla T_k(u)| \\ &\qquad - \int_\Omega [b(x) + \tfrac{1}{n}|f(x)|]g(u_n)w_n + \int_\Omega f(x)w_n \\ &\qquad - \int_\Omega a(x, T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u)). \end{aligned} \tag{3.7}$$

Our next step will be taking limits when $n \to \infty$ on the above inequality. First, see that as $|a(x, T_M(u_n), \nabla T_M(u_n))|$ is bounded in $L^{p'}(\Omega)$ by (1.2) and (3.2), and as $\chi_{\{|u_n|\geq k\}}|\nabla T_k(u)|$ converges strongly to zero in $L^p(\Omega)$ by Lebesgue Theorem, then

$$\lim_{n\to\infty} \int_{\{|u_n|\geq k\}} |a(x, T_M(u_n), \nabla T_M(u_n))|\,|\nabla T_k(u)| = 0. \tag{3.8}$$

Secondly, since $\{b(x)g(u_n)\}$ is bounded in $L^1(\Omega)$ by Lemma 2.6 and since $\{\frac{1}{n}|f(x)|g(u_n)\}$ is also bounded in $L^1(\Omega)$ because $\frac{1}{n}|f(x)g(u_n)| \leq |f(x)|$ for every $n \in \mathbb{N}$ by (1.5) and (1.7), Lebesgue Theorem easily implies that

$$\begin{aligned} &\lim_{n\to\infty} \left(- \int_\Omega [b(x) + \tfrac{1}{n}|f(x)|]g(u_n)w_n + \int_\Omega f(x)w_n \right) \\ &\quad = \int_\Omega [-b(x)g(u) + f(x)]T_{2k}(u - T_h(u)). \end{aligned} \tag{3.9}$$

Finally, since $a(x, T_k(u_n), \nabla T_k(u)) \to a(x, T_k(u), \nabla T_k(u))$ strongly in $L^{p'}(\Omega)$ by (1.2) and by Lebesgue Theorem, and since $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $L^p(\Omega)$, we deduce that

$$\lim_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u)) = 0. \tag{3.10}$$

Observe that the first integral of (3.7) is nonnegative by (1.3). So if we take limits when $n \to \infty$ in (3.7) and we apply (3.8), (3.9) and (3.10) we obtain that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))]\nabla(T_k(u_n) - T_k(u)) \\ &\leq \int_\Omega [-b(x)g(u) + f(x)]T_{2k}(u - T_h(u)). \end{aligned} \tag{3.11}$$

Now, see that $b(x)g(u) \in L^1(\Omega)$ by Lemma 2.6, so Lebesgue Theorem implies that

$$\lim_{h\to\infty} \int_\Omega [-b(x)g(u) + f(x)]T_{2k}(u - T_h(u)) = 0$$

and thus we can take limits when $h \to \infty$ in (3.11) to assure that

$$\lim_{n\to\infty} \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))]\nabla(T_k(u_n) - T_k(u)) = 0.$$

This allows us to apply Lemma 5 of [7] to conclude that

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega) \text{ for every } k > 0.$$

Step 4. u is the entropy solution of (P).

Let us take $T_k(u_n - \varphi)$ with $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $k > 0$ as test function in (3.1). Observe that if we define $L = k + \|\varphi\|_\infty$, then we have that $\nabla T_k(u_n - \varphi) = 0$ on the set $\{|u_n| > L\}$, so we can write

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \varphi) = \int_\Omega a(x, T_L(u_n), \nabla T_L(u_n))\nabla T_k(u_n - \varphi)$$

and thus (3.1) with this test function can be rewritten as

$$\begin{aligned} &\int_\Omega a(x, T_L(u_n), \nabla T_L(u_n))\nabla T_k(u_n - \varphi) \\ &\qquad + \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right]g(u_n)T_k(u_n - \varphi) = \int_\Omega f(x)T_k(u_n - \varphi). \end{aligned} \tag{3.12}$$

Since $T_L(u_n) \to T_L(u)$ strongly in $W_0^{1,p}(\Omega)$, then we have that $\nabla T_L(u_n) \to \nabla T_L(u)$ a.e. in Ω and, as a consequence of (1.2) and Lebesgue Theorem, we have that

$$a(x, T_L(u_n), \nabla T_L(u_n)) \to a(x, T_L(u), \nabla T_L(u)) \quad \text{in } L^{p'}(\Omega).$$

As we also have that $\nabla T_k(u_n - \varphi) \to \nabla T_k(u - \varphi)$ in $L^p(\Omega)$, we can assure that

$$\begin{aligned} \int_\Omega a(x, T_L(u_n), \nabla T_L(u_n))\nabla T_k(u_n - \varphi) &\to \int_\Omega a(x, T_L(u), \nabla T_L(u))\nabla T_k(u - \varphi) \\ &= \int_\Omega a(x, u, \nabla u)\nabla T_k(u - \varphi). \end{aligned}$$

If we use that $b(x)g(u_n) \to b(x)g(u)$ in $L^1(\Omega)$ by Lemma 2.6 and that $\frac{1}{n}|f(x)|g(u_n) \to 0$ in $L^1(\Omega)$ thanks to the (1.7) estimate, we can easily pass to the limit in (3.12) to obtain that

$$\int_\Omega a(x, u, \nabla u)\nabla T_k(u - \varphi) + \int_\Omega b(x)g(u)T_k(u - \varphi) = \int_\Omega f(x)T_k(u - \varphi),$$

so we can conclude that u is the entropy solution of (P). Finally, observe that due to the uniqueness of the entropy solution we can assert that the whole original sequence $\{u_n\}$ converges in measure to u. □

4 The semilinear case

In this section we prove the Theorem 1.2 and we give some additional remarks.

Proof of Theorem 1.2. First, let us show that the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Taking $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in (P_n) and in (P), we deduce that

$$\int_\Omega M(x)\nabla u_n \nabla u_n + \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right] g(u_n)u_n = \int_\Omega f(x)u_n$$
$$= \int_\Omega M(x)\nabla u \nabla u_n + \int_\Omega b(x)g(u)u_n.$$

Since $g(s)s \geq 0$ for every $s \in \mathbb{R}$ by (1.5), then the term $\int_\Omega \frac{1}{n}|f(x)|g(u_n)u_n$ is nonnegative and we can drop it to obtain that

$$\int_\Omega M(x)\nabla u_n \nabla u_n + \int_\Omega b(x)g(u_n)u_n \leq \int_\Omega M(x)\nabla u \nabla u_n + \int_\Omega b(x)g(u)u_n.$$

We can rewrite this expression as

$$\int_\Omega M(x)\nabla\left(u_n - \frac{u}{2}\right)\nabla\left(u_n - \frac{u}{2}\right) - \frac{1}{4}\int_\Omega M(x)\nabla u \nabla u$$
$$+ \int_\Omega b(x)[g(u_n) - g(u)](u_n - u) + \int_\Omega b(x)g(u_n)u - \int_\Omega b(x)g(u)u \leq 0.$$

Observe that we have used here the symmetry of the matrix $M(x)$ to obtain the identity $M(x)\nabla u_n \nabla u = M(x)\nabla u \nabla u_n$.

Now, as $b(x) \geq 0$ and g is increasing by (1.5), then the term $\int_\Omega b(x)[g(u_n) - g(u)](u_n - u)$ is nonnegative and we can drop it. If also we apply the ellipticity condition (1.8) of $M(x)$, we obtain that

$$\alpha \int_\Omega \left|\nabla\left(u_n - \frac{u}{2}\right)\right|^2 \leq \frac{1}{4}\int_\Omega M(x)\nabla u \nabla u - \int_\Omega b(x)g(u_n)u + \int_\Omega b(x)g(u)u. \tag{4.1}$$

Arguing as in the beginning of the proof of the Theorem 1.1, we can deduce that $\{u_n\}$ is bounded in some Marcinkiewicz space and thus we can apply Lemma 2.6 to assert that $\{b(x)g(u_n)\}$ is bounded in $L^1(\Omega)$. Thanks to this and to the fact that $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $b(x) \in L^1(\Omega)$, $M(x)$ is bounded by (1.9) and g is continuous, we can assure that the right hand side of (4.1) is bounded.

As a consequence, we obtain that $\{u_n - \frac{u}{2}\}$ is bounded in $H_0^1(\Omega)$ and, since $u \in H_0^1(\Omega)$, we can deduce that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Thanks to this bound there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and a function $v \in H_0^1(\Omega)$ such that $u_n \rightharpoonup v$ in $H_0^1(\Omega)$ and $u_n \to v$ a.e. in Ω.

Now, if we bear in mind that $b(x)g(u_n) \to b(x)g(v)$ in $L^1(\Omega)$ by Lemma 2.6 and that $\frac{1}{n}|f(x)g(u_n)| \leq |f(x)| \in L^1(\Omega)$ by (1.7) estimate, we can easily pass to the limit in

$$\int_\Omega M(x)\nabla u_n \nabla \varphi + \int_\Omega \left[b(x) + \tfrac{1}{n}|f(x)|\right] g(u_n)\varphi = \int_\Omega f(x)\varphi, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

to obtain that

$$\int_\Omega M(x)\nabla v \nabla \varphi + \int_\Omega b(x)g(v)\varphi = \int_\Omega f(x)\varphi, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

and thus it is proven that $v = u$, i.e., that v is the weak solution of (P). Moreover, due to the uniqueness of the solution u we can affirm that the whole original sequence $\{u_n\}$ converges weakly in $H_0^1(\Omega)$ to u. □

Observe that if we take $b(x) = 0$ in (P), then the assumption $u \in L^\infty(\Omega)$ is not necessary in the proof of this theorem. This allows us to state the following result.

Theorem 4.1. *Suppose that $a(x,s,\xi) = M(x)\xi$ with $M(x)$ a symmetric matrix satisfying (1.8) and (1.9). Assume also that $b(x) = 0$, that $f(x)$ verifies (1.4) and that g satisfies (1.5). If the weak solution $u \in H_0^1(\Omega)$ of (P) exists, then $\{u_n\}$, the sequence of weak solutions of (P_n) given by Theorem 2.1, verifies that*

$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega).$$

To end this paper, we state a remark related with the case in which f is a nonnegative function.

Remark 4.2. If $f \geq 0$ the proofs are easier and stronger results can be proven. This is mainly due to two facts: $\{u_n\}$ is nonnegative and increasing. The monotony of $\{u_n\}$ assures the existence of its a.e. limit and, by Theorem 1.1, this a.e. limit must be u, the entropy solution of (P).

Observe that this implies that $u_n \leq u$ a.e. in Ω for every $n \in \mathbb{N}$ and thus the assumption $u \in L^\infty(\Omega)$ on Theorem 1.2 implies that $\{u_n\}$ is bounded in $L^\infty(\Omega)$. This allows us not only to prove that theorem in simpler way, but also to show that

$$u_n \to u \quad \text{in } H_0^1(\Omega).$$

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Properties of Poincaré half-maps for planar linear systems and some direct applications to periodic orbits of piecewise systems

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Abstract. This paper deals with fundamental properties of Poincaré half-maps defined on a straight line for planar linear systems. Concretely, we focus on the analyticity of the Poincaré half-maps, their series expansions (Taylor and Newton–Puiseux) at the tangency point and at infinity, the relative position between the graph of Poincaré half-maps and the bisector of the fourth quadrant, and the sign of their second derivatives. All these properties are essential to understand the dynamic behavior of planar piecewise linear systems. Accordingly, we also provide some of their most immediate, but non-trivial, consequences regarding periodic orbits.

Keywords: piecewise planar linear systems, Poincaré half-maps, Taylor series expansion, Newton–Puiseux series expansion.

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1 Introduction

The study of the qualitative properties of distinguished solutions of piecewise linear differential systems rests mainly on the analysis of the features of Poincaré maps, which are defined as composition of transition maps between the separation manifolds. Sometimes these transition maps are called Poincaré half-maps. The linearity of the system in each zone invites to its integration, which automatically causes the emergence of a wide range of cases due to the nature

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of the different spectra of the matrices of the linear systems and the relative position between the equilibria, if any, and the separation manifolds. The number of cases to be studied is high even for planar systems with two zones of linearity. Moreover, the direct integration leads to different nonlinear equations where the flight time appears as a new variable.

Since the publication of the seminal work by Freire et al. [9], a large number of interesting papers have appeared in order to establish the dynamical behavior in planar piecewise linear systems with two zones of linearity and, in particular, to give conditions for the existence and stability of limit cycles and to provide an optimal bound for the number of coexisting limit cycles (see for instance, [11, 15–21]). None of these papers considers all the possible cases. Moreover, they are forced to use individualized approaches to study the different kind of functions that arise due to the distinct spectra of the matrices. This causes that a same result is usually expressed in different terms and, sometimes, it may be a hard task to obtain a common and brief statement for it. Thus, the use of individualized techniques for each case does not allow a unified view of several properties of the Poincaré half-maps and, when it does, more effort is required to complete the case-by-case study and to achieve independent statements of these cases.

This paper relies on a novel characterization of Poincaré half-maps for planar linear systems [2] which allows us to see the properties of these maps from a common point of view and to prove the results in a simple way, without the need of making particularized case-by-case studies. Accordingly, we will not have any of the disadvantages mentioned in the previous paragraphs because this novel characterization does not require integration of the systems and, therefore, the distinction of the spectra of the matrices is not needed. The strength of this approach can be seen in [3], where the uniqueness of limit cycles for continuous piecewise linear systems was provided in a simple and synthesized way.

In the framework of the study of Poincaré half-maps for planar linear systems, the most relevant properties are those related to the local behavior at tangency points between the flow and the Poincaré section, the behavior at infinity (obviously, in the case of the focus or center), and the sign of the derivatives. Some of these properties have been proven just for concrete cases. Even those which are valid for all situations have been proven in a large case-by-case study. This manuscript is primarily devoted to simplifying and unifying the proofs of these properties by considering all possible scenarios simultaneously. In addition, it will be stated an interesting fact about the relative position between the graphs of Poincaré half-maps and the bisector of the fourth quadrant. Among other things, from this property it is direct that Poincaré half-maps inherit the expansion/compression behavior of the flow of the planar linear system. Additionally, it is proven here that this relative position is also related to the (constant) sign of its second derivative.

As might be expected from the first two paragraphs of this introduction, all the previously commented properties have direct applications to planar piecewise linear systems; from the analysis of stability and bifurcations of equilibria, singularities, or the infinity, to the existence and characterization of periodic orbits and the obtention of uniform bounds to the number of limit cycles. In this work, some straightforward conclusions concerning the periodic behavior are obtained.

The paper is organized as follows. Section 2 presents the integral characterization of Poincaré half-maps for planar linear systems given in [2]. Two basic consequences of this characterization for the Poincaré half-maps are their analyticity and their understanding as solutions of a differential equation. In Section 3, we summarize the results on analyticity of the Poincaré half-maps given in [2] and obtain the Taylor and Newton–Puiseux series expansions at tangency points and infinity by means of the differential equation. Section 4 studies the relationship between the graphs of Poincaré half-maps and the bisector of the fourth quadrant, which is used to establish, in Section 5, the sign of the second derivatives of the Poincaré half-maps. Finally, Section 6 addresses the analysis of the periodic behavior of planar piecewise linear systems with two zones separated by a straight line. There, some direct consequences of the properties of Poincaré half-maps obtained in previous sections are stated.

2 Integral characterization for the Poincaré half-maps

Let us consider, for $\mathbf{x} = (x_1, x_2)^T$, the autonomous linear system

$$\dot{\mathbf{x}} = A\,\mathbf{x} + \mathbf{b} \tag{2.1}$$

where $A = (a_{ij})_{i,j=1,2}$ is a real matrix and $\mathbf{b} = (b_1, b_2)^T \in \mathbb{R}^2$. Let us choose, without loss of generality, the Poincaré section $\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$.

Notice that if the coefficient a_{12} vanishes, system (2.1) is uncoupled and a Poincaré half-map on section Σ cannot be defined. Hence, let us assume in this work that $a_{12} \neq 0$ (observability condition [6]). On the one hand, observe that, among other configurations, this condition removes the possibility of star-nodes to appear. On the other hand, under the assumption $a_{12} \neq 0$, the linear change of variable $x = x_1$, $y = a_{22}x_1 - a_{12}x_2 - b_1$, with $a = a_{12}b_2 - a_{22}b_1$, allows to write system (2.1) into the generalized Liénard form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} T & -1 \\ D & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix}, \tag{2.2}$$

where T and D stand for the trace and the determinant of matrix A, respectively. In the new coordinates, since $x_1 = x$, Poincaré section Σ remains the same.

The first equation of system (2.2) evaluated on the section Σ is reduced to $\dot{x}|_\Sigma = -y$. Therefore, the flow of the system crosses Σ from the half-plane $\Sigma^+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ to the half-plane $\Sigma^- = \{(x, y) \in \mathbb{R}^2 : x < 0\}$ when $y > 0$, from Σ^- to Σ^+ when $y < 0$, and it is tangent to Σ at the origin.

Since this work is devoted to Poincaré half-maps of system (2.2) corresponding to the section Σ and due to the fact that there is no possible return to section Σ when $a = D = 0$, we assume that $a^2 + D^2 \neq 0$ throughout this work. Note that this condition avoids the existence of a continuum of equilibria.

We are going to focus on the left Poincaré half-map (the one defined by the flow in the closed half-plane $\Sigma^- \cup \Sigma$ and the intersection points of its orbits with the Poincaré section Σ). Notice that the definition of the right Poincaré half-map and their corresponding results may be immediately obtained by the invariance of system (2.2) under the change $(x, y, a) \leftrightarrow (-x, -y, -a)$.

The left Poincaré half-map is usually defined in the following way. Let us consider $(0, y_0) \in \Sigma$ with $y_0 \geqslant 0$ and let $\Phi(t; y_0) = (\Phi_1(t; y_0), \Phi_2(t; y_0))$ the solution of system (2.2) that satisfies the initial condition $\Phi(0; y_0) = (0, y_0)$. The existence of a value $\tau(y_0) > 0$ such that $\Phi_1(\tau(y_0); y_0) = 0$ and $\Phi_1(t; y_0) < 0$ for every $t \in (0, \tau(y_0))$ allows to define the image of y_0 by the left Poincaré half-map as $P(y_0) = \Phi_2(\tau(y_0); y_0) \leqslant 0$. Moreover, the value $\tau(y_0)$ is called the left flight time.

Regarding the definition of the left Poincaré half-map at the origin, $P(0)$ cannot be defined as above when for every $\tau > 0$ there exists $t \in (0, \tau)$ such that $\Phi_1(t; 0) > 0$. However, it can be continuously extended as $P(0) = 0$ provided that for every $\varepsilon > 0$ there exist $y_0 \in (0, \varepsilon)$ and $y_1 \in (-\varepsilon, 0)$ such that $P(y_0) = y_1$. This finishes the usual definition of the left Poincaré half-map.

According to the above definition, it is natural to compute the flow of the system by means of explicit integrations. This leads to many case-by-case studies and forces the nonlinear appearance of the flight time. Here, we will use a characterization that avoids the computation of the flow, as it is done in [2]. For the sake of completeness, we give a brief summary of the main results and ideas of [2] that are going to be used in this paper.

The left Poincaré half-map P and its definition interval I are given in Theorem 19 and Corollary 21 of [2]. By using the quadratic polynomial function

$$W(y) = Dy^2 - aTy + a^2, \tag{2.3}$$

the *left Poincaré half-map* is the unique function $P : I \subset [0, +\infty) \longrightarrow (-\infty, 0]$ that, for every $y_0 \in I$, satisfies

$$\mathrm{PV}\left\{ \int_{P(y_0)}^{y_0} \frac{-y}{W(y)} dy \right\} = cT, \tag{2.4}$$

where c is given, in terms of the parameters, as follows: (i) $c = 0$ if $a > 0$, (ii) $c = \pi \left(D\sqrt{4D - T^2} \right)^{-1} \in \mathbb{R}$ if $a = 0$, and (iii) $c = 2\pi \left(D\sqrt{4D - T^2} \right)^{-1} \in \mathbb{R}$ if $a < 0$. Here, $\mathrm{PV}\{\cdot\}$ stands for the *Cauchy Principal Value* at the origin (see, for instance, [14]), which is defined as

$$\mathrm{PV}\left\{ \int_{y_1}^{y_0} \frac{-y}{W(y)} dy \right\} = \lim_{\varepsilon \searrow 0} \left(\int_{y_1}^{-\varepsilon} \frac{-y}{W(y)} dy + \int_{\varepsilon}^{y_0} \frac{-y}{W(y)} dy \right),$$

for $y_1 < 0 < y_0$.

As emphasized in [2], the interval I is essentially related with the roots of the quadratic polynomial function W. In the next remark, we shall briefly comment some of those relationships and other interesting properties of P which are proven in [2].

Remark 2.1. System (2.2), under the assumed condition $a^2 + D^2 \neq 0$, has invariant straight lines for several values of the parameters. These straight lines are either the invariant eigenspaces of equilibria (saddles, degenerate nodes or non-degenerate nodes) or the straight line $y = Tx + a/T$ in the case $T \neq 0$, $D = 0$ (what implies $a \neq 0$). In those cases, every invariant straight line intersects the Poincaré section Σ in a point $(0, \mu)$, where μ is a root of the quadratic polynomial function W given in (2.3). Moreover, when $I \subset [0, +\infty)$ is bounded, then the right endpoint of I is a real root of W and, in the same way, if $P(I)$ is bounded, then the left endpoint of $P(I)$ is also a real root of W. In Fig. 2.1 (a) and Fig. 2.1(b), we show two examples of bounded intervals I and/or $P(I)$, corresponding respectively to saddle and non-degenerate node configurations.

The interval I can be unbounded. For instance, if $4D - T^2 > 0$, then the equilibrium point of system (2.2) is a focus or a center, the intervals I and $P(I)$ are unbounded, and, obviously, $P(y_0)$ tends to $-\infty$ as $y_0 \to +\infty$. In this case, the intervals are $I = [0, +\infty)$ and $P(I) = (-\infty, 0]$, except when the equilibrium is a focus (i.e. $T \neq 0$) and it is located in the left half-plane $\{(x, y) \in \mathbb{R}^2 : x < 0\}$ (i.e. $a < 0$). In fact, when $T > 0$, the interval $P(I)$ is reduced to $(-\infty, \hat{y}_1]$, where $\hat{y}_1 = P(0)$ (see Fig. 2.1 (c)). Analogously, for $T < 0$, $I = [\hat{y}_0, +\infty)$ with $\hat{y}_0 = P^{-1}(0)$ (see Fig. 2.1 (d)).

Finally, the polynomial function W is strictly positive in each set $[P(y_0), 0) \cup (0, y_0]$, with $y_0 \in I$. Besides that, since $W(0) = a^2$, then $W(0) > 0$ for $a \neq 0$ and $W(0) = 0$ for $a = 0$.

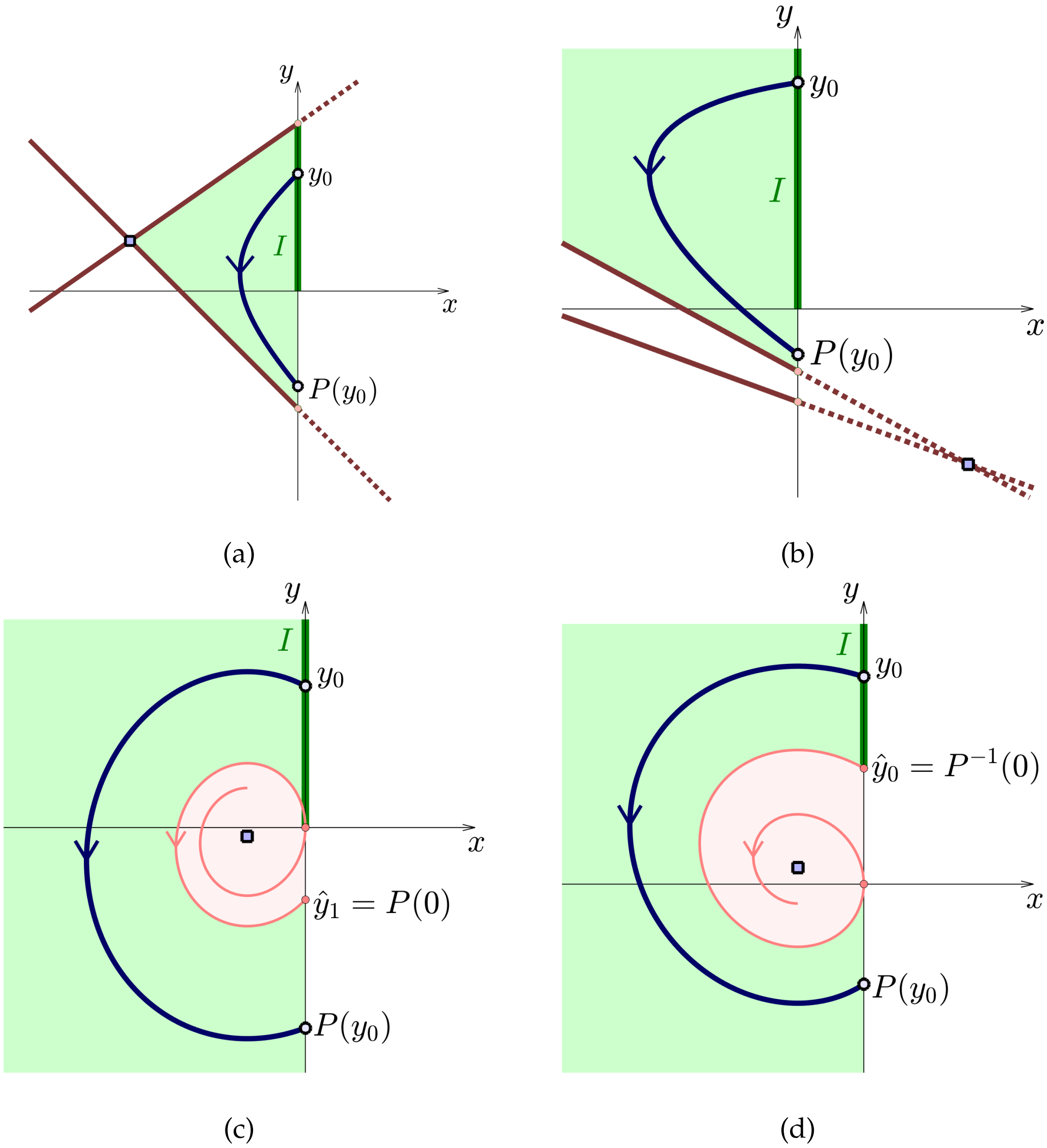

Figure 2.1: The left Poincaré half-map P and its interval of definition I for the cases: (a) saddle, (b) non-degenerate node, (c) unstable focus, and (d) stable focus.

It is worth mentioning that the integral given in (2.4) diverges when $a = 0$ and the Cauchy principal value is necessary to overcome this difficulty. Moreover, in this case, for $y_1 < 0 < y_0$, the Cauchy principal value at the origin is given by

$$\mathrm{PV}\left\{\int_{y_1}^{y_0} \frac{-y}{Dy^2}dy\right\} = \lim_{\varepsilon \searrow 0}\left(\int_{y_1}^{-\varepsilon} \frac{-y}{Dy^2}dy + \int_{\varepsilon}^{y_0} \frac{-y}{Dy^2}dy\right) = \frac{1}{D}\log\left|\frac{y_1}{y_0}\right|. \tag{2.5}$$

When $a \neq 0$, the integrating function $h(y) = -y/W(y)$ is continuous and, consequently, the Cauchy principal value just takes the value of the integral.

3 Analiticity and series expansions of Poincaré half-maps at the tangency point and its preimage, and at infinity

In this section, by means of the integral characterization and a subsequent differential equation, we shall compute the first coefficients of the Taylor expansion of the left Poincaré half-map P. Obviously, the used method does not depend on the spectrum of the matrix of the system. Before obtaining these coefficients it is necessary to determine the analyticity of the left Poincaré half-map P.

When $P(y_0) \neq 0$, it is well-known (see, for example, [7]) that the transversality between the flow of the system and the separation line Σ ensures the analyticity of P at y_0. The analyticity for the tangency point between the flow and Σ (that is, the origin) is more intricate and, in the literature, it has been approached with a case-by-case study (see some partial results at [8]). However, as follows from Corollary 24 of [2], the maps P and P^{-1} are real analytic functions in the open intervals $\text{int}(I)$ and $P(\text{int}(I))$, respectively, and at least one of the following statements is true:

(i) the map P is a real analytic function at the left endpoint of its domain,

(ii) the map P^{-1} is a real analytic function at the right endpoint of its domain.

When the equilibrium of system (2.2) is a center or a focus, the left Poincaré half-map P can be considered also at infinity. In addition, we shall obtain the first coefficients of the Taylor expansion of P around the infinity.

A first consequence from the definition of the left Poincaré half-map given in the integral form (2.4) is easily deduced by computing the derivative with respect to variable y_0 (see Remark 16 of [2]). Hence, one can see that the graph of the left Poincaré half-map P and its inverse function P^{-1}, oriented according to increasing y_0, are particular orbits of the cubic vector field

$$\begin{aligned} X(y_0, y_1) &= -\left(y_1 W(y_0), y_0 W(y_1)\right) \\ &= -\left(y_1\left(Dy_0^2 - aTy_0 + a^2\right), y_0\left(Dy_1^2 - aTy_1 + a^2\right)\right). \end{aligned}$$

In fact, the left Poincaré half-map P and its inverse function P^{-1} are solutions of the differential equation

$$y_1 W(y_0) dy_1 - y_0 W(y_1) dy_0 = 0. \tag{3.1}$$

The next proposition is a direct consequence of the results in [2] and allows to obtain the Taylor expansion of P around the origin when $a \neq 0$ and $P(0) = 0$. Notice that for $a = 0$, the existence of the left Poincaré half-map P implies $4D - T^2 > 0$. From Remark 2.1, the interval of definition of P is $I = [0, +\infty)$ and, for $y_0 \geqslant 0$, expression (2.4) can be written as

$$\text{PV}\left\{\int_{P(y_0)}^{y_0} \frac{-y}{Dy^2} dy\right\} = \frac{\pi T}{D\sqrt{4D - T^2}}.$$

Hence, by using the value for PV given in (2.5), the left Poincaré half-map P is given by

$$P(y_0) = -\exp\left(\frac{\pi T}{\sqrt{4D - T^2}}\right) y_0, \quad \text{for } y_0 \geqslant 0. \tag{3.2}$$

When $a \neq 0$, by denoting $\mathcal{I} = \{y \in \mathbb{R} : W(y) > 0\}$, from Theorem 14 of [2], it is deduced that the set

$$\mathcal{C}_0 = \left\{ (y_1, y_0) \in \mathcal{I}^2 : \int_{y_1}^{y_0} -y/W(y)dy = 0 \right\}$$

can be written in the form

$$\mathcal{C}_0 = \left\{ (y_1, y_0) \in \mathcal{I}^2 : (y_1 - y_0)(y_1 - \varphi_0(y_0) \right\},$$

where φ_0 is a real analytic function in $\mathcal{I}$ which is also an involution, that is, $(\varphi_0(\varphi_0(y_0))) = y_0$ for all $y_0 \in \mathcal{I}$. Now, by means of Corollary 21 of [2], it follows that the left Poincaré half-map P coincides with the function φ_0 restricted to the interval $\mathcal{I} \cap [0, \infty]$, provided $0 \in I$ and $P(0) = 0$. By an abuse of notation, we say that the Poincaré half-map P is an involution when $a \neq 0$, $0 \in I$, and $P(0) = 0$.

Proposition 3.1. *Assume that $a \neq 0$ and $0 \in I$. If $P(0) = 0$, then left Poincaré half-map P is a real analytic function in I, it is an involution and its Taylor expansion around the origin writes as*

$$\begin{aligned} P(y_0) = -y_0 - \frac{2Ty_0^2}{3a} - \frac{4T^2y_0^3}{9a^2} + \frac{2\left(9DT - 22T^3\right)y_0^4}{135a^3} \\ + \frac{4\left(27DT^2 - 26T^4\right)y_0^5}{405a^4} - \frac{2\left(27D^2T - 176DT^3 + 100T^5\right)y_0^6}{945a^5} + \mathcal{O}\left(y_0^7\right). \end{aligned} \tag{3.3}$$

Proof. From the hypotheses of the proposition and by means of Theorem 14 and Corollary 24 of [2], it is deduced that left Poincaré half-map P is a real analytic function in I and it is an involution. Hence, the derivative of P at the origin is $P'(0) = -1$.

Now, taking into account that P is a solution of the differential equation given in (3.1), it is easy to obtain, via undetermined coefficients, the Taylor expansion given in (3.3) and so the proof is concluded. □

Notice that the Taylor expansion around the origin given in Proposition 3.1 was already obtained in [23]. Although the calculations are not fully detailed in that work, the authors rely on the results given in [8], where the study requires different techniques depending on the situations. Before [23], the same series expansion was obtained in [10], by means of an inversion of the flight time, but only for the focus case.

When $0 \in I$ and $P(0) \neq 0$, the function P is a real analytic function at the origin and it is possible to obtain its Taylor expansion of P around the origin.

Proposition 3.2. *Assume that $0 \in I$. If $P(0) = \hat{y}_1 < 0$, then $a < 0, T > 0, 4D - T^2 > 0$, $\hat{y}_1$ is the right endpoint of the interval $P(I)$, and the left Poincaré half-map P is a real analytic function in I and its Taylor expansion around the origin writes as*

$$\begin{aligned} P(y_0) = \hat{y}_1 + \frac{W(\hat{y}_1)y_0^2}{2a^2\hat{y}_1} + \frac{TW(\hat{y}_1)y_0^3}{3a^3\hat{y}_1} - \frac{\left(a^2 + \left(D - 2T^2\right)\hat{y}_1^2\right)W(\hat{y}_1)y_0^4}{8a^4\hat{y}_1^3} \\ - \frac{T\left(5a^2 + \left(7D - 6T^2\right)\hat{y}_1^2 W(\hat{y}_1)\right)y_0^5}{30a^5\hat{y}_1^3} \\ + \frac{\left(9a^4 - 6a^3T\hat{y}_1 + 2a^2\left(9D - 13T^2\right)\hat{y}_1^2 + \left(9D^2 - 46DT^2 + 24T^4\right)\hat{y}_1^4\right)W(\hat{y}_1)y_0^6}{144a^6\hat{y}_1^5} \\ + \mathcal{O}\left(y_0^7\right). \end{aligned} \tag{3.4}$$

Proof. The expression given in (3.2) provides the left Poincaré half-map for the case $a = 0$. From there, one obtains $P(0) = 0$ when $a = 0$.

Suppose that $0 \in I$ and $P(0) = \hat{y}_1 < 0$. Then $a \neq 0$ and expression (2.4) leads us to

$$\int_{\hat{y}_1}^{0} \frac{-y}{W(y)} dy = cT.$$

From Remark 2.1, the polynomial W is strictly positive and, therefore, the left-hand term of the last expression is also strictly positive. If $a > 0$, from expression (2.4), $c = 0$ and this is impossible. Thus, it is deduced that $a < 0, T > 0$, and $4D - T^2 > 0$. Now, from Remark 2.1 again, the intervals I and $P(I)$ are unbounded and $P(y_0)$ tends to $-\infty$ as $y_0 \to +\infty$.

Next, let us prove that $\hat{y}_1$ is the right endpoint of the interval I. Let us consider $y_0 \geqslant 0$ and $y_1 \in (\hat{y}_1, 0]$. From the inequalities

$$\int_{y_1}^{y_0} \frac{-y}{W(y)} dy < \int_{\hat{y}_1}^{y_0} \frac{-y}{W(y)} dy \leqslant \int_{\hat{y}_1}^{0} \frac{-y}{W(y)} dy = cT$$

one can see that no point in the interval $y_1 \in (\hat{y}_1, 0]$ belongs to the interval $P(I)$ and so the right endpoint of interval $P(I)$ is $\hat{y}_1$.

The analyticity of P is a direct consequence of Theorem 14 and Corollary 21 of [2] and the Taylor expansion around the origin given in (3.4) follows from the method of undetermined coefficients applied to the differential equation (3.1). □

Remark 3.3. Note that the condition $P(0) = \hat{y}_1 < 0$ together with the linearity of the system implies that there exists an unstable focus equilibrium in the left half-plane Σ^- (see Fig. 2.1(c)) and so it is immediate that $a < 0, T > 0$, and $4D - T^2 > 0$. This is an alternative proof for the inequalities of Proposition 3.2. On the other hand, the endpoints of intervals I and $P(I)$ were also determined in Corollary 21 of [2] in a more generic way. For the sake of completeness, in the previous proof, we have included a different and specific reasoning for this case.

When there exists a point $\hat{y}_0 > 0$ such that $P(\hat{y}_0) = 0$, then left Poincaré half-map P is a non-analytic function at $\hat{y}_0$. However, in [2] it is proven that the inverse function P^{-1} is analytic at the origin and so it is possible, by means of an inversion, to get a Newton–Puiseux series expansion for the left Poincaré half-map P around $\hat{y}_0$. Some results about series inversion and Newton–Puiseux series can be found in [12] and the references therein. Also of interest are the results included in [1] concerning the expression of the solutions of differential equations as Newton–Puiseux series expansion and its convergence.

Proposition 3.4. *Assume that there exists a value $\hat{y}_0 > 0$ with $P(\hat{y}_0) = 0$. Then, $a < 0, T < 0$, $4D - T^2 > 0$, $\hat{y}_0$ is the left endpoint of the interval I, the inverse function P^{-1} is a real analytic function, and the left Poincaré half-map P admits the Newton–Puiseux serie expansion around the point $\hat{y}_0$ given by*

$$\begin{aligned} P(y_0) = {} & a\sqrt{\frac{2\hat{y}_0}{W(\hat{y}_0)}}(y_0 - \hat{y}_0)^{1/2} - \frac{aT}{3}\frac{2\hat{y}_0}{W(\hat{y}_0)}(y_0 - \hat{y}_0) \\ & + \frac{a^3}{72}\left(\frac{9D + 2T^2}{a^2} + \frac{9}{\hat{y}_0^2}\right)\left(\sqrt{\frac{2\hat{y}_0}{W(\hat{y}_0)}}\right)^3 (y_0 - \hat{y}_0)^{3/2} + \mathcal{O}\left((y_0 - \hat{y}_0)^2\right), \end{aligned} \tag{3.5}$$

which is valid for $y_0 \geqslant \hat{y}_0$.

Proof. Suppose that there exists a point $\hat{y}_0 > 0$ such that $P(\hat{y}_0) = 0$. An analogous reasoning to the first part of the proof of Proposition 3.2 leads to the inequalities $a < 0, T < 0$, and $4D - T^2 > 0$ and to the fact that $\hat{y}_0$ is the left endpoint of the interval I.

The inverse function P^{-1} satisfies $P^{-1}(0) = \hat{y}_0$ and, from differential equation (3.1), it follows that its derivative at the origin vanishes. This implies that P is a non-analytic function at $\hat{y}_0$. From Theorem 14 and Corollary 21 of [2], it follows that the inverse function P^{-1} is an analytic function at the origin and P^{-1} admits the Taylor expansion (3.4) by changing $\hat{y}_1$ by $\hat{y}_0$.

Now, the Newton–Puiseux series expansion of P is obtained by the inversion of the Taylor expansion of P^{-1}. Note that the direct inversion provides two possible series expansions but, since $P(y_0) \leqslant 0$ for all $y_0 \in I$, the valid one is that given in (3.5) and the proof is finished. □

An analogous comment to Remark 3.3 can be made about the inequalities of Proposition 3.4 and the left endpoint of I. The scenario described by the hypothesis stated in Proposition 3.4 is illustrated in Fig. 2.1(d).

Remark 3.5. The inversion used to obtain the Newton–Puiseux series expansion of P is equivalent to the computation of the Taylor expansion of $Q(z_0) := P(\hat{y}_0 + z_0^2)$ around $z_0 = 0$ and the subsequent change of z_0 by $\sqrt{y_0 - \hat{y}_0}$. In order to get this Taylor expansion it is enough to make the change of variable $y_0 \to \hat{y}_0 + z_0^2$ in the differential equation (3.1) to achieve a differential equation for the function Q.

Let us recall from Remark 2.1 that when $4D - T^2 > 0$ the domain I is unbounded with $P(y_0)$ tending to $-\infty$ as $y_0 \to +\infty$. Thus, the study of the left Poincaré half-map around the infinity is feasible. In fact, the first two terms of the Taylor expansion of left Poincaré half-map P around the infinity were already obtained in [9] by means of an expression of P parameterized by the flight time. In the following proposition, we present a simple method to get these and others terms.

Proposition 3.6. *Assume that $4D - T^2 > 0$.Then, the Taylor expansion of left Poincaré half-map P around the infinity writes as*

$$\begin{aligned} P(y_0) = &-\exp\left(\frac{\pi T}{\sqrt{4D-T^2}}\right) y_0 + \frac{aT}{D}\left(1+\exp\left(\frac{\pi T}{\sqrt{4D-T^2}}\right)\right) \\ &- \frac{a^2}{D}\sinh\left(\frac{\pi T}{\sqrt{4D-T^2}}\right)\cdot\frac{1}{y_0} \\ &- \frac{a^3 e^{-\frac{2\pi T}{\sqrt{4D-T^2}}}\left(-2+e^{\frac{\pi T}{\sqrt{4D-T^2}}}\right)\left(1+e^{\frac{\pi T}{\sqrt{4D-T^2}}}\right)^2 T}{6D^2}\cdot\frac{1}{y_0^2} + \mathcal{O}\left(\frac{1}{y_0^3}\right). \end{aligned}$$

Proof. Firstly, we shall prove the equality

$$\lim_{y_0 \to +\infty} \frac{P(y_0)}{y_0} = -\exp\left(\frac{\pi T}{\sqrt{4D-T^2}}\right). \tag{3.6}$$

If $a = 0$, then expression (3.2) leads us directly to equality (3.6).

If $a \neq 0$, taking into account that $W(y) > 0$ for $y \in \mathbb{R}$ (see Remark 2.1), then relationship (2.4) can be written as

$$\int_{P(y_0)}^{-y_0} \frac{-y}{W(y)}dy + \int_{-y_0}^{y_0} \frac{-y}{W(y)}dy = cT, \tag{3.7}$$

for $y_0 \in I$, being $c = 0$ for $a > 0$ and $c = 2\pi\left(D\sqrt{4D - T^2}\right)^{-1}$ for $a < 0$.

The change of variable $Y = 1/y$ applied to the first integral in expression (3.7) transforms it into

$$\int_{1/P(y_0)}^{-1/y_0} \frac{1}{a^2Y^3 - aTY^2 + DY} dY + \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy = cT$$

or, equivalently, into the expression

$$\int_{1/P(y_0)}^{-1/y_0} \frac{1}{DY} dY + \int_{1/P(y_0)}^{-1/y_0} \frac{a(T - aY)}{D\left(a^2Y^2 - aTY + D\right)} dY + \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy = cT.$$

That is,

$$\frac{P(y_0)}{y_0} = -\exp\left(DcT - D\int_{-y_0}^{y_0} \frac{-y}{W(y)} dy - \int_{1/P(y_0)}^{-1/y_0} \frac{a(T - aY)}{a^2Y^2 - aTY + D} dY\right).$$

Now, a direct integration provides

$$\lim_{y_0 \to +\infty} \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy = -\frac{\pi T \operatorname{sign}(a)}{D\sqrt{4D - T^2}} \tag{3.8}$$

and taking into account that

$$\lim_{y_0 \to +\infty} \int_{1/P(y_0)}^{-1/y_0} \frac{a(T - aY)}{a^2Y^2 - aTY + D} dY = 0,$$

the equality (3.6) follows.

Thus, the function $\tilde{P}$, defined by

$$\tilde{P}(Y_0) = \begin{cases} \dfrac{1}{P(1/Y_0)} & \text{if } Y_0 \neq 0 \text{ and } 1/Y_0 \in I, \\ 0 & \text{if } Y_0 = 0, \end{cases}$$

has derivative on the right at the origin and its value is

$$\alpha_1 := \frac{d\tilde{P}}{dY_0}\left(0^+\right) = -\exp\left(\frac{-\pi T}{\sqrt{4D - T^2}}\right). \tag{3.9}$$

Moreover, it is immediate to see that the function $\tilde{P}$ is a solution of differential equation

$$\left(a^2Y_0^2 - aTY_0 + D\right)Y_0 dY_1 - \left(a^2Y_1^2 - aTY_1 + D\right)Y_1 dY_0 = 0,$$

obtained from the differential equation (3.1) by means of the change of variables $(Y_0, Y_1) = (1/y_0, 1/y_1)$ (defined for $y_0 y_1 \neq 0$). From here, it is deduced that the function $\tilde{P}$ has derivatives on the right of all orders at $Y_0 = 0$ and, after a direct computation, one finds

$$\alpha_2 := \frac{d^2\tilde{P}}{dY_0^2}\left(0^+\right) = -\frac{2aT}{D} e^{-\frac{2\pi T}{\sqrt{4D-T^2}}} \left(e^{\frac{\pi T}{\sqrt{4D-T^2}}} + 1\right),$$

$$\alpha_3 := \frac{d^3\tilde{P}}{dY_0^3}\left(0^+\right) = \frac{3a^2}{D^2} e^{-\frac{3\pi T}{\sqrt{4D-T^2}}} \left(e^{\frac{\pi T}{\sqrt{4D-T^2}}} + 1\right)\left(-2T^2 e^{\frac{\pi T}{\sqrt{4D-T^2}}} + De^{\frac{\pi T}{\sqrt{4D-T^2}}} - D - 2T^2\right),$$

and

$$\alpha_4 := \frac{d^4\tilde{P}}{dY_0^4}(0^+)$$
$$= \frac{4a^3T}{D^3}e^{-\frac{4\pi T}{\sqrt{4D-T^2}}}\left(1+e^{\frac{\pi T}{\sqrt{4D-T^2}}}\right)^2\left(-8D+7De^{\frac{\pi T}{\sqrt{4D-T^2}}}-6T^2-6T^2e^{\frac{\pi T}{\sqrt{4D-T^2}}}\right). \tag{3.10}$$

Since the Taylor expansion of left Poincaré half-map P around the infinity is given by

$$P(y_0) = \frac{1}{\alpha_1}y_0 - \frac{\alpha_2}{2\alpha_1^2} + \frac{3\alpha_2^2-2\alpha_1\alpha_3}{12\alpha_1^3}\cdot\frac{1}{y_0} - \frac{3\alpha_2^3-4\alpha_1\alpha_2\alpha_3+\alpha_1^2\alpha_4}{24\alpha_1^4}\cdot\frac{1}{y_0^2}+\mathcal{O}\left(\frac{1}{y_0^3}\right), \tag{3.11}$$

the proof concludes by substituting expressions (3.9)–(3.10) into (3.11). □

4 The relative position between the graph of Poincaré half-maps and the bisector of the fourth quadrant

To study the relative position between the graph of the left Poincaré half-map and the bisector of the fourth quadrant, it is natural to analyze the sign of the difference $y_0 - (-P(y_0))$. In the next proposition, we show the relationship between this difference and the trace T. Notice that this relationship has been addressed via a case-by-case treatment (by distinguishing the spectrum of the matrix of the system) in the main results of chapter 4 of [22]. Here, we provide a concise proof by using the integral characterization of the left Poincaré half-map.

Proposition 4.1. *The left Poincaré half-map P satisfies the relationship*

$$\text{sign}\,(y_0 + P(y_0)) = -\text{sign}(T) \quad \textit{for } y_0 \in I\setminus\{0\}.$$

In addition, when $0 \in I$ and $P(0) \neq 0$ or when $T = 0$, the relationship also holds for $y_0 = 0$.

Proof. We will prove this proposition by distinguishing the cases $T = 0$ and $T \neq 0$.

For $T = 0$, the integral equation given in (2.4) is reduced to

$$PV\left\{\int_{P(y_0)}^{y_0}\frac{-y}{Dy^2+a^2}dy\right\} = 0, \quad \text{for } y_0 \in I.$$

By taking into account that the integrating function is an odd function, it is direct to see that $P(y_0) = -y_0$ for all $y_0 \in I$ and so the proposition is true for $T = 0$.

Now, we focus on the proof for the case $T \neq 0$ and we will consider the situations $a = 0$ and $a \neq 0$.

When $a = 0$, the left Poincaré half-map P is given by expression (3.2) and so the equality $\text{sign}\,(y_0 + P(y_0)) = -\text{sign}(T)$ holds for every $y_0 \in I$.

When $a \neq 0$, let us consider the interval

$$J = \{u \in \mathbb{R} : W(y) > 0, \qquad \forall y \in [-|u|, |u|]\}$$

and function $g : J \longrightarrow \mathbb{R}$ defined by

$$g(u) = \int_{-u}^{u}\frac{-y}{W(y)}dy,$$

where W is the polynomial function defined in (2.3).

Notice that function g satisfies $g(0) = 0$, its derivative is

$$g'(u) = \frac{-2aTu^2}{W(u)W(-u)}$$

and so $\operatorname{sign}(g'(u)) = -\operatorname{sign}(aT)$ for every $u \in J \setminus \{0\}$. Thus, $\operatorname{sign}(g(u)) = -\operatorname{sign}(aT)$ for every $u \in J \cap (0, +\infty)$ and $\operatorname{sign}(g(u)) = \operatorname{sign}(aT)$ for every $u \in J \cap (-\infty, 0)$.

Moreover, if $J = \mathbb{R}$ (i.e., when $4D - T^2 > 0$), then, from (3.8),

$$\lim_{u\to+\infty} g(u) = -\frac{\pi T \operatorname{sign}(a)}{D\sqrt{4D - T^2}}.$$

The existence of the left Poincaré half-map P for the case $a \neq 0$ implies $a > 0$ and $c = 0$ or $a < 0$ and $c = 2\pi\left(D\sqrt{4D - T^2}\right)^{-1} \in \mathbb{R}$. It is straightforward to see that these conditions together with the properties of function g lead to the equality

$$\operatorname{sign}(cT - g(u)) = \operatorname{sign}(T). \tag{4.1}$$

Let us consider $y_0 \in \operatorname{int}(I) \cap J$. From equality (2.4), one gets

$$cT = \int_{P(y_0)}^{y_0} \frac{-y}{W(y)} dy = \int_{P(y_0)}^{-y_0} \frac{-y}{W(y)} dy + \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy,$$

that is,

$$\int_{P(y_0)}^{-y_0} \frac{-y}{W(y)} dy = cT - g(y_0).$$

Thus, from (4.1),

$$\operatorname{sign}\left(\int_{P(y_0)}^{-y_0} \frac{-y}{W(y)} dy\right) = \operatorname{sign}(T) \neq 0$$

and, taking into account that $-y_0 \cdot P(y_0) \geqslant 0$, equality $\operatorname{sign}(y_0 + P(y_0)) = -\operatorname{sign}(T)$ holds for every $y_0 \in \operatorname{int}(I) \cap J$. Therefore, the conclusion follows by using the continuity of the left Poincaré half-map and the function $y_1(y_0) = -y_0$. □

The next result establishes, as a direct consequence of Proposition 4.1, the relationship between the graph of the left Poincaré half-map and the bisector of the fourth quadrant.

Corollary 4.2. *The following items are true.*

1. *If $T = 0$, then the graph of the left Poincaré half-map P of system* (2.2) *associated to section $\Sigma \equiv \{x = 0\}$, if it exists, is included in the bisector of the fourth quadrant.*

2. *If $T > 0$ (resp. $T < 0$), then the graph of left Poincaré half-map P of system* (2.2) *associated to section $\Sigma \equiv \{x = 0\}$, if it exists, is located below (resp. above) the bisector of the fourth quadrant except perhaps at the origin.*

5 The sign of the second derivative of Poincaré half-maps

From the differential equation given in (3.1), it is easy to obtain explicit expressions for the derivatives of P with respect to y_0. The first and second derivatives are shown in the next result. Its proof is a simple computation and so it is omitted.

Proposition 5.1. *The first and second derivatives of the left Poincaré half-map P with respect to y_0, in the interval* $\operatorname{int}(I)$, *are given by*

$$\frac{dP}{dy_0}(y_0) = \frac{y_0 W(P(y_0))}{P(y_0)W(y_0)}$$

and

$$\frac{d^2P}{dy_0^2}(y_0) = -\frac{a^2\left(y_0^2 - (P(y_0))^2\right) W(P(y_0))}{(P(y_0))^3\,(W(y_0))^2}. \tag{5.1}$$

As will be stated in the next section, some interesting applications to periodic behavior of piecewise linear systems come out from the signs of the first and the second derivatives of P. Note that the sign of the first derivative is obvious from (5.1), because $y_0P(y_0) < 0$ for $y_0 \in \operatorname{int}(I)$ and the polynomial W is positive (see Remark 2.1). Besides that, the sign of the second derivative of left Poincaré half-map P is an immediate consequence of expression given in (5.1) and Proposition 4.1.

Proposition 5.2. *The sign of the second derivative of left Poincaré half-map P is given by*

$$\operatorname{sign}\left(\frac{d^2P}{dy_0^2}(y_0)\right) = -\operatorname{sign}(a^2T) \quad \textit{for } y_0 \in \operatorname{int}(I).$$

Note that a^2 is written in the previous expression to include the case $a = 0$.

In previous works, the sign of the second derivative of the Poincaré half-maps has been addressed via case-by-case studies (see, for instance [22]), where distinguished analyses must be employed for different values of the parameters. Nevertheless, in Proposition 5.2, the integral characterization has allowed to obtain a closed expression for such a sign regardless the cases. As far as we know, this common expression has not been previously obtained in the literature.

6 Some immediate consequences in piecewise linear systems

The previous results established some fundamental properties of Poincaré half-maps defined on a straight line for planar linear systems. These properties are essential to understand the dynamic behavior of planar piecewise linear systems. This section is devoted to provide some immediate consequences regarding periodic behavior in piecewise linear systems with two zones separated by a straight line.

From Freire et. al in [10, Proposition 3.1], we known that any piecewise linear system with two zones separated by a straight line Σ for which a Poincaré map is well defined can be written in the following Liénard canonical form

$$\begin{cases} \dot{x} = T_L x - y \\ \dot{y} = D_L x - a_L \end{cases} \quad \text{for } x < 0, \qquad \begin{cases} \dot{x} = T_R x - y + b \\ \dot{y} = D_R x - a_R \end{cases} \quad \text{for } x > 0. \tag{6.1}$$

Note that the points $(0,0)$ and $(0,b)$ are the tangency points between Σ and, respectively, the flow of the left and right systems. When $b=0$ these points coincide and the flow of system (6.1) crosses the separation line transversally except at the origin. In this case, the system is called *sewing*. On the contrary, for $b \neq 0$ the flow of system (6.1) does not cross the separation line along the segment

$$\Sigma_s = \{(0, \mu + (1-\mu)b) \in \Sigma : \mu \in (0,1)\},$$

which is usually called the *sliding region*.

In order to analyze the behaviour of system (6.1) we consider two Poincaré half-maps associated to Σ, to wit, the *Forward Poincaré half-map* $y_L : I_L \subset [0,+\infty) \longrightarrow (-\infty, 0]$ and the *Backward Poincaré half-map* $y_R : I_R \subset [b,+\infty) \to (-\infty, b]$. The forward one goes in the positive direction of the flow and maps a point $(0,y_0)$, with $y_0 \geqslant 0$, to a point $(0, y_L(y_0))$. Analogously, the backward one goes in the negative direction of the flow and maps a point $(0,y_0)$, with $y_0 \geqslant b$, to $(0, y_R(y_0))$. Notice that y_L is defined by the left system and y_R is defined by the right system. Naturally, $y_L = P$ by taking $T = T_L, D = D_L$, and $a = a_L$ in system (2.2). In addition, taking into account the change $(t,x) \to -(t,x)$, one has $y_R(y_0) = P(y_0 - b) + b$ by taking $T = -T_R, D = D_R$, and $a = -a_R$.

Evidently, the intersections between the curves $y_1 = y_L(y_0)$ and $y_1 = y_R(y_0)$, for $y_0 \in \operatorname{Int}(I_L \cap I_R)$, are in bijective correspondence to crossing periodic solutions of (6.1).

From Proposition 4.1,

$$T_L = 0 \,(\text{resp. } T_R = 0) \Rightarrow y_L(y_0) = -y_0 \,(\text{resp. } y_R(y_0) = -y_0 + 2b), \tag{6.2}$$

when, of course, the map y_L (resp. y_R) exists. Therefore, the following result follows immediately.

Corollary 6.1. *Assume that $T_L^2 + T_R^2 = 0$. If $b \neq 0$, the system (6.1) does not have crossing periodic orbits. If $b = 0$ and $\operatorname{Int}(I_L \cap I_R) \neq \emptyset$, then it has a continuum of crossing periodic orbits.*

It is also possible to give some results for the case $T_L^2 + T_R^2 > 0$. From Corollary 4.2, if $T_L > 0$ (resp. $T_L < 0$), then the curve $y_1 = y_L(y_0)$, if it exists, is located below (resp. above) the straight line $y_1 = -y_0$ except perhaps at the origin. Analogously, if $T_R > 0$ (resp. $T_R < 0$), the curve $y_1 = y_R(y_0)$, if it exists, is located above (resp. below) the straight line $y_1 = -y_0 + 2b$ except perhaps at the point (b,b). Hence, also taking (6.2) into account, if $T_L > 0$, $T_R \geqslant 0$, and $b \geqslant 0$, then

$$y_L(y_0) < -y_0 \leqslant -y_0 + 2b \leq y_R(y_0).$$

Therefore, the graphs of y_L and y_R have no intersection points and so system (6.1) has no crossing periodic orbits. The following result about non-existence of periodic orbits follows immediately via a similar reasoning.

Corollary 6.2. *Assume that $T_L T_R \geqslant 0$ and that one of the following two non-exclusive hypotheses holds:*

1) $T_L \neq 0$ and $T_L b \geqslant 0$;

2) $T_R \neq 0$ and $T_R b \geqslant 0$.

Then, system (6.1) does not have crossing periodic orbits.

By merging the information of Corollaries 6.1 and 6.2, we get that if $T_L T_R \geqslant 0$ and $T_L b \geqslant 0$, then system (6.1) either does not have crossing periodic orbits or has a continuum of crossing periodic orbits. In other words, it does not have limit cycles.

Let us add some lines regarding the condition $T_L T_R \geqslant 0$ added in Corollary 6.2. Note that for the case $b = 0$ it is well known that $T_L T_R \leqslant 0$ is a necessary condition for the existence of crossing periodic orbits (see, for instance, [10]). However, when $b \neq 0$ they could exist even for $T_L T_R > 0$. Thus, the previous result allows to remove some cases where crossing periodic solutions do not exist.

The obtention of the previous results relies only on the relative location of the graphs of the Poincaré half-maps, which is easily determined in terms of the basic parameters T_L, T_R, and b, by using Proposition 4.1. Now, information about the shape of their graphs, revealed by Proposition 5.2, can be used to bound the number of limit cycles of piecewise linear systems (isolated crossing periodic solutions) in some generic cases. In fact, from Proposition 5.2, a simple expression is obtained for the sign of the second derivatives of the Poincaré half-maps,

$$\operatorname{sign}\left(\frac{d^2 y_L}{dy_0^2}(y_0)\right) = -\operatorname{sign}(a_L^2 T_L) \quad \text{and} \quad \operatorname{sign}\left(\frac{d^2 y_R}{dy_0^2}(y_0)\right) = \operatorname{sign}(a_R^2 T_R).$$

Therefore, the concavity of the functions y_L and y_R is established by $a_L^2 T_L$ and $a_R^2 T_R$, respectively. Thus, the following result for limit cycles follows immediately.

Corollary 6.3. *If $T_L\, T_R > 0$, then system* (6.1) *has at most two limit cycles.*

The upper bound given by the corollary above is reachable. Indeed, the last example provided by Han and Zhang in [13] satisfies $T_L\, T_R > 0$ and has two limit cycles near the origin.

Concerning the series expansions of Poincaré half-maps provided by Propositions 3.1-3.6, a natural application could consist in obtaining stability properties of some singular invariant sets of piecewise linear systems under suitable assumptions. For instance, Proposition 3.1 can provide whether the monodromic singularity at the separation line is attracting, repelling, or a center; analogously, Proposition 3.6 can provide whether the infinity is attracting, repelling, or a center in the case it is monodromic; finally, Propositions 3.2 and 3.4 can be used to study the stability of some fold-fold connections. Mixing the stability properties above, one can immediately get sufficient conditions for the existence of a limit cycle (forcing, for instance, the mondromic singularity at the discontinuity line and the infinity, in the monodromic case, to have the same stability).

7 Conclusions

In this paper we provided fundamental properties of Poincaré half-maps defined on a straight line for planar linear systems. Our analysis was based on a novel characterization of Poincaré half-maps [2], presented in Section 2. This characterization has proven to be an effective method to study these maps from a common point of view and to obtain results in a simple way, without the need of making particularized case-by-case studies.

We have focused on the analyticity of the Poincaré half-maps, their series expansions, the relative position between the graph of Poincaré half-maps and the bisector of the fourth quadrant, and the sign of their second derivatives. In what follows, we summarize the obtained results. In Section 3, we addressed the series expansion of a Poincaré half-map,

$P : I \subset [0, +\infty) \longrightarrow (-\infty, 0]$, around the extrema of its interval of definition I, namely: Propositions 3.1 and 3.2 provided the Taylor expansion of P around the origin when $0 \in I$; Proposition 3.4 provided the Newton–Puiseux series expansion of P around $\hat{y}_0 \in I$, where $\hat{y}_0 > 0$ satisfies $P(\hat{y}_0) = 0$; and Proposition 3.6 provided the Taylor series expansions of P around the infinity when $4D - T^2 > 0$. In Section 4, Proposition 4.1 and Corollary 4.2 established a relationship between the graphs of Poincaré half-maps and the bisector of the fourth quadrant depending only on the sign of the trace T. Finally, in Section 5, Proposition 5.1 provided expressions for the first and second derivative of the Poincaré half-maps and Proposition 5.2 determined the sign of the second derivative of the Poincaré half-maps.

All these properties are essential to understand the dynamic behavior of planar piecewise linear systems with two zones separated by a straight line (PPWLS, for short). Thus, in Section 6 we provided some immediate consequences regarding periodic behavior of such systems, namely: Corollary 6.1 established non-generic conditions for a PPWLS either not having periodic orbits and having a continuum of crossing periodic orbits; Corollary 6.2 gives generic conditions for a PPWLS not having periodic orbits; finally, Corollary 6.3 provided generic conditions for a PPWLS having at most two limit cycles.

The results obtained in this paper also allow deeper insights regarding periodic solutions for piecewise linear systems. For instance, in [4], the present results among others were of assistance in proving that PPWLS without sliding region (that is, $b = 0$) has at most one limit cycles. This result was obtained without unnecessary distinctions of spectra of the matrices. In addition, it is proven that this limit cycle, if exists, is hyperbolic and its stability is determined by a simple condition in terms of the parameters. Also, in [5], it was provide the existence of a uniform upper bound, L^*, for the maximum number of limit cycles of PPWLS. The present Proposition 4.1 helped to show that $L^* \leq 8$.

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On a Dirichlet boundary value problem for an Ermakov–Painlevé I equation. A Hamiltonian EPI system

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Abstract. Here, a proto-type Ermakov–Painlevé I equation is introduced and a homogeneous Dirichlet-type boundary value problem analysed. In addition, a novel Ermakov–Painlevé I system is set down which is reducible by an involutory transformation to the autonomous Ermakov–Ray–Reid system augmented by a single component Ermakov–Painlevé I equation. Hamiltonian such systems are delimited.

Keywords: Ermakov, Painlevé, Dirichlet boundary value problem, Hamiltonian system.

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1 Introduction

Ermakov in [13] in now classical work introduced a canonical nonlinear equation which has subsequently been established as the base member of two and, in general, multi-component nonlinear systems with diverse applications in both nonlinear physics and continuum mechanics [33]. Thus, in [16,17], what are now termed Ermakov–Ray–Reid systems were derived which admit a distinctive integral of motion together with concomitant nonlinear superposition principles. These two-component coupled systems arise notably in nonlinear optics as detailed in [14, 30, 31]. In [1], what constitutes a Ermakov–Ray–Reid system was derived in an application of a variational approach to the analysis of elliptic cloud evolution in a Bose-Einstein condensate.

In [24], a classical 2+1-dimensional rotating shallow water system with an underlying circular paraboloidal bottom topography was shown to admit an integrable subsystem of Ermakov–Ray–Reid type. The latter system in that context describes the time-evolution of the semi-axes of the elliptical moving shoreline on the paraboloidal basin. It is, in addition, Hamiltonian and this integral of motion allied with the admitted Ermakov invariant

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allows exact solution of the Ermakov–Ray–Reid system. The procedure adopted in [24] had its genesis in that applied in [18] to construct the general solution of an eight dimensional nonlinear dynamical system descriptive of the time-evolution of upper ocean warm-core elliptical eddies. Therein, representation of this system in terms of modulated versions of the divergence, spin, shear and normal deformation rates rendered the elliptic warm-core ring system analytically tractable. Importantly, a relevant class of exact solutions with a Ermakov connection therein termed pulsrodons was isolated which characteristically both rotate and pulsate periodically. Lyapunov stability of such pulsrodons and their duals was subsequently addressed via a Lagrangian treatment in [15]. In [34] pulsrodonic phenomena was exhibited in a 2+1-dimensional nonlinear system governing rotating homentropic magnetogasdynamics in a bounded region. In a related development [35], a 2+1-dimensional version of a non-isothermal gasdynamic system with origin in work of Dyson [12] on spinning gas clouds was investigated. It was established therein via an elliptic vortex ansatz that the system admits a Hamiltonian reduction to a particular Ermakov–Ray–Reid system when the adiabatic index $\sigma = 2$.

The preceding attest to the diverse physical applications of the two-component Ermakov–Ray–Reid systems. In the present context, such a system will be shown in an appendix to arise via reduction of a three-component hybrid Ermakov–Painlevé I system.

In [36], it was established that a symmetry reduction of a classical 2+1-dimensional N-layer hydrodynamic system leads naturally to a novel multi-component Ermakov-type system. Importantly, the latter was shown to be iteratively reducible to a system of $N-2$ linear equations augmented by a canonical Ermakov–Ray–Reid system. Moreover sequences of such systems were shown to be linked via Darboux transformations. Novel links between multi-component Ermakov systems and classes of many-body problems were subsequently established in [19]. In [29], Ermakov-type systems in two-dimensions were constructed and multi-wave solutions of a 2+1-dimensional modulated sine-Gordon equation thereby derived. Ermakov systems of arbitrary order and dimension were constructed in [42] which inherit key characteristics of the canonical Ermakov–Ray–Reid system.

The connection between the classical Painlevé I–VI equations and symmetry reduction of solitonic systems is well-documented (see e.g. [9] and literature cited therein). Indeed, the generic properties of solitonic equations associated with admittance of linear representations [2] and Bäcklund transformations [37,41] are likewise possessed by these Painlevé equations. It is remarked that such a Bäcklund transformation admitted by Painlevé II and its iteration have application not only in soliton theory but also in the analytic treatment of important boundary value problems for the celebrated Nernst–Planck system of ion transport [7,10,26].

In [20], wave packet representations inserted into a multi-component nonlinear Schrödinger system which incorporated a de-Broglie–Bohm quantum potential term resulted in novel hybrid Ermakov–Painlevé II reductions. Therein, a pair of Ermakov–Painlevé II equations was derived as a reduction of a nonlinear elastodynamic system governing the coupled stress associated with a class of shear motions. Hybrid Ermakov–Painlevé II-IV systems have subsequently been the subject of extensive investigation in [21,22,25,38]. In particular, physical applications of Ermakov–Painlevé II equations have been shown to arise in such diverse areas as cold plasma physics [28], Korteweg capillarity theory [27] and in multi-ion Nernst-Planck systems. In the latter context, Dirichlet-type boundary value problems were analysed in [3] for a Ermakov–Painlevé II reduction of such a three-ion electrolytic system. Hybrid Ermakov–Painlevé IV systems were originally derived via symmetry reduction of a multi-component resonant derivative nonlinear Schrödinger system in [21]. In subsequent work [25], Bäcklund transformations were applied to generate classes of exact solutions of the Ermakov–Painlevé IV system via the classical Painlevé IV equation. The forms of the prototype Ermakov–Painlevé II–IV equations have been set down explicitly in [4]. Two-point boundary value problems of Dirichlet-type for the single component base Ermakov–Painlevé IV equation were analysed in [5]. In addition, therein it was established that admitted Ermakov invariants can be used in the systematic generation of a coupled Ermakov–Painlevé IV system in terms of seed solutions of the canonical Painlevé IV equation.

The nonlinear coupled systems as introduced in [16, 17] that have come to be known as Ermakov–Ray–Reid systems adopt the form

$$\ddot{x} + \omega(t)x = \frac{1}{x^2 y}\Phi(y/x),$$
$$\ddot{y} + \omega(t)y = \frac{1}{xy^2}\Psi(x/y)$$

and admit the distinctive integral of motion

$$\mathrm{I} = \frac{1}{2}(x\dot{y} - y\dot{x})^2 + \int^{y/x} \Phi(z)dz + \int^{x/y} \Psi(w)dw$$

together with concomitant nonlinear superposition principles. The latter which are characteristic of the system are not of the type generic in soliton theory which are generated via invariance under Bäcklund transformations. The classical single component Ermakov equation of [13], namely

$$\ddot{\rho} + \omega(t)\rho = \delta/\rho^3$$

admits the nonlinear superposition principle

$$\rho = \sqrt{c_1\alpha^2(t) + 2c_2\alpha(t)\beta(t) + c_3\beta^2(t)}$$

wherein $\alpha(t), \beta(t)$ are two linearly independent solutions of

$$\ddot{\sigma} + \omega(t)\sigma = 0$$

with corresponding constant Wronskian $\mathcal{W} = \alpha\dot{\beta} - \beta\dot{\alpha}$ with constants c_1 such that

$$c_1c_3 - c_2^2 = \delta/\mathcal{W}^2 \, .$$

This result and its extensions are readily derived via Lie group methods [32, 40]. The preceding nonlinear superposition principle may be applied in the systematic reduction via reciprocal transformations of Ermakov-modulated solitonic systems to their canonical unmodulated counterparts [39].

In [6], the Ermakov–Ray–Reid system was reduced to its associated autonomous form via application of a novel class of involutory transformations. It was demonstrated thereby that the system admits an underlying linear structure albeit not of the type generic to solitonic systems.

Painlevé I has been derived in [8] via the classical Lie group procedure as a symmetry reduction of the solitonic Boussinesq equation. The latter arises in diverse physical applications such as long wave propagation in shallow water hydrodynamics, nonlinear lattice theory and plasma physics. Here a proto-type Ermakov–Painlevé I equation is introduced and a homogeneous Dirichlet-type boundary value problem analysed. In addition, a novel

Ermakov–Painlevé I system is set down which is reducible via an involutory transformation to the autonomous Ermakov–Ray–Reid system augmented by a single component hybrid Ermakov–Painlevé equation. Hamiltonian such systems are delimited.

The paper is organised as follows. The next section is devoted to the search of classical solutions to a homogeneous Dirichlet problem for a Ermakov–Painlevé I equation. Furthermore, the order of the zeros at the endpoints is analysed and an upper bound for the distance between distinct solutions is obtained. The main tool is the method of upper and lower solutions, combined with a Cantor diagonal argument. Finally, a two-component Ermakov–Painlevé I system with underlying Hamiltonian structure is set down and an associated Ermakov–Ray–Reid system constructed in the Appendix.

2 A Dirichlet problem

Here, a classical solution $\rho(t)$ of the Ermakov–Painlevé I equation

$$\rho''(t) = \left[5\left(\frac{\rho'(t)}{\rho(t)}\right)^2 - t\frac{\rho(t)^4}{4}\right]\rho(t) - \frac{3}{2\rho(t)^3} \tag{2.1}$$

is sought over the interval $(0,1)$ subject to the boundary conditions

$$\rho(0) = \rho(1) = 0. \tag{2.2}$$

It is seen that the EPI equation (2.1) is invariant under $\rho \to -\rho$ and in the sequel attention is restricted to solutions $\rho(t) > 0$ of the boundary value problem determined by (2.1)–(2.2).

Theorem 2.1. *Boundary value problem* (2.1)–(2.2) *has at least one solution* $\rho \in C[0,1] \cap C^2(0,1)$ *such that* $\rho(t) > 0$ *for* $t \in (0,1)$.

To establish this result, let us recall that the transformation $w = \rho^{-4}$ yields the standard Painlevé I equation

$$w''(t) = 6w(t)^2 + t.$$

The strategy shall consist in proving the existence of a monotone sequence $0 < w_1 < w_2 \ldots$ such that

$$\begin{aligned} w_n''(t) &= 6w_n(t)^2 + t, \quad t \in (0,1) \\ w_n(0) &= w_n(1) = n \end{aligned} \tag{2.3}$$

and set ρ as the limit of the sequence $\{w_n^{-1/4}\}$. However, it is not clear *a priori* whether or not the limit function $w(t) := \lim_{n\to\infty} w_n(t)$ is continuous and satisfies $w(t) < \infty$ for all $t \in (0,1)$. In order to circumvent this impediment, we shall give a location result with the aid of the method of upper and lower solutions. The following elementary result suffices in this regard (see e.g. [11, Ch. 2]):

Lemma 2.2. *Let* $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ *be continuous and let* $R, S > 0$. *Assume that the smooth functions* α, β *satisfy*

$$\alpha''(t) > f(t,\alpha(t)), \qquad \beta''(t) < f(t,\beta(t)) \qquad t \in (0,1)$$

$$\alpha(0) \le R \le \beta(0), \quad \alpha(1) \le S \le \beta(1),$$

and $0 \le \alpha(t) < \beta(t)$ *for all* $t \in (0,1)$*. Then the Dirichlet boundary value problem*

$$u''(t) = f(t, u(t)), \qquad u(0) = R, \quad u(1) = S$$

has at least one solution u *with* $\alpha(t) < u(t) < \beta(t)$ *for* $t \in (0,1)$*. If furthermore* f *is nondecreasing with respect to its second variable, then the boundary value problem has no other (positive) solutions.*

The next lemma provides an ordered couple (α_n, β_n) of positive lower and upper solutions for (2.3).

Lemma 2.3. *There exist unique* α_n, β_n *with* $0 < \alpha_n(t) < \beta_n(t) < n$ *for* $t \in (0,1)$ *such that*

$$\alpha_n''(t) = 6\alpha_n(t)^2 + 1, \qquad \beta_n''(t) = 6\beta_n(t)^2$$

$$\alpha_n(0) = \alpha_n(1) = \beta_n(0) = \beta_n(1) = n.$$

Moreover, $m_n := \min_{t\in[0,1]} \beta(t)$ *satisfies* $m_n = \beta\left(\frac{1}{2}\right)$ *and*

$$c \le m_n \le C$$

for constants $C > c > 0$ *independent of* n*.*

Proof. Let $u(t) := \left(t - \frac{1}{2}\right)^2$ and $v(t) \equiv n$, then, for $t \in (0,1)$,

$$u''(t) \equiv 2 > 6u(t)^2 + 1, \qquad v''(t) \equiv 0 < 6v(t)^2 < 6v(t)^2 + 1$$

and $0 \le u(t) < v(t)$. From Lemma 2.2, the existence and uniqueness of α_n between u and v follows. Next, the pair (α_n, n) is adopted as an ordered couple of a lower and an upper solution for the problem $\beta'' = 6\beta^2$ which, in turn, provides the existence and uniqueness of β_n, with $\alpha_n < \beta_n < n$.

Next, multiplication of the equality $\beta_n'' = 6\beta_n^2$ by β_n' and integration yields

$$\beta_n'(t)^2 = 4\beta_n(t)^3 + A$$

for some constant A. By virtue of convexity, it follows that β_n achieves a unique minimum value $m_n < n$ at some $t_0 \in (0,1)$. It is deduced that $A = -4m_n^3$ and

$$\beta_n'(t) = \begin{cases} -2\sqrt{\beta_n(t)^3 - m_n^3}, & t \le t_0 \\ 2\sqrt{\beta_n(t)^3 - m_n^3}, & t > t_0. \end{cases}$$

Thus, for $t \le t_0$ we obtain

$$-\int_0^t \frac{\beta_n'(s)ds}{\sqrt{\beta_n(s)^3 - m_n^3}} = 2t$$

and setting $u := \frac{\beta_n(s)}{m_n}$ it follows that

$$m_n^{-1/2} \int_{\beta_n(t)/m_n}^{n/m_n} \frac{du}{\sqrt{u^3 - 1}} = 2t.$$

Analogously, for $t > t_0$ it is seen that

$$m_n^{-1/2} \int_{\beta_n(t)/m_n}^{n/m_n} \frac{du}{\sqrt{u^3 - 1}} = 2(1 - t).$$

In particular, letting $t \to t_0$ it follows that $2t_0 = 2(1-t_0)$, that is, $t_0 = \frac{1}{2}$. Furthermore,

$$m_n^{-1/2} \int_1^{n/m_n} \frac{du}{\sqrt{u^3-1}} = 1,$$

whence

$$m_n^{1/2} = \int_1^{n/m_n} \frac{du}{\sqrt{u^3-1}} \le \int_1^{\infty} \frac{du}{\sqrt{u^3-1}} < \infty.$$

This gives the inequality $m_n \le C$ and, for $n \ge C+1$

$$m_n^{1/2} = \int_1^{n/m_n} \frac{du}{\sqrt{u^3-1}} \ge \int_1^{1+\frac{1}{C}} \frac{du}{\sqrt{u^3-1}},$$

so $m_n \ge c$ for some constant $c > 0$ independent of n. □

Remark 2.4. With regard to the preceding, the fact that the minimum m_n is achieved at $t_0 = \frac{1}{2}$ follows directly by noticing that β_n is symmetric, that is, $\beta_n(t) = \beta_n(1-t)$. Indeed, this is due to uniqueness since $\beta_n(1-t)'' = \beta_n''(1-t) = 6\beta_n(1-t)^2$ and $\beta_n(1-0) = \beta_n(1-1) = n$. A similar argument holds for α_n.

As a corollary, we obtain:

Lemma 2.5. *Boundary value problem for Painlevé I* (2.3) *has a unique positive solution w_n with $\alpha_n < w_n < \beta_n$.*

Next, we shall prove a monotonicity property.

Lemma 2.6. *The sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{w_n\}$ are strictly nondecreasing.*

Proof. The claim is here proved just for $\{w_n\}$. The other cases are analogous. Assume that $w_{n+1} - w_n$ achieves its absolute minimum at some $\hat{t}$. If $w_{n+1}(\hat{t}) < w_n(\hat{t})$, then $\hat{t} \in (0,1)$ and

$$0 \le (w_{n+1} - w_n)''(\hat{t}) = 6(w_{n+1} + w_n)(\hat{t})(w_{n+1} - w_n)(\hat{t}) < 0,$$

a contradiction. Furthermore, because $(w_{n+1} - w_n)'(\hat{t}) = 0$, it is deduced that the equality $w_{n+1}(\hat{t}) = w_n(\hat{t})$ cannot hold either, due to the uniqueness of solutions of the initial value problem for the equation $w'' = 6w^2 + t$. □

As a consequence of the preceding lemma, we may define the functions $\alpha, \beta, w : [0,1] \to [0, +\infty]$ as the respective pointwise limits of the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{w_n\}$. It is clear that $\alpha \le w \le \beta$; however, it remains to prove that $w(t)$ is finite and satisfies the Painlevé I equation for $t \in (0,1)$. With this in mind, it is noted that the monotone and bounded sequence $\{m_n\}$ converges to a value $m = \beta\left(\frac{1}{2}\right) \in (0, +\infty)$ and, for $t \in \left(0, \frac{1}{2}\right)$, the implicit formula

$$m_n^{-1/2} \int_{\beta_n(t)/m_n}^{n/m_n} \frac{du}{\sqrt{u^3-1}} = 2t.$$

implies, when passing to the limit, that

$$m^{-1/2} \int_{\beta(t)/m}^{\infty} \frac{du}{\sqrt{u^3-1}} = 2t.$$

This shows that $\beta(t)$ is finite and the same conclusion is obtained for $t \in \left(\frac{1}{2}, 1\right)$; thus, $w(t) < +\infty$ for all $t \in (0,1)$. Moreover, the previous identity also implies that β is smooth and satisfies the equality $\beta''(t) = 6\beta(t)^2$. Next, fix $n_0 > m$ and $a < b$ the (unique) values in $(0,1)$ such that $\alpha_{n_0}(a) = \alpha_{n_0}(b) = m$. This implies that $w_n(t) > m$ for all $n \geq n_0$ and $t \notin [a,b]$. In particular, if the absolute minimum of w_n is achieved at $t_n \in (0,1)$, then $t_n \in [a,b]$ and we may take a subsequence $t_{n_j} \to t_* \in [a,b]$. Thus, from the identity

$$w_n'(t) = \int_{t_n}^{t} (6w_n(s)^2 + s)\, ds$$

and by the monotone convergence theorem we obtain:

$$w_{n_j}'(t) \to \int_{t_*}^{t} (6w(s)^2 + s)\, ds.$$

Now writing $w_{n_j}(t) = w_{n_j}(t_*) + \int_{t_*}^{t} w_{n_j}'(s)\, ds$, it is immediately verified that w is smooth and satisfies the Painlevé I equation for all $t \in (0,1)$. Hence $\rho := w^{-1/4}$ is positive and satisfies (2.1) for $t \in (0,1)$. It remains to prove that $\rho(0^+) = \rho(1^-) = 0$. To this end, for arbitrary $M > 0$ fix $n_0 > M$ and $\delta > 0$ such that $\alpha_{n_0}(t) > M$ when $t < \delta$ or $t > 1-\delta$. Since $\{\alpha_n\}$ is increasing, it follows that $\alpha(t) > M$ when $t < \delta$ or $t > 1-\delta$; accordingly, it has been established that $\alpha(0^+) = \alpha(1^-) = +\infty$ and, consequently, ρ is extended continuously to a solution of (2.1)–(2.2).

2.1 Order of the zeros

This section is devoted to investigation of the behaviour of the classical positive solutions of (2.1)–(2.2) in the neighbourhood of the endpoints of the interval. With this in mind, set as before $w := \rho^{-4}$ satisfying the Painlevé I equation and let $t_{\min} \in (0,1)$ be the value in which the absolute minimum $w_{\min}$ of w is achieved. For $t \in (0, t_{\min})$, the inequalities $6w(t)^2 < w''(t) < 6w(t)^2 + 1$ yield

$$6w(t)^2 w'(t) > w''(t)w'(t) > 6w(t)^2 w'(t) + w'(t)$$

and, upon integration,

$$4w(t)^3 - 4w_{\min}^3 < w'(t)^2 < 4w(t)^3 + 2w(t) - [4w_{\min}^3 + 2w_{\min}].$$

Using the identity $\sqrt{A+B} - \sqrt{A} = \frac{B}{\sqrt{A+B}+\sqrt{A}}$ for $A, A+B > 0$, we may write

$$2w(t)^{3/2} - R(t) < -w'(t) < 2w(t)^{3/2} + S(t)$$

where, due to the continuity of the solution ρ, the positive functions R and S can be made arbitrarily small when t is close to 0. In fact, given $r \in (0,1)$ it suffices to fix $\delta_0 > 0$ such that $R(t), S(t) < 2rw(t)^{3/2}$ for all $t < \delta_0$. This implies, for $0 < t < \delta < \delta_0$,

$$1 - r < \left(w^{-1/2}\right)'(t) < 1 + r,$$

whence

$$(1-r)(\delta - t) < w^{-1/2}(\delta) - w^{-1/2}(t) < (1+r)(\delta - t)$$

and letting $t \to 0$ we obtain:

$$(1-r)\delta \leq w^{-1/2}(\delta) \leq (1+r)\delta$$

that is

$$\sqrt{(1-r)\delta} \le \rho(\delta) \le \sqrt{(1+r)\delta}.$$

Since r is arbitrary, we conclude that $\rho(\delta) \sim \sqrt{\delta}$ for small values of δ. Analogously, it is verified that $\rho(\xi) \sim \sqrt{1-\xi}$ when ξ is close to 1. The previous conclusions allow a more precise computation of the solution near the endpoints of the interval. Indeed, it is observed that the functions R and S behave respectively as

$$R(t) = R_0(t)w(t)^{-3/2}, \qquad S(t) = S_0(t)w(t)^{-1/2}$$

for some bounded positive functions R_0 and S_0, so for $0 < t < \delta$ sufficiently small it is obtained:

$$\sqrt{1 - O(\delta^4)} \le \frac{\rho(\delta)}{\sqrt{\delta}} \le \sqrt{1 + O(\delta^6)}.$$

In particular, this shows that $\rho(t) \sim \sqrt{t} + O(t^{5/2})$ as $t \sim 0$ and, analogously, $\rho(t) \sim \sqrt{1-t} + O((1-t)^{5/2})$ as $t \sim 1$.

2.2 The uniqueness problem

In this section, it is established that the solution given via Theorem 2.1 is maximal, that is, any other possible solution $\tilde{\rho}$ of (2.1)–(2.2) such that $\tilde{\rho} \neq \rho$ satisfies $\tilde{\rho}(t) < \rho(t)$ for all $t \in (0,1)$. Furthermore, if $\tilde{\rho}$ is the limit of a sequence of solutions of (2.1) that are strictly positive in $[0,1]$, then $\tilde{\rho} = \rho$. Accordingly, the solution obtained in the preceding sections is the only one that can be defined as the limit of approximate solutions of the non-homogeneous Dirichlet problem.

The proof of the previous assertions is deduced in a straightforward manner from the following:

Lemma 2.7. *Let $\rho_1, \rho_2 \in C^2(0,1)$ be distinct strictly positive solutions of (2.1). Then ρ_1 and ρ_2 cross each other at most in one value $t \in (0,1)$.*

Proof. Due to the uniqueness for the initial value problem, it is clear that all possible cross points are isolated. Suppose that $a < b$ are two consecutive cross points and, for example, that $\rho_1 < \rho_2$ in (a,b), then the corresponding functions $w_j := \rho_j^{-4}$ satisfy $w_1 > w_2$ and

$$(w_1 - w_2)'' = 6(w_1 + w_2)(w_1 - w_2) > 0$$

over (a,b), which contradicts the fact that $w_1 = w_2$ for $t = a, b$. □

Proposition 2.8. *Let ρ be a positive solution of (2.1)–(2.2) such that ρ is the limit of a sequence $\{\rho_n\}$ of solutions of (2.1) with $\rho_n > 0$ on $[0,1]$. If $\tilde{\rho}$ is any distinct positive solution of (2.1)–(2.2), then $\tilde{\rho}(t) < \rho(t)$ for all $t \in (0,1)$.*

Proof. Suppose that $\rho_n(t) < \tilde{\rho}(t)$ for some $t \in (0,1)$. Then, because $\rho_n(0)$ and $\rho_n(1)$ are strictly positive, it follows that ρ_n crosses $\tilde{\rho}$ in more than one point, a contradiction. This shows that $\rho_n(t) \ge \tilde{\rho}(t)$ for all t and, consequently, $\rho \ge \tilde{\rho}$. Furthermore, if $\rho(t) = \tilde{\rho}(t)$ for some t, then $\rho'(t) = \tilde{\rho}'(t)$, whence $\rho \equiv \tilde{\rho}$. □

In view of the latter result, it might be conjectured that the positive solution of (2.1)–(2.2) is, indeed, unique. However, our conclusions do not exclude the existence of “small” solutions. The next result provides a lower bound for such small solutions.

Proposition 2.9. *Let ρ be a positive solution of* (2.1)–(2.2) *and let β be defined as before. Then $\rho(t) > \beta(t)^{-1/4}$ for all $t \in (0,1)$.*

Proof. Observe, at the outset, that β is the unique positive solution of the problem $v''(t) = 6v(t)^2$ satisfying $v(0^+) = v(1^-) = +\infty$. Indeed, it is seen as before that v achieves its unique minimum at $t = \frac{1}{2}$, with

$$v\left(\tfrac{1}{2}\right)^{1/2} = \int_1^\infty \frac{du}{\sqrt{u^3 - 1}} = \beta\left(\tfrac{1}{2}\right)^{1/2}.$$

Since, furthermore, $v'\left(\frac{1}{2}\right) = 0 = \beta'\left(\frac{1}{2}\right)$, it is deduced that $v \equiv \beta$. Next, suppose that $\rho(t_0) \leq \beta(t_0)^{-1/4}$ for some $t_0 \in (0,1)$, then $w := \rho^{-4}$ satisfies

$$w''(t) = 6w(t)^2 + t > 6w(t)^2, \qquad w(0^+) = w(1^-) = +\infty.$$

On the other hand, setting $k \geq 1$ large enough, it is verified that

$$(\beta + kw)''(t) = 6\beta(t)^2 + 6kw(t)^2 + kt < 6[\beta(t) + kw(t)]^2.$$

Thus, $(w, \beta + kw)$ is an ordered couple of a lower and an upper solution for the problem $v'' = 6v^2$ and a diagonal argument proves the existence of a solution v with $w(t) < v(t) < \beta(t) + kw(t)$ for all $t \in (0,1)$. A contradiction then arises from the fact that $v \equiv \beta$. □

As a consequence, a somewhat sharp bound for the distance between distinct solutions is readily computed. Let w be the solution of the Painlevé I equation constructed in the proof of Theorem 2.1 and define $w_{\min}$ as the minimum value of w. As previously, w is a lower solution for the problem $v'' = 6v^2$ and setting $c > 0$ it is seen that

$$(w + c)''(t) = 6w(t)^2 + t < 6[w(t) + c]^2,$$

provided that $t < 6[c^2 + 2w(t)c]$. Thus, taking

$$c := \sqrt{w_{\min}^2 + \frac{1}{6}} - w_{\min}$$

it follows that $w(t) < \beta(t) < w(t) + c$ for all $t \in (0,1)$. For instance, a rough estimation shows that, since $\beta\left(\frac{1}{2}\right)$ is approximately equal to 5.9, then the optimal value of c is smaller than 0.015. In particular, this yields the bound

$$\beta(t)^{-1/4} < \rho(t) < [\beta(t) - c]^{-1/4} \qquad t \in (0,1)$$

for all possible solutions of (2.1)–(2.2).

Appendix. A Hamiltonian hybrid Ermakov–Painlevé I system

Just as the classical Ermakov equation of [13] constitutes the base one-component reduction of the Ermakov–Ray–Reid system of [16, 17], so the nonlinear Ermakov–Painlevé I equation, which is the subject of the present paper, may be embedded in a two-component hybrid Ermakov–Painlevé I system. Ermakov–Painlevé II–IV systems and their properties have been placed in a general solitonic context in [23]. Here, by way of illustration, a two-component Ermakov–Painlevé I system with underlying Hamiltonian structure is set down and an associated Ermakov–Ray–Reid system constructed.

Here, a Ermakov–Painlevé I system is introduced according to

$$\ddot{x} + \left[-5\left(\frac{\dot{\rho}}{\rho}\right)^2 + \frac{t\rho^4}{4} + \frac{3}{2\rho^4}\right] x = \frac{1}{x^2 y}\Phi(y/x),$$
$$\ddot{y} + \left[-5\left(\frac{\dot{\rho}}{\rho}\right)^2 + \frac{t\rho^4}{4} + \frac{3}{2\rho^4}\right] y = \frac{1}{xy^2}\Psi(x/y)$$

wherein ρ is governed by the single component EPI equation

$$\ddot{\rho} + \left[-5\left(\frac{\dot{\rho}}{\rho}\right)^2 + \frac{t\rho^4}{4} + \frac{3}{2\rho^4}\right] \rho = 0 \, .$$

Thus,

$$\rho\ddot{x} - \ddot{\rho}x = \frac{\rho}{x^2 y}\Phi(y/x), \quad \rho\ddot{y} - \ddot{\rho}y = \frac{\rho}{xy^2}\Psi(x/y) \, .$$

whence, on introduction of the involutory transformation

$$\left.\begin{aligned} x^* = x/\rho \, , \quad y^* = y/\rho \, , \\ dt^* = \rho^{-2} dt \\ \rho^* = 1/\rho \end{aligned}\right\} \quad \mathcal{R}$$

with $\mathcal{R}^2 = \mathrm{I}$, reduction is made to the canonical autonomous Ermakov–Ray–Reid system

$$x^*_{t^*t^*} = \frac{1}{x^{*2}y^*}\Phi(y^*/x^*), \quad y^*_{t^*t^*} = \frac{1}{x^* y^{*2}}\Psi(x^*/y^*) \, .$$

If the Ermakov–Painlevé I system has the J-parametric representation

$$\ddot{x} + \left[-5\left(\frac{\dot{\rho}}{\rho}\right)^2 + \frac{t\rho^4}{4} + \frac{3}{2\rho^4}\right] x = \frac{2}{x^3}J(y/x) + \frac{y}{x^4}J'(y/x)$$
$$\ddot{y} + \left[-5\left(\frac{\dot{\rho}}{\rho}\right)^2 + \frac{t\rho^4}{4} + \frac{3}{2\rho^4}\right] y = -\frac{1}{x^3}J'(y/x)$$

augmented by the canonical single component EPI equation, then application of the involutory transformation $\mathcal{R}$ produces the parametrisation of the canonical Hamiltonian Ermakov–Ray–Reid system as set down in a nonlinear optics context in [31]

$$x^*_{t^*t^*} = \frac{2}{x^{*3}}J(y^*/x^*) + \frac{y^*}{x^{*4}}J'(y^*/x^*) \, ,$$
$$y^*_{t^*t^*} = -\frac{1}{x^{*3}}J'(y^*/x^*) \, .$$

The latter admits the Hamiltonian integral of motion

$$\mathcal{H}^* = \frac{1}{2}\left[x^{*2}_{t^*} + y^{*2}_{t^*}\right] + \frac{1}{x^{*2}}J(y^*/x^*) \, .$$

which, together with the Ermakov invariant I^* allows the systematic integration of the canonical Hamiltonian system. It is remarked that such Ermakov–Ray–Reid systems with underlying Hamiltonian structure occur in diverse physical applications.

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Solutions for a quasilinear elliptic problem with indefinite nonlinearity with critical growth

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Abstract. We are interested in nonhomogeneous problems with a nonlinearity that changes sign and may possess a critical growth as follows

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) = \lambda|u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geq 2$, $1 < p \leq q < N$, $q < r \leq q^*$, $\lambda \in \mathbb{R}$ and function W is a weight function which changes sign in Ω. Using variational methods, we prove the existence of four solutions: two solutions which do not change sign and two solutions which change sign exactly once in Ω.

Keywords: subcritical and critical exponents, $p\&q$ Laplacian operator, indefinite problems.

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1 Introduction

The goal of this paper is to find nontrivial solutions for the problem

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) = \lambda|u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geq 2$, $1 < p \leq q < N$, $q < r \leq q^*$ and $\lambda \in \mathbb{R}$, where $q^* = \frac{Nq}{N-q}$ is the critical Sobolev exponent.

We introduce the hypotheses on the function a in the sequel.

(a_1) Function $a : [0,\infty) \to \mathbb{R}$ is of class C^1 and there exist constants $k_1, k_2 \geq 0$ such that

$$k_1 t^p + t^q \leq a(t^p)t^p \leq k_2 t^p + t^q, \quad \text{for all } t > 0;$$

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(a_2) Define, for $t \geq 0$, $A(t) = \int_0^t a(s)ds$. The mapping $t \mapsto A(t^p)$ is convex on $(0, \infty)$;

(a_3) The mapping $t \mapsto \frac{a(t^p)}{t^{q-p}}$ is nonincreasing on $(0, \infty)$.

(a_4) If $1 < p \leq q \leq 2 \leq N$, then the mapping $t \mapsto a(t)$ is nondecreasing for $t > 0$. If $2 \leq p \leq q < N$, the mapping $t \mapsto a(t)t^{p-2}$ is nondecreasing for $t > 0$.

As a direct consequence of (a_3), we obtain that the function a and its derivative a' satisfy

$$a'(t)t \leq \frac{(q-p)}{p} a(t) \quad \text{for all } t > 0. \tag{1.1}$$

Now, if we define the function $h(t) = a(t)t - \frac{q}{p}A(t)$, using (1.1) we can prove that function h is nonincreasing. Then,

$$\frac{1}{q} a(t)t \leq \frac{1}{p} A(t), \quad \text{for all } t \geq 0. \tag{1.2}$$

To illustrate the degree of generality of the kind of problems studied here, and with adequate hypotheses on the functions a, which will be made clear shortly, we present some examples of problems that are also interesting from a mathematical point of view and have a wide range of applications in physics and related sciences.

Problem 1: Let $a(t) = t^{\frac{q-p}{p}}$. In this case we are studying problem as

$$\begin{cases} -\Delta_q u = \lambda |u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

and it is related to the main result showed in [6]. See also the work [7].

Problem 2: Let $a(t) = 1 + t^{\frac{q-p}{p}}$. In this case we are studying problem as

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda |u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

Problem 3: Let $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$. In this case we are studying problem

$$\begin{cases} -\Delta_p u - \operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1 + |\nabla u|^p)^{\frac{p-2}{p}}} \right) = \lambda |u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

Problem 4: Let $a(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$. In this case, we are studying problem

$$\begin{cases} -\Delta_p u - \Delta_q u - \operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1 + |\nabla u|^p)^{\frac{p-2}{p}}} \right) = \lambda |u|^{q-2}u + W(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

Such class of problems arise from applications in physics and related sciences, such as biophysics, plasma physics and chemical reactions (for instance, see [16,17,24]).

The interest in studying nonlinear partial differential equations with $p\&q$ operator has increased because many applications arising in mathematical physics may be stated with an operator in this form. We refer the reader to the works [9–11, 15], where the authors have considered nonhomogeneous elliptic problems involving several type of function a.

Problems involving indefinite nonlinearities, that is, signal changing nonlinearities, have attracted the attention of many researchers over the past few decades, either because of their application in population dynamics describing the stationary behavior of a population in a heterogeneous environment (see [1, 19, 22, 23]) or because of their mathematical relevance. Researchers have studied this type of problem using: variational methods (see [2,3,8,12,20,21]), sub-supersolution method (see [12, 13, 20]) and Morse theory (see [2, 19]).

This paper deals with the class of problem (P_λ) that brings important characteristics, which are the nonlinearities that change signal (see the hypotheses on W below) with subcritical or critical growth and the generality of the operator that includes, for instance, $p-$Laplacian and $p\&q-$Laplacian operators. These characteristics provoke some behaviors in the geometry of the energy functional associated to problem (P_λ) which make it difficult to find nontrivial solutions. As far as we know, this is the only work that proves existence and multiplicity of ground state solutions of problem (P_λ) under our assumptions.

Let us consider a weight function $W : \Omega \to \mathbb{R}$ which changes sign in Ω. More specifically, function W satisfies

(W_1) $W \in L^\infty(\Omega)$ and the set $\Omega_+ := \{ x \in \Omega : W(x) > 0\}$ has positive measure.

It follows directly of (W_1) that

$$\lambda^* := \inf \left\{ \frac{\int_\Omega |\nabla u|^q \, dx}{\int_\Omega |u|^q \, dx} : u \in W_0^{1,q}(\Omega) \setminus \{0\} \text{ and } \int_\Omega W(x)|u|^r \, dx \geq 0 \right\} < +\infty. \tag{1.3}$$

We are going to require another important hypothesis on W. For this, let λ_1 be the first eigenvalue of the operator $(-\Delta_q)$ on Ω, with zero Dirichlet boundary condition, and let φ_1 be the first eigenfunction associated to λ_1. The weight function W satisfies only one of the following two hypotheses:

(W_2^+)

$$\int_\Omega W(x)|\varphi_1|^r \, dx > 0.$$

(W_2^-)

$$\int_\Omega W(x)|\varphi_1|^r \, dx < 0.$$

By the variational characterization of λ_1, we have

i) If the weight function W satisfies (W_1) and (W_2^+), then $\lambda^* = \lambda_1$.

ii) If the weight function W satisfies (W_1) and (W_2^-), then $\lambda^* > \lambda_1$.

We are now ready to state our first main result concerning the subcritical case.

Theorem 1.1. *Let $r < q^*$, a satisfying (a_1)–(a_4) and the weight function W satisfying (W_1), (W_2^+) or (W_2^-). Then,*

i) if $\lambda < \lambda_1$ and u is a nontrivial solution of (P_λ), then

$$\int_\Omega W(x)|u|^r dx > 0;$$

ii) *if* $\lambda \in (-\infty, \lambda^*)$, *then problem* ($P_\lambda$) *has two least energy solutions which do not change sign in* Ω. *Moreover, if* $\lambda < \lambda_1$, *these two solutions are ground state solutions;*

iii) *if* $\lambda \in (-\infty, \lambda^*)$, *then problem* ($P_\lambda$) *has two least energy nodal solutions which change sign exactly once in* Ω. *Moreover, if* $\lambda < \lambda_1$, *these two nodal solutions are nodal ground state solutions.*

Item (i) of Theorem 1.1 provides some interesting qualitative properties on nontrivial solutions of problem (P_λ). For example:

1) If $\Omega_0 := \{\, x \in \Omega : \ W(x) = 0\} \subset \Omega$ is a domain with smooth boundary and u is a nontrivial solution of (P_λ), then $u \neq 0$ $a.e$ in $\Omega \setminus \Omega_0$;

2) If $\Omega_+ := \{\, x \in \Omega : \ W(x) > 0\}$ and $\Omega_- := \{\, x \in \Omega : \ W(x) < 0\}$ have positive measure, and u is a nontrivial solution of (P_λ), then u must "belong" more to Ω_+ than Ω_-, that is,

$$\int_{\Omega_+} W(x)|u|^r dx > -\int_{\Omega_-} W(x)|u|^r dx > 0;$$

3) If Ω is a symmetric set and $W \in C(\Omega)$ is an odd function, then a nontrivial solution u of (P_λ) can be neither an even nor an odd function. In fact, otherwise

$$\int_{\Omega} W(x)|u|^r dx = \int_{\Omega_+} W(x)|u|^r dx + \int_{\Omega_-} W(x)|u|^r dx = 0;$$

To illustrate this, consider $\Omega = \left\{x \in \mathbb{R}^N : \ |x| < 2\pi\right\}$ and $W : \mathbb{R}^N \to \mathbb{R}$ given by $W(x) = \cos(|x|)$.

To show the existence of solutions to the problem in the critical case, we will need to add a new hypothesis on the weight function W. The new hypothesis is as follows.

(W_3) There exists an open set $\Omega_* \subset\subset \Omega_+$ such that $|\Omega_-| > |\Omega_*|$. Moreover, there exist positive numbers $\mathcal{W}_1$ and $\mathcal{W}_2$ such that

$$\mathcal{W}_1 \geq W(x) \geq \mathcal{W}_2 > \|W^-\|_\infty, \quad \forall\, x \in \Omega_*.$$

The above hypothesis is fundamental to overcome the lack of compactness generated by the critical exponent $r = q^*$. It is important to highlight that, up to our knowledge, (W_3) is a new hypothesis in the literature, which makes it one of the relevant points of this work.

To provide an example of a function that satisfies hypothesis (W_3), just consider $\Omega = \{x \in \mathbb{R}^N : 0 \leq |x| \leq 2\pi\}$, $\Omega_* = \left\{x \in \mathbb{R}^N : \frac{\pi}{4} \leq |x| \leq \frac{3\pi}{4}\right\}$ and $W : \Omega \to \mathbb{R}$, given by

$$W(x) = \begin{cases} \sin(|x|), & |x| \leq \pi, \\ \dfrac{\sin(|x|)}{2\sqrt{2}}, & \pi \leq |x| \leq 2\pi. \end{cases}$$

Now, let $S > 0$ be the best constant of the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{q^*}(\Omega)$. Our second main result, concerning the critical case, is the following.

Theorem 1.2. *Consider $r = q^*$ and $\lambda < \lambda_1$. Let a satisfy (a_1)–(a_4) and the weight function W satisfy (W_1), (W_3) and*

$$\int_\Omega W(x)|\varphi_1|^{q^*}\,dx < \frac{\frac{1}{N}\left(\mathcal{W}_2 - \|W^-\|_\infty\right)\left(\frac{S}{\mathcal{W}_1}\right)^{\frac{N}{q}}}{\frac{k_2}{k_1 p} + \frac{1}{N}}.$$

Then, there are two nontrivial solutions for problem (P_λ).

The paper is organized as follows: in Section 2, we will prove technical results and the first part of Theorem 1.1. In Section 3, we will demonstrate the second part of Theorem 1.1, namely, the existence of least energy solutions that do not change sign. Finally, in Section 4, we will establish the last part of Theorem 1.1, that is, the existence of least energy nodal solutions that change sign exactly once.

2 Variational framework and preliminary results

The natural space to look for weak solutions to problem (P_λ) is the Sobolev space $W_0^{1,q}(\Omega)$ with the associated norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^q dx\right)^{\frac{1}{q}}, \quad \text{for } u \in W_0^{1,q}(\Omega).$$

Since the approach is variational, consider the energy functional associated $J_\lambda : W_0^{1,q}(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) := \frac{1}{p}\int_\Omega A\left(|\nabla u|^p\right)\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \frac{1}{r}\int_\Omega W(x)|u|^r\,dx.$$

We know that J_λ is differentiable on $W_0^{1,q}(\Omega)$ and, for all $u, v \in W_0^{1,q}(\Omega)$,

$$J'_\lambda(u)v := \int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u \nabla v\,dx - \lambda\int_\Omega |u|^{q-2}uv\,dx - \int_\Omega W(x)|u|^{r-2}uv\,dx.$$

Thus, $u \in W_0^{1,q}(\Omega)$ is a critical point of J_λ if, and only if, u is a weak solution of problem (P_λ). Moreover, let us define the Nehari manifold

$$\mathcal{N}_\lambda := \left\{u \in W_0^{1,q}(\Omega) \ : \ J'_\lambda(u)u = 0\right\} \tag{2.1}$$

and the nodal Nehari set

$$\mathcal{N}_\lambda^\pm := \left\{u \in W_0^{1,q}(\Omega) \ : \ u^\pm \neq 0 \text{ and } J'_\lambda(u)u = 0\right\}, \tag{2.2}$$

where

$$u^+(x) := \max\{u(x),\, 0\} \quad \text{and} \quad u^-(x) := \min\{u(x),\, 0\}.$$

Notice that $u = u^+ + u^-$ and $\mathcal{N}_\lambda^\pm \subset \mathcal{N}_\lambda$.

Now we introduce some important subsets of $\mathcal{N}_\lambda$. Consider

$$\mathcal{M}_\lambda := \left\{u \in W_0^{1,q}(\Omega) : \ u \in \mathcal{N}_\lambda \text{ and } \int_\Omega W(x)|u|^r\,dx > 0\right\} \tag{2.3}$$

and

$$\mathcal{M}_\lambda^{\pm} := \left\{ u \in W_0^{1,q}(\Omega) : \ u^{\pm} \in \mathcal{N}_\lambda \text{ and } \int_\Omega W(x)|u^{\pm}|^r \, dx > 0 \right\}. \tag{2.4}$$

Since we want to use the method of minimization, we begin to study the behavior of the functional J_λ on $\mathcal{N}_\lambda$.

Proposition 2.1. *Assume that the function a satisfies (a_1)–(a_3). Then, there exist positive constants K_1, K_2 and K_3 such that the following properties hold:*

(i) $J_\lambda(u) \geq K_1\left(\frac{\lambda_1 - \lambda}{\lambda_1}\right)\|u\|^q$, *for all* $u \in \mathcal{N}_\lambda$.

(ii) $\|u\| \geq K_2\left(\frac{\lambda_1 - \lambda}{\lambda_1}\right)^{\frac{1}{r-q}}$, *for all* $u \in \mathcal{N}_\lambda$.

(iii) $\int_\Omega W(x)|u|^r \, dx \geq K_3\left(\frac{\lambda_1 - \lambda}{\lambda_1}\right)^{\frac{r}{r-q}}$, *for all* $u \in \mathcal{N}_\lambda$.

Proof. For every $u \in \mathcal{N}_\lambda$, by (1.2), we have

$$\begin{aligned}
J_\lambda(u) &= J_\lambda(u) - \frac{1}{r} J'_\lambda(u) u \\
&= \frac{1}{p}\int_\Omega A(|\nabla u|^p) dx - \frac{1}{r}\int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^p dx - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\int_\Omega |u|^q dx \\
&\geq \left(\frac{1}{q} - \frac{1}{r}\right)\int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^p \, dx - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\int_\Omega |u|^q dx.
\end{aligned}$$

Hence, by (a_1) and the Poincaré inequality,

$$J_\lambda(u) \geq \left(\frac{r-q}{qr}\right)\left(\frac{\lambda_1 - \lambda}{\lambda_1}\right)\left(\int_\Omega |\nabla u|^q dx\right). \tag{2.5}$$

Then item (i) follows.

We now prove item (ii). Taking $u \in \mathcal{N}_\lambda$, by (a_1) and the Poincaré inequality, one has

$$\begin{aligned}
\int_\Omega |\nabla u|^q \, dx \leq \int_\Omega a\left(|\nabla u|^p\right)|\nabla u|^p \, dx &= \lambda \int_\Omega |u|^q \, dx + \int_\Omega W(x)|u|^r \, dx \\
&\leq \frac{\lambda}{\lambda_1}\int_\Omega |\nabla u|^q \, dx + \int_\Omega W(x)|u|^r \, dx.
\end{aligned}$$

Hence,

$$\left(1 - \frac{\lambda}{\lambda_1}\right)\int_\Omega |\nabla u|^q \, dx \leq \int_\Omega W(x)|u|^r \, dx. \tag{2.6}$$

Finally, using that $W \in L^\infty(\Omega)$, the Sobolev embeddings and (2.6), there exists a positive constant C_1 such that

$$\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^q \leq C_1\|u\|^r.$$

This inequality proves item (ii).

Item (iii) follows directly from inequality contained in item (ii) and by (2.6). In fact,

$$K_2\left(1 - \frac{\lambda}{\lambda_1}\right) \leq \int_\Omega W(x)|u|^r \, dx.$$

□

The next result is a direct consequence of Proposition (2.1).

Corollary 2.2. *If $\lambda < \lambda_1$, then $\mathcal{N}_\lambda = \mathcal{M}_\lambda$ and $\mathcal{N}_\lambda^\pm = \mathcal{M}_\lambda^\pm$.*

Proof. By definition of $\mathcal{N}_\lambda$ and $\mathcal{M}_\lambda$, we get $\mathcal{M}_\lambda \subset \mathcal{N}_\lambda$ and $\mathcal{M}_\lambda^\pm \subset \mathcal{N}_\lambda^\pm$. The other inclusions follow from item (iii) of previous proposition. □

By the same arguments of Proposition 2.1, but using the definition of λ^* instead of Poincaré inequality, the next result follows.

Proposition 2.3. *Assume that function a satisfies (a_1)–(a_3). Then, there exist positive constants K_1, K_2 and K_3 such that the following properties hold:*

(i) $J_\lambda(u) \geq K_1\left(\frac{\lambda^*-\lambda}{\lambda^*}\right)\|u\|^q$, *for all $u \in \mathcal{M}_\lambda$.*

(ii) $\|u\| \geq K_2\left(\frac{\lambda^*-\lambda}{\lambda^*}\right)^{\frac{1}{r-q}}$, *for all $u \in \mathcal{M}_\lambda$.*

(iii) $\int_\Omega W(x)|u|^r\,dx \geq K_3\left(\frac{\lambda^*-\lambda}{\lambda^*}\right)^{\frac{r}{r-q}}$, *for all $u \in \mathcal{M}_\lambda$.*

Therefore, from Proposition 2.1 and Proposition 2.3, the following real numbers are well defined:

$$c_\lambda = \inf_{\mathcal{N}_\lambda} J_\lambda, \quad d_\lambda = \inf_{\mathcal{N}_\lambda^\pm} J_\lambda, \quad \tilde{c}_\lambda = \inf_{\mathcal{M}_\lambda} J_\lambda \quad \text{and} \quad \tilde{d}_\lambda = \inf_{\mathcal{M}_\lambda^\pm} J_\lambda. \tag{2.7}$$

Moreover, if $\lambda_1 > \lambda$, notice that Corollary 2.2 allows us called a solution of (P_λ) which is a minimizer of $\mathcal{M}_\lambda$ (or $\mathcal{M}_\lambda^\pm$) of ground state solution (or nodal ground state solution).

Lemma 2.4. *Consider $u \in W_0^{1,q}(\Omega) \setminus \{0\}$ such that $\int_\Omega W(x)|u|^r\,dx > 0$. Then, there exists a unique $t_u > 0$ satisfying*

$$J_\lambda(t_u u) := \max_{t\geq 0} J_\lambda(tu) > 0.$$

Moreover, if $J'_\lambda(u)u < 0$, then $t_u \in (0,1]$.

Proof. Let $u \in W_0^{1,q}(\Omega)\setminus\{0\}$ and $t \in (0,+\infty)$. So, by (a_1), we obtain

$$J_\lambda(tu) \leq k_2\frac{t^p}{p}\int_\Omega |\nabla u|^p\,dx + \frac{t^q}{q}\left(\int_\Omega |\nabla u|^q\,dx - \lambda\int_\Omega |u|^q\,dx\right) - \frac{t^r}{r}\int_\Omega W(x)|u|^r\,dx \tag{2.8}$$

and

$$J_\lambda(tu) \geq k_1\frac{t^p}{p}\int_\Omega |\nabla u|^p\,dx + \frac{t^q}{q}\left(\int_\Omega |\nabla u|^q\,dx - \lambda\int_\Omega |u|^q\,dx\right) - \frac{t^r}{r}\int_\Omega W(x)|u|^r\,dx. \tag{2.9}$$

Therefore,

$$\limsup_{t\to 0^+}\frac{J_\lambda(tu)}{t^p} > 0 \quad \text{and} \quad \limsup_{t\to+\infty}\frac{J_\lambda(tu)}{t^r} = -\frac{1}{r}\int_\Omega W(x)|u|^r\,dx. \tag{2.10}$$

Thus, since $\int_\Omega W(x)|u|^r\,dx > 0$, we ensure the existence of $t_u \in (0,+\infty)$ such that

$$J_\lambda(t_u u) := \max_{t\geq 0} J_\lambda(tu) > 0.$$

To guarantee that the value $t_u > 0$ is unique, let us prove that the equation $J'_\lambda(su)su = 0$ is satisfied only for $s = t_u$. Indeed, this equation is equivalent to

$$s^{r-q}\int_\Omega W(x)|u|^r\,dx + \lambda\int_\Omega |u|^q\,dx = \int_\Omega \frac{a\left(|\nabla(su)|^p\right)}{|\nabla(su)|^{q-p}}|\nabla u|^q\,dx.$$

By (a_3), the right-hand side of the equation above is a nonincreasing function on $s > 0$, while the left side, an increasing function on $s > 0$ provided $r > q$ and $\int_\Omega W(x)|u|^r\,dx > 0$. This shows the uniqueness of the value $t_u > 0$. With the same arguments, we obtain that $t_u > 1$ implies $J'_\lambda(u)u \geq 0$, and the proof of the lemma follows. □

Lemma 2.5. *If $q < r < q^*$ and $W : \Omega \to \mathbb{R}$ satisfies (W_1), then $\mathcal{M}^{\pm}_\lambda \neq \emptyset$ for all $\lambda \in \mathbb{R}$. Consequently, $\mathcal{M}_\lambda \neq \emptyset$ for all $\lambda \in \mathbb{R}$.*

Proof. From (W_1), we may consider two open balls B_1 and B_2 contained in Ω such that

$$B_1 \cap B_2 = \emptyset, \quad |B_1 \cap \Omega_+| > 0 \quad \text{and} \quad |B_2 \cap \Omega_+| > 0.$$

Arguing as in [6, Lemma 2.3], we have two negative solutions $u_1 \in C_0^\infty(B_1)$ and $u_2 \in C_0^\infty(B_2)$ such that

$$\int_\Omega W(x)|u_1|^r\,dx > 0 \quad \text{and} \quad \int_\Omega W(x)|u_2|^r\,dx > 0.$$

Then, by Lemma 2.4, there are $t_1,\ t_2 > 0$ such that $J'_\lambda(t_1u_1)t_1u_1 = 0$ and $J'_\lambda(t_2u_2)t_2u_2 = 0$. Using $B_1 \cap B_2 = \emptyset$, we have that

$$J'_\lambda(t_1u_1 + t_2u_2)(t_1u_1 + t_2u_2) = J'_\lambda(t_1u_1)t_1u_1 + J'_\lambda(t_2u_2)t_2u_2 = 0.$$

Hence $(t_1u_1 + t_2u_2) \in \mathcal{M}^{\pm}_\lambda$, which implies $\mathcal{M}^{\pm}_\lambda \neq \emptyset$. Since $\mathcal{M}^{\pm}_\lambda \subset \mathcal{M}_\lambda$, we have $\mathcal{M}_\lambda \neq \emptyset$. □

Proof of item (i) of Theorem 1.1

Proof. The proof follows directly from item (iii) of Proposition 2.1. □

3 Existence of two least energy solutions which do not change sign

In this section, we are going to show that $\tilde{c}_\lambda$ is attained by some function which is a solution of problem (P_λ). For our purposes, we write

$$J_\lambda(u) = \Phi_\lambda(u) - I(u), \quad \forall\, u \in W_0^{1,q}(\Omega),$$

where the functionals $\Phi_\lambda,\ I \in C^1(W_0^{1,q}(\Omega), \mathbb{R})$ are given by

$$\Phi_\lambda(u) := \frac{1}{p}\int_\Omega A(|\nabla u|^p)\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx \quad \text{and} \quad I(u) := \frac{1}{r}\int_\Omega W(x)|u|^r\,dx.$$

Let us consider the set $Y := \left\{u \in W_0^{1,q}(\Omega) : \int_\Omega W(x)|u|^r\,dx > 0\right\}$ which is an open cone of $W_0^{1,q}(\Omega)$, that is, $tu \in Y$ for every $t > 0$ and $u \in Y$.

We now present some properties of the functionals Φ_λ and I when $\lambda < \lambda^*$.

Lemma 3.1. *If $\lambda < \lambda^*$, then the following properties hold:*

(i) Φ_λ and $u \mapsto \Phi'_\lambda(u)u$ are weakly lower semicontinuous and $I'(u_n) \to I'(u)$ in $W_0^{1,q'}(\Omega)$ if $u_n \rightharpoonup u$ in $W_0^{1,q}(\Omega)$.

(ii) There exists $C_1 > 0$ such that $\Phi'_\lambda(u)u \geq C_1\|u\|^q$ for every $u \in \overline{Y}$ and $I'(u) = o(\|u\|^{q-1})$ as $u \to 0$ in $\overline{Y}$.

(iii) $I(u) = I'(u)u = 0$ *for every* $u \in \partial Y$.

(iv) $t \mapsto \frac{\Phi'_\lambda(tu)u}{t^{q-1}}$ *and* $t \mapsto \frac{I'(tu)u}{t^{q-1}}$ *are nonincreasing and increasing, respectively, in* $(0, +\infty)$ *and for every* $u \in Y$. *Moreover,*

$$\limsup_{t\to+\infty} \frac{\Phi_\lambda(tu)}{t^q} < \limsup_{t\to+\infty} \frac{I(tu)}{t^q} = +\infty.$$

(v) If $\lambda < \lambda_1$, *then* $I'(u)u \le 0 < \Phi'_\lambda(u)u$ *for all* $u \in W^{1,q}(\Omega) \setminus (Y \cup \{0\})$.

Proof. To prove (i), let us consider $(u_n) \subset W_0^{1,q'}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,q'}(\Omega)$. From (a_2), it follows that

$$\int_\Omega A(|\nabla u|^p)\, dx \le \liminf_{n\to+\infty} \int_\Omega A(|\nabla u_n|^p)\, dx. \tag{3.1}$$

$$\int_\Omega a(|\nabla u|^p)|\nabla u|^p\, dx \le \liminf_{n\to+\infty} \int_\Omega a(|\nabla u_n|^p)|\nabla u|^p\, dx, \tag{3.2}$$

Moreover, by Sobolev embeddings, (W_1) and, up to a subsequence, we get

$$\int_\Omega |u|^q\, dx = \lim_{n\to+\infty} \int_\Omega |u_n|^q\, dx \quad \text{and} \quad \int_\Omega W(x)|u|^r\, dx = \lim_{n\to+\infty} \int_\Omega W(x)|u_n|^r\, dx. \tag{3.3}$$

Hence, by (3.1), (3.2) and (3.3), the first item is proved.

To prove (ii), arguing as Proposition 2.3, we have

$$\Phi_\lambda(u) \ge \left(\frac{\lambda^* - \lambda}{\lambda^*}\right) \|u\|^q, \quad \forall\, u \in W_0^{1,q}(\Omega). \tag{3.4}$$

On the other hand, by (W_1),

$$|I'(u)v| \le \|W\|_\infty \left(\int_\Omega |v|^r\, dx\right)^{\frac{1}{r}} \left(\int_\Omega |u|^r\, dx\right)^{\frac{r-1}{r}},$$

and then, by Sobolev embeddings,

$$\frac{I'(u)}{\|u\|^{q-1}} \le C\|u\|^{r-q}, \quad u \ne 0. \tag{3.5}$$

From (3.4) and (3.5) the item (ii) holds. Since $\partial Y = \{0\}$, this shows that the item (iii) holds.

Now let us prove item (iv). Since $q < r$ and $u \in Y$, we obtain

$$\frac{d}{dt}\left[\frac{I'(tu)u}{t^{q-1}}\right] = (r-q)t^{r-q-1} \int_\Omega W(x)|u|^r\, dx > 0, \quad \forall\, t \in (0, +\infty),$$

which implies that $t \mapsto \dfrac{I'(tu)u}{t^{q-1}}$ is increasing in $(0, +\infty)$ and for every $u \in Y$. Moreover,

$$\limsup_{t\to+\infty} \frac{I(tu)}{t^q} = \limsup_{t\to+\infty} \frac{t^{r-q}}{r} \int_\Omega W(x)|u|^r\, dx = +\infty \tag{3.6}$$

On the other hand, note that

$$\frac{\Phi'_\lambda(tu)u}{t^{q-1}} = \int_\Omega \frac{a(|\nabla tu|^p)}{|\nabla tu|^{q-p}}|\nabla u|^q\, dx - \lambda \int_\Omega |u|^q\, dx$$

is a nonincreasing function by (a_3). Moreover, we also have

$$\limsup_{t\to+\infty} \frac{\Phi_\lambda(tu)}{t^{q-1}} \le \frac{1}{q}\left(\int_\Omega |\nabla u|^q\,dx - \lambda\int_\Omega |u|^q\,dx\right). \tag{3.7}$$

Then, by (3.6) and (3.7), we conclude the proof of item (iv).

To finish, if $u \in W^{1,q}(\Omega)\setminus(\Upsilon\cup\{0\})$, then $\int_\Omega W(x)|u|^r\,dx \le 0$. Hence, by (a_1) and $\lambda<\lambda_1$,

$$I'(u)u = \int_\Omega W(x)|u|^r\,dx \le 0 < \int_\Omega a(|\nabla u_n|^p)|\nabla u|^p\,dx - \lambda\int_\Omega |u|^q\,dx = \Phi'_\lambda(u)u$$

and the proof of the last item of the proposition is complete. □

Using the previous lemma and [14, Corollary 3.1], we obtain the next result.

Corollary 3.2. *If $\lambda<\lambda^*$, then there exists $v_\lambda\in\mathcal{M}_\lambda$ such that*

$$J_\lambda(v_\lambda) = \tilde{c}_\lambda := \inf_{u\in\mathcal{M}_\lambda} J_\lambda(u).$$

We now show that problem (P_λ) has two least energy solutions when $\lambda<\lambda^*$.

Proposition 3.3. *If $\lambda<\lambda^*$, then there exists a nontrivial function v_λ which is a least energy solution of (P_λ), and $\tilde{v}_\lambda := -v_\lambda$ is also a least energy solution of (P_λ). Moreover, if $\lambda<\lambda_1$ these solutions are ground state solutions.*

Proof. Let v_λ be the solution found in Corollary 3.2 and let us assume by contradiction that $v_\lambda^\pm \ne 0$. Since v_λ is a critical point of functional J_λ and the intersection of the support of the functions $v_\lambda^\pm$ is empty, we have that $v_\lambda^\pm\in\mathcal{N}_\lambda$. Hence,

$$c_\lambda \le J_\lambda(v_\lambda^\pm). \tag{3.8}$$

Since Proposition 2.3 holds, then either

$$\int_\Omega W(x)|v_\lambda^+|^r\,dx > 0 \quad \text{or} \quad \int_\Omega W(x)|v_\lambda^-|^r\,dx > 0.$$

Without loss of generality, we can assume that $\int_\Omega W(x)|v_\lambda^+|^r\,dx > 0$. Then, $v_\lambda^+\in\mathcal{M}_\lambda$ and, hence,

$$\tilde{c}_\lambda \le J_\lambda(v_\lambda^+). \tag{3.9}$$

Therefore, by (3.8) and (3.9),

$$c_\lambda + \tilde{c}_\lambda \le J_\lambda(v_\lambda^+) + J_\lambda(v_\lambda^-) = J_\lambda(v_\lambda) = \tilde{c}_\lambda.$$

This contradiction proves that the least energy solution does not change sign.

We may assume that v_λ is nonnegative. Then, setting $\tilde{v}_\lambda = -v_\lambda$, we have that

$$\tilde{c}_\lambda = J_\lambda(v_\lambda) = \frac{1}{p}\int_\Omega A\left(|\nabla(-v_\lambda)|^p\right)\,dx - \frac{\lambda}{q}\int_\Omega |(-v_\lambda)|^q\,dx - \frac{1}{r}\int_\Omega W(x)|(-v_\lambda)|^r\,dx = J_\lambda(\tilde{v}_\lambda).$$

Moreover, using that v_λ is a critical point of J_λ, we have for all $\varphi\in W_0^{1,q}(\Omega)$,

$$\int_\Omega a(|\nabla(-v_\lambda)|^p)|\nabla(-v_\lambda)|^{p-2}\nabla(-v_\lambda)\nabla\varphi\,dx = \lambda\int_\Omega |(-v_\lambda)|^{q-2}(-v_\lambda)\varphi\,dx + \int_\Omega W(x)|(-v_\lambda)|^{r-2}(-v_\lambda)\varphi\,dx.$$

Thus, $\tilde{v}_\lambda$ is a critical point of J_λ. Therefore, problem (P_λ) has a nonpositive solution and a nonnegative solution. Furthermore, when $\lambda<\lambda_1$, by Corollary 2.2, $\mathcal{M}_\lambda=\mathcal{N}_\lambda$. Thus, these solutions are ground state solutions of (P_λ). □

3.1 Proof of item (ii) of Theorem 1.1

Proof. The proof follows directly from Corollary 3.2 and Proposition 3.3. □

4 Existence of two nodal solutions

We begin this section by showing that $\tilde{d}_\lambda$ is attained by some function which is a least energy nodal solution of problem (P_λ).

Proposition 4.1. *If $\lambda < \lambda^*$, then there exists $\tilde{w}_\lambda \in \mathcal{M}_\lambda^\pm$ such that*

$$d_\lambda := J_\lambda(\tilde{w}_\lambda).$$

Proof. Let $(w_n) \subset \mathcal{M}_\lambda^\pm$ be a minimizing sequence, that is, a sequence satisfying

$$(w_n) \subset \mathcal{M}_\lambda^\pm \quad \text{and} \quad I_\lambda(w_n) = d_\lambda + o_n(1). \tag{4.1}$$

By item (i) of Lemma 2.1, we obtain that functional J_λ is coercive on $\mathcal{M}_\lambda^\pm$, and hence (w_n) is bounded in $W_0^{1,q}(\Omega)$. Then, by Sobolev embeddings and the continuity of the maps $w \mapsto w^+$ and $w \mapsto w^-$ are continuous from $L^r(\mathbb{R}^N)$ in $L^r(\mathbb{R}^N)$ (for details, see [4, Lemma 2.3] with suitable adaptations), there exists $w_\lambda \in W_0^{1,q}(\Omega)$ such that, up to a subsequence, we have

$$\begin{cases} w_n^\pm \rightharpoonup w_\lambda^\pm & \text{in } W_0^{1,q}(\Omega), \\ w_n^\pm \to w_\lambda^\pm & a.e. \text{ in } \Omega, \\ w_n^\pm \to w_\lambda^\pm & \text{in } L^s(\Omega),\ 1 \le s < q^*. \end{cases} \tag{4.2}$$

We claim that $w_\lambda^\pm \neq 0$ and $\int_\Omega W(x)|w_\lambda^\pm|^r\, dx > 0$. Indeed, using that $W \in L^\infty(\Omega)$ and item (iii) of Lemma 2.1, we obtain

$$\int_\Omega |w_\lambda^\pm|^r\, dx = \lim_{n\to\infty} \int_\Omega |w_n^\pm|^r\, dx \ge \frac{K_3}{\|W\|_\infty} > 0, \tag{4.3}$$

and

$$\int_\Omega W(x)|w_\lambda^\pm|^r\, dx = \lim_{n\to\infty} \int_\Omega W(x)|w_n^\pm|^r\, dx \ge K_3 > 0, \tag{4.4}$$

that proves our claim. Therefore, by Lemma 2.4, there exists $t_\lambda^\pm \in (0,+\infty)$ such that $t_\lambda^\pm w_\lambda^\pm \in \mathcal{M}_\lambda$.

We claim that $t_\lambda^\pm \in (0,1)$. In fact, by Fatou's Lemma and (4.2), we have

$$\begin{aligned} \int_\Omega a(|w_\lambda^\pm|^p)|w_\lambda^\pm|^p\, dx &\le \liminf_{n\to+\infty} \int_\Omega a(|w_n^\pm|^p)|w_n^\pm|^p\, dx \\ &= \lim_{n\to+\infty} \left(\lambda \int_\Omega |w_n^\pm|^q\, dx + \int_\Omega W(x)|w_n^\pm|^r\, dx \right) \\ &= \lambda \int_\Omega |w_\lambda^\pm|^q\, dx + \int_\Omega W(x)|w_\lambda^\pm|^r\, dx, \end{aligned}$$

that is, $J'_\lambda(w_\lambda^\pm)w_\lambda^\pm \le 0$. Hence, by Lemma 2.4, the claim follows.

Similarly, with the same arguments of Proposition 3.2, we obtain

$$
\begin{aligned}
J_\lambda(t_\lambda^\pm w_\lambda^\pm) &= J_\lambda(t_\lambda^\pm w_\lambda^\pm) - \frac{1}{q} J'_\lambda(t_\lambda^\pm w_\lambda^\pm) t_\lambda^\pm w_\lambda^\pm \\
&= \int_\Omega \left[\frac{1}{p} A(|\nabla(t_\lambda^\pm w_\lambda^\pm)|^p) - \frac{1}{q} a(|\nabla(t_\lambda^\pm w_\lambda^\pm)|^p) |\nabla(t_\lambda^\pm w_\lambda^\pm)|^p \right] dx \\
&\quad + \left(\frac{1}{q} - \frac{1}{r}\right) \int_\Omega W(x) |t_\lambda^\pm w_\lambda^\pm|^r \, dx \\
&\leq \int_\Omega \left[\frac{1}{p} A(|\nabla w_\lambda^\pm|^p) - \frac{1}{q} a(|\nabla u_\lambda^\pm|^p) |\nabla w_\lambda^\pm|^p \right] dx \\
&\quad + \left(\frac{1}{q} - \frac{1}{r}\right) \int_\Omega W(x) |w_\lambda^\pm|^r \, dx \\
&\leq \liminf_{n\to+\infty} \left[J_\lambda(w_n^\pm) - \frac{1}{q} J'_\lambda(w_n^\pm) w_n^\pm \right] = J_\lambda(w_n^\pm) + o_n(1).
\end{aligned} \tag{4.5}
$$

Then, setting $\tilde{w}_\lambda = t_\lambda^- w_\lambda^- + t_\lambda^+ w_\lambda^+$, from (4.4),

$$
\int_\Omega W(x) |\tilde{w}_\lambda|^r \, dx = \int_\Omega W(x) |t_\lambda^- w_\lambda^-|^r \, dx + \int_\Omega W(x) |t_\lambda^+ w_\lambda^+|^r \, dx \geq 2K_3 > 0,
$$

that is, $\tilde{w}_\lambda \in \mathcal{M}_\lambda^\pm$. Hence, using (4.5), we can conclude

$$
\begin{aligned}
d_\lambda = J_\lambda(\tilde{w}_\lambda) &= J_\lambda(t_\lambda^- w_\lambda^-) + J_\lambda(t_\lambda^+ w_\lambda^+) \\
&\leq J_\lambda(w_n^-) + J_\lambda(w_n^+) + o_n(1) = J_\lambda(w_n) + o_n(1) = d_\lambda.
\end{aligned}
$$

Thus, the level d_λ is attained by the function $\tilde{w}_\lambda \in \mathcal{M}_\lambda^\pm$. □

Corollary 4.2. *Let $\tilde{w}_\lambda$ be a minimizer found in Propositions 4.1. Then, $\tilde{w}_\lambda$ is a critical point of J_λ and has exactly two nodal domains.*

Proof. The proof that $\tilde{w}_\lambda \in \mathcal{M}_\lambda^\pm$ is a critical point of J_λ is done using a suitable quantitative deformation lemma and Brouwer's topological degree properties. It is done, with suitable modifications, as in [5, Lemma 4.3] and [5, Theorem 1.1]. To show that the nodal solution $\tilde{w}_\lambda$ has exactly two nodal domains, or in other words it changes sign exactly once, see for instance [5, pages 1230-1232] . □

Using the same arguments as in Proposition 3.3 one can immediately prove the following result.

Corollary 4.3. *If $\lambda < \lambda^*$, then there exists a function $\tilde{w}_\lambda$ which is a nodal least energy solution of (P_λ), and $\overline{w_\lambda} := -\tilde{w}_\lambda$ is also a nodal least energy solution of (P_λ). Moreover, if $\lambda < \lambda_1$, then these solutions are ground state solutions of (P_λ).*

4.1 Proof of item (iii) of Theorem 1.1.

Proof. It follows directly from Corollaries 4.2 and 4.3.

□

5 A nontrivial solution for the indefinite critical problem

In this section we consider the following critical problem

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u\right) = \lambda|u|^{q-2}u + W(x)|u|^{q^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geq 2$, $1 < p \leq q < N$ and $\lambda \in \mathbb{R}$, where $q^* = \frac{Nq}{N-q}$ is the critical Sobolev exponent. Here, we consider the associated functional $I_\lambda \in C^1(W_0^{1,q}(\Omega), \mathbb{R})$ given by

$$I_\lambda(u) := \frac{1}{p}\int_\Omega A\left(|\nabla u|^p\right)\, dx - \frac{\lambda}{q}\int_\Omega |u|^q\, dx - \frac{1}{r}\int_\Omega W(x)|u|^{q^*}\, dx.$$

Let us show that the associated functional to the indefinite critical problem has a mountain pass geometry.

Proposition 5.1. *The functional $I_\lambda : W_0^{1,q}(\Omega) \to \mathbb{R}$ satisfies the following properties:*

i) There exist positive numbers α and ρ such that

$$I_\lambda(u) \geq \rho, \quad \textit{for all } \|u\| = \rho.$$

ii) There exists a function $e \in W_0^{1,q}(\Omega)$ such that $\|e\| \geq \rho$ and

$$I_\lambda(e) < 0.$$

Proof. By (a_1) and the Poincaré inequality,

$$\begin{aligned} I_\lambda(u) &\geq \frac{k_1}{p}\int_\Omega |\nabla u|^p\, dx + \frac{1}{q}\int_\Omega |\nabla u|^q\, dx - \frac{\lambda}{q}\int_\Omega |u|^q\, dx - \int_\Omega W(x)|u|^{q^*}\, dx \\ &\geq \frac{1}{q}\left(\frac{\lambda_1 - \lambda}{\lambda_1}\right)\int_\Omega |\nabla u|^q\, dx - \int_\Omega W(x)|u|^{q^*}\, dx. \end{aligned}$$

Thus, by Sobolev embeddings and $W \in L^\infty(\Omega)$, there exists a positive constant C such that

$$I_\lambda(u) \geq \frac{1}{q}\left(\frac{\lambda_1 - \lambda}{\lambda_1}\right)\|u\|^q - C\|u\|^{q^*} = \|u\|^q\left[\frac{1}{q}\left(\frac{\lambda_1 - \lambda}{\lambda_1}\right) - C\|u\|^{q^*-q}\right].$$

Therefore, since $\lambda < \lambda_1$, we can choose $\|u\| = \rho$ small enough such that there exists $\alpha > 0$ satisfying

$$I_\lambda(u) \geq \rho, \quad \text{for all } \|u\| = \rho.$$

To prove item (ii), let us consider a nontrivial function $w \in C_0^\infty(\Omega_+) \setminus \{0\}$ and $t > 0$. Then, by (a_1),

$$\begin{aligned} I_\lambda(tw) &\leq \frac{k_2}{p}\int_\Omega |\nabla(tw)|^p\, dx + \frac{1}{q}\int_\Omega |\nabla(tw)|^q\, dx - \frac{\lambda}{q}\int_\Omega |tw|^q\, dx - \int_\Omega W(x)|tw|^{q^*}\, dx \\ &< t^{q^*}\left[t^{p-q^*}\frac{k_2}{p}\int_\Omega |\nabla w|^p\, dx + \frac{t^{q-q^*}}{q}\int_\Omega |\nabla w|^q\, dx - \int_\Omega W(x)|w|^{q^*}\, dx\right]. \end{aligned}$$

Hence, letting $t \to +\infty$,

$$\limsup_{t\to\infty} I_\lambda(tw) \leq -\infty.$$

□

Recall that, if E is a Banach space, $\Phi \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$ we say that Φ satisfies the Palais-Smale condition at level c (shortly: Φ satisfies $(PS)_c$) if every sequence $(u_n) \in E$ such that $\Phi(u_n) \to c$ in $\mathbb{R}$ and $\Phi'(u_n) \to 0$ in E', as $n \to \infty$, admits a subsequence that converges for a critical point of Φ. This sequence is called a $(PS)_c$ sequence for Φ.

Notice that Lemma 5.1 ensures us the existence of a $(PS)_{c_\lambda}$ sequence for the functional $I_\lambda : W_0^{1,q}(\Omega) \to \mathbb{R}$, where

$$c_\lambda = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} I_\lambda(\eta(t)) > 0,$$

and

$$\Gamma := \{\eta \in C([0,1], X) : \eta(0) = 0,\ I_\lambda(\eta(1)) < 0\}.$$

Lemma 5.2. *If $\lambda < \lambda_1$ and $(u_n) \subset W_0^{1,q}(\Omega)$ is a $(PS)_c$ sequence for I_λ, then (u_n) is bounded in $W_0^{1,q}(\Omega)$.*

Proof. Let $(u_n) \subset W_0^{1,q}(\Omega)$ be a $(PS)_c$ sequence for I_λ. Then, by (a_1) and Poincaré inequality for $W_0^{1,q}(\Omega)$,

$$\begin{aligned} c + o_n(1)\|u_n\| &= I_\lambda(u_n) - \frac{1}{q^*} I'_\lambda(u_n)u_n \\ &= \frac{1}{p}\int_\Omega A(|\nabla u_n|^p)\,dx - \frac{1}{q^*}\int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^p\,dx - \lambda\left(\frac{1}{q} - \frac{1}{q^*}\right)\int_\Omega |u_n|^q\,dx \\ &\geq \left(\frac{1}{q} - \frac{1}{q^*}\right)\left(k_2\int_\Omega |\nabla u_n|^p\,dx + \int_\Omega |\nabla u_n|^q\,dx\right) \\ &\quad - \frac{\lambda}{\lambda_1}\left(\frac{1}{q} - \frac{1}{q^*}\right)\int_\Omega |\nabla u_n|^q\,dx. \end{aligned}$$

Hence,

$$c + o_n(1)\|u_n\| \geq \left(1 - \frac{\lambda}{\lambda_1}\right)\left(\frac{1}{q} - \frac{1}{q^*}\right)\|u_n\|^q,$$

which implies that (u_n) is bounded in $W_0^{1,q}(\Omega)$. □

Lemma 5.3. *If $\lambda < \lambda_1$, then*

i) $\int_\Omega W(x)|\varphi_1|^{q^*}\,dx > 0.$

ii) $c_\lambda < \left(\frac{k_2}{k_1 p} + \frac{1}{N}\right)\int_\Omega W(x)|\varphi_1|^{q^*}\,dx$

Proof. Using Lemma 5.1, let us consider $t_\alpha > 0$ such that $J_\lambda(t_\alpha\varphi_1) = \max_{t\geq 0} I_\lambda(t\varphi_1)$. Then, by (a_1),

$$\begin{aligned} t_\alpha^{q^*}\int_\Omega W(x)|\varphi_1|^{q^*}\,dx &= \int_\Omega a(|\nabla(t_\alpha^p\varphi_1)|^p)|\nabla(t_\alpha^p\varphi_1)|^p\,dx - \lambda t_\alpha^q\int_\Omega |\varphi_1|^q\,dx \\ &\geq k_1 t_\alpha^p\int_\Omega |\nabla\varphi_1|^p\,dx + t_\alpha^q(\lambda_1 - \lambda)\int_\Omega |\varphi_1|^q\,dx > 0. \end{aligned} \tag{5.1}$$

This shows the first item. Moreover, with the same argument as in Lemma 2.4,

$$1 \geq t_\alpha \geq \left[\frac{k_1\int_\Omega |\nabla\varphi_1|^p\,dx}{\int_\Omega W(x)|\varphi_1|^{q^*}\,dx}\right]^{\frac{1}{q^*-q}} > 0.$$

Therefore, by (a_1) and (5.1),

$$\begin{aligned} c_\lambda &\le I_\lambda(t_\alpha\varphi_1) \\ &\le k_2\frac{t_\alpha^p}{p}\int_\Omega |\nabla\varphi_1|^p\,dx + \frac{t_\alpha^q}{q}(\lambda_1-\lambda)\int_\Omega |\varphi_1|^q\,dx - \frac{t_\alpha^{q^*}}{q^*}\int_\Omega W(x)|\varphi_1|^{q^*}\,dx \\ &\le \left(\frac{k_2}{k_1 p}+\frac{1}{q}-\frac{1}{q^*}\right)t_\alpha^{q^*}\int_\Omega W(x)|\varphi_1|^{q^*}\,dx \\ &< \left(\frac{k_2}{k_1 p}+\frac{1}{N}\right)\int_\Omega W(x)|\varphi_1|^{q^*}\,dx. \end{aligned} \tag{5.2}$$

The proof of the theorem is complete. □

Proposition 5.4. *If $\lambda < \lambda_1$ and*

$$\int_\Omega W(x)|\varphi_1|^{q^*}\,dx < \frac{\frac{1}{N}(\mathcal{W}_2 - \|W^-\|_\infty)\left(\frac{S}{\mathcal{W}_1}\right)^{\frac{N}{q}}}{\frac{k_2}{k_1 p}+\frac{1}{N}}, \tag{5.3}$$

where $\mathcal{W}_1$, $\mathcal{W}_2$ are positive constants given by (W_3), then I_λ has a nontrivial critical point.

Proof. By Proposition 5.1, let $(u_n)\subset W_0^{1,q}(\Omega)$ be a $(PS)_{c_\lambda}$-sequence for functional I_λ which is bounded in $W_0^{1,q}(\Omega)$ by Lemma 5.2. Then, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } W_0^{1,q}(\Omega), \\ u_n \to u & \text{strongly in } L^s(\Omega) \text{ for any } 1\le s<q^*, \\ u_n(x)\to u(x) & \text{for a.e. } x\in\Omega, \end{cases} \tag{5.4}$$

for some $u\in W_0^{1,q}(\Omega)$. From the Sobolev embeddings, we can conclude that u is a critical point of I_λ.

Now we are going to show that u is nontrivial. Suppose, by contradiction, that $u=0$ in $\Omega_* \subset\subset \Omega_+$, where Ω_* is a open set given by (W_3). Since (u_n) is bounded in $W_0^{1,q}(\Omega)$ and using the Lions's Concentration Compactness Principle [18], we may suppose that

$$|\nabla u_n|^q \rightharpoonup \mu \quad\text{and}\quad |u_n|^{q^*}\rightharpoonup \nu,$$

for some measures μ and ν. Hence, we obtain an at most countable index set Γ, sequences $(x_i)\subset\Omega_*$ and (μ_i), $(\nu_i)\subset(0,\infty)$ such that

$$\mu \ge |\nabla u|^q + \sum_{i\in\Gamma}\mu_i\delta x_i, \quad \nu = |u|^{q^*}+\sum_{i\in\Gamma}\nu_i\delta x_i \quad\text{and}\quad S\nu_i^{q/q^*}\le\mu_i, \tag{5.5}$$

for all $i\in\Gamma$, where δ_{x_i} is the Dirac mass at $x_i\in\Omega_*$ and $S>0$ is the best constant of the Sobolev embedding $W_0^{1,q}(\Omega)\hookrightarrow L^{q^*}(\Omega)$. Thus it is sufficient to show that $\{x_i\}_{i\in\Gamma}\cap\Omega_* = \emptyset$. Then we suppose, by contradiction, that $x_i\in\Omega_*$ for some $i\in\Gamma$. Consider $R>0$ and the function $\psi_R(x) := \psi(x_i - x)$, where $\psi\in C_0^\infty(\Omega_*,[0,1])$ is such that $\psi=1$ on $B_R(x_i)$, $\psi=0$ on $\Omega\backslash B_{2R}(x_i)$ and $|\nabla\psi|_\infty\le 2$. We suppose also that $R>0$ is chosen in such way that $I'_\mu(u_n)\psi_R u_n = o_n(1)$, we obtain

$$\begin{aligned} \int_\Omega \psi_R\, a(|\nabla u_n|^p)|\nabla u_n|^p\,dx = &-\int_\Omega u_n a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\psi_R\,dx \\ &+\lambda\int_\Omega |u_n|^q\psi_R\,dx + \int_\Omega W(x)|u_n|^{q^*}\psi_R dx + o_n(1). \end{aligned}$$

One gets from the weakly convergence $u_n \rightharpoonup u = 0$ that

$$\int_\Omega u_n a(|\nabla u_n|^p)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla \psi_R \, dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \lambda \int_\Omega |u_n|^q \psi_R \, dx = 0.$$

Consequently, by (5) and (a_1), as $n \to +\infty$,

$$\int_\Omega |\nabla u_n|^q \psi_R \, dx \le \int_\Omega a(|\nabla u_n|^p)|\nabla u_n|^p \psi_R \, dx = \int_\Omega W(x)|u_n|^{q^*}\psi_R dx + o_n(1).$$

Since ψ_R has compact support, taking $n \to \infty$ in the above expression, we have

$$\int_\Omega \psi_R d\mu \le \int_\Omega \psi_R W(x) d\nu,$$

which implies that

$$\mu_i \le \mathcal{W}_1 \nu_i,$$

where $\mathcal{W}_1 \ge W(x) \ge \mathcal{W}_2 > 0$ for all $x \in \Omega_* \subset\subset \Omega_+$. Since $\mu_i > 0$, then $x_i \in \Omega_*$. Therefore, from (5.5), we get

$$\left(\frac{S}{\mathcal{W}_1}\right)^{\frac{q^*}{q^*-q}} \le \nu_i. \tag{5.6}$$

On the other hand, (u_n) is a $(PS)_{c_\lambda}$-sequence for functional I_λ then, arguing as Proposition 2.3, we have

$$\int_\Omega W(x)|u_n|^{q^*} dx + o_n(1) > 0. \tag{5.7}$$

Since sequence $u_n \rightharpoonup u = 0$ weakly in $W_0^{1,q}(\Omega)$, $\psi_R \in C_0^\infty(\Omega_*; [0,1])$ and $|\Omega_-| \ge |\Omega_*|$, we obtain

$$\begin{aligned}
c_\lambda &= I_\lambda(u_n) - \frac{1}{q} I'_\lambda(u_n) u_n \psi_R + o_n(1) \\
&= \left(\frac{1}{q} - \frac{1}{q^*}\right) \int_\Omega W(x)|u_n|^{q^*} dx + o_n(1) \\
&= \frac{1}{N}\left[\int_{\Omega_+} W(x)|u_n|^{q^*} dx + \int_{\Omega_-} W(x)|u_n|^{q^*} dx\right] + o_n(1) \\
&\ge \frac{1}{N}\left[\int_{\Omega^*} W(x)|u_n|^{q^*} dx - \|W^-\|_\infty \int_{\Omega_-} |u_n|^{q^*} dx\right] + o_n(1) \\
&\ge \frac{1}{N}\left(\mathcal{W}_2 - \|W^-\|_\infty\right) \int_{\Omega^*} |u_n|^{q^*} dx + o_n(1) \\
&\ge \frac{1}{N}\left(\mathcal{W}_2 - \|W^-\|_\infty\right) \int_{\Omega^*} \psi_R |u_n|^{q^*} dx + o_n(1).
\end{aligned}$$

Therefore, using (5.5) and (5.6), we get

$$\begin{aligned}
c_\lambda &\ge \frac{1}{N}\left(\mathcal{W}_2 - \|W^-\|_\infty\right) \sum_{i\in\Gamma} \psi_R(x_i)\nu_i \\
&= \frac{1}{N}\left(\mathcal{W}_2 - \|W^-\|_\infty\right) \nu_i \\
&\ge \frac{1}{N}\left(\mathcal{W}_2 - \|W^-\|_\infty\right) \left(\frac{S}{\mathcal{W}_1}\right)^{\frac{N}{q}}.
\end{aligned}$$

Since (5.3) holds, we obtain a contradiction by Lemma 5.3. Hence, $u \in W_0^{1,q}(\Omega)$ is a nontrivial solution. $\square$

5.1 Proof of Theorem 1.2.

Proof. If $\lambda < \lambda_1$, by Proposition 5.1 and Lemma 5.2, we have that there exists a critical point u_λ of I_λ. Thus, if

$$\int_\Omega W(x)|\varphi_1|^{q^*}\,dx < \frac{\frac{1}{N}\left(\mathcal{W}_2 - \|W^-\|_\infty\right)\left(\frac{S}{\mathcal{W}_1}\right)^{\frac{N}{q}}}{\frac{k_2}{k_1 p} + \frac{1}{N}},$$

then, by Proposition 5.4, u_λ is nontrivial solution. Moreover, using the same arguments as in Proposition 3.3, one can immediately shows that $-u_\lambda$ is also a nontrivial solution. □

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On the analytic commutator for Λ–Ω differential systems

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Abstract. In this paper, we give the necessary and sufficient conditions for some Λ–Ω differential systems to have an analytic commutator, use these properties to judge the origin point of the Λ–Ω differential systems to be an isochronous center.

Keywords: analytic commutator, isochronous center, Λ–Ω system, center conditions.

2020 Mathematics Subject Classification: 34C07, 34C05, 34C25, 37G15.

1 Introduction

Consider differential systems of the form

$$\begin{cases} x' = -y + P, \\ y' = x + Q, \end{cases} \tag{1.1}$$

where $P = \sum_{k=2}^{\infty} P_k(x,y)$ and $Q = \sum_{k=2}^{\infty} Q_k(x,y)$ where, P_k and Q_k are homogeneous polynomials in x and y of degree k . If every orbit in a punctured neighbourhood of O is a nontrivial cycle then the origin point $O(0,0)$ is said to be a center. In particular, if every cycle in a punctured neighbourhood of O has the same period then this origin point is said to be an isochronous center. Christian Huygens is credited with being one of the first scholars to study isochronous systems in the XVII century, even before the development of the differential calculus. Huygens investigated the cycloidal pendulum, which has isochronous oscillations in opposition to the monotonicity of the period of the usual pendulum. It is probably the first example of a nonlinear isochrone. For more details see [10, 12]. However, it is far from being completely resolved, beside some specific families of vector fields [4, 7].

By [1], we know that for any analytic system (1.1), the existence of an analytic commutator with linear part $(x,y)^t$ is a necessary and sufficient condition for the origin to be an isochronous center. In [2, 3] Algaba and Reyes have studied a particular case of this family are the plane polynomial systems which have a center focus equilibrium at the origin and whose angular speed is constant. In these systems, the origin is the only finite equilibrium and if

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it is a center, it will be automatically isochronous. These systems, up to a linear change of variable, have the following form:

$$x' = -y + xH(x,y), \qquad y' = x + yH(x,y), \tag{1.2}$$

$H(0,0) = 0$. They pointed out that if (1.2) has an analytic commutator then it is in the form of $(U,V)^t = (xK(x,y), yK(x,y))^t$, where K and H are polynomials of the same degree. They also characterize the system $x' = -y + P_s(x,y) + xH(x,y)$, $y' = x + Q_s(x,y) + yH(x,y)$, where $H(x,y)$ is a polynomial with degree greater than or equal to s, if it has a polynomial commutator, then it is in the form of $(U,V)^t = (u_s(x,y) + xK(x,y), v_s(x,y) + yK(x,y))^t$, here P_s, Q_s, u_s, v_s are homogeneous polynomials of degree s.

There are only a few families of polynomial differential systems in which a complete classification of the isochronous centers is known, and almost all of them have polynomial commutator. The quadratic isochronous centers, characterized by Loud [17]. In Pleshkan [18], cubic isochronous centers with homogeneous nonlinear part are settled. In Christopher and Devlin[6], the isochronous centers of the Kukles family are obtained. Commutators of quadratic centers are computed in Sabatini [19]; commutators of cubic systems with homogeneous nonlinear part can be found in Gasull et al. [11]; commutators for the Kukles system can be seen in Volokitin and Ivanov [20]. The first example of a polynomial isochronous center without polynomial commutator is found in Devlin [8].

A center of (1.1) is called a **Weak Center** if the Poincaré–Liapunov first integral can be written as $H = \frac{1}{2}(x^2 + y^2)(1 + h.o.t.)$. By literature [13]-[16] we know that a center of an analytic or polynomial differential system (1.1) is a weak center if and only if it can be written as

$$\begin{cases} x' = -y\bar{\Lambda} + x\Omega, \\ y' = x\bar{\Lambda} + y\Omega, \end{cases} \tag{1.3}$$

where $\bar{\Lambda} = 1 + \Lambda(x,y)$ and $\Omega = \Omega(x,y)$ are analytic functions or polynomials such that $\Lambda(0,0) = \Omega(0,0) = 0$. The class of differential systems (1.3) is called the **Λ–Ω system**. The weak centers contain the uniform isochronous centers and the holomorphic isochronous centers [13], they also contain the class of centers studied by Alwash and Lloyd [5], but they do not coincide with all classes of isochronous centers [13], because in general weak centers are not isochronous.

In [14, 16] Llibre et al. put forward such conjecture.

Conjecture. The polynomial differential system of degree m

$$\begin{cases} x' = -y(1 + \mu(a_2x - a_1y)) + x((a_1x + a_2y) + \Phi_{m-1}(x,y)), \\ y' = x(1 + \mu(a_2x - a_1y)) + y((a_1x + a_2y) + \Phi_{m-1}(x,y)), \end{cases} \tag{1.4}$$

where $(\mu + m - 2)(a_1^2 + a_2^2) \neq 0$, and $\Phi_{m-1}(x,y)$ is a homogeneous polynomial of degree $m - 1$ has a weak center at the origin if and only if system (1.4) after a linear change of variables $(x,y) \to (X,Y)$ is invariant under the transformations $(X,Y,t) \to (-X,Y,-t)$. They have proved the conjecture holds for $m = 2,3,4,5,6$. And remarked that the only difficulty for proving conjecture for the Λ–Ω systems of degree m with $m > 6$ is the huge number of computations for obtaining the conditions that characterize the centers. In [21, 22] we use a method different from Llibre [14] and more simply, without huge number of computation, get the necessary and sufficient conditions for the origin point of Λ–Ω systems:

$$\begin{cases} x' = -y(1 + \mu(a_2x - a_1y)) + x(\nu(a_1x + a_2y) + \Psi_{m-1} + \Psi_{2m-1}), \\ y' = x(1 + \mu(a_2x - a_1y)) + y(\nu(a_1x + a_2y) + \Psi_{m-1} + \Psi_{2m-1}) \end{cases}$$

and

$$\begin{cases} x' = -y(1+\mu(a_2x-a_1y)) + x(\nu(a_1x+a_2y)+\Psi_2+\Psi_n), \\ y' = x(1+\mu(a_2x-a_1y)) + y(\nu(a_1x+a_2y)+\Psi_2+\Psi_n), \end{cases}$$

where $m > 2$, $n \geq 5$ and Ψ_k is a homogeneous polynomial of degree k, to be a center. Of special note is that the function Ω in the above two systems is a polynomial of missing some terms. For the polynomial differential system of higher degree, especially when the polynomial has no missing any term, it is difficult to derive the necessary conditions for the singular point being a center by either the Lyapunov's power series method or Poincaré's successor function method. Although according to Hilbert's finite basis theory, the necessary conditions must be obtained in finite steps, how much this finite number is very difficult to know [9]. To avoid finding this finite number, we will find the central conditions by determining when it has an analytic commutator.

In the following we will discuss the center problem for the Λ–Ω system (1.3) with Ω no missing any terms. Specifically, consider Λ–Ω differential systems:

$$\begin{cases} x' = -y(1+\mu(a_1y-a_2x)) + x(\lambda(a_1x+a_2y)+H(x,y)), \\ y' = x(1+\mu(a_1y-a_2x)) + y(\lambda(a_1x+a_2y)+H(x,y)) \end{cases} \tag{1.5}$$

and

$$\begin{cases} x' = -y(1+\mu(a_1y-a_2x)+\phi_2(x,y)) + x(\lambda(a_1x+a_2y)+\psi_2(x,y)+H(x,y)), \\ y' = x(1+\mu(a_1y-a_2x)+\phi_2(x,y)) + y(\lambda(a_1x+a_2y)+\psi_2(x,y)+H(x,y)), \end{cases} \tag{1.6}$$

where λ, μ, a_1, a_2 are real numbers such that $\mu(a_1^2+a_2^2) \neq 0$, $H(x,y) = \sum_{k=2}^{\infty} h_k(x,y)$ or $H(x,y) = \sum_{k=3}^{n} h_k(x,y)$, $h_k(x,y)$ is a homogeneous polynomial of degree k. We will give the necessary and sufficient conditions for these two families of differential systems to have a polynomial commutator or analytic commutator, apply the obtained results to judge the origin point of their to be a center (isochronous center, weak center).

2 Analytic commutator

As $a_1^2+a_2^2 \neq 0$, taking $X = a_1x+a_2y$, $Y = a_1y-a_2x$, the system (1.5) becomes

$$\begin{cases} X' = -Y(1+\mu Y) + X(\lambda X + H(X,Y)), \\ Y' = X(1+\mu Y) + Y(\lambda X + H(X,Y)). \end{cases}$$

For convenience, let us consider

$$\begin{cases} x' = -y(1+\mu y) + x(\lambda x + H(x,y)) = P(x,y), \\ y' = x(1+\mu y) + y(\lambda x + H(x,y)) = Q(x,y), \end{cases} \tag{2.1}$$

where $\mu \neq 0$, $H = \sum_{i=2}^{\infty} h_i(x,y)$, $h_i(x,y)$ is homogeneous polynomials of degree i. By [2], if system (2.1) has an analytic commutator, then either it has the form

$$(U,V)^t = (x + u_2 + xK(x,y),\ y + v_2 + yK(x,y))^t \tag{2.2}$$

or

$$(U,V)^t = (u_2 + xK(x,y),\ v_2 + yK(x,y))^t,$$

where $K = \sum_{i=2}^{\infty} k_i(x,y)$, $k_i(x,y)$ is homogeneous polynomials of degree i. In this paper, we are only interested in the center problem for system (2.1), therefore, we will only discuss when does (2.1) have a commutator in the form of (2.2)?

Lemma 2.1. *If $xh_{n-1} + yk_{n-1} = 0$, then*

$$u_2h_{n-1} - P_2k_{n-1} - x(h_{n-1\,x}u_2 + h_{n-1\,y}v_2 - k_{n-1\,x}P_2 - k_{n-1\,y}Q_2) = \mu(n-3)(x^2+y^2)k_{n-1}, \quad (2.3)$$

where $u_2 = 2\mu xy$, $v_2 = \mu(y^2 - x^2)$, $P_2 = \mu(x^2 - y^2)$, $Q_2 = 2\mu xy$.

Proof. As $xh_{n-1} + yk_{n-1} = 0$, $xh_{n-1\,x} = -h_{n-1} - yk_{n-1\,x}$, $xh_{n-1\,y} = -k_{n-1} - yk_{n-1\,y}$, thus

$$\begin{aligned} & x(h_{n-1\,x}u_2 + h_{n-1\,y}v_2 - k_{n-1\,x}P_2 - k_{n-1\,y}Q_2) \\ &= -u_2(h_{n-1} + yk_{n-1\,x}) - v_2(k_{n-1} + yk_{n-1\,y}) - k_{n-1\,x}P_2 - k_{n-1\,y}Q_2 \\ &= \mu(x^2+y^2)k_{n-1} - \mu(x^2+y^2)(xk_{n-1\,x} + yk_{n-1\,y}) \\ &= (2-n)\mu(x^2+y^2)k_{n-1}, u_2h_{n-1} - P_2k_{n-1} \\ &= 2\mu xyh_{n-1} - \mu(x^2-y^2)k_{n-1} = -\mu(x^2+y^2)k_{n-1}. \end{aligned}$$

Add the above two equations, it follows that equation (2.3) is valid. □

Lemma 2.2. *For n-th degree homogeneous polynomial functions $h_n(x,y)$ and $k_n(x,y)$, if they satisfy*

$$xh_n + yk_n = 0, \quad (2.4)$$

and

$$x(nh_n + yk_{n\,x} - xk_{n\,y}) = (n-3)\mu(x^2+y^2)k_{n-1}, \qquad (n = 3,4,\dots) \quad (2.5)$$

then

$$h_n = \sum_{j=0}^{n-3}(-1)^{j+1}\mu^{j+1}\lambda_{n-j}C_{n-3}^{j}x^{n-1-j}y^{j+1}, \qquad (n = 3,4,5,\dots.) \quad (2.6)$$

and

$$k_n = \sum_{j=0}^{n-3}(-1)^{j}\lambda_{n-j}\mu^{j}C_{n-3}^{j}x^{n-j}y^{j}, \qquad (n = 3,4,5,\dots), \quad (2.7)$$

where λ_i $(i = 3,4,\dots)$ are real numbers.

Proof. Based on the assumptions, when $n = 3$ we have

$$xh_3 + yk_3 = 0,\ 3h_3 + yk_{3\,x} - xk_{3\,y} = 0.$$

Putting $x = \cos\theta$, $y = \sin\theta$, the above equations become

$$\cos\theta h_3(\cos\theta, \sin\theta) + \sin\theta k_3(\cos\theta, \sin\theta) = 0,$$

$$\frac{dk_3(\cos\theta,\sin\theta)}{d\theta} = -3\tan\theta k_3(\cos\theta,\sin\theta),$$

solving these equations we deduce that

$$k_3(\cos\theta,\sin\theta) = \lambda_3\cos^3\theta.$$

Similarly, when $n = 4$ we obtain

$$\frac{dk_4(\cos\theta,\sin\theta)}{d\theta} = -4\tan\theta k_4(\cos\theta,\sin\theta) - \mu\sec\theta k_3(\cos\theta,\sin\theta),$$

solving this linear equation we have

$$k_4(\cos\theta,\sin\theta)=\cos^4\theta(\lambda_4-\lambda_3\mu\tan\theta).$$

Suppose that

$$k_n(\cos\theta,\sin\theta)=\cos^n\theta\sum_{j=0}^{n-3}(-1)^jC_{n-3}^j\lambda_{n-j}\mu^j\tan^j\theta. \tag{2.8}$$

Next we will prove that (2.8) is also true when n is replaced by $n+1$.

In fact, by assuming we obtain

$$\frac{dk_{n+1}(\cos\theta,\sin\theta)}{d\theta}=-(n+1)\tan\theta k_{n+1}-(n-2)\mu\sec\theta k_n(\cos\theta,\sin\theta).$$

Substituting (2.8) into the above yields

$$\frac{dk_{n+1}(\cos\theta,\sin\theta)}{d\theta}=-(n+1)\tan\theta k_{n+1}-(n-2)\cos^{n-1}\theta\left(\sum_{j=0}^{n-3}(-1)^jC_{n-3}^j\lambda_{n-j}\mu^{j+1}\tan^j\theta\right),$$

solving this linear equation we get

$$\begin{aligned}k_{n+1}&=\cos^{n+1}\theta\left(\lambda_{n+1}-(n-2)\int\cos^{-2}\theta\left(\sum_{j=0}^{n-3}(-1)^jC_{n-3}^j\lambda_{n-j}\mu^{j+1}j\tan^j\theta\right)d\theta\right)\\&=\cos^{n+1}\theta\left(\lambda_{n+1}-(n-2)\sum_{j=0}^{n-3}(-1)^jC_{n-3}^j\lambda_{n-j}\frac{1}{j+1}\mu^{j+1}\tan^{j+1}\theta\right)\\&=\cos^{n+1}\theta\sum_{j=0}^{n-2}(-1)^jC_{n-2}^j\lambda_{n+1-j}\mu^j\tan^j\theta.\end{aligned}$$

Therefore, by mathematical induction, the relation (2.8) is valid for any $n\geq 3$. So, the relations (2.6) and (2.7) are valid. □

Theorem 2.3. *The system* (2.1) *has an analytic commutator in the form of* (2.2), *if and only if*

$$\lambda=\mu,$$

$$u_2=2\mu xy,\quad v_2=\mu(y^2-x^2),$$

$$h_2=-\lambda_2xy,\quad k_2=\lambda_2x^2,$$

$$h_n=\sum_{j=0}^{n-3}(-1)^{j+1}\lambda_{n-j}C_{n-3}^jx^{n-1-j}y^{j+1},\quad (n=3,4,5,\dots)$$

$$k_n=\sum_{j=0}^{n-3}(-1)^j\lambda_{n-j}C_{n-3}^jx^{n-j}y^j,\qquad (n=3,4,5,\dots),$$

where λ_i $(i=2,3,4,\dots)$ are real numbers.

Moreover, the origin point of (2.1) *is a center and isochronous center.*

Proof. By [2], the vector (2.2) is an commutator of system (2.1) if and only if the Lie bracket vanishes, that is,

$$\begin{pmatrix} U_x & U_y \\ V_x & V_y \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \end{pmatrix} - \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} \cdot \begin{pmatrix} U \\ V \end{pmatrix} = 0, \tag{2.9}$$

expanding it

$$\Big(1 + u_{2x} + \sum_{i=2}^{\infty}(xk_i)_x\Big)\Big(-y + P_2 + \sum_{i=2}^{\infty} xh_i\Big) + \Big(u_{2y} + \sum_{i=2}^{\infty}(xk_i)_y\Big)\Big(x + Q_2 + \sum_{i=2}^{\infty} yh_i\Big)$$
$$= \Big(P_{2x} + \sum_{i=2}^{\infty}(xh_i)_x\Big)\Big(x + u_2 + \sum_{i=2}^{\infty} xk_i\Big) + \Big(-1 + P_{2y} + \sum_{i=2}^{\infty}(xh_i)_y\Big)\Big(y + v_2 + \sum_{i=2}^{\infty} yk_i\Big), \tag{2.10}$$

$$\Big(v_{2x} + \sum_{i=2}^{\infty}(yk_i)_x\Big)\Big(-y + P_2 + \sum_{i=2}^{\infty} xh_i\Big) + \Big(1 + v_{2y} + \sum_{i=2}^{\infty}(yk_i)_y\Big)\Big(x + Q_2 + \sum_{i=2}^{\infty} yh_i\Big)$$
$$= \Big(1 + Q_{2x} + \sum_{i=2}^{\infty}(yh_i)_x\Big)\Big(x + u_2 + \sum_{i=2}^{\infty} xk_i\Big) + \Big(Q_{2y} + \sum_{i=2}^{\infty}(yh_i)_y\Big)\Big(y + v_2 + \sum_{i=2}^{\infty} yk_i\Big), \tag{2.11}$$

where $P_2 = \lambda x^2 - \mu y^2$, $Q_2 = (\lambda + \mu)xy$.

From the terms of degree 2 of (2.10) and (2.11) equal to zero follows that

$$\begin{cases} v_2 = P_2 + yu_{2x} - xu_{2y}, \\ u_2 = -Q_2 + xv_{2y} - yv_{2x}. \end{cases} \tag{2.12}$$

Solving (2.12) we get

$$u_2 = (\lambda + \mu)xy,\ v_2 = \lambda y^2 - \mu x^2. \tag{2.13}$$

By the terms of degree 3 of (2.10) and (2.11) equal to zero we obtain

$$\begin{cases} u_{2x}P_2 + u_{2y}Q_2 - P_{2x}u_2 - P_{2y}v_2 = x(2h_2 + yk_{2x} - xk_{2y}), \\ v_{2x}P_2 + v_{2y}Q_2 - Q_{2x}u_2 - Q_{2y}v_2 = y(2h_2 + yk_{2x} - xk_{2y}). \end{cases} \tag{2.14}$$

The first equation of above multiplied by y minus the second equation multiplied by x, we deduce that

$$P_2(yu_{2x} - xv_{2x}) + Q_2(yu_{2y} - xv_{2y}) = u_2(yP_{2x} - xQ_{2x}) + v_2(yP_{2y} - xQ_{2y}),$$

which yields $\mu(\lambda - \mu) = 0$, in view of $\mu \neq 0$, then $\lambda = \mu$. Therefore,

$$u_2 = 2\mu xy, \qquad v_2 = \mu(y^2 - x^2), \qquad P_2 = \mu(x^2 - y^2), \qquad Q_2 = 2\mu xy. \tag{2.15}$$

Substituting (2.15) into (2.14) which follows that

$$2h_2 + yk_{2x} - xk_{2y} = 0. \tag{2.16}$$

From the terms of degree 4 of equations (2.10) and (2.11) equal to zero follows that

$$u_2h_2 - P_2k_2 = x(3h_3 + h_{2x}u_2 + h_{2y}v_2 - k_{2x}P_2 - k_{2y}Q_2 + yk_{3x} - xk_{3y}), \tag{2.17}$$
$$v_2h_2 - Q_2k_2 = y(3h_3 + h_{2x}u_2 + h_{2y}v_2 - k_{2x}P_2 - k_{2y}Q_2 + yk_{3x} - xk_{3y}). \tag{2.18}$$

Equation (2.17) multiplied by y minus (2.18) multiplied by x which implies that

$$h_2(yu_2 - xv_2) = k_2(yP_2 - xQ_2),$$

substituting (2.15) into the above equation we get

$$xh_2 + yk_2 = 0. \tag{2.19}$$

Solving equations (2.16) and (2.19) we deduce that

$$h_2 = -\lambda_2 xy,\ k_2 = \lambda_2 x^2, \tag{2.20}$$

where λ_2 is a constant. Substituting (2.20) into (2.17) we get

$$3h_3 + yk_{3\,x} - xk_{3\,y} = 0. \tag{2.21}$$

Similarly, by the terms of degree $n+1$ of equations (2.10) and (2.11) equal to zero we deduce that

$$u_2h_{n-1} - P_2k_{n-1} = x(nh_n + h_{n-1\,x}u_2 + h_{n-1\,y}v_2 - k_{n-1\,x}P_2 - k_{n-1\,y}Q_2 + yk_{n\,x} - xk_{n\,y}), \tag{2.22}$$

$$v_2h_{n-1} - Q_2k_{n-1} = y(nh_n + h_{n-1\,x}u_2 + h_{n-1\,y}v_2 - k_{n-1\,x}P_2 - k_{n-1\,y}Q_2 + yk_{n\,x} - xk_{n\,y}), \tag{2.23}$$

from these equations follow that

$$xh_{n-1} + yk_{n-1} = 0. \tag{2.24}$$

Using (2.24) and Lemma 2.1 we get

$$u_2h_{n-1} - P_2k_{n-1} - x(h_{n-1\,x}u_2 + h_{n-1\,y}v_2 - k_{n-1\,x}P_2 - k_{n-1\,y}Q_2) = \mu(n-3)(x^2+y^2)k_{n-1}.$$

Substituting this equation into (2.22) which yields that

$$x(nh_n + yk_{n\,x} - xk_{n\,y}) = (n-3)\mu(x^2+y^2)k_{n-1}. \tag{2.25}$$

Similarly, using the terms of degree $n+2$ of (2.10) and (2.11) equal to zero we obtain

$$xh_n + yk_n = 0. \tag{2.26}$$

By equations (2.25) and (2.26) and Lemma 2.1 imply that k_n, h_n can be expressed by (2.6) and (2.7).

By [1,2], the origin point of (2.1) is a center and isochronous center.

In summary, the proof is finished. □

Corollary 2.4. *If in the system (2.1), $H(x,y) = \sum_{i=2}^{n} h_i(x,y), h_i(x,y)\ (i = 2,1,\ldots,n)$ are homogeneous polynomials of degree i, and it has a polynomial commutator in the form of (2.2), if and only if,*

$$\lambda = \mu;\ \ u_2 = 2\mu xy,\ \ v_2 = \mu(y^2 - x^2);\ \ h_2 = -\lambda_2 xy,\ \ k_2 = \lambda_2 x^2;\ \ h_j = k_j = 0\ \ (j = 3,4,\ldots,n).$$

Proof. By the proof of Theorem 2.3 we get that

$$\lambda = \mu, \quad u_2 = 2\mu xy, \quad v_2 = \mu(y^2 - x^2), \quad h_2 = -\lambda_2 xy, \quad k_2 = \lambda_2 x^2$$

and

$$x(jh_j + yk_{j\,x} - xk_{j\,y}) = (j-3)\mu(x^2+y^2)k_{j-1} \qquad (j = 3,4,\ldots,n,n+1)$$

and

$$xh_j + yk_j = 0, \qquad (j = 3,4,\ldots,n,n+1).$$

Taking $j = n+1$, we get $k_n = 0$ and $h_n = 0$, substituting thus into the above equations with $j = n$ which implies that k_{n-1} and $h_{n-1} = 0$, as so on we can deduce $k_j = 0$ and $h_j = 0$, $(j = 3,4,\ldots,n)$. Thus the proof is completed. □

3 Polynomial commutator

In this section we will discuss when does the system (1.6) with $\lambda = \mu \neq 0$ and $h_3 \neq 0$, have a polynomial commutator in the form

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} x + u_2 + u_3 + xk_3(x,y) \\ y + v_2 + v_3 + yk_3(x,y) \end{pmatrix}, \tag{3.1}$$

where $u_l(x,y) = \sum_{i+j=l} u_{ij}x^iy^j$, $v_l(x,y) = \sum_{i+j=l} v_{ij}x^iy^j$, $(l = 2,3)$; $k_3 = \sum_{i+j=3} k_{ij}x^iy^j$.

Without losing generality, suppose that $\mu = 1$, otherwise taking $X = \mu x$, $Y = \mu y$.

First, let us consider system

$$\begin{cases} x' = -y(1+y) + P_3(x,y) + x(x + h_3(x,y)) = P(x,y), \\ y' = x(1+y) + Q_3(x,y) + y(x + h_3(x,y)) = Q(x,y), \end{cases} \tag{3.2}$$

where $P_3 = \sum_{i+j=3} p_{ij}x^iy^j$, $Q_3 = \sum_{i+j=3} q_{ij}x^iy^j$, $h_3 = \sum_{i+j=3} h_{ij}x^iy^j$.

Theorem 3.1. *The system (3.2) with $P_3 \cdot h_3 \neq 0$, has a polynomial commutator in the form of (3.1), if and only if*

$$p_{03}^2 - p_{30}(p_{12} + 2p_{30}) = 0, \tag{3.3}$$

$$p_{21}p_{30} + p_{12}p_{03} + 6p_{30}p_{03} = 0, \tag{3.4}$$

$$Q_3 = p_{03}x^3 - p_{12}x^2y + p_{21}xy^2 - p_{30}y^3, \tag{3.5}$$

$$u_2 = 2xy, \quad v_2 = y^2 - x^2, \tag{3.6}$$

$$\begin{aligned} u_3 &= -(p_{21} + 2p_{03})x^3 - p_{12}x^2y - 3p_{03}xy^2 - p_{30}y^3, \\ v_3 &= (2p_{30} + p_{12})x^3 - p_{21}x^2y + 3p_{30}xy^2 - p_{03}y^3, \end{aligned} \tag{3.7}$$

$$h_3 = (p_{12} + 3p_{30})x(-(p_{12} + 2p_{30})x^2 + (2p_{12} + 3p_{30})y^2) - (p_{21} + 3p_{03})y(p_{12}x^2 + p_{30}y^2), \tag{3.8}$$

$$k_3 = (p_{12}+2p_{30})x^2((p_{21}+3p_{03})x - 3(p_{12}+2p_{30})y) + p_{30}y^2(3(p_{21}+3p_{03})x - (p_{12}+3p_{30})y). \tag{3.9}$$

Moreover, the origin point of (3.2) is a center and isochronous center.

Proof. By (2.9), the vector (3.1) is a commutator of (3.2), if and only if

$$\begin{aligned} &(1 + u_{2x} + u_{3x} + (xk_3)_x)(-y + P_2 + P_3 + xh_3) + (u_{2y} + u_{3y} + (xk_3)_y)(x + Q_2 + Q_3 + yh_3) \\ &\quad = (P_{2x} + P_{3x} + (xh_3)_x)(x + u_2 + u_3 + xk_3) \\ &\qquad + (-1 + P_{2y} + P_{3y} + (xh_3)_y)(y + v_2 + v_3 + yk_3), \end{aligned} \tag{3.10}$$

$$\begin{aligned} &(v_{2x} + v_{3x} + (yk_3)_x)(-y + P_2 + P_3 + xh_3) + (1 + v_{2y} + v_{3y} + (yk_3)_y)(x + Q_2 + Q_3 + yh_3) \\ &\quad = (1 + Q_{2x} + Q_{3x} + (yh_3)_x)(x + u_2 + u_3 + xk_3) \\ &\qquad + (Q_{2y} + Q_{3y} + (yh_3)_y)(y + v_2 + v_3 + yk_3), \end{aligned} \tag{3.11}$$

where $P_2 = x^2 - y^2$, $Q_2 = 2xy$.

Similar to the proof of Theorem 2.3, from the terms of degree 2 of equations (3.10) and (3.11) equal to zero follows that (2.12) and (2.13) are valid. By the terms of degree 3 of equations (3.10) and (3.11) equal to zero we deduce that

$$\begin{cases} v_3 + xu_{3y} - yu_{3x} - 2P_3 = 0, \\ u_3 + yv_{3x} - xv_{3y} + 2Q_3 = 0. \end{cases} \tag{3.12}$$

Equating the same power of x and y of (3.12) which yields that

$$u_{12} = u_{30} + p_{21} - q_{30}, \quad u_{21} = u_{03} - p_{12} - q_{03}, \tag{3.13}$$

$$p_{21} - q_{12} + 3(p_{03} - q_{30}) = 0, \quad p_{12} + q_{21} + 3(p_{30} + q_{03}) = 0. \tag{3.14}$$

$$\begin{aligned} v_{30} &= -u_{03} + 2p_{30} + p_{12} + q_{03}, \quad v_{21} = u_{30} + 2q_{30}, \\ v_{12} &= -u_{03} - 2q_{03}, \quad v_{03} = u_{30} + p_{21} - q_{30} + 2p_{03}. \end{aligned} \tag{3.15}$$

By the terms of degree 4 of equations (3.10) and (3.11) equal to zero we get

$$u_{2\,x}P_3 + u_{2\,y}Q_3 + u_{3\,x}P_2 + u_{3\,y}Q_2 - u_3P_{2\,x} - v_3P_{2\,y} - u_2P_{3\,x} - v_2P_{3\,y} = x(3h_3 + yk_{3\,x} - xk_{3\,y}), \tag{3.16}$$

$$v_{2\,x}P_3 + v_{2\,y}Q_3 + v_{3\,x}P_2 + v_{3\,y}Q_2 - u_3Q_{2\,x} - v_3Q_{2\,y} - u_2Q_{3\,x} - v_2Q_{3\,y} = y(3h_3 + yk_{3\,x} - xk_{3\,y}). \tag{3.17}$$

Equation (3.16) multiplied by y minus (3.17) multiplied by x which implies that

$$\begin{aligned} &P_3(yu_{2\,x} - xv_{2\,x}) + Q_3(yu_{2\,y} - xv_{2\,y}) + P_2(yu_{3\,x} - xv_{3\,y}) + Q_2(yu_{3\,x} - xv_{3\,y}) \\ &\quad = u_3(yP_{2\,x} - xQ_{2\,x}) + v_3(yP_{2\,y} - xQ_{2\,y}) + u_2(yP_{3\,x} - xQ_{x\,x}) + v_2(yP_{3\,y} - xQ_{3\,y}). \end{aligned} \tag{3.18}$$

Comparing the coefficients of the same power of x and y on both sides of the equation (3.18) and (3.16) we obtain

$$u_{03} = p_{12} + q_{03} + q_{21}, \quad u_{30} = -p_{21} + q_{30} - 3p_{03}, \quad u_{30} = -3p_{21} - 2q_{30} + 2q_{12},$$

$$u_{03} = q_{03}, \quad 2u_{03} = -6p_{30} - 5p_{12} - 4q_{03} - 5q_{21}, \quad 2u_{30} = 2p_{21} + 3p_{03} - 4q_{12} - 7q_{30},$$

$$v_{30} = 2p_{30} - q_{21}, \quad u_{12} - 2v_{03} = -p_{03}, \quad 2v_{21} - 3u_{30} = 3p_{21} + 6q_{30} - 2q_{12},$$

$$2u_{21} - 3v_{12} - 6u_{03} = -2p_{12} + 3q_{03}, \quad 3v_{12} - 5v_{30} - 4u_{21} = -4p_{30} + 4p_{12} + 5q_{21} - 3q_{03},$$

$$3u_{30} + 4v_{03} - 4v_{21} - 5u_{12} = -3p_{21} + 5p_{03} + 4q_{12}.$$

According to the above equations and (3.13) and (3.15) we get

$$p_{03} - q_{30} = 0, \quad p_{30} + q_{03} = 0, \quad q_{12} - p_{21} = 0, p_{12} + q_{21} = 0,$$

$$u_{30} = -p_{21} - 2p_{03}, \quad u_{21} = -p_{12}, \quad u_{12} = -3p_{03}, \quad u_{03} = -p_{30},$$

$$v_{30} = p_{12} + 2p_{30}, \quad v_{21} = -p_{21}, \quad v_{12} = 3p_{03}, \quad v_{03} = -p_{03}.$$

Consequently, the relations (3.5) and (3.7) are valid.

Using (3.5)–(3.7) and (3.16) we deduce that

$$3h_3 + yk_{3\,x} - xk_{3\,y} = 0. \tag{3.19}$$

By this equation we get

$$k_3 = -(h_{21} + 2h_{03})x^3 + 3h_{30}x^2y - 3h_{03}xy^2 + (h_{12} + 2h_{30})y^3. \tag{3.20}$$

From the terms of degree 5 of equations (3.10) and (3.11) equal to zero which follows that

$$u_2h_3 - P_2k_3 + u_{3\,x}P_3 + u_{3\,y}Q_3 - P_{3\,x}u_3 - P_{3\,y}v_3 = x(h_{3\,x}u_2 + h_{3\,y}v_2 - k_{3\,x}P_2 - k_{3\,y}Q_2), \tag{3.21}$$

$$v_2h_3 - Q_2k_3 + v_{3\,x}P_3 + v_{3\,y}Q_3 - Q_{3\,x}u_3 - Q_{3\,y}v_3 = y(h_{3\,x}u_2 + h_{3\,y}v_2 - k_{3\,x}P_2 - k_{3\,y}Q_2). \tag{3.22}$$

Equation (3.21) multiplied by y minus (3.22) multiplied by x which implies that

$$\begin{aligned} &h_3(yu_2 - xv_2) - k_3(yP_2 - xQ_2) + P_3(yu_{3x} - xv_{3x}) + Q_3(yu_{3y} - xv_{3y}) \\ &\quad = u_3(yP_{3x} - xQ_{3x}) + v_3(yP_{3y} - xQ_{3y}). \end{aligned} \tag{3.23}$$

Comparing the coefficients of the same power of x and y on both sides of equations (3.21) and (3.23) and using (3.5)–(3.7) and (3.20) we get

$$\begin{aligned} 4h_{03} + h_{21} &= -2p_{21}p_{30} - p_{12}p_{03} - p_{21}p_{12}, \\ 2h_{12} + 5h_{30} &= 2p_{21}p_{03} + 3p_{12}p_{30} + p_{12}^2 + 6p_{03}^2, \\ h_{21} + 5h_{03} &= -3p_{30}p_{03} - p_{12}p_{21} - p_{03}p_{12} - 3p_{30}p_{21}, \\ 3h_{12} + 7h_{30} &= 6p_{03}^2 - 3p_{30}^2 + p_{12}^2 + 2p_{12}p_{30} + 2p_{21}p_{03}, \\ h_{03} &= -3p_{30}p_{03} - p_{30}p_{21}, \\ h_{12} + 2h_{30} &= -3p_{30}^2 - p_{12}p_{30}, \\ h_{30} &= 6p_{30}^2 + 6p_{03}^2 + 5p_{30}p_{12} + 2p_{21}p_{03} + p_{12}^2, \\ h_{12} + 4h_{30} &= 9p_{30}^2 + 12p_{03}^2 + 9p_{30}p_{12} + 4p_{21}p_{03} + 2p_{12}^2. \end{aligned}$$

Simplifying the above equations to obtain

$$h_{30} = 6p_{30}^2 + 6p_{03}^2 + 5p_{30}p_{12} + 2p_{21}p_{03} + p_{12}^2; \tag{3.24}$$

$$h_{21} = 2p_{21}p_{30} - p_{12}p_{03} - p_{21}p_{12} + 12p_{30}p_{03}, \tag{3.25}$$

$$h_{12} = -15p_{30}^2 - 12p_{03}^2 - 11p_{30}p_{12} - 4p_{21}p_{03} - 2p_{12}^2, \tag{3.26}$$

$$h_{03} = -3p_{30}p_{03} - p_{21}p_{30}. \tag{3.27}$$

By the terms of degree 6 of equations (3.10) and (3.11) equal to zero we get

$$2(u_3h_3 - P_3k_3) = x(h_{3x}u_3 + h_{3y}v_3 - k_{3x}P_3 - k_{3y}Q_3), \tag{3.28}$$

$$2(v_3h_3 - Q_3k_3) = y(h_{3x}u_3 + h_{3y}v_3 - k_{3x}P_3 - k_{3y}Q_3). \tag{3.29}$$

Equation (3.28) multiplied by y minus (3.29) multiplied by x we deduce that

$$h_3(yu_3 - xv_3) = k_3(yP_3 - xQ_3). \tag{3.30}$$

Equating the coefficients of the same power of x and y on both sides of the equation (3.28) we obtain

$$\begin{aligned} (p_{21} + 5p_{03})h_{30} - (p_{12} + 3p_{30})h_{21} - 2p_{30}h_{03} &= 0, \\ p_{12}h_{30} + (p_{12} + 2p_{30})h_{12} + (p_{21} + 3p_{03})h_{03} &= 0, \\ 2p_{21}h_{30} + (p_{21} - p_{03})h_{12} + (p_{12} - 3p_{30})h_{03} &= 0, \\ (2p_{12} + 3p_{30})h_{30} + (p_{12} + 2p_{30})h_{12} &= 0, \\ 2p_{03}h_{30} + p_{03}h_{12} + p_{30}h_{03} &= 0. \end{aligned}$$

As $h_3 \neq 0$, the determinant of the coefficient matrix of four equations of the above is equal to zero, that is

$$W_1 = p_{30}(p_{12} + 3p_{30})(p_{21}p_{30} + p_{12}p_{03} + 6p_{30}p_{03}) = 0. \tag{3.31}$$

Equating the coefficients of the same power of x and y on both sides of the equation (3.30) we get

$$\begin{aligned}(p_{12}+2p_{30})h_{30}+p_{03}h_{21}+2p_{03}h_{03}&=0,\\ p_{03}h_{30}-p_{30}h_{21}+2(p_{12}+p_{30})h_{03}&=0,\\ 2(p_{12}+p_{30})h_{30}+(p_{12}+2p_{30})h_{12}-p_{03}h_{03}&=0,\\ 2p_{03}h_{30}+p_{03}h_{12}+p_{30}h_{03}&=0.\end{aligned}$$

The determinant of the coefficient matrix of the above equations is equal to zero, that is

$$W_2=-(p_{30}(p_{12}+2p_{30})-p_{03}^2)^2=0. \tag{3.32}$$

By (3.31) and (3.32) and $P_3\cdot h_3\neq 0$, which implies that $p_{30}\cdot p_{03}\cdot(p_{12}+2p_{30})\neq 0$ and the relations (3.3) and (3.4) are valid and

$$h_{12}=-\frac{2p_{12}+2p_{30}}{p_{12}+2p_{30}}h_{30},\quad h_{21}=-\frac{p_{12}p_{30}}{p_{03}(p_{12}+2p_{30}}h_{30},\quad h_{03}=-\frac{p_{03}}{p_{12}+2p_{30}}h_{30}.$$

Using (3.4) and (3.5) and (3.24)–(3.27) we get

$$\begin{gathered}h_{30}=-(p_{12}+2p_{30})(p_{12}+3p_{30}),\quad h_{21}=-p_{12}(p_{21}+3p_{03}),\\ h_{12}=(p_{12}+3p_{30})(3p_{30}+2p_{12}),\quad h_{03}=-p_{30}(p_{21}+3p_{03}),\\ k_{30}=(p_{21}+3p_{03})(p_{12}+2p_{30}),\quad k_{21}=-3(p_{12}+2p_{30})(p_{12}+3p_{30}),\\ k_{12}=3p_{30}(p_{21}+3p_{03}),\quad k_{03}=-p_{30}(p_{12}+3p_{30}).\end{gathered}$$

Therefore, the functions u_3, v_3, h_3, k_3 are expressed by (3.7)–(3.9), respectively.

By [1] [2], the origin point of (3.2) is a center and isochronous center. □

Consider Λ–Ω system

$$\begin{cases}x'=-y(1+y+\phi_2(x,y))+x(x+\psi_2(x,y)+h_3(x,y)),\\ y'=x(1+y+\phi_2(x,y))+y(x+\psi_2(x,y)+h_3(x,y)),\end{cases} \tag{3.33}$$

where $\phi_2=a_{20}x^2+a_{11}xy+a_{02}y^2$, $\psi_2=b_{20}x^2+b_{11}xy+b_{02}y^2$, a_{ij}, b_{ij} are real numbers.

By Theorem 3.1, taking $P_3=-y\phi_2+x\psi_2$, $Q_3=x\phi_2+y\psi_2$, which follows the following corollary.

Corollary 3.2. *The system* (3.33) *has a polynomial commutator in the form of*

$$\begin{pmatrix}U\\V\end{pmatrix}=\begin{pmatrix}x+u_2+u_3+xk_3(x,y)\\ y+v_2+v_3+yk_3(x,y)\end{pmatrix}$$

if and only if

$$a_{20}+a_{02}=0,\ b_{20}+b_{02}=0,\ a_{20}^2-b_{20}^2+b_{20}a_{11}=0,\ b_{11}b_{20}-a_{11}a_{20}+4a_{20}b_{20}=0.$$

$$\begin{gathered}u_2=2xy,\ v_2=y^2-x^2,\\ u_3=-(b_{11}+a_{20})x^3+(b_{20}+a_{11})x^2y-3a_{20}xy^2-b_{20}y^3,\\ v_3=(b_{20}-a_{11})x^3-(b_{11}-a_{20})x^2y+3b_{20}xy^2-a_{20}y^3,\\ h_3=(2b_{20}-a_{11})((a_{11}-b_{20})x^3+(b_{20}-2a_{11})xy^2)+(b_{11}+2a_{20})((a_{11}+b_{20})x^2y-b_{20}y^3),\\ k_3=(2a_{20}+b_{11})((b_{20}-a_{11})x^3+3b_{20}xy^2)-(2b_{20}-a_{11})(3(b_{20}-a_{11})x^2y+b_{20}y^3).\end{gathered}$$

Moreover, the origin point of (3.33) *is a center and isochronous center and weak center.*

4 Examples

In Theorem 2.3, taking $\lambda = \mu = \lambda_i = 1$ $(i = 2, 3, \ldots)$ which yields the following example.

Example 4.1. Λ–Ω-differential system

$$\begin{cases} x' = -y(1+y) + x(x - xy - x^2 y \sum_{j=3}^{\infty}(x-y)^{j-3}), \\ y' = x(1+y) + y(x - xy - x^2 y \sum_{j=3}^{\infty}(x-y)^{j-3}) \end{cases} \tag{4.1}$$

has a commutator

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} x + 2xy + x(x^2 + x^3 \sum_{j=3}^{\infty}(x-y)^{j-3}) \\ y + y^2 - x^2 + y(x^2 + x^3 \sum_{j=3}^{\infty}(x-y)^{j-3}) \end{pmatrix}$$

and the origin point of (4.1) is a center and isochronous center and weak center.

In Theorem 3.1 taking $p_{30} = 1$, $p_{21} = -5$, $p_{12} = -1$, $p_{03} = 1$ which implies the following example.

Example 4.2. Differential system

$$\begin{cases} x' = -y(1+y) + x^3 - 5x^2y - xy^2 + y^3 + x(x - 2(x+y)(x^2 - y^2)), \\ y' = x(1+y) + x^3 + x^2y - 5xy^2 - y^3 + y(x - 2(x+y)(x^2 - y^2)) \end{cases} \tag{4.2}$$

has a commutator

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} x + 2xy + 3x^3 + x^2y - 3xy^2 - y^3 - 2x(x+y)^3 \\ y + y^2 - x^2 + x^3 + 5x^2y + 3xy^2 - y^3 - 2y(x+y)^3) \end{pmatrix} \tag{4.3}$$

and the origin point of (4.2) is a center and isochronous center.

In Corollary 3.1 taking $a_{20} = 1, a_{11} = 0, a_{02} = -1, b_{20} = 1, b_{11} = -4, b_{02} = -1$ we deduce the following example.

Example 4.3. Λ–Ω differential system

$$\begin{cases} x' = -y(1 + y + x^2 - y^2) + x(x + x^2 - y^2 - 4xy - 2(x+y)(x^2 - y^2)), \\ y' = x(1 + y + x^2 - y^2) + y(x + x^2 - y^2 - 4xy - 2(x+y)(x^2 - y^2)) \end{cases} \tag{4.4}$$

has a commutator (4.3) and the origin point of (4.4) is a center and weak center.

The above three examples have been verified and they are correct.

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Caffarelli–Kohn–Nirenberg inequality for biharmonic equations with inhomogeneous term and Rellich potential

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Abstract. In this article, multiplicity of nontrivial solutions for an inhomogeneous singular biharmonic equation with Rellich potential are studied. Firstly, a negative energy solution of the studied equations is achieved via the Ekeland's variational principle and Caffarelli–Kohn–Nirenberg inequality. Then by applying Mountain pass theorem lack of Palais–Smale conditions, the second solution with positive energy is also obtained.

Keywords: singular biharmonic equations, nontrivial solution, inhomogeneous, Caffarelli–Kohn–Nirenberg inequality.

Mathematics Subject Classification 35J35, 35J62, 35J75, 35D30.

1 Introduction

We investigate multiplicity of solutions for the following singular biharmonic equations with inhomogeneous terms

$$\begin{cases} \Delta^2 u - \mu \frac{u}{|x|^4} = \frac{|u|^{p_\alpha - 2}u}{|x|^\alpha} + \lambda f(x), & \text{in } \mathbb{R}^N, \\ u \in H_0^2(\mathbb{R}^N), \quad u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\Delta^2 u = \Delta(\Delta u)$, $N \geq 5$, $0 < \mu < \overline{\mu} := \frac{N^2(N-4)^2}{16}$, $p_\alpha = \frac{2(N-\alpha)}{N-4}$, $0 \leqslant \alpha < 4$, $f(x) \in H_0^{-2}(\mathbb{R}^N)$ is a given function and $f(x) \not\equiv 0$, $H_0^{-2}(\mathbb{R}^N)$ denotes the dual of $H_0^2(\mathbb{R}^N)$, the singular term $\frac{u}{|x|^4}$ comes from models in physics.

In the past decades, nonlinear elliptic equations involving biharmonic operator have received much attention due to their wide application to mechanical and physical models such as clamped plates, thin-elastic plates, and in the research of the Paneitz–Branson equation and the Willmore equation (see [11]). Under the framework of nonlinear function analysis, there are many results on qualitative properties, the existence and multiplicity of solutions for biharmonic equations with singular potential (see [1,7,9,12,14–16,19–22,25,26], and the references

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therein). At the beginning, Brezis and Nirenberg [4] considered the following problems:

$$\begin{cases} -\Delta u = \lambda u + u^{\frac{N+2}{N-2}}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, and let

$$S_\lambda = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \int_\Omega |u|^2 dx}{\int_\Omega |u|^{2^*} dx}, \qquad \lambda \in \mathbb{R},$$

where $2^* = \frac{2N}{N-2}$ as Sobolev critical exponent. They basically proved that S_λ is reachable when N and λ satisfy different conditions. Since the seminal work of Brezis and Nirenberg, the study of critical growth in semilinear and quasilinear problem have gradually become a hot subject. On the basis of (1.2), Jannelli [13] studied the following semilinear elliptic equations involving the Hardy terms and critical exponents, and obtained at least a nontrivial solution when $N \geq 3$ and

$$\lambda < \lambda_1(\mu) = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega (|\nabla u|^2 dx - \mu \frac{u^2}{|x|^2}) dx}{\int_\Omega |u|^2 dx}.$$

Furthermore, Wang and Zhou [24] considered the problem of [13] with $u^{2^*-1} + \lambda u$ being replaced by $\frac{u^{2*(s)-2}}{|x|^s} + h(x)$, where $N \geq 3$, $0 \leq \mu < \frac{(N-2)^2}{4}, 2^*(s) = \frac{2(N-s)}{N-2}, 0 \leq s < 2$, $h(x) \geq 0$. By using the upper and lower solution method and Mountain pass theorem, they proved the given problem has at least two nontrivial solutions.

Tarantello [23] studied the following semilinear elliptic equations involving inhomogeneous perturbation and critical exponential terms:

$$\begin{cases} -\Delta u = u^{2^*-2} u + f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

When $\|f\|$ is appropriately small, the author proved that problem (1.3) admits at least two solutions by applying the Mountain pass theorem and the Ekeland's variational principle.

By applying similar methods as in Ref. [23], Deng and Wang [8] studied the following nonlinear biharmonic problems with inhomogeneous perturbation terms and critical exponential terms:

$$\begin{cases} \Delta^2 u - \lambda u = |u|^{2_*-2} u + f(x), & \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \Omega}|_{\partial\Omega} = 0, \end{cases} \tag{1.4}$$

where $N \geq 5$ and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $2_* = \frac{2N}{N-4}$. They proved that problem (1.4) has at least two solutions when $\|f\|$ is appropriately small. Furthermore, they dealt with the non-existence of solutions for the above studied equation under some assumptions on the perturbation term f.

By using the strong Maximum principle and the Comparison principle, Ref. [17] discussed the existence and nonexistence results of the following semilinear biharmonic problems with the optimal exponent p:

$$\begin{cases} \Delta^2 u - \mu \frac{u}{|x|^4} = \lambda f(x) + u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = -\Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $p > 1, \mu > 0, \lambda > 0$ and $\Omega \subset \mathbb{R}^N (N > 4)$ is a smooth bounded domain and $0 \in \Omega$.

Mousomi Bhakta [2] considered the following elliptic problem with singular terms:

$$\begin{cases} \Delta^2 u - \mu \frac{u}{|x|^4} = \frac{|u|^{p_\alpha - 2} u}{|x|^\alpha}, & \text{in } \Omega, \\ u \in H_0^2(\Omega), \quad u > 0, & \text{in } \Omega, \end{cases} \tag{1.5}$$

when Ω is an open subset of $\mathbb{R}^N$ $(N \geq 5)$, some nonexistence of solutions results are obtained by applying Pohozaev identity and Nehari manifold, In addition, they further discussed the existence of positive solutions when $\alpha = 0$.

Through the analysis of the above mentioned studies, **a quite natural question to ask is whether the inhomogeneous biharmonic problem (1.1) possesses multiple nontrivial solution in $\mathbb{R}^N$?** As far as we know, when $\alpha \neq 0$ and $\Omega \neq \mathbb{R}^N$ in (1.5), the problem (1.5) does not have a solution. Thanks to lack of compactness of the functional energy, the author obtain that the non-existence result of solution in a bounded domain. Therefore, we consider adding a perturbation term to overcome this difficulty and prove that the energy function I of problem (1.1) admits at least two critical points. One is a negative energy solution obtained by using Ekeland's variational method in [10], and other is a positive energy solution achieved by applying Mountain pass theorem in [1] without Palais–Smale (PS) conditions. The main result of this paper is the following theorem.

Theorem 1.1. *Assume that $N \geq 5, 0 < \mu < \overline{\mu}, 0 \leq \alpha < 4, p_\alpha = \frac{2(N-\alpha)}{N-4}$,and $f(x) \in H_0^{-2}(\mathbb{R}^N)$ with $f(x) \not\equiv 0$. Then there exists a constant $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, the problem (1.1) admits at least two nontrivial solutions which one is of negative energy and the other solution with positive energy, if*

$$\|f\|_{H_0^{-2}(\mathbb{R}^N)} < \frac{p_\alpha - 2}{2\lambda(p_\alpha - 1)} \left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2(p_\alpha - 1)} \right)^{\frac{1}{p_\alpha - 2}},$$

where S_μ will be given in (2.3).

2 Preliminaries

This section will mainly give some preparation to the proof of Theorem 1.1.

Due to the fact that the space $H_0^2(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ in regard to the following norm

$$\|u\|_{H_0^2(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\Delta u|^2 dx \right)^{1/2}.$$

Note that $\mu < \bar{\mu}$ and by the following Rellich inequality [18]

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \geq \bar{\mu} \int_{\mathbb{R}^N} \frac{u^2}{|x|^4} dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{2.1}$$

where $\bar{\mu} = \frac{N^2(N-4)^2}{16}$ is optimal, then we can show that the norm

$$\|u\|_\mu = \left(\int_{\mathbb{R}^N} |\Delta u|^2 - \mu \frac{u^2}{|x|^4} dx \right)^{1/2}$$

is an equivalent norm to $\|u\|_{H_0^2(\mathbb{R}^N)}$.

From [3], we have the following Caffarelli–Kohn–Nirenberg (CKN) inequality

$$\int_{\mathbb{R}^N} |\Delta u|^2 dx \le C(N,\alpha)\left(\int_{\mathbb{R}^N} \frac{|u|^{p_\alpha}}{|x|^\alpha} dx\right)^{2/p_\alpha}, \qquad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{2.2}$$

where the constant $C(N,\alpha) > 0$. For each μ with $0 < \mu < \overline{\mu}$, the best Sobolev constant S_μ can be given by

$$S_\mu = \inf_{u\in H_0^2(\mathbb{R}^N), u(x)\not\equiv 0} \frac{\int_{\mathbb{R}^N}(|\Delta u|^2 - \mu\frac{u^2}{|x|^4})dx}{(\int_{\mathbb{R}^N} |x|^{-\alpha}|u|^{p_\alpha} dx)^{\frac{2}{p_\alpha}}}, \tag{2.3}$$

where S_μ is achieved in $\mathbb{R}^N$. By applying (2.1) and (2.2), we know $S_\mu > 0$.

To obtain our results, the energy function I of problem (1.1) can be defined by

$$I(u) = \frac{1}{2}\|u\|_\mu^2 - \frac{1}{p_\alpha}\int_{\mathbb{R}^N} \frac{|u|^{p_\alpha}}{|x|^\alpha}dx - \lambda\int_{\mathbb{R}^N} f(x)u dx, \qquad u \in H_0^2(\mathbb{R}^N). \tag{2.4}$$

According to $f \in H_0^{-2}(\mathbb{R}^N)$ and (2.1)–(2.2), it is easy to obtain that the energy function $I(u)$ is a well defined C^1 function in $H_0^2(\mathbb{R}^N)$.

A function $u \in H_0^2(\mathbb{R}^N)$ is said to be a weak solution of the equations (1.1) if u satisfies

$$\int_{\mathbb{R}^N}\left(\Delta u\Delta v - \mu\frac{uv}{|x|^4}\right)dx = \int_{\mathbb{R}^N} \frac{|u|^{p_\alpha-2}uv}{|x|^\alpha}dx + \lambda\int_{\mathbb{R}^N} f(x)uv dx \tag{2.5}$$

for any $v \in H_0^2(\mathbb{R}^N)$.

Definition 2.1. A sequence $\{u_n\}_{n=1}^\infty \subset H_0^2(\mathbb{R}^N)$ satisfy $I(u_n) \to c$ $(c \in \mathbb{R})$ and $I'(u_n) \to 0$ in $H_0^{-2}(\mathbb{R}^N)$ as $n \to \infty$. Then the sequence $\{u_n\}_{n=1}^\infty$ is called a $(PS)_c$ sequence.

Lemma 2.1. *Assume that the sequence$\{u_n\}_{n=1}^\infty$ in $H_0^2(\mathbb{R}^N)$ be a $(PS)_c$ sequence for the energy function I of problem (1.1) at level $c \in \mathbb{R}$. Then $u_n \rightharpoonup u$ in $H_0^2(\mathbb{R}^N)$ and $I'(u) = 0$.*

Proof. For n sufficiently large, there hold

$$\frac{1}{2}\|u_n\|_\mu^2 - \frac{1}{p_\alpha}\int_{\mathbb{R}^N} \frac{|u_n|^{p_\alpha}}{|x|^\alpha}dx - \lambda\int_{\mathbb{R}^N} f(x)u_n dx = c + o_n(1),$$

and

$$\|u_n\|_\mu^2 - \int_{\mathbb{R}^N} \frac{|u_n|^{p_\alpha}}{|x|^\alpha}dx - \lambda\int_{\mathbb{R}^N} f(x)u_n dx = o_n(1),$$

where $o_n(1)$ means that for $n \to \infty$, $o_n(1) \to 0$. Thus, there holds

$$\begin{aligned} c + o_n(1) &= I(u_n) - \frac{1}{p_\alpha}\langle I'(u_n), u_n\rangle \\ &\ge \left(\frac{1}{2} - \frac{1}{p_\alpha}\right)\|u_n\|_\mu^2 - \lambda\left(1 - \frac{1}{p_\alpha}\right)\|f\|_{H_0^{-2}(\mathbb{R}^N)}\|u_n\|_\mu, \end{aligned} \tag{2.6}$$

which means that $\{u_n\}_{n=1}^\infty$ is a bounded sequence in $H_0^2(\mathbb{R}^N)$. Up to a subsequence if necessary, there holds

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H_0^2(\mathbb{R}^N), \\ u_n \rightharpoonup u, & \text{in } L_{p_\alpha}(\mathbb{R}^N, |x|^{-\alpha}), \\ u_n \to u, & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{2.7}$$

Thus, it is easy to obtain that

$$\int_{\mathbb{R}^N} (\Delta u \Delta v - \mu \frac{uv}{|x|^4} - \frac{|u|^{p_\alpha - 2} uv}{|x|^\alpha} - \lambda f(x) uv) dx = 0$$

for all $v \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, which means $I'(u) = 0$. □

Lemma 2.2. *For some $c \in \mathbb{R}$, let $\{u_n\}_{n=1}^\infty$ in $H_0^2(\mathbb{R}^N)$ be a $(PS)_c$ sequence for the energy functional I, that is to say $I(u_n) \to c, I'(u_n) \to 0$ in $H_0^{-2}(\mathbb{R}^N)$ as $n \to \infty$. Then there is a $u_0 \in H_0^2(\mathbb{R}^N)$ such that $u_n \rightharpoonup u_0$ in $H_0^2(\mathbb{R}^N)$ holds and*

$$\text{either} \quad u_n \to u_0 \quad \text{or} \quad c \geq I(u_0) + \left(\frac{1}{2} - \frac{1}{p_\alpha}\right) S_\mu^{\frac{p_\alpha}{p_\alpha - 2}}.$$

Proof. It follows from (2.6) that $\{u_n\}_{n=1}^\infty$ is bounded in $H_0^2(\mathbb{R}^N)$. Due to boundedness of $\{u_n\}_{n=1}^\infty$, we know that the sequence $\{u_n\}_{n=1}^\infty$ possesses a weak convergent subsequence, still denoted by $\{u_n\}_{n=1}^\infty$, then we can get that $u_n \rightharpoonup u_0$ in $H_0^2(\mathbb{R}^N)$, and $u_n \to u_0$ a.e. in $\mathbb{R}^N$, as $n \to \infty$. Denote $w_n = u_n - u_0$, then we have $w_n \rightharpoonup 0$, as $n \to +\infty$.

On the basis of Brezis–Lieb Lemma (see [5]), we could obtain that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(\frac{|u_n|^{p_\alpha}}{|x|^\alpha} - \frac{|u_n - u_0|^{p_\alpha}}{|x|^\alpha} \right) dx = \int_{\mathbb{R}^N} \frac{|u_0|^{p_\alpha}}{|x|^\alpha} dx.$$

Therefore, there holds

$$I(u_n) - I(u_0) = \frac{1}{2} \|w_n\|_\mu^2 - \frac{1}{p_\alpha} \int_{\mathbb{R}^N} \frac{|w_n|^{p_\alpha}}{|x|^\alpha} dx + o_n(1). \tag{2.8}$$

And It follows from Lemma 2.1 that $I'(u_0) = 0$, combining with (2.8) we can infer that

$$\langle I'(u_n), u_n \rangle = \langle I'(u_n) - I'(u_0), w_n + u_0 \rangle = \|w_n\|_\mu^2 - \int_{\mathbb{R}^N} \frac{|w_n|^{p_\alpha}}{|x|^\alpha} dx + o_n(1).$$

In this situation, we may assume that

$$\lim_{n\to\infty} \|w_n\|_\mu^2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{|w_n|^{p_\alpha}}{|x|^\alpha} dx = \xi \geq 0.$$

Suppose $\xi > 0$, together with the definition of S_μ, we have $\xi \geq S_\mu^{\frac{p_\alpha}{p_\alpha - 2}}$. Furthermore, by (2.8), we obtain

$$c = I(u_0) + \left(\frac{1}{2} - \frac{1}{p_\alpha}\right) \xi \geq I(u_0) + \left(\frac{1}{2} - \frac{1}{p_\alpha}\right) S_\mu^{\frac{p_\alpha}{p_\alpha - 2}}.$$

This ends the proof of Lemma 2.2. □

3 Proof of Theorem 1.1

In this section, we first take advantage of some analytical skills and functional idea to prove that the functional I can admit a local minimizer, which is a nontrivial negative energy solution. After that we show the existence of a nontrivial solution with positive energy via using Mountain pass theorem without (PS) condition.

Lemma 3.1. *Suppose that* $0 < \mu < \overline{\mu} = \frac{N^2(N-4)^2}{16})$, $N \geq 5$, $0 \leq \alpha < 4$. *Then there exist constants* $\Lambda_1, \eta_0, \xi > 0$ *such that for every* $\lambda \in (0, \Lambda_1)$, *there holds*

$$I(u) \geq \xi > 0 \quad \textit{for } \|u\|_\mu = \eta_0. \tag{3.1}$$

Proof. From (2.4), Young inequality, and the definition of S_μ, we get

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_\mu^2 - \frac{1}{p_\alpha}\int_{\mathbb{R}^N} \frac{|u|^{p_\alpha}}{|x|^\alpha}dx - \lambda \int_{\mathbb{R}^N} f(x)udx \\ &\geq \frac{1}{2}\|u\|_\mu^2 - \frac{1}{p_\alpha} S_\mu^{-\frac{p_\alpha}{2}} \|u\|_\mu^{p_\alpha} - \lambda \|f\|_{H_0^{-2}(\mathbb{R}^N)}\|u\|_\mu \\ &= \|u\|_\mu \left(\frac{1}{2}\|u\|_\mu - \frac{1}{p_\alpha} S_\mu^{-\frac{p_\alpha}{2}} \|u\|_\mu^{p_\alpha - 1} - \lambda \|f\|_{H_0^{-2}(\mathbb{R}^N)}\right). \end{aligned} \tag{3.2}$$

Set

$$h(z) = \frac{1}{2}z - \frac{1}{p_\alpha} S_\mu^{-\frac{p_\alpha}{2}} z^{p_\alpha - 1}, \quad z \geq 0.$$

Then from $h'(z_0) = 0$, there holds

$$z_0 = \left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2p_\alpha - 2}\right)^{\frac{1}{p_\alpha - 2}},$$

which indicates that

$$\begin{aligned} h(z_0) &= \frac{1}{2}\left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2p_\alpha - 2}\right)^{\frac{1}{p_\alpha - 2}} - \frac{1}{p_\alpha} S_\mu^{-\frac{p_\alpha}{2}} \left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2p_\alpha - 2}\right)^{\frac{p_\alpha - 2 + 1}{p_\alpha - 2}} \\ &= \frac{p_\alpha - 2}{2p_\alpha - 2}\left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2p_\alpha - 2}\right)^{\frac{1}{p_\alpha - 2}} > 0. \end{aligned}$$

In order to obtain $h(z_0) > \lambda \|f\|_{H_0^{-2}(\mathbb{R}^N)}$, we could choose

$$\Lambda_1 := \frac{p_\alpha - 2}{2p_\alpha - 2}\left(\frac{p_\alpha S_\mu^{\frac{p_\alpha}{2}}}{2p_\alpha - 2}\right)^{\frac{1}{p_\alpha - 2}} / \|f\|_{H_0^{-2}(\mathbb{R}^N)}. \tag{3.3}$$

Due to $0 \leq \alpha < 4$, then $p_\alpha > 2$. Choosing $\eta_0 = z_0$ and $\xi = z_0(h(z_0) - \lambda\|f\|_{H_0^{-2}(\mathbb{R}^N)})$, it follows from (3.2) that there exists $\Lambda_1 > 0$ (be given in (3.3)) such that

$$I(u) \geq \xi > 0 \quad \text{for any } \|u\|_\mu = \eta_0, \text{ and } \lambda \in (0, \Lambda_1),$$

and the conclusion is achieved.

We now show that there exists a nontrivial solution with negative solution.

On account of the continuity of f on $\mathbb{R}^N$ and combining with $f \not\equiv 0$, we can choose $\phi \in C_0(\mathbb{R}^N \setminus \{0\})$ such that $\int_{\mathbb{R}^N} f(x)\phi dx > 0$. Then for $t > 0$ sufficiently small with $\|t\phi\|_\mu < \eta_0$, there holds

$$I(t\phi) = \frac{t^2}{2}\|\phi\|_\mu^2 - \frac{t^{p_\alpha}}{p_\alpha}\int_{\mathbb{R}^N} \frac{|\phi|^{p_\alpha}}{|x|^\alpha}dx - \lambda t \int_{\mathbb{R}^N} f(x)\phi dx < 0.$$

Therefore, we have

$$c_1 = \inf\{I(u) : u \in \overline{B}_{\eta_0}\} < 0, \quad \text{where} \quad \overline{B}_{\eta_0} = \{u \in H_0^2(\mathbb{R}^N), \|u\|_\mu < \eta_0\}.$$

According to the complete metric space $\overline{B}_{\eta_0}$ with respect to the norm of $H_0^2(\mathbb{R}^N)$, then applying the Ekeland's variational principle to $I(u)$ in $\overline{B}_{\eta_0}$ yields that there exist a $(PS)_{c_1}$ sequence $\{u_n\}_{n=1}^\infty$ in $\overline{B}_{\eta_0}$ and a $u_* \in H_0^2(\mathbb{R}^N)$ with $\|u_*\|_\mu < \eta_0$, such that $u_n \rightharpoonup u_*$.

We now turn to show that $u_n \to u_*$ in $H_0^2(\mathbb{R}^N)$ as $n \to \infty$. Otherwise, it follows from Lemma 2.2 that

$$c_1 \geq I(u_*) + \left(\frac{1}{2} - \frac{1}{p_\alpha}\right) S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} \geq c_1 + \left(\frac{1}{2} - \frac{1}{p_\alpha}\right) S^{\frac{p_\alpha}{p_\alpha-2}} > c_1,$$

which is a contradiction.

Then the above proof yields that u_* is a critical point of the functional I satisfying $c_1 = I(u_*) < 0$. Furthermore, it follows from (2.3) and (3.2) that

$$\begin{aligned}
c_1 &= \frac{p_\alpha - 2}{2p_\alpha}\|u_*\|_\mu^2 - \frac{p_\alpha - 1}{p_\alpha}\int_{\mathbb{R}^N} \lambda f(x) u_*(x) dx \\
&\geq \frac{p_\alpha - 2}{2p_\alpha}\|u_*\|_\mu^2 - \frac{\lambda(p_\alpha - 1)}{p_\alpha}\|f\|_{H_0^{-2}(\mathbb{R}^N)}\|u_*\|_\mu \\
&\geq \frac{(p_\alpha - 2)(p_\alpha - 1)^2\|\lambda f\|^2_{H_0^{-2}(\mathbb{R}^N)}}{2p_\alpha(p_\alpha - 2)^2} - \frac{\lambda(p_\alpha - 1)}{p_\alpha}\|f\|_{H_0^{-2}(\mathbb{R}^N)}\frac{p_\alpha - 1}{p_\alpha - 2}\|\lambda f\|_{H_0^{-2}(\mathbb{R}^N)} \\
&= -\frac{(p_\alpha - 1)^2}{2p_\alpha(p_\alpha - 2)}\lambda^2\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}.
\end{aligned}$$

Thus, we can deduce that the problem (1.1) possesses a nontrivial solution u_* with negative energy. □

Lemma 3.2. *Let constant $\Lambda_2 > 0$ such that*

$$(p_\alpha - 2)^2 S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \lambda^2(p_\alpha - 1)^2\|f\|^2_{H_0^{-2}(\mathbb{R}^N)} > 0, \quad \text{for any } \lambda \in (0, \Lambda_2). \tag{3.4}$$

Then there are a $\widetilde{u}(x) \in H_0^2(\mathbb{R}^N)$ and constant Λ_3 with $0 < \Lambda_3 \leq \Lambda_2$ such that

$$\sup_{t \geq 0} I(t\widetilde{u}) < \frac{p_\alpha - 2}{2p_\alpha} S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \frac{\lambda^2(p_\alpha - 1)^2}{2p_\alpha(p_\alpha - 2)}\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}, \quad \text{for all } \lambda \in (0, \Lambda_3). \tag{3.5}$$

Proof. From Theorem 2.1 of [2], we know that there is a nontrivial nonnegative solution as $\lambda = 0$ for problem(1.1), and then denote it as $\widetilde{z}(x)$. Next, we may choose $\widetilde{u}(x) = \widetilde{z}(x)$ if the function $f(x) \geq 0$ for each $x \in \mathbb{R}^N$, $\widetilde{u}(x) = -\widetilde{z}(x)$ if the function $f(x) \leq 0$ for each $x \in \mathbb{R}^N$, $\widetilde{u}(x) = \widetilde{z}(x - x_0)$ if there exists a point $x_0 \in \mathbb{R}^N$ satisfying $f(x_0) > 0$. We now claim that there holds

$$\int_{\mathbb{R}^N} f(x)\widetilde{u}(x) dx > 0. \tag{3.6}$$

Indeed, the inequality (3.6) holds obviously if the function $f(x) \geq 0$ or $f(x) \leq 0$ for each $x \in \mathbb{R}^N$. Now if there is a point $x_0 \in \mathbb{R}^N$ satisfying $f(x_0) > 0$, then by the continuity of the function f, we can deduce that there exists an open neighborhood $B(x_0, \tau) \subset \mathbb{R}^N$ of $x_0, \tau > 0$, such that the function $f(x) > 0$ for all $x \in B(x_0, \tau)$. Therefore, one can deduce from the definition of $\widetilde{z}(x - x_0)$, that

$$\int_{\mathbb{R}^N} f(x)\widetilde{z}(x - x_0) dx > 0.$$

To prove the inequality (3.5), we discuss the functions g and $\widetilde{g}$ defined by

$$g(t) := I(t\widetilde{u}) = \frac{t^2}{2}\|\widetilde{u}\|_\mu^2 - \frac{t^{p_\alpha}}{p_\alpha}\int_{\mathbb{R}^N}\frac{|\widetilde{u}|^{p_\alpha}}{|x|^\alpha}dx - \lambda t\int_{\mathbb{R}^N} f(x)\widetilde{u}dx,\ t \geq 0,$$

and

$$\widetilde{g}(t) := \frac{t^2}{2}\|\widetilde{u}\|_\mu^2 - \frac{t^{p_\alpha}}{p_\alpha}\int_{\mathbb{R}^N}\frac{|\widetilde{u}|^{p_\alpha}}{|x|^\alpha}dx,\ t \geq 0.$$

Obviously, there holds

$$g(0) = 0 < \frac{p_\alpha - 2}{2p_\alpha}S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \frac{\lambda^2(p_\alpha - 1)^2}{2p_\alpha(p_\alpha - 2)}\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}$$

for every $\lambda \in (0, \Lambda_2)$. Thus from the continuity of function g, there exists some $t_1 > 0$ sufficiently small, such that

$$\frac{p_\alpha - 2}{2p_\alpha}S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \frac{\lambda^2(p_\alpha - 1)^2}{2p_\alpha(p_\alpha - 2)}\|f\|^2_{H_0^{-2}(\mathbb{R}^N)} > g(t)$$

for all $t \in (0, t_1)$.

For another thing, by the definition of $\widetilde{g}$ there holds

$$\max_{t\geq 0}\widetilde{g}(t) = \frac{p_\alpha - 2}{2p_\alpha}S_\mu^{\frac{p_\alpha}{p_\alpha - 2}}.$$

This together with the definition of g, we have

$$\sup_{t\geq 0} I(t\widetilde{u}) < \left(\frac{1}{2} - \frac{1}{p_\alpha}\right)S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \lambda t_1\int_{\mathbb{R}^N} f(x)\widetilde{u}dx.$$

Choose $\lambda > 0$ satisfying that

$$\lambda t_1\int_{\mathbb{R}^N} f(x)\widetilde{u}dx > \frac{\lambda^2(p_\alpha - 1)^2}{2p_\alpha(p_\alpha - 2)}\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}.$$

Then from (3.6), one has

$$0 < \lambda < \frac{2p_\alpha(p_\alpha - 2)t_1\int_{\mathbb{R}^N} f(x)\widetilde{u}dx}{(p_\alpha - 1)^2\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}}.$$

Set

$$\Lambda_3 := \min\left\{\frac{2p_\alpha(p_\alpha - 2)t_1\int_{\mathbb{R}^N} f(x)\widetilde{u}dx}{(p_\alpha - 1)^2\|f\|^2_{H_0^{-2}(\mathbb{R}^N)}},\ \Lambda_2\right\}.$$

For all $\lambda \in (0, \Lambda_3)$, we conclude that

$$\sup_{t\geq 0} I(t\widetilde{u}) < \frac{p_\alpha - 2}{2p_\alpha}S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \frac{\lambda^2(p_\alpha - 1)^2}{2p_\alpha(p_\alpha - 2)}\|f\|^2_{H_0^{-2}(\mathbb{R}^N)},$$

and this ends the proof. □

Next we will show another critical point with positive energy of problem (1.1).

Since $I(t\widetilde{u}) \to -\infty$ as $t \to \infty$, then one may take $t^* > 0$ sufficiently large if necessary, such that $I(t^*\widetilde{z}) < 0$. Taking $\eta_0 > 0$, then Lemma 3.1 can show that $I|_{\partial B_{\eta_0}} \geq \xi > 0$ for every $\lambda \in (0, \Lambda_1)$. Set

$$\Gamma = \{\gamma \in C([0,1], H_0^2(\mathbb{R}^N)),\ \gamma(0) = 0,\ \gamma(1) = t^*\widetilde{u}\},$$

and

$$c_2 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)).$$

Then it follows from Mountain pass theorem without (PS) condition that there exists a $(PS)_{c_2}$ sequence $\{u_n\}_{n=1}^{\infty}$ in $H_0^2(\mathbb{R}^N)$ satisfying

$$I(u_n) \to c_2, \quad I'(u_n) \to 0, \quad \text{in } H_0^{-2}(\mathbb{R}^N)$$

as $n \to \infty$.

Furthermore, it follows from Lemma 2.1 that there exists a subsequence of $\{u_n\}_{n=1}^{\infty}$, still denoted by $\{u_n\}_{n=1}^{\infty}$, and a $u^* \in H_0^2(\mathbb{R}^N)$, such that $u_n \rightharpoonup u^*$, as $n \to \infty$. If $u_n \nrightarrow u^*$ as $n \to \infty$, then from Lemma 2.2 we can deduce that

$$c_2 \geq I(u^*) + \frac{p_\alpha - 2}{2p_\alpha} S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} \geq \frac{p_\alpha - 2}{2p_\alpha} S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \frac{\lambda^2 (p_\alpha - 1)^2}{2p_\alpha(p_\alpha - 2)} \|f\|^2_{H_0^{-2}(\mathbb{R}^N)}. \tag{3.7}$$

But Lemma 3.2 shows that

$$\sup_{t\geq 0} I(t\widetilde{u}) < \frac{p_\alpha - 2}{2p_\alpha} S_\mu^{\frac{p_\alpha}{p_\alpha - 2}} - \frac{\lambda^2 (p_\alpha - 1)^2}{2p_\alpha(p_\alpha - 2)} \|f\|^2_{H_0^{-2}(\mathbb{R}^N)}, \quad \text{for any } \lambda \in (0, \Lambda_3).$$

This together with (3.7) means that $u_n \to u^*$ in $H_0^2(\mathbb{R}^N)$, as $n \to \infty$. Taking $\lambda^* := \min\{\Lambda_1, \Lambda_3\}$, it is easy to show that for any $\lambda \in (0, \lambda^*)$, the functional I has the second critical point u^* satisfying $I(u^*) > 0$. Therefore the proof of Theorem 1.1 is finished.

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Bifurcations and Turing patterns in a diffusive Gierer–Meinhardt model

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Abstract. In this paper, the Hopf bifurcations and Turing bifurcations of the Gierer–Meinhardt activator-inhibitor model are studied. The very interesting and complex spatially periodic solutions and patterns induced by bifurcations are analyzed from both theoretical and numerical aspects respectively. Firstly, the conditions for the existence of Hopf bifurcation and Turing bifurcation are established in turn. Then, the Turing instability region caused by diffusion is obtained. In addition, to uncover the diffusion mechanics of Turing patterns, the dynamic behaviors are studied near the Turing bifurcation by using weakly nonlinear analysis techniques, and the type of spatial pattern was predicted by the amplitude equation. And our results show that the spatial patterns in the Turing instability region change from the spot, spot-stripe to stripe in order. Finally, the results of the analysis are verified by numerical simulations.

Keywords: Gierer–Meinhardt activator-inhibitor model, stability, Hopf bifurcation, Turing bifurcation, pattern.

2020 Mathematics Subject Classification: 34K18, 37G10, 35K57,35B36.

1 Introduction

In general, reaction-diffusion systems [4,14,15] are used to describe models in which the concentration of one or more substances diffuses in space and is affected by the diffusion and inter-conversion of substances. In 1952, A. M. Turing [23] mathematically proposed the conclusion that the homogeneous steady state in a reaction-diffusion system becomes destabilized under certain conditions, that is, the initial steady-state solution of the reaction-diffusion system becomes unstable due to the introduction of a diffusion term. This instability caused by diffusion is often referred to as Turing instability. Thereafter, Turing instability has received a great amount of attention from a wide range of scholars and has become a typical problem in the formation of spatio-temporal patterns [1,7,9,12,16,18,21,26]. The various results of pattern formation in the reaction-diffusion system are specified as follows. The Turing–Murray principle was proposed by James Murray [16], which investigated the reaction-diffusion systems

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of animal bodies and tails and their Turing instability. Schepers and Markus [21] demonstrated that cellular automata can produce Turing patterns in the activator-inhibitor system that is qualitatively consistent with various experiments in chemistry. A diffusion model with a Degn–Harrison reaction scheme is considered by Li et al. [12], and the local and global structure of the steady-state bifurcation is established by the technique of spatial decomposition and implicit function theorem. These works demonstrated that Turing patterns can emerge in a number of ecological and chemical systems.

To uncover the diffusion mechanism of Turing patterns and to examine the actual format of Turing patterns in the real world, we will select the activator-inhibitor model [6] proposed by Gierer and Meinhardt to study the typologies of Turing patterns. The activator-inhibitor model shows that two substances can resist each other's action, and can also be used to depict the formation of polar structures, animal structures, and periodic structures (dots on animals). In recent decades, a large literature has been devoted to the study of this system, as seen in[2, 11, 13, 20, 25] and the references therein, which can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} = \rho \dfrac{u^2}{v} - \mu_u u + D_u \dfrac{\partial^2 u}{\partial x^2} + \rho_u, \\ \dfrac{\partial v}{\partial t} = \rho u^2 - \mu_v v + D_v \dfrac{\partial^2 v}{\partial x^2} + \rho_v, \end{cases} \tag{1.1}$$

where

(i) u and v represent the concentration of activator and inhibitor respectively, D_u and D_v are their corresponding diffusion constants, and $\frac{\partial u}{\partial t}$ means the change in the concentration of the activator per unit of time.

(ii) $\rho_u > 0, \rho_v > 0$ represent the baseline yield of the activator and the inhibitor, separately, and μ_u, μ_v are the decay rate.

For the Gierer–Meinhardt system (1.1), Ruan [20] demonstrated that diffusion can cause homogeneous equilibrium solutions and homogeneous periodic solutions to become unstable. Liu et al. [13] investigated the multiple bifurcation analysis and spatiotemporal patterns in the one-dimensional Gierer–Meinhardt model. Wu et al. [25] performed a Hopf bifurcation analysis of this diffusion model and studied the direction and stability of Hopf bifurcation by standard central manifold theorem. Stability and Hopf bifurcation analysis on a simplified Gierer–Meinhardt model were studied by Asheghi [2], and the direction of the Hopf bifurcation was obtained by the normal form theory. The investigation conducted by Li et al. [11] pertained to the analysis of Turing patterns observed in a broad-spectrum Gierer–Meinhardt model of morphogenesis. In the particular case, when $\rho_u = \rho_v = 0$, a simple scale transformation model is as follows

$$\begin{cases} \frac{\partial u}{\partial t} = \sigma_1 \Delta u + \frac{u^2}{v} - \beta u & (x,y) \in \Omega, \quad t > 0, \\ \frac{\partial v}{\partial t} = \sigma_2 \Delta v + u^2 - v & (x,y) \in \Omega, \quad t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & (x,y) \in \partial\Omega, \quad t > 0, \end{cases} \tag{1.2}$$

where

(i) u and v stand for $u(x,y,t)$ and $v(x,y,t)$, $(x,y) \in \Omega \subset \mathbb{R}^2$, β denotes the decay rate of the activator.

(ii) $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a common Laplace operator in two-dimensional space, and $\partial\Omega$ represents the homogeneous Neumann boundary condition.

None of the above-mentioned literature deals with the formation of Turing patterns on two-dimensional space in the Gierer–Meinhardt model. However, for chemical systems, patterns on a two-dimensional plane will be more realistic, more intuitive, and abundant than those on a one-dimensional plane [17,24]. For the one-dimensional space, only spot patterns and strip patterns exist. However, in two dimensions, not only spots and strips but also patterns such as spot-strip coexistence and maze shapes may appear. To more clearly understand the mechanisms of pattern formation in Gierer–Meinhardt model, we will study the spatio-temporal evolution pattern of the system (1.2) in two dimensions space.

In this paper, the dynamical behaviors of the system (1.2) are studied by using the decay rate of activator β as a bifurcation parameter. The existing conditions of the Hopf bifurcations and the Turing bifurcations are established in turn. The very interesting and complex patterns (spot patterns, spot-stripe coexistence patterns, and stripe patterns) induced by the Turing bifurcation are analyzed from both theoretical and numerical aspects by a multi-scale method [3,5,27]. And our results show that the decay rate of the activator β can affect the dynamical behavior of the system (1.2). The system will occur Turing instability when the decay rate β is within a certain region, the impact of diffusion on the system will be diminished as the decay rate β increases.

The layout of this paper is organized as follows. In Section 2, the conditions for the existence of Hopf bifurcation and the Turing instability with spatial inhomogeneity are discussed analytically. In Section 3, the amplitude equation near the instability threshold is derived using weakly nonlinear analysis, and different solutions to the amplitude equation and its stability are investigated. And the correctness of the theoretical part of the analysis is verified by numerical simulations in space. In Section 4, finally, some conclusions and discussions are given.

2 Turing instability and bifurcation analysis

In this section, the conditions for the existence of Hopf bifurcation and the Turing instability are discussed.

The local system corresponding to the diffusion system (1.2) is

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{u^2}{v} - \beta u, \\ \frac{\partial v}{\partial t} = u^2 - v, \end{cases} \tag{2.1}$$

with a unique positive equilibrium

$$E_* = (U_*, V_*) = \left(\frac{1}{\beta}, \frac{1}{\beta^2}\right), \quad \beta > 0.$$

The Jacobian matrix computed. The Jacobi matrix taken at the positive equilibrium E_* is

$$A = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} \beta & -\beta^2 \\ \frac{2}{\beta} & -1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.2}$$

and the characteristic equation is as follows

$$\lambda^2 - \mathrm{tr}(A)\lambda + \mathrm{Det}(A) = 0, \tag{2.3}$$

with

$$tr(A) = a_{11} + a_{22} = \beta - 1,$$
$$Det(A) = a_{11}a_{22} - a_{21}a_{12} = \beta.$$

Theorem 2.1. *For the local system (2.1), when $0 < \beta < 1$, the positive equilibrium E_* is locally asymptotically stable, and the system (2.1) undergoes the Hopf bifurcation at $\beta = 1$.*

Proof. When $0 < \beta < 1$, obviously obtaining $tr(A) < 0$ and $Det(A) > 0$, hence, the positive equilibrium E_* is locally asymptotically stable. When $\beta = 1$, then $tr(A) = 0$ and $Det(A) > 0$, the system (2.1) undergoes Hopf bifurcation. Next, we verify the transversality condition for the Hopf bifurcation at $\beta = 1$

$$\left.\frac{dRe\lambda_0(\beta)}{d_\beta}\right|_{\beta=1} = \frac{1}{2} > 0.$$

According to the Poincaré–Andronov–Hopf bifurcation theorem [19], the system (2.1) undergoes a Hopf bifurcation when $\beta = 1$. □

Next, we study the diffusion-driven Turing instability of the diffusion system (1.2) under the basic assumption that the constant equilibrium $E_*(u_*, v_*)$ of the system (1.2) is asymptotically stable ($0 < \beta < 1$).

In order to study the linear stability of the constant equilibrium $E_*(u_*, v_*)$ of (1.2), we need to study the distribution of the roots of the characteristic equation of (1.2). The linearization of Equation (1.2) at the constant equilibrium point $E_*(u_*, v_*)$ is

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \sigma_1\Delta & 0 \\ 0 & \sigma_2\Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + A \begin{pmatrix} u \\ v \end{pmatrix}. \tag{2.4}$$

Assume the solution of (2.4) is that

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^* \\ v^* \end{pmatrix} + \begin{pmatrix} u_k \\ v_k \end{pmatrix} \exp(\lambda t + i(\mathbf{k} \cdot \mathbf{r})), \tag{2.5}$$

where $\mathbf{k}$ denotes the wave number with the expression $\mathbf{k} = (k_x, k_y)$, and satisfies $k = |\mathbf{k}|$. $\mathbf{r}$ is the spatial vector in two dimensions whose expression is $\mathbf{r} = (x, y)$. We can get the corresponding characteristic matrix is

$$A_k = \begin{pmatrix} a_{11} - \sigma_1 k^2 & a_{12} \\ a_{21} & a_{22} - \sigma_2 k^2 \end{pmatrix}.$$

The characteristic equation is

$$F_k(\lambda) = \lambda^2 - T_k\lambda + D_k = 0, \tag{2.6}$$

where

$$\begin{aligned} T_k &= \beta - 1 - k^2(\sigma_1 + \sigma_2), \\ D_k &= \sigma_1\sigma_2 k^4 - (-\sigma_1 + \sigma_2\beta)k^2 + \beta. \end{aligned} \tag{2.7}$$

Under Theorem 2.1, we have $0 < \beta < 1$, thus for any positive natural number k, there always exist $T_k < 0$. Then the instability condition of the positive equilibrium point $E_*(u_*, v_*)$ of the system (1.2) should be that: existing a $k > 0$ make $D_k < 0$. In other words, when $D_k < 0$

$(k > 0)$ is satisfied, there exists a diffusion-driven Turing instability. Since $\beta > 0$, the sufficient condition for $D_k < 0$ is that the following two conditions H_1 and H_2 hold

$$H_1 : -\sigma_1 + \sigma_2\beta > 0,$$

and

$$H_2 : (\sigma_1 - \sigma_2\beta)^2 - 4\beta\sigma_1\sigma_2 > 0.$$

Consider D_k as a quadratic function of k^2, the function D_k can obtain the minimum value at k_T, where $k_T^2 = \sqrt{\frac{\beta}{\sigma_1\sigma_2}}$. If H_1 and H_2 hold, then $\min D_{k_T} < 0$, which indicates the occurrence of Turing instability.

In the following, we choose β as the parameter to study the conditions that make H_1 and H_2 hold. Regarding the Turing instability of the system (1.2), we obtain the following results.

Theorem 2.2. *Assume that the positive equilibrium point E_* of the corresponding local system (2.1) is stable, which is given by Theorem 2.1. For the reaction-diffusion system (1.2)*

(I) if $\sigma_1 \geq \sigma_2$, there is no Turing instability;

(II) if $\sigma_1 < \sigma_2$, the following results are achieved

- *(i) when $\beta_T^{(2)} > 1$, there is no Turing instability;*
- *(ii) when $\beta_T^{(2)} < 1$, Turing instability occurs at $\beta \in (\beta_T^{(2)}, 1)$ and Turing bifurcation occurs at $\beta = \beta_T^{(2)}$,*

where

$$\beta_T^{(2)} = \frac{(3 + 2\sqrt{2})\sigma_1}{\sigma_2}.$$

Proof. (I) From Theorem 2.1, we know that the positive equilibrium point E_* is stable for $0 < \beta < 1$. Therefore, when $\sigma_1 \geq \sigma_2$, we have $\frac{\sigma_1}{\sigma_2} \geq 1 > \beta$, hence H_1 is not satisfied. The conclusion (I) is proved.

(II) Under the conditions of Theorem 2.1, it is easy to get H_1 equivalent to $\beta_* < \beta < 1$, where

$$\beta_* = \frac{\sigma_1}{\sigma_2}, \tag{2.8}$$

and H_2 is equivalent to the following condition

$$h(\beta) = \sigma_2^2\beta^2 - 6\sigma_1\sigma_2\beta + \sigma_1^2 > 0. \tag{2.9}$$

Let

$$Q_1 = (-6\sigma_1\sigma_2)^2 - 4\sigma_2^2\sigma_1^2 = 32\sigma_1^2\sigma_2^2, \tag{2.10}$$

obviously, $Q_1 > 0$. This means that $h(\beta) = 0$ has two positive roots, which are denoted as $\beta_T^{(1)}$ and $\beta_T^{(2)}$

$$0 < \beta_T^{(1)} = \frac{(3 - 2\sqrt{2})\sigma_1}{\sigma_2} < \beta_T^{(2)} = \frac{(3 + 2\sqrt{2})\sigma_1}{\sigma_2}, \tag{2.11}$$

and $h(\beta) > 0$ if only and if $0 < \beta < \beta_T^{(1)}$ and $\beta > \beta_T^{(2)}$. In addition, we can get

$$h(\beta_*) = \sigma_2^2\frac{\sigma_1^2}{\sigma_2^2} - 6\sigma_1\sigma_2\frac{\sigma_1}{\sigma_2} + \sigma_1^2 = -4\sigma_1^2 < 0,$$

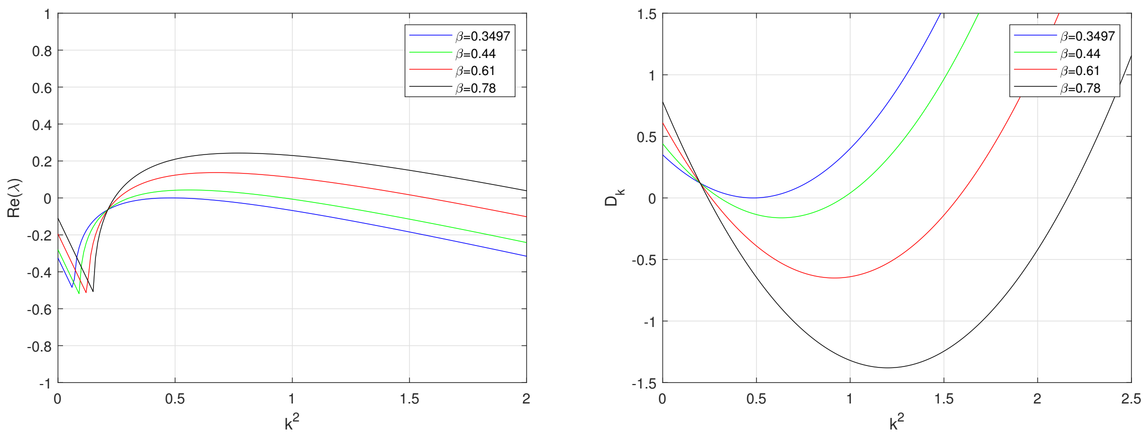
hence, we have the following inequality,

$$0 < \beta_T^{(1)} < \beta_* < \beta_T^{(2)}. \tag{2.12}$$

Therefore, H_1, H_2 are both satisfied for $\beta_T^{(2)} < \beta < 1$, not satisfied for $0 < \beta < \beta_T^{(1)}$, H_1, H_2. Then we can conclude that Turing instability occurs only in the region $\beta_T^{(2)} < \beta < 1$, which completes the proof of (ii) in Conclusion (II).

Furthermore, if $\beta_T^{(2)} > 1$, the positive equilibrium point E_* is unstable, hence, there is no Turing instability. The conclusion (i) in (II) is proved. □

To support the previous theoretical analysis, taking $\sigma_1 = 0.3$, $\sigma_2 = 5$, we can obtain $\beta_T^{(2)} = 0.3497$. According to Theorem 2.2, we know that Turing instability occurs for $\beta \in (\beta_T^{(2)}, 1)$. Therefore, to investigate the Turing pattern formation of system (1.2), we need to ensure that the control parameter $\beta \in (0.3497, 1)$. By increasing the value of parameter β in $(0.3497, 1)$, we can obtain the relationship between $Re(\lambda)$ and k^2 (see Figure 2.1(a)) and the relationship between D_k and k^2 (see Figure 2.1(b)), where $Re(\lambda)$ is the real part of λ. From Figure 2.1(a) and Figure 2.1(b), it is easy to see that $Re(\lambda) < 0$ and $D_k > 0$ always hold for $\beta < \beta_T^{(2)}$, which implies that there is no Turing instability. Therefore, $\beta > \beta_T^{(2)}$ is the necessary condition for Turing instability to occur.



(a) (b)

Figure 2.1: (a): the graph of the dispersion relation with respect to k^2 for different β; (b): the graph of $D_k(\beta)$ with respect to k^2 for different β.

In the following, we consider the Hopf bifurcation of the system (1.2) around $E(u_*, v_*)$. By Theorem 2.1, when $0 < \beta < 1$, then $T_0 = \beta - 1 < 0$ and $T_k = T_0 - k^2(\sigma_1 + \sigma_2) < 0$ for any $k \geq 0$. Let $k = \frac{n}{l}$, $n \in \mathbb{N}_0$, $l \in \mathbb{R}^+$. According to [8], n-mode Hopf bifurcation means that the characteristic equation (2.6) has a pair of purely imaginary roots, while the other roots have non-zero real parts and satisfy the corresponding transversal conditions.

Theorem 2.3. *Suppose one of the following conditions holds:*

(I) $0 < \beta \leq \beta_*$;

(II) $\beta > \beta_T^{(2)}$.

The system (1.2) *occurs 0-mode Hopf bifurcation at* $\beta = \beta_0^H = 1$, *where the characteristic* $F_0(\lambda) = 0$ *have a pair of purely imaginary roots and other roots of the characteristic* $F_k(\lambda) = 0$ $(k > 0)$ *have negative real parts. Where* β_* *and* $\beta_T^{(2)}$ *are defined in* (2.8) *and* (2.11).

Proof. Since $\frac{dT_0}{d\beta} = \frac{1}{2}$, then $T_0 = 0$ has a unique root $\beta = \beta_0^H = 1$, and obviously the transversal conditions satisfied. Moreover, $T_k < 0$ $(k \geq 1)$ and $D_0 = \beta > 0$. Since $-\sigma_1 + \sigma_2\beta \leq 0$, then $0 < \beta \leq \beta_*$. It is easy to obtain that $D_k > 0$ always holds for $0 < \beta \leq \beta_*$. When $-\sigma_1 + \sigma_2\beta > 0$, according (2.10) we know that $D_k > 0$ is equivalent to conditions $0 < \beta < \beta_T^{(1)}$ or $\beta > \beta_T^{(2)}$ holding. Combining (2.12), it is find that $\beta > \beta_T^{(2)}$ usable $D_k > 0$ always satisfied. Thus the system (1.2) occurs 0-mode Hopf bifurcation. □

In the next, we find the spatially inhomogeneous Hopf bifurcation for $n \in \mathbb{N}$. Define

$$\beta_n^H = 1 + \left(\frac{n}{l}\right)^2 (\sigma_1 + \sigma_2), \tag{2.13}$$

which is the root of $T_{\frac{n}{l}} = \beta - 1 - (\frac{n}{l})^2(\sigma_1 + \sigma_2) = 0$. There are the following conclusions.

Theorem 2.4. *Suppose one of the following conditions holds:*

(I) $0 < \beta \leq \beta_*$;

(II) $\beta > \beta_T^{(2)}$.

The system (1.2) *undergoes a n-mode Hopf bifurcation around* $E_*(u_*, v_*)$ *at* β_n^H *for* $n \in \mathbb{N}$, *where the characteristic equation* (2.6) *has a pair of purely imaginary roots, while all the other roots of* $F_j(\lambda) = 0$ $(j \neq \frac{n}{l})$ *have non-zero real parts. Where* β_* *and* $\beta_T^{(2)}$ *are defined in* (2.8) *and* (2.11).

Proof. To find the spatially inhomogeneous Hopf bifurcation points for $n \in \mathbb{N}$, we have to seek the roots of $(\frac{n}{l})^2(\sigma_1 + \sigma_2) + 1 = \beta$. Since $\frac{dT_{\frac{n}{l}}}{d\beta} = \frac{1}{2}$, then $T_{\frac{n}{l}} = 0$ has a unique root $\beta = \beta_n^H$ for $n \in \mathbb{N}$, and obviously the corresponding transversal conditions satisfied. Moreover, it is easy to get that $T_{\frac{n}{l}}$ is monotonically decreasing with respect to n, therefore $T_{\frac{j}{l}}(\beta_n^H) > 0$ for $j < n$ and $T_{\frac{j}{l}}(\beta_n^H) < 0$ for $j > n$. By the proof of Theorem 2.3, we know that $D_k > 0$ for one of the conditions in (I) or (II) holds. Thus the system undergoes n-mode Hopf bifurcation at β_n^H. □

In addition, to more intuitively understand Theorem Theorem 2.2–Theorem 2.4, taking $\sigma_1 = 0.4$, we plot the stability regions and the existing region of Turing instability in $\sigma_2 - \beta$ plane, as shown in Figure 2.2. According to Theorem 2.1–Theorem 2.4, in D_1, the positive equilibrium E_* is unstable and occurs Turing instability, and $\beta = \beta_T^{(2)}$ represents Turing bifurcation curve. In D_2, the positive equilibrium E_* is unstable but not occurs Turing instability. In D_3 and D_4 the positive equilibrium E_* is asymptotically stable. Moreover, we set $\sigma_1 = 0.4$, $\sigma_2 = 0.2$, then the 0-mode Hopf bifurcation will occurs at $\beta = \beta_0^H = 1$. Taking $\beta = 0.99 < \beta_0^H$, the system (1.2) can occur the spatially homogeneous periodic solutions (as shown in Figure 2.3). We set $\sigma_1 = 0.4$, $\sigma_2 = 3$, $n = 1$, $l = 8$, thus $\beta_1^H = 1.0531$. And the 1-mode Hopf bifurcation will occurs at $\beta = \beta_1^H$. Taking $\beta = 1.01 < \beta_1^H$, the system (1.2) can appear the spatially inhomogeneous periodic solution (as shown in Figure 2.4).

Remark 2.5. When $\beta \in (\beta_*, \beta_T^{(2)})$, at least one eigenvalue of D_k has positive real part, then the Hopf bifurcating periodic solutions are always unstable. Particularly, for 0-mode Hopf bifurcation, bifurcating periodic solutions are unstable in the interval AB in Figure 2.2.

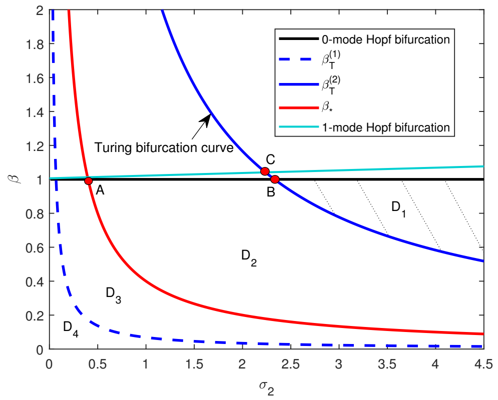


Figure 2.2: When $\sigma_1 = 0.4$, the Turing bifurcation curve and Hopf bifurcation curve in $\sigma_2 - \beta$ plane. D_1 is the Turing instability region, D_2 denotes unstable regions in which do not occurs Turing unstable, D_3 and D_4 are both stable regions. And B represents $(k_T, 0)$-mode Turing–Hopf bifurcation point, C stands for $(k_T, 1)$-mode Turing–Hopf bifurcation point.

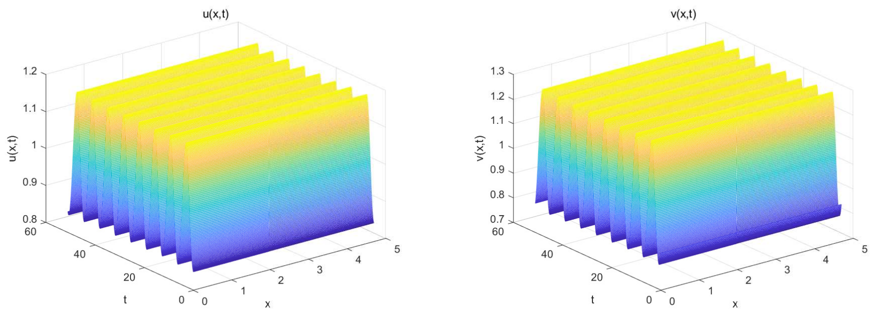


Figure 2.3: The spatially homogeneous periodic solution with $\sigma_1 = 0.4$, $\sigma_2 = 0.2$, $\beta = 0.99$. The initial values is (u_0, v_0)=(0.85, 0.85), and $0 \le x \le 5$, $0 \le t \le 60$.

Remark 2.6. In Figure 2.2, B and C denote the Turing–Hopf bifurcation points corresponding to the $(k_T, 0)$-mode and $(k_T, 1)$-mode, respectively. Point B is located at the coordinates (2.33,1), while point C is located at (2.24, 1.041). To investigate the dynamical behaviors that may occur near these points, we performed numerical simulations. Notably, in the vicinity of point B and C, we observe spatially homogeneous periodic solutions, non-constant steady-state solutions and spatially homogeneous quasi-periodic solutions. These observations are visually depicted in Figure 2.5. These results provide valuable insights into the behavior of the system near the Turing–Hopf bifurcation point.

This section focuses on the stability, Hopf bifurcation, and Turing instability regions of the diffusive Gierer–Meinhardt activator-inhibitor system (1.2) and obtains the conditions for the occurrence of Turing bifurcation, 0-mode Hopf bifurcation, k-mode Hopf bifurcation. As it is known, pattern formation can be induced by Turing instability. To uncover the diffusion mechanics of Turing patterns, this paper requires us to investigate and analyze the dynamic behavior of the Turing bifurcation. To solve this problem, we will employ the amplitude equation as an effective tool. In the next section, we will consider the amplitude equation of the system (1.2).

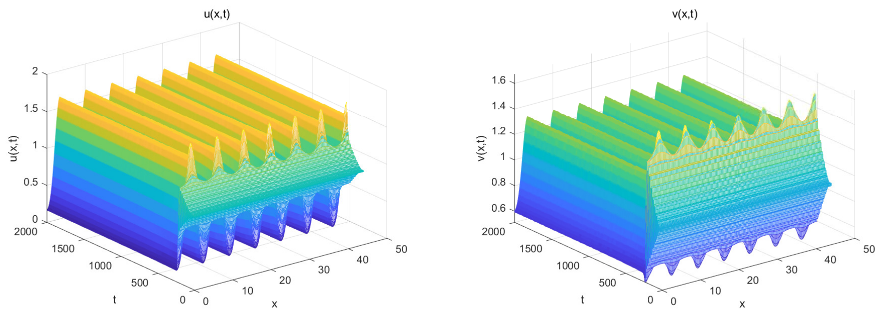


Figure 2.4: The spatially inhomogeneous periodic solution with $\sigma_1 = 0.4$, $\sigma_2 = 3$, $\beta = 1.01$. The initial values is (u_0, v_0)=(0.99, 0.99), and $0 \le x \le 50$, $0 \le t \le 2000$.

3 The amplitude equation and pattern formation

3.1 The amplitude equation of Turing bifurcation

In this subsection, in order to reveal the effect of diffusion on Turing patterns, the amplitude equation of the system (1.2) near the Turing bifurcation $\beta = \beta_T^{(2)}$ will be deduced by weakly nonlinear analysis [3,5,27]. To begin with, we consider the third order polynomial system of the system (1.2), which can be expressed as

$$\frac{\partial U}{\partial t} = IU + S(U,U), \tag{3.1}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad I = \begin{pmatrix} a_{11} + \sigma_1 \Delta & a_{12} \\ a_{21} & a_{22} + \sigma_2 \Delta \end{pmatrix},$$

and

$$S = \left(\begin{array}{c} f_{uu}u^2 + f_{uv}uv + f_{vv}v^2 \\ g_{uv}u^2 + g_{uv}uv + g_{uv}v^2 \end{array} \right) + \left(\begin{array}{c} f_{uuu}u^3 + f_{uuv}u^2v + f_{uvv}uv^2 + f_{vvv}v^3 \\ g_{uuu}u^3 + g_{uuv}u^2v + g_{uvv}uv^2 + g_{vvv}v^3 \end{array} \right) + o(4).$$

Applying perturbation techniques to the system (3.1), a small parameter ε is introduced near the critical value $\beta_T^{(2)}$ of the Turing bifurcation and satisfies the following form

$$\beta - \beta_T^{(2)} = \varepsilon\beta_1 + \varepsilon^2\beta_2 + \varepsilon^3\beta_3 + o\left(\varepsilon^3\right).$$

Meanwhile, the linear operator I can be decomposed into

$$I_\varepsilon = I_T + \left(\varepsilon\beta_1 + \varepsilon^2\beta_2 + \cdots\right) C, \tag{3.2}$$

where

$$I_T = \begin{pmatrix} a_{11}^T & a_{12}^T \\ a_{21}^T & a_{22}^T \end{pmatrix}, \tag{3.3}$$

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 1 & -2\beta_T^{(2)} \\ -\frac{2}{(\beta_T^{(2)})^2} & 0 \end{pmatrix},$$

with

$$a_{ij}^T = a_{ij}|_{\beta=\beta_T^{(2)}}, \quad c_{ij} = \frac{a_{ij}}{d\beta}.$$

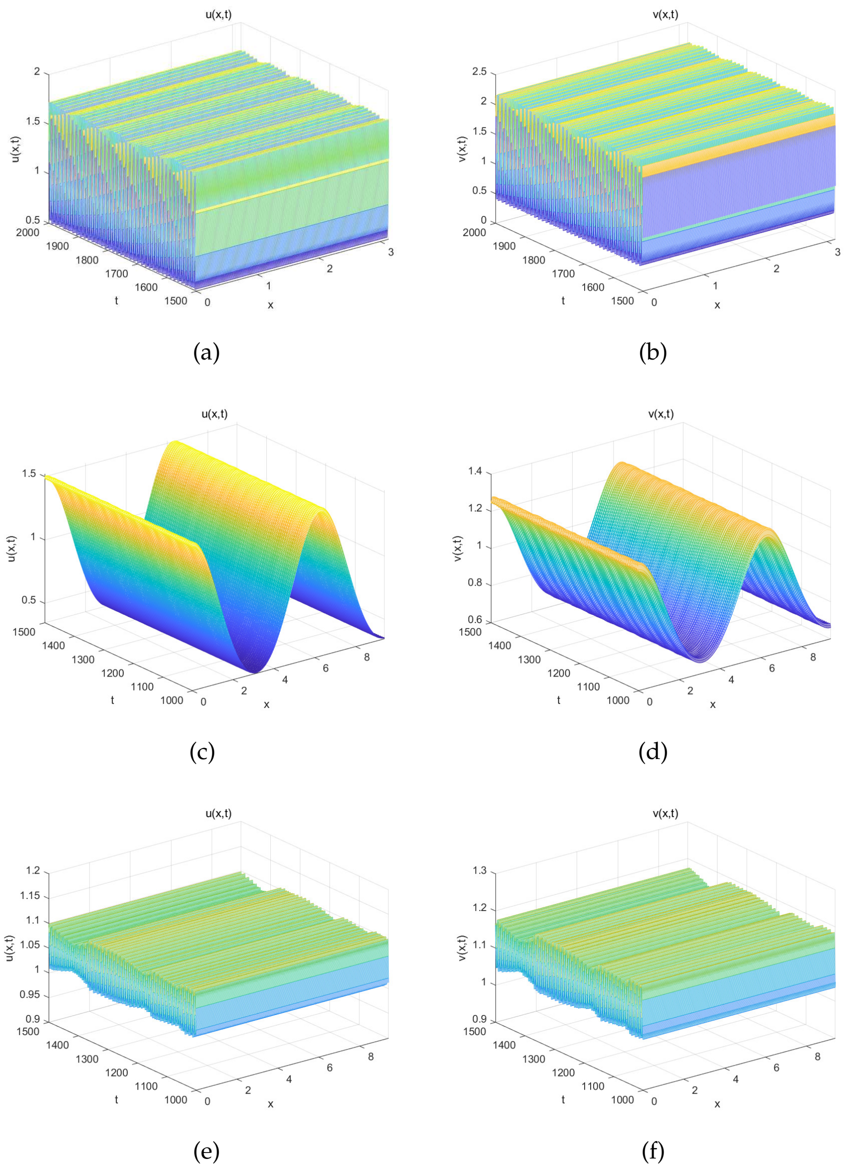


Figure 2.5: (a) and (b) are spatially homogeneous periodic solutions with $\sigma_1 = 0.4$, $\sigma_2 = 2.5$, $\beta = 1.002$, and $0 \le x \le 3$, $1500 \le t \le 2000$; (c) and (d) represent non-constant steady-state solutions with $\sigma_1 = 0.4$, $\sigma_2 = 2.43$, $\beta = 0.99$, and $0 \le x \le 8$, $1000 \le t \le 1500$; (e) and (f) correspond to spatially homogeneous quasi-periodic solutions with $\sigma_1 = 0.4$, $\sigma_2 = 2.25$, $\beta = 0.95$, and $0 \le x \le 8$, $1000 \le t \le 1500$. In all cases, the initial values for u and v are given by $(u_0, v_0) = (0.9 + 0.01\cos(2x), 0.9 + 0.01\cos(2x))$.

In addition, relating the variable U to the parameter ε can be written as

$$\mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix} = \varepsilon \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + \varepsilon^3 \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} + o\left(\varepsilon^3\right). \tag{3.4}$$

Substituting (3.2) and (3.4) into system (3.1), we obtain the following equation

$$\frac{\partial U}{\partial t} = \mathcal{I}_\varepsilon U + S(U, \varepsilon), \tag{3.5}$$

where

$$\mathcal{I}_\varepsilon = \begin{pmatrix} \sigma_1\Delta & 0 \\ 0 & \sigma_2\Delta \end{pmatrix} + I_\varepsilon, \quad S(U,\varepsilon) = \varepsilon^2 S_2 + \varepsilon^3 S_3 + o\left(\varepsilon^3\right), \tag{3.6}$$

particularly,

$$\mathcal{I}_0 = I_T + \begin{pmatrix} \sigma_1\Delta & 0 \\ 0 & \sigma_2\Delta \end{pmatrix}.$$

Accordingly, multiple time scales are introduced and the derivatives with respect to t are converted to

$$\frac{\partial}{\partial t} = \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + \varepsilon^3\frac{\partial}{\partial T_3} + o(\varepsilon^3). \tag{3.7}$$

Substitute (3.1)-(3.7) into (3.5), then deriving the coefficients of ε^j $(j = 1,2,3)$ satisfies the following equation

$O(\varepsilon)$:

$$\mathcal{I}_0 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = 0, \tag{3.8}$$

$O\left(\varepsilon^2\right)$:

$$\mathcal{I}_0 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \frac{\partial}{\partial T_1}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \beta_1 C \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - S_2, \tag{3.9}$$

$O\left(\varepsilon^3\right)$:

$$\mathcal{I}_0 \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \frac{\partial}{\partial T_1}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + \frac{\partial}{\partial T_2}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \beta_1 C \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} - \beta_2 C \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - S_3, \tag{3.10}$$

where

$$S_2 = \begin{pmatrix} s_{21} \\ s_{22} \end{pmatrix}, \quad S_3 = \begin{pmatrix} s_{31} \\ s_{32} \end{pmatrix},$$

with

$$\begin{aligned} s_{21} &= \frac{1}{2} f_{uu} u_1^2 + \frac{1}{2}\left(f_{uv} + f_{vu}\right) u_1 v_1 + \frac{1}{2} f_{vv} v_1^2, \\ s_{31} &= \frac{1}{6} f_{uuu} u_1^3 + \frac{1}{6} f_{vvv} v_1^3 + \frac{1}{6}\left(f_{uuv} + f_{uvu} + f_{vuu}\right) u_1^2 v_1 + f_{uu} u_2 u_1 + f_{vv} v_2 v_1 \\ &\quad + \frac{1}{6}\left(f_{uvv} + f_{vuv} + f_{vvu}\right) u_1 v_1^2 + \frac{1}{2}\left(f_{uv} + f_{vu}\right)\left(u_2 v_1 + u_1 v_2\right), \end{aligned}$$

s_{22} and s_{32} can be obtained by replacing f by g in s_{21} and s_{31}, and

$$\begin{aligned} &f_{uu} = \frac{2}{v}, && f_{uv} = -\frac{2u}{v^2}, && f_{uuv} = -\frac{2}{v^2}, && f_{uvv} = \frac{4u}{v^3}, \\ &f_{vu} = -\frac{2u}{v^2}, && f_{vv} = \frac{2u^2}{v^3}, && f_{vvv} = -\frac{6u^2}{v^4}, && g_{uu} = 2, \\ &f_{uvu} = -\frac{2}{v^2}, && f_{vuv} = \frac{4u}{v^3}, && f_{vvu} = \frac{4u}{v^3}, && f_{vuu} = -\frac{2}{v^2}. \end{aligned}$$

Firstly, we discuss the first order of ε, while $(u_1, v_1)^T$ is the linear combinations that belong to the eigenvectors corresponding to zero eigenvalues. The general solution of equation (3.9) can be composed in the following form

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \phi \\ 1 \end{pmatrix}\left(\sum_{j=1}^{3} M_j \exp\left(i\mathbf{k_j r}\right) + \sum_{j=1}^{3} \bar{M}_j \cdot \exp\left(-i\mathbf{k_j r}\right)\right), \tag{3.11}$$

where the wave numbers satisfy $\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}=0$, and $|\mathbf{k}|=k_T$. By substituting (3.11) into (3.8), we can get

$$\mathcal{I}_0\begin{pmatrix}\phi\\1\end{pmatrix}\exp(i\mathbf{k_j}\mathbf{r})=\begin{pmatrix}a_{11}-\sigma_1k_T^2 & a_{12}\\ a_{21} & a_{22}-\sigma_2k_T^2\end{pmatrix}\begin{pmatrix}\phi\\1\end{pmatrix}=0 \tag{3.12}$$

For convenience, we define

$$\mathcal{C}_k=\begin{pmatrix}a_{11}-\sigma_1k_T^2 & a_{12}\\ a_{21} & a_{22}-\sigma_2k_T^2\end{pmatrix}.$$

It is clear that $(\phi,1)^T$ is a zero eigenvector of $\mathcal{C}_k$, and by simple calculation, we can obtain $\phi=\frac{\sigma_2k_T^2-a_{22}}{a_{21}}$.

Using the Fredholm solvability condition for (3.10), the zero eigenvectors of the adjoint operator I_T^* of I_T is orthogonal to (3.10) right-hand side, and the eigenvector corresponding to the zero eigenvalues of I_T^* is

$$\begin{pmatrix}1\\\varphi\end{pmatrix}\cdot\exp(-i\mathbf{k_j}\mathbf{r})+c.c., \tag{3.13}$$

which follows

$$(1,\varphi)\cdot C_k^T=0, \tag{3.14}$$

with $\varphi=\frac{\sigma_1k_T^2-a_{11}}{a_{12}}$.

Using the Fredholm solvability condition to (3.10)

$$(1,\varphi)\cdot\left[\frac{\partial}{\partial T_1}\begin{pmatrix}u_1\\v_1\end{pmatrix}-\beta_1C\begin{pmatrix}u_1\\v_1\end{pmatrix}-S_2\right]=0. \tag{3.15}$$

By moving the term, we get the following formula

$$(1,\varphi)\cdot\frac{\partial}{\partial T_1}\begin{pmatrix}u_1\\v_1\end{pmatrix}=(1,\varphi)\cdot\left[\beta_1C\begin{pmatrix}u_1\\v_1\end{pmatrix}+S_2\right].$$

Using the orthogonality condition for (3.10), we can obtain the following equations

$$\begin{cases}(\phi+\varphi)\dfrac{\partial M_1}{\partial T_1}=\left(\phi-2\beta^T-\dfrac{2\phi\varphi}{(\beta^T)^2}\right)\beta_1M_1+2(1,\varphi)\begin{pmatrix}s_1\\s_2\end{pmatrix}\cdot\bar{M}_2\cdot\bar{M}_3,\\ (\phi+\varphi)\dfrac{\partial M_2}{\partial T_1}=\left(\phi-2\beta^T-\dfrac{2\phi\varphi}{(\beta^T)^2}\right)\beta_1M_2+2(1,\varphi)\begin{pmatrix}s_1\\s_2\end{pmatrix}\cdot\bar{M}_1\cdot\bar{M}_3,\\ (\phi+\varphi)\dfrac{\partial M_3}{\partial T_1}=\left(\phi-2\beta^T-\dfrac{2\phi\varphi}{(\beta^T)^2}\right)\beta_1M_3+2(1,\varphi)\cdot\begin{pmatrix}s_1\\s_2\end{pmatrix}\cdot\bar{M}_1\cdot\bar{M}_2,\end{cases} \tag{3.16}$$

where

$$s_1=\frac{f_{uu}}{2}\phi^2+\frac{(f_{uv}+f_{vu})}{2}\phi+\frac{f_{vv}}{2},\quad s_2=\frac{g_{uu}}{2}\phi^2+\frac{(g_{uv}+g_{vu})}{2}\phi+\frac{g_{vv}}{2}.$$

Suppose that the solution of (3.10) has the following form

$$\begin{aligned}\begin{pmatrix}u_2\\v_2\end{pmatrix}=&\begin{pmatrix}U_0\\V_0\end{pmatrix}+\sum_{j=1}^{3}\begin{pmatrix}U_j\\V_j\end{pmatrix}e^{i\mathbf{k_j}\mathbf{r}}+\sum_{j=1}^{3}\begin{pmatrix}U_{jj}\\V_{jj}\end{pmatrix}e^{i2\mathbf{k_j}\mathbf{r}}+\begin{pmatrix}U_{12}\\V_{12}\end{pmatrix}e^{i(\mathbf{k_1}-\mathbf{k_2})\mathbf{r}}\\&+\begin{pmatrix}U_{23}\\V_{23}\end{pmatrix}e^{i(\mathbf{k_3}-\mathbf{k_3})\mathbf{r}}+\begin{pmatrix}U_{31}\\V_{31}\end{pmatrix}e^{i(\mathbf{k_3}-\mathbf{k_1})\mathbf{r}}+\text{ c.c.}\end{aligned} \tag{3.17}$$

where *c.c* represents the complex conjugate of all the preceding terms. Substituting (3.17) into (3.10), we can derive that

$$\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \left(|M_1|^2 + |M_2|^2 + |M_3|^2\right), \quad U_j = \phi V_j,$$

$$\begin{pmatrix} U_{jj} \\ V_{jj} \end{pmatrix} = \begin{pmatrix} u_{11} \\ v_{11} \end{pmatrix} M_j^2, \quad \begin{pmatrix} U_{mn} \\ V_{mn} \end{pmatrix} = \begin{pmatrix} u_{mn} \\ v_{mn} \end{pmatrix} M_m \bar{M}_n,$$

where

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} -a_{22}s_1 + a_{12}s_2 \\ -a_{11}s_2 + a_{21}s_1 \end{pmatrix},$$

$$\begin{pmatrix} u_{11} \\ v_{11} \end{pmatrix} = \frac{1}{2} \frac{1}{(a_{11} - 4k_T^2\sigma_1)(a_{22} - 4k_T^2\sigma_2) - a_{12}a_{21}} \begin{pmatrix} -(a_{22} - 4k_T^2\sigma_2)s_1 + a_{12}s_2 \\ -(a_{11} - 4k_T^2\sigma_1)s_2 + a_{21}s_1 \end{pmatrix},$$

$$\begin{pmatrix} u_{mn} \\ v_{mn} \end{pmatrix} = \frac{1}{(a_{11} - 3k_T^2\sigma_1)(a_{22} - 3k_T^2\sigma_2) - a_{12}a_{21}} \begin{pmatrix} -(a_{22} - 3k_T^2\sigma_2)s_1 + a_{12}s_2 \\ -(a_{11} - 3k_T^2\sigma_1)s_2 + a_{21}s_1 \end{pmatrix}.$$

Using the Fredholm solvability condition to (3.10),

$$(1,\varphi) \cdot \left[\frac{\partial}{\partial T_1} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + \frac{\partial}{\partial T_2} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - \beta_1 C \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} - \beta_2 C \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - S_3 \right] = 0. \tag{3.18}$$

After simplification, we can obtain the following equations

$$\begin{cases} (\phi + \varphi)\left(\frac{\partial V_1}{\partial T_1} + \frac{\partial M_1}{\partial T_2}\right) = s_3\left(\beta_1 V_1 + \beta_2 M_1\right) + s_4\left(\bar{V}_2\bar{M}_3 + \bar{V}_3\bar{M}_2\right) + \\ \qquad \left(P_1 |M_1|^2 + P_2\left(|M_2|^2 + |M_3|^2\right)\right) M_1, \\ (\phi + \varphi)\left(\frac{\partial V_2}{\partial T_1} + \frac{\partial M_2}{\partial T_2}\right) = s_3\left(\beta_1 V_2 + \beta_2 M_2\right) + s_4\left(\bar{V}_1\bar{M}_3 + \bar{V}_3\bar{M}_1\right) + \\ \qquad \left(P_1 |M_2|^2 + P_2\left(|M_1|^2 + |M_3|^2\right)\right) M_2, \\ (\phi + \varphi)\left(\frac{\partial V_3}{\partial T_1} + \frac{\partial M_3}{\partial T_2}\right) = s_3\left(\beta_1 V_3 + \beta_2 M_3\right) + s_4\left(\bar{V}_1\bar{M}_2 + \bar{V}_2\bar{M}_1\right) + \\ \qquad \left(P_1 |M_3|^2 + P_2\left(|M_1|^2 + |M_2|^2\right)\right) M_3, \end{cases} \tag{3.19}$$

where

$$s_3 = \phi - 2\beta_T^{(2)} - \frac{2\phi\varphi}{(\beta_T^{(2)})^2},$$

$$s_4 = 2\,(1,\varphi) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

$$P_1 = \left(m_1 q_1 + m_2 b_1 + \frac{B_1}{2}\right) + \varphi\left(m_1 q_2 + m_2 b_2 + \frac{B_2}{2}\right),$$

$$P_2 = \left(n_1 q_1 + n_2 b_1 + B_1\right) + \varphi\left(n_1 q_2 + n_2 b_2 + B_2\right),$$

$$m_1 = u_0 + u_{11}, \quad m_2 = v_0 + v_{11},$$

$$n_1 = u_0 + u_{mn}, \quad n_2 = v_0 + v_{mn},$$

$$q_1 = f_{uu}\phi + \frac{1}{2}\left(f_{uv} + f_{vu}\right), \quad b_1 = f_{vv} + \frac{1}{2}\left(f_{uv} + f_{vu}\right)\phi,$$

$$q_2 = g_{uu}\phi + \frac{1}{2}\left(g_{uv} + g_{vu}\right), \quad b_2 = g_{vv} + \frac{1}{2}\left(g_{uv} + g_{vu}\right)\phi,$$

$$B_1 = f_{uuu}\phi^3 + (f_{uuv} + f_{uvu} + f_{vuu})\,\phi^2 + (f_{uvv} + f_{vuv} + f_{vvu})\,\phi + f_{vvv},$$
$$B_2 = g_{uuu}\phi^3 + (g_{uuv} + g_{uvu} + g_{vuu})\,\phi^2 + (g_{uvv} + g_{vuv} + g_{vvu})\,\phi + g_{vvv}.$$

The solution of the reaction-diffusion system (1.2) at the Turing instability critical point has the following form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \phi \\ 1 \end{pmatrix} \Big(\sum_{j=1}^{3} Z_j \exp\left(i\mathbf{k_j}\cdot\mathbf{r}\right) + \sum_{j=1}^{3} \bar{Z}_j \exp\left(-i\mathbf{k_j}\cdot\mathbf{r}\right)\Big). \tag{3.20}$$

Combining (3.4), (3.11), (3.17) and (3.20), the amplitude Z_j can be transformed into the following form $Z_j = \varepsilon M_j + \varepsilon^2 V_j + o(\varepsilon^3)$. Determined by the expressions of Z_j and Eqs. (3.7), (3.11), (3.16) and (3.19) we can obtain the equation for the amplitude corresponding to Z_1 as follows

$$\tau_0 \frac{\partial Z_1}{\partial t} = \mu Z_1 + d\bar{Z}_2\bar{Z}_3 - \left(w_1 |Z_1|^2 + w_2 |Z_2|^2 + |Z_3|^2\right) Z_1, \tag{3.21}$$

where

$$\tau_0 = \frac{\phi + \varphi}{s_3\beta_T^{(2)}}, \quad \mu = \frac{\beta - \beta_T^{(2)}}{\beta_T^{(2)}}, \quad d = \frac{s_4}{s_3\beta_T^{(2)}},$$
$$w_1 = -\frac{P_1}{s_3\beta_T^{(2)}}, \quad w_2 = -\frac{P_2}{s_3\beta_T^{(2)}}.$$

Analogously, we can derive two other amplitude equations

$$\begin{cases} \tau_0 \frac{\partial Z_2}{\partial t} = \mu Z_2 + d\bar{Z}_1\bar{Z}_3 - \left(w_1 |Z_1|^2 + w_2 \left(|Z_2|^2 + |Z_3|^2\right)\right) Z_2, \\ \tau_0 \frac{\partial Z_3}{\partial t} = \mu Z_3 + d\bar{Z}_1\bar{Z}_2 - \left(w_1 |Z_1|^2 + w_2 \left(|Z_2|^2 + |Z_3|^2\right)\right) Z_3. \end{cases} \tag{3.22}$$

Using the polar coordinate transform

$$Z_j = \rho_j \exp(i\varphi_j) \ (j = 1, 2, 3),$$

where $\rho = |Z_j|$, and φ_j is the polar angle. Then substituting (3.21) into (3.22), the system (3.22) becomes

$$\begin{cases} \tau_0 \frac{\partial \theta}{\partial t} = -d\frac{\rho_1^2\rho_2^2 + \rho_1^2\rho_3^2 + \rho_2^2\rho_2^2}{\rho_1\rho_2\rho_3} \sin\theta, \\ \tau_0 \frac{\partial \rho_1}{\partial t} = \mu\rho_1 + d\rho_2\rho_3\cos\theta - w_1\rho_1^3 - w_2\left(|\rho_2|^2 + |\rho_3|^2\right)\rho_1, \\ \tau_0 \frac{\partial \rho_2}{\partial t} = \mu\rho_2 + d\rho_1\rho_3\cos\theta - w_1\rho_2^3 - w_2\left(|\rho_1|^2 + |\rho_3|^2\right)\rho_2, \\ \tau_0 \frac{\partial \rho_3}{\partial t} = \mu\rho_3 + d\rho_2\rho_1\cos\theta - w_1\rho_3^3 - w_2\left(|\rho_1|^2 + |\rho_2|^2\right)\rho_3, \end{cases} \tag{3.23}$$

where

$$\theta = \theta_1 + \theta_2 + \theta_3.$$

From the first equation of the system (3.23), there are only two conditions to consider: $\theta = 0$ or π. The system (3.23) is stable for $\theta = 0$, $d > 0$ and $\theta = \pi$, $d < 0$. Hence, the system (3.23) can be reduced to the following form

$$\begin{cases} \tau_0 \frac{\partial \rho_1}{\partial t} = \mu\rho_1 + |d|\rho_2\rho_3 - w_1\rho_1^3 - w_2\left(\rho_2^2 + \rho_3^2\right)\rho_1, \\ \tau_0 \frac{\partial \rho_2}{\partial t} = \mu\rho_2 + |d|\rho_1\rho_3 - w_1\rho_2^3 - w_2\left(\rho_1^2 + \rho_3^2\right)\rho_2, \\ \tau_0 \frac{\partial \rho_3}{\partial t} = \mu\rho_3 + |d|\rho_1\rho_2 - w_1\rho_3^3 - w_2\left(\rho_1^2 + \rho_2^2\right)\rho_3. \end{cases} \tag{3.24}$$

As the results in the [17, 22, 24], by studying the existence and stability of the equilibrium points of the amplitude system (3.24), we know that the amplitude system (3.24) has five types of steady-state solutions with the following conclusions:

(1) The amplitude system (3.24) has an equilibrium $E_1 = (0, 0, 0)$, which is stable for $\mu < \mu_2$ and unstable for $\mu > \mu_2$;

(2) When $\mu w_1 > 0$, the amplitude system (3.24) has an equilibrium $E_2 = \left(\sqrt{\frac{\mu}{w_1}}, 0, 0\right)$, which is stable for $\mu > \mu_3$ with $w_2 > w_1 > 0$;

(3) When $w_1 + 2w_2 > 0$, $\mu_1 < \mu < 0$ or $w_1 + 2w_2 < 0$, $\mu < 0$, the system (3.24) has an equilibrium $E_3^{(0)} = (\rho_1^*, \rho_1^*, \rho_1^*)$ with $\rho_1^* = \frac{|d| - \sqrt{d^2 + 4\mu(w_1 + 2w_2)}}{2(w_1 + 2w_2)}$, which is always unstable;

(4) When $w_1 + 2w_2 > 0$, $\mu_1 < \mu$, the system (3.24) has an equilibrium $E_3^{(\pi)} = (\rho_2^*, \rho_2^*, \rho_2^*)$ with $\rho_2^* = \frac{|d| + \sqrt{d^2 + 4\mu(w_1 + w_2)}}{2(w_1 + 2w_2)}$, which is stable for $-2w_2 < w_1 \leq p_2$, $\mu_1 < \mu < \mu_4$ or $-\frac{1}{2}w_2 < w_2 < w_1$, $\mu_1 < \mu$;

(5) When $w_2 > w_1 > 0$, $\mu > \mu_3$ or $w_1 < w_2 < 0$, $\mu < \mu_3$, the system (3.24) has an equilibrium $E_4 = (\rho_3^*, \rho_4^*, \rho_4^*)$ with $\rho_3^* = \sqrt{\frac{|d|}{w_2 - w_1}}$ and $\rho_4^* = \sqrt{\frac{\mu - w_1 \rho_1^2}{w_1 + w_2}}$, which is always unstable;

where

$$\mu_1 = \frac{-d^2}{4(w_1 + 2w_2)}, \quad \mu_2 = 0, \quad \mu_3 = \frac{d^2 w_1}{(w_2 - w_1)^2}, \quad \mu_4 = \frac{2w_1 + w_2}{(w_2 - w_1)^2} d^2.$$

By Theorem 2.2, Turing instability occurs at $\beta \in (\beta_T^{(2)}, 1)$ for the system (1.2), that is $\mu = \frac{\beta - \beta_T^{(2)}}{\beta_T^{(2)}} > 0$. However, in this case, $E_3^{(0)}$ does not exist. According to the results in [17, 24], the existence and stability of the equilibria of the amplitude system (3.24) correspond to the type of spatial patterns of the original system (1.2). E_1 and $E_3^{(\pi)}$ correspond to the spot patterns, E_2 and E_4 correspond to the the stripe patterns and the mixed patterns, respectively. In addition, it is easy to know from the above discussion that $\mu_1 < \mu_2 < \mu_3 < \mu_4$. Consequently, one obtains the following results:

(1) The system (3.24) only has a equilibrium $E_3^{(\pi)}$ for $\mu_2 < \mu < \mu_3$, which is stable, therefore, the system (1.2) only appear spot patterns;

(2) When β crosses a critical value so that $\mu_3 < \mu < \mu_4$, the system (3.24) has two equilibria E_2 and $E_3^{(\pi)}$, correspondingly, the system (1.2) can occurs mixed patterns;

(3) When $\mu_4 < \mu$, the system (3.24) only has a equilibrium E_2, and then, stripe patterns will appear in the system (1.2).

Therefore, we are able to establish a connection between the initial reaction-diffusion equation and the amplitude equation presented in Table 3.1. This linkage not only sheds light on the underlying mechanisms of these mathematical models, but also provides a valuable theoretical framework for further research in this field of study [17].

In this subsection, we derive the amplitude equation (3.24) of the system (1.2) using the weakly nonlinear analysis method and obtain the conditions for the appearance of different Turing patterns. In the next subsection, we will verify theoretical analysis by numerical simulation.

Amplitude system (3.24)	The original system (1.2)
E_1	Spot pattern
E_2	Stripe pattern
$E_3^{(\pi)}$	Spot pattern
E_4	Mixed pattern

Table 3.1: The correspondence between the amplitude system and the original system.

3.2 Numerical simulations of pattern formation

In this subsection, we will perform numerical simulations to verify the last part of the theoretical analysis. Taking the Parameters $\sigma_1 = 0.5$, $\sigma_2 = 3.6$, then we have

$$k_T^2 = 0.6706, \quad \beta_T^{(2)} = 0.8095.$$

According to Theorem 2.2, Turing pattern will appear when $\beta \in (0.8095, 1)$. Then we choose $\beta = 0.99$, and with simple calculations, the following results can be obtained

$$d = -0.4661, \quad w_1 = 0.2653, \quad w_2 = 0.6939,$$

$$\mu_1 = -0.0329, \quad \mu_2 = 0, \quad \mu_3 = 0.3137, \quad \mu_4 = 1.4480, \quad \mu = 0.2230.$$

Hence, $\mu_2 < \mu < \mu_3$, the parameter values $\rho_2^* = 1.2414$, $-2w_2 = -1.3878 < w_1 < \rho_2$, and $E_3^{(\pi)} = (1.2414, 1.2414, 1.2414)$ represent a specific range of conditions that correspond to the fourth steady-state solution of the amplitude equation (as defined in (4)). Based on our previous analysis, the appearance of spot patterns in the reaction-diffusion system (1.2) is expected under these conditions (see Figure 3.1). Therefore, we can conclude that the formation of spot patterns in the system is likely to occur under the specified parameter values.

Next, choosing $\beta = 0.95$, we can obtain the following results

$$d = -0.4277, \quad w_1 = 1.7657, \quad w_2 = 3.2023,$$

$$\mu_1 = -0.0056, \quad \mu_2 = 0, \quad \mu_3 = 0.1565, \quad \mu_4 = 0.5968, \quad \mu = 0.1736.$$

And then get $\mu_3 < \mu < \mu_4$, $\rho_3^* = 0.5456$, $\rho_4^* = 0.0258$, $w_2 > w_1$, $\mu > \mu_3$, $E_4 = (0.5456, 0.0258, 0.0258)$, which falls within (5) of the steady-state solution of the amplitude equation. Based on the analysis in the previous section, this situation can induce the formation of the mixed patterns (the coexistence of spot patterns and stripe patterns) of the system (1.2) (see Figure 3.2).

In the following, reducing β to $\beta = 0.85$, by a series of calculations, we get

$$d = -0.0389, \quad w_1 = 1.1113, \quad w_2 = 2.4493,$$

$$\mu_1 = -0.0001, \quad \mu_2 = 0, \quad \mu_3 = 0.0009, \quad \mu_4 = 0.0039, \quad \mu = 0.0500.$$

And thus obtain $\mu > \mu_4$. The system (1.2) exhibits stripe patterns (see Figure 3.3), as predicted by previous theoretical findings, when the following conditions are met: $\rho_1 = 0.2121$, $\mu > \mu_3$, $w_2 > w_1 > 0$, $\mu w_1 = 0.0556 > 0$, and $E_2 = (0.2121, 0, 0)$. The corresponding steady-state solution of the amplitude equation is denoted as (2). From the above analysis, Table 3.2 was derived.

Parameters of the Amplitude Equation

$(\sigma_1, \sigma_2, \beta)$	d	w_1	w_2	μ_1	μ_2	μ_3	μ_4	μ	**type**
$(0.5, 3.6, 0.99)$	-0.4661	0.2653	0.6939	-0.0329	0	0.3137	1.4480	0.2230	**Spot**
$(0.5, 3.6, 0.95)$	-0.4277	1.7657	3.2023	-0.0056	0	0.1565	0.5968	0.1736	**Mixed**
$(0.5, 3.6, 0.85)$	-0.0389	1.1113	2.4493	-0.0001	0	0.0009	0.0039	0.0500	**Stripe**

Table 3.2: Different parameters and corresponding patterns.

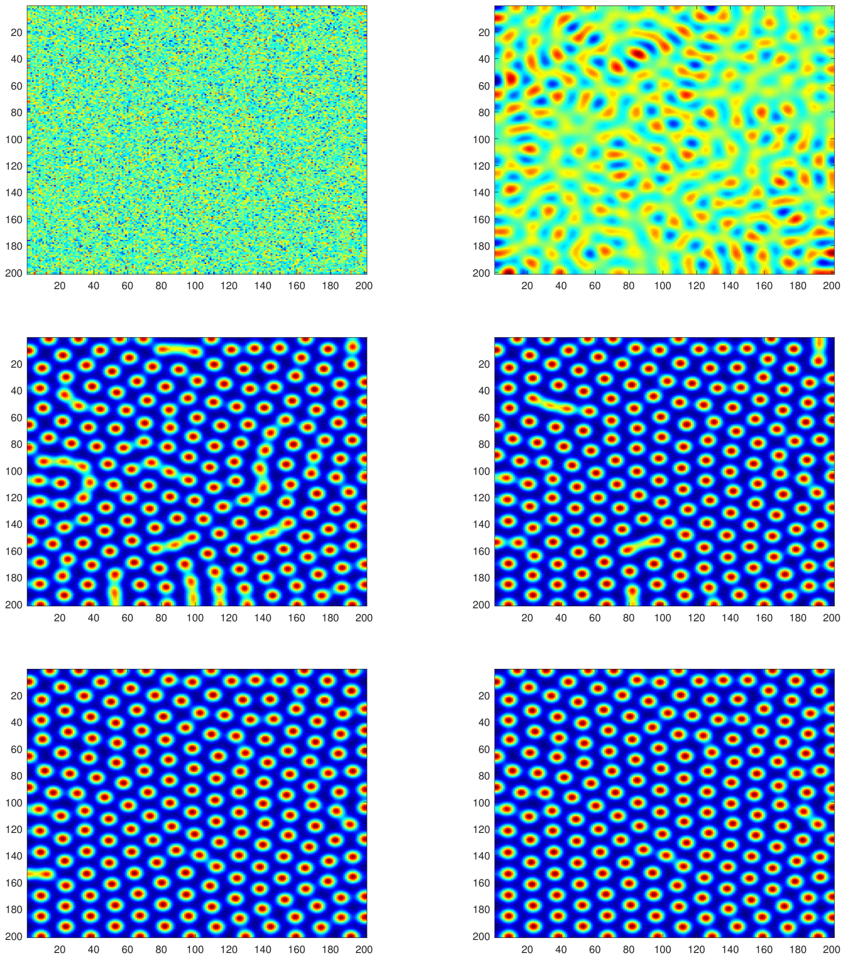


Figure 3.1: The evolutionary process of concentration of the activator u with $\sigma_1 = 0.5$, $\sigma_2 = 3.6$, $\beta = 0.99$ at $t = 0$, $t = 100,000$, $t = 750,000$, $t = 2,000,000$, $t = 2,500,000$, $t = 3,000,000$, respectively.

In order to solve the system of continuous reaction-diffusion equations (1.2) with MatLab, it is necessary to discretize the system (1.2) in space and time. Therefore, we choose $\Omega = [0, 200] \times [0, 200]$ as the discrete region, while choosing a time step $\Delta t = 0.0005$ and a space step $\Delta h = 0.5$. Since the concentration spatial pattern of the activator u is similar to the inhibitor v, we only show the concentration spatial pattern of the activator u, as shown in Figure 3.1–Figure 3.3. Next, numerical simulations are performed in the vicinity of the Turing bifurcation.

In Figure 3.1, $\beta = 0.99 \in (0.8095, 1)$ and then we have $\mu \in (\mu_2, \mu_3)$. The results show that the spot patterns and stripe patterns coexist as time t increases, but these patterns will gradually disappear as time t changes, eventually, the spot patterns will dominate the whole region. Theoretical and numerical results are kept consistent. Here, we take $t = 0$, $100,000$, $750,000$, $2,000,000$, $2,500,000$, and $3,000,000$, respectively, with the following initial values

$$\begin{cases} u(x,y,0) = u_* - 0.0002 \cdot \mathtt{randn}(200) \\ v(x,y,0) = u_* - 0.0002 \cdot \mathtt{randn}(200) \end{cases} \tag{3.25}$$

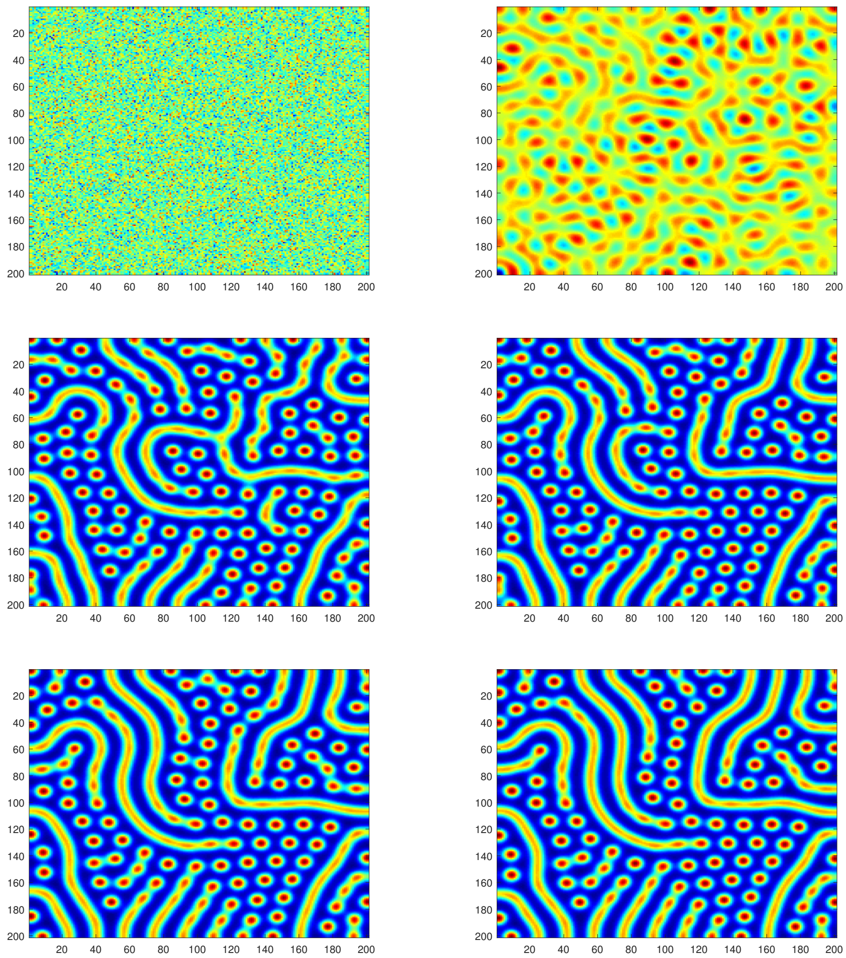


Figure 3.2: The evolutionary process of concentration of the activator u with $\sigma_1 = 0.5$, $\sigma_2 = 3.6$, $\beta = 0.95$ at $t = 0$, $t = 150,000$, $t = 750,000$, $t = 1,400,000$, $t = 2,000,000$, $t = 3,000,000$, respectively.

Figure 3.2 shows the spatial pattern evolution of the activator u at $t = 0$, $150,000$, $750,000$, $1,400,000$, $2,000,000$, and $3,000,000$ for the reaction-diffusion system (1.2) under the initial condition (3.25), and $\beta = 0.95 \in (0.8095, 1)$, $\mu \in (\mu_3, \mu_4)$. Based on the above theoretical analysis, in this case, there is the coexistence of the spot patterns and stripe patterns. Numerically, it can be seen that this random distribution leads to the coexistence of these two patterns and this coexistence does not change further with increasing time t.

Under the same initial value conditions as above, taking $t = 0$, $90,000$, $1,550,000$, $2,000,000$, $2,800,000$, and $3,000,000$, $\beta = 0.85 \in (0.8095, 1)$, then $\mu \in (\mu_4, \infty)$. With the increase of time t, the spot-stripe coexistence pattern starts to lose stability, and the stripe

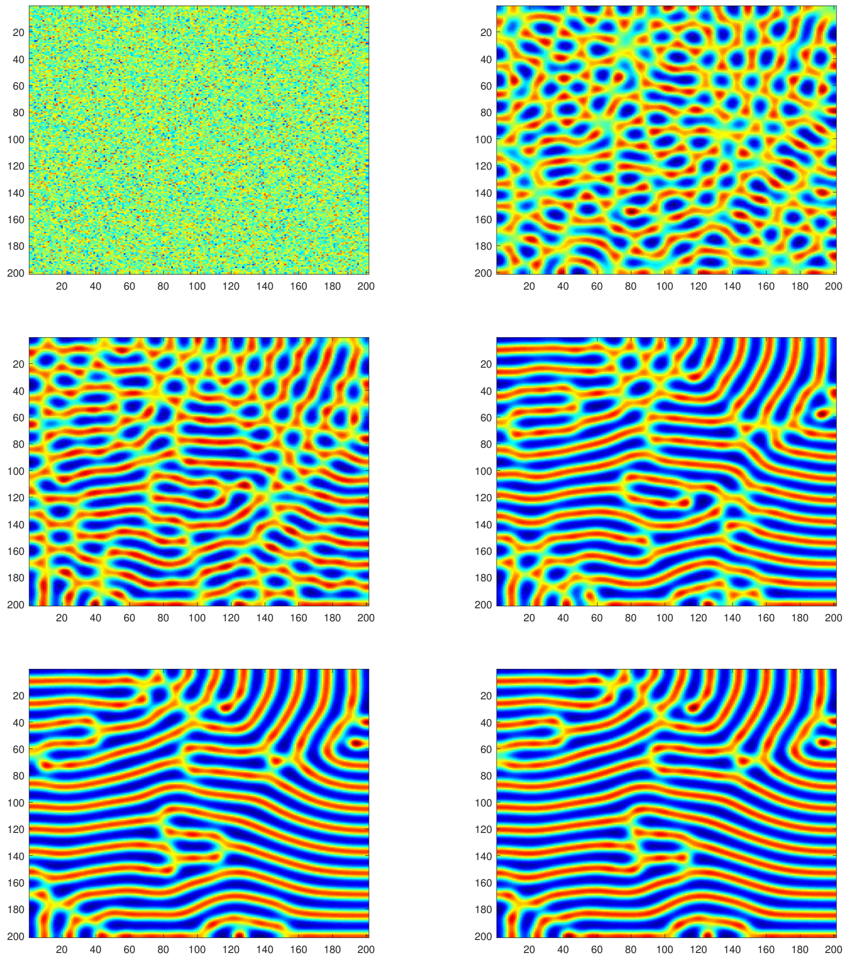


Figure 3.3: The evolutionary process of concentration of the activator u with $\sigma_1 = 0.5$, $\sigma_2 = 3.6$, $\beta = 0.85$ at $t = 0$, $t = 90,000$, $t = 1,550,000$, $t = 2,000,000$, $t = 2,800,000$, $t = 3,000,000$, respectively.

pattern appears and eventually stabilizes. The numerical simulation results (as shown in Figure 3.3) are inconsistent with the theoretical analysis.

From the results of numerical simulations, we can see that when the decay rate of the activator β decreases from 0.99, 0.95 to 0.85 in order, the type of activator concentration u pattern changes from spot patterns, spot-stripe patterns to stripe patterns in order. This indicates that the decay rate of the activator β affects the type of activator concentration u patterns. Therefore, in chemical reactions, we can adjust the decay rate of the activator β to make the concentration of the activator u tend to different patterns at dynamic equilibrium.

4 Conclusions

In this paper, the Hopf bifurcations, Turing instability, and pattern formation of Gierer–Meinhardt activator-inhibitor models with mutual resistance effects are investigated. The existence and stability of the positive equilibrium point E_* are analyzed firstly, which are influenced by the parameter β, indicating that the decay rate of the activator has an essential effect on the system. Then the conditions for the Hopf bifurcation as well as the Turing bifurcation are established theoretically, and the effects of parameter β on the Hopf bifurcation and Turing bifurcation are discussed numerically.

It is shown that under certain conditions, a diffusion-driven Turing instability occurs at the positive equilibrium point E_*. For a fixed σ_1, the Turing instability region in the $\beta - \sigma_2$ plane is surrounded by the Hopf bifurcation curve and the Turing bifurcation curve (see Figure 2.2). It can be concluded that there is no Turing instability for the higher decay rate of the activator.

For studying and analyzing the dynamic behavior near the Turing bifurcation, the corresponding amplitude equations are driven for the system (1.2) near the Turing bifurcation point by the weakly nonlinear analysis method, which can be used to predict the stability of the spatial pattern and its type. Based on theoretical analysis, the system will appear with spot patterns, mixed patterns, and stripe patterns, which can be verified by numerical simulations in the subsection 3.2. The results show that, with β as the adjustment parameter, the spatial patterns in the Turing instability region change from the spot patterns, and spot-stripe coexistence patterns to stripe patterns in order. These spatial patterns can not only simulate and explain the chemical oscillations between activator concentrations and inhibitor concentrations in a better way but they can also be applied to medical tests [10].

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An implicit system of delay differential algebraic equations from hydrodynamics

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Abstract. Direct spring operated pressure relief valves connected to a constantly charged vessel and a downstream pipe have a complex dynamics. The vessel-valve subsystem is described with an autonomous system of ordinary differential equations, while the presence of the pipe adds two partial differential equations to the mathematical model. The partial differential equations are transformed to a delay algebraic equation coupled to the ordinary differential equations. Due to a square root nonlinearity, the system is implicit. The linearized system can be transformed to a standard system of neutral delay differential equations (NDDEs) having more elaborated literature than the delay algebraic equations. First, the different forms of the mathematical model are presented, then the transformation of the linearized system is conducted. The paper aims at introducing this unusual mathematical model of an engineering system and inducing research focusing on the methodology to carry out bifurcation analysis in implicit NDDEs.

Keywords: difference equations, neutral delay differential equations, implicit differential equations, Hopf bifurcations, travelling wave solutions.

2020 Mathematics Subject Classification: 34H20, 37L10.

1 Introduction

The mathematical model in question describes the dynamics of a simplified mechanical model consisting of a vessel charged by a constant fluid flow rate, a direct spring operated pressure relief valve and a pipe delivering the fluid to the atmospheric pressure. Previous studies are presented in [7] and [8] mainly focus on the engineering aspects of the valve dynamics. The model construction and the derivation of the dimensionless mathematical model can be found in [7] in detail, but the various simplified forms of the mathematical model are summarized here with compressed parameters.

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The dimensionless mathematical model of the vessel-valve subsystem is an autonomous system of ordinary differential equations (ODEs):

$$\dot{y}_1(t) = y_2(t)\,, \tag{1.1}$$
$$\dot{y}_2(t) = -2\zeta y_2(t) - (y_1(t) + \delta) + y_3(t) - y_{\text{out}}\,, \tag{1.2}$$
$$\dot{y}_3(t) = \beta\big(q - y_1(t)\sqrt{y_3(t) - y_{\text{out}}}\,\big)\,, \tag{1.3}$$

with the state variables $y_{1,2,3} \in \mathbb{R}$, where dot refers to the derivative with respect to the dimensionless time t. The appearing parameters ζ, δ, β, q are positive according their physical meanings. The backpressure y_{out} is usually the atmospheric backpressure that is zero for the vessel-valve subsystem. In the followings, we investigate the dynamics of the open valve for which the conditions $y_1 > 0$ and $y_3 > 0$ fulfil. The system can be linearized around its trivial solution $y^*_{1,2,3}$, which is given as the only physically relevant non-negative solution of the following algebraic equations depending on the parameters q and δ:

$$y_1^{*3} + \delta y_1^{*2} - q^2 = 0\,, \quad y_3^* = y_1^* + \delta\,, \quad y_2^* = 0\,.$$

For small perturbations $\eta_{1,2,3} \in \mathbb{R}$ around the equilibrium, the linearised system is:

$$\dot{\eta}_1(t) = \eta_2(t)\,, \tag{1.4}$$
$$\dot{\eta}_2(t) = -2\zeta\eta_2(t) - \eta_1(t) + \eta_3(t)\,, \tag{1.5}$$
$$\dot{\eta}_3(t) = -\beta\left(\sqrt{y_3^*}\eta_1(t) + \frac{y_1^*}{2\sqrt{y_3^*}}\eta_3(t)\right). \tag{1.6}$$

The Navier–Stokes and continuity equations of the fluid flow through the pipe serve two dimensionless partial differential equations:

$$\dot{y}_4(x,t) = -\Gamma_1 y_5'(x,t)\,, \tag{1.7}$$
$$\dot{y}_5(x,t) = -\Gamma_2 y_4'(x,t)\,, \quad \Gamma_2 = \frac{1}{\Gamma_1(\tau/2)^2} \tag{1.8}$$

where the new coordinate x denotes the dimensionless space coordinate along the pipe such that $x = 1$ corresponds to the end of the pipe, while y_4 is the gauge pressure distribution and y_5 is the fluid velocity distribution along the pipe. The parameters Γ_1 and Γ_2 are also positive and can be calculated from the physical parameters of the system and the fluid. The time delay τ connects the parameters Γ_1 and Γ_2. The time delay is the time of the wave propagation along the pipe until the backward wave interacts with the valve dynamics. In this case, the valve backpressure is $y_{\text{out}} = y_4(0,t)$ in the ODEs (1.1)–(1.3). The corresponding boundary conditions are:

$$y_4(1,t) = 0\,, \tag{1.9}$$
$$y_5(0,t) = \Gamma_3 y_1(t)\sqrt{y_3(t) - y_4(0,t)}\,. \tag{1.10}$$

The first boundary condition prescribes that the fluid delivers to the zero overpressure. The second one expresses that the mass flow rate through the valve must be equal to the inlet mass flow rate to the pipe. The parameter $\Gamma_3 > 0$ can also be expressed from the physical parameters of the system and the flow.

The solutions of the partial differential equations in (1.7), (1.8) can be formulated in the so-called travelling wave solution form:

$$y_4(x,t) = f\Big(T - \frac{\tau}{2}X\Big) + g\Big(T + \frac{\tau}{2}X\Big)\,, \tag{1.11}$$

$$y_5(x,t) = \frac{2}{\Gamma_1 \tau}\Big(f\Big(T - \frac{\tau}{2}X\Big) - g\Big(T + \frac{\tau}{2}X\Big)\Big)\,, \tag{1.12}$$

where $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ are unknown functions. The travelling wave solutions can be substituted back into the boundary conditions. From the boundary condition (1.9), the connection

$$g(u) = -f\,(u - \tau)\,, \quad u \in \mathbb{R}. \tag{1.13}$$

is obtained, which can be further substituted into the boundary condition (1.10) leading to a nonlinear delay algebraic equation:

$$f(t) + f\,(t-\tau) = \phi y_1(t)\sqrt{y_3(t) - f(t) + f\,(t-\tau)}\,, \tag{1.14}$$

where $\phi > 0$ is called as pipe inlet parameter, and $\phi = \Gamma_1 \Gamma_3 \tau/2$. Note, that the differential algebraic equation is only one possible terminology for this kind of equation, see [3], while it can also be called as nonlinear difference equation, or shift map [4], or neutral renewal equation [1,2].

Finally, the whole system of delay differential algebraic equations (DDAEs) can be derived by considering the coupling between the ODEs and the algebraic equation with $y_4(0,t) = f(t) - f(t-\tau)$ from (1.11) and (1.13):

$$\dot{y}_1(t) = y_2(t)\,, \tag{1.15}$$

$$\dot{y}_2(t) = -2\zeta y_2(t) - (y_1(t) + \delta) + y_3(t) - f(t) + f(t-\tau)\,, \tag{1.16}$$

$$\dot{y}_3(t) = \beta\Big(q - \frac{1}{\phi}\big(f(t) + f(t-\tau)\big)\Big)\,, \tag{1.17}$$

$$f(t) = \phi y_1(t)\sqrt{y_3(t) - f(t) + f\,(t-\tau)} - f\,(t-\tau)\;. \tag{1.18}$$

This system is implicit, because of the square root nonlinearity in (1.18), thus, the algebraic equation cannot be arranged into standard explicit NDDE form [5,11]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\,y(t) - u(\,y(t-\tau))\Big) = v(\,y(t), y(t-\tau))\,, \tag{1.19}$$

where $y : \mathbb{R} \to \mathbb{R}^3$, $u : \mathbb{R}^3 \to \mathbb{R}^3$ and $v : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.

The derived implicit DDAE model is unusual in the field of engineering dynamics where the mathematical background is not yet elaborated. As it was referred to [1–4], even the terminology is not uniform and the numerical methods rarely handle neutral kind systems, especially implicit and/or algebraic ones. Paper [7] contains a thorough linear stability analysis, mode shape calculations and a numerical validation of the time-delay model of the vessel-valve-pipe system. The results presented in [8] mainly explain the nonlinear aspects of the safe operation of the pressure relief valves mounted on a vessel. Based on the results of [7], it is also shown how to use the attached downstream pipe consciously to avoid the harmful valve vibrations. At the end, it also presents nonlinear analysis of the time delay model of the vessel-valve-pipe system by means of DDE Biftool via creating an approximate system of retarded delay differential equations. Although the studies in [7,8] provided valuable information from engineering point of view, both analytical and numerical techniques have to be developed for the reliable handling of implicit nonlinear delay differential algebraic equations (DDAEs), which, to best of our knowledge, is missing from the literature.

2 Transformation to neutral delay differential equations

The presence of the pipe added one algebraic equation to the model of the vessel-valve subsystem. The trivial solution $y^*_{1,2,3}$ for the vessel valve system is not changed by the presence of the pipe, but additionally the trivial solution f^* appears for the DDAEs (1.15)–(1.18), extending the algebraic system of equations for the trivial solution as follows:

$$y_1^{*3} + \delta y_1^{*2} - q^2 = 0\,, \quad y_3^* = y_1^* + \delta\,, \quad y_2^* = 0\,, \quad f^* = \frac{\phi}{2}q\,.$$

The linearized system can be derived for the small perturbations $\eta_{1,2,3,4} \in \mathbb{R}$ around the trivial solution:

$$\dot{\eta}_1(t) = \eta_2(t)\,, \tag{2.1}$$

$$\dot{\eta}_2(t) = -2\zeta\eta_2(t) - \eta_1(t) + \eta_3(t) - \eta_4(t) + \eta_4(t-\tau)\,, \tag{2.2}$$

$$\dot{\eta}_3(t) = -\frac{\beta}{\phi}\left(\eta_4(t) + \eta_4(t-\tau)\right)\,, \tag{2.3}$$

$$\eta_4(t) = \phi\frac{y_1^*}{2\sqrt{y_3^*}}\left(\frac{2y_3^*}{y_1^*}\eta_1(t) + \eta_3(t) - \eta_4(t) + \eta_4(t-\tau)\right) - \eta_4(t-\tau)\,. \tag{2.4}$$

The vessel-valve dynamics around the equilibrium is described in the space of $\eta_{1,2,3}$ in (1.4)–(1.6). These state variables have actual physical meaning, while the pipe adds one new equation with the state variable η_4 originated in the wave propagation described by the function f, which cannot be directly associated to a physical measure. The most convenient interpretation from engineering point of view would be to present the vessel-valve-pipe system in a linear form in the space of the vessel-valve subsystem, thus in a delayed differential equation form with the variables $\eta_{1,2,3}$. This way, the vessel gauge pressure and the disk vibrations remain in the model, and the effect of the pipe would appear through the delayed terms only. Also, the equilibrium of the pipe is detached from the equilibrium of the vessel-valve subsystem, while the $\tau \to 0$ case must be the transition between the vessel-valve-pipe and vessel-valve model.

Indeed, the linearized system of DDAEs (2.1)–(2.4) can be transformed to a system of neutral delay differential equations (NDDEs) by means of some algebraic manipulation. The goal is to present a standard system of NNDEs in the space of the same state variables as in case of the vessel-valve subsystem presented in (1.4)–(1.6).

First, let us rearrange (2.4) to obtain

$$\left(1 + \phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\eta_4(t) + \left(1 - \phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\eta_4(t-\tau) = \phi\left(\sqrt{y_3^*}\eta_1(t) + \frac{y_1^*}{2\sqrt{y_3^*}}\eta_3(t)\right)\,. \tag{2.5}$$

Then take the linear combination of equation (2.2) and its τ-delayed form:

$$\begin{aligned}&\left(1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\dot{\eta}_2(t)+\left(1-\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\dot{\eta}_2(t-\tau)\\&=\left(1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\Big(-2\zeta\eta_2(t)-\eta_1(t)+\eta_3(t)-\eta_4(t)+\eta_4(t-\tau)\Big)\\&\quad+\left(1-\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\Big(-2\zeta\eta_2(t-\tau)-\eta_1(t-\tau)+\eta_3(t-\tau)-\eta_4(t-\tau)+\eta_4(t-2\tau)\Big).\end{aligned}\tag{2.6}$$

It can be done similarly for (2.3):

$$\begin{aligned}&\left(1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\dot{\eta}_3(t)+\left(1-\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\dot{\eta}_3(t-\tau)\\&=-\frac{\beta}{\phi}\left(\left(1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\Big(\eta_4(t)+\eta_4(t-\tau)\Big)+\left(1-\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\Big(\eta_4(t-\tau)+\eta_4(t-2\tau)\Big)\right).\end{aligned}\tag{2.7}$$

For both equations, (2.5) and its τ-delayed form appears. After their substitution, the following two equations are obtained in place of (2.2)–(2.3):

$$\begin{aligned}&\left(1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\dot{\eta}_2(t)+\left(1-\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\dot{\eta}_2(t-\tau)\\&=-\left(1+\phi\left(\frac{y_1^*}{2\sqrt{y_3^*}}+\sqrt{y_3^*}\right)\right)\eta_1(t)-2\zeta\left(1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\eta_2(t))+\eta_3(t)\\&\quad-\left(1-\phi\left(\frac{y_1^*}{2\sqrt{y_3^*}}+\sqrt{y_3^*}\right)\right)\eta_1(t-\tau)-2\zeta\left(1-\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\eta_2(t-\tau))+\eta_3(t-\tau)\end{aligned}\tag{2.8}$$

$$\begin{aligned}&\left(1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\dot{\eta}_3(t)+\left(1-\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\dot{\eta}_3(t-\tau)\\&=-\beta\left(\sqrt{y_3^*}(\eta_1(t)+\eta_1(t-\tau)\Big)+\frac{y_1^*}{2\sqrt{y_3^*}}\Big(\eta_3(t)+\eta_3(t-\tau))\right)\end{aligned}\tag{2.9}$$

Finally, the whole system of NDDEs is presented in a matrix form:

$$\dot{\eta}(t)=\mathrm{A}\eta(t)+\mathrm{B}\eta(t-\tau)+\mathrm{C}\dot{\eta}(t-\tau)\tag{2.10}$$

where $\eta=[\eta_1,\eta_2,\eta_3]^{\mathrm{T}}$, while $\mathrm{A},\mathrm{B},\mathrm{C}\in\mathbb{R}^{3x3}$ are given as follows:

$$\mathrm{C}=-\frac{1-\phi\frac{y_1^*}{2\sqrt{y_3^*}}}{1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}}\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix},\tag{2.11}$$

$$\mathrm{A}=\frac{1}{1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}}\begin{bmatrix}0&1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}&0\\-1-\phi\left(\sqrt{y_3^*}+\frac{y_1^*}{2\sqrt{y_3^*}}\right)&-2\zeta\left(1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)&1\\\beta\sqrt{y_3^*}&0&\beta\frac{y_1^*}{2\sqrt{y_3^*}}\end{bmatrix},\tag{2.12}$$

$$\mathrm{B}=\frac{1}{1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}}\begin{bmatrix}0&0&0\\-1+\phi\left(\sqrt{y_3^*}+\frac{y_1^*}{2\sqrt{y_3^*}}\right)&-2\zeta\left(1-\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)&1\\\beta\sqrt{y_3^*}&0&\beta\frac{y_1^*}{2\sqrt{y_3^*}}\end{bmatrix}.\tag{2.13}$$

As conclusion, a linear system of NDDEs is derived in the presence of the pipe with the state variables of the vessel-valve subsystem. It has a straight transition to the linearaized system of ODEs describing the dynamics of the vassel-valve subsystem in (1.4)–(1.6) with the limit of $\tau \to 0$. This limit case leads to the system

$$(\mathrm{I}-\mathrm{C})\dot{\eta}(t) = (\mathrm{A}+\mathrm{B})\eta(t) \tag{2.14}$$

with the identity matrix $\mathrm{I} \in \mathbb{R}^3$, which is, indeed, an equivalent system to (1.4)–(1.6).

2.1 Linear analysis of the DDAEs

The characteristic equation corresponding to the linearized system of NDDEs (2.1)–(2.4) can be derived as

$$\det(\mathrm{A}-\lambda\mathrm{I}+(\mathrm{B}+\lambda \mathrm{e}^{-\lambda\tau}\mathrm{C})) = 0 \tag{2.15}$$

leading to the form of

$$(a_3+b_3\mathrm{e}^{-\lambda\tau})(a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0+b_3\lambda^3\mathrm{e}^{-\lambda\tau}+b_2\lambda^2\mathrm{e}^{-\lambda\tau}+b_1\lambda\mathrm{e}^{-\lambda\tau}+b_0\mathrm{e}^{-\lambda\tau}) = 0\,, \tag{2.16}$$

with the coefficients

$$\begin{aligned}
a_3 &= 1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}\,, & b_3 &= 1-\phi\frac{y_1^*}{2\sqrt{y_3^*}}\,,\\
a_2 &= \beta\frac{y_1^*}{2\sqrt{y_3^*}}+2\zeta\left(1+\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\,, & b_2 &= \beta\frac{y_1^*}{2\sqrt{y_3^*}}+2\zeta\left(1-\phi\frac{y_1^*}{2\sqrt{y_3^*}}\right)\,,\\
a_1 &= 1+\phi\left(\sqrt{y_3^*}+\frac{y_1^*}{2\sqrt{y_3^*}}\right)+2\zeta\beta\frac{y_1^*}{2\sqrt{y_3^*}}\,, & b_1 &= 1-\phi\left(\sqrt{y_3^*}+\frac{y_1^*}{2\sqrt{y_3^*}}\right)+2\zeta\beta\frac{y_1^*}{2\sqrt{y_3^*}}\,,\\
a_0 &= \beta\left(\sqrt{y_3^*}+\frac{y_1^*}{2\sqrt{y_3^*}}\right)\,, & b_0 &= a_0\,.
\end{aligned} \tag{2.17}$$

Since we temporarily extended the interval of the initial data to $[-2\tau, 0]$ in the steps of derivation in (2.6) and (2.7), the term $a_3+b_3\mathrm{e}^{-\lambda\tau}$ appears as a multiplicator of $\sum_{i=0}^{3}(a_i\lambda^i+b_i\lambda^i\mathrm{e}^{-\lambda\tau})$. The 2τ-delayed term $\eta_4(t-2\tau)$ was eliminated in (2.8) and (2.9), so the final set of initial data is defined again on $[-\tau, 0]$ and the factorized characteristic equation does not contain exponential terms $\mathrm{e}^{-2\lambda\tau}$ explicitly. The linear stability analysis of the system of DDAEs can be carried out by direct substitution of an exponential trial solution into (1.15)–(1.18), see [7], leading to the characteristic equation $\sum_{i=0}^{3}(a_i\lambda^i+b_i\lambda^i\mathrm{e}^{-\lambda\tau}) = 0$ with the exact same coefficients as in (2.17); moreover, the roots of $a_3+b_3\mathrm{e}^{\lambda\tau} = 0$ assign the spectral abscissa of the essential spectrum [9]. Because of the neutral kind of the system, infinitely many characteristic roots may appear on the right-hand side of the complex plane causing the so-called essential loss of stability. The condition

$$\left|\frac{b_3}{a_3}\right| < 1 \tag{2.18}$$

guarantees that the spectral abscissa of the essential spectrum is located on the left side of the complex plane avoiding the possibility of the essential loss of stability. For the coefficients in (2.17), this condition always fulfils. The associated difference equation of the NDDE [9] in (2.10) is

$$\eta(t) - C\eta(t-\tau) = 0 \tag{2.19}$$

with the characteristic equation

$$\left(1 + \frac{b_3}{a_3}\mathrm{e}^{-\lambda\tau}\right)^2 = 0 \tag{2.20}$$

assigning the same spectral abscissa as the equation $\sum_{i=0}^{3}(a_i + b_i\mathrm{e}^{-\lambda\tau}) = 0$. Consequently, in case of the transformations resulting (2.10), the characteristic roots of the associate difference equation are false roots of the characteristic equation itself. If the condition (2.18) fulfils, it leads to the same stability properties of the linear system of DDAEs (1.15)–(1.18) and NDDEs (2.10).

2.2 Stability charts

The linear stability analysis detects the possible Hopf bifurcations in the system. The stability properties of the trivial solution can be determined for $\sum_{i=0}^{3}(a_i\lambda^i + b_i\lambda^i\mathrm{e}^{-\lambda\tau}) = 0$ through the so-called D-separation method [6, 10]. The critical roots are the poor imaginary complex conjugate $\pm\mathrm{i}\omega$, where $\omega \in \mathbb{R}^+$ is the angular frequency of the possibly emerging periodic solution, which corresponds to the self-excited vibration of the valve. In this case, two solutions exist for ω depending on the parameters ζ, δ, q. The stability boundaries $\beta_{\mathrm{cr}}(\tau;\omega)$ can be determined for both frequencies, depending on the time delay τ; for details and the closed form solutions, see [7]. As a practically useful result, stability charts like the one in Fig. 2.1 can be presented to characterise the effects of the various sizes of the pipe and the vessel on valve stability.

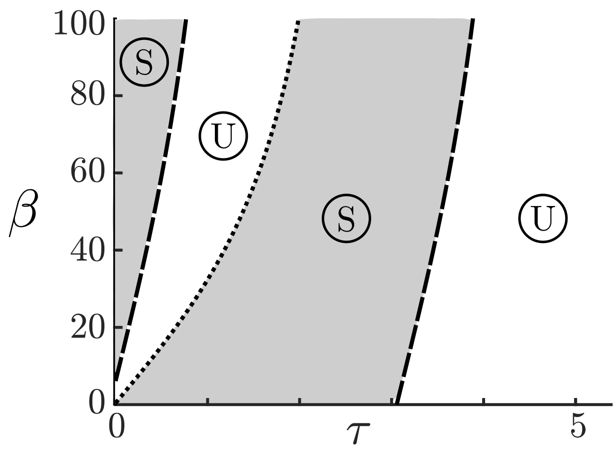

Figure 2.1: Example stability chart for $q = 6, \delta = 3$, $\zeta = 0.39$, $\phi = 48.2$. The gray domains are stable, the white domains are unstable. The two types of lines represent the stability boundaries with the two distinct vibration frequencies of the self-excited vibrations.

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Periodic solution of a bioeconomic fishery model by coincidence degree theory

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Abstract. In this article we use coincidence degree theory to study the existence of a positive periodic solutions to the following bioeconomic model in fishery dynamics

$$\begin{cases} \frac{dn}{dt} = n\left(r(t)\left(1-\frac{n}{K}\right) - \frac{q(t)E}{n+D}\right), \\ \frac{dE}{dt} = E\left(\frac{A(t)q(t)}{\alpha(t)}\frac{n}{n+D} - \frac{q^2(t)}{\alpha(t)}\frac{n^2E}{(n+D)^2} - c(t)\right), \end{cases}$$

where the functions r, q, A, c and α are continuous positive T-periodic functions. This is the model of a coastal fishery represented as a single site with $n(t)$ is the fish stock biomass, and $E(t)$ is the fishing effort. Examples are given to strengthen our results.

Keywords: periodic solution, coincidence degree theory, existence of solutions.

2020 Mathematics Subject Classification: 34C25, 34A34, 34A38.

1 Introduction

In [17], Moussaoui and Auger introduced following system of three ordinary differential equation describing the fishery dynamics with price depending on supply and demand

$$\begin{cases} \frac{dn}{d\tau} = \varepsilon\left(rn\left(1-\frac{n}{K}\right) - \frac{qnE}{n+D}\right) \\ \frac{dE}{d\tau} = \varepsilon\left(-cE + p\frac{qnE}{n+D}\right) \\ \frac{dp}{d\tau} = \phi p\left(P(p) - \frac{qnE}{n+D}\right) \end{cases} \tag{1.1}$$

$n(t)$ is the fish stock biomass, $E(t)$ is the fishing effort and ($p(t)$ is the price per unit of the resource at time t). Authors assumed that the price varies at a fast time scale τ, while fish growth and investment in the fishery by boat owners occur at a slow time scale $t = \varepsilon\tau$, with $\tau \ll 1$ being a small dimensionless parameter.

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The study of existence, uniqueness and asymptotic behavior of solutions of mathematical models can be found in all applied sciences in the recent years. Many of the mathematical models occur in terms of differential equations or a system of differential equations.The increasing expansion of branches of system of differential equations has attracted many researchers to study the dynamical nature of solutions, especially, on existence and uniqueness of solutions. One of the models that attracts the attention of researchers in applied science is the bioeconomic model, similar to classical bioeconomic models of fishery dynamics [1,3].

Using regression [17], we can transform model (1.1) into the following system of two differential equations.

$$\begin{cases} \frac{dn}{dt} = n\left(r\left(1-\frac{n}{K}\right)-\frac{qE}{n+D}\right) \\ \frac{dE}{dt} = E\left(\frac{Aq}{\alpha}\frac{n}{n+D}-\frac{q^2}{\alpha}\frac{n^2E}{(n+D)^2}-c\right). \end{cases} \tag{1.2}$$

Since the variation of environment, in particular the periodic variation of the environment, play an important role in many biological and ecological system, especially, in fish stock biomass and fishing effort, it is natural to study the existence and asymptotic behavior of periodic solutions of the model (1.2). From the application point of view, only positive periodic solutions are important. Hence, it is realistic to assume the periodicity of the coefficient functions in (1.2). Thus, assuming $r, q, A, c,$ and α to be positive T-periodic functions, we have the following nonautonomous model

$$\begin{cases} \frac{dn}{dt} = n\left(r(t)\left(1-\frac{n}{K}\right)-\frac{q(t)E}{n+D}\right) \\ \frac{dE}{dt} = E\left(\frac{A(t)q(t)}{\alpha(t)}\frac{n}{n+D}-\frac{q^2(t)}{\alpha(t)}\frac{n^2E}{(n+D)^2}-c(t)\right) \end{cases} \tag{1.3}$$

where $r,\ q,\ A,\ c,$ and α are continuous positive T-periodic functions with ecological meaning as n the fish stock biomass, E the fishing effort, r fish growth rate, K carrying capacity, q catchability per fishing effort unit, D half saturation level, A carrying capacity of the market or maximum demand and α slope of the linear demand function decreasing with the price.

Setting

$$f(t,n,E) = \frac{r(t)}{K}n^2 + \frac{q(t)En}{n+D}$$

and

$$g(t,n,E) = \frac{A(t)q(t)}{\alpha(t)}\frac{nE}{n+D} - \frac{q^2(t)}{\alpha(t)}\frac{n^2E^2}{(n+D)^2},$$

we can express (1.3) into the following systems of equations

$$\begin{cases} \frac{dn}{dt} = r(t)n(t) - f(t,n(t),E(t)) \\ \frac{dE}{dt} = -c(t)E(t) + g(t,n(t),E(t)). \end{cases} \tag{1.4}$$

System of equations of the form (1.4) with general f and g have been studied by many authors [2,11,14,20–25,28] using various types of fixed point theorems to study the existence of positive T-periodic of (1.4) when f and g are positive continuous functions. Further, they were applied to many mathematical models [11,14,20–25,28] to study the existence of positive T-periodic solutions. One may refer to [19] for applications of fixed point theorems [7,9,10,12] on the existence of positive periodic solutions of mathematical models. As far as our knowledge is concerned, there exist no results on the existence and uniqueness of positive T- periodic solutions of (1.3). We have used Mawhin's coincidence degree theory to study the existence of

T-periodic solution of (1.3). Although there exist hundreds of research articles in the literature on the use of Schauder's fixed point theorem and Krasnosel'skii's fixed point theorem, the use of Mawhin's coincidence degree theory to study the existence of positive T-periodic solutions of (1.3) is relatively scarce in the literature. Previous papers based on Mawhin's coincidence degree theory for different biological models are [4–6, 8, 15, 26, 27, 29].

In order to obtain our results, we assume $r(t)$, $q(t)$, $A(t)$, $c(t)$ and $\alpha(t)$ in (1.3) are all positive T-periodic functions. Further then we assume f and g are T-periodic functions with respect to the first variable.

This work has been divided into four sections. Section 1 is Introduction. Basic theory and Mawhin's coincidence degree theory is given in Section 2. Section 3 contains the main results of this paper. Examples are given to illustrate our results. Section 4 discusses the conclusion of this article.

2 Preliminaries

Before presenting our results on the existence of periodic solution of system (1.3), We provide the essentials of the coincidence degree theory. Let Z and W be the real Banach spaces, and Let $L : \mathrm{dom}(L) \subset Z \to W$ be Fredholm operator of index zero, If $P : Z \to Z$ and $Q : W \to W$ are two continuous projectors such that $\mathrm{Im}(P) = \mathrm{Ker}(L)$, $\mathrm{Ker}(Q) = \mathrm{Im}(L)$, $Z = \mathrm{Ker}(L) \oplus \mathrm{Ker}(P)$ and $W = \mathrm{Im}(L) \oplus \mathrm{Im}(Q)$, then the inverse operator of $L|_{\mathrm{dom}(L)\cap\mathrm{Ker}(P)} : \mathrm{dom}(L) \cap \mathrm{Ker}(P) \to \mathrm{Im}(L)$ exists and is denoted by K_p (generalized inverse operator of L). If Ω is an open bounded subset of Z such that $\mathrm{dom}(L) \cap \Omega \neq 0$, the mapping $N : Z \to W$ will be called L-compact on $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to Z$ is compact. The abstract equation $Lx = Nx$ is shown to be solvable in view of [16, Theorem 2.4 on p. 84].

Theorem 2.1 ([16]). *Let L be a Fredholm operator of index zero and let N be the L-compact on* $\overline{\Omega}$. *Assume the following conditions are satisfied:*

1) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(\mathrm{dom}(L)\backslash Ker(L)) \cap \partial\Omega] \times (0, 1)$;

2) $Nx \notin \mathrm{Im}(L)$ *for every* $x \in Ker(L) \cap \partial\Omega$;

3) $\deg(QN|_{\mathrm{Ker}(L)}, \mathrm{Ker}(L) \cap \Omega, 0) \neq 0$, *where* $Q : W \to W$ *is a projector as above with* $\mathrm{Im}(L) = \mathrm{Ker}(Q)$.

Then, the equation $Lx = Nx$ *has at least one solution in* $\mathrm{dom}(L) \cap \overline{\Omega}$.

3 Existence of the periodic solution

For the sake of convenience and simplicity, we use the notations:

$$\overline{f} = \frac{1}{T}\int_0^T f(t)dt, \qquad f^L = \min_{t\in[0,1]} f(t), \qquad f^M = \max_{t\in[0,1]} f(t),$$

where f is a continuous t-Periodic function.

Set:

$$m_\epsilon = A^M(K + D) + \epsilon, \quad g_\epsilon = K\left(1 - \frac{q^M m_0}{Dr^L}\right) - \epsilon, \quad h_\epsilon = \frac{\alpha^L}{(q^L)^2}\left(\frac{A^L q^L g_0}{\alpha^M(K + D)} - c^M\right) - \epsilon.$$

Also, there exist positive numbers L_i $(i = 1, 2, \ldots, 4)$ such that $L_2 \leq z_1(t) \leq L_1$, $L_4 \leq z_2(t) \leq L_3$, where L_i $(i = 1, 2 \ldots, 4)$ will be calculated as in the proof of following theorem.

Theorem 3.1. *Assume the following conditions hold:*

(A1) $q^M m_0 < Dr^L$,

(A2) $c^M \alpha^M (K + D) < A^L q^L g_0$,

(A3) $\overline{A}\overline{q} - \overline{\alpha}K < \overline{\alpha}\,\overline{c} < \overline{A}\overline{q} + D\overline{r}\,\overline{q}$.

Then, system (1.3) *has at least one positive T-periodic solution*

Proof. Firstly, we make a change of variables.

Consider

$$z_1(t) = \ln n(t) \Rightarrow n(t) = e^{z_1(t)},$$
$$z_2(t) = \ln E(t) \Rightarrow E(t) = e^{z_2(t)},$$

then system (1.3) becomes

$$\begin{cases} \frac{dz_1}{dt} = r(t)\left(1 - \frac{e^{z_1(t)}}{K}\right) - \frac{q(t)e^{z_2(t)}}{e^{z_1(t)}+D}, \\ \frac{dz_2}{dt} = \frac{A(t)q(t)}{\alpha(t)}\frac{e^{z_1(t)}}{e^{z_1(t)}+D} - \frac{q^2(t)}{\alpha(t)}\frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+D)^2} - c(t). \end{cases} \tag{3.1}$$

Define $Z = W = \{z = (z_1, z_2) \in (\mathbb{R}, \mathbb{R}^2) | z(t + T) = z(t)\}$, Z, W are both Banach spaces with the norm $\|\cdot\|$ as follows:

$$\|z\| = \max_{t\in[0,T]} \sum_{i=1}^{2} |z_i|, \quad z = (z_1, z_2) \in Z \text{ or } W.$$

For any $z = (z_1, z_2) \in Z$, the periodicity of system (3.1) implies

$$r(t)\left(1 - \frac{e^{z_1(t)}}{K}\right) - \frac{q(t)e^{z_2(t)}}{e^{z_1(t)} + D} = \Gamma_1(z, t),$$

$$\frac{A(t)q(t)}{\alpha(t)}\frac{e^{z_1(t)}}{e^{z_1(t)} + D} - \frac{q^2(t)}{\alpha(t)}\frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)} + D)^2} - c(t) = \Gamma_2(z, t),$$

are T-periodic functions. In fact

$$\Gamma_1(z(t + T), t + T) = r(t)\left(1 - \frac{e^{z_1(t)}}{K}\right) - \frac{q(t)e^{z_2(t)}}{e^{z_1(t)} + D}.$$

Obviously, $\Gamma_2(z, t)$ is also periodic function by similar way.

Define operators L, P, Q as follows, respectively

$$L : \text{dom}(L) \cap Z \to W, \quad Lz = \left(\frac{dz_1}{dt}, \frac{dz_2}{dt}\right),$$

$$P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Q\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{T}\int_0^T z_1(t)dt \\ \frac{1}{T}\int_0^T z_2(t)dt \end{pmatrix}, \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in Z = W,$$

where $\text{dom}(L) = \{z \in Z : z(t) \in C^1(\mathbb{R}, \mathbb{R}^2)\}$.

Define $N : Z \times [0,1] \to W$

$$N\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \Gamma_1(z,t) \\ \Gamma_2(z,t) \end{pmatrix}.$$

It is easy to see that

$$\text{Ker}(L) = \{z \in Z \mid z = c_0,\ c_0 \in \mathbb{R}^2\},$$

and

$$\text{Im}(L) = \left\{ z \in W \,\middle|\, \int_0^T z(t)dt = 0 \right\}$$

is closed in W. Furthermore, both P, Q are continuous projections satisfying

$$\text{Im}(P) = \text{Ker}(L), \qquad \text{Im}(L) = \text{Ker}(Q) = \text{Im}(I - Q).$$

For any $z \in W$, let $z_1 = z - Qz$, we can obtain that

$$\int_0^T z_1 dp = \int_0^T z(p)dp - \int_0^T \frac{1}{T}\int_0^T z(t)dtdp = 0,$$

so $z_1 \in \text{Im}(L)$. It follows that $W = \text{Im}(L) + \text{Im}(Q) = \text{Im}(L) + \mathbb{R}^2$. Since $\text{Im}(L) \cup \mathbb{R}^2 = 0$, we conclude that $W = \text{Im}(L) \oplus \mathbb{R}^3$, which means dim $\text{Ker}(L) =$ codim $\text{Im}(L) =$ dim $(\mathbb{R}^2) = 2$. Thus, L is a Fredholm operator of index zero, which implies that L has a unique generalized inverse operator.

Next we show that N is L-compact. Define the inverse of L as $K_P : \text{Im}(L) \to \text{Ker}(P) \cap \text{dom}(L)$ and is given by

$$K_P(z) = \int_0^t z(s)ds - \frac{1}{T}\int_0^T \int_0^t z(s)dsdt.$$

Therefore, for any $z(t) \in Z$, we have

$$QN\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{T}\int_0^T \Gamma_1(z,t)dt \\ \frac{1}{T}\int_0^T \Gamma_2(z,t)dt \end{pmatrix},$$

and

$$\begin{aligned} K_P(I-Q)Nz = & \int_0^t Nz(s)ds - \frac{1}{T}\int_0^T \int_0^t Nz(s)dsdt - \frac{1}{T}\int_0^t \int_0^T QNz(s)dtds \\ & + \frac{1}{T^2}\int_0^T \int_0^t \int_0^T QNz(s)dtdsdt \\ = & \int_0^t Nz(s)ds - \frac{1}{T}\int_0^T \int_0^t Nz(s)dsdt - \left(\frac{t}{T} - \frac{1}{2}\right)\int_0^T QNz(s)ds. \end{aligned}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Due to Z is Banach space, using the Arzelà–Ascoli theorem, we have that N is L-compact on $\overline{U}$ for any open bounded set $U \subset Z$. Next, in order to apply the coincidence degree theory, we need to construct an appropriate open bounded subset U. Therefore, the operator equation is defined by $Lz = \lambda Nz$, $\lambda \in (0,1)$, that is,

$$\begin{cases} \frac{dz_1}{dt} = \lambda \left[r(t)\left(1 - \frac{e^{z_1(t)}}{K}\right) - \frac{q(t)e^{z_2(t)}}{e^{z_1(t)}+D} \right], \\ \frac{dz_2}{dt} = \lambda \left[\frac{A(t)q(t)}{\alpha(t)} \frac{e^{z_1(t)}}{e^{z_1(t)}+D} - \frac{q^2(t)}{\alpha(t)} \frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+D)^2} - c(t) \right]. \end{cases} \tag{3.2}$$

We assume that $z \in (z_1, z_2)^T \in Z$ is a T-periodic solution of system (3.1) for any fixed $\lambda \in (0,1)$. Now, integrating system (3.1) from 0 to T leads to

$$\begin{cases} \bar{r}T = \int_0^T \left[\frac{r(t)e^{z_1(t)}}{K} + \frac{q(t)e^{z_2}}{e^{z_1}+D} \right] dt, \\ \bar{c}T = \int_0^T \left[\frac{A(t)q(t)}{\alpha(t)} \frac{e^{z_1(t)}}{e^{z_1(t)}+D} - \frac{q^2(t)}{\alpha(t)} \frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+D)^2} \right] dt. \end{cases} \tag{3.3}$$

Since $(z_1, z_2) \in Z$, there exist $\eta_i, \xi_i \in [0, T]$ such that

$$z_i(\eta_i) = \max_{t\in[0,T]} z_i(t), \qquad z_i(\xi_i) = \min_{t\in[0,T]} z_i(t), \qquad i = 1, 2.$$

Through simple analysis, we have,

$$\dot{z}_1(\eta_1) = \dot{z}_1(\xi_1) = 0, \qquad \dot{z}_2(\eta_2) = \dot{z}_2(\xi_2) = 0.$$

If we apply previous to (3.2), we obtain

$$r(\eta_1)\left(1 - \frac{e^{z_1(\eta_1)}}{K}\right) - \frac{q(\eta_1)e^{z_2(\eta_1)}}{e^{z_1(\eta_1)} + D} = 0, \tag{3.4}$$

$$-c(\eta_2) + \frac{A(\eta_2)q(\eta_2)}{\alpha(\eta_2)} \frac{e^{z_1(\eta_2)}}{e^{z_1(\eta_2)} + D} - \frac{q^2(\eta_2)}{\alpha(\eta_2)} \frac{e^{2z_1(\eta_2)}e^{z_2(\eta_2)}}{(e^{z_1(\eta_2)} + D)^2} = 0, \tag{3.5}$$

and

$$r(\xi_1)\left(1 - \frac{e^{z_1(\xi_1)}}{K}\right) - \frac{q(\xi_1)e^{z_2(\xi_1)}}{e^{z_1(\xi_1)} + D} = 0, \tag{3.6}$$

$$-c(\xi_2) + \frac{A(\xi_2)q(\xi_2)}{\alpha(\xi_2)} \frac{e^{z_1(\xi_2)}}{e^{z_1(\xi_2)} + D} - \frac{q^2(\xi_2)}{\alpha(\xi_2)} \frac{e^{2z_1(\xi_2)}e^{z_2(\xi_2)}}{(e^{z_1(\xi_2)} + D)^2} = 0. \tag{3.7}$$

From (3.4), we obtain

$$r(\eta_1) - \frac{r(\eta_1)e^{z_1(\eta_1)}}{K} > 0,$$

which implies that

$$z_1(\eta_1) < \ln(K) = L_1. \tag{3.8}$$

Considering (3.5) and (3.8), we get

$$\frac{q^2(\eta_2)}{\alpha(\eta_2)} \frac{e^{2z_1(\eta_2)}e^{z_2(\eta_2)}}{(e^{z_1(\eta_2)} + D)^2} + c(\eta_2) = \frac{A(\eta_2)q(\eta_2)}{\alpha(\eta_2)} \frac{e^{z_1(\eta_2)}}{e^{z_1(\eta_2)} + D}.$$

So, we can obtain

$$\frac{q^2(\eta_2)}{\alpha(\eta_2)} \frac{e^{2z_1(\eta_2)}e^{z_2(\eta_2)}}{(e^{z_1(\eta_2)} + D)^2} < \frac{A(\eta_2)q(\eta_2)}{\alpha(\eta_2)} \frac{e^{z_1(\eta_2)}}{e^{z_1(\eta_2)} + D},$$

or

$$\frac{qe^{z_1(\eta_2)}e^{z_2(\eta_2)}}{e^{z_1(\eta_2)}+D} < A(\eta_2),$$

or

$$e^{z_2(\eta_2)} < A^M(e^{z_1(\eta_2)}+D),$$

or

$$e^{z_2(\eta_2)} < A^M(K+D),$$

which gives

$$z_2(\eta_2) < \ln(A^M(K+D)) = \ln m_0 = L_3. \tag{3.9}$$

From (3.6) and (3.9), we can obtain

$$r(\xi_1) - \frac{r(\xi_1)e^{z_1(\xi_1)}}{K} - \frac{q(\xi_1)m_0}{D} < 0,$$

then,

$$\frac{e^{z_1(\xi_!)}}{K} > 1 - \frac{q^M m_0}{Dr^L}$$

which implies that

$$z_1(\xi_1) > \ln\left(K\left(1-\frac{q^M m_0}{Dr^L}\right)\right) = \ln(g_0) = L_2. \tag{3.10}$$

In view of (3.7) and (3.10), we have

$$\frac{q^2(\xi_2)}{\alpha(\xi_2)}\frac{e^{2z_1(\xi_2)}e^{z_2(\xi_2)}}{(e^{z_1(\xi_2)}+D)^2} = \frac{A(\xi_2)q(\xi_2)}{\alpha(\xi_2)}\frac{e^{z_1(\xi_2)}}{e^{z_1(\xi_2)}+D} - c(\xi_2).$$

Thus,

$$\frac{q^2(\xi_2)e^{z_2(\xi_2)}}{\alpha(\xi_2)} > \frac{A^L q^L g_0}{\alpha^M(K+D)} - c^M,$$

or

$$e^{z_2(\xi_2)} > \frac{\alpha^L}{(q^2)^L}\left(\frac{A^L q^L g_0}{\alpha^M(K+D)} - c^M\right),$$

that is

$$z_2(\xi_2) > \ln\left(\frac{\alpha^L}{(q^2)^L}\left(\frac{A^L q^L g_0}{\alpha^M(K+D)} - c^M\right)\right) = \ln(h_0) = L_4. \tag{3.11}$$

Finally, from (3.8), (3.9), (3.10), (3.11), we get

$$|z_1(t)| < \max\{|L_1|, |L_2|\} = \Lambda_1,$$

$$|z_2(t)| < \max\{|L_3|, |L_4|\} = \Lambda_2.$$

where Λ_1, Λ_2 is independent of λ. Denote $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$ where Λ_3 is taken sufficiently large such that each solution (z_1^*, z_2^*) of following system

$$\begin{cases} \bar{r} - \frac{\bar{r}}{K}e^{z_1(t)} - \frac{\bar{q}e^{z_2(t)}}{e^{z_1(t)}+D} = 0, \\ \frac{\overline{Aq}}{\alpha(t)}\frac{e^{z_1(t)}}{e^{z_1(t)}+D} - e^{z_1(t)} + \frac{\overline{q}^2}{\alpha(t)}\frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+D)^2} - \bar{c} = 0, \end{cases} \tag{3.12}$$

satisfies $|z_1^*| + |z_2^*| < \Lambda$. Now we consider $\Omega = \{(z_1, z_2)^T \in Z : \|(z_1, z_2)\| < \Lambda\}$ then it is clear that Ω satisfies the first condition of Theorem 2.1.

For the second condition of Theorem 2.1, we prove that $QN(z_1, z_2)^T \neq (0, 0)^T$ for each $(z_1, z_2) \in \partial\Omega \cap \mathrm{Ker}(L)$. When $(z_1, z_2)^T \in \partial\Omega \cap \mathrm{Ker}(L) = \partial\Omega \cap \mathbb{R}^2$, $(z_1, z_2)^T$ is a constant vector in $\mathbb{R}^2$ and $|z_1| + |z_2| = \Lambda$. If the system (3.12) has a solution, then

$$QN\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{r} - \frac{\bar{r}}{K}e^{z_1(t)} - \frac{\bar{q}e^{z_2(t)}}{e^{z_1(t)}+D} \\ \frac{\overline{Aq}}{\bar{\alpha}}\frac{e^{z_1(t)}}{e^{z_1(t)}+D} - e^{z_1(t)} + \frac{\bar{q}^2}{\bar{\alpha}}\frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+D)^2} - \bar{c} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since, (3.12) does not have solution then, it is evident that $QN(z_1, z_2)^T \neq 0$, thus the second condition of Theorem 2.1 is satisfied. Finally, we prove that the last condition of Theorem 2.1 is satisfied, to do so, we define the following mapping $\Psi_\mu : \mathrm{dom}(L) \times [0, 1] \to Z$

$$\Psi_\mu\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{r} - \frac{\bar{r}}{K}e^{z_1(t)} - \frac{\bar{q}e^{z_2(t)}}{\mu e^{z_1(t)}+D} \\ \frac{\overline{Aq}}{\bar{\alpha}}\frac{e^{z_1(t)}}{e^{z_1(t)}+\mu D} - e^{z_1(t)} + \frac{\bar{q}^2}{\bar{\alpha}}\frac{e^{2z_1(t)}e^{z_2(t)}}{(e^{z_1(t)}+\mu D)^2} - \bar{c} \end{pmatrix}.$$

By using the invariance property of homotopy in topological degree theory, we get,

$$\begin{aligned}
&\deg(QN(z_1, z_2)^T, \Omega \cap \mathrm{Ker}(L), (0, 0)^T) \\
&\quad = \deg(\Psi(z_1, z_2, 1)^T, \Omega \cap \mathrm{Ker}(L), (0, 0)^T) \\
&\quad = \deg(\Psi(z_1, z_2, \mu)^T, \Omega \cap \mathrm{Ker}(L), (0, 0)^T) \\
&\quad = \deg(\Psi(z_1, z_2, 0)^T, \Omega \cap \mathrm{Ker}(L), (0, 0)^T) \\
&\quad = \deg\left(\bar{r} - \frac{\bar{r}}{K}e^{z_1(t)} - \frac{\bar{q}e^{z_2(t)}}{D}, \frac{\overline{Aq}}{\bar{\alpha}} - e^{z_1(t)} + \frac{\bar{q}^2}{\bar{\alpha}}e^{z_2(t)} - \bar{c}, \Omega \cap \mathrm{Ker}(L), (0, 0)^T\right)
\end{aligned}$$

Furthermore, the system of algebraic equation

$$\begin{cases} \bar{r} - \frac{\bar{r}}{K}x - \frac{\bar{q}y}{D} = 0, \\ \frac{\overline{Aq}}{\bar{\alpha}} - x + \frac{\bar{q}^2}{\bar{\alpha}}y - \bar{c} = 0, \end{cases} \tag{3.13}$$

has a unique solution (x^*, y^*), where $x^* = \bar{r}(1 - \frac{\bar{\alpha}K + \bar{\alpha}\bar{c} - \overline{Aq}}{\bar{\alpha}K + D\bar{r}q}) > 0$ and $y^* = \frac{D\bar{r}(\bar{\alpha}K + \bar{\alpha}\bar{c} - \overline{Aq})}{\bar{q}(\bar{\alpha}K + D\bar{r}\bar{q})} > 0$. Thus,

$$\begin{aligned}
\deg\{QN(z_1, z_2)^T, \Omega \cap \mathrm{Ker}(L), (0, 0)^T\} &= \begin{vmatrix} -\frac{\bar{r}}{K}x^* & -\frac{q}{D}y^* \\ -1 & \frac{q^2}{\alpha}y^* \end{vmatrix} \\
&= \mathrm{sgn}\left[-\bar{q}\left(\frac{\bar{q}\bar{r}x^*y^*}{\bar{\alpha}K} - \frac{y^*}{D}\right)\right] \\
&= -1 \neq 0.
\end{aligned}$$

Now, all the conditions in Theorem 2.1 have been verified. This implies that system (3.1) has at least one T-periodic solution. Consequently, system (1.3) has at least one positive T-periodic solution. The theorem is proved. □

Corollary 3.1. *If $qA(K + D) < Dr$, $c\alpha(K + D) < AqK(1 - \frac{qA(K+D)}{Dr})$, and $Aq - \alpha K < \alpha c <$*

$\overline{A}\overline{q} + D\overline{r}\overline{q}$ holds, then the system (1.2) has a positive T-periodic solution.

Example 3.1. By Corollary 3.1, the system of equations

$$\begin{cases} \frac{dn}{dt} = n\left(8\left(1-\frac{n}{1}\right) - \frac{E}{n+1/2}\right) \\ \frac{dE}{dt} = E\left(\frac{1.5\times 1}{1.5}\frac{n}{n+1/2} - \frac{1^2}{1.5}\frac{n^2 E}{(n+1/2)^2} - \frac{1}{4}\right). \end{cases} \tag{3.14}$$

has a positive periodic solution.

Example 3.2. Consider the system

$$\begin{cases} \frac{dn}{dt} = n\left((21+\cos t)\left(1-\frac{n}{1.1}\right) - \frac{(1.15+\frac{1}{10}\cos t)E}{n+\frac{1}{4}}\right) \\ \frac{dE}{dt} = E\left(\frac{(1.7+\frac{1}{10}\sin t)(1.15+\frac{1}{10}cost)}{(1.5+\frac{1}{10}\cos t)}\frac{n}{n+\frac{1}{4}} - \frac{(1.15+\frac{1}{10}\cos t)^2}{(1.5+\frac{1}{10}\cos t)}\frac{n^2E}{(n+\frac{1}{4})^2} - \left(\frac{1}{4}+\frac{1}{10}\sin t\right)\right) \end{cases} \tag{3.15}$$

It is easy to obtain $\overline{q} = 1.15,\ \overline{r} = 21,\ \overline{A} = 1.7,\ \overline{\alpha} = 1.5,\ \overline{c} = 0.25,\ D = 0.25, K = 1.1,\ q^L = 1.05,\ q^M = 1.25,\ r^L = 20,\ A^L = 1.6,\ A^M = 1.8,\ \alpha^M = 1.6,\ c^M = 0.35,\ m_0 = 1.8(1.1+0.25) = 2.43,\ \ g_0 = 1.1\left(1 - \frac{1.25\times 2.43}{0.25\times 20}\right) = 0.43175$. Consequently, we obtain

$$q^M m_0 = 3.0375 < Dr^L = 5,$$

$$c^M \alpha^M (K+D) = 0.7 < A^L q^L g_0 = 0.72534,$$

and

$$\overline{A}\overline{q} = 0.305 < \overline{\alpha}\overline{c} = 0.525 < \overline{A}\overline{q} + D\overline{r}\overline{q} = 7.9925.$$

It is clear that assumptions (A1), (A2), (A3) are satisfied. Hence, according to Theorem 3.1, system (3.15) has at least one positive T-periodic solution.

4 Conclusion

Using Mawhin's coincidence degree theory, we have established sufficient conditions for the existence of positive periodic solutions of the model (1.3). By formulating the model as a system of differential equations and introducing appropriate transformations, we were able to apply the coincidence degree theory and obtain our main results. The conditions (A1), (A2), and (A3) played a crucial role in establishing the existence of periodic solutions.

Set $r(t) \equiv r,\ q(t) \equiv q,\ A(t) \equiv A,\ \alpha(t) \equiv \alpha$ and $c(t) \equiv c$ be constants; then (1.3) reduces to

$$\begin{cases} \frac{dn}{dt} = n\left(r\left(1-\frac{n}{K}\right) - \frac{qE}{n+D}\right) \\ \frac{dE}{dt} = E\left(\frac{Aq}{\alpha}\frac{n}{n+D} - \frac{q^2}{\alpha}\frac{n^2E}{(n+D)^2} - c\right). \end{cases} \tag{4.1}$$

In a recent paper, Moussaoui and Auger [17], studied the equilibrium points of (4.1). They proved that if

$$Aq < \alpha c, \tag{4.2}$$

then the system (4.1) has no positive equilibrium point provided that

$$D < K, \qquad \frac{\alpha}{q} < \frac{K}{2} \tag{4.3}$$

holds. It is worth noting that our conditions (A1), (A2), and (A3) are different from the condition (4.2). By Theorem 3.1, the system (4.1) has positive T-periodic solution. We note that the condition (4.3) can be satisfied for large K. On the other hand, by Theorem 1 b) of [17], the system (4.1) has a unique positive equilibrium, which is a positive T-periodic solution of (4.1). Our Theorem 3.1 strengthens this observation.

While this research paper has successfully addressed the existence of positive periodic solutions for the bioeconomic fishery model, there are several avenues for further exploration. It would be interesting to study global attractivity and uniqueness of the solution for the system investigated in this paper. Another promising direction is to examine problem (1.3) by introducing a delay in the system, such as incorporating a time lag in fish stock biomass. Conducting further investigations in these areas have potential implications for understanding and managing fisheries dynamics.

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New monotonicity properties and oscillation of n-order functional differential equations with deviating argument

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Abstract. In this paper, we offer new technique for investigation of the even order linear differential equations of the form

$$y^{(n)}(t) = p(t)y(\tau(t)). \tag{E}$$

We establish new criteria for bounded and unbounded oscillation of (E) which improve a number of related ones in the literature. Our approach essentially involves establishing stronger monotonicities for the positive solutions of (E) than those presented in known works. We illustrate the improvement over known results by applying and comparing our technique with the other known methods on the particular examples.

Keywords: higher order differential equations, delay, advanced argument, monotonicity, oscillation.

2020 Mathematics Subject Classification: 34K11, 34C10.

1 Introduction

We consider the general higher order differential equation with deviating argument

$$y^{(n)}(t) = p(t)y(\tau(t)). \tag{E}$$

Throughout the paper, it is assumed that n is even and the following conditions hold

(H_1) $p(t) \in C^1([t_0,\infty))$, $p(t) > 0$,

(H_2) $\tau(t) \in C^1([t_0,\infty))$, $\tau'(t) > 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

By a proper solution of Eq. (E) we mean a function $y : [T_y,\infty) \to R$ which satisfies (E) for all sufficiently large t and $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We make the standing hypothesis that (E) does possess proper solutions.

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As is customary, a proper solution $y(t)$ of (E) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its proper solutions oscillate.

Oscillation phenomena appear in different models from real world applications; see, for instance, the papers [15–17] for models from mathematical biology where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. The problem of establishing oscillation criteria for differential equations with deviating arguments has been a very active research area over the past decades (see [1]–[18]) and several references and reviews of known results can be found in the monographs by Agarwal et al. [1], Došlý and Rehák [5] and Ladde et al. [18].

It is known that the set $\mathcal{N}$ of all nonoscillatory solutions of (E) has the following decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_n,$$

where $y(t) \in \mathcal{N}_\ell$ means that there exists $t_0 \geq T_y$ such that

$$\begin{aligned} y(t)y^{(i)}(t) > 0 \quad &\text{on } [t_0, \infty) \text{ for } 0 \leq i \leq \ell, \\ (-1)^i y(t)y^{(i)}(t) > 0 \quad &\text{on } [t_0, \infty) \text{ for } \ell \leq i \leq n. \end{aligned} \tag{1.1}$$

Such a $y(t)$ is said to be a solution of degree ℓ.

Following Kiguradze [7], we say that equation (E) enjoys property (B) if $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_n$. The reason for such definition is the observation that (E) with $\tau(t) \equiv t$ always possesses solutions of degrees 0 and n, that is $\mathcal{N}_0 \neq \emptyset$ and $\mathcal{N}_n \neq \emptyset$. The situation when $\tau(t) \not\equiv t$ is different. In fact, it may happen that $\mathcal{N}_0 = \emptyset$ or $\mathcal{N}_n = \emptyset$ when the deviation $|t - \tau(t)|$ is sufficiently large. This remarkable fact was first observed by Ladas et al. [18]. Later Koplatadze and Chanturia [11] have shown that (E) does not allow solutions of degree 0 if $\tau(t) \leq t$ and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} (s - \tau(t))^{n-1} p(s)\, \mathrm{d}s > (n-1)! \tag{1.2}$$

and (E) does not allow solutions of degree n provided that $\tau(t) \geq t$ and

$$\limsup_{t\to\infty} \int_{t}^{\tau(t)} (\tau(t) - s)^{n-1} p(s)\, \mathrm{d}s > (n-1)!. \tag{1.3}$$

On the other hand, Koplatadze et al. [12] proved that (E) enjoys property (B) if $\tau(t) \leq t$ and

$$\limsup_{t\to\infty} \left\{ \tau(t) \int_t^\infty s^{n-3} \tau(s) p(s)\, \mathrm{d}s + \int_{\tau(t)}^{t} s(\tau(s))^{n-2} p(s)\, \mathrm{d}s + \frac{1}{\tau(t)} \int_0^{\tau(t)} s^2 (\tau(s))^{n-2} p(s)\, \mathrm{d}s \right\} > 2(n-2)! \tag{1.4}$$

or $\tau(t) \geq t$ and

$$\limsup_{t\to\infty} \left\{ \tau(t) \int_t^\infty s^{n-3} \tau(s) p(s)\, \mathrm{d}s + \int_{t}^{\tau(t)} s^{n-2} \tau(s) p(s)\, \mathrm{d}s + \frac{1}{\tau(t)} \int_0^{t} s^{n-2} (\tau(s))^2 p(s)\, \mathrm{d}s \right\} > 2(n-2)!. \tag{1.5}$$

Therefore conditions (1.2)–(1.5) yield stronger asymptotic behavior than property (B) claims, namely (1.3) together with (1.5) guarantees that $\mathcal{N} = \mathcal{N}_0$ for (E) with $\tau(t) \geq t$, i.e., every unbounded solution is oscillatory, while (1.2) together with (1.4) are sufficient for $\mathcal{N} = \mathcal{N}_n$ for (E) with $\tau(t) \le t$, i.e., roughly speaking every bounded solution is oscillatory.

In this paper, we establish new technique that essentially improves (1.2) and (1.3), which leads to qualitative better criteria for bounded or unbounded oscillation of (E). Our approach essentially involves establishing stronger monotonicities for the positive solutions of (E) than those presented in known works.

2 Main results

Now we are introduce new monotonicity for nonoscillatory solution $y(t) \in \mathcal{N}_0$ of (E).

Lemma 2.1. *Assume that* $y(t) \in \mathcal{N}_0$ *and*

$$\int_{t_0}^{\infty} p(s)s^{n-1}\,\mathrm{d}s = \infty. \tag{2.1}$$

Then $\lim_{t\to\infty} y(t) = 0$.

Proof. Assume on the contrary that $y(t)$ is an eventually positive solution of (E) such that $y(t) \in \mathcal{N}_0$, and $\lim_{t\to\infty} y(t) = \ell > 0$. Then $y(\tau(t)) > \ell$, eventually, let us say for $t \ge t_1$. An integration of (E) from t to ∞ yields

$$-y^{(n-1)}(t) \ge \int_t^{\infty} p(s)y(\tau(s))\,\mathrm{d}s \ge \ell \int_t^{\infty} p(s)\,\mathrm{d}s.$$

Integrating again from t to ∞ and changing the order of integration, we have

$$y^{(n-2)}(t) \ge \ell \int_t^{\infty}\int_u^{\infty} p(s)\,\mathrm{d}s\,\mathrm{d}u = \ell \int_t^{\infty} p(s)(s-t)\,\mathrm{d}s.$$

Repeating this procedure, we are led to

$$y(t_1) \ge \ell \int_{t_1}^{\infty} \frac{p(s)(s-t_1)^{n-1}}{(n-1)!}\,\mathrm{d}s, \tag{2.2}$$

where the last integration was from t_1 to ∞. Condition (2.2) contradicts (2.1) and we conclude that $y(t) \to 0$ as $t \to \infty$. □

Corollary 2.2. *For* $y(t) \in N_0$, *it follows from* $y(t) \to 0$ *as* $t \to \infty$ *that* $y'(t) \to 0$, $y''(t) \to 0, \dots, y^{(n-1)}(t) \to 0$ *as* $t \to \infty$.

To simplify our notation we introduce the following couple of functions

$$\alpha_0(t) = -\frac{p'(t)}{p(t)} + \tau'(t)\int_{\tau(t)}^{t} p(s)\frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s,$$
$$\beta_0'(t) = \alpha_0(t).$$

Theorem 2.3. *Let* $y(t) \in \mathcal{N}_0$, $\tau(t) \le t$ *and* (2.1) *hold. Then*

$$|y(\tau(t))|p(t)e^{\beta_0(t)} \quad \textit{is decreasing.}$$

Proof. Assume that $y(t) \in \mathcal{N}_0$ is an eventually positive solution of (*E*). It follows from (*E*) that

$$y^{(n+1)}(t) = p'(t)y(\tau(t)) + p(t)y'(\tau(t))\tau'(t). \tag{2.3}$$

In view of Corollary 2.2, an integration of (*E*) from t to ∞ yields

$$-y^{(n-1)}(t) = \int_t^\infty p(s)y(\tau(s))\,\mathrm{d}s.$$

Integrating again from t to ∞ and changing the order of integration, we have

$$y^{(n-2)}(t) = \int_t^\infty \int_u^\infty p(s)y(\tau(s))\,\mathrm{d}s\,\mathrm{d}s = \int_t^\infty p(s)y(\tau(s))(s-t)\,\mathrm{d}s.$$

Repeated reusing of this procedure yields

$$-y'(t) = \int_t^\infty p(s)y(\tau(s))\frac{(s-t)^{n-2}}{(n-2)!}\,\mathrm{d}s.$$

Since $y(\tau(t))$ is decreasing, this implies

$$\begin{aligned} -y'(\tau(t)) &= \int_{\tau(t)}^\infty p(s)y(\tau(s))\frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s \\ &\geq \int_{\tau(t)}^t p(s)y(\tau(s))\frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s \\ &\geq y(\tau(t))\int_{\tau(t)}^t p(s)\frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

Setting (2.4) into (2.3) and taking (*E*) into account, one gets

$$\begin{aligned} y^{(n+1)}(t) &\leq y(\tau(t))\left[p'(t) - p(t)\tau'(t)\int_{\tau(t)}^t p(s)\frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s\right] \\ &= y^{(n)}(t)\left[\frac{p'(t)}{p(t)} - \tau'(t)\int_{\tau(t)}^t p(s)\frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s\right]. \end{aligned}$$

Therefore

$$y^{(n+1)}(t) + \alpha_0(t)y^{(n)}(t) \leq 0$$

which means that

$$\left(e^{\beta_0(t)}y^{(n)}(t)\right)' \leq 0$$

and we conclude that $e^{\beta_0(t)}y^{(n)}(t)$ is decreasing, which is, in view of (*E*), equivalent to the fact that $p(t)y(\tau(t))e^{\beta_0(t)}$ is decreasing. □

Employing the above-mentioned monotonicity we are prepared to present criterion for bounded oscillation of (*E*).

Theorem 2.4. *Assume that* (2.1) *holds,* $\tau(t) \leq t$*, and*

$$\limsup_{t\to\infty} p(t)e^{\beta_0(t)}\int_{\tau(t)}^t e^{-\beta_0(s)}(s-\tau(t))^{n-1}\,\mathrm{d}s > (n-1)!. \tag{2.5}$$

Then $\mathcal{N}_0 = \emptyset$. *If in addition* (1.4) *holds, then all nonoscillatory solutions of* (*E*) *are of degree* n*, i.e.,* $\mathcal{N} = \mathcal{N}_n$.

Proof. We argue by contradiction. Assume that (E) possesses an eventually positive solution $y(t) \in \mathcal{N}_0$. Integrating (E) from u to t ($u \le t$) and using the monotonicity of $p(t)y(\tau(t))e^{\beta_0(t)}$, we have

$$-y^{(n-1)}(u) \ge \int_u^t p(s)y(\tau(s))\,\mathrm{d}s \ge y(\tau(t))p(t)e^{\beta_0(t)} \int_u^t e^{-\beta_0(s)}\,\mathrm{d}s.$$

Integrating the above inequality from u to t and changing the order of integration leads to

$$\begin{aligned} y^{(n-2)}(u) &\ge y(\tau(t))p(t)e^{\beta_0(t)} \int_u^t \int_x^t e^{-\beta_0(s)}\,\mathrm{d}s\,\mathrm{d}x \\ &= y(\tau(t))p(t)e^{\beta_0(t)} \int_u^t e^{-\beta_0(s)}(s-u)\,\mathrm{d}s. \end{aligned} \tag{2.6}$$

Proceeding in the same way $(n-2)$-times, we finally get

$$y(u) \ge y(\tau(t))p(t)e^{\beta_0(t)} \int_u^t e^{-\beta_0(s)} \frac{(s-u)^{n-1}}{(n-1)!}\,\mathrm{d}s.$$

Setting $u = \tau(t)$, we obtain

$$y(\tau(t)) \ge y(\tau(t))p(t)e^{\beta_0(t)} \int_{\tau(t)}^t e^{-\beta_0(s)} \frac{(s-\tau(t))^{n-1}}{(n-1)!}\,\mathrm{d}s$$

which is contraction with (2.5) and we conclude, that class N_0 is empty. Moreover, thanks to (1.4) every nonoscillatory solution of (E) is of degree n. □

Example 2.5. Consider the delay differential equation

$$y^{(n)}(t) = p_0 y(t-\tau), \quad p_0 > 0, \quad \tau > 0. \tag{E_{x1}}$$

It is easy to see that (1.4) holds true. Since $\alpha_0(t) = \frac{p_0}{(n-1)!}\tau^{n-1} = \omega$ and $\beta_0(t) = \omega t$, condition (2.5) takes the form

$$\lim_{t\to\infty} p_0 e^{\omega t} \int_{t-\tau}^t e^{-\omega s} \frac{(s-t+\tau)^{n-1}}{(n-1)!}\,\mathrm{d}s > 1 \tag{2.7}$$

which after substitution $s - t + \tau = x$ reduces to

$$\frac{p_0}{(n-1)!} e^{\omega\tau} \int_0^\tau e^{-\omega x} x^{n-1}\,\mathrm{d}s > 1.$$

Let us denote

$$I(n) = e^{\omega\tau} \int_0^\tau e^{-\omega x} x^{n-1}\,\mathrm{d}s.$$

Then

$$I(n) = -\frac{\tau^{n-1}}{\omega} + \frac{n-1}{\omega} I(n-1), \quad I(1) = -\frac{1}{\omega} + \frac{e^{\omega\tau}}{\omega}$$

which implies

$$I(n) = \frac{(n-1)!e^{\omega\tau}}{\omega^n} - \frac{\tau^{n-1}}{\omega} - \frac{(n-1)\tau^{n-2}}{\omega^2} - \cdots - \frac{(n-1)!}{\omega^n}.$$

Therefore, (2.7) is equivalent to

$$\frac{p_0}{(n-1)!}\left[\frac{(n-1)!e^{\omega\tau}}{\omega^n} - \frac{\tau^{n-1}}{\omega} - \frac{(n-1)\tau^{n-2}}{\omega^2} - \cdots - \frac{(n-1)!}{\omega^n}\right] > 1. \tag{2.8}$$

By Theorem 2.4 condition (2.8) guarantees that every nonoscillatory solution of (E) is of degree n or in other words, every bounded solution of (E) is oscillatory. If $p_0 = \left(\frac{\pi(4k+1-(-1)^{n/2})}{2\tau}\right)^n$ where k is a positive integer such that (2.8) holds, then a bounded oscillatory solution of (E) is $y(t) = \sin(\sqrt[n]{p_0})t$.

Example 2.6. We consider the delayed Euler differential equation

$$y^{(n)}(t) = \frac{p_0}{t^n} y(\lambda t), \quad p_0 > 0, \quad \lambda \in (0,1). \tag{E_{x2}}$$

It is easy to see that (1.4) reduces to

$$p_0 \left(\lambda^2 - \lambda^{n-2} \ln \lambda + \lambda^{n-3}\right) > 2(n-2)!. \tag{2.9}$$

On the other hand,

$$\alpha_0(t) = \frac{1}{t} \left[\frac{p_0(1-\lambda)^{n-1}}{(n-1)!} + n \right].$$

Using notation

$$\frac{p_0(1-\lambda)^{n-1}}{(n-1)!} + n = \delta_0,$$

we obtain

$$\beta_0(t) = \delta_0 \ln t.$$

Therefore (2.5) is equivalent to

$$\limsup_{t\to\infty} \frac{p_0 t^{\delta_0 - n}}{(n-1)!} \int_{\lambda t}^{t} \frac{(s-\lambda t)^{n-1}}{s^{\delta_0}} \, ds > 1.$$

Since

$$\int_{\lambda t}^{t} \frac{(s-\lambda t)^{n-1}}{s^{\delta_0}} \, ds = \sum_{i=0}^{n-1} \frac{(n-1)!(-\lambda t)^i s^{n-\delta_0 - i}}{(n-1-i)!\, i!\, (n-\delta_0 - i)} \Bigg|_{\lambda t}^{t},$$

condition (2.5) takes the form

$$p_0 \sum_{i=0}^{n-1} (-1)^i \frac{\lambda^i - \lambda^{n-\delta_0}}{(n-1-i)!\, i!\, (n-\delta_0-i)} > 1 \tag{2.10}$$

which guarantees that $\mathcal{N}_0 = \emptyset$ for (E_{x2}). If in addition (2.9) holds, then every nonoscillatory solution of (E_{x2}) is of degree n.

For $n = 2$ $(n = 4)$ and $\lambda = 0.5$ condition (2.10) is satisfied when

$$p_0 > 3.3198 \quad (p_0 > 135.77)$$

while (1.2) requires $p_0 > 5.1774$ $(p_0 > 226.58)$. So our progress is significant.

Now we turn our attention to bounded oscillation of (E). We set

$$\alpha_n(t) = \frac{p'(t)}{p(t)} + \tau'(t) \int_t^{\tau(t)} p(s) \frac{(\tau(t)-s)^{n-2}}{(n-2)!} \, ds,$$
$$\beta_n'(t) = \alpha_n(t).$$

Theorem 2.7. *Let $y(t) \in \mathcal{N}_n$, $\tau(t) \geq t$. Then*

$$|y(\tau(t))| p(t) e^{-\beta_n(t)} \quad \textit{is increasing.}$$

Proof. Assume that $y(t) \in \mathcal{N}_n$ is an eventually positive solution of (E). An integration of (E) from t_1 to t yields

$$y^{(n-1)}(t) \geq \int_{t_1}^{t} p(s)y(\tau(s))\,\mathrm{d}s.$$

Integrating the last inequality from t_1 to t and and changing the order of integration, we obtain

$$y^{(n-2)}(t) \geq \int_{t_1}^{t}\int_{t_1}^{u} p(s)y(\tau(s))\,\mathrm{d}s\,\mathrm{d}u = \int_{t_1}^{t} p(s)y(\tau(s))(t-s)\,\mathrm{d}s.$$

Repeating this procedure, we have

$$y'(t) \geq \int_{t_1}^{t} p(s)y(\tau(s))\frac{(t-s)^{n-2}}{(n-2)!}\,\mathrm{d}s.$$

Consequently,

$$\begin{aligned} y'(\tau(t)) &\geq \int_{t}^{\tau(t)} p(s)y(\tau(s))\frac{(\tau(t)-s)^{n-2}}{(n-2)!}\,\mathrm{d}s \\ &\geq y(\tau(t))\int_{t}^{\tau(t)} p(s)\frac{(\tau(t)-s)^{n-2}}{(n-2)!}\,\mathrm{d}s, \end{aligned} \tag{2.11}$$

where we have used that $y(\tau(t))$ is increasing. By combining inequalities (2.3) and (2.11), we conclude that

$$y^{(n+1)}(t) \geq y(\tau(t))\left[p'(t) + p(t)\tau'(t)\int_{t}^{\tau(t)} p(s)\frac{(\tau(t)-s)^{n-2}}{(n-2)!}\,\mathrm{d}s\right]$$

which in view of (E) implies

$$y^{(n+1)}(t) \geq y^{(n)}(t)\left[\frac{p'(t)}{p(t)} + \tau'(t)\int_{t}^{\tau(t)} p(s)\frac{(\tau(t)-s)^{n-2}}{(n-2)!}\,\mathrm{d}s\right],$$

that is

$$y^{(n+1)}(t) - \alpha_n(t)y^{(n)}(t) \geq 0.$$

Consequently,

$$\left(e^{-\beta_n(t)}y^{(n)}(t)\right)' \geq 0$$

and we conclude that $e^{-\beta_n(t)}y^{(n)}(t)$ is increasing, which is in view of (E) means that $p(t)y(\tau(t))e^{-\beta_n(t)}$ is increasing function. The proof is completed. □

We use the above-mentioned monotonicity to establish criterion for unbounded oscillation of (E).

Theorem 2.8. *Let* $\tau(t) \geq t$ *and*

$$\limsup_{t\to\infty} p(t)e^{-\beta_n(t)}\int_{t}^{\tau(t)} e^{\beta_n(s)}(\tau(t)-s)^{n-1}\,\mathrm{d}s > (n-1)!, \tag{2.12}$$

then $\mathcal{N}_n = \emptyset$. *If in addition* (1.5) *holds, then all nonoscillatory solutions of* (E) *are of degree* 0, *i.e.,* $\mathcal{N} = \mathcal{N}_0$.

Proof. Assume on the contrary that (E) possesses an eventually positive solution $y(t) \in \mathcal{N}_n$. Integrating (E) from t to u $(t \le u)$ and using the monotonicity of $p(t)y(\tau(t))e^{-\beta_n(t)}$, we have

$$y^{(n-1)}(u) \ge y(\tau(t))p(t)e^{-\beta_n(t)} \int_t^u e^{\beta_n(s)}\,\mathrm{d}s.$$

Integrating again from t to u and changing order of integration, we get

$$\begin{aligned} y^{(n-2)}(u) &\ge y(\tau(t))p(t)e^{-\beta_n(t)} \int_t^u \int_t^x e^{\beta_n(s)}\,\mathrm{d}s\,\mathrm{d}x \\ &= y(\tau(t))p(t)e^{-\beta_n(t)} \int_t^u e^{\beta_n(s)}(u-s)\,\mathrm{d}s. \end{aligned} \tag{2.13}$$

Proceeding in the same way $(n-2)$-times, we finally obtain

$$y(u) \ge y(\tau(t))p(t)e^{-\beta_n(t)} \int_t^u e^{\beta_n(s)} \frac{(u-s)^{n-1}}{(n-1)!}\,\mathrm{d}s.$$

Setting $u = \tau(t)$, we have

$$y(\tau(t)) \ge y(\tau(t))p(t)e^{-\beta_n(t)} \int_t^{\tau(t)} e^{\beta_n(s)} \frac{(\tau(t)-s)^{n-1}}{(n-1)!}\,\mathrm{d}s.$$

This contradiction establishes the desired result and the proof is completed. □

Example 2.9. Consider the advanced differential equation

$$y^{(n)}(t) = p_0 y(t+\tau), \quad p_0 > 0, \quad \tau > 0. \tag{E_{x3}}$$

It is easy to see that (1.5) holds, $\alpha_n(t) = \frac{p_0}{(n-1)!}\tau^{n-1} = \omega$ and $\beta_n(t) = \omega t$. Condition (2.12) yields

$$\lim_{t\to\infty} p_0 e^{-\omega t} \int_t^{t+\tau} e^{\omega s} \frac{(t+\tau-s)^{n-1}}{(n-1)!}\,\mathrm{d}s > 1. \tag{2.14}$$

Employing substitution $t + \tau - s = x$, one gets

$$\frac{p_0}{(n-1)!} e^{\omega\tau} \int_0^\tau e^{-\omega x} x^{n-1}\,\mathrm{d}s > 1.$$

Proceeding exactly as in Example 2.5 we are led to (2.8) which by Theorem 2.8 ensures that every nonoscillatory solution of (E_{x3}) is of degree 0 or in other words, every unbounded solution (if exists) of (E_{x3}) is oscillatory.

Example 2.10. We consider the advanced Euler differential equation

$$y^{(n)}(t) = \frac{p_0}{t^n} y(\lambda t), \quad p_0 > 0, \quad \lambda > 1. \tag{E_{x4}}$$

Simple calculation shows that (1.5) reduces to

$$p_0\lambda\,(2 + \ln\lambda) > 2(n-2)! \tag{2.15}$$

and

$$\alpha_n(t) = \frac{1}{t}\left[\frac{p_0(\lambda-1)^{n-1}}{(n-1)!} - n\right].$$

Les us denote

$$\frac{p_0(\lambda-1)^{n-1}}{(n-1)!} - n = \delta_n > 0,$$

Then

$$\beta_n = \delta_n \ln t.$$

Therefore (2.12) is equivalent to

$$\limsup_{t\to\infty} \frac{p_0 t^{-\delta_n - n}}{(n-1)!} \int_t^{\lambda t} s^{\delta_n} (\lambda t - s)^{n-1}\, \mathrm{d}s > 1.$$

On the other hand, as

$$\int_t^{\lambda t} \frac{(\lambda t - s)^{n-1}}{s^{-\delta_n}}\, \mathrm{d}s = -\sum_{i=0}^{n-1} \frac{(n-1)!(-\lambda t)^i s^{n+\delta_n - i}}{(n-1-i)\,!\,i\,!(n+\delta_n-1)} \Bigg|_t^{\lambda t}$$

condition (2.12), which guaranties $N_n = \emptyset$ for equation (E_{x4}), takes the form

$$p_0 \sum_{i=0}^{n-1} (-1)^{i+1} \frac{\lambda^{n+\delta_n} - \lambda^i}{(n-1-i)!\, i!\, (n+\delta_n - 1)} > 1. \tag{2.16}$$

Moreover, if (2.15) holds, then every nonoscillatory solution of (E_{x2}) is of degree 0. To see the progress which our criteria brings, let us consider $n = 2$ $(n = 4)$ and $\lambda = 1.5$. The condition (2.16) is satisfied when

$$p_0 > 6.56 \quad (p_0 > 304.48)$$

while (1.3) requires $p_0 > 10.58$ $(p_0 > 535.64)$.

Remark 2.11. In this paper, we have introduce new technique for investigation of monotonicity for nonoscillatory solutions of higher order differential equations. The monotonicities obtained have been applied to establish new criteria for all solutions to be of degree 0 or to be of degree n.

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Homoclinic solutions for a class of asymptotically autonomous Hamiltonian systems with indefinite sign nonlinearities

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Abstract. In this paper, we obtain the multiplicity of homoclinic solutions for a class of asymptotically autonomous Hamiltonian systems with indefinite sign potentials. The concentration-compactness principle is applied to show the compactness. As a byproduct, we obtain the uniqueness of the positive ground state solution for a class of autonomous Hamiltonian systems and the best constant for Sobolev inequality which are of independent interests.

Keywords: multiple homoclinic solutions, asymptotically autonomous Hamiltonian systems, indefinite sign nonlinearities, best constant for Sobolev inequality, the Concentration-Compactness Principle.

2020 Mathematics Subject Classification: 34C37, 37J06.

1 Introduction and main results

In this paper, we consider the following second-order planar Hamiltonian systems

$$\ddot{u}(t) + \nabla V(t, u(t)) = 0, \tag{1.1}$$

where $V : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a C^1-map. We say that a solution $u(t)$ of problem (1.1) is nontrivial homoclinic (to 0) if $u \not\equiv 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \pm\infty$. Subsequently, $\nabla V(t, x)$ denotes the gradient with respect to the x variable, $(\cdot, \cdot) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^2$ and $|\cdot|$ is the induced norm.

Hamiltonian system is a classical model in celestial mechanics, fluid mechanics and so on. Since its importance in physic fields, searching for the solutions of the Hamiltonian systems has attracted much attention of mathematicians since Poincaré. In a remarkable paper [31], the periodic solutions are firstly obtained for (1.1) with prescribed energy and prescribed period cases respectively via variational methods by Rabinowitz. However, to show homoclinic solutions via variational methods seems difficult since the lack of compactness for

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the Sobolev embedding. In order to regain the compactness, different strategies are adopted. In 1990, Rabinowitz [32] considered (1.1) with the following potentials

$$V(t,x) = -\frac{1}{2}(a(t)x,x) + W(t,x),$$

where $a(t)$ and $W(t,x)$ are T-periodic in t and homoclinic solution are obtained as the limit of a sequence of $2kT$-periodic solutions. Without periodic hypothesis, Rabinowitz and Tanaka [23] assumed the least eigenvalues of $a(t)$ go to infinity as $t \to \infty$. Under this condition, Omana and Willem [28] obtained a compact embedding theorem and showed the multiplicity of homoclinic solutions for problem (1.1). Without periodic or coercive hypothesis, there are still some other conditions proposed to obtain the nontrivial homoclinic solutions. In 2007, Lv and Tang [24] assumed that $V(t,x)$ is even in t and obtained one homoclinic solution for (1.1) as the limit of the solutions of nil-boundary-value problems. In 2010, Wu, Wu and Tang [43] showed that (1.1) possesses at least one nontrivial homoclinic solution if there is a nontrivial perturbation. In detail, they considered the following systems

$$\ddot{u}(t) - L(t)u(t) - \nabla W(t,u(t)) = f(t). \tag{1.2}$$

When $f \not\equiv 0$, the authors showed the existence of nontrivial homoclinic solutions for (1.2) without periodic nor coercive conditions on L and W.

As we know, the growth of W is crucial in determining the geometric structure of the corresponding functional and the boundedness of the almost critical sequence. Three typical growth cases are superquadratic, subquadratic and asymptotically quadratic cases. The following Ambrosetti–Rabinowitz-type condition is a classical superquadratic condition.

(AR) there exists a constant $v > 2$ such that

$$0 < vW(t,x) \leq (\nabla W(t,x), x)$$

for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \{0\}$.

In 1991, Rabinowitz and Tanaka [33] also obtained the homoclinic solutions for (1.1) under the following non-quadratic condition

(RT) $s^{-1}(\nabla W(t,sx), x)$ is an increasing function of $s \in (0,1]$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

As shown in [25], condition (RT) implies that

(MS) there exists $\theta \geq 1$ such that

$$\theta \widetilde{W}(t,x) \geq \widetilde{W}(t,sx)$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and $s \in [0,1]$, where $\widetilde{W}(t,x) = (\nabla W(t,x), x) - 2W(t,x)$.

With (MS) Lv and Tang [25] obtained infinitely many homoclinic solutions for (1.1). Besides, many superquadratic conditions are introduced. In 2004, Ou and Tang [29] considered the following superquadratic condition

(OT) $W(t,x)/|x|^2 \to +\infty$ as $|x| \to \infty$ uniformly in $t \in \mathbb{R}$.

Based on above results, Ding and Lee [8] introduced the following superquadratic condition

(DL) $\widetilde{W}(t,x) > 0$ if $x \neq 0$, and there exist $\epsilon \in (0,1)$ and $c > 0$ such that

$$\widetilde{W}(t,x) \geq c\frac{(\nabla W(t,x),x)}{|x|^{2-\epsilon}} \quad \text{for } |x| \text{ large enough.}$$

There are also some other superquadratic growth conditions introduced by many mathematicians. The readers are referred to [6,15,18,22,23,29,30,39,40,43–48] for more details.

In this paper, we mainly consider the asymptotically autonomous potentials without periodic, coercive, even assumption or perturbations. In 1999, Carrião and Miyagaki [5] showed the existence of homoclinic for problem (1.1) by assuming that $V(t,x)$ converges to $V_\infty(x)$ as $|t| \to +\infty$ and $V_\infty(x)$ satisfies the (AR) condition. The asymptotically autonomous Hamiltonian systems has also been considered by Lv, Xue and Tang [26] with asymptotically quadratic potentials. They showed the existence of homoclinic solutions for systems (1.1) with $a(t) \equiv$ *const*. being small enough. In another paper, Lv, etc. [27] also obtained ground state homoclinic orbits for a class of asymptotically periodic second-order Hamiltonian systems. Their results generalized the conclusions in [1,5] by replacing the (AR) condition with strict monotonic conditions on $W(t,x)$.

In this paper, we mainly consider the combined nonlinearities. In [6,26,36,37,44,46], the authors also considered the following concave-convex potentials

$$V(t,x) = -\frac{1}{2}(a(t)x,x) + \lambda F(t,x) + G(t,x),$$

where $a(t)$ is coercive, i.e. $a(t) \to +\infty$ as $t \to \infty$, $F(t,x)$ is subquadratic and $G(t,x)$ is superquadratic in $x \in \mathbb{R}^N$. The coercivity of $a(t)$ is an important assumption which can guarantee the compactness of Sobolev embedding.

In [46], Yang, Chen and Sun assumed that $a(t)$ is coercive, $F(t,x) = m(t)|x|^\gamma$ and $G(t,x) = d|x|^p$ with $m \in L^{\frac{2}{2-\gamma}}(\mathbb{R},\mathbb{R}^+)$ and $1 < \gamma < 2$, $d \geq 0$, $p > 2$. This result is generalized by Chen and He [6] with the following generalized superquadratic condition

(CH) There exist $\rho > 2$ and $1 < \delta < 2$ such that

$$\rho G(t,x) - (\nabla G(t,x),x) \leq h(t)|x|^\delta, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N$$

where $h : \mathbb{R} \to \mathbb{R}^+$ is a positive continuous function such that $h \in L^{\frac{2}{2-\delta}}(\mathbb{R},\mathbb{R}^+)$.

Obviously, (CH) is weaker than (AR) since $h(t) > 0$ for all $t \in \mathbb{R}$. In [42], Wu, Tang and Wu generalized the above results by relaxing the conditions on G. However, $a(t)$ is also required to be coercive.

Without coercive assumption, there are also some other papers concerning on this case with the steep well potentials (see [36,37]). In [36], the nonlinearities are the combination of subquadratic and asymptotic quadratic nonlinearities. While in [37], the nonlinearities are the combination of superquadratic and subquadratic nonlinearities. In [46], Ye and Tang obtained infinitely many homoclinic solutions for systems (1.1) with

$$V(t,x) = -\frac{1}{2}(a(t)x,x) + \frac{h(t)}{p}|x|^p + \frac{d(t)}{\nu}|x|^\nu, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $a(t) \geq 0$ and

$$\begin{cases} h \in L^{2/(2-p)}(\mathbb{R},\mathbb{R}^+) \\ d \in L^\infty(\mathbb{R},\mathbb{R}) \\ 1 < p < 2 < \nu. \end{cases}$$

By assuming $h(t) > 0$, the authors in [46] constructed a sequence of negative critical values. However, in [5,26,27], $W(t,x)$ is assumed to be non-negative in $\mathbb{R} \times \mathbb{R}^N$. A natural question is whether (1.1) possesses homoclinic solutions if $W(t,x)$ change signs without periodic or coercive assumptions. In this paper, we partially give some answers to this question. Precisely, we consider the sign-changing and asymptotically autonomous potentials, which have not been considered before as we know. Hence, we cannot obtain our results as the authors did in [6,26,36,37,44,46]. Concentration-compactness principle(CCP) is adopted to show the compactness. The crucial step in using the (CCP) is to exclude the dichotomy case by estimating the critical values. This can be easily done if W satisfies the following monotonic condition

(MC) the mapping $\tau \to (\frac{\nabla W(t,\tau x)}{\tau}, x)$ is strictly increasing in $\tau \in (0,1]$ for all $x \neq 0$ and $t \in \mathbb{R}$.

However, condition (MC) is not valid for our potentials. Hence we need more delicate estimates for the critical values to show the contradictions. The constant for the Sobolev inequality plays an important role in obtaining our results. In the next section, we show the best constant for the Sobolev inequality.

2 Best constant for the Sobolev inequality

Let's make it clear that $L^p(\mathbb{R},\mathbb{R}^m)$ and $H^1(\mathbb{R},\mathbb{R}^m)$ are the Banach spaces of functions on $\mathbb{R}$ valued in $\mathbb{R}^m$ under the norms

$$\|u\|_p := \left(\int_{\mathbb{R}} |u|^p dt\right)^{1/p}$$

and

$$\|u\| = \|u\|_{H^1} = \left(\int_{\mathbb{R}} \left(|\dot{u}|^2 + |u|^2\right) dt\right)^{1/2}.$$

Moreover, let $L^\infty(\mathbb{R},\mathbb{R}^m)$ be the Banach space of essentially bounded measurable functions from $\mathbb{R}$ into $\mathbb{R}^m$ under the norm

$$\|u\|_\infty := \operatorname{ess\,sup}\{|u(t)| : t \in \mathbb{R}\}.$$

As we know, for any $m > 1$, $H^1(\mathbb{R},\mathbb{R}^m)$ can be embedded into $L^\nu(\mathbb{R},\mathbb{R}^m)$ continuously for any $\nu \in [2,+\infty]$. Then we have the following Sobolev inequality

$$\|u\|_\nu \leq C_\nu \|u\| \quad \text{for all } u \in H^1(\mathbb{R},\mathbb{R}^m), \tag{2.1}$$

where C_ν is the best constant which is defined in the following proof. This inequality is important in using variational methods to show the existence and multiplicity of differential equations. However, since the best constant for the Sobolev inequality seems not important in previous studies of Hamiltonian systems, as we know, there is no paper concerning on the best constant of Sobolev inequality for (2.1). In this section, we show the best constant for (2.1).

There have been many papers concerning on the best constant for the Sobolev inequality in $H^1(\mathbb{R},\mathbb{R})$ (see [2–4,12]). In a remarkable paper, Talenti [38] obtained the best constant for

Sobolev inequality in $H^1(\mathbb{R}^N,\mathbb{R})$ with $N>1$. In 1983, Weinstein obtained the best constant for the following Gagliardo-Nirenberg-Sobolev inequalities

$$\|u\|_\nu^\nu \le C_*\|\nabla u\|_2^{\frac{N(\nu-2)}{2}}\|u\|_2^{\nu-\frac{N(\nu-2)}{2}} \quad \text{for } u\in H^1(\mathbb{R}^N,\mathbb{R}), \tag{2.2}$$

where $N\ge 2$, $2<\nu<\frac{2N}{N-2}$, $C_*=\frac{\nu}{2\|\mathcal{G}\|_2^{\nu-2}}$ is the best constant for (2.2) and $\mathcal{G}$ is the unique positive solution for the following scalar field equation

$$-\frac{N(\nu-2)}{4}\Delta u+\left(1+\frac{\nu-2}{4}(2-N)\right)u=|u|^{\nu-2}u, \quad x\in\mathbb{R}^N.$$

In a recent paper, Dolbeault, etc. [10] considered the best constant for the one-dimensional Gagliardo–Nirenberg–Sobolev inequalities in $H^1(\mathbb{R},\mathbb{R}^m)(m=1)$ and obtained

$$\frac{1}{M_{GN}(\nu)}=\inf_{y\in H^1(\mathbb{R},\mathbb{R})\setminus\{0\}}\frac{\left(\int_{\mathbb{R}}|y'|^2dt\right)^{\frac{\nu-2}{4\nu}}\left(\int_{\mathbb{R}}|y|^2dt\right)^{\frac{\nu+2}{4\nu}}}{\left(\int_{\mathbb{R}}|y|^\nu dt\right)^{\frac{1}{\nu}}}, \tag{2.3}$$

where $M_{GN}(\nu)$ is defined as

$$M_{GN}(\nu)=4^{-\frac{1}{\nu}}\left(\frac{(\nu+2)^{\nu+2}}{(\nu-2)^{\nu-2}}\right)^{\frac{1}{4\nu}}\left(\frac{2\sqrt{\pi}\Gamma\left(\frac{2}{\nu-2}\right)}{(\nu+2)\Gamma\left(\frac{2}{\nu-2}+\frac{1}{2}\right)}\right)^{\frac{\nu-2}{2\nu}}. \tag{2.4}$$

Moreover, $M_{GN}(\nu)$ is attained at $v_\star$, which is the unique optimal function up to translations, multiplication by a constant and scalings, defined as

$$v_\star(t)=\frac{1}{(\cosh t)^{\frac{2}{\nu-2}}}.$$

The following computation is made by the authors in [10]. For the reader's convenience, we write them here.

$$\int_{\mathbb{R}}|v_\star|^2dt=\frac{\sqrt{\pi}\Gamma\left(\frac{2}{\nu-2}\right)}{\Gamma\left(\frac{2}{\nu-2}+\frac{1}{2}\right)}, \qquad \int_{\mathbb{R}}|v_\star|^\nu dt=\frac{4}{\nu+2}\int_{\mathbb{R}}|v_\star|^2dt$$

and

$$\int_{\mathbb{R}}\left|v_\star'\right|^2dt=\frac{4}{(\nu-2)(\nu+2)}\int_{\mathbb{R}}|v_\star|^2dt.$$

Subsequently, we consider the case $m>1$. For any $u(t)=(u_1(t),\dots,u_m(t))\in H^1(\mathbb{R},\mathbb{R}^m)\setminus\{0\}$, set

$$y(t)=|u(t)|=\sqrt{\sum_{i=1}^m u_i^2(t)}\in H^1(\mathbb{R},\mathbb{R})\setminus\{0\}. \tag{2.5}$$

Then we have

$$[y']^2=\frac{\left(\sum_{i=1}^m u_iu_i'\right)^2}{\sum_{i=1}^m u_i^2}. \tag{2.6}$$

For any $\nu>2$, let

$$\mathfrak{R}=\inf_{u_1,\dots,u_m\in H^1(\mathbb{R},\mathbb{R})\setminus\{0\}}\frac{\left(\int_{\mathbb{R}}\left(\sum_{i=1}^m u_i'^2\right)dt\right)^{\frac{\nu-2}{4\nu}}}{\left(\int_{\mathbb{R}}\frac{\left(\sum_{i=1}^m u_iu_i'\right)^2}{\sum_{i=1}^m u_i^2}dt\right)^{\frac{\nu-2}{4\nu}}}. \tag{2.7}$$

On one hand, if we choose $u_1 = \ldots = u_m$, it is easy to see that $\mathfrak{R} \le 1$. On the other hand, we can also deduce that $\mathfrak{R} \ge 1$ since

$$\left(\sum_{i=1}^m u_i u_i'\right)^2 \le \left(\sum_{i=1}^m u_i^2\right)\left(\sum_{i=1}^m u_i'^2\right),$$

which implies $\mathfrak{R} = 1$. Therefore, by (2.3), (2.5)–(2.7), one has

$$\begin{aligned}
&\inf_{u\in H^1(\mathbb{R},\mathbb{R}^m)\setminus\{0\}} \frac{\left(\int_{\mathbb{R}}|\dot u|^2dt\right)^{\frac{\nu-2}{4\nu}}\left(\int_{\mathbb{R}}|u|^2dt\right)^{\frac{\nu+2}{4\nu}}}{\left(\int_{\mathbb{R}}|u|^\nu\,dt\right)^{\frac{1}{\nu}}}\\
&= \inf_{u_1,\ldots,u_m\in H^1(\mathbb{R},\mathbb{R})\setminus\{0\}} \frac{\left(\int_{\mathbb{R}}\left(\sum_{i=1}^m u_i'^2\right)dt\right)^{\frac{\nu-2}{4\nu}}}{\left(\int_{\mathbb{R}}\frac{\left(\sum_{i=1}^m u_iu_i'\right)^2}{\sum_{i=1}^m u_i^2}dt\right)^{\frac{\nu-2}{4\nu}}}\frac{\left(\int_{\mathbb{R}}\frac{\left(\sum_{i=1}^m u_iu_i'\right)^2}{\sum_{i=1}^m u_i^2}dt\right)^{\frac{\nu-2}{4\nu}}\left(\int_{\mathbb{R}}\left(\sum_{i=1}^m u_i^2\right)dt\right)^{\frac{\nu+2}{4\nu}}}{\left(\int_{\mathbb{R}}\left(\sum_{i=1}^m u_i^2\right)^{\frac{\nu}{2}}dt\right)^{\frac{1}{\nu}}}\\
&\ge \inf_{u_1,\ldots,u_m\in H^1(\mathbb{R},\mathbb{R})\setminus\{0\}} \frac{\left(\int_{\mathbb{R}}\left(\sum_{i=1}^m u_1'^2\right)dt\right)^{\frac{\nu-2}{4\nu}}}{\left(\int_{\mathbb{R}}\frac{\left(\sum_{i=1}^m u_iu_i'\right)^2}{\sum_{i=1}^m u_i^2}dt\right)^{\frac{\nu-2}{4\nu}}}\inf_{y\in H^1(\mathbb{R},\mathbb{R})\setminus\{0\}}\frac{\left(\int_{\mathbb{R}}|y'|^2dt\right)^{\frac{\nu-2}{4\nu}}\left(\int_{\mathbb{R}}|y|^2dt\right)^{\frac{\nu+2}{4\nu}}}{\left(\int_{\mathbb{R}}|y|^\nu dt\right)^{\frac{1}{\nu}}}\\
&= \frac{1}{M_{GN}(\nu)}.
\end{aligned}$$

Hence, for any $\nu > 2$

$$\left(\int_{\mathbb{R}}|u|^\nu dt\right)^{\frac{1}{\nu}} \le M_{GN}(\nu)\left(\int_{\mathbb{R}}|\dot u|^2dt\right)^{\frac{\nu-2}{4\nu}}\left(\int_{\mathbb{R}}|u|^2dt\right)^{\frac{\nu+2}{4\nu}} \quad \text{for all } u\in H^1(\mathbb{R},\mathbb{R}^m),$$

where $M_{GN}(\nu)$ is the best constant defined in (2.4) and attained at

$$\mathcal{V} = (k_1,\ldots,k_m)v_\star \tag{2.8}$$

with $k_i \ge 0$ and $k_1^2+\ldots+k_m^2 = 1$. Moreover, for any $\Delta \subset \mathbb{R}$, there holds

$$\left(\int_{\Delta}|u|^\nu dt\right)^{\frac{1}{\nu}} \le M_{GN}(\nu)\left(\int_{\Delta}|\dot u|^2dt\right)^{\frac{\nu-2}{4\nu}}\left(\int_{\Delta}|u|^2dt\right)^{\frac{\nu+2}{4\nu}} \quad \text{for all } u\in H_0^1(\Delta,\mathbb{R}^m) \tag{2.9}$$

and $M_{GN}(\nu)$ is the best constant which can be attained if and only if $\Delta = \mathbb{R}$. For any $u \in H^1(\mathbb{R},\mathbb{R}^m)\setminus\{0\}$ and $\tau > 0$, let $q_\tau(t) = u(\tau t)$ with

$$\mathcal{Q}_\tau(u) = \frac{\left(\int_{\mathbb{R}}\left(|\dot q_\tau|^2+|q_\tau|^2\right)dt\right)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}}|q_\tau|^\nu dt\right)^{\frac{1}{\nu}}}$$

and

$$\tau_u = \sqrt{\frac{(\nu-2)\int_{\mathbb{R}}|u|^2dt}{(\nu+2)\int_{\mathbb{R}}|\dot u|^2dt}}.$$

It is easy to see that

$$\inf_{\tau>0}\mathcal{Q}_\tau(u) = \mathcal{Q}_{\tau_u}(u) \le \mathcal{Q}_1(u) = \frac{\|u\|}{\|u\|_\nu} \tag{2.10}$$

and

$$\inf_{u\in H^1(\mathbb{R},\mathbb{R}^m)\setminus\{0\}} \mathcal{Q}_{\tau_u}(u) = \inf_{u\in H^1(\mathbb{R},\mathbb{R}^m)\setminus\{0\}} \left(\frac{\nu+2}{\nu-2}\right)^{\frac{\nu-2}{4\nu}} \left(\frac{2\nu}{\nu+2}\right)^{\frac{1}{2}} \frac{\left(\int_{\mathbb{R}}|\dot{u}|^2dt\right)^{\frac{\nu-2}{4\nu}}\left(\int_{\mathbb{R}}|u|^2dt\right)^{\frac{\nu+2}{4\nu}}}{\left(\int_{\mathbb{R}}|u|^\nu dt\right)^{\frac{1}{\nu}}}$$
$$= \frac{1}{M_{GN}(\nu)}\left(\frac{\nu+2}{\nu-2}\right)^{\frac{\nu-2}{4\nu}} \left(\frac{2\nu}{\nu+2}\right)^{\frac{1}{2}}. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$\inf_{u\in H^1(\mathbb{R},\mathbb{R}^m)\setminus\{0\}} \frac{\|u\|}{\|u\|_\nu} \geq \frac{1}{M_{GN}(\nu)}\left(\frac{\nu+2}{\nu-2}\right)^{\frac{\nu-2}{4\nu}} \left(\frac{2\nu}{\nu+2}\right)^{\frac{1}{2}} \tag{2.12}$$

and

$$\frac{\|\mathcal{V}_{\tau_\mathcal{V}}\|}{\|\mathcal{V}_{\tau_\mathcal{V}}\|_\nu} = \frac{1}{M_{GN}(\nu)}\left(\frac{\nu+2}{\nu-2}\right)^{\frac{\nu-2}{4\nu}} \left(\frac{2\nu}{\nu+2}\right)^{\frac{1}{2}}. \tag{2.13}$$

Then, we infer that (2.1) holds and

$$C_\nu = M_{GN}(\nu)\left(\frac{\nu-2}{\nu+2}\right)^{\frac{\nu-2}{4\nu}} \left(\frac{\nu+2}{2\nu}\right)^{\frac{1}{2}} \tag{2.14}$$

is the best constant. Moreover, we also need to consider the best constant when $\nu = +\infty$. It follows from (2.14) that $C_\nu \to \frac{1}{\sqrt{2}}$ as $\nu \to +\infty$. It has been shown by Janczewska in [16] that $C_\infty = \frac{1}{\sqrt{2}}$, which is the best constant for (2.1) when $\nu = \infty$.

3 Solutions for the limit systems

In this section, we consider the solutions for the limit systems of (1.1). In the rest of this paper, we only consider the systems in $\mathbb{R}^2$. The potential V is defined as

$$V(t,x) = -\frac{1}{2}a(t)|x|^2 + \lambda F(t,x) + d(t)|x|^\nu,$$

where $a, d \in C(\mathbb{R},\mathbb{R})$, $\lambda > 0$, $\nu > 2$ and the following conditions hold

($V1$) there exists $a_0 > 0$ such that $a(t) \geq a_0$ for all $t \in \mathbb{R}$;

($V2$) there exist a_∞, $d_\infty > 0$ such that $a(t) \to a_\infty$ and $d(t) \to d_\infty$ as $|t| \to +\infty$;

($V3$) $\|a\|_\infty \geq 1$, $d(0) = \|d\|_\infty$;

($V4$) $F(t,0) = 0$ and $F(t,x) \in C^1(\mathbb{R}\times\mathbb{R}^2,\mathbb{R})$;

($V5$) for any $(t,x) \in \mathbb{R}\times\mathbb{R}^2$, there exist $1 < r_1 \leq r_2 < 2$ such that

$$|\nabla F(t,x)| \leq b_1(t)|x|^{r_1-1} + b_2(t)|x|^{r_2-1},$$

where $b_1(t) \in L^{\beta_1}(\mathbb{R},\mathbb{R}^+)$ and $b_2(t) \in L^{\beta_2}(\mathbb{R},\mathbb{R}^+)$ for some $\beta_1 \in (1,\frac{2}{2-r_1})$ and $\beta_2 \in (1,\frac{2}{2-r_2})$;

($V6$) there exist $\bar{t} \in \mathbb{R}$, $\bar{r} \in (1,2)$ and $b_0 > 0$ such that $F(\bar{t},x) > b_0|x|^{\bar{r}}$ for all $x \in \mathbb{R}^2$.

Here $Q_\nu : \mathbb{R} \to \mathbb{R}^+$ is the **unique positive ground state** solution(up to translations) for the following equation

$$\ddot{u}(t) - u(t) + u^{\nu-1}(t) = 0 \quad \text{for } t \in \mathbb{R}. \tag{3.1}$$

Let us consider the following systems

$$\begin{cases} \Delta u_j - a_\infty u_j + \nu d_\infty \left(\sum_{i=1}^m u_i^2\right)^{\frac{\nu-2}{2}} u_j = 0 \text{ in } \mathbb{R}^n, \\ \dot{u}_j(0) = 0, \quad j = 1, \ldots, m, \\ u_j(y) \to 0 \text{ as } |y| \to +\infty. \end{cases} \tag{3.2}$$

The existence of solutions for systems (3.2) has been considered by many mathematicians via the variational methods. A solution $(u_1, \ldots, u_m)$ for (3.2) is said to be **positive** if $u_1, \ldots, u_m > 0$. When $m = 1$, (3.2) reduces to a differential equation and the uniqueness of positive ground state solution for (3.2) has been shown by M. K. Kwong [19] with $\nu > 2$. The readers are also referred to [17,34] for more general cases.

When $m > 1$, (3.2) is related to the coupled nonlinear Schrödinger equations. In last decades, there have been many mathematicians devoting themselves to the uniqueness of positive solutions for the coupled nonlinear Schrödinger equations and obtained many significant results (see [9,19,26,41]). In a recent paper [41], Wei and Yao considered the following systems

$$\begin{cases} \ddot{u}(r) + \frac{n-1}{\nu}\dot{u}(r) - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0, & \text{in } [0,\infty) \\ \ddot{v}(r) + \frac{n-1}{r}\dot{v}(r) - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0, & \text{in } [0,\infty) \\ u(r), v(r) > 0 \quad \text{in } [0,\infty) \\ \dot{u}(0) = \dot{v}(0) = 0, \text{ and } u(r), v(r) \to 0 \text{ as } r \to \infty. \end{cases} \tag{3.3}$$

When $\lambda_1 = \lambda_2 = \lambda$ with $0 \le \beta \notin [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$, they showed the uniqueness of positive solutions for system (3.3), defined as

$$(u_0, v_0) = \left(\sqrt{\frac{\lambda(\beta - \mu_2)}{\beta^2 - \mu_1\mu_2}} w_0(\sqrt{\lambda}x), \sqrt{\frac{\lambda(\beta - \mu_1)}{\beta^2 - \mu_1\mu_2}} w_0(\sqrt{\lambda}x)\right)$$

where w_0 is the unique positive solution of

$$\Delta w - w + w^3 = 0 \quad \text{in } \mathbb{R}, \qquad w(0) = \max_{x\in\mathbb{R}^N} w(x), \qquad w(x) \to 0 \text{ as } |x| \to \infty.$$

q When $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \beta$, it has also been shown in [41] that all the positive solutions of system (3.3) have the following form

$$(u(x), v(x)) = \left(\sqrt{\frac{\lambda}{\beta}} w(\sqrt{\lambda}x)\cos\theta, \sqrt{\frac{\lambda}{\beta}} w(\sqrt{\lambda}x)\sin\theta\right), \qquad \theta \in (0, \pi/2).$$

For the high dimension cases, i.e. $n = 2, 3$ and $m = 2$, the readers are referred to another paper by Dai, Tian and Zhang [11]. However, the case $n = 1$ is not considered. We can see that (3.2) reduces to (3.3) if $\nu = 4$. Motivated by above papers, we obtain the uniqueness of solutions for (3.2) when $m = 2$, $n = 1$ and $\nu > 2$. More precisely, we obtain the following lemma.

Lemma 3.1. *Suppose $m = 2$, $n = 1$, a_∞, $d_\infty > 0$ and $\nu > 2$. Then system (3.2) possesses at least one positive solution. Let $\mathcal{U}_\nu : \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}^+$ be a positive solution for systems (3.2), then there exits $\omega \in (0, \pi/2)$ such that*

$$\mathcal{U}_\nu = \left(\frac{a_\infty}{\nu d_\infty}\right)^{\frac{1}{\nu-2}} Q_\nu\left(\sqrt{a_\infty}t\right)(\cos\omega, \sin\omega) \tag{3.4}$$

and $\mathcal{U}_\nu$ is the ground state solution for (3.2).

Proof. Since $n = 1$, the critical exponent equals to $+\infty$. The existence of positive solutions for the subcritical problems have been considered in [7, 13, 35, 41]. Subsequently, we only show (3.4) holds and $\mathcal{U}_\nu$ is the ground state solution for (3.2). Let

$$\mathcal{M}_\nu(t) = \left(\frac{a_\infty}{\nu d_\infty}\right)^{-\frac{1}{\nu-2}} \mathcal{U}_\nu\left(\frac{t}{\sqrt{a_\infty}}\right).$$

Then $\mathcal{M}_\nu = (\mathcal{M}_1(t), \mathcal{M}_2(t))$ is the positive solution for the following system

$$\begin{cases} \ddot{u}_j(t) - u_j(t) + \left(u_1^2(t) + u_2^2(t)\right)^{\frac{\nu-2}{2}} u_j(t) = 0, \quad j = 1, 2, \text{ for } t \in \mathbb{R}, \\ \dot{u}_1(0) = \dot{u}_2(0) = 0, \\ u_1(t), u_2(t) \to 0 \quad \text{as } |t| \to \infty, \end{cases}$$

which implies

$$\ddot{\mathcal{M}}_1 - \mathcal{M}_1 + (\mathcal{M}_1^2 + \mathcal{M}_2^2)^{\frac{\nu-2}{2}} \mathcal{M}_1 = 0, \tag{3.5}$$

$$\ddot{\mathcal{M}}_2 - \mathcal{M}_2 + (\mathcal{M}_1^2 + \mathcal{M}_2^2)^{\frac{\nu-2}{2}} \mathcal{M}_2 = 0. \tag{3.6}$$

Subtracting (3.5) by (3.6), one infers that

$$\frac{d}{dt}\left(\dot{\mathcal{M}}_1 \mathcal{M}_2 - \mathcal{M}_1 \dot{\mathcal{M}}_2\right) = 0,$$

which implies

$$\dot{\mathcal{M}}_1 \mathcal{M}_2 - \mathcal{M}_1 \dot{\mathcal{M}}_2 = C \quad \text{for some } C \in \mathbb{R}.$$

Since $\dot{\mathcal{M}}_1(0) = \dot{\mathcal{M}}_2(0) = 0$, we obtain

$$\dot{\mathcal{M}}_1 \mathcal{M}_2 - \mathcal{M}_1 \dot{\mathcal{M}}_2 = 0 \quad \text{for all } t \in \mathbb{R}.$$

By the ordinary differential equation theory, one can deduce

$$\mathcal{M}_1 = K\mathcal{M}_2 \quad \text{for some } K > 0. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\ddot{\mathcal{M}}_2 - \mathcal{M}_2 + (K^2 + 1)^{\frac{\nu-2}{2}} \mathcal{M}_2^{\nu-1} = 0.$$

Letting $T(t) = \left(K^2 + 1\right)^{\frac{1}{2}} \mathcal{M}_2(t)$, we see $T(t) > 0$ satisfies (3.1). By the uniqueness, one has $T = Q_\nu$, which implies $\mathcal{M}_2 = \left(K^2 + 1\right)^{-\frac{1}{2}} Q_\nu$. Then it follows that

$$\mathcal{M}_\nu(t) = Q_\nu(t)\left(K^2 + 1\right)^{-\frac{1}{2}}(K, 1),$$

which implies (3.4). We also show that $\mathcal{U}_\nu$ is a ground state solution for systems (3.2). Actually, the corresponding functional of (3.2) is defined as

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}} \left(|\dot{u}|^2 + a_\infty |u|^2\right) dt - d_\infty \int_{\mathbb{R}} |u|^\nu dt.$$

Set $\mathcal{N} = \{u \in H^1(\mathbb{R}, \mathbb{R}^2) \setminus \{0\} : \langle I'_\infty(u), u\rangle = 0\}$ and $c_\infty = \inf_{u\in\mathcal{N}} I_\infty(u)$. Moreover, the corresponding functional of (3.1) is defined as

$$J_\infty(q) = \frac{1}{2}\int_{\mathbb{R}} \left(|\dot{q}|^2 + |q|^2\right) dt - \frac{1}{\nu}\int_{\mathbb{R}} |q|^\nu dt.$$

Let $\mathfrak{N} = \{q \in H^1(\mathbb{R}, \mathbb{R}) \setminus \{0\} : \langle J'_\infty(q), q\rangle = 0\}$ and $\mathfrak{C}_\infty = \inf_{q\in\mathfrak{N}} J_\infty(q)$. By the definition of Q_ν, we deduce that

$$J_\infty(Q_\nu) = \mathfrak{C}_\infty.$$

Obviously, for any $q(t) \in \mathfrak{N}$ and $e \in \mathbb{R}^2$ with $|e| = 1$, we have that $\left(\frac{a_\infty}{\nu d_\infty}\right)^{\frac{1}{\nu-2}} q(\sqrt{a_\infty} t)e \in \mathcal{N}$. In turn, for any $u(t) \in \mathcal{N}$ we have $\left(\frac{\nu d_\infty}{a_\infty}\right)^{\frac{1}{\nu-2}} \left|u\left(\frac{t}{\sqrt{a_\infty}}\right)\right| \in \mathfrak{N}$. Therefore, we infer that $c_\infty = a^{\frac{\nu+2}{2(\nu-2)}} \left(\frac{1}{\nu d_\infty}\right)^{\frac{2}{\nu-2}} \mathfrak{C}_\infty$. Moreover, it follows from (3.4) and the definition of Q_ν that

$$\int_{\mathbb{R}} \left(|\dot{Q}_\nu|^2 + |Q_\nu|^2\right) dt = \int_{\mathbb{R}} |Q_\nu|^\nu dt \tag{3.8}$$

and

$$c_\infty = I_\infty(\mathcal{U}_\nu) = \frac{\nu-2}{2\nu}\left(\frac{1}{\nu d_\infty}\right)^{\frac{2}{\nu-2}} a_\infty^{\frac{\nu+2}{2(\nu-2)}} \int_{\mathbb{R}} |Q_\nu|^\nu dt. \tag{3.9}$$

□

Remark 3.2. When $\nu = 4$, Theorem 3.1 reduces to the results in [41].

4 Main results

In this section, we prove our main result.

Theorem 4.1. *Suppose that $\nu > 2$, $(V1)$–$(V5)$ hold. Then there exist $\lambda_0, d_0 > 0$ such that problem (1.1) possesses at least one homoclinic solution for all $\lambda \in (0, \lambda_0)$ and $d_\infty \in (0, d_0)$. Moreover, (1.1) possesses another homoclinic solution if $(V6)$ holds.*

Remark 4.2. In [36,37], Sun and Wu also considered (1.1) with mixed nonlinearities. In both papers, the infimum of $a(t)$ cannot be attained at infinity, which is different from our result.

Remark 4.3. In Theorem 4.1, there are no periodic, coercive or symmetric assumptions on $a(t)$, which is different from the results in [6,14,32,39,44]. According to our conditions, both of the superquadratic and subquadratic parts of V can change signs, then we can not obtain the compactness as the authors did in [46].

Remark 4.4. In [26,27], $W(t,x)$ is required to satisfy

$$(\nabla W(t,x), x) \geq (\nabla W^\infty(x), x) \geq 0 \quad \text{for all } t \in \mathbb{R}, x \in \mathbb{R}^N, \tag{4.1}$$

and

$$(\nabla W(t,x),x) \geq 2W(x) \quad \text{for all } t \in \mathbb{R}, x \in \mathbb{R}^N, \tag{4.2}$$

where W^∞ is the limit function of W as $t \to \infty$. In our theorem, we have

$$W(t,x) = \lambda F(t,x) + d(t)|x|^\nu. \tag{4.3}$$

Since $F(t,x)$ and $d(t)$ can change signs, we infer that (4.1) and (4.2) are not valid for (4.3). Moreover, since (4.1), (4.2) and (MC) hold in [27], the authors can show that for any $u \in H^1(\mathbb{R},\mathbb{R}^2)$, there exists unique $s_u > 0$ such that $s_u u \in \mathcal{L}$ and $\sup_{s\geq 0} I(su) = I(s_u u)$, where $\mathcal{L} = \{u \in H^1(\mathbb{R},\mathbb{R}^2) \setminus \{0\} : \langle I'(u),u\rangle = 0\}$. This conclusion is crucial in using the (CCP) to show the contradictions. However, we can not obtain this conclusion by our conditions. Therefore, the Nehari-manifold method is not applicable for our theorem.

4.1 Preliminaries

The corresponding functional of (1.1) is defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}} \left(|\dot{u}|^2 + a(t)|u|^2\right) dt - \lambda \int_{\mathbb{R}} F(t,u)dt - \int_{\mathbb{R}} d(t)|u|^\nu dt. \tag{4.4}$$

Lemma 4.5. *Under* $(V1)$–$(V5)$*,* I *is of* C^1 *class and weakly lower semi-continuous. Moreover, we have*

$$\langle I'(u),v\rangle = \int_{\mathbb{R}} ((\dot{u},\dot{v}) + a(t)(u,v))dt - \lambda \int_{\mathbb{R}} (\nabla F(t,u),v)dt - \nu \int_{\mathbb{R}} d(t)|u|^{\nu-2}(u,v)dt,$$

which implies that

$$\langle I'(u),u\rangle = \int_{\mathbb{R}} \left(|\dot{u}|^2 + a(t)|u|^2\right) dt - \lambda \int_{\mathbb{R}} (\nabla F(t,u),u)dt - \nu \int_{\mathbb{R}} d(t)|u|^\nu dt.$$

Proof. The proof is similar to Lemma 2.3 in [6]. □

Lemma 4.6. *The critical points of* I *are homoclinic solutions for problem* (1.1)*.*

Proof. Since $\|a\|_\infty \geq a(t) > a_0 > 0$, the proof is similar to Lemma 3.1 in [49]. □

We will show the existence of two critical points of I by the Mountain Pass Theorem and the following critical point lemma respectively.

Lemma 4.7 (Lu [22])**.** *Let* X *be a real reflexive Banach space and* $\Omega \subset X$ *be a closed bounded convex subset of* X*. Suppose that* $\varphi : X \to \mathbb{R}$ *is a weakly lower semi-continuous (w.l.s.c. for short) functional. If there exists a point* $x_0 \in \Omega \setminus \partial\Omega$ *such that*

$$\varphi(x) > \varphi(x_0), \quad \forall\, x \in \partial\Omega,$$

then there must be an $x^* \in \Omega \setminus \partial\Omega$ *such that*

$$\varphi(x^*) = \inf_{x\in\Omega} \varphi(x).$$

4.2 The Mountain Pass Structure

In this section, we mainly show the Mountain Pass structure of I and obtain some crucial estimates.

Lemma 4.8. *Suppose the conditions of Theorem 4.1 hold, then there exist $\varrho_0, \overline{\alpha} > 0$ such that $I|_{\partial\mathcal{S}_{\varrho_0}} \geq \overline{\alpha}$, where $\mathcal{S}_{\varrho_0} = \{u \in H^1 : \|u\| \leq \varrho_0\}$.*

Proof. By $(V4)$ and $(V5)$, we can deduce that

$$|(\nabla F(t,x),x)| \leq b_1(t)|x|^{r_1} + b_2(t)|x|^{r_2} \tag{4.5}$$

and

$$|F(t,x)| \leq \frac{1}{r_1}b_1(t)|x|^{r_1} + \frac{1}{r_2}b_2(t)|x|^{r_2} \tag{4.6}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^2$. By (2.1), (4.4), (4.6) and $(V1)$, for all $u \in \partial\mathcal{S}_\varrho$, we have

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_{\mathbb{R}} \left(|\dot{u}|^2 + a(t)|u|^2\right) dt - \lambda \int_{\mathbb{R}} F(t,u)dt - \int_{\mathbb{R}} d(t)|u|^\nu dt \\
&\geq \frac{\min\{1,a_0\}}{2}\|u\|^2 - \lambda\left(\frac{1}{r_1}\int_{\mathbb{R}} b_1(t)|u|^{r_1}dt + \frac{1}{r_2}\int_{\mathbb{R}} b_2(t)|u|^{r_2}dt\right) - C_\nu^\nu\|d\|_\infty\|u\|^\nu \\
&\geq \frac{\min\{1,a_0\}}{2}\|u\|^2 - \lambda\left(\frac{1}{r_1}C_{r_1\beta_1^*}^{r_1}\|b_1\|_{\beta_1}\|u\|^{r_1} + \frac{1}{r_2}C_{r_2\beta_2^*}^{r_2}\|b_2\|_{\beta_2}\|u\|^{r_2}\right) - C_\nu^\nu\|d\|_\infty\|u\|^\nu.
\end{aligned}$$

For any $\varrho > 0$, set

$$h(\varrho) = \frac{\min\{1,a_0\}}{2}\varrho^2 - \|d\|_\infty C_\nu^\nu \varrho^\nu.$$

It is easy to see that $h'(\varrho_0) = 0$ and ϱ_0 is the unique critical point of h defined as

$$\varrho_0 = \left(\frac{\min\{1,a_0\}}{\nu C_\nu^\nu\|d\|_\infty}\right)^{\frac{1}{\nu-2}}.$$

Then there exists $\overline{\lambda}_1 > 0$ such that for any $\lambda \in (0,\overline{\lambda}_1)$ with $\|u\| = \varrho_0$, we have

$$I(u) \geq \frac{1}{2}h(\varrho_0) \doteq \overline{\alpha}.$$

We obtain our conclusion. □

Lemma 4.9. *Suppose the conditions of Theorem 4.1 hold, then for λ small enough, there exists $e_0 \in H^1$ such that $\|e_0\| > \varrho_0$ and $I(e_0) \leq \overline{\alpha}$, where $\varrho_0, \overline{\alpha}$ are defined in Lemma 4.8.*

Proof. it follows from the definition of Q_ν, (3.8) and (2.14) that

$$\begin{aligned}
J_\infty(Q_\nu) = \mathcal{E}(\nu) &= \inf_{u\in H^1(\mathbb{R},\mathbb{R})\setminus\{0\}} \frac{(\nu-2)\left(\|\dot{u}\|_2^2 + \|u\|_2^2\right)^{\frac{\nu}{\nu-2}}}{2\nu\|u\|_\nu^{\frac{2\nu}{\nu-2}}} \\
&= \frac{(\nu-2)\left(\|\dot{Q}_\nu\|_2^2 + \|Q_\nu\|_2^2\right)^{\frac{\nu}{\nu-2}}}{2\nu\|Q_\nu\|_\nu^{\frac{2\nu}{\nu-2}}} \\
&= \frac{\nu-2}{2\nu}\int_{\mathbb{R}} |Q_\nu|^\nu dt \\
&\geq \frac{\nu-2}{2\nu}C_\nu^{-\frac{2\nu}{\nu-2}},
\end{aligned}$$

which implies

$$\int_{\mathbb{R}} |Q_\nu|^\nu dt \geq C_\nu^{-\frac{2\nu}{\nu-2}}.$$

It follows from $(V3)$ that, there exist $T > 0$ such that $|d(t) - \|d\|_\infty| \leq \varepsilon_0$ for all $t \in (-T, T)$. For any $u \in H_0^1((-T,T), \mathbb{R}^2)$, let

$$L(u) = \frac{\left(\int_{-T}^{T} \left(|\dot{u}|^2 + |u|^2\right) dt\right)^{\frac{\nu}{\nu-2}}}{\left(\int_{-T}^{T} |u|^\nu dt\right)^{\frac{2}{\nu-2}}}.$$

Let $\chi \in H_0^1((-T,T), \mathbb{R}^2)$ and

$$\overline{u}(t) = (\chi(\sqrt{\|a\|_\infty + \varepsilon_0} t), 0),$$

which implies $\overline{u} \in H^1(\mathbb{R}, \mathbb{R}^2)$. For any $t \in (-T, T)$, it follows from $\|a\|_\infty \geq 1$ that $\frac{t}{\sqrt{\|a\|_\infty + \varepsilon_0}} \in (-T, T)$. Then

$$\begin{aligned}
I(\theta\overline{u}) \leq{}& \frac{\theta^2\sqrt{\|a\|_\infty + \varepsilon_0}}{2} \int_{-T}^{T} \left(|\dot{\chi}|^2 + |\chi|^2\right) dt - \frac{\theta^\nu}{\sqrt{\|a\|_\infty + \varepsilon_0}} \int_{-T}^{T} d\left(\frac{t}{\sqrt{\|a\|_\infty + \varepsilon_0}}\right) |\chi|^\nu dt \\
& - \lambda \int_{-T}^{T} F(t, \theta\overline{u}) dt \\
\leq{}& \frac{\theta^2\sqrt{\|a\|_\infty + \varepsilon_0}}{2} \int_{-T}^{T} \left(|\dot{\chi}|^2 + |\chi|^2\right) dt - \frac{(\|d\|_\infty - \varepsilon_0)\theta^\nu}{\sqrt{\|a\|_\infty + \varepsilon_0}} \int_{-T}^{T} |\chi|^\nu dt \\
& + \lambda \left(\frac{\theta^{r_1}}{r_1} \|a\|_\infty^{-\frac{1}{2\beta_1^*}} C_{r_1\beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|\chi\|^{r_1} + \frac{\theta^{r_2}}{r_2} \|a\|_\infty^{-\frac{1}{2\beta_2^*}} C_{r_2\beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|\chi\|^{r_2} \right).
\end{aligned}$$

Choose $\theta_0 > 0$ large enough such that $I(\theta_0\overline{u}) < 0$ and $\theta_0\|\overline{u}\| > \varrho_0$. Letting $e_0 = \theta_0\overline{u}$, we see that there exists $\overline{\lambda}_2 \in (0, \overline{\lambda}_1)$ such that for any $\lambda \in (0, \overline{\lambda}_2)$, $I(e_0) < 0$ and $\|e_0\| > \varrho_0$. We obtain the conclusion of this lemma. □

By the Mountain Pass theorem, there exists a sequence $\{u_n\}$ and $c \geq \overline{\alpha}$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) \mid g(0) = O, \ \ g(1) = e_0\}.$$

such that

$$I(u_n) \to c \tag{4.7}$$

and for any $v \in H^1\left(\mathbb{R}, \mathbb{R}^2\right)$

$$\begin{aligned}
o(1)\|v\| = \langle I'(u_n), v\rangle ={}& \int_{\mathbb{R}} ((\dot{u}_n, \dot{v}) + a(t)(u_n, v)) dt - \lambda \int_{\mathbb{R}} (\nabla F(t, u_n), v) dt \\
& - \nu \int_{\mathbb{R}} d(t)|u_n|^{\nu-2}(u_n, v) dt.
\end{aligned} \tag{4.8}$$

Next, we show an important relation between c and c_∞, which is crucial in the following concentration compactness study.

Lemma 4.10. *Suppose λ and d_∞ are small enough, then*

$$c_\infty - c \geq 2\lambda\left(\frac{r_1+\nu}{\nu r_1}C^{r_1}_{r_1\beta_1^*}\|b_1\|_{\beta_1}\|v_n\|^{r_1} + \frac{r_1+\nu}{\nu r_2}C^{r_2}_{r_2\beta_2^*}\|b_2\|_{\beta_2}\|v_n\|^{r_2}\right). \tag{4.9}$$

Proof. First, we estimate the critical value of I along the sequence $\{u_n\}$. For $s \in [0,1]$, set

$$g_0(s) = se_0 = s\theta_0\overline{u},$$

which implies $g_0(s) \in \Gamma$. It follows from the definition of c that

$$\begin{aligned}
c &= \inf_{g\in\Gamma}\max_{s\in[0,1]} I(g(s)) \\
&\leq \max_{s\in[0,1]} I(g_0(s)) \\
&\leq \max_{s\in[0,1]}\left[\frac{(s\theta_0)^2\sqrt{\|a\|_\infty+\varepsilon_0}}{2}\int_{-T}^{T}(|\dot\chi|^2+|\chi|^2)\,dt - \frac{(\|d\|_\infty-\varepsilon_0)(s\theta_0)^\nu}{\sqrt{\|a\|_\infty+\varepsilon_0}}\int_{-T}^{T}|\chi|^\nu dt\right] \\
&\quad + \lambda\left(\frac{\theta_0^{r_1}}{r_1}\|a\|_\infty^{-\frac{1}{2\beta_1^*}}C^{r_1}_{r_1\beta_1^*}\|b_1\|_{\beta_1}\|\chi\|^{r_1} + \frac{\theta_0^{r_2}}{r_2}\|a\|_\infty^{-\frac{1}{2\beta_2^*}}C^{r_2}_{r_2\beta_2^*}\|b_2\|_{\beta_2}\|\chi\|^{r_2}\right) \\
&\leq \frac{\nu-2}{2\nu}\left(\frac{1}{\nu(\|d\|_\infty-\varepsilon_0)}\right)^{\frac{2}{\nu-2}}(\|a\|_\infty+\varepsilon_0)^{\frac{\nu+2}{2(\nu-2)}}L(\chi) \\
&\quad + \lambda\left(\frac{\theta_0^{r_1}}{r_1}\|a\|_\infty^{-\frac{1}{2\beta_1^*}}C^{r_1}_{r_1\beta_1^*}\|b_1\|_{\beta_1}\|\chi\|^{r_1} + \frac{\theta_0^{r_2}}{r_2}\|a\|_\infty^{-\frac{1}{2\beta_2^*}}C^{r_2}_{r_2\beta_2^*}\|b_2\|_{\beta_2}\|\chi\|^{r_2}\right).
\end{aligned} \tag{4.10}$$

Moreover, there exists $d_0 > 0$ small enough such that for any $d_\infty \in (0, d_0)$, one has

$$\begin{aligned}
a_\infty^{\frac{\nu+2}{2(\nu-2)}}\left(\frac{1}{d_\infty}\right)^{\frac{2}{\nu-2}}\int_{\mathbb{R}}|Q_\nu|^\nu dt &\geq a_\infty^{\frac{\nu+2}{2(\nu-2)}}\left(\frac{1}{d_\infty}\right)^{\frac{2}{\nu-2}}C_\nu^{-\frac{2\nu}{\nu-2}} \\
&= a_\infty^{\frac{\nu+2}{2(\nu-2)}}\left(\frac{1}{d_\infty}\right)^{\frac{2}{\nu-2}}\left(M_{GN}(\nu)\left(\frac{\nu-2}{\nu+2}\right)^{\frac{\nu-2}{4\nu}}\left(\frac{\nu+2}{2\nu}\right)^{\frac{1}{2}}\right)^{-\frac{2\nu}{\nu-2}} \\
&> \left(\frac{1}{\|d\|_\infty-\varepsilon_0}\right)^{\frac{2}{\nu-2}}(\|a\|_\infty+\varepsilon_0)^{\frac{\nu+2}{2(\nu-2)}}L(\chi).
\end{aligned}$$

By (3.9) and (4.10), there exists $\overline{\lambda}_3 \in (0, \overline{\lambda}_2)$ such that for any $\lambda \in (0, \overline{\lambda}_3)$ and $\varepsilon_0 >$ small enough

$$\begin{aligned}
c_\infty - c &\geq \frac{\nu-2}{2\nu}\left(\frac{1}{\nu}\right)^{\frac{2}{\nu-2}}\left(a_\infty^{\frac{\nu+2}{2(\nu-2)}}\left(\frac{1}{d_\infty}\right)^{\frac{2}{\nu-2}}\int_{\mathbb{R}}|Q_\nu|^\nu dt\right. \\
&\quad \left. - \left(\frac{1}{\|d\|_\infty-\varepsilon_0}\right)^{\frac{2}{\nu-2}}(\|a\|_\infty+\varepsilon_0)^{\frac{\nu+2}{2(\nu-2)}}L(\chi)\right) \\
&\quad - \lambda\left(\frac{\theta_0^{r_1}}{r_1}a_\infty^{-\frac{1}{2\beta_1^*}}C^{r_1}_{r_1\beta_1^*}\|b_1\|_{\beta_1}\|\chi\|^{r_1} + \frac{\theta_0^{r_2}}{r_2}a_\infty^{-\frac{1}{2\beta_2^*}}C^{r_2}_{r_2\beta_2^*}\|b_2\|_{\beta_2}\|\chi\|^{r_2}\right) \\
&> 2\lambda\left(\frac{r_1+\nu}{\nu r_1}C^{r_1}_{r_1\beta_1^*}\|b_1\|_{\beta_1}(4\mathfrak{D})^{r_1} + \frac{r_1+\nu}{\nu r_2}C^{r_2}_{r_2\beta_2^*}\|b_2\|_{\beta_2}(4\mathfrak{D})^{r_2}\right) \\
&\geq 2\lambda\left(\frac{r_1+\nu}{\nu r_1}C^{r_1}_{r_1\beta_1^*}\|b_1\|_{\beta_1}\|v_n\|^{r_1} + \frac{r_1+\nu}{\nu r_2}C^{r_2}_{r_2\beta_2^*}\|b_2\|_{\beta_2}\|v_n\|^{r_2}\right).
\end{aligned}$$

We obtain our conclusion. ∎

4.3 The compactness property

In this section, we show that $\{u_n\}$ converges to a nontrivial solution for problem (1.1). We will utilize the concentration-compactness principle by P. L. Lions [20] to obtain the compactness.

Lemma 4.11 (See [20, Lemma1.1])**.** *Let $\{\rho_n\}$ be a sequence of nonnegative L^1 functions on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \rho_n(t)dt = \kappa$, where κ is a fixed constant. Then there exists a subsequence which we still denote by $\{\rho_n\}$, satisfying one of the three following possibilities:*

(i) (Vanishing): for all $R > 0$, it follows

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{B_R(y)} \rho_n dt = 0;$$

(ii) (Compactness): there exists $\{y_n\} \subset \mathbb{R}$ such that, for any $\varepsilon > 0$, there exists $R > 0$ satisfying

$$\int_{B_R(y_n)} \rho_n dt \geq \kappa - \varepsilon;$$

(iii) (Dichotomy): there exist $\alpha \in (0, \kappa), \rho_n^1 \geq 0, \rho_n^2 \geq 0$, and $\rho_n^1, \rho_n^2 \in L^1(\mathbb{R})$ such that

- *(a)* $\left\|\rho_n - (\rho_n^1 + \rho_n^2)\right\|_{L^1} \to 0$ *as* $n \to \infty$;
- *(b)* $\int_{\mathbb{R}} \rho_n^1 dt \to \alpha$ *as* $n \to \infty$;
- *(c)* $\int_{\mathbb{R}} \rho_n^2 dt \to \kappa - \alpha$ *as* $n \to \infty$;
- *(d)* $\operatorname{dist}\left(\operatorname{supp} \rho_n^1, \operatorname{supp} \rho_n^2\right) \to \infty$ *as* $n \to \infty$.

Lemma 4.12 (See [21])**.** *Let $\{u_n\}$ be bounded sequence in $L^q(\mathbb{R})$ for $1 \leq q < +\infty$ such that $\{\dot{u}_n\}$ is bounded in $L^p(\mathbb{R})$ for $1 < p \leq +\infty$. If there exists $R > 0$ such that*

$$\sup_{y\in\mathbb{R}} \int_{B_R(y)} |u_n|^q dt \to 0 \quad \text{as } n \to \infty,$$

then $u_n \to 0$ in $L^r(\mathbb{R})$ for all $r \in (q, +\infty)$.

First, we show the boundedness of $\|u_n\|$. It follows from (4.4), (4.5), (4.7), (4.8) and (4.10) that

$$\begin{aligned}
&\nu c + o(1) \\
&\quad \geq \nu I(u_n) - \langle I'(u_n), u_n \rangle \\
&\quad = \left(\frac{\nu}{2} - 1\right) \int_{\mathbb{R}} \left(|\dot{u}_n|^2 + a(t)|u_n|^2\right) dt - \lambda \int_{\mathbb{R}} \left((\nabla F(t, u_n), u_n) - \nu F(t, u_n)\right) dt \\
&\quad \geq \min\{1, a_0\} \left(\frac{\nu}{2} - 1\right) \|u_n\|^2 - \lambda \left(\frac{r_1 + \nu}{r_1} C_{r_1\beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|u_n\|^{r_1} + \frac{r_2 + \nu}{r_2} C_{r_2\beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|u_n\|^{r_2} \right).
\end{aligned}$$

Hence there exists $\mathfrak{D} > 0$ such that

$$\|u_n\| \leq \mathfrak{D} \quad \text{for all } n \in \mathbb{N}. \tag{4.11}$$

Without loss of generality, we assume that

$$\lim_{n\to\infty} \|u_n\| = \sqrt{\kappa}. \tag{4.12}$$

We have that $\kappa > 0$. If not, assuming by contradiction that $\|u_n\| \to 0$, there will be a contradiction. It follows from $\|u_n\| \to 0$ that $\|u_n\|_\infty \to 0$. It is easy to see that

$$\left|\lambda \int_{\mathbb{R}} F(t,u_n)dt + \int_{\mathbb{R}} d(t)|u_n|^\nu dt\right| dt \to 0 \quad \text{as } n \to \infty,$$

which contradicts to (4.7). Then (4.12) holds.

Lemma 4.13. *The sequence $\{u_n\}$ converges to a nontrivial function u_0 in $H^1(\mathbb{R},\mathbb{R}^2)$, which is the homoclinic solution for systems (1.1).*

Proof. In order to prove this lemma, we consider three cases of behavior for $\{u_n\}$, which are classified in Lemma 4.11. Set $\rho_n(t) = |\dot{u}_n(t)|^2 + |u_n(t)|^2$. The proof is divided into three steps.

Step 1: Vanishing does not occur.

Suppose by contradiction, for all $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{B_R(y)} \rho_n dt = 0.$$

We deduce from Lemma 4.12 that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |u_n|^\nu dt = 0. \tag{4.13}$$

By (4.8), for n large enough, we can conclude that

$$\int_{\mathbb{R}} \left(|\dot{u}_n|^2 + a(t)|u_n|^2\right) dt \le \lambda \int_{\mathbb{R}} (\nabla F(t,u_n),u_n)dt + \nu \int_{\mathbb{R}} d(t)|u_n|^\nu dt + \frac{1}{2}c. \tag{4.14}$$

It follows from (4.4), (4.5), (4.6), (4.7), (4.8), (4.13) and (4.14) that there exists $\overline{\lambda}_4 \in (0,\overline{\lambda}_3)$ such that for any $\lambda \in (0,\overline{\lambda}_4)$

$$\begin{aligned}
0 &< \frac{1}{2}c \\
&\le I(u_n) \\
&= \frac{1}{2}\int_{\mathbb{R}} \left(|\dot{u}_n|^2 + a(t)|u_n|^2\right) dt - \lambda \int_{\mathbb{R}} F(t,u_n)dt - \int_{\mathbb{R}} d(t)|u_n|^\nu dt \\
&\le \frac{1}{2}\left(\lambda \int_{\mathbb{R}} (\nabla F(t,u_n),u_n)dt + \nu \int_{\mathbb{R}} d(t)|u_n|^\nu dt + \frac{1}{2}c\right) - \lambda \int_{\mathbb{R}} F(t,u_n)dt - \int_{\mathbb{R}} d(t)|u_n|^\nu dt \\
&\le \lambda \left(\frac{r_1+2}{2r_1}\|b_1\|_{\beta_1}\|u_n\|_{r_1\beta_1^*}^{r_1} + \frac{r_1+2}{2r_2}\|b_2\|_{\beta_2}\|u_n\|_{r_2\beta_2^*}^{r_2}\right) + \|d\|_\infty \left(\frac{\nu}{2}-1\right)\int_{\mathbb{R}} |u_n|^\nu dt + \frac{1}{4}c \\
&\to \frac{1}{4}c \quad \text{as } n \to \infty,
\end{aligned}$$

which is a contradiction. Then we see that vanishing case does not occur.

Step 2: Dichotomy does not occur.

There exist $R_0 > 0$ and sequences $\{y_n\} \subset \mathbb{R}, \{R_n\} \subset \mathbb{R}^+$, with $R_0 < R_1 < \cdots < R_n < R_{n+1} \to \infty$, $\Omega_n = B_{R_n}(y_n) \setminus B_{R_0}(y_n)$ such that

$$\int_{\Omega_n} \rho_n dt \to 0, \quad \int_{B_{R_0}(y_n)} \rho_n dt \to \alpha \quad \text{and} \quad \int_{\mathbb{R}\setminus B_{2R_n}(y_n)} \rho_n dt \to \kappa - \alpha \tag{4.15}$$

as $n \to \infty$. Set $\xi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ with $0 \le \xi \le 1$, $\xi(s) \equiv 1$ for $s \le 1$; $\xi(s) \equiv 0$ for $s \ge 2$ and $|\dot{\xi}(s)| \le 2$. Let

$$v_n(t) = \xi\left(\frac{|t-y_n|}{R_0}\right)u_n(t) \quad \text{and} \quad w_n(t) = \left(1 - \xi\left(\frac{|t-y_n|}{R_n}\right)\right)u_n(t).$$

On one hand, we can easily deduce that

$$\begin{aligned}
\|w_n\|^2 &= \int_{\mathbb{R}} |\dot{w}_n|^2 dt + \int_{\mathbb{R}} |w_n|^2 dt \\
&= \int_{\mathbb{R}} \left(\frac{1}{R_n^2} \left| \dot{\xi}\left(\frac{|t-y_n|}{R_n}\right) u_n \right|^2 + \left| \left(1 - \xi\left(\frac{|t-y_n|}{R_n}\right)\right) \dot{u}_n \right|^2 \right) dt \\
&\quad - \frac{2}{R_n} \int_{\mathbb{R}} \dot{\xi}\left(\frac{|t-y_n|}{R_n}\right)\left(1 - \xi\left(\frac{|t-y_n|}{R_n}\right)\right)(u_n, \dot{u}_n) dt + \int_{\mathbb{R}} \left| \left(1 - \xi\left(\frac{|t-y_n|}{R_n}\right)\right) u_n \right|^2 dt \\
&\ge \int_{\mathbb{R} \setminus B_{2R_n}(y_n)} \rho_n dt - \frac{2}{R_n} \|u_n\|^2,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|w_n\|^2 \ge \kappa - \alpha.$$

On the other hand, it can be easily deduce from (4.15) that

$$\int_{\Omega_n} d(t)|u_n|^\nu dt \to 0, \quad \int_{\mathbb{R}} [(\dot{u}_n, \dot{w}_n) - |\dot{w}_n|^2] dt \to 0 \quad \text{as } n \to \infty, \tag{4.16}$$

and

$$\int_{B_{R_n}(y_n)} (\nabla F(t, u_n), w_n) dt \to 0, \quad \int_{B_{R_n}(y_n)} (\nabla F(t, w_n), w_n) dt \to 0 \quad \text{as } n \to \infty. \tag{4.17}$$

Then one has

$$\begin{aligned}
\|w_n\|^2 &= \int_{\mathbb{R}} \left(\frac{1}{R_n^2} \left| \dot{\xi}\left(\frac{|t-y_n|}{R_n}\right) u_n \right|^2 + \left| \left(1 - \xi\left(\frac{|t-y_n|}{R_n}\right)\right) \dot{u}_n \right|^2 \right) dt \\
&\quad - \frac{2}{R_n} \int_{\mathbb{R}} \dot{\xi}\left(\frac{|t-y_n|}{R_n}\right)\left(1 - \xi\left(\frac{|t-y_n|}{R_n}\right)\right)(u_n, \dot{u}_n) dt + \int_{\mathbb{R}} \left| \left(1 - \xi\left(\frac{|t-y_n|}{R_n}\right)\right) u_n \right|^2 dt \\
&\le \frac{4}{R_n^2} \int_{\Omega_n} |u_n|^2 dt + \frac{2}{R_n} \|u_n\|^2 + \int_{\Omega_n} \rho_n dt + \int_{\mathbb{R} \setminus B_{2R_n}(y_n)} \rho_n dt,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|w_n\|^2 = \kappa - \alpha. \tag{4.18}$$

Subsequently, for any $u \in H^1(\mathbb{R}, \mathbb{R}^2)$ and $t \in \mathbb{R}$, set

$$G(t, u) = |\dot{u}|^2 + a(t)|u|^2 - \lambda(\nabla F(t, u), u) - \nu d(t)|u|^\nu.$$

Hence, it follows from the definition of v, w and (4.16) that

$$\begin{aligned}
&\int_{\mathbb{R}} |F(t,u_n) - F(t,v_n) - F(t,w_n)|\, dt \\
&= \int_{\Omega_n} |F(t,u_n) - F(t,v_n) - F(t,w_n)|\, dt \\
&\le \int_{\Omega_n} |F(t,u_n)| dt + \int_{\Omega_n} |F(t,v_n)| dt + \int_{\Omega_n} |F(t,w_n)| dt \\
&\le \frac{1}{r_1}\int_{\Omega_n} b_1(t)|u_n|^{r_1} dt + \frac{1}{r_2}\int_{\Omega_n} b_2(t)|u_n|^{r_2} dt \\
&\quad + \frac{1}{r_1}\int_{\Omega_n} b_1(t)|v_n|^{r_1} dt + \frac{1}{r_2}\int_{\Omega_n} b_2(t)|v_n|^{r_2} dt \\
&\quad + \frac{1}{r_1}\int_{\Omega_n} b_1(t)|w_n|^{r_1} dt + \frac{1}{r_2}\int_{\Omega_n} b_2(t)|w_n|^{r_2} dt \\
&\le \frac{3}{r_1}\int_{\Omega_n} b_1(t)|u_n|^{r_1} dt + \frac{3}{r_2}\int_{\Omega_n} b_2(t)|u_n|^{r_2} dt \\
&\le 3\left(\frac{\|b_1\|_{\beta_1}}{r_1} + \frac{\|b_2\|_{\beta_2}}{r_2}\right)\left(\left(\int_{\Omega_n} |u_n|^{r_1\beta_1^*} dt\right)^{\frac{1}{\beta_1^*}} + \left(\int_{\Omega_n} |u_n|^{r_2\beta_2^*} dt\right)^{\frac{1}{\beta_2^*}}\right) \\
&\to 0 \quad \text{as } n \to \infty
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
\left|\int_{\mathbb{R}} d(t)\, ||u_n|^{\nu} - |v_n|^{\nu} - |w_n|^{\nu}|\, dt\right| &= \int_{\Omega_n} d(t)\, ||u_n|^{\nu} - |v_n|^{\nu} - |w_n|^{\nu}|\, dt \\
&\le 3\|d\|_\infty \|u_n\|_\infty^{\nu-2}\int_{\Omega_n} |u_n|^2 dt \\
&\le 3 \times 2^{-\frac{\nu-2}{2}} \mathfrak{D}^{\nu-2}\|d\|_\infty \int_{\Omega_n} |u_n|^2 dt \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{4.20}$$

Furthermore, we can deduce that

$$\begin{aligned}
|\|u_n\|^2 - \|v_n\|^2 - \|w_n\|^2| \le \int_{\Omega_n} ||\dot{u}_n|^2 - |\dot{v}_n|^2 - |\dot{w}_n|^2|\, dt + \int_{\Omega_n} ||u_n|^2 - |v_n|^2 - |w_n|^2|\, dt \\
\to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{4.21}$$

Together with (4.19), (4.20) and (4.21), we have

$$I(u_n) \ge I(v_n) + I(w_n) - o(1). \tag{4.22}$$

The discussion for this step is divided into two cases.

Case 1. $\{y_n\} \subset \mathbb{R}$ is bounded.

First, we show the following claim.

Claim 1: $I(w_n) \ge c_\infty - o(1)$.
By $(V2)$, for any $\varepsilon > 0$, there exists $r_\infty > 0$ such that

$$|a(t) - a_\infty| \le \varepsilon$$

for all $|t| \geq r_\infty$. Since $\{y_n\}$ is bounded, then there exists $\overline{y} > \underline{y} > 0$ such that $\{y_n\} \subset [\underline{y}, \overline{y}]$ for all $n \in \mathbb{N}$ and $\min\{R_n - \overline{y}, R_n + \underline{y}\} \to +\infty$ as $n \to \infty$. By the definition of w_n, for n large enough, we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}} (a_\infty - a(t))|w_n|^2 dt\right| &\leq \int_{\mathbb{R}\setminus B_{R_n}(y_n)} |a_\infty - a(t)| \left(1 - \xi\left(\frac{|t-y_n|}{R_n}\right)\right)^2 |u_n|^2 dt \\ &\leq \left(\int_{-\infty}^{\overline{y}-R_n} + \int_{\underline{y}+R_n}^{+\infty}\right) |a_\infty - a(t)||u_n|^2 dt \\ &\leq 2\varepsilon \int_{\mathbb{R}} |u_n|^2 dt \\ &\leq 2\varepsilon \mathfrak{D}^2.\end{aligned}$$

By the arbitrariness of ε, we can see that

$$\left|\int_{\mathbb{R}} (a_\infty - a(t))|w_n|^2 dt\right| \to 0 \quad \text{as } n \to \infty. \tag{4.23}$$

Similarly, we have

$$\left|\int_{\mathbb{R}} (d_\infty - d(t))|w_n|^\nu dt\right| \to 0 \quad \text{as } n \to \infty. \tag{4.24}$$

Moreover, we have

$$\begin{aligned}\left|\int_{\mathbb{R}} F(t, w_n) dt\right| &\leq \frac{1}{r_1}\int_{\mathbb{R}} b_1(t)|w_n|^{r_1} dt + \frac{1}{r_2}\int_{\mathbb{R}} b_2(t)|w_n|^{r_2} dt \\ &\leq \frac{1}{r_1}\left(\int_{\mathbb{R}\setminus B_{R_n}(y_n)} |b_1|^{\beta_1} dt\right)^{\frac{1}{\beta_1}} \left(\int_{\mathbb{R}\setminus B_{R_n}(y_n)} |w_n|^{r_1\beta_1^*} dt\right)^{\frac{1}{\beta_1^*}} \\ &\quad + \frac{1}{r_2}\left(\int_{\mathbb{R}\setminus B_{R_n}(y_n)} |b_2|^{\beta_2} dt\right)^{\frac{1}{\beta_2}} \left(\int_{\mathbb{R}\setminus B_{R_n}(y_n)} |w_n|^{r_2\beta_2^*} dt\right)^{\frac{1}{\beta_2^*}} \\ &\leq \frac{1}{r_1}\left(\left(\int_{-\infty}^{\overline{y}-R_n} + \int_{\underline{y}+R_n}^{+\infty}\right) |b_1|^{\beta_1} dt\right)^{\frac{1}{\beta_1}} \left(\int_{\mathbb{R}\setminus B_{R_n}(y_n)} |u_n|^{r_1\beta_1^*} dt\right)^{\frac{1}{\beta_1^*}} \\ &\quad + \frac{1}{r_2}\left(\left(\int_{-\infty}^{\overline{y}-R_n} + \int_{\underline{y}+R_n}^{+\infty}\right) |b_2|^{\beta_2} dt\right)^{\frac{1}{\beta_2}} \left(\int_{\mathbb{R}\setminus B_{R_n}(y_n)} |u_n|^{r_2\beta_2^*} dt\right)^{\frac{1}{\beta_2^*}} \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{4.25}$$

Similarly,

$$\left|\int_{\mathbb{R}} (\nabla F(t, w_n), w_n) dt\right| \to 0 \quad \text{as } n \to \infty. \tag{4.26}$$

Combining (4.23) and (4.25), we can obtain

$$I(w_n) \geq I_\infty(w_n) - o(1). \tag{4.27}$$

It follows from (4.23), (4.24), (4.26) that

$$\begin{aligned}&|\langle I'(w_n), w_n\rangle - \langle I'_\infty(w_n), w_n\rangle| \\ &\quad \leq \int_{\mathbb{R}} |a_\infty - a(t)||w_n|^2 dt + \lambda \int_{\mathbb{R}} |(\nabla F(t, w_n), w_n)|\, dt + \nu \int_{\mathbb{R}} |d_\infty - d(t)||w_n|^\nu dt \\ &\quad \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{4.28}$$

We can also infer from (4.16), (4.17) that

$$\begin{aligned}
&\left|\langle I'(u_n), w_n\rangle - \langle I'(w_n), w_n\rangle\right| \\
&\quad\le \int_{\mathbb{R}} [(\dot{u}_n, \dot{\omega}_n) - |\dot{\omega}_n|^2]dt + \lambda \left(\left| \int_{\Omega_n} (\nabla F(t, \omega_n), \omega_n) dt \right| + \left| \int_{\Omega_n} (\nabla F(t, u_n), \omega_n) dt \right| \right) \\
&\qquad + \nu \|d\|_\infty \left(\left| \int_{\Omega_n} (1-\xi)^\nu |\omega_n|^\nu dt \right| + \left| \int_{\Omega_n} (1-\xi) |u_n|^\nu dt \right| \right) \\
&\quad\to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{4.29}$$

Together with (4.8), (4.18), (4.28) and (4.29), one has

$$\langle I'_\infty(w_n), w_n\rangle \to 0 \quad \text{as } n \to \infty. \tag{4.30}$$

It follows from (4.18) that

$$\int_{\mathbb{R}} (|\dot{w}_n|^2 + a_\infty |w_n|^2) dt \ge \frac{\min\{1, a_\infty\}}{2} (\kappa - \alpha) > 0$$

for n large enough. Letting

$$A_n = \frac{\langle I'_\infty(w_n), w_n\rangle}{\int_{\mathbb{R}} (|\dot{w}_n|^2 + a_\infty |w_n|^2) dt}$$

and

$$\sigma_n = \left(\frac{1}{1 - A_n} \right)^{\frac{1}{\nu-2}},$$

we deduce that $A_n \to 0$ and $\sigma_n \to 1$ as $n \to \infty$. Setting $z_n = \sigma_n w_n(t)$, we have

$$\begin{aligned}
\langle I'_\infty(z_n), z_n\rangle &= \sigma_n^2 \left(\int_{\mathbb{R}} |w_n|^2 dt + \int_{\mathbb{R}} a_\infty |w_n|^2 dt - \sigma_n^{\nu-2} \nu d_\infty \int_{\mathbb{R}} |w_n|^\nu dt \right) \\
&= \sigma_n^2 \left(1 - \sigma_n^{\nu-2}(1 - A_n)\right) \left(\int_{\mathbb{R}} |w_n|^2 dt + \int_{\mathbb{R}} a_\infty |w_n|^2 dt \right) \\
&= 0,
\end{aligned}$$

which implies $z_n \in \mathcal{N}$. Furthermore, we have

$$\begin{aligned}
I_\infty(z_n) &= \frac{\sigma_n^2}{2} \left(\int_{\mathbb{R}} |w_n|^2 dt + \int_{\mathbb{R}} a_\infty |w_n|^2 dt \right) - \sigma_n^\nu d_\infty \int_{\mathbb{R}} |w_n|^\nu dt \\
&= \frac{\sigma_n^2 - \sigma_n^\nu}{2} \left(\int_{\mathbb{R}} |w_n|^2 dt + \int_{\mathbb{R}} a_\infty |w_n|^2 dt \right) + \sigma_n^\nu I_\infty(w_n) \\
&\ge c_\infty,
\end{aligned}$$

which implies

$$\begin{aligned}
I_\infty(w_n) &\ge \frac{\sigma_n^\nu - \sigma_n^2}{2\sigma_n^\nu} \left(\int_{\mathbb{R}} |w_n|^2 dt + \int_{\mathbb{R}} a_\infty |w_n|^2 dt \right) + \frac{1}{\sigma_n^\nu} c_\infty \\
&\ge c_\infty - o(1).
\end{aligned}$$

By (4.27), we can finish the proof of **Claim 1**.

Similar to (4.28), (4.29) and (4.30), we get $\langle I'(v_n), v_n\rangle \to 0$ as $n \to \infty$. By the definition of v_n and (4.11), we have

$$\|v_n\| \le 4\|u_n\| \le 4\mathfrak{D}.$$

Therefore,

$$\begin{aligned}
I(v_n) &= \frac{1}{2}\int_{\mathbb{R}} |\dot{v}_n|^2 dt + \frac{1}{2}\int_{\mathbb{R}} a(t)|v_n|^2 dt - \lambda \int_{\mathbb{R}} F(t, v_n) dt - \int_{\mathbb{R}} d(t)|v_n|^\nu dt \\
&= \frac{1}{2}\left(\int_{\mathbb{R}} |\dot{v}_n|^2 dt + \int_{\mathbb{R}} a(t)|v_n|^2 dt\right) - \lambda \int_{\mathbb{R}} F(t, v_n) dt \\
&\quad - \frac{1}{\nu}\left(\left(\int_{\mathbb{R}} |\dot{v}_n|^2 dt + \int_{\mathbb{R}} a(t)|v_n|^2 dt\right) - \lambda \int_{\mathbb{R}} (\nabla F(t, v_n), v_n)\, dt - \langle I'(v_n), v_n\rangle\right) \\
&\geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\left(\int_{\mathbb{R}} |\dot{v}_n|^2 dt + \int_{\mathbb{R}} a(t)|v_n|^2 dt\right) \\
&\quad + \lambda\left(\frac{1}{\nu}\int_{\mathbb{R}} (\nabla F(t, v_n), v_n)\, dt - \int_{\mathbb{R}} F(t, v_n) dt\right) + o(1) \\
&\geq -\lambda\left(\frac{r_1+\nu}{\nu r_1} C_{r_1\beta_1^*}^{r_1}\|b_1\|_{\beta_1}\|v_n\|^{r_1} + \frac{r_1+\nu}{\nu r_2} C_{r_2\beta_2^*}^{r_2}\|b_2\|_{\beta_2}\|v_n\|^{r_2}\right) + o(1).
\end{aligned} \tag{4.31}$$

It follows from (4.7), (4.22), (4.31) and **Claim 1** that

$$\lambda\left(\frac{r_1+\nu}{\nu r_1} C_{r_1\beta_1^*}^{r_1}\|b_1\|_{\beta_1}\|v_n\|^{r_1} + \frac{r_1+\nu}{\nu r_2} C_{r_2\beta_2^*}^{r_2}\|b_2\|_{\beta_2}\|v_n\|^{r_2}\right) \geq c_\infty - c - o(1). \tag{4.32}$$

This is an obvious contradiction to Lemma 4.10 when $\lambda > 0$ and $d_\infty > 0$ are small enough. Then the dichotomy does not occur when $\{y_n\}$ is bounded.

Case 2: $\{y_n\} \subset \mathbb{R}$ is unbounded. Then, passing to a subsequence if necessary, we can assume that $|y_n| \to \infty$ as $n \to \infty$. In this case, we can choose a suitable sequence $\{R_n\} \subset \mathbb{R}$ such that $R_n \pm y_n \to +\infty$ as $n \to \infty$ and arguing similarly as above. Then we conclude that dichotomy does not occur when $\{y_n\}$ is unbounded.

Step 3: Compactness.

It can be see from Theorem 4.1 that there exists $\{y_n\} \subset \mathbb{R}$ such that, for any $\varepsilon > 0$, there exists $R_1 > 0$ satisfying

$$\int_{B_{R_1}(y_n)} \rho_n dt \geq \kappa - \varepsilon. \tag{4.33}$$

Since $\int_{\mathbb{R}} \rho_n dt = \kappa$, then we have

$$\int_{\mathbb{R}\setminus B_{R_1}(y_n)} \rho_n dt \leq \varepsilon$$

for all $n \in N$. If $\{y_n\}$ is unbounded, similar to the arguments in *Step* 2, we can obtain a contradiction. Then we conclude that $\{y_n\}$ is bounded. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}, \mathbb{R}^2)$, there exists u_0 in H^1 such that $u_n \rightharpoonup u_0$. It follows from the continuity of the embedding $H^1(\mathbb{R}, \mathbb{R}^2) \hookrightarrow L^\nu(\mathbb{R}, \mathbb{R}^2)$ for any $\nu \in [2, +\infty]$ that there exists $R_2 > 0$ such that

$$\int_{\mathbb{R}\setminus B_{R_2}(0)} |u_n|^\nu dt \leq \varepsilon \quad \text{and} \quad \int_{\mathbb{R}\setminus B_{R_2}(0)} |u_0|^\nu dt \leq \varepsilon. \tag{4.34}$$

It is clear that $u_n \to u_0$ in $L^\nu(B_{R_2}(0), \mathbb{R}^2)$ and it follows from (4.33) and (4.34) that

$$\begin{aligned}
\int_{\mathbb{R}} |u_n - u_0|^\nu dt &= \int_{B_{R_2}(0)} |u_n - u_0|^\nu dt + \int_{\mathbb{R}\setminus B_{R_2}(0)} |u_n - u_0|^\nu dt \\
&\leq \int_{B_{R_2}(0)} |u_n - u_0|^\nu dt + 2^{\nu-1}\left(\int_{\mathbb{R}\setminus B_{R_2}(0)} |u_0|^\nu dt + \int_{\mathbb{R}\setminus B_{R_2}(0)} |u_n|^\nu dt\right) \\
&\leq \int_{B_{R_2}(0)} |u_n - u_0|^\nu dt + 2^\nu \varepsilon,
\end{aligned}$$

which implies that

$$u_n \to u_0 \quad \text{as } n \to \infty \quad \text{in } L^\nu(\mathbb{R}) \quad \text{for any } \nu \in [2, +\infty).$$

On one hand, by Lebesgue Dominated Convergence Theorem, we can deduce that

$$\int_{\mathbb{R}} |u_n|^{\nu-2}(u_n, u_0)dt \to \int_{\mathbb{R}} |u_0|^\nu dt \quad \text{as } n \to \infty.$$

By $\langle I'(u_n), u_0\rangle \to 0$ as $n \to \infty$, we obtain

$$\begin{aligned} o(1) &= \langle I'(u_n), u_n - u_0\rangle \\ &= \|u_n - u_0\|^2 + \int_{\mathbb{R}} (\dot{u}_0, \dot{u}_n - \dot{u}_0)dt + \int_{\mathbb{R}} a(t)(u_0, u_n - u_0)dt - \int_{\mathbb{R}} |u_n - u_0|^2 dt \\ &\quad - \lambda \int_{\mathbb{R}} (\nabla F(t, u_n), u_n - u_0)dt - \nu \int_{\mathbb{R}} d(t)|u_n|^{\nu-2}(u_n, u_n - u_0)dt. \end{aligned} \tag{4.35}$$

On one hand, for $i = 1, 2$, set

$$\Delta_{i,1} = (1, 2], \qquad \Delta_{i,2} = \left(\frac{2}{3 - r_i}, \frac{2}{2 - r_i}\right).$$

It is easy to see that $\left(1, \frac{2}{2-r_i}\right) = \Delta_{i,1} \bigcup \Delta_{i,2}$ and $\Delta_{i,1} \bigcap \Delta_{i,2} \neq \emptyset$. Hence, we deduce that there exists $\eta_i \in [2, +\infty)$ such that $\frac{1}{\beta_i} + \frac{r_i - 1}{\xi_i} + \frac{1}{\eta_i} = 1$. Moreover, let

$$\xi_i = \begin{cases} +\infty & \text{if } \beta_i \in \Delta_{i,1}, \\ 2 & \text{if } \beta_i \in \Delta_{i,2} \setminus \Delta_{i,1}. \end{cases}$$

By $(V5)$, we show

$$\begin{aligned} \int_{\mathbb{R}} (\nabla F(t, u_n), u_n - u_0)dt &\leq \int_{\mathbb{R}} \sum_{i=1,2} b_i(t)(|u_n|^{r_i - 1} + |u_0|^{r_i - 1})dt \\ &\leq \sum_{i=1,2} \|b_i\|_{\beta_i} (\|u_n\|_{\xi_i}^{r_i - 1} + \|u_0\|_{\xi_i}^{r_i - 1}) \|u_n - u_0\|_{\eta_i} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

On the other hand, it is easy to see

$$\begin{aligned} \left| \int_{\mathbb{R}} d(t)|u_n|^{\nu-2}(u_n, u_n - u_0)dt \right| &\leq \|d\|_\infty \int_{\mathbb{R}} |u_n|^{\nu-1} |u_n - u_0| \, dt \\ &\leq \|d\|_\infty \|u_n\|_{2(\nu-1)}^{\nu-1} \|u_n - u_0\|_2^2 \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

We conclude from (4.35) that $\|u_n - u_0\| \to 0$ as $n \to \infty$, which implies that u_0 is a homoclinic solution for problem (1.1). □

4.4 Proof of Theorem 4.1

In this section, we look for the second homoclinic solution corresponding to negative critical value with the following lemma.

Lemma 4.14 (See [22]). *Let X be a real reflexive Banach space and $\Omega \subset X$ be a closed bounded convex subset of X. Suppose that $\varphi : X \to \mathbb{R}$ is a weakly lower semi-continuous (w.l.s.c. for short) functional. If there exists a point $x_0 \in \Omega \setminus \partial\Omega$ such that*

$$\varphi(x) > \varphi(x_0), \qquad \forall\, x \in \partial\Omega$$

then there must be a $x^* \in \Omega \setminus \partial\Omega$ such that

$$\varphi(x^*) = \inf_{x\in\Omega} \varphi(x).$$

It follows from $(V4)$ and $(V6)$ that there exists $\delta > 0$ such that

$$F(t,x) > \frac{1}{2} b_0 |x|^{r_0} \tag{4.36}$$

for all $t \in (\bar{t} - \delta, \bar{t} + \delta)$ and $x \in \mathbb{R}^2$. Choose $\psi \in C_0^\infty((t_0 - \delta, t_0 + \delta), \mathbb{R}^2) \setminus \{0\}$. It follows from (4.36) and $r_0 \in (0,2)$ that

$$\begin{aligned} I(\vartheta\psi) &= \frac{\vartheta^2}{2}\|\psi\|^2 - \lambda \int_{\mathbb{R}} F(t,\psi)dt - \vartheta^\nu \int_{\mathbb{R}} d(t)|\psi|^\nu dt \\ &\le \frac{\vartheta^2}{2}\|\psi\|^2 - \lambda b_0 \vartheta^{r_0} \int_{t_0-\delta}^{t_0+\delta} |\psi|^{r_0} dt - \vartheta^\nu d_\infty \int_{\mathbb{R}} |\psi|^\nu dt \\ &< 0 \end{aligned}$$

for $\vartheta > 0$ small enough. By Lemma 4.14, we can see there exists a critical point of I corresponding to negative critical value. □

Data availability statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Mild solutions, variation of constants formula, and linearized stability for delay differential equations

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Abstract. The method and the formula of variation of constants for ordinary differential equations (ODEs) is a fundamental tool to analyze the dynamics of an ODE near an equilibrium. It is natural to expect that such a formula works for delay differential equations (DDEs), however, it is well-known that there is a conceptual difficulty in the formula for DDEs. Here we discuss the variation of constants formula for DDEs by introducing the notion of a *mild solution*, which is a solution under an initial condition having a discontinuous history function. Then the *principal fundamental matrix solution* is defined as a matrix-valued mild solution, and we obtain the variation of constants formula with this function. This is also obtained in the framework of a Volterra convolution integral equation, but the treatment here gives an understanding in its own right. We also apply the formula to show the principle of linearized stability and the Poincaré–Lyapunov theorem for DDEs, where we do not need to assume the uniqueness of a solution.

Keywords: delay differential equations, discontinuous history functions, fundamental matrix solution, variation of constants formula, principle of linearized stability, Poincaré–Lyapunov theorem.

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Contents

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1 Introduction

Studies concerning with the variation of constants formula for delay differential equations (DDEs) have a long history of over fifty years. Nevertheless, the reason why we try to discuss the variation of constants formula in this paper is that such a consideration gives rise to a conceptual difficulty that is peculiar to the theory of DDEs. Specifically, it is usual to discuss DDEs within the scope of continuous history functions, but a class of discontinuous history functions emerges as initial conditions when we try to obtain the variation of constants formula. In connection with this, a matrix-valued solution having a certain discontinuous matrix-valued function as the initial condition is called the *fundamental matrix solution*. However, it is quite difficult to understand why the solution is called the "fundamental matrix solution" when compared with the theory of ordinary differential equations (ODEs).

This conceptual difficulty has arisen in the theoretical development about the variation of constants formula in the texts [18] and [19] by Jack Hale. In the revised edition [22], the theoretical development is rewritten based on the consideration in [34]. There also exist studies to understand the conceptual difficulty of the variation of constants formula for DDEs within the framework of Functional Analysis (e.g., see [7], [12], and [13]). In this framework, it is essential that the Banach space of continuous functions on closed and bounded interval endowed with the supremum norm is not reflexive, and the theory is constructed by using the so called "sun-star calculus". See [14] for the details. See also [36] for a survey article.

The idea of discussing the variation of constants formula for DDEs in this paper is to define a solution under an initial condition having a discontinuous history function as a *mild solution*. This concept comes from the analogy of the notion of mild solutions of abstract linear evolution equations, and its terminology also originates from this. It can be said that the notion of mild solutions is to elevate the technique to exchange the order of integration to a concept.

The dependence of the derivative $\dot{x}(t)$ of an unknown function x on the past value of x is abstracted to the concept of *retarded functional differential equations* (RFDEs). In this paper, we consider an autonomous linear RFDE

$$\dot{x}(t) = Lx_t \quad (t \geq 0) \tag{1.1}$$

for a continuous linear map $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $n \geq 1$ is an integer, and $r > 0$ is a constant, which are fixed throughout this paper. The derivative of x at 0 is interpreted as the right-hand derivative. We are using the following notations:

- $C([-r,0],\mathbb{K}^n)$ denotes the Banach space of all continuous functions from $[-r,0]$ to $\mathbb{K}^n$ endowed with the supremum norm $\|\cdot\|$. Here a norm $|\cdot|$ on $\mathbb{K}^n$, which is not necessarily the Euclidean norm, is fixed throughout this paper.
- For each $t \geq 0$, $x_t\colon [-r,0] \to \mathbb{K}^n$ is a continuous function defined by

  $$x_t(\theta) := x(t+\theta) \quad (\theta \in [-r,0])$$

  when $x\colon [-r,\infty) \to \mathbb{K}^n$ is continuous. See also Definition 2.1.

In addition to the linear RFDE (1.1), we also consider a non-homogeneous linear RFDE

$$\dot{x}(t) = Lx_t + g(t) \quad (\text{a.e. } t \geq 0) \tag{1.2}$$

for some $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{K}^n)$. Here $\mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{K}^n)$ denotes the linear space of all locally Lebesgue integrable functions from $[0,\infty)$ to $\mathbb{K}^n$ defined almost everywhere. See also the notations given below. We refer the reader to [32] and [30] as references of the theory of Lebesgue integration for scalar-valued functions.

To study these differential equations, the following expression of L by a *Riemann–Stieltjes integral*

$$L\psi = \int_{-r}^{0} \mathrm{d}\eta(\theta)\,\psi(\theta) \tag{1.3}$$

for $\psi \in C([-r,0],\mathbb{K}^n)$ is useful. Here $\eta\colon [-r,0] \to M_n(\mathbb{K})$ is an $n \times n$ matrix-valued function of bounded variation. The above representability is ensured by a corollary of the Riesz representation theorem (see Corollary B.3). It is a useful convention that the domain of definition of η is extended to $(-\infty,0]$ by letting

$$\eta(\theta) := \eta(-r)$$

for $\theta \in (-\infty,-r]$. See Appendix A for the Riemann–Stieltjes integrals with respect to matrix-valued functions. For the use of Riemann–Stieltjes integrals in the context of RFDEs, see [19, Chapters 6 and 7], [34, Chapter 2], [24, Chapter 4], [22, Chapters 6 and 7], and [14, Chapter I], for example.

This paper is organized as follows:

In Section 2, we introduce the notion of a history segment x_t for a discontinuous function $x\colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$. By using this, we also introduce the notion of a mild solution to the linear RFDE (1.1) under an initial condition

$$x_0 = \phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n). \tag{1.4}$$

Here $\mathcal{M}^1([-r,0],\mathbb{K}^n)$ consists of elements of $\mathcal{L}^1([-r,0],\mathbb{K}^n)$ that are defined at 0. Roughly speaking, a function $x\colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ is said to be a mild solution of (1.1) under the initial condition (1.4) if it satisfies

$$x(t) = \phi(0) + L\int_0^t x_s \, \mathrm{d}s \qquad (t \geq 0).$$

Here $\int_0^t x_s \, \mathrm{d}s \in C([-r,0],\mathbb{K}^n)$ is defined by

$$\left(\int_0^t x_s \, \mathrm{d}s\right)(\theta) := \int_0^t x(s+\theta) \, \mathrm{d}s \qquad (\theta \in [-r,0]).$$

See Definitions 2.5 and 2.7 for the details. After proving the existence and uniqueness of a mild solution of the linear RFDE (1.1) under the initial condition (1.4), we define the *principal fundamental matrix solution* of (1.1) as a matrix-valued mild solution $X^L\colon [-r,\infty) \to M_n(\mathbb{K})$ under the initial condition $X_0^L = \hat{I}$. Here $\hat{I}\colon [-r,0] \to M_n(\mathbb{K})$ is a discontinuous function defined by

$$\hat{I}(\theta) := \begin{cases} O & (\theta \in [-r,0)), \\ I & (\theta = 0). \end{cases} \tag{1.5}$$

In Section 3, we derive a differential equation

$$\dot{X}^L(t) = \int_{-t}^0 \mathrm{d}\eta(\theta) \, X^L(t+\theta)$$

satisfied by the principal fundamental matrix solution X^L of (1.1). In the derivation, it is useful to use the notions of *Volterra operator* and *Riemann–Stieltjes convolution*. See Subsection 3.1 for the definitions and Subsection 3.3 for the fundamental properties. The above differential equation is the key to obtain a variation of constants formula.

In Section 4, we consider the non-homogeneous linear RFDE (1.2). To study a mild solution of (1.2) under the initial condition (1.4), we also consider an integral equation

$$x(t) = \phi(0) + L\int_0^t x_s \, \mathrm{d}s + G(t) \qquad (t \geq 0) \tag{1.6}$$

for a continuous function $G\colon [0,\infty) \to \mathbb{K}^n$ with $G(0) = 0$. We show that the above integral equation has a unique solution $x^L(\cdot\,;\phi, G)$ under the initial condition (1.4).

In Section 5, we consider a non-homogeneous linear RFDE

$$\dot{x}(t) = Lx_t + f(t) \qquad (t \geq 0) \tag{1.7}$$

for a continuous function $f\colon [0,\infty) \to \mathbb{K}^n$ to motivate the use of the convolution for locally Riemann integrable functions. We show that the function $x(\cdot\,; f)\colon [-r,\infty) \to \mathbb{K}^n$ defined by $x(\cdot\,; f)_0 = 0$ and

$$x(t; f) := \int_0^t X^L(t-u) f(u)\,\mathrm{d}u \tag{1.8}$$

for $t \geq 0$ is a solution to Eq. (1.7) after developing the results of convolution for locally Riemann integrable functions. See Subsection 5.2 for the developments.

In Section 6, we study the non-homogeneous linear RFDE (1.2) under the initial condition (1.4) and find a variation of constants formula expressed by the principal fundamental matrix solution X^L. For this purpose, we indeed consider the integral equation (1.6) for some continuous function $G\colon [0,\infty) \to \mathbb{K}^n$ with $G(0) = 0$. One of the main results of this paper is that the solution $x^L(\cdot\,; \phi, G)$ of (1.6) under the initial condition (1.4) satisfies

$$x^L(t; \phi, G) = X^L(t)\phi(0) + \left[G^L(t; \phi) + G(t)\right] + \int_0^t \dot{X}^L(t-u)\left[G^L(u; \phi) + G(u)\right]\mathrm{d}u \tag{1.9}$$

for all $t \geq 0$. Here $\dot{X}^L(t)$ denotes the derivative of the locally absolutely continuous function $X^L|_{[0,\infty)}$ at $t \geq 0$ (when it exists), and $G^L(\cdot\,; \phi)\colon [0,\infty) \to \mathbb{K}^n$ is a function determined by the initial history function ϕ. See Subsection 6.2 for the detail of the derivation of the function $G^L(\cdot\,; \phi)$. We note that before we obtain the variation of constants formula (1.9), we show that

$$x^L(t; 0, G) = G(t) + \int_0^t \dot{X}^L(t-u) G(u)\,\mathrm{d}u \tag{1.10}$$

holds for all $t \geq 0$. Then the derivation of (1.9) is performed by defining a function

$$z^L(\cdot\,; \phi)\colon [-r,\infty) \to \mathbb{K}^n$$

by $z^L(\cdot\,; \phi)_0 = 0$ and

$$z^L(t; \phi) := x^L(t; \phi, 0) - X^L(t)\phi(0) \tag{1.11}$$

for $t \geq 0$ and showing that $z := z^L(\cdot\,; \phi)$ satisfies an integral equation

$$z(t) = L\int_0^t z_s\,\mathrm{d}s + G^L(t; \phi) \qquad (t \geq 0), \tag{1.12}$$

because (1.12) shows that

$$z^L(\cdot\,; \phi) = x^L(\cdot\,; 0, G^L(\cdot\,; \phi))$$

holds. Here we need to know the regularity of the function $G^L(\cdot\,; \phi)$, which is discussed in Subsection 6.3.

In Section 7, we discuss the exponential stability of the principal fundamental matrix solution X^L of the linear RFDE (1.1) and the uniform exponential stability of the C_0-semigroup $\left(T^L(t)\right)_{t\geq 0}$ on the Banach space $C([-r,0],\mathbb{K}^n)$ defined by

$$T^L(t)\phi := x^L(\cdot\,; \phi, 0)_t \tag{1.13}$$

for $(t,\phi) \in [0,\infty) \times C([-r,0],\mathbb{K}^n)$. We show that X^L is α-exponentially stable if and only if $\left(T^L(t)\right)_{t\geq 0}$ is uniformly α-exponentially stable. See Theorems 7.3 and 7.4 for the details.

In Section 8, we apply the obtained variation of constants formulas to a proof of the stability part of the principle of linearized stability and Poincaré–Lyapunov theorem for RFDEs. This is indeed an appropriate modification of the proof for ODEs. However, the given proof makes clear the importance of the principal fundamental matrix solution. In the statement, we do not need to assume the uniqueness of a solution. Therefore, this should be compared with the proof relying on the nonlinear semigroup theory.

We have five appendices. In Appendix A, we collect results on Riemann–Stieltjes integrals for matrix-valued functions that are needed for this paper. In Appendix B, we give a proof of the representability of L by a Riemann–Stieltjes integral (1.3) because there does not seem to be any proof of the representability in the literature. In Appendix C, we discuss Gronwall's inequality and its variants used in the context of RFDEs. In Appendix D, we give lemmas that are used in the fixed point argument in this paper. In Appendix E, we continue to discuss the convolution. The contents of this appendix will not be used in this paper, but it will be useful to share the proofs of results on the convolution for matrix-valued locally Lebesgue integrable functions in the literature of RFDEs.

Notations

Throughout this paper, the following notations will be used.

- Let $E = (E, \|\cdot\|)$ be a Banach space. For each subset $I \subset \mathbb{R}$, let $C(I, E)$ denote the linear space of all continuous functions from I to E. When the subset I is a closed and bounded interval, the linear space $C(I, E)$ is considered as the Banach space of continuous functions endowed with the supremum norm $\|\cdot\|$ given by

  $$\|f\| := \sup_{x \in I} \|f(x)\|$$

  for $f \in C(I, E)$.

- For each pair of Banach spaces $E = (E, \|\cdot\|)$ and $F = (F, \|\cdot\|)$, let $\mathcal{B}(E, F)$ denote the linear space of all continuous linear maps (i.e., all bounded linear operators) from E to F. For each $T \in \mathcal{B}(E, F)$, its operator norm is denoted by $\|T\|$. Then $\mathcal{B}(E, F)$ is considered as the Banach space of continuous linear maps endowed with the operator norm. When $F = E$, $\mathcal{B}(E, F)$ is also denoted by $\mathcal{B}(E)$.

- An $n \times n$ matrix $A \in M_n(\mathbb{K})$ is considered as a continuous linear map on the Banach space $\mathbb{K}^n$ endowed with the given norm $|\cdot|$. The operator norm of A is denoted by $|A|$. The linear space $M_n(\mathbb{K})$ of all $n \times n$ matrices is considered as the Banach space of matrices endowed with the operator norm.

- Let $d \geq 1$ be an integer, X be a measurable set of $\mathbb{R}^d$, and $Y = \mathbb{K}^n$ or $M_n(\mathbb{K})$.
  - We say that a function $f \colon X \supset \operatorname{dom}(f) \to Y$ is a *Lebesgue integrable function defined almost everywhere* if (i) $\operatorname{dom}(f)$ is measurable, (ii) $X \setminus \operatorname{dom}(f)$ has measure 0, and (iii) $f|_{\operatorname{dom}(f)} \colon \operatorname{dom}(f) \to Y$ is Lebesgue integrable, i.e., it is measurable and

    $$\|f\|_1 := \int_X |f(x)| \,\mathrm{d}x := \int_{\operatorname{dom}(f)} |f(x)| \,\mathrm{d}x$$

    is finite. We note that the function $\operatorname{dom}(f) \ni x \mapsto |f(x)| \in [0, \infty)$ is also measurable by the continuity of the norm $|\cdot|$, and the above integral is the unsigned Lebesgue integral.

– Let $\mathcal{L}^1(X,Y)$ be the set of all Lebesgue integrable functions from X to Y defined almost everywhere. For $f \in \mathcal{L}^1(X,Y)$, let

$$\int_X f(x)\,\mathrm{d}x := \int_{\mathrm{dom}(f)} f(x)\,\mathrm{d}x.$$

Then one can prove that

$$\left|\int_X f(x)\,\mathrm{d}x\right| \le \int_{\mathrm{dom}(f)} |f(x)|\,\mathrm{d}x = \|f\|_1$$

holds.

– For $f, g \in \mathcal{L}^1(X,Y)$, the addition $f+g \colon X \supset \mathrm{dom}(f) \cap \mathrm{dom}(g) \to Y$ is defined by

$$(f+g)(x) := f(x) + g(x)$$

for $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. Then $f+g \in \mathcal{L}^1(X,Y)$. The scalar multiplication αf for $\alpha \in \mathbb{K}$ is also defined, and it holds that $\alpha f \in \mathcal{L}^1(X,Y)$.

- Let X be an interval of $\mathbb{R}$ and $Y = \mathbb{K}^n$ or $M_n(\mathbb{K})$. Let $\mathcal{L}^1_{\mathrm{loc}}(X,Y)$ be the set of all functions $f \colon X \supset \mathrm{dom}(f) \to Y$ satisfying (i) $\mathrm{dom}(f)$ is measurable, (ii) $X \setminus \mathrm{dom}(f)$ has measure 0, and (iii) for each closed and bounded interval I contained in X, the restriction $f|_I \colon I \supset \mathrm{dom}(f) \cap I \to Y$ belongs to $\mathcal{L}^1(I,Y)$.

2 Mild solutions and fundamental matrix solutions

2.1 Definitions

2.1.1 History segments and memory space

We first make clear the notion of history segments in our setting.

Definition 2.1. Let $x \colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ be a function. For each $t \ge 0$, we define a function $x_t \colon [-r,0] \supset \mathrm{dom}(x_t) \to \mathbb{K}^n$ by

$$\mathrm{dom}(x_t) := \{\theta \in [-r,0] : t+\theta \in \mathrm{dom}(x)\},$$
$$x_t(\theta) := x(t+\theta) \qquad (\theta \in \mathrm{dom}(x_t)).$$

We call x_t the *history segment* of x at t.

We note that $\mathrm{dom}(x_t)$ is expressed by

$$\mathrm{dom}(x_t) = (\mathrm{dom}(x) - t) \cap [-r,0],$$

where $\mathrm{dom}(x)$ is not necessarily equal to $[-r,\infty)$.

In this paper, we need discontinuous initial history functions. For this purpose, we adopt the following space of history functions.

Definition 2.2 (cf. [10])**.** We define a linear subspace $\mathcal{M}^1([-r,0],\mathbb{K}^n)$ of $\mathcal{L}^1([-r,0],\mathbb{K}^n)$ by

$$\mathcal{M}^1([-r,0],\mathbb{K}^n) := \left\{\phi \in \mathcal{L}^1([-r,0],\mathbb{K}^n) : 0 \in \mathrm{dom}(\phi)\right\}$$

and call it the *memory space* of $\mathcal{L}^1$-type. We consider $\mathcal{M}^1([-r,0],\mathbb{K}^n)$ as a seminormed space endowed with the seminorm $\|\cdot\|_{\mathcal{M}^1} \colon \mathcal{M}^1([-r,0],\mathbb{K}^n) \to [0,\infty)$ defined by

$$\|\phi\|_{\mathcal{M}^1} := \|\phi\|_1 + |\phi(0)|.$$

Remark 2.3. Let $1 \le p < \infty$ and E be a Banach space. The memory space of $\mathcal{L}^1$-type should be compared with a Banach space $M^p([-r,0],E)$ introduced by Delfour and Mitter [10]. It is isomorphic to the product Banach space

$$L^p([-r,0],E) \oplus E.$$

See also [3], [8], and references therein for the use of the product space.

Definition 2.4. For each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, we will call a function $x\colon [-r,\infty) \supset \operatorname{dom}(x) \to \mathbb{K}^n$ a *continuous prolongation* of ϕ if it satisfies the following properties: (i) $x_0 = \phi$, (ii) $[0,\infty) \subset \operatorname{dom}(x)$, and (iii) $x|_{[0,\infty)}$ is continuous.

For a continuous prolongation x of ϕ,

$$\operatorname{dom}(x) = \operatorname{dom}(\phi) \cup [0,\infty)$$

holds.

2.1.2 Mild solutions

The following is the notion of a mild solution, whose introduction is one of the contributions of this paper. We use the expression of L by the Riemann–Stieltjes integral (1.3)

$$L\psi = \int_{-r}^{0} \mathrm{d}\eta(\theta)\,\psi(\theta)$$

for $\psi \in C([-r,0],\mathbb{K}^n)$.

Definition 2.5 (cf. [38])**.** Let $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ be given. We say that a function $x\colon [-r,\infty) \supset \operatorname{dom}(x) \to \mathbb{K}^n$ is a *mild solution* of the linear RFDE (1.1) under the initial condition $x_0 = \phi$ if the following conditions are satisfied: (i) x is a continuous prolongation of ϕ and (ii) for all $t \ge 0$,

$$x(t) = \phi(0) + \int_{-r}^{0} \mathrm{d}\eta(\theta) \left(\int_0^t x(s+\theta)\,\mathrm{d}s \right) \tag{2.1}$$

holds. Here $\int_0^t x(s+\theta)\,\mathrm{d}s$ is a Lebesgue integral.

Since

$$\int_0^t x(s+\theta)\,\mathrm{d}s = \int_{\theta}^{t+\theta} x(s)\,\mathrm{d}s,$$

the integrand in Eq. (2.1) is continuous with respect to $\theta \in [-r,0]$. Therefore, the integral in (2.1) is meaningful as a Riemann–Stieltjes integral. Eq. (2.1) is also expressed by

$$x(t) = \phi(0) + \int_{-r}^{0} \mathrm{d}\eta(\theta) \left(\int_{\theta}^{0} \phi(s)\,\mathrm{d}s \right) + \int_{-r}^{0} \mathrm{d}\eta(\theta) \left(\int_0^{t+\theta} x(s)\,\mathrm{d}s \right), \tag{2.2}$$

where the third term of the right-hand side may depend on ϕ.

Remark 2.6. Eq. (2.1) appeared at [38, (5.19) in Corollary 5.13] after developing a nonlinear semigroup theory for some class of RFDEs. Compared with this approach, the method of this paper is considered to be taking the notion of mild solutions as a starting point.

2.1.3 Notation $\int_0^t x_s \, \mathrm{d}s$

For ease of notation, we introduce the following.

Definition 2.7. Let $x \in \mathcal{L}^1_{\mathrm{loc}}([-r,\infty),\mathbb{K}^n)$ be given. For each $t \geq 0$, we define $\int_0^t x_s \, \mathrm{d}s \in C([-r,0],\mathbb{K}^n)$ by

$$\left(\int_0^t x_s \, \mathrm{d}s\right)(\theta) := \int_0^t x_s(\theta) \, \mathrm{d}s = \int_\theta^{t+\theta} x(s) \, \mathrm{d}s$$

for $\theta \in [-r,0]$.

We note that $\int_0^t x_s \, \mathrm{d}s \in C([-r,0],\mathbb{K}^n)$ introduced above is not an integral of a vector-valued function

$$[0,t] \ni s \mapsto x_s \in \mathcal{X}$$

for some function space $\mathcal{X}$.

2.2 $\int_0^t x_s \, \mathrm{d}s$ and its properties

We have the following lemma.

Lemma 2.8. *If $x \in \mathcal{L}^1_{\mathrm{loc}}([-r,\infty),\mathbb{K}^n)$, then*

$$[0,\infty) \ni t \mapsto \int_0^t x_s \, \mathrm{d}s \in C([-r,0],\mathbb{K}^n) \tag{2.3}$$

is continuous.

Proof. We define a function $y \colon [-r,\infty) \to \mathbb{K}^n$ by

$$y(t) = \int_{-r}^t x(s) \, \mathrm{d}s$$

for $t \geq -r$. Then y is continuous, and

$$y(t+\theta) = \int_\theta^{t+\theta} x(s) \, \mathrm{d}s + \int_{-r}^\theta x(s) \, \mathrm{d}s$$

holds for all $t \geq 0$ and all $\theta \in [-r,0]$. This shows that the function (2.3) is continuous if and only if

$$[0,\infty) \ni t \mapsto y_t \in C([-r,0],\mathbb{K}^n)$$

is continuous. Since the continuity of this function is ensured by the uniform continuity of y on any closed and bounded interval, the conclusion is obtained. □

When $x \in C([-r,\infty),\mathbb{K}^n)$, the Riemann integral

$$(\mathrm{R}) \int_0^t x_s \, \mathrm{d}s \in C([-r,0],\mathbb{K}^n)$$

of the continuous function

$$[0,t] \ni s \mapsto x_s \in C([-r,0],\mathbb{K}^n)$$

exists. See Graves [16, Section 2] for the definition of the Riemann integrability of functions on closed and bounded intervals taking values in normed spaces. We now show that when $x \in C([-r,\infty),\mathbb{K}^n)$, the Riemann integral $(\mathrm{R}) \int_0^t x_s \, \mathrm{d}s$ coincides with $\int_0^t x_s \, \mathrm{d}s$ introduced in Definition 2.7. More generally, one can prove the following result.

Lemma 2.9. *Let E be a Banach space, $[a,b]$ and $[c,d]$ be closed and bounded intervals of $\mathbb{R}$, and $f\colon [a,b] \times [c,d] \to E$ be a continuous function. For each $y \in [c,d]$, let $f(\cdot, y) \in C([a,b], E)$ be defined by*

$$f(\cdot, y)(x) := f(x,y)$$

for $x \in [a,b]$. Then

$$\left(\int_c^d f(\cdot, y)\, \mathrm{d}y\right)(x) = \int_c^d f(x,y)\, \mathrm{d}y$$

holds for all $x \in [a,b]$. Here $\int_c^d f(\cdot, y)\, \mathrm{d}y$ is the Riemann integral of the continuous function $[c,d] \ni y \mapsto f(\cdot, y) \in C([a,b], E)$.

We note that the continuity of $[c,d] \ni y \mapsto f(\cdot, y) \in C([a,b], E)$ is a consequence of the uniform continuity of f.

Proof of Lemma 2.9. We fix $x \in [a,b]$. Let $T\colon C([a,b], E) \to E$ be the evaluation map defined by

$$Tg := g(x)$$

for $g \in C([a,b], E)$. Since T is a bounded linear operator, we have

$$\left(\int_c^d f(\cdot, y)\, \mathrm{d}y\right)(x) = T\int_c^d f(\cdot, y)\, \mathrm{d}y = \int_c^d Tf(\cdot, y)\, \mathrm{d}y,$$

where the last term is equal to $\int_c^d f(x,y)\, \mathrm{d}y$. This completes the proof. □

As an application of Lemma 2.9, the following result can be obtained.

Theorem 2.10. *If $x \in C([-r, \infty), \mathbb{K}^n)$, then*

$$(\mathrm{R})\int_0^t x_s\, \mathrm{d}s = \int_0^t x_s\, \mathrm{d}s$$

holds for all $t \geq 0$.

Proof. Let $t > 0$ be given. We consider a function $f\colon [-r, 0] \times [0,t] \to \mathbb{K}^n$ defined by

$$f(\theta, s) := x(s+\theta).$$

Then the function $f(\cdot, s)$ is equal to x_s. By applying Lemma 2.9 with this f,

$$\left[(\mathrm{R})\int_0^t x_s\, \mathrm{d}s\right](\theta) = \int_0^t x(s+\theta)\, \mathrm{d}s$$

holds for all $\theta \in [-r, 0]$. Since the right-hand side is equal to $\left(\int_0^t x_s\, \mathrm{d}s\right)(\theta)$, this shows the conclusion. □

Remark 2.11. When $x \in C([-r, \infty), \mathbb{K}^n)$, Theorem 2.10 yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t x_s\, \mathrm{d}s = x_t \in C([-r, 0], \mathbb{K}^n)$$

holds by the fundamental theorem of calculus for vector-valued functions.

We have the following corollary.

Corollary 2.12. *Let F be a Banach space over $\mathbb{K}$ and $T\colon C([-r,0],\mathbb{K}^n) \to F$ be a bounded linear operator. If $x \in C([-r,\infty),\mathbb{K}^n)$, then*

$$T\int_0^t x_s \,\mathrm{d}s = \int_0^t Tx_s \,\mathrm{d}s \tag{2.4}$$

holds for all $t \geq 0$. Here the right-hand side is the Riemann integral of the continuous function $[0,t] \ni s \mapsto Tx_s \in F$.

Proof. From Theorem 2.10,

$$T\int_0^t x_s \,\mathrm{d}s = T\left[(\mathrm{R})\int_0^t x_s \,\mathrm{d}s\right] = \int_0^t Tx_s \,\mathrm{d}s$$

holds since T is a bounded linear operator. □

Remark 2.13. Corollary 2.12 yields the following: Let $x\colon [-r,\infty) \to \mathbb{K}^n$ be a continuous function satisfying $x_0 = \phi \in C([-r,0],\mathbb{K}^n)$. Since $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ is a bounded linear operator, x is a mild solution of the linear RFDE (1.1) with the initial history function ϕ if and only if it satisfies

$$x(t) = \phi(0) + \int_0^t Lx_s \,\mathrm{d}s$$

for all $t \geq 0$. This shows that a mild solution coincides with a solution in the usual sense when the initial history function ϕ is continuous.

2.3 Existence and uniqueness of a mild solution

By using the contraction mapping principle with an *a priori* estimate, we will prove the unique existence of a mild solution of the linear RFDE (1.1) under an initial condition (1.4)

$$x_0 = \phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n).$$

We note that a solution of (1.1) in the usual sense is also a mild solution (see Remark 2.13).

We will use the following notation.

Notation 1. For each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, let $\bar{\phi}\colon \mathrm{dom}(\phi) \cup [0,\infty) \to \mathbb{K}^n$ be the function defined by

$$\bar{\phi}(t) := \begin{cases} \phi(t) & (t \in \mathrm{dom}(\phi)), \\ \phi(0) & (t \geq 0). \end{cases} \tag{2.5}$$

$\bar{\phi}$ is a constant prolongation of ϕ.

Theorem 2.14. *For any $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, the linear RFDE (1.1) has a unique mild solution under the initial condition $x_0 = \phi$.*

In the following, we give a proof based on an *a priori* estimate. See Chicone [5, Subsection 2.1] for a similar argument.

Proof of Theorem 2.14. We divide the proof into the following steps.

Step 1: Reduction to a continuous unknown function and derivation of an *a priori* estimate. For a continuous prolongation $x\colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ of ϕ, we consider the function $y\colon [-r,\infty) \to \mathbb{K}^n$ defined by

$$y(t) := \begin{cases} x(t) - \bar{\phi}(t) & (t \in \mathrm{dom}(x)), \\ 0 & (t \notin \mathrm{dom}(x)). \end{cases}$$

Then y is a continuous function satisfying $y_0 = 0$. The problem of finding a mild solution $x\colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ of the linear RFDE (1.1) under the initial condition $x_0 = \phi$ is reduced to find a continuous function $y\colon [-r,\infty) \to \mathbb{K}^n$ satisfying $y_0 = 0$ and

$$y(t) = L\int_0^t (y+\bar{\phi})_s \,\mathrm{d}s = \int_0^t Ly_s \,\mathrm{d}s + L\int_0^t \bar{\phi}_s \,\mathrm{d}s \tag{2.6}$$

for all $t \geq 0$. Here Corollary 2.12 is used. By noticing the following estimate from above

$$\left| \int_0^t \bar{\phi}(s+\theta)\,\mathrm{d}s \right| \leq \|\phi\|_1 + t|\phi(0)|$$

for $t \geq 0$ and $\theta \in [-r,0]$, a continuous function $y\colon [-r,\infty) \to \mathbb{K}^n$ satisfying $y_0 = 0$ and Eq. (2.6) must satisfy

$$|y(t)| \leq \|L\|(\|\phi\|_1 + t|\phi(0)|) + \int_0^t \|L\|\|y_s\|\,\mathrm{d}s$$

for all $t \geq 0$. By applying Lemma C.4,

$$\|y_t\| \leq \|L\|(\|\phi\|_1 + t|\phi(0)|)\mathrm{e}^{\|L\|t}$$

holds for all $t \geq 0$.

Step 2: Setting of function space. For each $\gamma > \|L\|$, Step 1 indicates that for a continuous function $y\colon [-r,\infty) \to \mathbb{K}^n$ satisfying $y_0 = 0$ and Eq. (2.6), we have

$$\mathrm{e}^{-\gamma t}\|y_t\| \leq \|L\|(\|\phi\|_1 + t|\phi(0)|)\mathrm{e}^{(\|L\|-\gamma)t}.$$

Here the right-hand side converges to 0 as $t \to \infty$. Therefore,

$$\|y\|_\gamma := \sup_{t\geq 0}\left(\mathrm{e}^{-\gamma t}\|y_t\|\right) = \sup_{t\geq 0}\left(\mathrm{e}^{-\gamma t}|y(t)|\right) < \infty$$

holds (see Lemma D.1 for the detail). For each $\gamma > \|L\|$, let Y_γ be the linear subspace of $C([-r,\infty),\mathbb{K}^n)$ given by

$$Y_\gamma := \left\{ y \in C([-r,\infty),\mathbb{K}^n) : y_0 = 0,\ \|y\|_\gamma < \infty \right\},$$

which is considered as a normed space endowed with the norm $\|\cdot\|_\gamma$. Then Y_γ is a Banach space (see Lemma D.2). We fix $\gamma > \|L\|$ arbitrarily, and let $Y := Y_\gamma$ and $\|\cdot\|_Y := \|\cdot\|_\gamma$.

Step 3: Reduction to fixed point problem. We define a transformation $T\colon Y \to C([-r,\infty),\mathbb{K}^n)$ by $(Ty)_0 = 0$ and

$$(Ty)(t) := \int_0^t Ly_s\,\mathrm{d}s + L\int_0^t \bar{\phi}_s\,\mathrm{d}s \quad (t \geq 0).$$

We now claim that $T(Y) \subset Y$ holds. Let $y \in Y$ be given. In the same way as in Step 1,

$$|Ty(t)| \le \|L\|(\|\phi\|_1 + t|\phi(0)|) + \|L\| \int_0^t \|y_s\| \,\mathrm{d}s$$

holds for all $t \ge 0$. Since $\mathrm{e}^{-\gamma t}\|L\|(\|\phi\|_1 + t|\phi(0)|) \to 0$ as $t \to \infty$, we only need to show

$$\sup_{t\ge 0} \mathrm{e}^{-\gamma t} \int_0^t \|y_s\| \,\mathrm{d}s < \infty$$

in order to obtain $Ty \in Y$. By the assumption of $y \in Y$, $\|y_t\| \le \|y\|_Y \mathrm{e}^{\gamma t}$ holds for all $t \ge 0$. Therefore, we have

$$\int_0^t \|y_s\| \,\mathrm{d}s \le \|y\|_Y \int_0^t \mathrm{e}^{\gamma s} \,\mathrm{d}s \le \frac{\|y\|_Y}{\gamma} \mathrm{e}^{\gamma t} \qquad (t \ge 0),$$

which implies $\sup_{t\ge 0} \mathrm{e}^{-\gamma t} \int_0^t \|y_s\| \,\mathrm{d}s < \infty$. Thus, $Ty \in Y$ is concluded.

Step 4: Application of contraction mapping principle. We now claim that the mapping $T\colon Y \to Y$ is a contraction. For any $y^1, y^2 \in Y$,

$$\mathrm{e}^{-\gamma t}\left|Ty^1(t) - Ty^2(t)\right| \le \mathrm{e}^{-\gamma t}\|L\| \int_0^t \left\|y^1_s - y^2_s\right\| \,\mathrm{d}s$$

holds. Since we have

$$\begin{aligned}\left\|y^1_s - y^2_s\right\| &= \mathrm{e}^{\gamma s} \cdot \mathrm{e}^{-\gamma s}\left\|(y^1 - y^2)_s\right\| \\ &\le \mathrm{e}^{\gamma s}\left\|y^1 - y^2\right\|_Y\end{aligned}$$

for the integrand in the right-hand side,

$$\begin{aligned}\mathrm{e}^{-\gamma t}\|L\| \int_0^t \left\|y^1_s - y^2_s\right\| \,\mathrm{d}s &\le \frac{\|L\|}{\gamma}(1 - \mathrm{e}^{-\gamma t})\left\|y^1 - y^2\right\|_Y \\ &\le \frac{\|L\|}{\gamma}\left\|y^1 - y^2\right\|_Y\end{aligned}$$

is concluded. Therefore, $T\colon Y \to Y$ is a contraction. By applying the contraction mapping principle, there exists a unique $y_* \in Y$ such that

$$Ty_* = y_*.$$

The function $x_*\colon [-r, \infty) \supset \mathrm{dom}(\phi) \cup [0, \infty) \to \mathbb{K}^n$ defined by

$$x_*(t) := y(t) + \bar{\phi}(t) \qquad (t \in \mathrm{dom}(\phi) \cup [0, \infty))$$

is a mild solution of the linear RFDE (1.1) under the initial condition $x_0 = \phi$. The uniqueness follows by the above discussion. □

We hereafter use the following notation.

Notation 2. For each $\phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$, we denote the unique mild solution of the linear RFDE (1.1) under the initial condition $x_0 = \phi$ by $x^L(\cdot\,; \phi)\colon\ \mathrm{dom}(\phi) \cup [0, \infty) \to \mathbb{K}^n$.

We have the following corollary.

Corollary 2.15. *Let $\alpha, \beta \in \mathbb{K}$ and $\phi, \psi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ be given. Then for all $t \geq 0$,*

$$x^L(t;\alpha\phi + \beta\psi) = \alpha x^L(t;\phi) + \beta x^L(t;\psi) \tag{2.7}$$

holds.

Proof. Let $\chi := \alpha\phi + \beta\psi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ and $x\colon [-r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ be the function defined by

$$\mathrm{dom}(x) := \mathrm{dom}(\chi) \cup [0,\infty), \quad x(t) := \alpha x^L(t;\phi) + \beta x^L(t;\psi).$$

Since the map L and the Lebesgue integration are linear, x is a mild solution of the linear RFDE (1.1) under the initial condition $x_0 = \chi$ by the definition of mild solutions (see Definition 2.5). Therefore, (2.7) is a consequence of Theorem 2.14. □

2.4 Fundamental matrix solutions

Since ODEs are special DDEs, it is natural to expect that the notions of fundamental systems of solutions and fundamental matrix solutions for linear ODEs are meaningful for DDEs in some way. However, the solution space of the linear RFDE (1.1) is infinite-dimensional. Therefore, it is impossible to define these notions to (1.1) as a simple generalization.

A key to this consideration is to focus on a "finite-dimensionality". For this purpose, we consider an "instantaneous input" as an initial history function. We will use the following notation.

Definition 2.16. For each $\xi \in \mathbb{K}^n$, we define a function $\hat{\xi}\colon [-r,0] \to \mathbb{K}^n$ by

$$\hat{\xi}(\theta) := \begin{cases} 0 & (\theta \in [-r,0)), \\ \xi & (\theta = 0). \end{cases}$$

$\hat{0}$ is the constant function whose value is identically equal to the zero vector $0 \in \mathbb{K}^n$.

Since $\hat{\xi} \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ for each $\xi \in \mathbb{K}^n$, one can consider the mild solution

$$x^L(\cdot;\hat{\xi})\colon [-r,\infty) \to \mathbb{K}^n$$

of the linear RFDE (1.1) under the initial condition $x_0 = \hat{\xi}$ from Theorem 2.14. Then Corollary 2.15 yields that the subset $\mathcal{S}$ given by

$$\mathcal{S} := \left\{ x^L(\cdot;\hat{\xi})\colon [-r,\infty) \to \mathbb{K}^n : \xi \in \mathbb{K}^n \right\}$$

forms a linear space. We have the following lemma.

Lemma 2.17. *Let $\xi_1, \dots, \xi_m \in \mathbb{K}^n$ be vectors and let $x_j := x^L(\cdot;\hat{\xi}_j)\colon [-r,\infty) \to \mathbb{K}^n$ for each $j \in \{1,\dots,m\}$. Then the following properties are equivalent:*

(a) The system of vectors $\xi_1, \dots, \xi_m$ is linearly independent.

(b) The system of functions $x_1, \dots, x_m$ is linearly independent.

Here the system of functions $x_1, \dots, x_m$ is said to be *linearly independent* if for any scalars $\alpha_1, \dots, \alpha_m$, $\alpha_1 x_1 + \dots + \alpha_m x_m = 0$ implies $\alpha_1 = \dots = \alpha_m = 0$.

Proof of Lemma 2.17. (a) $\Rightarrow$ (b): Since $\alpha_1 x_1 + \cdots + \alpha_m x_m = 0$ implies

$$\alpha_1 \xi_1 + \cdots + \alpha_m \xi_m = (\alpha_1 x_1 + \cdots + \alpha_m x_m)(0) = 0,$$

this part follows by the definition of linear independence for functions.

(b) $\Rightarrow$ (a): We suppose $\alpha_1 \xi_1 + \cdots + \alpha_m \xi_m = 0$ for $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$. Since this implies

$$\alpha_1 \hat{\xi}_1 + \cdots + \alpha_m \hat{\xi}_m = \hat{0},$$

(2.7) yields

$$\alpha_1 x_1 + \cdots + \alpha_m x_m = 0.$$

Therefore, we have $\alpha_1 = \cdots = \alpha_m = 0$ by the assumption (b).

This completes the proof. □

Theorem 2.18. *The linear space $\mathcal{S}$ is n-dimensional.*

Proof. Let $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ be a basis of $\mathbb{K}^n$. From Lemma 2.17, the system of functions

$$x^L(\cdot; \hat{\boldsymbol{b}}_1), \ldots, x^L(\cdot; \hat{\boldsymbol{b}}_n) \in \mathcal{S}$$

is linearly independent. Furthermore, for any $x_1, \ldots, x_{n+1} \in \mathcal{S}$, the system of functions is linearly dependent from Lemma 2.17 because the system $x_1(0), \ldots, x_{n+1}(0) \in \mathbb{K}^n$ of vectors is linearly dependent. Therefore, the statement holds. □

Theorem 2.18 naturally leads us to the following definition.

Definition 2.19 (cf. [18], [19])**.** We call a basis of the n-dimensional linear space $\mathcal{S}$ a *fundamental system of solutions* to the linear RFDE (1.1). Equivalently, a fundamental system of solutions is the linear independent system

$$x^L(\cdot; \hat{\boldsymbol{b}}_1), \ldots, x^L(\cdot; \hat{\boldsymbol{b}}_n) \colon [-r, \infty) \to \mathbb{K}^n$$

for some basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ of $\mathbb{K}^n$. We call a matrix-valued function having a fundamental system of solutions as its column vectors a *fundamental matrix solution*. In particular, we call the fundamental matrix solution

$$X \colon [-r, \infty) \to M_n(\mathbb{K})$$

satisfying $X(0) = I$ the *principal fundamental matrix solution*. Here I denotes the identity matrix.

The above definition is considered as a natural generalization of the corresponding definition for linear ODEs (see [6, Definition 2.12 in Section 2.1 of Chapter 2]). See also [37, Definition 5.10] for a related definition.

We hereafter use the following notation.

Notation 3. Let $X^L \colon [-r, \infty) \to M_n(\mathbb{K})$ denote the principal fundamental matrix solution of the linear RFDE (1.1). By the above definition,

$$X^L(\cdot) = \Big(x^L(\cdot; \hat{\boldsymbol{e}}_1) \ \cdots \ x^L(\cdot; \hat{\boldsymbol{e}}_n) \Big) \tag{2.8}$$

holds. Here $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ denotes the standard basis of $\mathbb{K}^n$.

Remark 2.20. Let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{K}^n$ be given. From (2.8), we have

$$X^L(\cdot)\xi = \xi_1 x^L(\cdot; \hat{e}_1) + \cdots + \xi_n x^L(\cdot; \hat{e}_n).$$

Here the right-hand side is equal to $x^L(\cdot; \xi_1\hat{e}_1 + \cdots + \xi_n\hat{e}_n)$ from (2.7). Therefore,

$$X^L(\cdot)\xi = x^L(\cdot; \hat{\xi})$$

holds.

Remark 2.21. We consider an autonomous linear ODE

$$\dot{x} = Ax \tag{2.9}$$

for some $A \in M_n(\mathbb{K})$. For a system of global solutions $y_1, \dots, y_m \colon \mathbb{R} \to \mathbb{K}^n$ to the linear ODE (2.9), the following statements are equivalent:

(a) For any $t \in \mathbb{R}$, $y_1(t), \dots, y_m(t) \in \mathbb{K}^n$ is linearly independent.

(b) For some $t_0 \in \mathbb{R}$, $y_1(t_0), \dots, y_m(t_0) \in \mathbb{K}^n$ is linearly independent.

(c) The system of functions $y_1, \dots, y_m$ is linearly independent.

The nontrivial part is (c) $\Rightarrow$ (a), which is proved by the principle of superposition and by the unique existence of a solution of (2.9) under an initial condition

$$x(t_0) = \xi \in \mathbb{K}^n.$$

Compared with this situation, the linear independence of vectors $x_1(t_0), \dots, x_m(t_0) \in \mathbb{K}^n$ for each $t_0 > 0$ is not necessarily guaranteed for the functions $x_1, \dots, x_m$ in Lemma 2.17 under the assumption that (a) or (b) in Lemma 2.17 holds. This should be compared with an example given by Popov [29], which is a three dimensional system of linear DDEs whose solution values are contained in a hyperplane of $\mathbb{R}^3$ after a certain amount of time has elapsed. See also [19, Section 3.5] and [22, Section 3.5].

2.5 Remarks

2.5.1 Consideration by Delfour

The definition of a mild solution in Definition 2.5 is also related to the consideration by Delfour [8]. In that paper, the author considered a continuous linear map

$$L \colon W^{1,p}((-r, 0), \mathbb{R}^n) \to \mathbb{R}^n$$

for some $p \in [1, \infty)$. Here $W^{1,p}((-r, 0), \mathbb{R}^n)$ is the Sobolev space (e.g., see Brezis [4, Section 8.2]). The author used the integral representation of L given by

$$L\phi := \int_{-r}^{0} [A_1(\theta)\phi(\theta) + A_2(\theta)\phi'(\theta)] \, \mathrm{d}\theta, \tag{2.10}$$

where $A_1, A_2 \colon (-r, 0) \to M_n(\mathbb{R})$ are $n \times n$ real matrix-valued q-integrable functions with $(1/p) + (1/q) = 1$. For the first term of the right-hand side of (2.10), we have

$$\int_0^t \left(\int_{-r}^0 A_1(\theta) x(s + \theta) \, \mathrm{d}\theta \right) \mathrm{d}s = \int_{-r}^0 A_1(\theta) \left(\int_0^t x(s + \theta) \, \mathrm{d}s \right) \mathrm{d}\theta$$

under the exchange of order of integration. Here we have replaced ϕ with x_s and have integrated from 0 to t with respect to s. In view of the above equality, it can be said that the concept of mild solutions in Definition 2.5 is also hidden in [8]. Theorem 2.14 and its proof should be compared with the existence and uniqueness result in [8].

2.5.2 Mild solutions for linear differential difference equations

We consider an autonomous linear *differential difference equation*

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} B_k x(t - \tau_k) \quad (t \geq 0) \tag{2.11}$$

for $n \times n$ matrices $A, B_1, \dots, B_m \in M_n(\mathbb{K})$ and $\tau_1, \dots, \tau_m \in (0, r]$. We refer the reader to [2] as a general reference of the theory of differential difference equations.

The linear DDE (2.11) can be expressed in the form of the linear RFDE (1.1) by defining a continuous linear map $L \colon C([-r, 0], \mathbb{K}^n) \to \mathbb{K}^n$ by

$$L\psi = A\psi(0) + \sum_{k=1}^{m} B_k \psi(-\tau_k) \tag{2.12}$$

for $\psi \in C([-r, 0], \mathbb{K}^n)$. Let $\phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$ be given and $x := x^L(\cdot; \phi)$ for the above continuous linear map L. By the definition of mild solutions (see Definitions 2.5 and 2.7), x satisfies

$$\begin{aligned} x(t) &= \phi(0) + L \int_0^t x_s \, \mathrm{d}s \\ &= \phi(0) + A \int_0^t x(s) \, \mathrm{d}s + \sum_{k=1}^{m} B_k \int_{-\tau_k}^{t-\tau_k} x(s) \, \mathrm{d}s \end{aligned}$$

for all $t \geq 0$. Since the last term is equal to

$$\phi(0) + \int_0^t Ax(s) \, \mathrm{d}s + \sum_{k=1}^{m} \int_{-\tau_k}^{t-\tau_k} B_k x(s) \, \mathrm{d}s,$$

x also satisfies

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} B_k x(t - \tau_k) \quad (\text{a.e. } t \geq 0)$$

by the Lebesgue differentiation theorem (see Subsection 3.1).

3 Differential equation satisfied by principal fundamental matrix solution

In this section, we consider the linear RFDE (1.1)

$$\dot{x}(t) = Lx_t \quad (t \geq 0)$$

for a continuous linear map $L \colon C([-r, 0], \mathbb{K}^n) \to \mathbb{K}^n$. We choose a matrix-valued function $\eta \colon [-r, 0] \to M_n(\mathbb{K})$ of bounded variation so that L is represented as the Riemann–Stieltjes integral (1.3)

$$L\psi = \int_{-r}^{0} \mathrm{d}\eta(\theta) \, \psi(\theta)$$

for $\psi \in C([-r, 0], \mathbb{K}^n)$. We recall that the domain of definition of η is extended to $(-\infty, 0]$ by letting $\eta(\theta) := \eta(-r)$ for $\theta \in (-\infty, -r]$. We will use the following notation.

Notation 4. Let $\check{\eta}\colon [0,\infty) \to M_n(\mathbb{K})$ be the function given by

$$\check{\eta}(u) := -\eta(-u)$$

for $u \in [0,\infty)$.

In this paper, a function defined on $[0,\infty)$ is said to be of *locally bounded variation* if it is of bounded variation on any closed and bounded interval of $[0,\infty)$. A function of locally bounded variation is also called a *locally BV function*. Then the above function $\check{\eta}$ is a function of locally bounded variation whose value is constant on $[r,\infty)$. It is related to the reversal formula for Riemann–Stieltjes integrals (see Theorem A.9).

It will turn out that the notions of Volterra operator and Riemann–Stieltjes convolution are useful to deduce a differential equation that is satisfied by the principal fundamental matrix solution $X^L\colon [-r,\infty) \to M_n(\mathbb{K})$ of the linear RFDE (1.1).

3.1 Definitions

Definition 3.1. For each $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$, let $Vf\colon [0,\infty) \to M_n(\mathbb{K})$ be the function defined by

$$(Vf)(t) = \int_0^t f(s)\,\mathrm{d}s. \tag{3.1}$$

Here the right-hand side is a Lebesgue integral. We call V the *Volterra operator*.

For details related to the Volterra operator as a linear operator on $C([0,T],\mathbb{K})$ for each $T > 0$, see [31]. By using the Lebesgue differentiation theorem (e.g., see [32, Theorem 1.3 in Section 1 of Chapter 3]) component-wise, it holds that Vf is *locally absolutely continuous* (i.e., locally absolutely continuous on any closed and bounded interval of $[0,\infty)$), differentiable almost everywhere on $[0,\infty)$, and

$$(Vf)'(t) = f(t)$$

holds for almost all $t \in [0,\infty)$.

Definition 3.2. For each function $\alpha\colon [0,\infty) \to M_n(\mathbb{K})$ of locally bounded variation and for each continuous function $f\colon [0,\infty) \to M_n(\mathbb{K})$, we define a function $\mathrm{d}\alpha * f\colon [0,\infty) \to M_n(\mathbb{K})$ by

$$(\mathrm{d}\alpha * f)(t) := \int_0^t \mathrm{d}\alpha(u) f(t-u).$$

Here the right-hand side is a Riemann–Stieltjes integral. This function is called a *Riemann–Stieltjes convolution*.

See [31, Definition 10.3 in Section 10.3] for the scalar-valued case. The above definition should be compared with the treatment in [34, Eq. (2.13) in Chapter 2] and [14, Corollary 2.5 in Section I.2 of Appendix I], where an appearing integral is not a Riemann–Stieltjes integral but a Lebesgue–Stieltjes integral.

3.2 Motivation

The following lemma motivates the use of Volterra operator and Riemann–Stieltjes convolution.

Lemma 3.3. *If $x \in \mathcal{L}^1_{\mathrm{loc}}([-r,\infty),\mathbb{K}^n)$ satisfies $x_0 = \hat{\xi}$ for some $\xi \in \mathbb{K}^n$, then*

$$L\int_0^t x_s \,\mathrm{d}s = \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x(s)\,\mathrm{d}s\right) = \int_0^t \mathrm{d}\check{\eta}(u)\left(\int_0^{t-u} x(s)\,\mathrm{d}s\right)$$

holds for all $t \geq 0$.

Proof. Let $t > 0$ be fixed. By the assumption, $\int_\theta^0 x(s)\,\mathrm{d}s = 0$ holds for all $\theta \in [-r,0]$. Therefore, we have

$$L\int_0^t x_s \,\mathrm{d}s = \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_\theta^{t+\theta} x(s)\,\mathrm{d}s\right) = \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x(s)\,\mathrm{d}s\right).$$

We examine the last term by dividing the consideration into the following cases:

- Case: $t \in [0,r)$. In this case, $t+\theta \geq 0$ is equivalent to $\theta \geq -t$ for each $\theta \in [-r,0]$. Since $\int_0^{t+\theta} x(s)\,\mathrm{d}s = 0$ for $\theta \in [-r,-t)$,

$$L\int_0^t x_s \,\mathrm{d}s = \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x(s)\,\mathrm{d}s\right)$$

  holds by the additivity of Riemann–Stieltjes integrals on sub-intervals.

- Case: $t \geq r$. In this case, $t+\theta \geq 0$ holds for all $\theta \in [-r,0]$. Since η is constant on $[-t,-r]$,

$$L\int_0^t x_s \,\mathrm{d}s = \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x(s)\,\mathrm{d}s\right)$$

  holds.

Therefore, the expressions of $L\int_0^t x_s\,\mathrm{d}s$ are obtained in combination with the reversal formula for Riemann–Stieltjes integrals (see Theorem A.9). □

3.3 Properties of Volterra operator and Riemann–Stieltjes convolution

Throughout this subsection, let $\alpha\colon [0,\infty) \to M_n(\mathbb{K})$ be a function of locally bounded variation and $f\colon [0,\infty) \to M_n(\mathbb{K})$ be a continuous function.

3.3.1 Continuity and local integrability

The following is a simple result about the continuity of Riemann–Stieltjes convolution.

Lemma 3.4. *If $f(0) = O$, then $\mathrm{d}\alpha * f$ is continuous.*

Proof. We extend the domain of definition of f to $\mathbb{R}$ by defining $f(t) := f(0) = O$ for $t \leq 0$. Then the obtained function $f\colon \mathbb{R} \to M_n(\mathbb{K})$ is continuous. Let $s,t \in [0,\infty)$ be given so that $s < t$. By the additivity of Riemann–Stieltjes integrals on sub-intervals,

$$\begin{aligned}(\mathrm{d}\alpha * f)(s) &= \int_0^s \mathrm{d}\alpha(u)\,f(s-u) \\ &= \int_0^t \mathrm{d}\alpha(u)\,f(s-u) - \int_s^t \mathrm{d}\alpha(u)\,f(s-u)\end{aligned}$$

holds. Since

$$\int_s^t \mathrm{d}\alpha(u)\,f(s-u) = [\alpha(t)-\alpha(s)]f(0) = O,$$

we have

$$(\mathrm{d}\alpha * f)(t) - (\mathrm{d}\alpha * f)(s) = \int_0^t \mathrm{d}\alpha(u)\,[f(t-u) - f(s-u)].$$

By combining this and the uniform continuity of f on closed and bounded intervals, the continuity of $\mathrm{d}\alpha * f$ is obtained. □

See [31, Lemma 10.4 in Section 10.3] for the corresponding result for scalar-valued functions. In this paper, we say that a function is *locally Riemann integrable* if it is Riemann integrable on any closed and bounded interval.

Theorem 3.5. *$\mathrm{d}\alpha * f$ is a sum of a continuous function and a function of locally bounded variation. Consequently, $\mathrm{d}\alpha * f$ is locally Riemann integrable.*

Proof. By using $f = (f - f(0)) + f(0)$, we have

$$\mathrm{d}\alpha * f = \mathrm{d}\alpha * (f - f(0)) + \mathrm{d}\alpha * f(0). \tag{3.2}$$

The first term in the right-hand side is continuous from Lemma 3.4. The second term is of locally bounded variation since

$$(\mathrm{d}\alpha * f(0))(t) = [\alpha(t) - \alpha(0)]f(0)$$

holds for all $t \geq 0$. Therefore, the conclusion holds. □

Remark 3.6. Theorem 3.5 yields that $V(\mathrm{d}\alpha * f)$ makes sense. Furthermore if α is continuous, then (3.2) shows that $\mathrm{d}\alpha * f$ is also continuous.

3.3.2 Riemann–Stieltjes convolution under Volterra operator

The Riemann–Stieltjes convolution and Volterra operator are related in the following way.

Theorem 3.7. *The equality*

$$V(\mathrm{d}\alpha * f) = \mathrm{d}\alpha * Vf \tag{3.3}$$

holds. Consequently, $\mathrm{d}\alpha * Vf$ is locally absolutely continuous, differentiable almost everywhere, and satisfies

$$(\mathrm{d}\alpha * Vf)'(t) = (\mathrm{d}\alpha * f)(t)$$

holds for almost all $t \geq 0$.

For the proof, we need the following theorem. It contains the result on iterated Riemann integrals for continuous functions on rectangles as a special case.

Theorem 3.8. *Let $[a,b]$ and $[c,d]$ be closed and bounded intervals of $\mathbb{R}$. If $f\colon [a,b] \times [c,d] \to M_n(\mathbb{K})$ is continuous and $\alpha\colon [a,b] \to M_n(\mathbb{K})$ is a function of bounded variation, then*

$$\int_a^b \mathrm{d}\alpha(x) \left(\int_c^d f(x,y)\,\mathrm{d}y \right) = \int_c^d \left(\int_a^b \mathrm{d}\alpha(x)\, f(x,y) \right) \mathrm{d}y \tag{3.4}$$

holds.

See also [39, Theorem 15a in Section 15 of Chapter I]. We will give the proof in Appendix A.7.

Proof of Theorem 3.7. We extend the domain of definition of f to $\mathbb{R}$ by defining $f(t) := f(0)$ for $t \le 0$. By the proof of Lemma 3.4, we have

$$V(\mathrm{d}\alpha * f)(t) = \int_0^t \left(\int_0^t \mathrm{d}\alpha(u) f(s-u) \right) \mathrm{d}s - \int_0^t [\alpha(t) - \alpha(s)] f(0) \,\mathrm{d}s$$

for $t \ge 0$, where

$$\int_0^t \left(\int_0^t \mathrm{d}\alpha(u) f(s-u) \right) \mathrm{d}s = \int_0^t \mathrm{d}\alpha(u) \left(\int_0^t f(s-u) \,\mathrm{d}s \right)$$

holds from Theorem 3.8. The last term is expressed by

$$(\mathrm{d}\alpha * Vf)(t) + \int_0^t \mathrm{d}\alpha(u) \int_{-u}^0 f(s) \,\mathrm{d}s$$

by using the Volterra operator and the Riemann–Stieltjes convolution. Since $\int_{-u}^0 f(s)\,\mathrm{d}s = uf(0)$ for $u \in [0, t]$, the proof is complete by showing

$$\int_0^t [\alpha(t) - \alpha(s)] \,\mathrm{d}s = \int_0^t u \,\mathrm{d}\alpha(u).$$

This is indeed true because

$$\begin{aligned} \int_0^t u \,\mathrm{d}\alpha(u) &= [u\alpha(u)]_{u=0}^t - \int_0^t \alpha(u) \,\mathrm{d}u \\ &= t\alpha(t) - \int_0^t \alpha(u) \,\mathrm{d}u \end{aligned}$$

holds by the integration by parts formula for Riemann–Stieltjes integrals. See Theorem A.10 for the detail. □

The following is a corollary of Theorem 3.7. It will not be used in the sequel.

Corollary 3.9. *Furthermore, if f is continuously differentiable, then $\mathrm{d}\alpha * f$ is expressed by*

$$\mathrm{d}\alpha * f = (\alpha - \alpha(0)) f(0) + V\big(\mathrm{d}\alpha * f'\big).$$

Consequently, $\mathrm{d}\alpha * f$ is of locally bounded variation, differentiable almost everywhere, and satisfies

$$(\mathrm{d}\alpha * f)'(t) = \alpha'(t) f(0) + (\mathrm{d}\alpha * f')(t)$$

for almost all $t \ge 0$.

Proof. By the fundamental theorem of calculus, $f = f(0) + Vf'$ holds. By combining this and (3.3), the expression of $\mathrm{d}\alpha * f$ is obtained. Since $V(\mathrm{d}\alpha * f')$ is locally absolutely continuous, it is also of locally bounded variation. Therefore, the expression of $\mathrm{d}\alpha * f$ yields that $\mathrm{d}\alpha * f$ is of locally bounded variation. The remaining properties are consequences of the fact that matrix-valued functions of bounded variation are differentiable almost everywhere. This is obtained by applying the corresponding result[1] for real-valued functions component-wise. □

[1] See [32, Theorem 3.4 in Subsection 3.1 of Chapter 3], for example.

3.4 Differential equation and principal fundamental matrix solution

As an application of Theorem 3.7, one can derive a differential equation that is satisfied by $x^L(\,\cdot\,;\hat{\xi})$ for each $\xi \in \mathbb{K}^n$.

Theorem 3.10. *Let $x := x^L(\,\cdot\,;\hat{\xi})$ for some $\xi \in \mathbb{K}^n$. Then x satisfies*

$$x(t) = \xi + \int_{-t}^{0} \mathrm{d}\eta(\theta) \left(\int_{0}^{t+\theta} x(s)\,\mathrm{d}s \right) = \xi + \big(\mathrm{d}\check{\eta} * Vx|_{[0,\infty)}\big)(t) \tag{3.5}$$

for all $t \geq 0$. Furthermore, $x|_{[0,\infty)}$ is locally absolutely continuous, differentiable almost everywhere, and satisfies

$$\dot{x}(t) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\, x(t+\theta) = \big(\mathrm{d}\check{\eta} * x|_{[0,\infty)}\big)(t) \tag{3.6}$$

for almost all $t \in [0,\infty)$.

Proof. By definition, x satisfies

$$x(t) = \xi + L \int_{0}^{t} x_s \,\mathrm{d}s$$

for all $t \geq 0$, and $x|_{[0,\infty)}$ is continuous. Then Eq. (3.5) is a consequence of Lemma 3.3. Theorem 3.7 and Eq. (3.5) yield that

$$x(t) = \xi + V\big(\mathrm{d}\check{\eta} * x|_{[0,\infty)}\big)(t)$$

holds for all $t \geq 0$. Therefore, it holds that $x|_{[0,\infty)}$ is locally absolutely continuous, differentiable almost everywhere, and satisfies

$$\dot{x}(t) = \big(\mathrm{d}\check{\eta} * x|_{[0,\infty)}\big)(t)$$

for almost all $t \geq 0$. The remaining expression in Eq. (3.6) is a consequence of the reversal formula for Riemann–Stieltjes integrals. □

The following theorem also holds.

Theorem 3.11. *Let $x \in \mathcal{L}^1_{\mathrm{loc}}([-r,\infty),\mathbb{K}^n)$ be given so that $x_0 = \hat{\xi}$ for some $\xi \in \mathbb{K}^n$. If x satisfies (3.5) for all $t \geq 0$, then $x = x^L(\,\cdot\,;\hat{\xi})$.*

Proof. From Lemma 3.3, x satisfies

$$x(t) = \xi + L \int_{0}^{t} x_s \,\mathrm{d}s$$

for all $t \geq 0$. From Lemma 2.8 and by the continuity of L, the right-hand side is continuous with respect to $t \geq 0$. Therefore, x is a mild solution of the linear RFDE (1.1) under the initial condition $x_0 = \hat{\xi}$. By the uniqueness (see Theorem 2.14), the conclusion is obtained. □

We obtain the following result as a direct consequence of Theorem 3.10 and (2.8). We omit the proof.

Theorem 3.12 (cf. [34])**.** *The principal fundamental matrix solution $X^L \colon [-r,\infty) \to M_n(\mathbb{K})$ of the linear RFDE (1.1) satisfies*

$$X^L(t) = I + \int_{-t}^{0} \mathrm{d}\eta(\theta) \left(\int_{0}^{t+\theta} X^L(s)\,\mathrm{d}s \right) = I + \big(\mathrm{d}\check{\eta} * VX^L|_{[0,\infty)}\big)(t) \tag{3.7}$$

for all $t \geq 0$*. Furthermore,* $X^L|_{[0,\infty)}$ *is locally absolutely continuous, differentiable almost everywhere, and satisfies*

$$\dot{X}^L(t) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\, X^L(t+\theta) = \big(\mathrm{d}\check{\eta} * X^L|_{[0,\infty)}\big)(t) \tag{3.8}$$

for almost all $t \in [0,\infty)$*.*

Roughly speaking, Eq. (3.8) is used as the defining equation of the fundamental matrix solution in [34, (9.1) in Chapter 9].

4 Non-homogeneous linear RFDEs

In this section, we study a non-homogeneous linear RFDE (1.2)

$$\dot{x}(t) = Lx_t + g(t) \qquad (\text{a.e. } t \geq 0)$$

for a continuous linear map $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ and some $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{K}^n)$.

4.1 Non-homogeneous linear RFDE and mild solutions

It is natural to define the notion of mild solutions to Eq. (1.2) in the following way.

Definition 4.1. Let $t_0 \geq 0$ and $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ be given. We say that a function $x\colon [t_0 - r,\infty) \supset \mathrm{dom}(x) \to \mathbb{K}^n$ is a *mild solution* of Eq. (1.2) under the initial condition $x_{t_0} = \phi$ if the following conditions are satisfied: (i) $x_{t_0} = \phi$, (ii) $[t_0,\infty) \subset \mathrm{dom}(x)$, (iii) $x|_{[t_0,\infty)}$ is continuous, and (iv) for all $t \geq t_0$,

$$x(t) = \phi(0) + L\int_{t_0}^{t} x_s\,\mathrm{d}s + \int_{t_0}^{t} g(s)\,\mathrm{d}s$$

holds.

We note that $\int_{t_0}^{t} x_s\,\mathrm{d}s \in C([-r,0],\mathbb{K}^n)$ is defined by

$$\left(\int_{t_0}^{t} x_s\,\mathrm{d}s\right)(\theta) := \int_{t_0}^{t} x(s+\theta)\,\mathrm{d}s = \int_{t_0+\theta}^{t+\theta} x(s)\,\mathrm{d}s$$

for $\theta \in [-r,0]$, and

$$\mathrm{dom}(x) = (t_0 + \mathrm{dom}(\phi)) \cup [t_0,\infty) = t_0 + (\mathrm{dom}(\phi) \cup [0,\infty))$$

holds for a mild solution of Eq. (1.2) under the initial condition $x_{t_0} = \phi$.

Lemma 4.2. *Let* $t_0 \geq 0$ *and* $\phi \in C([-r,0],\mathbb{K}^n)$ *be given. If* $x\colon [t_0 - r,\infty) \to \mathbb{K}^n$ *is a mild solution of Eq. (1.2) under the initial condition* $x_{t_0} = \phi$*, then* x *satisfies*

$$\dot{x}(t) = Lx_t + g(t)$$

for almost all $t \geq t_0$*.*

Proof. By the translation, we may assume $t_0 = 0$. Since $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ is a bounded linear operator,

$$x(t) = \phi(0) + \int_{0}^{t} Lx_s\,\mathrm{d}s + \int_{0}^{t} g(s)\,\mathrm{d}s$$

holds for all $t \geq 0$ from Corollary 2.12. Then the fundamental theorem of calculus and the Lebesgue differentiation theorem yield that $x|_{[0,\infty)}$ is differentiable almost everywhere and

$$\dot{x}(t) = Lx_t + g(t)$$

holds for almost all $t \geq 0$. □

Remark 4.3. Let $\mathbb{K} = \mathbb{R}$. We assume that $\mathrm{dom}(g) = [0, \infty)$ and consider the function $F\colon [0, \infty) \times C([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ defined by

$$F(t, \phi) := L\phi + g(t).$$

Then F satisfies the Carathéodory condition. See [19, Section 2.6 of Chapter 2] and [22, Section 2.6 of Chapter 2] for the detail of the Carathéodory condition for RFDEs. Lemma 4.2 shows that a mild solution $x\colon [t_0 - r, \infty) \to \mathbb{R}^n$ of Eq. (1.2) under the initial condition $x_{t_0} = \phi \in C([-r, 0], \mathbb{R}^n)$ is a solution (in the Carathéodory sense).

4.2 Integral equation with a general forcing term

More generally, for a given $t_0 \geq 0$ and a given continuous function $G\colon [t_0, \infty) \to \mathbb{K}^n$ with $G(t_0) = 0$, we can discuss a solution of the following integral equation

$$x(t) = \phi(0) + L \int_{t_0}^{t} x_s \, \mathrm{d}s + G(t) \qquad (t \geq t_0) \tag{4.1}$$

under an initial condition $x_{t_0} = \phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$. Here the assumption $G(t_0) = 0$ is natural because the right-hand side of (4.1) is equal to

$$\phi(0) + G(t_0)$$

at $t = t_0$. The notion of a solution of (4.1) can be defined in a similar way as in Definition 4.1. The following theorem holds.

Theorem 4.4. *Let $t_0 \geq 0$ be given. Suppose that $G\colon [t_0, \infty) \to \mathbb{K}^n$ is a continuous function with $G(t_0) = 0$. Then for any $\phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$, Eq. (4.1) has a unique solution under the initial condition $x_{t_0} = \phi$.*

The following proof should be compared with the proof of Theorem 2.14.

Proof of Theorem 4.4. By the translation, it is sufficient to consider the case $t_0 = 0$. We will solve the integral equation locally and will connect the obtained local solutions. For this purpose, we need to consider an integral equation under the initial condition $x_\sigma = \psi$ for each $\sigma \geq 0$ and each $\psi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$. Here an appropriate forcing term is given by

$$G(t, \sigma) := G(t) - G(\sigma)$$

for $t \geq \sigma$. Then we are going to consider an integral equation

$$x(t) = \psi(0) + L \int_{\sigma}^{t} x_s \, \mathrm{d}s + G(t, \sigma) \qquad (t \geq \sigma) \tag{4.2}$$

under the initial condition $x_\sigma = \psi$. The remainder of the proof is divided into the following steps.

Step 1: Existence and uniqueness of a local solution. We fix the above σ and ψ. By defining a continuous function $y\colon [-r,\infty) \to \mathbb{K}^n$ by

$$y(s) := \begin{cases} x(\sigma+s) - \bar{\psi}(s) & (\sigma + s \in \mathrm{dom}(x)), \\ 0 & (\sigma + s \notin \mathrm{dom}(x)), \end{cases}$$

Eq. (4.2) is transformed into

$$y(s) = \int_0^s L y_u \, \mathrm{d}u + L \int_0^s \bar{\psi}_u \, \mathrm{d}u + G(\sigma + s, \sigma) \quad (s \geq 0),$$

which is an integral equation under the initial condition $y_0 = 0$. We choose a constant $a > 0$ so that

$$\|L\| a < 1$$

and consider a closed subset Y of the Banach space $C([-r,a],\mathbb{K}^n)$ given by

$$Y := \{y \in C([-r,a],\mathbb{K}^n) : y_0 = 0\}.$$

Furthermore, we define a transformation $T\colon Y \to Y$ by $(Ty)_0 = 0$ and

$$(Ty)(s) := \int_0^s L y_u \, \mathrm{d}u + L \int_0^s \bar{\psi}_u \, \mathrm{d}u + G(\sigma + s, \sigma) \quad (s \geq 0).$$

Then it holds that T is contractive, and the application of the contraction mapping principle yields the unique existence of a fixed point y_* of T. By defining a function $x_*\colon [\sigma - r, \sigma + a] \to \mathbb{K}^n$ by

$$x_*(\sigma + s) := y_*(s) + \bar{\psi}(s) \quad (s \in \mathrm{dom}(\psi) \cup [0,a]),$$

it is concluded that x_* is a solution of Eq. (4.2). We note that such a local solution is unique by the choice of the above a.

Step 2: Existence and uniqueness of a (global) solution. We note that the time $a > 0$ of existence of a local solution to Eq. (4.2) in Step 1 does not depend on the considered integral equation (4.2) and the specified initial condition $x_\sigma = \psi$. In this step, we will show that by connecting these local solutions, we obtain a global solution. For this purpose, for each $\sigma \geq 0$ and each $\psi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, let

$$x(\,\cdot\,;\sigma,\psi)\colon [\sigma - r, \sigma + a] \to \mathbb{K}^n$$

be the obtained unique solution of Eq. (4.2) under an initial condition $x_\sigma = \psi$. We fix σ and ψ. Let

$$x := x(\,\cdot\,;\sigma,\psi) \quad \text{and} \quad y := x(\,\cdot\,;\sigma + a, x_{\sigma+a}).$$

We now claim that the function $z\colon [\sigma - r, \sigma + 2a] \to \mathbb{K}^n$ defined by

$$z(t) := \begin{cases} x(t) & (t \in [\sigma - r, \sigma + a]) \\ y(t) & (t \in [\sigma + a - r, \sigma + 2a]) \end{cases}$$

is a solution to Eq. (4.2). We note that this definition makes sense because $y_{\sigma+a} = x_{\sigma+a}$. To show the claim, it is sufficient to consider the case $t \in [\sigma + a, \sigma + 2a]$. In this case, we have

$$z(t) = y(t) = x_{\sigma+a}(0) + L \int_{\sigma+a}^t y_s \, \mathrm{d}s + G(t, \sigma + a),$$

where

$$x_{\sigma+a}(0) = x(\sigma + a) = \phi(0) + L \int_{\sigma}^{\sigma+a} x_s \, \mathrm{d}s + G(\sigma + a, \sigma).$$

In the above equations, one can replace y_s and x_s with z_s. Therefore, in view of

$$G(t, \sigma + a) + G(\sigma + a, \sigma) = G(t, \sigma),$$

it holds that z is a solution of Eq. (4.2) under the initial condition $x_{t_0} = \phi$.

By repeating the above procedure, a global solution of the original integral equation (4.1) is obtained. By the uniqueness of each local solution, such a global solution is unique. □

Remark 4.5. Let $\mathbb{K} = \mathbb{R}$. In [21], Hale and Meyer studied the following equation

$$x(t) = \phi(0) + g(t, x_t) - g(t_0, \phi) + \int_{t_0}^{t} f(s, x_s) \, \mathrm{d}s + \int_{t_0}^{t} h(s) \, \mathrm{d}s$$

under an initial condition $x_{t_0} = \phi \in C([-r, 0], \mathbb{R}^n)$ for each $t_0 \in \mathbb{R}$. Here

$$f, g \colon \mathbb{R} \times C([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$$

are continuous maps with the properties that

$$C([-r, 0], \mathbb{R}^n) \ni \phi \mapsto f(t, \phi) \in \mathbb{R}^n \quad \text{and} \quad C([-r, 0], \mathbb{R}^n) \ni \phi \mapsto g(t, \phi) \in \mathbb{R}^n$$

are linear for each $t \in \mathbb{R}$, and $h \colon \mathbb{R} \to \mathbb{R}^n$ is a locally Lebesgue integrable function. In [21, Theorem 1 in Chapter II], it is shown that the above problem has a unique solution under an additional assumption of the non-atomicity of g at 0. See [21, Chapter I] for the detail of this condition. The proof of Theorem 4.4 should be compared with [21, Proof of Theorem 1 in Chapter II].

We hereafter use the following notation.

Notation 5. Let $G \colon [0, \infty) \to \mathbb{K}^n$ be a continuous function with $G(0) = 0$ and

$$\phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$$

be given. The unique solution of Eq. (1.6)

$$x(t) = \phi(0) + L \int_0^t x_s \, \mathrm{d}s + G(t) \qquad (t \geq 0)$$

is denoted by $x^L(\cdot; \phi, G) \colon [-r, \infty) \to \mathbb{K}^n$. Then $x^L(\cdot; \phi, 0) = x^L(\cdot; \phi)$.

We obtain the following corollary. It will be a basics of considering a variation of constants formula for Eq. (1.6).

Corollary 4.6. *For any $\phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$ and any continuous function $G \colon [0, \infty) \to \mathbb{K}^n$ with $G(0) = 0$,*

$$x^L(\cdot; \phi, G) = x^L(\cdot; \phi, 0) + x^L(\cdot; 0, G)$$

holds.

Proof. Let $x := x^L(\cdot\,;\phi,0) + x^L(\cdot\,;0,G)$. Then x satisfies $x_0 = \phi$. Furthermore, we have

$$\begin{aligned} x(t) &= x^L(t;\phi,0) + x^L(t;0,G) \\ &= \phi(0) + L\int_0^t x^L(\cdot\,;\phi,0)_s \,\mathrm{d}s + L\int_0^t x^L(\cdot\,;0,G)_s \,\mathrm{d}s + G(t) \end{aligned}$$

for all $t \ge 0$. Since the last term is equal to

$$\phi(0) + L\int_0^t x_s \,\mathrm{d}s + G(t)$$

by the linearity of L, Theorem 4.4 yields $x = x^L(\cdot\,;\phi,G)$. □

In the same way as in Theorems 3.10 and 3.11 under Theorem 4.4, we obtain the following theorems. The proof can be omitted.

Theorem 4.7. *Let $G\colon [0,\infty) \to \mathbb{K}^n$ be a continuous function with $G(0) = 0$ and $x := x^L(\cdot\,;\hat{\xi},G)$ for some $\xi \in \mathbb{K}^n$. Then x satisfies*

$$x(t) = \xi + \int_{-t}^{0} \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x(s)\,\mathrm{d}s\right) + G(t) = \xi + \left(\mathrm{d}\check{\eta} * Vx|_{[0,\infty)}\right)(t) + G(t) \tag{4.3}$$

for all $t \ge 0$.

Theorem 4.8. *Let $G\colon [0,\infty) \to \mathbb{K}^n$ be a continuous function with $G(0) = 0$ and*

$$x \in \mathcal{L}^1_{\mathrm{loc}}([-r,\infty),\mathbb{K}^n)$$

be given so that $x_0 = \hat{\xi}$ for some $\xi \in \mathbb{K}^n$. If x satisfies (4.3) for all $t \ge 0$, then $x = x^L(\cdot\,;\hat{\xi},G)$.

5 Convolution and Volterra operator

5.1 A motivation to introduce convolution

5.1.1 Variation of constants formula for non-homogeneous linear ODEs

As a motivation to introduce convolution for locally Riemann integrable functions on $[0,\infty)$, we first recall the variation of constants formula for a non-homogeneous linear ODE

$$\dot{x} = Ax + f(t) \tag{5.1}$$

for an $n \times n$ matrix $A \in M_n(\mathbb{K})$ and a continuous function $f\colon \mathbb{R} \to \mathbb{K}^n$. The unique global solution $x^A(\cdot\,;t_0,\xi,f)\colon \mathbb{R} \to \mathbb{K}^n$ of Eq. (5.1) satisfying an initial condition $x(t_0) = \xi \in \mathbb{K}^n$ is expressed by

$$x^A(t;t_0,\xi,f) = \mathrm{e}^{tA}\left[\mathrm{e}^{-t_0A}\xi + \int_{t_0}^{t} \mathrm{e}^{-uA} f(u)\,\mathrm{d}u\right] \qquad (t \in \mathbb{R}) \tag{5.2}$$

with the matrix exponential. This is the *variation of constants formula* for (5.1), which is obtained by finding an equation of $y = y(t)$ under the change of variable $x(t) = \mathrm{e}^{tA}y(t)$. Indeed, the function y must satisfy an initial condition $y(t_0) = \mathrm{e}^{-t_0A}\xi$ and

$$\dot{y}(t) = \mathrm{e}^{-tA} f(t) \qquad (t \in \mathbb{R}).$$

This procedure to derive the formula (5.2) corresponds to replacing a constant vector $v \in \mathbb{K}^n$ in the general solution

$$x(t) = \mathrm{e}^{tA}v$$

for the linear ODE (2.9) with a vector-valued function $y = y(t)$. This is the reason for the terminology of the variation of constants formula.

The above method to derive (5.2) should be called the *method of variation of constants*. Unfortunately, this method does not exist for a non-homogeneous linear RFDE (1.2) because the solution space of the linear RFDE (1.1) is infinite-dimensional and (1.1) does not have a general solution. Even if the method itself does not exist for (1.2), a formula similar to (5.2) if it exists will be useful to analyze the dynamics of RFDEs near equilibria. For this purpose, a form

$$x^A(t;\xi,f) = \mathrm{e}^{tA}\xi + \int_0^t \mathrm{e}^{(t-u)A} f(u)\,\mathrm{d}u, \tag{5.3}$$

which is equivalent to (5.2) is helpful. Here the initial time t_0 is set to 0, and it has been omitted in $x^A(t;\xi,f)$. The first term of the right-hand side of (5.3) is the solution of the linear ODE (2.9) under the initial condition $x(0) = \xi$. Therefore, the second term of the right-hand side of (5.3) is the solution of (5.1) under the initial condition $x(0) = 0$. This can be checked directly by differentiating the second term as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{(t-u)A} f(u)\,\mathrm{d}u &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{tA}\int_0^t \mathrm{e}^{-uA} f(u)\,\mathrm{d}u\right] \\ &= f(t) + A\mathrm{e}^{tA}\int_0^t \mathrm{e}^{-uA} f(u)\,\mathrm{d}u.\end{aligned}$$

We note that this gives another proof of (5.3).

5.1.2 Convolution and non-homogeneous linear RFDEs

For a continuous linear map $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ and a continuous function $f\colon [0,\infty) \to \mathbb{K}^n$, we consider the non-homogeneous linear RFDE (1.7)

$$\dot{x}(t) = Lx_t + f(t) \qquad (t \geq 0).$$

Since $\mathbb{R} \ni t \mapsto \mathrm{e}^{tA} \in M_n(\mathbb{K})$ is the *principal fundamental matrix solution* of the linear ODE (2.9) in the sense that it is a matrix solution to (2.9) and $\mathrm{e}^{tA}|_{t=0}$ is the identity matrix, it is natural to ask whether the function $x(\cdot\,;f)\colon [-r,\infty) \to \mathbb{K}^n$ defined by $x(\cdot\,;f)_0 = 0$ and (1.8)

$$x(t;f) := \int_0^t X^L(t-u)f(u)\,\mathrm{d}u$$

for $t \geq 0$ is a solution to Eq. (1.7). Here $X^L\colon [-r,\infty) \to M_n(\mathbb{K})$ is the principal fundamental matrix solution of the linear RFDE (1.1)

$$\dot{x}(t) = Lx_t \qquad (t \geq 0).$$

In Theorem 3.12, we obtained the differential equation that is satisfied by X^L. However, one can not directly prove that the function $x(\cdot\,;f)$ is a solution to (1.7) by differentiating the right-hand side of (1.8) as in the case of the non-homogeneous linear ODE (5.1) because one cannot take the term $X^L(t)$ out of the integral. This comes from the property that initial value problems of RFDEs cannot be solved backward in general. Therefore, one needs to treat the integral of the right-hand side of (1.8) as it is.

Such an integral is a convolution for locally (Riemann) integrable functions, which should be distinguished from the convolution for integrable functions. The convolution for locally integrable functions has been used in the literature of DDEs. For example, see [2, Chapter 1] with the context of the Laplace transform. The convolution is also used in [34] and [14], however, the detail has been omitted there.

5.2 Convolution and Riemann–Stieltjes convolution

In this subsection, we study a convolution of the following type.

Definition 5.1. For each pair of locally Riemann integrable functions $f, g\colon [0,\infty) \to M_n(\mathbb{K})$, we define a function $g * f\colon [0,\infty) \to M_n(\mathbb{K})$ by

$$(g * f)(t) := \int_0^t g(t-u)f(u)\,\mathrm{d}u = \int_0^t g(u)f(t-u)\,\mathrm{d}u$$

for $t \geq 0$. Here the above integrals are Riemann integrals. We call the function $g * f$ the *convolution* of g and f.

See [31, Section 5.3] for the convolution of continuous functions. We note that when f is a constant function,

$$(g * f)(t) = \int_0^t g(u)f(0)\,\mathrm{d}u = (Vg)(t)f(0) \tag{5.4}$$

holds for all $t \geq 0$. In the same way, $g * f = g(0)Vf$ holds when g is constant.

Lemma 5.2 (cf. [31]). *Let $f, g\colon [0,\infty) \to M_n(\mathbb{K})$ be locally Riemann integrable functions. If f is continuous, then $g * f$ is a sum of a continuous function and a locally absolutely continuous function.*

Proof. By using (5.4),

$$g * f = g * (f - f(0)) + (Vg)f(0)$$

holds. Therefore, the conclusion is obtained by showing that $g * f$ is continuous when $f(0) = O$. We extend the domain of definition of f to $\mathbb{R}$ by defining $f(t) := f(0) = O$ for $t \leq 0$. Let $s, t \in [0,\infty)$ be given so that $s < t$. By the same reasoning as in the proof of Lemma 3.4, we have

$$(g * f)(t) - (g * f)(s) = \int_0^t g(u)[f(t-u) - f(s-u)]\,\mathrm{d}u.$$

By combining this and the uniform continuity of f on closed and bounded intervals, the continuity of $g * f$ is obtained. □

5.2.1 Convolution of locally BV functions and continuous functions

By using Theorem 3.7, one can obtain the following result on the regularity of convolution.

Theorem 5.3 (cf. [33]). *If $f\colon [0,\infty) \to M_n(\mathbb{K})$ is continuous and $g\colon [0,\infty) \to M_n(\mathbb{K})$ is of locally bounded variation, then*

$$g * f = g(0)Vf + \mathrm{d}g * (Vf) = V(g(0)f + \mathrm{d}g * f) \tag{5.5}$$

holds. Consequently, the convolution $g * f$ is locally absolutely continuous, differentiable almost everywhere, and satisfies

$$(g * f)'(t) = g(0)f(t) + (\mathrm{d}g * f)(t)$$

for almost all $t \geq 0$.

The above result is considered as the finite-dimensional version of [33, Theorem 3.2] (i.e., the case that the Banach space X in [33, Theorem 3.2] is finite-dimensional) except for the equality

$$g * f = g(0)Vf + \mathrm{d}g * (Vf).$$

In the following, we give a simpler proof of Theorem 5.3 based on Theorem 3.7.

Proof of Theorem 5.3. Since Vf is continuously differentiable and $(Vf)(0) = O$,

$$\begin{aligned}[\mathrm{d}g * (Vf)](t) &= [g(u)(Vf)(t-u)]_{u=0}^{t} + \int_0^t g(u)f(t-u)\,\mathrm{d}u \\ &= -g(0)(Vf)(t) + (g * f)(t)\end{aligned}$$

holds for all $t \geq 0$ by the integration by parts formula for Riemann–Stieltjes integrals and from Theorem A.19. By combining the obtained equality

$$g * f = g(0)(Vf) + \mathrm{d}g * (Vf)$$

and Theorem 3.7, the equality (5.5) is obtained. □

Remark 5.4. Let $f\colon [0,\infty) \to M_n(\mathbb{K})$ be a continuous function and $g\colon [0,\infty) \to M_n(\mathbb{K})$ be a function of locally bounded variation. By defining a function $V(\mathrm{d}g)\colon [0,\infty) \to M_n(\mathbb{K})$ by

$$V(\mathrm{d}g)(t) := g(t) - g(0)$$

for $t \geq 0$, we have

$$V(\mathrm{d}g * f) = \mathrm{d}g * (Vf) = V(\mathrm{d}g) * f$$

from Theorems 3.7 and 5.3. This formula is easy to remember. We note that the above definition of $V(\mathrm{d}g)$ is reasonable because

$$\int_0^t \mathrm{d}g(u) = g(t) - g(0)$$

holds for all $t \geq 0$.

We have the following corollaries.

Corollary 5.5. *If $f\colon [0,\infty) \to M_n(\mathbb{K})$ is continuous and $g\colon [0,\infty) \to M_n(\mathbb{K})$ is of locally bounded variation, then*

$$V(g * f) = g * (Vf)$$

holds.

Proof. From Theorems 5.3 and 3.7, we have

$$V(g * f) = g(0)\big(V^2 f\big) + \mathrm{d}g * \big(V^2 f\big),$$

where $V^2 f := V(Vf)$. Since the right-hand side is equal to $g * (Vf)$ from Theorem 5.3, the equality is obtained. □

Corollary 5.6. *Let $f\colon [0,\infty) \to M_n(\mathbb{K})$ be a continuous function and $g\colon [0,\infty) \to M_n(\mathbb{K})$ be a function of locally bounded variation. Then the following statements hold:*

1. *If g is continuous or $f(0) = O$, then $g * f$ is continuously differentiable and*

$$(g * f)' = g(0)f + \mathrm{d}g * f$$

holds.

2. *If g is locally absolutely continuous, then $g * f$ is continuously differentiable and*

$$(g * f)' = g(0)f + g' * f$$

holds. Here $g' * f\colon [0,\infty) \to M_n(\mathbb{K})$ is the function defined by

$$(g' * f)(t) := \int_0^t g'(t-u)f(u)\,\mathrm{d}u = \int_0^t g'(u)f(t-u)\,\mathrm{d}u$$

for $t \geq 0$, where the integrals are Lebesgue integrals.

Proof. 1. Under the assumption, $\mathrm{d}g * f$ is continuous from Lemma 3.4 and Remark 3.6. Therefore, the conclusion follows by the formula (5.5).

2. The continuous differentiability of $g * f$ follows by the statement 1. When g is locally absolutely continuous,

$$\mathrm{d}g * f = g' * f$$

holds from Theorem A.20. □

5.2.2 Associativity of Riemann–Stieltjes convolution

For the proof of Theorem 5.9 below, we need the following result.

Theorem 5.7 (refs. [17], [31]). *Let $\alpha\colon [0,\infty) \to M_n(\mathbb{K})$ be a function of locally bounded variation. Then for any continuous functions $f, g\colon [0,\infty) \to M_n(\mathbb{K})$,*

$$\mathrm{d}\alpha * (g * f) = (\mathrm{d}\alpha * g) * f \tag{5.6}$$

holds.

Remark 5.8. Both sides of Eq. (5.6) are meaningful from Lemma 5.2 and Theorem 3.5. Eq. (3.3) is a special case of (5.6) since we have

$$Vf = f * \mathcal{I} = \mathcal{I} * f$$

for any $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. Here $\mathcal{I}\colon [0,\infty) \to M_n(\mathbb{K})$ denotes the constant function whose value is equal to the identity matrix I.

The above is a result on the associativity for Riemann–Stieltjes convolutions. The corresponding statements in a more general setting are given in [17, Section 6 in Chapter 3]. See also [31, Proposition D.9 in Appendix D] for a similar result to Theorem 5.7.

One can prove Theorem 5.7 by the same reasoning as in the proof of Theorem 3.7, however, we give an outline of the proof for reader's convenience.

Outline of the proof of Theorem 5.7. We extend the domain of definition of g to $\mathbb{R}$ by defining $g(t) := g(0)$ for $t \leq 0$. Then the obtained function $g\colon \mathbb{R} \to M_n(\mathbb{K})$ is continuous. Let $t > 0$ be fixed. By the proof of Lemma 3.4, we have

$$(\mathrm{d}\alpha * g)(u) = \int_0^t \mathrm{d}\alpha(v)\, g(u-v) - [\alpha(t) - \alpha(u)]g(0)$$

for $u \in [0,t]$. Therefore, $[(\mathrm{d}\alpha * g) * f](t)$ is expressed as

$$[(\mathrm{d}\alpha * g) * f](t) = \int_0^t \left(\int_0^t \mathrm{d}\alpha(v)\, g(u-v) \right) f(t-u)\,\mathrm{d}u - \int_0^t [\alpha(t) - \alpha(u)]g(0)f(t-u)\,\mathrm{d}u.$$

Since

$$[0,t] \times [0,t] \ni (u,v) \mapsto g(u-v)f(t-u) \in M_n(\mathbb{K})$$

is continuous, the first term of the right-hand side becomes

$$\int_0^t \mathrm{d}\alpha(v) \left(\int_0^t g(u-v)f(t-u)\,\mathrm{d}u \right)$$

from Theorem 3.8. Here the integrand also becomes

$$\int_{-v}^{t-v} g(u)f(t-u-v)\,\mathrm{d}u = (g * f)(t-v) + \int_{-v}^{0} g(0)f(t-u-v)\,\mathrm{d}u.$$

Then the proof is complete by showing

$$\int_0^t [\alpha(t) - \alpha(u)]g(0)f(t-u)\,\mathrm{d}u = \int_0^t \mathrm{d}\alpha(v) \left(\int_{-v}^{0} g(0)f(t-u-v)\,\mathrm{d}u \right).$$

One can prove this by using the integration by parts formula for Riemann–Stieltjes integrals. □

5.3 A formula for non-homogeneous equations with trivial initial history

Let $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ be a continuous linear map. We recall that for a continuous map $G\colon [0,\infty) \to \mathbb{K}^n$ with $G(0) = 0$, the function $x^L(\cdot;0,G)\colon [-r,\infty) \to \mathbb{K}^n$ denotes the unique solution of an integral equation

$$x(t) = L \int_0^t x_s\,\mathrm{d}s + G(t) \qquad (t \geq 0) \tag{5.7}$$

under the initial condition $x_0 = 0$.

In this subsection, as an application of the results in Subsection 5.2, we show that the function $x(\cdot;f)\colon [-r,\infty) \to \mathbb{K}^n$ defined by $x(\cdot;f)_0 = 0$ and (1.8) is a solution to the non-homogeneous linear RFDE (1.7).

Theorem 5.9 (cf. [35])**.** *Let $f\colon [0,\infty) \to \mathbb{K}^n$ be a continuous function. Then*

$$x^L(t;0,Vf) = \int_0^t X^L(t-u)f(u)\,\mathrm{d}u \tag{5.8}$$

holds for all $t \geq 0$.

We note that $x^L(\cdot;0,Vf)$ is a solution to Eq. (1.7) (see Lemma 4.2).

Proof of Theorem 5.9. Let $x := x(\cdot\,; f)|_{[0,\infty)}$ and $X := X^L|_{[0,\infty)}$. Since X is locally absolutely continuous (see Theorem 3.12) and $X(0) = I$,

$$x = X * f = Vf + \dot{X} * (Vf)$$

holds from Corollary 5.6. For the term $\dot{X} * (Vf)$, we have

$$\begin{aligned}\dot{X} * (Vf) &= (\mathrm{d}\check{\eta} * X) * (Vf) \\ &= \mathrm{d}\check{\eta} * [X * (Vf)] \\ &= \mathrm{d}\check{\eta} * V(X * f)\end{aligned}$$

from Theorems 3.12, 5.7, and Corollary 5.5. This shows that $x(\cdot\,; f)$ satisfies

$$x(t; f) = \big(\mathrm{d}\check{\eta} * Vx(\cdot\,; f)|_{[0,\infty)}\big)(t) + (Vf)(t)$$

for all $t \geq 0$. Therefore, the equality (5.8) is obtained from Theorem 4.8. □

The above proof of Theorem 5.9 is different from the proofs in the literature (e.g., see [35, Section 4]).

6 Variation of constants formula

Let $L\colon C([-r, 0], \mathbb{K}^n) \to \mathbb{K}^n$ be a continuous linear map and $X^L\colon [-r, \infty) \to M_n(\mathbb{K})$ be the principal fundamental matrix solution of the linear RFDE (1.1)

$$\dot{x}(t) = Lx_t \qquad (t \geq 0).$$

In this section, we obtain a "variation of constants formula" for the non-homogeneous linear RFDE (1.2)

$$\dot{x}(t) = Lx_t + g(t) \qquad (\text{a.e. } t \geq 0)$$

for some $g \in \mathcal{L}^1_{\mathrm{loc}}([0, \infty), \mathbb{K}^n)$ expressed by X^L. In view of Corollary 4.6, we will divide our consideration into the following steps:

- Step 1: To find a formula for the mild solution $x^L(\cdot\,; 0, Vg)$ of Eq. (1.2) under the initial condition $x_0 = 0$.
- Step 2: To find a formula for the mild solution $x^L(\cdot\,; \phi, 0)$ of Eq. (1.1) under the initial condition $x_0 = \phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$.

Then the full formula for the mild solution of (1.2) under the initial condition $x_0 = \phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$ is obtained by combining the above formulas. In Step 1, for a given continuous function $G\colon [0, \infty) \to \mathbb{K}^n$ with $G(0) = 0$, we indeed consider the integral equation (5.7)

$$x(t) = L\int_0^t x_s \,\mathrm{d}s + G(t) \qquad (t \geq 0)$$

under the initial condition $x_0 = 0$ and try to find a formula for the solution $x^L(\cdot\,; 0, G)$ expressed by X^L.

Remark 6.1. Since $x_0 = 0 = \hat{0}$, Eq. (5.7) is equivalent to

$$x(t) = \int_{-t}^{0} \mathrm{d}\eta(\theta) \left(\int_{0}^{t+\theta} x(s)\,\mathrm{d}s \right) + G(t) \quad (t \geq 0)$$

from Lemma 3.3.

The following is the main result of this section.

Theorem 6.2. *Let $G\colon [0,\infty) \to \mathbb{K}^n$ be a continuous function with $G(0) = 0$ and $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ be given. Then the solution $x^L(\cdot;\phi,G)$ of the integral equation* (1.6)

$$x(t) = \phi(0) + L\int_0^t x_s\,\mathrm{d}s + G(t) \quad (t \geq 0)$$

under the initial condition $x_0 = \phi$ satisfies (1.9)

$$x^L(t;\phi,G) = X^L(t)\phi(0) + \left[G^L(t;\phi) + G(t)\right] + \int_0^t \dot{X}^L(t-u)\left[G^L(u;\phi) + G(u)\right]\mathrm{d}u$$

for all $t \geq 0$.

We will call the formula (1.9) the *variation of constants formula* for Eq. (1.6). The definition of the function $G^L(\cdot;\phi)\colon [0,\infty) \to \mathbb{K}^n$ for $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ will be given later. For this definition, the expression of L by the Riemann–Stieltjes integral (1.3)

$$L\psi = \int_{-r}^{0} \mathrm{d}\eta(\theta)\,\psi(\theta)$$

for $\psi \in C([-r,0],\mathbb{K}^n)$ is a key tool.

6.1 Motivation: Naito's consideration

We first concentrate our consideration to the case that $g \in C([0,\infty),\mathbb{K}^n)$ and $\phi \in C([-r,0],\mathbb{K}^n)$. From Theorem 5.9, we only need to find a formula for $x^L(\cdot;\phi,0)$ in this case.

Naito [26, Theorem 6.5] has discussed an expression of the form

$$x(t) = \phi(0) + \int_0^t X(t-u)L\bar{\phi}_u\,\mathrm{d}u \quad (t \geq 0).$$

In the above formula, $x\colon [-r,\infty) \to \mathbb{K}^n$ is the solution of the linear RFDE (1.1) under the initial condition $x_0 = \phi \in C([-r,0],\mathbb{K}^n)$, and $\bar{\phi}\colon [-r,\infty) \to \mathbb{K}^n$ is the function defined by

$$\bar{\phi}(t) := \begin{cases} \phi(t) & (t \in [-r,0]), \\ \phi(0) & (t \geq 0). \end{cases}$$

See also Notation 1. Although the study of [26] is in the setting of infinite retardation, we are now interpreting this in the setting of finite retardation (i.e., the history function space is $C([-r,0],\mathbb{K}^n)$). We note that the matrix-valued function $X\colon [0,\infty) \to M_n(\mathbb{K})$ is defined by using the inverse Laplace transform. See [26] for the detail. See also [27], where an interpretation of the matrix-valued function X is given.

In our setting, a formula expressed by the principal fundamental matrix solution X^L

$$x^L(t;\phi,0) = \phi(0) + \int_0^t X^L(t-u)L\bar{\phi}_u\,\mathrm{d}u \quad (t \geq 0) \tag{6.1}$$

is true. To see this, let $y(t) := x^L(t;\phi,0) - \bar{\phi}(t)$ for $t \in [-r,\infty)$. Then the function $y\colon [-r,\infty) \to \mathbb{K}^n$ satisfies $y_0 = 0$ and

$$\dot{y}(t) = Ly_t + L\bar{\phi}_t \qquad (t \geq 0).$$

See also the proof of Theorem 2.14. Since the function $[0,\infty) \ni t \mapsto L\bar{\phi}_t \in \mathbb{K}^n$ is continuous, we obtain

$$y(t) = \int_0^t X^L(t-u)L\bar{\phi}_u \,\mathrm{d}u \qquad (t \geq 0)$$

by applying Theorem 5.9.

6.2 Derivation of a general forcing term

The formula (6.1) is not sufficient for the application to the linearized stability. See Section 8 for the detail of the application of the variation of constants formula to the linearized stability. We now introduce the following function.

Notation 6. For each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, we define a function $z^L(\cdot\,;\phi)\colon [-r,\infty) \to \mathbb{K}^n$ by $z^L(\cdot\,;\phi)_0 = 0$ and (1.11)

$$z^L(t;\phi) := x^L(t;\phi,0) - X^L(t)\phi(0)$$

for $t \geq 0$.

Remark 6.3. Since

$$z^L(0;\phi) = \phi(0) - X^L(0)\phi(0) = 0,$$

the function $z^L(\cdot\,;\phi)$ is continuous. In view of $X^L(\cdot)\phi(0) = x^L\Big(\cdot\,;\widehat{\phi(0)},0\Big)$, we also have

$$z^L(t;\phi) = x^L\Big(t;\phi - \widehat{\phi(0)},0\Big) \qquad (t \geq 0)$$

from Corollary 2.15. We note that this equality is not valid for $t \in [-r,0)$ because $z^L(\cdot\,;\phi)_0 = 0$.

From the expression (2.2) for a mild solution, the function $z^L(\cdot\,;\phi)$ satisfies

$$z^L(t;\phi) = \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_\theta^0 \phi(s)\,\mathrm{d}s\right) + \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} x^L\Big(s;\phi - \widehat{\phi(0)},0\Big)\,\mathrm{d}s\right)$$

for all $t \geq 0$. The second term of the right-hand side is further calculated as follows:

- When $t \in [0,r)$, $\theta \in [-r,0]$ satisfies $t + \theta \geq 0$ if and only if $\theta \in [-t,0]$. Since

  $$x^L\Big(s;\phi - \widehat{\phi(0)},0\Big) = \phi(s)$$

  for $s \in \mathrm{dom}(\phi) \setminus \{0\}$, the second term is decomposed by

  $$\int_{-r}^{-t} \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \phi(s)\,\mathrm{d}s\right) + \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} z^L(s;\phi)\,\mathrm{d}s\right)$$

  by the additivity of Riemann–Stieltjes integrals on sub-intervals.

- When $t \geq r$, the second term is equal to $\int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} z^L(s;\phi)\,\mathrm{d}s\right)$.

This leads to the following definition.

Definition 6.4. For each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, we define a function $G^L(\cdot;\phi)\colon [0,\infty) \to \mathbb{K}^n$ by

$$G^L(t;\phi) := \int_{-r}^{0} \mathrm{d}\eta(\theta) \left(\int_{\theta}^{0} \phi(s)\,\mathrm{d}s \right) + \int_{-r}^{-t} \mathrm{d}\eta(\theta) \left(\int_{0}^{t+\theta} \phi(s)\,\mathrm{d}s \right)$$

for $t \in [0,r)$ and

$$G^L(t;\phi) := \int_{-r}^{0} \mathrm{d}\eta(\theta) \left(\int_{\theta}^{0} \phi(s)\,\mathrm{d}s \right)$$

for $t \in [r,\infty)$.

By definition, $G^L(0;\phi) = 0$ holds. Summarizing the above discussion, we obtain the following lemma.

Lemma 6.5. *For each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$, the function $z := z^L(\cdot;\phi)$ is a solution of an integral equation* (1.12)

$$z(t) = L \int_0^t z_s\,\mathrm{d}s + G^L(t;\phi) \qquad (t \geq 0)$$

under the initial condition $z_0 = 0$.

6.3 Regularity of the general forcing term

To study Eq. (1.12), it is important to reveal the regularity of the function $G^L(\cdot;\phi)$ for each $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$.

6.3.1 Forcing terms for continuous initial histories

Before we tackle this problem, we find a differential equation satisfied by $z := z^L(\cdot;\phi)$ for $\phi \in C([-r,0],\mathbb{K}^n)$. It should be noted that this is not straightforward because (1.11) is only valid for $t \geq 0$.

Let $x := x^L(\cdot;\phi,0)$ and $\tilde{x} := x^L\left(\cdot;\widehat{\phi(0)},0\right)$. In view of

$$Lz_t = \int_{-r}^{0} \mathrm{d}\eta(\theta)\,z(t+\theta) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\,z(t+\theta)$$

for each $t \geq 0$, we express the linear RFDE (1.1) as

$$\dot{x}(t) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\,x(t+\theta) + \int_{-r}^{-t} \mathrm{d}\eta(\theta)\,\phi(t+\theta)$$

by using the additivity of Riemann–Stieltjes integrals on sub-intervals. Here we are interpreting that the second term of the right-hand side is equal to 0 when $t \geq r$. More precisely, we introduce the following.

Definition 6.6 (cf. [3], [9], [25])**.** For each $\phi \in C([-r,0],\mathbb{K}^n)$, we define a function

$$g^L(\cdot;\phi)\colon [0,\infty) \to \mathbb{K}^n$$

by

$$g^L(t;\phi) := \int_{-r}^{-t} \mathrm{d}\eta(\theta)\,\phi(t+\theta)$$

for $t \in [0,r)$ and $g^L(t;\phi) = 0$ for $t \geq r$. Here the right-hand side is a Riemann–Stieltjes integral.

We note that similar concepts have appeared in the literature. See [3, (3.1) and (3.2)], [9, (2.7) and (2.13)], and [25, Lemma 1.10], for example.

From Theorem 3.10, $\tilde{x}$ satisfies

$$\dot{\tilde{x}}(t) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\,\tilde{x}(t+\theta)$$

for almost all $t \geq 0$. In combination with the above consideration, z satisfies

$$\begin{aligned}\dot{z}(t) &= \dot{x}(t) - \dot{\tilde{x}}(t) \\ &= \int_{-t}^{0} \mathrm{d}\eta(\theta)\,z(t+\theta) + g^L(t;\phi)\end{aligned}$$

for almost all $t \geq 0$. Here the property that $t + \theta \geq 0$ for all $\theta \in [-t, 0]$ is used.

In summary, we have the following statement.

Lemma 6.7. *For each $\phi \in C([-r,0],\mathbb{K}^n)$, $z := z^L(\cdot;\phi)$ is locally absolutely continuous, differentiable almost everywhere, and*

$$\dot{z}(t) = Lz_t + g^L(t;\phi) \tag{6.2}$$

holds for almost all $t \geq 0$.

We note that since $g^L(\cdot;\phi)$ is not necessarily continuous, Theorem 5.9 is not sufficient to obtain an expression of $z = z^L(\cdot;\phi)$ by X^L.

6.3.2 Relationship with the forcing terms

Comparing (1.12) and (6.2), it is natural to expect that

$$G^L(t;\phi) = \int_0^t g^L(s;\phi)\,\mathrm{d}s \tag{6.3}$$

holds for all $t \geq 0$ when $\phi \in C([-r,0],\mathbb{K}^n)$. We now justify this relationship.

Lemma 6.8. *Suppose $\phi \in C([-r,0],\mathbb{K}^n)$. Then*

$$g^L(t;\phi) = L\bar{\phi}_t - [\eta(0) - \eta(-t)]\phi(0) \tag{6.4}$$

holds for all $t \geq 0$. Consequently, $g^L(\cdot;\phi)$ is a locally Riemann integrable function vanishing at $[r,\infty)$.

Proof. When $t \geq r$,

$$L\bar{\phi}_t = \int_{-r}^{0} \mathrm{d}\eta(\theta)\,\phi(0) = [\eta(0) - \eta(-r)]\phi(0)$$

holds. Therefore, the right-hand side of (6.4) is equal to 0 for all $t \geq r$. We next consider the case $t \in [0, r)$. In this case, we have

$$\begin{aligned}g^L(t;\phi) &= \int_{-r}^{-t} \mathrm{d}\eta(\theta)\,\bar{\phi}(t+\theta) \\ &= \int_{-r}^{0} \mathrm{d}\eta(\theta)\,\bar{\phi}(t+\theta) - \int_{-t}^{0} \mathrm{d}\eta(\theta)\,\bar{\phi}(t+\theta)\end{aligned}$$

by the additivity of Riemann–Stieltjes integrals on sub-intervals. Since

$$\int_{-t}^{0} \mathrm{d}\eta(\theta)\,\bar{\phi}(t+\theta) = \int_{-t}^{0} \mathrm{d}\eta(\theta)\,\phi(0) = [\eta(0) - \eta(-t)]\phi(0),$$

the expression (6.4) is obtained. Since $[0,\infty) \ni t \mapsto L\bar{\phi}_t \in \mathbb{K}^n$ is continuous and $[0,\infty) \ni t \mapsto \eta(-t)\phi(0)$ is of locally bounded variation, the local Riemann integrability of $g^L(\cdot;\phi)$ follows by the expression (6.4). ☐

Remark 6.9. The expression (6.4) also shows that $g^L(\cdot;\phi)$ is continuous if $\phi(0) = 0$. This should be compared with [9, Theorem 2.1(ii) and Remark 2.1].

The following theorem reveals a connection between $G^L(\cdot;\phi)$ and $g^L(\cdot;\phi)$.

Theorem 6.10. *Let $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ be given. Then for all $t \geq 0$,*

$$G^L(t;\phi) = L\int_0^t \bar{\phi}_s \, \mathrm{d}s - \int_0^t [\eta(0) - \eta(-s)]\phi(0)\,\mathrm{d}s \tag{6.5}$$

holds.

Proof. For the first term of the definition of $G^L(t;\phi)$, we have

$$\begin{aligned}\int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_\theta^0 \phi(s)\,\mathrm{d}s\right) &= \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_\theta^0 \bar{\phi}(s)\,\mathrm{d}s\right)\\ &= \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_\theta^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right) - \int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right),\end{aligned}$$

where the first term of the last equation is equal to $L\int_0^t \bar{\phi}_s\,\mathrm{d}s$. The remainder of the proof is divided into the cases $t \in [0,r]$ and $t \in (r,\infty)$ in order to study the term $\int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right)$.

Case 1: $t \in [0,r]$**.** When $t = r$, we have

$$\int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right) = \int_{-r}^0 \mathrm{d}\eta(\theta)\,(r+\theta)\phi(0)$$

because $r + \theta \geq 0$ for all $\theta \in [-r,0]$. We next consider the case $t \in [0,r)$. In this case,

$$\begin{aligned}&\int_{-r}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right)\\ &= \int_{-r}^{-t} \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \phi(s)\,\mathrm{d}s\right) + \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right)\end{aligned}$$

holds by the additivity of Riemann–Stieltjes integrals on sub-intervals and by the property that $t + \theta \leq 0$ for all $\theta \in [-r,-t]$. Here the second term of the right-hand side is equal to

$$\int_{-t}^0 \mathrm{d}\eta(\theta)\,(t+\theta)\phi(0).$$

Therefore, the definition of $G^L(t;\phi)$ yields

$$\begin{aligned}G^L(t;\phi) &= L\int_0^t \bar{\phi}_s\,\mathrm{d}s - \int_{-t}^0 \mathrm{d}\eta(\theta)\left(\int_0^{t+\theta} \bar{\phi}(s)\,\mathrm{d}s\right)\\ &= L\int_0^t \bar{\phi}_s\,\mathrm{d}s - \int_{-t}^0 \mathrm{d}\eta(\theta)\,(t+\theta)\phi(0)\end{aligned}$$

including the case $t = r$. The proof is complete in view of

$$\begin{aligned}\int_{-t}^0 (t+\theta)\,\mathrm{d}\eta(\theta) &= [(t+\theta)\eta(\theta)]_{\theta=-t}^0 - \int_{-t}^0 \eta(\theta)\,\mathrm{d}\theta\\ &= t\eta(0) - \int_0^t \eta(-s)\,\mathrm{d}s,\end{aligned}$$

where the integration by parts formula for Riemann–Stieltjes integrals is used.

Case 2: $t \in (r, \infty)$**.** Since we have shown that (6.5) holds for $t = r$,

$$G^L(t;\phi) = L\int_0^r \bar{\phi}_s \, \mathrm{d}s - \int_0^r [\eta(0) - \eta(-s)]\phi(0)\, \mathrm{d}s$$

holds for all $t \geq r$. Here the property that $G^L(\cdot;\phi)$ is constant on $[r, \infty)$ is used. Then the proof is complete by showing that the right-hand side of (6.5) is constant on $[r, \infty)$. For this purpose, we calculate

$$L\int_0^t \bar{\phi}_s \, \mathrm{d}s - L\int_0^r \bar{\phi}_s \, \mathrm{d}s.$$

By the linearity of L, it is calculated as

$$\begin{aligned}\int_{-r}^0 \mathrm{d}\eta(\theta) \left(\int_{r+\theta}^{t+\theta} \bar{\phi}(s)\, \mathrm{d}s\right) &= \int_{-r}^0 \mathrm{d}\eta(\theta)\,(t-r)\phi(0) \\ &= (t-r)[\eta(0) - \eta(-r)]\phi(0).\end{aligned}$$

Since η is constant on $(-\infty, -r]$, the last value is expressed as

$$\int_r^t [\eta(0) - \eta(-s)]\phi(0)\, \mathrm{d}s.$$

This shows that

$$L\int_0^t \bar{\phi}_s \, \mathrm{d}s = L\int_0^r \bar{\phi}_s \, \mathrm{d}s + \int_r^t [\eta(0) - \eta(-s)]\phi(0)\, \mathrm{d}s,$$

which also implies that the right-hand side of (6.5) is equal to

$$L\int_0^r \bar{\phi}_s \, \mathrm{d}s - \int_0^r [\eta(0) - \eta(-s)]\phi(0)\, \mathrm{d}s$$

for all $t \geq r$. □

Remark 6.11. $G^L(t;\phi)$ is also expressed as

$$G^L(t;\phi) = \int_{-r}^0 [\eta(\theta) - \eta(\theta - t)]\phi(\theta)\, \mathrm{d}\theta.$$

See [14, Section I.2 of Chapter I] for the detail. See also [34, Remark 2.10(iii) in Chapter 2]. In this paper, we do not need the above expression.

By combining the obtained results, we obtain the following result on the regularity of $G^L(\cdot;\phi)$. See also [34, Remark 2.10(ii) in Chapter 2].

Theorem 6.12. *For any $\phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$, the function $G^L(\cdot;\phi)$ is continuous with $G^L(0;\phi) = 0$. Furthermore, if $\phi \in C([-r, 0], \mathbb{K}^n)$, then it is locally absolutely continuous, differentiable almost everywhere, and*

$$\dot{G}^L(t;\phi) = g^L(t;\phi)$$

holds for almost all $t \in [0, \infty)$. Here $\dot{G}^L(t;\phi)$ denotes the derivative of $G^L(\cdot;\phi)$ at t.

Proof. Let $\phi \in \mathcal{M}^1([-r, 0], \mathbb{K}^n)$ be given. Then (6.5) yields the continuity of $G^L(\cdot;\phi)$ from Lemma 2.8 and by the continuity of L. The property $G^L(0;\phi) = 0$ follows by definition. We next assume $\phi \in C([-r, 0], \mathbb{K}^n)$. Since $\bar{\phi} \colon [-r, \infty) \to \mathbb{K}^n$ is continuous, Theorem 6.10, Corollary 2.12, and Lemma 6.8 show that (6.3)

$$G^L(t;\phi) = \big(Vg^L(\cdot;\phi)\big)(t)$$

holds for all $t \geq 0$. This yields the properties of $G^L(\cdot;\phi)$. □

6.4 Derivation of the variation of constants formula

6.4.1 Formulas for trivial initial histories

For the derivation of the variation of constants formula, we use the following result.

Theorem 6.13. *Let $[a, b]$ be a closed and bounded interval of $\mathbb{R}$. If $F, G\colon [a, b] \to M_n(\mathbb{K})$ are absolutely continuous, then*

$$\int_a^b F'(x)G(x)\,\mathrm{d}x = [F(x)G(x)]_{x=a}^b - \int_a^b F(x)G'(x)\,\mathrm{d}x$$

holds.

This should be called the integration by parts formula for matrix-valued absolutely continuous functions. We note that the above result also holds when $G\colon [a, b] \to \mathbb{K}^n$ is an absolutely continuous function. Since the Lebesgue integral of a matrix-valued function is defined component-wise, Theorem 6.13 can be obtained by the corresponding result for scalar-valued functions in combination with the linearity of Lebesgue integration. We note that the result for scalar-valued functions is mentioned in [30, Exercise 14 of Chapter 7]. One can also give a direct proof based on the matrix product.

By using the local absolute continuity of $X^L|_{[0,\infty)}$ (see Theorem 3.12), Theorem 6.13 shows that

$$\begin{aligned}\int_0^t X^L(t-u)g(u)\,\mathrm{d}u &= [X^L(t-u)(Vg)(u)]_{u=0}^t + \int_0^t \dot{X}^L(t-u)(Vg)(u)\,\mathrm{d}u \\ &= (Vg)(t) + \int_0^t \dot{X}^L(t-u)(Vg)(u)\,\mathrm{d}u\end{aligned}$$

holds for any $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), \mathbb{K}^n)$. Here $X^L(0) = I$ and $(Vg)(0) = 0$ are also used. The following theorem is motivated by this.

Theorem 6.14. *Let $G\colon [0,\infty) \to \mathbb{K}^n$ be a continuous function with $G(0) = 0$. Then (1.10)*

$$x^L(t; 0, G) = G(t) + \int_0^t \dot{X}^L(t-u)G(u)\,\mathrm{d}u$$

holds for all $t \geq 0$.

Proof. Let $X := X^L|_{[0,\infty)}$. We define a function $x\colon [-r,\infty) \to \mathbb{K}^n$ by $x_0 = 0$ and

$$x(t) := G(t) + \big(\dot{X} * G\big)(t)$$

for $t \geq 0$. By applying Corollary 5.6 in combination with the fundamental theorem of calculus, we have $Vx|_{[0,\infty)} = X * G$. Here $(X * G)(0) = 0$ is also used. Furthermore, we have

$$x(t) = G(t) + [\mathrm{d}\check{\eta} * (X * G)](t) \quad (t \geq 0)$$

from Theorems 3.12 and 5.7. Therefore, x satisfies

$$x(t) = G(t) + \big(\mathrm{d}\check{\eta} * Vx|_{[0,\infty)}\big)(t)$$

for all $t \geq 0$. This implies that (1.10) holds by applying Theorem 4.8. □

The following corollary is obtained from Theorem 6.14 by using the discussion before Theorem 6.14. It is an extension of Theorem 5.9.

Corollary 6.15 (cf. [18], [19])**.** *Let $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), \mathbb{K}^n)$. Then*

$$x^L(t; 0, Vg) = \int_0^t X^L(t-u)g(u)\,\mathrm{d}u \tag{6.6}$$

holds for all $t \geq 0$.

6.4.2 Formulas for homogeneous equations

We next find an expression of $x^L(\cdot;\phi,0)$ by X^L as an application of Theorem 6.14.

Theorem 6.16. *Let $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$. Then*

$$x^L(t;\phi,0) = X^L(t)\phi(0) + G^L(t;\phi) + \int_0^t \dot{X}^L(t-u)G^L(u;\phi)\,\mathrm{d}u \tag{6.7}$$

holds for all $t \geq 0$.

Proof. From Lemma 6.5 and Theorem 6.14 together with Theorem 6.12,

$$z^L(t;\phi) = G^L(t;\phi) + \int_0^t \dot{X}^L(t-u)G^L(u;\phi)\,\mathrm{d}u$$

holds for all $t \geq 0$. Then the formula (6.7) is obtained in view of

$$z^L(t;\phi) = x^L(t;\phi,0) - X^L(t)\phi(0)$$

for $t \geq 0$. □

Remark 6.17. The above proof of Theorem 6.16 is considered to be a reorganization of [14, Section I.2 of Chapter I]. It leads us to the understanding of the variation of constants formula for non-homogeneous linear RFDEs that does not rely on the theory of Volterra convolution integral equations.

We have the following corollary.

Corollary 6.18 (cf. [25])**.** *Let $\phi \in C([-r,0],\mathbb{K}^n)$. Then*

$$x^L(t;\phi,0) = X^L(t)\phi(0) + \int_0^t X^L(t-u)g^L(u;\phi)\,\mathrm{d}u \tag{6.8}$$

holds for all $t \geq 0$.

Proof. From Theorem 6.12,

$$G^L(\cdot;\phi) = V\big(g^L(\cdot;\phi)\big)$$

holds. Therefore, the formula (6.8) is obtained from (6.7) by using the integration by parts formula for matrix-valued absolutely continuous functions. □

Corollary 6.18 should be compared with [25, Theorem 1.11], where the inverse Laplace transform is used to obtain a formula.

6.4.3 Derivation of the main result of this section

Theorem 6.2 is a combination of Theorems 6.14 and 6.16 in view of Corollary 4.6. Therefore, the proof can be omitted. The following is a corollary of Theorem 6.2, which is a combination of Corollaries 6.15 and 6.18 in view of Corollary 4.6. The proof can be omitted.

Corollary 6.19. *If $\phi \in C([-r,0],\mathbb{K}^n)$ and $G = Vg$ for some $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{K}^n)$, then*

$$x^L(t;\phi,G) = X^L(t)\phi(0) + \int_0^t X^L(t-u)[g^L(u;\phi) + g(u)]\,\mathrm{d}u$$

holds for all $t \geq 0$.

6.5 Variation of constants formula for linear differential difference equations

We apply Theorem 6.16 to an autonomous linear differential difference equation (2.11)

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{m} B_k x(t - \tau_k) \qquad (t \geq 0)$$

for $n \times n$ matrices $A, B_1, \ldots, B_m \in M_n(\mathbb{K})$ and $\tau_1, \ldots, \tau_m \in (0, r]$. We recall that the linear DDE (2.11) can be expressed in the form of the linear RFDE (1.1) by defining a continuous linear map $L\colon C([-r,0], \mathbb{K}^n) \to \mathbb{K}^n$ by (2.12)

$$L\psi = A\psi(0) + \sum_{k=1}^{m} B_k \psi(-\tau_k)$$

for $\psi \in C([-r,0], \mathbb{K}^n)$.

For the above mentioned application, we need to calculate the function $G^L(\cdot\,;\phi)$ for each $\phi \in \mathcal{M}^1([-r,0], \mathbb{K}^n)$ based on Definition 6.4. By the linearity of $L \mapsto G^L(\cdot\,;\phi)$, this can be reduced to the calculation of $G^{L_k}(\cdot\,;\phi)$ for each $k \in \{0, \ldots, m\}$, where $L_k\colon C([-r,0], \mathbb{K}^n) \to \mathbb{K}^n$ is the continuous linear map given by

$$L_0\psi := A\psi(0),$$

and

$$L_k\psi := B_k\psi(-\tau_k)$$

for $k \in \{1, \ldots, m\}$. We have the following lemma.

Lemma 6.20. *Let $\phi \in \mathcal{M}^1([-r,0], \mathbb{K}^n)$ be given. Then the following statements hold:*

1. $G^{L_0}(\cdot\,;\phi) = 0$.
2. *For each $k \in \{1, \ldots, m\}$, $G^{L_k}(0;\phi) = 0$ and*

$$G^{L_k}(t;\phi) = \begin{cases} B_k \int_{-\tau_k}^{t-\tau_k} \phi(s)\,\mathrm{d}s & (t \in (0, \tau_k]), \\ B_k \int_{-\tau_k}^{0} \phi(s)\,\mathrm{d}s & (t \in (\tau_k, \infty)). \end{cases}$$

holds.

Proof. 1. Let $\eta_0\colon [-r,0] \to M_n(\mathbb{K})$ be the matrix-valued function given by

$$\eta_0(\theta) := \begin{cases} O & (-r \leq \theta < 0), \\ A & (\theta = 0). \end{cases}$$

Then L_0 is expressed as

$$L_0\psi = \int_{-r}^{0} \mathrm{d}\eta_0(\theta)\,\psi(\theta)$$

for $\psi \in C([-r,0], \mathbb{K}^n)$. Therefore, the definition of $G^L(\cdot\,;\phi)$ yields the conclusion.

2. Let $k \in \{1, \ldots, m\}$ be fixed and $\eta_k\colon [-r,0] \to M_n(\mathbb{K})$ be the matrix-valued function given by

$$\eta_k(\theta) := \begin{cases} O & (-r \leq \theta \leq -\tau_k), \\ B_k & (-\tau_k < \theta \leq 0). \end{cases}$$

Then L_k is expressed as

$$L_k\psi = \int_{-r}^{0} \mathrm{d}\eta_k(\theta)\,\psi(\theta)$$

for $\psi \in C([-r,0],\mathbb{K}^n)$. By the definition of L_k, we have

$$\int_{-r}^{0} \mathrm{d}\eta_k(\theta)\left(\int_{\theta}^{0} \phi(s)\,\mathrm{d}s\right) = B_k \int_{-\tau_k}^{0} \phi(s)\,\mathrm{d}s.$$

Furthermore, the integral $\int_{-r}^{-t} \mathrm{d}\eta_k(\theta)\left(\int_0^{t+\theta} \phi(s)\,\mathrm{d}s\right)$ is calculated as

$$\int_{-r}^{-t} \mathrm{d}\eta_k(\theta)\left(\int_0^{t+\theta} \phi(s)\,\mathrm{d}s\right) = \begin{cases} B_k \int_0^{t-\tau_k} \phi(s)\,\mathrm{d}s & (t \in [0,\tau_k]), \\ 0 & (t \in (\tau_k,\infty)). \end{cases}$$

By combining the above expressions, the conclusion is obtained. □

Theorem 6.21 (cf. [19], [22]). *Let $L\colon C([-r,0],\mathbb{K}^n) \to \mathbb{K}^n$ be the continuous linear map given by* (2.12). *Then for any $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$,*

$$x^L(t;\phi,0) = X^L(t)\phi(0) + \sum_{k=1}^{m} \int_{-\tau_k}^{0} X^L(t-\tau_k-\theta)B_k\phi(\theta)\,\mathrm{d}\theta \tag{6.9}$$

holds for all $t \geq 0$.

Proof. Let $\phi \in \mathcal{M}^1([-r,0],\mathbb{K}^n)$ be given. From Lemma 6.20,

$$G^L(\cdot;\phi) = \sum_{k=1}^{m} G^{L_k}(\cdot;\phi)$$

is locally absolutely continuous. Therefore, Theorem 6.16 and the integration by parts formula for absolutely continuous functions yield that

$$x^L(t;\phi,0) = X^L(t)\phi(0) + \sum_{k=1}^{m} \int_0^t X^L(t-u)\dot{G}^{L_k}(u;\phi)\,\mathrm{d}u$$

holds for all $t \geq 0$. We now fix $k \in \{1,\dots,m\}$ and find an expression of the integral

$$\int_0^t X^L(t-u)\dot{G}^{L_k}(u;\phi)\,\mathrm{d}u.$$

Lemma 6.20 shows that $\dot{G}^{L_k}(t;\phi) = B_k\phi(t-\tau_k)$ holds for almost all $t \in [0,\tau_k]$, and $\dot{G}^{L_k}(t;\phi) = 0$ holds for all $t \in (\tau_k,\infty)$. Then the integral is expressed as follows:

- When $t \in [0,\tau_k]$, the integral becomes

$$\begin{aligned}\int_0^t X^L(t-u)B_k\phi(u-\tau_k)\,\mathrm{d}u &= \int_{-\tau_k}^{t-\tau_k} X^L(t-\tau_k-\theta)B_k\phi(\theta)\,\mathrm{d}\theta \\ &= \int_{-\tau_k}^{0} X^L(t-\tau_k-\theta)B_k\phi(\theta)\,\mathrm{d}\theta\end{aligned}$$

  because $t-\tau_k \leq 0$.

- When $t \in (\tau_k, \infty)$, the integral becomes

$$\int_0^{\tau_k} X^L(t-u)B_k\phi(u-\tau_k)\,\mathrm{d}u = \int_{-\tau_k}^0 X^L(t-\tau_k-\theta)B_k\phi(\theta)\,\mathrm{d}\theta.$$

This completes the proof. □

Remark 6.22. Suppose $\phi \in C([-r,0],\mathbb{R})$ and $m = 1$. In [19, Theorem 6.1 in Section 1.6] and [22, Theorem 6.1 in Section 1.6], (6.9) is obtained by using the Laplace transform. See also [28, Theorem 4.2], where (6.9) is obtained under different assumptions for a linear evolution equation with commensurate delays.

6.6 Remarks on definitions of "fundamental matrix"

6.6.1 Definition by Hale

Let $\mathbb{K} = \mathbb{R}$. In [18, Theorem 16.3 and Corollary 16.1] and [19, Theorem 2.1 and Corollary 2.1 in Chapter 6], a matrix-valued function $X\colon [0,\infty) \to M_n(\mathbb{R})$ is defined by using the property that for every $t \geq 0$,

$$\mathcal{L}^1_{\mathrm{loc}}([0,\infty),\mathbb{R}^n) \ni g \mapsto x^L(t;0,Vg) \in \mathbb{R}^n$$

is a bounded linear operator to show

$$x^L(t;0,Vg) = \int_0^t X(t-u)g(u)\,\mathrm{d}u \qquad (t \geq 0).$$

Furthermore, by the formal exchange of order of integration, the function X is interpreted as a "matrix-valued solution" to the linear RFDE (1.1). Indeed, Hale argued that X satisfies (i) $X_0 = \hat{I}$, (ii) $X|_{[0,\infty)}$ is locally absolutely continuous, and (iii) X satisfies

$$\dot{X}(t) = \int_{-r}^0 \mathrm{d}\eta(\theta)\,X(t+\theta)$$

for almost all $t \in [0,\infty)$. Here $\hat{I}\colon [-r,0] \to M_n(\mathbb{R})$ is defined by (1.5)

$$\hat{I}(\theta) := \begin{cases} O & (\theta \in [-r,0)), \\ I & (\theta = 0). \end{cases}$$

However, the above integral does not make sense in general because X is not continuous.

6.6.2 Volterra convolution integral equations and fundamental matrix solutions

Let $x := x^L(\cdot;\hat{\xi})|_{[0,\infty)}$ for some $\xi \in \mathbb{K}^n$ and suppose $\eta(0) = O$. By using the integration by parts formula for Riemann–Stieltjes integrals and Theorem A.19 in (3.5)

$$x = \xi + \mathrm{d}\check{\eta} * Vx,$$

we have

$$\begin{aligned} x(t) &= \xi + [\check{\eta}(u)(Vx)(t-u)]_{u=0}^t + \int_0^t \check{\eta}(u)x(t-u)\,\mathrm{d}u \\ &= \xi + (\check{\eta} * x)(t) \end{aligned}$$

for all $t \geq 0$. Here $(Vx)(0) = 0$ is also used. The above calculation shows that the function $x \colon [0, \infty) \to \mathbb{K}^n$ satisfies

$$x = \check{\eta} * x + \xi,$$

which is a Volterra convolution integral equation with the kernel function $\check{\eta}$ and with the constant forcing term ξ. Therefore, $X := X^L|_{[0,\infty)}$ satisfies

$$X = \check{\eta} * X + I.$$

This means that the restriction $X = X^L|_{[0,\infty)}$ is the *fundamental matrix solution* for the Volterra convolution integral equation with the kernel function $\check{\eta}$ under the assumption that $\eta(0) = O$. For an approach by the Volterra convolution integral equation, see [14, Section I.2 of Chapter I].

7 Exponential stability of principal fundamental matrix solution

For a continuous linear map $L \colon C([-r, 0], \mathbb{K}^n) \to \mathbb{K}^n$, we consider a linear RFDE (1.1)

$$\dot{x}(t) = Lx_t \quad (t \geq 0).$$

Let $X^L \colon [-r, \infty) \to M_n(\mathbb{K})$ be the principal fundamental matrix solution. We use the following terminology.

Definition 7.1. We say that the principal fundamental matrix solution X^L is *exponentially stable* if there exist constants $M \geq 1$ and $\alpha > 0$ such that

$$\left| X^L(t) \right| \leq M e^{-\alpha t} \tag{7.1}$$

holds for all $t \geq 0$. We also say that X^L is *α-exponentially stable.*

In the following calculations, it is useful to extend the domain of definition of X^L to $\mathbb{R}$ by letting $X^L(t) := O$ for $t \in (-\infty, -r)$.

Lemma 7.2. *If X^L is α-exponentially stable for some $\alpha > 0$, then there exists a constant $M \geq 1$ such that*

$$\sup_{\theta \in [-r, 0]} \left| X^L(t + \theta) \right| \leq M e^{-\alpha t}$$

holds for all $t \in \mathbb{R}$.

Proof. By the assumption, one can choose a constant $M_0 \geq 1$ so that

$$\left| X^L(t) \right| \leq M_0 e^{-\alpha t}$$

holds for all $t \geq 0$. Since the statement is trivial when $t \leq 0$, we only have to consider the case $t > 0$. Let $\theta \in [-r, 0]$. When $t + \theta \geq 0$, we have

$$\left| X^L(t + \theta) \right| \leq M_0 e^{-\alpha(t+\theta)} \leq M_0 e^{\alpha r} e^{-\alpha t}.$$

The above estimate also holds when $t + \theta < 0$ because $X^L(t + \theta) = O$ in this case. Therefore, the conclusion is obtained. □

Theorem 7.3 (cf. [19], [22]). *If X^L is α-exponentially stable for some $\alpha > 0$, then the C_0-semigroup $\big(T^L(t)\big)_{t\geq 0}$ on $C([-r,0],\mathbb{K}^n)$ defined by* (1.13)

$$T^L(t)\phi := x^L(\cdot\,;\phi,0)_t$$

for $(t,\phi) \in [0,\infty) \times C([-r,0],\mathbb{K}^n)$ is uniformly α-exponentially stable, i.e., there exists a constant $M \geq 1$ such that for all $t \geq 0$,

$$\left\|T^L(t)\right\| \leq M\mathrm{e}^{-\alpha t}$$

holds.

Proof. By applying Lemma 7.2, we choose a constant $M_0 \geq 1$ so that

$$\sup_{\theta\in[-r,0]}\left|X^L(t+\theta)\right| \leq M_0\mathrm{e}^{-\alpha t}$$

holds for all $t \in \mathbb{R}$. Since the statement is trivial when $t = 0$, we only have to consider the case $t > 0$. Let $\theta \in [-r,0]$ and $\phi \in C([-r,0],\mathbb{K}^n)$ be given. Then

$$\left[T^L(t)\phi\right](\theta) = \begin{cases} X^L(t+\theta)\phi(0) + \int_0^{t+\theta} X^L(t+\theta-u)g^L(u;\phi)\,\mathrm{d}u & (t+\theta\geq 0), \\ \phi(t+\theta) & (t+\theta\in[-r,0]) \end{cases}$$

holds from Corollary 6.18 (see Definition 6.6 for the definition of $g^L(t;\phi)$). We divide the consideration into the following cases.

Case 1: $t+\theta \geq 0$**.** For the first term of the right-hand side,

$$\left|X^L(t+\theta)\phi(0)\right| \leq M_0\mathrm{e}^{-\alpha t}|\phi(0)| \leq M_0\mathrm{e}^{-\alpha t}\|\phi\|$$

holds. For the second term,

$$\begin{aligned}\left|\int_0^{t+\theta} X^L(t+\theta-u)g^L(u;\phi)\,\mathrm{d}u\right| &\leq \int_0^{t+\theta}\left|X^L(t-u+\theta)\right|\left|g^L(u;\phi)\right|\mathrm{d}u \\ &\leq \int_0^{t+\theta} M_0\mathrm{e}^{-\alpha(t-u)}\left|g^L(u;\phi)\right|\mathrm{d}u\end{aligned}$$

holds from Lemma 7.2. Since

$$\left|g^L(t;\phi)\right| = \left|\int_{-r}^{-t}\mathrm{d}\eta(\theta)\,\phi(t+\theta)\right| \leq \mathrm{Var}(\eta)\|\phi\|$$

holds for all $t\in[0,r)$ (see Lemma A.4) and $g^L(t;\phi) = 0$ for all $t \geq r$, we have

$$\begin{aligned}\int_0^{t+\theta} M_0\mathrm{e}^{-\alpha(t-u)}\left|g^L(u;\phi)\right|\mathrm{d}u &\leq \int_0^r M_0\mathrm{e}^{-\alpha(t-u)}\,\mathrm{Var}(\eta)\|\phi\|\,\mathrm{d}u \\ &= M_0\left(\int_0^r \mathrm{e}^{\alpha u}\,\mathrm{d}u\right)\mathrm{Var}(\eta)\mathrm{e}^{-\alpha t}\|\phi\|.\end{aligned}$$

We note that

$$\int_0^r \mathrm{e}^{\alpha u}\,\mathrm{d}u = \frac{1}{\alpha}(\mathrm{e}^{\alpha r}-1)$$

holds.

Case 2: $t + \theta < 0$**.** In this case, we have

$$|\phi(t+\theta)| \le e^{-\alpha(t+\theta)}|\phi(t+\theta)| \le e^{\alpha r}e^{-\alpha t}\|\phi\|.$$

By combining the estimates obtained in Cases 1 and 2, one can choose a constant $M \ge 1$ so that

$$\sup_{\theta\in[-r,0]} \left|\left[T^L(t)\phi\right](\theta)\right| \le M e^{-\alpha t}\|\phi\|$$

holds for all $(t,\phi) \in [0,\infty) \times C([-r,0],\mathbb{K}^n)$. This completes the proof. □

The converse of Theorem 7.3 also holds.

Theorem 7.4 (cf. [19], [22])**.** *If* $\left(T^L(t)\right)_{t\ge 0}$ *is uniformly* α*-exponentially stable for some* $\alpha > 0$*, then* X^L *is* α*-exponentially stable.*

Proof. By the assumption, we choose a constant $M_0 \ge 1$ so that

$$\left\|T^L(t)\right\| \le M_0 e^{-\alpha t}$$

holds for all $t \ge 0$. We fix $\xi \in \mathbb{K}^n$ and let

$$\phi_\xi := x^L(\cdot\,;\hat{\xi})_r \in C([-r,0],\mathbb{K}^n).$$

Then the map $\mathbb{K}^n \ni \xi \mapsto \phi_\xi \in C([-r,0],\mathbb{K}^n)$ is linear from Corollary 2.7. Since $X^L(\cdot)\xi = x^L(\cdot\,;\hat{\xi})$, we have

$$\|\phi_\xi\| = \sup_{t\in[0,r]} \left|x^L(t;\hat{\xi})\right| \le \left(\sup_{t\in[0,r]}\left|X^L(t)\right|\right)\cdot|\xi|.$$

This yields that the linear operator $\mathbb{K}^n \ni \xi \mapsto \phi_\xi \in C([-r,0],\mathbb{K}^n)$ is bounded.

We now show that X^L is α-exponentially stable by dividing into the following cases.

Case 1: $t \ge r$**.** From Theorem 2.14, we have

$$x^L(t;\hat{\xi}) = x^L(t-r;\phi_\xi),$$

where the right-hand side is equal to $[T^L(t-r)\phi_\xi](0)$. Therefore,

$$\left|X^L(t)\xi\right| \le \left\|T^L(t-r)\right\|\|\phi_\xi\|$$

holds. Since $\left\|T^L(t-r)\right\| \le M_0 e^{\alpha r}e^{-\alpha t}$, we obtain

$$\left|X^L(t)\right| \le \left(M_0 e^{\alpha r}\sup_{t\in[0,r]}\left|X^L(t)\right|\right)\cdot e^{-\alpha t}$$

by combining the above estimate on $\|\phi_\xi\|$.

Case 2: $t \in [0,r]$**.** In this case, $\left|X^L(t)\right|$ is estimated by

$$\left|X^L(t)\right| \le \left(e^{\alpha r}\sup_{t\in[0,r]}\left|X^L(t)\right|\right)\cdot e^{-\alpha t}.$$

Here $1 = e^{-\alpha t}e^{\alpha t}$ is used.

By combining the above estimates, the conclusion is obtained. □

See [19, Lemmas 6.1, 6.2, and 6.3 in Chapter 6] and [22, Lemmas 5.1, 5.2, and 5.3 in Chapter 6] for related results. We note that the statements of Theorems 7.3 and 7.4 are included in these results, where the detailed proofs are not given.

8 Principle of linearized stability and Poincaré–Lyapunov theorem

Throughout this section, let $L\colon C([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$ be a continuous linear map. We consider a non-autonomous RFDE

$$\dot{x}(t) = Lx_t + f(t, x_t) \tag{8.1}$$

for some continuous map

$$f\colon \mathbb{R} \times C([-r,0],\mathbb{R}^n) \supset \operatorname{dom}(f) \to \mathbb{R}^n.$$

Let $X^L\colon [-r,\infty) \to M_n(\mathbb{R})$ be the principal fundamental matrix solution of the linear RFDE (1.1)

$$\dot{x}(t) = Lx_t \quad (t \geq 0)$$

and $\left(T^L(t)\right)_{t\geq 0}$ be the C_0-semigroup on $C([-r,0],\mathbb{R}^n)$ generated by (1.1). See also Section 7.

We recall the definition of a solution to the RFDE (8.1). For each $(t_0,\phi) \in \operatorname{dom}(f)$ and each $T > 0$, a continuous function

$$x\colon [t_0 - r, t_0 + T] \to \mathbb{R}^n$$

is called a *solution* of (8.1) under an initial condition $x_{t_0} = \phi$ if the following conditions are satisfied: (i) $x_{t_0} = \phi$, (ii) $(t, x_t) \in \operatorname{dom}(f)$ for all $t \in [t_0, t_0 + T]$, and (iii) $x|_{[t_0,t_0+T]}$ is differentiable and satisfies the RFDE (8.1) for all $t \in [t_0, t_0 + T]$. Here the derivative of x at t_0 and $t_0 + T$ are understood as the right-hand derivative at t_0 and the left-hand derivative at $t_0 + T$, respectively.

8.1 Variation of constants formula and nonlinear equations

Theorem 8.1. *Let $(t_0,\phi) \in \operatorname{dom}(f)$ and $T > 0$ be given. Then for a continuous function $x\colon [t_0 - r, t_0 + T] \to \mathbb{R}^n$ with the properties (i) $x_{t_0} = \phi$ and (ii) $(t, x_t) \in \operatorname{dom}(f)$ for all $t \in [t_0, t_0 + T]$, x is a solution of the RFDE (8.1) under the initial condition $x_{t_0} = \phi$ if and only if x satisfies*

$$x(t) = x^L(t - t_0; \phi, 0) + \int_{t_0}^{t} X^L(t - u) f(u, x_u)\,\mathrm{d}u$$

for all $t \in [t_0, t_0 + T]$.

We note that the above statement is not a simple application of Corollaries 4.6 and 6.15 because there is no method of variation of constants for RFDEs (see Subsection 5.1).

Proof of Theorem 8.1. Let $x\colon [t_0 - r, t_0 + T] \to \mathbb{R}^n$ be a continuous function with the properties (i) and (ii) in Theorem 8.1. Then it is a solution of the RFDE (8.1) under the initial condition $x_{t_0} = \phi$ if and only if

$$x(t) = \phi(0) + \int_{t_0}^{t} [Lx_s + f(s, x_s)]\,\mathrm{d}s$$

holds for all $t \in [t_0, t_0 + T]$. Let $z\colon [-r, T] \to \mathbb{R}^n$ be the function defined by

$$z(s) := x(t_0 + s) - x^L(s; \phi)$$

for $s \in [-r, T]$. Then z satisfies $z_0 = 0$ and an integral equation

$$z(s) = \int_0^s Lz_u\,\mathrm{d}u + \int_0^s f(t_0 + u, x_{t_0+u})\,\mathrm{d}u$$

for $s \in [0, T]$. Since $[0, T] \ni u \mapsto f(t_0 + u, x_{t_0+u}) \in \mathbb{R}^n$ is continuous, $z|_{[0,T]}$ is expressed by

$$z(s) = \int_0^s X^L(s-u) f(t_0 + u, x_{t_0+u}) \,\mathrm{d}u \quad (s \in [0, T])$$

from Theorem 5.9 or Corollary 6.15. Therefore, we have

$$x(t_0 + s) = x^L(s; \phi) + \int_0^s X^L(s-u) f(t_0 + u, x_{t_0+u}) \,\mathrm{d}u$$

for $s \in [0, T]$. The expression of x is obtained by the change of variable $t_0 + s = t$. □

8.2 Stability part of principle of linearized stability

In this subsection, we consider a continuous map

$$h \colon \mathbb{R} \times C([-r, 0], \mathbb{R}^n) \supset \mathbb{R} \times U_0 \to \mathbb{R}^n$$

for some open neighborhood U_0 of 0 in $C([-r, 0], \mathbb{R}^n)$ with the property that $h(t, \phi) = o(\|\phi\|)$ as $\|\phi\| \to 0$ uniformly in t. This means that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $(t, \phi) \in \mathbb{R} \times U_0$, $\|\phi\| < \delta$ implies

$$|h(t, \phi)| \le \varepsilon \|\phi\|.$$

In the following theorem, we suppose that $\mathrm{dom}(f) = \mathbb{R} \times U_0$ and $f(t, \phi) = h(t, \phi)$ holds for all $(t, \phi) \in \mathbb{R} \times U_0$ in the RFDE (8.1). Then (8.1) is considered as a perturbation of the linear RFDE (1.1). Since $f(t, 0) = 0$ holds for all $t \in \mathbb{R}$, (8.1) has the zero solution. The statement in the following theorem is the stability part of the *principle of linearized stability* for RFDEs.

Theorem 8.2 (cf. [14])**.** *If X^L is exponentially stable, then there exist $M \ge 1$, $\beta > 0$, and a neighborhood U of 0 in $C([-r, 0], \mathbb{R}^n)$ such that for every $t_0 \in \mathbb{R}$, every $\phi \in U$, and every non-continuable solution x of* (8.1) *under the initial condition $x_{t_0} = \phi$, x is defined for all $t \ge t_0$ and satisfies*

$$\|x_t\| \le M \mathrm{e}^{-\beta(t - t_0)} \|\phi\|$$

for all $t \ge t_0$.

Remark 8.3. See [2, Chapter 11] for the corresponding result for differential difference equations. See [11] for the general result of the principle of linearized stability in the context of nonlinear semigroups. See also [14, Chapter VII] for a general treatment of the principle of linearized stability and its application to RFDEs under the local Lipschitz continuity of h.

In the proof of Theorem 8.2 given below, the Peano existence theorem and the continuation of solutions for RFDEs play key roles. See [19, Chapter 2] and [22, Chapter 2] for the fundamental theory of RFDEs.

Proof of Theorem 8.2. From Lemma 7.2 and Theorem 7.3, we choose constants $M \ge 1$ and $\alpha > 0$ so that

$$\sup_{\theta \in [-r, 0]} \left| X^L(t + \theta) \right| \le M \mathrm{e}^{-\alpha t} \quad (t \in \mathbb{R})$$

and

$$\left\| T^L(t) \right\| \le M \mathrm{e}^{-\alpha t} \quad (t \ge 0)$$

hold. We also choose an $\varepsilon > 0$ so that

$$-\beta := M\varepsilon - \alpha < 0.$$

We divide the proof into the following steps.

Step 1: Choice of a neighborhood of 0 and a non-continuable solution. Since $f(t,\phi) = o(\|\phi\|)$ as $\|\phi\| \to 0$ uniformly in t, there exists a $\widetilde{\delta} > 0$ for this $\varepsilon > 0$ with the following properties:

(i) For all $\phi \in C([-r,0],\mathbb{R}^n)$, $\|\phi\| < \widetilde{\delta}$ implies $\phi \in U_0$.

(ii) $\|\phi\| < \widetilde{\delta}$ implies $|f(t,\phi)| \le \varepsilon\|\phi\|$ for all $t \in \mathbb{R}$.

Let $\delta := \widetilde{\delta}/M$. We define open sets U and $\widetilde{U}$ by

$$U := \{\phi \in C([-r,0],\mathbb{R}^n) : \|\phi\| < \delta\},$$
$$\widetilde{U} := \left\{\phi \in C([-r,0],\mathbb{R}^n) : \|\phi\| < \widetilde{\delta}\right\}.$$

Then

$$U \subset \widetilde{U} \subset U_0$$

holds. From now on, we fix $t_0 \in \mathbb{R}$ and $\phi \in U$ and proceed with the discussion. By applying the Peano existence theorem for RFDEs, the RFDE

$$\dot{x}(t) = L|_{\widetilde{U}}(x_t) + f|_{\mathbb{R}\times\widetilde{U}}(t, x_t) \tag{8.2}$$

has a solution under the initial condition $x_{t_0} = \phi$. Let x be a non-continuable solution of the RFDE (8.2) under this initial condition. Then its domain of definition is written as $[t_0 - r, t_0 + T)$ for some $T \in (0,\infty]$.

Step 2: Estimate by Gronwall's inequality. Let $t \in [t_0, t_0 + T)$ and $\theta \in [-r, 0]$. By applying Theorem 8.1,

$$x(t) = x^L(t - t_0; \phi, 0) + \int_{t_0}^{t} X^L(t-u) f(u, x_u)\,\mathrm{d}u \qquad (t \in [t_0, t_0 + T))$$

holds for this non-continuable solution $x\colon [t_0 - r, t_0 + T) \to \mathbb{R}^n$. When $t + \theta \ge t_0$, we have

$$\begin{aligned}
|x(t+\theta)| &\le \left|x^L(t+\theta-t_0;\phi,0)\right| + \int_{t_0}^{t+\theta} \left|X^L(t-u+\theta)\right| |f(u,x_u)|\,\mathrm{d}u \\
&\le \left\|T^L(t-t_0)\phi\right\| + \int_{t_0}^{t} M\mathrm{e}^{-\alpha(t-u)} |f(u,x_u)|\,\mathrm{d}u \\
&\le M\mathrm{e}^{-\alpha(t-t_0)}\|\phi\| + \int_{t_0}^{t} M\mathrm{e}^{-\alpha t}\mathrm{e}^{\alpha u}\varepsilon\|x_u\|\,\mathrm{d}u.
\end{aligned}$$

When $t + \theta < t_0$, the estimate

$$|x(t+\theta)| \le M\mathrm{e}^{-\alpha(t-t_0)}\|\phi\| + \int_{t_0}^{t} M\mathrm{e}^{-\alpha t}\mathrm{e}^{\alpha u}\varepsilon\|x_u\|\,\mathrm{d}u$$

also holds in view of

$$|x(t+\theta)| = |\phi(t - t_0 + \theta)| = \left|[T^L(t-t_0)\phi](\theta)\right| \le M\mathrm{e}^{-\alpha(t-t_0)}\|\phi\|.$$

These estimates yield

$$e^{\alpha(t-t_0)}\|x_t\| \le M\|\phi\| + \int_{t_0}^{t} M\varepsilon e^{\alpha(u-t_0)}\|x_u\|\,du,$$

and we obtain

$$e^{\alpha(t-t_0)}\|x_t\| \le M\|\phi\| e^{M\varepsilon(t-t_0)}$$

by applying Gronwall's inequality (see Lemma C.1). This means that

$$\|x_t\| \le M\|\phi\| e^{-\beta(t-t_0)} \tag{8.3}$$

holds for all $t \in [t_0, t_0+T)$.

Step 3: Proof by contradiction. We next show that T is equal to ∞, i.e., the non-continuable solution x is defined on $[t_0 - r, \infty)$. We suppose $T < \infty$ and derive a contradiction. Since $\|x_t\| < \widetilde{\delta}$ holds for all $t \in [t_0, t_0+T)$, we have

$$\begin{aligned}|\dot{x}(t)| &\le \|L\|\|x_t\| + |f(t, x_t)| \\ &\le (\|L\| + \varepsilon)\widetilde{\delta} \\ &< \infty.\end{aligned}$$

This shows that $x|_{[t_0, t_0+T)}$ is Lipschitz continuous. In particular, $x|_{[t_0, t_0+T)}$ is uniformly continuous, and therefore, the limit $\lim_{t \uparrow t_0+T} x(t)$ exists. Since this yields the existence of the limit

$$\lim_{t \uparrow t_0+T} x_t =: \psi \in C([-r, 0], \mathbb{R}^n),$$

we have

$$\|\psi\| \le M\|\phi\| e^{-\beta T} < M\delta = \widetilde{\delta},$$

i.e., $\psi \in \widetilde{U}$, by taking the limit as $t \uparrow t_0 + T$ in the inequality (8.3). Then the RFDE (8.2) has a solution under the initial condition $x_{t_0+T} = \psi$ by the Peano existence theorem for RFDEs, and one can construct a continuation of x. It contradicts the property that x is non-continuable. Therefore, T should be infinity.

The above steps yield the conclusion. □

The above proof of Theorem 8.2 is an appropriate modification of the stability part of the principle of linearized stability for ODEs (e.g., see [6, Section 2.3]). It also should be compared with [35, Theorem 2 and its proof]. We note that the continuity of the higher-order term f in the RFDE (8.1) is sufficient for the proof.

8.3 Poincaré–Lyapunov theorem for RFDEs

In this subsection, we consider the continuous map $h\colon \mathbb{R} \times C([-r, 0], \mathbb{R}^n) \supset \mathbb{R} \times U_0 \to \mathbb{R}^n$ used in Subsection 8.2 and a map

$$N\colon \mathbb{R} \times C([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$$

with the following properties:

- For each $t \in \mathbb{R}$, the map $N(t) \colon C([-r,0],\mathbb{R}^n) \to \mathbb{R}^n$ defined by
$$N(t)\phi := N(t,\phi)$$
for $\phi \in C([-r,0],\mathbb{R}^n)$ is a bounded linear operator.
- $\mathbb{R} \ni t \mapsto N(t) \in \mathcal{B}(C([-r,0],\mathbb{R}^n),\mathbb{R}^n)$ is continuous.
- $\lim_{t\to\infty} \|N(t)\| = 0$ holds.

We note that the map N with the above properties is continuous.

Lemma 8.4. *Suppose that f satisfies $\mathrm{dom}(f) = \mathbb{R} \times U_0$ and $f(t,\phi) = N(t)\phi + h(t,\phi)$ for all $(t,\phi) \in \mathbb{R} \times U_0$. Then for every $\varepsilon > 0$, there exist $a \in \mathbb{R}$ and $\widetilde{\delta} > 0$ such that*
$$|f(t,\phi)| \le \varepsilon\|\phi\|$$
holds for all $t \ge a$ and all $\|\phi\| < \widetilde{\delta}$.

Proof. Let $\varepsilon > 0$ be given. Then we can choose $a \in \mathbb{R}$ and $\widetilde{\delta} > 0$ with the following properties:

- $\|N(t)\| < \varepsilon/2$ holds for all $t \ge a$.
- For all $\phi \in C([-r,0],\mathbb{R}^n)$, $\|\phi\| < \widetilde{\delta}$ implies $\phi \in U_0$.
- $|h(t,\phi)| \le (\varepsilon/2)\|\phi\|$ holds for all $t \in \mathbb{R}$ and all $\|\phi\| < \widetilde{\delta}$.

Then for all $t \ge a$ and all $\|\phi\| < \widetilde{\delta}$,
$$|f(t,\phi)| \le \|N(t)\|\|\phi\| + |h(t,\phi)| \le \varepsilon\|\phi\|$$
holds. □

Lemma 8.5. *Suppose that f satisfies $\mathrm{dom}(f) = \mathbb{R} \times U_0$ and $f(t,\phi) = N(t)\phi + h(t,\phi)$ for all $(t,\phi) \in \mathbb{R} \times U_0$. Let $\sigma \in \mathbb{R}$ be given. Then for every $\varepsilon > 0$, there exist $a \in \mathbb{R}$, $\widetilde{\delta} > 0$, and a continuous function $R \colon [\sigma,\infty) \to (0,\infty)$ with the following properties: (i) $R(t) \le \varepsilon$ for all $t \ge a$, (ii) there exists an $R_0 > \varepsilon$ such that $R(t) \le R_0$ holds for all $t \in [\sigma, a]$, and (iii)*
$$|f(t,\phi)| \le R(t)\|\phi\|$$
holds for all $t \ge \sigma$ and all $\|\phi\| < \widetilde{\delta}$.

Proof. Let $\varepsilon > 0$ be given. In the same way as in the proof of Lemma 8.4, we choose $a > 0$ and $\widetilde{\delta} > 0$. When $\sigma \ge a$, the condition (ii) in Lemma 8.5 is vacuous, and one can choose the constant function whose value is equal to ε as a function R. When $\sigma < a$, we choose $R_0 > \varepsilon$ so that
$$\sup_{t\in[\sigma,\infty)} \|N(t)\| + \frac{\varepsilon}{2} < R_0.$$
We note that $\sup_{t\in[\sigma,\infty)} \|N(t)\| < \infty$ holds because $\|N(t)\| < \varepsilon/2$ for all $t \ge a$ and $t \mapsto \|N(t)\|$ is continuous. Then the continuous function $R \colon [\sigma,\infty) \to (0,\infty)$ given by
$$R(t) := \|N(t)\| + \frac{\varepsilon}{2}$$
satisfies the properties (i), (ii), and (iii). □

Theorem 8.6. *Suppose that the map f in the RFDE (8.1) satisfies* $\mathrm{dom}(f) = \mathbb{R} \times U_0$ *and the conclusion of Lemma 8.5. If X^L is exponentially stable, then for each given $\sigma \in \mathbb{R}$, there exist $M \geq 1$, $\beta > 0$, and a neighborhood U of 0 in $C([-r,0],\mathbb{R}^n)$ with the following property: for every $t_0 \geq \sigma$, every $\phi \in U$, and every non-continuable solution x of (8.1) under the initial condition $x_{t_0} = \phi$, x is defined for all $t \geq t_0$ and satisfies*

$$\|x_t\| \leq M\mathrm{e}^{-\beta(t-t_0)}\|\phi\|$$

for all $t \geq t_0$.

Proof. From Lemma 7.2 and Theorem 7.3, we choose constants $M_0 \geq 1$ and $\alpha > 0$ so that

$$\sup_{\theta\in[-r,0]} \left|X^L(t+\theta)\right| \leq M_0\mathrm{e}^{-\alpha t} \qquad (t \in \mathbb{R})$$

and

$$\left\|T^L(t)\right\| \leq M_0\mathrm{e}^{-\alpha t} \qquad (t \geq 0)$$

hold. We also choose an $\varepsilon > 0$ so that

$$-\beta := M_0\varepsilon - \alpha < 0.$$

For this $\varepsilon > 0$, we choose the $a \in \mathbb{R}$, $\widetilde{\delta} > 0$, and the continuous function $R\colon [\sigma,\infty) \to (0,\infty)$ in Lemma 8.5. We divide the proof into the following cases: (I) $\sigma \geq a$, (II) $\sigma < a$.

When (I) $\sigma \geq a$, the completely same argument as in the proof of Theorem 8.2 is valid by choosing $M_0 := M$. See also Lemma 8.4. Therefore, we only have to consider the case (II) $\sigma < a$. In this case, we further divide the proof into the following steps.

Step 1: Choice of a neighborhood of 0 and a non-continuable solution. Let

$$M := M_0\mathrm{e}^{M_0(R_0-\varepsilon)(a-\sigma)} \quad \text{and} \quad \delta := \frac{\widetilde{\delta}}{M}.$$

We define open sets U and $\widetilde{U}$ by

$$U := \{\phi \in C([-r,0],\mathbb{R}^n) : \|\phi\| < \delta\},$$
$$\widetilde{U} := \left\{\phi \in C([-r,0],\mathbb{R}^n) : \|\phi\| < \widetilde{\delta}\right\}.$$

Since $M > M_0 \geq 1$,

$$U \subset \widetilde{U} \subset U_0$$

holds. We now fix $t_0 \geq \sigma$ and $\phi \in U$, and let $x\colon [t_0 - r, t_0 + T) \to \mathbb{R}^n$ be a non-continuable solution of the RFDE (8.2)

$$\dot{x}(t) = L|_{\widetilde{U}}(x_t) + f|_{\mathbb{R}\times\widetilde{U}}(t, x_t)$$

under the initial condition $x_{t_0} = \phi$.

Step 2: Estimate by Gronwall's inequality. A similar argument as in Step 2 of the proof of Theorem 8.2 yields that

$$\mathrm{e}^{\alpha(t-t_0)}\|x_t\| \leq M_0\|\phi\| \exp\left(\int_{t_0}^t M_0R(u)\,\mathrm{d}u\right)$$

holds for all $t \in [t_0, t_0 + T)$ by Gronwall's inequality (see Lemma C.1). This is just obtained by replacing M and ε with M_0 and $R(u)$, respectively. The above inequality means that

$$\|x_t\| \le M_0 \|\phi\| \exp\left(\int_{t_0}^{t} [M_0 R(u) - \alpha] \, \mathrm{d}u \right)$$

holds for all $t \in [t_0, t_0 + T)$. We now estimate

$$C(t) := \exp\left(\int_{t_0}^{t} [M_0 R(u) - \alpha] \, \mathrm{d}u \right)$$

from above for $t \in [t_0, t_0 + T)$ by dividing into the following cases:

- Case: $t_0 + T < a$. Since $t < a$, we have

  $$C(t) \le \mathrm{e}^{(M_0 R_0 - \alpha)(t - t_0)}$$

  by the property (ii) in Lemma 8.5. Here the right-hand side is equal to

  $$\mathrm{e}^{M_0 (R_0 - \varepsilon)(t - t_0)} \mathrm{e}^{-\beta (t - t_0)}$$

  by the choice of β. In view of $\sigma \le t_0 \le t < a$, the above is further estimated from above by

  $$\mathrm{e}^{M_0 (R_0 - \varepsilon)(a - \sigma)} \mathrm{e}^{-\beta (t - t_0)}.$$

  Therefore, inequality (8.3)

  $$\|x_t\| \le M \|\phi\| \mathrm{e}^{-\beta (t - t_0)}$$

  holds for all $t \in [t_0, t_0 + T)$ with $M = M_0 \mathrm{e}^{M_0 (R_0 - \varepsilon)(a - \sigma)}$.

- Case: $t_0 + T \ge a$. The integral in $C(t)$ is estimated from above by

  $$\int_{t_0}^{a} (M_0 R_0 - \alpha) \, \mathrm{d}u + \int_{a}^{t} (M_0 \varepsilon - \alpha) \, \mathrm{d}u = (M_0 R_0 - \alpha)(a - t_0) + (-\beta)(t - a).$$

  Here $-\beta = M_0 \varepsilon - \alpha$ is used. In view of $t - a = (t - t_0) + (t_0 - a)$, the above value becomes

  $$(M_0 R_0 - \alpha + \beta)(a - t_0) + (-\beta)(t - t_0) = M_0 (R_0 - \varepsilon)(a - t_0) + (-\beta)(t - t_0).$$

  The last term is also estimated from above by

  $$M_0 (R_0 - \varepsilon)(a - \sigma) + (-\beta)(t - t_0)$$

  because of $t_0 \ge \sigma$ and $M_0 (R_0 - \varepsilon) > 0$. Therefore, inequality (8.3) holds for all $t \in [t_0, t_0 + T)$ with $M = M_0 \mathrm{e}^{M_0 (R_0 - \varepsilon)(a - \sigma)}$.

Step 3: Proof by contradiction. We next show that T is equal to ∞, i.e., the non-continuable solution x is defined on $[t_0 - r, \infty)$. We suppose $T < \infty$ and derive a contradiction. Since $\|x_t\| < \widetilde{\delta}$ holds for all $t \in [t_0, t_0 + T)$, we have

$$\begin{aligned} |\dot{x}(t)| &\le \|L\| \|x_t\| + |f(t, x_t)| \\ &\le (\|L\| + R(t)) \widetilde{\delta} \\ &< \infty. \end{aligned}$$

We note that the continuous function R is bounded. The remainder of the proof is completely same as in Step 3 of the proof of Theorem 8.2.

The above steps yield the conclusion. □

As a consequence of Theorem 8.6 and Lemma 8.5, the following *Poincaré–Lyapunov theorem* for RFDEs is obtained. See [6, Exercise 2.79] for the theorem for ODEs. In the theorem, we suppose that $\mathrm{dom}(f) = \mathbb{R} \times U_0$ and $f(t,\phi) = N(t)\phi + h(t,\phi)$ holds for all $(t,\phi) \in \mathbb{R} \times U_0$ in the RFDE (8.1).

Theorem 8.7. *If X^L is exponentially stable, then for each given $\sigma \in \mathbb{R}$, there exist $M \geq 1$, $\beta > 0$, and a neighborhood U of 0 in $C([-r,0],\mathbb{R}^n)$ with the following property: for every $t_0 \geq \sigma$, every $\phi \in U$, and every non-continuable solution x of the RFDE (8.1) under the initial condition $x_{t_0} = \phi$, x is defined for all $t \geq t_0$ and satisfies*

$$\|x_t\| \leq M e^{-\beta(t-t_0)} \|\phi\|$$

for all $t \geq t_0$.

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A Riemann–Stieltjes integrals with respect to matrix-valued functions

Throughout this appendix, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $n \geq 1$ be an integer, and $[a,b]$ be a closed and bounded interval of $\mathbb{R}$. In this appendix, we study Riemann–Stieltjes integrals with respect to matrix-valued functions. We refer the reader to [39, Chapter 1] and [31, Appendix D] as references of Riemann–Stieltjes integrals for scalar-valued functions. See also [24, Section 3.1] and [14, Section I.1 in Appendix I].

A.1 Definitions

Definition A.1. A finite sequence $(x_k)_{k=0}^m$ for some integer $m \geq 1$ satisfying

$$a = x_0 < x_1 < \cdots < x_m = b$$

is called a *partition* of $[a,b]$. This is also denoted by a symbol $P : a = x_0 < x_1 < \cdots < x_m = b$. For a finite sequence $\xi := (\xi_k)_{k=1}^m$ satisfying

$$x_{k-1} \leq \xi_k \leq x_k \quad (k \in \{1,\dots,m\}),$$

we call a pair (P,ξ) a *tagged partition* of $[a,b]$. For the tagged partition (P,ξ), let

$$|(P,\xi)| := |P| := \max_{1 \leq k \leq m} (x_k - x_{k-1}),$$

which is called the *norm* of (P,ξ).

The above terminology of tagged partition comes from [15].

Definition A.2. Let $f, \alpha \colon [a,b] \to M_n(\mathbb{K})$ be functions. For a tagged partition (P,ξ) of $[a,b]$ given in Definition A.1, let

$$S(f;\alpha,(P,\xi)) := \sum_{k=1}^{m} [\alpha(x_k) - \alpha(x_{k-1})] f(\xi_k).$$

We call $S(f;\alpha,(P,\xi))$ the *Riemann–Stieltjes sum* of f with respect to α under the tagged partition (P,ξ).

Definition A.3. Let $f, \alpha \colon [a,b] \to M_n(\mathbb{K})$ be functions. We say that f is *Riemann–Stieltjes integrable with respect to* α if there exists a $J \in M_n(\mathbb{K})$ with the following property: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all tagged partition (P,ξ) of $[a,b]$, $|(P,\xi)| < \delta$ implies

$$|S(f;\alpha,(P,\xi)) - J| < \varepsilon.$$

We note that such a J is unique if it exists. It is called the *Riemann–Stieltjes integral of f with respect to α* and is denoted by $\int_a^b \mathrm{d}\alpha(x)\, f(x)$.

A.1.1 Remarks

Remark A.4. One can also consider a sum

$$\sum_{k=1}^{m} f(\xi_k)[\alpha(x_k) - \alpha(x_{k-1})],$$

which is different from $S(f;\alpha,(P,\xi))$ in general. If a limit of the above sum as $|(P,\xi)| \to 0$ exists in the sense of Definition A.3, we will write the limit as $\int_a^b f(x)\, \mathrm{d}\alpha(x)$. By taking the transpose,

$$\left(\int_a^b \mathrm{d}\alpha(x)\, f(x) \right)^{\mathrm{T}} = \int_a^b f(x)^{\mathrm{T}}\, \mathrm{d}\alpha(x)^{\mathrm{T}}$$

holds. Here A^{T} denotes the transpose of a matrix $A \in M_n(\mathbb{K})$. When $n = 1$,

$$\int_a^b \mathrm{d}\alpha(x)\, f(x) = \int_a^b f(x)\, \mathrm{d}\alpha(x)$$

holds.

Remark A.5. The notions of the Riemann–Stieltjes sum $S(f;\alpha,(P,\xi))$ and the Riemann–Stieltjes integrability of f with respect to α are also defined for functions

$$f \colon [a,b] \to \mathbb{K}^n \quad \text{and} \quad \alpha \colon [a,b] \to M_n(\mathbb{K}).$$

In this case, the sum $S(f;\alpha,(P,\xi))$ and the integral $\int_a^b \mathrm{d}\alpha(x)\, f(x)$ belong to $\mathbb{K}^n$.

A.2 Reduction to scalar-valued case

Since the linear space $M_n(\mathbb{K})$ is finite-dimensional, the operator norm $|\cdot|$ on $M_n(\mathbb{K})$ is equivalent to the norm $|\cdot|_2$ on $M_n(\mathbb{K})$ defined by

$$|A|_2 := \sqrt{\sum_{i,j \in \{1,\dots,n\}} |a_{i,j}|^2}, \tag{A.1}$$

where $a_{i,j}$ is the (i,j)-component of the matrix $A \in M_n(\mathbb{K})$. This means that the notion of convergence in $M_n(\mathbb{K})$ can be treated component-wise.

Lemma A.6. *Let $f, \alpha\colon [a,b] \to M_n(\mathbb{K})$ be functions. Then the following properties are equivalent:*

(a) f is Riemann–Stieltjes integrable with respect to α.

(b) For each column vector $f_j\colon [a,b] \to \mathbb{K}^n$ of $f = (f_1 \ \cdots \ f_n)$, it is Riemann–Stieltjes integrable with respect to α.

Furthermore,

$$\int_a^b \mathrm{d}\alpha(x) f(x) = \left(\int_a^b \mathrm{d}\alpha(x) f_1(x) \ \cdots \ \int_a^b \mathrm{d}\alpha(x) f_n(x) \right)$$

holds when one of the above properties are satisfied.

The proof is based on the definition of the matrix product and on the property that the operator norm $|\cdot|$ is equivalent to the norm $|\cdot|_2$ given in (A.1). Therefore, we omit the proof.

Lemma A.7. *Let $f\colon [a,b] \to \mathbb{K}^n$ and $\alpha\colon [a,b] \to M_n(\mathbb{K})$ be functions with $f = (f_1, \dots, f_n)$ and $\alpha = (\alpha_{i,j})_{i,j\in\{1,\dots,n\}}$. If $f_j\colon [a,b] \to \mathbb{K}$ is Riemann–Stieltjes integrable with respect to $\alpha_{i,j}\colon [a,b] \to \mathbb{K}$ for every $i, j \in \{1, \dots, n\}$, then so is f with respect to α. Furthermore,*

$$\int_a^b \mathrm{d}\alpha(x) f(x) = \left(\sum_{j=1}^n \int_a^b f_j(x) \, \mathrm{d}\alpha_{i,j}(x) \right)_{i=1}^n$$

holds.

Proof. By the definition of the product of a matrix and a vector, the i-th component of

$$S(f; \alpha, (P, \xi)) \in \mathbb{K}^n$$

is equal to

$$\sum_{j=1}^n S(f_j; \alpha_{i,j}, (P, \xi)).$$

Therefore, the conclusion is obtained by the triangle inequality. □

The converse of Lemma A.7 does not necessarily hold as the following example shows.

Example A.8. Let $n = 2$ and $g, \beta\colon [a,b] \to \mathbb{K}$ be given functions. Let

$$f := (g, -g)\colon [a,b] \to \mathbb{K}^2 \quad \text{and} \quad \alpha := (\beta)_{i,j\in\{1,2\}}\colon [a,b] \to M_2(\mathbb{K}),$$

i.e., $f_1 = f$, $f_2 = -f$, and $\alpha_{i,j} = \beta$. Then the Riemann–Stieltjes sum of f with respect to α is equal to 0 under any tagged partition of $[a,b]$. This means that f is Riemann–Stieltjes integrable with respect to α for any pair (g, β) of functions.

In view of the above example, the Riemann–Stieltjes integration of vector-valued functions with respect to matrix-valued functions is not completely reduced to that for scalar-valued functions. However, it is often useful to reduce the integration to scalar-valued case in view of Lemma A.7.

A.3 Fundamental results

The following are fundamental results on Riemann–Stieltjes integrals for matrix-valued functions.

A.3.1 Reversal formula

Theorem A.9. *Let $f, \alpha \colon [a,b] \to M_n(\mathbb{K})$ be functions. We define functions $\bar{f}, \bar{\alpha} \colon [-b,-a] \to M_n(\mathbb{K})$ by*

$$\bar{f}(y) := f(-y), \quad \bar{\alpha}(y) := \alpha(-y)$$

for $y \in [-b,-a]$. If f is Riemann–Stieltjes integrable with respect to α, then so is $\bar{f}$ with respect to $\bar{\alpha}$. Furthermore,

$$\int_{-b}^{-a} \mathrm{d}\bar{\alpha}(y)\, \bar{f}(y) = -\int_a^b \mathrm{d}\alpha(x)\, f(x) \tag{A.2}$$

holds.

We call Eq. (A.2) the *reversal formula* for Riemann–Stieltjes integrals. The proof is obtained by returning to the definition of Riemann–Stieltjes integrals. Therefore, it can be omitted.

A.3.2 Integration by parts formula

The following is the *integration by parts formula* for Riemann–Stieltjes integrals with respect to matrix-valued functions.

Theorem A.10. *Let $f, \alpha \colon [a,b] \to M_n(\mathbb{K})$ be functions. If f is Riemann–Stieltjes integrable with respect to α, then so is α with respect to f. Furthermore,*

$$\int_a^b \mathrm{d}\alpha(x)\, f(x) = [\alpha(x) f(x)]_{x=a}^b - \int_a^b \alpha(x)\, \mathrm{d}f(x)$$

holds. Here $[\alpha(x) f(x)]_{x=a}^b := \alpha(b) f(b) - \alpha(a) f(a)$.

The proof is basically same as the proof for the case $n = 1$ (i.e., the scalar-valued case). See [31, Proposition D.3] for the proof of this case. See also [39, Theorems 4a and 4b in Chapter 1].

A.4 Integrability

A.4.1 Matrix-valued functions of bounded variation

We first recall the definition of matrix-valued functions of bounded variation.

Definition A.11. Let $\alpha \colon [a,b] \to M_n(\mathbb{K})$ be a function. For each partition $P : a = x_0 < x_1 < \cdots < x_m = b$ of $[a,b]$, let

$$\operatorname{Var}(\alpha; P) := \sum_{k=1}^{m} |\alpha(x_k) - \alpha(x_{k-1})|,$$

which is called the *variation* of α under the partition P. The value

$$\operatorname{Var}(\alpha) := \sup \{ \operatorname{Var}(\alpha; P) : P \text{ is a partition of } [a,b] \}$$

is called the *total variation* of α. α is said to be *of bounded variation* if $\operatorname{Var}(\alpha) < \infty$.

Since the operator norm $|\cdot|$ on $M_n(\mathbb{K})$ and the norm $|\cdot|_2$ on $M_n(\mathbb{K})$ given in (A.1) are equivalent, a matrix-valued function $\alpha \colon [a,b] \to M_n(\mathbb{K})$ is of bounded variation if and only if each component function $\alpha_{i,j} \colon [a,b] \to \mathbb{K}$ is of bounded variation.

Remark A.12. Let $\alpha\colon [a,b] \to M_n(\mathbb{K})$ be a function. Then for any $c \in (a,b)$,

$$\operatorname{Var}\bigl(\alpha|_{[a,c]}\bigr) + \operatorname{Var}\bigl(\alpha|_{[c,b]}\bigr) = \operatorname{Var}(\alpha) \tag{A.3}$$

holds. This equality is obtained from

$$\operatorname{Var}\bigl(\alpha|_{[a,c]}; P_1\bigr) + \operatorname{Var}\bigl(\alpha|_{[c,b]}; P_2\bigr) = \operatorname{Var}(\alpha; P),$$

where P_1 is a partition of $[a,c]$, P_2 is a partition of $[c,b]$, and P is the partition of $[a,b]$ obtained by joining P_1 and P_2.

Lemma A.13. *Let $f, \alpha\colon [a,b] \to M_n(\mathbb{K})$ be functions. If f is Riemann–Stieltjes integrable with respect to α, then*

$$\left| \int_a^b \mathrm{d}\alpha(x) f(x) \right| \le \operatorname{Var}(\alpha) \cdot \sup_{x \in [a,b]} |f(x)| \tag{A.4}$$

holds.

Proof. Let (P, ξ) be a tagged partition of $[a,b]$ given in Definition A.1. Since $|AB| \le |A||B|$ holds for any $A, B \in M_n(\mathbb{K})$, we have

$$|S(f; \alpha, (P,\xi))| \le \sum_{k=1}^{m} |\alpha(x_k) - \alpha(x_{k-1})||f(\xi_k)| \le \operatorname{Var}(\alpha) \cdot \sup_{x \in [a,b]} |f(x)|.$$

Then the remaining proof is essentially same as the scalar-valued case. □

Remark A.14. In the completely similar way, (A.4) also holds for any function $f\colon [a,b] \to \mathbb{K}^n$ which is Riemann–Stieltjes integrable with respect to α. This can also be seen from Lemma A.13 because for any $A \in M_n(\mathbb{K})$ of the form

$$A = (a \; 0 \; \cdots \; 0) \quad (a \in \mathbb{K}^n, 0 \in \mathbb{K}^n),$$

$|A| = |a|$ holds.

A.4.2 Integrability of matrix-valued functions

The following is a fundamental theorem on the Riemann–Stieltjes integrability for scalar-valued functions.

Theorem A.15. *Let $f, \alpha\colon [a,b] \to \mathbb{K}$ be functions. If f is continuous and α is of bounded variation, then f is Riemann–Stieltjes integrable with respect to α.*

See [31, Theorem D.1] for a proof, which is valid for the case $\mathbb{K} = \mathbb{C}$ because it does not use the order structure. By using Theorem A.15, one can obtain the following.

Theorem A.16. *Let $f, \alpha\colon [a,b] \to M_n(\mathbb{K})$ be functions. If f is continuous and α is of bounded variation, then f is Riemann–Stieltjes integrable with respect to α.*

Proof. From Lemma A.6, the problem is reduced to the Riemann–Stieltjes integrability of each column vector of f with respect to α. From Lemma A.7, it is sufficient to show that each component $f_{i,j}\colon [a,b] \to \mathbb{K}$ of f is Riemann–Stieltjes integrable with respect to each component $\alpha_{i,j}\colon [a,b] \to \mathbb{K}$ of α. Since each $f_{i,j}$ is continuous and each $\alpha_{i,j}$ is of bounded variation, the conclusion is obtained from Theorem A.15. □

The following is the result on additivity of Riemann–Stieltjes integrals with respect to matrix-valued functions on sub-intervals.

Theorem A.17. *Let $f\colon [a,b] \to M_n(\mathbb{K})$ be a continuous function and $\alpha\colon [a,b] \to M_n(\mathbb{K})$ be a function of bounded variation. Then for any $c \in (a,b)$,*

$$\int_a^b \mathrm{d}\alpha(x)\, f(x) = \int_a^c \mathrm{d}\alpha(x)\, f(x) + \int_c^b \mathrm{d}\alpha(x)\, f(x)$$

holds.

The proof is same as that for the case $n = 1$. See [31, Proposition D.2] for the proof. We note that the statement can be proved by considering partitions of $[a,b]$ with $c \in (a,b)$ as an intermediate point.

Remark A.18. In Theorem A.17, the assumptions that f is continuous and α is of bounded variation are essential because these assumptions ensure the existence of three integrals (see (A.3) and Theorem A.16). Without these assumptions, the integral in the left-hand side does not necessarily exist even if the integrals in the right-hand side exist. Such a situation will occur when the functions f and α share a discontinuity at c. See [39, Section 5 in Chapter I] for the detail.

A.5 Integration with respect to continuously differentiable functions

The following theorem shows a relationship between Riemann–Stieltjes integrals and Riemann integrals.

Theorem A.19. *Let $f\colon [a,b] \to M_n(\mathbb{K})$ be a Riemann integrable function and $\alpha\colon [a,b] \to M_n(\mathbb{K})$ be a continuously differentiable function. Then f is Riemann–Stieltjes integrable with respect to α, and*

$$\int_a^b \mathrm{d}\alpha(x)\, f(x) = \int_a^b \alpha'(x) f(x)\, \mathrm{d}x$$

holds. Here the right-hand side is a Riemann integral.

Since the above statement is not mentioned in [39] and [31] even for the case $n = 1$, we now give an outline of the proof.

Outline of the proof of Theorem A.19. Let (P, ξ) be a tagged partition of $[a,b]$ given in Definition A.1. Let

$$S(\alpha' f; (P,\xi)) := \sum_{k=1}^{m} (x_k - x_{k-1}) \alpha'(\xi_k) f(\xi_k).$$

Since

$$\alpha(x_k) - \alpha(x_{k-1}) = \int_{x_{k-1}}^{x_k} \alpha'(t)\, \mathrm{d}t$$

holds for each $k \in \{1, \dots, m\}$ by the fundamental theorem of calculus, we have

$$S(f; \alpha, (P,\xi)) - S(\alpha' f; (P,\xi)) = \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} [\alpha'(t) - \alpha'(\xi_k)]\, \mathrm{d}t \cdot f(\xi_k).$$

From this, we also have

$$\left| S(f; \alpha, (P,\xi)) - S(\alpha' f; (P,\xi)) \right| \le \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \left| \alpha'(t) - \alpha'(\xi_k) \right| \mathrm{d}t \cdot |f(\xi_k)|.$$

By combining this and the uniform continuity of α', one can obtain the conclusion. □

When $n = 1$ and $\mathbb{K} = \mathbb{R}$, one can use the mean value theorem for the proof of Theorem A.19.

A.6 Integration with respect to absolutely continuous functions

The following theorem should be compared with Theorem A.19.

Theorem A.20. *Let $f\colon [a,b] \to M_n(\mathbb{K})$ be a continuous function and $\alpha\colon [a,b] \to M_n(\mathbb{K})$ be an absolutely continuous function. Then*

$$\int_a^b \mathrm{d}\alpha(x)\, f(x) = \int_a^b \alpha'(x) f(x)\, \mathrm{d}x$$

holds. Here the right-hand side is a Lebesgue integral.

See [39, Theorem 6a in Chapter I] for the proof of the scalar-valued case. We note that the existence of the Riemann–Stieltjes integral in the left-hand side is ensured by Theorem A.16 because the absolutely continuous function α is of bounded variation. We also note that the function $[a,b] \ni x \mapsto \alpha'(x) f(x) \in M_n(\mathbb{K})$ is Lebesgue integrable because it is measurable and

$$\int_a^b |\alpha'(x) f(x)|\, \mathrm{d}x \le \int_a^b |\alpha'(x)||f(x)|\, \mathrm{d}x \le \|\alpha'\|_1 \|f\| < \infty$$

holds.

Since it is interesting to compare the proof of Theorem A.19 and the proof of Theorem A.20, we now give an outline of the proof.

Outline of the proof of Theorem A.20. Let (P, ξ) be a tagged partition of $[a,b]$ given in Definition A.1. Since $\alpha = \alpha(0) + V\alpha'$,

$$S(f; \alpha, (P, \xi)) = \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \alpha'(t)\, \mathrm{d}t \cdot f(\xi_k)$$

holds. Therefore, we have

$$\int_a^b \alpha'(x) f(x)\, \mathrm{d}x - S(f; \alpha, (P, \xi)) = \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \alpha'(t)[f(t) - f(\xi_k)]\, \mathrm{d}t.$$

In combination with the uniform continuity of f, the conclusion is obtained by taking the limit as $|(P, \xi)| \to 0$. □

A.7 Proof of the theorem on iterated integrals

In this subsection, we give a proof of Theorem 3.8.

Proof of Theorem 3.8. We define a bounded linear operator $T\colon C([a,b], M_n(\mathbb{K})) \to M_n(\mathbb{K})$ by

$$Tg := \int_a^b \mathrm{d}\alpha(x)\, g(x)$$

for $g \in C([a,b], M_n(\mathbb{K}))$. From Lemma 2.9, the left-hand side of (3.4) is equal to

$$T \int_c^d f(\cdot, y)\, \mathrm{d}y,$$

which is also equal to $\int_c^d Tf(\cdot, y)\, \mathrm{d}y$ since T is a bounded linear operator. By the definition of T, this integral is equal to the right-hand side of (3.4). This completes the proof. □

B Riesz representation theorem

Throughout this appendix, let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $[a, b]$ be a closed and bounded interval of $\mathbb{R}$.

The following is the cerebrated *Riesz representation theorem*.

Theorem B.1. *For any continuous linear functional $A \colon C([a, b], \mathbb{K}) \to \mathbb{K}$, there exists a function $\alpha \colon [a, b] \to \mathbb{K}$ with the following properties: (i) $\operatorname{Var}(\alpha) = \|A\|$, (ii) every $f \in C([a, b], \mathbb{K})$ is Riemann–Stieltjes integrable with respect to α, and (iii)*

$$A(f) = \int_a^b f(x)\,\mathrm{d}\alpha(x)$$

holds for all $f \in C([a, b], \mathbb{K})$.

In a proof of Theorem B.1 (e.g., see discussions on [31, Chapter 9]), we construct such a function α by using a continuous linear extension

$$\bar{A} \colon B([a, b], \mathbb{K}) \to \mathbb{K}$$

of A with $\|\bar{A}\| = \|A\|$. Here $B([a, b], \mathbb{K})$ denotes the linear space of all bounded functions from $[a, b]$ to $\mathbb{K}$ endowed with the supremum norm. Its existence is ensured by the Hahn–Banach extension theorem in normed spaces (see [40, Theorem 1 in Section 5 of Chapter IV]). See also [1, Section 4 of Chapter IV].

Remark B.2. The Riemann-Stieltjes integrability of any $f \in C([a, b], \mathbb{K})$ with respect to the constructed function α is also obtained in the proof. This should be compared with Theorem A.15.

The following is a corollary of Theorem B.1.

Corollary B.3. *For any integer $n \geq 1$ and any continuous linear map $A \colon C([a, b], \mathbb{K}^n) \to \mathbb{K}^n$, there exists a function $\alpha \colon [a, b] \to M_n(\mathbb{K})$ of bounded variation such that*

$$A(f) = \int_a^b \mathrm{d}\alpha(x)\, f(x)$$

holds for all $f \in C([a, b], \mathbb{K}^n)$.

Corollary B.3 has been used in the literature of RFDEs (e.g., see [18], [19], [22], and [14]). We now give the proof of Corollary B.3 because it is not given in these references.

Proof of Corollary B.3. Let $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ be the standard basis of $\mathbb{K}^n$. For each $g \in C([a, b], \mathbb{K})$ and each $j \in \{1, \dots, n\}$, let $g\boldsymbol{e}_j \in C([a, b], \mathbb{K}^n)$ be defined by

$$(g\boldsymbol{e}_j)(x) := g(x)\boldsymbol{e}_j$$

for $x \in [a, b]$. For each $i, j \in \{1, \dots, n\}$, we define a functional $A_{i,j} \colon C([a, b], \mathbb{K}) \to \mathbb{K}$ by

$$A_{i,j}(g) := A(g\boldsymbol{e}_j)_i.$$

Here y_i denotes the i-th component of $y \in \mathbb{K}^n$. Since $A_{i,j}$ is a continuous linear functional, one can choose a function $\alpha_{i,j} \colon [a, b] \to \mathbb{K}$ of bounded variation so that

$$A_{i,j}(g) = \int_a^b g(x)\,\mathrm{d}\alpha_{i,j}(x)$$

holds for all $g \in C([a,b],\mathbb{K})$ from Theorem B.1. By using $f = \sum_{j=1}^{n} f_j \boldsymbol{e}_j$ for $f = (f_1, \dots, f_n)$, we have

$$A(f)_i = \sum_{j=1}^{n} A(f_j \boldsymbol{e}_j)_i = \sum_{j=1}^{n} A_{i,j}(f_j) = \sum_{j=1}^{n} \int_a^b f_j(x)\,\mathrm{d}\alpha_{i,j}(x).$$

From Lemma A.7, this yields that

$$A(f) = \int_a^b \mathrm{d}\alpha(x)\,f(x)$$

holds for all $f \in C([a,b],\mathbb{K}^n)$ by defining a matrix-valued function $\alpha\colon [a,b] \to M_n(\mathbb{K})$ of bounded variation by $\alpha := (\alpha_{i,j})_{i,j}$. This completes the proof. □

C Variants of Gronwall's inequality

Throughout this appendix, let $[a,b]$ be a closed and bounded interval of $\mathbb{R}$.

C.1 Gronwall's inequality and its generalization

The following is known as Gronwall's inequality.

Lemma C.1 (ref. [20]). *Let $\alpha \in \mathbb{R}$ be a constant and $\beta\colon [a,b] \to [0,\infty)$ be a continuous function. If a continuous function $u\colon [a,b] \to \mathbb{R}$ satisfies*

$$u(t) \le \alpha + \int_a^t \beta(s)u(s)\,\mathrm{d}s$$

for all $t \in [a,b]$, then

$$u(t) \le \alpha \exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right)$$

holds for all $t \in [a,b]$.

Outline of the proof. To use a technique for scalar homogeneous linear ODEs, let

$$v(t) := \int_a^t \beta(s)u(s)\,\mathrm{d}s.$$

Then the given inequality becomes

$$\dot{v}(t) \le \beta(t)[v(t) + \alpha] \qquad (t \in [a,b]),$$

where the non-negativity of β is used. Since the left-hand side is the derivative of the function $t \mapsto v(t) + \alpha$, it is natural to consider the derivative of

$$t \mapsto \exp\left(-\int_a^t \beta(s)\,\mathrm{d}s\right)[v(t) + \alpha].$$

Then it holds that this function is monotonically decreasing, which yields the conclusion. □

The following is a generalized version of Gronwall's inequality.

Lemma C.2 (refs. [19], [20], [22])**.** *Let $\alpha\colon [a,b] \to \mathbb{R}$ and $\beta\colon [a,b] \to [0,\infty)$ be given continuous functions. If a continuous function $u\colon [a,b] \to \mathbb{R}$ satisfies*

$$u(t) \le \alpha(t) + \int_a^t \beta(s)u(s)\,\mathrm{d}s$$

for all $t \in [a,b]$, then

$$u(t) \le \alpha(t) + \int_a^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s$$

holds for all $t \in [a,b]$. Furthermore, if α is monotonically increasing, then

$$u(t) \le \alpha(t)\exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right)$$

holds.

By letting $v(t) := \int_a^t \beta(s)u(s)\,\mathrm{d}s$, one can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp\left(-\int_a^t \beta(s)\,\mathrm{d}s\right)v(t) \le \exp\left(-\int_a^t \beta(s)\,\mathrm{d}s\right)\beta(t)\alpha(t).$$

Then the first inequality is obtained by integrating both sides in combination with $u(t) \le \alpha(t) + v(t)$. See [20, Section I.6], [19, Lemma 3.1 in Section 1.3], and [22, Lemma 3.1 in Section 1.3] for the detail of the proof.

C.2 Gronwall's inequality and RFDEs

In this subsection, let $r > 0$ and $E = (E, \|\cdot\|)$ be a normed space. For each continuous function $u\colon [a-r,b] \to E$ and each $t \in [a,b]$, let $u_t \in C([-r,0],E)$ be defined by

$$u_t(\theta) := u(t+\theta) \qquad (\theta \in [-r,0]).$$

It holds that the function $[a,b] \ni t \mapsto u_t \in C([-r,0],E)$ is continuous.

In the context of RFDEs, it is often convenient to use the following result rather than to use Gronwall's inequality directly.

Lemma C.3 (cf. [23])**.** *Let $\alpha \in \mathbb{R}$ be a constant and $\beta\colon [a,b] \to [0,\infty)$ be a given continuous function. If a continuous function $u\colon [a-r,b] \to E$ satisfies*

$$\|u(t)\| \le \alpha + \int_a^t \beta(s)\|u_s\|\,\mathrm{d}s$$

for all $t \in [a,b]$, then

$$\|u_t\| \le \max\{\|u_a\|, \alpha\}\exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right)$$

holds for all $t \in [a,b]$.

This should be compared with [23, Lemma 2.1]. We note that the argument of the proof has appeared in [19, Theorem 1.1 in Chapter 6] and [22, Theorem 1.1 in Chapter 6].

A generalization of Lemma C.3 is possible by using Lemma C.2.

Lemma C.4. *Let $\alpha\colon [a,b] \to \mathbb{R}$ and $\beta\colon [a,b] \to [0,\infty)$ be given continuous functions. If a continuous function $u\colon [a-r,b] \to E$ satisfies*

$$\|u(t)\| \le \alpha(t) + \int_a^t \beta(s)\|u_s\|\,\mathrm{d}s$$

for all $t \in [a,b]$ and α is monotonically increasing, then

$$\|u_t\| \le \max\{\|u_a\|, \alpha(t)\}\exp\left(\int_a^t \beta(s)\,\mathrm{d}s\right)$$

holds for all $t \in [a,b]$.

Proof. Let $t \in [a,b]$ be fixed and $\theta \in [-r,0]$ be given. When $t+\theta \ge a$, we have

$$\begin{aligned}\|u(t+\theta)\| &\le \alpha(t+\theta) + \int_a^{t+\theta} \beta(s)\|u_s\|\,\mathrm{d}s \\ &\le \alpha(t) + \int_a^t \beta(s)\|u_s\|\,\mathrm{d}s.\end{aligned}$$

Here the property that α is monotonically increasing and the non-negativity of β are used. When $t+\theta \le a$, we have

$$\|u(t+\theta)\| \le \|u_a\|.$$

By combining the above inequalities, we obtain

$$\|u_t\| \le \max\{\|u_a\|, \alpha(t)\} + \int_a^t \beta(s)\|u_s\|\,\mathrm{d}s.$$

Since the functions $[a,b] \ni t \mapsto \|u_t\| \in [0,\infty)$ and $[a,b] \ni t \mapsto \max\{\|u_a\|, \alpha(t)\} \in \mathbb{R}$ are continuous, the conclusion is obtained by applying Lemma C.2. □

D Lemmas on fixed point argument

Let $E = (E, \|\cdot\|)$ be a normed space and $r > 0$ be a constant. For each $\gamma > 0$, let

$$Y_\gamma := \left\{ y \in C([-r,\infty), E) : y_0 = 0,\ \|y\|_\gamma < \infty \right\}$$

be a normed space endowed with the norm $\|\cdot\|_\gamma$ given by

$$\|y\|_\gamma := \sup_{t\ge 0}\,(\mathrm{e}^{-\gamma t}\|y_t\|) < \infty.$$

For the notation $\|y_t\|$, see Subsection C.2.

Lemma D.1. *For any continuous function $y\colon [-r,\infty) \to E$ with $y_0 = 0$,*

$$\|y\|_\gamma = \sup_{t\ge 0}\,(\mathrm{e}^{-\gamma t}\|y(t)\|)$$

holds.

Proof. Since $\|y(t)\| \le \|y_t\|$ holds for all $t \ge 0$,

$$\sup_{t\ge 0} (\mathrm{e}^{-\gamma t}\|y(t)\|) \le \|y\|_\gamma$$

holds. The reverse inequality also follows in view of

$$\mathrm{e}^{-\gamma t}\|y(t+\theta)\| = \mathrm{e}^{-\gamma(t+\theta)}\|y(t+\theta)\| \cdot \mathrm{e}^{\gamma\theta} \le \sup_{t\ge 0} (\mathrm{e}^{-\gamma t}\|y(t)\|)$$

for $t \ge 0$ and $\theta \in [-r, 0]$. Here $y_0 = 0$ and $\mathrm{e}^{\gamma\theta} \le 1$ are used. □

Lemma D.2. *If E is a Banach space, then Y_γ is also a Banach space.*

Proof. Let $(y^k)_{k=1}^\infty$ be a Cauchy sequence in Y_γ. We choose $\varepsilon > 0$. Then for all sufficiently large $k, \ell \ge 1$, we have $\left\|y^k - y^\ell\right\|_\gamma \le \varepsilon$. From Lemma D.1, this means that for all sufficiently large $k, \ell \ge 1$,

$$\left\|y^k(t) - y^\ell(t)\right\| \le \varepsilon \mathrm{e}^{\gamma t}$$

holds for all $t \ge 0$. This implies that $(y^k(t))_{k=1}^\infty$ is a Cauchy sequence for each $t \ge 0$, and therefore, $(y^k)_{k=1}^\infty$ has the limit function $y \colon [-r, \infty) \to E$ with $y_0 = 0$. Since the above relation shows that the convergence of $(y^k)_{k=1}^\infty$ to y is uniform on each closed and bounded interval of $\mathbb{R}$ by taking the limit as $\ell \to \infty$, the limit function y is continuous. Then it is concluded that

$$\left\|y^k - y\right\|_\gamma \le \varepsilon$$

holds for all sufficiently large $k \ge 1$, which implies that $(y^k)_{k=1}^\infty$ converges to y in Y_γ. □

E Convolution continued

In this appendix, we discuss the convolution for functions in $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. The purpose is to share results on the convolution and their proofs in the literature of RFDEs. The results discussed here extend the results in Subsection 5.2, but they will not be used in this paper. See also [25, Proposition A.4, Theorems A.5, A.6, A.7 in Appendix A].

E.1 Convolution for locally essentially bounded functions and locally Lebesgue integrable functions

We first recall that a function $g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ is said to be *locally essentially bounded* if

$$\operatorname*{ess\,sup}_{t\in[0,T]} |g(t)| := \inf\{M > 0 : |g(t)| \le M \text{ holds for almost all } t \in [0,T]\}$$

is finite for all $T > 0$. Let

$$\mathcal{L}^\infty_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K})) := \left\{g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K})) : g \text{ is locally essentially bounded}\right\},$$

which is a linear subspace of $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. As in Definition 5.1, we introduce the following.

Definition E.1. For each $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ and each $g \in \mathcal{L}^\infty_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$, we define a function $g * f \colon [0,\infty) \to M_n(\mathbb{K})$ by

$$(g * f)(t) := \int_0^t g(t-u)f(u)\,\mathrm{d}u = \int_0^t g(u)f(t-u)\,\mathrm{d}u$$

for $t \geq 0$. Here the integrals are Lebesgue integrals. The function $g * f$ is called the *convolution* of g and f.

We note that

$$|(g * f)(t)| \leq \operatorname*{ess\,sup}_{u \in [0,t]} |g(u)| \cdot \int_0^t |f(u)|\,\mathrm{d}u$$

holds for all $t \geq 0$. The following result should be compared with Lemma 5.2.

Lemma E.2. *Let $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ and $g \in \mathcal{L}^\infty_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. Then $g * f$ is continuous.*

Outline of the proof. We show the continuity of $g * f$ on $[0, T]$ for each fixed $T > 0$. We define a function $\tilde{f} \colon \mathbb{R} \to M_n(\mathbb{K})$ by

$$\tilde{f}(t) := \begin{cases} f(t) & (t \in \mathrm{dom}(f) \cap [0,T]), \\ O & (\text{otherwise}). \end{cases}$$

Then $\tilde{f} \in \mathcal{L}^1(\mathbb{R}, M_n(\mathbb{K}))$, and

$$(g * f)(t) = \int_0^t g(u)\tilde{f}(t-u)\,\mathrm{d}u$$

holds for all $t \in [0, T]$. We fix $t_0 \in [0, T]$. By the reasoning as in the proof of Lemma 3.4, we have

$$(g * f)(t) - (g * f)(t_0) = \int_0^{t_0} g(u)\left[\tilde{f}(t-u) - \tilde{f}(t_0-u)\right]\mathrm{d}u + \int_{t_0}^t g(u)\tilde{f}(t-u)\,\mathrm{d}u$$

for all $t \in [0, T]$. Therefore, the continuity of $g * f$ on $[0, T]$ is obtained by Hölder's inequality, the continuity of the translation in $\mathcal{L}^1$, and the integrability of $\tilde{f}$. □

E.2 Convolution for locally Lebesgue integrable functions

The notion of convolution in Definition E.1 is not satisfactory in the sense that the condition on f and g is not symmetry. To introduce the notion of convolution for functions in $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$, we need the following.

Theorem E.3. *Let $f, g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ be given. Then the following statements hold:*

1. *For almost all $t > 0$, $u \mapsto g(t-u)f(u)$ belongs to $\mathcal{L}^1([0,t], M_n(\mathbb{K}))$.*
2. *The function $g * f$ defined by*

$$(g * f)(t) := \int_0^t g(t-u)f(u)\,\mathrm{d}u$$

for almost all $t \geq 0$ belongs to $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$.

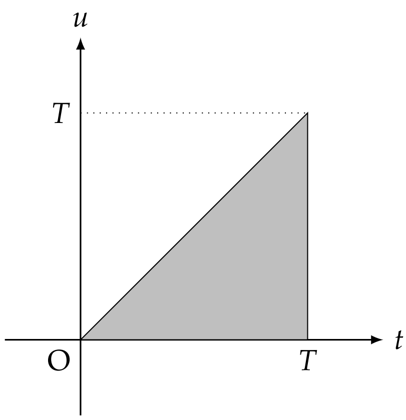

Figure E.1: The light gray region is the subset A.

3. *For all $T \geq 0$,*
$$\int_0^T |(g * f)(t)| \, \mathrm{d}t \leq \left(\int_0^T |g(t)| \, \mathrm{d}t \right) \cdot \left(\int_0^T |f(t)| \, \mathrm{d}t \right)$$
holds.

In the following, we give a direct proof of Theorem E.3 by using Fubini's theorem and Tonelli's theorem for functions on the Euclidean space $\mathbb{R}^d$. See [32, Theorems 3.1 and 3.2 in Section 3 of Chapter 2] for these statements and their proofs.

A direct proof of Theorem E.3. Let $\bar{f} \colon \mathbb{R} \to M_n(\mathbb{K})$ be the function defined by
$$\bar{f}(t) := \begin{cases} f(t) & (t \in \operatorname{dom}(f)), \\ O & (t \in \mathbb{R} \setminus \operatorname{dom}(f)). \end{cases}$$
In the same way, we define the function $\bar{g} \colon \mathbb{R} \to M_n(\mathbb{K})$. Then $\bar{f}, \bar{g} \colon \mathbb{R} \to M_n(\mathbb{K})$ are locally Lebesgue integrable functions.

Let $T > 0$ be fixed. The remainder of the proof is divided into the following steps.

Step 1: Setting of triangle region and function. We consider a closed set A of $\mathbb{R}^2$ given by
$$A := \left\{ (t, u) \in \mathbb{R}^2 : t \in [0, T], \, u \in [0, t] \right\}.$$
See Fig. E.1 for the picture of A. Then the characteristic function $\mathbf{1}_A$ is measurable and
$$\mathbf{1}_A(t, u) = \mathbf{1}_{[0,T]}(t) \mathbf{1}_{[0,t]}(u) = \mathbf{1}_{[0,T]}(u) \mathbf{1}_{[u,T]}(t)$$
holds for all $(t, u) \in \mathbb{R}^2$. We define a function $h \colon \mathbb{R}^2 \to M_n(\mathbb{K})$ by
$$h(t, u) := \mathbf{1}_A(t, u) \bar{g}(t - u) \bar{f}(u).$$
Then h is measurable because
$$\mathbb{R}^2 \ni (t, u) \mapsto \bar{g}(t - u) \in M_n(\mathbb{K}) \quad \text{and} \quad \mathbb{R}^2 \ni (t, u) \mapsto \bar{f}(u) \in M_n(\mathbb{K})$$
are measurable.[2] This implies that the function $\mathbb{R}^2 \ni (t, u) \mapsto |h(t, u)| \in [0, \infty)$ is also measurable.

Step 2: Application of Tonelli's theorem. By applying Tonelli's theorem, the following statements hold:

[2]See [32, Corollary 3.7 and Proposition 3.9 in Section 3 of Chapter 2] for the results of scalar-valued case.

- For almost all $u \in \mathbb{R}$, the function $\mathbb{R} \ni t \mapsto |h(t,u)| \in [0,\infty)$ is measurable.
- For almost all $t \in \mathbb{R}$, the function $\mathbb{R} \ni u \mapsto |h(t,u)| \in [0,\infty)$ is measurable.
- The functions

$$u \mapsto \int_{\mathbb{R}} |h(t,u)|\,\mathrm{d}t \in [0,\infty], \quad t \mapsto \int_{\mathbb{R}} |h(t,u)|\,\mathrm{d}u \in [0,\infty]$$

  are measurable functions defined almost everywhere.
- We have

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(t,u)|\,\mathrm{d}t \right) \mathrm{d}u = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(t,u)|\,\mathrm{d}u \right) \mathrm{d}t = \int_{\mathbb{R}^2} |h(t,u)|\,\mathrm{d}(t,u)$$

  including the possibility that all the unsigned Lebesgue integrals are ∞.

Step 3: Application of Fubini's theorem. By Step 2, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |h(t,u)|\,\mathrm{d}(t,u) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(t,u)|\,\mathrm{d}t \right) \mathrm{d}u \\ &\le \int_0^T \left(\int_u^T |\bar{g}(t-u)|\,\mathrm{d}t \right) |\bar{f}(u)|\,\mathrm{d}u \\ &\le \left(\int_0^T |g(t)|\,\mathrm{d}t \right) \cdot \left(\int_0^T |f(t)|\,\mathrm{d}t \right). \end{aligned}$$

Since the last term is finite, it holds that h is integrable. By applying Fubini's theorem component-wise, the following statements hold:

- For almost all $u \in \mathbb{R}$, the function $\mathbb{R} \ni t \mapsto h(t,u) \in M_n(\mathbb{K})$ is Lebesgue integrable.
- For almost all $t \in \mathbb{R}$, the function $\mathbb{R} \ni u \mapsto h(t,u) \in M_n(\mathbb{K})$ is Lebesgue integrable.
- The functions

$$u \mapsto \int_{\mathbb{R}} h(t,u)\,\mathrm{d}t \in M_n(\mathbb{K}), \quad t \mapsto \int_{\mathbb{R}} h(t,u)\,\mathrm{d}u \in M_n(\mathbb{K})$$

  belong to $\mathcal{L}^1(\mathbb{R}, M_n(\mathbb{K}))$.
- The equalities

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(t,u)\,\mathrm{d}t \right) \mathrm{d}u = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(t,u)\,\mathrm{d}u \right) \mathrm{d}t = \int_{\mathbb{R}^2} h(t,u)\,\mathrm{d}(t,u)$$

  hold.

Step 4: Conclusion. For each $t \in [0,T]$,

$$h(t,u) = g(t-u)f(u)$$

holds for almost all $u \in [0,t]$. Therefore, for almost all $t \in [0,T]$, the function $u \mapsto g(t-u)f(u)$ belongs to $\mathcal{L}^1([0,t], M_n(\mathbb{K}))$. Furthermore, we have

$$\int_{\mathbb{R}} h(t,u)\,\mathrm{d}u = \int_0^t g(t-u)f(u)\,\mathrm{d}u$$

for almost all $t \in [0, T]$, and it holds that the function

$$t \mapsto \int_0^t g(t-u) f(u) \, \mathrm{d}u$$

is a Lebesgue integrable function defined almost everywhere on $[0, T]$. Since $T > 0$ is arbitrary, the statements 1 and 2 hold. The statement 3 also holds because we have

$$\begin{aligned}
\int_0^T |(g * f)(t)| \, \mathrm{d}t &\le \int_0^T \left(\int_0^t |g(t-u) f(u)| \, \mathrm{d}u \right) \mathrm{d}t \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(t,u)| \, \mathrm{d}u \right) \mathrm{d}t \\
&\le \left(\int_0^T |g(t)| \, \mathrm{d}t \right) \cdot \left(\int_0^T |f(t)| \, \mathrm{d}t \right),
\end{aligned}$$

where the calculation in Step 3 is used.

This completes the proof. □

Another proof of Theorem E.3. Let $T > 0$ be fixed. We define $\tilde{f} \colon \mathbb{R} \to M_n(\mathbb{K})$ by

$$\tilde{f}(t) := \begin{cases} f(t) & (t \in \mathrm{dom}(f) \cap [0, T]), \\ O & (\text{otherwise}). \end{cases}$$

In the same way, we define the function $\tilde{g} \colon \mathbb{R} \to M_n(\mathbb{K})$. Since $\tilde{f}, \tilde{g} \colon \mathbb{R} \to M_n(\mathbb{K})$ are Lebesgue integrable functions, one can prove the following statements as in the scalar-valued case:[3]

1′. For almost all $t \in \mathbb{R}$, the function $u \mapsto \tilde{g}(t-u)\tilde{f}(u)$ is a Lebesgue integrable function defined almost everywhere.

2′. The function $\tilde{g} \star \tilde{f}$ defined by

$$(\tilde{g} \star \tilde{f})(t) := \int_{\mathbb{R}} \tilde{g}(t-u) \tilde{f}(u) \, \mathrm{d}u$$

for almost all $t \in \mathbb{R}$ belongs to $\mathcal{L}^1(\mathbb{R}, M_n(\mathbb{K}))$.

3′. An estimate

$$\int_{\mathbb{R}} \left| (\tilde{g} \star \tilde{f})(t) \right| \mathrm{d}t \le \|\tilde{g}\|_1 \cdot \|\tilde{f}\|_1$$

holds.

1. For each $t \in [0, T]$, we have

$$\tilde{g}(t-u)\tilde{f}(u) = g(t-u) f(u)$$

for almost all $u \in [0, t]$. By combining this and the above statement 1′, it holds that for almost all $t \in [0, T]$, $u \mapsto g(t-u) f(u)$ is a Lebesgue integrable function defined almost everywhere on $[0, t]$. Since $T > 0$ is arbitrary, the statement 1 holds.

2. By the definitions of $\tilde{f}$ and $\tilde{g}$,

$$(\tilde{g} \star \tilde{f})(t) = \int_0^t \tilde{g}(t-u) \tilde{f}(u) \, \mathrm{d}u = \int_0^t g(t-u) f(u) \, \mathrm{d}u$$

[3] See [32, Exercise 21 in Chapter 2] and [30, 8.13 and 8.14 of Chapter 8] for the scalar-valued case.

holds for all $t \in \operatorname{dom}(\tilde{g} \star \tilde{f}) \cap [0, T]$. Since $T > 0$ is arbitrary, this shows that

$$t \mapsto \int_0^t g(t-u) f(u) \, \mathrm{d}u$$

is a measurable function defined almost everywhere on $[0, \infty)$ from the statement 2′. Furthermore, we also have

$$\int_0^T \left| \int_0^t g(t-u) f(u) \, \mathrm{d}u \right| \mathrm{d}t = \int_0^T \left| (\tilde{g} \star \tilde{f})(t) \right| \mathrm{d}t < \infty.$$

Since $T > 0$ is arbitrary, the statement 2 holds.

3. By combining the proof of the statement 2 and the inequality in the statement 3′, we have

$$\int_0^T |(g * f)(t)| \, \mathrm{d}t \le \|\tilde{g}\|_1 \cdot \|\tilde{f}\|_1$$

Here

$$\|\tilde{f}\|_1 = \int_0^T |f(t)| \, \mathrm{d}t, \quad \|\tilde{g}\|_1 = \int_0^T |g(t)| \, \mathrm{d}t$$

holds since $f(t) = g(t) = O$ for $t \in (-\infty, 0) \cup (T, \infty)$. Therefore, the inequality in the statement 3 is obtained. □

The above proof of Theorem E.3 is not given in [34], [14], [25], and [17]. Based on Theorem E.3, we introduce the following.

Definition E.4. Let $f, g \in \mathcal{L}^1_{\mathrm{loc}}([0, \infty), M_n(\mathbb{K}))$. We call $g * f \in \mathcal{L}^1_{\mathrm{loc}}([0, \infty), M_n(\mathbb{K}))$ in Theorem E.3 defined by

$$(g * f)(t) := \int_0^t g(t-u) f(u) \, \mathrm{d}u = \int_0^t g(u) f(t-u) \, \mathrm{d}u$$

the *convolution* of f and g.

E.3 Convolution under Volterra operator

The convolution for functions in $\mathcal{L}^1_{\mathrm{loc}}([0, \infty), M_n(\mathbb{K}))$ and the Volterra operator are related in the following way.

Theorem E.5. *For any pair of $f, g \in \mathcal{L}^1_{\mathrm{loc}}([0, \infty), M_n(\mathbb{K}))$,*

$$V(g * f) = (Vg) * f = g * (Vf) \tag{E.1}$$

holds.

The above theorem is an extension of Corollary 5.5.

Proof of Theorem E.5. For each $t > 0$,

$$V(g * f)(t) = \int_0^t \left(\int_0^s g(s-u) f(u) \, \mathrm{d}u \right) \mathrm{d}s$$

holds by the definition of convolution for functions in $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. By applying Fubini's theorem in a similar way as in the direct proof of Theorem E.3, the right-hand side is calculated as

$$\begin{aligned}\int_0^t \left(\int_0^s g(s-u)f(u)\,\mathrm{d}u\right)\mathrm{d}s &= \int_0^t \left(\int_u^t g(s-u)\,\mathrm{d}s\right) f(u)\,\mathrm{d}u \\ &= \int_0^t (Vg)(t-u)f(u)\,\mathrm{d}u,\end{aligned}$$

where the last term is equal to $[(Vg)*f](t)$. Therefore, the integration by parts formula for matrix-valued absolutely continuous functions (see Theorem 6.13) yields

$$\begin{aligned}[(Vg)*f](t) &= [(Vg)(t-u)(Vf)(u)]_{u=0}^{t} + \int_0^t g(t-u)(Vf)(u)\,\mathrm{d}u \\ &= [g*(Vf)](t),\end{aligned}$$

where $(Vg)(0) = (Vf)(0) = O$ is used. This completes the proof. □

Remark E.6. Eq. (E.1) is a special case of the *associativity of convolution*

$$(h*g)*f = h*(g*f) \tag{E.2}$$

for $f, g, h \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ because

$$(f*\mathcal{I})(t) = (\mathcal{I}*f)(t) = \int_0^t f(s)\,\mathrm{d}s = (Vf)(t) \quad (t \geq 0)$$

holds for any $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. Here $\mathcal{I}\colon [0,\infty) \to M_n(\mathbb{K})$ denote the constant function whose value is equal to the identity matrix.

The following is a result on the regularity of convolution. It should be compared with Theorem 5.3.

Theorem E.7. *Let $f \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$ and $g\colon [0,\infty) \to M_n(\mathbb{K})$ be a locally absolutely continuous function. Then $g*f$ is expressed by*

$$g*f = V\big(g(0)f + g'*f\big). \tag{E.3}$$

Consequently, $g*f$ is locally absolutely continuous, differentiable almost everywhere, and satisfies

$$(g*f)'(t) = g(0)f(t) + (g'*f)(t)$$

for almost all $t \geq 0$.

We note that for a locally absolutely continuous function $g\colon [0,\infty) \to M_n(\mathbb{K})$, the derivative g' belongs to $\mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. Therefore, the convolution $g'*f$ makes sense from Theorem E.3.

Proof of Theorem E.7. Since $g = g(0) + Vg'$, we obtain

$$g*f = g(0)Vf + (Vg')*f = g(0)Vf + V(g'*f)$$

by using Theorem E.5. This yields the expression (E.3) because the Volterra operator is linear. The remaining properties of $g*f$ are derived by the properties of Volterra operator. □

See also [17, 7.4 Corollary in Chapter 3] for related results.

Remark E.8. From Theorem E.7, we have

$$(Vg) * f = V(g * f)$$

for $f, g \in \mathcal{L}^1_{\mathrm{loc}}([0,\infty), M_n(\mathbb{K}))$. We also have

$$g * (Vf) = V(g * f)$$

in a similar way.

We note that the statement 2 of Corollary 5.6 also follows by Lemma E.2 and Theorem E.7.

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The Dirichlet problem in an unbounded cone-like domain for second order elliptic quasilinear equations with variable nonlinearity exponent

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Abstract. In this paper we consider the Dirichlet problem for quasi-linear second-order elliptic equation with the $m(x)$-Laplacian and the strong nonlinearity on the right side in an unbounded cone-like domain. We study the behavior of weak solutions to the problem at infinity and we find the sharp exponent of the solution decreasing rate. We show that the exponent is related to the least eigenvalue of the eigenvalue problem for the Laplace–Beltrami operator on the unit sphere.

Keywords: $m(x)$-Laplacian, elliptic equation, unbounded domain, cone-like domain.

2020 Mathematics Subject Classification: 35J20, 35J25, 35J70.

1 Introduction

In recent years there has been an increasing interest in the study of various mathematical problems with variable exponent, see e.g. [4, 16, 17, 21–23, 28, 29] and references therein. The basic properties of variable exponent function spaces were derived by O. Kováčik and J. Rákosník in [18] and (by different methods) by X.-L. Fan and D. Zhao in [14]. For a comprehensive survey concerning Lebesgue and Sobolev spaces with variable exponent we refer to [12].

Differential equations and variational problems with $m(x)$-growth conditions arise from the study of elastic mechanics, oscillation problem, electrorheological fluids [11, 24, 25], image restoration [10], thermistor problem [31] and other. Moreover, the motion of a compressible fluid in a nonhomogeneous anisotropic porous medium obeys to nonlinear the Darcy law [3]. The model of electrorheological fluids considered in [25] includes an integral of the symmetric part of gradient in a variable power which is caused by the action of an electromagnetic field. A similar structure of energy is also presented in the thermorheological model proposed in [30] for fluids with the stress tensor depending on the temperature. This system can be referred to as a coupled Boussinesq type sytem for a non-Newtonian fluid.

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Our interest is in the studying of the behavior of weak solutions to the Dirichlet problem with boundary condition on the lateral surface of a cone-like unbounded domain at infinity. For other results in unbounded and bounded cone-like domains we refer to [5–8,27]. We refer also to some very recent works dealing with complementary aspects [20,26]. These works can provide some ideas for further investigations in the cone-like domain too. For putting more emphasis on the effects of a gradient dependent reaction in the principal equation we refer to [15,19].

This paper is organized as follows. At first, we formulate the Dirichlet problem in an unbounded cone-like domain for second order elliptic quasilinear equations with variable nonlinearity exponent. Then, we introduce notations and function spaces that are used in the following sections. The main result, Theorem 1.2, is also formulated. In Section 2 we formulate an eigenvalue problem for the Laplace–Beltrami operator on the unit sphere, a Friedrichs–Wirtinger type inequality and some auxiliary inequalities and lemmas. In the next sections local estimate of the weighted Dirichlet integral and local estimate of weak solutions at infinity are investigated. Finally in Section 5 the power modulus of continuity near the infinity for weak solutions is considered.

Let $B_1(\mathcal{O})$ be the unit ball in $\mathbb{R}^n$, $n \geq 2$ with center at the origin $\mathcal{O}$ and $G \subset \mathbb{R}^n \setminus B_1(\mathcal{O})$ be an unbounded domain with the smooth boundary ∂G. We assume that $G \supset G_R$, where G_R is a cone-like domain, $G_R = \{x = (r, \omega) \in \mathbb{R}^n \mid r \in (R, \infty),\ \omega \in \Omega \subset S^{n-1},\ n \geq 2\}$, $R \gg 1$, S^{n-1} is the unit sphere (see Figure 1.1).

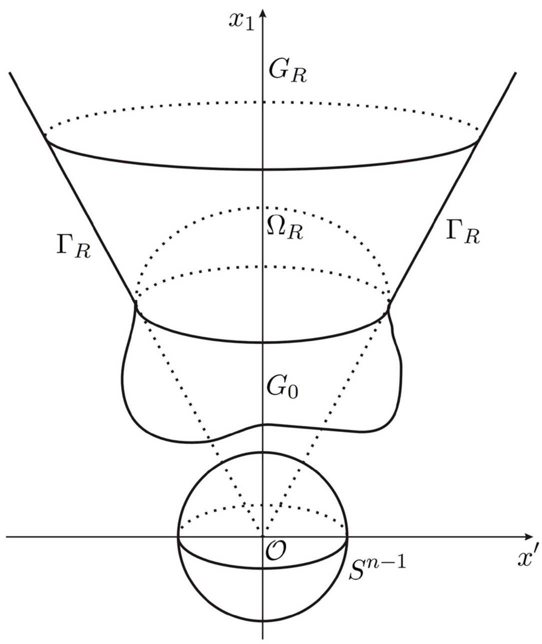

Figure 1.1: An unbounded cone-like domain

We consider the following Dirichlet problem for a quasi-linear elliptic equation **with the variable growth exponent**:

$$\begin{cases} -\frac{d}{dx_i}(|\nabla u|^{m(x)-2}u_{x_i}) + b(x,u,\nabla u) = 0, & x \in G_R, \\ u(x) = 0, & x \in \Gamma_R, \\ u(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{QL}$$

The following conditions will be needed throughout the paper:

(i) $1 < \inf\{m(x) : x \in G_R\} = m_- \le m(x) \le m_+ = \sup\{m(x) : x \in G_R\} < \infty$;

(ii) the function $m(x)$ is Hölder continuous in $\overline{G_R}$, i.e. there exist a positive constant M and an exponent $\alpha \in (0,1)$ such that

$$|m(x) - m(+\infty)| \le M|x|^{-\alpha}, \quad \forall x \in \overline{G_R},$$

where $m(+\infty) = \lim_{|x|\to+\infty} m(x) = 2$;

(iii) $b(x,u,\xi)$ is a Carathéodory function $G_R \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and

$$|b(x,u,\xi)| \le \mu(|u|+1)^{-1}|\xi|^{m(x)}, \quad 0 \le \mu < \frac{1}{m_+} < 1;$$

(iv) $\partial\Omega \in C^{1+\gamma}$, $\gamma \in (0,1)$.

We introduce the following notations:

- $\mathcal{C}$: a rotational cone $\{x_1 > r\cos\frac{\omega_0}{2}\}$;
- $\partial\mathcal{C}$: the lateral surface of $\mathcal{C}$: $\{x_1 = r\cos\frac{\omega_0}{2}\}$;
- Ω : a domain on the unit sphere S^{n-1} with smooth boundary $\partial\Omega$ obtained by the intersection of the cone $\mathcal{C}$ with the sphere S^{n-1};
- $\partial\Omega = \partial\mathcal{C} \cap S^{n-1}$;
- $G_a^b = \{(r,\omega) \mid a < r < b;\ \omega \in \Omega\} \cap G$: the layer in $\mathbb{R}^n$;
- $\Gamma_a^b = \{(r,\omega) \mid a < r < b;\ \omega \in \partial\Omega\} \cap \partial G$: the lateral surface of layer G_a^b, $\Gamma_\varrho = \Gamma_\varrho^\infty$

and the class of functions

$$W_{\text{loc}}(G_R) = \{u : u \in W_0^{1,1}(G_R,\Gamma_R),\ |\nabla u|^{m(x)} \in L_1(G_R),\ \forall R \gg 1\},$$

where $W_0^{1,1}(G_R,\Gamma_R)$ is the Sobolev space of those functions with zero trace on Γ_R that, together with all their first order distributional derivatives, are L^1-integrable in G_R.

We denote $W_0^1(\Omega) \equiv W_0^{1,2}(\Omega)$.

Definition 1.1. A function $u(x) \in W_{\text{loc}}(G_R)$ such that $u(x) \to 0$ as $|x| \to \infty$ is said to be a weak solution of problem (QL) provided the integral identity

$$\int_{G_R} \Big(|\nabla u|^{m(x)-2} u_{x_i}\eta_{x_i} + b(x,u,\nabla u)\eta(x) \Big)\, dx = 0 \qquad (II)$$

holds for all test functions $\eta(x) \in W_{\text{loc}}(G_R)$ such that $\eta(x) \to 0$ as $|x| \to \infty$.

We use the Sobolev embedding theorem for functions $\varphi \in W_0^{1,q}(G_1^2)$:

$$\left(\int_{G_1^2} |\varphi|^{\tilde{n}q} dx' \right)^{\frac{1}{\tilde{n}}} \le C \int_{G_1^2} |\nabla'\varphi|^q dx', \quad \tilde{n} = \frac{n}{n-1}, \ \forall q \ge 1, \qquad (1.1)$$

where $x' = \frac{1}{\varrho}x$, $\varrho > R$. Our main theorem is the following:

Theorem 1.2. *Let u be a weak solution of problem (QL), $l = \max\{m(x) : x \in \overline{G}_\varrho^{2\varrho}\}$, λ_- be as in (2.4) and assumption (i)–(iv) be satisfied. Then there exist $R \gg 1$ and a positive constant C such that*

$$|u(x)| \le C \cdot |x|^{\lambda_-(1-\mu)} \quad \forall x \in G_R. \qquad (1.2)$$

2 Preliminaries

2.1 Eigenvalue problem

We consider the eigenvalue problem for the Laplace–Beltrami operator Δ_ω on the unit sphere

$$\begin{cases} \Delta_\omega \psi + \vartheta\psi = 0, & \omega \in \Omega; \\ \psi(\omega) = 0, & \omega \in \partial\Omega, \end{cases} \tag{EVP}$$

which consists of the determination of all values ϑ (eigenvalues) for which (EVP) has non-zero weak solutions $\psi(\omega) \neq 0$ (eigenfunctions).

Definition 2.1. A function ψ is said to be a weak solution of problem (EVP) provided that $\psi \in W_0^1(\Omega)$ and satisfies the integral identity

$$\int_\Omega \left(\frac{1}{q_i} \frac{\partial\psi}{\partial\omega_i} \frac{\partial\eta}{\partial\omega_i} - \vartheta\psi\eta \right) d\Omega = 0$$

for all $\eta(\omega) \in W_0^1(\Omega)$.

Throughout the paper we need only the least positive eigenvalue:

$$\vartheta_* := \inf_{\psi \in W_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_\omega \psi|^2 d\Omega}{\int_\Omega |\psi|^2 d\Omega}.$$

For the existence problem of the least positive eigenvalue to problem (EVP) see for example Section 8.2.3 [9].

2.2 The Friedrichs–Wirtinger type inequality

From the definition of $\vartheta_*(\Omega)$ we obtain the following Friedrichs–Wirtinger type inequality:

Theorem 2.2. *For all $\psi \in W_0^1(\Omega)$ the inequality*

$$\int_\Omega |\psi|^2 d\Omega \leq \frac{1}{\vartheta_*} \int\limits_\Omega |\nabla_\omega \psi|^2 d\Omega \tag{2.1}$$

holds with the sharp constant $\frac{1}{\vartheta_*}$.

Corollary 2.3. *Let $v(x) \in W_0^1(G_R)$. Then for any $\varrho > R$ and for all α*

$$\int_{G_\varrho} r^\alpha |v|^2 dx \leq \frac{1}{\vartheta_*} \int_{G_\varrho} r^{\alpha+2} |\nabla v|^2 dx \tag{2.2}$$

provided that the integral on the right is finite.

Proof. Consider the inequality (2.1) for the function $u(r, \omega)$. Multiplying it by $r^{\alpha+n-1}$ and integrating over $r \in (\varrho, \infty)$, we obtain the desired inequality. □

2.3 Auxiliary integro-differential inequalities

Lemma 2.4 (see Lemma 2.9 in [27]). *Let G_R be an unbounded cone-like domain and $\nabla u(\varrho, \cdot) \in L_2(\Omega)$ for almost all $\varrho \in (R, \infty)$. Suppose also that*

$$U(\varrho) = \int_{G_\varrho} r^{2-n} |\nabla u|^2 dx < \infty.$$

Then

$$\int_\Omega \left(\varrho u \frac{\partial u}{\partial r} + \frac{n-2}{2} u^2 \right) \bigg|_{r=\varrho} d\Omega \geq -\frac{\varrho}{2\lambda_-} U'(\varrho), \tag{2.3}$$

where λ_- is a negative number connected with ϑ_* by the equality

$$\lambda_- = \frac{2 - n - \sqrt{(n-2)^2 + 4\vartheta_*}}{2}. \tag{2.4}$$

Theorem 2.5 (see Theorem 2.10 in [27]). *Suppose that $U(\varrho)$ is a monotonically decreasing, nonnegative differentiable function defined on $[R, \infty)$, $R \gg 1$, satisfying*

$$\begin{cases} U'(\varrho) + P(\varrho) U(\varrho) - Q(\varrho) \leq 0, & \varrho > R, \\ U(R) \leq U_0, \end{cases} \tag{CP}$$

where $P(\varrho), Q(\varrho)$ are nonnegative continuous functions defined on $[R, \infty)$ and U_0 is a constant. Then

$$U(\varrho) \leq U_0 \exp\left(-\int_R^\varrho P(\sigma) d\sigma \right) + \int_R^\varrho Q(t) \exp\left(-\int_t^\varrho P(\sigma) d\sigma \right) dt.$$

Now our aim is to estimate the gradient modulus of the problem (QL) solutions at infinity.

Lemma 2.6. *Let $u(x)$ be a weak solution of (QL) and assumptions (i)–(iv) hold. Then*

$$|\nabla u(x)| \leq M_1' |x|^{-1}, \quad \forall x \in G_R, \ R \gg 1. \tag{2.5}$$

We consider the solution u to the problem (QL) in the domain $G_{\frac{\varrho}{2}}^{\varrho} \subset G_R$, $\varrho > R$. We make the change of variables $x = \varrho x'$. Then the function $z(x') = u(\varrho x')$ satisfies the problem

$$\begin{cases} -\frac{d}{dx_i'} (\varrho^{m_- - m(\varrho x')} |\nabla' z|^{m(\varrho x') - 2} z_{x_i'}) + \varrho^{m_-} b(\varrho x', z, \varrho^{-1} \nabla' z) = 0, & x' \in G_{\frac{1}{2}}^1, \\ z(x') = 0, & x' \in \Gamma_{\frac{1}{2}}^1. \end{cases} \tag{QL'}$$

We verify that function $d(x') = \varrho^{m_- - m(\varrho x')}$ is Hölder continuous at infinity.

First of all, by the mean value Lagrange theorem, we have

$$|\varrho^{m_- - m(\varrho x')} - \varrho^{m_- - m(+\infty)}| = |m(+\infty) - m(\varrho x')| \cdot \varrho^t \ln \varrho,$$

where t is a negative number between $m_- - m(\varrho x')$ and $m_- - m(+\infty)$. Hence and by the Hölder assumption (ii), we get

$$|d(x') - d(+\infty)| = |\varrho^{m_- - m(\varrho x')} - \varrho^{m_- - m(+\infty)}| \leq M |x'|^{-\alpha} \varrho^{-\alpha} \ln \varrho.$$

Now, using first derivative test, we can conclude that

$$|a|^\delta |\ln |a|| \leq \frac{1}{\delta e}, \quad |a| < 1, \ \forall \delta > 0. \tag{2.6}$$

Thus, we obtain the required

$$|d(x') - d(+\infty)| \le \frac{M}{\alpha e}|x'|^{-\alpha}.$$

Further, assumptions (i), (iii) yield:

$$\varrho^{m_-}|b(\varrho x', z, \varrho^{-1}\nabla' z)| \le \mu|\nabla' z|^{m(\varrho x')}, \quad \varrho \gg 1$$

and therefore we can apply the X. Fan Theorem 1.2 and Remark 5.2 [13] about a priori estimate of the gradient modulus of the problem (QL') solution

$$\max_{x' \in G^1_{\frac{1}{2}}} |\nabla' z| \le M'_1.$$

Returning to variable x and function $u(x)$, we obtain

$$|\nabla u| \le M'_1 \varrho^{-1}, \quad x \in G^{\varrho}_{\frac{\varrho}{2}}, \ \varrho > R.$$

Setting now $|x| = \frac{2}{3}\varrho$ we obtain the required (2.5).

Lemma 2.7. *Let u be a weak solution of problem (QL) and assumptions (i)–(iv) be satisfied. Then we have:*

$$\int_{G_{2R}} r^{2-n}|\nabla u|^{m(x)}dx < \infty, \qquad \int_{G_{2R}} r^{2-n}|\nabla u|^2 dx < \infty \tag{2.7}$$

$$\lim_{\mathcal{N}\to+\infty} \mathcal{N}^{-1} \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2} u \frac{\partial u}{\partial r} dx = 0. \tag{2.8}$$

Proof. At first we will show the convergence of the first integral. We set $r_k = 2^k \cdot R$, $k = 0, 1, 2, \ldots$ and let $\eta_k \in C_0^\infty(G_{r_k})$ with the following properties:

$$\begin{cases} 0 \le \eta_k \le 1, \ |\nabla \eta_k| \le c \cdot r_k^{-1} & x \in G_{r_k} \\ \eta_k = 1 & x \in G_{r_{k+1}}. \end{cases}$$

We choose $\eta = u\eta_k^{m^+}$ as a test function in (II). Then we obtain:

$$\int_{G_{r_k}} |\nabla u|^{m(x)} \eta_k^{m_+} dx = -\int_{G_{r_k}} \left(m_+ u |\nabla u|^{m(x)-2} \nabla u \nabla \eta_k \cdot \eta_k^{m_+-1} + b(x, u, u_x) \cdot u \cdot \eta_k^{m_+} \right) dx. \tag{2.9}$$

Next, using the Young inequality, with $q = \frac{m(x)}{m(x)-1}$, $q' = m(x)$, we get

$$m_+|u||\nabla u|^{m(x)-1} \cdot |\nabla \eta_k| \cdot \eta_k^{m_+-1} = \left(m_+|u||\nabla \eta_k| \eta_k^{\frac{m_+-m(x)}{m(x)}} \right) \cdot \left(|\nabla u|^{m(x)-1} \eta_k^{\frac{m_+(m(x)-1)}{m(x)}} \right)$$
$$\le \frac{m_+^{m(x)}}{m(x)} |u|^{m(x)} |\nabla \eta_k|^{m(x)} \eta_k^{m_+-m} + \frac{m(x)-1}{m(x)} |\nabla u|^{m(x)} \eta_k^{m_+}.$$

Thus, from (2.9) we get

$$\int_{G_{r_k}} |\nabla u|^m \eta_k^{m_+} dx \le c(m_-, m_+) \int_{G_{r_k}} |u|^m |\nabla \eta_k|^m \eta_k^{m_+-m(x)} dx + m_+ \int_{G_{r_k}} |b(x, u, u_x)||u| \eta_k^{m_+} dx.$$

Next, using assumption (iii), the inequality above yields

$$(1-m_+\mu)\int_{G_{r_k}}|\nabla u|^m\eta_k^{m_+}dx\leq c(m_-,m_+)\int_{G_{r_k}}|u|^m|\nabla\eta_k|^{m(x)}\eta_k^{m_+-m(x)}dx.$$

In view of the choice of η_k, we get

$$(1-m_+\mu)\int_{G_{r_{k+1}}^{r_{k+2}}}|\nabla u|^m dx\leq\tilde{c}_1(m_-,m_+)\int_{G_{r_k}^{r_{k+1}}}|u|^m r^{-m}dx. \tag{2.10}$$

We use the fact [2] that any solution u is Hölder continuous in G_R:

$$|u|\leq H_0|x|^{-\alpha_0},\quad \forall x\in G_R.$$

Hence, by assumption (ii) and because

$$\lim_{r\to+\infty}r^{r^{-\alpha}}=1, \tag{2.11}$$

we can estimate

$$\begin{aligned}|u|^{m(x)}&\leq(H_0+1)^{m_+}r^{-2\alpha_0}r^{\alpha_0(2-m)}=(H_0+1)^{m_+}r^{-2\alpha_0}r^{\alpha_0(m(+\infty)-m(x))}\\&\leq(H_0+1)^{m_+}r^{-2\alpha_0}r^{\alpha_0Mr^{-\alpha}}\leq C(H_0,M,\alpha_0,\alpha,m_+)\cdot r^{-2\alpha_0},\quad x\in G_R;\\ r^{-m(x)}&=r^{-2}\cdot r^{2-m(x)}\leq r^{-2}\cdot r^{Mr^{-\alpha}}\leq C(M,\alpha)r^{-2}.\end{aligned} \tag{2.12}$$

Hence

$$|u|^{m(x)}\cdot r^{-m(x)}\leq C(H_0,M,\alpha,\alpha_0,m_+)r^{-2-2\alpha_0},\quad x\in G_R. \tag{2.13}$$

In this way, from (2.10)

$$\int_{G_{r_{k+1}}^{r_{k+2}}}|\nabla u|^m dx\leq C_2(H_0,M,\alpha,\alpha_0,m_\pm)\int_{G_{r_k}^{r_{k+1}}}r^{-2(\alpha_0+1)}dx. \tag{2.14}$$

Multiplying both sides of (2.14) by r_k^{2-n}, by the definition of r_k, we find

$$\int_{G_{r_{k+1}}^{r_{k+2}}}r^{2-n}|\nabla u|^m dx\leq C_2\int_{G_{r_k}^{r_{k+1}}}r^{-2\alpha_0-n}dx.$$

Summing up above inequalities for all $k=0,1,2,\dots$, we obtain

$$\int_{G_{2R}}r^{2-n}|\nabla u|^{m(x)}\leq C_2\int_{G_R}r^{-2\alpha_0-n}dx\leq C_2|\Omega|\int_R^\infty r^{-2\alpha_0-1}dr=C_3\cdot R^{-2\alpha_0}. \tag{2.15}$$

Thus, the convergence of the first integral in (2.7) is proved.

Now we observe that, in virtue of (2.5), (ii) and (2.11), we get

$$|\nabla u|^2=|\nabla u|^{m(x)}|\nabla u|^{2-m(x)}\leq C|\nabla u|^{m(x)}r^{Mr^{-\alpha}}\leq C|\nabla u|^{m(x)},$$

which, by (2.15), yields the convergence of the second integral in (2.7).

We shall prove (2.8). Applying the Young inequality with $q=\frac{m(x)}{m(x)-1}$, $q'=m(x)$ we have

$$\left| \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n} |\nabla u|^{m(x)-2} u \frac{\partial u}{\partial r} dx \right|$$
$$\le \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n} |\nabla u|^{m(x)-1} |u| dx$$
$$= \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} \left(r^{(3-n)\frac{m(x)-1}{m(x)}} |\nabla u|^{m(x)-1} \right) \cdot \left(r^{\frac{3-n-m(x)}{m(x)}} |u| \right) dx$$
$$\le \left(\mathcal{N} + \frac{1}{\mathcal{N}} \right) \left(\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n} |\nabla u|^{m(x)} dx + \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n-m(x)} |u|^{m(x)} dx \right).$$

We can estimate the first integral using (2.5) and (2.12) in the following way:

$$\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n} |\nabla u|^{m(x)} dx \le c \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n-m} dx \le C(M, M_1', \alpha, |\Omega|) \int_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}} \frac{1}{r} dr = C \ln \left(1 + \frac{1}{\mathcal{N}^2} \right),$$

while the second integral using (2.13):

$$\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n-m(x)} |u|^{m(x)} dx \le C \int_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}} r^{2-n} \cdot r^{-2-2\alpha_0} \cdot r^{n-1} dr \le C \mathcal{N}^{-2\alpha_0}.$$

From above inequalities we get

$$\lim_{\mathcal{N}\to+\infty} \mathcal{N}^{-1} \left| \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n} |\nabla u|^{m(x)-2} u \frac{\partial u}{\partial r} dx \right|$$
$$\le \lim_{\mathcal{N}\to+\infty} C \cdot \left(1 + \frac{1}{\mathcal{N}^2} \right) \cdot \left\{ \ln \left(1 + \frac{1}{\mathcal{N}^2} \right) + \mathcal{N}^{-2\alpha_0} \right\} = 0,$$

which is the required (2.8). □

We indicate another consequence of the integral identity (*II*) for solutions u to the problem (*QL*) which is essentially used in the further consideration.

Lemma 2.8. *If assumptions* (i)–(iv) *are satisfied, then*

$$\int_{G_\varrho} r^{2-n} |\nabla u|^{m(x)} + (2-n) \int_{G_\varrho} r^{1-n} |\nabla u|^{m(x)-2} u \frac{\partial u}{\partial r} dx$$
$$+ \int_{G_\varrho} r^{2-n} u b(x, u, u_x) dx = -\varrho^{2-n} \int_{\Omega_\varrho} |\nabla u|^{m(x)-2} u \frac{\partial u}{\partial r} d\Omega_\varrho, \ \forall \varrho \ge 4R \gg 1. \quad (2.16)$$

Proof. Let $\mathcal{N} > \varrho \ge 4R$. On $[R, \infty)$ we consider a Lipschitz piecewise linear function $\eta_{\mathcal{N}}(t)$ defined by

$$\eta_{\mathcal{N}}(t) = \begin{cases} 0, & \text{if } t \in [4R, \varrho] \cup [\mathcal{N} + \frac{1}{\mathcal{N}}, \infty), \\ 1, & \text{if } t \in [\varrho + \frac{1}{\mathcal{N}}, \mathcal{N}], \\ \mathcal{N}(t-\varrho), & \text{if } t \in [\varrho, \varrho + \frac{1}{\mathcal{N}}], \\ \mathcal{N}(\mathcal{N} - t) + 1, & \text{if } t \in [\mathcal{N}, \mathcal{N} + \frac{1}{\mathcal{N}}] \end{cases}$$

$$\Longrightarrow \quad \eta_{\mathcal{N}}'(t) = \begin{cases} 0, & \text{if } t \in [4R, \varrho) \cup (\varrho + \frac{1}{\mathcal{N}}, \mathcal{N}) \cup (\mathcal{N} + \frac{1}{\mathcal{N}}, \infty), \\ \mathcal{N}, & \text{if } t \in (\varrho, \varrho + \frac{1}{\mathcal{N}}), \\ -\mathcal{N}, & \text{if } t \in (\mathcal{N}, \mathcal{N} + \frac{1}{\mathcal{N}}) \end{cases}$$

and take a test function $\eta(x) = r^{2-n}\eta_{\mathcal{N}}(r)u(x)$ in the integral identity (II). Calculating

$$\eta_{x_i} = r^{2-n}\eta_{\mathcal{N}}(r)u_{x_i} + u(x)\cdot\left((2-n)r^{1-n}\frac{x_i}{r}\eta_{\mathcal{N}}(r) + r^{2-n}\frac{x_i}{r}\eta'_{\mathcal{N}}(r)\right),$$

we arrive at the equality

$$\begin{aligned}
&\int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}}\left(r^{2-n}|\nabla u|^{m(x)} + (2-n)r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r} + r^{2-n}ub(x,u,u_x)\right)\mathcal{N}(r-\varrho)dx\\
&\quad+\int_{G^{\mathcal{N}}_{\varrho+\frac{1}{\mathcal{N}}}}\left(r^{2-n}|\nabla u|^{m(x)} + (2-n)r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r} + r^{2-n}ub(x,u,u_x)\right)dx\\
&\quad+\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}}\left(r^{2-n}|\nabla u|^{m(x)} + (2-n)r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r} + r^{2-n}ub(x,u,u_x)\right)\cdot[\mathcal{N}(\mathcal{N}-r)+1]\,dx\\
&= -\mathcal{N}\int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx + \mathcal{N}\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx.
\end{aligned}$$

First of all we observe that by assumption (iii) we have $ub(x,u,u_x) \le \mu|\nabla u|^{m(x)}$.
In virtue of (2.7) it is clearly that

$$\begin{aligned}
&\lim_{\mathcal{N}\to+\infty}\int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}\cup G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}dx = 0,\\
&\lim_{\mathcal{N}\to+\infty}\int_{G^{\mathcal{N}}_{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}dx = \int_{G_\varrho} r^{2-n}|\nabla u|^{m(x)}dx.
\end{aligned}\tag{2.17}$$

Since

$$\begin{aligned}
&0 \le \int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}\cdot\mathcal{N}(r-\varrho)dx \le \int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}dx,\\
&0 \le \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}\cdot[\mathcal{N}(\mathcal{N}-r)+1]\,dx \le \int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}dx,
\end{aligned}$$

by (2.17), we get

$$\lim_{\mathcal{N}\to+\infty}\int_{G_\varrho^{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}\cdot\mathcal{N}(r-\varrho)dx = \lim_{\mathcal{N}\to+\infty}\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)}\cdot[\mathcal{N}(\mathcal{N}-r)+1]\,dx = 0.$$

Applying now the Young inequality with $q = \frac{m(x)}{m(x)-1}$, $q' = m(x)$ we have

$$\begin{aligned}
\left|\int_{G_\varrho} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx\right| &\le \int_{G_\varrho} r^{1-n}|\nabla u|^{m(x)-1}|u|dx\\
&= \int_{G_\varrho}\left(r^{(2-n)\frac{m(x)-1}{m(x)}}|\nabla u|^{m(x)-1}\right)\cdot\left(r^{\frac{2-m(x)-n}{m(x)}}|u|\right)dx\\
&\le \int_{G_\varrho} r^{2-n}|\nabla u|^{m(x)}dx + \int_{G_\varrho} r^{2-m(x)-n}|u|^{m(x)}dx\\
&\le C\varrho^{-2\alpha_0}, \qquad \varrho\in(R,\infty),
\end{aligned}$$

by (2.13) and (2.15). Consequently

$$\lim_{\mathcal{N}\to+\infty}\int_{G^{\mathcal{N}}_{\varrho+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx = \int_{G_\varrho} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx.$$

Now we consider the integral

$$\left|\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}\left[\mathcal{N}(\mathcal{N}-r)+1\right]dx\right| \le \frac{1}{\mathcal{N}}\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2}|u|\left|\frac{\partial u}{\partial r}\right|dx$$

and hence, by (2.8)

$$\lim_{\mathcal{N}\to+\infty}\int_{G_{\mathcal{N}}^{\mathcal{N}+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}\left[\mathcal{N}(\mathcal{N}-r)+1\right]dx = 0.$$

Next, because of (2.18),

$$\lim_{\mathcal{N}\to+\infty}\int_{G_{\varrho}^{\varrho+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx = 0,$$

and therefore we can apply the L'Hospital rule:

$$\begin{aligned}\varrho\cdot\lim_{\mathcal{N}\to+\infty}\mathcal{N}\cdot\int_{G_{\varrho}^{\varrho+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx &= \varrho\cdot\lim_{\mathcal{N}\to+\infty}\frac{\int_{\varrho}^{\varrho+\frac{1}{\mathcal{N}}}\left(\int_{\Omega}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}\right)d\Omega dr}{\mathcal{N}^{-1}}\\ &= \varrho^{2-n}\int_{\Omega_\varrho}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}d\Omega\varrho\end{aligned}$$

and

$$\lim_{\mathcal{N}\to+\infty}\mathcal{N}\cdot\int_{G_{\varrho}^{\varrho+\frac{1}{\mathcal{N}}}} r^{2-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}dx = \varrho^{2-n}\int_{\Omega_\varrho}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}d\Omega\varrho.$$

Hence

$$\lim_{\mathcal{N}\to+\infty}\int_{G_{\varrho}^{\varrho+\frac{1}{\mathcal{N}}}} r^{1-n}|\nabla u|^{m(x)-2}u\frac{\partial u}{\partial r}\mathcal{N}(r-\varrho)dx = 0.$$

□

3 Local estimate of the weighted Dirichlet integral

Theorem 3.1. *Let u be a weak solution of problem (QL) and assumptions (i)–(iv) be satisfied. Let λ_- be as in (2.4). Then there exist $R \gg 1$ and a constant $C > 0$ such that*

$$\int_{G_\varrho} r^{2-n}|\nabla u|^2 dx \le C\varrho^{2\lambda_-(1-\mu)}, \quad \forall \varrho > R.$$

Proof. We rewrite the inequality (2.16) in the form:

$$\begin{aligned}U(\varrho) = \int_{G_\varrho} r^{2-n}|\nabla u|^2 dx &= \int_{G_\varrho} r^{2-n}(|\nabla u|^2 - |\nabla u|^{m(x)})dx - \int_{G_\varrho} r^{2-n}ub(x,u,u_x)dx\\ &+ (2-n)\int_{G_\varrho} r^{1-n}(1-|\nabla u|^{m(x)-2})uu_r dx + (n-2)\int_{G_\varrho} r^{1-n}uu_r dx\\ &- \varrho^{2-n}\int_{\Omega_\varrho}(|\nabla u|^{m(x)-2}-1)uu_r d\Omega_\varrho - \varrho^{2-n}\int_{\Omega_\varrho} uu_r d\Omega_\varrho.\end{aligned} \tag{3.1}$$

Now, we observe that

$$\int_{G_\varrho} r^{1-n}uu_r dx = -\frac{1}{2}\varrho^{1-n}\int_{\Omega_\varrho} u^2 d\Omega\varrho. \tag{3.2}$$

In fact, we get

$$\int_{G_\varrho^{\mathcal{N}}} r^{1-n} u u_r dx = \int_\Omega \int_\varrho^{\mathcal{N}} u u_r dr d\Omega = \frac{1}{2}\int_\Omega \int_\varrho^{\mathcal{N}} \frac{\partial u^2}{\partial r} dr d\Omega = \frac{1}{2}\int_\Omega (u^2(\mathcal{N},\omega) - u^2(\varrho,\omega)) d\Omega$$
$$= \frac{1}{2}\int_\Omega u^2(\mathcal{N},\omega) d\Omega - \frac{1}{2}\varrho^{1-n}\int_{\Omega_\varrho} u^2 d\Omega_\varrho.$$

Passing to the limit $\mathcal{N} \to +\infty$ we obtain (3.2).

By assumption (iii), we get

$$\left|\int_{G_\varrho} r^{2-n} u b(x,u,u_x) dx\right| \le \mu \int_{G_\varrho} r^{2-n} |\nabla u|^2 dx + \mu \int_{G_\varrho} r^{2-n} \left||\nabla u|^{m(x)} - |\nabla u|^2\right| dx.$$

Hence and from (3.1), (3.2) it follows that

$$(1-\mu)U(\varrho) \le (1+\mu)\int_{G_\varrho} r^{2-n}\left||\nabla u|^{m(x)} - |\nabla u|^2\right| dx$$
$$+ (n-2)\int_{G_\varrho} r^{1-n}\left|1 - |\nabla u|^{m(x)-2}\right| |u||u_r| dx - \frac{n-2}{2}\varrho^{1-n}\int_{\Omega_\varrho} u^2 d\Omega_\varrho$$
$$+ \varrho^{2-n}\int_{\Omega_\varrho}\left||\nabla u|^{m(x)-2} - 1\right| |u||u_r| d\Omega_\varrho - \varrho^{2-n}\int_{\Omega_\varrho} u u_r d\Omega_\varrho. \tag{3.3}$$

Let us estimate the integrals:

$$I_1(\varrho) = \int_{G_\varrho} r^{2-n}\left||\nabla u|^{m(x)} - |\nabla u|^2\right| dx,$$
$$I_2(\varrho) = \int_{G_\varrho} r^{1-n}\left|1 - |\nabla u|^{m(x)-2}\right| |u||u_r| dx,$$
$$I_3(\varrho) = \int_{\Omega_\varrho}\left||\nabla u|^{m(x)-2} - 1\right| |u||u_r| d\Omega_\varrho.$$

To estimate them we set

$$F_1 = \{x : \ x \in \overline{G_\varrho}, \ |\nabla u| < |x|^\gamma\},$$
$$F_2 = \{x : \ x \in \overline{G_\varrho}, \ |x|^\gamma \le |\nabla u| \le M_1' |x|^{-1}\},$$

where the constant $\gamma < -1$ will be defined above.

By assumption (ii) and (2.11) for any $x \in F_1$, we get

$$|\nabla u|^2 + |\nabla u|^m < |x|^{2\gamma} + |x|^{\gamma(m-2)} \cdot |x|^{2\gamma}$$
$$\le |x|^{2\gamma} + |x|^{-\gamma M |x|^{-\alpha}} \cdot |x|^{2\gamma} \le C_1(M,\gamma,\alpha) \cdot |x|^{2\gamma}. \tag{3.4}$$

In this way

$$\int_{F_1} r^{2-n}\left||\nabla u|^2 - |\nabla u|^{m(x)}\right| dx \le C_2 \cdot \varrho^{2\gamma+2}.$$

Next, (ii) yields for $x \in F_2$, that

$$|\nabla u|^2 + |\nabla u|^{m(x)} = |\nabla u|^2 (1 + |\nabla u|^{m(x)-2})$$
$$\le |\nabla u|^2 (1 + |x|^{-M\gamma |x|^{-\alpha}}) \le C_3(M,\alpha)|\nabla u|^2, \tag{3.5}$$

because

$$(m(x)-2)\ln|\nabla u| \le -M|x|^{-\alpha}\ln|\nabla u| \le -M|x|^{-\alpha}\ln|x|^{\gamma}.$$

Hence, once again in virtue of (ii) and by the inequality

$$\left||z|^{t_2}-|z|^{t_1}\right| \le \frac{1}{2}|t_2-t_1|(|z|^{t_1}+|z|^{t_2})|\ln|z||, \; z\in\mathbb{R}\setminus\{0\},\; t_1\ge 0,\; t_2\ge 0 \tag{3.6}$$

(see Proposition 2.1 in [1]), we obtain

$$\left||\nabla u|^2-|\nabla u|^{m(x)}\right| \le \frac{1}{2}|m(x)-2|(|\nabla u|^{m(x)}+|\nabla u|^2)|\ln|\nabla u|| \le \frac{MC_3}{2}|x|^{-\alpha}|\nabla u|^2|\ln|\nabla u||.$$

Applying inequality (2.6) with $\delta=-\frac{\alpha}{2\gamma}$, we get

$$|\ln|\nabla u|| \le |\ln|x|^{\gamma}| \le \frac{-2\gamma}{\alpha e}|x|^{\frac{\alpha}{2}} \quad x\in F_2. \tag{3.7}$$

Eventually, we find that

$$I_1 \le C_4\varrho^{-\frac{\alpha}{2}}U(\varrho)+C\varrho^{2\gamma+2}. \tag{3.8}$$

Integrals I_2 and I_3 are estimated similarly. Arguing as in (3.4), (3.5), we establish that

$$|\nabla u|+|\nabla u|^{m(x)-1} \le C|x|^{\gamma} \quad \forall x\in F_1, \tag{3.9}$$

$$|\nabla u|+|\nabla u|^{m(x)-1} \le C|\nabla u| \quad \forall x\in F_2. \tag{3.10}$$

From (3.9) and by our assumption about Hölder continuity we get

$$\int_{F_1} r^{1-n}\left||1-|\nabla u|^{m(x)-2}\right||u||u_r|dx \le C\int_{F_1} r^{1-n}|x|^{\gamma}|u|dx \le C\varrho^{\gamma-\alpha_0+1}, \tag{3.11}$$

$$\int_{\Omega_\varrho\cap F_1}\left||1-|\nabla u|^{m(x)-2}\right||u||u_r|d\Omega\varrho \le C\varrho^{\gamma-\alpha_0+n-1}. \tag{3.12}$$

Repeating steps (3.6)–(3.7) and using (3.10), we have

$$\left||\nabla u|^{m(x)-1}-|\nabla u|\right| \le C_5|\nabla u||x|^{-\frac{\alpha}{2}} \tag{3.13}$$

on the set F_2. Thus

$$\begin{aligned}
\int_{F_2} r^{1-n}\left|1-|\nabla u|^{m(x)-2}\right||u_r||u|dx &\le C\int_{F_2} r^{1-n-\frac{\alpha}{2}}|\nabla u||u|dx \\
\le C\varrho^{-\frac{\alpha}{2}}\int_{G_\varrho} r^{1-n}|\nabla u||u|dx &= C\varrho^{-\frac{\alpha}{2}}\int_{G_\varrho}\left(r^{1-\frac{n}{2}}|\nabla u|\right)\cdot\left(r^{-\frac{n}{2}}|u|\right)dx \\
\le C\varrho^{-\frac{\alpha}{2}}\left(\int_{G_\varrho} r^{2-n}|\nabla u|^2dx\right)^{1/2}\cdot\left(\int_{G_\varrho} r^{-n}u^2dx\right)^{1/2} &\le C\varrho^{-\frac{\alpha}{2}}\cdot\frac{1}{\vartheta_*}\int_{G_\varrho} r^{2-n}|\nabla u|^2dx
\end{aligned}$$

in virtue of the Hardy–Wirtinger inequality (2.2), where ϑ_* is the smallest positive eigenvalue of the Dirichlet problem for the Laplace–Beltrami operator in the domain Ω. Using (3.11), we obtain the estimate

$$I_2 \le C\varrho^{-\frac{\alpha}{2}}\cdot\frac{1}{\vartheta_*}U(\varrho)+C\varrho^{\gamma-\alpha_0+1}. \tag{3.14}$$

Now, by (3.13), we have

$$\int_{\Omega_\varrho\cap F_2}\left||\nabla u|^{m(x)-2}-1\right||u||u_r|d\Omega_\varrho \le C\varrho^{-\frac{\alpha}{2}}\int_{\Omega_\varrho}|u_r||u|d\Omega_\varrho.$$

Taking into account (3.12) we find that

$$I_3 \le C\varrho^{-\frac{\alpha}{2}} \int_{\Omega_\varrho} |u_r||u| d\Omega_\varrho + C\varrho^{\gamma-\alpha_0+n-1}. \tag{3.15}$$

Thus, inserting (3.8), (3.14), (3.15) into (3.3), we obtain

$$\begin{aligned}\left(1-\mu-C\varrho^{-\frac{\alpha}{2}}\right) U(\varrho) \le C\varrho^{1-\frac{\alpha}{2}} \int_\Omega |u_r||u| d\Omega \\ -\frac{n-2}{2}\int_\Omega u^2 d\Omega - \varrho\int_\Omega u u_r d\Omega + C(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1}).\end{aligned} \tag{3.16}$$

Now we can use Lemma 2.4. Hence, (3.16) takes the following form

$$\left(1-\mu-C\varrho^{-\frac{\alpha}{2}}\right) U(\varrho) \le \frac{\varrho}{2\lambda_-}U'(\varrho) + C\varrho^{1-\frac{\alpha}{2}}\int_\Omega |\nabla u||u| d\Omega + C(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1}).$$

Applying the Cauchy inequality and (2.1), we have

$$\varrho\int_\Omega |u||\nabla u| d\Omega \le \frac{1}{2}\int_\Omega (\varrho^2|\nabla u|^2 + |u|^2) d\Omega \le -c_1(\vartheta_*)\varrho U'(\varrho).$$

Thus we get

$$\left(1-\mu-C\varrho^{-\frac{\alpha}{2}}\right) U(\varrho) \le \frac{\varrho}{2\lambda_-}(1+\tilde{C}\varrho^{-\frac{\alpha}{2}})U'(\varrho) + C(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1})$$

or

$$U'(\varrho) - \frac{2\lambda_-}{\varrho}\cdot\frac{1-\mu-C\varrho^{-\frac{\alpha}{2}}}{1+\tilde{C}\varrho^{-\frac{\alpha}{2}}}U(\varrho) + 2\lambda_- C\cdot\frac{\varrho^{2\gamma+1}+\varrho^{\gamma-\alpha_0}}{1+\tilde{C}\varrho^{\frac{-\alpha}{2}}} \le 0.$$

In this way we have the Cauchy problem (CP) with

$$\begin{aligned}P(\varrho) &= -\frac{2\lambda_-}{\varrho}\cdot\frac{1-\mu-C\varrho^{-\frac{\alpha}{2}}}{1+\tilde{C}\varrho^{-\frac{\alpha}{2}}},\\ Q(\varrho) &= -2\lambda_- C\cdot\frac{\varrho^{2\gamma+1}+\varrho^{\gamma-\alpha_0}}{1+\tilde{C}\varrho^{-\frac{\alpha}{2}}}.\end{aligned}$$

Now we show that $U(R) \le U_0 = \text{const}$. We can rewrite inequality (3.16) in the following form

$$\begin{aligned}\left((1-\mu)-C\varrho^{-\frac{\alpha}{2}}\right) U(\varrho) \le (1+C\varrho^{-\frac{\alpha}{2}})\varrho^{2-n}\int_{\Omega_\varrho} |\nabla u||u| d\Omega_\varrho \\ + \frac{n-2}{2}\varrho^{1-n}\int_{\Omega_\varrho} u^2 d\Omega_\varrho + C(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1}).\end{aligned}$$

Hence

$$\frac{1-\widetilde{C}\varrho^{-\frac{\alpha}{2}}}{1+C\varrho^{-\frac{\alpha}{2}}}U(\varrho) \le \frac{1}{1-\mu}\varrho^{2-n}\int_{\Omega_\varrho} |\nabla u||u| d\Omega_\varrho + \frac{n-2}{2(1-\mu)}\varrho^{1-n}\int_{\Omega_\varrho} u^2 d\Omega_\varrho + \frac{\widetilde{C}(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1})}{1+C\varrho^{-\frac{\alpha}{2}}}.$$

Since $\gamma < -1$ for sufficiently large $\varrho \ge 1$, we have

$$\frac{1-\widetilde{C}\varrho^{-\frac{\alpha}{2}}}{1+C\varrho^{-\frac{\alpha}{2}}} \ge 1-\varrho^{-\frac{\alpha}{4}} \quad \text{and} \quad \frac{\widetilde{C}(\varrho^{2\gamma+2}+\varrho^{\gamma-\alpha_0+1})}{1+C\varrho^{-\frac{\alpha}{2}}} \le \varrho^{\gamma+1}.$$

In this way

$$(1-\varrho^{-\frac{\alpha}{4}})U(\varrho) \le \frac{1}{1-\mu}\int_{\Omega}(\varrho|\nabla u||u| + \frac{n-2}{2}u^2)d\Omega + \varrho^{\gamma+1}.$$

Hence, from (2.5) it follows that $U(R) < \infty$.

All assumptions of Theorem 2.5 are satisfied. Since

$$-P(\varrho) = \frac{2\lambda_-(1-\mu)}{\varrho} - \frac{2\lambda_-(1-\mu)c_2\varrho^{-1-\frac{\alpha}{2}}}{1+\tilde{C}\varrho^{-\frac{\alpha}{2}}} \le \frac{2\lambda_-(1-\mu)}{\varrho} - 2\lambda_-(1-\mu)c_2\varrho^{-1-\frac{\alpha}{2}}$$

it follows that

$$-\int_R^{\varrho} P(\sigma)d\sigma \le 2\lambda_-(1-\mu)\int_R^{\varrho}\left(\frac{1}{\sigma} - c_2\sigma^{-\frac{\alpha}{2}-1}\right)d\sigma \le \ln\left(\frac{\varrho}{R}\right)^{2\lambda_-(1-\mu)} + c_3(\lambda_-,\mu,R,\vartheta_*)$$

which yields

$$\exp\left(-\int_R^{\varrho} P(\sigma)d\sigma\right) \le c_4\cdot\left(\frac{\varrho}{R}\right)^{2\lambda_-(1-\mu)}.$$

Next, because

$$Q(\varrho) \le -2C\lambda_-(\varrho^{2\gamma+1} + \varrho^{\gamma-\alpha_0}),$$

choosing $\gamma = -1 + 2\lambda_-(1-\mu)$ we have:

$$\int_R^{\varrho} Q(t)\exp\left(-\int_t^{\varrho} P(\sigma)d\sigma\right)dt$$
$$\le -2\lambda_-\cdot c_5\cdot\varrho^{2\lambda_-(1-\mu)}\int_R^{\varrho}(t^{-1-\alpha_0} + t^{2\lambda_-(1-\mu)-1})dt \le c_6\varrho^{2\lambda_-(1-\mu)}.$$

Eventually, by Theorem 2.5 we get

$$U(\varrho) \le C\varrho^{2\lambda_-(1-\mu)}.$$

□

4 Local estimate at infinity

The weak solution of problem (QL) is locally bounded at infinity. More precisely, we have

Theorem 4.1. *Let u be a weak solution of problem* (QL) *and assumptions* (i)–(iv) *be satisfied. Then for any $k<0$, $\varkappa \in (1,2)$, $\varrho > R$ with $R \gg 1$ the inequality*

$$\sup_{x\in G^{2\varrho}_{\varrho\varkappa}} |u| \le C^*\left(\varrho^{-\frac{n}{t}}\|u\|_{t,G^{2\varrho}_{\varrho}} + \varrho^k\right),$$

holds, where constant C^* depends only $m_+, m_-, \mu, M, M_1', \alpha, R, k, n, \varkappa$.

Proof. Set

$$l = \max_{\overline{G}^{2\varrho}_{\varrho}} m(x).$$

Let us consider the case $t \ge l > 1$. We make the coordinate transformation $x = \varrho x'$, $\varrho > R$ in the integral identity (II). Let $v(x') = u(\varrho x')$. We choose a test function η as

$$\eta(\varrho x') = v(x')\overline{v}^{t-l}(x')\zeta^l(|x'|),$$

where $\overline{v} = |v| + \varrho^k$ with a certain $k < 0$, $\zeta(|x'|) \in C_0^\infty([1,2])$ with the property that $0 \le \zeta(x') \le 1$ for $x' \in [1,2]$. Then (II) takes the following form

$$\int_{G_1^2} \Big[\overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \varrho^{-m(\varrho x')} \left(1 + (t-l)\frac{|v|}{\overline{v}}\right) \zeta^l + l v \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')-2} \varrho^{-m(\varrho x')} \zeta^{l-1} v_{x'_i} \zeta_{x'_i} + v\overline{v}^{t-l} b(\varrho x', v, \varrho^{-1} v_{x'}) \zeta^l \Big] dx' = 0.$$

Now, in virtue of $(t-l)\frac{|v|}{\overline{v}} \ge 0$, it follows that

$$\int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \varrho^{-m(\varrho x')} \zeta^l \le l \int_{G_1^2} |v| \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')-1} \varrho^{-m(\varrho x')} \zeta^{l-1} |\nabla' \zeta| dx' + \int_{G_1^2} |v| \overline{v}^{t-l} |b(\varrho x', v, \varrho^{-1} v_{x'})| \zeta^l dx'.$$

Now, by assumption (iii) regarding that $|v| < \overline{v}$ and in virtue of $\varrho^{-m(\varrho x')} \ge \varrho^{-l}$ we obtain

$$(1-\mu) \int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \zeta^l dx' \le l \int_{G_1^2} \overline{v}^{t-l+1} |\nabla' v|^{m(\varrho x')-1} \varrho^{l-m(\varrho x')} \zeta^{l-1} |\nabla' \zeta| dx'. \tag{4.1}$$

Next, by assumption (ii) we can estimate for all $x', x'_2 \in G_1^2$:

$$l - m(\varrho x') = m(\varrho x'_2) - m(\varrho x') \le M\varrho^{-\alpha}(|x'_2|^{-\alpha} + |x'|^{-\alpha}) \le 2M\varrho^{-\alpha}. \tag{4.2}$$

This estimation, with regard to (2.11) implies that

$$\varrho^{l-m(\varrho x')} \le \varrho^{2M\varrho^{-\alpha}} \le C.$$

For estimating the integral from the right-hand side of (4.1), we apply the Young inequality with $p = \frac{m(\varrho x')}{m(\varrho x')-1}$, $q = m(\varrho x')$, $\delta = \frac{\tilde{\delta}}{l}$:

$$\overline{v} |\nabla' v|^{m(\varrho x')-1} \zeta^{-1} |\nabla' \zeta| = \left(|\nabla' v|^{m(\varrho x')-1}\right)\left(\overline{v}\zeta^{-1}|\nabla'\zeta|\right) \le \frac{\tilde{\delta}}{l} |\nabla' v|^{m(\varrho x')} + \left(\frac{\tilde{\delta}}{l}\right)^{1-m(\varrho x')} \cdot \overline{v}^{m(\varrho x')} \zeta^{-m(\varrho x')} |\nabla' \zeta|^{m(\varrho x')}.$$

Hence, (4.1) takes the following form:

$$(1-\mu-\tilde{\delta}) \int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \zeta^l dx' \le \int_{G_1^2} \tilde{\delta}^{1-m(\varrho x')} \cdot l^{m(\varrho x')} \cdot \overline{v}^{t-l+m(\varrho x')} \zeta^{l-m(\varrho x')} |\nabla' \zeta|^{m(\varrho x')} dx'.$$

Choosing $\tilde{\delta} = \frac{1-\mu}{2}$, we get

$$\int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \zeta^l dx' \le \int_{G_1^2} \left(\frac{2l}{1-\mu}\right)^{m(\varrho x')} \overline{v}^{t-l+m(\varrho x')} \zeta^{l-m(\varrho x')} |\nabla' \zeta|^{m(\varrho x')} dx'.$$

Now we observe that $\zeta^{l-m(\varrho x')} \le 1$ for $x' \in G_1^2$, because $0 \le \zeta \le 1$ and $\left(\frac{2l}{1-\mu}\right)^{m(\varrho x')} \le \left(\frac{2l}{1-\mu}\right)^l$. By these arguments, we obtain

$$\int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^{m(\varrho x')} \zeta^l dx' \le C_1 \int_{G_1^2} \overline{v}^{t-l+m(\varrho x')} |\nabla' \zeta|^{m(\varrho x')} dx', \tag{4.3}$$

where $C_1 = \left(\frac{2l}{1-\mu}\right)^l$. Now our aim is to estimate the integral from the left hand side. For this purpose we write

$$|\nabla' v|^l = |\nabla' v|^{m(\varrho x')} \cdot |\nabla' v|^{l-m(\varrho x')}.$$

If $|\nabla' v| \le 1$, then $|\nabla' v|^l \le |\nabla' v|^{m(\varrho x')}$. Let $1 < |\nabla' v| \le M_1'$. Hence, by (4.2):

$$|\nabla' v|^{l-m(\varrho x')} \le |\nabla' v|^{2M\varrho^{-\alpha}} \le M_1'^{2M\varrho^{-\alpha}} \le C(M, M_1', \alpha, R).$$

Thus

$$|\nabla' v|^l \le C|\nabla' v|^{m(\varrho x')}. \tag{4.4}$$

Further, in virtue of $\overline{v} \ge \varrho^k, k < 0$, by (2.11), (4.2):

$$\overline{v}^{m(\varrho x')-l} \le \varrho^{k(m-l)} \le \varrho^{-2M\varrho^{-\alpha}k} \le C(M, k, \alpha). \tag{4.5}$$

From (4.3), (4.4) and (4.5) it follows that

$$\int_{G_1^2} \overline{v}^{t-l} |\nabla' v|^l \zeta^l dx' \le C \int_{G_1^2} \overline{v}^t |\nabla' \zeta|^{m(\varrho x')} dx'. \tag{4.6}$$

Applying now the Sobolev embedding theorem's formula (1.1) for $\varphi = \overline{v}^{\frac{t}{l}}\zeta,\ q = l$, we obtain

$$\|\overline{v}^t \zeta^l\|_{\tilde{n}, G_1^2} \le C \int_{G_1^2} \left(t^l \overline{v}^{t-l} |\nabla' v|^l \zeta^l + \overline{v}^t |\nabla' \zeta|^l \right) dx', \quad \tilde{n} = \frac{n}{n-1}. \tag{4.7}$$

Eventually, from (4.6), (4.7):

$$\|\overline{v}^t \zeta^l\|_{\widetilde{n}, G_1^2} \le C t^l \int_{G_1^2} \overline{v}^t (|\nabla' \zeta|^{m(\varrho x')} + |\nabla' \zeta|^l) dx'. \tag{4.8}$$

For any $\varkappa \in (1, 2)$ we define sets $G'_{(j)} \equiv G^2_{\varkappa - (\varkappa - 1)2^{-j}},\ j = 0, 1, 2, \dots$ We see at once that

$$G^2_\varkappa \equiv G'_{(\infty)} \subset \dots \subset G'_{(j+1)} \subset G'_{(j)} \subset \dots \subset G'_{(0)} \equiv G^2_1.$$

Now we consider the sequence of cut-off functions $\zeta_j(x') \in C^\infty(G'_{(j)})$ such that

$$0 \le \zeta_j(x') \le 1 \text{ in } G'_{(j)} \quad \text{and} \quad \zeta_j(x') \equiv 1 \text{ in } G'_{(j+1)},$$
$$\zeta_j(x') \equiv 0 \quad \text{for } 1 < |x'| < \varkappa - 2^{-j}(\varkappa - 1);$$
$$|\nabla \zeta_j'| \le \frac{2^{j+1}}{\varkappa - 1} \quad \text{for } \varkappa - 2^{-j}(\varkappa - 1) < |x'| < \varkappa - 2^{-j-1}(\varkappa - 1)$$

and the number sequence $t_j = t\widetilde{n}^j,\ j = 0, 1, 2, \dots$. We rewrite the inequality (4.8) replacing ζ by ζ_j and t by t_j. As a result, by virtue of properties of functions ζ_j, we obtain

$$\left(\int_{G'_{(j+1)}} \overline{v}^{\widetilde{n} t_j} dx' \right)^{\frac{1}{\widetilde{n}}} \le C t_j^l \int_{G'_{(j)}} \overline{v}^{t_j} \left(\frac{2^{j+1}}{\varkappa - 1} \right)^l dx'.$$

Hence, taking t_j-th root we get

$$\|\overline{v}\|_{t_{j+1}, G'_{(j+1)}} \le \left(\frac{C}{\varkappa - 1} \right)^{\frac{l}{t_j}} t_j^{\frac{l}{t_j}} 2^{\frac{(j+1)l}{t_j}} \|\overline{v}\|_{t_j, G'_{(j)}}.$$

After iteration process we find

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \le \left(\frac{Ct}{\varkappa-1}\right)^{l\sum_{j=0}^{\infty}\frac{1}{t_j}} \left(\frac{n}{n-1}\right)^{l\sum_{j=0}^{\infty}\frac{j}{t_j}} 2^{l\sum\limits_{j=0}^{\infty}\frac{j+1}{t_j}} \|\overline{v}\|_{t,G_1^2}.$$

The series $\sum_{j=0}^{\infty}\frac{j}{t_j}, \sum_{j=0}^{\infty}\frac{j+1}{t_j}$ are convergent according to the d'Alembert ratio test, while the series $\sum_{j=0}^{\infty}\frac{1}{t_j} = \frac{1}{t}\cdot\sum_{j=0}^{\infty}\left(\frac{n-1}{n}\right)^j = \frac{n}{t}$ as a geometric series. Hence, letting $j\to\infty$, we obtain

$$\sup_{x\in G_{\varkappa}^2} \overline{v} \le \frac{C^*}{(\varkappa-1)^{\frac{l\cdot n}{t}}}\|\overline{v}\|_{t,G_1^2}.$$

Thus, by the definition of $\overline{v}$, we obtain the required estimate.

5 The power modulus of continuity near infinity for weak solutions

By Theorem 4.1 with $t=2$, we have

$$\sup_{x\in G_{\frac{3}{2}\varrho}^{2\varrho}} |u| \le C^*\left(\varrho^{-\frac{n}{2}}\|u\|_{2,G_{\varrho}^{2\varrho}} + \varrho^k\right).$$

We can observe that

$$\varrho^{-\frac{n}{2}}\|u\|_{2,G_{\varrho}^{2\varrho}} \le 2^{\frac{n}{2}}\left(\int_{G_{\varrho}^{2\varrho}} r^{-n}u^2dx\right)^{\frac{1}{2}}.$$

Then, by (2.2) we get

$$\sup_{x\in G_{\frac{3}{2}\varrho}^{2\varrho}} |u| \le C^*\cdot\left\{\left(\int_{G_{\varrho}^{2\varrho}} r^{-n}u^2dx\right)^{\frac{1}{2}} + \varrho^k\right\} \le \widetilde{C^*}\cdot\left\{\left(\int_{G_{\varrho}^{2\varrho}} r^{2-n}|\nabla u|^2dx\right)^{\frac{1}{2}} + \varrho^k\right\}.$$

Next, by Theorem 3.1, choosing $k=\lambda_-(1-\mu)$ we obtain

$$\sup_{x\in G_{\frac{3}{2}\varrho}^{2\varrho}} |u(x)| \le C\varrho^{\lambda_-(1-\mu)}.$$

Putting now $|x| = \frac{7}{4}\varrho$ we eventually obtain the desired estimate (1.2). □

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A class of singularly perturbed Robin boundary value problems in critical case

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Abstract. This paper discusses a class of nonlinear singular perturbation problems with Robin boundary values in critical cases. By using the boundary layer function method and successive approximation theory, the corresponding asymptotic expansions of small parameters are constructed, and the existence of uniformly efficient smooth solutions is proved. Meanwhile, we give a concrete example to prove the validity of our results.

Keywords: critical case, singular perturbation, boundary function method, approximate solution, diagonalization technique.

2020 Mathematics Subject Classification: 34B15, 34E10, 34E15.

1 Introduction

Singular perturbation problems with small parameters have been used in many fields such as chemical kinetics [14], semiconductor simulation [16], and radio engineering [5, 18, 20]. The singular perturbed differential equation considered in this paper is obtained by transforming and dimensionless the differential equation controlling the enzyme kinetic reaction [4, 15, 18]. Using boundary layer function method [18], the first or higher order approximate solution of the problem can provide stronger theoretical support for obtaining the approximate value of enzyme concentration, substrate concentration and intermediate enzyme mixture concentration.

It is well known that the reduction equations for singularly perturbed systems usually have isolated roots. However, the degenerate equation of singularly perturbed problem considered in this paper has no isolated root. Instead, it has a series of solutions that depend on one or more parameters. This case will be called the critical case [17]. Compared with the singularly perturbed problem in the non-critical case, the singularly perturbed problem in the critical case is not only difficult to find, but also very complicated in the calculation process. Its complexity lies in the need to solve the following three difficulties in the calculation process: first, the zero-degree regular approximation solution is an unknown arbitrary function,

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which needs to be obtained through the following conditions; secondly, the solution process of the zero-order boundary layer is also very complicated. Finally, appropriate diagonalization should be found for the subsequent k-order boundary layer part to reduce the coupling degree of the equation.

The research methods of singular perturbation critical problems mainly include boundary layer function method [17, 18], that is, the solution is separated by fast scale and slow scale. The fast scale is the boundary layer part, and the slow scale is the regular part. The two parts are solved separately to construct the formal asymptotic solution. In addition, unlike the boundary layer function method, in the literature [9], the authors use orthogonal projectors on $\ker A(t)$ and $\ker A(t)'$(the prime denotes the transposition) for an asymptotic approximation, where $A(t)$ is a singular matrix in front of the unknown function at the right end of the singular perturbation equation. Through the theory of boundary layer function method, Vasil'eva and Butuzov [17] were the first to study initial value problems for singularly perturbed systems in the critical case. Subsequently, Vasil'eva and Adelaida [19], Dontchev and Veliov [3], Wang [21], Karandjulov [8], Kurina and Thi Hoai [10] generalized the results of singularly perturbed in the critical case. As far as we know, only Dirichlet or Neumann boundary value conditions are discussed in the above problems, and Robin boundary value conditions are not studied. In general, Robin boundary value conditions are a combination of Dirichlet and Neumann boundary value conditions. The singularly perturbed Robin boundary value problem in the critical case will result in that the initial value or boundary value corresponding to the differential equation of any order asymptotic term cannot be given directly, but must be obtained indirectly through certain techniques. In the past several decades, authors of [1, 2, 6, 7, 11, 12, 22, 23] discussed the singularly perturbed Robin boundary value problem for various noncritical cases.

However, until now, from what we understand, there is no literature talking about the singular perturbation problem in critical cases with Robin boundary value conditions seriously so far. Motivated by these issues, we fill in the gaps of this class of problems, giving corresponding asymptotic expansions and numerical examples in this paper.

The structure of the paper is as follows. In Section 2 we discuss singularly perturbed critical cases with Robin boundary value conditions. In the next section, we determine all terms of the asymptotic expansion of the system (2.1)–(2.2) using boundary layer function theory. Based on the successive approximation principle, section 4 proves the existence, uniqueness, and remainder estimation of the solutions to problems (2.1)–(2.2). Section 5 illustrates our results with an example. The last section gives concluding remarks.

2 Problem formulation

We consider a class of nonlinear singularly perturbed systems in critical case

$$\begin{cases} \mu \dfrac{dx}{dt} = A(y, \mu z, t)z + \mu B(y, z, t), \\ \mu \dfrac{dy}{dt} = C(x, y, t) + \mu D(y, z, t), \qquad a \le t \le b, \\ \mu \dfrac{dz}{dt} = F(\mu x, y, t)z + \mu H(y, z, t), \end{cases} \tag{2.1}$$

then, add the corresponding Robin initial and boundary value conditions as follows

$$\begin{gathered} y(a,\mu) - \mu y'(a,\mu) = y^0, \qquad x(b,\mu) + x'(b,\mu) = x^1, \\ z(a,\mu) - \mu z'(a,\mu) = z^0. \end{gathered} \tag{2.2}$$

Where x, y, and z are scalar functions, and $0 < \mu \ll 1$ is a small parameter. y^0, x^1, and z^0 are given known initial boundary values.

The following assumptions are theoretically some basic assumptions of questions (2.1)–(2.2).

- $[H_1]$ Suppose that the functions A, B, C, D, F, and H are sufficiently smooth for $a \le t \le b, |x| \le l, |y| \le l$, and $|z| \le l$, where l is some real numbers.
- $[H_2]$ Suppose that $C_x \neq 0, C_y < 0, F < 0$.
- $[H_3]$ Suppose that the degradation equation of the system (2.1) is

$$\begin{cases} A(\bar{y}, 0, t)\bar{z} = 0, \\ C(\bar{x}, \bar{y}, t) = 0, \\ F(0, \bar{y}, t)\bar{z} = 0, \end{cases} \tag{2.3}$$

we suppose that the system (2.3) has a series of solutions

$$\bar{x}(t) = \omega(t), \qquad \bar{y}(t) = \beta(\omega(t), t), \qquad \bar{z}(t) = 0, \tag{2.4}$$

where $\omega(t)$ is an arbitrary scalar function and $\beta(\omega(t), t)$ is a function with respect to $\omega(t)$.

Set

$$W = \begin{pmatrix} A(y, \mu z, t)z + \mu B(y, z, t) \\ C(x, y, t) + \mu D(y, z, t) \\ F(\mu x, y, t)z + \mu H(y, z, t) \end{pmatrix}, \qquad u = (x, y, z)^T;$$

Then the Jacobian matrix W_u at the equilibrium point $x = x^*, y = y^*, z = z^*$ has the following form

$$W_u = \begin{pmatrix} 0 & 0 & A \\ C_x & C_y & 0 \\ 0 & 0 & F \end{pmatrix}.$$

Therefore, it is easy to get the eigenvalue $\lambda \equiv 0$ of the matrix W_u and the other two eigenvalues $\lambda_1 = C_y < 0, \lambda_2 = F < 0$. According to the eigenvalues corresponding to the equilibrium point, $\lambda \equiv 0$ corresponds to the critical case, while the other two eigenvalues $\lambda_{1,2} < 0$ correspond to the stable equilibrium position. At the same time, the above case is called critically stable case in singular perturbation problems. Therefore, we can use the boundary layer function method [18] in the stable case, and because $\lambda_{1,2} < 0$, it is easy to find that the boundary function decays exponentially. Given the assumption that the condition $[H_2]$ and the initial value condition (2.2) hold, the solutions $x(t, \mu), y(t, \mu)$ and $z(t, \mu)$ generally only produce boundary layers near $t = a$.

3 Construction of asymptotic solution

The asymptotic approximation of problems (2.1)–(2.2) constructed by the boundary layer function method [18] will have the following form

$$u(t,\mu)=\begin{cases} x=\sum_{k=0}^{\infty}\mu^k[\bar{x}_k(t)+L_kx(\tau_0)],\\ y=\sum_{k=0}^{\infty}\mu^k[\bar{y}_k(t)+L_ky(\tau_0)],\\ z=\sum_{k=0}^{\infty}\mu^k[\bar{z}_k(t)+L_kz(\tau_0)],\end{cases}\tag{3.1}$$

where $\tau_0=\frac{t-a}{\mu}$; $\bar{x}_k(t)$, $\bar{y}_k(t)$, and $\bar{z}_k(t)(a<t\le b)$ are coefficients of regular terms; $L_kx(\tau_0)$, $L_ky(\tau_0)$, and $L_kz(\tau_0)(\tau_0\ge 0)$ are coefficients of boundary layer terms at $t=a$; By the initial and boundary value conditions, we obtain

$$\begin{aligned} \bar{x}_0(b)+\bar{x}_0'(b)&=x^1, & \bar{x}_k(b)+\bar{x}_k'(b)&=0,\\ \bar{y}_0(a)+L_0y(0)-L_0y'(0)&=y^0, & \bar{y}_k(a)+L_ky(0)&=\bar{y}_{k-1}'(a)+L_ky'(0),\\ \bar{z}_0(a)+L_0z(0)-L_0z'(0)&=z^0, & \bar{z}_k(a)+L_kz(0)&=\bar{z}_{k-1}'(a)+L_kz'(0). \end{aligned}$$

Set

$$\begin{aligned} \bar{x}(t,\mu)&=\sum_{i=0}^{k}\mu^i\bar{x}_k(t), & Lx(\tau_0,\mu)&=\sum_{i=0}^{k}\mu^iL_kx(\tau_0);\\ \bar{y}(t,\mu)&=\sum_{i=0}^{k}\mu^i\bar{y}_k(t), & Ly(\tau_0,\mu)&=\sum_{i=0}^{k}\mu^iL_ky(\tau_0);\\ \bar{z}(t,\mu)&=\sum_{i=0}^{k}\mu^i\bar{z}_k(t), & Lz(\tau_0,\mu)&=\sum_{i=0}^{k}\mu^iL_kz(\tau_0). \end{aligned}\tag{3.2}$$

Substituting equations (3.2) into equations (2.1), the following form can be obtained according to scale separation:

$$\begin{aligned} \mu\frac{d\bar{x}}{dt}+\frac{dLx}{d\tau_0}&=\bar{A}+\mu\bar{B}+LA+\mu LB,\\ \mu\frac{d\bar{y}}{dt}+\frac{dLy}{d\tau_0}&=\bar{C}+\mu\bar{D}+LC+\mu LD,\\ \mu\frac{d\bar{z}}{dt}+\frac{dLz}{d\tau_0}&=\bar{F}+\mu\bar{H}+LF+\mu LH, \end{aligned}$$

with

$$\begin{aligned} \bar{A}+\mu\bar{B}&=A(\bar{y}(t,\mu),\mu\bar{z}(t,\mu),t)\bar{z}(t,\mu)+\mu B(\bar{y}(t,\mu),\bar{z}(t,\mu),t),\\ \bar{C}+\mu\bar{D}&=C(\bar{x}(t,\mu),\bar{y}(t,\mu),t)+\mu D(\bar{y}(t,\mu),\bar{z}(t,\mu),t),\\ \bar{F}+\mu\bar{H}&=F(\mu\bar{x}(t,\mu),\bar{y}(t,\mu),t)\bar{z}(t,\mu)+\mu H(\bar{y}(t,\mu),\bar{z}(t,\mu),t),\\ LA&=A(\bar{y}(\mu\tau_0+a,\mu)+Ly(\tau_0,\mu),\mu\bar{z}(\mu\tau_0+a,\mu)+\mu Lz(\tau_0,\mu),\mu\tau_0+a)(Lz(\tau_0,\mu)\\ &\quad+\bar{z}(\mu\tau_0+a,\mu))-A(\bar{y}(\mu\tau_0+a,\mu),\mu\bar{z}(\mu\tau_0+a,\mu),\mu\tau_0+a)\bar{z}(\mu\tau_0+a,\mu),\\ \mu LB&=\mu B(\bar{y}(\mu\tau_0+a,\mu)+Ly(\tau_0,\mu),\bar{z}(\mu\tau_0+a,\mu)+Lz(\tau_0,\mu),\mu\tau_0+a)\\ &\quad-\mu B(\bar{y}(\mu\tau_0+a,\mu),\bar{z}(\mu\tau_0+a,\mu),\mu\tau_0+a), \end{aligned}$$

$$\begin{aligned}
LC &= (\bar{x}(\mu\tau_0 + a, \mu) + Lx(\tau_0, \mu), \bar{y}(\mu\tau_0 + a, \mu) + Ly(\tau_0, \mu), \mu\tau_0 + a) \\
&\quad - C(\bar{x}(\mu\tau_0 + a, \mu), \bar{y}(\mu\tau_0 + a, \mu), \mu\tau_0 + a), \\
\mu LD &= \mu D(\bar{y}(\mu\tau_0 + a, \mu) + Ly(\tau_0, \mu), \bar{z}(\mu\tau_0 + a, \mu) + Lz(\tau_0, \mu), \mu\tau_0 + a) \\
&\quad - \mu D(\bar{y}(\mu\tau_0 + a, \mu), \bar{z}(\mu\tau_0 + a, \mu), \mu\tau_0 + a), \\
LF &= F(\mu\bar{x}(\mu\tau_0 + a, \mu) + \mu Lx(\tau_0, \mu), \bar{y}(\mu\tau_0 + a, \mu) + Ly(\tau_0, \mu), \mu\tau_0 + a)(Lz(\tau_0, \mu) \\
&\quad + \bar{z}(\mu\tau_0 + a, \mu)) - F(\mu\bar{x}(\mu\tau_0 + a, \mu), \bar{y}(\mu\tau_0 + a, \mu), \mu\tau_0 + a)\bar{z}(\mu\tau_0 + a, \mu), \\
\mu LH &= \mu H(\bar{y}(\mu\tau_0 + a, \mu) + Ly(\tau_0, \mu), \bar{z}(\mu\tau_0 + a, \mu) + Lz(\tau_0, \mu), \mu\tau_0 + a) \\
&\quad - \mu H(\bar{y}(\mu\tau_0 + a, \mu), \bar{z}(\mu\tau_0 + a, \mu), \mu\tau_0 + a).
\end{aligned}$$

Secondly, according to the two scales t and τ_0, the equations of the regular part and the boundary layer part are written respectively:

$$\begin{cases}
\mu \dfrac{d\bar{x}}{dt} = \bar{A} + \mu\bar{B}, \\
\mu \dfrac{d\bar{y}}{dt} = \bar{C} + \mu\bar{D}, \\
\mu \dfrac{d\bar{z}}{dt} = \bar{F} + \mu\bar{H},
\end{cases} \tag{3.3}$$

$$\begin{cases}
\dfrac{dLx}{d\tau_0} = LA + \mu LB, \\
\dfrac{dLy}{d\tau_0} = LC + \mu LD, \\
\dfrac{dLz}{d\tau_0} = LF + \mu LH.
\end{cases} \tag{3.4}$$

Finally, the right-hand sides of equations (3.3) and (3.4) are expanded into a power series of μ, and then, according to the same power of μ at both ends of equations (3.3) and (3.4), the equations for the regular terms $\bar{u}_k(t)$ $(k \geqslant 0)$ and the boundary layer terms $L_k u(t)$ $(k \geqslant 0)$ are written, respectively.

We consider the zero-order regular part of the asymptotic solution of the form of problems (2.1)–(2.2) and obtain the zero-order regular parts $\bar{u}_0(t)$ is the same as the degenerate problem (2.3)–(2.4)

$$\begin{aligned}
A(\bar{y}_0, 0, t)\bar{z}_0 &= 0, \\
C(\bar{x}_0, \bar{y}_0, t) &= 0, \\
F(0, \bar{y}_0, t)\bar{z}_0 &= 0,
\end{aligned} \tag{3.5}$$

the root of the system of degradation equations (3.5) is

$$\bar{x}_0(t) = \omega(t), \qquad \bar{y}_0(t) = \beta(\omega(t), t), \qquad \bar{z}_0(t) = 0. \tag{3.6}$$

The equations for $L_0 u(\tau_0)$ are:

$$\begin{aligned}
\frac{dL_0 x(\tau_0)}{d\tau_0} &= A(\beta(\omega(a), a) + L_0 y(\tau_0), 0, a) L_0 z(\tau_0), \\
\frac{dL_0 y(\tau_0)}{d\tau_0} &= C(\omega(a) + L_0 x(\tau_0), \beta(\omega(a), a) + L_0 y(\tau_0), a) - C(\omega(a), \beta(\omega(a), a), a), \\
\frac{dL_0 z(\tau_0)}{d\tau_0} &= F(0, \beta(\omega(a), a) + L_0 y(\tau_0), a) L_0 z(\tau_0),
\end{aligned} \tag{3.7}$$

with the initial and boundary conditions

$$\begin{aligned} L_0y(0) &= L_0y'(0) - \beta(\omega(a), a) + y^0, \qquad \omega(b) + \omega'(b) = x^1, \\ L_0z(0) &= L_0z'(0) + z^0, \qquad\qquad\qquad\qquad L_0u(+\infty) = 0, \end{aligned} \tag{3.8}$$

where $\omega(a)$ is unknown, and the initial value of $L_0u(0)$ is arbitrary. We will use this arbitrariness to ensure that $L_0u(\tau_0)$ decays exponentially and satisfies $L_0u(+\infty) = 0$.

From (3.7)–(3.8), we have

$$\begin{aligned} L_0z(\tau_0) &= \frac{z^0 e^{\int_0^{\tau_0} F(0,\beta(\omega(a),a)+L_0y(s),a)ds}}{1 - F(0, \beta(\omega(a), a) + L_0y(0), a)}; \\ L_0x(\tau_0) &= \int_{+\infty}^{\tau_0} A(\beta(\omega(a), a) + L_0y(s), 0, a) \frac{z^0 e^{\int_0^{s} F(0,\beta(\omega(a),a)+L_0y(p),a)dp}}{1 - F(0, \beta(\omega(a), a) + L_0y(0), a)} ds \\ &= \varphi(L_0y(\tau_0), \omega(a), L_0y(0), a). \end{aligned} \tag{3.9}$$

At this time, substituting (3.9) into (3.7) the second differential equation and adding the initial boundary value condition (3.8), the following form can be obtained

$$\begin{cases} \dfrac{dL_0y(\tau_0)}{d\tau_0} = C(\omega(a) + \varphi, \beta(\omega(a), a) + L_0y(\tau_0), a) - C(\omega(a), \beta(\omega(a), a), a), \\ L_0y(0) = L_0y'(0) - \beta + y^0, \qquad L_0y(+\infty) = 0. \end{cases} \tag{3.10}$$

$[H_4]$ Suppose that equation (3.10) has a root $L_0y(\tau_0)$, which is denoted by

$$L_0y(\tau_0) = \Phi_0(\tau_0, \omega(a), L_0y(0), a). \tag{3.11}$$

However, neither $\omega(a)$ nor $L_0y(0)$ is known. Therefore, $L_0y(\tau_0)$ is not know. After $\omega(t)$ is found, $\omega(a)$ and $L_0y(0)$ can be determined and thus known. For the currently unknown functions $\omega(t)$ and $\beta(\omega, t)$, we need to determine them in the first approximation equation of the regular part $\bar{u}_1(t)$.

For $\bar{u}_1(t)$, we get

$$\begin{aligned} 0 &= F(0, \beta(\omega, t), t)\bar{z}_1 + H(\beta(\omega, t), 0, t), \\ \frac{d\omega(t)}{dt} &= A(\beta(\omega, t), 0, t)\bar{z}_1 + B(\beta(\omega, t), 0, t), \\ \frac{d\beta(\omega, t)}{dt} &= C_x(\omega, \beta(\omega, t), t)\bar{x}_1 + D(\beta(\omega, t), 0, t) + C_y(\omega, \beta(\omega, t), t)\bar{y}_1, \end{aligned} \tag{3.12}$$

the system of differential equations (3.12) can be rewritten in the following expression

$$\begin{aligned} \bar{z}_1(t) &= -\frac{\bar{H}}{\bar{F}} = \zeta(\omega, t), \qquad \frac{d\omega(t)}{dt} = \bar{A}\bar{z}_1 + \bar{B}, \\ \bar{x}_1(t) &= \frac{\beta' - C_y\bar{y}_1 - D}{C_x}, \end{aligned} \tag{3.13}$$

where $\bar{A}, \bar{B}$ and $\bar{H}$ are all taken value at the point $(\beta(\omega, t), 0, t)$, $\bar{F}$ is taken value at the point $(0, \beta(\omega, t), t)$. Using (3.13) and by taking (3.5) into account, we can determine $\omega(b) = \omega^1$. According to the existence of the solution of the boundary value problem, the second first-order differential equation in (3.13), and the known boundary value condition $\omega(b) = \omega^1$, there is a solution $\omega(t)$ for $a \le t \le b$. Hence, both $\omega(a)$ and $\bar{y}_0 = \beta(\omega, t)$ can be determined. At this time, $\bar{u}_0(t)$ can be completely determined.

Substituting (3.10) into (3.8) the first initial value condition, we can determine $L_0y(0) = \varphi^0$. Therefore, $L_0y(\tau_0)$ can be obtained by $[H_4]$. By (3.9) and $[H_4]$, we obtain $L_0z(\tau_0)$ and $L_0x(\tau_0)$. At this time, $L_0u(\tau_0)$ are completely determined. For the first approximation of the regular part of the system (2.1)–(2.2), equations (3.12) has only been determined $\bar{z}_1(t)$, while $\bar{x}_1(t)$ and $\bar{y}_1(t)$ need to be determined by the equation system of the regular parts $\bar{u}_2(t)$ and the corresponding boundary value conditions. Next, we need to first determine the first-order boundary layer terms $L_1u(\tau_0)$ of the asymptotic solution.

The equations for $L_1u(\tau_0)$ are:

$$\begin{aligned}
\frac{dL_1x}{d\tau_0} &= \widehat{A}(\beta(\omega,a) + L_0y(\tau_0), 0, a)L_1z(\tau_0) + \widehat{A}_y L_0z(\tau_0)L_1y(\tau_0) \\
&\quad + \widehat{A}_y L_0z\bar{y}_1(a) + \phi_1, \\
\frac{dL_1y}{d\tau_0} &= \tilde{C}_x(\omega(a) + L_0x(\tau_0), \beta(\omega,a) + L_0y(\tau_0), a)L_1x(\tau_0) \\
&\quad + \tilde{C}_y L_1y(\tau_0) + (\tilde{C}_x - \bar{C}_x)\bar{x}_1(a) + (\tilde{C}_y - \bar{C}_y)\bar{y}_1(a) + \phi_2, \\
\frac{dL_1z}{d\tau_0} &= \breve{F}(0, \beta(\omega,a) + L_0y(\tau_0), a)L_1z(\tau_0) + \breve{F}_y L_0z(\tau_0)L_1y(\tau_0) \\
&\quad + \breve{F}_y L_0z\bar{y}_1(a) + \phi_3,
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
\phi_1 &= \left(\widehat{A} - \bar{A}\right)\bar{z}_1 + \left[\widehat{A}_z L_0z + \widehat{A}_y\beta'\tau_0 + \widehat{A}_t\tau_0\right]L_0z + \hat{B} - \bar{B}, \\
\phi_2 &= \left[\left(\tilde{C}_x - \bar{\bar{C}}_x\right)\omega'(a) + \left(\tilde{C}_y - \bar{\bar{C}}_y\right)\beta' + \left(\tilde{C}_t - \bar{\bar{C}}_t\right)\right]\tau_0 + \hat{D} - \bar{D}, \\
\phi_3 &= \left(\breve{F} - \bar{\bar{\bar{F}}}\right)\bar{z}_1 + \left[\breve{F}_z(\omega(a) + L_0x) + \left(\breve{F}_y\beta' + \breve{F}_t\right)\tau_0\right]L_0z + \hat{H} - \bar{H}.
\end{aligned}$$

Here $\bar{B}, \bar{A}, \bar{D}, \bar{H}$ take values at the point $(\beta(\omega,a), 0, a)$, $\bar{\bar{C}}_x, \bar{\bar{C}}_y, \bar{\bar{C}}_t$ take values at the point $(\omega(a), \beta(\omega,a), a)$, $\bar{\bar{\bar{F}}}$ take values at the point $(0, \beta(\omega,a), a)$,$\breve{F}, \breve{F}_y, \breve{F}_z, \breve{F}_t$ take values at the point $(0, \beta(\omega,a) + L_0y(\tau_0), a)$,$\hat{B}, \hat{D}, \hat{H}$ take values at the point $(\beta(\omega,a) + L_0y(\tau_0), L_0z(\tau_0), a)$, $\tilde{C}_x$, $\tilde{C}_y$, $\tilde{C}_t$ take values at the point $(\omega(a) + L_0x(\tau_0), \beta(\omega,a) + L_0y(\tau_0), a)$ and $\widehat{A}, \widehat{A}_y, \widehat{A}_z, \widehat{A}_t$ take values at the point $(\beta(\omega,a) + L_0y(\tau_0), 0, a)$.

The initial and boundary value conditions corresponding to $L_1u(\tau_0)$ are

$$\begin{aligned}
&\bar{y}_1(a) + L_1y(0) = \bar{y}_0'(a) + L_1y'(0), \qquad \bar{z}_1(a) + L_1z(0) = L_1z'(0), \\
&L_1u(+\infty) = 0, \qquad \bar{x}_1(b) + \bar{x}_1'(b) = 0.
\end{aligned} \tag{3.15}$$

Introduce a diagonal transformation

$$L_1x(\tau_0) = \delta_1 + \frac{L_0x}{L_0z}\delta_3, \qquad L_1y(\tau_0) = \delta_2, \qquad L_1z(\tau_0) = \delta_3, \tag{3.16}$$

we can get

$$\begin{aligned}
\frac{d\delta_1}{d\tau_0} &= (\widehat{A}_y L_0z - \breve{F}_y L_0x)\delta_2 + \phi_4(\bar{y}_1(a), \tau_0), \\
\frac{d\delta_2}{d\tau_0} &= \tilde{C}_x\left(\delta_1 + \frac{L_0x}{L_0z}\delta_3\right) + \tilde{C}_y\delta_2 + \left(\tilde{C}_x - \bar{\bar{C}}_x\right)\bar{x}_1(a) \\
&\quad + \left(\tilde{C}_y - \bar{\bar{C}}_y\right)\bar{y}_1(a) + \phi_2, \\
\frac{d\delta_3}{d\tau_0} &= \breve{F}\delta_3 + \breve{F}_y L_0z\delta_2 + \breve{F}_y L_0z\bar{y}_1(a) + \phi_3,
\end{aligned} \tag{3.17}$$

where $\phi_4(\bar{y}_1(a),\tau_0) = (\widehat{A}_y L_0 z - \breve{F}_y L_0 x)\bar{y}_1(a) + \phi_1 - \frac{L_0 x}{L_0 z}\phi_3$. The initial and boundary conditions of δ_1, δ_2 and δ_3 are

$$\begin{aligned} \bar{y}_1(a) + \delta_2(0) = \bar{y}_0'(a) + \delta_2{}'(0), \qquad & \bar{z}_1(a) + \delta_3(0) = \delta_3{}'(0), \\ \delta_i(+\infty) = 0 (i = 1,2,3), \qquad & \bar{x}_1(b) + \bar{x}_1'(b) = 0. \end{aligned} \tag{3.18}$$

We introduce a new transformation $\delta_1 = \delta_4 + \frac{\widehat{A}_y L_0 z - \breve{F}_y L_0 x}{\breve{F}_y L_0 z}\delta_3$ so that the equation (3.17) changes to the following form

$$\begin{aligned} \frac{d\delta_2}{d\tau_0} &= \tilde{C}_y \delta_2 + \frac{\widehat{A}_y}{\breve{F}_y}\delta_3 + \tilde{C}_x \delta_4 + \phi_6, \\ \frac{d\delta_3}{d\tau_0} &= \breve{F}\delta_3 + \breve{F}_y L_0 z \delta_2 + \breve{F}_y L_0 z \bar{y}_1(a) + \phi_3, \\ \frac{d\delta_4}{d\tau_0} &= \left[\left(\frac{\widehat{A}_y L_0 z - \breve{F}_y L_0 x}{\breve{F}_y L_0 z} \right)' + \frac{(\widehat{A}_y L_0 z - \breve{F}_y L_0 x)\breve{F}}{\breve{F}_y L_0 z} \right] \delta_3 + \phi_5, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} \phi_5 &= \phi_4 - (\widehat{A}_y L_0 z - \breve{F}_y L_0 x)\bar{y}_1(a) + \frac{\widehat{A}_y L_0 z - \breve{F}_y L_0 x}{\breve{F}_y L_0 z}\phi_3, \\ \phi_6 &= (\tilde{C}_x - \bar{\bar{C}}_x)\bar{x}_1(a) + (\tilde{C}_y - \bar{\bar{C}}_y)\bar{y}_1(a) + \phi_2. \end{aligned}$$

Let us introduce the transformation $\delta_2 = \delta_5 - \frac{\breve{F}}{\breve{F}_y L_0 z}\delta_3$ again, the system of differential equations (3.19) changed to the equations (3.20)

$$\begin{aligned} \frac{d\delta_3}{d\tau_0} &= \breve{F}_y L_0 z \delta_5 + \phi_8, \\ \frac{d\delta_4}{d\tau_0} &= \left[\left(\frac{\widehat{A}_y L_0 z - \breve{F}_y L_0 x}{\breve{F}_y L_0 z} \right)' + \frac{(\widehat{A}_y L_0 z - \breve{F}_y L_0 x)\breve{F}}{\breve{F}_y L_0 z} \right] \delta_3 + \phi_5, \\ \frac{d\delta_5}{d\tau_0} &= \left[\frac{\widehat{A}_y \tilde{C}_x L_0 z - L_0 z \breve{F}}{\breve{F}_y L_0 z} + \left(\frac{\breve{F}}{\breve{F}_y L_0 z} \right)' \right] \delta_3 + \tilde{C}_x \delta_4 + (\tilde{C}_y + \breve{F})\delta_5 + \phi_7, \end{aligned} \tag{3.20}$$

where $\phi_7 = \phi_6 + \breve{F}\bar{y}_1(a) + \frac{\breve{F}}{\breve{F}_y L_0 z}\phi_3, \phi_8 = \breve{F}_y L_0 z \bar{y}_1(a) + \phi_3$. From the above two transformations, we find that the initial value condition (3.20) has the following form

$$\begin{aligned} &\bar{y}_1(a) + \delta_5(0) = \bar{y}_0'(a) + \delta_5{}'(0) - \left(\frac{\breve{F}}{\breve{F}_y L_0 z} \right)' \delta_3(0) - \frac{\breve{F}}{\breve{F}_y L_0 z}\bar{z}_1(a), \\ &\bar{z}_1(a) + \delta_3(0) = \delta_3{}'(0), \delta_i(+\infty) = 0 \quad (i = 3,4,5), \qquad \bar{x}_1(b) + \bar{x}_1'(b) = 0. \end{aligned} \tag{3.21}$$

We find that the right end of the first two equations of equation system (3.20) contains only one unknown function, which greatly reduces the coupling of the right end of the original equation system (3.14). Thus, by the system of equations (3.20) and the initial value condition (3.21), $\delta_3(\tau_0)$ and $\delta_4(\tau_0)$ can be written as

$$\begin{aligned}
\delta_3(\tau_0) &= \int_{+\infty}^{\tau_0} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds, \\
\delta_4(\tau_0) &= \int_{+\infty}^{\tau_0} \left[\left(\frac{\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p)}{\breve{F}_y(p)L_0z(p)} \right)' \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \right. \\
&+ \frac{(\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p))\breve{F}(p)}{\breve{F}_y(p)L_0z(p)} \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\
&\left. + \phi_5(\bar{y}_1(a), p) \right] dp.
\end{aligned} \tag{3.22}$$

At the same time, we substitute the expression (3.22) into the last equation of the system of equations (3.20), and the right end contains only the unknown function $\delta_5(\tau_0)$. At this point, the entire equation is a first-order integral differential equation for $\delta_5(\tau_0)$. First, we get the integral equation for the initial value condition $\delta_5(0)$

$$\begin{aligned}
\delta_5(0) &= \frac{(\widehat{A}_y(0)L_0z(0) - \breve{F}_y(0)L_0x(0))\breve{F}(0)}{\breve{F}_y(0)L_0z(0)(1 - \tilde{C}_y(0) - \breve{F}(0))} \int_{+\infty}^{0} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\
&+ \int_{+\infty}^{0} \left[\left(\frac{\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p)}{\breve{F}_y(p)L_0z(p)} \right)' \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \right. \\
&+ \frac{(\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p))\breve{F}(p)}{\breve{F}_y(p)L_0z(p)} \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\
&\left. + \phi_5(\bar{y}_1(a), p) \right] dp \frac{\tilde{C}_x(0)}{(1 - \tilde{C}_y(0) - \breve{F}(0))} + \frac{\bar{y}_0'(a) - \bar{y}_1(a) + \phi_7(0, \bar{y}_1(a))}{(1 - \tilde{C}_y(0) - \breve{F}(0))}.
\end{aligned} \tag{3.23}$$

$[H_5]$ Suppose that the integral equation (3.23) can be converted to $\delta_5(0) = \varsigma(a, \bar{y}_1(a))$.

Next, we write the integral differential equation for $\delta_5(\tau_0)$

$$\begin{aligned}
\delta_5(\tau_0) &= \int_0^{\tau_0} \left\{ \left[\frac{\widehat{A}_y\tilde{C}_xL_0z - L_0z\breve{F}}{\breve{F}_yL_0z} + \left(\frac{\breve{F}}{\breve{F}_yL_0z} \right)' \right] \int_{+\infty}^{q} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \right. \\
&+ \tilde{C}_x \int_{+\infty}^{q} \left[\left(\frac{\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p)}{\breve{F}_y(p)L_0z(p)} \right)' \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \right. \\
&+ \frac{(\widehat{A}_y(p)L_0z(p) - \breve{F}_y(p)L_0x(p))\breve{F}(p)}{\breve{F}_y(p)L_0z(p)} \int_{+\infty}^{p} \breve{F}_y(s)L_0z(s)\delta_5(s) + \phi_8(\bar{y}_1(a), s)ds \\
&\left. \left. + \phi_5(\bar{y}_1(a), p) \right] dp + (\tilde{C}_y(q) + \breve{F}(q))\delta_5(q) + \phi_7(q) \right\} dq + \varsigma(a, \bar{y}_1(a)).
\end{aligned} \tag{3.24}$$

$[H_6]$ Suppose that the integral equation (3.24) has a unique solution and can be expressed as

$$\delta_5(\tau_0) = \Phi_1(\tau_0, \bar{y}_1(a), a), \tag{3.25}$$

where $\bar{y}_1(a)$ is unknown. At this time, we can determine $\bar{y}_1(t)$ from the equation of the regular parts $\bar{u}_2(t)$.

For $\bar{u}_2(t)$, we get

$$\begin{aligned}
\frac{d\bar{x}_1}{dt} &= \bar{A}\bar{z}_2 + \bar{A}_y\bar{y}_1\bar{z}_1 + \bar{B}_y\bar{y}_1 + \bar{B}_z\bar{z}_1, \\
\frac{d\bar{y}_1}{dt} &= \bar{\bar{\bar{C}}}_x\bar{x}_2 + \bar{\bar{\bar{C}}}_y\bar{y}_2 + g_2, \\
\frac{d\bar{z}_1}{dt} &= \bar{\bar{\bar{\bar{F}}}}\bar{z}_2 + \bar{\bar{\bar{\bar{F}}}}_y\bar{y}_1\bar{z}_1 + \bar{\bar{\bar{\bar{F}}}}_x\bar{x}_0\bar{z}_1 + \bar{H}_y\bar{y}_1 + \bar{H}_z\bar{z}_1,
\end{aligned} \tag{3.26}$$

where g_2 is known to be a composite function. As for the unknown function $\bar{x}_1(t)$, the determination of $\bar{x}_1(t)$ is entirely similar to the of $\bar{x}_0(t) = \omega(t)$. Therefore, utilizing (3.26) and by taking (3.21) into account, we can determine $\bar{x}_1(b) = \alpha^1$. According to the existence of the solution of the boundary value problem, the first differential equation in (3.26), and the known boundary value condition $\bar{x}_1(b) = \alpha^1$, there is a solution $\bar{x}_1(t)$ for $a \leq t \leq b$, thence, both $\bar{x}_1(t)$ and $\bar{y}_1(t)$ can be determined. So far, $\bar{u}_1(t)$ can be completely determined.

Therefore, $\bar{y}_1(a)$ is known, $\delta_5(\tau_0)$ can be obtained by $[H_6]$. At the same time, $\delta_3(\tau_0)$ and $\delta_4(\tau_0)$ are determined. At this point, we go backwards along the diagonalization transformation to determine $L_1x(\tau_0)$, $L_1y(\tau_0)$, and $L_1z(\tau_0)$.

Next, the coefficients $\bar{u}_k(t)$ and $L_ku(\tau_0)(k \geq 2)$ of the higher-order asymptotic solutions are similar to the first-order asymptotic solutions $\bar{u}_1(t)$ and $L_1u(\tau_0)$. At this point, we need to write the equations and initial-boundary value conditions to determine $\bar{u}_{k+1}(t)$ and $L_ku(\tau_0)$.

The equations for $L_ku(\tau_0)$ are:

$$\begin{aligned}
\frac{dL_kx}{d\tau_0} &= \overset{\frown}{A}L_kz(\tau_0) + \overset{\frown}{A}_yL_0z(\tau_0)L_ky(\tau_0) + \overset{\frown}{A}_yL_0z\bar{y}_k(a) + Q_k, \\
\frac{dL_ky}{d\tau_0} &= \tilde{C}_xL_kx(\tau_0) + \tilde{C}_yL_ky(\tau_0) + (\tilde{C}_x - \bar{C}_x)\bar{x}_k(a) + (\tilde{C}_y - \bar{C}_y)\bar{y}_k(a) + I_k, \\
\frac{dL_kz}{d\tau_0} &= \breve{F}L_kz(\tau_0) + \breve{F}_yL_0z(\tau_0)L_ky(\tau_0) + \breve{F}_yL_0z\bar{y}_k(a) + S_k,
\end{aligned} \tag{3.27}$$

where Q_k, I_k, and S_k are all known composite functions. The initial and boundary conditions of $L_ku(\tau_0)$ are:

$$\begin{aligned}
\bar{x}_k(b) + \bar{x}_k'(b) = 0, \qquad L_ku(+\infty) = 0,& \\
L_ky(0) = \bar{y}_{k-1}(a) + L_ky'(0) - \bar{y}_k(a),& \\
L_kz(0) = \bar{z}_{k-1}(a) + L_kz'(0) - \bar{z}_k(a).&
\end{aligned} \tag{3.28}$$

The equations for $\bar{u}_{k+1}(t)$ are:

$$\begin{aligned}
\frac{d\bar{x}_k}{dt} &= \bar{A}\bar{z}_{k+1} + \bar{A}_y\bar{y}_k\bar{z}_1 + \bar{B}_y\bar{y}_k + m_{k+1}, \\
\frac{d\bar{y}_k}{dt} &= \bar{\bar{\bar{C}}}_x\bar{x}_{k+1} + \bar{\bar{\bar{C}}}_y\bar{y}_{k+1} + g_{k+1}, \\
\frac{d\bar{z}_k}{dt} &= \bar{\bar{\bar{\bar{F}}}}\bar{z}_{k+1} + \bar{\bar{\bar{\bar{F}}}}_y\bar{y}_k\bar{z}_1 + \bar{H}_y\bar{y}_k + f_{k+1},
\end{aligned} \tag{3.29}$$

where m_{k+1}, g_{k+1}, and f_{k+1} are all known composite functions. The process of solving these problems is almost identical to the case of $k = 1$, so we will not repeat it here. In this way, the asymptotic expansion (3.1) can be completely determined.

4 The existence of the solution and the remainder estimate

We first introduce a curve L_0 in the space of the variables (u, t). The curve L_0 is composed of the following

$$L_{01} = \{(u,t) : \bar{u}_0(a) + L_0 u(\tau_0), \tau_0 \geq 0, t = a\}, \qquad L_{02} = \{(u,t) : \bar{u}_0(t), a \leq t \leq b\},$$

we denote the projection of L_0 onto the space of the variables (u, t) by $\tilde{L}_0$.

[H_7] Suppose that the functions A, B, C, D, F, and H have continuous partial derivatives concerning each argument up to order $(n+2)$ inclusive in some δ-tube of $\tilde{L}_0$.

We use $U_n(t, \mu)$ to denote the first $n+1$ terms of the series (3.1) and

$$U_n = \sum_{k=0}^{n} \mu^k [\bar{u}_k(t) + L_k u(\tau_0)]. \tag{4.1}$$

Theorem 4.1 ([18,21]). *When conditions* $[H_1] \sim [H_7]$ *are met, there must be constants* $\mu_0 > 0$ *and* $c > 0$, *so that when* $\mu \in (0, \mu_0]$, *the solutions* $x(t,\mu), y(t,\mu)$ *and* $z(t,\mu)$ *of the problems* (2.1) *and* (2.2) *are lying in a* $c\delta$*-tube of* L_0, *is unique and satisfies the inequality*

$$|u(t,\mu) - U_n(t,\mu)| \leq c\mu^{n+1}, \qquad a \leq t \leq b. \tag{4.2}$$

Proof. Let $\xi = x - X_{n+1}, \eta = y - Y_{n+1}, \rho = z - Z_{n+1}$, where (x, y, z) is an exact solution of the problem (2.1), (2.2), and $(X_{n+1}, Y_{n+1}, Z_{n+1})$ is the partial sum of (4.1). Substituting $x = \xi + X_{n+1}, y = \eta + Y_{n+1}, z = \rho + Z_{n+1}$ into (2.1), (2.2), the equations of the remainder (ξ, η, ρ) is obtained

$$\begin{cases} \mu \dfrac{d\xi}{dt} = A(\eta + Y_{n+1}, \mu(\rho + Z_{n+1}), t)(\rho + Z_{n+1}) - \mu \dfrac{dX_{n+1}}{dt} \mu B(\eta + Y_{n+1}, \rho + Z_{n+1}, t), \\ \mu \dfrac{d\eta}{dt} = C(\xi + X_{n+1}, \eta + Y_{n+1}, t) - \mu \dfrac{dY_{n+1}}{dt} + \mu D(\eta + Y_{n+1}, \rho + Z_{n+1}, t), \\ \mu \dfrac{d\rho}{dt} = F(\mu(\xi + X_{n+1}), \eta + Y_{n+1}, t)(\rho + Z_{n+1}) - \mu \dfrac{dZ_{n+1}}{dt} + \mu H(\eta + Y_{n+1}, \rho + Z_{n+1}, t), \end{cases} \tag{4.3}$$

separating the linear part of the zeroth approximation, we obtain for (ξ, η, ρ) the boundary value problem on the intervals $[a, b]$, respectively, namely,

$$\begin{cases} \mu \dfrac{d\xi}{dt} = A(\bar{y}_0 + L_0 y, 0, t)\rho + A_y(\bar{y}_0 + L_0 y, 0, t)\eta + G_1(\eta, \rho, t, \mu), \\ \mu \dfrac{d\eta}{dt} = C_x(\bar{x}_0 + L_0 x, \bar{y}_0 + L_0 y, t)\xi + C_y(\bar{x}_0 + L_0 x, \bar{y}_0 + L_0 y, t)\eta + G_2(\xi, \eta, t, \mu), \\ \mu \dfrac{d\rho}{dt} = F(0, \bar{y}_0 + L_0 y, t)\rho + F_y(0, \bar{y}_0 + L_0 y, t)\eta + G_3(\xi, \eta, \rho, t, \mu). \end{cases} \tag{4.4}$$

The functions G_1, G_2 and G_3 are

$$\begin{aligned}G_1(\eta,\rho,t,\mu) &= A(\eta+Y_{n+1},\mu(\rho+Z_{n+1}),t)(\rho+Z_{n+1})-A(\bar{y}_0+L_0y,0,t)\rho\\&\quad - A_y(\bar{y}_0+L_0y,0,t)\eta-\mu\frac{dX_{n+1}}{dt}+\mu B(\eta+Y_{n+1},\rho+Z_{n+1},t),\\G_2(\xi,\eta,t,\mu) &= C(\xi+X_{n+1},\eta+Y_{n+1},t)+\mu D(\eta+Y_{n+1},\rho+Z_{n+1},t)-\mu\frac{dY_{n+1}}{dt}\\&\quad - C_x(\bar{x}_0+L_0x,\bar{y}_0+L_0y,t)\xi - C_y(\bar{x}_0+L_0x,\bar{y}_0+L_0y,t)\eta,\\G_3(\xi,\eta,\rho,t,\mu) &= F(\mu(\xi+X_{n+1}),\eta+Y_{n+1},t)(\rho+Z_{n+1})-F(0,\bar{y}_0+L_0y,t)\rho\\&\quad + \mu H(\eta+Y_{n+1},\rho+Z_{n+1},t)-\mu\frac{dZ_{n+1}}{dt}-F_y(0,\bar{y}_0+L_0y,t)\eta.\end{aligned}$$

G_1, G_2 and G_3, which we define having the following two important properties:

I. $|G_{1,2}(0,0,t,\mu)| \le c\mu^{n+2}, |G_3(0,0,0,t,\mu)| \le c\mu^{n+2}$, where $a \le t \le b, 0 < \mu \le \mu_0$;

II. For all $\varepsilon = O(\mu) > 0$, there are constants $\delta = \delta(\varepsilon)$ and $\mu_0 = \mu_0(\varepsilon)$ so that as long as $|\xi_1| \le \delta, |\xi_2| \le \delta, |\eta_1| \le \delta, |\eta_2| \le \delta, |\rho_1| \le \delta, |\rho_2| \le \delta, 0 < \mu \le \mu_0$, then for

$$\begin{aligned}|G_1(\eta_1,\rho_1,t,\mu)-G_1(\eta_2,\rho_2,t,\mu)| &\le \varepsilon(|\eta_1-\eta_2|+|\rho_1-\rho_2|),\\|G_2(\xi_1,\eta_1,t,\mu)-G_2(\xi_2,\eta_2,t,\mu)| &\le \varepsilon(|\xi_1-\xi_2|+|\eta_1-\eta_2|),\\|G_3(\xi_1,\eta_1,\rho_1,t,\mu)-G_3(\xi_2,\eta_2,\rho_2,t,\mu)| &\le \varepsilon(|\xi_1-\xi_2|+|\eta_1-\eta_2|+|\rho_1-\rho_2|).\end{aligned}$$

Let $\xi = v + \frac{L_0x}{L_0z}\rho$, convert ξ to v and substitute it in (4.4), we have

$$\begin{cases}\mu\dfrac{dv}{dt} = \left(A_y(\bar{y}_0+L_0y,0,t)-\dfrac{L_0x}{L_0z}F_y(0,\bar{y}_0+L_0y,t)\right)\eta\\ \qquad + G_1(\eta,\rho,t,\mu)-\dfrac{L_0x}{L_0z}G_3\left(v+\dfrac{L_0x}{L_0z}\rho,\eta,\rho,t,\mu\right),\\ \mu\dfrac{d\eta}{dt} = C_x(\bar{x}_0+L_0x,\bar{y}_0+L_0y,t)\dfrac{L_0x}{L_0z}\rho + C_x(\bar{x}_0+L_0x,\bar{y}_0+L_0y,t)v\\ \qquad + C_y(\bar{x}_0+L_0x,\bar{y}_0+L_0y,t)\eta + G_2\left(v+\dfrac{L_0x}{L_0z}\rho,\eta,t,\mu\right),\\ \mu\dfrac{d\rho}{dt} = F(0,\bar{y}_0+L_0y,t)\rho+F_y(0,\bar{y}_0+L_0y,t)\eta+G_3\left(v+\dfrac{L_0x}{L_0z}\rho,\eta,\rho,t,\mu\right).\end{cases} \tag{4.5}$$

The initial value condition for $v(t,\mu)$ is of the same type as that for $\xi(t,\mu)$. That is, $v(a,\mu) = O(\mu^{n+2})$. Then, the first differential equation in (4.5) can be rewritten as an integral equation

$$\begin{aligned}v(t,\mu) &= O(\mu^{n+2}) + \int_0^t \mu^{-1}\left(G_1(\eta,\rho,s,\mu)-\frac{L_0x}{L_0z}G_3(\eta,\rho,v,s,\mu)\right)ds\\&\quad + \int_0^t \mu^{-1}\left(A_y(\bar{y}_0+L_0y,0,s)-\frac{L_0x}{L_0z}F_y(0,\bar{y}_0+L_0y,s)\right)\eta(s)ds\\&= H_1(\rho,\eta,v,t,\mu).\end{aligned} \tag{4.6}$$

It is not difficult to prove that the integral operator $H_1(\rho,\eta,v,t,\mu)$ has a compression coefficient of $O(u)$ for ρ,η and v, and satisfies $H_1(0,0,0,t,\mu) = O(\mu^{n+1})$.

At this time, we consider the right ends $C_xv + G_2$ and G_3 of the last two equations of equation (4.5) as non-homogeneous terms, and write them into the equivalent integral equations. We write the Green's function of the first two equations in (4.5) as $\gamma(t,s,\mu)$. Under the boundary condition $\eta(a,\mu) = \eta(b,\mu) = 0$, γ is satisfied with the estimate $\gamma(t,s,\mu) = O\big(\exp\big(\frac{\kappa|s-t|}{\mu}\big)\big), a \le t \le b$. By the conditions $\eta(a,\mu) = O(\mu^{n+2})$ and $\eta(b,\mu) = O(\mu^{n+2})$, thus, the last two equations of equation (4.5) can be replaced with the following integral equations

$$\begin{pmatrix} \eta(t,\mu) \\ \rho(t,\mu) \end{pmatrix} = O(\mu^{n+2}) + \frac{1}{\mu}\int_0^t \gamma(t,s,\mu)\begin{pmatrix} C_x v + G_2 \\ G_3 \end{pmatrix} ds = \begin{pmatrix} \lambda_1(\rho,\eta,v,t,\mu) \\ \lambda_2(\rho,\eta,v,t,\mu) \end{pmatrix}, \tag{4.7}$$

under (4.6) in (4.7), we get

$$\begin{aligned} \eta(t,\mu) &= \lambda_1(\rho,\eta,v,t,\mu) \equiv H_2(\rho,\eta,v,t,\mu), \\ \rho(t,\mu) &= \lambda_2(\rho,\eta,v,t,\mu) \equiv H_3(\rho,\eta,v,t,\mu). \end{aligned} \tag{4.8}$$

Among them, the integral operator H_2, H_3 is similar to H_1.

By using the successive approximation method for the system (4.6), (4.8), we can prove that when parameter μ_0 is sufficient small, there is a unique solution $\xi = \eta = \rho = 0$ in the δ-tube, and meet the estimation of the solution $\xi = O(\mu^{n+1}), \eta = O(\mu^{n+1})$, and $\rho = O(\mu^{n+1})$. Therefore, $\xi = x - X_{n+1}, \eta = y - Y_{n+1}$ and $\rho = z - Z_{n+1}$ are all of the order $O(\mu^{n+1})$.

As a result of $u(t,\mu) - U_n(t,\mu) = O(\mu^{n+1})$, we obtain the inequality (4.2). This completes the proof. □

5 Illustrative example

Example 5.1. Consider the following system:

$$\begin{aligned} &\mu\frac{dx}{dt} = (\mu z - 1)z + \mu y, \qquad \mu\frac{dy}{dt} = x - y + \mu y, \\ &\mu\frac{dz}{dt} = (\mu x - 2)z + \mu y, \qquad 0 \le t \le 1, \end{aligned} \tag{5.1}$$

with the initial and boundary conditions

$$\begin{aligned} &y(0,\mu) - \mu y'(0,\mu) = \frac{1}{\sqrt{e}} - \frac{3}{2}, \qquad x(1,\mu) + x'(1,\mu) = \frac{3}{2}, \\ &z(0,\mu) - \mu z'(0,\mu) = 3. \end{aligned} \tag{5.2}$$

According to (2.1)–(2.2), we have $A = \mu z - 1, C = x - y, F = \mu x - 2, B = D = H = y, y^0 = \frac{1}{\sqrt{e}} - \frac{3}{2}, x^1 = \frac{3}{2}$ and $z^0 = 3$. It is easy to see that $C_x \neq 0, C_y < 0, F < 0$. Then the condition $[H_2]$ is satisfied.

By calculation, we can get

$$W_u = \begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

therefore, it is easy to get the eigenvalue $\lambda \equiv 0$ of the matrix W_u and the other two eigenvalues $\lambda_1 = -1 < 0, \lambda_2 = -2 < 0$. Thus, we get the case where the critical condition is stable.

By taking $\mu = 0$, we obtain the solution of the degenerated problem on the interval $[0,1]$, that is,

$$\bar{x}_0(t) = \omega(t), \qquad \bar{y}_0(t) = \omega(t), \qquad \bar{z}_0(t) = 0.$$

For $L_0u(\tau_0)$ and $\bar{u}_1(t)$, we have

$$\frac{dL_0x}{d\tau_0} = -L_0z, \qquad \frac{dL_0y}{d\tau_0} = L_0x - L_0y, \qquad \frac{dL_0z}{d\tau_0} = -2L_0z, \tag{5.3}$$
$$\frac{d\omega}{dt} = \bar{y}_0 - \bar{z}_1, \qquad \frac{d\bar{y}_0}{dt} = \bar{x}_1 - \bar{y}_1 + \bar{y}_0, \qquad \bar{y}_0 = 2\bar{z}_1, \tag{5.4}$$

and zero-order initial boundary value conditions

$$\begin{gathered} \omega(1) + \omega'(1) = \frac{3}{2}, \qquad L_0z(0) = L_0z'(0) + 3, \\ L_0y(0) = L_0y'(0) - \bar{y}_0(0) + \frac{1}{\sqrt{e}} - \frac{3}{2}. \end{gathered} \tag{5.5}$$

After calculation, we can get

$$\begin{aligned} L_0x(\tau_0) = \frac{1}{2}L_0z(\tau_0) = \frac{1}{2}e^{-2\tau_0}, \qquad & L_0y(\tau_0) = -\frac{1}{2}e^{-2\tau_0}, \\ \bar{x}_0(t) = \bar{y}_0(t) = e^{\frac{1}{2}(t-1)}, \qquad & \bar{z}_1(t) = \frac{1}{2}e^{\frac{1}{2}(t-1)}. \end{aligned} \tag{5.6}$$

Therefore, $L_0u(\tau_0)$ satisfies an exponential decay estimation $|L_0u(\tau_0)| \le C_0e^{-\kappa\tau_0}$, for C_0 and κ are positive constants.

The system for $L_1u(\tau_0)$ and $\bar{u}_2(t)$, we have

$$\begin{aligned} &\frac{dL_1x}{d\tau_0} = L_0^2z - L_1z + L_0y, \qquad \frac{dL_1y}{d\tau_0} = L_1z - L_1y + L_0y, \\ &\frac{dL_1z}{d\tau_0} = L_0xL_0z - 2L_1z + L_0y. \end{aligned} \tag{5.7}$$

$$\begin{aligned} &\frac{d\bar{x}_1}{dt} = -\bar{z}_2 + \bar{y}_1, \qquad \frac{d\bar{y}_1}{dt} = \bar{x}_2 - \bar{y}_2 + \bar{y}_1, \\ &\frac{d\bar{z}_1}{dt} = \bar{x}_0\bar{z}_1 - 2\bar{z}_2 + \bar{y}_1, \end{aligned} \tag{5.8}$$

and first-order initial boundary value conditions

$$\begin{aligned} &\bar{y}_1(0) + L_1y(0) = \bar{y}_0'(0) + {L_1}'y(0), \qquad \bar{x}_1(1) + \bar{x}_1'(1) = 0, \\ &\bar{z}_1(0) + L_1z(0) = {L_1}'z(0). \end{aligned} \tag{5.9}$$

Through calculation, we can get

$$\begin{aligned} \bar{x}_1(t) &= -\frac{1}{2}e^{(t-1)} + \frac{3}{8}te^{\frac{1}{2}(t-1)} - \frac{1}{24}e^{\frac{1}{2}(t-1)}, \\ \bar{y}_1(t) &= -\frac{1}{2}e^{(t-1)} + \frac{3}{8}te^{\frac{1}{2}(t-1)} + \frac{11}{24}e^{\frac{1}{2}(t-1)}, \qquad \bar{z}_1(t) = \frac{1}{2}e^{\frac{1}{2}(t-1)}, \\ L_1x(\tau_0) &= \left(\frac{1}{8} - \frac{1}{4\sqrt{e}}\right)e^{-2\tau_0} - \frac{1}{4}\tau_0e^{-2\tau_0} - \frac{5}{16}e^{-4\tau_0}, \\ L_1z(\tau_0) &= -\frac{1}{2\sqrt{e}}e^{-2\tau_0} - \frac{1}{2}\tau_0e^{-2\tau_0} - \frac{1}{4}e^{-4\tau_0}, \\ L_1y(\tau_0) &= \left(\frac{1}{4e} - \frac{5}{48\sqrt{e}} - \frac{103}{96}\right)e^{-\tau_0} + \left(\frac{5}{8} + \frac{1}{4\sqrt{e}} + \frac{1}{4}\tau_0\right)e^{-2\tau_0} + \frac{5}{48}e^{-4\tau_0}. \end{aligned} \tag{5.10}$$

It is easy to get that $L_1u(+\infty) = 0$. Thus, $L_1u(\tau_0)$ satisfies an exponential decay estimation. Therefore, we construct the first-order approximate solution $u = (x\ y\ z)^T$ of the system (5.1)–(5.2). Namely, we have

$$u(t,\mu) = \begin{cases} x(t,\mu) = \bar{x}_0(t) + L_0x(\tau_0) + \mu(\bar{x}_1(t) + L_1x(\tau_0)) + O(\mu^2), \\ y(t,\mu) = \bar{y}_0(t) + L_0y(\tau_0) + \mu(\bar{y}_1(t) + L_1y(\tau_0)) + O(\mu^2), \\ z(t,\mu) = \bar{z}_0(t) + L_0z(\tau_0) + \mu(\bar{z}_1(t) + L_1z(\tau_0)) + O(\mu^2), \end{cases} \tag{5.11}$$

where $\tau_0 = \dfrac{t-0}{\mu}, 0 \le t \le 1$.

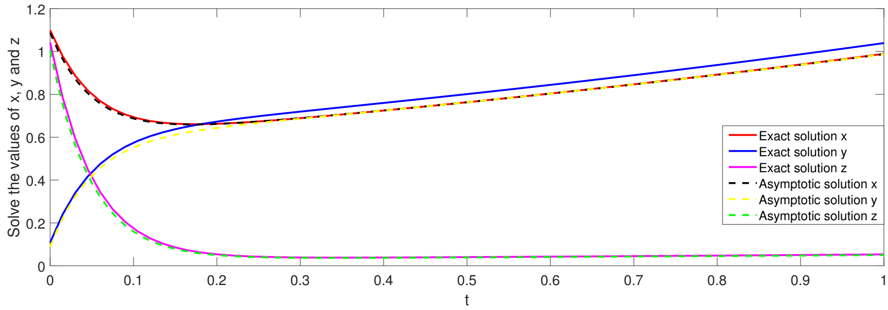

Figure 5.1: Contrast of the exact with the approximate solutions to (5.1) with boundary condition (5.2) for values of the perturbing parameter $\mu = 0.1$.

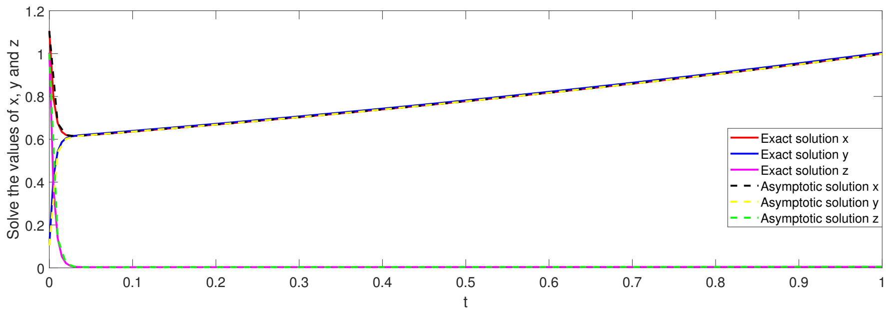

Figure 5.2: Contrast of the exact with the approximate solutions to (5.1) with boundary condition (5.2) for values of the perturbing parameter $\mu = 0.01$.

Therefore, these results were obtained using the boundary layer function method in [18]. The results obtained by Matlab are given in Figures 5.1 and 5.2. Different line types corresponding to exact and approximate solutions have been marked in the figure. These graphs show that an asymptotic solution is closer to the exact one if we use higher-order asymptotics. If we use the small parameter μ to be smaller, the formal asymptotic solution is more approximate to the exact solution. The image of the solution also better illustrates the nature of the exponential decay of the boundary function.

6 Conclusive remarks

This paper studies a class of nonlinear critical singular perturbation problems with Robin initial boundary value, the results show that how to obtain the zero-order approximate solution and simplify the first-order boundary layer term equation is the key to obtaining the approximate solution of the system. In this paper, the successive approximation method is used to prove the remainder estimation and the existence and uniqueness of the solution.

Finally, in the process of researching the system (2.1)–(2.2), the following situations were discovered.

Remark 6.1. When $F > 0$ and $F \neq 1$ in the interval $[a, b]$ of t, the system (2.1) only produces the right layer, but the solution method is similar to the left boundary layer. And when the F has a zero point in the interval $[a, b]$, the stability will be in the interval $[a, b]$ has changed, thus forming a very complicated nonlinear turning point problem. Since the linear turning point problem is already very complicated, as the nonlinear strength increases, the nonlinear turning point problem will become more complicated. So this type of problem will be very challenging. At this time, the boundary layer function method will no longer be applicable, and it is necessary to find a suitable method to solve it.

Remark 6.2. When the coefficients in front of $y'(t, \mu)$ and $z'(t, \mu)$ in Robin's initial boundary value condition (2.2) do not contain the small parameter μ when the derivative of t is calculated for the boundary layer function, the μ^{-1} term will appear. So the right end of the corresponding boundary condition should have the μ^{-1} term to perform matching, and in the process of using the boundary layer function method to solve the problem, it should be set to the following form

$$\begin{cases} y = \sum_{k=-1}^{\infty} \left[\mu^{k+1}\bar{y}_{k+1}(t, \mu) + \mu^k L_k y(\tau_0, \mu)\right], \\ z = \sum_{k=-1}^{\infty} \left[\mu^{k+1}\bar{z}_{k+1}(t, \mu) + \mu^k L_k z(\tau_0, \mu)\right]. \end{cases} \tag{6.1}$$

But when y and z are substituted into the system (2.1) at this time, A, B, C, D, H, and F can't perform Taylor expansion when $\mu = 0$, which contradicts the boundary layer function method. Therefore, the weak nonlinear problem generally does not produce an infinite initial boundary value problem.

Remark 6.3. When the initial boundary values (2.2) are all Robin initial conditions, the form is as follows

$$\begin{aligned} &y(a, \mu) - \mu y'(a, \mu) = y^0, \qquad x(a, \mu) + \mu x'(a, \mu) = x^0, \\ &z(a, \mu) - \mu z'(a, \mu) = z^0, \end{aligned} \tag{6.2}$$

the process of solving the n-order equations of problem (2.1)–(2.2) does not change, but the corresponding initial conditions and boundary value conditions change. Through calculations, we found that the basic process of the solution does not change significantly, but from first finding the value of $\bar{x}(t)$ at $t = b$, it becomes possible to find the value of $\bar{y}(t)$ at $t = a$, and $L_0 y(\tau_0)$ at the value of $\tau_0 = 0$.

Remark 6.4. For system 2.1, because the functions B, D and H at the right end are preceded by a small parameter μ. Therefore, no matter the form is $(z, y, t), (x, y, t)$ or (x, z, t), system 2.1 is still a critically stable situation, and the corresponding asymptotic solution form and solution method will not be changed.

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Global algebraic Poincaré–Bendixson annulus for the Rayleigh equation

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Abstract. We consider the Rayleigh equation $\ddot{x} + \lambda(\dot{x}^2/3 - 1)\dot{x} + x = 0$ depending on the real parameter λ and construct a Poincaré–Bendixson annulus $\mathcal{A}_\lambda$ in the phase plane containing the unique limit cycle Γ_λ of the Rayleigh equation for all $\lambda > 0$. The novelty of this annulus consists in the fact that its boundaries are algebraic curves depending on λ. The polynomial defining the interior boundary represents a special Dulac–Cherkas function for the Rayleigh equation which immediately implies that the Rayleigh equation has at most one limit cycle. The outer boundary is the diffeomorphic image of the corresponding boundary for the van der Pol equation. Additionally we present some equations which are linearly topologically equivalent to the Rayleigh equation and provide also for these equations global algebraic Poincaré–Bendixson annuli.

Keywords: limit cycle, Rayleigh equation, Dulac–Cherkas function, Poincaré–Bendixson annulus, topologically equivalent Rayleigh systems.

2020 Mathematics Subject Classification: 34C05, 34C07, 34C23.

1 Introduction

The British physicist and Nobel prize winner J. W. Strutt, better known as Lord Rayleigh, published fundamental results to a broad spectrum of physical phenomena. In his monograph “Theory of Sounds” [18] he used the linear differential equation with constant coefficients

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + n^2x = 0$$

for the description of acoustic oscillations of a clarinet. The nonlinear modification of this equation

$$\frac{d^2x}{dt^2} + \lambda\left[\left(\frac{dx}{dt}\right)^2\Big/3 - 1\right]\frac{dx}{dt} + x = 0, \tag{1.1}$$

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where λ is a real parameter, is known under the name Rayleigh equation [3, 17]. Its corresponding system

$$\begin{aligned}\frac{dx}{dt} &= -y,\\ \frac{dy}{dt} &= x + \lambda y - \lambda\frac{y^3}{3},\end{aligned} \tag{1.2}$$

which is invariant under the transformation $t \to -t, y \to -y, \lambda \to -\lambda$ has been studied by several authors [1,2,8,9,13,15,21–23].

The existence of a limit cycle (isolated closed orbit) of a planar autonomous system is established usually by the construction of an annulus $\mathcal{A}$ in the phase plane with the following properties: (i). $\mathcal{A}$ contains no equilibrium of the system under consideration. (ii). The boundary of $\mathcal{A}$ consists of two simple closed curves (in what follows called ovals) such that any trajectory of the considered system meeting the boundary of $\mathcal{A}$ will enter $\mathcal{A}$ either for increasing or for decreasing t. An annulus with the properties (i) and (ii) is called a Poincaré–Bendixson annulus since the application of the Poincaré–Bendixson theorem [5, 16] to that annulus provides the existence of at least one limit cycle in $\mathcal{A}$. The crucial problem in that approach is the construction of the ovals forming the boundary of $\mathcal{A}$. In numerous publications (see e.g. [5, 6, 14, 16, 19, 20]) these ovals consist of piecewise smooth curves constructed in a sophisticated way. In this paper we are concerned with the construction of such ovals which are differentiable curves having only a finite number of points where the trajectories of the underlying system touch the ovals. We call such ovals as crossing ovals. Recently, two papers have been published [7, 10] in which a procedure for the construction of algebraic crossing ovals for planar polynomial systems is described. For both papers it is characteristic that they need the approximation of at least one orbit by a polynomial in t. In what follows, we present an approach to construct algebraic crossing ovales for the Rayleigh system (1.2) and some of its topologically equivalent systems, which is completely different from that one presented in the cited papers [7, 10].

The structure of our paper is as follows: in Section 2 we describe a method for the construction of an algebraic crossing oval for a class of polynomial systems. For this reason we introduce the concept of Dulac–Cherkas functions including one method for their construction. Section 3 is devoted to the construction of a crossing oval by means of a diffeomorphically equivalent system. In Section 4 we derive some linearly diffeomorphically equivalent systems to the Rayleigh system (1.2) and present the corresponding Poincaré–Bendixson annuli.

2 Construction of an interior boundary for a Poincaré–Bendixson annulus of the Rayleigh system (1.2)

Our approach to construct an interior boundary for a Poincaré–Bendixson annulus for system (1.2) is based on the use of a Dulac–Cherkas function. For this reason we introduce in the next subsection the definition of a Dulac–Cherkas function and compose some of its properties.

2.1 Definition and properties of Dulac–Cherkas functions

We consider the planar differential system

$$\frac{dx}{dt} = P(x, y, \lambda), \qquad \frac{dy}{dt} = Q(x, y, \lambda) \tag{2.1}$$

under the assumption

(A) Let $\mathcal{G}$ be an open subset of $\mathbb{R}^2$, let Λ be some open interval, let $P, Q \in C^{1\ \ 0}_{(x,y)\ \lambda}(\mathcal{G}\times\Lambda, \mathbb{R})$.

We denote by X the vector field defined by (2.1). First we recall the definition of a Dulac function.

Definition 2.1. Suppose the assumption (A) to be valid. A function B belonging to the class $C^{1\ \ 0}_{(x,y)\ \lambda}(\mathcal{G}\times\Lambda, \mathbb{R})$ is called a Dulac function of system (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$ if the expression

$$\operatorname{div}(BX) \equiv \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} \equiv (\operatorname{grad} B, X) + B \operatorname{div} X$$

does not change sign in $\mathcal{G}$ and vanishes only on a set $\mathcal{N}_\lambda \subset \mathcal{G}$ of measure zero for $\lambda \in \Lambda$.

The class of Dulac functions has been generalized by L. A. Cherkas in 1997 (see [4]). The corresponding generalized Dulac function, which is called Dulac–Cherkas function nowadays, is defined as follows.

Definition 2.2. Suppose the assumption (A) to be valid. A function $\Psi \in C^{1\ \ 0}_{(x,y)\ \lambda}(\mathcal{G}\times\Lambda, \mathbb{R})$ is called a Dulac–Cherkas function of system (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$ if there exists a real number $\kappa \neq 0$ such that

$$\Phi := (\operatorname{grad} \Psi, X) + \kappa\Psi \ \operatorname{div}\ X > 0 \quad (< 0) \quad \text{in } \mathcal{G} \text{ for } \lambda \in \Lambda. \tag{2.2}$$

Remark 2.3. Condition (2.2) can be relaxed by assuming that Φ may vanish in $\mathcal{G}$ on a set $\mathcal{N}_\lambda$ of measure zero, and that no oval of this set is a limit cycle.

Remark 2.4. In case $\kappa = 1$, Ψ is a Dulac function.

For the sequel we introduce the subset $\mathcal{W}_\lambda$ of $\mathcal{G}$ defined by

$$\mathcal{W}_\lambda := \{(x,y) \in \mathcal{G} : \Psi(x,y,\lambda) = 0\}. \tag{2.3}$$

From the Definition 2.2 we get immediately

Lemma 2.5. *Suppose the assumption (A) to be valid. Let Ψ be a Dulac–Cherkas function of system (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$. Then any oval of $\mathcal{W}_\lambda$ having only a finite number of points where $(\operatorname{grad}\Psi, X)$ vanishes is a crossing oval for system (2.1) and can be used as a boundary for a Poincaré–Bendixson annulus.*

The following theorem is a special case of a more general result established in [11].

Theorem 2.6. *Suppose the assumption (A) to be valid. Let $\mathcal{G}$ be a simply connected region, let Ψ be a Dulac–Cherkas function of (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$ such that $\mathcal{W}_\lambda$ contains exactly one oval $\mathcal{O}_\lambda$ in $\mathcal{G}$. Then in the case $\kappa < 0$ system (2.1) has for $\lambda \in \Lambda$ at most one limit cycle in $\mathcal{G}$, and if it exists, it surrounds $\mathcal{W}_\lambda$ and is hyperbolic.*

This theorem implies

Corollary 2.7. *Under the assumptions of Theorem 2.6 the oval $\mathcal{O}_\lambda$ can be used as interior boundary for a Poincaré–Bendixson annulus of system (2.1) provided it is a crossing oval.*

The problem how to construct a Dulac–Cherkas function for the Rayleigh system (1.2) will be treated in the next subsection. We note that the presented procedure can be applied to a more general class of planar polynomial systems.

2.2 Construction of Dulac–Cherkas functions for system (1.2)

We consider system (1.2) in $\mathbb{R}^2$ for $\lambda > 0$. The corresponding vector field X reads

$$X(x,y,\lambda) := (-y, x + \lambda y - \lambda y^3/3). \tag{2.4}$$

We look for a Dulac–Cherkas function in the form

$$\Psi(x,y,\lambda) := \Psi_0(y,\lambda) + \Psi_1(y,\lambda)x + \Psi_2(y,\lambda)x^2, \tag{2.5}$$

where we assume that for all $\lambda > 0$ the function Ψ_2 is not identically zero.

Using (2.4) and (2.5) we obtain for the function Φ defined in (2.2) the representation

$$\Phi(x,y,\lambda,\kappa) = \sum_{k=0}^{3} \Phi_k(y,\lambda,\kappa)x^k, \tag{2.6}$$

where the functions Φ_k are defined by the relations

$$\Phi_0 := -\Psi_1 y + \Psi_0'\lambda(y - y^3/3) + \kappa\lambda\Psi_0(1-y^2), \tag{2.7}$$
$$\Phi_1 := -2\Psi_2 y + \Psi_0' + \Psi_1'\lambda(y - y^3/3) + \kappa\lambda\Psi_1(1-y^2), \tag{2.8}$$
$$\Phi_2 := \Psi_1' + \Psi_2'\lambda(y - y^3/3) + \kappa\lambda\Psi_2(1-y^2), \tag{2.9}$$
$$\Phi_3 := \Psi_2', \tag{2.10}$$

where the symbol $'$ indicates the differentiation with respect to y. One approach to guarantee that Φ is a definite function in $\mathbb{R}^2$ for $\lambda > 0$ is to require Φ_k to be identically zero for $1 \le k \le 3$ and that Φ_0 is definite. Applying this approach we get from (2.10) the linear differential equation

$$\Psi_2' = 0, \tag{2.11}$$

such that it holds

$$\Psi_2(y,\lambda,\kappa) \equiv c_2 \neq 0. \tag{2.12}$$

Taking into account (2.11) and (2.12) we obtain from (2.9)

$$\Psi_1' + \kappa c_2\lambda(1-y^2) = 0 \tag{2.13}$$

whose solution reads

$$\Psi_1(y,\lambda) = -\kappa c_2\lambda(y - y^3/3) + c_1. \tag{2.14}$$

Taking into account (2.14), (2.13) and (2.12) we get from (2.8)

$$\Psi_0' = 2c_2 y + (1+\kappa)\kappa c_2\lambda^2(1-y^2)(y-y^3/3) - \kappa c_1\lambda(1-y^2). \tag{2.15}$$

Setting

$$\kappa = -1, \qquad c_1 = 0 \tag{2.16}$$

we have by (2.14)

$$\Psi_1(y,\lambda) = c_2\lambda(y - y^3/3) \tag{2.17}$$

and the differential equation (2.15) reads

$$\Psi_0' = 2c_2 y \tag{2.18}$$

whose solution has the form

$$\Psi_0(y,\lambda) = c_2 y^2 + c_0. \tag{2.19}$$

Using (2.16)–(2.19) we obtain from (2.7)

$$\Phi_0(y,\lambda,-1) = \frac{2}{3}\lambda c_2\left(y^4 + \frac{3c_0}{2c_2}y^2 - \frac{3c_0}{2c_2}\right). \tag{2.20}$$

Now we have to determine c_0 and c_2 such that $\Phi_0(y,\lambda,-1)$ is a definite function and that the corresponding Dulac–Cherkas function Ψ has the property that its zero-level set $\mathcal{W}_\lambda$ contains an oval surrounding the origin. Setting $c_0 = -\frac{8}{3}c_2$, where by (2.12) $c_0 \neq 0$ holds, we have

$$\Phi_0(y,\lambda,-1) = \frac{2}{3}\lambda c_2(y^4 - 4y^2 + 4) \tag{2.21}$$

which has for $\lambda > 0$ the same sign for all y and vanishes only at $y = \pm\sqrt{2}$. Thus it holds

Lemma 2.8. *The polynomial*

$$\Psi(x,y,\lambda) := c_2\left(x^2 + y^2 - \frac{8}{3} + \lambda x\left(y - \frac{y^3}{3}\right)\right) \tag{2.22}$$

is a Dulac–Cherkas function for system (1.2) *in* $\mathbb{R}^2$ *for* $\lambda > 0$.

2.3 Construction of an interior boundary for a Poincaré–Bendixson annulus of system (1.2)

The set $\mathcal{W}_\lambda$ of the Dulac–Cherkas function Ψ in (2.22) is defined by

$$\mathcal{W}_\lambda := \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - \frac{8}{3} + \lambda x\left(y - \frac{y^3}{3}\right) = 0\right\}. \tag{2.23}$$

First we note that $\mathcal{W}_0$ is the circle $x^2 + y^2 = 8/3$. From (2.23) we get further that for all $\lambda > 0$ the set $\mathcal{W}_\lambda$ is symmetric with respect to the origin and that the intersection of $\mathcal{W}_\lambda$ with the straight lines $y = \pm\sqrt{3}$ is empty for any $\lambda > 0$. For the following we denote by $\mathcal{S}_{2\sqrt{3}}$ in $\mathbb{R}^2$ the strip symmetric to the x-axis and with thickness $2\sqrt{3}$. We obtain from (2.23) the result

Lemma 2.9. *The set* $\mathcal{W}_\lambda$ *defined in* (2.23) *consists in* $\mathbb{R}^2$ *for* $\lambda > 0$ *of three different branches: the oval* $\mathcal{I}_\lambda$ *surrounding the origin and located in the strip* $\mathcal{S}_{2\sqrt{3}}$*, the unbounded branch* $\mathcal{W}_\lambda^1$ *located in the first quadrant in the region* $y > \sqrt{3}$ *and the symmetric branch* $\mathcal{W}_\lambda^3$ *in the third quadrant in the region* $y < -\sqrt{3}$ *.*

Figure 2.1 shows the branches of $\mathcal{W}_\lambda$ for $\lambda = 1.3$.

In order to prove that the oval $\mathcal{I}_\lambda$ is a crossing oval, we note that we have by (2.2)

$$(\mathrm{grad}\,\Psi, X)\big|_{\Psi=0} = \Phi\big|_{\Psi=0} = \Phi_0\big|_{\Psi=0}.$$

According to (2.21) there exist four points on $\mathcal{I}_\lambda$, where the vector field X touches the oval $\mathcal{I}_\lambda$. Therefore, $\mathcal{I}_\lambda$ is a crossing oval and we get from Corollary 2.7

Theorem 2.10. *The oval* $\mathcal{I}_\lambda$ *represents for* $\lambda > 0$ *an interior boundary for a Poincaré–Bendixson annulus of system* (1.2).

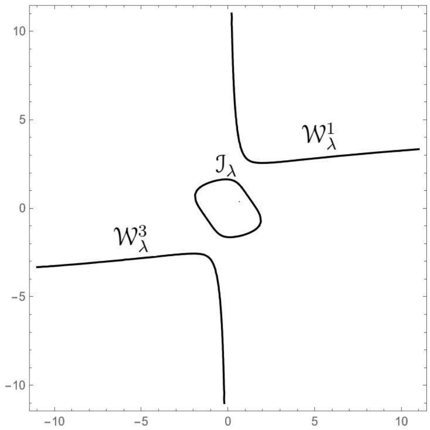

Figure 2.1: Three branches of the set $\mathcal{W}_\lambda$ including the oval $\mathcal{I}_\lambda$ for $\lambda = 1.3$.

3 Construction of an outer boundary for a Poincaré–Bendixson annulus of the Rayleigh system (1.2)

For the construction of an outer boundary of a Poincaré–Bendixson annulus for the Rayleigh system (1.2) a similar but more sophisticated procedure could be applied as it has been used for the van der Pol system in our paper [12]. In what follows we describe another approach based on the concept of diffeomorphically equivalent systems. In the following subsection we present the definition of topological equivalence of phase portraits and some important consequence.

3.1 Definition of topological equivalence and some important consequences

Our basic assumption reads as follows

($\tilde{A}$) Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be open subsets of $\mathbb{R}^2$, let Λ be some open interval, let $P_1, Q_1 \in C^{1}_{(x,y)}{}^{0}_{\lambda}(\mathcal{G}_1 \times \Lambda, \mathbb{R}), P_2, Q_2 \in C^{1}_{(x,y)}{}^{0}_{\lambda}(\mathcal{G}_2 \times \Lambda, \mathbb{R})$.

Consider the topological structure of the trajectories of the system

$$\frac{dx}{dt} = P_1(x,y,\lambda), \qquad \frac{dy}{dt} = Q_1(x,y,\lambda) \tag{3.1}$$

in $\mathcal{G}_1$ and the topological structure of the trajectories of the system

$$\frac{dx}{d\tau} = P_2(x,y,\lambda), \qquad \frac{dy}{d\tau} = Q_2(x,y,\lambda) \tag{3.2}$$

in $\mathcal{G}_2$.

Definition 3.1. Suppose assumption ($\tilde{A}$) to be valid. Let Λ_1 be a subinterval of Λ. The systems (3.1) and (3.2) are called topologically equivalent for $\lambda \in \Lambda_1$ if for $\lambda \in \Lambda_1$ there is a homeomorphism h_λ mapping $\mathcal{G}_1$ onto $\mathcal{G}_2$ and which maps the trajectories of system (3.1) onto the trajectories of system (3.2) and there is a strictly increasing homeomorphism g_λ mapping $\mathbb{R}$ onto itself such that $\tau = g_\lambda(t)$. If h_λ is a diffeomorphism then the systems are called diffeomorphically equivalent.

The following result is a consequence of the well known fact that the composition of a local diffeomorphism with a diffeomorphism is still a local diffeomorphism.

Theorem 3.2. *Suppose that the assumption* $(\tilde{A})$ *is valid and that the systems* (3.1) *and* (3.2) *are diffeomorphically equivalent for* $\lambda \in \Lambda_1$. *Let* $\mathcal{O}_\lambda$ *be a crossing oval for system* (3.1) *for* $\lambda \in \Lambda_1$. *Then the image of* $\mathcal{O}_\lambda$ *under the diffeomorphism* d_λ *is a crossing oval for system* (3.2) *for* $\lambda \in \Lambda_1$.

In order to be able to apply Theorem 3.2 for the construction of an outer boundary for a Poincaré–Bendixson annulus for the Rayleigh system (1.2) we use the following lemma.

Lemma 3.3. *The van der Pol system*

$$\begin{aligned} \frac{du}{dt} &= -v, \\ \frac{dv}{dt} &= u - \lambda(u^2-1)v \end{aligned} \tag{3.3}$$

is for $\lambda > 0$ *diffeomorphically equivalent to the Rayleigh system* (1.2).

Proof. Applying the diffeomorphism d_λ mapping $\mathbb{R}^2$ onto itself defined by

$$\begin{aligned} x &= -v + \lambda\left(\frac{u^3}{3} - u\right), \\ y &= u \end{aligned} \tag{3.4}$$

we get from (3.3)

$$\begin{aligned} \frac{dx}{dt} &= -y, \\ \frac{dy}{dt} &= x + \lambda y - \lambda\frac{y^3}{3}, \end{aligned} \tag{3.5}$$

which coincides with the Rayleigh system (1.2). □

3.2 Construction of an outer boundary for a Poincaré–Bendixson annulus of the Rayleigh system (1.2)

In the paper [12] we have proved the following result

Theorem 3.4. *For* $\lambda > 0$, *the oval*

$$\mathcal{V}_\lambda := \left\{(u,v) \in \mathbb{R}^2 : v^2 + \lambda v u\left(2 - \frac{u^2}{3}\right) + (1+\lambda^2)u^2 - \frac{7}{12}\lambda^2 u^4 + \frac{\lambda^2}{18}u^6 - 8 - 3\lambda - 18\lambda^2 = 0\right\} \tag{3.6}$$

is a crossing oval forming an outer boundary of a global algebraic Poincaré–Bendixson annulus for the van der Pol system (3.3).

According to Lemma 3.3, the van der Pol system (3.3) is for $\lambda > 0$ diffeomorphically equivalent to the Rayleigh system (1.2), where the corresponding diffeomorphism d_λ is defined in (3.4). By Theorem 3.2, the image of the crossing oval $\mathcal{V}_\lambda$ for the van der Pol system (3.3) under the diffeomorphism d_λ is for $\lambda > 0$ a crossing oval $\mathcal{O}_\lambda$ of the Rayleigh system (1.2). From (3.4) and (3.6) we get

$$\mathcal{O}_\lambda := \Big\{ (x,y) : \Big(-x + \lambda\Big(\frac{y^3}{3} - y\Big)\Big)^2 + \lambda\Big(-x + \lambda\Big(\frac{y^3}{3} - y\Big)\Big)\Big(2y - \frac{y^3}{3}\Big) + (1+\lambda^2)y^2 - \frac{7}{12}\lambda^2 y^4 + \frac{1}{18}\lambda^2 y^6 - 8 - 3\lambda - 18\lambda^2 = 0. \Big\} \tag{3.7}$$

It can be verified that the derivative of $\mathcal{O}_\lambda$ along system (1.2) is negative on $\mathcal{O}_\lambda$ except at four points. Thus we have the result

Theorem 3.5. *The algebraic oval $\mathcal{O}_\lambda$ defined in (3.7) is for $\lambda > 0$ an algebraic crossing oval of the Rayleigh system (1.2) forming the outer boundary of a Poincaré–Bendixson annulus. Together with the algebraic oval $\mathcal{I}_\lambda$ it determines a global algebraic Poincaré–Bendixson annulus $\mathcal{A}_\lambda$ containing the unique limit cycle Γ_λ of the Rayleigh system (1.2).*

Figure 3.1 shows the Poincaré–Bendixson annulus $\mathcal{A}_\lambda$ with the limit cycle Γ_λ of system (1.2) for $\lambda = 0.1$ and $\lambda = 1.3$.

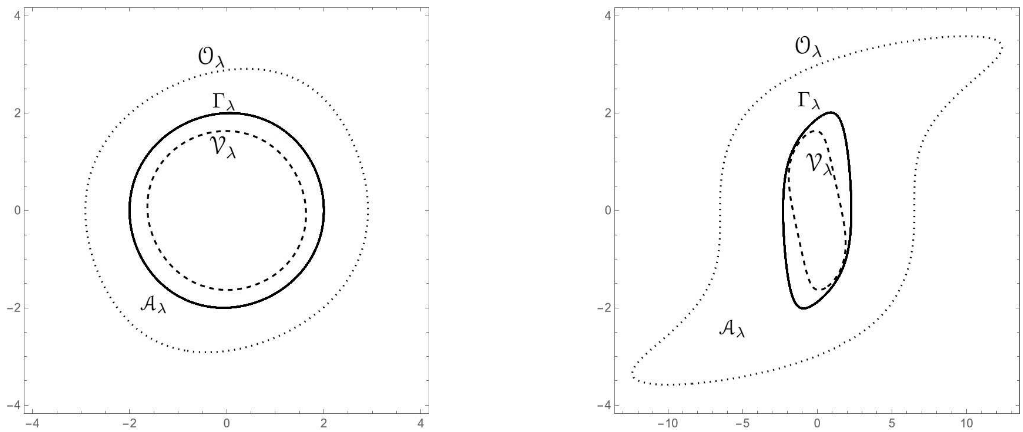

Figure 3.1: Annulus $\mathcal{A}_\lambda$ with the limit cycle Γ_λ of system (1.2) for $\lambda = 0.1$ (left) and $\lambda = 1$ (right).

4 Global algebraic Poincaré–Bendixson annuli for systems diffeomorphically equivalent to the Rayleigh system

If we apply for $\lambda > 0$ the linear diffeomorphism

$$u = \sqrt{\lambda}x, \quad v = \sqrt{\lambda}y \tag{4.1}$$

to the Rayleigh system (1.2) we obtain the system

$$\begin{aligned} \frac{du}{dt} &= -v, \\ \frac{dv}{dt} &= u + \lambda v - \frac{v^3}{3}, \end{aligned} \tag{4.2}$$

which is diffeomorphically equivalent to system (1.2) for $\lambda > 0$. Thus, system (4.2) has for $\lambda > 0$ a unique limit cycle $\bar{\Gamma}_\lambda$. According to Theorem 3.2 we obtain a global algebraic Poincaré–Bendixson annulus for system (4.2) by applying the diffeomorphism (4.1) to the ovals $\mathcal{I}_\lambda$ and $\mathcal{O}_\lambda$. It holds

Theorem 4.1. *The algebraic ovals*

$$\bar{\mathcal{I}}_\lambda := \left\{(u,v) \in \mathcal{S}_{2\sqrt{3\lambda}} : u^2 + v^2 + uv\left(\lambda - \frac{v^2}{3}\right) - \frac{8}{3}\lambda = 0\right\} \tag{4.3}$$

and

$$\begin{aligned}\bar{\mathcal{O}}_\lambda := \Bigg\{&(u,v) \in \mathbb{R}^2 : \left(-u + \left(\frac{v^3}{3} - \lambda v\right)\right)^2 + \left(-u + \left(\frac{v^3}{3} - \lambda v\right)v\left(2\lambda - \frac{v^2}{3}\right)\right. \\ &+ (1+\lambda^2)v^2 - \frac{7}{12}\lambda v^4 + \frac{1}{18}v^6 - 8\lambda - 3\lambda^2 - 18\lambda^3 = 0\Bigg\}\end{aligned} \tag{4.4}$$

form a global algebraic Poincaré–Bendixson annulus $\bar{A}_\lambda$ containing the unique limit cycle $\bar{\Gamma}_\lambda$ of system (4.2).

If λ tends to zero we get from (4.3) and (4.4) that both ovals shrink to the origin which reflects the property of system (4.2) that the limit cycle $\bar{\Gamma}_\lambda$ bifurcates from the origin when λ passes zero (Andronov–Hopf bifurcation). This distinguishes system (4.2) from the Rayleigh system where the limit cycle Γ_λ bifurcates from the circle $x^2 + y^2 = 2$ when λ passes zero.

Figure 4.1 shows the Poincaré–Bendixson annulus $\bar{A}_\lambda$ with the limit cycle $\bar{\Gamma}_\lambda$ of system 4.2 for $\lambda = 0.1$ and $\lambda = 1$.

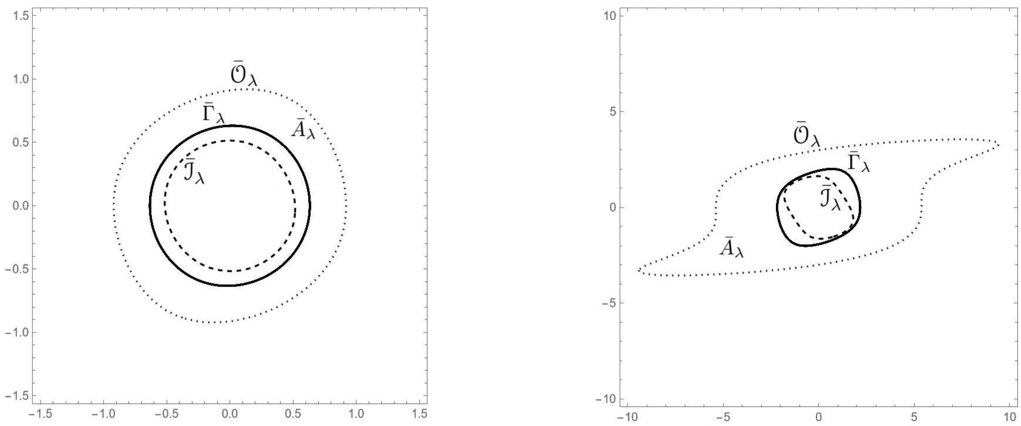

Figure 4.1: Annulus $\bar{A}_\lambda$ with the limit cycle $\bar{\Gamma}_\lambda$ of system (4.2) for $\lambda = 0.1$ (left) and $\lambda = 1$ (right).

If we apply for $\lambda > 0$ the linear diffeomorphism

$$x = \lambda u, \quad y = v, \quad t = \lambda\tau \tag{4.5}$$

to the Rayleigh system (1.2) and use the notation $\varepsilon = 1/\lambda^2$ we obtain the topologically equivalent system

$$\begin{aligned}\frac{du}{d\tau} &= -v, \\ \varepsilon\frac{dv}{d\tau} &= u + v - \frac{v^3}{3}\end{aligned} \tag{4.6}$$

which is a singularly perturbed system in case of small ε. Thus, the unique limit cycle $\hat{\Gamma}_\varepsilon$ represents a relaxation oscillation for small ε. If we apply the linear diffeomorphism (4.5) to the ovals $\mathcal{I}_\lambda$ and $\mathcal{O}_\lambda$ we obtain a global algebraic Poincaré–Bendixson annulus $\hat{\mathcal{A}}_\varepsilon$ for system (4.6).

Theorem 4.2. *The algebraic ovals*

$$\hat{\mathcal{I}}_\varepsilon := \left\{(u,v) \in \mathcal{S}_{2\sqrt{3}} : u^2 + \varepsilon v^2 + uv\left(1 - \frac{v^2}{3}\right) - \frac{8}{3}\varepsilon = 0\right\} \tag{4.7}$$

and

$$\hat{\mathcal{O}}_\varepsilon := \Bigg\{(u,v) \in \mathbb{R}^2 : \left(-u + v\left(\frac{v^2}{3} - 1\right)\right)(-u+v) + (1+\varepsilon)v^2 - \frac{7}{12}v^4 + \frac{1}{18}v^6 - 8\varepsilon - 3\sqrt{\varepsilon} - 18 = 0\Bigg\} \tag{4.8}$$

form a global algebraic Poincaré–Bendixson annulus $\hat{\mathcal{A}}_\varepsilon$ containing the unique limit cycle $\hat{\Gamma}_\varepsilon$ of system (4.2).

Figure 4.2 shows the Poincaré–Bendixson annulus $\hat{\mathcal{A}}_\varepsilon$ with the limit cycle $\hat{\Gamma}_\varepsilon$ of system (4.6) for $\varepsilon = 0.1$ and $\varepsilon = 2$.

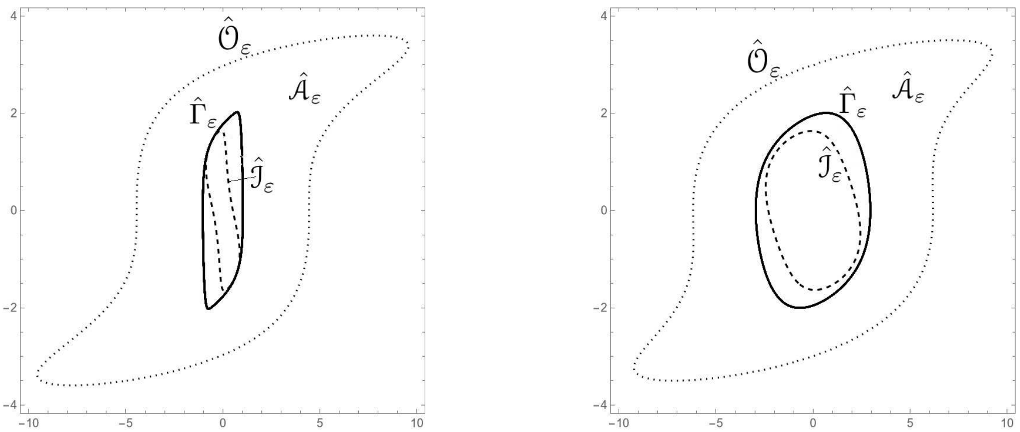

Figure 4.2: Annulus $\hat{\mathcal{A}}_\varepsilon$ with limit cycle $\hat{\Gamma}_\varepsilon$ of system (4.6) for $\varepsilon = 0.1$ (left) and $\varepsilon = 2$ (right).

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Global bifurcation of positive solutions for a class of superlinear elliptic systems

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Abstract. We are concerned with the global bifurcation of positive solutions for semilinear elliptic systems of the form

$$\begin{cases} -\Delta u = \lambda f(u,v) & \text{in } \Omega, \\ -\Delta v = \lambda g(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter, $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with smooth boundary $\partial\Omega$. We establish the existence of an unbounded branch of positive solutions, emanating from the origin, which is bounded in positive λ-direction. The nonlinearities $f,\ g \in C^1(\mathbb{R} \times \mathbb{R}, (0,\infty))$ are nondecreasing for each variable and have superlinear growth at infinity. The proof of our main result is based upon bifurcation theory. In addition, as an application for our main result, when f and g subject to the upper growth bound, by a technique of taking superior limit for components, then we may show that the branch must bifurcate from infinity at $\lambda = 0$.

Keywords: elliptic systems, positive solutions, superlinear growth, bifurcation.

2020 Mathematics Subject Classification: 35B09, 35B32, 35J57.

1 Introduction

Let B be the unit ball in $\mathbb{R}^N$. D. D. Joseph and T. S. Lundgren [12] considered

$$\begin{cases} -\Delta u = \lambda e^u, & x \in B, \\ u = 0, & x \in \partial B \end{cases} \tag{1.1}$$

and found a very interesting phenomenon that the behaviour of the connected component of positive solutions of (1.1) heavily depends on the dimension N, see Figure 1.1 below.

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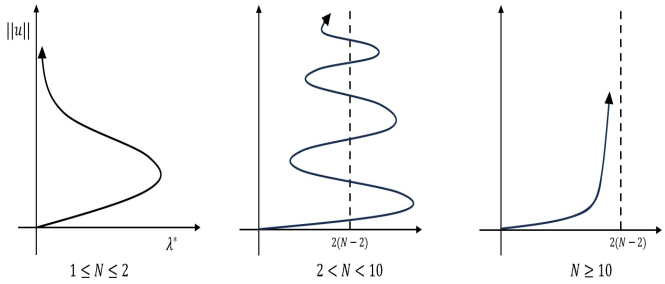

Figure 1.1: Global continua for (1.1) depend on N

Fourth order analogue of (1.1), a biharmonic elliptic problem

$$\begin{cases} \Delta^2 u = \lambda e^u, & x \in B, \\ u = |\nabla u| = 0, & x \in \partial B \end{cases} \tag{1.2}$$

has been extensively studied by several authors, see G. Arioli, F. Gazzola, H.-C. Grunau, E. Mitidieri [2] and A. Ferrero, H.-C. Grunau [8] and the references therein.

For elliptic systems, Ph. Clément, D. G. de Figueiredo and E. Mitidieri [7] investigated the existence of positive solution of a Dirichlet problem for

$$-\Delta u = f(v), \qquad -\Delta v = f(u) \tag{1.3}$$

in a bounded convex domain Ω of $\mathbb{R}^N$ with smooth boundary. Furthermore, the authors considered the existence of nontrivial solutions for the biharmonic equation subject to Navier boundary conditions. Namely

$$\Delta^2 u = g(u) \quad \text{in } \Omega, \qquad u = \Delta u = 0 \quad \text{on } \partial\Omega.$$

This problem is a special case of (1.3) when $f(v) = v$.

Recently, M. Chhetri, P. Girg [6] considered the elliptic system

$$\begin{cases} -\Delta u = \lambda \hat{f}(v) & \text{in } \Omega, \\ -\Delta v = \lambda \hat{g}(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter and $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with $C^{2,\eta}$-boundary $\partial\Omega$ for some $\eta \in (0,1)$. The nonlinearities $\hat{f}$, $\hat{g} : \mathbb{R} \to (0, +\infty)$ are nondecreasing continuous functions and have superlinear growth at infinity, i.e.

$$\lim_{s \to +\infty} \frac{\hat{f}(s)}{s} = +\infty = \lim_{s \to +\infty} \frac{\hat{g}(s)}{s}.$$

Then the authors established the global structure of positive solutions for system (1.4).

Of course the natural question is whether or not we may show the global structure of positive solutions for the more general system

$$\begin{cases} -\Delta u = \lambda f(u,v) & \text{in } \Omega, \\ -\Delta v = \lambda g(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter. We make the following assumptions throughout the paper.

(H1) $f,\ g \in C^1(\mathbb{R}\times\mathbb{R},(0,+\infty))$ are nondecreasing for each variable and there exists a $\tau > 0$, satisfy

$$\min\left\{\frac{\partial f}{\partial s}(s,t),\frac{\partial f}{\partial t}(s,t)\right\} > \frac{f(s,t)}{s+t} \quad \text{for } (s,t)\in\mathbb{R}^2\backslash B_\tau,$$

$$\min\left\{\frac{\partial g}{\partial s}(s,t),\frac{\partial g}{\partial t}(s,t)\right\} > \frac{g(s,t)}{s+t} \quad \text{for } (s,t)\in\mathbb{R}^2\backslash B_\tau,$$

where $B_\tau := \{(s,t)\in\mathbb{R}^2 : |s|^2+|t|^2 \le \tau\}$;

(H2)
$$\lim_{s+t\to+\infty}\frac{f(s,t)}{s+t} = \lim_{s+t\to+\infty}\frac{g(s,t)}{s+t} = +\infty.$$

Notice that the functions satisfying (H1)–(H2) are easy to illustrate, for example $f(s,t) = (s+t)^5+1$, $g(s,t) = (s+t)^3+2$.

For system of equations with $\lambda = 1$, see [9–11] for $N = 2$ and [1,3,4] for $N \ge 3$, where existence results were discussed, but no any information about the *Connectivity Properties* of positive solution set are provided. In fact, these positive solutions of (1.5) may not lie on one bifurcating set.

It is the purpose of this paper to show the existence of a unbounded component of positive solutions of (1.5) by use bifurcation theory and a technique of taking *superior limit for components*, see [15,16].

In order to better state our result, we will briefly introduce the following notations that were defined in more detail in [6] and extend the use of these definitions to all systems throughout the paper.

Let

$$E := [W_0^{1,2}(\Omega)\cap W^{2,r}(\Omega)]^2 \quad \text{and} \quad X := [L^r(\Omega)]^2$$

be Banach spaces endowed with norms

$$\|(w_1,w_2)\|_E := \|w_1\|_{W^{2,r}(\Omega)} + \|w_2\|_{W^{2,r}(\Omega)} \quad \text{and} \quad \|(w_1,w_2)\|_X := \|w_1\|_{L^r(\Omega)} + \|w_2\|_{L^r(\Omega)},$$

respectively for $r > N$. By assumption $r > N$, $W^{2,r}(\Omega)$ is continuously imbedded into $C^{1,\eta}(\overline{\Omega})$ for some $\eta \in (0,1)$. Thus there exists $\xi_* > 0$ such that $\|\omega\|_{L^\infty(\Omega)} \le \xi_*\|\omega\|_{W^{2,r}(\Omega)}$ holds for all $\omega \in W^{2,r}(\Omega)$. By a solution of (1.5) we mean $(\lambda,(u,v)) \in \mathbb{R}\times E$ which solves (1.5) in the strong sense, that is, $(u,v) \in W^{2,r}(\Omega)\times W^{2,r}(\Omega)$ and $(\lambda,(u,v))$ satisfies (1.5) almost everywhere in Ω. Now define $\mathcal{S} := \{(\lambda,(u,v)) \in \mathbb{R}\times E : (\lambda,(u,v)) \text{ solution of (1.5)}\}$.

Definition 1.1 ([6])**.**

(1) By a *continuum* of solutions of (1.5) we mean a subset of $\mathcal{S}$ which is closed and connected.

(2) By a *component* of solutions set $\mathcal{S}$ we mean a continuum which is maximal with respect to inclusion ordering.

(3) $\lambda_\infty \in \mathbb{R}$ is *a bifurcation point from infinity* if the solution set $\mathcal{S}$ contains a sequence $(\lambda_n,(u_n,v_n))$ such that $\lambda_n \to \lambda_\infty$ and $\|(u_n,v_n)\|_E \to +\infty$ as $n \to +\infty$.

The main result of the paper is the following.

Theorem 1.2. *Let (H1)–(H2) hold, then there exists an unbounded component $\mathcal{C} \subset \mathcal{S}$ satisfying the following:*

(a) For any $\lambda \in (0, \lambda^*)$, $(\lambda, (u, v)) \in \mathcal{C}$ is positive, i.e. $u > 0$ and $v > 0$.

(b) If $\lambda = 0$, then $(0, (0, 0))$ is the unique element belonging to $\mathcal{C}$.

(c) $\mathrm{Proj}_{\lambda \in [0,+\infty)} \mathcal{C} := \{\lambda \in [0, +\infty) : \exists\, (u, v) \in E$ with $(\lambda, (u, v)) \in \mathcal{C}\} \subset [0, \lambda^*)$.

(d) There exists a sequence of positive solutions $\{(\lambda_n, (u_n, v_n))\} \in \mathcal{C}$ satisfying $\lambda_n \in (0, \lambda^*)$ for all $n \in \mathbb{N}$ and $\lim_{n \to +\infty} \|(u_n, v_n)\|_E \to +\infty$.

Remark 1.3. In the special case Ω is convex, M. Chhetri, P. Girg [6] show that the only bifurcation point of positive solutions of (1.4) from infinity with $\hat{f}(v) = v^p$, $\hat{g}(u) = u^q$ at $\lambda = 0$ under the critical hyperbola condition

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}. \tag{1.6}$$

The proof of this result in [6] is deeply depend on the uniform priori bound. For system (1.4), condition (1.6) is optimal for obtaining the priori estimate when $\hat{f}(v) = v^p$, $\hat{g}(u) = u^q$. But for the more general system (1.5), the conditions for obtaining a priori bound are more complicated, and the proof will be more difficult if we want to obtain the similar results. We give the specific proof in Section 5.

The rest of paper is arranged as follows: In Section 2 we present the nonexistence result of (1.5). Section 3 is devoted to asymptotically positively homogeneous system by using a global continuation principle. In Section 4, we prove Theorem 1.2. In final section, as an application of Theorem 1.2, by applying some priori estimates, see [17], we attempt to understand the structure of the resulting continua of positive solutions.

2 Statement of the nonexistence result

Let $\mu_1 > 0$ be the principal eigenvalue of

$$\begin{cases} -\Delta\varphi = \mu\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\varphi_1 \in W_0^{1,2}(\Omega)$ be the corresponding eigenfunction, then $\frac{\partial \varphi_1}{\partial \vec{n}} < 0$ on Ω, where $\vec{n}$ is the outward unit normal on $\partial\Omega$. Without loss of generality, we normalize the eigenfunction such that $\varphi_1 > 0$ in Ω.

We shall prove the nonexistence result.

Theorem 2.1. *Suppose there exist a_1, a_2, α_1, $\alpha_2 > 0$ such that*

$$f(s,t) > a_1(s+t) + \alpha_1, \qquad g(s,t) > a_2(s+t) + \alpha_2, \qquad \forall\, (s,t) \in \mathbb{R}^2. \tag{2.1}$$

Then for $\lambda \geq \lambda^* := \frac{\mu_1}{2a^*}$, there are no solutions for (1.5), where $a^* := \min\{a_1, a_2\}$.

Remark 2.2. We notice that in order to obtain the nonexistence result of solutions for (1.5), it suffices to show that f and g satisfy (2.1), which is weaker than (H2).

Proof. By (H1), since $f,\ g$ are positive functions, all solutions $(\lambda,(u,v))$ of (1.5) with $\lambda > 0$ must satisfy $u,v > 0$ in Ω by the maximum principle. Let $(\lambda,(u,v))$ be a solution of (1.5) with $\lambda > 0$. Then

$$\begin{aligned} -\Delta(u+v) &> \lambda(a_1(u+v)+\alpha_1)+\lambda(a_2(u+v)+\alpha_2) \\ &> 2\lambda a^*(u+v)+2\lambda\alpha^* = 2\lambda a^*\Big(u+v+\frac{\alpha^*}{a^*}\Big) \quad \text{in } \Omega, \end{aligned}$$

where $a^* := \min\{a_1,a_2\}$, $\alpha^* := \min\{\alpha_1,\alpha_2\}$. Therefore, we have

$$-\Delta\Big(u+v+\frac{\alpha^*}{a^*}\Big) > 2\lambda a^*\Big(u+v+\frac{\alpha^*}{a^*}\Big) \quad \text{in } \Omega.$$

Denoting $w := u+v+\frac{\alpha^*}{a^*}$, we see that $w>0$ on $\overline{\Omega}$ and

$$-\Delta w > 2\lambda a^* w \quad \text{in } \Omega.$$

Since $-\Delta\varphi_1 = \mu_1\varphi_1$ in $\Omega,\ \varphi_1 = 0$ on $\partial\Omega$. We have

$$\int_\Omega(\varphi_1\Delta w - w\Delta\varphi_1)dx < \int_\Omega(-2\lambda a^* w\varphi_1 + w\mu_1\varphi_1)dx = (-2\lambda a^*+\mu_1)\int_\Omega(w\varphi_1)dx. \tag{2.2}$$

On the other hand, since $\varphi_1 = 0$ on $\partial\Omega$, $\inf_{\partial\Omega} w > 0$ and $\frac{\partial\varphi_1}{\partial\vec{n}} < 0$ on $\partial\Omega$, we get

$$\int_\Omega(\varphi_1\Delta w - w\Delta\varphi_1)dx = \int_{\partial\Omega}(\varphi_1\nabla w - w\nabla\varphi_1)\cdot\vec{n}\,dS = -\int_{\partial\Omega} w\nabla\varphi_1\cdot\vec{n}\,dS. \tag{2.3}$$

Then we have

$$-\int_{\partial\Omega} w\nabla\varphi_1\cdot\vec{n}\,dS \geq -\inf_{\partial\Omega} w\int_{\partial\Omega}\frac{\partial\varphi_1}{\partial\vec{n}}dS > 0. \tag{2.4}$$

It follows from (2.2), (2.3) and (2.4) that for (u,v) to be a solution of (1.5) for $\lambda > 0$, we must have $\lambda < \frac{\mu_1}{2a^*}$. Therefore, (1.5) has no solution for $\lambda \geq \lambda^* := \frac{\mu_1}{2a^*}$. □

3 Asymptotically positively homogeneous system

In order to discuss the auxiliary result, we mention some properties of the following eigenvalue problem

$$\begin{cases} -\Delta w_1 = \lambda[a_{11}(x)w_1 + a_{12}(x)w_2] & \text{in } \Omega, \\ -\Delta w_2 = \lambda[a_{21}(x)w_1 + a_{22}(x)w_2] & \text{in } \Omega, \\ w_1 = w_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $a_{ij} : \overline{\Omega} \to (0,\infty)$ are continuous function($i,\ j = 1,\ 2$). It follows from [14, Theorem 4.1] that (3.1) has exactly one positive principal eigenvalue λ_1 and associated eigenfunction (χ_1,ψ_1) is positive in Ω.

Remark 3.1. If a_{ij} = constant ($i,\ j = 1,\ 2$), the principal eigenvalue of (3.1) and the corresponding eigenfunction are related to μ_1 and φ_1 (μ_1 and φ_1 are defined in the Section 2.)

For example, let $a_{11} = 2,\ a_{12} = 9,\ a_{21} = 4,\ a_{22} = 2$, the principal eigenvalue of (3.1) is $\frac{\mu_1}{8}$ and the associated eigenfunction is $(\frac{3}{2}\varphi_1,\varphi_1)$. The detailed calculation method is shown in Appendix 2 of [5].

Now, let us consider an asymptotically positively homogeneous system

$$\begin{cases} -\Delta u = \lambda[a_{11}(x)u^+ + a_{12}(x)v^+] + \lambda\widetilde{f}(u,v) & \text{in } \Omega, \\ -\Delta v = \lambda[a_{21}(x)u^+ + a_{22}(x)v^+] + \lambda\widetilde{g}(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $s^+ := \max\{s, 0\}$, $a_{ij}(i,\ j = 1,\ 2)$ are as in (3.1) and $\lambda \in \mathbb{R}$ is the bifurcation parameter. $\widetilde{f},\ \widetilde{g} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

(F1) $\widetilde{f},\ \widetilde{g}$ are continuous and bounded functions;

(F2) $a_{11}s^+ + a_{12}t^+ + \widetilde{f}(s,t) > 0,\ a_{21}s^+ + a_{22}t^+ + \widetilde{g}(s,t) > 0$ for all $(s,t) \in \mathbb{R} \times \mathbb{R}$.

By a solution of (3.2) we mean $(\lambda, (u, v)) \in \mathbb{R} \times E$ which solves (3.2) in the strong sense. Now let $\mathcal{T} := \{(\lambda, (u, v)) : (\lambda, (u, v))$ solution of (3.2)$\}$. We shall prove the following bifurcation result.

Theorem 3.2. *Let (F1)–(F2) hold. Then λ_1 is the only bifurcation point from infinity for (3.2). Moreover, there exists a component $\mathcal{X} \subset \mathcal{T}$ bifurcating from infinity at λ_1 and satisfies:*

(i) for $\lambda > 0$ and $(\lambda, (u, v)) \in \mathcal{X}$, then $u > 0$ and $v > 0$;

(ii) for $\lambda = 0$, $(u, v) = (0, 0)$ is the unique solution of (3.2) and $(0, (0, 0)) \in \mathcal{X}$;

(iii) $\mathrm{Proj}_\lambda\, \mathcal{X} := \{\lambda \in \mathbb{R} : \exists\, (u, v) \in E$ with $(\lambda, (u, v)) \in \mathcal{X}\}$ is bounded from above and unbounded from below.

To prove this theorem, we use a variant of Krasnoselskii's necessary condition for bifurcation from infinity (Lemma 3.4), Theorem 2.1 and the global continuation principle of Leray and Schauder (Lemma 3.3) below.

Lemma 3.3 ([19]). *Let Y be a Banach space with $Y \neq \{0\}$ and let $F : Y \to Y$ be compact. Then the solution component $\hat{\mathcal{C}} \subset \mathbb{R} \times Y$ of the equation*

$$x = \lambda F(x)$$

which contains $(0, 0) \in \mathbb{R} \times Y$ is unbounded as are both subsets

$$\hat{\mathcal{C}}_\pm := \hat{\mathcal{C}} \cap (\mathbb{R}_\pm \times Y),$$

where $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0]$.

System (3.2) is equivalent to

$$(u, v) = \lambda L^+(u, v) + \lambda H(u, v), \tag{3.3}$$

where $L^+ : E \to E$ is defined by

$$(u, v) \mapsto (-\Delta)^{-1}(a_{11}(x)u^+ + a_{12}(x)v^+, a_{21}(x)u^+ + a_{22}(x)v^+)$$

and $H : E \to E$ is defined by

$$(u, v) \mapsto (-\Delta)^{-1}(\widetilde{f}(u, v), \widetilde{g}(u, v)).$$

Notice that L^+ is not linear but both L^+ and H are continuous and compact. Moreover, since $\widetilde{f}$ and $\widetilde{g}$ are bounded, H satisfies

$$\lim_{\|(u,v)\|_E\to+\infty}\frac{\|H(u,v)\|_E}{\|(u,v)\|_E}=0. \tag{3.4}$$

For asymptotically linear problem, a necessary condition for bifurcation from infinity was established in [13]. Inspired by this work, we prove the following lemma to show that the unique possible bifurcation point from infinity for (3.3) is λ_1.

Lemma 3.4. *If λ_∞ is a bifurcation point from infinity for (3.3), then $\lambda_\infty=\lambda_1$. Moreover, for any sequence $(\lambda_j,(u_j,v_j))\in\mathbb{R}\times E$ with $\lambda_j\to\lambda_1$ and $\|(u_j,v_j)\|_E\to+\infty$ as $j\to+\infty$. There exists a subsequence $(\lambda_{j_k},(u_{j_k},v_{j_k}))$ of $(\lambda_j,(u_j,v_j))$ such that*

$$\lim_{j_k\to+\infty}\frac{(u_{j_k},v_{j_k})}{\|(u_{j_k},v_{j_k})\|_E}=\frac{(\chi_1,\psi_1)}{\|(\chi_1,\psi_1)\|_E}, \tag{3.5}$$

where the convergence is in $C^{1,\eta}(\overline{\Omega})\times C^{1,\eta}(\overline{\Omega})$ for some $\eta\in(0,1)$.

Proof. Now by the same argument in the proof of [6, Proposition 3.1], with obvious changes, we may deduce the desired results. Let $(\lambda_j,(u_j,v_j))\in\mathbb{R}\times E$ be solutions of (3.2) such that $\|(u_j,v_j)\|_E\to+\infty$ and $\lambda_j\to\lambda_\infty$. Then $(\widehat{u}_j,\widehat{v}_j)=\frac{(u_j,v_j)}{\|(u_j,v_j)\|_E}$ satisfies

$$\widehat{u}_j=\lambda_j(-\Delta)^{-1}\left(a_{11}(x)\widehat{u}_j^+ + a_{12}(x)\widehat{v}_j^+ + \frac{\widetilde{f}(u_j,v_j)}{\|(u_j,v_j)\|_E}\right),$$
$$\widehat{v}_j=\lambda_j(-\Delta)^{-1}\left(a_{21}(x)\widehat{u}_j^+ + a_{22}(x)\widehat{v}_j^+ + \frac{\widetilde{g}(u_j,v_j)}{\|(u_j,v_j)\|_E}\right),$$

or equivalently satisfied

$$(\widehat{u}_j,\widehat{v}_j)=\lambda_jL^+(\widehat{u}_j,\widehat{v}_j)+\lambda_j\frac{H(u_j,v_j)}{\|(u_j,v_j)\|_E}.$$

It then follows from (3.4) that the right hand side is bounded in X (independent of j). Hence $\|\widehat{u}_j\|_{W^{2,r}(\Omega)}$ and $\|\widehat{v}_j\|_{W^{2,r}(\Omega)}$ are bounded (independent of j) and so are $\|\widehat{u}_j\|_{C^{1,\eta}(\overline{\Omega})}$ and $\|\widehat{v}_j\|_{C^{1,\eta}(\overline{\Omega})}$ for some $\eta\in(0,1)$. Since $C^{1,\eta'}(\overline{\Omega})\hookrightarrow C^{1,\eta}(\overline{\Omega})$ compactly for $\eta'\in(0,\eta)$, passing to subsequences, $\widehat{u}_j\to\widehat{u}$, $\widehat{v}_j\to\widehat{v}$ in $C^{1,\eta'}(\overline{\Omega})$. Therefore, $(\lambda_\infty,(\widehat{u},\widehat{v}))$ satisfies

$$-\Delta\widehat{u}=\lambda_\infty[a_{11}(x)\widehat{u}^+ + a_{12}(x)\widehat{v}^+]\quad\text{in }\Omega, \tag{3.6}$$

$$-\Delta\widehat{v}=\lambda_\infty[a_{21}(x)\widehat{u}^+ + a_{22}(x)\widehat{v}^+]\quad\text{in }\Omega, \tag{3.7}$$
$$\widehat{u}=\widehat{v}=0\quad\text{on }\partial\Omega.$$

Suppose $\lambda_\infty\le 0$. Since $\widehat{u}^+\ge 0$ and $\widehat{v}^+\ge 0$, it follows by applying the maximum principle to (3.6) that $\widehat{u}\equiv 0$ and hence repeating the same argument using (3.7) we get $\widehat{v}\equiv 0$ as well. This leads to a contradiction since $\|(\widehat{u},\widehat{v})\|_E=1$.

For $\lambda_\infty>0$, we distinguish two cases: $\widehat{v}^+\equiv 0$ and $\widehat{v}^+\not\equiv 0$. In the first case, if $\widehat{u}^+\equiv 0$, from (3.6), using the maximum principle, a contradiction as before. If $\widehat{u}^+\not\equiv 0$, then it follows from (3.7) and $a_{21}>0$ that $\widehat{v}>0$ in Ω. However, this contradicts $\widehat{v}^+\equiv 0$.

In the case $\widehat{v}^+ \not\equiv 0$, we may get $\widehat{u}^+ \not\equiv 0$ in Ω from (3.6) by the maximum principle, which in turn implies $\widehat{u} > 0$ and $\widehat{v} > 0$ in Ω from (3.6) and (3.7) by the maximum principle again. Thus $\lambda_\infty > 0$ and $\widehat{u}$, $\widehat{v} > 0$ in Ω satisfy the linear eigenvalue problem (3.1). However, we already discussed that (3.1) has precisely one eigenvalue λ_1 with componentwise positive eigenfunction (χ_1, ψ_1). Therefore, it must be that $\lambda_\infty = \lambda_1$ and

$$(\widehat{u}, \widehat{v}) = \frac{(\chi_1, \psi_1)}{\|(\chi_1, \psi_1)\|_E}. \qquad \square$$

Now we will complete the proof of Theorem 3.2.

Proof. (3.3) satisfies the hypotheses of Lemma 3.3 with $F := L^+ + H$. Then there exist unbounded continua

$$\mathcal{X}_\pm \subset \widehat{\mathcal{T}} := \{(\lambda, (u, v)) \in \mathbb{R} \times E : (\lambda, (u, v)) \text{ is a solution of (3.2)}\}$$

containing $(0, (0, 0))$. By Theorem 2.1,

$$\mathcal{X}_+ \subset ([0, \lambda^*) \times E)$$

and thus $\mathcal{X}_+$ must be unbounded in the Banach space E-direction. Then $\mathcal{X} := \mathcal{X}_+ \cup \mathcal{X}_-$ is a component containing $(0, (0, 0))$. By Lemma 3.4, λ_1 is the only bifurcation point from infinity from (3.3) and $\mathcal{X}_+$ is unbounded in the E-direction, hence $\mathcal{X}_+$ must bifurcate from infinity at λ_1. By similar argument in [6], we will verify that $\mathcal{X}$ satisfies the properties (i)-(iii).

It follows from assumption (F2) that u, $v > 0$ in Ω whenever $(\lambda, (u, v)) \in \mathcal{X}$ and $\lambda > 0$. This implies part (i). For $\lambda = 0$, $(u, v) = (0, 0)$ is the only solution of (3.2) and $(0, (0, 0)) \in \mathcal{X}$. Hence part (ii) holds. Applying Lemma 3.3, we see that $\mathcal{X}_-$ must be unbounded in $\mathbb{R} \times E$. However, by part (ii) and the fact that λ_1 is the unique bifurcation point from infinity for (3.3), we see that $\mathcal{X}_-$ must be unbounded in the negative λ-direction. Hence $(-\infty, \lambda_1) \subset \mathrm{Proj}_\lambda \mathcal{X}$. $\square$

4 Proof of main result

For $n \in \mathbb{N}$, let

$$\mathcal{A}_n^f := \{(s, t) \in \mathbb{R}_+^2 : f(s, t) = n(s + t)\},$$

where $\mathbb{R}_+ := [0, +\infty)$. We shall show that $\mathcal{A}_n^f$ contains a curve Γ_n^f which can be (globally) parametrized as the graph of a decreasing function.

First, we state and prove several preliminary results.

Lemma 4.1. *There exists $n_0 > 0$ such that for any $n > n_0$, $\mathcal{A}_n^f \neq \emptyset$.*

Proof. It follows from (H1) and (H2) that there exists $n_0 > 0$ such that for any $n > n_0$, there exists $(s_n^*, s_n^*) \in \mathbb{R}_+^2$ with $f(s_n^*, s_n^*) = 2ns_n^*$. Consequently, $\mathcal{A}_n^f \neq \emptyset$. $\square$

For given $\theta \in [0, +\infty)$, let

$$t = \theta s, \quad s \in [0, +\infty).$$

Obviously, (H1) and (H2) imply that $\lim\limits_{s \to \infty} \frac{f(s, \theta s)}{s + \theta s} = \infty$. For any $n > n_0$, denote

$$\mathcal{A}_{n,\theta}^f := \Big\{(s, \theta s) : \ (s, \theta s) \in \mathcal{A}_n^f\Big\}.$$

Fix $n > n_0$, analogous to proof of Lemma 4.1, it is easy to see $\mathcal{A}_{n,\theta}^f \neq \emptyset$.

Lemma 4.2. *Fix $n > n_0$, there exists $M_n > 0$ independent of $\theta \in [0, +\infty)$ such that*

$$\sup\left\{s + \theta s : (s, \theta s) \in \mathcal{A}_n^f\right\} \leq M_n.$$

Proof. Suppose on the contrary that there exists a sequence $\{(s_k, \theta_k s_k)\} \in \mathcal{A}_n^f$ such that $\lim_{k\to\infty}(s_k + \theta_k s_k) = \infty$. Then it follows from (H2) that

$$\lim_{(s_k+\theta_k s_k)\to+\infty} \frac{f(s_k, \theta_k s_k)}{s_k + \theta_k s_k} = +\infty.$$

This contradicts the fact that $f(s_k, \theta_k s_k) = n(s_k + \theta_k s_k)$. □

Fix $\theta \in [0, +\infty)$, define

$$\gamma_n(\theta) := \max\left\{s \in \mathbb{R}_+ : (s, \theta s) \in \mathcal{A}_{n,\theta}^f\right\}.$$

Lemma 4.3. *For any $M > 0$ and $\theta \in [0, +\infty)$, there exists $n_1 > n_0 > 0$ such that $(\gamma_n(\theta))^2 + (\theta\gamma_n(\theta))^2 > M$ for any $n > n_1$.*

Proof. Suppose on the contrary that there exists a sequence $\{(\theta_n, \gamma_n(\theta_n))\}$ such that $(\gamma_n(\theta_n))^2 + (\theta_n\gamma_n(\theta_n))^2$ is bounded for any $n > n_1$. After taking a subsequence if necessary, we have

$$(\theta_n, \gamma_n(\theta_n)) \to (\theta_*, \gamma_*) \quad \text{as } n \to \infty$$

in $\mathbb{R}_+^2$. Since $(\gamma_n(\theta_n), \theta_n\gamma_n(\theta_n)) \in \mathcal{A}_n^f$, then

$$f(\gamma_n(\theta_n), \theta_n\gamma_n(\theta_n)) = n(\gamma_n(\theta_n) + \theta_n\gamma_n(\theta_n)).$$

It is easy to verify that

$$f(\gamma_*, \theta_*\gamma_*) = n(\gamma_* + \theta_*\gamma_*) \to \infty \quad \text{as } n \to \infty,$$

this contradicts the fact that $f(\gamma_*, \theta_*\gamma_*)$ is bounded since $\theta_* \in [0, +\infty)$. □

For $(s, t) \in \mathbb{R}_+^2$, denote

$$F(s, t) := f(s, t) - n(s + t).$$

Let

$$\widetilde{s} := \sup\{s \in \mathbb{R}_+ : (s, 0) \in \mathcal{A}_{n,0}^f\}.$$

Then Lemma 4.2 implies $\widetilde{s} < \infty$.

Lemma 4.4. *There exists $n_2 > n_1 > 0$ such that for $n > n_2$, there exists a decreasing function $t = \Gamma_n^f(s)$ for $s \in (0, \widetilde{s})$, which joins the point $(\widetilde{s}, 0)$ to a point $(0, \widehat{t})$ for some $\widehat{t} < \infty$.*

Proof. For given $\theta \in (0, +\infty)$, we know that $(\gamma_n(\theta), \theta\gamma_n(\theta)) \in \mathcal{A}_n^f$. By (H1), we have

$$\min\left\{f_t(\gamma_n(\theta), \theta\gamma_n(\theta)), f_s(\gamma_n(\theta), \theta\gamma_n(\theta))\right\} > \frac{f(\gamma_n(\theta), \theta\gamma_n(\theta))}{\gamma_n(\theta) + \theta\gamma_n(\theta)}$$

if n sufficiently large. Therefore,

$$\begin{aligned}F_t(\gamma_n(\theta),\theta\gamma_n(\theta)) &= f_t(\gamma_n(\theta),\theta\gamma_n(\theta)) - n\\ &> \frac{f(\gamma_n(\theta),\theta\gamma_n(\theta))}{\gamma_n(\theta)+\theta\gamma_n(\theta)} - n\\ &= \frac{n(\gamma_n(\theta)+\theta\gamma_n(\theta))}{\gamma_n(\theta)+\theta\gamma_n(\theta)} - n\\ &= 0\end{aligned}$$

for sufficiently large n. By similar argument, we can obtain $F_s(\gamma_n(\theta),\theta\gamma_n(\theta)) > 0$ for sufficiently large n. Consequently, applying the implicit function existence theorem, there exists a unique curve $t = \Gamma_n^f(s)$ in $(\gamma_n(\theta)-\delta, \gamma_n(\theta)+\delta)$ for sufficiently small $\delta > 0$, and

$$(\Gamma_n^f(s))' = -\frac{F_s(s,t)}{F_t(s,t)} = -\frac{f_s(s,t)-n}{f_t(s,t)-n} < 0$$

for sufficiently large n. Thus there exists $n_2 > n_1 > 0$ such that for $n > n_2$, $\Gamma_n^f(s)$ is a decreasing function for $s \in (\gamma_n(\theta)-\delta, \gamma_n(\theta)+\delta)$. By the standard extension method, we may get a decreasing function $\Gamma_n^f(\cdot)$ defined on $(0,\widetilde{s})$.

By Lemma 4.2, we have $\{(\gamma_n(\theta),\theta\gamma_n(\theta)) : \theta \in (0,+\infty)\} \subset \mathcal{A}_n^f$ is bounded. This together with the fact that $\Gamma_n^f(s)$ is decreasing, we can deduce that $\lim_{\theta\to 0+}(\gamma_n(\theta),\theta\gamma_n(\theta)) = (\widehat{s},0)$ for $\widehat{s} \in (0,+\infty)$ and $\lim_{\theta\to+\infty}(\gamma_n(\theta),\theta\gamma_n(\theta)) = (0,\widehat{t})$ for some $\widehat{t} < \infty$. Obviously, we have $\widehat{s} = \widetilde{s}$ by the definition of $\widetilde{s}$. □

It is easy to see that there exists n_*, n^* such that for all $n \geq n_*$, Γ_n^f divide $\mathbb{R}_+^2$ into two parts

$$\mathbb{R}_+^2 = \Omega_n^f \cup \Gamma_n^f \cup U_n^f, \quad \Omega_n^f \cap U_n^f = \emptyset,$$

and for all $n \geq n^*$, Γ_n^g divide $\mathbb{R}_+^2$ into two parts

$$\mathbb{R}_+^2 = \Omega_n^g \cup \Gamma_n^g \cup U_n^g, \quad \Omega_n^g \cap U_n^g = \emptyset,$$

where Ω_n^f, Ω_n^g are bounded, and U_n^f, U_n^g are unbounded.

4.1 Approximation problems

Fix $n \in \mathbb{N}$ and define $f_n(t,s)$, $g_n(t,s) : \mathbb{R}^2 \to (0,\infty)$ by

$$f_n(s,t) = \begin{cases} f(s,t), & (s,t) \in \Omega_n^f,\\ n(s+t), & (s,t) \in U_n^f \cup \Gamma_n^f,\end{cases}$$

$$g_n(s,t) = \begin{cases} g(s,t), & (s,t) \in \Omega_n^g,\\ n(s+t), & (s,t) \in U_n^g \cup \Gamma_n^g.\end{cases}$$

Then f_n and g_n are continuous functions on $\mathbb{R}^2$.

For each $n \in \mathbb{N}$, we consider the following problem

$$\begin{cases} -\Delta u = \lambda f_n(u,v) & \text{in } \Omega,\\ -\Delta v = \lambda g_n(u,v) & \text{in } \Omega,\\ u = v = 0 & \text{on } \partial\Omega,\end{cases} \tag{4.1}$$

which approaches (1.5) as $n \to \infty$. We will use Theorem 3.2 to treat (4.1) and thus we rewrite (4.1) in the form of (3.2) as

$$\begin{cases} -\Delta u = \lambda[nu^+ + nv^+] + \lambda \widetilde{f}_n(u,v) & \text{in } \Omega, \\ -\Delta v = \lambda[nu^+ + nv^+] + \lambda \widetilde{g}_n(u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where

$$\widetilde{f}_n(s,t) := f_n(s,t) - ns^+ - nt^+, \qquad \widetilde{g}_n(s,t) := g_n(s,t) - ns^+ - nt^+.$$

We note that $\widetilde{f}_n$ and $\widetilde{g}_n$ are bounded in $\mathbb{R}^2$. Indeed, since f_n is nondecreasing for each variable and $f_n(s,t) = f(s,t) > 0$ for $(s,t) \in \overline{\Omega}_n^f$, we get

$$|\widetilde{f}_n(s,t)| \le \sup_{(s,t)\in\mathbb{R}^2} |f_n(s,t) - n(s^+ + t^+)| \le \sup_{(s,t)\in\Omega_n^f} |f_n(s,t) - n(s^+ + t^+)| + f(0,0) = \text{constant},$$

where the constant is independent of s, t and depends on n. We can repeat the same argument for $\widetilde{g}_n$. Since $f_n(s,t)$, $g_n(s,t) > 0$, it is easy to see that (4.2) satisfies the hypotheses of Theorem 3.2 with $a_{11} = n$, $a_{12} = n$, $a_{21} = n$, $a_{22} = n$, $\widetilde{f} = \widetilde{f}_n$, $\widetilde{g} = \widetilde{g}_n$, and $\lambda_1 = \lambda_{1,A}$, where

$$A = \begin{pmatrix} n & n \\ n & n \end{pmatrix}.$$

Then by Theorem 3.2, $\lambda_{1,A}$ is the unique bifurcation point from infinity for (4.2) and there exists a component $\mathcal{C}_n$ of positive solutions of (4.2) bifurcating from infinity at $\lambda_{1,A}$ satisfying the properties (i)–(iii) of Theorem 3.2. In particular, $(0,(0,0)) \in \mathcal{C}_n$, $\mathcal{C}_n$ is bounded above by $\lambda^* \times E$ (λ^* is as in Theorem 2.1) and $\mathcal{C}_n$ does not cross $\{0\} \times \mathbb{E}$ except through the point $(0,(0,0))$.

4.2 Passing to the limit

We first state some properties of the superior limit of a certain infinity collection of connected sets.

Definition 4.5 ([18]). Let X be a Banach space and $\{C_n : n = 1,2,\dots\}$ be a certain infinite collection of subsets of X. Then the superior limit $\mathcal{D}$ of $\{C_n\}$ is defined by

$$\mathcal{D} := \limsup_{n\to\infty} C_n = \{x \in X : \exists\{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \to x\}.$$

Lemma 4.6 ([15]). *Let X be a Banach space and let $\{C_n\}$ be a family of closed connected subsets of X. Assume that:*

(i) there exist $z_n \in C_n$, $n = 1,2,\dots$, and $z^* \in X$, such that $z_n \to z^*$;

(ii) $r_n = \sup\{\|x\| : x \in C_n\} = \infty$;

(iii) for every $R > 0$, $(\bigcup_{n=1}^{\infty}) \cap B_R$ is a relatively compact set of X, where $B_R = \{x \in X : \|x\| \le R\}$.

Then there exists an unbounded component $\mathcal{C}$ in $\mathcal{D}$ and $z^* \in \mathcal{C}$.

By means of the corresponding auxiliary equations (4.2), we obtained a sequence of unbounded components $\mathcal{C}_n$, and this enables us to find an unbounded component $\mathcal{C}$ satisfying

$$\mathcal{C} \subset \limsup_{n\to\infty} \mathcal{C}_n.$$

It following from the existence of Γ_n^f and Γ_n^g that $f_n(s,t) = f(t,s)$, $g_n(s,t) = g(t,s)$ for $n \to \infty$. Thus $(\lambda,(u,v)) \in \mathcal{C}$ solves the original problem (1.5) when $n \to \infty$. Now we verify $\{\mathcal{C}_n\}$ satisfying the assumptions of Lemma 4.6. By the definition of continuum and component, $\mathcal{C}_n$ is closed.

Since all of $\mathcal{C}_n$ contain $(0,(0,0))$, we can choose $z_n \in \mathcal{C}_n$ such that $z_n = (0,(0,0))$ for $n = 1,2,\ldots$ Clearly, $z_n \to z^* = (0,(0,0))$, the assumption (i) of Lemma 4.6 is satisfied.

By unboundedness of $\mathcal{C}_n$, obviously, we have

$$r_n = \sup\{|\lambda| + \|(u,v)\|_E : (\lambda,(u,v)) \in \mathcal{C}_n\} = +\infty.$$

(iii) in Lemma 4.6 can be deduced directly from the Arzelà–Ascoli theorem and the definition of f_n, g_n. Therefore, the superior limit of $\mathcal{C}_n$ contains a component $\mathcal{C}$.

It follows from (H1) that $u,\ v > 0$ in Ω for $\lambda > 0$ whenever $(\lambda,(u,v)) \in \mathcal{C}$, which establishes part (a), Clearly, $(0,(0,0)) \in \mathcal{C}$, which together with the maximum principle establish part (b). Part (c) follows from Theorem 2.1. By construction of $\mathcal{C}$, there exists a sequence $(\lambda_n,(u_n,v_n)) \in \mathcal{C}$ such that $0 < \lambda_n < \lambda^*$ (λ^* is as in Theorem 2.1) and $u_n > 0,\ v_n > 0$ for all $n \in \mathbb{N}$, and $\|(u_n,v_n)\|_E \to +\infty$ as $n \to +\infty$. Thus $\mathcal{C}$ is unbounded in the Banach space E. This establishes part (d) and completes the proof of Theorem 1.2.

5 Application of Theorem 1.2

The unbounded component $\mathcal{C}$ from Theorem 1.2 may bifurcate from infinity at any or all $\lambda \in [0,\lambda^*]$. Next, As an application of Theorem 1.2, we will show that, under additional assumptions on f and g, the component $\mathcal{C}$ must approach towards the hyperplane $\lambda = 0$ as the norm $\|(u,v)\|_E$ grows large.

5.1 Main result

Theorem 5.1. *Let (H1)–(H2) hold. Assume*

$$f(u,v) \le C_1(1 + v^{p_1} + u^{p_2}), \qquad u,\ v \ge 0,\ x \in \Omega, \tag{5.1}$$
$$g(u,v) \le C_1(1 + u^{q_1} + v^{q_2}), \qquad u,\ v \ge 0,\ x \in \Omega, \tag{5.2}$$
$$f(u,v) + g(u,v) \ge \kappa(u+v) - C_1, \qquad u,\ v \ge 0,\ x \in \Omega, \tag{5.3}$$

here $p_1,\ q_1 > 0$, $p_1q_1 > 1$, $p_2,\ q_2 \ge 1$, $C_1 > 0$ *and* $\kappa > \mu_1$ (μ_1 *defined in Section 2). Define*

$$\alpha = \frac{2(p_1+1)}{p_1q_1 - 1}, \qquad \beta = \frac{2(q_1+1)}{p_1q_1 - 1}.$$

If

$$\max\{\alpha,\beta\} > N - 1 \tag{5.4}$$

and

$$p_2,\ q_2 < \frac{N+1}{N-1}. \tag{5.5}$$

Then $\lambda_\infty = 0$ *is the unique bifurcation point from infinity in* $[0,\lambda^*]$*, for the component* $\mathcal{C} \subseteq \mathcal{S}$ *from Theorem 1.2. More specifically,* *(i) There exists a sequence of positive solutions $(\lambda_n, (u_n, v_n)) \in \mathcal{C}$ with $\lambda_n \in [0, \lambda^*)$ for all $n \in \mathbb{N}$ such that $\|(u_n, v_n)\|_E \to +\infty$ and $\lambda_n \to 0^+$ as $n \to +\infty$.*

(ii) Any sequence $(\lambda_n, (u_n, v_n)) \in \mathcal{C}$ such that $\|(u_n, v_n)\|_E \to +\infty$ as $n \to \infty$ and $\lambda_n > 0$ must satisfy $\lambda_n \to 0^+$ as $n \to +\infty$.

It is worth noting that for problem (1.5) satisfying (H1), (H2) and (5.1)–(5.5), the solutions $(\lambda, (u, v)) \in \mathbb{R} \times E$.

Lemma 5.2. *Assume (5.1)–(5.5) hold. Let $\lambda_n \in \mathbb{R}$ be a sequence with $\lambda_1 < \lambda^*$ such that $\lambda_n \searrow 0^+$ as $n \to +\infty$. Then for each $n \in \mathbb{N}$, there exists $C_n := C(\lambda_n)$ such that any solution $(\lambda, (u, v))$ of (1.5) satisfies*

$$\|u\|_{L^\infty(\Omega)},\ \|v\|_{L^\infty(\Omega)} \leq C_n, \quad \text{for all } \lambda \in [\lambda_n, \lambda^*)$$

and $C_n \to +\infty$ as $n \to +\infty$.

Proof. We begin by observing that under above hypotheses uniform a priori bound result holds [17, Theorem 1.1] for positive solutions of (1.5). By retracing the proof of [17, Theorem 2.1, Proposition 3.1] with λf and λg (in place of f and g) we will establish the dependence of the uniform bounds on λ.

First, let $b^* > \lambda^*$, we consider the system (1.5) with $\lambda \in [\lambda_n, b^*]$ under the assumptions

$$|\lambda f(u, v)| \leq b^* C_1(|v|^{p_1} + |u|^{p_2}) + b^* h_2(x), \qquad u,\ v \in \mathbb{R},\ x \in \Omega, \tag{5.6}$$

$$|\lambda g(u, v)| \leq b^* C_1(|u|^{q_1} + |v|^{q_2}) + b^* h_2(x), \qquad u,\ v \in \mathbb{R},\ x \in \Omega, \tag{5.7}$$

$$a\lambda f(u, v) + b\lambda g(u, v) \geq \kappa\lambda_n(au + bv) - b^* C_1, \qquad u,\ v \geq 0,\ x \in \Omega, \tag{5.8}$$

with $p_1,\ q_1 > 0,\ p_1 q_1 > 1,\ p_2,\ q_2 \geq 1,\ h_2 \in L^\gamma(\Omega),\ \gamma > \frac{N}{2},\ a,\ b > 0, \kappa\lambda_n > \mu_1$ and $C_1 \geq 0$. By [17, Theorem 2.1, Proposition 3.1], we know any nonnegative solution of (1.5) satisfies

$$\|u\|_{L^\infty(\Omega)},\ \|v\|_{L^\infty(\Omega)} \leq C_n, \quad \text{for all } \lambda \in [\lambda_n, b^*].$$

The constant C_n depends only on $p_1,\ q_1,\ p_2,\ q_2,\ \gamma,\ C_1$ and the norms of h_2 in $h_2 \in L^\gamma(\Omega)$.

Next, we show $C_n \to +\infty$ as $n \to +\infty$. In fact, by Theorem 1.2, we know that there exists a sequence of positive solutions $(\lambda_n, (u_n, v_n)) \in \mathcal{C}$ such that $\lambda_n \in (0, \lambda^*)$ for all $n \in \mathbb{N}$ and $\|(u_n, v_n)\|_E \to +\infty$ as $n \to +\infty$.

For $n \in \mathbb{N}$, let

$$\sup\{\|(u, v)\|_E : (\lambda, (u, v)) \in \mathcal{C},\ \lambda_n < \lambda < \lambda_*\} =: B_n,$$

then clearly

$$B_n \leq C_n.$$

For $(\lambda_n, (u_n, v_n)) \in \mathcal{C}$, we have

$$\lim_{n \to +\infty} \|(u_n, v_n)\|_E = \infty,$$

therefore,

$$\lim_{n \to +\infty} B_n = \infty,$$

thus

$$\lim_{n \to +\infty} C_n = \infty.$$

□

Now we will complete the proof of Theorem 5.1.

Proof of Theorem 5.1. By Lemma 5.2, the component $\mathcal{C}$ must bifurcate from infinity at $\lambda = 0$ and by construction $(0,(0,0)) \in \mathcal{C}$. Part (i) follows from the construction of $\mathcal{C}$ and the fact that $\mathcal{C}$ cannot cross the hyperplane $\{0\} \times E$.

Let $\{\lambda_n,(u_n,v_n)\} \in \mathcal{C}$ with $\|(u_n,v_n)\|_E \to +\infty$ as $n \to +\infty$ and $\lambda_n > 0$ for all $n \in \mathbb{N}$. Suppose to the contrary that $\lambda_n \to \tilde{\lambda} > 0$ as $n \to +\infty$. By Lemma 5.2,

$$\|u_n\|_{L^\infty(\Omega)},\ \|v_n\|_{L^\infty(\Omega)} \le C_{\tilde{\lambda}} < +\infty$$

for all $\lambda \in [\frac{\tilde{\lambda}}{2},\lambda^*)$, a contradiction to $\|(u_n,v_n)\|_E \to +\infty$ as $n \to +\infty$. Hence part (ii) follows. This completes the proof of Theorem 5.1. □

5.2 Examples

Let $f(u,v) = (u+v)^\tau + 1$, $g(u,v) = (u+2v)^\tau + 1$, where $\tau \in (1, \frac{N+1}{N-1})$.

It is easy to see f, g satisfy (H1) and (H2). When C_1 is large enough, there exist $\tau < p_2, q_2 < \frac{N+1}{N-1}$, $p_1, q_1 > \tau$ and $p_1 q_1 > 1$, such that (5.1)–(5.5) hold. Then there exists an unbounded continuum $\mathcal{C}$ and $\lambda_\infty = 0$ is the unique bifurcation point from infinity in $[0,\lambda^*]$.

Such as, when $N = 3$. Let $f(u,v) = (u+v)^{\frac{3}{2}} + 1$, $g(u,v) = (u+2v)^{\frac{3}{2}} + 1$, then (H1) and (H2) hold. We set $p_2, q_2 = \frac{7}{4}$, $p_1 = \frac{7}{5}$, $q_1 = \frac{16}{7}$, then (5.1)–(5.5) hold.

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1/2-Laplacian problem with logarithmic and exponential nonlinearities

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Abstract. In this paper, based on a suitable fractional Trudinger-–Moser inequality, we establish sufficient conditions for the existence result of least energy sign-changing solution for a class of one-dimensional nonlocal equations involving logarithmic and exponential nonlinearities. By using a main tool of constrained minimization in Nehari manifold and a quantitative deformation lemma, we consider both subcritical and critical exponential growths. This work can be regarded as the complement for some results of the literature.

Keywords: 1/2-Laplacian operator, logarithmic nonlinearity, exponential nonlinearity, sign-changing solutions.

2020 Mathematics Subject Classification: 35R11, 35J60, 35B33.

1 Introduction

In the present paper, we investigate the existence of least energy sign-changing solution for a Dirichlet problem driven by the 1/2-Laplacian operator of the following type:

$$\begin{cases} (-\Delta)^{1/2}u = |u|^{p-2}u \ln |u|^2 + \mu f(u) & \text{in } (0,1), \\ u = 0 & \text{in } \mathbb{R}\backslash(0,1), \end{cases} \tag{1.1}$$

where $2 < p < \infty$, μ is a positive parameter and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function with exponential subcritical or critical growth in the sense of the fractional Trudinger–Moser inequality. The nonlocal operator $(-\Delta)^{1/2}$ defined on smooth functions by

$$(-\Delta)^{1/2}u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^2}\,dy, \qquad \forall x \in \mathbb{R}. \tag{1.2}$$

Recently, a great attention has been focused on the study of nonlocal operators $(-\Delta)^s_p$, $p > 1$, $s \in (0,1)$. These arise in thin obstacle problems, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, water waves, etc. See for instance [8].

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It is natural to work on the *Sobolev–Slobodeckij* space

$$X := W_0^{1/2,2}(0,1) = \left\{u \in H^{1/2}(\mathbb{R}) : u = 0 \text{ a.e. in } \mathbb{R}\backslash(0,1)\right\}$$

with respect to the Gagliardo semi-norm

$$\|u\| := [u]_{H^{1/2}(\mathbb{R})} = \left[\int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2}\, \mathrm{d}x\, \mathrm{d}y\right]^{\frac{1}{2}}.$$

The problem of type (1.1) with exponential growth nonlinearity is motivated from the fractional Trudinger–Moser inequality, which specialized the results of Iannizzotto, Squassina [14, Corollary 2.4] to the space X: there exists $0 < \omega \le \pi$ such that for all $0 < \alpha < 2\pi\omega$, we can find $K_\alpha > 0$ such that

$$\int_0^1 e^{\alpha u^2}\, \mathrm{d}x \le K_\alpha, \qquad \text{for all } u \in X,\ \|u\| \le 1. \tag{1.3}$$

For more information, we refer the readers to Ozawa [21, Theorem 1], and Kozono, Sato & Wadade [17, Theorem 1.1], and do Ó, Medeiros & Severo [11, Theorem 1.1]. Therefore, from this result we have naturally associated notions of subcriticality and criticality, namely: we say that a function $f : \mathbb{R} \to \mathbb{R}$ has *subcritical* growth if

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = 0, \qquad \forall \alpha > 0,$$

and f has *critical* growth if there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = 0, \qquad \forall \alpha > \alpha_0$$

and

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = +\infty, \qquad \forall \alpha < \alpha_0.$$

We assume the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is a function with exponential growth in the sense of Trudinger–Moser inequality. More precisely, the function f satisfies the following conditions:

(f_1) $f \in C^1(\mathbb{R}, \mathbb{R})$ and there exists $\alpha_0 \ge 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ +\infty, & \text{if } \alpha < \alpha_0; \end{cases}$$

(f_2) $\lim_{t\to 0} \frac{|f(t)|}{|t|} = 0$;

(f_3) there exists $\theta > p$ such that

$$0 < \theta F(t) \le t f(t) \quad \text{for } t \in \mathbb{R}\backslash\{0\},$$

where $F(t) = \int_0^t f(s)ds$;

(f_4) $tf'(t) \ge (p-1)f(t)$ for $t > 0$ and $tf'(t) \le (p-1)f(t)$ for $t < 0$.

Similar conditions were also used in [28]. Here we'd like to highlight that the result in this work can be applied for the model nonlinearity $f(t) = |t|^{\theta-2}te^{\alpha_0 t^2}$, $t \in \mathbb{R}$.

Remark 1.1. The condition (f_4) implies that $H(s) = sf(s) - pF(s)$ is a nonnegative function, increasing in $|s|$ with

$$sH'(s) = s^2 f'(s) - (p-1)f(s)s \geq 0, \quad \text{for any } |s| > 0.$$

The problem driven by the 1/2-Laplacian operator was earlier considered in [14] (see also [13]), where the authors studied the existence of mountain-pass weak solutions to the problem

$$-\frac{1}{2\pi}\int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^2}\,\mathrm{d}y = f(u), \qquad u \in W_0^{1/2,2}(-1,1).$$

We also mention [10, 11] for other investigations in the one dimensional case on the whole space $\mathbb{R}$, facing the problem of the lack of compactness. In particular in [11], the existence of ground state solutions for the problem

$$-\frac{1}{2\pi}\int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^2}\,\mathrm{d}y + u = f(u), \qquad u \in W_0^{1/2,2}(\mathbb{R})$$

was proved, where f is a Trudinger–Moser critical growth nonlinearity. In [7], Böer and Miyagaki investigated the existence and multiplicity of nontrivial solutions for the Choquard logarithmic equation

$$(-\Delta)^{1/2}u + u + \left(\ln|\cdot| * |u|^2\right)u = f(u), \quad \text{in } \mathbb{R},$$

for the nonlinearity f with exponential critical growth.

For local quasilinear problems of the following type

$$\begin{cases} -\Delta_N u = f(u), & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where the nonlinearity $f(u)$ behaves like $\exp\left(\alpha|u|^{N/N-1}\right)$, as $|u| \to \infty$, have been analyzed in literature, see [1,9,18,27] and the references therein.

On the other hand, the signed and sign-changing solutions for elliptic equations with logarithmic nonlinearities were investigated. There is an extensive bibliography on this subject. See, for instance, Ji, Szulkin [15], Alves, Ji [2–4], Tian [23], Wen, Tang & Chen [25], Truong [24], Liang, Rădulescu [19], and the references therein.

After a careful bibliography review, we have found only a paper is due to Zhang et al. [28], which is dealing with the existence of sign-changing solutions for the local quasilinear N-Laplacian problem with logarithmic and exponential critical nonlinearities

$$\begin{cases} -\Delta_N u = |u|^{p-2}u\ln|u|^2 + \mu f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In that interesting paper, the authors applied the constrained minimization in Nehari manifold and the quantitative deformation lemma, and obtained the existence of least energy sign-changing solution.

Motivated by above works, especially by [14,22,28], the main goal of this paper is to show the existence of least energy sign-changing solutions for problem (1.1). To the author's knowledge, in the framework of the Sobolev–Slobodeckij spaces $W_0^{1/2,2}(0,1)$, fractional counterparts of the local quasilinear N-Laplacian problem (1.4) were not previously tackled in the literature. This is precisely the goal of this manuscript.

We give our problem a variational formulation by setting for all $u \in X$

$$I_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{2}{p^2}\int_0^1 |u|^p \, dx - \frac{1}{p}\int_0^1 |u|^p \ln|u|^2 \, dx - \mu \int_0^1 F(u) \, dx.$$

Observe that

$$\lim_{|t|\to 0} \frac{|t|^{p-1}\ln|t|^2}{|t|} = 0,$$
$$\lim_{|t|\to\infty} \frac{|t|^{p-1}\ln|t|^2}{|t|^{r-1}} = 0, \quad \text{for all } r \in (p,\infty),$$

since $p > 2$. Then for any $\epsilon > 0$, there exists a positive constant $C_1 = C_1(\epsilon)$ such that

$$|t|^{p-1}\ln|t|^2 \le \epsilon|t| + C_1|t|^{r-1}, \quad \text{for all } t \in \mathbb{R}. \tag{1.5}$$

By (f_1), for all $\alpha \ge \alpha_0$ there exists $c_2 > 0$ such that

$$|f(t)| \le c_2 e^{\alpha t^2}, \quad \text{for all } t \in \mathbb{R}. \tag{1.6}$$

For given $\epsilon > 0$, (f_2) implies that there exists $\delta > 0$ such that for all $|t| < \delta$ we have $F(t) \le \frac{\epsilon}{2}|t|^2$. Fix $q > 2, 0 < \alpha < 2\pi\omega$ and $r > 1$ such that $r\alpha < 2\pi\omega$ as well. By (1.6) there exists $C_\epsilon > 0$ such that for all $|t| \ge \delta$ we have $F(t) \le C_\epsilon |t|^q e^{\alpha t^2}$. Summarizing, we obtain

$$|F(t)| \le \frac{\epsilon}{2}|t|^2 + C_\epsilon|t|^q e^{\alpha t^2}, \qquad \forall t \in \mathbb{R}. \tag{1.7}$$

Using (1.5), (1.7), the Sobolev embedding theorem and the fractional Trudinger–Moser inequality (1.3), one can verify that I_μ is well defined, of class $C^1(X,\mathbb{R})$ and

$$\begin{aligned}\langle I'_\mu(u), v\rangle &= \langle u, v\rangle_X - \int_0^1 |u|^{p-2}uv\ln|u|^2 \, dx - \mu\int_0^1 f(u)v \, dx \\ &= \int_{\mathbb{R}^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2} \, dx \, dy - \int_0^1 |u|^{p-2}uv\ln|u|^2 \, dx - \mu\int_0^1 f(u)v \, dx\end{aligned}$$

for all $u, v \in X$. From now on, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X' and X. Clearly, the critical points of I_μ are exactly the weak solutions of problem (1.1).

We call u a *least energy sign-changing solution* to problem (1.1) if $u^\pm \neq 0$ and

$$I_\mu(u) = \inf\left\{I_\mu(v) : v^\pm \neq 0, I'_\mu(v) = 0\right\},$$

where $v^+ = \max\{v(x), 0\}$ and $v^- = \min\{v(x), 0\}$. By a simple calculation, for any $u = u^+ + u^-$ with $u^\pm \neq 0$, we obtain

$$\begin{aligned}\|u\|^2 &= \|u^+\|^2 + \|u^-\|^2 + 2H(u), \\ I_\mu(u) &= I_\mu(u^+) + I_\mu(u^-) + H(u) > I_\mu(u^+) + I_\mu(u^-), \\ \langle I'_\mu(u), u^\pm\rangle &= \langle I'_\mu(u^\pm), u^\pm\rangle + H(u) > \langle I'_\mu(u^\pm), u^\pm\rangle,\end{aligned}$$

where

$$H(u) = -\int_0^1\int_0^1 \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^2} \, dx \, dy > 0.$$

Therefore, the methods used to seek sign-changing solutions of the local problems do not work to problem (1.1) due to the presence of the nonlocal operator $(-\Delta)^{1/2}$. And so, a careful analysis is necessary in a lot of estimates. Inspired by [6], our strategy consists in finding sign-changing solutions which minimize the corresponding energy functional I_μ among the set of all sign-changing solutions to problem (1.1). To this end, we define the sign-changing Nehari set as

$$\mathcal{M}_\mu := \left\{ u \in X : \langle I'_\mu(u), u^+ \rangle = \langle I'_\mu(u), u^- \rangle = 0, u^\pm \neq 0 \right\}.$$

Note that $u^\pm \in X$ and $u = u^+ + u^-$. Clearly, any sign-changing solution of problem (1.1) lies in the set $\mathcal{M}_\mu$.

Here are our main results.

Theorem 1.2. (Subcritical case). *Assume that conditions (f_2)–(f_4) and (f_1) with $\alpha_0 = 0$ hold. Then problem (1.1) admits a least energy sign-changing solution $u_\mu \in \mathcal{M}_\mu$ for $\mu > 0$ satisfying $I_\mu(u_\mu) = m_\mu$, where $m_\mu = \inf_{u \in \mathcal{M}_\mu} I_\mu(u)$.*

Theorem 1.3. (Critical case). *Assume that conditions (f_2)–(f_4) and (f_1) with $\alpha_0 > 0$ hold. Then there exists $\mu^* > 0$ such that problem (1.1) has a least energy sign-changing solution $u_\mu \in \mathcal{M}_\mu$ for $\mu \geq \mu^*$ satisfying $I_\mu(u_\mu) = m_\mu$.*

The inequality (3.4) or (4.3) plays a crucial role to show that the minimum m_μ of the associated energy functional I_μ is achieved. In the subcritical case, (3.4) holds due to the positive number α can take arbitrary small, thus we can conclude that Lemma 3.2 for all $\mu > 0$. However, in the critical case, we can't prove directly that (4.3) holds by the fractional Trudinger–Moser inequality (1.3) since $\alpha > \alpha_0$ for some positive number α_0. Based on this reason, we need to further analyze the asymptotic property of m_μ, by utilize Lemmas 2.3(ii) and 2.4, we can find a threshold $\mu^* > 0$ such that (4.3) holds for all $\mu \geq \mu^*$. Thus, we can conclude that Lemma 4.2 for all $\mu \geq \mu^*$. It is quite natural to ask whether in the critical case a least energy sign-changing solution exists even for $\mu \in (0, \mu^*)$. This is the issue we need to further consider in the future. Our initial idea is below: to do that, based on works such as [29], we insert an additional condition that makes possible to get a boundedness for the integral involving the exponential term. By utilize an argument similar to [29], we will try to pull the energy of sign-changing solutions down below some critical value to recover the compactness which urges us to prove that m_μ can be achieved by some $u_\mu \in \mathcal{M}_\mu$. Finally, followed the idea used in [30, Theorem 1.1], we shall prove that u_μ is indeed a least energy sign-changing solution of problem (1.1).

This paper is organized as follows. In Section 2, we show some technical lemmas and estimates in both subcritical and critical cases. Then we give the proofs of Theorem 1.2 and Theorem 1.3 in Section 3 and 4, respectively.

2 Technical lemmas

In this section, we present some extra framework information and provide very useful technical results.

We start remembering the operator $(-\Delta)^{1/2}$, of a smooth function $u : \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathcal{F}\left((-\Delta)^{1/2} u\right)(\xi) = |\xi| \mathcal{F}(u)(\xi),$$

where $\mathcal{F}$ denotes the Fourier transform, that is,

$$\mathcal{F}(\phi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi\cdot x}\phi(x)\,\mathrm{d}x$$

for functions ϕ in the Schwartz class. Also $(-\Delta)^{1/2}u$ can be equivalently represented as (1.2).

Now, we turn our attention to the Hilbert space

$$H^{1/2}(\mathbb{R}) = \left\{ u \in L^2(\mathbb{R}) : \int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y < \infty \right\}$$

endowed with the norm

$$\|u\|_{H^{1/2}(\mathbb{R})} = \left(\|u\|^2_{L^2(\mathbb{R})} + [u]^2_{H^{1/2}(\mathbb{R})} \right)^{\frac{1}{2}},$$

where $\|\cdot\|_{L^s(\mathbb{R})}$ denotes the standard $L^s(\mathbb{R})$ norm for any $s \geq 1$. We know that $(H^{1/2}(\mathbb{R}), \|\cdot\|_{H^{1/2}(\mathbb{R})})$ is a Hilbert space. Also, in light of [8, Proposition 3.6], we have

$$\|(-\Delta)^{1/4}u\|_{L^2(\mathbb{R})} = (2\pi)^{-\frac{1}{2}}[u]_{H^{1/2}(\mathbb{R})}, \quad \text{for all } u \in H^{1/2}(\mathbb{R}),$$

and, sometimes, we identify these two quantities by omitting the normalization constant $1/2\pi$.

It follows from Proposition 2.2 in [14] to that there exists $\lambda_1 > 0$ such that for all $u \in X$

$$\|u\|_{L^2(0,1)} \leq \lambda_1^{-\frac{1}{2}}\|u\|. \tag{2.1}$$

Moreover, equality holds for some $u \in X$ with $\|u\|_{L^2(0,1)} = 1$. Due to the inequality (2.1), we can prove further $(X, \|\cdot\|)$ is a Hilbert space, where $\|\cdot\|$ is induced by an inner product, defined for all $u, v \in X$ by

$$\langle u, v\rangle_X = \int_{\mathbb{R}^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Hereafter, we assume throughout, unless otherwise mentioned, that the function f satisfies conditions (f_1) to (f_4). Now, fix $u \in X$ with $u^{\pm} \neq 0$, and we define the function $\Psi_u : [0,\infty)\times[0,\infty) \to \mathbb{R}$ and mapping $T_u : [0,\infty)\times[0,\infty) \to \mathbb{R}^2$ as

$$\Psi_u(a,b) = I_\mu\left(au^+ + bu^-\right) \tag{2.2}$$

and

$$T_u(a,b) = \left(\langle I'_\mu\left(au^+ + bu^-\right), au^+\rangle, \langle I'_\mu\left(au^+ + bu^-\right), bu^-\rangle \right) = (g_1(a,b), g_2(a,b)). \tag{2.3}$$

Lemma 2.1. *For each $u \in X$ with $u^{\pm} \neq 0$, there exists an unique pair $(a_u, b_u) \in (0,\infty)\times(0,\infty)$ such that*

$$a_u u^+ + b_u u^- \in \mathcal{M}_\mu.$$

In particular, the set $\mathcal{M}_\mu$ is nonempty. Moreover, for all $a, b \geq 0$ with $(a,b) \neq (a_u, b_u)$

$$I_\mu\left(au^+ + bu^-\right) < I_\mu\left(a_u u^+ + b_u u^-\right)$$

holds.

Proof. First we will work to obtain the existence result. From (f_1) and (f_2), given $\epsilon > 0$, there exists a positive constant $C_2 = C_2(\epsilon)$ such that

$$f(t)t \leq \epsilon |t|^2 + C_2 |t|^q e^{\alpha t^2} \quad \text{for all } \alpha > \alpha_0, q > 2. \tag{2.4}$$

Now, given $u \in X$ with $u^{\pm} \neq 0$, it follows from (1.5), (2.4), the Sobolev embedding theorem, the Hölder inequality and the fractional Trudinger–Moser inequality (1.3) that when $s, s' > 1$ with $1/s + 1/s' = 1$ and small $a > 0$ with $\alpha s \left\|au^+\right\|^2 \leq 2\pi\omega$

$$\begin{aligned} g_1(a,b) &= \langle I'_\mu \left(au^+ + bu^-\right), au^+ \rangle \\ &= \left\|au^+\right\|^2 + abH(u) - \int_0^1 \left|au^+\right|^p \ln\left|au^+\right|^2 \, dx - \mu \int_0^1 f\left(au^+\right) au^+ \, dx \\ &\geq \left\|au^+\right\|^2 - \epsilon \int_0^1 \left|au^+\right|^2 \, dx - C_1 \int_0^1 \left|au^+\right|^r \, dx \\ &\quad - \mu\epsilon \int_0^1 \left|au^+\right|^2 \, dx - \mu C_2 \int_0^1 \left|au^+\right|^q e^{\alpha |au^+|^2} \, dx \\ &\geq \left\|au^+\right\|^2 - \epsilon C_3 \left\|au^+\right\|^2 - C_1 C_4 \left\|au^+\right\|^r - \mu\epsilon C_3 \left\|au^+\right\|^2 \\ &\quad - \mu C_2 \left(\int_0^1 \left|au^+\right|^{qs'} \, dx \right)^{\frac{1}{s'}} \left(\int_0^1 e^{\alpha s \|au^+\|^2 (|au^+|/\|au^+\|)^2} \, dx \right)^{\frac{1}{s}} \\ &\geq \left(1 - \epsilon C_3 - \mu\epsilon C_3\right) \left\|au^+\right\|^2 - C_1 C_4 \left\|au^+\right\|^r - \mu C_2 K_{\alpha s \|au^+\|^2} C_5 \left\|au^+\right\|^q \end{aligned} \tag{2.5}$$

holds. Choose $\epsilon > 0$ sufficiently small such that $1 - \epsilon C_3 - \mu\epsilon C_3 > 0$ and then it is easy to see that $\langle I'_\mu \left(au^+ + bu^-\right), au^+ \rangle > 0$ for small $a > 0$ and all $b > 0$ by $r, q > 2$. In turn, we can also obtain that $\langle I'_\mu \left(au^+ + bu^-\right), bu^- \rangle > 0$ for $b > 0$ small enough and all $a > 0$. Hence, it is evident that there exists $\delta_1 > 0$ such that

$$\langle I'_\mu \left(\delta_1 u^+ + bu^-\right), \delta_1 u^+ \rangle > 0, \qquad \langle I'_\mu \left(au^+ + \delta_1 u^-\right), \delta_1 u^- \rangle > 0 \tag{2.6}$$

for all $a, b > 0$.

On the other hand, recall the elementary inequality

$$2t^p - pt^p \ln t^2 \leq 2 \tag{2.7}$$

for all $t \in (0, \infty)$. From (f_3), we can deduce that there exist $C_{\theta,1}, C_{\theta,2} > 0$ such that

$$F(t) \geq C_{\theta,1} |t|^\theta - C_{\theta,2}. \tag{2.8}$$

Now, choose $a = \delta_2^* > \delta_1$ with δ_2^* large enough and it follows from (2.7), (2.8) and $2 < p < \theta$ that

$$\begin{aligned} g_1(\delta_2^*, b) &= \langle I'_\mu \left(\delta_2^* u^+ + bu^-\right), \delta_2^* u^+ \rangle \\ &\leq \left\|\delta_2^* u^+\right\|^2 + \delta_2^* bH(u) + \int_0^1 \left(\frac{2}{p} - \frac{2}{p} \left|\delta_2^* u^+\right|^p \right) dx - \mu\theta \int_0^1 C_{\theta,1} \left|\delta_2^* u^+\right|^\theta dx + \mu\theta C_{\theta,2} \\ &\leq 0 \end{aligned}$$

for $b \in [\delta_1, \delta_2^*]$. With the similar argument, we can choose sufficiently large $b = \delta_2^* > \delta_1$ such that $\langle I'_\mu \left(au^+ + \delta_2^* u^-\right), \delta_2^* u^- \rangle \leq 0$ holds for $a \in [\delta_1, \delta_2^*]$.

Hence, let $\delta_2 > \delta_2^*$ be large enough. Then we obtain that

$$\langle I'_\mu \left(\delta_2 u^+ + bu^-\right), \delta_2 u^+ \rangle < 0, \qquad \langle I'_\mu \left(au^+ + \delta_2 u^-\right), \delta_2 u^- \rangle < 0 \tag{2.9}$$

for all $a, b \in [\delta_1, \delta_2]$. Combining (2.6) and (2.9) with Miranda's Theorem [5], there exists at least one point pair $(a_u, b_u) \in (0,\infty) \times (0,\infty)$ such that $T_u(a_u, b_u) = (0,0)$, that is, $a_u u^+ + b_u u^- \in \mathcal{M}_\mu$.

Next we will prove the uniqueness of the pair (a_u, b_u). In fact, it is sufficient to show that if $u \in \mathcal{M}_\mu$ and $a_0 u^+ + b_0 u^- \in \mathcal{M}_\mu$ with $a_0 > 0$ and $b_0 > 0$, then $(a_0, b_0) = (1,1)$. Assume that $u \in \mathcal{M}_\mu$ and $a_0 u^+ + b_0 u^- \in \mathcal{M}_\mu$. We thus obtain that $\langle I'_\mu(a_0 u^+ + b_0 u^-), a_0 u^+\rangle = 0$, $\langle I'_\mu(a_0 u^+ + b_0 u^-), b_0 u^-\rangle = 0$, and $\langle I'_\mu(u), u^\pm\rangle = 0$, namely

$$\left\|a_0 u^+\right\|^2 + a_0 b_0 H(u) = \int_0^1 \left|a_0 u^+\right|^p \ln\left|a_0 u^+\right|^2 \,\mathrm{d}x + \mu \int_0^1 f\left(a_0 u^+\right) a_0 u^+ \,\mathrm{d}x, \tag{2.10}$$

$$\left\|b_0 u^-\right\|^2 + b_0 a_0 H(u) = \int_0^1 \left|b_0 u^-\right|^p \ln\left|b_0 u^-\right|^2 \,\mathrm{d}x + \mu \int_0^1 f\left(b_0 u^-\right) b_0 u^- \,\mathrm{d}x, \tag{2.11}$$

$$\left\|u^+\right\|^2 + H(u) = \int_0^1 \left|u^+\right|^p \ln\left|u^+\right|^2 \,\mathrm{d}x + \mu \int_0^1 f\left(u^+\right) u^+ \,\mathrm{d}x, \tag{2.12}$$

$$\left\|u^-\right\|^2 + H(u) = \int_0^1 \left|u^-\right|^p \ln\left|u^-\right|^2 \,\mathrm{d}x + \mu \int_0^1 f\left(u^-\right) u^- \,\mathrm{d}x. \tag{2.13}$$

Without loss of generality, we may assume that $0 < a_0 \le b_0$. Thus, form (2.11), we get

$$\left\|b_0 u^-\right\|^2 + b_0^2 H(u) \ge \int_0^1 \left|b_0 u^-\right|^p \ln\left|b_0 u^-\right|^2 \,\mathrm{d}x + \mu \int_0^1 f\left(b_0 u^-\right) b_0 u^- \,\mathrm{d}x. \tag{2.14}$$

Combining (2.14) and (2.13), we deduce that

$$\int_0^1 \left|u^-\right|^p \ln\left|u^-\right|^2 \,\mathrm{d}x - \int_0^1 \frac{\left|b_0 u^-\right|^p \ln\left|b_0 u^-\right|^2}{b_0^2} \mathrm{d}x \ge \mu \int_0^1 \frac{f\left(b_0 u^-\right) b_0 u^-}{b_0^2} \,\mathrm{d}x - \mu \int_0^1 f\left(u^-\right) u^- \mathrm{d}x,$$

that is,

$$\int_0^1 \left(\left|u^-\right|^{p-2} \ln\left|u^-\right|^2 - \left|b_0 u^-\right|^{p-2} \ln\left|b_0 u^-\right|^2\right) \left|u^-\right|^2 \,\mathrm{d}x \ge \mu \int_0^1 \left(\frac{f\left(b_0 u^-\right)}{b_0 u^-} - \frac{f\left(u^-\right)}{u^-}\right) \left(u^-\right)^2 \mathrm{d}x.$$

It follows from (f_4) and $p > 2$ that $t \mapsto \frac{f(t)}{t}$ and $t \mapsto t^{p-2} \ln t^2$ are increasing for $t > 0$. If $b_0 > 1$, the left hand side of the above inequality is negative, which is absurd due to the right hand side is positive. Therefore, we obtain $a_0 \le b_0 \le 1$. Similarly, from (2.10), (2.12) and $0 < a_0 \le b_0$, one has

$$\int_0^1 \left(\left|u^+\right|^{p-2} \ln\left|u^+\right|^2 - \left|a_0 u^+\right|^{p-2} \ln\left|a_0 u^+\right|^2\right) \left|u^+\right|^2 \,\mathrm{d}x \le \mu \int_0^1 \left(\frac{f\left(a_0 u^+\right)}{a_0 u^+} - \frac{f\left(u^+\right)}{u^+}\right) \left(u^+\right)^2 \mathrm{d}x.$$

Thus, we can deduce that $a_0 \ge 1$. So $a_0 = b_0 = 1$.

To complete the proof of this lemma, it remains to show that (a_u, b_u) is the unique maximum point of Ψ_μ in $[0,\infty) \times [0,\infty)$. It follows from (2.7), (2.8), the Hölder inequality, the elementary inequality and $\theta > p > 2$ that

$$\begin{aligned}
\Psi_u(a,b) &= I_\mu\left(a u^+ + b u^-\right) \\
&= \frac{1}{2}\left\|a u^+ + b u^-\right\|^2 + \frac{2}{p^2} \int_0^1 \left|a u^+ + b u^-\right|^p \,\mathrm{d}x \\
&\quad - \frac{1}{p} \int_0^1 \left|a u^+ + b u^-\right|^p \ln\left|a u^+ + b u^-\right|^2 \,\mathrm{d}x - \mu \int_0^1 F\left(a u^+ + b u^-\right) \mathrm{d}x \\
&\le a^2 \left\|u^+\right\|^2 + b^2 \left\|u^-\right\|^2 + \frac{2}{p^2} - \mu C_{\theta,1} a^\theta \int_0^1 \left|u^+\right|^\theta \,\mathrm{d}x - \mu C_{\theta,1} b^\theta \int_0^1 \left|u^-\right|^\theta \,\mathrm{d}x + 2\mu C_{\theta,2},
\end{aligned}$$

which implies that $\lim_{|(a,b)|\to\infty} \Psi_u(a,b) = -\infty$. Therefore, it suffices to show that the maximum point of Ψ_u cannot be achieved on the boundary of $[0,\infty)\times[0,\infty)$. Suppose, by contradiction, that $(0,b)$ with $b\geq 0$ is a maximum point of Ψ_u. Then from (2.5), we have

$$a\frac{\mathrm{d}}{\mathrm{d}a}\left[I_\mu\left(au^+ + bu^-\right)\right] = \langle I'_\mu\left(au^+ + bu^-\right), au^+\rangle > 0$$

for small $a>0$, which means that Ψ_u is increasing with respect to a if $a>0$ is small enough. This yields a contradiction. Similarly, we can deduce that Ψ_u cannot achieve its global maximum on $(a,0)$ with $a\geq 0$. □

Lemma 2.2. *For any $u\in X$ with $u^\pm\neq 0$ such that $\langle I'_\mu(u), u^\pm\rangle \leq 0$, the unique maximum point (a_u, b_u) of Ψ_u on $[0,\infty)\times[0,\infty)$ satisfies $0<a_u, b_u\leq 1$.*

Proof. Here we will only prove $0<a_u\leq 1$. The proof of $0<b_u\leq 1$ is the same. For $u\in X$ with $u^\pm\neq 0$, by Lemma 2.1, there exist unique a_u and b_u such that $a_uu^+ + b_uu^- \in \mathcal{M}_\mu$. Without loss of generality, we may assume that $a_u\geq b_u>0$. Since $a_uu^+ + b_uu^- \in \mathcal{M}_\mu$. Then, we have that

$$\left\|a_uu^+\right\|^2 + a_u^2H(u) \geq \int_0^1 \left|a_uu^+\right|^p \ln\left|a_uu^+\right|^2\,\mathrm{d}x + \mu\int_0^1 f\left(a_uu^+\right)a_uu^+\,\mathrm{d}x. \tag{2.15}$$

Moreover, by $\langle I'_\mu(u), u^\pm\rangle \leq 0$, we have that

$$\left\|u^+\right\|^2 + H(u) \leq \int_0^1 \left|u^+\right|^p \ln\left|u^+\right|^2\,\mathrm{d}x + \mu\int_0^1 f\left(u^+\right)u^+\,\mathrm{d}x. \tag{2.16}$$

Therefore, from (2.15) and (2.16), it follows that

$$\int_0^1 \frac{\left|a_uu^+\right|^p\ln\left|a_uu^+\right|^2}{a_u^2}\,\mathrm{d}x - \int_0^1 \left|u^+\right|^p\ln\left|u^+\right|^2\mathrm{d}x \leq \mu\int_0^1 f\left(u^+\right)u^+\mathrm{d}x - \mu\int_0^1\frac{f\left(a_uu^+\right)a_uu^+}{a_u^2}\,\mathrm{d}x,$$

that is,

$$\int_0^1\left(\left|a_uu^+\right|^{p-2}\ln\left|a_uu^+\right|^2 - \left|u^+\right|^{p-2}\ln\left|u^+\right|^2\right)\left|u^+\right|^2\mathrm{d}x \leq \mu\int_0^1\left(\frac{f\left(u^+\right)}{u^+} - \frac{f\left(a_uu^+\right)}{a_uu^+}\right)\left(u^+\right)^2\mathrm{d}x.$$

Now, we suppose, by contradiction, that $a_u>1$. Since (f_4) and $p>2$, then $t\mapsto\frac{f(t)}{t}$ and $t\mapsto t^{p-2}\ln t^2$ are increasing for $t>0$, which implies that the last inequality is impossible. Thus, we conclude $0<a_u\leq 1$. □

Lemma 2.3. *For all $u\in\mathcal{M}_\mu$, there exists a positive number ρ independent of u such that*

(i) $\|u^\pm\|\geq\rho$*;*

(ii) $I_\mu(u)\geq\left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2$.

Proof. (i) We only prove that there exists a positive constant ρ independent of u such that $\|u^+\|\geq\rho$ for all $u\in\mathcal{M}_\mu$ and the result for $\|u^-\|$ is similar. By contradiction, for arbitrary small $\varepsilon>0$, there exists $\{u_\varepsilon\}\subset\mathcal{M}_\mu$ such that $\|u_\varepsilon^+\|<\varepsilon$. Letting $\varepsilon=1/n$ for large enough $n\in\mathbb{N}$, thus, we can suppose that there exists a sequence $\{u_n\}\subset\mathcal{M}_\mu$ such that $u_n^+\to 0$ in X. Since $\langle I'_\mu(u_n), u_n^+\rangle = 0$ holds. Then it follows from (1.5) and (2.4) that

$$\begin{aligned}\left\|u_n^+\right\|^2 &\leq \left\|u_n^+\right\|^2 + H(u_n) = \int_0^1\left|u_n^+\right|^p\ln\left|u_n^+\right|^2\,\mathrm{d}x + \mu\int_0^1 f\left(u_n^+\right)u_n^+\,\mathrm{d}x\\ &\leq \epsilon\int_0^1\left|u_n^+\right|^2\,\mathrm{d}x + C_1\int_0^1\left|u_n^+\right|^r\,\mathrm{d}x + \mu\epsilon\int_0^1\left|u_n^+\right|^2\,\mathrm{d}x + \mu C_2\int_0^1\left|u_n^+\right|^q e^{\alpha\left|u_n^+\right|^2}\,\mathrm{d}x.\end{aligned} \tag{2.17}$$

Let $s > 1$ with $1/s + 1/s' = 1$. Since $u_n^+ \to 0$ in X, then there exists $n_0 \in \mathbb{N}$ such that $\alpha s \|u_n^+\|^2 \le 2\pi\omega$ for all $n \ge n_0$. From Hölder's inequality and the fractional Trudinger–Moser inequality (1.3), we have

$$\int_0^1 |u_n^+|^q \exp\left(\alpha |u_n^+|^2\right) \, \mathrm{d}x \le \left(\int_0^1 |u_n^+|^{qs'} \, \mathrm{d}x\right)^{\frac{1}{s'}} \left(\int_0^1 e^{\alpha s \|u_n^+\|^2 (|u_n^+|/\|u_n^+\|)^2} \, \mathrm{d}x\right)^{\frac{1}{s}}$$
$$\le K_{\alpha s \|u_n^+\|^2} \left(\int_0^1 |u_n^+|^{qs'} \, \mathrm{d}x\right)^{\frac{1}{s'}}.$$

Combining (2.17) with the last inequality, we can deduce from the Sobolev embedding theorem that when $n \ge n_0$

$$\|u_n^+\|^2 \le (\epsilon + \mu\epsilon) C_6 \|u_n^+\|^2 + C_1 C_7 \|u_n^+\|^r + \mu C_2 K_{\alpha s \|u_n^+\|^2} C_8 \|u_n^+\|^q. \tag{2.18}$$

Choose appropriate $\epsilon > 0$ such that $1 - \mu\epsilon C_6 - \epsilon C_6 > 0$. Noticing that $2 < p < r$ and $2 < q$, we can deduce that (2.18) contradicts $u_n^+ \to 0$ in X.

(ii) Given $u \in \mathcal{M}_\mu$, by the definition of $\mathcal{M}_\mu$ and (f_3) we obtain

$$\begin{aligned} I_\mu(u) &= I_\mu(u) - \frac{1}{p}\langle I'_\mu(u), u\rangle \\ &= \frac{1}{2}\|u\|^2 + \frac{2}{p^2}\int_0^1 |u|^p \, \mathrm{d}x - \mu \int_0^1 F(u) \, \mathrm{d}x - \frac{1}{p}\|u\|^2 + \mu \frac{1}{p}\int_0^1 f(u)u \, \mathrm{d}x \\ &\ge \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|^2. \end{aligned}$$

Thus, we finish the proof. □

Lemma 2.3 tells that $I_\mu(u) > 0$ for all $u \in \mathcal{M}_\mu$. Therefore, I_μ is bounded below in $\mathcal{M}_\mu$, which means that $m_\mu = \inf_{u \in \mathcal{M}_\mu} I_\mu(u)$ is well-defined. The following lemma is about the asymptotic property of m_μ.

Lemma 2.4. *Let $m_\mu = \inf_{u \in \mathcal{M}_\mu} I_\mu(u)$, then $\lim_{\mu \to \infty} m_\mu = 0$.*

Proof. Fix $u \in X$ with $u^\pm \ne 0$. Then, by Lemma 2.1, for each $\mu > 0$ there exists a point pair (a_μ, b_μ) such that $a_\mu u^+ + b_\mu u^- \in \mathcal{M}_\mu$. Let

$$\mathcal{T}_u := \left\{(a_\mu, b_\mu) \in [0, \infty) \times [0, \infty) : T_u(a_\mu, b_\mu) = (0, 0), \mu > 0\right\},$$

where T_u is defined in (2.3).

Since $a_\mu u^+ + b_\mu u^- \in \mathcal{M}_\mu$, by assumption (f_3), (2.7) and (2.8), we have

$$\begin{aligned} \|a_\mu u^+\|^2 + \|b_\mu u^-\|^2 + 2a_\mu b_\mu H(u) &= \int_0^1 |a_\mu u^+ + b_\mu u^-|^p \ln |a_\mu u^+ + b_\mu u^-|^2 \, \mathrm{d}x \\ &\quad + \mu \int_0^1 f(a_\mu u^+ + b_\mu u^-)(a_\mu u^+ + b_\mu u^-) \, \mathrm{d}x \\ &\ge \frac{2a_\mu^p}{p}\int_0^1 |u^+|^p \, \mathrm{d}x + \frac{2b_\mu^p}{p}\int_0^1 |u^-|^p \, \mathrm{d}x - \frac{2}{p} - \mu\theta C_{\theta,2} \\ &\quad + \mu\theta C_{\theta,1} a_\mu^\theta \int_0^1 |u^+|^\theta \, \mathrm{d}x + \mu\theta C_{\theta,1} b_\mu^\theta \int_0^1 |u^-|^\theta \, \mathrm{d}x. \end{aligned}$$

From $\theta > p > 2$, it follows that the set $\mathcal{T}_\mu$ is bounded. Hence, if $\{\mu_n\} \subset (0, \infty)$ satisfies $\mu_n \to \infty$ as $n \to \infty$, then, up to a subsequence, there exist $\bar{a}, \bar{b} \geq 0$ such that $a_{\mu_n} \to \bar{a}$ and $b_{\mu_n} \to \bar{b}$.

Claim that $\bar{a} = \bar{b} = 0$. Suppose, by contradiction, that $\bar{a} > 0$ or $\bar{b} > 0$. For each $n \in \mathbb{N}$, $a_{\mu_n} u^+ + b_{\mu_n} u^- \in M_{\mu_n}$, we have $\langle I'_{\mu_n} \left(a_{\mu_n} u^+ + b_{\mu_n} u^-\right), a_{\mu_n} u^+ + b_{\mu_n} u^- \rangle = 0$, namely

$$\begin{aligned} \left\|a_{\mu_n} u^+ + b_{\mu_n} u^-\right\|^2 = & \int_0^1 \left|a_{\mu_n} u^+ + b_{\mu_n} u^-\right|^p \ln\left|a_{\mu_n} u^+ + b_{\mu_n} u^-\right|^2 \mathrm{d}x \\ & + \mu_n \int_0^1 f\left(a_{\mu_n} u^+ + b_{\mu_n} u^-\right)\left(a_{\mu_n} u^+ + b_{\mu_n} u^-\right) \mathrm{d}x. \end{aligned} \tag{2.19}$$

Note that $a_{\mu_n} u^+ \to \bar{a} u^+$ and $b_{\mu_n} u^- \to \bar{b} u^-$ in X, by (1.5) and the Lebesgue dominated convergence theorem, we have that

$$\int_0^1 \left|a_{\mu_n} u^+ + b_{\mu_n} u^-\right|^p \ln\left|a_{\mu_n} u^+ + b_{\mu_n} u^-\right|^2 \mathrm{d}x \to \int_0^1 \left|\bar{a} u^+ + \bar{b} u^-\right|^p \ln\left|\bar{a} u^+ + \bar{b} u^-\right|^2 \mathrm{d}x. \tag{2.20}$$

Once $\mu_n \to \infty$ as $n \to \infty$ and $\left\{a_{\mu_n} u^+ + b_{\mu_n} u^-\right\}$ is bounded in X, from (2.19), (2.20) and (f_3), it follows that

$$\begin{aligned} \left\|\bar{a} u^+ + \bar{b} u^-\right\|^2 = & \int_0^1 \left|\bar{a} u^+ + \bar{b} u^-\right|^p \ln\left|\bar{a} u^+ + \bar{b} u^-\right|^2 \mathrm{d}x \\ & + \left(\lim_{n\to\infty} \mu_n\right) \lim_{n\to\infty} \int_0^1 f\left(a_{\mu_n} u^+ + b_{\mu_n} u^-\right)\left(a_{\mu_n} u^+ + b_{\mu_n} u^-\right) \mathrm{d}x, \end{aligned}$$

which is impossible. Thus, $\bar{a} = \bar{b} = 0$, i.e., $a_{\mu_n} \to 0$ and $b_{\mu_n} \to 0$ as $n \to \infty$. Finally, by (f_3) and (2.19), we have $0 \leq m_\mu = \inf_{\mathcal{M}_\mu} I_\mu(u) \leq I_{\mu_n}\left(a_{\mu_n} u^+ + b_{\mu_n} u^-\right) \to 0$, from which we conclude the fact that $m_\mu \to 0$ as $\mu \to \infty$. □

Subsequently, we will prove that if the minimum of I_μ on $\mathcal{M}_\mu$ is achieved in some $u_0 \in \mathcal{M}_\mu$, then u_0 is a critical point of I_μ. The proof of this lemma follows from some arguments used in [12, 19], including the quantitative deformation lemma and Brouwer degree in $\mathbb{R}$.

Lemma 2.5. *If $u_0 \in \mathcal{M}_\mu$ satisfies $I_\mu(u_0) = m_\mu$, then $I'_\mu(u_0) = 0$.*

Proof. Since $u_0 \in \mathcal{M}_\mu$, we have $\langle I'_\mu(u_0), u_0^+\rangle = \langle I'_\mu(u_0), u_0^-\rangle = 0$. By Lemma 2.1, for $(\alpha, \beta) \in (\mathbb{R}_+ \times \mathbb{R}_+) \backslash (1,1)$, we have

$$I_\mu\left(\alpha u_0^+ + \beta u_0^-\right) < I_\mu\left(u_0^+ + u_0^-\right) = m_\mu. \tag{2.21}$$

Arguing by contradiction. We assume that $I'_\mu(u_0) \neq 0$. For the continuity of I'_μ, there exists $\iota, \delta > 0$ such that

$$\left\|I'_\mu(v)\right\| \geq \iota, \qquad \text{for all } \|v - u_0\| \leq 3\delta. \tag{2.22}$$

Choose $\tau \in (0, \min\{1/2, \delta/(\sqrt{2}\|u_0\|)\})$. Let $D = (1-\tau, 1+\tau) \times (1-\tau, 1+\tau)$ and $g(\alpha, \beta) = \alpha u_0^+ + \beta u_0^-$ for all $(\alpha, \beta) \in D$. By virtue of (2.21), it is easy to see that

$$\overline{m}_\mu := \max_{\partial D} I_\mu \circ g < m_\mu. \tag{2.23}$$

Indeed, let $\epsilon := \min\{(m_\mu - \overline{m}_\mu)/3, \iota\delta/8\}$, $S_\delta := B(u_0, \delta)$ and $I_\mu^c := I_\mu^{-1}((-\infty, c])$. And according to the quantitative deformation lemma [26, Lemma 2.3], there exists a deformation $\eta \in C([0,1] \times X, X)$ such that:

(i) $\eta(1,v)=v$, if $v\notin I_\mu^{-1}\left([m_\mu-2\epsilon,m_\mu+2\epsilon]\right)\cap S_{2\delta}$,

(ii) $\eta\left(1,I_\mu^{m_\mu+\epsilon}\cap S_\delta\right)\subset I_\mu^{m_\mu-\epsilon}$,

(iii) $I_\mu(\eta(1,v))\le I_\mu(v)$, for all $v\in X$.

Since $I_\mu(g(\alpha,\beta))\le m_\mu$ and $g(\alpha,\beta)\in S_\delta$ for $(\alpha,\beta)\in\overline{D}$, then it follows from (ii) that

$$\max_{(\alpha,\beta)\in\overline{D}} I_\mu(\eta(1,g(\alpha,\beta)))\le m_\mu-\epsilon. \tag{2.24}$$

In this way, we obtain a contradiction to (2.24) from the definition of m_μ if we could prove that $\eta(1,g(D))\cap\mathcal{M}_\mu$ is nonempty. Thus we complete the proof of this lemma. To do this, we first define

$$\begin{aligned}\bar{g}(\alpha,\beta)&:=\eta(1,g(\alpha,\beta)),\\ \Psi_0(\alpha,\beta)&=\left(\langle I_\mu'(g(\alpha,\beta)),u_0^+\rangle,\langle I_\mu'(g(\alpha,\beta)),u_0^-\rangle\right)\\ &=\left(\langle I_\mu'\left(\alpha u_0^++\beta u_0^-\right),u_0^+\rangle,\langle I_\mu'\left(\alpha u_0^++\beta u_0^-\right),u_0^-\rangle\right)\\ &:=\left(\varphi_u^1(\alpha,\beta),\varphi_u^2(\alpha,\beta)\right),\end{aligned}$$

and

$$\Psi_1(\alpha,\beta):=\left(\frac{1}{\alpha}\langle I_\mu'(\bar{g}(\alpha,\beta)),(\bar{g}(\alpha,\beta))^+\rangle,\frac{1}{\beta}\langle I_\mu'(\bar{g}(\alpha,\beta)),(\bar{g}(\alpha,\beta))^-\rangle\right).$$

Moreover, a straightforward calculation, based on $u_0\in\mathcal{M}_\mu$, shows that

$$\begin{aligned}\left.\frac{\partial\varphi_u^1(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)}&=\|u_0^+\|^2-(p-1)\int|u_0^+|^p\ln|u_0^+|^2\,\mathrm{d}x-2\int|u_0^+|^p\,\mathrm{d}x\\ &\quad-\mu\int_0^1 f'\left(u_0^+\right)|u_0^+|^2\,\mathrm{d}x\\ &=(2-p)\int_0^1|u_0^+|^p\ln|u_0^+|^2\,\mathrm{d}x+\mu\int_0^1 f\left(u_0^+\right)u_0^+\,\mathrm{d}x\\ &\quad-2\int_0^1|u_0^+|^p\,\mathrm{d}x-\mu\int_0^1 f'\left(u_0^+\right)\left(u_0^+\right)^2\,\mathrm{d}x-H(u)\end{aligned}$$

and

$$\left.\frac{\partial\varphi_u^1(\alpha,\beta)}{\partial\beta}\right|_{(1,1)}=H(u).$$

Similarly,

$$\left.\frac{\partial\varphi_u^2(\alpha,\beta)}{\partial\alpha}\right|_{(1,1)}=H(u)$$

and

$$\begin{aligned}\left.\frac{\partial\varphi_u^2(\alpha,\beta)}{\partial\beta}\right|_{(1,1)}&=\|u_0^-\|^2-(p-1)\int|u_0^-|^p\ln|u_0^-|^2\,\mathrm{d}x-2\int|u_0^-|^p\,\mathrm{d}x\\ &\quad-\mu\int_0^1 f'\left(u_0^-\right)|u_0^-|^2\,\mathrm{d}x\\ &=(2-p)\int_0^1|u_0^-|^p\ln|u_0^-|^2\,\mathrm{d}x+\mu\int_0^1 f\left(u_0^-\right)u_0^-\,\mathrm{d}x\\ &\quad-2\int_0^1|u_0^-|^p\,\mathrm{d}x-\mu\int_0^1 f'\left(u_0^-\right)\left(u_0^-\right)^2\,\mathrm{d}x-H(u).\end{aligned}$$

Let

$$H = \begin{bmatrix} \left.\frac{\partial \varphi_u^1(\alpha,\beta)}{\partial \alpha}\right|_{(1,1)}, & \left.\frac{\partial \varphi_u^2(\alpha,\beta)}{\partial \alpha}\right|_{(1,1)} \\ \left.\frac{\partial \varphi_u^1(\alpha,\beta)}{\partial \beta}\right|_{(1,1)}, & \left.\frac{\partial \varphi_u^2(\alpha,\beta)}{\partial \beta}\right|_{(1,1)} \end{bmatrix}.$$

Then we deduce that $\det H \neq 0$. Therefore, Ψ_0 is a C^1 function with the point pair $(1,1)$ being the unique isolated zero point in D. By using the Brouwer's degree in $\mathbb{R}$, we deduce that $\deg(\Psi_0, D, 0) = 1$.

Now, it follows from (2.24) and (i) that $g(\alpha,\beta) = \bar{g}(\alpha,\beta)$ on ∂D. For the boundary dependence of Brouwer's degree (see [20, Theorem 4.5]), there holds $\deg(\Psi_1, D, 0) = \deg(\Psi_0, D, 0) = 1$. Therefore, there exists some $(\bar{\alpha}, \bar{\beta}) \in D$ such that

$$\eta(1, g(\bar{\alpha}, \bar{\beta})) \in \mathcal{M}_\mu.$$

So we obtain a contradiction to (2.24). □

3 Subcritical case

Lemma 3.1 (Subcritical case)**.** *If $\{u_n\} \subset \mathcal{M}_\mu$ is a minimizing sequence for m_μ, then there exists some $u \in X$ such that*

$$\int_0^1 f\left(u_n^\pm\right) u_n^\pm \,\mathrm{d}x \to \int_0^1 f\left(u^\pm\right) u^\pm \,\mathrm{d}x \;\; and \;\; \int_0^1 F\left(u_n^\pm\right) \,\mathrm{d}x \to \int_0^1 F\left(u^\pm\right) \,\mathrm{d}x.$$

Proof. We will only prove the first result. Since the second limit is a direct consequence of the first one, we omit it here.

Let sequence $\{u_n\} \subset \mathcal{M}_\mu$ be a minimizing sequence such that $\lim_{n\to\infty} I_\mu(u_n) = m_\mu$. Thus, $\{u_n\}$ is bounded in X by Lemma 2.3. It follows from Proposition 2.2 in [14] to that $\{u_n\}$ is bounded in $H^{1/2}(\mathbb{R})$ as well. By [8, Theorem 7.1 and Theorem 6.10], passing to a subsequence we may assume that $u_n \rightharpoonup u$ weakly in both X and $H^{1/2}(\mathbb{R})$, and that $u_n \to u$ in $L^q(0,1)$ for all $q \geq 1$ and $u_n(x) \to u(x)$ a.e. in $(0,1)$. Thus,

$$\begin{aligned} u_n^\pm &\rightharpoonup u^\pm \quad \text{weakly in } X, \\ u_n^\pm &\to u^\pm \quad \text{in } L^q(0,1) \text{ for } q \in [1,\infty), \\ u_n^\pm &\to u^\pm \quad \text{a.e. in } (0,1). \end{aligned} \tag{3.1}$$

Note that by (2.4), we have

$$f\left(u_n^\pm(x)\right) u_n^\pm(x) \leq \epsilon \left|u_n^\pm(x)\right|^2 + C_2 \left|u_n^\pm(x)\right|^q e^{\alpha |u_n^\pm(x)|^2} =: h\left(u_n^\pm(x)\right), \tag{3.2}$$

for all $\alpha > \alpha_0 = 0$ and $q > 2$. It is sufficient to prove that sequence $\{h(u_n^\pm)\}$ is convergent in $L^1(0,1)$.

Choosing $s, s' > 1$ with $1/s + 1/s' = 1$, by (3.1), we get that

$$|u_n^\pm|^q \to |u^\pm|^q \quad \text{in } L^{s'}(0,1). \tag{3.3}$$

In particular, there exists $c_5 > 0$ such that $\|u_n^\pm\|^2 \leq c_5$ for all $n \in \mathbb{N}$. Choosing $0 < \alpha < 2\pi\omega/sc_5$, by the fractional Trudinger–Moser inequality (1.3), we get

$$\int_0^1 e^{\alpha s |u_n^\pm|^2} \,\mathrm{d}x \leq \int_0^1 e^{\alpha s c_5 (u_n^\pm/\|u_n^\pm\|)^2} \,\mathrm{d}x \leq K_{\alpha s c_5}. \tag{3.4}$$

By reflexivity of $L^s(0,1)$, passing to a subsequence, we have

$$e^{\alpha|u_n^\pm|^2} \rightharpoonup e^{\alpha|u^\pm|^2} \quad \text{weakly in } L^s(0,1). \tag{3.5}$$

Hence, by (3.3), (3.5) and [16, Lemma 4.8, Chapter 1], we conclude that

$$\int_0^1 f\left(u_n^\pm\right) u_n^\pm \,\mathrm{d}x \to \int_0^1 f\left(u^\pm\right) u^\pm \,\mathrm{d}x.$$

This completes the proof. □

Lemma 3.2 (Subcritical case)**.** *There exists some $u_\mu \in \mathcal{M}_\mu$ such that $I_\mu(u_\mu) = m_\mu$.*

Proof. As indicated earlier that $m_\mu > 0$. In what follows, we only need to show that m_μ is achieved. By the definition of $m_\mu = \inf_{u\in\mathcal{M}_\mu} I_\mu(u)$, there exists a sequence $\{u_n\} \subset \mathcal{M}_\mu$ such that

$$\lim_{n\to\infty} I_\mu\left(u_n\right) = m_\mu.$$

On the one hand, (3.1) and the Vitali convergence theorem yield that

$$\lim_{n\to\infty} \int_0^1 |u_n|^p \ln|u_n|^2 \,\mathrm{d}x \to \int_0^1 |u|^p \ln|u|^2 \,\mathrm{d}x. \tag{3.6}$$

On the other hand, it follows from (3.1) that $u_n \to u$ in $L^p(0,1)$, we have

$$\lim_{n\to\infty} \int_0^1 |u_n|^p \,\mathrm{d}x \to \int_0^1 |u|^p \,\mathrm{d}x. \tag{3.7}$$

Lemma 2.1 implies $I_\mu\left(\alpha u_n^+ + \beta u_n^-\right) \le I_\mu\left(u_n\right)$ for all $\alpha, \beta \ge 0$. So, by using the Brezis–Lieb Lemma, Fatou's Lemma, (3.6), (3.7) and Lemma 3.1, we get

$$\begin{aligned}
\liminf_{n\to\infty} I_\mu\left(\alpha u_n^+ + \beta u_n^-\right) \ge{}& \frac{\alpha^2}{2}\lim_{n\to\infty}\left(\|u_n^+ - u^+\|^2 + \|u^+\|^2\right) + \frac{\beta^2}{2}\lim_{n\to\infty}\left(\|u_n^- - u^-\|^2 + \|u^-\|^2\right) \\
&+ \alpha\beta\liminf_{n\to\infty} H\left(u_n\right) - \mu\int_0^1 F\left(\alpha u^+\right)\mathrm{d}x - \mu\int_0^1 F\left(\beta u^-\right)\mathrm{d}x \\
&+ \frac{2}{p^2}\int_0^1 |\alpha u^+ + \beta u^-|^p \,\mathrm{d}x - \frac{1}{p}\int_0^1 |\alpha u^+ + \beta u^-|^p \ln|\alpha u^+ + \beta u^-|^2 \,\mathrm{d}x \\
\ge{}& I_\mu\left(\alpha u^+ + \beta u^-\right) + \frac{\alpha^2}{2}A_1 + \frac{\beta^2}{2}A_2,
\end{aligned}$$

where $A_1 = \lim_{n\to\infty}\|u_n^+ - u^+\|^2$, $A_2 = \lim_{n\to\infty}\|u_n^- - u^-\|^2$. So, for all $\alpha \ge 0$ and all $\beta \ge 0$, one has that

$$m_\mu \ge I_\mu\left(\alpha u^+ + \beta u^-\right) + \frac{\alpha^2}{2}A_1 + \frac{\beta^2}{2}A_2. \tag{3.8}$$

Firstly, we prove that $u^\pm \ne 0$. Here we only prove $u^+ \ne 0$ since $u^- \ne 0$ is analogous, by contradiction, we assume $u^+ = 0$. Hence, let $\beta = 0$ in (3.8) and we have that

$$m_\mu \ge \frac{\alpha^2}{2}A_1 \quad \text{for all } \alpha \ge 0. \tag{3.9}$$

If $A_1 = 0$, that is, $u_n^+ \to u^+$ in X. Lemma 2.3(i) implies $\|u^+\| > 0$, which contradicts supposition. If $A_1 > 0$, by (3.9) we get $m_\mu \ge \frac{\alpha^2}{2}A_1$ for all $\alpha \ge 0$, which is a contradiction by Lemma 2.4. That is, we deduce that $u^+ \ne 0$.

Lastly, we prove that m_μ is achieved. By Lemma 2.1, there exists $(s_u, t_u) \in (0,\infty)\times(0,\infty)$ such that $u_\mu := s_u u^+ + t_u u^- \in \mathcal{M}_\mu$, that is,

$$\langle I'_\mu\left(s_u u^+ + t_u u^-\right), s_u u^+\rangle = 0 = \langle I'_\mu\left(s_u u^+ + t_u u^-\right), t_u u^-\rangle.$$

We now claim that $0 < s_u, t_u \le 1$. Indeed, by $\{u_n\} \subset \mathcal{M}_\mu$, we have $\langle I'_\mu(u_n), u_n^\pm\rangle = 0$, that is,

$$\|u_n^\pm\|^2 + H(u_n) = \int_0^1 |u_n^\pm|^p \ln|u_n^\pm|^2 \, \mathrm{d}x + \mu\int_0^1 f\left(u_n^\pm\right) u_n^\pm \, \mathrm{d}x.$$

Therefore, by the weak lower semicontinuity of norm, Fatou's lemma, (3.6), and Lemma 3.1 we have

$$\|u^\pm\|^2 + H(u) \le \int_0^1 |u^\pm|^p \ln|u^\pm|^2 \, \mathrm{d}x + \mu\int_{\mathbb{R}^3} f\left(u^\pm\right) u^\pm \, \mathrm{d}x.$$

That is,

$$\langle I'_\mu(u), u^\pm\rangle \le \liminf_{n\to\infty}\langle I'_\mu(u_n), u_n^\pm\rangle = 0. \tag{3.10}$$

By (3.10) and similar to the proof in Lemma 2.2, we have $s_u, t_u \le 1$.

Our next step is show that $I_\mu(u_\mu) = m_\mu$. Remark 1.1 shows that $H(s) := sf(s) - pF(s)$ is a nonnegative function, increasing in $|s|$. Hence, by the weaker lower semicontinuity of norm, (3.7), Remark 1.1, $\mu > 0$ and Lemma 3.1, we get

$$\begin{aligned}
m_\mu \le I_\mu(u_\mu) &= I_\mu(u_\mu) - \frac{1}{p}\langle I'_\mu(u_\mu), u_\mu\rangle \\
&= \left(\frac{1}{2}-\frac{1}{p}\right)\|u_\mu\|^2 + \frac{2}{p^2}\int_0^1 |u_\mu|^p \, \mathrm{d}x + \frac{\mu}{p}\int_0^1 \left[f(u_\mu)u_\mu - pF(u_\mu)\right] \mathrm{d}x \\
&= \left(\frac{1}{2}-\frac{1}{p}\right)\|s_u u^+\|^2 + \left(\frac{1}{2}-\frac{1}{p}\right)\|t_u u^-\|^2 + 2\left(\frac{1}{2}-\frac{1}{p}\right)s_u t_u H(u) \\
&\quad + \frac{2}{p^2}s_u^p\int_0^1 |u^+|^p \, \mathrm{d}x + \frac{2}{p^2}t_u^p\int_0^1 |u^-|^p \, \mathrm{d}x \\
&\quad + \frac{\mu}{p}\int_0^1 \left(f\left(s_u u^+\right)s_u u^+ - pF\left(s_u u^+\right)\right) \mathrm{d}x \\
&\quad + \frac{\mu}{p}\int_0^1 \left(f\left(t_u u^-\right)t_u u^- - pF\left(t_u u^-\right)\right) \mathrm{d}x \\
&\le \left(\frac{1}{2}-\frac{1}{p}\right)\|u\|^2 + \frac{2}{p^2}\int_0^1 |u|^p \, \mathrm{d}x + \frac{\mu}{p}\int_0^1 \left(f(u)u - pF(u)\right) \mathrm{d}x \\
&\le \liminf_{n\to\infty}\left[\left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2 + \frac{2}{p^2}\int_0^1 |u_n|^p \, \mathrm{d}x + \frac{\mu}{p}\int_0^1 \left(f(u_n)u_n - pF(u_n)\right) \mathrm{d}x\right] \\
&= \liminf_{n\to\infty}\left(I_\mu(u_n) - \frac{1}{p}\langle I'_\mu(u_n), u_n\rangle\right) \\
&= \liminf_{n\to\infty} I_\mu(u_n) = m_\mu,
\end{aligned}$$

and if $s_u < 1$ or $t_u < 1$, then the above inequality is strict. Hence, it follows that $s_u = t_u = 1$. Thus, $u_\mu \in \mathcal{M}_\mu$ and $I_\mu(u_\mu) = m_\mu$. This completes the proof. □

Proof of Theorem 1.2. From Lemma 2.5 and Lemma 3.2, we deduce that u_μ is a least energy sign-changing solution for problem (1.1). □

4 Critical case

Lemma 4.1 (Critical case). *There exists $\mu^* > 0$ such that if $\mu \geq \mu^*$ and $\{u_n\} \subset \mathcal{M}_\mu$ is a minimizing sequence for m_μ, then*

$$\int_0^1 f\left(u_n^\pm\right) u_n^\pm \,\mathrm{d}x \to \int_0^1 f\left(u^\pm\right) u^\pm \,\mathrm{d}x \quad \textit{and} \quad \int_0^1 F\left(u_n^\pm\right) \,\mathrm{d}x \to \int_0^1 F\left(u^\pm\right) \,\mathrm{d}x$$

hold for some $u \in X$.

Proof. Arguing as in Lemma 3.1, it is sufficient to prove that $\{h\left(u_n^\pm\right)\}$ is convergent in $L^1(0,1)$ for appropriate $\mu > 0$, where $\{h\left(u_n^\pm(x)\right)\}$ is defined in (3.2).

Let sequence $\{u_n\} \subset \mathcal{M}_\mu$ satisfy $\lim_{n\to\infty} I_\mu\left(u_n\right) = m_\mu$ and $\nu > 0$. Since Lemma 2.3(ii) and Lemma 2.4, there exists $\mu^* > 0$ such that when $\mu \geq \mu^*$, there holds

$$\limsup_{n\to\infty} \|u_n^\pm\|^2 < \frac{\pi\omega}{\alpha_0 + \nu}. \tag{4.1}$$

Now, considering $s, s' > 1$ with $1/s + 1/s' = 1$ and s close to 1 , we get that

$$|u_n^\pm|^q \to |u^\pm|^q \quad \text{in } L^{s'}(0,1). \tag{4.2}$$

Moreover, choosing $\alpha = \alpha_0 + \nu$, from (4.1), we get that

$$\int_0^1 e^{\alpha s|u_n^\pm(x)|^2} \mathrm{d}x = \int_0^1 e^{(\alpha_0+\nu)s|u_n^\pm(x)|^2} \mathrm{d}x \leq \int_0^1 e^{\pi\omega s(|u_n^\pm|/\|u_n^\pm\|)^2} \mathrm{d}x.$$

It follows from $s > 1$ close to 1 and the fractional Trudinger–Moser inequality (1.3) that there exists $K_{\pi\omega s} > 0$ such that

$$\int_0^1 e^{\alpha s|u_n^\pm(x)|^2} \,\mathrm{d}x \leq K_{\pi\omega s}. \tag{4.3}$$

Since $e^{\alpha|u_n^\pm(x)|^2} \to e^{\alpha|u^\pm(x)|^2}$ a.e. in $(0,1)$. From (4.3) and [16, Lemma 4.8, Chapter 1], we obtain that

$$e^{\alpha|u_n^\pm|^2} \rightharpoonup e^{\alpha|u^\pm|^2} \quad \text{weakly in } L^s(0,1). \tag{4.4}$$

Hence, by (4.2), (4.4) and [16, Lemma 4.8, Chapter 1] again, we conclude that

$$\int_0^1 f\left(u_n^\pm\right) u_n^\pm \,\mathrm{d}x \to \int_0^1 f\left(u^\pm\right) u^\pm \,\mathrm{d}x.$$

Hence, we complete the proof. □

Lemma 4.2 (Critical case). *If $\mu \geq \mu^*$, then there exists some $u_\mu \in \mathcal{M}_\mu$ such that $I_\mu(u_\mu) = m_\mu$.*

Proof. By an argument similar to Lemma 3.2, replacing Lemma 3.1 by Lemma 4.1, we can obtain the same conclusion. □

Proof of Theorem 1.3. From Lemma 2.5 and Lemma 4.2, we deduce that u_μ is a least energy sign-changing solution for problem (1.1). □

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Ground state solution for fractional problem with critical combined nonlinearities

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Abstract. This paper is concerned with the following nonlocal problem with combined critical nonlinearities

$$(-\Delta)^s u = -\alpha|u|^{q-2}u + \beta u + \gamma|u|^{2_s^*-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \mathbb{R}^N\backslash\Omega,$$

where $s \in (0,1)$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ is a bounded $C^{1,1}$ domain with Lipschitz boundary, α is a positive parameter, $q \in (1,2)$, β and γ are positive constants, and $2_s^* = 2N/(N-2s)$ is the fractional critical exponent. For $\gamma > 0$, if $N \geqslant 4s$ and $0 < \beta < \lambda_{1,s}$, or $N > 2s$ and $\beta \geqslant \lambda_{1,s}$, we show that the problem possesses a ground state solution when α is sufficiently small.

Keywords: fractional problem, ground state solution, critical combined nonlinearities.

2020 Mathematics Subject Classification: 35A15, 35D30, 35J60.

1 Introduction

In this paper, we study ground state solution for the following fractional equation

$$\begin{cases} (-\Delta)^s u = -\alpha|u|^{q-2}u + \beta u + \gamma|u|^{2_s^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where $s \in (0,1)$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ is a bounded $C^{1,1}$ domain with Lipschitz boundary, $\alpha > 0$ is a parameter, $q \in (1,2)$, β and γ are positive constants, and $2_s^* = 2N/(N-2s)$ is the fractional critical exponent. The equation (1.1) is driven by the fractional Laplacian $(-\Delta)^s$ and exhibits combined nonlinearities and linear perturbation. $(-\Delta)^s$ is the nonlocal operator defined as follows

$$(-\Delta)^s u(x) := 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N\backslash B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where $B_\varepsilon(x)$ denotes the open ball centered at x and of radius $\varepsilon > 0$. The operator $(-\Delta)^s$ arises in physics, biology, chemistry and finance and can be seen as the infinitesimal generators of

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Lévy stable diffusion process [3,4]. And, $\left(-\Delta+m^2\right)^{\frac{1}{2}}$ appears naturally in quantum mechanics, where m is the mass of the particle under consideration [35]. The study of nonlinear equations involving a fractional Laplacian has attracted much attention from many mathematicians working in different fields. We refer to [5,9,12,14–17,19,23–25,27–33,36,38–43] for more details on the fractional operator and applications.

From [42] we get that the spectrum of $(-\Delta)^s$ on $X_0^s(\Omega)$ consists of a sequence of eigenvalues $\{\lambda_{j,s}\}$ satisfying

$$0<\lambda_{1,s}<\lambda_{2,s}\leqslant\lambda_{3,s}\leqslant\ldots\leqslant\lambda_{j,s}\leqslant\lambda_{j+1,s}\leqslant\ldots,\quad \lambda_{j,s}\to\infty\quad\text{as } j\to\infty,$$

where the space $X_0^s(\Omega)$ is given in [40].

For the problem (1.1), when $\alpha=0$ and $\gamma=1$, the equation is a fractional critical problem with linear perturbation term. For the critical problem, due to a lack of compactness occurs, there are serious difficulties when we try to find critical points by variational methods. Motivated by the pioneering work of Brezis and Nirenberg [8], the nonlocal fractional counterpart of the Laplacian equations involving critical nonlinearity were studied in [38–43], their model is the equation

$$\begin{cases}(-\Delta)^s u=\beta u+|u|^{2_s^*-2}u & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\backslash\Omega.\end{cases}\tag{1.2}$$

Servadei and Valdinoci have showed that problem (1.2) admits a nontrivial solution in the following case:

(i) $N>4s$ and $\beta>0$;

(ii) $N=4s$ and $\beta\neq\lambda_{k,s}, k=1,2,\ldots$;

(iii) $2s<N<4s$ and β is sufficiently large.

Moreover, the multiplicity result of (1.2) was proved by Fiscella et al. [24], where it was shown the number of solutions is at least twice the multiplicity of the $\lambda_{k,s}$, provided that β lies in a suitable neighborhood of $\lambda_{k,s}$, the authors also gave an estimate of the length of this neighborhood. Figueiredo et al. [23] proved the problem (1.2) has at least $cat_\Omega(\Omega)$ nontrivial solutions if $N\geqslant 4s$ and β is sufficiently small. For interesting results on the fractional Brezis–Nirenberg problem, we refer to [12,27] and the references therein.

For the problem (1.1), when $\alpha<0$, $\beta=0$ and $\gamma=1$, the equation contains a sublinear term $|u|^{q-2}u$ and a critical superlinear term $|u|^{2_s^*-2}u$, it belongs to the class of problems with competing nonlinearities, for instance sublinear-superlinear. An early example in this direction was given in [26] for the p-Laplacian operator. Other results for the classical Laplacian operator can be found in [1,6,13]. More generally, the problem with completely nonlinear operators has been studied in [10]. And we observed that Barrios et al. [5] have studied the critical fractional problem with concave-convex power nonlinearities, where they considered the following problem

$$\begin{cases}(-\Delta)^s u=-\alpha u^{q-1}+u^{2_s^*-1},\ u>0 & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\backslash\Omega.\end{cases}\tag{1.3}$$

Main results show the existence and multiplicity of solutions to problem (1.3) for different values of α. To be more precise, assume that $N>2s$, then there is $\alpha_3<0$, such that problem (1.3):

(i) has no solution for $\alpha < \alpha_3$;

(ii) if $\alpha = \alpha_3$ there exists at least one solution;

(iii) for $\alpha_3 < \alpha < 0$, there are at least two solutions, one of them is a minimal solution.

We refer to [15, 16, 21, 30] and references therein for more fractional problem with competing nonlinearities.

For the problem (1.1), if u in the critical term is the positive part of u, the problem becomes a nonlocal Dirichlet problem with asymmetric nonlinearities, that is

$$\begin{cases} (-\Delta)^s u = -\alpha |u|^{q-2}u + \beta u + \gamma (u^+)^{2_s^*-1} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.4}$$

Miyagaki et al. [36] studied the existence of at least three nontrivial solutions for problem (1.4). The corresponding local problem was studied by de Paiva and Presoto [37]. The study of equations with critical exponent and asymmetric nonlinearities was initiated by De Figueiredo and Yang [18] to investigate Ambrosetti–Prodi type problems involving critical growth. The Ambrosetti–Prodi type problems have a strong physical meaning because it appears in quantum mechanics models with asymmetric nonlinearities, see for instance [9, 11, 20, 28] and references therein. It can be seen from [36, Theorem 6], the two constant sign solutions of (1.4) are solutions for two corresponding auxiliary problems which are similar to problem (1.1). So solution of the problem (1.1) is valuable to study the Ambrosetti–Prodi type problem.

Motivated by the above works, in this paper, we consider the existence of ground state solutions of (1.1) which is affected by combined nonlinearities and linear perturbation. Our first main result can be stated as follows.

Theorem 1.1. *Let $\gamma > 0$, then there exists $\alpha_1 > 0$, such that for any $\alpha \in (0, \alpha_1)$, problem (1.1) has a ground state solution u_{m_α}, provided that*

- *$N \geqslant 4s$ and $0 < \beta < \lambda_{1,s}$ or*
- *$N > 2s$ and $\beta \geqslant \lambda_{1,s}$.*

It is well known that ground state solutions have important applications. For instance, to obtain the optimal constant in the Sobolev inequality and the interpolation estimates of the Gagliardo–Nirenberg inequality. To possess a global solution of nonlinear Schrödinger equation when L^2-norm of the initial value is sufficiently small. To overcome the loss of compactness when we consider some Schrödinger equation with potential and so on. There are several ways to get the ground state solution. The one in Theorem 1.1 is found by looking for the point at which infimum of the functional on Nehari manifold is attainable. Furthermore, under the same assumptions, we show that the functional possesses mountain pass geometry. By estimate of the minimax level, we have the following theorem.

Theorem 1.2. *Assume that the hypotheses of Theorem 1.1 are satisfied, problem (1.1) has a mountain pass ground state solution u_{c_α}.*

It is observed that there are some differences between the cases $\alpha = 0$ and $\alpha > 0$. Indeed, assume that $2s < N < 4s$. In case of $\alpha = 0$, the problem (1.1) translates into problem (1.2). Servadei et al. [39] have showed that problem (1.2) has a nontrivial solution when β is sufficiently large. If $\alpha > 0$ is small enough, owing to influence of sublinear term, Theorem 1.1

and Theorem 1.2 state that the problem (1.1) has solutions as long as $\beta \geqslant \lambda_{1,s}$ holds. Suppose that $N = 4s$, the problem (1.1) has solutions for any $\beta > 0$, which is also different from $\beta \neq \lambda_{k,s}, k = 1, 2, \ldots$ when $\alpha = 0$.

There are some similarities between the cases $\alpha < 0$ and $\alpha > 0$. Note that $\alpha < 0$ in problem (1.3), Barrios et al. [5] indicate that problem (1.3) has solutions when α is close to zero. For problem (1.1), if $\alpha < 0$ and $\beta \geqslant \lambda_{1,s}$, then it is easy to verify that it has no nontrivial solution since the corresponding Nehari manifold is empty, and it is unknown whether the Nehari manifold is nonempty in the case $0 < \beta < \lambda_{1,s}$. Thus in the present paper we study the case of $\alpha > 0$. Even though the sign of sublinearity in problem (1.1) is opposite to that of problem (1.3), Theorem 1.1 and 1.2 show that the problem (1.1) has ground state solution when α is small enough.

It can be seen from the comparison above, Theorem 1.1 and 1.2 are not only effective supplement to the main results of Barrios et al. [5], but also have some differences with Servadei et al. [39]. To the best of our knowledge, these results are novel and meaningful.

Since the problem (1.1) is affected by sublinearity, linearity and critical superlinearity at the same time, we have a different situation from (1.2) or (1.3). The minimax principle used by Servadei et al. in [38–43] and the method of obtaining the minimal solution in [5] cannot be applied directly to problem (1.1). Some other techniques and methods are used. In the proof of Theorem 1.1, an abstract result for existence of constrained extrema is used. So it is necessary to obtain that the infimum of the functional on the Nehari manifold is strictly less than admissible threshold for the (PS) condition. To confirm this result, a crucial point is to show a sufficiently small upper bound for the quotient

$$\frac{\|u_\varepsilon\|^2 - \beta \|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2_s^*}^2} \tag{1.5}$$

when $\varepsilon > 0$ is sufficiently small, and u_ε is given in [43]. The estimation of (1.5) in Lemma 3.1 is meticulous. In the proof of Theorem 1.2, due to influence of the sublinear term, it seems impossible to prove that the functional has mountain pass geometry directly according to the structures of the functional and the properties of $X_0^s(\Omega)$. We prove that 0 is the local minimum point of the functional in a special subspace of $X_0^s(\Omega)$.

The organization of this paper is as follows: In Section 2, we introduce some notations and preliminary lemmas which are needed later. Section 3 and Section 4 are devoted to the proof of Theorems 1.1–1.2, respectively.

2 Preliminaries

In this section, we recall a few notions and results that will be used later on. Throughout the paper, $|A|$ denotes the N-dimensional Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$, $L^r(\Omega)$ is usual Lebesgue space endowed with the norm $\|\cdot\|_r$ for $1 \leq r < \infty$. We recall that the Gagliardo seminorm of a measurable function $u : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$[u]_s := \left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy \right)^{1/2}$$

provided the integral is finite. The fractional Sobolev space $H^s\left(\mathbb{R}^N\right)$ is introduced in [19] as

$$H^s\left(\mathbb{R}^N\right) := \left\{ u \in L^2\left(\mathbb{R}^N\right) : [u]_s < \infty \right\}$$

endowed with the norm $\|u\|_{H^s} = \left(\|u\|_2^2 + [u]_s^2\right)^{1/2}$ making it a Hilbert space. The relevant space to problem (1.1) is the closed subspace of $H^s(\mathbb{R}^N)$ given by

$$X_0^s(\Omega) := \left\{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\},$$

this Hilbert space was introduced in [40] with the scalar product

$$\langle u, v \rangle_{X_0^s(\Omega)} = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dxdy$$

inducing the equivalent norm $\|\cdot\| = [\cdot]_s$.

It is known from [19], the following embedding results hold true:

$$\begin{aligned} &X_0^s(\Omega) \hookrightarrow L^v(\Omega) \quad \text{compactly for any } v \in [1, 2_s^*), \\ &X_0^s(\Omega) \hookrightarrow L^{2_s^*}(\Omega) \quad \text{continuously.} \end{aligned} \tag{2.1}$$

And the constant

$$S_s = \inf_{u \in H^s(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy}{\left(\int_{\mathbb{R}^N} |u(x)|^{2_s^*} dx\right)^{2/2_s^*}} \tag{2.2}$$

is finite, by [14, Theorem 1.1] we know that S_s is attained by the function $(1/\|\tilde{u}\|_{2_s^*})\tilde{u}$ with $\tilde{u}(x) = (1 + |x|^2)^{-\frac{N-2s}{2}}, x \in \mathbb{R}^N$. For every $\varepsilon > 0$, we shall use the family of functions $\{U_\varepsilon\}$ introduced in [43] as

$$U_\varepsilon(x) = \varepsilon^{-\frac{N-2s}{2}} \frac{1}{\|\tilde{u}\|_{2_s^*}} \tilde{u}\left(\frac{x}{\varepsilon S_s^{1/(2s)}}\right), \quad x \in \mathbb{R}^N,$$

which is a solution of problem $(-\Delta)^s u = |u|^{2_s^*-2}u$, in $\mathbb{R}^N$. Without loss of generality, we suppose that $0 \in \Omega$, let us fix $\delta > 0$ such that $B_{4\delta} \subset \Omega$, and let $\eta \in C^\infty(\mathbb{R}^N)$ be such that $0 \leqslant \eta \leqslant 1$ in $\mathbb{R}^N$, $\eta \equiv 1$ in B_δ and $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_{2\delta}$, where $B_\delta = B(0, \delta)$. We denote by u_ε the following function

$$u_\varepsilon(x) = \eta(x) U_\varepsilon(x). \tag{2.3}$$

It is obvious that $u_\varepsilon \in X_0^s(\Omega)$, and the following estimates on the function u_ε were proved in [43, Proposition 21 and 22],

$$\|u_\varepsilon\|^2 \leqslant S_s^{\frac{N}{2s}} + O\left(\varepsilon^{N-2s}\right), \tag{2.4}$$

$$\|u_\varepsilon\|_{2_s^*}^{2_s^*} = S_s^{\frac{N}{2s}} + O\left(\varepsilon^N\right), \tag{2.5}$$

$$\|u_\varepsilon\|_2^2 \geqslant \begin{cases} C_s \varepsilon^{2s} + O\left(\varepsilon^{N-2s}\right) & \text{if } N > 4s, \\ C_s \varepsilon^{2s} |\log \varepsilon| + O\left(\varepsilon^{2s}\right) & \text{if } N = 4s, \end{cases} \tag{2.6}$$

as $\varepsilon \to 0$, for some positive constant C_s depending on s.

The Euler functional $\mathcal{I}_\alpha : X_0^s(\Omega) \to \mathbb{R}$ corresponding to problem (1.1) is given by

$$\mathcal{I}_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dxdy + \frac{\alpha}{q} \int_\Omega |u|^q dx - \frac{\beta}{2} \int_\Omega u^2 dx - \frac{\gamma}{2_s^*} \int_\Omega |u|^{2_s^*} dx. \tag{2.7}$$

It is easy to verify that $I_\alpha \in C^1(X_0^s(\Omega))$ with

$$\langle \mathcal{I}'_\alpha(u), v \rangle = \langle u, v \rangle_{X_0^s(\Omega)} + \alpha \int_\Omega |u|^{q-2} uvdx - \beta \int_\Omega uvdx - \gamma \int_\Omega |u|^{2_s^*-2} uvdx, \tag{2.8}$$

for $v \in X_0^s(\Omega)$. A direct computation shows that weak solution of (1.1) is critical point of I_α.

We say that the functional $\mathcal{I}_\alpha$ satisfies the Palais–Smale ((PS) for short) condition at level $c \in \mathbb{R}$ if any sequence $\{u_j\} \subset X_0^s(\Omega)$ such that

$$\mathcal{I}_\alpha(u_j) \to c \tag{2.9}$$

and

$$\sup\left\{\left|\left\langle \mathcal{I}_\alpha'(u_j), \varphi\right\rangle\right| : \varphi \in X_0^s(\Omega), \|\varphi\| = 1\right\} \to 0 \quad \text{as } j \to \infty \tag{2.10}$$

admits a subsequence which is convergent in $X_0^s(\Omega)$.

Now, we are ready to prove that the functional $\mathcal{I}_\alpha$ satisfies the (PS) condition in a suitable energy range involving the best fractional critical Sobolev constant S_s given in (2.2).

Lemma 2.1. *Assume that $1 < q < 2$, β and γ are positive constants, and $\alpha > 0$. Then the functional $\mathcal{I}_\alpha$ satisfies the (PS) condition at any level $c < \frac{s}{N}(S_s)^{\frac{N}{2s}}\gamma^{\frac{2s-N}{2s}}$.*

Proof. Let $\{u_j\} \subset X_0^s(\Omega)$ be a (PS) sequence for $\mathcal{I}_\alpha$, first of all, we show the $\{u_j\}$ is bounded in $X_0^s(\Omega)$. In fact, by (2.9) and (2.10), there is $\kappa > 0$ such that $\left|\mathcal{I}_\alpha(u_j)\right| \le \kappa$, $\left|\left\langle \mathcal{I}_\alpha{}'(u_j), u_j\right\rangle\right| \le \kappa\|u_j\|$. Taking into account that $1 < q < 2 < 2_s^*$, we have

$$\begin{aligned}
\kappa\left(1 + \|u_j\|\right) &\geqslant \mathcal{I}_\alpha(u_j) - \frac{1}{2}\left\langle \mathcal{I}_\alpha'(u_j), u_j\right\rangle \\
&= \alpha\left(\frac{1}{q} - \frac{1}{2}\right)\int_\Omega |u_j(x)|^q\,dx + \gamma\left(\frac{1}{2} - \frac{1}{2_s^*}\right)\int_\Omega |u_j(x)|^{2_s^*}\,dx \\
&\geqslant \frac{\gamma s}{N}\|u_j\|_{2_s^*}^{2_s^*}.
\end{aligned}$$

For $\bar{\kappa} := \frac{N\kappa}{\gamma s} > 0$, hence,

$$\|u_j\|_{2_s^*}^{2_s^*} \leqslant \bar{\kappa}\left(1 + \|u_j\|\right) \quad \text{for } j \in \mathbb{N}. \tag{2.11}$$

Thus, by the Hölder inequality and (2.11), we get

$$\|u_j\|_2^2 \leqslant |\Omega|^{\frac{2s}{N}}\|u_j\|_{2_s^*}^2 \leqslant \bar{\kappa}^{\frac{2}{2_s^*}}|\Omega|^{\frac{2s}{N}}\left(1 + \|u_j\|\right)^{\frac{2}{2_s^*}} \leqslant \hat{\kappa}\left(1 + \|u_j\|\right) \tag{2.12}$$

with $\hat{\kappa} := \bar{\kappa}^{\frac{2}{2_s^*}}|\Omega|^{\frac{2s}{N}}$. Thus, by (2.11) and (2.12) we conclude that

$$\begin{aligned}
\kappa \geqslant \mathcal{I}_\alpha(u_j) &= \frac{1}{2}\|u_j\|^2 + \frac{\alpha}{q}\int_\Omega |u_j|^q dx - \frac{\beta}{2}\int_\Omega u_j^2 dx - \frac{\gamma}{2_s^*}\int_\Omega |u_j|^{2_s^*} dx \\
&\geqslant \frac{1}{2}\|u_j\|^2 - \frac{\beta}{2}\int_\Omega u_j^2 dx - \frac{\gamma}{2_s^*}\int_\Omega |u_j|^{2_s^*} dx \\
&\geqslant \frac{1}{2}\|u_j\|^2 - \left(\frac{\beta}{2}\hat{\kappa} + \frac{\gamma}{2_s^*}\bar{\kappa}\right)\left(1 + \|u_j\|\right).
\end{aligned}$$

Hence, $\{u_j\}$ is bounded in $X_0^s(\Omega)$.

Consequently, passing to a subsequence if necessary, we may assume that

$$\begin{aligned}
&u_j \rightharpoonup u_\infty \quad \text{in } X_0^s(\Omega), \qquad u_j \to u_\infty \quad \text{in } L^2(\Omega), \\
&u_j \to u_\infty \quad \text{in } L^q(\Omega) \quad \text{and} \quad u_j \to u_\infty \quad \text{for a.e. } x \in \Omega \text{ with some } u_\infty \in X_0^s(\Omega).
\end{aligned} \tag{2.13}$$

Next, we show that u_∞ is a solution of (1.1) and $\mathcal{I}_\alpha(u_\infty) \geqslant 0$. Indeed, for any $\varphi \in X_0^s(\Omega)$, by (2.1) and (2.13), we have that

$$\int_\Omega |u_j(x)|^{2_s^*-2}u_j(x)\varphi(x)dx \to \int_\Omega |u_\infty(x)|^{2_s^*-2}u_\infty(x)\varphi(x)dx, \tag{2.14}$$

and

$$\begin{gathered}\int_{\Omega}\left|u_j(x)\right|^{q-2}u_j(x)\varphi(x)dx\to\int_{\Omega}\left|u_\infty(x)\right|^{q-2}u_\infty(x)\varphi(x)dx,\\ \int_{\Omega}u_j(x)\varphi(x)dx\to\int_{\Omega}u_\infty(x)\varphi(x)dx.\end{gathered}\tag{2.15}$$

Thus, by (2.13), (2.14) and (2.15), we conclude that

$$\langle\mathcal{I}'_\alpha(u_j),\varphi\rangle\to\langle\mathcal{I}'_\alpha(u_\infty),\varphi\rangle.$$

In view of (2.10), we get

$$\langle\mathcal{I}'_\alpha(u_\infty),\varphi\rangle=0,\tag{2.16}$$

namely, u_∞ is a solution of (1.1). Taking $\varphi=u_\infty$ as a test function in (2.16), we get

$$\|u_\infty\|^2=-\alpha\int_{\Omega}|u_\infty|^q dx+\beta\int_{\Omega}u_\infty^2dx+\gamma\int_{\Omega}|u_\infty|^{2_s^*}dx,$$

then $1<q<2<2_s^*$ implies that

$$\mathcal{I}_\alpha(u_\infty)=\alpha\left(\frac{1}{q}-\frac{1}{2}\right)\int_{\Omega}|u_\infty(x)|^q\,dx+\gamma\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\int_{\Omega}|u_\infty(x)|^{2_s^*}\,dx\geqslant 0.\tag{2.17}$$

Finally, we show that $\{u_j\}$ converges to u_∞ in $X_0^s(\Omega)$. Note that $\{u_j\}$ is bounded in $X_0^s(\Omega)$, by (2.13), (2.1) and Brezis–Lieb lemma [7, Theorem 1], for $p\in(1,2_s^*]$, we have

$$\int_{\Omega}|u_j|^p dx-\int_{\Omega}|u_j-u_\infty|^p dx=\int_{\Omega}|u_\infty|^p dx+o(1)\tag{2.18}$$

The boundedness of $\{u_j\}$ in $X_0^s(\Omega)$, (2.1), (2.10), (2.13), (2.16) and (2.18) imply that

$$\begin{aligned}o(1)&=\langle\mathcal{I}'_\alpha(u_j)-\mathcal{I}'_\alpha(u_\infty),u_j-u_\infty\rangle\\&=\|u_j-u_\infty\|^2+\alpha\int_{\Omega}(|u_j|^{q-2}u_j-|u_\infty|^{q-2}u_\infty)(u_j-u_\infty)dx-\beta\int_{\Omega}|u_j-u_\infty|^2dx\\&\quad-\gamma\int_{\Omega}(|u_j|^{2_s^*-2}u_j-|u_\infty|^{2_s^*-2}u_\infty)(u_j-u_\infty)dx\\&=\|u_j-u_\infty\|^2+\alpha\int_{\Omega}|u_j-u_\infty|^q dx-\beta\int_{\Omega}|u_j-u_\infty|^2dx-\gamma\int_{\Omega}|u_j-u_\infty|^{2_s^*}dx+o(1),\end{aligned}$$

thus, by (2.13), we deduce that

$$\|u_j-u_\infty\|^2-\gamma\int_{\Omega}|u_j-u_\infty|^{2_s^*}dx=o(1).\tag{2.19}$$

Since the sequence $\{\|u_j\|\}$ is bounded, we may assume that $\|u_j-u_\infty\|^2\to L$ as $j\to+\infty$, in view of (2.19), $\int_\Omega|u_j(x)-u_\infty(x)|^{2_s^*}dx\to\frac{L}{\gamma}$ as $j\to+\infty$. So taking into account (2.2), we get $\left(\frac{L}{\gamma}\right)^{\frac{2}{2_s^*}}S_s\leqslant L$, then $L=0$ or $L\geqslant(S_s)^{\frac{N}{2s}}\gamma^{\frac{2s-N}{2s}}$. Assume that $L\geqslant(S_s)^{\frac{N}{2s}}\gamma^{\frac{2s-N}{2s}}$. Since $u_j\rightharpoonup u_\infty$, we have

$$\|u_j-u_\infty\|^2=\|u_j\|^2-\|u_\infty\|^2+o(1).\tag{2.20}$$

So (2.13), (2.20) and (2.18) yield

$$\begin{aligned}\mathcal{I}_\alpha(u_j)&=\frac{1}{2}\|u_j\|^2+\frac{\alpha}{q}\int_{\Omega}|u_j|^q dx-\frac{\beta}{2}\int_{\Omega}u_j^2dx-\frac{\gamma}{2_s^*}\int_{\Omega}|u_j|^{2_s^*}dx\\&=I_\alpha(u_\infty)+\frac{1}{2}\|u_j-u_\infty\|^2-\frac{\gamma}{2_s^*}\int_{\Omega}|u_j-u_\infty|^{2_s^*}dx+o(1).\end{aligned}\tag{2.21}$$

By (2.21), (2.19) and (2.17), we obtain

$$c=\lim_{j\to\infty}\mathcal{I}_\alpha\left(u_j\right)=I_\alpha\left(u_\infty\right)+\frac{1}{2}L-\frac{\gamma}{2_s^*}\frac{L}{\gamma}\geqslant\frac{s}{N}L\geqslant\frac{s}{N}(S_s)^{\frac{N}{2s}}\gamma^{\frac{2s-N}{2s}}$$

which contradicts the condition $c<\frac{s}{N}(S_s)^{\frac{N}{2s}}\gamma^{\frac{2s-N}{2s}}$. Thus $L=0$, and so $\|u_j-u_\infty\|\to 0$ as $j\to+\infty$. □

The manifold we are interested in this paper is the Nehari manifold associated with $I_\alpha(u)$, given by

$$\mathcal{N}_\alpha:=\left\{u\in X_0^s(\Omega)\backslash\{0\}:\langle\mathcal{I}_\alpha'(u),u\rangle=0\right\}.$$

First of all, we point out that $\mathcal{N}_\alpha$ is not empty.

Lemma 2.2. *$\mathcal{N}_\alpha\neq\varnothing$. Precisely, for every $u\in X_0^s(\Omega)\backslash\{0\}$, then there exists a unique $t_u\in(0,+\infty)$, such that $t_uu\in\mathcal{N}_\alpha$.*

Proof. Fix $u\in X_0^s(\Omega)\backslash\{0\}$, we consider the function $\varphi:[0,+\infty)\to\mathbb{R}$

$$\varphi(t):=\langle\mathcal{I}_\alpha'(tu),tu\rangle=t^2\|u\|^2+\alpha t^q\int_\Omega|u|^q dx-\beta t^2\int_\Omega u^2dx-\gamma t^{2_s^*}\int_\Omega|u|^{2_s^*}dx=t^q\phi(t),$$

where

$$\phi(t)=\alpha\int_\Omega|u|^q dx+t^{2-q}(\|u\|^2-\beta\int_\Omega u^2dx)-\gamma t^{2_s^*-q}\int_\Omega|u|^{2_s^*}dx.$$

We have that $\phi\in C^1([0,+\infty))$ with $\phi(0)=\alpha\int_\Omega|u|^q dx>0$ and $\lim_{t\to+\infty}\phi(t)=-\infty$. In the case of $0<\beta<\lambda_{1,s}$, we have $\|u\|^2-\beta\int_\Omega u^2dx>0$, ϕ has a unique maximum point

$$t_0=\left(\frac{(2-q)(\|u\|^2-\beta\int_\Omega u^2dx)}{(2_s^*-q)\gamma\int_\Omega|u|^{2_s^*}dx}\right)^{\frac{1}{2_s^*-2}},$$

ϕ increases on $[0,t_0)$ and decreases on $(t_0,+\infty)$. In the case of $\beta\geqslant\lambda_{1,s}$, we can get that $\|u\|^2-\beta\int_\Omega u^2dx\leqslant 0$ and ϕ decreases on $[0,+\infty)$. Thus there is only one zero point in $(0,+\infty)$ to ϕ, namely, there exists a unique $t_u\in(0,+\infty)$, such that $t_uu\in\mathcal{N}_\alpha$. □

The $\mathcal{N}_\alpha$ is a natural constraint for the functional $\mathcal{I}_\alpha$, since every constrained critical point of $\mathcal{I}_\alpha$ on $\mathcal{N}_\alpha$ is indeed a critical point of $\mathcal{I}_\alpha$. Precisely, the following result holds true.

Lemma 2.3. *$\mathcal{I}_\alpha$ is bounded from below on $\mathcal{N}_\alpha$. And u is a critical point of $\mathcal{I}_\alpha$ constrained to $\mathcal{N}_\alpha$ if and only if u is a nontrivial critical point of $\mathcal{I}_\alpha$.*

Proof. Notice that on $\mathcal{N}_\alpha$ the functional $\mathcal{I}_\alpha$ reads as follows

$$\mathcal{I}_\alpha(u)=\alpha\left(\frac{1}{q}-\frac{1}{2}\right)\int_\Omega|u|^q dx+\gamma\left(\frac{1}{2}-\frac{1}{2_s^*}\right)\int_\Omega|u|^{2_s^*}dx,$$

thank to $1<q<2<2_s^*$, so that $\inf_{u\in\mathcal{N}_\alpha}\mathcal{I}_\alpha(u)\geqslant 0$.

It is obvious that every nontrivial critical point of $\mathcal{I}_\alpha$ belongs to $\mathcal{N}_\alpha$. Let us show the converse. In the sequel we will denote by $G_\alpha:X_0^s(\Omega)\to\mathbb{R}$, the functional given by

$$G_\alpha(u):=\langle\mathcal{I}_\alpha'(u),u\rangle=\|u\|^2+\alpha\int_\Omega|u|^q dx-\beta\int_\Omega u^2dx-\gamma\int_\Omega|u|^{2_s^*}dx.$$

It is easy to verify that $G_\alpha \in C^1\left(X_0^s(\Omega)\right)$ and

$$\left\langle G'_\alpha(u), u\right\rangle = 2\|u\|^2 + q\alpha \int_\Omega |u|^q dx - 2\beta \int_\Omega u^2 dx - \gamma 2_s^* \int_\Omega |u|^{2_s^*} dx,$$

so that, taking into account the definition of $\mathcal{N}_\alpha$, we have

$$\left\langle G'_\alpha(u), u\right\rangle = \alpha(q-2) \int_\Omega |u|^q dx + \gamma(2-2_s^*) \int_\Omega |u|^{2_s^*} dx < 0 \quad \text{for } u \in \mathcal{N}_\alpha. \tag{2.22}$$

Let u be a constrained critical point of $\mathcal{I}_\alpha$ on $\mathcal{N}_\alpha$, namely $u \in \mathcal{N}_\alpha$ and

$$\mathcal{I}'_\alpha(u) = \eta G'_\alpha(u) \tag{2.23}$$

for some $\eta \in \mathbb{R}$. Note that (2.23) yields

$$\left\langle \mathcal{I}'_\alpha(u), u\right\rangle = \eta \left\langle G'_\alpha(u), u\right\rangle. \tag{2.24}$$

Taking into account the fact that $u \in \mathcal{N}_\alpha$ and (2.22), by (2.24) we deduce that $\eta = 0$. Hence, again by (2.23), we get $\mathcal{I}'_\alpha(u) = 0$. □

We say that the functional $\mathcal{I}_\alpha$ constrained on $\mathcal{N}_\alpha$ satisfies the (PS) condition at level $c \in \mathbb{R}$ if any sequence $\{u_j\} \subset \mathcal{N}_\alpha$ such that (2.9) holds and there exists $\{\eta_j\} \subset \mathbb{R}$ with

$$\sup\left\{\left|\left\langle \mathcal{I}'_\alpha\left(u_j\right) - \eta_j G'_\alpha\left(u_j\right), \varphi\right\rangle\right| : \varphi \in X_0^s(\Omega), \|\varphi\| = 1\right\} \to 0 \tag{2.25}$$

as $j \to +\infty$ admits a subsequence which is convergent in $X_0^s(\Omega)$.

By Lemma 2.1 we know that the functional $\mathcal{I}_\alpha$ satisfies the (PS) condition at level $c < \frac{s}{N}\left(S_s\right)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}$. Now, we are ready to show that the functional $\mathcal{I}_\alpha$ constrained on $\mathcal{N}_\alpha$ satisfies the (PS) condition at the same level.

Lemma 2.4. *Assume that $1 < q < 2$, β and γ are positive constants, and $\alpha > 0$. Then the functional $\mathcal{I}_\alpha$ constrained on $\mathcal{N}_\alpha$ satisfies the (PS) condition at any level $c < \frac{s}{N}\left(S_s\right)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}$.*

Proof. Let $\{u_j\} \subset \mathcal{N}_\alpha$ be a sequence such that (2.9) holds and there exists $\{\eta_j\} \subset \mathbb{R}$ for which (2.25) is satisfied. First of all, we claim that $\{u_j\}$ is bounded in $L^q(\Omega)$ and $L^{2_s^*}(\Omega)$. Indeed, by (2.9) there exists a positive constant M such that

$$\left|\mathcal{I}_\alpha\left(u_j\right)\right| \leqslant M, \tag{2.26}$$

for any $j \in \mathbb{N}$. By (2.26) and the fact that $u_j \in \mathcal{N}_\alpha$, we obtain that

$$\begin{aligned} M &\geqslant \mathcal{I}_\alpha\left(u_j\right) \\ &= \mathcal{I}_\alpha\left(u_j\right) - \frac{1}{2}\left\langle \mathcal{I}'_\alpha\left(u_j\right), u_j\right\rangle \\ &= \alpha\left(\frac{1}{q} - \frac{1}{2}\right) \int_\Omega |u_j|^q dx + \gamma\left(\frac{1}{2} - \frac{1}{2_s^*}\right) \int_\Omega |u_j|^{2_s^*} dx, \end{aligned}$$

thus $\{u_j\}$ is bounded in $L^q(\Omega)$ and $L^{2_s^*}(\Omega)$. Hence, taking into account (2.22), we conclude that $\{\left\langle G'_\alpha\left(u_j\right), u_j\right\rangle\}$ is bounded in $\mathbb{R}$ and there exists $\theta \in (-\infty, 0]$ such that, up to a subsequence

$$\left\langle G'_\alpha\left(u_j\right), u_j\right\rangle \to \theta, \quad \text{as } j \to \infty. \tag{2.27}$$

Now, suppose that $\theta < 0$. Then, by (2.25), the fact that $u_j \in \mathcal{N}_\alpha$ and (2.27) we deduce that $\eta_j \to 0$ as $j \to \infty$. Hence, again by (2.25) we obtain that (2.10) holds. So, $\{u_j\} \subset \mathcal{N}_\alpha$ is a PS sequence for the functional $\mathcal{I}_\alpha$, the assertion of Lemma 2.4 follows from Lemma 2.1.

Finally, suppose that $\theta = 0$. By (2.22) and (2.27) we get that

$$\int_\Omega |u_j|^q dx \to 0 \quad \text{and} \quad \int_\Omega |u_j|^{2_s^*} dx \to 0, \quad \text{as } j \to \infty,$$

since $u_j \in \mathcal{N}_\alpha$, we get that $\|u_j\| \to 0$ as $j \to \infty$. Thus, $u_j \to 0$ in $X_0^s(\Omega)$ as $j \to \infty$. □

In order to obtain a ground state solution of (1.1), here we will use a theory which is introduced by Ambrosetti and Malchiodi in [2, Theorem 7.12].

Lemma 2.5. *Let E be a Banach space and $J \in C^{1,1}(E, \mathbb{R})$. If there exist $G \in C^{1,1}(E, \mathbb{R})$ such that $M = G^{-1}(0)$ with $G'(u) \neq 0$ for any $u \in M$. Moreover, suppose that J is bounded from below on M and satisfies $(PS)_m$ condition, where*

$$m := \inf_{u \in M} J(u) > -\infty.$$

Then the infimum m is achieved. Precisely, there is $z \in M$ such that $J(z) = m$ and $\nabla_M J(z) = 0$.

3 Proof of Theorem 1.1

In order to show that the equation (1.1) has a ground state solution, it suffices to verify that the infimum of $\mathcal{I}_\alpha$ on $\mathcal{N}_\alpha$ is attainable, in which the estimation of the energy of $\mathcal{I}_\alpha$ on $\mathcal{N}_\alpha$ is essential. Now, we have the following result.

Lemma 3.1. *Suppose that $\gamma > 0$, Then there exists $\alpha_1 > 0$, such that for any $\alpha \in (0, \alpha_1)$, there holds the estimate*

$$\inf_{u \in \mathcal{N}_\alpha} \mathcal{I}_\alpha(u) < \frac{s}{N} (S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}, \tag{3.1}$$

provided that

- *$N \geqslant 4s$ and $0 < \beta < \lambda_{1,s}$, or*
- *$N > 2s$ and $\beta \geqslant \lambda_{1,s}$.*

Proof. In order to prove (3.1) it is enough to show that there exists $u_0 \in \mathcal{N}_\alpha$ such that

$$\mathcal{I}_\alpha(u_0) < \frac{s}{N} (S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}. \tag{3.2}$$

Firstly, let us consider the case $0 < \beta < \lambda_{1,s}$. Let $\varepsilon > 0$ and u_ε be as in (2.3). By Lemma 2.2 there exists $t_\varepsilon > 0$ such that $t_\varepsilon u_\varepsilon \in \mathcal{N}_\alpha$, namely, that is

$$\left\langle \mathcal{I}'_\alpha (t_\varepsilon u_\varepsilon), t_\varepsilon u_\varepsilon \right\rangle = \alpha t_\varepsilon^q \int_\Omega |u_\varepsilon|^q dx + t_\varepsilon^2 \left(\|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx \right) - \gamma t_\varepsilon^{2_s^*} \int_\Omega |u_\varepsilon|^{2_s^*} dx = 0. \tag{3.3}$$

Then, in view of $0 < \beta < \lambda_{1,s}$ and (2.4), we obtain that

$$0 < \|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx \leqslant \|u_\varepsilon\|^2 \leqslant S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s}). \tag{3.4}$$

It follows from Hölder's inequality and (2.5) that

$$0 < \int_\Omega |u_\varepsilon|^q dx \leqslant |\Omega|^{\frac{2_s^*-q}{2_s^*}} \|u_\varepsilon\|_{2_s^*}^q \leqslant |\Omega|^{\frac{2_s^*-q}{2_s^*}} \left(S_s^{\frac{N}{2s}} + O\left(\varepsilon^N\right)\right)^{\frac{q}{2_s^*}}. \tag{3.5}$$

So (3.4) and (3.5) imply that there exists $K > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \max \left\{ \int_\Omega |u_\varepsilon|^q dx, \|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx, \int_\Omega |u_\varepsilon|^{2_s^*} dx \right\} \leqslant K. \tag{3.6}$$

Thank to $1 < q < 2 < 2_s^*$, by (3.3) and (3.6) we conclude that there exists $t_0 > 0$ such that

$$t_\varepsilon \in (0, t_0) \quad \text{for } \varepsilon \in (0, \varepsilon_0). \tag{3.7}$$

Let the function $f : [0, +\infty) \to \mathbb{R}$ given by

$$f(t) := \frac{1}{2} t^2 \left(\|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx \right) - \frac{\gamma}{2_s^*} t^{2_s^*} \int_\Omega |u_\varepsilon|^{2_s^*} dx,$$

then f admits the maximum point

$$t_{max} = \left(\frac{\|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx}{\gamma \int_\Omega |u_\varepsilon|^{2_s^*} dx} \right)^{\frac{1}{2_s^*-2}}$$

with the maximum value

$$f(t_{max}) = \frac{s}{N} \gamma^{\frac{2s-N}{2s}} \left(\frac{\|u_\varepsilon\|^2 - \beta \|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2_s^*}^2} \right)^{\frac{N}{2s}}. \tag{3.8}$$

We note that

$$\mathcal{I}_\alpha(t_\varepsilon u_\varepsilon) = \frac{\alpha}{q} t_\varepsilon^q \int_\Omega |u_\varepsilon|^q dx + \frac{1}{2} t_\varepsilon^2 \left(\|u_\varepsilon\|^2 - \beta \int_\Omega u_\varepsilon^2 dx \right) - \frac{\gamma}{2_s^*} t_\varepsilon^{2_s^*} \int_\Omega |u_\varepsilon|^{2_s^*} dx. \tag{3.9}$$

From (3.9) and (3.8) it turns out

$$\mathcal{I}_\alpha(t_\varepsilon u_\varepsilon) \leqslant \frac{\alpha}{q} t_\varepsilon^q \int_\Omega |u_\varepsilon|^q dx + \frac{s}{N} \gamma^{\frac{2s-N}{2s}} \left(\frac{\|u_\varepsilon\|^2 - \beta \|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2_s^*}^2} \right)^{\frac{N}{2s}}. \tag{3.10}$$

Suppose that $N > 4s$, in view of (2.4)–(2.6), and by using the mean value theorem for the function $(1+t)^{\frac{N-2s}{N}}$, we find that

$$\begin{aligned}\frac{\|u_\varepsilon\|^2-\beta\|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2_s^*}^2} &\leqslant \frac{\left(S_s^{\frac{N}{2s}}+O\left(\varepsilon^{N-2s}\right)\right)-\beta\left(C_s\varepsilon^{2s}+O\left(\varepsilon^{N-2s}\right)\right)}{\left(S_s^{\frac{N}{2s}}+O\left(\varepsilon^{N}\right)\right)^{\frac{N-2s}{N}}}\\ &= S_s+\frac{S_s^{\frac{N}{2s}}\left(1-\left(1+S_s^{-\frac{N}{2s}}O\left(\varepsilon^{N}\right)\right)^{\frac{N-2s}{N}}\right)+O\left(\varepsilon^{N-2s}\right)}{\left(S_s^{\frac{N}{2s}}+O\left(\varepsilon^{N}\right)\right)^{\frac{N-2s}{N}}}\\ &\quad-\varepsilon^{2s}\frac{\beta\left(C_s+O\left(\varepsilon^{N-4s}\right)\right)}{\left(S_s^{\frac{N}{2s}}+O\left(\varepsilon^{N}\right)\right)^{\frac{N-2s}{N}}}\\ &= S_s+\frac{O\left(\varepsilon^{N}\right)+O\left(\varepsilon^{N-2s}\right)}{\left(S_s^{\frac{N}{2s}}+O\left(\varepsilon^{N}\right)\right)^{\frac{N-2s}{N}}}-\varepsilon^{2s}\frac{\beta\left(C_s+O\left(\varepsilon^{N-4s}\right)\right)}{\left(S_s^{\frac{N}{2s}}+O\left(\varepsilon^{N}\right)\right)^{\frac{N-2s}{N}}}\\ &< S_s\end{aligned}\tag{3.11}$$

with $\varepsilon>0$ sufficiently small. Now assume that $N=4s$, in this case, by (2.4)–(2.6), we get

$$\begin{aligned}\frac{\|u_\varepsilon\|^2-\beta\|u_\varepsilon\|_2^2}{\|u_\varepsilon\|_{2_s^*}^2} &\leqslant \frac{\left(S_s^2+O\left(\varepsilon^{2s}\right)\right)-\beta\left(C_s\varepsilon^{2s}|\log\varepsilon|+O\left(\varepsilon^{2s}\right)\right)}{\left(S_s^2+O\left(\varepsilon^{4s}\right)\right)^{\frac{1}{2}}}\\ &= S_s+\varepsilon^{2s}\frac{\left(O\left(\varepsilon^{2s}\right)+O(1)\right)-\beta\left(C_s|\log\varepsilon|+O(1)\right)}{\left(S_s^2+O\left(\varepsilon^{4s}\right)\right)^{\frac{1}{2}}}\\ &< S_s\end{aligned}\tag{3.12}$$

when $\varepsilon>0$ is small enough, since $|\log\varepsilon|\to+\infty$ as $\varepsilon\to 0$.

So we can choose $\varepsilon>0$ sufficiently small such that (3.11), (3.12) and $\varepsilon<\varepsilon_0$ hold. For this ε, let $N\geqslant 4s$, $u_0=t_\varepsilon u_\varepsilon$. By (3.6), (3.7) and (3.10), then there is $\alpha_4>0$, if $0<\alpha<\alpha_4$, such that (3.2) holds.

Secondly, in the case of $\beta\geqslant\lambda_{1,s}$. Fix $u\in X_0^s(\Omega)\backslash\{0\}$, by Lemma 2.2, there exists a unique $t_u\in(0,+\infty)$, such that

$$\left\langle\mathcal{I}'_\alpha\left(t_u u\right),t_u u\right\rangle=0,\tag{3.13}$$

Hölder inequality and (3.13) imply that

$$\gamma t_u^{2_s^*}\int_\Omega|u|^{2_s^*}dx\leqslant\alpha t_u^q\int_\Omega|u|^q dx\leqslant\alpha|\Omega|^{\frac{2_s^*-q}{2_s^*}}t_u^q\|u\|_{2_s^*}^q,$$

thus

$$t_u\leqslant\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2_s^*-q}}\frac{|\Omega|^{\frac{1}{2_s^*}}}{\|u\|_{2_s^*}}.\tag{3.14}$$

Hence, by the fact that $\beta\geqslant\lambda_{1,s}$, (3.14) and Hölder's inequality we conclude that

$$\mathcal{I}_\alpha(t_u u)\leqslant\frac{\alpha}{q}t_u^q\int_\Omega|u|^q dx\leqslant\frac{\alpha}{q}\left(\frac{\alpha}{\gamma}\right)^{\frac{q}{2_s^*-q}}\frac{|\Omega|^{\frac{q}{2_s^*}}}{\|u\|_{2_s^*}^q}|\Omega|^{\frac{2_s^*-q}{2_s^*}}\|u\|_{2_s^*}^q=\frac{\alpha^{\frac{2_s^*}{2_s^*-q}}}{q\gamma^{\frac{q}{2_s^*-q}}}|\Omega|$$

So we choose $u_0 = t_u u$, there exists $\alpha_5 > 0$ such that (3.2) holds provided that $0 < \alpha < \alpha_5$.

Let $\alpha_1 = \min\{\alpha_4, \alpha_5\}$, Assume $N \geqslant 4s$ and $0 < \beta < \lambda_{1,s}$, or $N > 2s$ and $\beta \geqslant \lambda_{1,s}$. Then there exists $u_0 \in \mathcal{N}_\alpha$, if $\alpha \in (0, \alpha_1)$, such that (3.2) holds. □

Finally we are ready to apply the above lemmas to prove the first main result.

Proof of Theorem 1.1. Taking into account the definitions of $\mathcal{I}_\alpha$ and $\mathcal{N}_\alpha$, it is easy to verify that $\mathcal{I}_\alpha, G_\alpha \in C^{1,1}(X_0^s(\Omega))$, whose proof is similar to that of [22, 8.5.2 Theorem 3]. Lemma 2.3 imply that

$$\mathcal{N}_\alpha = G_\alpha^{-1}(0),\ \langle G'_\alpha(u), u\rangle < 0 \text{ for } u \in \mathcal{N}_\alpha \text{ and } \inf_{u\in\mathcal{N}_\alpha} \mathcal{I}_\alpha(u) \geqslant 0.$$

By Lemma 3.1, we know that there exists $\alpha_1 > 0$ such that

$$m_\alpha := \inf_{u\in\mathcal{N}_\alpha} \mathcal{I}_\alpha(u) < \frac{s}{N}(S_s)^{\frac{N}{2s}}\gamma^{\frac{2s-N}{2s}}$$

for $\alpha \in (0, \alpha_1)$ provided that $N \geqslant 4s$ and $0 < \beta < \lambda_{1,s}$, or $N > 2s$ and $\beta \geqslant \lambda_{1,s}$. In view of Lemma 2.4, we deduce that the functional $\mathcal{I}_\alpha$ constrained on $\mathcal{N}_\alpha$ satisfies the $(PS)_{m_\alpha}$ condition. According to Lemma 2.5, let E and M be $X_0^s(\Omega)$ and $\mathcal{N}_\alpha$ respectively, then there exists $u_{m_\alpha} \in \mathcal{N}_\alpha$ such that

$$\mathcal{I}_\alpha(u_{m_\alpha}) = m_\alpha \quad \text{and} \quad \mathcal{I}'_\alpha|_{\mathcal{N}_\alpha}(u_{m_\alpha}) = 0.$$

Moreover, Lemma 2.3 implies that $\mathcal{I}'_\alpha(u_{m_\alpha}) = 0$, thus u_{m_α} is a ground state solution of problem (1.1). □

4 Proof of Theorem 1.2

Proof. We first prove that the functional $\mathcal{I}_\alpha$ has mountain pass geometry when the conditions of Theorem 1.1 are satisfied. Let $\alpha_1 > 0$ be given in Theorem 1.1, it suffices to show that the following assertions hold provided that $0 < \alpha < \alpha_1$.

(i) there are $\rho, r > 0$ such that for $u \in X_0^s(\Omega)$ with $\|u\| = \rho$, we have $\mathcal{I}_\alpha(u) \geqslant r$.

(ii) there exists $e \in X_0^s(\Omega)$ such that $\|e\| > \rho$ and $\mathcal{I}_\alpha(e) < 0$.

We claim that $u \equiv 0$ is a strict local minimizer of the functional $\mathcal{I}_\alpha$. In virtue of [31, Theorem 1.1], it suffices to prove this claim in the space $C_s^0(\bar\Omega) \cap X_0^s(\Omega)$, where

$$C_s^0(\bar\Omega) = \left\{ w \in C^0(\bar\Omega) : \|w\|_{C_s^0} := \left\| \frac{w}{\delta^s} \right\|_{L^\infty} < \infty \right\}$$

with $\delta(x) := \text{dist}(x, \partial\Omega)$. Notice that $\sup_{x\in\Omega} \delta(x) \leqslant \text{diam}(\Omega)$, then for any $u \in C_s^0(\bar\Omega) \cap X_0^s(\Omega)$ we have that

$$\int_\Omega u^2 dx = \int_\Omega \left(\frac{|u|}{\delta^s}\right)^{2-q} (\delta^s)^{2-q} |u|^q dx \leq C_1 \|u\|_{C_s^0}^{2-q} \int_\Omega |u|^q dx \tag{4.1}$$

and

$$\int_\Omega |u|^{2_s^*} dx = \int_\Omega \left(\frac{|u|}{\delta^s}\right)^{2_s^*-q} (\delta^s)^{2_s^*-q} |u|^q dx \leq C_2 \|u\|_{C_s^0}^{2_s^*-q} \int_\Omega |u|^q dx \tag{4.2}$$

with positive constants C_1 and C_2. From (4.1) and (4.2) we obtain

$$\mathcal{I}_\alpha(u) \geq \frac{1}{2}\|u\|^2 + \left(\frac{\alpha}{q} - \frac{\beta C_1}{2}\|u\|_{C_s^0}^{2-q} - \frac{\gamma C_2}{2_s^*}\|u\|_{C_s^0}^{2_s^*-q}\right) \int_\Omega |u|^q dx. \tag{4.3}$$

Since β and γ are positive constants and $1 < q < 2 < 2_s^*$, by (4.3) we deduce that $u \equiv 0$ is a strict local minimizer of $\mathcal{I}_\alpha$ in $C_s^0(\bar{\Omega}) \cap X_0^s(\Omega)$ for any $\alpha > 0$. Thus the assertion (i) holds.

Next, we show that the assertion (ii) is true. Let u_{m_α} be the ground state solution obtained in Theorem 1.1. For $t > 0$, we have

$$\mathcal{I}_\alpha(tu_{m_\alpha}) = \frac{t^2}{2}(\|u_{m_\alpha}\|^2 - \beta \int_\Omega u_{m_\alpha}^2 dx) + \frac{\alpha}{q} t^q \int_\Omega |u_{m_\alpha}|^q dx - \frac{\gamma}{2_s^*} t^{2_s^*} \int_\Omega |u_{m_\alpha}|^{2_s^*} dx. \tag{4.4}$$

For any $\alpha \in (0, \alpha_1)$, thanks to $1 < q < 2 < 2_s^*$ and (4.4), there is $t_0 \in (0, +\infty)$ sufficiently large such that $\|t_0 u_{m_\alpha}\| > \rho$ and $\mathcal{I}_\alpha(t_0 u_{m_\alpha}) < 0$. So we complete the proof of (ii) by choosing $e = t_0 u_{m_\alpha}$.

Set the minimax value

$$c_\alpha := \inf_{h \in \Gamma} \max_{t \in [0,1]} I_\alpha(h(t)),$$

where

$$\Gamma = \{h \in C([0,1], X_0^s(\Omega)) : h(0) = 0 \text{ and } h(1) = e\}$$

where $e = t_0 u_{m_\alpha}$ is given in (ii). By Lemma 2.2 and Lemma 3.1, we have that

$$c_\alpha \leqslant \max_{t \in [0, t_0]} \mathcal{I}_\alpha(tu_{m_\alpha}) = \mathcal{I}_\alpha(u_{m_\alpha}) < \frac{s}{N}(S_s)^{\frac{N}{2s}} \gamma^{\frac{2s-N}{2s}}.$$

So, the functional $\mathcal{I}_\alpha$ possesses mountain path geometry, by Lemma 2.1, the functional $\mathcal{I}_\alpha$ satisfies the (PS) condition at the level c_α. Therefore, in view of the Mountain Pass theorem, we conclude that c_α is a critical value of $\mathcal{I}_\alpha$. According to (i), we have $c_\alpha \geqslant r > 0$, even it is obvious that $c_\alpha = \mathcal{I}_\alpha(u_{m_\alpha})$. Hence problem (1.1) has a ground state solution u_{c_α} with $I_\alpha(u_{c_\alpha}) = c_\alpha$. □

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Infinite memory effects on the stabilization of a biharmonic Schrödinger equation

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Abstract. This paper deals with the stabilization of the linear biharmonic Schrödinger equation in an n-dimensional open bounded domain under Dirichlet–Neumann boundary conditions considering three infinite memory terms as damping mechanisms. We show that depending on the smoothness of initial data and the arbitrary growth at infinity of the kernel function, this class of solution goes to zero with a polynomial decay rate like t^{-n} depending on assumptions about the kernel function associated with the infinite memory terms.

Keywords: Biharmonic Schrödinger equation, well-posedness, infinite memory, stabilization.

2020 Mathematics Subject Classification: 35B40, 35B45.

1 Introduction

1.1 Problem setting

The fourth-order nonlinear Schrödinger equation (4NLS) or biharmonic cubic nonlinear Schrödinger equation

$$i\partial_t y + \Delta y - \Delta^2 y = \lambda |y|^2 y, \tag{1.1}$$

has been introduced by Karpman [12] and Karpman and Shagalov [13] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Equation (1.1) arises in many scientific fields such as quantum mechanics, nonlinear optics, and plasma physics, and has been intensively studied with fruitful references (see [2, 12, 16] and references therein).

Over the past twenty years, equation (1.1) has been deeply studied from a different mathematical viewpoint, including linear settings which can be written generically as

$$i\partial_t y + \alpha\Delta y - \beta\Delta^2 y = f, \tag{1.2}$$

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with $\alpha, \beta \geq 0$ and different types of boundary conditions. For example, considering the problem (1.2) several authors treated this equation, see, for instance, [1, 10, 17, 19, 20, 22] and the references therein. Inspired by these results for the linear problem associated with the 4NLS, a mathematical viewpoint problem is to study the well-posedness and stabilization for solutions of the system (1.2) in an appropriate framework.

So, consider the equation (1.2) when $\alpha = \beta = 1$ in a n-dimensional open bounded subset of $\mathbb{R}^n$. Our goal is to consider an initial boundary value problem (IBVP) associated with (1.2) when the source term f is viewed as an infinite memory term:

$$f = -(-1)^j i \int_0^\infty f(s) \Delta^j y(x, t-s) ds.$$

Thus, the goal of this manuscript is to deal with the following system

$$\begin{cases} i\partial_t y(x,t) + \Delta y(x,t) - \Delta^2 y(x,t) + (-1)^j i \int_0^\infty f(s)\Delta^j y(x,t-s)ds = 0, & (x,t) \in \Omega \times \mathbb{R}_+, \\ y(x,t) = \nabla y(x,t) = 0, & (x,t) \in \Gamma \times \mathbb{R}_+^*, \\ y(x,-t) = y_0(x,t), & (x,t) \in \Omega \times \mathbb{R}_+, \end{cases} \tag{1.3}$$

where $j \in \{0, 1, 2\}$, $\Omega \subset \mathbb{R}^n$ is a n-dimensional open bounded domain with a smooth boundary Γ, and $f : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}$ is the kernel (or relaxation) function. We point out that for each j the memory term present in (1.3) is modified.

In (1.3), the memory kernel f satisfies the following assumptions:

Assumption 1. *Consider $f \in C^2(\mathbb{R}_+)$. For some positive constant c_0, we have the following conditions*

$$f' < 0, \quad 0 \leq f'' \leq -c_0 f', \quad f(0) > 0 \quad \text{and} \quad \lim_{s\to\infty} f(s) = 0. \tag{1.4}$$

Under the Assumption 1, let us introduce the following energy functionals associated with the solutions of (1.3)

$$E_j(t) = \frac{1}{2}\left(\|y\|^2 + \int_0^\infty g(s) \|\Delta^{\frac{j}{2}} \eta^t\|^2 ds \right), \tag{1.5}$$

with $j \in \{0, 1, 2\}$ and $g = -f'$, so $g \in C^1(\mathbb{R}_+)$, g is non-negative and

$$g_0 := \int_0^\infty g(s) ds = f(0) \in \mathbb{R}_+^*.$$

It is worth mentioning that the abuse of notation $\Delta^{\frac{j}{2}}$ in (1.5) means the identity operator for $j = 0$, the ∇ operator for $j = 1$ and the Laplacian operator for $j = 2$.

Therefore, taking into account the action of the infinite memory term in (1.3), the following issue will be addressed in this article:

Problem 1.1. *Does $E(t) \longrightarrow 0$, as $t \to \infty$? If so, can we provide a decay rate?*

It should be noted that the answer to the above question is crucial in the understanding of the behavior of the solutions to the fourth-order Schrödinger system when it is subject to an infinite memory term. In other words:

Problem 1.2. *Are the solutions to our problem stable despite the action of the memory term? If yes, then how robust is the stabilization property of the solutions?*

1.2 Historical background

Distributed systems with memory have a long history and have been first introduced in viscoelasticity by Maxwell, Boltzmann, and Volterra [3, 4, 15, 18]. In the context of heat processes with finite dimension speed, these systems have been introduced by Cattaneo [7] (a previous work of Maxwell had been forgotten).

In our context, to our knowledge, there is no result considering the system (1.3) in n–dimensional case. However, considering the fourth-order Schrödinger system

$$i\partial_t u + \Delta^2 u = 0, \tag{1.6}$$

there are interesting results in the sense of control problems in a bounded domain of $\mathbb{R}$ or $\mathbb{R}^n$ and, more recently, on a periodic domain $\mathbb{T}$ and manifolds, which we will summarize below.

The first result about the exact controllability of the linearized fourth order Schrödinger equation (1.6) on a bounded domain Ω of $\mathbb{R}^n$ is due to Zheng and Zhou in [21]. In this work, using an L^2-Neumann boundary control, the authors proved that the solution is exactly controllable in $H^s(\Omega)$, $s = -2$, for an arbitrarily small time. They used Hilbert Uniqueness Method (HUM) (see, for instance, [9, 14]) combined with the multiplier techniques to get the main result of the article. More recently, in [22], Zheng proved a global Carleman estimate for the fourth-order Schrödinger equation posed on a finite domain. The Carleman estimate is used to prove the Lipschitz stability for an inverse problem associated with the fourth-order Schrödinger system.

Still, on control theory Wen *et al.* in two works [19, 20], studied well-posedness and control problems related to the equation (1.6) on a bounded domain of $\mathbb{R}^n$, for $n \geq 2$. In [19], they considered the Neumann boundary controllability with collocated observation. With this result in hand, the stabilization of the closed-loop system under proportional output feedback control holds. Recently, the same authors, in [20], gave positive answers when considering the equation with hinged boundary by either moment or Dirichlet boundary control and collocated observation, respectively.

To get a general outline of the control theory already done for the system (1.6), two interesting problems were studied recently by Aksas and Rebiai [1] and Gao [10]: Uniform stabilization and stochastic control problem, in a smooth bounded domain Ω of $\mathbb{R}^n$ and on the interval $I = (0, 1)$ of $\mathbb{R}$, respectively. In the first work, by introducing suitable dissipative boundary conditions, the authors proved that the solution decays exponentially in $L^2(\Omega)$ when the damping term is effective on a neighborhood of a part of the boundary. The results are established by using multiplier techniques and compactness/uniqueness arguments. Regarding the second work, the author showed Carleman estimates for forward and backward stochastic fourth order Schrödinger equations which provided the proof of the observability inequality, unique continuation property, and, consequently, the exact controllability for the forward and backward stochastic system associated with (1.6).

Recently, the first author [5] showed the global stabilization and exact controllability properties of the 4NLS

$$\begin{cases} i\partial_t u + \partial_x^2 u - \partial_x^4 u = \lambda |u|^2 u + f(x,t), & (x,t) \in \mathbb{T} \times \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{1.7}$$

on a periodic domain $\mathbb{T}$ with internal control supported on an arbitrary sub-domain of $\mathbb{T}$. More precisely, by certain properties of propagation of compactness and regularity in

Bourgain spaces, for the solution of the associated linear system, the authors proved that system (1.7) is globally exponentially stabilizable, considering $f(x,t) = -ia^2(x)u$. This property together with the local exact controllability ensures that 4NLS is globally exactly controllable on $\mathbb{T}$.

Lastly, the first author showed in [6] the global controllability and stabilization properties for the fractional Schrödinger equation on d-dimensional compact Riemannian manifolds without boundary (M,g),

$$\begin{cases} i\partial_t u + \Lambda_g^\sigma u + P'(|u|^2)u - a(x)(1-\Delta_g)^{-\frac{\sigma}{2}}a(x)\partial_t u = 0, & \text{on } M \times \mathbb{R}_+, \\ u(x,0) = u_0(x), & x \in M. \end{cases} \tag{1.8}$$

Under the suitable assumption of the damping term $a(x)$ they proved their result using microlocal analysis, being precise, they can prove propagation of regularity which together with the so-called Geometric Control Condition and Unique Continuation Property, shows the main results of the article. Is important to mention that when $\sigma = 4$ they have the equation (1.6).

1.3 Notations

Before presenting the main result let us give some notations and definitions. In what follows, the variables x,t, and s will be suppressed, except when there is ambiguity and, throughout this article, C will denote a constant that can be different from one step to the next in the proofs presented here. We will use the notations $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$ to denote, respectively, the complex inner product in $L^2(\Omega)$ and its associated standard norm, namely

$$\langle u,v\rangle = \mathrm{Re}\left(\int_\Omega u(x)\overline{v}(x)dx\right) \quad \text{and} \quad \|u\| = \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

Now, consider the following approximation

$$\eta^t(x,s) = \int_{t-s}^t y(x,\tau)d\tau \quad \text{and} \quad \eta^0(x,s) = \int_0^s y_0(x,\tau)d\tau, \qquad x \in \Omega, s,t \in \mathbb{R}_+.$$

This approximation ensures that η^t satisfies

$$\begin{cases} \partial_t\eta^t(x,s) + \partial_s\eta^t(x,s) = y(x,t), & x \in \Omega,\ s,t \in \mathbb{R}_+, \\ \eta^t(x,s) = 0, & x \in \Gamma,\ s,t \in \mathbb{R}_+, \\ \eta^t(x,0) = 0, & x \in \Omega,\ t \in \mathbb{R}_+. \end{cases} \tag{1.9}$$

To express the memory integral in (1.3) in terms of η^t, we will denote $g := -f'$. Thus, according to (1.4), we have $g \in C^1(\mathbb{R}_+)$ and

$$g > 0, \quad 0 \le -g' \le c_0 g, \quad g_0 = \int_0^\infty g(s)ds = f(0) > 0 \tag{1.10}$$

and

$$\lim_{s\to\infty} g(s) = 0. \tag{1.11}$$

Now on, rewrite (1.3) into

$$i\partial_t y(x,t) + \Delta y(x,t) - \Delta^2 y(x,t) + i(-1)^j \int_0^\infty g(s)\Delta^j \eta^t(x,s)ds = 0. \tag{1.12}$$

Define the following sets

$$H_j = \begin{cases} L^2(\Omega), & \text{if } j = 0, \\ H_0^1(\Omega), & \text{if } j = 1, \\ H_0^2(\Omega), & \text{if } j = 2, \end{cases}$$

with natural inner product

$$\langle v, w\rangle_{H_j} = \begin{cases} \langle v(s), w(s)\rangle, & \text{if } j = 0, \\ \langle \nabla v(s), \nabla w(s)\rangle, & \text{if } j = 1, \\ \langle \Delta v(s), \Delta w(s)\rangle, & \text{if } j = 2 \end{cases}$$

and norm

$$\|v\|_{H_j} = \begin{cases} \|v(s)\|, & \text{if } j = 0, \\ \|\nabla v(s)\|, & \text{if } j = 1, \\ \|\Delta v(s)\|, & \text{if } j = 2, \end{cases}$$

respectively[1]. Consider

$$U = (y, \eta^t)^T \quad \text{and} \quad U_0(x,s) = (y_0(x,0), \eta^0(x,s))^T$$

where

$$y \in L^2(\Omega) \quad \text{and} \quad \eta^t \in L_j$$

with

$$L_j = L_g^2(\mathbb{R}_+; H_j) := \left\{ v : \mathbb{R}_+ \longrightarrow H_j; \int_0^\infty g(s)\|v(s)\|_{H_j}^2 ds < +\infty \right\}.$$

Define the energy space as follows

$$\mathcal{H}_j = L^2(\Omega) \times L_j, \qquad j \in \{0,1,2\},$$

with inner product and norm

$$\langle (v_1, v_2), (w_1, w_2)\rangle_{\mathcal{H}_j} = \langle v_1, w_1\rangle + \langle v_2, w_2\rangle_{L_j}$$

and

$$\|(v(s), w(s))\|_{\mathcal{H}_j} = \left(\|v(s)\|^2 + \|w(s)\|_{L_j}^2 \right)^{\frac{1}{2}},$$

respectively. Therefore, the systems (1.3) and (1.9) can be seen as the following initial value problem (IVP)

$$\begin{cases} \partial_t U(t) = \mathcal{A}_j U \\ U(0) = U_0. \end{cases} \tag{1.13}$$

Here, the operator $\mathcal{A}_j$ is defined by

$$\mathcal{A}_j(U) = \begin{pmatrix} i\Delta y - i\Delta^2 y + (-1)^{j+1} \int_0^\infty g(s) \Delta^j \eta^t(\cdot, s) ds \\ \\ y - \eta_s^t \end{pmatrix} \tag{1.14}$$

with domain

$$D(\mathcal{A}_j) = \{ U \in \mathcal{H}_j; \mathcal{A}_j(U) \in \mathcal{H}_j, y \in H_0^2(\Omega), \eta^t(x,0) = 0 \}. \tag{1.15}$$

[1]Here $\langle \nabla v(s), \nabla w(s)\rangle := \sum_{k=1}^n \langle \partial_{x_k} v, \partial_{x_k} w\rangle$ and $\|\nabla v(s)\|^2 = \sum_{k=1}^n \|\partial_{x_k} v(s)\|^2$.

Remark 1.3. Observe that for the fourth-order Schrödinger equation, the natural domain to be considered is $H_0^2(\Omega) \cap H^4(\Omega)$. However, since we are working with a more general operator, namely operator defined in (1.14) and (1.15), we need to impose $\mathcal{A}_j(U) \in \mathcal{H}_j$. However, note that the inclusion below

$$H_0^2(\Omega) \cap H^4(\Omega) \times \{\eta^t \in L_j : (-1)^{j+1} \int_0^\infty g(s)\Delta^j \eta^t(\cdot,s)ds \in L^2(\Omega), \eta^t(x,0) = 0\} \subset D(\mathcal{A}_j).$$

is verified. So, the operator $\mathcal{A}_j(U)$ is well-defined.

1.4 Main result

As mentioned, some valuable efforts in the last years focus on the well-posedness and stabilization problem for the fourth-order Schrödinger system. So, in this article, we present a new way to ensure that, in some sense, the Problems 1.1 and 1.2 can be solved for the system (1.3) in n-dimensional case. To do that, we use the ideas contained in [11], so additionally to the Assumption 1 we have also assumed the memory kernel satisfying the following:

Assumption 2. *Assume there is a positive constant α_0 and a strictly convex increasing function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^*)$ satisfying*

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} G'(t) = \infty \tag{1.16}$$

such that

$$g' \leq -\alpha_0 g \tag{1.17}$$

or

$$\int_0^\infty \frac{s^2 g(s)}{G^{-1}(-g'(s))} ds + \sup_{s\in\mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < \infty. \tag{1.18}$$

Additionally, when (1.17) is not verified, we will assume that y_0 satisfies,

$$\sup_{t\in\mathbb{R}_+} \max_{k\in\{0,\dots,n+1\}} \int_t^\infty \frac{g(s)}{G^{-1}(-g'(s))} \left\| \int_0^{s-t} \Delta^{\frac{j}{2}} \partial_s^k y_0(\cdot,\tau)d\tau \right\|^2 ds < \infty. \tag{1.19}$$

for $j \in \{0, 1, 2\}$.

The next theorem is the main result of the article.

Theorem 1.4. *Assume (1.10) and that the Assumption 2 holds. Let $n \in \mathbb{N}^*$, $U_0 \in D(\mathcal{A}_j^{2n})$ when $j = 0$, and $U_0 \in D(\mathcal{A}_j^{2n+2})$ when $j \in \{1, 2\}$. Thus, there exists positive constants $\alpha_{j,n}$ such that the energy (1.5) associated with (1.13) satisfies*

$$E_j(t) \leq \alpha_{j,n} G_n\left(\frac{\alpha_{j,n}}{t}\right), \qquad t \in \mathbb{R}_+^*,\ j \in \{0, 1, 2\}. \tag{1.20}$$

Here, G_n is defined, recursively, as follows:

$$G_m(s) = G_1(sG_{m-1}(s)), \qquad m = 2, 3, \dots, n,\ G_1 = G_0^{-1}, \tag{1.21}$$

where $G_0(s) = s$ if (1.17) is verified, and $G_0(s) = sG'(s)$ if (1.18) holds.

Remark 1.5. Let us give some remarks about the Assumption 2.

i. Thanks to the relation (1.18), we have that (1.19) is valid, for example, if

$$\|\Delta^{\frac{j}{2}}\partial_s^k y_0\|^2, \qquad k = 0, 1, \ldots, n+1,$$

is bounded with respect to s.

ii. There are many class of function g satisfying (1.10), (1.11), (1.16), (1.17), (1.18), and (1.19). For example, those that converge exponentially to zero as

$$g_1(s) := d_1 e^{-q_1 s} \tag{1.22}$$

or those that converge at a slower rate, like

$$g_2(s) := d_2(1+s)^{-q_2} \tag{1.23}$$

with d_1, $q_1, d_2 > 0$, and $q_2 > 3$. Additionally, we point out that conditions (1.10) and (1.17) are satisfied for g_1 defined by (1.22) with $c_0 = \alpha_0 = q_1$, since

$$g_1'(s) = -q_1 d_1 e^{-q_1 s} = -q_1 g_1(s).$$

However, the conditions (1.10) and (1.18) are satisfied for g_2 given by (1.23) with $c_0 = q_2$ and $G(s) = s^p$, for $p > \frac{q_2+1}{q_2-3}$.

Remark 1.6. Now, we will present the following remarks related to the main result of the article.

i. When (1.17) is verified, note that $G_n(0) = 0$, so (1.20) implies

$$\lim_{t\to\infty} E_j(t) \le \alpha_{j,1} G_1\left(\frac{\alpha_{j,1}}{t}\right) = 0. \tag{1.24}$$

Since we have that $D(\mathcal{A}_j^2)$ is dense in $\mathcal{H}_j$, when $j = 0$, and $D(\mathcal{A}_j^4)$ is dense in $\mathcal{H}_j$ when $j = 1, 2$ (see Lemma A.1 in A), we have that (1.24) is valid for any $U_0 \in \mathcal{H}_j$. Therefore, in this case, (1.21) gives $G_n(s) = s^n$ and from (1.20) we get

$$E_j(t) \le \alpha_{j,n}\left(\frac{\alpha_{j,n}}{t}\right)^n = \frac{(\alpha_{j,n})^{n+1}}{t^n} = \beta_{j,n} t^{-n}, \tag{1.25}$$

showing that the energy (1.5) associated with the solutions of the system (1.13) have a polynomial decay rate.

ii. Given (1.18) verified, the relation of (1.20) is weaker than the previous case. For example, when $g = g_2$ defined by (1.23), we see that $G(s) = s^p$ with $p > \frac{q_2+1}{q_2-3}$ satisfies the Assumption 2. Moreover,

$$G_0(s) = sG'(s) = ps^p, \; G_1(s) = \sqrt[p]{\frac{s}{p}},$$

$$G_2(s) = G_1(sG_1(s)) = \sqrt[p]{\frac{s\sqrt[p]{\frac{s}{p}}}{p}} = \left(\frac{s}{p}\right)^{\frac{1}{p}+\frac{1}{p^2}},$$

$$G_3(s) = G_1(sG_2(s)) = \sqrt[p]{\frac{s}{p}\left(\frac{s}{p}\right)^{\frac{1}{p}+\frac{1}{p^2}}} = \left(\frac{s}{p}\right)^{\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}}$$

and so,

$$G_n(s) = \left(\frac{s}{p}\right)^{\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^n}} = \left(\frac{s}{p}\right)^{p_n},$$

where $p_n = \sum_{m=1}^{n} p^{-m} = \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^n}$. Therefore, the energy (1.5) associated with the solutions of the system (1.13) satisfies

$$E_j(t) \leq \alpha_{j,n}\left(\frac{1}{p}\frac{\alpha_{j,n}}{t}\right)^{p_n} = \beta_{j,n}t^{-p_n},$$

with $\beta_{j,n} = \alpha_{j,n}(\frac{\alpha_{j,n}}{p})^{p_n} > 0$, showing that the decay rate of (1.20) is arbitrarily near of t^{-n}, when $p \to 1$, that is, $p_n \to n$ when $q_2 \to \infty$.

1.5 Novelty and structure of the work

Among the main novelties introduced in this article, we give an affirmative answer to the Problems 1.1 and 1.2, providing a further step toward a better understanding of the stabilization problem for the linear system associated with (1.1) in the n-dimensional case. Here, we have used the multipliers method and some arguments devised in [11].

Since we are working with a mixed dispersion we can consider three different memory kernels acting as damping control to stabilize equation (1.3) in contrast to [5], for example, where interior damping is required and no memory is taken into consideration, in a one-dimensional case. Moreover, if we also compare with the linear Schrödinger equation (see e.g. [8]) we have more kernels acting to decay the solution of the equation (1.3) since we have more regularity with the mixed dispersion, which is a gain due the bi-Laplacian operator.

In addition to this, recently, using another approach, the authors in [6] showed that the system (1.8) is stable, however considering a damping mechanism and some important assumptions such as the Geometric Control Condition (GCC) and Unique Continuation Property (UCP). Here, we are not able to prove that the solutions decay exponentially, however, with the approach of this article, the (GCC) and (UCP) are not required. The drawback is that we only provide that the energy of the system (1.3), with memory terms, decays in some sense as explained in the Remark 1.6.

A natural issue is how to deal with the 4NLS system given in (1.1). The main point is that we are not able to use Strichartz estimates or Bourgain spaces to obtain more regularity for the solution of the problem with memory terms, therefore, Theorem 1.4 for the system (1.1) with memory terms remains open.

Now, let us present the outline of our paper. In Section 2 we prove a series of lemmas that are paramount to prove the main result of the article. With the previous section in hand, Theorem 1.4 is shown in Section 3. Finally, for the sake of completeness, in Appendix A, we present the existence of a solution for the system (1.13) in the energy space $\mathcal{H}_j$.

2 Auxiliary results

In this section, we will give some auxiliary lemmas that help us to prove the main result of the article. In this way, the first result shows identities for the derivatives of E_j given by (1.5).

Lemma 2.1. *Suppose the Assumption 1. Then, the energy functional satisfies*

$$E_j'(t) = \frac{1}{2}\int_0^\infty g'(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds, \qquad j \in \{0,1,2\}. \tag{2.1}$$

Proof. Observe that (2.1) is a direct consequence of (A.3), and the result follows. □

Next, we will give a H^1-estimate for the solution of (1.12).

Lemma 2.2. *There exist positive constants $c_{k,j}$, $j \in \{0,1,2\}$ and $k \in \{1,2\}$ such that the following inequality*

$$\|\nabla y\|^2 \leq c_{1,j}\|\eta^t\|_{L_j}^2 + c_{2,j}\int_\Omega [\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Im}(y_t)\mathrm{Re}(y)]\, dx, \tag{2.2}$$

holds.

Proof. We use the multipliers method to prove (2.2). First, multiplying the equation (1.12) by $\overline{y}$, integrating over Ω and taking the real part we get

$$-\mathrm{Im}\left(\int_\Omega y_t\overline{y}dx\right) - \|\nabla y\|^2 - \|\Delta y\|^2 + \mathrm{Re}\left((-1)^j i\int_0^\infty g(s)\int_\Omega \Delta^j\eta^t\overline{y}dxds\right) = 0, \tag{2.3}$$

taking into account the boundary conditions in (1.3) and (1.9), for $y(t,\cdot) \in H_0^2(\Omega)$, for all $t \in \mathbb{R}^+$.

Note that the last term of the left-hand side of (2.3) can be bounded using the generalized Young's inequality giving

$$\begin{aligned}\left|(-1)^j i\int_0^\infty g(s)\int_\Omega \Delta^j\eta^t\overline{y}dxds\right| &= \left|i\langle\eta^t, y\rangle_{L_j}\right| \leq \|\eta^t\|_{L_j}\|y\|_{L_j} \leq \epsilon\|y\|_{L_j}^2 + C(\epsilon)\|\eta^t\|_{L_j}^2 \\ &= \underbrace{g_1\epsilon}_{=:\delta}\|\Delta^{\frac{j}{2}}y\|^2 + C(\epsilon)\|\eta^t\|_{L_j}^2 = \delta\|\Delta^{\frac{j}{2}}y\|^2 + C(\delta)\|\eta^t\|_{L_j}^2,\end{aligned} \tag{2.4}$$

for any $\delta > 0$. In addition to that, the first term of the left-hand side of (2.3) can be viewed as

$$\mathrm{Im}\left(\int_\Omega y_t\overline{y}dx\right) = \int_\Omega (\mathrm{Re}(y)\mathrm{Im}(y_t) - \mathrm{Re}(y_t)\mathrm{Im}(y))\, dx. \tag{2.5}$$

So, replacing (2.4) and (2.5) in (2.3), yields

$$\begin{aligned}\|\nabla y\|^2 &\leq \int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\, dx - \|\Delta y\|^2 + \delta\|\Delta^{\frac{j}{2}}y\|^2 + C(\delta)\|\eta^t\|_{L_j}^2 \\ &\leq \int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\, dx + \delta\|\Delta^{\frac{j}{2}}y\|^2 + C(\delta)\|\eta^t\|_{L_j}^2.\end{aligned} \tag{2.6}$$

We now split the remainder of the proof into three cases.

Case 1. $j = 0$

Poincaré's inequality in (2.6) gives

$$\|\nabla y\|^2 \leq \int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\, dx + \delta c_*\|\nabla y\|^2 + C(\delta)\|\eta^t\|_{L_j}^2. \tag{2.7}$$

Picking $\delta = \frac{1}{2c_*} > 0$ in (2.7) yields

$$\frac{1}{2}\|\nabla y\|^2 \leq \int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\, dx + C(\delta)\|\eta^t\|_{L_j}^2,$$

showing (2.2) with $c_{1,0} = 2C(\delta)$ and $c_{2,0} = 2$.

Case 2. $j = 1$

In this case (2.6) is giving by

$$\|\nabla y\|^2 \leq \int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\, dx + \delta\|\nabla y\|^2 + C(\delta)\|\eta^t\|^2_{L_j}$$

and taking $\delta = \frac{1}{2} > 0$, the inequality (2.2) holds with $c_{1,1} = 2C(\delta)$ and $c_{2,1} = 2$.

Case 3. $j = 2$

Finally, just take any $\delta > 0$ such that $\delta < 1$. Therefore, using (2.6) we get (2.2) for $c_{1,2} = C(\delta)$ and $c_{2,2} = 1$, achieving the result. □

We need now define the following higher-order energy functionals

$$E_{j,k}(t) = \frac{1}{2}\left\|\partial_t^k U\right\|^2_{\mathcal{H}_j}, \tag{2.8}$$

for $U_0 \in D(\mathcal{A}_j^{2n+2})$ in the case when $j = 1, 2$, and $U_0 \in D(\mathcal{A}_0^{2n})$ with $n \in \mathbb{N}^*$. This is possible thanks to the Theorem A.2 in A that guarantees $U \in C^k(\mathbb{R}_+; D(\mathcal{A}_j^{4-k}))$ for $k \in \{1, 2, 3, 4\}$ when $j \in \{1, 2\}$, and that $U \in C^k(\mathbb{R}_+; D(\mathcal{A}_j^{2-k}))$ for $k \in \{1, 2\}$ when $j = 0$. In addition to that, the linearity of the operator $\mathcal{A}_j$ together with (2.1) gives

$$E'_{j,k}(t) = \frac{1}{2}\int_0^\infty g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds. \tag{2.9}$$

With this in hand, let us control the last term of the right-hand side of (2.2) in terms of the $E'_{j,1}$ and the L_j-norms of the $\Delta^{\frac{j}{2}}\eta^t_{tt}$.

Lemma 2.3. *The following estimate is valid*

$$\int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\, dx \leq \epsilon\|\nabla y\|^2 + c_\epsilon \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t_{tt}\|^2 ds - c_\epsilon E'_{j,1}(t), \tag{2.10}$$

for any $\epsilon > 0$.

Proof. Differentiating (1.9) with respect to t, multiplying the result by $g(s)$, and integrating on $[0, \infty)$ we have

$$y_t = \frac{1}{g_0}\int_0^\infty g(s)\left(\eta^t_{tt}(s,x) + \eta^t_{st}(s,x)\right) ds,$$

taking into account the third relation in (1.10). So, we get

$$\begin{aligned}
\mathcal{I} &:= \int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\, dx \\
&= \int_\Omega \mathrm{Re}\left(\frac{1}{g_0}\int_0^\infty g(s)\left(\eta^t_{tt} + \eta^t_{st}\right) ds\right)\mathrm{Im}(y) dx \\
&\quad - \int_\Omega \mathrm{Re}(y)\mathrm{Im}\left(\frac{1}{g_0}\int_0^\infty g(s)\left(\eta^t_{tt} + \eta^t_{st}\right) ds\right) dx.
\end{aligned} \tag{2.11}$$

Now, let us bound the right-hand side of (2.11). To do that, reorganize the terms of the (RHS) and note that

$$\begin{aligned}(RHS) = &\frac{1}{g_0}\int_0^\infty g(s)\int_\Omega \left(\mathrm{Re}\left(\eta_{tt}^t\right)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}\left(\eta_{tt}^t\right)\right)dxds\\ &+\frac{1}{g_0}\int_0^\infty(-g'(s))\int_\Omega\left(\mathrm{Re}\left(\eta_t^t\right)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}\left(\eta_t^t\right)\right)dxds\\ \le&\frac{1}{g_0}\int_0^\infty g(s)\int_\Omega |y||\eta_{tt}^t|dxds + \frac{1}{g_0}\int_0^\infty(-g'(s))\int_\Omega |y||\eta_t^t|dxds\\ \le&\frac{1}{g_0}\int_0^\infty g(s)\|y\|\|\eta_{tt}^t\|ds + \frac{1}{g_0}\int_0^\infty(-g'(s))\|y\|\|\eta_t^t\|ds.\end{aligned} \tag{2.12}$$

The generalized Young inequality gives for any $\delta > 0$ that

$$\|y\|\|\eta_t^t\| \le \delta\|y\|^2 + C_\delta\|\eta_t^t\|^2$$

and

$$\|y\|\|\eta_{tt}^t\| \le \delta\|y\|^2 + C_\delta\|\eta_{tt}^t\|^2.$$

Substituting both inequalities into (2.12) yields

$$\begin{aligned}(RHS) \le &\delta\frac{1}{g_0}\int_0^\infty g(s)\|y\|^2ds + C_\delta\frac{1}{g_0}\int_0^\infty g(s)\|\eta_{tt}^t\|^2ds\\ &+\delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|y\|^2ds + C_\delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|\eta_t^t\|^2ds.\end{aligned} \tag{2.13}$$

Now replacing (2.13) into (2.11) we have

$$\begin{aligned}\mathcal{I} \le &\delta\frac{1}{g_0}\int_0^\infty g(s)\|y\|^2ds + C_\delta\frac{1}{g_0}\int_0^\infty g(s)\|\eta_{tt}^t\|^2ds\\ &+\delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|y\|^2ds + C_\delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|\eta_t^t\|^2ds\\ =&\delta\left(1+\frac{1}{g_0}\left(\int_0^\infty(-g'(s))ds\right)\right)\|y\|^2 + C_\delta\frac{1}{g_0}\int_0^\infty g(s)\|\eta_{tt}^t\|^2ds\\ &+C_\delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|\eta_t^t\|^2ds\\ \le& c_*\delta\left(1+\frac{1}{g_0}\left(\int_0^\infty(-g'(s))ds\right)\right)\|\nabla y\|^2 + c_{**}C_\delta\frac{1}{g_0}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta_{tt}^t\|^2ds\\ &+c_{**}C_\delta\frac{1}{g_0}\int_0^\infty(-g'(s))\|\Delta^{\frac{j}{2}}\eta_t^t\|^2ds,\end{aligned} \tag{2.14}$$

thanks to Poincaré inequality. Here,

$$c_{**} = \begin{cases}1, & \text{if } j=0,\\ c_*, & \text{if } j=1,\\ c_*^2, & \text{if } j=2,\end{cases} \tag{2.15}$$

and $c_* > 0$ is the Poincaré constant. Finally, taking $k=1$ in (2.9), we see that (2.14) leads to (2.10) with $\epsilon = c_*\delta\big(1+\frac{1}{g_0}\big(\int_0^\infty(-g'(s))ds\big)\big)$ and $c_\epsilon = c_{**}C_\delta\frac{1}{g_0}$. □

Now, just in the case $j = 2$, we need an estimate H^2-for the solution of (1.12) similar to the estimate (2.2). This estimate is reported in the following lemma.

Lemma 2.4. *When $j = 2$, there exist positive constants $c_{k,2}$, $k \in \{1,2\}$, such that the following inequality*

$$\|\Delta y\|^2 \leq c_{1,2}\|\eta^t\|_{L_2}^2 + c_{2,2}\int_\Omega [\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Im}(y_t)\mathrm{Re}(y)]\, dx \tag{2.16}$$

holds.

Proof. Multiplying equation (1.12) by $\overline{y}$, integrating over we have

$$0 = i\int_\Omega y_t\overline{y}dx - \|\nabla y\|^2 - \|\Delta y\|^2 + i\int_0^\infty g(s)\int_\Omega \Delta^2\eta^t\overline{y}dxds,$$

since the boundary conditions (1.3) and (1.9) are verified and $y(t,\cdot) \in H_0^2(\Omega)$ for all $t \in \mathbb{R}^+$. Now, taking the real part in the previous equality give us

$$-\mathrm{Im}\left(\int_\Omega y_t\overline{y}dx\right) - \|\nabla y\|^2 - \|\Delta y\|^2 + \mathrm{Re}\left(i\int_0^\infty g(s)\int_\Omega \Delta^2\eta^t\overline{y}dxds\right) = 0. \tag{2.17}$$

Taking into account that

$$\mathrm{Im}\left(\int_\Omega y_t\overline{y}dx\right) = \int_\Omega (\mathrm{Re}(y)\mathrm{Im}(y_t) - \mathrm{Re}(y_t)\mathrm{Im}(y))\, dx \tag{2.18}$$

and, thanks to the generalized Young inequality, we have that

$$\begin{aligned}\left|i\int_0^\infty g(s)\int_\Omega \Delta^2\eta^t\overline{y}dxds\right| &= |i\langle\eta^t, y\rangle_{L_2}| \leq \|y\|_{L_2}\|\eta^t\|_{L_2} \\ &\leq \underbrace{g_1\epsilon}_{=:\delta}\|\Delta y\|^2 + C(\epsilon)\|\eta^t\|_{L_2}^2 = \delta\|\Delta y\|^2 + C(\delta)\|\eta^t\|_{L_2}^2.\end{aligned} \tag{2.19}$$

We get, putting (2.18) and (2.19) into (2.17), that

$$\|\Delta y\|^2 \leq \int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\, dx + \delta\|\Delta y\|^2 + C(\delta)\|\eta^t\|_{L_2}^2. \tag{2.20}$$

Finally, pick $\delta = \dfrac{1}{2} > 0$ in (2.20) to get

$$\frac{1}{2}\|\Delta y\|^2 \leq \int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\, dx + C(\delta)\|\eta^t\|_{L_2}^2,$$

showing (2.16) with $c_{1,2} = 2C(\delta)$ and $c_{2,2} = 2$. □

As a consequence of (2.10), the last term of the right-hand side of (2.16) can be bounded as follows.

Lemma 2.5. *For any $\epsilon > 0$, we have the following inequality*

$$\int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\, dx \leq \epsilon\|\Delta y\|^2 + c_\epsilon\int_0^\infty g(s)\|\Delta\eta_{tt}^t\|^2 ds - c_\epsilon E_{2,1}'(t). \tag{2.21}$$

Proof. Using the Poincaré inequality in the first term of the right-hand side of (2.10), and taking $\epsilon = c^*\epsilon$, where c^* is the Poincaré constant, the result follows. □

The next lemma combines the previous one to get an estimate in H_j for solutions of (1.12).

Lemma 2.6. *There exist a positive constant $c = c(j) > 0$, with $j \in \{1,2\}$ such that*

$$\|\Delta^{\frac{j}{2}} y\|^2 \leq c(E_j(0)) + E_{j,1}(0) + E_{j,2}(0)). \tag{2.22}$$

Proof. Pick $\epsilon = \dfrac{1}{2c_{2,j}}$ in (2.10) and (2.21) when $j = 1$ and $j = 2$, respectively. So we have

$$\int_\Omega (\mathrm{Re}(y_t)\mathrm{Im}(y) - \mathrm{Re}(y)\mathrm{Im}(y_t))\,dx \leq \frac{1}{2c_{2,j}}\|\Delta^{\frac{j}{2}} y\|^2 + c_\epsilon \int_0^\infty g(s)\|\Delta^{\frac{j}{2}} \eta^t_{tt}\|^2 ds - c_\epsilon E'_{j,1}(t).$$

Replacing the previous inequality in (2.2) and in (2.16) for $j = 1$ and $j = 2$, respectively, we get that

$$\|\Delta^{\frac{j}{2}} y\|^2 \leq c_{1,j}\|\eta^t\|^2_{L_j} + \frac{1}{2}\|\Delta^{\frac{j}{2}} y\|^2 + c_{2,j}c_\epsilon \int_0^\infty g(s)\|\Delta^{\frac{j}{2}} \eta^t_{tt}\|^2 ds - c_{2,j}c_\epsilon E'_{j,1}(t). \tag{2.23}$$

Therefore, the properties (1.10) for the function g, together to the fact that $E_{j,k}$, given in (2.8), is non-increasing and (2.9) give us

$$\begin{aligned}
\|\Delta^{\frac{j}{2}} y\|^2 \leq& 2c_{1,j}\|\eta^t\|^2_{L_j} + 2c_{2,j}c_\epsilon \int_0^\infty g(s)\|\Delta^{\frac{j}{2}} \eta^t_{tt}\|^2 ds - 2c_{2,j}c_\epsilon E'_{j,1}(t) \\
\leq& c_{j,4}\left(E_j(t) + E_{j,1}(t) + E_{j,2}(t)\right) \\
\leq& c\left(E_j(0) + E_{j,1}(0) + E_{j,2}(0)\right)
\end{aligned}$$

where $c = c(j) := c_{j,4} = \max\{4c_{1,j}, 4c_{2,j}c_\epsilon, 2c_0c_{2,j}c_\epsilon\}$, for $j \in \{1,2\}$, proving the lemma. □

Before presenting the main result of this section, the next result ensures that the following norms $\|\Delta^{\frac{j}{2}}\eta^t\|$, $\|\eta^t\|$, and $\|\eta^t_{tt}\|$ can be controlled by the generalized energies $E_{k,j}(0)$ and the initial condition y_0, for $t \geq s \geq 0$. The result is the following one.

Lemma 2.7. *Considering the hypothesis of the Lemma 2.6, the following inequality holds*

$$\|\Delta^{\frac{j}{2}}\eta^t\|^2 \leq M_{j,0}(t,s), \tag{2.24}$$

where

$$M_{j,0}(t,s) := \begin{cases} c\left(E_j(0) + E_{j,1}(0) + E_{j,2}(0)\right), \text{ if } 0 \leq s \leq t, \\ \left\|\displaystyle\int_0^{s-t} \Delta^{\frac{j}{2}} y_0(\cdot,\tau)d\tau\right\|^2 + 2s^2c\left(E_j(0) + E_{j,1}(0) + E_{j,2}(0)\right), \text{ if } s > t \geq 0. \end{cases} \tag{2.25}$$

Additionally, for $j = 0$, we have

$$\|\eta^t\|^2 \leq M_{0,0}(t,s) := \begin{cases} 2s^2E_0(0), \text{ if } 0 \leq s \leq t, \\ 2\left\|\displaystyle\int_0^{s-t} y_0(\cdot,\tau)d\tau\right\|^2 + 4s^2E_0(0), \text{ if } s > t \geq 0 \end{cases} \tag{2.26}$$

and

$$\|\eta^t_{tt}\|^2 \leq M_{0,2}(t,s) := \begin{cases} 2s^2E_{0,2}(0), \text{ if } 0 \leq s \leq t, \\ 2\left\|\displaystyle\int_0^{s-t} \partial^2_\tau y_0(\cdot,\tau)d\tau\right\|^2 + 4s^2E_{0,2}(0), \text{ if } s > t \geq 0. \end{cases} \tag{2.27}$$

Proof. Let us first prove (2.24). Hölder inequality and (2.22), for$j \in \{1,2\}$, gives that

$$\|\Delta^{\frac{j}{2}}\eta^t\|^2 = \left\|\int_{t-s}^{t} \Delta^{\frac{j}{2}}y(\cdot,\tau)d\tau\right\|^2 \leq \left(\int_{t-s}^{t} 1\cdot\|\Delta^{\frac{j}{2}}y(\cdot,\tau)\|d\tau\right)^2 \leq s\left(\int_{t-s}^{t} \|\Delta^{\frac{j}{2}}y(\cdot,\tau)\|^2 d\tau\right)$$
$$\leq s^2 c\left(E_j(0)+E_{j,1}(0)+E_{j,2}(0)\right),$$

for $t \geq s \geq 0$. Analogously,

$$\|\Delta^{\frac{j}{2}}\eta^t\|^2 = \left\|\int_{t-s}^{t} \Delta^{\frac{j}{2}}y(\cdot,\tau)d\tau\right\|^2 \leq 2\left\|\int_{0}^{s-t} \Delta^{\frac{j}{2}}y_0(\cdot,\tau)d\tau\right\|^2 + 2s^2c\left(E_j(0)+E_{j,1}(0)+E_{j,2}(0)\right),$$

when $s > t \geq 0$. Consequently, (2.24) is verified.

Now, for $j = 0$, since $\|y\|^2$ is part of E_0 (see (1.5)), and the energy E_0 is non-increasing, we observe, using Hölder inequality, that

$$\|\eta^t\|^2 = \left\|\int_{t-s}^{t} y(\cdot,\tau)d\tau\right\|^2 \leq \left(\int_{t-s}^{t} 1\cdot\|y(\cdot,\tau)\|d\tau\right)^2 \leq s\int_{t-s}^{t}\|y(\cdot,\tau)\|^2 d\tau$$
$$\leq s\int_{t-s}^{t} 2E_0(\tau)d\tau \leq s\int_{t-s}^{t} 2E_0(0)d\tau = 2s^2E_0(0),$$

for $t \geq s \geq 0$. On the other hand,

$$\|\eta^t\|^2 = \left\|\int_{0}^{s-t} y_0(\cdot,\tau)d\tau + \int_0^t y(\cdot,\tau)d\tau\right\|^2 \leq 2\left\|\int_{0}^{s-t} y_0(\cdot,\tau)d\tau\right\|^2 + 2\left\|\int_0^t y(\cdot,\tau)d\tau\right\|^2$$
$$\leq 2\left\|\int_{0}^{s-t} y_0(\cdot,\tau)d\tau\right\|^2 + 4E_0(0)s^2,$$

for $s > t \geq 0$. Thus, (2.26) follows.

Finally, let us prove (2.27). To do that, observe that (1.13) is linear and $V = \partial_t^2 U$ is solution for (1.13) with initial condition $V(0)(x,s) = (\partial_t^2 y_0(x,0), \zeta^0(x,s))$, where

$$\zeta^0(x,s) = \int_0^s \partial_\tau^2 y_0(x,\tau)d\tau.$$

Thanks to relation (2.25), for $j \in \{1,2\}$, we get that

$$M_{j,2}(t,s) := \begin{cases} c_{j,5}\left(E_{j,2}(0)+E_{j,3}(0)+E_{j,4}(0)\right), \text{ if } 0 \leq s \leq t, \\ 2\left\|\int_0^{s-t} \Delta^{\frac{j}{2}}\partial_\tau^2 y_0(\cdot,\tau)d\tau\right\|^2 \\ \qquad +2s^2c_{j,5}\left(E_{j,2}(0)+E_{j,3}(0)+E_{j,4}(0)\right), \text{ if } s > t \geq 0, \end{cases} \tag{2.28}$$

and so,

$$\|\eta_{tt}^t\|^2 \leq M_{j,2}(t,s).$$

Therefore, inequality (2.27) follows using the previous inequality with $j = 0$, and thanks to the relation (2.26), the result is proved. □

The next result is the key lemma to establish the stabilization result for the biharmonic Schrödinger system (1.3).

Lemma 2.8. *There exist positive constants $d_{j,k}$, for each $j \in \{0,1,2\}$ and each $k \in \{0,2\}$ such that the following inequality holds*

$$\frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)} \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k \eta^t\|^2 ds \leq -d_{j,k}E'_{j,k}(t) + d_{j,k}G_0(\epsilon_0 E_j(t)), \tag{2.29}$$

for any $\epsilon_0 > 0$. Here, $E_{j,0} = E_j$, $E'_{j,0} = E'_j(0)$ and G_0 defined as in Theorem 1.4.

Proof. Suppose, first, that the relation (1.17) is satisfied. So, thanks to the relation (2.9), we have

$$E'_{j,k} = \frac{1}{2}\int_0^\infty g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k \eta^t\|^2 ds \leq -\frac{1}{2}\alpha_0 \int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k \eta^t\|^2 ds,$$

for each $j \in \{0,1,2\}$ and each $k \in \{0,2\}$, that is,

$$\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k \eta^t\|^2 ds \leq -\frac{2}{\alpha_0}E'_{j,k},$$

showing (2.29) for each $d_{j,k} = \dfrac{2}{\alpha_0}$ and $G_0(s) = s$.

On the other hand, suppose now that (1.18) and (1.11) are verified. Let us assume, without loss of generality, that $E_j(t) > 0$ and $g' < 0$ in $\mathbb{R}_+$. Let $\tau_{j,k}(t,s), \theta_j(t,s), j \in \{0,1,2\},\ k \in \{0,2\}$ and ϵ_0 be a positive real number which will be fixed later on, and $K(s) = \frac{s}{G^{-1}(s)}$, for $s > 0$. Assumption 2 implies that

$$\lim_{s\to 0^+} K(s) = \lim_{s\to 0^+}\frac{s}{G^{-1}(s)} = \lim_{\tau = G^{-1}(s)\to 0^+}\frac{G(\tau)}{\tau} = G'(0) = 0.$$

Additionally, thanks to the continuity of K we have $K(0) = 0$.

We claim that the function K is non-decreasing. Indeed, since G is convex we have that G^{-1} is concave and $G^{-1}(0) = 0$, implying that

$$K(s_1) = \frac{s_1}{G^{-1}\left(\frac{s_1}{s_2}s_2 + \left(1 - \frac{s_1}{s_2}\right)\cdot 0\right)} \leq \frac{s_1}{\frac{s_1}{s_2}G^{-1}(s_2)} = \frac{s_2}{G^{-1}(s_2)} = K(s_2),$$

for $0 \leq s_1 < s_2$, proving the claim.

Now, note that thanks to the fact that K is non-decreasing and by (2.24), (2.26), (2.28), and (2.27), we get

$$K\left(-\theta_{j,k}(t,s)g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right) \leq K\left(-\theta_{j,k}(t,s)g'(s)M_{j,k}(t,s)\right). \tag{2.30}$$

Inequality (2.30) yields that

$$\begin{aligned}
\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds &= \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}(t,s)}G^{-1}(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2)\\
&\quad \times \frac{\tau_{j,k}(t,s)G'\left(\epsilon_0 E_j(t)\right)g(s)}{-\theta_{j,k}g'(s)}K\left(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right) ds\\
&\leq \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}(t,s)}G^{-1}(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2)\\
&\quad \times \frac{\tau_{j,k}(t,s)G'\left(\epsilon_0 E_j(t)\right)g(s)}{-\theta_{j,k}g'(s)}K\left(-M_{j,k}\theta_{j,k}g'(s)\right) ds\\
&\leq \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}(t,s)}G^{-1}(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2)\\
&\quad \times \frac{M_{j,k}(t,s)\tau_{j,k}(t,s)G'\left(\epsilon_0 E_j(t)\right)g(s)}{G^{-1}\left(-M_{j,k}\theta_{j,k}g'(s)\right)} ds.
\end{aligned} \tag{2.31}$$

Denote the dual function of G by $G^*(s) = \sup_{\tau\in\mathbb{R}_+}\{s\tau - G(\tau)\}$, for $s \in \mathbb{R}_+$. From the Assumption 2 we have

$$G^*(s) = s(G')^{-1}(s) - G((G')^{-1}(s)), \qquad s \in \mathbb{R}_+.$$

Observe also that

$$s_1 s_2 \leq G(s_1) + G^*(s_2), \qquad \forall s_1, s_2 \in \mathbb{R}_+,$$

in particular

$$s_1 = G^{-1}\left(-\theta_{j,k}(t,s)g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right)$$

and

$$s_2 = \frac{M_{j,k}\tau_{j,k}G'(\epsilon_0 E_j(t))g(s)}{-M_{j,k}(t,s)g'(s)\theta_{j,k}}.$$

Therefore, we obtain, by using the previous equality in (2.31), that

$$\begin{aligned}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq & \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}(t,s)}\left(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right)ds \\ & + \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}}G^*\left(\frac{M_{j,k}\tau_{j,k}G'\left(\epsilon_0 E_j(t)\right)g(s)}{G^{-1}\left(-M_{j,k}\theta_{j,k}g'(s)\right)}\right)ds.\end{aligned}$$

Using that $G^*(s) \leq s(G')^{-1}(s)$, we get

$$\begin{aligned}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq & \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\left[\int_0^\infty \frac{1}{\tau_{j,k}}\left(-\theta_{j,k}g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right)ds\right. \\ & \left.+\int_0^\infty \frac{1}{\tau_{j,k}}\frac{M_{j,k}\tau_{j,k}G'\left(\epsilon_0 E_j(t)\right)g(s)}{G^{-1}\left(-M_{j,k}\theta_{j,k}g'(s)\right)}(G')^{-1}\left(\frac{M_{j,k}\tau_{j,k}G'\left(\epsilon_0 E_j(t)\right)g(s)}{G^{-1}\left(-M_{j,k}\theta_{j,k}g'(s)\right)}\right)ds\right].\end{aligned}$$

Pick $\theta_{j,k} = \frac{1}{M_{j,k}}$, to ensure that

$$\begin{aligned}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq & \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}M_{j,k}}\left(-g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right)ds \\ & + \int_0^\infty \frac{M_{j,k}g(s)}{G^{-1}\left(-g'(s)\right)}(G')^{-1}\left(\frac{M_{j,k}\tau_{j,k}G'\left(\epsilon_0 E_j(t)\right)g(s)}{G^{-1}\left(-g'(s)\right)}\right)ds.\end{aligned}$$

Thanks to the fact that $(G')^{-1}$ is non-decreasing we get,

$$\begin{aligned}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq & \frac{1}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \frac{1}{\tau_{j,k}M_{j,k}}\left(-g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right)ds \\ & + \int_0^\infty \frac{M_{j,k}g(s)}{G^{-1}\left(-g'(s)\right)}(G')^{-1}\left(m_0 M_{j,k}\tau_{j,k}G'\left(\epsilon_0 E_j(t)\right)\right)ds,\end{aligned}$$

where $m_0 = \sup_{s\in\mathbb{R}_+}\frac{g(s)}{G^{-1}(-g'(s))}$. Note that (1.18) and (1.19), yields that

$$m_1 = \sup_{s\in\mathbb{R}_+}\int_0^\infty \frac{M_{j,k}(s,t)g(s)}{G^{-1}\left(-g'(s)\right)}ds < \infty.$$

Thus, using that $\tau_{j,k}(t,s) = \frac{1}{m_0 M_{j,k}(t,s)}$ and relation (2.1), we have that

$$\begin{aligned}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq & -\frac{m_0}{G'\left(\epsilon_0 E_j(t)\right)}\int_0^\infty \left(g'(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2\right) ds \\ & + \left(\epsilon_0 E_j(t)\right)\int_0^\infty \frac{M_{j,k}g(s)}{G^{-1}\left(-g'(s)\right)} ds \\ = & -\frac{2m_0}{G'\left(\epsilon_0 E_j(t)\right)}E'_{j,k}(t) + \epsilon_0 m_1 E_j(t).\end{aligned}$$

Finally, multiplying the previous inequality by $G'\left(\epsilon_0 E_j(t)\right) = \frac{G_0\left(\epsilon_0 E_j(t)\right)}{\epsilon_0 E_j(t)}$ gives

$$\frac{G_0\left(\epsilon_0 E_j(t)\right)}{\epsilon_0 E_j(t)}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\partial_t^k\eta^t\|^2 ds \leq -2m_0 E'_{j,k}(t) + m_1 G_0\left(\epsilon_0 E_j(t)\right),$$

which taking $d_{j,k} = \max\{2m_0, m_1\}$, ensures (2.29), showing the lemma. □

3 Proof of Theorem 1.4

Let us split the proof into two cases: a) $n = 1$ and b) $n > 1$.

a) $n = 1$

Poincaré's inequality gives us

$$\|y\|^2 \leq c_*\|\nabla y\|^2 \leq c_*^2\|\Delta y\|^2,$$

where $c_* > 0$ is the Poincaré constant. Summarizing,

$$\|y\|^2 \leq c_{**}\|\Delta^{\frac{j}{2}} y\|^2,$$

for c_{**} defined by (2.15). From the definition of E_j given by (1.5) we found that

$$\frac{2}{\epsilon_0 c_{**}}G_0(\epsilon_0 E_j(t)) \leq \frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\|\Delta^{\frac{j}{2}} y\|^2 + \frac{1}{c_{**}}\frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds.$$

Thanks to the inequality (2.23), we have

$$\begin{aligned}\frac{2}{\epsilon_0 c_{**}}G_0(\epsilon_0 E_j(t) \leq & \frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\left(2c_{1,j}\|\eta^t\|^2_{L_j} + 2c_{2,j}c_\epsilon\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t_{tt}\|^2 ds - 2c_{2,j}c_\epsilon E'_{j,1}(t)\right) \\ & + \frac{1}{c_{**}}\frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds \\ = & \frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\left(2c_{2,j}c_\epsilon\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t_{tt}\|^2 ds - 2c_{2,j}c_\epsilon E'_{j,1}(t)\right) \\ & + \left(2c_{1,j} + \frac{1}{c_{**}}\right)\frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}\int_0^\infty g(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds. \qquad (3.1)\end{aligned}$$

Combining (3.1) with (2.29), gives

$$\begin{aligned}\frac{2}{\epsilon_0 c_{**}}G_0(\epsilon_0 E_j(t)) \leq & -2c_{2,j}c_\epsilon d_{j,2}E'_{j,2}(t) + 2c_{2,j}c_\epsilon d_{j,2}G_0(\epsilon_0 E_j(t)) - 2c_{2,j}\frac{G_0(\epsilon_0 E_j(t))}{\epsilon_0 E_j(t)}c_\epsilon E'_{j,1}(t) \\ & - d_{j,0}\left(2c_{1,j} + \frac{1}{c_{**}}\right)E'_j(t) + d_{j,0}\left(2c_{1,j} + \frac{1}{c_{**}}\right)G_0(\epsilon_0 E_j(t)).\end{aligned}$$

So,

$$\begin{aligned}&\left(\frac{2}{\epsilon_0 c_{**}}-2c_{2,j}c_\epsilon d_{j,2}-d_{j,0}\left(2c_{1,j}+\frac{1}{c_{**}}\right)\right)G_0(\epsilon_0 E_j(t))\\&\quad\le -2c_{2,j}c_\epsilon d_{j,2}E'_{j,2}(t)-2c_{2,j}\frac{G_0(\epsilon_0E_j(t))}{\epsilon_0E_j(t)}c_\epsilon E'_{j,1}(t)-d_{j,0}\left(2c_{1,j}+\frac{1}{c_{**}}\right)E'_j(t).\end{aligned}\tag{3.2}$$

Observe that $H_0(s)=\frac{G_0(s)}{s}$ is non-decreasing and E_j is non-increasing for each j, thus $\frac{G_0(\epsilon_0E_j(t))}{\epsilon_0E_j(t)}$ is non-increasing for each j, and therefore by (3.2) we get

$$\begin{aligned}&\left(\frac{2}{\epsilon_0 c_{**}}-2c_{2,j}c_\epsilon d_{j,2}-d_{j,0}\left(2c_{1,j}+\frac{1}{c_{**}}\right)\right)G_0(\epsilon_0 E_j(t))\\&\quad\le -2c_{2,j}c_\epsilon d_{j,2}E'_{j,2}(t)-2c_{2,j}\frac{G_0(\epsilon_0E_j(0))}{\epsilon_0E_j(0)}c_\epsilon E'_{j,1}(t)-d_{j,0}\left(2c_{1,j}+\frac{1}{c_{**}}\right)E'_j(t).\end{aligned}\tag{3.3}$$

For $\epsilon_0>0$ small enough we have

$$c_1=\left(\frac{2}{\epsilon_0 c_{**}}-2c_{2,j}c_\epsilon d_{j,2}-d_{j,0}\left(2c_{1,j}+\frac{1}{c_{**}}\right)\right)>0.$$

Thus, dividing (3.3) by $c_1>0$ yields that

$$G_0(\epsilon_0E_j(t))\le -c_2\left(E'_j(t)+E'_{j,1}(t)+E'_{j,2}(t)\right),\tag{3.4}$$

where

$$c_2=\max\left\{\frac{2c_{2,j}c_\epsilon d_{j,2}}{c_1},\frac{2c_{2,j}\frac{G_0(\epsilon_0E_j(0))}{\epsilon_0E_j(0)}c_\epsilon}{c_1},\frac{d_{j,0}\left(2c_{1,j}+\frac{1}{c_{**}}\right)}{c_1}\right\}.$$

Now, integrating (3.4) on $[0,t]$, $t\in\mathbb{R}^*_+$, and observing that $G_0(\epsilon_0E_j(t))$ is non-increasing gives

$$\begin{aligned}tG_0(\epsilon_0E_j(t))&=\int_0^t G_0(\epsilon_0E_j(t))ds\le\int_0^t G_0(\epsilon_0E_j(s))ds\le -c_2\int_0^t\left(E'_j(s)+E'_{j,1}(s)+E'_{j,2}(s)\right)ds\\&=-c_2\left(E_j(t)+E_{j,1}(t)+E_{j,2}(t)\right)+c_2\left(E_j(0)+E_{j,1}(0)+E_{j,2}(0)\right)\\&\le c_2\left(E_j(0)+E_{j,1}(0)+E_{j,2}(0)\right)=:c_3.\end{aligned}$$

Because G_0 is invertible and non-decreasing, we deduce that

$$E_j(t)\le\frac{1}{\epsilon_0}(G_0)^{-1}\left(\frac{c_3}{t}\right)=\frac{1}{\epsilon_0}G_1\left(\frac{c_3}{t}\right)\le\alpha_{j,1}G_1\left(\frac{\alpha_{j,1}}{t}\right),$$

for $\alpha_{j,1}=\max\left\{\frac{1}{\epsilon_0},c_3\right\}$, showing (1.20) when $n=1$.

a) $n>1$

Suppose, for induction hypothesis, that for some $n\in\mathbb{N}^*$, we have that (1.20) is verified when $U_0\in D(\mathcal{A}_j^{2n+2})$ for $j\in\{1,2\}$ and $U_0\in D(\mathcal{A}_j^{2n})$ for $j=0$. For $j\in\{1,2\}$, let us take $U_0\in D(\mathcal{A}_j^{2(n+1)+2})$ and for $j=0$, take $U_0\in D(\mathcal{A}_j^{2(n+1)})$. So when $j\in\{1,2\}$ we have

$$U_0\in D(\mathcal{A}_j^{2(n+1)+2})\subset D(\mathcal{A}_j^{2n+2}),\qquad U_t(0)\in D(\mathcal{A}_j^{2(n+1)+1})\subset D(\mathcal{A}_j^{2n+2}),$$
$$\text{and}\quad U_{tt}(0)\in D(\mathcal{A}_j^{2n+2}).$$

Now, for $j = 0$, we found

$$U_0 \in D(\mathcal{A}_j^{2(n+1)}) \subset D(\mathcal{A}_j^{2n}), \qquad U_t(0) \in D(\mathcal{A}_j^{2n+1}) \subset D(\mathcal{A}_j^{2n}) \quad \text{and} \quad U_{tt}(0) \in D(\mathcal{A}_j^{2n}).$$

So, it follows from the induction hypothesis that: there exists $\alpha_{j,n}$ such that

$$E_j(t) \leq \alpha_{j,n} G_n\left(\frac{\alpha_{j,n}}{t}\right), \qquad \forall t \in \mathbb{R}_+^*.$$

Now, since U_t and U_{tt} are solution of (1.13) with initial conditions $U_t(0) \in D(\mathcal{A}_j^{2n+2})$ and $U_{tt}(0) \in D(\mathcal{A}_j^{2n+2})$, respectively, the induction hypothesis guarantees the existence of $\beta_{n,t} > 0$ and $\gamma_{n,t} > 0$, such that

$$E_{j,1}(t) \leq \beta_{j,n} G_n\left(\frac{\beta_{j,n}}{t}\right), \quad \forall t \in \mathbb{R}_+^* \quad \text{and} \quad E_{j,2}(t) \leq \gamma_{j,n} G_n\left(\frac{\gamma_{j,n}}{t}\right), \quad \forall t \in \mathbb{R}_+^*,$$

respectively. Thus, as $G_n's$ are non-decreasing for $\tilde{d}_{j,n} = \max\{3\alpha_{j,n}, 3\beta_{j,n}, 3\gamma_{j,n}\}$, we get

$$E_j(t) + E_{j,1}(t) + E_{j,2}(t) \leq \tilde{d}_{j,n} G_n\left(\frac{\tilde{d}_{j,n}}{t}\right).$$

Finally, how $t \in [T, 2T]$, we have

$$G_0(\epsilon_0 E_j(2T)) \leq G_0(\epsilon_0 E_j(t))$$

and from (3.4) we found the following

$$\begin{aligned} TG_0(\epsilon_0 E_j(2T)) \leq \int_T^{2T} G_0(\epsilon_0 E_j(t))dt &\leq -c_2 \int_T^{2T} \left(E_j'(t) + E_{j,1}'(t) + E_{j,2}'(t)\right) dt \\ &= -c_2\left(E_j(2T) + E_{j,1}(2T) + E_{j,2}(2T)\right) + c_2\left(E_j(T) + E_{j,1}(T) + E_{j,2}(T)\right) \\ &\leq c_2\left(E_j(T) + E_{j,1}(T) + E_{j,2}(T)\right) \leq c_2 \tilde{d}_{j,n} G_n\left(\frac{\tilde{d}_{j,n}}{T}\right) \leq d_{j,n} G_n\left(\frac{d_{j,n}}{T}\right), \end{aligned}$$

where $d_{j,n} = \max\{c_2\tilde{d}_{j,n}, \tilde{d}_{j,n}\}$. Moreover, as G_0 is non-decreasing, $G_1 = G_0^{-1}$ is also non-decreasing. Therefore,

$$E_j(2T) \leq \frac{1}{\epsilon_0} G_0^{-1}\left(\frac{2d_{j,n}}{2T} G_n\left(\frac{2d_{j,n}}{2T}\right)\right) = \frac{1}{\epsilon_0} G_1\left(\tilde{s} G_n(\tilde{s})\right) = \frac{1}{\epsilon_0} G_{n+1}(\tilde{s}) = \alpha_{j,n+1} G_{n+1}\left(\frac{\alpha_{j,n+1}}{2T}\right),$$

where $\alpha_{j,n+1} := \max\left\{\frac{1}{\epsilon_0}, 2d_{j,n}\right\}$. In other words, there is $\alpha_{j,n+1} > 0$ such that (1.20) holds for $n+1$. By the principle of induction we have that (1.20) is verified for all $n \in \mathbb{N}^*$, showing Theorem 1.4. □

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A Well-posedness *via* semigroup theory

This section is devoted to proving that the system (1.13) is well-posed in the energy space $\mathcal{H}_j$. To do that, first, let us present some properties of $\mathcal{A}_j$, defined by (1.14)–(1.15) and its adjoin $\mathcal{A}_j^*$ defined by

$$\mathcal{A}_j^*(V) = \begin{pmatrix} -i\Delta v + i\Delta^2 v + (-1)^j \int_0^\infty g(s)\Delta^j \zeta^t(\cdot,s)ds \\ \\ \zeta_s^t + \dfrac{g'(s)}{g(s)}\zeta^t - v \end{pmatrix} \tag{A.1}$$

with

$$D(\mathcal{A}_j^*) = \{V = (v,\zeta^t) \in \mathcal{H}_j; \mathcal{A}^*{}_j(V) \in \mathcal{H}_j, v \in H_0^2(\Omega), \zeta^t(x,0) = 0\}, \tag{A.2}$$

for $j \in \{0,1,2\}$. So, our first result in this section ensures that $\mathcal{A}_j$ (resp. $\mathcal{A}_j^*$) is dissipative, and $D(\mathcal{A}_j)$ (resp. $D(\mathcal{A}_j^*)$) is dense in the energy space[2].

Lemma A.1. *$\mathcal{A}_j$ and $\mathcal{A}_j^*$ are dissipative. Moreover, $D(\mathcal{A}_j)$ and $D(\mathcal{A}_j^*)$ are dense in $\mathcal{H}_j$, for $j \in \{0,1,2\}$.*

Proof. Indeed, let $(y,\eta^t) \in D(\mathcal{A}_j)$ so

$$\langle \mathcal{A}_j(y,\eta^t),(y,\eta^t)\rangle = -\mathrm{Re}\left(\int_0^\infty g(s)\int_\Omega \Delta^{\frac{j}{2}}\eta_s^t \Delta^{\frac{j}{2}}\overline{\eta^t}dxds\right).$$

As

$$\Delta^{\frac{j}{2}}\eta_s^t\Delta^{\frac{j}{2}}\overline{\eta^t} = \frac{1}{2}(|\Delta^{\frac{j}{2}}\eta^t|^2)_s + i\mathrm{Im}(\Delta^{\frac{j}{2}}\eta_s^t\Delta^{\frac{j}{2}}\overline{\eta^t}),$$

integration by parts over variable s, ensures that

$$\begin{aligned}\langle \mathcal{A}_j(y,\eta^t),(y,\eta^t)\rangle =& -\mathrm{Re}\left(\int_0^\infty g(s)\int_\Omega\left(\frac{1}{2}(|\Delta^{\frac{j}{2}}\eta^t|^2)_s + i\mathrm{Im}(\Delta^{\frac{j}{2}}\eta_s^t\Delta^{\frac{j}{2}}\overline{\eta^t})\right)dxds\right)\\ =& \frac{1}{2}\mathrm{Re}\left(\int_0^\infty g'(s)\int_\Omega |\Delta^{\frac{j}{2}}\eta^t|^2 dxds\right)\\ =& \frac{1}{2}\int_0^\infty g'(s)\|\Delta^{\frac{j}{2}}\eta^t\|^2 ds \le 0,\end{aligned} \tag{A.3}$$

since (1.10) is verified. So, $\mathcal{A}_j$ is dissipative. Similarly, $\mathcal{A}_j^*$ defined by (A.1) is dissipative.

Now, let us prove that $D(\mathcal{A}_j)$ is dense on $\mathcal{H}_j$. Since we showed that $\mathcal{A}_j$ is dissipative, we need to prove that the image of $I - \mathcal{A}_j$ is $\mathcal{H}_j$, since $\mathcal{H}_j$ is reflexive. To do that, pick $(f_1, f_2) \in \mathcal{H}_j = L^2(\Omega) \times L_g^2(\mathbb{R}_+; H_0^j(\Omega))$, we claim that there exists $(y,\eta^t) \in D(\mathcal{A}_j)$ such that

$$(y,\eta^t) - (i\Delta y - i\Delta^2 y + (-1)^{j+1}\int_0^\infty g(s)\Delta^j\eta^t(\cdot,s)ds, y - \eta_s^t) = (f_1, f_2).$$

Or equivalently, we claim that there exits $(y,\eta^t) \in D(\mathcal{A}_j)$ satisfying

$$\begin{cases} y - i\Delta y + i\Delta^2 y + (-1)^j \int_0^\infty g(s)\Delta^j\eta^t(\cdot,s)ds = f_1 \\ \eta^t - y + \eta_s^t = f_2. \end{cases} \tag{A.4}$$

[2]Now on, we will use the following Poincaré inequality $\|y\|^2 \le c_*\|\nabla y\|^2$, $y \in H_0^1(\Omega)$, where $c_* > 0$ is the Poincaré constant.

Indeed, multiplying the second equation of (A.4) by e^s and integrating over s, we get

$$\eta^t(x,s) = (1-e^{-s})y + \int_0^s e^{\tau-s} f_2(\tau)d\tau = (1-e^{-s})y + f_3(s). \tag{A.5}$$

Since $f_2 \in L^2_g(\mathbb{R}_+; H^j_0(\Omega))$, taking $f_3 = \int_0^s e^{\tau-s} f_2(\tau)d\tau$ we have

$$\begin{aligned}
\int_0^\infty g(s)\|\Delta^{\frac{j}{2}} f_3(s)\|^2 ds &= \int_0^\infty g(s)e^{-2s} \int_\Omega \left| \int_0^s e^\tau \Delta^{\frac{j}{2}} f_2(\tau)d\tau \right|^2 dxds \\
&\leq \int_0^\infty g(s)e^{-s} \int_\Omega \int_0^s e^\tau |\Delta^{\frac{j}{2}} f_2(\tau)|^2 d\tau dxds \\
&\leq \int_0^\infty \int_0^s g(s)e^{-s}e^\tau \|\Delta^{\frac{j}{2}} f_2(\tau)\|^2 d\tau ds \\
&= \int_0^\infty \int_\tau^\infty g(s)e^{-s}e^\tau \|\Delta^{\frac{j}{2}} f_2(\tau)\|^2 dsd\tau \\
&\leq \int_0^\infty \int_\tau^\infty g(\tau)e^{-s}e^\tau \|\Delta^{\frac{j}{2}} f_2(\tau)\|^2 dsd\tau \\
&= \|f_2\|^2_{L^2_g(\mathbb{R}_+;H^j_0(\Omega))} < +\infty,
\end{aligned}$$

that is, $f_3 \in L^2_g(\mathbb{R}_+; H^j_0(\Omega))$. Now, for $y \in H^2_0(\Omega)$ holds that

$$\int_0^\infty g(s)\|(1-e^{-s})\Delta^{\frac{j}{2}} y\|^2 ds = \|\Delta^{\frac{j}{2}} y\|^2 \int_0^\infty g(s)(1-e^{-s})^2 ds \leq \|\Delta^{\frac{j}{2}} y\|^2 g_1 < +\infty,$$

since

$$g_1 := \int_0^\infty g(s)(1-e^{-s})ds \leq \int_0^\infty g(s)ds = g_0.$$

So $(1-e^{-s})y \in L^2_g(\mathbb{R}_+; H^j_0(\Omega))$. Therefore, for $y \in H^2_0(\Omega)$, choosing η^t as in (A.5), follows that $\eta^t \in L^2_g(\mathbb{R}_+; H^j_0(\Omega))$ and, so $\eta^t(x,0) = 0$. Thanks to (A.4) we get

$$\eta^t_s = f_2 - \eta^t + y \in L^2_g(\mathbb{R}_+; H^j_0(\Omega)).$$

Finally, let us prove that $y \in H^2_0(\Omega)$ satisfies

$$y - i\Delta y + i\Delta^2 y + (-1)^j \int_0^\infty g(s)\Delta^j \eta^t(\cdot,s)ds = f_1, \tag{A.6}$$

for $\eta^t = (1-e^{-s})y + f_3$. This is equivalent to obtain $y \in H^2_0(\Omega)$ satisfying the following elliptic equation

$$y - i\Delta y + i\Delta^2 y + (-1)^j g_1 \Delta^j y = f_1 - (-1)^j \int_0^\infty g(s)\Delta^j f_3(\cdot,s)ds, \tag{A.7}$$

which is a direct consequence of the Lax–Milgram theorem. Therefore, $(y,\eta^t) \in D(\mathcal{A}_j)$ is a strong solution of $(I-\mathcal{A}_j)(y,\eta^t) = (f_1,f_2)$ and $I-\mathcal{A}_j$ is surjective, showing the result. Similarly, it is shown that $D(\mathcal{A}^*_j)$ defined by (A.2) is dense in $\mathcal{H}_j$. □

The main result of this section is a consequence of the Lemma A.1 and can be read as follows.

Theorem A.2. *Suppose that Assumption 1 and (1.9) are verified. Thus, for each $j \in \{0,1,2\}$, the linear operator $\mathcal{A}_j$ defined by (1.14) is the infinitesimal generator of a semigroup of class C_0 and, for each $n \in \mathbb{N}$ and $U_0 \in D(\mathcal{A}^n_j)$, the system (1.13) has unique solution in the class $U \in \bigcap_{k=0}^n C^k(\mathbb{R}_+; D(\mathcal{A}^{n-k}_j))$.*

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The family of cubic differential systems with two real and two complex distinct infinite singularities and invariant straight lines of the type $(3, 1, 1, 1)$

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Abstract. In this article we consider the class $\mathbf{CSL}_7^{2r2c\infty}$ of non-degenerate real planar cubic vector fields, which possess two real and two complex distinct infinite singularities and invariant straight lines of total multiplicity 7, including the line at infinity. The classification according to the configurations of invariant lines of systems possessing invariant straight lines was given in articles published from 2014 up to 2022. We continue our investigation for the family $\mathbf{CSL}_7^{2r2c\infty}$ possessing configurations of invariant lines of type $(3, 1, 1, 1)$ and prove that there are exactly 42 distinct configurations of this type. Moreover we construct all the orbit representatives of the systems in this class with respect to affine group of transformations and a time rescaling.

Keywords: cubic vector fields, invariant straight lines, infinite and finite singularities, multiplicity of invariant lines, configurations of invariant straight lines, multiplicity of singularity.

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1 Introduction and statement of the Main Theorem

We consider here real polynomial differential systems

$$\frac{dx}{dt} = p(x, y), \qquad \frac{dy}{dt} = q(x, y), \tag{1.1}$$

where p, q are polynomials in x, y with real coefficients, i.e. $p, q \in \mathbb{R}[x, y]$. We call degree of a system (1.1) $\max(\deg(p), \deg(q))$. A *cubic* system (1.1) is of degree three. We say that a system (1.1) is non-degenerate if the polynomials $p(x, y)$ and $q(x, y)$ are co-prime, i.e. $\gcd(p, q) =$ constant.

Let

$$\mathbf{X} = p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y}$$

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be the polynomial vector field corresponding to a system (1.1).

In [17] Darboux introduced the notion of an algebraic invariant curve for differential equations on the complex plane. An algebraic curve $f(x,y) = 0$ with $f(x,y) \in \mathbb{C}[x,y]$ is an invariant curve of a system of the form (1.1) where $p(x,y), q(x,y) \in \mathbb{C}[x,y]$ if and only if there exists $K[x,y] \in \mathbb{C}[x,y]$ such that

$$\mathbf{X}(f) = p(x,y)\frac{\partial f}{\partial x} + q(x,y)\frac{\partial f}{\partial y} = f(x,y)K(x,y)$$

is an identity in $\mathbb{C}[x,y]$. Since $\mathbb{R} \subset \mathbb{C}$, any system (1.1) over $\mathbb{R}$ generates a system of differential equations over $\mathbb{C}$. Using the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}_2(\mathbb{C})$, $(x,y) \mapsto [x:y:1] = [X:Y:Z]$, $(x = X/Z, y = Y/Z$ and $Z \neq 0)$, we can compactify the differential equation $q(x,y)dy - p(x,y)dx = 0$ to an associated differential equation over the complex projective plane. In fact the theory of Darboux in [17] is done for differential equations on the complex projective plane.

We compactify the space of all the polynomial differential systems (1.1) of degree n on $\mathbb{S}^{N-1}$ with $N = (n+1)(n+2)$ by multiplying the coefficients of each systems with $1/(\sum(a_{ij}^2 + b_{ij}^2))^{1/2}$, where a_{ij} and b_{ij} are the coefficients of the polynomials $p(x,y)$ and $q(x,y)$, respectively.

Definition 1.1 ([36]). (1) We say that an invariant curve $\mathcal{L} : f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ for a polynomial system (S) of degree n has *multiplicity* m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}$, $N = (n+1)(n+2)$, such that each (S_k) has m distinct invariant curves $\mathcal{L}_{1,k} : f_{1,k}(x,y) = 0, \ldots, \mathcal{L}_{m,k} : f_{m,k}(x,y) = 0$ over $\mathbb{C}$, $\deg(f) = \deg(f_{i,k}) = r$, converging to $\mathcal{L}$ as $k \to \infty$, in the topology of $P_{R-1}(\mathbb{C})$, with $R = (r+1)(r+2)/2$ and this does not occur for $m+1$.

(2) We say that the line at infinity $\mathcal{L}_\infty : Z = 0$ of a polynomial system (S) of degree n has *multiplicity* m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}$, $N = (n+1)(n+2)$, such that each (S_k) has $m-1$ distinct invariant lines $\mathcal{L}_{1,k} : f_{1,k}(x,y) = 0, \ldots, \mathcal{L}_{m,k} : f_{m-1,k}(x,y) = 0$ over $\mathbb{C}$, converging to the line at infinity $\mathcal{L}_\infty$ as $k \to \infty$, in the topology of $P_2(\mathbb{C})$ and this does not occur for m.

In this work we consider a particular case of invariant algebraic curves, namely the invariant straight lines of systems (1.1). A straight line over $\mathbb{C}$ is the locus $\{(x,y) \in \mathbb{C}^2 | f(x,y) = 0\}$ of an equation $f(x,y) = ux + vy + w = 0$ with $(u,v) \neq (0,0)$ and $(u,v,w) \in \mathbb{C}^3$. We note that by multiplying the equation by a non-zero complex number λ, the locus of the equation does not change. So that we have an injection from the lines in $\mathbb{C}^2$ to the points in $\mathbb{P}_2(\mathbb{C}) \backslash \{[0:0:1]\}$. This injection induces a topology on the set of lines in $\mathbb{C}^2$ from the topology of $\mathbb{P}_2(\mathbb{C})$ and hence we can talk about a sequence of lines convergent to a line in $\mathbb{C}^2$.

For an invariant line $f(x,y) = ux + vy + w = 0$ we denote $\hat{a} = (u,v,w) \in \mathbb{C}^3$ and by $[\hat{a}] = [u:v:w]$ the corresponding point in $\mathbb{P}_2(\mathbb{C})$. We say that a sequence of straight lines $f_i(x,y) = 0$ converges to a straight line $f(x,y) = 0$ if and only if the sequence of points $[\hat{a}_i]$ converges to $[\hat{a}] = [u:v:w]$ in the topology of $\mathbb{P}_2(\mathbb{C})$.

In view of the above definition of an invariant algebraic curve of a system (1.1), a line $f(x,y) = ux + vy + w = 0$ over $\mathbb{C}$ is an invariant line if and only if it there exists $K(x,y) \in \mathbb{C}[x,y]$ which satisfies the following identity in $\mathbb{C}[x,y]$:

$$\mathbf{X}(f) = up(x,y) + vq(x,y) = (ux + vy + w)K(x,y).$$

We point out that if we have an invariant line $f(x,y) = 0$ over $\mathbb{C}$ it could happen that multiplying the equation by a number $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the coefficients of the new equation become real, i.e. $(u\lambda, v\lambda, w\lambda) \in \mathbb{R}^3$. In this case, along with the line $f(x,y) = 0$ sitting in $\mathbb{C}^2$ we also have an associated real line, sitting in $\mathbb{R}^2$ defined by $\lambda f(x,y) = 0$.

Note that, since a system (1.1) is with real coefficients, if its associated complex system has a complex invariant straight line $ux + vy + w = 0$, then its conjugate complex invariant straight line $\bar{u}x + \bar{v}y + \bar{w} = 0$ is also invariant.

A line in $\mathbb{P}_2(\mathbb{C})$ is the locus in $\mathbb{P}_2(\mathbb{C})$ of an equation $F(X,Y,Z) = uX + vY + wZ = 0$ where $(u,v,w) \in \mathbb{C}^3$ and $F(X,Y,Z) \in \mathbb{C}[X,Y,Z]$. The line $Z = 0$ in $\mathbb{P}_2(\mathbb{C})$ is called the line at infinity of the affine plane $\mathbb{C}^2$. This line is an invariant manifold of the complex differential equation on $\mathbb{P}_2(\mathbb{C})$. Clearly the lines in $\mathbb{P}_2(\mathbb{C})$ are in a one-to-one correspondence with points $[u : v : w] \in \mathbb{P}_2(\mathbb{C})$ and thus we have a topology on the set of lines in $\mathbb{P}_2(\mathbb{C})$. We can thus talk about a sequence of lines in $\mathbb{P}_2(\mathbb{C})$ convergent to a line in $\mathbb{P}_2(\mathbb{C})$.

To a line $f(x,y) = ux + vy + w = 0$, $(u,v) \neq (0,0)$, $f \in \mathbb{C}[x,y]$, we associate its projective completion $F(X,Y,Z) = uX + vY + wZ = 0$ under the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}_2(\mathbb{C})$, $(x,y) \mapsto [x : y : 1] = [X,Y,Z]$ indicated above.

We first remark that in the above definition we made an abuse of language. Indeed, we talk about complex invariant lines of real systems. However we already said that to a real system one can associate a complex systems and to a differential equation $q(x,y)dy - p(x,y)dx = 0$ corresponds a differential equation in $\mathbb{P}_2(\mathbb{C})$.

We remark that Definition 1.1 is a particular case of the definition of geometric multiplicity given in [16], and namely the "strong geometric multiplicity" with the restriction, that the corresponding perturbations are cubic systems.

The set **CS** of cubic differential systems depends on 20 parameters and for this reason people began by studying particular subclasses of **CS**. Some of these subclasses are on cubic systems having invariant straight lines.

We mention here some papers on polynomial differential systems possessing invariant straight lines. For quadratic systems see [8, 18, 31, 32, 36–40] and [41]; for cubic systems see [4, 5, 7, 9–14, 23, 25–27, 33, 34, 44] and [45]; for quartic systems see [43] and [47].

The existence of sufficiently many invariant straight lines of planar polynomial systems could be used for proving the integrability of such systems. During the past 15 years several articles were published on this theme (see for example [13, 14, 37, 39]).

According to [1, 16], for a non-degenerate polynomial differential system of degree m, the maximum number of invariant straight lines including the line at infinity and taking into account their multiplicities is $3m$. This bound is always reached (see [16]).

In particular, the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems with a finite number of infinite singularities is 9. In [25] the authors classified all cubic systems possessing the maximum number of invariant straight lines taking into account their multiplicities according to their *configurations of invariant lines*. The notion of configuration of invariant lines for a polynomial differential system was first introduced in [36].

Definition 1.2 ([40])**.** Consider a real planar polynomial differential system (1.1). We call *configuration of invariant straight lines* of this system, the set of (complex) invariant straight lines (which may have real coefficients), including the line at infinity, of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

In [25] the authors used a weaker notion, not taking into account the multiplicities of real singularities. They detected 23 such configurations. Moreover, in [25] the necessary and sufficient conditions for the realization of each one of 23 configurations detected, are determined using invariant polynomials with respect to the action of *the group of affine transformations* $(Aff(2,\mathbb{R}))$ *and time rescaling* (i.e. $Aff(2,\mathbb{R})\times\mathbb{R}^*$)). In [4] the author detected another class of cubic systems whose configuration of invariant lines was not detected in [25].

If two polynomial systems are equivalent under the action of the affine group and time rescaling, clearly they must have the same kinds of configurations of invariant lines. But it could happen that two distinct polynomial systems which are non-equivalent modulo the action of the affine group and time rescaling have "the same kind of configurations" of straight lines. We need to say when two configurations are considered equivalent.

Definition 1.3. Suppose we have two cubic systems $(S),(S')$ both with a finite number of singularities, finite and infinite, a finite set of invariant straight lines $\mathcal{L}_i: f_i(x,y)=0$, $i=1,\ldots,k$, of (S) (respectively $\mathcal{L}'_i: f'_i(x,y)=0$, $i=1,\ldots,k'$, of (S')). We say that the two configurations C,C' of invariant lines, including the line at infinity, of these systems are equivalent if there is a one-to-one correspondence ϕ between the lines of C and C' such that:

(i) ϕ sends an affine line (real or complex) to an affine line and the line at infinity to the line at infinity conserving the multiplicities of the lines and also sends an invariant line with coefficients in $\mathbb{R}$ to an invariant line with coefficients in $\mathbb{R}$;

(ii) for each line $\mathcal{L}: f(x,y)=0$ we have a one-to-one correspondence between the real singular points on $\mathcal{L}$ and the real singular points on $\phi(\mathcal{L})$ conserving their multiplicities and their order on these lines;

(iii) we have a one-to-one correspondence ϕ_∞ between the real singular points at infinity on the (real) lines at infinity of (S) and (S') such that when we list in a counterclockwise sense the real singular points at infinity on (S) starting from a point p on the Poincaré disc, $p_1=p$,...,p_k, ϕ_∞ preserves the multiplicities of the singular points and preserves or reverses the orientation;

(iv) consider the total curves

$$\mathcal{F}: \prod F_j(X,Y,Z)^{m_i}Z^m=0, \quad \mathcal{F}': \prod F'_j(X,Y,Z)^{m'_i}Z^{m'}=0$$

where $F_i(X,Y,Z)=0$ (respectively $F'_i(X,Y,Z)=0$) are the projective completions of $\mathcal{L}_i$ (respectively $\mathcal{L}'_i$) and m_i,m'_i are the multiplicities of the curves $F_i=0,F'_i=0$ and m,m' are respectively the multiplicities of $Z=0$ in the first and in the second system. Then, there is a one-to-one correspondence ψ between the real singularities of the curves $\mathcal{F}$ and $\mathcal{F}'$ conserving their multiplicities as singular points of the total curves.

Remark 1.4. In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [36]. Thus we denote by "(a,b)" the maximum number a (respectively b) of infinite (respectively finite) singularities which can be obtained by perturbation of a multiple infinite singular point.

The configurations of invariant straight lines which were detected for some families of systems (1.1), were instrumental for determining the phase portraits of those families. For example, in [37,39] it was proved that we have a total of 57 distinct configurations of invariant lines for quadratic systems with invariant lines of total multiplicity greater than or equal to 4. These 57 configurations lead to the existence of 135 topologically distinct phase portraits. In [33,34,44,45] it was proved that cubic systems with invariant lines of total parallel multiplicity six or seven (the notion of "parallel multiplicity" could be found in [45]) have 113 topologically distinct phase portraits. This was done by using the various possible configurations of invariant lines of these systems.

In what follows we define some algebraic-geometric notions which will be needed in order to describe the invariants used for distinguishing configurations of invariant lines.

Let V be an irreducible algebraic variety of dimension n over a field K.

Definition 1.5. A cycle of dimension r or r-cycle on V with coefficients in an Abelian group G is a formal sum $\Sigma_W n_W W$, where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in G$, and only a finite number of n_W are non-zero. The support of a cycle C is the set $\mathrm{Supp}(C) = \{W | n_W \neq 0\}$. An $(n-1)$-cycle is called a divisor $\mathcal{D}$.

Definition 1.6. We call type of a divisor $\mathcal{D}$ the set of all ordered couples (m, s_m) where m is an integer appearing as a coefficient in the divisor $\mathcal{D}$ and s_m is the number of occurrences in $\mathcal{D}$ of the coefficient m.

Clearly the notion of *type of a divisor* is an affine invariant.

These notions (see [21]) which occur frequently in algebraic geometry, were used for classification purposes of planar quadratic differential systems by Pal and Schlomiuk [29], [35] and by Llibre and Schlomiuk in [24]. They are also helpful here as we indicate below.

We apply the preceding notions to planar polynomial differential systems (1.1). We denote by $\mathbf{PSL}_{n,\mathfrak{L}}$ the class of all non-degenerate planar polynomial differential systems of degree n with a finite number of infinite singularities and possessing invariant lines, including the line at infinity, of total multiplicity $\mathfrak{L}$.

We define here below an important divisor which is used in this work and which we call *the parallelism divisor*. Consider a system in $(S) \in \mathbf{PSL}_{n,\mathfrak{L}}$. Let $p_1, p_2, \ldots, p_s$ be the set of all the real singular points at infinity of (S). Let j_k, $k \in \{1, \ldots, s\}$ be the total multiplicity of all invariant affine lines which cut the line at infinity at p_k. Let i_k, $k \in \{1, \ldots, s\}$ be the maximum number of distinct invariant affine lines which can appear from the line at infinity in a perturbation of (S) in the class $\mathbf{PSL}_{n,\mathfrak{L}}$ and which cut the line at infinity at p_k.

Definition 1.7. We call parallelism divisor on $Z = 0$ with coefficients in $\mathbb{Z}^2$ the divisor $D_L(S; Z)$ defined as follows:

$$D_L(S; Z) = \sum_{k=1}^{s} \binom{i_k}{j_k} p_k.$$

Observation 1.8. In this definition we spell out the affine part j_k (the finite parallelism index) as well as the infinite part expressed by i_k (the infinite parallelism index). We could form another divisor on the line at infinity, namely $\sum_{k=1}^{s} (i_k + j_k) p_k$ whose coefficients are the total parallelism indices.

Definition 1.9. We define the parallelism type of the configuration (or simply type of the configuration) of invariant lines occurring for a cubic polynomial system (S), the sequence of non-zero numbers, $\tau_k = i_k + j_k$, $k \in \{1, \ldots, s\}$ attached to $D_L(S; Z)$, listed according to descending magnitudes:

$$\mathfrak{T} = (\tau_1, \tau_2, \ldots, \tau_l), \quad 1 \leq l \leq s.$$

Clearly $\mathfrak{T}$ is an affine invariant of systems in the class $\mathbf{PSL}_{n,\mathfrak{L}}$ and of their configurations of invariant lines.

Notation 1.10. As already used in the Abstract $\mathbf{CSL}_7^{2r2c\infty}$ is meant to be the class of non-degenerate cubic systems with invariant lines of total multiplicity seven which have two real and two complex distinct singularities at infinity.

As we have two real and two complex infinite singularities and the total multiplicity of the invariant lines (including the line at infinity) must be 7, then the cubic systems in $\mathbf{CSL}_7^{2r2c\infty}$ could only have one of the following four possible types of configurations of invariant lines:

$$(i)\ \mathfrak{T}=(3,3);\quad (ii)\ \mathfrak{T}=(3,1,1,1);\quad (iii)\ \mathfrak{T}=(2,2,2);\quad (iv)\ \mathfrak{T}=(2,2,1,1). \qquad (1.2)$$

Remark 1.11. We remark that the cubic systems in $\mathbf{CSL}_7^{2r2c\infty}$ possessing the configurations of invariant lines of the type $\mathfrak{T}=(3,3)$ were already investigated in [6], where the existence of 14 distinct configurations *Config. 7.1a* −*Config. 7.14a* of this type are determined.

In this article we classify the subfamily of cubic systems in $\mathbf{CSL}_7^{2r2c\infty}$, possessing configurations of invariant line of the type $(3,1,1,1)$, according to the relation of equivalence of configurations. We denote this subfamily by $\mathbf{CSL}_{(3,1,1,1)}^{2r2c\infty}$.

Our main result is the following one.

Main Theorem.

(A) *A non-degenerate cubic system (1.1) belongs to the class* $\boldsymbol{CSL}_{(3,1,1,1)}^{2r2c\infty}$ *if and only if* $\mathcal{D}_1<0$, $\mathcal{V}_4=\mathcal{U}_2=0$ *and one of the following set of conditions holds:*

- (A_1) *If* $\mathcal{D}_7\neq0$, $\mathcal{D}_8\neq0$, $\chi_1=0$, $\mathcal{D}_6\neq0$ *then* $\chi_3=\chi_6=0$.
- (A_2) *If* $\mathcal{D}_7\neq0$, $\mathcal{D}_8\neq0$, $\chi_1=0$, $\mathcal{D}_6=0$ *then* $\chi_2=\chi_3=0$.
- (A_3) *If* $\mathcal{D}_7\neq0$, $\mathcal{D}_8\neq0$, $\chi_1\neq0$, $\mathcal{D}_4\neq0$ *then* $\chi_7=\chi_8=\chi_9=\chi_{10}$ *and either* $\mathcal{D}_5\neq0$, $\chi_{11}=0$ *or* $\mathcal{D}_5=\chi_{12}=0$.
- (A_4) *If* $\mathcal{D}_7\neq0$, $\mathcal{D}_8\neq0$, $\chi_1\neq0$, $\mathcal{D}_4=0$ *then* $\chi_4=\chi_5=\chi_7=\chi_9=\chi_{13}=\chi_{14}=0$.
- (A_5) *If* $\mathcal{D}_7\neq0$, $\mathcal{D}_8=0$, $\mathcal{D}_6\neq0$, $\mathcal{D}_4\neq0$ *then* $\chi_1=\chi_3=\chi_6=0$.
- (A_6) *If* $\mathcal{D}_7\neq0$, $\mathcal{D}_8=0$, $\mathcal{D}_6\neq0$, $\mathcal{D}_4=0$ *then* $\chi_1=\chi_3=\chi_8=\chi_{16}=0$, $\chi_{15}\neq0$.
- (A_7) *If* $\mathcal{D}_7\neq0$, $\mathcal{D}_8=0$, $\mathcal{D}_6=0$ *then* $\chi_1=\chi_2=\chi_4=\chi_6=\chi_{17}=0$, $\chi_{11}\neq0$, $\zeta_4\leq0$.
- (A_8) *If* $\mathcal{D}_7=0$, $\widetilde{\chi}_1\neq0$ *then* $\chi_1=\chi_2=\chi_3=0$.
- (A_9) *If* $\mathcal{D}_7=0$, $\widetilde{\chi}_1=0$, $\widetilde{\chi}_2\neq0$ *then* $\chi_1=\chi_3=\chi_6=0$.

If $\mathcal{D}_7=\widetilde{\chi}_1=\widetilde{\chi}_2=0$ *then a cubic system (1.1) could not belong to the class* $\boldsymbol{CSL}_{(3,1,1,1)}^{2r2c\infty}$.

(B) *Assume that a non-degenerate cubic system (1.1) belongs to the class* $\boldsymbol{CSL}_{(3,1,1,1)}^{2r2c\infty}$, *i.e. one of the sets of conditions provided by statement* (A) *holds. Then this system possesses one of the configurations Config. 7.1b – Config. 7.42b, presented in Figure 1.1. Moreover the necessary and sufficient conditions for the realization of each one the mentioned configurations are given in Diagrams from Figures 1.2, 1.3 and 1.4, correspondingly.*

(C) *In Figure 1.1 are given all the configurations that could occur for systems in the class* $\boldsymbol{CSL}_{(3,1,1,1)}^{2r2c\infty}$. *We prove that all these configurations are realizable within* $\boldsymbol{CSL}_{(3,1,1,1)}^{2r2c\infty}$ *(see the examples given in the proof of the statement* (A)*) and that these 42 configurations are distinct. This proof is done in Subsection 3.3 using geometric invariants and it is presented in the corresponding diagram from Figures 3.1.*

Notation 1.12. We give here the directions for reading the pictures representing the configurations. An invariant line with multiplicity $k>1$ will appear in a configuration in bold face and will have next to it the number k. Real invariant straight lines are represented by continuous lines, whereas complex invariant straight lines are represented by dashed lines. The multiplicities of the real singular points of the system located on the invariant lines, will be indicated next to the singular points.

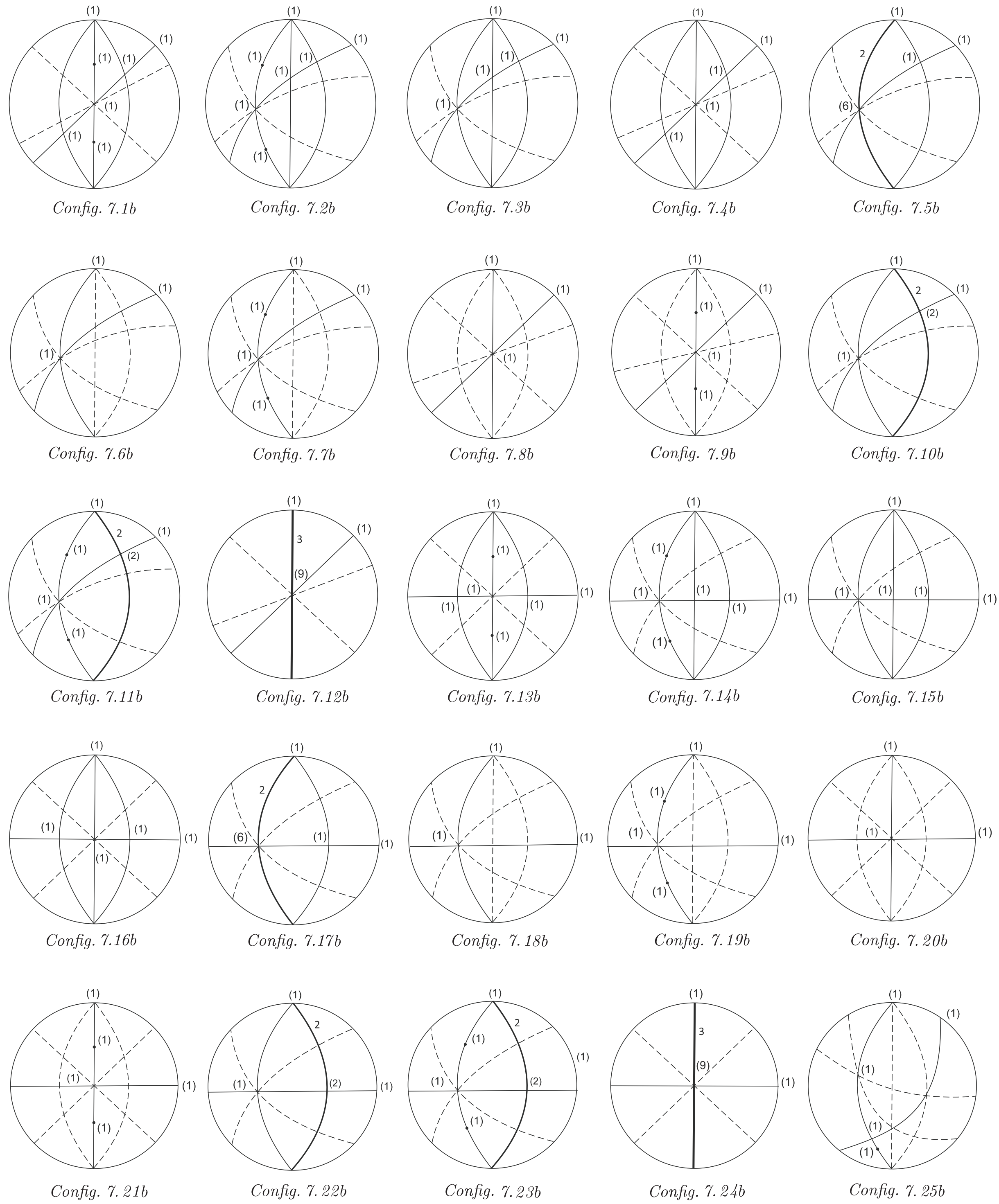

Figure 1.1: The configurations of invariant lines for cubic systems in the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$

Since a configuration of invariant lines of a system (1.1) could contain simultaneously real and complex invariant lines, there appears the problem of indicating these lines simultaneously on a picture in the Poincaré disc in order to capture and see schematically this phenomenon. So in order to fix the positions of real lines with respect to the complex ones in

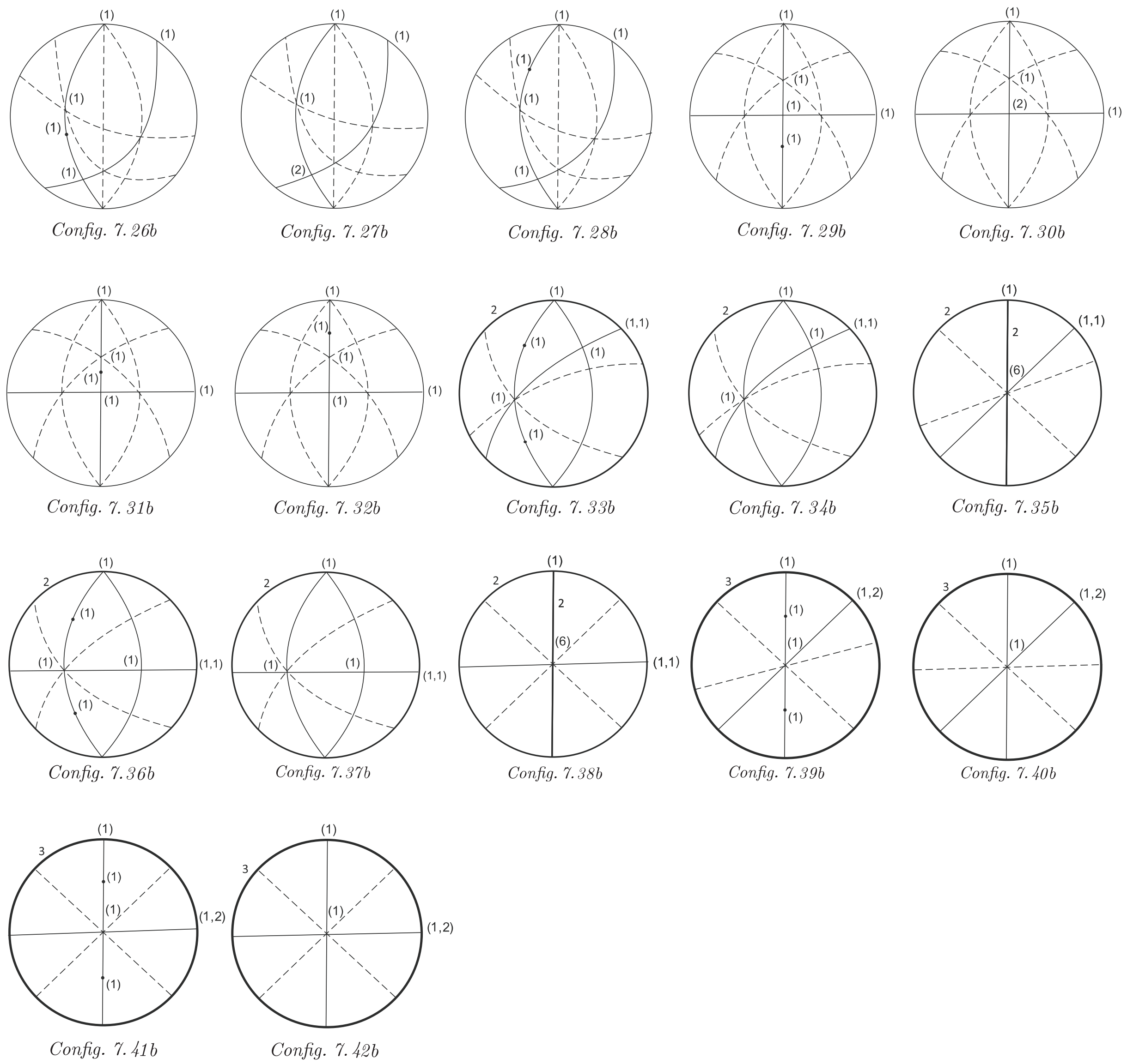

Figure 1.1 (*cont.*): The configurations of invariant lines for cubic systems in the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$

a coherent way, we present here the following mode of representing complex invariant lines of systems (1.1) along with the real invariant ones on the Poincaré disc.

Convention. Assume that a system (1.1) possesses an invariant line with complex coefficients such that we cannot multiply all its coefficients by a non-zero complex number and obtain real coefficients. Then clearly the corresponding conjugate line is also invariant for this system having the same property. Suppose that such invariant lines are:

$$L : Ax + By + C = 0, \qquad \bar{L} : \bar{A}x + \bar{B}y + \bar{C} = 0, \quad A, B, C \in \mathbb{C}, \quad (A, B) \neq (0, 0).$$

These lines are affine lines in $\mathbb{C}^2$ ($\cong \mathbb{R}^4$) and hence planes in $\mathbb{R}^4$.

Without loss of generality, due to the change $x \leftrightarrow y$ we may assume $B \neq 0$ and then the lines become:

$$y = (a \pm bi)x + (c \pm di) = (ax + c) \pm i(bx + d), \quad (a, b, c, d) \in \mathbb{R}^4, \; b^2 + d^2 \neq 0. \qquad (1.3)$$

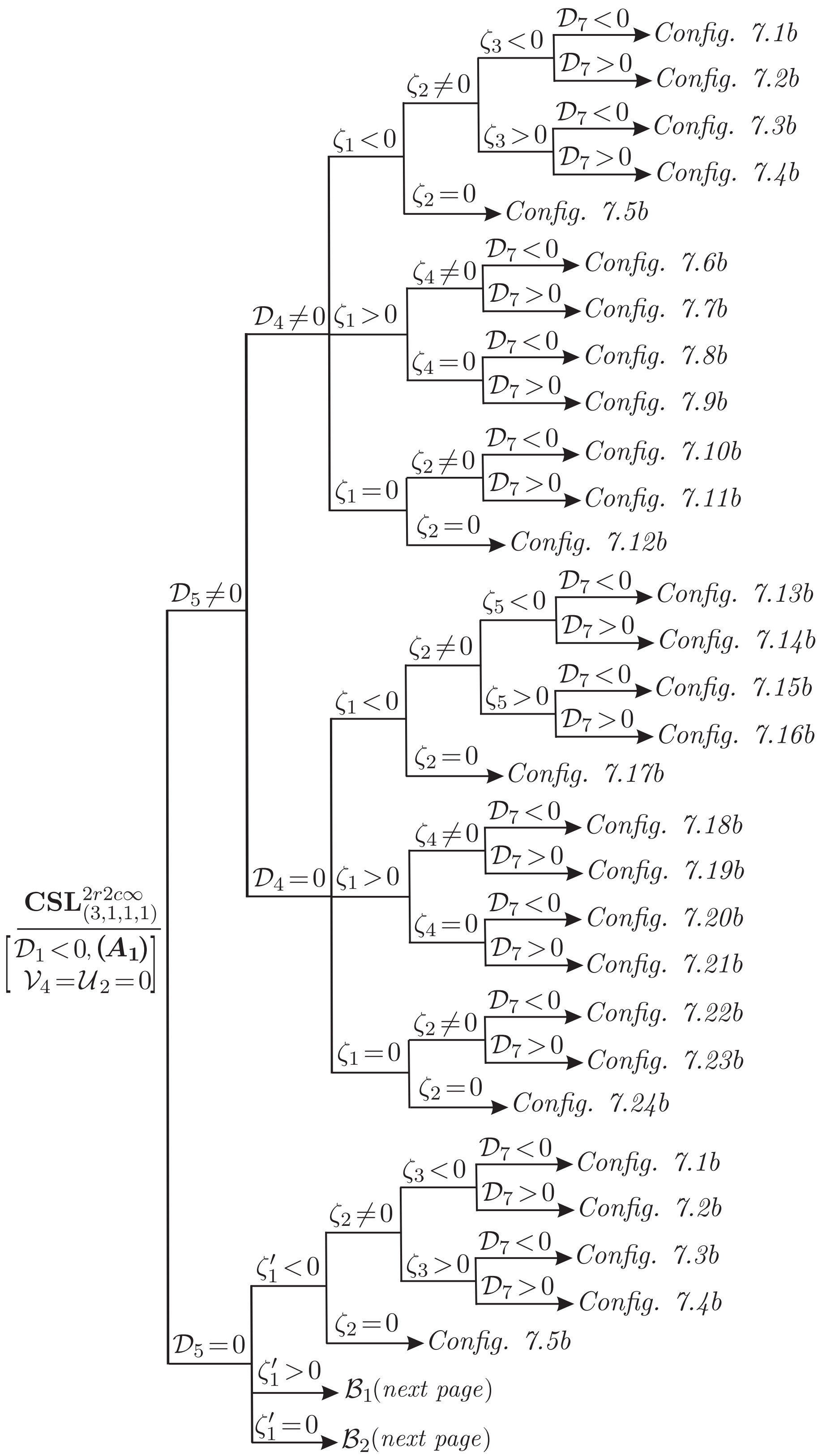

Figure 1.2: Diagram of the configurations for the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$: statement $(\boldsymbol{A_1})$

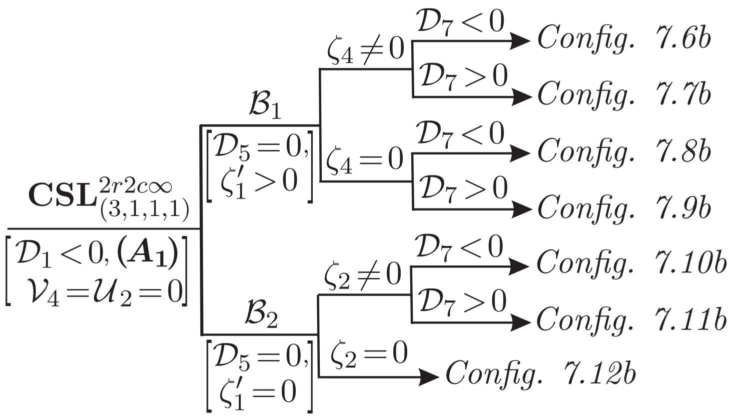

Figure 1.2 (*cont.*): Diagram of the configurations for the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$: statement $(\boldsymbol{A_1})$

We associate to the lines (1.3) over $\mathbb{C}$ the two lines with real coefficients: the real line $l = \mathcal{R}(L,\bar{L})$: $y = ax + c$ as well as its complexification Cl defined by the same equation but letting x, y run over the complex plane. The real line l can be drawn on the Poincaré disk. Consider now the two cases $b \neq 0$ and $b = 0$.

Case $b \neq 0$. In this case the two lines (1.3) intersect at the real point $M_0 = (-d/b, -(ad - bc)/b) \in \mathbb{R}^2$ that also lies on the real line $l \subset Cl$. Being at the intersection of the two complex invariant lines (1.3) , the real point M_0 is a singular point for systems (1.1). To signal the presence of the complex lines (1.3) we make the convention to represent them on the Poincaré disk as two dashed lines both passing through M_0. Thus the real line l will appear inside two of the four curvilinear triangles described by the dashed lines and parts of the circle at infinity. We denote this domain by $\mathcal{D}$.

Suppose now that the system S has a real invariant line l' also passing through M_0 and consider its complexification $L' = Cl'$ that is also an invariant line.

We assume that our system is included in a family of systems possessing the invariant lines (1.3) and the line L′. If the parameters b and d tend simultaneously to zero, then it is clear that the two complex lines tend to the complexification of the real line $y = ax + c$. Then clearly this line is an invariant line that is a multiple line of multiplicity two or three. We now distinguish two subcases: $l' = l$ or $l' \neq l$.

Subcase $l' = l$. In this case two complex invariant lines (1.3) coalesced with the invariant line L' and hence this is a triple line. In this case we will draw the real line l' inside the domain $\mathcal{D}$.

Subcase $l' \neq l$. In this case if both b and d tend to zero then the lines (1.3) will tend to a double line, the complexification of the real line $y = ax + c$. In this case we draw the line l' outside $\mathcal{D}$.

Case $b = 0$. In this case the lines (1.3) intersect at infinity at the real point $[1 : a : 0]$. The real line $l : y = ax + c$ passes also through this point. We draw by dashed lines these two complex lines placing inside the domain delimited by them and denoted by $\mathcal{D}'$ the real line l. Suppose the line L' passes through the same point at infinity $[1 : a : 0]$. We make the following convention:

If $l' = l$ then we will draw l' inside the domain $\mathcal{D}'$. If $l' \neq l$ then we will draw l' outside the domain $\mathcal{D}'$.

The work is organized as follows. In Section 2 we give some preliminary results needed

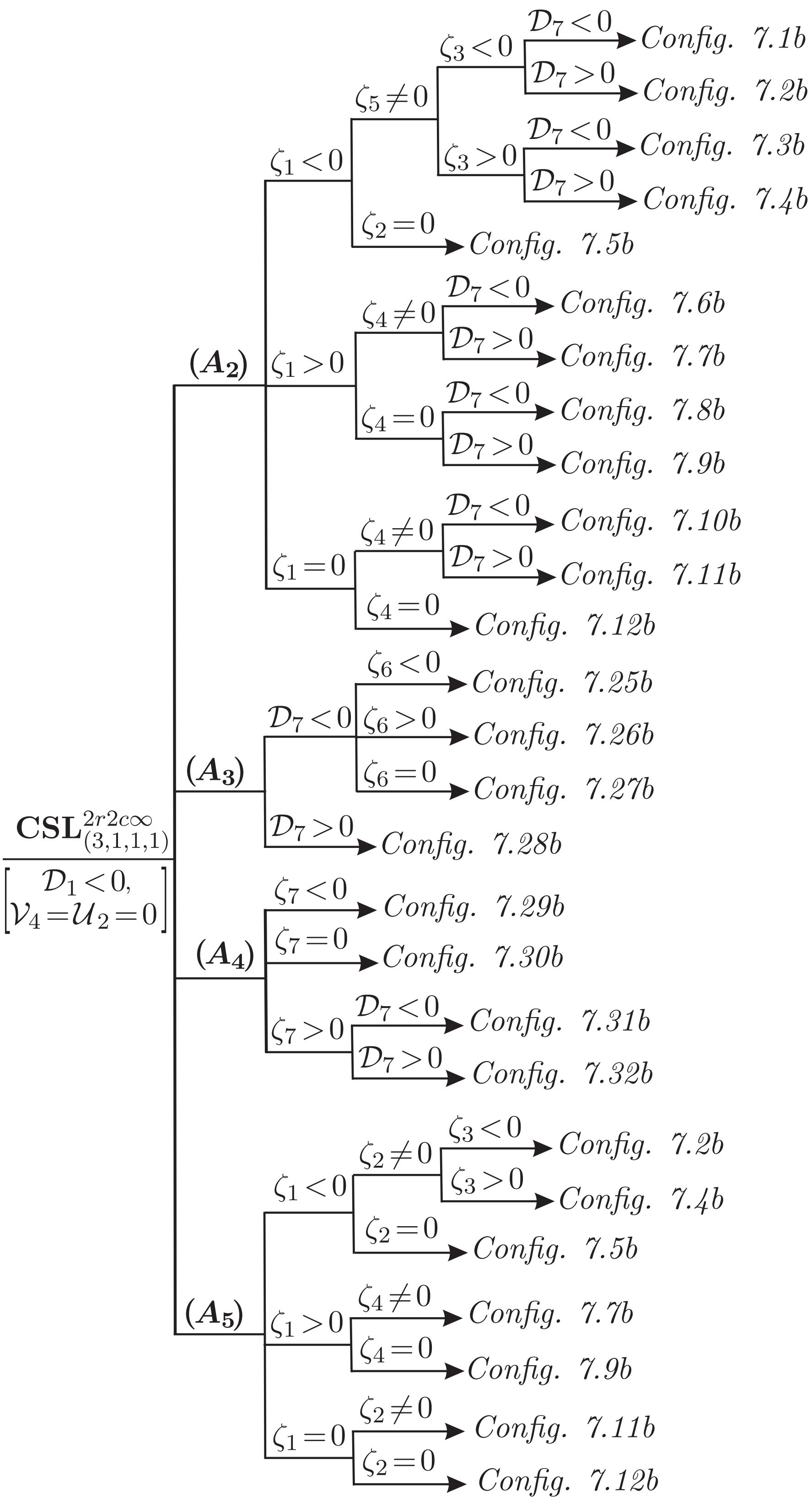

Figure 1.3: Diagram of the configurations for the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$: statements (A_2)–(A_5)

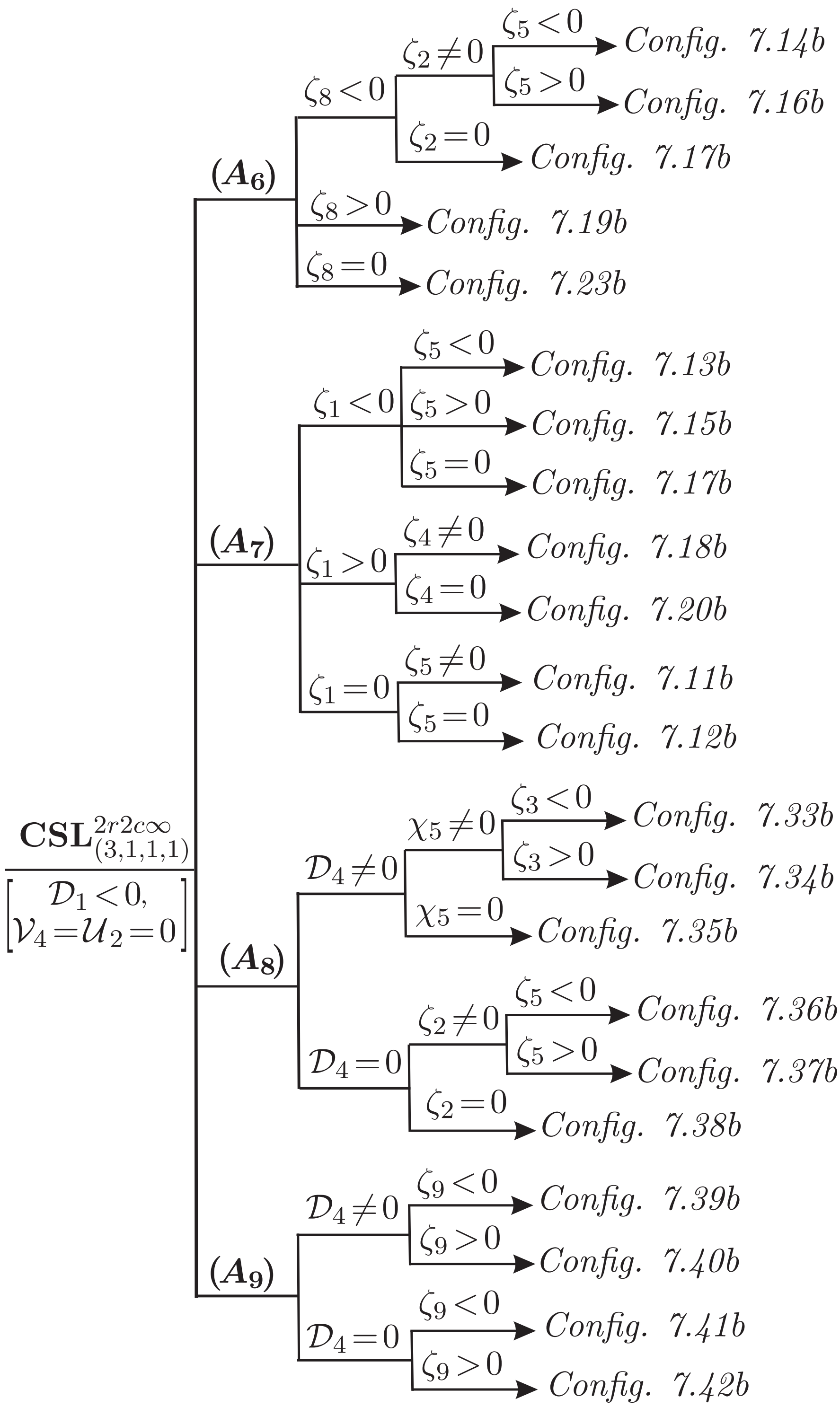

Figure 1.4: Diagram of the configurations for the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$: statements **$(A_6)$–$(A_9)$**

for this paper. In Section 3 we prove our Main Theorem considering the family of cubic systems possessing invariant lines in the configuration of the type $(3,1,1,1)$ and having two real and two complex distinct infinite singularities. More exactly, in Subsection 3.1 we prove the statement (A) of the Main Theorem, constructing the canonical systems and determining the corresponding configurations which these systems could possess. Moreover, the necessary and sufficient conditions for the realization of each one the obtained configurations are determined. In Subsection 3.2 we prove the statement (B) of the Main Theorem. Using the geometric invariants, we prove that all the 42 detected configurations of invariant lines for the class of cubic systems in $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$ are distinct according to Definition 1.3.

2 Preliminaries

Consider real cubic systems, i.e. systems of the form:

$$\begin{aligned} \dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(a,x,y),\\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(a,x,y) \end{aligned} \tag{2.1}$$

with variables x and y and real coefficients. The polynomials p_i and q_i $(i = 0,1,2,3)$ are homogeneous polynomials of degree i in x and y:

$$\begin{aligned} &p_0 = a_{00}, \quad p_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3,\\ &p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2,\\ &q_0 = b_{00}, \quad q_3(x,y) = b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3,\\ &q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

Let $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ be the 20-tuple of the coefficients of systems (2.1) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03}, x, y]$.

2.1 The main invariant polynomials associated to configurations of invariant lines

It is known that on the set of polynomial systems (1.1), in particular on the set **CS** of all cubic differential systems (2.1), acts the group $Aff(2,\mathbb{R})$ of affine transformation on the plane [40]. For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **CS**. We can identify the set **CS** of systems (2.1) with a subset of $\mathbb{R}^{20}$ via the map $\mathbf{CS} \longrightarrow \mathbb{R}^{20}$ which associates to each system (2.1) the 20-tuple $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ of its coefficients.

For the definitions of an affine or GL-comitant or invariant as well as for the definition of a T-comitant and CT-comitant we refer the reader to [36]. Here we shall only construct the necessary affine invariant polynomials which are needed to detect the existence of invariant lines for the class of cubic systems with four real distinct infinite singularities and with exactly seven invariant straight lines including the line at infinity and including multiplicities.

Let us consider the polynomials

$$\begin{aligned} C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y], \quad i = 0,1,2,3,\\ D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y], \quad i = 1,2,3. \end{aligned}$$

As it was shown in [42] the polynomials

$$\{C_0(a,x,y),\ C_1(a,x,y),\ C_2(a,x,y),\ C_3(a,x,y),\ D_1(a),\ D_2(a,x,y)\ D_3(a,x,y)\} \tag{2.2}$$

of degree one in the coefficients of systems (2.1) are GL-comitants of these systems.

Notation 2.1. Let $f, g \in \mathbb{R}[a,x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called the transvectant of index k of (f,g) (cf. [20,28]).

Theorem 2.2 ([46]). *Any GL-comitant of systems (2.1) can be constructed from the elements of the set (2.2) by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(f,g)^{(k)}$.*

Let us apply a translation $x = x' + x_0$, $y = y' + y_0$ to the polynomials $P(a,x,y)$ and $Q(a,x,y)$. We obtain $\tilde{P}(\tilde{a}(a,x_0,y_0),x',y') = P(a,x'+x_0,y'+y_0)$, $\tilde{Q}(\tilde{a}(a,x_0,y_0),x',y') = Q(a,x'+x_0,y'+y_0)$. We construct the following polynomials

$$\Omega_i(a,x_0,y_0) \equiv \text{Res}_{x'}\Big(C_i\big(\tilde{a}(a,x_0,y_0),x',y'\big), C_0\big(\tilde{a}(a,x_0,y_0),x',y'\big)\Big)/(y')^{i+1},$$
$$\Omega_i(a,x_0,y_0) \in \mathbb{R}[a,x_0,y_0], \quad (i=1,2,3)$$

and we denote

$$\tilde{\mathcal{G}}_i(a,x,y) = \Omega_i(a,x_0,y_0)|_{\{x_0=x,\ y_0=y\}} \in \mathbb{R}[a,x,y] \quad (i=1,2,3).$$

Remark 2.3. We note that the polynomials $\tilde{\mathcal{G}}_1(a,x,y)$, $\tilde{\mathcal{G}}_2(a,x,y)$ and $\tilde{\mathcal{G}}_3(a,x,y)$ are affine comitants of systems (2.1) and are homogeneous polynomials in the coefficients $a_{00},\dots,b_{03}$ and non-homogeneous in x,y and

$$\deg_a \tilde{\mathcal{G}}_1 = 3, \qquad \deg_a \tilde{\mathcal{G}}_2 = 4, \qquad \deg_a \tilde{\mathcal{G}}_3 = 5,$$
$$\deg_{(x,y)} \tilde{\mathcal{G}}_1 = 8, \qquad \deg_{(x,y)} \tilde{\mathcal{G}}_2 = 10, \qquad \deg_{(x,y)} \tilde{\mathcal{G}}_3 = 12.$$

Notation 2.4. Let $\mathcal{G}_i(a,X,Y,Z)$ $(i=1,2,3)$ be the homogenization of $\tilde{\mathcal{G}}_i(a,x,y)$, i.e.

$$\mathcal{G}_1(a,X,Y,Z) = Z^8\tilde{\mathcal{G}}_1(a,X/Z,Y/Z),$$
$$\mathcal{G}_2(a,X,Y,Z) = Z^{10}\tilde{\mathcal{G}}_2(a,X/Z,Y/Z),$$
$$\mathcal{G}_3(a,X,Y,Z) = Z^{12}\tilde{\mathcal{G}}_3(a,X/Z,Y/Z),$$

and $\mathcal{H}(a,X,Y,Z) = \gcd\Big(\mathcal{G}_1(a,X,Y,Z),\ \mathcal{G}_2(a,X,Y,Z),\ \mathcal{G}_3(a,X,Y,Z)\Big)$ in $\mathbb{R}[a,X,Y,Z]$.

The geometrical meaning of these affine comitants is given by the two following lemmas (see [25]):

Lemma 2.5. *The straight line $\mathcal{L}(x,y) \equiv ux+vy+w = 0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ is an invariant line for a cubic system (2.1) if and only if the polynomial $\mathcal{L}(x,y)$ is a common factor of the polynomials $\tilde{\mathcal{G}}_1(x,y)$, $\tilde{\mathcal{G}}_2(x,y)$ and $\tilde{\mathcal{G}}_3(x,y)$ over $\mathbb{C}$, i.e.*

$$\tilde{\mathcal{G}}_i(x,y) = (ux+vy+w)\widetilde{W}_i(x,y) \quad (i=1,2,3),$$

where $\widetilde{W}_i(x,y) \in \mathbb{C}[x,y]$.

Lemma 2.6. *Consider a cubic system (2.1) and let $\boldsymbol{a} \in \mathbb{R}^{20}$ be its 20-tuple of coefficients.*

1) *If* $\mathcal{L}(x,y) \equiv ux + vy + w = 0$, $u,v,w \in \mathbb{C}$, $(u,v) \neq (0,0)$ *is an invariant straight line of multiplicity* k *for the system associated to* $\boldsymbol{a}$ *then* $[\mathcal{L}(x,y)]^k \mid \gcd(\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2, \tilde{\mathcal{G}}_3)$ *in* $\mathbb{C}[x,y]$*, i.e. there exist* $W_i(a,x,y) \in \mathbb{C}[x,y]$ $(i = 1,2,3)$ *such that*

$$\tilde{\mathcal{G}}_i(\boldsymbol{a},x,y) = (ux+vy+w)^k W_i(a,x,y), \quad i = 1,2,3.$$

2) *If the line* $l_\infty : Z = 0$ *is of multiplicity* $k > 1$ *then* $Z^{k-1} \mid \gcd(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$*, i.e. we have* $Z^{k-1} \mid H(\boldsymbol{a}, X, Y, Z)$*.*

Consider the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ constructed in [3] and acting on $\mathbb{R}[a,x,y]$, where

$$\begin{aligned}
\mathbf{L}_1 = {} & 3a_{00}\frac{\partial}{\partial a_{10}} + 2a_{10}\frac{\partial}{\partial a_{20}} + a_{01}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{02}\frac{\partial}{\partial a_{12}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{21}} + a_{20}\frac{\partial}{\partial a_{30}} \\
& + 3b_{00}\frac{\partial}{\partial b_{10}} + 2b_{10}\frac{\partial}{\partial b_{20}} + b_{01}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{02}\frac{\partial}{\partial b_{12}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{21}} + b_{20}\frac{\partial}{\partial b_{30}}, \\
\mathbf{L}_2 = {} & 3a_{00}\frac{\partial}{\partial a_{01}} + 2a_{01}\frac{\partial}{\partial a_{02}} + a_{10}\frac{\partial}{\partial a_{11}} + \frac{1}{3}a_{20}\frac{\partial}{\partial a_{21}} + \frac{2}{3}a_{11}\frac{\partial}{\partial a_{12}} + a_{02}\frac{\partial}{\partial a_{03}} \\
& + 3b_{00}\frac{\partial}{\partial b_{01}} + 2b_{01}\frac{\partial}{\partial b_{02}} + b_{10}\frac{\partial}{\partial b_{11}} + \frac{1}{3}b_{20}\frac{\partial}{\partial b_{21}} + \frac{2}{3}b_{11}\frac{\partial}{\partial b_{12}} + b_{02}\frac{\partial}{\partial b_{03}}.
\end{aligned}$$

Using this operator and the affine invariant $\mu_0 = \text{Resultant}_x\big(p_3(a,x,y), q_3(a,x,y)\big)/y^9$ we construct the following polynomials

$$\mu_i(a,x,y) = \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0), \quad i = 1,\ldots,9,$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0) = \mu_0$.

These polynomials are in fact comitants of systems (2.1) with respect to the group $GL(2,\mathbb{R})$ (see [3]). The polynomial $\mu_i(a,x,y)$, $i \in \{0,1,\ldots,9\}$ is homogeneous of degree 6 in the coefficients of systems (2.1) and homogeneous of degree i in the variables x and y. The geometrical meaning of these polynomial is revealed in the next lemma.

Lemma 2.7 ([2,3])**.** *Assume that a cubic system* (S) *with coefficients* $\boldsymbol{a} \in \mathbb{R}^{20}$ *belongs to the family* (2.1)*. Then:*

(i) The total multiplicity of all finite singularities of this system equals $9-k$ *if and only if for every* $i \in \{0,1,\ldots,k-1\}$ *we have* $\mu_i(\boldsymbol{a},x,y) = 0$ *in the ring* $\mathbb{R}[x,y]$ *and* $\mu_k(\boldsymbol{a},x,y) \neq 0$*. In this case the factorization* $\mu_k(\boldsymbol{a},x,y) = \prod_{i=1}^{k}(u_i x - v_i y) \neq 0$ *over* $\mathbb{C}$ *indicates the coordinates* $[v_i : u_i : 0]$ *of singularities at infinity which in perturbations generate finite singularities of the system* (S)*. Moreover the number of distinct factors in this factorization is less than or equal to four (the maximum number of infinite singularities of a cubic system) and the multiplicity of each one of the factors* $u_i x - v_i y$ *gives us the number of the finite singularities of the system* (S) *which have coalesced with the infinite singular point* $[v_i : u_i : 0]$*.*

(ii) The point $M_0(0,0)$ *is a singular point of multiplicity* k $(1 \leq k \leq 9)$ *for the cubic system* (S) *if and only if for every* i *such that* $0 \leq i \leq k-1$ *we have* $\mu_{9-i}(\boldsymbol{a},x,y) = 0$ *in* $\mathbb{R}[x,y]$ *and* $\mu_{9-k}(\boldsymbol{a},x,y) \neq 0$*.*

(iii) The system (S) *is degenerate (i.e.* $\gcd(p,q) \neq \text{const}$*) if and only if* $\mu_i(\boldsymbol{a},x,y) = 0$ *in* $\mathbb{R}[x,y]$ *for every* $i = 0,1,\ldots,9$*.*

In order to define the invariant polynomials we need, we first construct the following comitants of second degree with respect to the coefficients of initial systems (2.1):

$$
\begin{array}{lll}
S_1 = (C_0, C_1)^{(1)}, & S_{10} = (C_1, C_3)^{(1)}, & S_{19} = (C_2, D_3)^{(1)}, \\
S_2 = (C_0, C_2)^{(1)}, & S_{11} = (C_1, C_3)^{(2)}, & S_{20} = (C_2, D_3)^{(2)}, \\
S_3 = (C_0, D_2)^{(1)}, & S_{12} = (C_1, D_3)^{(1)}, & S_{21} = (D_2, C_3)^{(1)}, \\
S_4 = (C_0, C_3)^{(1)}, & S_{13} = (C_1, D_3)^{(2)}, & S_{22} = (D_2, D_3)^{(1)}, \\
S_5 = (C_0, D_3)^{(1)}, & S_{14} = (C_2, C_2)^{(2)}, & S_{23} = (C_3, C_3)^{(2)}, \\
S_6 = (C_1, C_1)^{(2)}, & S_{15} = (C_2, D_2)^{(1)}, & S_{24} = (C_3, C_3)^{(4)}, \\
S_7 = (C_1, C_2)^{(1)}, & S_{16} = (C_2, C_3)^{(1)}, & S_{25} = (C_3, D_3)^{(1)}, \\
S_8 = (C_1, C_2)^{(2)}, & S_{17} = (C_2, C_3)^{(2)}, & S_{26} = (C_3, D_3)^{(2)}, \\
S_9 = (C_1, D_2)^{(1)}, & S_{18} = (C_2, C_3)^{(3)}, & S_{27} = (D_3, D_3)^{(2)}.
\end{array}
$$

We shall use here the following invariant polynomials constructed in [25] and [10]:

$$
\begin{aligned}
\mathcal{D}_1(a) &= 6S_{24}^3 - \left[(C_3, S_{23})^{(4)}\right]^2, \\
\mathcal{D}_2(a,x,y) &= -S_{23}, \\
\mathcal{D}_3(a,x,y) &= (S_{23},\, S_{23})^{(2)} - 6C_3(C_3,\, S_{23})^{(4)}, \\
\mathcal{D}_4(a) &= (C_3,\, \mathcal{D}_2)^{(4)}, \\
\mathcal{D}_5(a) &= A_1 - A_2, \\
\mathcal{D}_6(a) &= 3A_1 + A_2, \\
\mathcal{D}_7(a) &= -A_1 - 3A_2, \\
\mathcal{D}_8(a) &= 2A_1^3 - 9A_6^2 + 2A_1A_{10} + A_{16}, \\
\mathcal{V}_1(a,x,y) &= S_{23} + 2D_3^2, \\
\mathcal{V}_2(a,x,y) &= S_{26}, \\
\mathcal{V}_3(a,x,y) &= 6S_{25} - 3S_{23} - 2D_3^2, \\
\mathcal{V}_4(a,x,y) &= C_3\left[(C_3, S_{23})^{(4)} + 36\,(D_3, S_{26})^{(2)}\right], \\
\mathcal{V}_5(a,x,y) &= 6T_1(9A_5-7A_6) + 2T_2(4T_{16} - T_{17}) - 3T_3(3A_1+5A_2)+3A_2T_4+36T_5^2-3T_{44}, \\
\mathcal{U}_1(a) &= S_{24} - 4S_{27}, \\
\mathcal{U}_2(a,x,y) &= 6\,(S_{23} - 3S_{25}, S_{26})^{(1)} - 3S_{23}(S_{24} - 8S_{27}) - 24S_{26}^2 \\
&\quad + 2C_3\,(C_3, S_{23})^{(4)} + 24D_3\,(D_3, S_{26})^{(1)} + 24D_3^2S_{27},
\end{aligned}
$$

In order to characterize the cubic systems belonging to the class $\mathbf{CSL}_{(3,1,1,1)}^{2r2c\infty}$ we define here the following new invariant polynomials:

$$
\begin{aligned}
\chi_1(a,x,y) &= T_{13} - 2T_{11}, \\
\chi_2(a,x,y) &= 8A_3T_2 + 22A_4T_2 + 15T_{57} + 9T_{60} - 21T_{62} + 6T_{63} + 9T_{65}, \\
\chi_3(a,x,y) &= 2T_1T_8T_{15} + 2T_5T_{74} + T_5T_{75}, \\
\chi_4(a) &= A_7 + A_8 - A_9, \\
\chi_5(a) &= A_7, \\
\chi_6(a,x,y) &= 30(6A_3T_1^2 + 9T_5T_6 - 3T_4T_9 - T_2T_{26}) - T_1(29T_2T_{14} + 32T_2T_{15} - 108T_{36} - 45T_{42}), \\
\chi_7(a,x,y) &= T_{12} - T_{13},
\end{aligned}
$$

$$\chi_8(a,x,y) = 10A_3T_2 + 30A_4T_2 - 6T_{59} + 15T_{60} + 15T_{57} - 31T_{62} + 17T_{63} + 5T_{64} + 5T_{65},$$
$$\chi_9(a,x,y) = 6T_5(3T_{11} - 4T_{13}) + 10T_3T_{18} + 6T_4T_{18} - 3T_2T_{48} + 2T_2(T_{49} + T_{50}) + 22T_1T_{71} + T_{86},$$
$$\begin{aligned}\chi_{10}(a,x,y) = {} & 880A_3T_1(101T_2T_6 - 36T_1T_9) + 337920(T_{11} - T_{13})(T_{74} + T_{75}) - 880(5T_2^2 + 27T_3)T_9^2 \\ & - 528T_9(120A_4T_1^2 + 11658T_5T_6 + 50T_2T_{26} - 60T_1T_{37} + 25T_{76} - 80T_{78}) \\ & - 44T_{19}(21442T_2T_{15} - 259854T_{36} + 42588T_{37} + 59307T_{42} - 42888T_{38}) \\ & - 2640T_{26}(128T_{25} - 3T_{23} + 10T_{24} + 24T_{26}) + 24T_4T_6(344426T_{14} - 921997T_{15}) \\ & - 3T_6(345752T_3T_{15} - 1006720T_{80} - 1019038T_{81} + 969523T_{82} + 2177623T_{83} - 11264T_{84}),\end{aligned}$$
$$\begin{aligned}\chi_{11}(a,x,y) = {} & 360A_7T_2 + 3066T_{110} - 270T_{111} + 148T_{113} - 1895T_{114} + 2675T_{115} - 1176T_{116} + 3090T_{117} \\ & - 540T_{118} - 680T_{119} + 155T_{120} + 1375T_{121},\end{aligned}$$
$$\begin{aligned}\chi_{12}(a,x,y) = {} & 18T_2^2T_9 - T_2(36T_{23} + 324T_{24} - 737T_{26}) - T_1(108T_2T_{15} - 6(460T_{36} - 629T_{37} - 656T_{42})) \\ & - 3(29028T_5T_6 + 54T_3T_9 - 629T_4T_9 + 96T_{78}),\end{aligned}$$
$$\begin{aligned}\chi_{13}(a,x,y) = {} & -60(2A_{14} + 47A_{15})T_2 - 12180A_4T_{17} + 30A_3(47T_{16} + 51T_{17}) - 105A_1(T_{57} + 12T_{63}) \\ & - A_2(1200T_{60} - 174T_{59} - 255T_{57} - 5754T_{62} + 2403T_{63} - 4435T_{64} + 7820T_{65}),\end{aligned}$$
$$\chi_{14}(a,x,y) = 3T_1T_8T_{15} - 3T_5T_{75},$$
$$\chi_{15}(a,x,y) = T_8,$$
$$\chi_{16}(a,x,y) = 96T_6T_8 + 12T_{133} + 9T_{135} + 28T_2T_{74} + 21T_2T_{75},$$
$$\chi_{17}(a,x,y) = 9T_6T_9(174T_1T_9 + 193T_{19}) + T_6^2(77T_2T_9 + 1164T_1T_{14} - 69T_{23} - 57T_{24}) - 696T_{74}^2,$$
$$\widetilde{\chi}_1(a,x,y) = T_{13} - 2T_{12},$$
$$\widetilde{\chi}_2(a,x,y) = 3T_2T_6 + 2T_1T_9 + T_{19},$$
$$\zeta_1(a,x,y) = (A_1 - A_2)(5A_3T_2 + 25A_4T_2 - 9T_{59} + 15T_{57} - 39T_{62} + 33T_{63}),$$
$$\begin{aligned}\zeta_1'(a,x,y) = {} & 972T_1(A_8T_2 + 6T_{107}) - 5832T_5(5T_{36} + T_{38}) + 27(14904T_{11}^2 + 216T_{10}T_{15} - 16344T_8T_{18} \\ & - 7T_2^2T_{59} - 81T_3T_{59} + 18T_4T_{59}),\end{aligned}$$
$$\zeta_2(a) = 432A_2A_4 - 162A_{12} - 81A_{13} - 27A_{14} - 648A_{15},$$
$$\zeta_3(a) = A_7(2A_1A_9 - 3A_4A_6),$$
$$\zeta_4(a,x,y) = T_{59},$$
$$\zeta_5(a) = 36A_1^2A_4^2 - (A_{12} - 4A_{13} - A_{14} - 2A_{15})^2,$$
$$\zeta_6(a) = 8A_1^2A_2 + 58A_2^3 - 29A_6^2 + 82A_2A_{10} + 245A_{16},$$
$$\zeta_7(a) = -(5A_1 + 3A_2),$$
$$\zeta_8(a) = A_4(2A_3 + 3A_4),$$
$$\zeta_9(a) = -T_9T_{17},$$

where

$$A_1 = S_{24}/288, \quad A_2 = S_{27}/72, \quad A_3 = \big(72D_1A_2 + (S_{22},D_2)^{(1)}\big)/24,$$
$$\begin{aligned}A_4 = {} & \big[9D_1S_{24} - 2592D_1A_2 + 36(S_{11},D_3)^{(2)} + 24(S_{18},D_2)^{(1)} - 8(S_{14},D_3)^{(2)} - 8(S_{20},D_2)^{(1)} \\ & - 32(S_{22},D_2)^{(1)}\big]/2^7/3^3, \quad A_6 = \big(S_{26},D_3\big)^{(2)}/2^5/3^3, \quad A_7 = (T_9,C_3)^{(4)}/2^5/3^2,\end{aligned}$$
$$A_8 = (T_{14},D_3)^{(2)}/12, \quad A_9 = (T_{15},D_3)^{(2)}/12, \quad A_{10} = [\![S_{23},D_3)^{(2)},D_3)^{(2)}/2^9/3^4,$$
$$A_{12} = [\![T_9,C_3)^{(3)},D_3)^{(2)}/2^6/3^3, \quad A_{13} = [\![T_9,C_3)^{(2)},C_3)^{(4)}/2^7/3^3,$$
$$A_{14} = [\![T_9,D_3)^{(2)},D_3)^{(2)}/2^5/3^2, \quad A_{15} = [\![T_{14},C_3)^{(2)},D_3)^{(2)}/2^5/3^2,$$
$$A_{16} = [\![S_{23},C_3)^{(1)},D_3)^{(2)},D_3)^{(2)},D_3)^{(2)}/5/2^{13}/3^7$$

are affine invariants, whereas the polynomials

$$T_1 = C_3, \quad T_2 = D_3, \quad T_3 = S_{23}/18, \quad T_4 = S_{25}/6, \quad T_5 = S_{26}/72,$$
$$T_6 = \big[3C_1(D_3^2 - 9T_3 + 18T_4) - 2C_2(2D_2D_3 - S_{17} + 2S_{19} - 6S_{21}) + {}$$
$$+ 2C_3(2D_2^2 - S_{14} + 8S_{15})\big]/2^4/3^2, \quad T_7 = (S_{23}, C_3)^{(1)}/72,$$
$$T_8 = \big[5D_2(D_3^2 + 27T_3 - 18T_4) + 20D_3S_{19} + 12\big(S_{16}, D_3\big)^{(1)} - 8D_3S_{17}\big]/5/2^5/3^3,$$
$$T_9 = \big[9D_1(9T_3 - 18T_4 - D_3^2) + 2D_2(D_2D_3 - 3S_{17} - S_{19} - 9S_{21}) + 18\big(S_{15}, C_3\big)^{(1)} -$$
$$- 6C_2(2S_{20} - 3S_{22}) + 18C_1S_{26} + 2D_3S_{14}\big]/2^4/3^3, \quad T_{10} = (S_{23}, D_3)^{(1)}/2^5/3^3,$$
$$T_{11} = \big[\big(D_3^2 - 9T_3 + 18T_4, C_2\big)^{(2)} - 6\big(D_3^2 - 9T_3 + 18T_4, D_2\big)^{(1)} - 12\big(S_{26}, C_2\big)^{(1)} +$$
$$+ 12D_2S_{26} + 432C_2(A_1 - 5A_2)\big]/2^7/3^4,$$
$$T_{12} = \big[\big(D_3^2 + 15T_3 - 6T_4, C_2\big)^{(2)} - 6\big(D_3^2 - 3T_3 + 12T_4, D_2\big)^{(1)} - 4\big(S_{26}, C_2\big)^{(1)} +$$
$$+ 10D_2S_{26} - 720(A_1 + 3A_2)C_2\big]/2^7/3^3,$$
$$T_{13} = \big[\big(D_3^2 + 27T_3 - 18T_4, C_2\big)^{(2)} - 216\big(T_4, D_2\big)^{(1)} + 48D_3S_{22} + 36D_2S_{26} - 432C_2(3A_1 + 17A_2)\big]/2^7/3^4,$$
$$T_{14} = \big[\big(8S_{19} + 9S_{21}, D_2\big)^{(1)} - D_2(8S_{20} + 3S_{22}) + 18D_1S_{26} + 1296C_1A_2\big]/2^4/3^3,$$
$$T_{15} = \big[72(S_{19}, D_2)^{(1)} - (S_{17}, C_2)^{(2)} + 16(S_{21}, D_2)^{(1)} - 3(D_3^2, C_1)^{(2)} + 27(T_3, C_1)^{(2)} - 54(T_4, C_1)^{(2)}$$
$$+ 36D_1S_{26} + 2160C_1A_1 + 4752C_1A_2 - 16D_2S_{22} + 4(S_{14}, D_3)^{(1)} - 68(S_{15}, D_3)^{(1)} - 8(S_{14}, C_3)^{(2)}$$
$$- 8(S_{15}, C_3)^{(2)} - 4D_2S_{18}\big]/2^6/3^3, \quad T_{16} = (S_{23}, D_2)^{(2)}/2^6/3^3, \quad T_{17} = (S_{26}, D_3)^{(1)}/2^5/3^3,$$
$$T_{18} = \big[4\big(D_3^2 + 6T_4, C_2\big)^{(3)} + 2(C_2D_3, C_3)^{(4)} - 9D_2(96A_2 + S_{24}),$$
$$T_{19} = \big(T_6, C_3\big)^{(1)}/2, \quad T_{23} = \big(T_6, C_3\big)^{(2)}/6, \quad T_{24} = \big(T_6, D_3\big)^{(1)}/6,$$
$$T_{25} = \big[16\big[(C_2, D_3)^{(1)}\big]^2 + 5184C_1(3A_2C_3 - T_5D_3) + D_2^2(D_3^2 - 81T_3 - 54T_4) + 4D_2(648T_5C_2$$
$$+ 3D_3S_{17} + 2D_3S_{19} - 18C_3S_{20})\big]/2^6/3^4, \quad T_{26} = \big(T_9, C_3\big)^{(1)}/4, \quad T_{36} = \big(T_6, D_3\big)^{(2)}/12,$$
$$T_{37} = \big(T_9, C_3\big)^{(2)}/12, \quad T_{38} = \big(T_9, D_3\big)^{(1)}/6, \quad T_{42} = \big(T_{14}, C_3\big)^{(1)}/2,$$
$$T_{44} = \big((S_{23}, C_3)^{(1)}, D_3\big)^{(2)}/5/2^6/3^3, \quad T_{50} = \big(T_{12}, D_3\big)^{(1)}/6,$$
$$T_{57} = \big(T_9, D_3\big)^{(2)}/12, \quad T_{59} = \big(T_6, C_3\big)^{(4)}/2^4/3^2, \quad T_{60} = \big(T_9, C_3\big)^{(3)}/72,$$
$$T_{62} = \big(T_{14}, C_3\big)^{(2)}/6, \quad T_{63} = \big(T_{15}, C_3\big)^{(2)}/6, \quad T_{64} = \big(T_{15}, D_3\big)^{(1)}/6, \quad T_{65} = \big(T_{14}, D_3\big)^{(1)}/6,$$
$$T_{74} = \big[18(27T_3C_2D_1 - 54T_4C_2D_1 - 64T_6D_2 - 3T_3C_1D_2 + 6T_4C_1D_2)D_3 - 6(9C_2D_1 - C_1D_2)D_3^3$$
$$+ 27C_0D_3^4 + D_3^2(-486T_3C_0 + 972T_4C_0 + 108C_3D_1D_2 - 8C_2D_2^2 - 54C_3S_8 + 108C_3S_9$$
$$+ 27C_2S_{11} - 27C_2S_{12} + 4C_2S_{14} - 32C_2S_{15} + 54D_1S_{16} - 3C_1S_{17} + 6C_1S_{19} - 18C_1S_{21})$$
$$- 972(T_3 - 2T_4)C_3D_1D_2 + 72(T_3 - 2T_4)C_2D_2^2 - 486C_3(128T_{11}C_1 - T_3S_8 + 2T_4S_8$$
$$+ 2T_3S_9 - 4T_4S_9) + 20736T_{11}C_2^2 - 9C_2(T_3 - 2T_4)(27S_{11} - 27S_{12} + 4S_{14} - 32S_{15})$$
$$+ 2187(T_3 - 2T_4)^2C_0 + 576T_6(S_{17} - 2S_{19} + 6S_{21}) - 27T_3(18D_1S_{16} - C_1S_{17} + 2C_1S_{19}$$
$$- 6C_1S_{21}) + 54T_4(18D_1S_{16} - C_1S_{17} + 2C_1S_{19} - 6C_1S_{21})\big]/2^8/3^4,$$
$$T_{75} = \big[-18(40C_3D_2 + 137S_{16})\big(D_3^2, C_1\big)^{(1)} - 48(4C_3D_2 - 3C_2D_3 - S_{16})\big(S_{14}, C_3\big)^{(1)}$$
$$- 768C_3D_2\big(S_{15}, C_3\big)^{(1)} + 9(16C_3D_2 - 9C_2D_3 + 5S_{16})\big(S_{23}, C_1\big)^{(1)} - 648C_0D_3^4$$
$$+ 162(C_2D_3 + 3S_{16})\big(S_{25}, C_1\big)^{(1)} + 144(9C_2D_1 + 2C_1D_2)D_3^3 - 12D_3^2(32C_2D_2^2 - 18C_3S_8$$
$$+ 9C_2S_{11} - 54C_2S_{12} + 24C_2S_{14} - 96C_2S_{15} - 324D_1S_{16} - 6C_1S_{17} + 12C_1S_{19} - 18C_0S_{23}$$

$$+216C_0S_{25})+8D_3(64C_3D_2^3+64C_3D_2S_{14}+16D_2^2S_{16}+12S_{14}S_{16}-96S_{15}S_{16}-36C_2^2S_{18}$$
$$-96C_2D_2S_{19}+108C_2^2S_{20}+240C_2D_2S_{21}-297C_2D_1S_{23}-24C_1D_2S_{23}+1134C_2D_1S_{25})$$
$$+62208C_3(3T_{13}C_1-16T_8D_1)+2(1728C_3D_1D_2+32C_2D_2^2+18C_3S_8+4176C_3S_9-9C_2S_{11}$$
$$-1395C_2S_{12}-16C_2S_{14}+96C_2S_{15}-108D_1S_{16}-18C_1S_{17}-60C_1S_{19}+2160C_1S_{21})S_{23}$$
$$+54C_0S_{23}^2+32(5832T_{13}C_1C_3-31104T_8C_3D_1-34992T_8S_{10}-3C_3S_{14}S_{17}-4D_2S_{16}S_{17}$$
$$+3C_2S_{17}^2+12C_2C_3D_2S_{18}-3C_2S_{16}S_{18}+16C_3D_2^2S_{19}-2C_3S_{14}S_{19}+16C_3S_{15}S_{19}$$
$$+24D_2S_{16}S_{19}-12C_2S_{19}^2-36C_2C_3D_2S_{20}+9C_2S_{16}S_{20}-48C_3D_2^2S_{21}-12C_3S_{14}S_{21}$$
$$-24D_2S_{16}S_{21}+12C_2S_{17}S_{21})-36(288C_3D_1D_2+474C_3S_8+528C_3S_9-237C_2S_{11}$$
$$-255C_2S_{12}-180D_1S_{16}-86C_1S_{17}+156C_1S_{19}+276C_1S_{21})S_{25}-1944C_0S_{25}^2]/2^{11}/3^4,$$

$$T_{76}=[\![T_6,C_3)^{(2)},C_3)^{(1)}/36,\ T_{78}=\big(T_{25},C_3\big)^{(1)}/2,\ T_{80}=[\![T_9,C_3)^{(2)},C_3)^{(1)}/144,$$
$$T_{81}=[\![T_6,C_3)^{(3)},C_3)^{(1)}/2^6/3^2,\ T_{82}=[\![T_6,C_3)^{(2)},D_3)^{(1)}/2^3/3^3,\ T_{83}=[\![T_6,C_3)^{(1)},D_3)^{(2)}/24,$$
$$T_{84}=\big(T_{25},C_3\big)^{(2)}/6,\ T_{86}=[\![T_{11},C_3)^{(2)},C_3)^{(1)}/36,\ T_{107}=[\![T_9,D_3)^{(2)},D_3)^{(1)}/432,$$
$$T_{110}=[\![T_6,C_3)^{(4)},C_3)^{(2)}/2^7/3^3,\ T_{111}=[\![T_6,C_3)^{(3)},D_3)^{(2)}/2^7/3^3,\ T_{113}=[\![T_{14},C_3)^{(2)},C_3)^{(2)}/72,$$
$$T_{114}=[\![T_{14},C_3)^{(2)},D_3)^{(1)}/72,\ T_{115}=[\![T_{14},C_3)^{(1)},D_3)^{(2)}/72,\ T_{116}=[\![T_{15},C_3)^{(2)},C_3)^{(2)}/72,$$
$$T_{117}=[\![T_6,D_3)^{(2)},D_3)^{(2)}/2^5/3^3,\ T_{118}=[\![T_9,C_3)^{(3)},C_3)^{(2)}/2^5/3^3,\ T_{119}=\big(T_{25},C_3\big)^{(4)}/2^4/3^2,$$
$$T_{120}=[\![T_9,C_3)^{(3)},D_3)^{(1)}/2^5/3^3,\ T_{121}=[\![T_9,C_3)^{(2)},D_3)^{(2)}/2^4/3^3,\ T_{133}=\big(T_{74},C_3\big)^{(1)},$$
$$T_{135}=\big(T_{75},C_3\big)^{(1)},$$

are T-comitants of cubic systems (2.1) (see for details [36]). In the above list the bracket "$[\![$" means a succession of two or up to four parentheses "(" depending on the row in which it appears.

We note that these invariant polynomials are the elements of the polynomial basis of T-comitants up to degree six constructed by Iu. Calin [15].

2.2 Preliminary results

In order to determine the degree of the common factor of the polynomials $\tilde{\mathcal{G}}_i(a,x,y)$ for $i=1,2,3$, we shall use the notion of the k^{th} *subresultant* of two polynomials with respect to a given indeterminate (see for instance, [22,28]).

Following [25] we consider two polynomials

$$f(z)=a_0z^n+a_1z^{n-1}+\cdots+a_n,\qquad g(z)=b_0z^m+b_1z^{m-1}+\cdots+b_m,$$

in the variable z of degree n and m, respectively.

We say that the k–th *subresultant* (see for example, [28]) with respect to variable z of the two polynomials $f(z)$ and $g(z)$ is the $(m+n-2k)\times(m+n-2k)$ determinant

$$R_z^{(k)}(f,g)=\begin{vmatrix} a_0 & a_1 & a_2 & \dots & \dots & a_{m+n-2k-1}\\ 0 & a_0 & a_1 & \dots & \dots & a_{m+n-2k-2}\\ 0 & 0 & a_0 & \dots & \dots & a_{m+n-2k-3}\\ \dots & \dots & \dots & \dots & \dots & \dots\dots\dots\\ 0 & 0 & b_0 & \dots & \dots & b_{m+n-2k-3}\\ 0 & b_0 & b_1 & \dots & \dots & b_{m+n-2k-2}\\ b_0 & b_1 & b_2 & \dots & \dots & b_{m+n-2k-1}\end{vmatrix}\begin{matrix}\left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}(m-k)\text{ times}\\ \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}(n-k)\text{ times}\end{matrix}\tag{2.3}$$

in which there are $m-k$ rows of a's and $n-k$ rows of b's, and $a_i = 0$ for $i > n$, and $b_j = 0$ for $j > m$.

For $k = 0$ we obtain the standard resultant of two polynomials. In other words we can say that the k–th subresultant with respect to the variable z of the two polynomials $f(z)$ and $g(z)$ can be obtained by deleting the first and the last k rows and the first and the last k columns from its resultant written in the form (2.3) when $k = 0$.

The geometrical meaning of the subresultants is based on the following lemma.

Lemma 2.8 (see [22,28]). *Polynomials $f(z)$ and $g(z)$ have precisely k roots in common (considering their multiplicities) if and only if the following conditions hold:*

$$R_z^{(0)}(f,g) = R_z^{(1)}(f,g) = R_z^{(2)}(f,g) = \cdots = R_z^{(k-1)}(f,g) = 0 \neq R_z^{(k)}(f,g).$$

For the polynomials in more than one variables it is easy to deduce from Lemma 2.8 the following result.

Lemma 2.9. *Two polynomials $\tilde{f}(x_1, x_2, \ldots, x_n)$ and $\tilde{g}(x_1, x_2, \ldots, x_n)$ have a common factor of degree k with respect to the variable x_j if and only if the following conditions are satisfied:*

$$R_{x_j}^{(0)}(\tilde{f},\tilde{g}) = R_{x_j}^{(1)}(\tilde{f},\tilde{g}) = R_{x_j}^{(2)}(\tilde{f},\tilde{g}) = \cdots = R_{x_j}^{(k-1)}(\tilde{f},\tilde{g}) = 0 \neq R_{x_j}^{(k)}(\tilde{f},\tilde{g}),$$

where $R_{x_j}^{(i)}(\tilde{f},\tilde{g}) = 0$ in $\mathbb{R}[x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_n]$.

In paper [25] all the possible configurations of invariant lines are determined in the case, when the total multiplicity of these lines (including the line at infinity) equals nine. All possible configurations of invariant lines in the case when the total multiplicity of these lines (including the line at infinity) equals eight, are determined in [5,9–12].

In the above mentioned articles, several lemmas are proved concerning the number of triplets and/or couples of parallel invariant straight lines which could have a cubic system. Taking together these lemmas produce the following theorem.

Theorem 2.10. *If a cubic system (2.1) possesses a given number of triplets or/and couples of invariant parallel lines real or/and complex, then the following conditions are satisfied, respectively:*

(i)	*two triplets*	$\Rightarrow$	$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$;
(ii)	*one triplet and one couple*	$\Rightarrow$	$\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$;
(iii)	*one triplet*	$\Rightarrow$	$\mathcal{V}_4 = \mathcal{U}_2 = 0$;
(iv)	*3 couples*	$\Rightarrow$	$\mathcal{V}_3 = 0$;
(v)	*2 couples*	$\Rightarrow$	$\mathcal{V}_5 = 0$.

Remark 2.11. The above conditions depend only on the coefficients of the cubic homogeneous parts of the systems (2.1).

We rewrite the systems (2.1) using a different notation for the coefficients::

$$\begin{aligned}\dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + px^3 + 3qx^2y + 3rxy^2 + sy^3 \equiv p(x,y),\\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3 \equiv q(x,y).\end{aligned} \tag{2.4}$$

Let $L(x,y) = Ux + Vy + W = 0$ be an invariant straight line of this family of cubic systems. Then, we have

$$Up(x,y) + Vq(x,y) = (Ux + Vy + W)(Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F),$$

and this identity provides the following 10 relations:

$$\begin{array}{ll} Eq_1 = (p-A)U + tV = 0, & Eq_6 = (2h-E)U+(2m-D)V-2BW=0, \\ Eq_2 = (3q-2B)U + (3u-A)V = 0, & Eq_7 = kU + (n-E)V - CW = 0, \\ Eq_3 = (3r-C)U + (3v-2B)V = 0, & Eq_8 = (c-F)U + eV - DW = 0 \\ Eq_4 = (s-C)U + Vw = 0, & Eq_9 = dU + (f-F)V - EW = 0, \\ Eq_5 = (g-D)U + lV - AW = 0, & Eq_{10} = aU + bV - FW = 0. \end{array} \tag{2.5}$$

It is well known that in the case of the non-singular infinite invariant line the infinite singularities (real or complex) of systems (2.4) are determined by the linear factors of the polynomial

$$C_3 = yp_3(x,y) - xq_3(x,y).$$

Remark 2.12. Let $C_3 = \prod_{i=1}^{4}(\alpha_i x + \beta_i y)$, $i = 1,2,3,4$. Then $[\beta_i : \alpha_i : 0]$ are the singular points at infinity. Hence the invariant affine lines must be of the form $Ux + Vy + W = 0$ with (U,V) among (α_i, β_i). In this case, for any fixed (α_i, β_i), for a specific value of W, six equations among (2.5) become linear with respect to the parameters $\{A, B, C, D, E, F\}$ (with the corresponding non-zero determinant) and we can determine their values, which annihilate some of the equations (2.5). So in what follows, for each direction given by (α_i, β_i), we will examine only the non-zero equations containing the last parameter W.

For the proof of the Main Theorem it is useful to consider the following homogeneous cubic systems associated to systems (2.4):

$$x' = p_3(x,y), \quad y' = q_3(x,y). \tag{2.6}$$

Clearly in the case of two real and two complex distinct infinite singularities the polynomial $C_3(x,y)$ has four distinct linear factors over $\mathbb{C}$: two of them being real and two complex. The following remark concerning the associated homogeneous cubic systems (2.6) is useful.

Remark 2.13. Assume that a cubic system (2.4) in $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$ possesses invariant lines of total multiplicity three in a real direction. Then the corresponding associated homogeneous cubic systems (2.6) has one invariant line of multiplicity three in the same direction.

Indeed, if a system (2.4) possesses a triplet of parallel invariant lines (distinct or coinciding) in a real direction then via an affine transformation this system could be brought to the form

$$\dot{x} = x[(x+b)^2 + u], \quad \dot{y} = q(a,x,y).$$

It is clear that if $u < 0$ (respectively $u > 0$) then we have three real (respectively one real and two complex) all distinct invariant lines. In the case $u = 0$ we either have one simple and one double invariant lines if $b \neq 0$, or one triple invariant line if $b = 0$. It remains to observe that in all four cases the corresponding associated homogeneous cubic systems possess the invariant line $x = 0$ of multiplicity at least three.

According to [9, 25] (see also [30]) we have the following result.

Lemma 2.14. *A cubic system (2.4) has 2 real and two complex all distinct infinite singularities if and only if the condition $\mathcal{D}_1 < 0$ holds. Moreover its associated homogeneous cubic systems (2.6) could be brought via a linear transformation to the canonical form*

$$(S_{II}) \quad \begin{cases} x' = (1+u)x^3 + (s+v)x^2y + rxy^2, \\ y' = -sx^3 + ux^2y + vxy^2 + (r-1)y^3. \end{cases} \quad C_3 = x(sx+y)(x^2+y^2), \tag{2.7}$$

3 The proof of the Main Theorem

Considering Lemma 2.14 we deduce that for the systems in the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$ the condition $\mathcal{D}_1 < 0$ holds and these systems could be brought via a linear transformation to the family of systems

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + (1+u)x^3 + (s+v)x^2y + rxy^2, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + ny^2 - sx^3 + ux^2y + vxy^2 + (r-1)y^3 \end{aligned} \tag{3.1}$$

with $C_3 = x(sx+y)(x^2+y^2)$. In what follows we examine cubic systems possessing configurations of invariant lines of the type $\mathfrak{T} = (3,1,1,1)$.

3.1 The proof of the statement (A)

The configurations of the type $\mathcal{T} = (3,1,1,1)$ could only have one triplet of parallel invariant lines and clearly in the case of two real and two complex infinite singularities such triplet could be only in a real direction.

3.1.1 Construction of the associated homogeneous systems

Since systems with the configuration of the type $\mathcal{T} = (3,1,1,1)$ could only possess one triplet of parallel invariant lines, according to Theorem 2.10 the conditions $\mathcal{V}_4 = \mathcal{U}_2 = 0$ are necessary for systems (3.1). Taking the corresponding associated homogeneous systems (2.7) we force the conditions $\mathcal{V}_4 = \mathcal{U}_2 = 0$.

We observe that the invariant polynomial $\mathcal{U}_2$ is a homogeneous polynomial of degree four in x and y. So we shall use the following notation:

$$\mathcal{U}_2 = \sum_{j=0}^{4} \mathcal{U}_{2j} x^{4-j} y^j.$$

On the other hand a straightforward computation of the value of $\mathcal{V}_4$ for systems (2.7) yields

$$\begin{aligned} \mathcal{V}_4 &= 9216\,\widehat{\mathcal{V}}_4\, C_3(x,y), \quad \text{where} \\ \widehat{\mathcal{V}}_4 &= 6r^2s + r(2su - 9s - 3v) + (s+v)(sv - 3u). \end{aligned}$$

As for systems (2.7) we have $C_3 = x(sx+y)(x^2+y^2) \neq 0$, we conclude that the condition $\mathcal{V}_4 = 0$ for these systems is equivalent to $\widehat{\mathcal{V}}_4 = 0$.

For systems (2.7) we evaluate

$$\mathcal{U}_2 = 3 \cdot 2^{12} \sum_{j=0}^{4} \widehat{\mathcal{U}}_{2j} x^{4-j} y^j,$$

where $\widehat{\mathcal{U}}_{2j}$ are polynomials in the parameters r, s, u and v. We have

$$\widehat{\mathcal{U}}_{24} = r(9u - 12ru + 4r^2u - 3sv + 2rsv - rv^2) = 0$$

and we consider two cases: $r \neq 0$ and $r = 0$.

1: *The case* $r \neq 0$. Then we must have

$$9u - 12ru + 4r^2u - 3sv + 2rsv - rv^2 = (3-2r)^2u + v(-3s + 2rs - rv) = 0$$

and we examine two subcases: $3-2r\neq 0$ and $3-2r=0$.

1.1: *The subcase* $3-2r\neq 0$. Then the condition $\widehat{\mathcal{U}}_{24}=0$ gives $u=\frac{3sv-2rsv+rv^2}{(3-2r)^2}$ and we obtain:

$$\widehat{\mathcal{U}}_{23}=\frac{3r(3s-2rs+v)\big[(3-2r)^2+v^2)}{2r-3}=(3-2r)\widehat{\mathcal{V}}_4=0.$$

Since $r(3-2r)\neq 0$ the above condition gives $v=(2r-3)s$ and this implies $\mathcal{U}_2=\widehat{\mathcal{V}}_4=0$. Therefore we get the family of systems

$$\begin{aligned}\dot{x}&=(1-s^2+rs^2)x^3+2(-1+r)sx^2y+rxy^2,\\ \dot{y}&=-sx^3+(-1+r)s^2x^2y+(-3+2r)sxy^2+(-1+r)y^3.\end{aligned}\tag{3.2}$$

1.2: *The subcase* $3-2r=0$. We get $r=3/2$ and therefore the condition $\widehat{\mathcal{U}}_{24}=0$ gives $v=0$. Then we obtain $\widehat{\mathcal{V}}_4=0$ and

$$\widehat{\mathcal{U}}_{20}=-3(s^2-2u)\big[4s^2+(3+2u)^2\big]/4=0$$

and we discuss two possibilities: $s^2-2u=0$ or $s=3+2u=0$.

1.2.1: *The possibility* $s^2-2u=0$. We have $u=s^2/2$ and then $\mathcal{U}_2=\widehat{\mathcal{V}}_4=0$. In this case we arrive at the family of systems

$$\begin{aligned}\dot{x}&=(1+s^2/2)x^3+sx^2y+3xy^2/2,\\ \dot{y}&=-sx^3+1/2s^2x^2y+y^3/2.\end{aligned}\tag{3.3}$$

We observe that the above family of systems is a subfamily of (3.2) defined by the value $r=3/2$.

1.2.2: *The possibility* $s=3+2u=0$. In this case we get again $\mathcal{U}_2=\widehat{\mathcal{V}}_4=0$ and we obtain the system

$$\dot{x}=-x^3/2+3xy^2/2,\quad \dot{y}=-3x^2y/2+y^3/2.\tag{3.4}$$

However for this system we calculate (see the definition of the polynomial $H(X,Y,Z)$ on the page 14, Notation 2.4):

$$H(X,Y,Z)=\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)=3XY(X^2+Y^2)^3/4.$$

So system (3.4) possesses two triple invariant lines $x\pm iy=0$ and by Remark 2.13, systems (3.1) could have triplets of parallel invariant lines only in these two directions. However since these lines will be complex, we deduce that systems (3.1) with the associated homogeneous cubic system (3.4) could not possess invariant lines with the configuration of the type $\mathfrak{T}=(3,1,1,1)$.

2: *The case* $r=0$. Then we calculate

$$\widehat{\mathcal{U}}_{23}=3(s+v)(3u-sv),\quad \widehat{\mathcal{V}}_4=-(s+v)(3u-sv)$$

and we examine two subcases: $s+v=0$ or $s+v\neq 0$ and $3u-sv=0$.

2.1: *The subcase* $s+v=0$. Then $v=-s$ and this implies $\mathcal{U}_2=\widehat{\mathcal{V}}_4=0$. Therefore we get the family of systems (we set new parameters and variables: $s=s_1$, $u=u_1$, $x=x_1$, $y=y_1$)

$$\dot{x}_1=(1+u_1)x_1^3,\quad \dot{y}_1=-s_1x_1^3+u_1x_1^2y_1-s_1x_1y_1^2-y_1^3.\tag{3.5}$$

In this case we observe that the systems (3.2) via the transformation

$$x_1 = -(sx + y), \quad y_1 = -x + sy, \quad t_1 = -t/(s^2 + 1)$$

can be transformed to systems (3.5) after additional change of the parameters: $s = s_1$ and $s^2 - r(1 + s^2) = u_1$.

2.2: *The subcase* $3u - sv = 0$ *and* $s + v \neq 0$. Then $u = sv/3$ and we calculate

$$\widehat{\mathcal{U}}_{22} = -(s + v)(3s + v)(9 + v^2).$$

Since $s + v \neq 0$ we get $v = -3s$ and this implies $\mathcal{U}_2 = \widehat{\mathcal{V}}_4 = 0$. In this case we obtain the family of systems

$$\dot{x} = (1 - s^2)x^3 - 2sx^2y, \quad \dot{y} = -sx^3 - s^2x^2y - 3sxy^2 - y^3. \tag{3.6}$$

We observe that the above family of systems is a subfamily of (3.2) defined by the value $r = 0$.

So we have proved the next lemma.

Lemma 3.1. *If for a homogeneous cubic system (2.7) the conditions $\mathcal{V}_4 = \mathcal{U}_2 = 0$ hold then this system could be brought via a linear transformation and time rescaling to the form (3.5) with one exception: when the conditions $s = v = 0$ and $r = -u = 3/2$ (which imply $\mathcal{V}_4 = \mathcal{U}_2 = 0$) then we get the system (3.4) that has two triple complex invariant lines $x \pm iy = 0$.*

Thus according to this lemma forcing the conditions $\mathcal{V}_4 = \mathcal{U}_2 = 0$ to be satisfied for systems (3.1) we obtain two families of systems. The first one with the associated homogeneous cubic systems of the form (3.5) and due to an additional translation having the parameter $n = 0$ in the quadratic parts of systems (3.1):

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 + (1 + u)x^3, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy - sx^3 + ux^2y - sxy^2 - y^3, \end{aligned} \tag{3.7}$$

The second family has the associated homogeneous cubic systems of the form (3.4) and applying an additional translation we can assume that two parameters vanish: $m = 0$ and $n = 0$. As a result we arrive at the following family of systems:

$$\begin{aligned} \dot{x} &= a + cx + dy + gx^2 + 2hxy + ky^2 - x^3/2 + 3xy^2/2, \\ \dot{y} &= b + ex + fy + lx^2 - 3x^2y/2 + y^3/2. \end{aligned} \tag{3.8}$$

As it was mentioned above, by Remark 2.13 systems (3.8) could not possess invariant lines with the configuration of the type $\mathfrak{T} = (3, 1, 1, 1)$. And later (see Lemma 3.25) will be proved that none of the sets of conditions provided by the statement $A)$ of Main Theorem could be satisfied for systems (3.8).

Next we prove the following lemma which is the first step in the classification of the configuration of systems in the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$.

Lemma 3.2. *Assume that for a non-degenerate cubic system (2.4) the conditions $\mathcal{D}_1 < 0$ and $\mathcal{V}_4 = \mathcal{U}_2 = 0$ hold. Then the infinite invariant line $Z = 0$ of this system has the multiplicity indicated below if and only if the corresponding conditions are satisfied, respectively:*

(i) one $\Leftrightarrow \mathcal{D}_7 \neq 0$;

(*ii*) *two* $\Leftrightarrow \mathcal{D}_7 = 0$ *and* $\widetilde{\chi}_1 \neq 0$;

(*iii*) *three* $\Leftrightarrow \mathcal{D}_7 = \widetilde{\chi}_1 = 0$ *and* $\widetilde{\chi}_2 \neq 0$;

(*iv*) *four* $\Leftrightarrow \mathcal{D}_7 = \widetilde{\chi}_1 = \widetilde{\chi}_2 = 0$.

Moreover the maximum multiplicity which could have the line at infinity of a non-degenerate system with $\mathcal{D}_1 < 0$ *and* $\mathcal{V}_4 = \mathcal{U}_2 = 0$ *is four.*

Proof. First of all we mention that by Lemma 2.14 the condition $\mathcal{D}_1 < 0$ implies the existence of 2 real and 2 complex infinite singularities.

On the other hand as it was mentioned earlier, according to Lemma 3.1 the conditions $\mathcal{V}_4 = \mathcal{U}_2 = 0$ lead either to the family of systems (3.7) or to (3.8).

According to Lemma 2.6 if the invariant line $Z = 0$ is of multiplicity $k > 1$ then Z^{k-1} is a common factor of the invariant polynomials $\mathcal{G}_i(a, X, Y, Z)$, $i = 1, 2, 3$ defined in Notation 2.4 of the manuscript. So the existence of such a common factor of the above mentioned three polynomials is a necessary condition for the invariant line $Z = 0$ of systems (3.7) to be of the multiplicity k.

For systems (3.7) calculations yield:

$$\begin{aligned}
\mathcal{G}_1(X,Y,Z) &= (1+u)X^3(sX+Y)(X^2+Y^2)(uX^2-2sXY-3Y^2) + Z\big[\Psi_1(X,Y,Z)\big],\\
\mathcal{G}_2(X,Y,Z) &= (1+u)X^3(sX+Y)(X^2+Y^2)\big[(s^2+2u+2u^2)X^4-4suX^3Y\\
&\qquad + (s^2-3-6u)X^2Y^2+4sXY^3+3Y^4\big] + Z\big[\Psi_2(X,Y,Z)\big],\\
\mathcal{G}_3(X,Y,Z) &= 24(1+u)X^3(sX+Y)(uX^2-Y^2)(X^2+Y^2)\big[(1+s^2+2u+u^2)X^4\\
&\qquad -2suX^3Y+(s^2-1-2u)X^2Y^2+2sXY^3+Y^4\big] + Z\big[\Psi_3(X,Y,Z)\big],
\end{aligned} \tag{3.9}$$

where $\Psi_j(X, Y, Z)$ $(j = 1, 2, 3)$ are some polynomials in X, Y and Z.

Evidently Z will be a common factor of the polynomials $\mathcal{G}_i(X, Y, Z)$ (i.e. $\mathcal{G}_i(X, Y, 0) = 0$ for each $i = 1, 2, 3$) if and only if $u + 1 = 0$. Since the condition $\mathcal{D}_7 \neq 0$ implies $u + 1 \neq 0$ we deduce that Z could not be the needed common factor and hence the infinite invariant line $Z = 0$ for systems (3.7) is of multiplicity one.

On the other hand for systems (3.8) we calculate

$$\mathcal{G}_1(X,Y,Z) = \ 6XY(X^2+Y^2)^3 + Z\big[\Phi(X,Y,Z)\big], \quad \mathcal{D}_7 = 4 \neq 0,$$

where $\Phi(X, Y, Z)$ is a polynomials in X, Y and Z. So we can see that the polynomial $\mathcal{G}_1(X, Y, Z)$ could not have as a factor Z and hence all three polynomials $\mathcal{G}_i(X, Y, Z)$ $i = 1, 2, 3$ could not have the common factor Z. So we arrive at the following remark.

Remark 3.3. The family of systems (3.8) could not have the infinite invariant line $Z = 0$ of multiplicity greater than one.

Thus we conclude that in the case $\mathcal{D}_7 \neq 0$ a non-degenerate cubic system with $\mathcal{D}_1 < 0$ and $\mathcal{V}_4 = \mathcal{U}_2 = 0$ has the line at infinity of multiplicity one. This completes the proof of the statement (i) of the lemma.

(ii) Assume now that the condition $\mathcal{D}_7 = 0$ holds and taking into account Remark 3.3 we consider the family of systems (3.7). In this case the condition $\mathcal{D}_7 = 0$ gives us $u = -1$ and considering (3.9) we deduce that Z is a common factor of the polynomials $\mathcal{G}_i(X, Y, Z)$, $i = 1, 2, 3$. We claim that the invariant line $Z = 0$ of systems (3.7) has multiplicity at least two. For this it is sufficient to apply the following perturbation to systems (3.7) with $u = -1$ (remaining in the class of cubic systems):

$$\dot{x} = (a + cx + dy + gx^2 + 2hxy + ky^2)(1 + \varepsilon x), \quad \dot{y} = q(x, y), \quad |\varepsilon| \ll 1.$$

It is clear that the perturbed systems possess the invariant line $\varepsilon x + 1 = 0$ which coalesces with infinite one when ε tends to zero. So we deduce that the invariant line $Z = 0$ is of multiplicity at least 2 and in order to determine exactly its multiplicity we calculate:

$$\begin{aligned} \mathcal{G}_1(X,Y,Z)/Z =& -(sX+Y)(X^2+Y^2)\big[(g-2hs)X^4 + 2(g-k)sX^3Y \\ &+ (3g-k+2hs)X^2Y^2 + 4hXY^3 + kY^4\big] + Z\big[\Psi_1'(X,Y,Z)\big], \\ \mathcal{G}_2(X,Y,Z)/Z =& (sX+Y)^2(X^2+Y^2)^2\big[(g-k)sX^3 + (3g-k+2hs)X^2Y \\ &+ 6hXY^2 + 2kY^3\big] + Z\big[\Psi_2'(X,Y,Z)\big], \\ \mathcal{G}_3(X,Y,Z)/Z =& -24(sX+Y)^3(X^2+Y^2)^3(gX^2+2hXY+kY^2) + Z\big[\Psi_3'(X,Y,Z)\big], \end{aligned} \tag{3.10}$$

where $\Psi_j'(X,Y,Z)$ $(j = 1,2,3)$ are some polynomials in X, Y and Z.

We observe that each one of the polynomials $\mathcal{G}_i(X,Y,Z)/Z$, $i = 1,2,3$ has the factor Z if and only if $k = h = g = 0$. This condition is governed by the invariant polynomials $\widetilde{\chi}_1$ because for systems (3.7) with $u = -1$ we have

$$\text{Coefficient}[\widetilde{\chi}_1, xy^2] = -8ks/3, \quad \text{Coefficient}[\widetilde{\chi}_1, y^3] = 2k(s^2-3)/9$$

and clearly the condition $\widetilde{\chi}_1 = 0$ implies $k = 0$. Then we calculate

$$\widetilde{\chi}_1 = 2x^2\big[2(h+2gs-3hs^2)x + (3g-8hs-gs^2)y\big]/9 = 0,$$

and we determine that for $s = 0$ we get $h = g = 0$. If $s \neq 0$ we obtain:

$$h + 2gs - 3hs^2 = 0 \Rightarrow g = \frac{h(3s^2-1)}{2s} \Rightarrow 3g - 8hs - gs^2 = -\frac{3h(1+s^2)^2}{2s} = 0 \Rightarrow h = g = 0.$$

So in the case $\widetilde{\chi}_1 \neq 0$ we have $k^2 + h^2 + g^2 \neq 0$ and therefore the invariant line $Z = 0$ has the multiplicity exactly two.

(iii) Admit now that the conditions $\mathcal{D}_7 = 0$ and $\widetilde{\chi}_1 = 0$ are satisfied. This implies $u = -1$ and $k = h = g = 0$ and considering (3.10) we deduce that Z^2 is a common factor of the polynomials $\mathcal{G}_i(X,Y,Z)$, $i = 1,2,3$. We claim that the invariant line $Z = 0$ of systems (3.7) has the multiplicity at least three. For this it is sufficient to apply to (3.7) with $u = -1$ and $k = h = g = 0$ the following perturbation (remaining in the class of cubic systems):

$$\dot{x} = (a + cx + dy)(1 + \varepsilon_1 x + \varepsilon_2 x^2), \quad \dot{y} = q(x, y)$$

with $|\varepsilon_i| \ll 1$ $(i = 1,2)$. Clearly the perturbed systems possess the two invariant lines defined by the equation $1 + \varepsilon_1 x + \varepsilon_2 x^2 = 0$ which coalesces with infinite one when ε_1 and ε_2 tend to zero. So we deduce that the invariant line $Z = 0$ is of multiplicity at least 3 and in order to determine precisely its multiplicity we calculate:

$$\begin{aligned} \mathcal{G}_1(X,Y,Z)/Z^2 =& -(sX+Y)(X^2+Y^2)\big[(c-ds)X^3 + 2csX^2Y + (3c+ds)XY^2 \\ &+ 2dY^3\big] + Z\big[\Psi_1''(X,Y,Z)\big], \\ \mathcal{G}_2(X,Y,Z)/Z^2 =& (sX+Y)^2(sX+3Y)(cX+dY)(X^2+Y^2)^2 + Z\big[\Psi_2''(X,Y,Z)\big], \\ \mathcal{G}_3(X,Y,Z)/Z^2 =& -24(sX+Y)^3(cX+dY)(X^2+Y^2)^3 + Z\big[\Psi_3''(X,Y,Z)\big], \end{aligned} \tag{3.11}$$

where $\Psi''_j(X,Y,Z)$ $(j=1,2,3)$ are some polynomials in X,Y and Z. We observe that each of the polynomials $\mathcal{G}_i(X,Y,Z)/Z^2$, $i=1,2,3$ has as a factor Z if and only if $c=d=0$. This condition is governed by the invariant polynomials $\widetilde{\chi}_2$ because for of systems (3.7) $u=-1$ and $k=h=g=0$ we have

$$\widetilde{\chi}_2 = 4x^2(sx+y)(cx+dy)(x^2+y^2)\big[(3s^2-1)x^2+8sxy+(3-s^2)y^2\big]/3.$$

Evidently the condition $\widetilde{\chi}_2=0$ is equivalent to $c=d=0$. So in the case $\widetilde{\chi}_2\neq 0$ we have $c^2+d^2\neq 0$ and therefore we deduce that the invariant line $Z=0$ has the multiplicity exactly three.

(iv) Admit now that the conditions $\mathcal{D}_7=0$ (i.e. $u=-1$) and $\widetilde{\chi}_1=\widetilde{\chi}_2=0$ which implies $k=h=g=d=c=0$. Then considering (3.11) we deduce that Z^3 is a common factor of the polynomials $\mathcal{G}_i(X,Y,Z)$, $i=1,2,3$. We claim that the invariant line $Z=0$ of systems (3.7) is of multiplicity at least four. For this it is sufficient to apply to (3.7) with $u=-1$ and $k=h=g=d=c=0$ the following perturbation (remaining in the class of cubic systems):

$$\dot{x} = a(1+\varepsilon_1 x+\varepsilon_2 x^2+\varepsilon_3 x^3), \quad \dot{y}=q(x,y)$$

with $|\varepsilon_i|\ll 1$ $(i=1,2,3)$. Clearly the perturbed systems possess the three parallel invariant lines defined by the equation $1+\varepsilon_1 x+\varepsilon_2 x^2+\varepsilon_3 x^3=0$ which coalesce with the infinite one when ε_i $(i=1,2,3)$ tend to zero. So we deduce that the invariant line $Z=0$ is of multiplicity at least 4 and in order to determine precisely its multiplicity we calculate:

$$\mathcal{G}_1(X,Y,Z)/Z^3 = -(sX+Y)(X^2+Y^2)(X^2+2sXY+3Y^2)+Z\big[\Psi'''_1(X,Y,Z)\big],$$

where $\Psi'''_1(X,Y,Z)$ $(j=1,2,3)$ is a polynomial in X,Y and Z. As we can see the polynomial $\mathcal{G}_1(X,Y,Z)/Z^3$ could not have Z as a factor and therefore we deduce that the maximum multiplicity of the invariant line $Z=0$ for systems (3.7) equals four.

As all the cases are examined we conclude that Lemma 3.2 is proved. □

Thus considering Lemma 3.1 as well as Lemma 3.25 (which will be proved later) in what follows we consider the family of systems (3.7), i.e. the systems

$$\begin{aligned}\dot{x} &= a+cx+dy+gx^2+2hxy+ky^2+(1+u)x^3,\\ \dot{y} &= b+ex+fy+lx^2+2mxy-sx^3+ux^2y-sxy^2-y^3,\end{aligned} \tag{3.12}$$

for which we have $C_3(x,y)=x(sx+y)(x^2+y^2)$. For the corresponding associated homogeneous cubic systems we calculate (see the definition of the polynomial $H(X,Y,Z)$ on the page 14, Notation 2.4):

$$H(X,Y,Z)=\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)=(1+u)X^3(sX+Y)(X^2+Y^2). \tag{3.13}$$

So by Remark 2.13, systems (3.12) could have one triplet of parallel invariant lines in the direction $x=0$. However for some values of the parameters u and s the common divisor $\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$ could contain additional factors (see Notation 2.4 and Lemma 2.6). We prove the following lemma.

Lemma 3.4. *Systems (3.12) could possess a triplet of parallel invariant lines in the real direction $sx+y=0$ if and only if $s=u=0$.*

Proof. Consider the corresponding homogeneous cubic systems associated to (3.12):

$$\dot{x} = (1+u)x^3, \quad \dot{y} = -sx^3 + ux^2y - sxy^2 - y^3. \tag{3.14}$$

It was shown above that for these systems the value of $H(X,Y,Z)$ is given in (3.13). Since the factor $(sX+Y)$ in $\gcd(\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3)$ depends on Y, according to Lemma 2.9 in order to increase its multiplicity up to 3 it is necessary

$$R_Y^{(0)}(\mathcal{G}_2/H,\, \mathcal{G}_1/H) = R_Y^{(1)}(\mathcal{G}_2/H,\, \mathcal{G}_1/H) = 0.$$

We calculate

$$R_Y^{(1)}(\mathcal{G}_2/H,\, \mathcal{G}_1/H) = 6s(9+s^2)X^3 = 0$$

which implies $s=0$. Then for systems (3.14) with $s=0$ we obtain

$$R_Y^{(0)}(\mathcal{G}_2/H,\, \mathcal{G}_1/H) = 9u^2(3+u)^2X^8 = 0.$$

Therefore this condition gives $u(3+u)=0$. If $u=-3$ we get the homogeneous system

$$\dot{x} = -2x^3, \quad \dot{y} = -y(3x^2+y^2),$$

for which we have $H(X,Y,Z) = 6X^3Y(X^2+Y^2)^2$, i.e. considering Remark 2.13 we could not have a triplet of parallel invariant lines in he direction $y=0$.

Assuming $u=0$ we get the homogeneous system

$$\dot{x} = x^3, \quad \dot{y} = -y^3, \tag{3.15}$$

for which we have $H(X,Y,Z) = 3X^3Y^3(X^2+Y^2)$ and this completes the proof of the lemma. □

3.1.2 Construction of the cubic systems possessing configuration of the type $\mathcal{T} = (3,1,1,1)$

In what follows we examine systems (3.12) considering each one of the cases provided by Lemma 3.2.

1: The case $\mathcal{D}_7 \neq 0$. Then by Lemma 3.2 the infinite invariant line $Z=0$ of systems (3.12) is of multiplicity one and hence, we have to detect the conditions for the existence of invariant affine lines of total multiplicity six. Moreover these lines have to be in the configuration of the type $(3,1,1,1)$. Since the existence of a triplet of parallel invariant lines in the real direction $x=0$ for systems (3.12) is a generic case we begin with the study of this case.

Considering the equations (2.5) and Remark 2.12 for systems (3.12) in the case of the direction $x=0$ we obtain the following non-vanishing equations Eq_i:

$$Eq_7 = k, \quad Eq_9 = d - 2hW, \quad Eq_{10} = a - cW + gW^2 - (1+u)W^3. \tag{3.16}$$

It is clear that these three equations can have three common solutions if and only if $k=d=h=0$ and since $\mathcal{D}_7 = (u+1) \neq 0$ we obtain that in this case the equation $Eq_{10}=0$ has three solutions. They could be real or/and complex, distinct or coinciding. This means that systems (3.12) have in the direction $x=0$ a triplet of parallel invariant lines.

Next we have to determine the conditions for the existence of three invariant lines in three distinct directions: one real $(sx+y=0)$ and two complex $(x \pm iy = 0)$. Since the coefficients of systems (3.12) are real it is clear that for the complex directions it is sufficient to examine only one of them: $x + iy = 0$. In this case we have $U = 1$, $V = i$ and considering (2.5) and Remark 2.12 we obtain

$$\begin{aligned} Eq_7 &= 2m - g - il + (3 + u - 2is)W, \\ Eq_9 &= e + i(f - c) - 2\big[l + i(m - g)\big]\big)W - \big[3s + i(3 + 2u)\big]W^2, \\ Eq_{10} &= a + ib - (c + ie)W + (g + il)W^2 - (1 + u - is)W^3. \end{aligned} \tag{3.17}$$

Calculations yield

$$Res_W(Eq_7, Eq_9) = H_1 + iH_2, \quad Res_W(Eq_7, Eq_{10}) = H_3 + iH_4$$

where

$$\begin{aligned} H_1 =& -\big[4s^2 - (3 + u)^2\big]e + \big[4(f - c)(3 + u)s + (g^2 - 4m^2)s - l^2 s - 2l(3g - 3m + mu)\big], \\ H_2 =& \big[4s^2 - (3 + u)^2\big](c - f) + 2gm(u - 3) - 3l^2 + 3g^2 - 12es + 2lgs - 4m^2 u - 4esu\big] \\ H_3 =& -a(3 + u)\big[12s^2 - (3 + u)^2\big] - 2bs\big[4s^2 - 3(3 + u)^2\big] + (cg - le - 2cm)\big[4s^2 - (3 + u)^2\big] \\ & - l^3 s - 2cl^2(3g - 3m + mu) + 2(g - 2m)(g^2 - gm - 2m^2 - 6es + gmu - 2m^2 u - 2esu) \\ & - ls(12c - 3g^2 + 4gm + 4m^2 + 4cu), \\ H_4 =& 2as\big[4s^2 - 3(3+u)^2\big] - b(3+u)\big[12s^2 - (3+u)^2\big] + (cl + eg - 2em)\big[4s^2 - (3+u)^2\big] - 2l^3 \\ & - (g - 2m)(g^2 - 12c - 4m^2 - 4cu)s + 2l(3g^2 - 6gm - 6es + 2gmu - 4m^2 u - 2esu) \\ & + l^2(3g - 2m)s. \end{aligned} \tag{3.18}$$

It is clear that for the existence of a common solution of equations $Eq_7 = Eq_9 = Eq_{10} = 0$ with respect to W it is necessary and sufficient $H_1 = H_2 = H_3 = H_4 = 0$.

Solving the system of equations $H_1 = H_2 = 0$ with respect to the parameters e and f we obtain:

$$\begin{aligned} e =& \frac{1}{\big[4s^2 + (3+u)^2\big]^2}\Big[l^2 s(u^2 - 27 - 4s^2 - 6u) + 2l\big[m(3-u)(4s^2 - (3+u)^2) + gu(4s^2 + 18 + 3u) \\ & + 27g\big] + s(g - 2m)(27g - 18m + 4gs^2 + 8ms^2 + 6gu + 12mu - gu^2 + 6mu^2)\Big], \\ f =& c + \frac{1}{\big[4s^2 + (3+u)^2\big]^2}\Big[l^2(27 + 18u + 4s^2 u + 3u^2) + 2ls(27g - 36m + 4gs^2 + 6gu - gu^2 + 4mu^2) \\ & - (g - 2m)\big[4s^2(6m + gu) + (3 + u)^2(3g + 2mu)\big]\Big] \end{aligned} \tag{3.19}$$

and evidently we could do this only in the case $4s^2 + (3 + u)^2 \neq 0$.

On the other hand for systems (3.12) we have

$$\mathcal{D}_8 = -8(s^2 - u)\big[4s^2 + (3 + u)^2\big]/27 \tag{3.20}$$

and as we will see later the condition $s^2 - u = 0$ is also essential.

So in what follows we have to consider two subcases: $\mathcal{D}_8 \neq 0$ and $\mathcal{D}_8 = 0$.

1.1: *The subcase* $\mathcal{D}_8 \neq 0$, i.e. $4s^2 + (3 + u)^2 \neq 0$. Considering this condition we examine all the needed directions.

(i) *The direction* $x+iy=0$. In this case we have the conditions (3.19) and solving the system of equations $H_3=H_4=0$ with respect to the parameters a and b we obtain:

$$a=\frac{(g-2m)(3+u)-2ls}{\big[4s^2+(3+u)^2\big]^3}\Big[c(4s^2+(3+u)^2)^2+\big[4ls-2g(3+u)+4m(3+u)\big]$$
$$\times\big[g(3+2s^2+u)+(1+u)(ls+3m+mu)\big]\Big],$$
$$b=\frac{2(g-2m)s+l(3+u)}{\big[4s^2+(3+u)^2\big]^3}\Big[c\big[4s^2+(3+u)^2\big]^2-2(g-2m)\big[2gs^2(1+u)+m(8s^2+(3+u)^3)\big]$$
$$+2l^2(3+u)(3+2s^2+u)+8lsm(1+u)(3+u)+2lgs\big[4s^2-(u-1)(3+u)\big]\Big]. \tag{3.21}$$

Thus if for systems (3.12) the conditions $k=d=h=0$, (3.19) and (3.21) are satisfied then these systems possess five invariant affine lines: three in the direction $x=0$ and two in the complex directions $x\pm iy$.

(ii) *The direction* $sx+y=0$. Then we have $U=s$, $V=1$ and considering (2.5) and the above conditions we obtain

$$Eq_5=l+gs-2ms+(s^2-u)W \tag{3.22}$$

and since the condition $\mathcal{D}_8\neq 0$ implies $s^2-u\neq 0$ we deduce that the above equation is linear with respect to the parameter W. Then the condition $Eq_5=0$ gives $W=(l+gs-2ms)/(u-s^2)$ and we calculate:

$$Eq_8=\frac{2(1+s^2)\,\widehat{H}\,H_5}{(s^2-u)^2\big[4s^2+(3+u)^2\big]^2},\quad Eq_{10}=\frac{(1+s^2)\,\widehat{H}\,H_6}{(s^2-u)^3\big[4s^2+(3+u)^2\big]^3}, \tag{3.23}$$

where

$$\widehat{H}=s(g-2m)(9+u)+l(9-2s^2+3u),$$
$$H_5=ls(9+s^2)(1+u)+m(1+u)\big[s^2(u-9)-3u(3+u)\big]+g\big[2s^4+u^2(3+u)+s^2(9+5u)\big],$$
$$H_6=c(s^2-u)^2\big[4s^2+(3+u)^2\big]^2+l^2(2s^2-9-3u)(1+u)(9+7s^2+2s^4+3u-s^2u)$$
$$+g^2\big[6s^4(u^2-9-4u)-4s^6(3+u)-u^2(3+u)^3-s^2(81+99u+55u^2+13u^3)\big]$$
$$+2m\big[4gs^6(3+u)+4ls^5(1+u)(3+u)+gu^2(3+u)^3-8ls^3(1+u)(u^2-9+2u)$$
$$+6ls(1+u)(3+u)(9+4u+u^2)-gs^4(u^3-81-45u+13u^2)$$
$$+2gs^2(81+126u+76u^2+20u^3+u^4)\big]+4m^2s^2(1+u)(9+u)(s^2u-9-3s^2-7u-2u^2)$$
$$+2lgs\big[4s^6+s^4(3-10u-u^2)+2s^2(2u^2+u^3-18-23u)-(3+u)(27+39u+12u^2+2u^3)\big]. \tag{3.24}$$

We observe that the equations $Eq_8=Eq_{10}=0$ imply either $\widehat{H}=0$ or $H_5=H_6=0$ and we examine both possibilities.

First we observe that if for systems (3.12) the conditions of the existence of a triplet in the direction $x=0$ are satisfied (i.e. $k=d=h=0$) then for these systems we have $\chi_1=-\widehat{H}(g,l,m,s,u)x^3/9$. Therefore we conclude that the condition $\widehat{H}=0$ is equivalent to $\chi_1=0$ in the case under consideration.

1.1.1: *The possibility* $\chi_1=0$, i.e. $\widehat{H}=0$. We observe that the polynomial $\widehat{H}$ is linear with respect to the parameter l with the coefficient $2s^2-3(u+3)$ in front.

On the other hand for systems (3.12) we have

$$\mathcal{D}_6 = 4\big[2s^2 - 3(u+3)\big]/9$$

and therefore we have to consider two cases: $\mathcal{D}_6 \neq 0$ and $\mathcal{D}_6 = 0$.

1.1.1.1: *The case $\mathcal{D}_6 \neq 0$.* Then $9 - 2s^2 + 3u \neq 0$ and we calculate $l = \frac{(g-2m)s(9+u)}{2s^2-3u-9}$. So considering conditions (3.19) and (3.21) we arrive at the following lemma.

Lemma 3.5. *Assume that for a system (3.12) the conditions*

$$u+1 \neq 0, \quad (s^2-u)\big[4s^2+(3+u)^2\big] \neq 0, \quad \varkappa \equiv 2s^2 - 3(u+3) \neq 0. \tag{3.25}$$

hold. Then this system possesses invariant lines in the configuration $(3,1,1,1)$ *if and only if the following conditions are satisfied:*

$$\begin{aligned} k = d = h = 0, \quad l = \frac{s}{\varkappa}(u+9)(g-2m),\\ e = \frac{s}{\varkappa^2}(g-2m)\big[g(s^2-27) + 2m(s^2-3u+18)\big],\\ f = c + \frac{3}{\varkappa^2}(g-2m)\big[3g(s^2-3) - 2m(s^2+3u)\big],\\ a = -\frac{3}{\varkappa^3}(g-2m)\big[c\,\varkappa^2 + 6(g-2m)(gs^2-3g-3m-3mu)\big],\\ b = \frac{s}{\varkappa^3}(g-2m)\big[c\,\varkappa^2 + 2(g-2m)(4gs^2-2ms^2-9mu-27m)\big]. \end{aligned} \tag{3.26}$$

Next we construct the invariant conditions corresponding to (3.26).

Lemma 3.6. *Assume that for a cubic system (3.12) the conditions $\chi_1 = 0$ and $\mathcal{D}_6\mathcal{D}_7\mathcal{D}_8 \neq 0$ hold. Then this system has invariant lines in the configuration $(3,1,1,1)$ if and only if the conditions $\chi_3 = \chi_6 = 0$ are satisfied.*

Proof. For systems (3.12) we have $\mathcal{D}_4 = 2304s(9+s^2)$ and we examine two possibilities: $\mathcal{D}_4 \neq 0$ and $\mathcal{D}_4 = 0$.

a) *The possibility $\mathcal{D}_4 \neq 0$.* For systems (3.12) we calculate

$$\text{Coefficient}[\chi_1, y^3] = k(1+u)$$

and since $\mathcal{D}_7 = 4(1+u) \neq 0$ the condition $\chi_1 = 0$ implies $k = 0$. Then we get the conditions

$$\text{Coefficient}[\chi_1, xy^2] = \frac{2}{9}h(s^2+3u) = 0, \quad \text{Coefficient}[\chi_1, x^2y] = \frac{4}{9}hs(u-3) = 0$$

and since $s \neq 0$ (due to $\mathcal{D}_4 = 2304s(9+s^2) \neq 0$) we obtain $h = 0$. In this case we calculate

$$\chi_1 = \frac{1}{9}\big[l(-9+2s^2-3u) - (g-2m)s(9+u)\big]x^3 = 0$$

which implies $l = \frac{s(u+9)(g-2m)}{2s^2-3(u+3)}$. Thus the condition $\chi_1 = 0$ for systems (3.12) gives us the conditions on the parameters k, h and l from (3.26).

Next assuming that these conditions are satisfied we examine the other conditions from (3.26). Evaluating the invariant polynomial χ_6 we obtain

$$\text{Coefficient}[\chi_6, xy^7] = 10d(s^2-9-6u), \quad \text{Coefficient}[\chi_6, x^2y^6] = \frac{10}{3}ds(81+23s^2-42u)$$

and since $s \neq 0$ we claim that the vanishing of these coefficients implies $d = 0$. Indeed supposing $d \neq 0$ we get $s^2 - 9 - 6u = 0$ which gives $u = (s^2-9)/6$. Then we obtain $81 + 23s^2 - 42u = 16(9+s^2) \neq 0$ and the contradiction we obtained proves our claim.

Thus $d = 0$ and calculations yield

$$\begin{aligned}
\text{Coefficient}[\chi_6, x^3y^5] &= 110\Big[e(s^2-9-6u) - fs(3+s^2-2u) + cs(3+s^2-2u) \\
&\quad + \frac{4s}{\varkappa}(g-2m)(g-m)(9+s^2)\Big], \\
\text{Coefficient}[\chi_6, x^4y^4] &= \frac{10s}{3}\Big[e(7s^2-927-330u) - fs(21+39s^2-110u) + cs(21+39s^2-110u) \\
&\quad + \frac{4s}{\varkappa^2}(g-2m)(9+s^2)(927m-711g+86gs^2-62ms^2-165gu+165mu)\Big]
\end{aligned}$$

and we observe that the above polynomials are linear with respect to the parameters e and f with the corresponding determinant $-32fs(9+s^2)^2\varkappa^3 \neq 0$. So forcing these polynomials to vanish we get

$$\begin{aligned}
e &= \frac{s}{\varkappa^2}(g-2m)\big[g(s^2-27) + 2m(s^2-3u+18)\big], \\
f &= c + \frac{3}{\varkappa^2}(g-2m)\big[3g(s^2-3) - 2m(s^2+3u)\big].
\end{aligned}$$

Thus provided the condition $\chi_1 = 0$ is fulfilled, the condition $\chi_6 = 0$ for systems (3.12) gives us the conditions on the parameters d, e and f from (3.26).

So it remains to determine the invariant polynomials which are responsible for the conditions on the parameters a and b given in (3.26). Evaluating the invariant polynomial χ_3 for systems (3.12) for which the conditions on the parameters k, d, h, l, e and f are given in (3.26) we have:

$$\chi_3 = \frac{2x^5}{27\varkappa^3}\psi_1\psi_2\psi_3\big[\psi_4\, x + \psi_5\, y\big],$$

where

$$\begin{aligned}
\psi_1 &= s(u-3)x + (s^2+3u)y, \\
\psi_2 &= (u^2-3s^2)x^2 - 4s(3+u)xy + (s^2-9-6u)y^2, \\
\psi_3 &= (6s^2+3u+u^2)x^2 + 2s(9+u)xy + (9-2s^2+3u)y^2, \\
\psi_4 &= -b\varkappa^3 + s(g-2m)\big[c\,\varkappa^2 + 2(g-2m)(4gs^2-2ms^2-9mu-27m)\big], \\
\psi_5 &= a\varkappa^3 + 3(g-2m)\big[c\,\varkappa^2 + 6(g-2m)(gs^2-3g-3m-3mu)\big].
\end{aligned}$$

It is not too difficult to see that due to $s \neq 0$ the condition $\psi_1\psi_2\psi_3 \neq 0$ holds. Therefore the condition $\chi_3 = 0$ is equivalent to $\psi_4 = \psi_5 = 0$ and solving these equations with respect to the parameters a and b we get the expressions for these parameters given in (3.26). This completes the proof of Lemma 3.6 as well as the statement (A_1) of the Main Theorem in the case $\mathcal{D}_4 \neq 0$. □

b) *The possibility* $\mathcal{D}_4 = 0$. Then $s = 0$ and we observe that the conditions (3.26) become of the form:

$$\begin{aligned}
k = d = h = l = e = b = 0, \quad f &= c - \frac{1}{(3+u)^2}(g-2m)(3g+2mu), \\
a &= \frac{1}{(3+u)^3}(g-2m)\big[c(3+u)^2 - 2(g-2m)(g+m+mu)\big],
\end{aligned} \tag{3.27}$$

For systems (3.12) with $s = 0$ we calculate

$$\chi_1 = -\frac{1}{9}[3l(u+3)+2hu^2]x^3 - \frac{1}{9}k(u-3)ux^2y + \frac{2}{3}huxy^2 + k(u+1)y^3$$

and due to the condition $\mathcal{D}_7\mathcal{D}_8 = 32u(1+u)(3+u)^2/27 \neq 0$ we deduce that the condition $\chi_1 = 0$ is equivalent to $k = h = l = 0$.

On the other hand for systems (3.12) with $s = k = h = l = 0$ we calculate

$$\text{Coefficient}[\chi_6, xy^7] = -30d(3+2u), \quad \text{Coefficient}[\chi_6, x^3y^5] = 180d - 10(3+2u)(42d+33e+40du)$$

and evidently the condition $\chi_6 = 0$ implies $d = 0$. Then we calculate again

$$\text{Coefficient}[\chi_6, x^3y^5] = -330e(3+2u), \quad \text{Coefficient}[\chi_6, x^5y^3] = 5e\big[81+(3+2u)(10u-99)\big]$$

and we observe that in this case the condition $\chi_6 = 0$ implies $e = 0$. We finally calculate

$$\chi_6 = 60u\big[(c-f)(3+u)^2 - (g-2m)(3g+2mu)\big]x^6y^2$$

and since $u(3+u) \neq 0$ the condition $\chi_6 = 0$ yields

$$f = c - \frac{1}{(3+u)^2}(g-2m)(3g+2mu),$$

i.e. we get the condition for the parameter f given in (3.27).

Next assuming the above mentioned conditions are fulfilled for systems (3.12) we calculate

$$\chi_3 = \frac{2u}{9(3+u)^2}x^5y(ux^2+3y^2)(u^2x^2-9y^2-6uy^2)$$
$$\times\Big\{b(3+u)^3x - \big[a(3+u)^3 - (g-2m)(c(3+u)^2 - 2(g-2m)(g+m+mu)\big]y\Big\}\big].$$

Therefore due to $u(3+u) \neq 0$ the condition $\chi_3 = 0$ implies

$$b = 0, \quad a = \frac{1}{(3+u)^3}(g-2m)\big[c(3+u)^2 - 2(g-2m)(g+m+mu)\big],$$

i.e. we get the two conditions for the parameters b and a given in (3.27). This completes the proof of our claim and hence the statement (A_1) the Main Theorem is valid also in the case $\mathcal{D}_4 = 0$.

1.1.1.2: *The case* $\mathcal{D}_6 = 0$. This implies $9 - 2s^2 + 3u = 0$ and we have $u = (2s^2 - 9)/3$. Then we obtain:

$$\widehat{H} \equiv s(g-2m)(9+u) + l(9-2s^2+3u) = 2(g-2m)s(9+s^2)/3 = 0$$

i.e. we get $s(g-2m) = 0$. On the other hand for $u = (2s^2-9)/3$ we calculate (see (3.20))

$$\mathcal{D}_8 = -8(s^2-u)\big[4s^2+(3+u)^2\big]/27 = -\frac{32}{729}s^2\left(s^2+9\right)^2 \neq 0.$$

So $s \neq 0$ and this implies $g = 2m$. Considering (3.19) and (3.21) we arrive at the next lemma.

Lemma 3.7. *Assume that for a system* (3.12) *the conditions* $\chi_1 = 0$, $\mathcal{D}_7\mathcal{D}_8 \neq 0$ *and* $\mathcal{D}_6 = 0$ *hold. Then this system possesses invariant lines in the configuration* $(3,1,1,1)$ *if and only if the following conditions are satisfied:*

$$\begin{aligned} &k = d = h = 0, \quad u = (2s^2-9)/3, \quad g = 2m,\\ &e = \frac{3l\left[3l(s^2-27)+4ms(9+s^2)\right]}{4s(9+s^2)^2},\\ &f = \frac{81l^2(s^2-3)+36lms(9+s^2)+4cs^2(9+s^2)^2}{4s^2(9+s^2)^2},\\ &a = -\frac{9l\left[27l^2(s^2-3)+18lms(9+s^2)+2cs^2(9+s^2)^2\right]}{4s^3(9+s^2)^3},\\ &b = \frac{3l\left[18l^2s+9lm(9+s^2)+cs(9+s^2)^2\right]}{2s(9+s^2)^3}. \end{aligned} \tag{3.28}$$

Next we determine the invariant conditions equivalent to those provided in the above lemma. More exactly we prove the following lemma.

Lemma 3.8. *Assume that for a system* (3.12) *the conditions* $\chi_1 = 0$, $\mathcal{D}_7\mathcal{D}_8 \neq 0$ *and* $\mathcal{D}_6 = 0$ *hold. Then this system has invariant lines in the configuration* $(3,1,1,1)$ *if and only if the conditions* $\chi_2 = \chi_3 = 0$ *are satisfied.*

Proof. For systems (3.12) with $u = (2s^2-9)/3$ we calculate:

$$\mathcal{D}_7 = \frac{8}{3}(s^2-3), \quad \text{Coefficient}[\chi_1, y^3] = 2k(s^2-3)/3, \quad \text{Coefficient}[\chi_1, xy^2]\big|_{k=0} = 2h(s^2-3)/3.$$

So it is clear that due to $\mathcal{D}_8 \neq 0$ (i.e. $s \neq 0$) the condition $\chi_1 = 0$ implies $k = h = 0$ and then calculations yield:

$$\chi_1 = -2(g-2m)s(9+s^2)x^3/27, \quad \mathcal{D}_8 = -\frac{32}{729}s^2(9+s^2)^2 \neq 0.$$

So we conclude that the condition $\chi_1 = 0$ for systems (3.12) with $\mathcal{D}_6 = 0$ (i.e. $u = (2s^2-9)/3$) is equivalent to $k = h = 0$ and $g = 2m$. Assuming that these conditions are fulfilled for systems (3.12) we obtain:

$$\text{Coefficient}[\chi_2, y^2] = 56ds(9+s^2)/3 = 0 \quad \Leftrightarrow \quad d = 0$$

and then we calculate

$$\chi_2 = -\frac{8}{9}\varphi_1' x^2 + \frac{16}{3}\varphi_2' xy,$$

where

$$\begin{aligned} \varphi_1' &= 36es(3+s^2) - 8fs^4 + 81l^2 - 36lms + 8cs^4,\\ \varphi_2' &= 9e(s^2-3) - fs(-27+s^2) - 18clm - 27cs + cs^3. \end{aligned}$$

We observe that the polynomials φ_1' and φ_2' are linear with respect to the parameters e and f with the corresponding determinant $36s^2(9+s^2)^2 \neq 0$ and therefore the equations $\varphi_1' = \varphi_2' = 0$ give us

$$\begin{aligned} e &= \frac{3l}{4s(9+s^2)^2}\left[3l(s^2-27)+4ms(9+s^2)\right],\\ f &= \frac{1}{4s^2(9+s^2)^2}\left[81l^2(s^2-3)+36lms(9+s^2)+4cs^2(9+s^2)^2\right]. \end{aligned}$$

Thus provided $\chi_1 = 0$ is fulfilled, the condition $\chi_2 = 0$ for systems (3.12) gives us the conditions on the parameters d, e and f from (3.28).

Next evaluating the invariant polynomial χ_3 for systems (3.12) for which the conditions on the parameters k, d, h, u, g, e and f are given in (3.28) we have:

$$\chi_3 = \frac{2x^6(sx+3y)}{6561s^2(9+s^2)^2}\hat{\psi}_1\hat{\psi}_2\big[-2s^2\hat{\psi}_3\,x+\hat{\psi}_4\,y\big],$$

where

$$\begin{aligned}
\hat{\psi}_1 &= 2s(s^2-9)x+9(s^2-3)y,\\
\hat{\psi}_2 &= (81-63s^2+4s^4)x^2-24s^3xy-27(s^2-3)y^2,\\
\hat{\psi}_3 &= -2bs(9+s^2)^3+3l\big[18l^2s+9lm(9+s^2)+cs(9+s^2)^2\big],\\
\hat{\psi}_4 &= -4as^3(9+s^2)^3-9l\big[27l^2(s^2-3)+18lms(9+s^2)+2cs^2(9+s^2)^2\big].
\end{aligned}$$

It is not too difficult to see that due to $s(s^2-3)\neq 0$ the condition $\hat{\psi}_1\hat{\psi}_2\neq 0$ holds. Therefore the condition $\chi_3 = 0$ is equivalent to $\hat{\psi}_3 = \hat{\psi}_4 = 0$ and solving these equations with respect to the parameters a and b we get the expressions for these parameters given in (3.28). This completes the proof of Lemma 3.8 as well as the statement (A_2) of the Main Theorem. □

1.1.2: *The possibility* $\chi_1 \neq 0$. Then considering (3.23) in order to have invariant lines of total multiplicity seven we must force $H_5 = H_6 = 0$. Taking into account (3.24) we consider two cases: $s\neq 0$ and $s = 0$ and this condition is governed by the invariant polynomial $\mathcal{D}_4 = 2304s(9+s^2)$.

1.1.2.1: *The case* $\mathcal{D}_4 \neq 0$. Then $s\neq 0$ and solving the equations $H_5 = H_6 = 0$ with respect to the parameters c and l we obtain:

$$\begin{aligned}
c = {}&\frac{1}{s^2(9+s^2)^2(1+u)}\Big[-27m^2(s^2-3)(1+u)^2+6gm(s^2-3)(1+u)(s^2+3u)\\
&-g^2(2s^4u-27s^2-7s^4-6s^2u-9u^2+3s^2u^2)\Big],\\
l = {}&\frac{1}{s(9+s^2)(1+u)}\Big[m(1+u)(9s^2+9u-s^2u+3u^2)-g(9s^2+2s^4+5s^2u+3u^2+u^3)\Big].
\end{aligned}$$

Thus considering the conditions $k = d = h = 0$ and the conditions for the parameters e and f from (3.19) as well as for the parameters a and b from (3.21) and the above conditions we conclude that altogether these conditions guarantee the existence of common solutions of the equations (2.5) for each one of the four directions for invariant lines of systems (3.12). So we arrive at the following lemma.

Lemma 3.9. *Assume that for a system* (3.12) *the conditions* $\chi_1\mathcal{D}_7\mathcal{D}_8 \neq 0$ *and* $\mathcal{D}_4 \neq 0$ *hold. Then this system possesses invariant lines in the configuration* $(3,1,1,1)$ *if and only if the following conditions* *are satisfied:*

$$\begin{aligned}
&k=d=h=0,\\
&f=\frac{1}{s^2\left(s^2+9\right)^2(u+1)^2}\big[3m^2(1+u)^2(27-9s^2+s^4+27u-3s^2u+9u^2)\\
&\quad+18gm(1+u)(2s^2+s^4-3u-3u^2-u^3)-g^2(s^2+3u)(2s^2+s^4-3u-3u^2-u^3)\big],\\
&e=\frac{1}{s\left(s^2+9\right)^2(u+1)^2}\big[m^2(3u-18-s^2)(1+u)^2(s^2+3u)\\
&\quad+6gm(1+u)(6s^2+s^4+9u+4s^2u+9u^2-u^3)\\
&\quad+g^2(-27s^2-11s^4-s^6-24s^2u-4s^4u-18u^2-8s^2u^2-12u^3+u^4)\big],\\
&a=\frac{g(3+s^2+2u)-6m(1+u)}{s^2\left(s^2+9\right)^2(u+1)^2}\big[9m^2(1+u)^2-6gmu(1+u)+g^2(s^2+u^2)\big],\\
&b=-\frac{m(s^2-3u)(1+u)+g(s^2+u^2)}{s^3\left(s^2+9\right)^3(u+1)^3}\big[m^2(1+u)^2(81+81s^2+2s^4+81u-3s^2u+18u^2)\\
&\quad+g^2(3+s^2+2u)(9s^2+2s^4+5s^2u+3u^2+u^3)\\
&\quad-2gm(1+u)(36s^2+10s^4+27u+33s^2u+27u^2+s^2u^2+6u^3)\big],\\
&c=\frac{1}{s^2(9+s^2)^2(1+u)}\big[27m^2(3-s^2)(1+u)^2+6gm(s^2-3)(1+u)(s^2+3u)\\
&\quad+g^2(27s^2+7s^4+6s^2u-2s^4u+9u^2-3s^2u^2)\big],\\
&l=-\frac{1}{s(9+s^2)(1+u)}\big[m(1+u)(s^2u-9s^2-9u-3u^2)+g(9s^2+2s^4+5s^2u+3u^2+u^3)\big]
\end{aligned}\tag{3.29}$$

Next we determine the invariant conditions equivalent to those provided by the above lemma. More exactly we prove the following lemma.

Lemma 3.10. *Assume that for a system* (3.12) *the conditions* $\chi_1\mathcal{D}_7\mathcal{D}_8\neq 0$ *and* $\mathcal{D}_4\neq 0$ *hold. Then this system has invariant lines in the configuration* $(3,1,1,1)$ *if and only if the conditions* $\chi_7=\chi_8=\chi_9=\chi_{10}=0$ *and either* $\mathcal{D}_5\neq 0$ *and* $\chi_{11}=0$ *or* $\mathcal{D}_5=\chi_{12}=0$ *are satisfied.*

Proof. For systems (3.12) we calculate:

$$\chi_7=\frac{1}{9}(hx+ky)\big[(3s^2+3u+2u^2)x^2-2s(u-3)xy-(s^2+3u)y^2\big].$$

We claim that the condition $\chi_7=0$ is equivalent to $k=h=0$. Indeed assume that $\chi_7=0$ and $k^2+h^2\neq 0$. Then we must have $3s^2+3u+2u^2=s(u-3)=s^2+3u=0$. However since $s\neq 0$ (due to $\mathcal{D}_4\neq 0$) we obtain $u=3$ and this leads to a contradiction $s^2+9=0$. So our claim is proved and we conclude that the condition $\chi_7=0$ gives $k=h=0$ from (3.29).

Assuming that for systems (3.12) the conditions $k=h=0$ hold we calculate

$$\chi_8=\frac{160}{9}d\big[s(3s^2-9+4u^2)x^2+2(6s^2+9u+s^2u+6u^2)xy+s(9+s^2)y^2\big]$$

and due to $s\neq 0$ we deduce that the condition $\chi_8=0$ is equivalent to $d=0$.

Next we evaluate the invariant polynomial χ_9 for systems (3.12) with the conditions $k=h=d=0$:

$$\begin{aligned}\chi_9=-\frac{16}{9}\big[&ls(9+s^2)(1+u)+m(1+u)(s^2u-9s^2-9u-3u^2)\\&+g(9s^2+2s^4+5s^2u+3u^2+u^3)\big]x^5.\end{aligned}$$

Evidently forcing the condition $\chi_9 = 0$ to be fulfilled we get the condition for the parameter l given in (3.29).

Assuming that for systems (3.12) the conditions under the parameters k, h, d and l provided by Lemma 3.9 are fulfilled we calculate

$$\chi_{11} = -\frac{40}{9s^2(9+s^2)^2(1+u)^2}\Big[\big(u_{11}c + u_{12}e - u_{13}f + \widetilde{U}(g,m,s,u)\big)x^2 + \big(u_{21}c + u_{22}e - u_{23}f + \widetilde{V}(g,m,s,u)\big)xy + \big(u_{31}c + u_{32}e - u_{33}f + \widetilde{W}(g,m,s,u)\big)y^2\Big],$$

where

$$\begin{aligned}
u_{11} &= s^3(9+s^2)^2(1+u)^2(427s^2u + 912s^2 - 905u^2 - 237u + 2277),\\
u_{12} &= s^2(9+s^2)^2(1+u)^2(529s^2u + 1830s^2 - 2103u^2 - 7479u - 6831),\\
u_{13} &= s^3(9+s^2)^2(1+u)^2(529s^2u + 606s^2 - 599u^2 - 1155u + 2277),\\
u_{21} &= 2s^2(9+s^2)^2(1+u)^2(332s^4 - 613s^2u + 996s^2 - 306u^2 - 459u),\\
u_{22} &= 2s^3(9+s^2)^2(1+u)^2(383s^2 - 1686u - 1611),\\
u_{23} &= 2s^2(9+s^2)^2(1+u)^2(383s^4 - 307s^2u + 996s^2 + 153u^2 - 459u),\\
u_{31} &= 843s^3(9+s^2)^2(1+u)^2(3+s^2-2u),\\
u_{32} &= 843s^2(9+s^2)^2(1+u)^2(s^2-6u-9),\\
u_{33} &= 843s^3(9+s^2)^2(1+u)^2(3+s^2-2u)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{U} = 2s\Big[&m^2(1+u)^2(s^2+3u)(1058s^4u + 1824s^4 - 1740s^2u^2 + 2025s^2u + 20601s^2 + 459u^3\\
&- 8775u^2 - 55080u - 73872) - 3gm(1+u)(427s^6u^2 + 281s^6u + 924s^6 + 376s^4u^3 + 3404s^4u^2\\
&+ 4065s^4u - 6777s^4 - 1657s^2u^4 - 3456s^2u^3 - 9999s^2u^2 - 50841s^2u - 61479s^2 + 306u^5\\
&- 12159u^4 - 78084u^3 - 137052u^2 - 61479u) + g^2(612s^8 + 427s^6u^3 - 643s^6u^2 - 1805s^6u\\
&+ 1713s^6 + 2658s^4u^3 - 12786s^4u^2 - 54540s^4u - 35424s^4 - 1128s^2u^5 - 5259s^2u^4 - 21321s^2u^3\\
&- 112509s^2u^2 - 220725s^2u - 122958s^2 + 153u^6 - 9234u^5 - 59724u^4 - 112428u^3 - 61479u^2)\Big],\\
\widetilde{V} = 2\Big[&m^2(1+u)^2(s^2+3u)(1532s^6 - 1686s^4u + 6894s^4 - 1377s^2u^2 - 17928s^2u - 24867s^2\\
&+ 1377u^3 + 8262u^2 + 12393u) - 6gm(1+u)(332s^8u - 434s^8 + 383s^6u^2 + 1150s^6u - 4599s^6\\
&- 1379s^4u^3 - 1074s^4u^2 - 8055s^4u - 18630s^4 - 153s^2u^4 - 9198s^2u^3 - 26919s^2u^2 - 10368s^2u\\
&+ 459u^5 + 2754u^4 + 4131u^3) + g^2(664s^8u^2 - 2656s^8u - 1484s^8 + 153s^6u^3 + 2298s^6u^2\\
&- 30507s^6u - 25308s^6 - 1992s^4u^4 - 3378s^4u^3 - 19341s^4u^2 - 99360s^4u - 70389s^4 - 13338s^2u^4\\
&- 37287s^2u^3 - 4212s^2u^2 + 12393s^2u + 459u^6 + 2754u^5 + 4131u^4)\Big],\\
\widetilde{W} = 1686&(g-m)s(9+s^2)(1+u)\big[g(s^4u - 5s^4 - 6s^2u - 18s^2 - 3u^3 - 9u^2)\\
&- m(2s^2 - 9 - 3u)(1+u)(s^2+3u)\big].
\end{aligned}$$

We observe that the condition $\chi_{11} = 0$ yields the equations

$$\text{Coefficient}[\chi_{11}, x^2] = \text{Coefficient}[\chi_{11}, xy] = \text{Coefficient}[\chi_{11}, y^2] = 0 \tag{3.30}$$

which are linear with respect to the parameters c, e and f. Calculating the corresponding determinant $\det||u_{ij}||$ $(i,j = 1,2,3)$ we obtain

$$\det||u_{ij}|| = 26311716s^7(s^2+3u)(9+s^2)^7(1+u)^6(s^2-u)\big[4s^2 + (u+3)^2\big].$$

On the other hand for systems (3.12) we have

$$\mathcal{D}_5 = \frac{4}{9}(s^2+3u), \ \ \mathcal{D}_4 = 2304s(9+s^2), \ \ \mathcal{D}_7 = 4(1+u), \ \ \mathcal{D}_8 = -8(s^2-u)\big[4s^2+(3+u)^2\big]/27$$

and since $\mathcal{D}_4\mathcal{D}_7\mathcal{D}_8 \neq 0$ we conclude that in the case $\mathcal{D}_5 \neq 0$ we get $\det||u_{ij}|| \neq 0$.

So assuming $\mathcal{D}_5 \neq 0$ and solving the system of equations (3.30) with respect to the parameters c, e and f we get exactly the conditions provided by Lemma 3.9 for these three parameters.

We examine now the case $\mathcal{D}_5 = 0$ when the invariant polynomial χ_{11} could not be used for the determining the conditions under parameters c, e and f.

So assume that for systems (3.12) the conditions on the parameters k, h, d and l provided by Lemma 3.9 are fulfilled and in addition the condition $\mathcal{D}_5 = 0$ holds. This implies $s^2+3u=0$ (i.e. $u=-s^2/3$) and we calculate

$$\chi_{12} = \frac{4x^2}{729(s^2-3)^2}(\phi_1' x^6+\phi_2' x^5y+\phi_3' y^4y^2+\phi_4' x^3y^3+\phi_5' x^2y^4+\phi_6' xy^5+\phi_7' y^6), \quad \mathcal{D}_7 = -\frac{4}{3}(s^2-3),$$

where

$$\phi_7' = -2751246(s^2-3)^2(3f-3c+g^2-2gm+cs^2-fs^2).$$

Since $\mathcal{D}_7 \neq 0$ (i.e. $s^2-3 \neq 0$) the condition $\phi_7' = 0$ gives us

$$f = \frac{1}{s^2-3}(cs^2-3c+g^2-2gm) \qquad (3.31)$$

and then we calculate

$$\phi_6' = 655371(s^2-3)\big[9e(s^2-3)^2 - gs(18m-27g+gs^2-6ms^2)\big].$$

Therefore due to $s^2-3 \neq 0$ the condition $\phi_6' = 0$ implies

$$e = \frac{gs}{7(s^2-3)^2}(gs^2-27g-6ms^2+18m)$$

and for these values of the parameters f and e we obtain

$$\chi_{12} = -432(s^2-3)x^8(sx+3y)^2\big[cs^2(s^2-3)(9+s^2)^2 - 9m^2(s^2-3)^3 + g^2s^2(9+s^2)^2\big].$$

Again since $s^2-3 \neq 0$ as well as $s \neq 0$ (due to $\mathcal{D}_4 \neq 0$) the condition $\chi_{12} = 0$ yields

$$c = \frac{1}{s^2(s^2-3)(9+s^2)^2}\big[9m^2(s^2-3)^3 - g^2s^2(9+s^2)^2\big].$$

So considering (3.31) we obtain

$$f = \frac{m}{s^2(s^2-3)(9+s^2)^2}\big[9m(s^2-3)^3 - 2gs^2(9+s^2)^2\big].$$

Comparing the conditions obtained for the parameters c, e and f above with (3.29) for $u = -s^2/3$ we conclude that they coincide.

Thus from the conditions (3.29) it remains to construct the invariant analog for the conditions on the parameters a and b and this will be done independently on the value of the invariant polynomial $\mathcal{D}_5$.

So evaluating the invariant polynomial χ_{10} for systems (3.12) for which the conditions on the parameters k, d, h, l, c, e and f are given in (3.29) we have:

$$\chi_{10} = \frac{112640x^6}{9s^4(9+s^2)^4(1+u)^4}\psi_1'\psi_2'\psi_3'\big[9m(1+u)-g(s^2+3u)\big]\big[\psi_4' x+\psi_5' y\big],$$

where

$$\begin{aligned}
\psi_1' &= (2s^4+s^2u^2-3s^2u-2u^3-9u^2-9u)\,x - s(s^2+9)(u+1)y,\\
\psi_2' &= (3s^2+2u^2+3u)\,x^2-2s(u-3)xy-(s^2+3u)\,y^2,\\
\psi_3' &= (3s^2-u^2)x^2+4s(3+u)xy-(s^2-9-6u)y^2,\\
\psi_4' &= bs^3(9{+}s^2)^3(1{+}u)^3+\big[m(s^2{-}3u)(1{+}u)+g(s^2{+}u^2)\big]\big[m^2(1{+}u)^2(81{+}81s^2{+}2s^4{+}81u\\
&\quad -3s^2u{+}18u^2)+g^2(3{+}s^2{+}2u)(9s^2{+}2s^4{+}5s^2u{+}3u^2{+}u^3)-2gm(1{+}u)(36s^2{+}10s^4\\
&\quad +27u+33s^2u+27u^2+s^2u^2+6u^3)\big],\\
\psi_5' &= -s(9+s^2)(1+u)\Big[as^2(9+s^2)^2(1+u)^2-(3g-6m+gs^2+2gu-6mu)\big[g^2s^2\\
&\quad +(gu-3m-3mu)^2\big]\Big].
\end{aligned}$$

We observe that the polynomial χ_{10} contains as a factor the expression $9m(1+u)-g(s^2+3u)$ which is different from zero due to the condition $\chi_1 \neq 0$ because in the case under consideration we have:

$$\chi_1 = \frac{(s^2-u)}{9s(9+s^2)(1+u)}\big[4s^2+(u+3)^2\big]\big[9m(1+u)-g(s^2+3u)\big].$$

It is evidently due to $\mathcal{D}_4\mathcal{D}_7\mathcal{D}_8 \neq 0$ that the condition $\chi_1 \neq 0$ is equivalent to $9m(1+u)-g(s^2+3u) \neq 0$.

Thus due to $\chi_1 \neq 0$ we conclude that the condition $\chi_{10}=0$ is equivalent to $\psi_4' = \psi_5' = 0$, because for $s \neq 0$ the condition $\psi_1'\psi_2'\psi_3' \neq 0$ holds. Solving the equations $\psi_4' = \psi_5' = 0$ with respect to the parameters a and b we get exactly the expressions for these parameters given in (3.29).

So Lemma 3.10 is proved and this means that the statement (A_3) of the Main Theorem is also proved. □

1.1.2.2: *The case* $\mathcal{D}_4 = 0$. Then $s=0$ and considering (3.24) and systems (3.12) with the conditions (3.19) and (3.21) we obtain

$$\begin{gathered}
\mathcal{D}_8 = 8u(3+u)^2/27, \quad \widehat{H} = 3l(3+u), \quad H_5 = -u(3+u)(3m-gu+3mu),\\
H_6 = -(3+u)^2\big[9l^2(1+u)+g(g-2m)u^2(3+u)-cu^2(3+u)^2\big].
\end{gathered}$$

Since in this case the conditions $\widehat{H} \neq 0$ and $\mathcal{D}_8 \neq 0$ imply $lu(3+u) \neq 0$, solving the equations $H_5 = H_6 = 0$ with respect to the parameters c and g we obtain:

$$c = \frac{3(1+u)\big[3l^2+m^2(3+u)^2\big]}{u^2(3+u)^2}, \quad g = \frac{3m(1+u)}{u}.$$

So we arrive at the next lemma.

Lemma 3.11. *Assume that for a system (3.12) the conditions $\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\mathcal{D}_4 = 0$ hold. Then this system possesses invariant lines in the configuration $(3, 1, 1, 1)$ if and only if the following conditions are satisfied:*

$$
\begin{aligned}
& k = d = h = s = 0, \quad e = \frac{2lm}{u}, \quad g = \frac{3m(1+u)}{u},\\
& f = \frac{1}{u^2(3+u)^2}\big[m^2u(3+u)^2 + 3l^2(3+3u+u^2)\big],\\
& c = \frac{3(1+u)}{u^2(3+u)^2}\big[3l^2 + m^2(3+u)^2\big], \quad l(3+u) \neq 0,\\
& a = \frac{m(1+u)}{u^3(3+u)^2}\big[9l^2 + m^2(3+u)^2\big],\\
& b = \frac{l}{u^2(3+u)^2}\big[m^2(3+u)^2 + l^2(3+2u)\big],
\end{aligned}
\tag{3.32}
$$

Next we determine the invariant conditions equivalent to those provided in the above lemma. More exactly we prove the following lemma.

Lemma 3.12. *Assume that for a cubic system (3.12) the conditions $\chi_1\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\mathcal{D}_4 = 0$ hold. Then this system possesses invariant lines in the configuration $(3, 1, 1, 1)$ if and only if the conditions $\chi_4 = \chi_5 = \chi_7 = \chi_9 = \chi_{13} = \chi_{14} = 0$ are satisfied.*

Proof. For systems (3.12) the condition $\mathcal{D}_4 = 2304s(9+s^2) = 0$ gives $s = 0$ and we calculate:

$$\chi_7 = \frac{1}{9}u(hx+ky)\big((3x^2+2ux^2-3y^2)\big).$$

Is evidently due to $\mathcal{D}_8 \neq 0$ the condition $\chi_7 = 0$ is equivalent to $k = h = 0$. Assuming these conditions to be satisfied for systems (3.12) as well as the condition $s = 0$ we calculate

$$
\begin{aligned}
\chi_4 &= -2du(1+u), \quad \chi_5 = -2lm - du + eu,\\
\chi_9 &= -\frac{3376}{27}u(3+u)(gu-3m-3mu)x^5, \quad \mathcal{D}_7 = 4(1+u).
\end{aligned}
$$

Therefore due to $\mathcal{D}_7\mathcal{D}_8 \neq 0$ (i.e. $u(u+1)(u+3) \neq 0$) we conclude that the condition $\chi_4 = 0$ is equivalent to $d = 0$ and in this case the condition $\chi_5 = 0$ gives us $e = 2lm/u$. Moreover the condition $\chi_9 = 0$ implies $g = 3m(1+u)/u$. So we get the conditions for the parameters d, e and g given in (3.32).

Next provided these conditions are satisfied for systems (3.12) and evaluating the invariant polynomial χ_{13} we have:

$$\chi_{13} = \frac{120}{u}(\theta_1' x^2 + \theta_2' y^2)$$

where

$$
\begin{aligned}
\theta_1' = & -cu^2(138+178u+69u^2) - fu^2(-138-145u+17u^2) + 3l^2u(11+17u)\\
& + 2m^2(207+405u+298u^2+112u^3),\\
\theta_2' = & -3cu^2(3+2u) - 3f(-3+u)u^2 + 3\big[3l^2u + m^2(9+12u+7u^2)\big].
\end{aligned}
$$

We observe that the equations $\theta_1' = \theta_2' = 0$ are linear with respect to the parameters c and f and the corresponding determinant equals $105u^5(3+u)^2 \neq 0$. Solving these equations we obtain:

$$c = \frac{3(1+u)}{u^2(3+u)^2}\big[3l^2 + m^2(3+u)^2\big], \quad f = \frac{1}{u^2(3+u)^2}\big[m^2u(3+u)^2 + 3l^2(3+3u+u^2)\big],$$

i.e. we get exactly the values for the parameters c and f presented in (3.32).

Thus from the conditions (3.32) it remains to construct the invariant analog for the conditions under parameters a and b. Evaluating the invariant polynomial χ_{14} for systems (3.12) for which the conditions on the parameters k, d, h, s, e, g, c and f are given in (3.32) we have:

$$\chi_{14} = \frac{2(u+1)x^5y}{u^2(3+u)^2}(ux^2-3y^2)\big[u^2x^2-3(3+2u)y^2\big]\big[u\theta_1'' x+\theta_2'' y\big],$$

where

$$\begin{aligned}\theta_1'' &= -bu^2(3+u)^2 + l\big[m^2(3+u)^2 + l^2(3+2u)\big],\\ \theta_2'' &= au^3(3+u)^2 - m(1+u)(9l^2+9m^2+6m^2u+m^2u^2).\end{aligned}$$

It is clear that due to $u(u+1)(u+3)\neq 0$ the condition $\chi_{14}=0$ is equivalent to $\theta_1''=\theta_2''=0$ and this implies exactly the conditions for the parameters a and b given in (3.32).

It remains to observe that the condition $l(3+u)\neq 0$ from (3.32) is equivalent to $\chi_1 = -l(3+u)x^3/3\neq 0$. This completes the proof of Lemma 3.12 as well as of the statement (A_4) of the Main Theorem. □

1.2: *The subcase* $\mathcal{D}_8=0$, i.e. $(s^2-u)\big[4s^2+(3+u)^2\big]=0$.

Remark 3.13. For systems (3.12) the condition $\mathcal{D}_8=0=\mathcal{D}_6$ is equivalent to $4s^2+(3+u)^2=0$.

Indeed for systems (3.12) we have $\mathcal{D}_6=4(2s^2-9-3u)/9$. Assume that $\mathcal{D}_8=\mathcal{D}_6=0$ but $4s^2+(3+u)^2\neq 0$. Then we get $u=s^2$ and this implies $\mathcal{D}_6=-4(9+s^2)/9\neq 0$. This contradiction proves the validity of the above remark. So in what follows we examine two possibilities: $\mathcal{D}_6\neq 0$ and $\mathcal{D}_6=0$.

1.2.1: *The possibility* $\mathcal{D}_6\neq 0$. Then the condition $\mathcal{D}_8=0$ yields $s^2-u=0$. We mention that earlier (up to **1.1:** *The subcase* $\mathcal{D}_8\neq 0$, see page 29) we have investigated the directions $x=0$ and $x\pm iy=0$. So now we examine the remaining direction for the invariant lines, i.e. $sx+y=0$.

Thus we have $u=s^2$ and considering (3.22) for the direction $sx+y=0$ the condition $Eq_5=0$ gives $l=s(2m-g)$. In this case for the equations $Eq_8=0$ and $Eq_{10}=0$ we obtain

$$\begin{aligned}Eq_8 &= \frac{2(2m-g)s+(9+s^2)W}{(9+s^2)^2}\Phi_1(g,m,s,W)=0,\\ Eq_{10} &= \frac{2(2m-g)s+(9+s^2)W}{(9+s^2)^2}\Phi_2(g,m,s,W)=0,\end{aligned}$$

where $\Phi_i(g,m,s,W)$ is a polynomial in the parameters g,m and s and it is of degree i with respect to the variable W.

As we can see the above equations have a common solution in variable W, i.e. in the direction $sx+y=0$ we have one invariant line and altogether we get six invariant affine lines.

Thus considering (3.19) and (3.21) in the case $u=s^2$ as well as the condition $l=s(2m-g)$ we arrive at the following lemma.

Lemma 3.14. *Assume that for a system* (3.12) *the conditions* $\mathcal{D}_7\mathcal{D}_6\neq 0$ *and* $\mathcal{D}_8=0$ *hold. Then this system possesses invariant lines in the configuration* $(3,1,1,1)$ *if and only if the following conditions* *are satisfied:*

$$
\begin{aligned}
&k = d = h = 0, \quad l = s(2m-g), \quad u = s^2,\\
&f = c + \frac{3(g-2m)}{(s^2+9)^2}\,(3gs^2 - 9g - 8ms^2),\\
&e = \frac{s(g-2m)}{(s^2+9)^2}\,(gs^2 - 27g - 4ms^2 + 36m),\\
&a = \frac{3(g-2m)}{(s^2+9)^3}\,\big[c(9+s^2)^2 + 6(g-2m)(gs^2 - 3g - 3m - 3ms^2)\big],\\
&b = -\,\frac{s(g-2m)}{(s^2+9)^3}\,\big[c(9+s^2)^2 + 2(g-2m)(4gs^2 - 27m - 11ms^2)\big].
\end{aligned}
\tag{3.33}
$$

In order to detect the corresponding invariant conditions we consider two cases: $\mathcal{D}_4 \neq 0$ and $\mathcal{D}_4 = 0$.

1.2.1.1: *The case* $\mathcal{D}_4 \neq 0$. We observe that the conditions (3.33) can be obtained as a particular case from the conditions (3.26) by setting $u = s^2$ (i.e. we allow the condition $\mathcal{D}_8 = 0$ to be satisfied).

On the other hand in the proof of Lemma 3.6 we did not use the condition $s^2 - u \neq 0$ and this means that Lemma 3.6 is valid in the case $\mathcal{D}_8 = 0$ too. Therefore we deduce that the statement (A_5) of the Main Theorem is true.

1.2.1.2: *The case* $\mathcal{D}_4 = 0$. Then $s = 0$ and we have $u = 0 = s$. Then according to Lemma 3.4 we could have a triplet of invariant lines either in the direction $x = 0$ or in the direction $y = 0$. Therefore we have to construct the affine invariant conditions taking into consideration the second possibility for the existence of a triplet.

In the first case (i.e. when a triplet of invariant lines is in the direction $x = 0$) we have constructed the corresponding conditions which coincide with (3.33) for $s = 0$. Now we have to determine the conditions on the parameters of systems (3.12) in order to possess invariant lines in the configuration $(3,1,1,1)$ with the triplet in the direction $y = 0$.

So we have to examine each one of the directions for the invariant lines in this case.

(i) *The direction* $y = 0$. Considering the equations (2.5) and Remark 2.12 for systems (3.12) with $s = u = 0$ in the case of the direction $y = 0$ we obtain the following non-vanishing equations containing the parameter W:

$$Eq_5 = l, \quad Eq_8 = e - 2mW, \quad Eq_{10} = b - fW + W^3.$$

So it is evident that for the existence of a triplet the conditions $l = e = m = 0$ have to be satisfied.

(ii) *The direction* $x + iy = 0$. In this case we have $U = 1$, $V = i$ and considering (2.5), Remark 2.12 and the conditions $u = s = l = e = m = 0$ we obtain

$$
\begin{gathered}
Eq_7 = k - g - 2ih + 3W,\\
Eq_9 = d + i(f-c) - 2(h - ig)W - 3iW^2,\\
Eq_{10} = a + Ib - cW + gW^2 - W^3.
\end{gathered}
$$

Calculations yield

$$Res_W(Eq_7, Eq_9) = V_1 + iV_2, \quad Res_W(Eq_7, Eq_{10}) = V_3 + iV_4$$

$$\begin{aligned} V_1 &= 3d - 2gh - 2hk, \quad V_2 = 3f - 3c + g^2 - k^2 \\ V_3 &= 27a - 9cg + 2g^3 + 9ck - 3g^2k - 12h^2k + k^3, \\ V_4 &= 27b - 18ch + 6g^2h + 8h^3 - 6hk^2. \end{aligned} \tag{3.34}$$

It is clear that for the existence of a common solution of equations $Eq_7 = Eq_9 = Eq_{10} = 0$ with respect to W it is necessary and sufficient $V_1 = V_2 = V_3 = V_4 = 0$. Solving these equations we get

$$f = \frac{1}{3}(3c - g^2 + k^2), \quad d = \frac{2h}{3}(g + k), \quad b = \frac{2h}{27}(9c - 3g^2 - 4h^2 + 3k^2),$$
$$a = \frac{1}{27}\left[9c(g - k) - 2g^3 + 3(g^2 + 4h^2)k - k^3\right]$$

(iii) *The direction* $x = 0$. In this case considering the above already detected conditions and (2.5) as well as Remark 2.12 we obtain $Eq_7 = k = 0$. Hence $k = 0$ and we calculate the remaining non-vanishing equations:

$$Eq_9 = \frac{2}{3}h(g - 3W), \quad Eq_{10} = -\frac{1}{27}(g - 3W)(-9c + 2g^2 + 6gW - 9W^2).$$

As we can see the equations $Eq_9 = 0$ and $Eq_{10} = 0$ have a common solution $W = g/3$.

Thus we conclude that the following lemma is valid.

Lemma 3.15. *Assume that for a system (3.12) the conditions $\mathcal{D}_7\mathcal{D}_6 \neq 0$ and $\mathcal{D}_8 = \mathcal{D}_4 = 0$ hold. Then this system possesses invariant lines in the configuration $(3, 1, 1, 1)$ if and only if one of the following sets of the conditions is satisfied:*

– for a triplet in the direction $x = 0$:

$$\begin{aligned} &u = s = k = d = h = l = e = b = 0, \quad f = c + \frac{g(2m - g)}{3}, \\ &a = -\frac{g - 2m}{27}\left(2g^2 - 9c - 2gm - 4m^2\right). \end{aligned} \tag{3.35}$$

– for a triplet in the direction $y = 0$:

$$\begin{aligned} &u = s = k = l = e = m = 0, \quad d = \frac{2gh}{3}, \quad f = c - \frac{g^2}{3}, \\ &a = \frac{g}{27}(9c - 2g^2), \quad b = -\frac{2h}{27}(-9c + 3g^2 + 4h^2). \end{aligned} \tag{3.36}$$

We point out that in order to construct the equivalent invariant conditions for a system (3.12) to possess invariant lines in the configuration $(3, 1, 1, 1)$ we have to take into considerations both sets of conditions: (3.35) (when the triplet is in the direction $x = 0$) and (3.36) (when the triplet is in the direction $y = 0$).

First of all we recall that for systems (3.12) the conditions $\mathcal{D}_8 = 0$ and $\mathcal{D}_6 \neq 0$ yields $s^2 - u = 0$ (see page 41) and $\mathcal{D}_4 = 0$ gives $s = 0$, i.e. we have for systems (3.12) $s = u = 0$.

Considering these conditions we evaluate the invariant polynomial χ_1 for systems (3.12):

$$\chi_1 = -lx^3 + ky^3$$

and evidently the condition $\chi_1 = 0$ implies $k = l = 0$. We observe that these conditions are included in (3.35) as well as in (3.36). Then we calculate

$$\begin{aligned} \chi_3 &= 2(mx + hy)x^3y^3(x^2 + y^2)\left[3ex^2 - 2(3c - 3f - g^2 + 2gm)xy - (3d - 2gh)y^2\right], \\ \chi_8 &= -960hmxy \end{aligned}$$

and we prove the next lemma.

Lemma 3.16. *Assume that for a system (3.12) are satisfied either the conditions (3.35) or (3.36) and in addition we have $h = m = 0$. Then this system possess invariant lines of total multiplicity 9.*

Proof. Supposing $h = m = 0$ a straightforward calculation shows us that the conditions (3.35) coincide with (3.36) and have:

$$u = s = k = d = h = l = e = b = 0, \quad f = c - g^2/3, \quad a = -\frac{g}{27}\left(2g^2 - 9c\right).$$

The above conditions lead to the family of systems

$$\dot{x} = -\frac{1}{27}(g + 3x)\left(2g^2 - 9c - 6gx - 9x^2\right),$$
$$\dot{y} = \frac{1}{3}y\left(3c - g^2 - 3y^2\right),$$

which evidently possess two triplets of parallel invariant lines: one in the direction $x = 0$ and another in the direction $y = 0$. Moreover in addition these systems possess the following two complex invariant lines: $g + 3(x \pm iy) = 0$ and this completes the proof of the lemma. □

Thus we conclude that in the case of the conditions (3.35) or (3.36) the conditions $h^2 + m^2 \neq 0$ must hold. It remains to observe that this condition is governed by the invariant polynomial χ_{15} because for systems (3.12) with $s = u = k = l = 0$ we have $\chi_{15} = x^2y^2(mx + hy)$.

So in what follows we assume that the condition $\chi_{15} \neq 0$ holds, i.e. $h^2 + m^2 \neq 0$. Then the condition $\chi_3 = 0$ implies

$$e = 0, \quad f = \frac{1}{3}(3c - g^2 + 2gm), \quad d = \frac{2gh}{3},$$

whereas the condition $\chi_8 = 0$ implies $hm = 0$.

In the case $h = 0$ we get $e = d = 0$ and $f = (3c - g^2 + 2gm)/3$ and we observe that we obtain exactly the conditions from (3.35) provided for the parameters h, e, d and f.

On the other hand if $m = 0$ we obtain $e = 0$, $d = 2gh/3$ and $f = c - g^2/3$, i.e. we obtain exactly the conditions from (3.36) provided for the parameters m, e, d and f.

We examine each one of the cases mentioned above.

α) Assume first that for systems (3.12) all the conditions (3.35) are satisfied except the conditions on the parameters a and b. Then for these systems we calculate

$$\chi_{16} = -12x^5y^5\left[(27a - 9cg + 2g^3 + 18cm - 6g^2m + 8m^3)x + 27by\right]$$

and we determine that the condition $\chi_{16} = 0$ implies

$$a = \frac{1}{27}(g - 2m)(9c - 2g^2 + 2gm + 4m^2), \quad b = 0.$$

So we obtain exactly the conditions on the parameters a and b given in (3.35).

β) Suppose now that for systems (3.12) all the conditions (3.36) are satisfied excepting the conditions on the parameters a and b. Then for these systems we calculate

$$\chi_{16} = -12x^5y^5\left[(27a - 9cg + 2g^3)x + (27b - 18ch + 6g^2h + 8h^3)y\right]$$

and we obtain that the condition $\chi_{16} = 0$ implies in this case

$$a = \frac{1}{27}g(9c - 2g^2), \quad b = \frac{2}{27}h(9c - 3g^2 - 4h^2).$$

So we obtain exactly the conditions on the parameters a and b given in (3.36).

Thus we have proved the following lemma.

Lemma 3.17. *Assume that for a cubic system* (3.12) *the conditions* $\mathcal{D}_6\mathcal{D}_7 \neq 0$ *and* $\mathcal{D}_8 = \mathcal{D}_4 = 0$ *hold. Then this system possesses invariant lines in the configuration* $(3, 1, 1, 1)$ *if and only if the conditions* $\chi_1 = \chi_3 = \chi_8 = \chi_{16} = 0$ *and* $\chi_{15} \neq 0$ *are satisfied.*

From this lemma the validity of the statement (A_6) of the Main Theorem follows.

1.2.2: *The possibility* $\mathcal{D}_6 = 0$. Since $\mathcal{D}_8 = 0$, according to Remark 3.13 the condition $4s^2 + (3 + u)^2 = 0$ holds. Then $s = 0$, $u = -3$ and by Lemma 3.4 we conclude that a triplet could be only in the direction $x = 0$. So considering the condition $k = d = h = 0$ which guarantees the existence of a triplet of parallel invariant lines in the direction $x = 0$ we examine the directions $sx + y = 0$ (which becomes $y = 0$) and $x + iy = 0$.

a) *The direction* $y = 0$. Considering (2.5) and Remark 2.12 we obtain

$$Eq_5 = l + 3W, \quad Eq_8 = e - 2mW, \quad Eq_{10} = b - fW + W^3.$$

Therefore the condition $Eq_5 = 0$ yields $W = -l/3$ and then we obtain

$$Eq_8 = (3e + 2lm)/3 = 0, \quad Eq_{10} = (27b - l^3 + 9lf)/27 = 0.$$

Solving these equations with respect to the parameters b and e we get

$$b = l(l^2 - 9f)/27, \quad e = -2lm/3 \tag{3.37}$$

and these conditions guarantee the existence of one invariant line in the direction $y = 0$.

b) *The direction* $x + iy = 0$. In this case taking into account the conditions $k = d = h = 0$ and (3.37) we obtain

$$Eq_7 = 2m - g - il, \quad Eq_9 = -2lm/3 + i(f - c) - 2\big[l + i(m - g)\big]\big)W + 3iW^2,$$
$$Eq_{10} = a + il(l^2 - 9f)/27 - (c - 2lmi/3)W + (g + il)W^2 + 2W^3.$$

Clearly the condition $Eq_7 = 0$ implies $l = 0$ and $g = 2m$ and therefore we have

$$Eq_9 = i(-c + f + 2mW + 3W^2), \quad Eq_{10} = a - cW + 2mW^2 + 2W^3.$$

Calculations yield

$$\begin{aligned} &Res_W(Eq_9, Eq_{10}) = i\big[27a^2 + 2am(9c + 4m^2) - (c - f)(c^2 + 4cf + 4f^2 + 4fm^2)\big] \equiv iH', \\ &Res_W^{(2)}(Eq_9, Eq_{10}) = 3c + 6f + 4m^2. \end{aligned} \tag{3.38}$$

Thus the condition $H' = 0$ implies the existence of at least one common solution $W = W_0$ of the equations $Eq_9 = 0$ and $Eq_{10} = 0$. Moreover in this case the condition $Res_W^{(2)}(Eq_9, Eq_{10}) \neq 0$ must hold (i.e. $3c + 6f + 4m^2 \neq 0$), otherwise we get that the mentioned equations have two common solutions and therefore the corresponding systems do not belong to the class $\mathbf{CSL}^{2r2c\infty}_{(3,1,1,1)}$.

We observe that the polynomial H' is quadratic with respect to the parameter a and we calculate

$$\text{Discrim}[H', a] = 4(3c - 3f + m^2)(3c + 6f + 4m^2)^2.$$

Since the condition $3c + 6f + 4m^2 \neq 0$ has to be fulfilled (see the previous paragraph) we deduce that the condition $3c - 3f + m^2 \geq 0$ must hold.

So we have proved the following lemma.

Lemma 3.18. *Assume that for a system* (3.12) *the conditions* $\mathcal{D}_7 \neq 0$ *and* $\mathcal{D}_8 = \mathcal{D}_6 = 0$ *hold. Then this system possesses invariant lines in the configuration* $(3,1,1,1)$ *if and only if the following conditions are satisfied:*

$$\begin{gathered} s=0,\ u=-3,\ k=d=h=e=l=b=0,\ g=2m, \\ 27a^2+2am(9c+4m^2)-(c-f)(c^2+4cf+4f^2+4fm^2)=0, \\ 3c-3f+m^2\geq 0,\quad 3c+6f+4m^2\neq 0. \end{gathered} \tag{3.39}$$

Next we determine the invariant conditions equivalent to those provided in the above lemma. More exactly we prove the following lemma.

Lemma 3.19. *Assume that for a cubic system* (3.12) *the conditions* $\mathcal{D}_7 \neq 0$ *and* $\mathcal{D}_8 = \mathcal{D}_6 = 0$ *hold. Then this system possesses invariant lines in the configuration* $(3,1,1,1)$ *if and only if the conditions* $\chi_1 = \chi_2 = \chi_4 = \chi_6 = \chi_{17} = 0$, $\chi_{11} \neq 0$ *and* $\zeta_4 \leq 0$ *are satisfied.*

Proof. As it was mentioned earlier (see Remark 3.13) the conditions $\mathcal{D}_8 = \mathcal{D}_6 = 0$ imply for systems (3.12) $s = 0$ and $u = -3$. Then for these systems we calculate:

$$\chi_1 = -2(hx+ky)(x^2+y^2) = 0 \quad \Leftrightarrow \quad h=k=0.$$

Herein calculations yield

$$\text{Coefficient}[\chi_6, xy^7] = 90d, \quad \text{Coefficient}[\chi_6, x^4y^4] = -720l^2$$

and evidently the condition $\chi_6 = 0$ implies $d = 0$ and $l = 0$. Then we calculate again

$$\chi_6 = 90x^3y\left[15ex^4 + 6(g-2m)^2x^3y + 26ex^2y^2 + 11ey^4\right]$$

and clearly the condition $\chi_6 = 0$ yields $e = 0$ and $g = 2m$. Then considering the above detected conditions we obtain

$$\chi_{11} = 4080(3c+6f+4m^2)xy$$

and we deduce that the condition $3c + 6f + 4m^2 \neq 0$ fixed in (3.39) is equivalent to $\chi_{11} \neq 0$.

Next we calculate

$$\zeta_4 = -(3c-3f+m^2)(13x^2+3y^2)$$

and clearly the condition $3c - 3f + m^2 \geq 0$ given in (3.39) is equivalent to $\zeta_4 \leq 0$.

Thus all the conditions provided by Lemma 3.18 are defined by the corresponding invariant polynomials except the conditions $b = 0$ and $H' = 0$ (see (3.38)). These conditions are governed by the invariant polynomials χ_{17} which being evaluated for systems (3.12) under the conditions (3.39) (except for $b = 0$ and $H' = 0$) has the form

$$\chi_{17} = -18792x^8(x^2+y^2)^4\left[27b^2x^2 - 2b(27a+9cm+4m^3)xy + H'y^2\right].$$

The condition $\chi_{17} = 0$ is evidently equivalent to $b = 0 = H'$ and this completes the proof of Lemma 3.19 as well as the proof of the statement (A_7) of the Main Theorem. □

2: The case $\mathcal{D}_7 = 0$**.** Then $u = -1$ and by Lemma 3.4 we could not have a triplet of parallel invariant lines in the direction $y = 0$. Since for the direction $x = 0$ we have the equations

$$Eq_7 = k, \quad Eq_9 = d - 2hW, \quad Eq_{10} = a - cW + gW^2. \tag{3.40}$$

we arrive at the conditions $k = d = h = 0$ and considering $u = -1$ we have

$$\widetilde{\chi}_1 = 2gx^2\left[4sx + (3 - s^2)y\right]/9.$$

According to Lemma 3.2 we consider two subcases: $\widetilde{\chi}_1 \neq 0$ and $\widetilde{\chi}_1 = 0$.

2.1: *The subcase* $\widetilde{\chi}_1 \neq 0$. We prove the following lemma.

Lemma 3.20. *Assume that for a system* (3.12) *the conditions* $\mathcal{D}_7 = 0$ *and* $\widetilde{\chi}_1 \neq 0$ *hold. Then this system possesses invariant lines in the configuration* $(3, 1, 1, 1)$ *if and only if the conditions* $\chi_1 = \chi_2 = \chi_3 = 0$ *are satisfied.*

Proof. As it was mentioned above the condition $\widetilde{\chi}_1 \neq 0$ implies $g \neq 0$ and according to Lemma 3.2 the infinite line $Z = 0$ of systems (3.12) with the conditions $u = -1$ has the multiplicity exactly 2. Moreover in the direction $x = 0$ we have two parallel invariant affine lines (due to $g \neq 0$).

Thus we have to examine the remaining three directions: $x \pm iy = 0$ and $sx + y = 0$.

(i) *The direction* $x + iy = 0$. We repeat the examinations of the corresponding equations (3.17) for this particular case (i.e. $u = -1$) and considering (3.18) we arrive at the equations

$$H_i|_{\{u=-1\}} = 0, \quad i = 1, 2, 3, 4.$$

Solving these equations with respect o the parameters a, b, e and f we obtain the values of these parameters given in (3.19) and (3.21) for this particular case $u = -1$. More precisely we get the following conditions:

$$\begin{aligned}
a &= \frac{1}{4(1+s^2)^2}(g - 2m - ls)\left[2c(1+s^2) - g(g - 2m - ls)\right],\\
b &= \frac{1}{4(1+s^2)^2}(l + gs - 2ms)\left[l^2 - 2(g - 2m)m + lgs + 2c(1+s^2)\right],\\
e &= \frac{1}{4(1+s^2)^2}\left[(g - 2m)s(5g - 6m + gs^2 + 2ms^2) - l^2s(5 + s^2) + 2l(3g - 4m - gs^2 + 4ms^2)\right],\\
f &= \frac{1}{4(1+s^2)^2}\left[4c(1+s^2)^2 + l^2(3-s^2) + 2ls(5g-8m+gs^2) + (g-2m)(2m-3g+gs^2-6ms^2)\right].
\end{aligned} \tag{3.41}$$

(ii) *The direction* $sx + y = 0$. Considering (3.22) and $u = -1$ for this direction we obtain

$$Eq_5 = l + gs - 2ms + (s^2 + 1)W = 0,$$

which yields $W = -(l + gs - 2ms)/(s^2 + 1)$. Then considering (3.23) we obtain

$$\begin{aligned}
Eq_8 &= \frac{g}{2(1+s^2)}\left[4(g - 2m)s + l(3 - s^2)\right],\\
Eq_{10} &= \frac{1}{4(1+s^2)^2}(2c - g^2 + 2gm + lgs + 2cs^2)\left[4(g - 2m)s + l(3 - s^2)\right].
\end{aligned} \tag{3.42}$$

Since $g \neq 0$ the condition $Eq_8 = 0$ gives $4(g - 2m)s + l(3 - s^2) = 0$, which implies also $Eq_{10} = 0$, and we consider two possibilities: $\mathcal{D}_4 \neq 0$ and $\mathcal{D}_4 = 0$.

2.1.1: *The possibility* $\mathcal{D}_4 \neq 0$. Then $s \neq 0$ and we obtain $m = (3l + 4gs - ls^2)/(8s)$. So taking into consideration (3.41) we obtained that a system possesses invariant lines in the configuration $(3,1,1,1)$ if and only if the following conditions are satisfied:

$$\begin{aligned} &u=-1, \quad k=d=h=0, \quad e=-\frac{l(21l-8gs+ls^2)}{64s},\\ &f=\frac{24lgs+64cs^2+3l^2(s^2-3)}{64s^2}, \quad a=-\frac{3l(3lg+8cs)}{64s^2},\\ &b=\frac{l\big[12lgs+32cs^2+l^2(9+s^2)\big]}{256s^2}, \quad m=\frac{(3l+4gs-ls^2)}{8s}. \end{aligned} \tag{3.43}$$

Considering the conditions provided by Lemma 3.20 for systems (3.12) with $u=-1$ we calculate:

$$\text{Coefficient}[\chi_1, xy^2]=2\big[4ks+h(s^2-3)\big]/9, \quad \text{Coefficient}[\chi_1, x^2y]=4\big[k(3s^2-1)-4hs\big]/9$$

and due to $s\neq 0$ the condition $\chi_1=0$ implies $k=h=0$. Then we obtain

$$\chi_1=2\big[l(s^2-3)-4(g-2m)s\big]/9=0$$

which gives us $m=(3l+4gs-ls^2)/(8s)$. So we deduce that forcing $\chi_1=0$ we get the conditions for the parameters k, h and m given in (3.43).

Herein calculations yield

$$\text{Coefficient}[\chi_2, y^2]=56ds(9+s^2)/3=0 \quad \Leftrightarrow \quad d=0$$

due to $s\neq 0$. Then we obtain $\chi_2=\hat{\varphi}_1x^2/6+\hat{\varphi}_2xy/s$, where

$$\begin{aligned} \hat{\varphi}_1 &=128fs^2-64es(15+s^2)-128cs^2-l(9+s^2)(33l-8gs+ls^2),\\ \hat{\varphi}_2 &=16es(s^2-3)-16fs^2(5+s^2)+16cs^2(5+s^2)+l(9+s^2)(4gs-3l+ls^2). \end{aligned}$$

We observe that the equations $\hat{\varphi}_1=0$ and $\hat{\varphi}_2=0$ are real with respect to the parameters e and f and the corresponding determinant equals $1024s^3(9+s^2)^2\neq 0$ due to $s\neq 0$. Solving these equations we get exactly the expressions for the parameters e and f given in (3.43).

Next supposing that for systems (3.12) the conditions on the parameters u,k,h,d,e,m and f given in (3.43) are satisfied, we calculate

$$\chi_3=-\frac{x^5}{1728s^2}\hat{\varphi}_3\hat{\varphi}_4^2(\hat{\varphi}_5x+\hat{\varphi}_6y),$$

where

$$\begin{aligned} \hat{\varphi}_3 &=4sx+(3-s^2)y, \quad \hat{\varphi}_4=(3s^2-1)x^2+8sxy+(3-s^2)y^2,\\ \hat{\varphi}_5 &=-256bs^2+l^3(9+s^2)+4ls(3lg+8cs),\\ \hat{\varphi}_6 &=4\big[64as^2+3l(3lg+8cs)\big]. \end{aligned}$$

We observe that due to $s\neq 0$ we have $\hat{\varphi}_3\hat{\varphi}_4\neq 0$ and therefore the condition $\chi_3=0$ is equivalent to $\hat{\varphi}_5=\hat{\varphi}_6=0$. Solving these equations with respect to the parameters a and b we get exactly the expressions given in (3.43) for these parameters. This completes the proof of Lemma 3.20 in the case $\mathcal{D}_4\neq 0$.

2.1.2: *The possibility* $\mathcal{D}_4=0$. Then $s=0$ and the condition (see (3.42))

$$Eq_8=\frac{g}{2(1+s^2)}\big[4(g-2m)s+l(3-s^2)\big]=0$$

implies $l = 0$. Then considering (3.41) in the case $s = 0$ we arrive at the following conditions:

$$\begin{gathered} u=-1, \quad k=d=h=e=b=s=l=0, \\ f=\frac{1}{4}(4c-3g^2+8gm-4m^2), \quad a=-\frac{1}{4}(g-2m)(-2c+g^2-2gm). \end{gathered} \tag{3.44}$$

Next for systems (3.12) with $u = -1$ and $s = 0$ we calculate

$$\chi_1 = -2x(3lx^2 + hx^2 + 2kxy + 3hy^2)/9$$

and evidently the condition $\chi_1 = 0$ is equivalent to $k = h = l = 0$. Then calculations yield

$$\chi_2 = -16(2d+3e)xy, \quad \text{Coefficient}[\chi_3, x^4y^7] = -4dg/3,$$
$$\text{Coefficient}[\chi_3, x^6y^5] = 2(6b-2dg-3eg+6em)/3$$

and clearly the conditions $\chi_2 = \chi_3 = 0$ implies to $d = e = b = 0$. Herein we obtain

$$\chi_3 = 2x^5y(x^2-3y^2)(\widehat{\psi}_1x^2+\widehat{\psi}_2y^2)/9,$$

where

$$\begin{aligned} \widehat{\psi}_1 &= -2a+3cg-2fg-2g^3-2cm+6g^2m-4gm^2, \\ \widehat{\psi}_2 &= 6a-cg-2fg+6cm-2g^2m+4gm^2. \end{aligned}$$

So the condition $\chi_3 = 0$ implies $\widehat{\psi}_1 = \widehat{\psi}_2 = 0$ and solving these two equations with respect to the parameters a and f we obtain exactly the expressions given in (3.44) for these parameters. This complete the proof of Lemma 3.20 as well as the proof of the statement (A_8) of the Main Theorem. □

2.2: *The subcase* $\widetilde{\chi}_1 = 0$. Then $g = 0$ and according to Lemma 3.2 we examine two possibilities: $\widetilde{\chi}_2 \neq 0$ and $\widetilde{\chi}_2 = 0$. Since for systems (3.12) with the conditions $u = -1$ and $k = d = h = g = 0$ we have

$$\widetilde{\chi}_2 = 4cx^3(sx+y)(x^2+y^2)\big[(3s^2-1)x^2+8sxy+(3-s^2)y^2\big]/3$$

we deduce that the condition $\widetilde{\chi}_2 = 0$ is equivalent to $c = 0$.

2.2.1: *The possibility* $\widetilde{\chi}_2 \neq 0$. Then by Lemma 3.2 systems (3.12) possess a triple infinite invariant line $Z = 0$ and since $\widetilde{\chi}_2 \neq 0$ implies $c \neq 0$ we deduce that in the direction $x = 0$ systems (3.12) possess only one invariant line, which is real.

So we have to examine the remaining three directions: $x \pm iy = 0$ and $sx + y = 0$.

(i) *The direction* $x + iy = 0$. We repeat the examinations of the corresponding equations (3.17) for this particular case $u = -1$ and $g = 0$ and considering (3.18) we arrive at the equations

$$H_i|_{\{u=-1,g=0\}} = 0, \quad i = 1,2,3,4.$$

Solving these equations with respect o the parameters a, b, e and f we get the values of these parameters given in (3.19) and (3.21) for this particular case $u = -1$ and $g = 0$. More precisely we get the following conditions:

$$\begin{aligned} a &= -\frac{c(2m+ls)}{2(1+s^2)}, \quad b=\frac{(l-2ms)}{4(1+s^2)^2}\Big[l^2+4m^2+2c(1+s^2)\Big], \\ e &= -\frac{1}{4(1+s^2)^2}\Big[4m^2s(-3+s^2)-8lm(-1+s^2)+l^2s(5+s^2)\Big], \\ f &= c-\frac{1}{4(1+s^2)^2}\Big[16lms+4m^2(1-3s^2)+l^2(-3+s^2)\Big]. \end{aligned} \tag{3.45}$$

(ii) *The direction* $sx+y=0$. Considering (3.22) and the conditions $u=-1$ and $g=0$ for this direction we obtain

$$Eq_5 = l-2ms+(s^2+1)W=0,$$

i.e. $W=-(l-2ms)/(s^2+1)$. Then considering (3.23) we obtain

$$Eq_8=0,\quad Eq_{10}=\frac{c}{2(1+s^2)}\big[l(3-s^2)-8ms\big]. \tag{3.46}$$

Since $c\neq 0$ the condition $Eq_{10}=0$ gives $l(3-s^2)-8ms=0$ and we consider two cases: $s\neq 0$ and $s=0$. As it was mentioned earlier these conditions are governed by the invariant polynomial $\mathcal{D}_4=2304s(9+s^2)$.

2.2.1.1: *The case* $\mathcal{D}_4\neq 0$. Then $s\neq 0$ and we obtain $m=l(3-s^2)/(8s)$ and considering the conditions $u=-1$, $k=d=h=g=0$ and (3.45) we arrive at the following lemma.

Lemma 3.21. *Assume that for a system* (3.12) *the conditions* $\mathcal{D}_7=\widetilde{\chi}_1=0$, $\widetilde{\chi}_2\neq 0$ *and* $\mathcal{D}_4\neq 0$ *hold. Then this system possesses invariant lines in the configuration* $(3,1,1,1)$ *if and only if the following conditions are satisfied:*

$$\begin{aligned}&u=-1,\ k=d=h=g=0,\quad e=-\frac{l^2(21+s^2)}{64s},\\ &f=\frac{64cs^2+3l^2(s^2-3)}{64s^2},\quad a=-\frac{3c\,l}{8s},\\ &b=\frac{l\big[32cs^2+l^2(9+s^2)\big]}{256s^2},\quad m=\frac{l(3-s^2)}{8s}.\end{aligned} \tag{3.47}$$

Next we determine the invariant conditions equivalent to those provided by the above lemma. More exactly we prove the following lemma.

Lemma 3.22. *Assume that for a system* (3.12) *the conditions* $\mathcal{D}_7=\widetilde{\chi}_1=0$, $\widetilde{\chi}_2\neq 0$ *and* $\mathcal{D}_4\neq 0$ *hold. Then this system possesses invariant lines in the configuration* $(3,1,1,1)$ *if and only if the conditions* $\chi_1=\chi_3=\chi_6=0$ *are satisfied.*

Proof. Clearly the condition $\mathcal{D}_7=0$ imply $u=-1$. Then for systems (3.12) we calculate

$$\text{Coefficient}[\widetilde{\chi}_1,xy^2]=-(8ks)/3$$

and clearly due to $\mathcal{D}_4\neq 0$ (i.e. $s\neq 0$) the condition $\widetilde{\chi}_1=0$ implies $k=0$. Then we calculate

$$\widetilde{\chi}_1=2x^2\big[2(h+2gs-3hs^2)x+(3g-8hs-gs^2)y\big]/9=0$$

and we claim that the condition $\widetilde{\chi}_1=0$ implies $g=h=0$. Indeed assuming $h+2gs-3hs^2=0$ we get $g=\frac{h(3s^2-1)}{2s}$ and then

$$\widetilde{\chi}_1=-\frac{h(s^2+1)^2}{2s}=0\ \Rightarrow\ h=0\ \Rightarrow\ g=0$$

and this completes the proof of our claim.

Thus the condition $\widetilde{\chi}_1=0$ for systems (3.12) with $u=-1$ gives us $k=h=g=0$. Then calculations yield

$$\text{Coefficient}[\chi_6,x^2y^6]=10ds(123+23s^2)/3=0\ \Rightarrow\ d=0$$

due to $s \neq 0$. Herein we obtain

$$\begin{aligned}
\text{Coefficient}[\chi_6, x^3y^5] &= 110\big[e(s^2-3) - fs(5+s^2) - (6lm - 5cs - 4m^2s - cs^3)\big] \\
\text{Coefficient}[\chi_6, x^4y^4] &= \frac{10}{3}\big[es(7s^2-597) - fs^2(131+39s^2) - 216l^2 - 42lms \\
&\quad + 131cs^2 + 124m^2s^2 + 39cs^4\big]
\end{aligned}$$

and we observe that the above polynomials are linear with respect to the parameters e and f with the corresponding determinant $-35200s^2(9+s^2)^2/3 \neq 0$. So forcing these polynomials to vanish we get

$$\begin{aligned}
e &= \frac{1}{4s(9+s^2)^2}(2ms-3l)(45l+6ms+9ls^2+2ms^3), \\
f &= \frac{1}{4s^2(9+s^2)^2}\big[4cs^2(9+s^2)^2 - 3(3l-2ms)(42ms-9l+3ls^2+2ms^3)\big],
\end{aligned} \tag{3.48}$$

and then calculations yield

$$\begin{aligned}
\chi_6 = &-\frac{20}{s^2(9+s^2)^2}(8ms-3l+ls^2)^2x^5\big[16s^2(6+s^2)x^3 + s(63+30s^2-s^4)x^2y \\
&+ 9(14s^2-3+s^4)xy^2 + 12s(9+s^2)y^3\big].
\end{aligned}$$

Due to $s \neq 0$ the condition $\chi_6 = 0$ evidently implies $8ms - 3l + ls^2 = 0$, and considering (3.48) we determine:

$$e = -\frac{l^2(21+s^2)}{64s}, \quad f = \frac{64cs^2 + 3l^2(s^2-3)}{64s^2}, \quad m = \frac{l(3-s^2)}{8s}.$$

So we obtain exactly the expressions for the parameters e, f and m given in (3.47).

Next considering that all the conditions from (3.47) are satisfied except the conditions for the parameters a and b we calculate:

$$\begin{aligned}
\chi_3 = &-\frac{1}{1728s^2}x^5(4sx+3y-s^2y)\big[(3s^2-1)x^2 + 8sxy + (3-s^2)y^2\big]^2 \\
&\times \big[(9l^3 - 256bs^2 + 32cls^2 + l^3s^2)x + 32s(3cl+8as)y\big].
\end{aligned}$$

Therefore due to $s \neq 0$ it is simple to determine that the condition $\chi_3 = 0$ gives us exactly the expressions for the parameters a and b given in (3.47). This completes the proof of Lemma 3.22. □

2.2.1.2: *The case* $\mathcal{D}_4 = 0$. Then $s = 0$ and the condition $Eq_{10} = 0$ (see (3.46)) gives us $Eq_{10} = 3cl/2 = 0$ and due to $c \neq 0$ this implies $l = 0$. In this case considering the conditions $u = -1$, $k = d = h = g = s = 0$ and (3.45) we arrive at the following lemma.

Lemma 3.23. *Assume that for a system* (3.12) *the conditions* $\mathcal{D}_7 = \widetilde{\chi}_1 = 0$, $\widetilde{\chi}_2 \neq 0$ *and* $\mathcal{D}_4 = 0$ *hold. Then this system possesses invariant lines in the configuration* $(3,1,1,1)$ *if and only if the following conditions are satisfied:*

$$u = -1,\ s = 0, \quad k = d = h = g = l = b = e = 0, \quad a = -cm, \quad f = c - m^2. \tag{3.49}$$

Now we determine the invariant conditions equivalent to those provided by the above lemma. We claim that Lemma 3.22 which was proved for $\mathcal{D}_4 \neq 0$ is also true in the case $\mathcal{D}_4 = 0$.

Indeed it is clear that the conditions $\mathcal{D}_7 = \mathcal{D}_4 = 0$ imply $u = -1$ and $s = 0$. Then for systems (3.12) we calculate

$$\widetilde{\chi}_1 = 2\big[2hx^3 + (3g + 2k)x^2y - 3ky^3\big]/9$$

and evidently the condition $\widetilde{\chi}_1 = 0$ implies $k = h = g = 0$. Then calculations yield

$$\chi_1 = -2lx^3/3 = 0 \quad \Rightarrow \quad l = 0$$

and we obtain

$$\chi_6 = -10xy\big[(10d-129e)x^6+72(c-f-m^2)x^5y+(25d+42e)x^4y^2-3(16d-33e)x^2y^4+9dy^6\big]/3.$$

It is clear that the condition $\chi_6 = 0$ implies $d = e = 0$ and $f = c - m^2$ and we observe that all the conditions given in (3.49) are obtained except for the parameters a and b.

Finally we calculate

$$\chi_3 = 4\big[bx - (a + cm)y\big]x^5y(x^2 - 3y^2)^2/9$$

and evidently the condition $\chi_3 = 0$ gives us $b = 0$ and $a = -cm$. This complete the proof of the statement (A_9) of the Main Theorem.

2.2.2: *The possibility* $\widetilde{\chi}_2 = 0$. We prove the following lemma.

Lemma 3.24. *If for a system* (3.12) *the conditions* $\mathcal{D}_7 = \widetilde{\chi}_1 = \widetilde{\chi}_2 = 0$ *hold then this system could hot have a configuration of the type* $(3,1,1,1)$.

Proof. Assume that for a system (3.12) the conditions provided by this lemma are fulfilled. As we already know the condition $\mathcal{D}_7 = 0$ implies $u = -1$ and by Lemma 3.4 we could not have a triplet of parallel invariant lines in the direction $y = 0$. Since for the direction $x = 0$ we have the equations (see (3.40))

$$Eq_7 = k, \quad Eq_9 = d - 2hW, \quad Eq_{10} = a - cW + gW^2.$$

we arrive at the conditions $k = d = h = 0$ and considering $u = -1$ we have

$$\widetilde{\chi}_1 = 2gx^2\big[4sx + (3 - s^2)y\big]/9.$$

Therefore the condition $\widetilde{\chi}_1 = 0$ implies $g = 0$ and evaluating the invariant polynomial $\widetilde{\chi}_2$ for systems (3.12) with $u = -1$ and $k = d = h = g = 0$ we get

$$\widetilde{\chi}_2 = 4cx^3(sx + y)(x^2 + y^2)\big[(3s^2 - 1)x^2 + 8sxy + (3 - s^2)y^2\big]/3.$$

It is clear that the condition $\widetilde{\chi}_2 = 0$ implies $c = 0$ and hence we arrive at the family of systems

$$\dot{x} = a, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy - sx^3 - x^2y - sxy^2 - y^3. \tag{3.50}$$

Suppose the contrary, that these systems possess the configuration invariant lines of the type $(3,1,1,1)$. Therefore we must have two complex invariant lines: one in the direction $x + iy = 0$ and another in the direction $x - iy = 0$.

Thus considering the equations (2.5) and Remark 2.12 for the direction $x + iy = 0$ we obtain $U = 1$, $V = i$ and

$$Eq_7 = 2m - il + 2(1 - is)W, \quad Eq_9 = e + if - 2(l + im)W - (i + 3s)W^2,$$
$$Eq_{10} = a + ib - ieW + ilW^2 + isW^3.$$

We calculate $Res_W(Eq_7, Eq_9) = \widehat{H}_1 + i\widehat{H}_2$ where

$$\widehat{H}_1 = 8lm - s(l^2 - 8f + 4m^2) - 4e(s^2 - 1),$$
$$\widehat{H}_2 = 4m^2 - 3l^2 - 8es - 4f(s^2 - 1).$$

So solving the system of equations $\widehat{H}_1 = 0$ and $\widehat{H}_2 = 0$ which are linear with respect to the parameters e and f we obtain:

$$e = -\frac{4m^2s(s^2 - 3) - 8lm(s^2 - 1) + l^2s(5 + s^2)}{4(1 + s^2)^2},$$
$$f = -\frac{16lms + 4m^2(1 - 3s^2) + l^2(s^2 - 3)}{4(1 + s^2)^2}.$$

Then calculations yield

$$Res_W(Eq_7, Eq_{10}) = \frac{(s + i)^3}{(1 + s^2)^2}\Big[(l^2 + 4m^2)(l - 2ms) - 4b(1 + s^2)^2 + 4ia(1 + s^2)^2\Big] = 0$$

and since the parameters of the systems are real this condition implies $a = 0$. However in this case systems (3.50) become degenerate and this completes the proof of the Lemma 3.24. □

Since all the possibilities for cases provided by the statement (A) of the Main Theorem are examined we conclude that this statement is proved.

As we mentioned earlier (see page 24) we have to prove the following lemma.

Lemma 3.25. *None of the sets of the conditions (A_1)–(A_9) could be satisfied for systems (3.8).*

Proof. For systems (3.8) calculations yield:

$$\mathcal{D}_4 = 0, \quad \mathcal{D}_6 = -4, \quad \mathcal{D}_7 = 4, \quad \mathcal{D}_8 = 0$$

and comparing with the sets of the conditions provided by the statement (A) of the Main Theorem we conclude that all the sets of the conditions (A_i) for $i = 1, 2, \ldots, 5, 7, \ldots, 9$ could not be satisfied for systems (3.8). It remains to prove that set of of the conditions (A_6) could also not be satisfied for this family of systems.

Indeed for systems (3.8) we have

$$\chi_1 = \frac{1}{4}\big[-(l + 2h)x^3 + 3(g - k)x^2y + 3(cl + 2h)xy^2 + (k - g)y^3\big]$$

and therefore the condition $\chi_1 = 0$ provided by the statement (A_6) gives $l = -2h$ and $g = k$. Then we calculate

$$\chi_8 = -240(h^2 + k^2)(x^2 + y^2), \qquad \chi_{15} = -(kx - hy)(x^2 + y^2)^2/4$$

and since according to the statement (A_6) we must have $\chi_8 = 0$ and $\chi_{15} \neq 0$ we evidently get a contradiction and this completes the proof of the lemma. □

3.2 The proof of the statement (B) of the Main Theorem

In this section we examine step by step each one of the statements (A_i) $(i = 1, \ldots, 9)$ of Main Theorem and determine the possible configurations of invariant lines, correspondingly. Moreover we find out necessary and sufficient affine invariant conditions for the realization of each one of the configurations found.

3.2.1 The statement (A_1)

It was shown in the proof of the statement (A) of the Main Theorem that the affine invariant conditions provided by the statement (A_1) for the family of systems (3.12) lead to the conditions (3.26).

It is not too difficult to determine that in this cases we arrive at the family of systems

$$\begin{aligned}
\dot{x} = &\Big[x - \frac{3}{\varkappa}(g-2m)\Big]\Big[c + \frac{6}{\varkappa^2}(g-2m)\left(gs^2 - 3g - 3mu - 3m\right)\\
&+ \frac{2}{\varkappa}(gs^2 - 3g - 3m - 3mu)\,x + (1+u)x^2\Big] \equiv\; L_1(x)L_{2,3}(x),\\
\dot{y} = &\frac{s}{\varkappa^2}(g-2m)\Big[c\varkappa^2 + 2(g-2m)\big[4gs^2 - m(27+2s^2+9u)\big]\Big]\\
&+ \frac{s}{\varkappa^2}(g-2m)\big[g\left(s^2-27\right) + 2m\left(s^2-3u+18\right)\big]\,x + \frac{s}{\varkappa}(g-2m)(9+u)\,x^2\\
&+ \Big[c + \frac{3}{\varkappa^2}(g-2m)\big[3g\left(s^2-3\right) - 2m\left(s^2+3u\right)\big]\Big]\,y + 2mxy - sx^3 + ux^2y - sxy^2 - y^3,
\end{aligned} \tag{3.51}$$

where $\varkappa = 2s^2 - 3(u+3) \neq 0$ and $s \neq 0$ or $s = 0$ depending on the value of the invariant polynomial $\mathcal{D}_4$.

We need to determine if the two lines defined by the equation $L_{2,3} = 0$ are real or complex and in the case when they are real, if one of them coincides with the invariant line $L_1 = 0$. So we calculate

$$\text{Discrim}\,[L_{2,3}, x] = -\frac{4}{\varkappa^2}\lambda(c,g,m,s,u),$$
$$Res_x(L_1, L_{2,3}) = \frac{1}{\varkappa^2}\mu(c,g,m,s,u)$$

where

$$\begin{aligned}
&\lambda = c(1+u)\varkappa^2 - \big[g(s^2-3) - 3m(1+u)\big]\big[g(s^2-9-6u) + 9m(1+u)\big],\\
&\mu = c\varkappa^2 + 3(g-2m)\big[g(4s^2-9+3u) - 18m(1+u)\big].
\end{aligned} \tag{3.52}$$

We observe that

$$\text{sign}\left(\text{Discrim}\,[L_{2,3}, x]\right) = -\text{sign}\,(\lambda),$$

i.e. the invariant lines $L_{2,3} = 0$ are real (respectively complex; coinciding) if $\lambda < 0$ (respectively $\lambda > 0$; $\lambda = 0$). The invariant line $L_1 = 0$ coincides with one of the lines $L_{2,3} = 0$ if and only if $\mu = 0$.

Evaluating for systems (3.51) the invariant polynomials ζ_1 and ζ_2 we obtain:

$$\zeta_1 = \frac{80}{3\varkappa^2}(s^2+3u)^2x^2\lambda, \quad \zeta_2 = 8\mu, \quad \mathcal{D}_5 = 4(s^2+3u)/9$$

and therefore the condition $\zeta_2 = 0$ is equivalent to $\mu = 0$. On the other hand we have $\text{sign}\,(\lambda) = \text{sign}\,(\zeta_1)$ only if $\mathcal{D}_5 \neq 0$. So in what follows we examine two possibilities: $\mathcal{D}_5 \neq 0$ and $\mathcal{D}_5 = 0$.

1: The possibility $\mathcal{D}_5 \neq 0$**.** Then the sign of λ is governed by the invariant polynomial ζ_1 and we prove the next proposition.

Proposition 3.26. *Assume that for a system* (3.51) *the conditions* $\mathcal{D}_6\mathcal{D}_7\mathcal{D}_8 \neq 0$ *and* $\mathcal{D}_5 \neq 0$ *hold. Then this system possesses one of the configurations of invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$\mathcal{D}_4 \neq 0, \zeta_1 < 0, \zeta_2 \neq 0, \zeta_3 < 0, \mathcal{D}_7 < 0$	$\Leftrightarrow$	*Config. 7.1b;*
$\mathcal{D}_4 \neq 0, \zeta_1 < 0, \zeta_2 \neq 0, \zeta_3 < 0, \mathcal{D}_7 > 0$	$\Leftrightarrow$	*Config. 7.2b;*
$\mathcal{D}_4 \neq 0, \zeta_1 < 0, \zeta_2 \neq 0, \zeta_3 > 0, \mathcal{D}_7 < 0$	$\Leftrightarrow$	*Config. 7.3b;*
$\mathcal{D}_4 \neq 0, \zeta_1 < 0, \zeta_2 \neq 0, \zeta_3 > 0, \mathcal{D}_7 > 0$	$\Leftrightarrow$	*Config. 7.4b;*
$\mathcal{D}_4 \neq 0, \zeta_1 < 0, \zeta_2 = 0$	$\Leftrightarrow$	*Config. 7.5b;*
$\mathcal{D}_4 \neq 0, \zeta_1 > 0, \zeta_4 \neq 0, \mathcal{D}_7 < 0$	$\Leftrightarrow$	*Config. 7.6b;*
$\mathcal{D}_4 \neq 0, \zeta_1 > 0, \zeta_4 \neq 0, \mathcal{D}_7 > 0$	$\Leftrightarrow$	*Config. 7.7b;*
$\mathcal{D}_4 \neq 0, \zeta_1 > 0, \zeta_4 = 0, \mathcal{D}_7 < 0$	$\Leftrightarrow$	*Config. 7.8b;*
$\mathcal{D}_4 \neq 0, \zeta_1 > 0, \zeta_4 = 0, \mathcal{D}_7 > 0$	$\Leftrightarrow$	*Config. 7.9b;*
$\mathcal{D}_4 \neq 0, \zeta_1 = 0, \zeta_2 \neq 0, \mathcal{D}_7 < 0$	$\Leftrightarrow$	*Config. 7.10b;*
$\mathcal{D}_4 \neq 0, \zeta_1 = 0, \zeta_2 \neq 0, \mathcal{D}_7 > 0$	$\Leftrightarrow$	*Config. 7.11b;*
$\mathcal{D}_4 \neq 0, \zeta_1 = 0, \zeta_2 = 0$	$\Leftrightarrow$	*Config. 7.12b;*
$\mathcal{D}_4 = 0, \zeta_1 < 0, \zeta_2 \neq 0, \zeta_5 < 0, \mathcal{D}_7 < 0$	$\Leftrightarrow$	*Config. 7.13b;*
$\mathcal{D}_4 = 0, \zeta_1 < 0, \zeta_2 \neq 0, \zeta_5 < 0, \mathcal{D}_7 > 0$	$\Leftrightarrow$	*Config. 7.14b;*
$\mathcal{D}_4 = 0, \zeta_1 < 0, \zeta_2 \neq 0, \zeta_5 > 0, \mathcal{D}_7 < 0$	$\Leftrightarrow$	*Config. 7.15b;*
$\mathcal{D}_4 = 0, \zeta_1 < 0, \zeta_2 \neq 0, \zeta_5 > 0, \mathcal{D}_7 > 0$	$\Leftrightarrow$	*Config. 7.16b;*
$\mathcal{D}_4 = 0, \zeta_1 < 0, \zeta_2 = 0$	$\Leftrightarrow$	*Config. 7.17b;*
$\mathcal{D}_4 = 0, \zeta_1 > 0, \zeta_4 \neq 0, \mathcal{D}_7 < 0$	$\Leftrightarrow$	*Config. 7.18b;*
$\mathcal{D}_4 = 0, \zeta_1 > 0, \zeta_4 \neq 0, \mathcal{D}_7 > 0$	$\Leftrightarrow$	*Config. 7.19b;*
$\mathcal{D}_4 = 0, \zeta_1 > 0, \zeta_4 = 0, \mathcal{D}_7 < 0$	$\Leftrightarrow$	*Config. 7.20b;*
$\mathcal{D}_4 = 0, \zeta_1 > 0, \zeta_4 = 0, \mathcal{D}_7 > 0$	$\Leftrightarrow$	*Config. 7.21b;*
$\mathcal{D}_4 = 0, \zeta_1 = 0, \zeta_2 \neq 0, \mathcal{D}_7 < 0$	$\Leftrightarrow$	*Config. 7.22b;*
$\mathcal{D}_4 = 0, \zeta_1 = 0, \zeta_2 \neq 0, \mathcal{D}_7 > 0$	$\Leftrightarrow$	*Config. 7.23b;*
$\mathcal{D}_4 = 0, \zeta_1 = 0, \zeta_2 = 0$	$\Leftrightarrow$	*Config. 7.24b.*

Proof. Following the conditions provided by the above proposition we consider two cases: $\mathcal{D}_4 \neq 0$ and $\mathcal{D}_4 = 0$.

1.1: The case $\mathcal{D}_4 \neq 0$**.** We examine three subcases: $\zeta_1 < 0$, $\zeta_1 > 0$ and $\zeta_1 = 0$.

a) *The subcase* $\zeta_1 < 0$ *(*$\mathcal{D}_5 \neq 0$*).* Then $\lambda < 0$ and we may use a new parameter v setting $\lambda = -v^2 < 0$. Since we have $\mathcal{D}_6\mathcal{D}_7 \neq 0$ (i.e. $(1+u)\varkappa \neq 0$) considering (3.52) we obtain

$$c = \frac{1}{(1+u)\varkappa^2}\left[g^2(s^2-3)(s^2-6u-9) - 27m^2(1+u)^2 + 6gm(1+u)(s^2+3u) - v^2\right]. \tag{3.53}$$

This leads to the following family of systems

$$\dot{x} = (1+u)\Big[x - \frac{3(g-2m)}{\varkappa}\Big]\Big[x + \frac{gs^2-3g-3m-3mu-v}{\varkappa(1+u)}\Big]\times$$
$$\Big[x + \frac{gs^2-3g-3m-3mu+v}{(1+u)}\Big],$$

$$\dot{y} = -\frac{(g-2m)s}{(1+u)\varkappa^3}\Big[(2gm(27+7s^2)(1+u) - m^2(1+u)(81+8s^2+9u)$$
$$- g^2\big[s^4+2s^2(u-2)+9(3+2u)\big]+v^2)\Big] + \frac{(g-2m)s}{\varkappa^2}(36m-27g+gs^2+2ms^2-6mu)\,x$$
$$+\frac{1}{\varkappa^2(1+u)}\big[g^2(s^2-3)(s^2+3u)-18gm(s^2-3)(1+u)+3m^2(1+u)(4s^2-9+3u)-v^2\big]\,y$$
$$+\frac{s}{\varkappa}(g-2m)(9+u)\,x^2+2mxy-sx^3+ux^2y-sxy^2-y^3. \tag{3.54}$$

On the other hand for the value of c given in (3.53) we calculate

$$\mu = \frac{1}{1+u}(\gamma^2-v^2), \quad \gamma = g(s^2+3u)-9m(1+u) \tag{3.55}$$

and since the condition $\mu = 0$ leads to the coalescence of two invariant lines of the triplet, we examine two possibilities: $\zeta_2 \neq 0$ and $\zeta_2 = 0$.

a.1) *The possibility $\zeta_2 \neq 0$.* Then $\mu \neq 0$, i.e. $(\gamma - v)(\gamma + v) \neq 0$ and setting a new parameter $a = \frac{\gamma+v}{\gamma-v}$ we observe that $a-1 \neq 0$. Indeed calculation yields: $a-1 = \frac{2v}{\gamma-v} \neq 0$ due to $v \neq 0$. So from the relation $a = \frac{\gamma+v}{\gamma-v}$ we can determine the parameter m as follows:

$$m = \frac{g(a-1)(s^2+3u)-(a+1)v}{9(a-1)(1+u)}.$$

Then we can apply to systems (3.54) the following transformation (we recall that $\varkappa = 2s^2 - 3(u+3) \neq 0$):

$$x_1 = \alpha x - \frac{\nu}{6v}, \quad y_1 = \alpha y + \frac{s\nu}{18v}, \quad t_1 = \frac{t}{\alpha^2},$$
$$\alpha = -\frac{(a-1)(1+u)\varkappa}{2v}, \quad \nu = (a-1)g\varkappa - 2(1+a)v.$$

This transformation brings these systems to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned}\dot{x} &= (1+u)x(x-1)(x-a),\\ \dot{y} &= (1+u)\big[a-x(a+1)\big]y - sx^3+ux^2y-sxy^2-y^3,\end{aligned} \tag{3.56}$$

for which the parameters s and u satisfy the conditions (3.25).

We detect that systems (3.56) possess six distinct invariant affine straight lines

$$L_1: x=0,\ L_2: x=1,\ L_3: x=a,\ L_4: y=-sx, \quad L_{5,6}: y=\pm ix \tag{3.57}$$

and the following nine finite singularities:

$$M_1(0,0),\ M_{2,3}\big(0,\pm\sqrt{a(1+u)}\,\big),\ M_4(1,-s),\ M_{5,6}(1,\pm i),\ M_7(a,-as),\ M_{8,9}(a,\pm ia). \tag{3.58}$$

For systems (3.56) calculations yield

$$\zeta_1 = -\frac{20}{3}(a-1)^2(u+1)^2x^2\left(s^2+3u\right)^2, \quad \zeta_2 = 8a\varkappa^2(1+u),$$
$$\mathcal{D}_5 = \frac{4}{9}\left(s^2+3u\right), \quad \mathcal{D}_6 = \frac{4}{9}\big[2s^2-3(u+3)\big] \equiv \frac{4}{9}\varkappa, \quad \mathcal{D}_7 = 4(1+u).$$

Since the conditions $\mathcal{D}_5 \neq 0$ and $\mathcal{D}_6\mathcal{D}_7 \neq 0$ hold we obtain that the conditions $\zeta_1 < 0$ and $\zeta_2 \neq 0$ imply for the parameter a of systems (3.56) the condition $a(a-1) \neq 0$.

We observe that the singular points M_2 and M_3 could be real (if $a(1+u) > 0$) or complex (if $a(1+u) < 0$). On the other hand due to the condition $a(a-1) \neq 0$ we conclude that the line $L_3 : x = a$ could neither coincide with L_1 (for $a = 0$) nor with L_2 (for $a = 1$). So considering the condition $a(a-1) \neq 0$ we examine the following two cases: $a(1+u) > 0$ and $a(1+u) < 0$.

a.1.1) *The case* $a(1+u) > 0$. So the singular points $M_{2,3}\big(0, \pm\sqrt{a(1+u)}\,\big)$ are real and we observe that all the singularities (3.58) are located at the intersection of the invariant lines, except for $M_{2,3}\big(0, \pm\sqrt{a(1+u)}\,\big)$ which lie on the line $x = 0$ and are symmetric with respect to the origin of coordinates. Moreover fixing the position of all invariant lines and moving only the singularities $M_{2,3}$ we could not obtain new configurations. So the distinct configurations depend on the position of the invariant lines.

We deduce that only two lines are not fixed, and namely: $L_3 : x = a$, $L_4 : y = -sx$. Moreover four of them (i.e. L_1, L_4 and $L_{5,6}$) intersect at the same point $(0,0)$. Since this point lies on the line L_1, considering the triplet of parallel invariant lines (L_1, L_2 and L_3) we deduce that we could get different configurations depending on the position of the line $L_3 = a$. More precisely, if $a < 0$ then L_3 is located on the left of L_1 and if $a > 0$ then L_3 is located on the right of L_1.

Regarding the invariant line $L_4 : y = -sx$ we make the following remark.

Remark 3.27. Considering our Convention on page 8 we deduce that the invariant line $y = -sx$ coincides with the projection of the complex invariant lines $y = \pm ix$ on the plan (x, y) if and only if $s = 0$.

Since in the case under consideration we have $s \neq 0$ it is not too difficult to determine that systems (3.56) possess the configuration of invariant lines *Config. 7.1b* if $a < 0$ and *Config. 7.2b* if $a > 0$.

a.1.2) *The case* $a(1+u) < 0$. Then the singular points $M_{2,3}\big(0, \pm\sqrt{a(1+u)}\,\big)$ are complex and on the invariant line L_1 there are no real singularities except $M_1(0,0)$. So applying the same argument as in the previous case above we obtain the following two configurations of invariant lines for systems (3.56): *Config. 7.3b* if $a > 0$ and *Config. 7.4b* if $a < 0$.

Thus we obtain that in the case $\zeta_1 < 0$ and $\zeta_2 \neq 0$ systems (3.56) could possess only four distinct configurations *Config. 7.1b - Config. 7.4b*.

Next we determine the corresponding invariant conditions for distinguishing these configurations of invariant lines. We evaluate for systems (3.56) the next invariant polynomials:

$$\zeta_3 = -2(a-1)^2 as^2(9+s^2)^2(1+u)^3/81, \quad \mathcal{D}_7 = 4(1+u) \neq 0, \quad \mathcal{D}_4 = 2304s(9+s^2) \neq 0$$

and due to $a(a-1)(u+1)s \neq 0$ we have $\operatorname{sign}(\zeta_3) = -\operatorname{sign}(a(u+1))$ and $\operatorname{sign}(\mathcal{D}_7) = \operatorname{sign}(u+1)$.

Considering the conditions on the parameters a, u determined above which define the configurations *Config. 7.1b - Config. 7.4b* for systems (3.56) in the case $\zeta_1 < 0$ and $\zeta_2 \neq 0$ as well as the expressions for the invariant polynomials given above we obtain the following affine invariant conditions for distinguishing these configurations (as well as the corresponding examples of their realization):

$$\begin{array}{lcll}
\zeta_3 < 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.1b} & (a=-1,\, u=-2,\, s=1); \\
\zeta_3 < 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.2b} & (a=2,\, u=2,\, s=1); \\
\zeta_3 > 0,\, \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.3b} & (a=2,\, u=-2,\, s=1); \\
\zeta_3 > 0,\, \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.4b} & (a=-1,\, u=1,\, s=1).
\end{array}$$

a.2) *The possibility* $\zeta_2 = 0$. Then $\mu = 0$ and considering (3.55) we get $(\gamma - v)(\gamma + v) = 0$. We may assume $\gamma - v = 0$ due to change $v \to -v$ and setting $v = \gamma \neq 0$ in systems (3.54) we arrive at the family of systems

$$\begin{aligned}
\dot{x} =& (1+u)\Big[x - \frac{3(g-2m)}{\varkappa}\Big]^2\Big[x - \frac{12m(1+u) - g(2s^2+3u-3)}{\varkappa(1+u)}\Big],\\
\dot{y} =& -\frac{(g-2m)s}{(1+u)\varkappa^3}\Big[(2gm(27+7s^2)(1+u) - m^2(1+u)(81+8s^2+9u)\\
&- g^2\big[s^4+2s^2(u-2)+9(3+2u)\big]+v^2)\Big] + \frac{(g-2m)s}{\varkappa^2}(36m-27g+gs^2+2ms^2-6mu)\,x\\
&+\frac{1}{\varkappa^2(1+u)}\big[g^2(s^2-3)(s^2+3u) - 18gm(s^2-3)(1+u)+3m^2(1+u)(4s^2-9+3u)-v^2\big]\,y\\
&+\frac{s}{\varkappa}(g-2m)(9+u)\,x^2 + 2mxy - sx^3 + ux^2y - sxy^2 - y^3,
\end{aligned}$$

where $\varkappa = 2s^2 - 3(u+3) \neq 0$. Since $\gamma = g(s^2+3u) - 9m(1+u) \neq 0$ then applying the transformation

$$x_1 = \alpha x + \frac{3(g-2m)(1+u)}{2\gamma}, \quad y_1 = \alpha y - \frac{s(g-2m)(1+u)}{2\gamma},$$
$$t_1 = \frac{t}{\alpha^2}, \quad \alpha = \frac{\varkappa(1+u)}{2\gamma}$$

we obtain the following 2-parameter family of systems:

$$\begin{aligned}
\dot{x} &= (1+u)x^2(x-1),\\
\dot{y} &= -(1+u)xy - sx^3 + ux^2y - sxy^2 - y^3.
\end{aligned} \tag{3.59}$$

We observe that this family of systems is a subfamily of (3.56) defined by the condition $a = 0$. So the above systems possess invariant lines (3.57) among which only five are distinct, because the line $L_4 \equiv L_1 : x = 0$ is double. These systems have only 4 distinct finite singularities because setting $a = 0$ in (3.58) we obtain that the five singularities $M_{2,3}(0, \pm\sqrt{a(1+u)}\,)$, $M_7(a, -as)$ and $M_{8,9}(a, \pm ia)$ coalesce with the singular point $M_1(0,0)$. As a result we get a singular point of multiplicity 6 which is a point of intersection of four invariant lines: L_1, L_4 and $L_{5,6}$. Therefore considering Remark 3.27 due to the condition $s \neq 0$ we obtain the unique configuration of singularities given by *Config. 7.5b*.

b) *The subcase* $\zeta_1 > 0$ $(\mathcal{D}_5 \neq 0)$. Then $\lambda > 0$ and we set $\lambda = v^2 > 0$. Since $(1+u)(9-2s^2+3u) \neq 0$ we obtain

$$c = \frac{1}{(1+u)(9-2s^2+3u)^2}\big[g^2(s^2-3)(s^2-6u-9) - 27m^2(1+u)^2 + 6gm(1+u)(s^2+3u) + v^2\big].$$

This leads to the following family of systems

$$\begin{aligned}
\dot{x} =& (1+u)\Big[x - \frac{3(g-2m)}{\varkappa}\Big]\Big[\frac{\big[g(s^2-3) - 3m(1+u)\big]^2 + v^2}{\varkappa^2(1+u)^2}\\
&+ \frac{2(gs^2-3g-3m-3mu)}{\varkappa(1+u)}\,x + x^2\Big],
\end{aligned}$$

$$\dot{y} = \frac{(g-2m)s}{\varkappa^3(1+u)}\big[m^2(1+u)(81+8s^2+9u) - 2gm(27+7s^2)(1+u) + g^2(s^4+2s^2(u-2)$$
$$+9(3+2u)) + v^2\big] + \frac{(g-2m)s}{\varkappa^2}(36m-27g+36m+gs^2+2ms^2-6mu)\,x$$
$$+\frac{1}{\varkappa^2(1+u)}\big[g^2(s^2-3)(s^2+3u)+3m^2(1+u)(4s^2+3u-9)$$
$$-18gm(s^2-3)(1+u)+v^2\big]\,y + \frac{s}{\varkappa}(g-2m)(9+u)\,x^2 + 2mxy - sx^3 + ux^2y - sxy^2 - y^3. \tag{3.60}$$

In order to simplify these systems we need to use a transformation which depends on the condition: either $\gamma \neq 0$ or $\gamma = 0$ (we recall that $\gamma = g(s^2+3u) - 9m(1+u)$).

On the other hand for systems (3.60) we calculate:

$$\zeta_4 = \frac{\gamma^2}{\varkappa^2}\big[(4s^2-13)x^2 - 2sxy - 3y^2\big]$$

and therefore the condition $\gamma = 0$ is equivalent to $\zeta_4 = 0$.

b.1) *The possibility* $\zeta_4 \neq 0$. Then $\gamma \neq 0$ and setting a new parameter $a = \frac{v}{\gamma} \neq 0$ we can determine the parameter m as follows:

$$m = \frac{ags^2 + 3agu - v}{9a(1+u)}.$$

Then we can apply the following transformation

$$x_1 = \alpha x - \frac{\nu}{3v}, \quad y_1 = \alpha y + \frac{s\nu}{9v}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = -\frac{a(1+u)\varkappa}{v}, \quad \nu = ag\varkappa - 2v,$$

which brings systems (3.60) to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned} \dot{x} &= (1+u)x\big[(x-1)^2 + a^2\big],\\ \dot{y} &= (1+a^2)(1+u)y - 2(1+u)xy - sx^3 + ux^2y - sxy^2 - y^3. \end{aligned} \tag{3.61}$$

For the above systems calculations yield

$$\zeta_1 = \frac{80}{3}a^2(1+u)^2(s^2+3u)^2x^2, \quad \mathcal{D}_7 = 4(1+u)$$

and clearly the condition $\zeta_1 > 0$ implies $a \neq 0$ and we must have $u+1 \neq 0$ (i.e. $\mathcal{D}_7 \neq 0$), otherwise we get degenerate systems.

We determine that systems (3.61) possess six distinct invariant affine straight lines

$$L_1 : x = 0,\ L_{2,3} : x = 1 \pm ia,\ L_4 : y = -sx, \quad L_{5,6} : y = \pm ix$$

and the following nine finite singularities:

$$\begin{aligned} &M_1(0,0),\ M_{2,3}\big(0, \pm\sqrt{(1+a^2)(1+u)}\ \big),\ M_{4,5}\big(1+ia, \pm(i-a)\big),\\ &M_{6,7}\big(1-ia, \pm(i+a)\big),\ M_{8,9}\big(1 \pm ia, -s \mp is\big). \end{aligned} \tag{3.62}$$

We observe that the singular points M_2 and M_3 could be real (if $1+u > 0$) or complex (if $1+u < 0$), but they could not coincide due to $1+u \neq 0$. So we consider two cases: $1+u < 0$ and $1+u > 0$, taking into account that sign $(1+u) = $ sign $(\mathcal{D}_7)$.

b.1.1) *The case* $\mathcal{D}_7 < 0$. Then $1+u<0$ and therefore the singular points $M_{2,3}$ are complex and the unique real finite singular point of systems (3.61) is $M_1(0,0)$ which is the point of intersection of four invariant lines: L_1, L_4 and $L_{5,6}$. As a result taking into consideration Remark 3.27 due to the condition $s \neq 0$ we obtain the unique configuration *Config. 7.6b*.

b.1.2) *The case* $\mathcal{D}_7 > 0$. Then $1+u>0$ and hence the singular points $M_{2,3}$ are real. We observe that all the singularities (3.62) are located at the intersection of the invariant lines, except for $M_{2,3}$ which lie on the line $x=0$ and are symmetric with respect to the origin of coordinates. Therefore considering Remark 3.27 and the condition $s \neq 0$ we arrive at the configuration of invariant lines given by *Config. 7.7b*.

So we have proved that if $\zeta_1 > 0$, $\zeta_4 \neq 0$ and $\mathcal{D}_4 \neq 0$ systems (3.54) possess the configuration *Config. 7.6b* $(a=1,\ u=-2,\ s=1)$ if $\mathcal{D}_7 < 0$ and *Config. 7.7b* $(a=1,\ u=1,\ s=1)$ if $\mathcal{D}_7 > 0$.

b.2) *The possibility* $\zeta_4 = 0$. This implies $\gamma = 0$ and considering (3.55) the condition $\gamma = 0$ gives us

$$m = \frac{g(s^2+3u)}{9(1+u)}$$

Then we can apply the transformation

$$x_1 = \alpha x + \frac{g\varkappa}{3v}, \quad y_1 = \alpha y - \frac{sg\varkappa}{9v}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = \frac{(1+u)\varkappa}{v}.$$

which brings systems (3.60) to the following family of systems (we keep the old notations for the variables):

$$\dot{x} = (1+u)x(x^2+1), \quad \dot{y} = (1+u)y - sx^3 + ux^2y - sxy^2 - y^3 \tag{3.63}$$

with $1+u \neq 0$. We determine that systems (3.63) possess six distinct invariant affine straight lines

$$L_1: x=0,\ L_{2,3}: x=\pm i,\ L_4: y=-sx, \quad L_{5,6}: y=\pm ix$$

and the following nine finite singularities:

$$M_1(0,0),\ M_{2,3}(0,\pm\sqrt{1+u}\ ),\ M_{4,5}(\pm i,1),\ M_{6,7}(\pm i,-1),\ M_{8,9}(\pm i,\mp is).$$

We observe that the singular points M_2 and M_3 could be real (if $1+u>0$) or complex (if $1+u<0$), but they could not coincide due to $1+u \neq 0$.

So, similarly as in the case of systems (3.61) we have two real and four complex invariant lines. However in this case considering our Convention (see page 8) we determine that the real invariant line $x=0$ coincides with the projection of the complex invariant lines $L_{2,3}: x=\pm i$ on the plane (x,y). As it was mentioned earlier the invariant line $L_4: y=-sx$ coincides with the projection of the complex invariant lines $L_{5,6}: y=\pm ix$ on the plane (x,y) (see our Convention on page 8) if and only if $s=0$. Therefore due to the condition $s \neq 0$ we arrive at the configuration *Config. 7.8b* if $u<-1$ and at *Config. 7.9b* if $u>-1$.

Since sign$\,(u+1) =$ sign$\,(\mathcal{D}_7)$ we deduce that for $\zeta_1 > 0$, $\zeta_4 \neq 0$ and $\mathcal{D}_4 \neq 0$ systems (3.54) possess the configuration *Config. 7.8b* $(u=-2,\ s=1)$ if $\mathcal{D}_7 < 0$ and *Config. 7.9b* $(u=1,\ s=1)$ if $\mathcal{D}_7 > 0$.

c) *The subcase* $\zeta_1 = 0$ *(*$\mathcal{D}_5 \neq 0$*)*. This implies $\lambda = 0$ and considering (3.52) and solving the equation $\lambda = 0$ with respect to the parameter c, it is clear that we get (3.53) for $v=0$. This leads to the systems (3.54) with $v=0$, which we denote by $(3.54)_{\{v=0\}}$.

In this case for systems (3.54)$_{\{v=0\}}$ we have

$$\zeta_2 = \frac{8\gamma^2}{(1+u)}$$

and we again consider two possibilities: $\zeta_2 \neq 0$ and $\zeta_2 = 0$.

c.1) *The possibility* $\zeta_2 \neq 0$. This implies $\gamma \neq 0$ and via the transformation

$$x_1 = \alpha x + \frac{3\nu}{\gamma}, \quad y_1 = \alpha y - \frac{s\nu}{\gamma}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = -\frac{(1+u)\varkappa}{\gamma}, \quad \nu = (g-2m)(1+u)$$

systems (3.54)$_{\{v=0\}}$ can be brought to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned} \dot{x} &= (1+u)x(x-1)^2, \\ \dot{y} &= (1+u)(1-2x)y - sx^3 + ux^2y - sxy^2 - y^3. \end{aligned} \tag{3.64}$$

We observe that this family of systems is a subfamily of (3.56) defined by the condition $a = 1$. So systems (3.64) possess invariant lines (3.57) among which only five are distinct. More exactly the line $L_3 \equiv L_2 : x = 1$ is double.

These systems have 6 distinct finite singularities because setting $a = 1$ in (3.58) we obtain that the real singularity M_7 coalesces with the real singularity M_4, whereas the complex singularity M_8 (respectively M_9) coalesces with the complex singularity M_5 (respectively M_6). Moreover we observe that the simple singular point $M_1(0,0)$ is the point of intersection of four invariant lines: L_1, L_4 and $L_{5,6}$. Therefore considering Remark 3.27 and the condition $s \neq 0$ we arrive at the configuration *Config. 7.10b* if $u+1 < 0$ and at configuration *Config. 7.11b* if $u+1 > 0$.

So since sign$\,(u+1) = $ sign$\,(\mathcal{D}_7)$ we conclude that in the case $\zeta_1 = 0$, $\zeta_2 \neq 0$ and $\mathcal{D}_4 \neq 0$ systems (3.54) possess the configuration *Config. 7.10b* $(u = -2,\ s = 1)$ if $\mathcal{D}_7 < 0$ and *Config. 7.11b* $(u = 1,\ s = 1)$ if $\mathcal{D}_7 > 0$.

c.2) *The possibility* $\zeta_2 = 0$. This implies $\gamma = 0$ and considering (3.55) we determine $m = \frac{g(s^2+3u)}{9(1+u)}$. In this case systems (3.54) with $v = 0$ for this value of the parameter m become the systems

$$\begin{aligned} \dot{x} =& \frac{(g+3x+3ux)^3}{27(1+u)^2}, \\ \dot{y} =& -\frac{g^3s(27+2s^2+9u)}{729(1+u)^3} - \frac{g^2s(27+s^2+6u)}{81(1+u)^2}x + \frac{g^2(s^2+3u)}{27(1+u)^2}y \\ & -\frac{gs(9+u)}{9(1+u)}x^2 + \frac{2g(s^2+3u)}{9(1+u)}xy - sx^3 + ux^2y - sxy^2 - y^3, \end{aligned}$$

and after the transformation

$$x_1 = x + \frac{g}{3(1+u)}, \quad y_1 = y - \frac{gs}{9(1+u)}, \quad t_1 = t$$

we arrive at the homogeneous systems

$$\dot{x} =(1+u)x^3, \quad \dot{y} = -sx^3 + ux^2y - sxy^2 - y^3, \quad 1+u \neq 0. \tag{3.65}$$

These systems possess the invariant lines

$$L_{1,2,3} : x = 0,\ L_4 : y = -sx, \quad L_{5,6} : y = \pm ix$$

and the unique finite singularities $M_1(0,0)$ of the multiplicity nine. As a result, taking into consideration Remark 3.27 and the condition $s \neq 0$ we obtain the unique configuration *Config. 7.12b*.

1.2: The case $\mathcal{D}_4 = 0$. Then we arrive at the family of systems (3.51) with $s = 0$.

So we could follow step by step the investigations given earlier for systems (3.51) but now considering the condition $s = 0$. This condition is essential because considering Remark 3.27 we could obtain new configurations of invariant lines. More exactly we have the following remark.

Remark 3.28. We observe that in the case $s \neq 0$ we have constructed 6 canonical forms of systems (3.51) depending on the the values of the invariant polynomials ζ_1, ζ_2 and ζ_4. And the algorithm of the construction does not depends on the value of parameter s. More precisely we have the following canonical systems and their corresponding form in the case $s = 0$:

$$\begin{array}{lclcl}
\zeta_1 < 0, \zeta_2 \neq 0 & \Rightarrow & (3.56) & \overset{s=0}{\Rightarrow} & (3.56)_{\{s=0\}};\\
\zeta_1 < 0, \zeta_2 = 0 & \Rightarrow & (3.59) & \overset{s=0}{\Rightarrow} & (3.59)_{\{s=0\}};\\
\zeta_1 > 0, \zeta_4 \neq 0 & \Rightarrow & (3.61) & \overset{s=0}{\Rightarrow} & (3.61)_{\{s=0\}};\\
\zeta_1 > 0, \zeta_4 \neq 0 & \Rightarrow & (3.63) & \overset{s=0}{\Rightarrow} & (3.63)_{\{s=0\}};\\
\zeta_1 = 0, \zeta_2 \neq 0 & \Rightarrow & (3.64) & \overset{s=0}{\Rightarrow} & (3.64)_{\{s=0\}};\\
\zeta_1 = 0, \zeta_2 = 0 & \Rightarrow & (3.65) & \overset{s=0}{\Rightarrow} & (3.65)_{\{s=0\}}.
\end{array}$$

As it was shown in the case $s \neq 0$ all the canonical systems enumerated above possess among their invariant lines the following three ones: $L_4 : y = -sx$ (or $y = 0$ if $s = 0$) and $L_{5,6} : y = \pm ix$. Considering Remark 3.27 the positions of these three invariant lines in configurations in the case $s = 0$ are different from that in the case $s \neq 0$.

So considering the above remark we conclude that in the case $s = 0$ systems (3.51) possess also 12 configurations of invariant lines which are distinct from those in the case $s \neq 0$. In order to determine the corresponding affine invariant conditions we evaluate for systems $(3.51)_{\{s=0\}}$ the invariant polynomials which distinguished the configurations *Config. 7.1b* – *Config. 7.12b*.

Considering Remark 3.28 we observe that the invariant polynomials ζ_1, ζ_2 and ζ_4 were used for constructing the canonical forms mentioned in this remark. On the other hand the invariant polynomials $\mathcal{D}_7$ and ζ_3 were applied for distinguishing the configurations *Config. 7.1b* – *Config. 7.12b* (see the statement of Proposition 3.26, case $\mathcal{D}_4 \neq 0$). Evaluating these two polynomials for systems $(3.51)_{\{s=0\}}$ we have

$$\mathcal{D}_7 = 4(1+u), \quad \zeta_3 = 0$$

and hence the invariant polynomial ζ_3 could not be used for systems $(3.51)_{\{s=0\}}$.

On the other hand we observe that this invariant polynomial is applied only in the case of systems (3.56) and in the case $s \neq 0$ it is responsible for the sign of the expression $a(u+1)$ because for systems (3.56) we have

$$\zeta_3 = -2(a-1)^2 as^2 (9+s^2)^2 (1+u)^3/81.$$

Therefore for these systems in the case $s = 0$ we need another invariant polynomial and we define the invariant ζ_5 which for systems $(3.56)_{\{s=0\}}$ has the value

$$\zeta_5 = -144(a-1)^2 a(1+u)^3$$

and clearly if $\zeta_5 \neq 0$ then $\operatorname{sign}(\zeta_5) = -\operatorname{sign}(a(u+1))$.

Thus considering Remark 3.28 and the first part of the statement of Proposition 3.26 corresponding to the case $D_4 \neq 0$ as well as our Convention on page 8 and Remark 3.27, in the case $D_4 = 0$ we arrive at the configurations *Config. 7.13b* – *Config. 7.24b*. For the realization of each one of these configurations it is sufficient to take the corresponding examples presented in the proof of the case $\mathcal{D}_4 \neq 0$ and substitute $s = 1$ by $s = 0$. Thus we conclude that Proposition 3.26 is completely proved. □

2: The possibility $\mathcal{D}_5 = 0$**.** In this case we get $u = -s^2/3$ and then $\zeta_1 = 0$. So we have to detect another invariant polynomial which governs the sign of the polynomial λ. We observe that in this particular case we have

$$\lambda = 3(s^2-3)^2(3c - g^2 + m^2 - cs^2),$$

where $s^2 - 3 \neq 0$ due to $\mathcal{D}_7 = -4(s^2-3)/3 \neq 0$.

On the other hand for systems (3.51) with $u = -s^2/3$ we calculate

$$\zeta_1' = 64s^2(9+s^2)^2(3c - g^2 + m^2 - cs^2)x^6, \quad \mathcal{D}_8 = -32s^2(9+s^2)^2/729. \tag{3.66}$$

Therefore due to $\mathcal{D}_8 \neq 0$ we have $s \neq 0$ and we conclude that in the case $\mathcal{D}_5 = 0$ we have $\operatorname{sign}(\lambda) = \operatorname{sign}(\zeta_1')$.

We prove the following proposition.

Proposition 3.29. *Assume that for a system* (3.51) *the conditions* $\mathcal{D}_6\mathcal{D}_7\mathcal{D}_8 \neq 0$ *and* $\mathcal{D}_5 = 0$ *hold. Then this system possesses one of the configurations of invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\zeta_1' < 0,\ \zeta_2 \neq 0,\ \zeta_3 < 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.1b};\\
\zeta_1' < 0,\ \zeta_2 \neq 0,\ \zeta_3 < 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.2b};\\
\zeta_1' < 0,\ \zeta_2 \neq 0,\ \zeta_3 > 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.3b};\\
\zeta_1' < 0,\ \zeta_2 \neq 0,\ \zeta_3 > 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.4b};\\
\zeta_1' < 0,\ \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.5b};\\
\zeta_1' > 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.6b};\\
\zeta_1' > 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.7b};\\
\zeta_1' > 0,\ \zeta_4 = 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.8b};\\
\zeta_1' > 0,\ \zeta_4 = 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.9b};\\
\zeta_1' = 0,\ \zeta_2 \neq 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.10b};\\
\zeta_1' = 0,\ \zeta_2 \neq 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.11b};\\
\zeta_1' = 0,\ \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.12b}.
\end{array}$$

Proof. First of all we observe that for systems (3.51) with $u = -s^2/3$ according to (3.66) the condition $\mathcal{D}_8 \neq 0$ implies $s \neq 0$ (i.e. $\mathcal{D}_4 \neq 0$).

On the other hand, the proof of Proposition 3.26 for the case $\mathcal{D}_4 \neq 0$ was performed for the condition $u = -s^2/3$ inclusively, because this condition is not essential for the proof. Therefore a system (3.51) with $u = -s^2/3$ could possess only one of the configurations *Config. 7.1b*–*Config. 7.12b* provided by Proposition 3.26 in the case $\mathcal{D}_4 \neq 0$. We claim that each one of these 12 configurations is realizable in the case $u = -s^2/3$.

Indeed for systems (3.51) with $u = -s^2/3$ we have

$$\zeta_1 = 0, \quad \zeta_2 = -72(s^2-3)(3c-g^2+m^2-cs^2),$$
$$\zeta_3 = -\frac{8s^2(9+s^2)^2}{243(-3+s^2)}(3c-g^2+m^2-cs^2)(3c-g^2+4m^2-cs^2),$$
$$\zeta_4 = m^2(4s^2x^2-13x^2-2sxy-3y^2), \quad \mathcal{D}_7 = -4(s^2-3)/3, \quad \mathcal{D}_8 = -32s^2(9+s^2)^2/729$$

and to prove the compatibility of the conditions provided by Proposition 3.29 it is sufficient to present the examples of the realizations of the corresponding configurations for systems (3.54) with $u = -s^2/3$ in terms of the parameters $(c,g,m,s) = (c_0,g_0,m_0,s_0)$ with $s_0 \neq 0$. So we have

Config. 7.1b: $(c_0,g_0,m_0,s_0) = (1,1,0,-2)$;
Config. 7.2b: $(c_0,g_0,m_0,s_0) = (-1,0,1,-1)$;
Config. 7.3b: $(c_0,g_0,m_0,s_0) = (2,0,1,-2)$;
Config. 7.4b: $(c_0,g_0,m_0,s_0) = (0,1,0,-1)$;
Config. 7.5b: $(c_0,g_0,m_0,s_0) = (4,0,1,2)$;
Config. 7.6b: $(c_0,g_0,m_0,s_0) = (-1,0,-2,-2)$;
Config. 7.7b: $(c_0,g_0,m_0,s_0) = (-1,0,-2,-1)$;
Config. 7.8b: $(c_0,g_0,m_0,s_0) = (-1,0,0,-2)$;
Config. 7.9b: $(c_0,g_0,m_0,s_0) = (1,0,0,-1)$;
Config. 7.10b: $(c_0,g_0,m_0,s_0) = (0,1,1,-2)$;
Config. 7.11b: $(c_0,g_0,m_0,s_0) = (0,1,1,-1)$;
Config. 7.12b: $(c_0,g_0,m_0,s_0) = (0,0,0,1)$.

This completes the proof of Proposition 3.29. □

3.2.2 The statement (A_2)

As it was shown in the proof of statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_2) for the family of systems (3.12) lead to the conditions (3.28).

Assuming these conditions to be fulfilled for systems (3.12) we arrive at the family of systems

$$\begin{aligned}
\dot{x} =& \left[x - \frac{9l}{2s(9+s^2)}\right]\Big[\frac{27l^2(s^2-3)+18lms(9+s^2)+2cs^2(9+s^2)^2}{2s^2(9+s^2)^2} \\
&+ \frac{3l(s^2-3)+2ms(9+s^2)}{s(9+s^2)}\,x + 2(s^2-3)x^2/3\Big] \equiv L_1^{(1)}(x)L_{2,3}^{(1)}(x), \\
\dot{y} =& \frac{3l\left[18l^2s+9lm(9+s^2)+cs(9+s^2)^2\right]}{2s(9+s^2)^3} + \frac{3l\left[3l(s^2-27)+4ms(9+s^2)\right]}{4s(9+s^2)^2}\,x \\
&+ \frac{81l^2(s^2-3)+36lms(9+s^2)+4cs^2(9+s^2)^2}{4s^2(9+s^2)^2}\,y + lx^2 + 2mxy \\
&- sx^3 + (2s^2-9)x^2y/3 - sxy^2 - y^3,
\end{aligned} \tag{3.67}$$

for which we have

$$\mathcal{D}_7 = \frac{8}{3}\,(s^2-3) \neq 0, \quad \mathcal{D}_8 = -\frac{32}{729}s^2(9+s^2)^2 \neq 0.$$

We need to determine if the two lines defined by the equation $L^{(1)}_{2,3} = 0$ are real or complex and in the case when they are real, if one of them coincides with the invariant line $L^{(1)}_1 = 0$. So we calculate

$$\text{Discrim}\,[L^{(1)}_{2,3}, x] = -\frac{1}{3s^2(9+s^2)^2}\lambda^{(1)}(c,l,m,s),$$
$$Res_x(L^{(1)}_1, L^{(1)}_{2,3}) = \frac{1}{2s^2(9+s^2)^2}\mu^{(1)}(c,l,m,s)$$

where

$$\begin{aligned}\lambda^{(1)} &= 81l^2(s^2-3)^2 + 36lms(s^2-3)(9+s^2) + 4s^2(9+s^2)^2(2cs^2-6c-3m^2),\\ \mu^{(1)} &= 81l^2(s^2-3) + 36lms(9+s^2) + 2cs^2(9+s^2)^2.\end{aligned} \tag{3.68}$$

We observe that

$$\text{sign}\left(\text{Discrim}\,[L^{(1)}_{2,3}, x]\right) = -\text{sign}\,(\lambda^{(1)}),$$

i.e. the invariant lines $L^{(1)}_{2,3} = 0$ are real (respectively complex; coinciding) if $\lambda^{(1)} < 0$ (respectively $\lambda^{(1)} > 0$; $\lambda^{(1)} = 0$). Moreover, the invariant line $L^{(1)}_1 = 0$ coincides with one of the lines $L^{(1)}_{2,3} = 0$ if and only if $\mu^{(1)} = 0$.

On the other hand for systems (3.67) calculations yield:

$$\zeta_1 = \frac{20\lambda^{(1)}\left(s^2-3\right)^2 x^2}{s^2\left(s^2+9\right)^2}, \quad \chi_5 = -\frac{\mu^{(1)}}{9s\left(s^2+9\right)}$$

and hence due to $\mathcal{D}_7 \neq 0$ we have $\text{sign}\,(\lambda^{(1)}) = \text{sign}\,(\zeta_1)$. Moreover we observe that the condition $\mu^{(1)} = 0$ is equivalent to $\chi_5 = 0$.

Proposition 3.30. *Assume that for a system* (3.67) *the condition* $\mathcal{D}_7\mathcal{D}_8 \neq 0$ *holds. Then this system possesses one of the configurations of invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\zeta_1 < 0,\ \zeta_5 \neq 0,\ \zeta_3 < 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.1b ;}\\
\zeta_1 < 0,\ \zeta_5 \neq 0,\ \zeta_3 < 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.2b;}\\
\zeta_1 < 0,\ \zeta_5 \neq 0,\ \zeta_3 > 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.3b ;}\\
\zeta_1 < 0,\ \zeta_5 \neq 0,\ \zeta_3 > 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.4b;}\\
\zeta_1 < 0,\ \zeta_5 = 0 & \Leftrightarrow & \textit{Config. 7.5b;}\\
\zeta_1 > 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.6b;}\\
\zeta_1 > 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.7b;}\\
\zeta_1 > 0,\ \zeta_4 = 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.8b;}\\
\zeta_1 > 0,\ \zeta_4 = 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.9b;}\\
\zeta_1 = 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 < 0 & \Leftrightarrow & \textit{Config. 7.10b;}\\
\zeta_1 = 0,\ \zeta_4 \neq 0,\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.11b;}\\
\zeta_1 = 0,\ \zeta_4 = 0 & \Leftrightarrow & \textit{Config. 7.12b.}
\end{array}$$

Proof. We examine three cases: $\zeta_1 < 0$, $\zeta_1 > 0$ and $\zeta_1 = 0$.

a) *The case* $\zeta_1 < 0$. This implies $\lambda^{(1)} < 0$ and we may set $\lambda^{(1)} = -3v^2 < 0$. We observe that the polynomial $\lambda^{(1)}$ is linear with respect to the parameter c with the coefficient $8s^2(s^2 - 3)(9+s^2)^2 \neq 0$ (due to $\mathcal{D}_7\mathcal{D}_8 \neq 0$).

Thus solving the equation $\lambda^{(1)} = -3v^2$ we obtain

$$c = -\frac{3}{8s^2(s^2-3)(9+s^2)^2}\Big[\big[9l(s^2-3) - 2ms(9+s^2)\big]\big[3l(s^2-3) + 2ms(9+s^2)\big] + v^2\Big]. \tag{3.69}$$

This leads to the following family of systems

$$\begin{aligned}
\dot{x} =& \frac{2(s^2-3)}{3}\Big[x - \frac{9l}{2s(9+s^2)}\Big]\Big[x + \frac{3(18ms - 9l + 3ls^2 + 2ms^3 - v)}{4s(s^2-3)(9+s^2)}\Big]\times\\
&\Big[x + \frac{3(18ms - 9l + 3ls^2 + 2ms^3 + v)}{4s(s^2-3)(9+s^2)}\Big],\\
\dot{y} =& \frac{9l}{16s^2(s^2-3)(9+s^2)^3}\big[12lms(s^2-3)(9+s^2) + 4m^2(9s+s^3)^2\\
&+ 3l^2(6s^2 - 81 + 7s^4) - v^2\big] + \frac{3l}{4s(9+s^2)^2}(36ms - 81l + 3ls^2 + 4ms^3)\,x\\
&+ \frac{3}{8s^2(s^2-3)(9+s^2)^2}\big[27l^2(s^2-3)^2 + 12lms(s^2-3)(9+s^2)\\
&+ 4m^2(9s+s^3)^2 - v^2\big]\,y + lx^2 + 2mxy - sx^3 + (2s^2-9)x^2y/3 - sxy^2 - y^3.
\end{aligned} \tag{3.70}$$

On the other hand for the value of c given in (3.69) we calculate

$$\mu^{(1)} = \frac{3\big[(\gamma^{(1)})^2 - v^2\big]}{4(s^2-3)}, \quad \gamma^{(1)} = 9l(s^2-3) + 2ms(9+s^2) \tag{3.71}$$

and since the condition $\mu^{(1)} = 0$ (i.e. $\chi_5 = 0$) leads to the coalescence of two invariant lines of the triplet, we examine two possibilities: $\chi_5 \neq 0$ and $\chi_5 = 0$.

a.1) *The possibility* $\chi_5 \neq 0$*.* Then $\mu^{(1)} \neq 0$, i.e. $(\gamma^{(1)} - v)(\gamma^{(1)} + v) \neq 0$ and setting a new parameter $a = \frac{\gamma^{(1)}+v}{\gamma^{(1)}-v}$ we observe that $a - 1 \neq 0$. Indeed calculation yields: $a - 1 = \frac{2v}{\gamma^{(1)}-v} \neq 0$ due to $v \neq 0$. So from the relation $a = \frac{\gamma^{(1)}+v}{\gamma^{(1)}-v}$ we can determine the value of the parameter m:

$$m = \frac{-9l(a-1)(s^2-3) + (a+1)v}{2s(a-1)(9+s^2)}.$$

Then we can apply the following transformation

$$\begin{gathered}
x_1 = \alpha x + \frac{3\nu}{v}, \quad y_1 = \alpha y - \frac{s\nu}{v}, \quad t_1 = \frac{t}{\alpha^2},\\
\alpha = \frac{2(1-a)s(s^2-3)(9+s^2)}{3v}, \quad \nu = l(a-1)(s^2-3),
\end{gathered}$$

which brings systems (3.70) to the family of systems (we keep the old notations for the variables)

$$\begin{aligned}
\dot{x} &= \frac{2}{3}(s^2-3)x(x-1)(x-a),\\
\dot{y} &= \frac{2}{3}(s^2-3)\big[a - x(a+1)\big]y - sx^3 + \frac{1}{3}(2s^2-9)x^2y - sxy^2 - y^3.
\end{aligned} \tag{3.72}$$

It remains to observe that this family of systems is a subfamily of systems (3.56) defined by the condition $u = (2s^2-9)/3$. This family was investigated earlier and since $u = (2s^2-9)/3$ is not a point of bifurcation, we deduce that there are no new configurations. However we have to determine the conditions for the realization of the corresponding configurations of invariant lines in this case.

For systems (3.72) we calculate:

$$\zeta_1 = -\frac{80}{3}(a-1)^2\left(s^2-3\right)^4 x^2, \quad \chi_5 = -\frac{4}{27}as\left(s^2-3\right)\left(s^2+9\right),$$
$$\zeta_3 = -\frac{16}{2187}(a-1)^2 a\left(s^2-3\right)^3 s^2\left(s^2+9\right)^2.$$

Since the condition $\mathcal{D}_7\mathcal{D}_8 \neq 0$ is satisfied we conclude that the condition $\zeta_1 < 0$ gives us $a-1 \neq 0$ and the condition $\chi_5 \neq 0$ implies $a \neq 0$.

As it was shown earlier systems (3.56) in the case $a(a-1) \neq 0$ and $s \neq 0$ could possess only 4 configurations *Config. 7.1b* – *Config. 7.4b*. More precisely for systems (3.56) we have obtained the following configurations when the corresponding conditions are satisfied, respectively:

Config. 7.1b $\Leftrightarrow$ $a(a-1) > 0,\ a < 0$;
Config. 7.2b $\Leftrightarrow$ $a(a-1) > 0,\ a > 0$;
Config. 7.3b $\Leftrightarrow$ $a(a-1) < 0,\ a > 0$;
Config. 7.4b $\Leftrightarrow$ $a(a-1) < 0,\ a < 0$.

On the other hand for systems (3.72) we have $s \neq 0$ due to $\mathcal{D}_8 \neq 0$ and furthermore we have

$$\operatorname{sign}\left(a(u+1)\right) = \operatorname{sign}\left(a(s^2-3)\right) = \operatorname{sign}\left(\zeta_3\right), \quad \operatorname{sign}\left(u+1\right) = \operatorname{sign}\left(s^2-3\right) = \operatorname{sign}\left(\mathcal{D}_7\right).$$

Therefore we conclude that in the case $\zeta_1 < 0$ and $\chi_5 \neq 0$ the statement of Proposition 3.30 is valid.

a.2) *The possibility* $\chi_5 = 0$. Then $\mu^{(1)} = 0$ and considering (3.71) we get $(\gamma^{(1)} - v)(\gamma^{(1)} + v) = 0$. So we may assume $\gamma^{(1)} - v = 0$ due to change $v \to -v$ and setting $v = \gamma^{(1)} \neq 0$ in systems (3.70) we arrive at the family of systems

$$\begin{aligned}
\dot{x} =& \frac{2(s^2-3)}{3}\left[x - \frac{9l}{2s(9+s^2)}\right]^2\left[x + \frac{3(9ms-9l+3ls^2+ms^3)}{s(s^2-3)(9+s^2)}\right],\\
\dot{y} =& -\frac{27l^2(18ms-27l+5ls^2+2ms^3)}{4s^2(9+s^2)^3} + \frac{3l}{4s(9+s^2)^2}(36ms-81l+3ls^2+4ms^3)\,x\\
&-\frac{9l}{4s^2(9+s^2)^2}\left[36ms-27l+9ls^2+4ms^3\right]y + lx^2 + 2mxy\\
&-sx^3 + (2s^2-9)x^2y/3 - sxy^2 - y^3
\end{aligned}$$

So since $\gamma^{(1)} = 9l(s^2-3) + 2ms(9+s^2) \neq 0$ then applying the transformation

$$x_1 = \alpha x + \frac{3l(s^2-3)}{\gamma^{(1)}}, \quad y_1 = \alpha y - \frac{ls(s^2-3)}{\gamma^{(1)}}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = -\frac{2s(s^2-3)(9+s^2)}{3\gamma^{(1)}},$$

we obtain the following 1-parameter family of systems:

$$\begin{aligned}
\dot{x} &= \frac{2}{3}(s^2-3)x^2(x-1),\\
\dot{y} &= -\frac{2}{3}(s^2-3)xy - sx^3 + \frac{1}{3}(2s^2-9)x^2y - sxy^2 - y^3.
\end{aligned}$$

We again observe that the above family of systems is a subfamily of systems (3.59) defined by the condition $u = (2s^2 - 9)/3$. This family was investigated earlier and it was shown that it possesses the unique configuration *Config. 7.5b* including the case $u = (2s^2 - 9)/3$.

b) *The case* $\zeta_1 > 0$. Then $\lambda^{(1)} > 0$ and we set $\lambda^{(1)} = 3v^2 > 0$ and since $s(s^2-3) \neq 0$ (due to $\mathcal{D}_7\mathcal{D}_8 \neq 0$) we obtain

$$c = -\frac{3}{8s^2(s^2-3)(9+s^2)^2}\Big[\big[9l(s^2-3) - 2ms(9+s^2)\big]\big[3l(s^2-3) + 2ms(9+s^2)\big] - v^2\Big].$$

This leads to the following family of systems

$$\begin{aligned} \dot{x} =&\Big[x - \frac{9l}{2s(9+s^2)}\Big]\Big[\frac{9v^2 + \big[9l(s^2-3) + 2s(9+s^2)(3m+2(s^2-3)x)\big]^2}{24s^2(s^2-3)(9+s^2)^2}\Big],\\ \dot{y} =&\frac{9l}{16s^2(s^2-3)(9+s^2)^3}\big[12lms(s^2-3)(9+s^2) + 4m^2(9s+s^3)^2\\ &+3l^2(6s^2-81+7s^4)+v^2\big] + \frac{3l}{4s(9+s^2)^2}(36ms - 81l + 3ls^2 + 4ms^3)\,x\\ &+\frac{3}{8s^2(s^2-3)(9+s^2)^2}\big[27l^2(s^2-3)^2 + 12lms(s^2-3)(9+s^2)\\ &+4m^2(9s+s^3)^2 + v^2\big]\,y + lx^2 + 2mxy - sx^3 + (2s^2-9)x^2y/3 - sxy^2 - y^3. \end{aligned} \tag{3.73}$$

In order to simplify these systems we need to use a transformation which depends on the condition: either $\gamma^{(1)} \neq 0$ or $\gamma^{(1)} = 0$. Since for the above systems we have

$$\zeta_4 = \frac{\left(\gamma^{(1)}\right)^2}{4s^2(9+s^2)^2}\big[(4s^2-13)x^2 - 2sxy - 3y^2\big]$$

we conclude that the condition $\gamma^{(1)} = 0$ is equivalent to $\zeta_4 = 0$. So we discuss two possibilities: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

b.1) *The possibility* $\zeta_4 \neq 0$. This implies $\gamma^{(1)} \neq 0$ and setting a new parameter $a = \frac{v}{\gamma^{(1)}} \neq 0$ we have $v = a\gamma^{(1)}$. Then we can apply to systems (3.73) the following transformation

$$x_1 = \alpha x + \frac{6l(s^2-3)}{\gamma^{(1)}},\quad y_1 = \alpha y - \frac{2ls(s^2-3)}{\gamma^{(1)}},\quad t_1 = \frac{t}{\alpha^2},\quad \alpha = -\frac{4s(s^2-3)(9+s^2)}{3\gamma^{(1)}},$$

which brings these systems to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned} \dot{x} &= \frac{2}{3}(s^2-3)x\big[(x-1)^2 + a^2\big],\\ \dot{y} &= \frac{2}{3}(s^2-3)(1+a^2)y - \frac{4}{3}(s^2-3)xy - sx^3 + \frac{1}{3}(2s^2-9)x^2y - sxy^2 - y^3. \end{aligned} \tag{3.74}$$

It remains to observe that this family of systems is a subfamily of systems (3.61) defined by the condition $u = (2s^2 - 9)/3$. The family (3.61) was investigated earlier and it was proved the existence of only two configurations of the invariant lines: *Config. 7.6b* if $\mathcal{D}_7 < 0$ and *Config. 7.7b* if $\mathcal{D}_7 > 0$.

Since for systems (3.74) we have $\mathcal{D}_7 = 8\left(s^2-3\right)/3$ we deduce that both configurations are also realizable in the case under consideration.

b.2) *The possibility* $\zeta_4 = 0$. This implies $\gamma^{(1)} = 0$ and considering (3.71) the condition $\gamma^{(1)} = 0$ gives

$$m = -\frac{9l(s^2-3)}{2s(9+s^2)}$$

Then we can apply to systems (3.73) the transformation

$$x_1 = \alpha x - \frac{6l(s^2-3)}{v}, \quad y_1 = \alpha y + \frac{2ls(s^2-3)}{v}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = \frac{4s(s^2-3)(9+s^2)}{3v},$$

which brings these systems to the following family of systems (we keep the old notations for the variables):

$$\dot{x} = \frac{2}{3}(s^2-3)x(x^2+1), \quad \dot{y} = \frac{2}{3}(s^2-3)y - sx^3 + \frac{1}{3}(2s^2-9)x^2y - sxy^2 - y^3. \tag{3.75}$$

It is easy to observe that this family is a subfamily of systems (3.63) defined by the condition $u = (2s^2-9)/3$. The family (3.63) was investigated earlier and we have proved the existence of two configurations: *Config. 7.8b* if $\mathcal{D}_7 < 0$ and *Config. 7.9b* if $\mathcal{D}_7 > 0$. So by the same reasons as in the possibility ***b.1)*** above we conclude that in the case $\zeta_1 > 0$ and $\zeta_4 = 0$ the statement of Proposition 3.30 is valid.

c) *The case* $\zeta_1 = 0$. This implies $\lambda^{(1)} = 0$ and considering (3.68) and solving the equation $\lambda^{(1)} = 0$ with respect to the parameter c, it is clear that we get (3.69) for $v = 0$. This leads to the systems (3.70) with $v = 0$, which we denote by $(3.70)_{\{v=0\}}$ and for these systems we have

$$\zeta_4 = \frac{\left(\gamma^{(1)}\right)^2}{4s^2(9+s^2)^2}\left[(4s^2-13)x^2 - 2sxy - 3y^2\right]$$

and we again consider two possibilities: $\zeta_4 \neq 0$ and $\zeta_4 = 0$.

c.1) *The possibility* $\zeta_4 \neq 0$. Then $\gamma^{(1)} \neq 0$ and via the transformation

$$x_1 = \alpha x + \frac{6l(s^2-3)}{\gamma^{(1)}}, \quad y_1 = \alpha y - \frac{2ls(s^2-3)}{\gamma^{(1)}}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = -\frac{4s(s^2-3)(9+s^2)}{3\gamma^{(1)}},$$

systems $(3.70)_{\{v=0\}}$ can be brought to the following family of systems (we keep the old notations for the variables):

$$\begin{aligned} \dot{x} &= \frac{2}{3}(s^2-3)x(x-1)^2, \\ \dot{y} &= \frac{2}{3}(s^2-3)y - \frac{4}{3}(s^2-3)xy - sx^3 + \frac{1}{3}(2s^2-9)x^2y - sxy^2 - y^3. \end{aligned} \tag{3.76}$$

We observe that this family of systems is a subfamily of systems (3.64) defined by the condition $u = (2s^2-9)/3$. This family was investigated earlier and we have detected *Config. 7.10b* if $\mathcal{D}_7 < 0$ and *Config. 7.11b* if $\mathcal{D}_7 > 0$. Clearly we get the same configurations in the case $u = (2s^2-9)/3$, i.e. when $\mathcal{D}_6 = 0$.

c.2) *The possibility* $\zeta_4 = 0$. Then $\gamma^{(1)} = 0$ and considering (3.71) we determine $m = -\frac{9l(s^2-3)}{2s(9+s^2)}$. In this case systems (3.70) with $v = 0$ for this value of the parameter m after the transformation

$$x_1 = x - \frac{9l}{2s(9+s^2)}, \quad y_1 = y + \frac{3ls}{2(9+s^2)}, \quad t_1 = t$$

we will be brought to the homogeneous systems (3.65) with $u = (2s^2-9)/3$. However these systems are already examined and we found only the configuration *Config. 7.12b*.

Since all the cases are examined we conclude that Proposition 3.30 is proved. □

3.2.3 The statement (A_3)

According to the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by this statement for the family of systems (3.12) lead to the conditions (3.29).

Next we determine the canonical form of the systems (3.12) subject to the conditions (3.29). Assuming these conditions to be fulfilled for systems (3.12) we arrive at the following family of systems

$$\begin{aligned}\dot{x} = &(1+u)\Big[x + \frac{3g-6m+gs^2+2gu-6mu}{(9+s^2)(1+u)}\Big]\Big[\frac{g^2s^2+\big[gu-3m(1+u)\big]^2}{s^2(9+s^2)(1+u)^2}\\ &-\frac{2(gu-3g-3m-3mu)}{(9+s^2)(1+u)}\,x + x^2\Big] \equiv\ (1+u)L_1^{(2)}(x)L_{2,3}^{(2)}(x),\\ \dot{y} =&\widetilde{Q}(x,y),\end{aligned} \tag{3.77}$$

where the polynomial $\widetilde{Q}(x,y)$ depends on the parameters g,m,s and u and it is determined by the conditions (3.29). According to the statement (A_3) of the Main Theorem for the above systems the conditions $\mathcal{D}_7\mathcal{D}_8\mathcal{D}_4 \neq 0$ and $\chi_1 \neq 0$ must hold. So calculations yield:

$$\begin{gathered}\mathcal{D}_7 = 4(1+u) \neq 0, \quad \mathcal{D}_8 = -8(s^2-u)\big[4s^2+(3+u)^2\big]/27 \neq 0, \quad \mathcal{D}_4 = 2304s(9+s^2) \neq 0,\\ \chi_1 = \frac{1}{9s(9+s^2)(1+u)}(s^2-u)\big[9m(1+u)-g(s^2+3u)\big]\big[4s^2+(u+3)^2\big] \neq 0.\end{gathered}$$

Considering the first equation of systems (3.77) we observe that

$$\text{Discrim}\,[L_{2,3}^{(2)},x] = -\frac{4(\gamma^{(2)})^2}{s^2(9+s^2)^2}, \quad Res_x(L_1^{(2)},L_{2,3}^{(2)}) = \frac{(s^2+1)(\gamma^{(2)})^2}{s^2(9+s^2)^2}$$

where

$$\gamma^{(2)} = 9m(1+u)-g(s^2+3u) \neq 0$$

due to the condition $\chi_1 \neq 0$.

Thus we deduce that the invariant lines $L_{2,3}^{(2)} = 0$ are complex and they could not coalesce. Moreover all three invariant lines are distinct.

Since $\gamma^{(2)} \neq 0$ applying the transformation

$$\begin{aligned}x_1 &= \alpha x - \frac{3g-6m+gs^2+2gu-6mu}{\gamma^{(2)}}, \quad \alpha = -\frac{(u+1)(9+s^2)}{\gamma^{(2)}},\\ y_1 &= \alpha y + \frac{m(1+u)(9+s^2+6u)-gu(3+s^2+2u)}{s\gamma^{(2)}}, \quad t_1 = \frac{t}{\alpha^2},\end{aligned}$$

to systems (3.77) we arrive at the family of systems

$$\begin{aligned}\dot{x} &= (1+u)x\big[(x-1)^2+\frac{1}{s^2}\big], \quad s(1+u) \neq 0,\\ \dot{y} &= (1+u)^2x/s-(1+u)(2+s^2+u)y/s^2+(s^2-2u-u^2)x^2/s\\ &\quad +(3+s^2+2u)y^2/s - sx^3+ux^2y-sxy^2-y^3.\end{aligned} \tag{3.78}$$

We determine that systems (3.78) possess six distinct invariant affine straight lines

$$L_1: x=0,\ L_{2,3}: x=1\pm i/s,\ L_4: y=-sx+(s^2+u+2)/s, \quad L_{5,6}: y=\pm ix+(1+u)/s$$

and the following nine finite singularities:

$$M_1(0,0),\ M_2\big(0,(1+u)/s\big),\ M_3\big(0,(2+s^2+u)/s\big),\ M_{4,5}\big(1\pm i/s,\ 1/s\mp i\big),\\ M_{6,7}\big(1\pm i/s,\ u/s\pm i\big),\ M_{8,9}\big(1\pm i/s,\ (2+u)/s\mp i\big), \tag{3.79}$$

which due to $s(1+u)\neq 0$ are all distinct except for the case $2+s^2+u=0$ which implies the coalescence of the real singular point M_3 with M_1.

We observe that all the singularities (3.79) are located at the intersections of the invariant lines, except for the real singularity $M_1(0,0)$ and the complex singularities $M_{4,5}\big(1\pm i/s,\ 1/s\mp i\big)$. Moreover we have exactly three real singularities, which are all located on the invariant line $x=0$. We note that the real singularity M_2 (respectively M_3) is the point of intersection of the invariant line L_1 with the two complex lines $L_{5,6}$ (respectively with the real line L_4).

On the other hand the complex singularity M_6 (respectively M_7) is a point of intersection of two invariant lines L_2 and L_5 (respectively L_3 and L_6), whereas the complex singularity M_8 (respectively M_9) is a point of intersection of three invariant lines L_2, L_4 and L_6 (respectively L_3, L_4 and L_5).

So, considering the fact that we have exatly three real finite singularities M_1, M_2 and M_3 and all of them are located on the invariant line $x=0$ we conclude that we could obtain three distinct configurations of invariant lines defined by the distinct positions of the free point M_1 with respect to the other two real singularities (M_2 and M_3).

In order to describe the positions of the finite real singularities located on the same invariant line we use the following notations.

Notation 3.31. Assume that two finite real singular points $\widetilde{M}_1(x_1,y_1)$ and $\widetilde{M}_2(x_2,y_2)$ of a cubic system are located on the real invariant line $ax+by+c=0$ of this system. Then:

(α) in the case $a\neq 0$ we say that the singular point $\widetilde{M}_1$ is located *below* (respectively *above*) or coincides with, the singularity $\widetilde{M}_2$ if $y_1\leq y_2$ (respectively $y_2<y_1$) and we denote this position by $\widetilde{M}_1\preceq\widetilde{M}_2$ (respectively $\widetilde{M}_2\prec\widetilde{M}_1$);

(β) in the case $a=0$ (then $y_1=y_2$) we say that the singular point $\widetilde{M}_1$ is located *on the left* (respectively *on the right*) or coincides with, the singularity $\widetilde{M}_2$ if $x_1\leq x_2$ (respectively $x_2<x_1$) and we again denote this position by $\widetilde{M}_1\preceq\widetilde{M}_2$ (respectively $\widetilde{M}_2\prec\widetilde{M}_1$).

Since $y_3-y_2=(1+s^2)/s$ it is easy to determine that the positions of the real singularities on the line $x=0$ are determined by the following conditions:

$$M_2\prec M_1\ \Leftrightarrow\ (1+u)s<0;\quad M_3\preceq M_1\ \Leftrightarrow\ (2+u+s^2)s\leq 0;\quad M_3\preceq M_2\ \Leftrightarrow\ s<0.$$

Therefore considering these conditions we obtain the following conditions for the realization of the corresponding configurations of invariant lines:

$$\begin{array}{llll}
1+u<0,\ 2+u+s^2<0 & \Rightarrow\ M_2\prec M_3\prec M_1 & \Rightarrow\ \textit{Config. 7.25b}; & (s=1,\ u=-7/2)\\
1+u<0,\ 2+u+s^2>0 & \Rightarrow\ M_2\prec M_1\prec M_3 & \Rightarrow\ \textit{Config. 7.26b}; & (s=1,\ u=-3/2)\\
1+u<0,\ 2+u+s^2=0 & \Rightarrow\ M_2\prec M_3\equiv M_1 & \Rightarrow\ \textit{Config. 7.27b}; & (s=1,\ u=-3)\\
1+u>0 & \Rightarrow\ M_1\prec M_2\prec M_3 & \Rightarrow\ \textit{Config. 7.28b}. & (s=1,\ u=0)
\end{array}$$

In order to determine the corresponding invariant conditions we evaluate for systems (3.78) the following invariant polynomials:

$$\mathcal{D}_4=2304s(9+s^2),\quad \mathcal{D}_7=4(1+u),\quad \zeta_6=8(2+u+s^2)\big[4s^2+(u-1)^2\big].$$

So due to the condition $s \neq 0$ (as $\mathcal{D}_4 \neq 0$) we have sign $(\zeta_6) =$ sign $(2+u+s^2)$ and sign $(\mathcal{D}_7) =$ sign $(1+u)$.

Considering the conditions for the configurations of invariant lines presented above we arrive at the following proposition.

Proposition 3.32. *Assume that for a system (3.77) the condition $\mathcal{D}_7\mathcal{D}_8\chi_1 \neq 0$ and $\mathcal{D}_4 \neq 0$ holds. Then this system possesses one of following four configurations of invariant lines if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl} \mathcal{D}_7 < 0,\ \zeta_6 < 0 & \Leftrightarrow & \textit{Config. 7.25b;} \\ \mathcal{D}_7 < 0,\ \zeta_6 > 0 & \Leftrightarrow & \textit{Config. 7.26b;} \\ \mathcal{D}_7 < 0,\ \zeta_6 = 0 & \Leftrightarrow & \textit{Config. 7.27b;} \\ \mathcal{D}_7 > 0 & \Leftrightarrow & \textit{Config. 7.28b.} \end{array}$$

3.2.4 The statement (A_4)

As it was shown in the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_4) for the family of systems (3.12) lead to the conditions (3.32).

Assuming these conditions to be fulfilled for systems (3.12) we arrive at the family of systems

$$\begin{aligned} \dot{x} &= (1+u)\Big(x+\frac{m}{u}\Big)\Big[\frac{9l^2}{u^2(3+u)^2}+\Big(x+\frac{m}{u}\Big)^2\Big], \\ \dot{y} &= \frac{(l+uy)}{u^2}\Big[\frac{l^2(3+2u)}{(3+u)^2}+(ux+m)^2+ly-uy^2\Big]. \end{aligned} \tag{3.80}$$

According to the statement (A_4) of the Main Theorem for the above systems the conditions $\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\chi_1 \neq 0$ must hold. So calculations yield:

$$\mathcal{D}_7 = 4(1+u) \neq 0, \quad \mathcal{D}_8 = 8u(3+u)^2/27 \neq 0, \quad \chi_1 = -l(3+u)x^3/3 \neq 0$$

and hence for systems (3.80) the condition $lu(3+u) \neq 0$ holds. Then via the transformation

$$x_1 = \alpha x + \frac{m(3+u)}{3l}, \quad y_1 = \alpha y + \frac{3+u}{3}, \quad t_1 = \frac{t}{\alpha^2}, \quad \alpha = \frac{u(3+u)}{3l}$$

systems (3.80) can be brought to the systems

$$\dot{x} = (1+u)x(x^2+1), \quad \dot{y} = y\big[-2-u+(3+u)y+ux^2-y^2\big], \tag{3.81}$$

for which we have $\mathcal{D}_7 = 4(1+u) \neq 0$ and $\mathcal{D}_8 = 8u(3+u)^2/27 \neq 0$. Therefore for the above systems the condition $u(1+u)(3+u) \neq 0$ is satisfied.

We determine that systems (3.81) possess six distinct invariant affine straight lines

$$L_1: x=0,\ L_{2,3}: x=\pm i,\ L_4: y=0, \quad L_{5,6}: y=1\pm ix$$

and the following nine finite singularities:

$$M_1(0,0),\ M_2(0,1),\ M_3(0,2+u),\ M_{4,5}(\pm i,0),\ M_{6,7}(\pm i,2),\ M_{8,9}(\pm i,1+u).$$

We observe that we could have multiple singularities for some values of the parameter u. More exactly, in the case $u=-2$ the singular point M_3 coalesces with M_1 and we obtain a double singular point $(0,0)$. On the other hand we determine that for $u=1$ the complex singular point $M_8(i,1+u)$ (respectively $M_9(-i,1+u)$ coalesces with the complex singular point $M_6(i,2)$ (respectively $M_7(-i,2)$. As a result we get two double complex singular points, however according to Definition 1.2 this fact is irrelevant for a configuration because we take into consideration only real singularities located on the invariant lines.

We remark that we have only three real singularities and all of them are located on the invariant line $x=0$. Two among these real singularities are fixed: $M_1(0,0)$ (which is a point of the intersection of the invariant lines L_1 and L_4) and $M_2(0,1)$ (which is a point of the intersection of the invariant lines L_1, L_5 and L_6). The singular point $M_3(0,u+2)$ depends on the parameter u and hence could change its position with respect to the singularities M_1 and M_2.

Thus, since we have $M_1 \prec M_2$, taking into consideration our Convention (see page 8) we conclude that the position of $M_3(0,u+2)$ leads to the following four distinct configurations of invariant lines:

$$
\begin{array}{rcccl}
u<-2 & \Rightarrow & M_3 \prec M_1 \prec M_2 & \Rightarrow & \textit{Config. 7.29b;}\\
u=-2 & \Rightarrow & M_3 = M_1 \prec M_2 & \Rightarrow & \textit{Config. 7.30b;}\\
-2<u<-1 & \Rightarrow & M_1 \prec M_3 \prec M_2 & \Rightarrow & \textit{Config. 7.31b;}\\
u>-1 & \Rightarrow & M_1 \prec M_2 \prec M_3 & \Rightarrow & \textit{Config. 7.32b.}
\end{array}
$$

On the other hand for systems (3.81) we have

$$\mathcal{D}_7 = 4(1+u), \quad \zeta_7 = 4(2+u)$$

and evidently we arrive at the following proposition.

Proposition 3.33. *Assume that for a system (3.80) the conditions $\mathcal{D}_7\mathcal{D}_8 \neq 0$ and $\chi_1 \neq 0$ hold. Then this system possesses one of the following four configurations of the invariant lines if and only if the corresponding conditions are satisfied, respectively:*

$$
\begin{array}{lcl}
\zeta_7<0 & \Leftrightarrow & \textit{Config. 7.29b;}\\
\zeta_7=0 & \Leftrightarrow & \textit{Config. 7.30b;}\\
\zeta_7>0,\ \mathcal{D}_7<0 & \Leftrightarrow & \textit{Config. 7.31b;}\\
\zeta_7>0,\ \mathcal{D}_7>0 & \Leftrightarrow & \textit{Config. 7.32b.}
\end{array}
$$

3.2.5 The statement (A_5)

According to the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_5) for the family of systems (3.12) lead to the conditions (3.33).

Remark 3.34. We observe that the conditions (3.33) can be obtained as a particular case from the conditions (3.26) by setting $u=s^2$ (i.e. we allow the condition $\mathcal{D}_8=0$ to be satisfied). This means that we could follow all the steps we have done in the case of the conditions (3.26) if these steps do not depend on the condition $u=s^2$.

Thus applying the conditions (3.33) to systems (3.12) we arrive at the family of systems $(3.51)_{\{u=s^2\}}$ which is a subfamily of (3.51) defined by the condition $u=s^2$.

We remark that all the configurations of the family (3.51) were investigated and Proposition 3.26 provides the necessary and sufficient affine invariant conditions for the realization of each one of the possible 12 possible configurations in the case $\mathcal{D}_4 \neq 0$.

Thus we have to determine which sets of the conditions provided by Proposition 3.26 (for $\mathcal{D}_4 \neq 0$) are compatible in the case $u = s^2 \neq 0$. We prove the following proposition.

Proposition 3.35. *Assume that for a system* $(3.51)_{\{u=s^2\}}$ *the conditions* $\mathcal{D}_7\mathcal{D}_6 \neq 0$ *and* $\mathcal{D}_4 \neq 0$ *hold. Then this system possesses one of the configurations of the invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\zeta_1 < 0,\ \zeta_2 \neq 0,\ \zeta_3 < 0 & \Leftrightarrow & \textit{Config. 7.2b;}\\
\zeta_1 < 0,\ \zeta_2 \neq 0,\ \zeta_3 > 0 & \Leftrightarrow & \textit{Config. 7.4b;}\\
\zeta_1 < 0,\ \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.5b;}\\
\zeta_1 > 0,\ \zeta_4 \neq 0 & \Leftrightarrow & \textit{Config. 7.7b;}\\
\zeta_1 > 0,\ \zeta_4 = 0 & \Leftrightarrow & \textit{Config. 7.9b;}\\
\zeta_1 = 0,\ \zeta_2 \neq 0 & \Leftrightarrow & \textit{Config. 7.11b;}\\
\zeta_1 = 0,\ \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.12b.}
\end{array}$$

Proof. Evaluating for systems $(3.54)_{\{u=s^2\}}$ the invariant polynomials ζ_1, ζ_2, ζ_3, ζ_4, $\mathcal{D}_4$ and $\mathcal{D}_7$ which are involved in Proposition 3.26 (for $\mathcal{D}_4 \neq 0$) we obtain:

$$\zeta_1 = \frac{1280s^4}{3(s^2+9)^2}\kappa_1 x^2, \quad \zeta_2 = 8\kappa_2, \quad \zeta_3 = \frac{8s^2}{81(s^2+9)^2}\kappa_1\kappa_2, \quad \mathcal{D}_4 = 2304s(9+s^2)$$
$$\zeta_4 = \frac{1}{(s^2+9)^2}\kappa_3^2\big[(4s^2-13)x^2 - 2sxy - 3y^2\big], \quad \mathcal{D}_7 = 4(1+s^2), \quad \mathcal{D}_6 = -4(9+s^2)/9,$$

where

$$\begin{aligned}
\kappa_1 &= c(1+s^2)(9+s^2)^2 + (9g - 9m + 5gs^2 - 9ms^2)(gs^2 - 3g - 3m - 3ms^2),\\
\kappa_2 &= c(9+s^2)^2 + 3(g-2m)(7gs^2 - 9g - 18m - 18ms^2),\\
\kappa_3 &= 9m - 4gs^2 + 9ms^2.
\end{aligned}$$

As we can see the condition $\mathcal{D}_7 > 0$ holds. Therefore we conclude that the configurations *Configs. 7.1b, 7.3b, 7.6b, 7.8b, 7.10b* which correspond to the case $\mathcal{D}_7 < 0$ and are realizable for systems (3.54) (see Proposition 3.26), could not be realizable for systems $(3.54)_{\{u=s^2\}}$.

To prove the compatibility of other conditions provided by Proposition 3.26 it is sufficient to present the examples of the realizations of the corresponding configurations for systems $(3.54)_{\{u=s^2\}}$ in terms of the parameters $(c, g, m, s) = (c_0, g_0, m_0, s_0)$ with $s_0 \neq 0$. So we have

$$\begin{array}{ll}
\textit{Config. 7.2b}: & (c_0, g_0, m_0, s_0) = (-1, 1, 1, -1);\\
\textit{Config. 7.4b}: & (c_0, g_0, m_0, s_0) = (-2, 1, 1, -1);\\
\textit{Config. 7.5b}: & (c_0, g_0, m_0, s_0) = (-57/50, 1, 1, -1);\\
\textit{Config. 7.7b}: & (c_0, g_0, m_0, s_0) = (0, 1, 1, -1);\\
\textit{Config. 7.9b}: & (c_0, g_0, m_0, s_0) = (1, 0, 0, -1);\\
\textit{Config. 7.11b}: & (c_0, g_0, m_0, s_0) = (0, -3, 1, 1);\\
\textit{Config. 7.12b}: & (c_0, g_0, m_0, s_0) = (0, 0, 0, 1)
\end{array}$$

This completes the proof of Proposition 3.35. □

3.2.6 The statement (A_6)

As it was shown in the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_6) for the family of systems (3.12) according to Lemma 3.15 lead either to the conditions

$$\begin{aligned} &u=s=k=d=h=l=e=b=0, \quad f=c+\frac{g(2m-g)}{3},\\ &a=-\frac{g-2m}{27}\left(2g^2-9c-2gm-4m^2\right). \end{aligned} \tag{3.82}$$

(for a triplet in the direction $x=0$), or to the conditions

$$\begin{aligned} &u=s=k=l=e=m=0, \quad d=\frac{2gh}{3}, \quad f=c-\frac{g^2}{3},\\ &a=\frac{g}{27}\left(9c-2g^2\right), \quad b=-\frac{2h}{27}(-9c+3g^2+4h^2) \end{aligned} \tag{3.83}$$

(for a triplet in the direction $y=0$).

It is not too difficult to detect that when conditions (3.82) are satisfied then (3.12) become the systems

$$\begin{aligned} \dot{x}&=\frac{1}{27}(g-2m+3x)(9c-2g^2+2gm+4m^2+6gx+6mx+9x^2),\\ \dot{y}&=\frac{1}{3}y\left(3c-g^2+2gm+6mx-3y^2\right). \end{aligned} \tag{3.84}$$

On the other hand, if conditions (3.83) are satisfied then we arrive at the family of systems

$$\begin{aligned} \dot{x}&=-\frac{1}{27}(g+3x)\left(-9c+2g^2-6gx-18hy-9x^2\right),\\ \dot{y}&=-\frac{1}{27}(2h+3y)\left(-9c+3g^2+4h^2-6hy+9y^2\right). \end{aligned} \tag{3.85}$$

We claim that systems (3.84) and (3.85) are affinely equivalent. Indeed since some parameters of the two systems coincide we set for systems (3.85) free parameters $\tilde{c}=c$, $\tilde{g}=g$ and $\tilde{h}=h$. Then the transformation

$$x_1=y-\tilde{g}/3, \quad y_1=-x-\tilde{g}/3, \quad t_1=-t$$

leads to the systems

$$\begin{aligned} \dot{x}_1&=\frac{1}{27}(g_1-2m_1+3x)\left(9c_1-2g_1^2+2g_1m_1+4m_1^2+6g_1x_1+6m_1x_1+9x_1^2\right),\\ \dot{y}_1&=\frac{1}{3}y_1\left(3c_1-g_1^2+2g_1m_1+6m_1x-3y_1^2\right) \end{aligned}$$

with $g_1=\tilde{g}$, $m_1=-\tilde{h}$ and $c_1=(-3\tilde{c}+2\tilde{g}^2)/3$. In other words we have obtained exactly systems (3.84) with new parameters c_1, g_1, m_1. This completes the proof of our claim.

Thus in this case either the conditions (3.82) or (3.83) are satisfied in both cases using an affine transformation and time rescaling we arrive at the same family of systems (3.84).

We observe that the family of systems (3.84) is a subfamily of (3.51) defined by the condition $u=s=0$. We have shown that systems (3.51) possess three parallel invariant lines in the direction $x=0$ and the kind of these lines (real, complex, distinct or coinciding) are determined by the polynomials λ and μ given in (3.52). For the particular case $u = s = 0$ (i.e. for systems (3.84)) these polynomials become

$$\lambda\big|_{\{u=s=0\}} = 27(3c - g^2 + m^2), \quad \mu\big|_{\{u=s=0\}} = 27(3c - g^2 + 4m^2).$$

On the other hand we observe that the sign of the polynomial λ as well as the the value of the polynomial μ are governed by the invariant polynomials ζ_1 and ζ_2 which for systems (3.51) have the form (see page 54)

$$\zeta_1 = \frac{80}{3\varkappa^2}(s^2 + 3u)^2 x^2 \lambda, \quad \zeta_2 = 8\mu.$$

As we can see for $u = s = 0$ the invariant ζ_1 vanishes, i.e. it could not be used to define the sign of $\lambda\big|_{\{u=s=0\}}$, i.e. the sign of the polynomial $3c - g^2 + m^2$.

Thus we have to define another invariant polynomial which captures the sign of $3c - g^2 + m^2$. Such a polynomial could be ζ_8 which for systems (3.84) has the value

$$\zeta_8 = 8m^2(3c - g^2 + m^2).$$

On the other hand according to the conditions provided by the statement (A_6) of the Main Theorem the condition $\chi_{15} \neq 0$ must hold. For systems (3.84) we calculate $\chi_{15} = mx^3y^2 \neq 0$, i.e. $m \neq 0$ and we have

$$\text{sign}\,(\zeta_8) = \text{sign}\,(3c - g^2 + m^2) = \text{sign}\,(\lambda\big|_{\{u=s=0\}}).$$

Thus substituting the invariant polynomial ζ_1 (which vanishes) by ζ_8 we could determine which sets of the conditions provided by Proposition 3.26 are compatible in the case $\mathcal{D}_8 = \mathcal{D}_4 = 0$ (i.e. $u = s = 0$).

Proposition 3.36. *Assume that for a system (3.84) the condition $\chi_{15} \neq 0$ (i.e. $m \neq 0$) holds. Then this system possesses one of the configurations of the invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\zeta_8 < 0,\, \zeta_2 \neq 0,\, \zeta_5 < 0 & \Leftrightarrow & \textit{Config. 7.14b;} \\
\zeta_8 < 0,\, \zeta_2 \neq 0,\, \zeta_5 > 0 & \Leftrightarrow & \textit{Config. 7.16b;} \\
\zeta_8 < 0,\, \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.17b;} \\
\zeta_8 > 0 & \Leftrightarrow & \textit{Config. 7.19b;} \\
\zeta_8 = 0 & \Leftrightarrow & \textit{Config. 7.23b.}
\end{array}$$

Proof. Considering Proposition 3.26 we evaluate for systems (3.84) the invariant polynomials ζ_8 (instead of ζ_1), ζ_2, ζ_4, ζ_5 and $\mathcal{D}_7$ which are involved in Proposition 3.26 in the case $\mathcal{D}_4 = 0$. The calculations yield:

$$\zeta_8 = 8m^2(3c - g^2 + m^2), \quad \zeta_2 = 216(3c - g^2 + 4m^2), \quad \zeta_4 = -m^2(13x^2 + 3y^2),$$
$$\zeta_5 = -64(3c - g^2 + 4m^2)(3c - g^2 + m^2), \quad \mathcal{D}_4 = 0, \quad \mathcal{D}_7 = 4, \quad \mathcal{D}_6 = -4.$$

As we can see the conditions $\mathcal{D}_7 > 0$ and $\zeta_4 \neq 0$ (due to $\chi_{15} \neq 0$, i.e. $m \neq 0$) hold. Therefore we conclude that the configurations *Configs. 7.13b, 7.15b, 7.18b, 7.20b, 7.21b, 7.22b* which correspond to the case $\mathcal{D}_7 < 0$ (or $\zeta_4 = 0$) and are realizable for systems (3.54) (see Proposition 3.26) could not be realizable for systems (3.84). Moreover the configuration *Configs. 7.24b* is defined by the conditions $\zeta_8 = \zeta_2 = 0$, however these conditions are incompatible with $\chi_{15} \neq 0$. Indeed, assuming $\zeta_8 = 0$ we get $c = (g^2 - m^2)/3$ and then $\zeta_2 = 648m^2 \neq 0$ (due to $\chi_{15} \neq 0$). Hence *Configs. 7.24b* could also not be realizable for systems (3.84).

To prove the compatibility of other conditions provided by Proposition 3.36 it is sufficient to present the examples of the realization of the corresponding configurations for systems (3.84) in terms of the parameters $(c, g, m) = (c_0, g_0, m_0)$. So we have

$$\begin{aligned}
&\textit{Config. 7.14b}: && (c_0, g_0, m_0) = (-3/2, 1, -1);\\
&\textit{Config. 7.16b}: && (c_0, g_0, m_0) = (-1/2, 1, -1);\\
&\textit{Config. 7.17b}: && (c_0, g_0, m_0) = (-1, 1, -1);\\
&\textit{Config. 7.19b}: && (c_0, g_0, m_0) = (1, 1, -1);\\
&\textit{Config. 7.23b}: && (c_0, g_0, m_0) = (0, 1, 1).
\end{aligned}$$

This completes the proof of Proposition 3.36. □

3.2.7 The statement (A_7)

According to the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_7) for the family of systems (3.12) lead to the conditions (3.39). We observe that these conditions contain the equality $H' = 0$ where the polynomial

$$H' = 27a^2 + 2am(9c + 4m^2) - (c - f)(c^2 + 4cf + 4f^2 + 4fm^2)$$

is quadratic with respect to parameter a. So in order to construct the canonical form of systems (3.12) subject to conditions (3.39) we have to examine this polynomial. We observe that

$$\text{Discrim}[H', a] = 4(3c - 3f + m^2)(3c + 6f + 4m^2)^2$$

and since according to the conditions (3.39) we must have $3c + 6f + 4m^2 \neq 0$ and $3c - 3f + m^2 \geq 0$ we set a new parameter v as follows: $3c - 3f + m^2 = v^2 \geq 0$. Then we obtain $f = (3c + m^2 - v^2)/3$ and this implies

$$H' = \left[27a + 9cm + 4m^3 + 3(3c + 2m^2)v - 2v^3\right]\left[27a + 9cm + 4m^3 - 3(3c + 2m^2)v + 2v^3\right]/27 = 0.$$

Due to the change $v \to -v$ we may assume that the first factor vanishes and we obtain

$$a = -(m + v)(9c + 4m^2 + 2mv - 2v^2)/27.$$

This leads to the family of systems

$$\begin{aligned}
\dot{x} &= (3x - m - v)(9c + 4m^2 + 2mv - 2v^2 + 12mx - 6vx - 18x^2)/27\\
&\equiv \frac{1}{27}\tilde{L}_1(x)\tilde{L}_{2,3}(x),\\
\dot{y} &= y(3c + m^2 - v^2 + 6mx - 9x^2 - 3y^2)/3.
\end{aligned} \tag{3.86}$$

We need to determine if the two lines defined by the equation $\tilde{L}_{2,3} = 0$ are real or complex and in the case when they are real, if one of them coincides with the invariant line $\tilde{L}_1 = 0$ or not. So we calculate

$$\text{Discrim}\,[\tilde{L}_{2,3}, x] = 108(6c + 4m^2 - v^2) \equiv 108\tilde{\lambda}, \quad Res_x(\tilde{L}_1, \tilde{L}_{2,3}) = 27(3c + 2m^2 - 2v^2) \equiv 27\tilde{\mu} \tag{3.87}$$

and clearly the invariant lines $\tilde{L}_{2,3} = 0$ are real (respectively complex; coinciding) if $\tilde{\lambda} > 0$ (respectively $\tilde{\lambda} < 0$; $\tilde{\lambda} = 0$). Moreover the invariant line $\tilde{L}_1 = 0$ coincides with one of the lines $\tilde{L}_{2,3} = 0$ if and only if $\tilde{\mu} = 0$.

On the other hand for systems (3.86) we calculate

$$\zeta_1 = -720\tilde{\lambda}x^2, \quad \zeta_5 = 64\tilde{\lambda}\tilde{\mu}$$

and evidently we have $\text{sign}\,(\zeta_1) = -\text{sign}\,(\tilde{\lambda})$ and in the case $\zeta_1 \neq 0$ the condition $\tilde{\mu} = 0$ is equivalent to $\zeta_5 = 0$.

Proposition 3.37. *Assume that for a system (3.86) the condition $\chi_{11} \neq 0$ holds. Then this system possesses one of the configurations of the invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\zeta_1 < 0, \zeta_5 < 0 & \Leftrightarrow & \textit{Config. 7.13b;} \\
\zeta_1 < 0, \zeta_5 > 0 & \Leftrightarrow & \textit{Config. 7.15b;} \\
\zeta_1 < 0, \zeta_5 = 0 & \Leftrightarrow & \textit{Config. 7.17b;} \\
\zeta_1 > 0, \zeta_4 \neq 0 & \Leftrightarrow & \textit{Config. 7.18b;} \\
\zeta_1 > 0, \zeta_4 = 0 & \Leftrightarrow & \textit{Config. 7.20b;} \\
\zeta_1 = 0, \zeta_5 \neq 0 & \Leftrightarrow & \textit{Config. 7.22b;} \\
\zeta_1 = 0, \zeta_5 = 0 & \Leftrightarrow & \textit{Config. 7.24b.}
\end{array}$$

Proof. Considering the above proposition we consider three cases: $\zeta_1 < 0$, $\zeta_1 > 0$ and $\zeta_1 = 0$.

a) *The case $\zeta_1 < 0$.* This implies $\tilde{\lambda} > 0$ and we may set $\tilde{\lambda} = 3w^2 > 0$. Then we obtain $c = (v^2 + 3w^2 - 4m^2)/6$ and this leads to the factorization

$$\tilde{L}_{2,3} = -(2m - v - 3w - 6x)(2m - v + 3w - 6x)/2$$

and since the condition $\tilde{\mu} = 0$ implies the coalescence of two invariant lines from the triplet we examine two subcases: $\zeta_5 \neq 0$ and $\zeta_5 = 0$.

a.1) *The subcase $\zeta_5 \neq 0$.* Then $\tilde{\mu} \neq 0$ and for the value of the parameter c given above we calculate: $\tilde{\mu} = 3(w - v)(w + v)/2 \neq 0$ and we can apply to systems (3.86) the transformation

$$x_1 = \frac{2}{(w-v)}\,x - \frac{2(m+v)}{3(w-v)}, \quad y_1 = \frac{2}{(w-v)}\,y, \quad t_1 = t(w-v)^2/4.$$

Then setting an additional parameter $a = (v + w)/(v - w) \neq 0, 1$ (because $\tilde{\mu} \neq 0$ and $a - 1 = 2w/(v - w) \neq 0$), we arrive at the following family of systems (we keep the old notations for the variables):

$$\begin{aligned} \dot{x} &= -2x(x-1)(x-a), \\ \dot{y} &= y(-2a + 2x + 2ax - 3x^2 - y^2). \end{aligned} \tag{3.88}$$

with $a(a - 1) \neq 0$. It remains to observe that this family of systems is a subfamily of systems (3.56) defined by the conditions $u = -3$ and $s = 0$. The canonical form (3.56) was obtained from (3.51) via an affine transformation and time rescaling in the case $\zeta_1 < 0$ and $\zeta_2 \neq 0$ (which imply $\lambda > 0$ and $\mu \neq 0$, respectively) and therefore all the invariant lines from the triplet are real and distinct.

In the proof of Proposition 3.26 it was shown that systems (3.56) with $s = 0$ and $a(a - 1)(u + 1) \neq 0$ possess the following configurations of invariant lines if and only if the corresponding conditions are satisfied, respectively:

$$\begin{array}{lcl} a(1+u)>0,\, 1+u<0 & \Leftrightarrow & \textit{Config. 7.13b}\,; \\ a(1+u)>0,\, 1+u>0 & \Leftrightarrow & \textit{Config. 7.14b}; \\ a(1+u)<0,\, 1+u<0 & \Leftrightarrow & \textit{Config. 7.15b}\,; \\ a(1+u)<0,\, 1+u>0 & \Leftrightarrow & \textit{Config. 7.16b}. \end{array}$$

Since for the systems (3.88) is a subfamily of systems (3.56) defined by the conditions $u = -3$ and $s = 0$ we have $1 + u = -2 < 0$. Therefore we conclude that systems (3.88) could not possess configurations *Config. 7.14b* and *Config. 7.16b*.

On the other hand for these systems we have

$$\zeta_5 = -1152a(a-1)^2 \quad \Rightarrow \quad \text{sign}\,(\zeta_5) = -\text{sign}\,(a)$$

and hence we arrive at *Config. 7.13b* if $\zeta_5 < 0$ and at *Config. 7.15b* if $\zeta_5 > 0$. So we deduce that in the case $\zeta_1 < 0$ and $\zeta_5 \neq 0$ systems Proposition 3.37 is true.

a.2) *The subcase* $\zeta_5 = 0$. Then $\tilde{\mu} = 0$ and we get $(w - v)(w + v) = 0$. We may assume $w - v = 0$ due to change $w \to -w$. So setting $v = w \neq 0$ we obtain $c = -2(m - w)(m + w)/3$ and therefore systems (3.86) become as systems

$$\begin{aligned} \dot{x} &= 2(m - 2w - 3x)(m + w - 3x)^2/27, \\ \dot{y} &= y\big[-(m - w)(m + w)y/3 + 2mxy - 3x^2y - y^3\big]. \end{aligned}$$

We observe that the above systems via the transformation

$$x_1 = -\frac{1}{w}x + \frac{m+w}{3w}, \quad y_1 = -\frac{1}{w}y, \quad t_1 = tw^2$$

can be brought to the system

$$\dot{x} = -2x^2(x-1), \quad \dot{y} = y(2x - 3x^2 - y^2).$$

This system is contained in the family (3.59) for $u = -3$ and $s = 0$. Since systems (3.59) in the case $s = 0$ possess the unique configuration of invariant line given by *Config. 7.17b* we conclude that Proposition 3.37 is true also in the case $\zeta_1 < 0$ and $\zeta_5 = 0$.

b) *The case* $\zeta_1 > 0$. This implies $\tilde{\lambda} < 0$ and we may set $\lambda = -3w^2 < 0$. So we obtain $c = (v^2 - 3w^2 - 4m^2)/6$ and this leads to the family of systems

$$\begin{aligned} \dot{x} &= (m + v - 3x)\big[9w^2 + (-2m + v + 6x)^2\big]/54, \\ \dot{y} &= -y(2m^2 + v^2 + 3w^2 - 12mx + 18x^2 + 6y^2)/6, \end{aligned} \tag{3.89}$$

for which we examine two subcases: $v \neq 0$ and $v = 0$. These conditions are governed by the invariant polynomial $\zeta_4 = -v^2(13x^2 + 3y^2)$.

b.1) *The subcase* $\zeta_4 \neq 0$. Then $v \neq 0$ and via the transformation

$$x_1 = -\frac{2}{v}x + \frac{2(m+w)}{3v}, \quad y_1 = -\frac{2}{v}y, \quad t_1 = tv^2/4$$

after the additional setting of the parameter $a = w/v \neq 0$ systems (3.89) can be brought to the systems

$$\dot{x} = -2x\big[(x-1)^2 + a^2\big], \quad \dot{y} = y(-2 - 2a^2 + 4x - 3x^2 - y^2). \tag{3.90}$$

So we get a subfamily of systems (3.61) defined by the conditions $u = -3$ and $s = 0$. We observe that systems (3.61) in the case $s = 0$ possess 2 configurations: *Config. 7.18b* if $u + 1 < 0$

and *Config. 7.19b* if $1+u>0$. However for systems (3.90) we have $1+u=-2<0$ and therefore we obtain the unique configuration *Config. 7.18b*.

b.2) *The subcase* $\zeta_4=0$. Then $v=0$ and since $w\neq 0$ in this case we apply to systems (3.89) the transformation

$$x_1=-\frac{2}{w}x-\frac{2m}{3w},\quad y_1=-\frac{2}{w}y,\quad t_1=tw^2/4$$

obtaining the following system

$$\dot{x}=-2x(1+x^2),\quad \dot{y}=-y(2+3x^2+y^2).$$

which is contained in the family (3.63) for $u=-3$ and $s=0$. Since for this system we have $\mathcal{D}_4=0$, $\zeta_1>0$, $\zeta_4=0$ and $\mathcal{D}_7=-8<0$, according to Proposition 3.26 we deduce that the above system possesses the unique configuration given by *Config. 7.20b*.

c) *The case* $\zeta_1=0$. This implies $\tilde{\lambda}=0$ and considering (3.87) we obtain $c=(v^2-4m^2)/6$ and this leads to the systems

$$\begin{aligned}\dot{x}&=(2m-v-6x)^2(m+v-3x)/54,\\ \dot{y}&=-y(2m^2+v^2-12mx+18x^2+6y^2)/6,\end{aligned}$$

for which we calculate $\zeta_4=-v^2(13x^2+3y^2)$.

c.1) *The subcase* $\zeta_4\neq 0$. Then $v\neq 0$ and via the transformation

$$x_1=-\frac{2}{v}x+\frac{2(m+v)}{3v},\quad y_1=-\frac{2}{v}y,\quad t_1=tv^2/4$$

we arrive at the following system

$$\dot{x}=-2(x-1)^2x,\quad \dot{y}=y(-2+4x-3x^2-y^2)$$

which belongs to the family (3.63) for $u=-3$ and $s=0$ already examined. We observe that systems (3.63) in the case $s=0$ possess 2 configurations: *Config. 7.22b* if $u+1<0$ and *Config. 7.23b* if $u+1>0$. However for the above system we have $1+u=-2<0$ and therefore we obtain the unique configuration *Config. 7.22b*.

c.1) *The subcase* $\zeta_4=0$. Then $v=0$ and we get the systems

$$\dot{x}=2(m-3x)^3/27,\quad \dot{y}=-y(m^2-6mx+9x^2+3y^2)/3$$

which via the transformation $x_1=x-m/3$, $y_1=y$, $t_1=t$ will be brought to the homogeneous systems

$$\dot{x}=-2x^3,\quad \dot{y}=-y(3x^2+y^2).$$

This system belongs to the family (3.65) for $u=-3$ and $s=0$ already examined and in the case $s=0$ it was determined that we have the unique configuration *Config. 7.24b*.

As all the cases are examined we deduce that Proposition 3.37 is proved. □

3.2.8 The statement (A_8)

We prove the following proposition.

Proposition 3.38. *Assume that for a system (3.12) the conditions provided by the statement (A_8) of the Main Theorem are satisfied. Then this system possesses one of the configurations of the invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl}
\mathcal{D}_4 \neq 0,\, \chi_5 \neq 0,\, \zeta_3 < 0 & \Leftrightarrow & \textit{Config. 7.33b;}\\
\mathcal{D}_4 \neq 0,\, \chi_5 \neq 0,\, \zeta_3 > 0 & \Leftrightarrow & \textit{Config. 7.34b;}\\
\mathcal{D}_4 \neq 0,\, \chi_5 = 0 & \Leftrightarrow & \textit{Config. 7.35b;}\\
\mathcal{D}_4 = 0,\, \zeta_2 \neq 0,\, \zeta_5 < 0 & \Leftrightarrow & \textit{Config. 7.36b;}\\
\mathcal{D}_4 = 0,\, \zeta_2 \neq 0,\, \zeta_5 > 0 & \Leftrightarrow & \textit{Config. 7.37b;}\\
\mathcal{D}_4 = 0,\, \zeta_2 = 0 & \Leftrightarrow & \textit{Config. 7.38b.}
\end{array}$$

Proof. As it was proved in the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_8) for the family of systems (3.12) lead to the conditions (3.43) in the case $\mathcal{D}_4 \neq 0$ and to the conditions (3.44) in the case $\mathcal{D}_4 = 0$. So we consider two cases: $\mathcal{D}_4 \neq 0$ and $\mathcal{D}_4 = 0$.

1: The case $\mathcal{D}_4 \neq 0$. Then for the family of systems (3.12) the conditions (3.43) are satisfied and we arrive at the systems

$$\begin{aligned}
\dot{x} &= \frac{1}{64s^2}(8sx-3l)(8gsx+3lg+8cs) \equiv \frac{1}{64s^2}L_1^{(1)}L_2^{(1)},\\
\dot{y} &= \frac{l}{256s^2}(9l^2+12lgs+32cs^2+l^2s^2) - \frac{l}{64s}(21l-8gs+ls^2)x + lx^2\\
&\quad + \frac{1}{64s^2}(3l^2s^2-9l^2+24lgs+64cs^2)y - \frac{1}{4s}(ls^2-3l-4gs)xy - sx^3 - x^2y - sxy^2 - y^3.
\end{aligned} \tag{3.91}$$

Next we investigate if the invariant lines $L_1^{(1)} = 0$ and $L_2^{(1)} = 0$ could coincide. So we calculate

$$Res_x(L_1^{(1)}, L_2^{(1)}) = 16s(3lg+4cs) \equiv 16s\mu^{(1)}$$

and since $s \neq 0$ we conclude that these two parallel invariant lines could coincide if and only if $\mu^{(1)} = 0$. We determine that this condition is governed by the invariant polynomial χ_5 because for systems (3.91) we have

$$\chi_5 = -(3lg+4cs)(9+s^2)/18.$$

***a)** The case* $\chi_5 \neq 0$. Then $\mu^{(1)} \neq 0$ and due to $gs \neq 0$ via the transformation

$$x_1 = -\frac{4gs}{\mu^{(1)}}x + \frac{3lg}{2\mu^{(1)}}, \quad y_1 = -\frac{4gs}{\mu^{(1)}}y - \frac{lgs}{2\mu^{(1)}}, \quad t_1 = \frac{\left[\mu^{(1)}\right]^2}{16g^2s^2}t,$$

after the additional setting of a new parameter $a = -\frac{4g^2s}{\mu^{(1)}}$ we arrive at the systems

$$\dot{x} = ax(x-1), \quad \dot{y} = -ay + axy - sx^3 - x^2y - sxy^2 - y^3 \tag{3.92}$$

for which we have $\chi_5 = 2as(9+s^2)/9 \neq 0$, i.e. $as \neq 0$.

We determine that systems (3.92) possess five distinct invariant affine straight lines

$$L_1 : x = 0, \quad L_2 : x = 1, \quad L_3 : y = -sx, \quad L_{4,5} : y = \pm ix$$

and by Lemma 3.2 the line at infinity is of multiplicity 2. On the other hand these systems possess the following six finite singularities:

$$M_1(0,0),\; M_{2,3}(0, \pm\sqrt{-a}\,),\; M_{4,5}(1, \pm i), \quad M_6(1, -s).$$

We observe that the singular points $M_{2,3}$ could be real (if $a<0$) or complex (if $a>0$), but they could not coincide due to $a\neq 0$. We draw attention to the fact that all these finite singularities are simple, because three finite singular points coalesced with infinite singularities.

Indeed considering Lemma 2.7 for systems (3.92) we calculate

$$\mu_0=\mu_1=\mu_2=0,\quad \mu_3=a^3(sx+y)(x^2+y^2)\neq 0.$$

So by Lemma 2.7 (see statement (i)) considering the factorization of the invariant polynomial μ_3 we deduce that one real finite singular point coalesced with the real infinite singularity $N[1:-s:0]$ which becomes of the multiplicity $(1,1)$ (see Remark 1.4). And simultaneously two complex finite singularities coalesced with the complex infinite singularities located at the intersection of the complex lines $y=\pm ix$ with the line at infinity $Z=0$ (however according to Definition 1.2 of a configuration, we do not consider the complex singularities).

On the other hand all the invariant lines of systems (3.92) are fixed, except for the invariant line $L_3: y=-sx$. Moreover we will determine according to our Convention (see page 8) the position of this line with respect to the complex lines $L_{4,5}: y=\pm ix$. Since $s\neq 0$, according to Remark 3.27 the invariant line $y=-sx$ does not coincide with the projection of the complex invariant lines $y=\pm ix$ on the plane (x,y).

We remark that the singular point $M_1(0,0)$ is a point of intersection of four invariant lines: L_1, L_3, L_4 and L_5 and that in the case $a<0$ the real singular points $M_{2,3}(0,\pm\sqrt{-a}\,)$, located on the invariant line $x=0$, are symmetric with respect to the origin of coordinates. As a result we arrive at the following two distinct configurations of invariant lines for systems (3.92) with $as\neq 0$: *Config. 7.33b* if $a<0$ and *Config. 7.34b* if $a>0$.

On the other hand for systems (3.92) we calculate $\zeta_3 = 2a^3s^2(9+s^2)^2/81$ and hence $\operatorname{sign}(a)=\operatorname{sign}(\zeta_3)$. So we deduce that systems (3.92) possess the configuration *Config. 7.33b* if $\zeta_3<0$ and *Config. 7.34b* if $\zeta_3>0$.

b) *The case* $\chi_5=0$. This implies $\mu^{(1)}=0$ and this means that the invariant line $L_1^{(1)}$ coalesces with $L_2^{(1)}$ and we have a double invariant line in the direction $x=0$. The condition $\mu^{(1)}=0$ yields $3lg+4cs=0$, i.e. $c=-3lg/(4s)$. In this case systems (3.91) can be brought via the transformation

$$x_1=\frac{1}{g}x-\frac{3l}{8gs},\quad y_1=\frac{1}{g}y+\frac{l}{8g},\quad t_1=g^2t,$$

to the family of systems

$$\dot{x}=x^2,\quad \dot{y}=xy-sx^3-x^2y-sxy^2-y^3 \tag{3.93}$$

with $s\neq 0$ (due to $\mathcal{D}_4\neq 0$). We determine that the above systems possess four distinct invariant affine straight lines

$$L_{1,2}: x=0,\quad L_3: y=-sx,\quad L_{4,5}: y=\pm ix.$$

We observe that the line $x=0$ as well as the line at infinity are of multiplicity 2 (see Lemma 3.2). On the other hand these systems possess the unique singularity $M_1(0,0)$ which is of the multiplicity six. Indeed considering Lemma 2.7 for systems (3.93) we calculate

$$\mu_0=\mu_1=\mu_2=0,\quad \mu_3=(sx+y)(x^2+y^2)\neq 0,\quad \mu_4=\mu_5=\mu_6=\mu_7=\mu_8=\mu_9=0.$$

Therefore by Lemma 2.7 (see statement (ii)) the above finite singularity has multiplicity six. On the other hand by the same arguments which we provided for systems (3.92) we deduce that the infinite singularity $N[1:-s:0]$ is of the multiplicity $(1,1)$.

So taking into account the condition $s \neq 0$ and Remark 3.27 as well as the fact that all the invariant affine lines of systems (3.93) intersect at the same singular point $M_1(0,0)$ (of multiplicity 6) we arrive at the unique configuration *Config. 7.35b.*

2: The case $\mathcal{D}_4 = 0$. Then for the family of systems (3.12) the conditions (3.44) are satisfied and we arrive at the systems

$$\begin{aligned} \dot{x} &= \frac{1}{4}(g-2m+2x)(2c-g^2+2gm+2gx) \equiv \frac{1}{4}L_1^{(2)}L_2^{(2)}, \\ \dot{y} &= \frac{1}{4}(4c-3g^2+8gm-4m^2)y+2mxy-x^2y-y^3. \end{aligned} \tag{3.94}$$

We calculate

$$Res_x(L_1^{(2)}, L_2^{(2)}) = 4(c-g^2+2gm) \equiv 4\mu^{(2)}$$

and clearly the parallel invariant lines $L_1^{(1)} = 0$ and $L_2^{(1)} = 0$ could coincide if and only if $\mu^{(2)} = 0$.

On the other hand for systems (3.94) we have $\zeta_2 = 288\mu^{(2)}$ and therefore the condition $\mu^{(2)} = 0$ is equivalent to $\zeta_2 = 0$.

***a)** The case $\zeta_2 \neq 0$.* Then since $g \neq 0$ (due to $\widetilde{\chi}_1 = 2gx^2y/3 \neq 0$) via the transformation

$$x_1 = -\frac{g}{\mu^{(2)}}x - \frac{g(g-2m)}{2\mu^{(2)}}, \quad y_1 = -\frac{g}{\mu^{(2)}}y, \quad t_1 = \frac{[\mu^{(2)}]^2}{g^2}t,$$

after the additional setting of a new parameter $a = -\frac{g^2}{\mu^{(2)}}$ we arrive at the systems

$$\dot{x} = ax(x-1), \quad \dot{y} = -ay+axy-x^2y-y^3.$$

So we get a subfamily of systems (3.92) defined by the condition $s = 0$ and considering the investigation of systems (3.92) and Remark 3.27 we deduce that the above systems possess the configuration *Config. 7.36b* if $a < 0$ and *Config. 7.37b* if $a > 0$.

We observe that in the case $s = 0$ the invariant polynomial ζ_3 vanishes because it contains as a factor s^2. In this case for determining the sign of the parameter a we apply the invariant ζ_5 that for the above systems has the value $\zeta_5 = -144a^3$. Hence we have sign $(a) = -$sign (ζ_5) and consequently we get the configuration *Config. 7.36b* if $\zeta_5 > 0$ and *Config. 7.37b* if $\zeta_5 < 0$.

***b)** The case $\zeta_2 = 0$.* Then $\mu^{(2)} = 0$ and this means that the invariant line $L_1^{(2)}$ coalesces with $L_2^{(2)}$ and we have a double invariant line in the direction $x = 0$. The condition $\mu^{(2)} = 0$ yields $c = g(g-2m)$ and then systems(3.94) via the transformation

$$x_1 = \frac{x}{g} + \frac{g-2m}{2g}, \quad y_1 = \frac{y}{g}y, \quad t_1 = g^2 t,$$

can be brought to the system

$$\dot{x} = x^2, \quad \dot{y} = xy - x^2y - y^3,$$

which belongs to the family (3.93) defined by the condition $s = 0$. Considering Remark 3.27 we deduce that the above system possesses the configuration *Config. 7.38b.*

Since all the cases are examined we conclude that Proposition 3.38 is proved. □

3.2.9 The statement (A_9)

We prove the following proposition.

Proposition 3.39. *Assume that for a system (3.12) the conditions provided by the statement (A_9) of the Main Theorem are satisfied. Then this system possesses one of the configurations of the invariant lines presented below if and only if the corresponding conditions are satisfied, respectively:*

$$\begin{array}{lcl} \mathcal{D}_4 \neq 0,\, \zeta_9 < 0 & \Leftrightarrow & \textit{Config. 7.39b;} \\ \mathcal{D}_4 \neq 0,\, \zeta_9 > 0 & \Leftrightarrow & \textit{Config. 7.40b;} \\ \mathcal{D}_4 = 0,\, \zeta_9 < 0 & \Leftrightarrow & \textit{Config. 7.41b;} \\ \mathcal{D}_4 = 0,\, \zeta_9 > 0 & \Leftrightarrow & \textit{Config. 7.42b;} \end{array}$$

Proof. According to the proof of the statement (A) of the Main Theorem the affine invariant conditions provided by the statement (A_9) for the family of systems (3.12) lead either to the conditions (3.47) in the case $\mathcal{D}_4 \neq 0$ or to the conditions (3.49). So we examine these two cases.

1: The case $\mathcal{D}_4 \neq 0$. Then we have the conditions (3.47) and in this case we arrive at the systems

$$\begin{aligned} \dot{x} &= cx - \frac{3cl}{8s}, \\ \dot{y} &= \frac{l}{256s^2}(9l^2 + 32cs^2 + l^2s^2) - \frac{l^2}{64s}(21 + s^2)\,x + \frac{1}{64s^2}(3l^2s^2 - 9l^2 + 64cs^2)\,y \\ &\quad + lx^2 - \frac{l}{4s}(s^2 - 3)\,xy - sx^3 - x^2y - sxy^2 - y^3. \end{aligned}$$

For these systems we have

$$\widetilde{\chi}_2 = 4cx^3(sx+y)(x^2+y^2)\big[(3s^2-1)x^2 + 8sxy + (3-s^2)y^2\big]/3, \quad \mathcal{D}_4 = 2304s(9+s^2)$$

and therefore the condition $\widetilde{\chi}_2\mathcal{D}_4 \neq 0$ implies $cs \neq 0$. Then the above systems could be brought via the transformation

$$x_1 = x - \frac{3l}{8s}, \quad y_1 = y + \frac{l}{8}, \quad t_1 = t$$

to the following family of systems

$$\dot{x} = cx, \quad \dot{y} = cy - sx^3 - x^2y - sxy^2 - y^3 \tag{3.95}$$

with $cs \neq 0$. We determine that systems (3.95) possess four distinct invariant affine straight lines

$$L_1 : x = 0, \quad L_2 : y = -sx, \quad L_{3,4} : y = \pm ix.$$

Moreover the line at infinity has multiplicity 3 (see Lemma 3.2, statement (iii)). On the other hand these systems possess the following three singularities:

$$M_1(0,0),\ M_{2,3}(0, \pm\sqrt{c}\,)$$

and the singular points M_2 and M_3 could be real (if $c > 0$) or complex (if $c < 0$). We draw attention to the fact that all these finite singularities are simple, because six finite singularities coalesced with infinite singularities.

Indeed considering Lemma 2.7 for systems (3.95) we calculate

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \quad \mu_6 = -c^3(sx+y)^2(x^2+y^2)^2 \neq 0.$$

So by Lemma 2.7 (see statement (i)) considering the factorization of the invariant polynomial μ_6 we deduce that two real finite singular point coalesced with the real infinite singularity $N_1[1 : -s : 0]$ and this infinite singularity becomes of the multiplicity $(2,1)$ (see Remark 1.4), whereas four complex finite singularities coalesced with complex singularities at infinity. More exactly, two of them with $N[1 : +i : 0]$ and other two with $\bar{N}[1 : -i : 0]$. However according to Definition 1.2 this fact is irrelevant for a configuration.

So taking into account our Convention (see page 8) and the fact that all the invariant affine lines of systems (3.95) intersect at the same singular point $M_1(0,0)$ (of multiplicity 6) we arrive at the following two configurations:

$$\textit{Config. 7.39b} \Leftrightarrow \quad c > 0; \quad \textit{Config. 7.40b} \Leftrightarrow \quad (c < 0).$$

On the other hand for systems (3.95) we calculate

$$\zeta_9 = -2cx^2\big[(3s^2-1)x^2 + 8sxy + (3-s^2)y^2\big]^2/27$$

and hence sign $(\zeta_9) = -$sign (c). Therefore we get *Config. 7.39b* if $\zeta_9 < 0$ and *Config. 7.40b* if $\zeta_9 > 0$.

2: The case $\mathcal{D}_4 = 0$. Then $s = 0$ and in this case the conditions (3.49) hold for systems (3.12). In this case we arrive at the systems

$$\dot{x} = -cm + cx, \quad \dot{y} = (c-m^2)y + 2mxy - x^2y - y^3$$

applying the transformation $(x,y,t) \mapsto (x+m, y, t)$ we arrive at the systems (3.95) with $s = 0$.

Thus considering our Convention (see page 8) and the sign of the invariant polynomial ζ_9 we arrive at the configuration of invariant lines given by *Config. 7.41b* if $\zeta_9 < 0$ and by *Config. 7.42b* if $\zeta_9 > 0$. This completes the proof of Proposition 3.39. □

Since all the cases provided by the statement (A) are examined we conclude that the statement (B) of the Main Theorem is proved completely.

3.3 Geometric invariants and the proof of the statement (C)

In this subsection we complete the proof of the Main Theorem by showing that all 42 configurations of invariant lines we constructed are non-equivalent according to Definition 1.3. For this we define the invariants that split the configurations of this family into the 42 *distinct* ones. We would like these invariants to be among those best suited for describing the geometric phenomena that are specific to this class.

The basic algebraic-geometric definitions of use here are the notion of an integer valued r-cycle and its type i.e. we take $G = Z$ in the Definitions 1.5 and 1.6 and we have:

Definition 3.40. Let V be an irreducible algebraic variety of dimension n over a field K. A cycle of dimension r or r-cycle on V is a formal sum $\sum_W m(W)W$ where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $m(W) \in \mathbb{Z}$, and only a finite number of $m(W)$'s are non-zero. We call degree of an r-cycle the sum $\sum_W$. An $(n-1)$-cycle is called a divisor.

Definition 3.41. We call type of an r-cycle the set of all ordered couples (n_1, n_2) where n_1 is a coefficient, $n_1 = m(W)$ appearing in the $r - cyle$ and n_2 is the number of W's in the cycle whose coefficient is $m(W)$.

We denote the type of an r-cycle C by $\mathcal{T}(C)$. We use the following notations:

$$\mathbf{CS} = \left\{ (S) \,\middle|\, \begin{array}{l} (S) \text{ is a system (2.1) such that } \gcd(P(x,y), Q(x,y)) = 1 \\ \text{and } \max\big(\deg(P(x,y)), \deg(Q(x,y))\big) = 3 \end{array} \right\};$$

$$\mathbf{CSL} = \left\{ (S) \in \mathbf{CS} \,\middle|\, \begin{array}{l} (S) \text{ possesses at least one invariant affine line or} \\ \text{the line at infinity with multiplicity at least two} \end{array} \right\}.$$

Notation 3.42. Let

$$\begin{aligned}
\widetilde{P}(X,Y,Z) &= p_0(\boldsymbol{a})Z^2 + p_1(\boldsymbol{a},X,Y)Z + p_2(\boldsymbol{a},X,Y);\\
\widetilde{Q}(X,Y,Z) &= q_0(\boldsymbol{a})Z^2 + q_1(\boldsymbol{a},X,Y)Z + q_2(\boldsymbol{a},X,Y);\\
\widetilde{C}(X,Y,Z) &= Y\widetilde{P}(X,Y,Z) - X\widetilde{Q}(X,Y,Z);\\
\sigma(p,q) &= \{w \in \mathbb{R}^2) \mid p(w) = q(w) = 0\};\\
\mathbf{D}_S(\widetilde{P},\widetilde{Q}) &= \sum_{w\in\sigma(\widetilde{P},\widetilde{Q})} I_w(\widetilde{P},\widetilde{Q})w;\\
\mathbf{D}_S(\widetilde{C},Z) &= \sum_{w\in\{Z=0\}} I_w(\widetilde{C},Z)w \quad \text{if} \quad Z \nmid \widetilde{C}(X,Y,Z);\\
\mathbf{D}_S(\widetilde{P},\widetilde{Q};Z) &= \sum_{w\in\{Z=0\}} I_w(\widetilde{P},\widetilde{Q})w;\\
\widehat{\mathbf{D}}_S(\widetilde{P},\widetilde{Q},Z) &= \sum_{w\in\{Z=0\}} \Big(I_w(\widetilde{C},Z),\, I_w(\widetilde{P},\widetilde{Q})\Big)w,
\end{aligned}$$

where $I_w(F,G)$ is the intersection number (see [19]) of the curves defined by homogeneous polynomials $F,\ G \in \mathbb{C}[X,Y,Z]$ and $\deg(F), \deg(G) \geq 1$.

The set $\sigma(p,q)$ is thus formed by the finite (or affine) singularities of a polynomial system defined by $p(x,y), q(x,y)$. The multiplicity of a finite singular point w is the number $I_w(p,q)$ which is the intersection number of the affine curves defined by p and q. The total multiplicity of a point at infinity, i.e. located on $Z = 0$ is $I_w(\tilde{P},\tilde{Q})$ and it is the sum $I_w(\widetilde{C},Z) + I_w(\widetilde{P},\widetilde{Q})$ of the two multiplicities appearing in the last divisor above. A complex projective line $uX + vY + wZ = 0$ in $\mathbf{P}_2(\mathbb{C})$ is invariant for a system (S) if it either coincides with $Z = 0$ or it is the projective completion of an invariant affine line $ux + vy + w = 0$.

Notation 3.43. Let $(S) \in \mathbf{CSL}$. Let us denote

$$\mathbf{IL}(S) = \left\{ \; l \;\middle|\; \begin{array}{l} l \text{ is a line in } \mathbf{P}_2(\mathbb{C}) \text{ such} \\ \text{that } l \text{ is invariant for } (S) \end{array} \right\};$$

$$M(l) = \text{the multiplicity of the invariant line } l \text{ of } (S).$$

In defining $M(l)$ we assume, of course, that (S) has a finite number of invariant lines.

Remark 3.44. We note that the line $L_\infty : Z = 0$ is included in $\mathbf{IL}(S)$ for any $(S) \in \mathbf{CSL}$.

Assuming we have a finite number of invariant lines, let $l_i\ :\ f_i(x,y) = ax + by + c = 0$, $i = 1, \ldots, k$, be all the distinct invariant affine lines (real or complex) of a system $(S) \in \mathbf{CSL}$. Let $L_i\ :\ \mathcal{F}_i(X,Y,Z) = aX + bY + cZ = 0$ be the complex projective completion of l_i. Let M_i be the multiplicity of the line L_i and let M be the multiplicity of the line at infinity $Z = 0$.

Notation 3.45.

$$\mathcal{G}\ :\ \prod_i \mathcal{F}_i(X,Y,Z)^{M_i}Z^M = 0; \quad Sing\,\mathcal{G} = \{w \in \mathcal{G} \mid w \text{ is a singular point of } \mathcal{G}\}\,;$$

$$m(w) = \text{the multiplicity of the point } w\text{, as a point of } \mathcal{G}.$$

We call $\mathcal{G}$ *the total curve.*

Suppose that a system (2.1) possesses a finite number of invariant lines $L_1, \dots L_k$, including the line at infinity. Sometimes it is convenient to consider in our discussion a number of these invariant lines say $L_{i_1}, \dots L_{i_l}$ of a system (S). We call *marked* system (S) by the lines $L_{i_1}, \dots L_{i_l}$ the object denoted by $(S, L_{i_1}, \dots L_{i_l})$ of the system (S) in which we singled out the lines $L_{i_1}, \dots L_{i_l}$. We shall consider invariants attached to such marked systems.

Because in this paper we are concerned with triplets of parallel lines, *the affine plane* clearly plays an important role. This needs to be reflected in our choice of invariants. We now define an invariant that captures the most basic geometric distinctions of the configurations in this family:

Definition 3.46. Let $\mathcal{M}$ be the ordered couple $(M_{Aff}, M(l_\infty))$, where M_{Aff} is the maximum multiplicity of the invariant affine lines of the system and $M(l_\infty)$ is the multiplicity of the line at infinity. Clearly $\mathcal{M}$ is an invariant.

Using $\mathcal{M}$ we split the 42 configurations in 6 classes: three with $M(l_\infty) = 1$ and three with $M(l_\infty) > 1$.

We describe now the way the invariant $\mathcal{M}$ captures the geometry of the configurations related to the parallel lines by letting $\mathcal{M}$ run through all its six possible values: the generic case and five limiting cases:

$\mathcal{M} = (1,1)$ This is ***the generic case*** with 3 (distinct) parallel lines;

$\mathcal{M} = (2,1)$ is ***a first limit case*** of the preceding one, where two of the three parallel lines coalesced yielding just two parallel lines, one of them double;

$\mathcal{M} = (3,1)$ is ***a second limit case*** where the three parallel lines coalesced yielding a triple line;

$\mathcal{M} = (1,2)$ is ***a third limit case*** where a line of the triplet coalesced with the line at infinity yielding a double line at infinity;

$\mathcal{M} = (1,3)$ is ***a fourth limit case*** where two lines of the triplet disappeared at infinity yielding a triple line at infinity;

$\mathcal{M} = (2,2)$ is ***a fifth limit case*** when one one line of the triplet went to infinity and the other two lines of the triplet coalesced.

It is clear that we also need to define invariants that relate to the real singularities of the systems located on the configurations. We first observe that all the real singularities of the systems are located on the invariant lines of the configurations, occasionally even on a single line.

We encapsulate in two zero-cycles $C^{\mathbb{R}}_{Sing} = \sum_w \nu(w)w$ and $C^{R}_{\mathcal{G}} = \sum_w m(w)w$ the multiplicity properties of the real singularities of the systems located on the configurations. In the first cycle we denoted by $\nu(w)$ the multiplicity of the real singular point w and in the second cycle we denoted by $m(w)$ the multiplicity of the real singular point w this time regarded as a simple or multiple point of the total curve $\mathcal{G}$. We denote their respective types by $\mathcal{T}^{\mathbb{R}}_{Sing}$ and $\mathcal{T}^{\mathbb{R}}_{\mathcal{G}}$. In view of the geometry of the systems we actually only need to consider the restriction of these two invariants on the affine plane and we denote them by $\mathcal{T}^{\mathbb{R},aff}_{Sing}$ and $\mathcal{T}^{\mathbb{R},aff}_{\mathcal{G}}$. If anyone of these two invariants, say $\mathcal{T}^{\mathbb{R},aff}_{\mathcal{G}}$ yields the same value for two or more configurations, to be

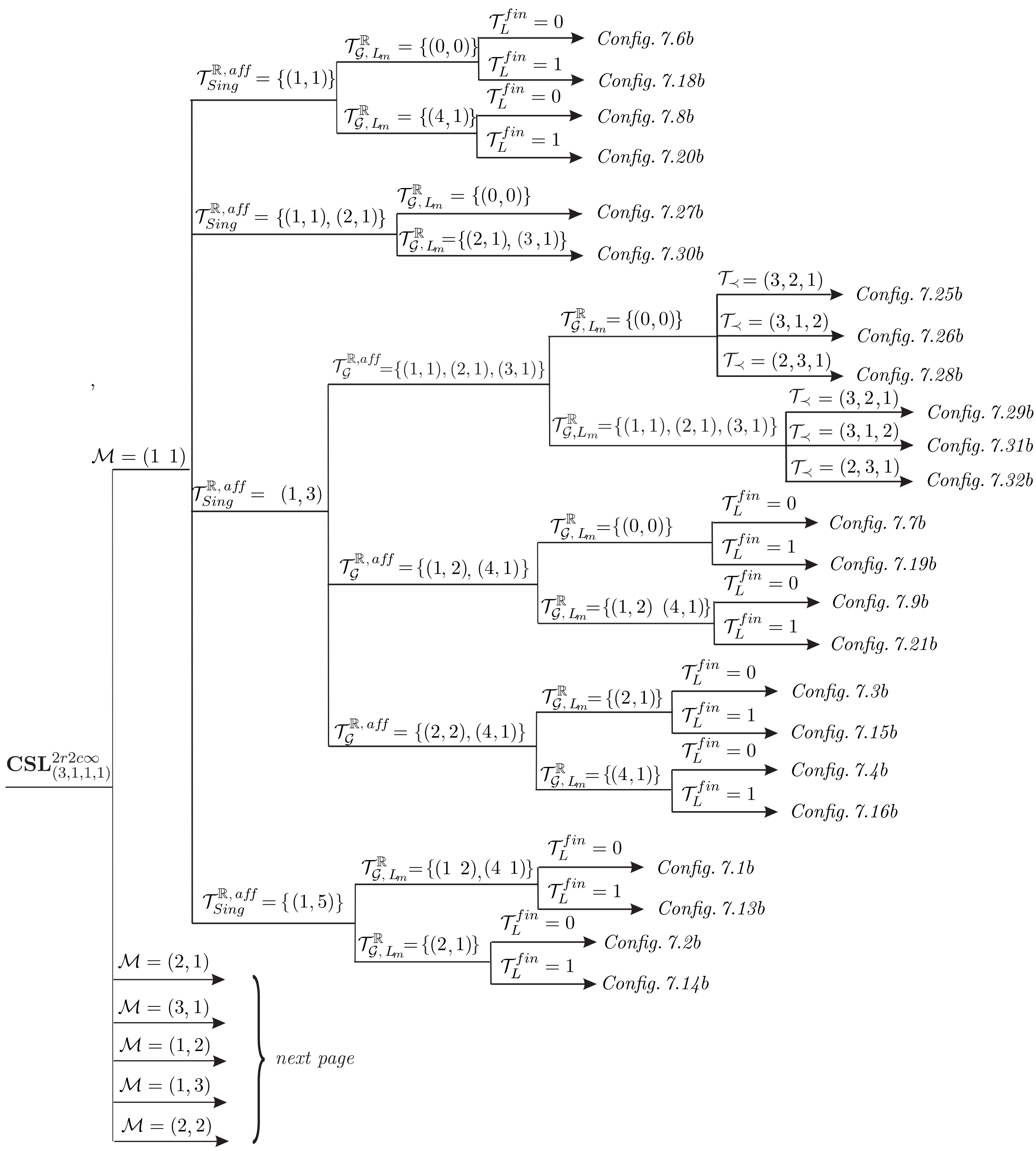

Figure 3.1: Diagram of non-equivalent configurations

able to distinguish we shall need to restrict its value to a single affine line L and in this case the resulting invariant will be denoted by $\mathcal{T}^{\mathbb{R}}_{\mathcal{G},L}$.

Assume that for a marked system $(S, L_r, L_c, \bar{L}_c)$ with a real invariant line L_r and a complex invariant line L_c together with its conjugate line $\bar{L}_c$ these three invariant lines intersect at the same real point which could be finite or infinite.

Considering our Convention (see page 8) we define an invariant $\mathcal{T}^{fin}_{L}$ for such marked systems $(S, L_r, L_c, \bar{L}_c)$ in the case when the intersection point is finite:

$\mathcal{T}^{fin}_{L} = 1$ if and only if the real invariant line L_r coincides with the line $\mathcal{R}(L_c, \bar{L}_c) : y = ax + c$

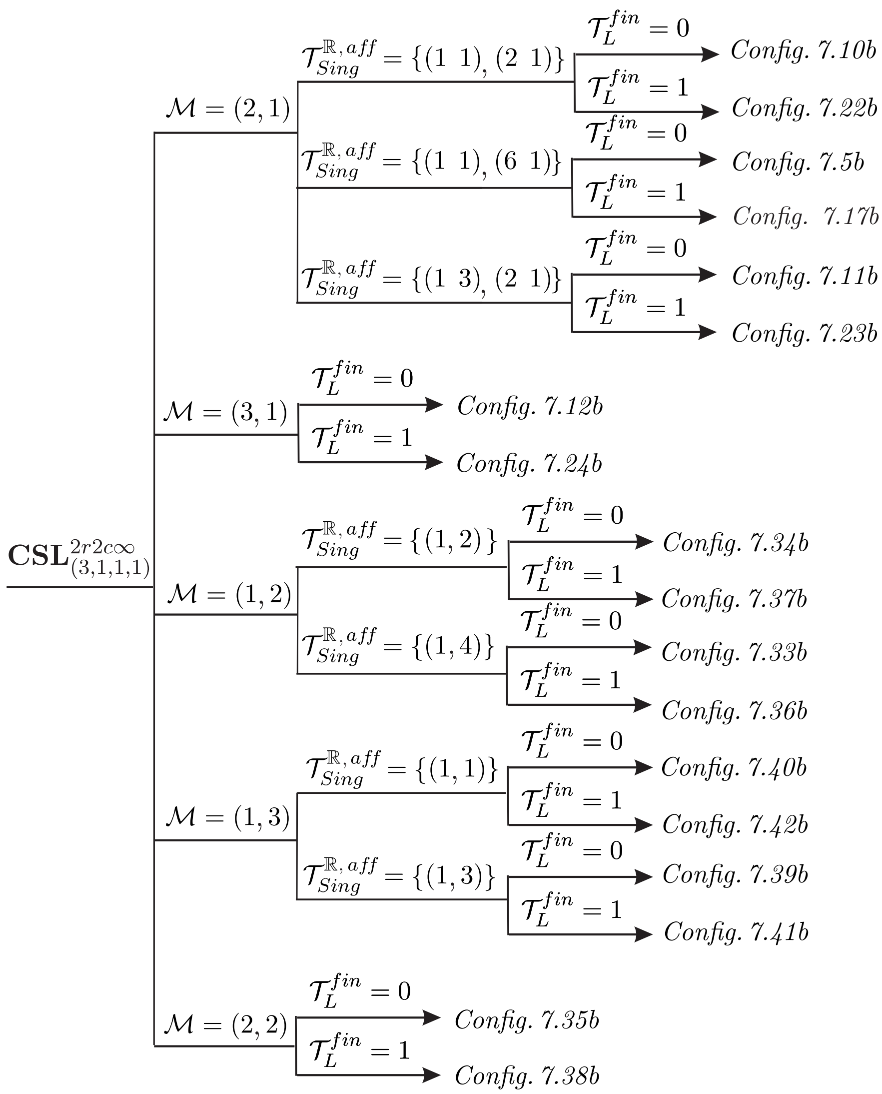

Figure 3.1 (*cont.*): Diagram of non-equivalent configurations

defined in our Convention on page 6;

$\mathcal{T}^{fin}_{L}=0$ if and only if the the real invariant line L_r does not coincide with $\mathcal{R}(L_c,\bar{L}_c)$.

Let us now consider the generic case $\mathcal{M}=(1,1)$ which is the more complex one. This class contains 30 configurations i.e. all *Config. 7.jb* with $j\leq 32$ with two exceptions: *Config. 7.12b* and *Config. 7.22b*. To distinguish the corresponding configurations the first one of the invariants we use is $\mathcal{T}^{\mathbb{R},aff}_{Sing}$ and its values for this class are: $\mathcal{T}^{\mathbb{R},aff}_{Sing}$: $\{(1,1)\},\{(1,1),(2,1)\},\{(1,3)\},\ \{(1,5)\}$. For the second case we then only need to apply $\mathcal{T}^{\mathbb{R}}_{\mathcal{G},L_m}$ while for the first and last case to distinguish further the configurations we need to apply first $\mathcal{T}^{\mathbb{R}}_{\mathcal{G},L_m}$, where L_m is the middle line in the triplet of parallel lines and secondly the invariant $\mathcal{T}^{fin}_{L}$. In the third case, i.e. $\mathcal{T}^{\mathbb{R},aff}_{Sing}=\{(1,3)\}$ we first use $\mathcal{T}^{\mathbb{R},aff}_{\mathcal{G}}$ which has three values and for two of them $\mathcal{T}^{\mathbb{R}}_{\mathcal{G},L_m}$ together with $\mathcal{T}^{fin}_{L}$ distinguish the configurations. For the value $\mathcal{T}^{\mathbb{R},aff}_{\mathcal{G}}=\{(1,1),(2,1),(3,1)\}$ we need a new invariant which we denote by $\mathcal{T}_{\prec}$ and define as follows:

We first observe that for all six configurations occurring for $\mathcal{T}^{\mathbb{R},aff}_{\mathcal{G}}=\{(1,1),(2,1),(3,1)\}$ all real affine singularities are located on a single real affine line and they are three in number determining a closed interval on this line. Based on this observation we introduce this new invariant. We consider these three real singular points and their associated multiplicities as simple or multiple points of the curve $\mathcal{G}$. We first note that the maximum multiplicity of the three points in all six cases is either 3 or 4 and this maximum multiplicity corresponds to a uniquely determined point. We then list the multiplicities $m(w)$ in an ordered sequence in the following way. If we have an end point of the segment determined by the three points which is of maximum multiplicity, we initiate the sequence with its multiplicity and we folow with the multiplicity of the middle point and end with the multiplicity of the other end point of the segment. If none of the end points has maximum multiplicity then we start with the multiplicity of the end point of maximum multiplicity among the two and follow with the multiplicity of the middle point and finally with the multiplicity of the other end point. In case the two end points have equal multiplicity we start with the common multiplicity followed by the multiplicity of the middle point and end with the common multiplicity of the end points. This order is clearly preserved as the multiplicities are preserved. So this is an invariant which we denote by $\mathcal{T}_{\prec}$. The case $\mathcal{T}_{\mathcal{G}}^{\mathbb{R},aff} = \{(1,1),(2,1),(3,1)\}$ is the only one where this invariant occurs. For the remaining values of $\mathcal{M}$ to distinguish the configurations the two invariants $\mathcal{T}_{Sing}^{\mathbb{R},aff}$ and $\mathcal{T}_{L}^{fin}$ do the job as we see in the bifurcation diagram for the configurations which gives all the explicit calculations of the invariants (see Figure 5).

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Qualitative analysis on the diffusive Holling–Tanner predator–prey model

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Abstract. We consider the diffusive Holling–Tanner predator–prey model subject to the homogeneous Neumann boundary condition. We first apply Lyapunov function method to prove some global stability results of the unique positive constant steady-state. And then, we derive a non-existence result of positive non-constant steady-states by a novel approach that can also be applied to the classical Sel'kov model to obtain the non-existence of positive non-constant steady-states if $0 < p \leq 1$.

Keywords: Holling–Tanner predator–prey model, Sel'kov model, global stability, Lyapunov function method.

2020 Mathematics Subject Classification: 34C20, 34C25, 92E20.

1 Introduction

In this paper, we consider the diffusive Holling–Tanner predator–prey model:

$$\begin{cases} u_t - d_1\Delta u = au - u^2 - \dfrac{uv}{m+u}, & x \in \Omega, t > 0, \\ v_t - d_2\Delta v = bv - \dfrac{v^2}{\gamma u}, & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) > 0, & x \in \overline{\Omega}. \end{cases} \tag{1.1}$$

Here u and v are the density of prey and predator, respectively, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, and the parameters $d_1, d_2, a, b, m, \gamma$ are positive constants. The initial data u_0 and v_0 are $C^1(\overline{\Omega})$ functions satisfying $\partial_\nu u_0 = \partial_\nu v_0 = 0$ on $\partial\Omega$. The model describes real ecological interactions of various populations such as lynx and hare, sparrow and sparrow hawk (cf. [7, 13, 15]), and the Neumann boundary condition means that no species can pass across the boundary $\partial\Omega$. We note that problem (1.1) has a unique positive global solution, see the Appendix for the proof.

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It is easy to verify that system (1.1) has a unique positive equilibrium $\mathbf{E}_* = (u_*, v_*)$, where

$$u_* = \frac{1}{2}\left[a - m - b\gamma + \sqrt{(a-m-b\gamma)^2 + 4am}\right] \quad \text{and} \quad v_* = b\gamma u_*.$$

System (1.1) had been extensively investigated, see [1, 2, 4, 5, 9–11] and the references therein. In particular, Peng and Wang [10], Chen and Shi [1], Duan, Niu and Wei [2], and Qi and Zhu [11] proved some stability results that are collected as follows.

Theorem 1.1. *Suppose $d_1, d_2, a, m, b, \gamma$ are positive constants. Then the following statements hold.*

- *(a) (See [10]). The positive equilibrium $\mathbf{E}_*$ is locally asymptotically stable if $m^2 + 2(a+b\gamma)m + a^2 - 2ab\gamma > 0$.*
- *(b) (See [1]). The positive equilibrium $\mathbf{E}_*$ is globally asymptotically stable if $m > b\gamma$.*
- *(c) (See [2]). The positive equilibrium $\mathbf{E}_*$ is globally asymptotically stable if $u_* \geq m$ and $m \geq a - u_*$.*
- *(d) (See [11]). $\lim_{t\to+\infty}(u(x,t), v(x,t)) = \mathbf{E}_*$ uniformly on $\overline{\Omega}$ if $d_1 = d_2$ and $\gamma^{-1} > \frac{a}{m+a}$.*

Motivated by the above works in [1,2,10,11], in the present paper, we first study the global stability of the positive equilibrium $\mathbf{E}_*$, and obtain the following result.

Theorem 1.2. *Suppose $d_1, d_2, a, m, b, \gamma$ are positive constants. Then the positive equilibrium $\mathbf{E}_*$ is globally asymptotically stable if $m > \max\{M_1, M_2\}$, where*

$$M_1 = \frac{ab\gamma}{a + b\gamma} \quad \text{and} \quad M_2 = \frac{1}{2}\left[(b\gamma - 2a)_+ + \sqrt{b\gamma(b\gamma - 2a)_+}\right].$$

Here $s_+ = \max\{0, s\}$.

Obviously, $M_1, M_2 < b\gamma$. Then Theorem 1.2 is an improvement to Theorem 1.1(b). Since $a - u_* = \frac{v_*}{m+u_*} = \frac{b\gamma u_*}{m+u_*}$, we see that $a - u_* < m \Leftrightarrow b\gamma u_* < m(m+u_*)$, so $a - u_* < m \Leftrightarrow m > M_1$ according to Lemma 2.1(a). On the other hand, since the condition $m \leq u_*$ implies $\frac{am}{a+2m} < u_*$, it also implies $m > M_2$ because $\frac{am}{a+2m} < u_* \Leftrightarrow m > M_2$ according to Lemma 2.1(b). Thus, Theorem 1.2 is also an improvement to Theorem 1.1(c).

Note that for fixed a, b and m, every global result in Theorems 1.1 and 1.2 excludes the case where γ is large. In this paper, we prove the following result that covers the case.

Theorem 1.3. *Suppose $d_1, d_2, a, m, b, \gamma$ are positive constants with $d_1 = d_2, b > a, m > M_1 = \frac{ab\gamma}{a+b\gamma}$, and*

$$2am\left[a + 2m + \frac{2m(b-a)}{m+a}\gamma\right]^{-1} < a - m - b\gamma + \sqrt{(a-m-b\gamma)^2 + 4am}. \tag{1.2}$$

Then the positive equilibrium $\mathbf{E}_*$ is globally asymptotically stable.

Remark 1.4. Let $m > a$ and $b > \frac{2am}{m-a}$. Then $m > M_1$ and (1.2) hold for any sufficiently large γ. Indeed, we have

$$\begin{aligned}
&\lim_{\gamma\to+\infty} \gamma\left[a - m - b\gamma + \sqrt{(a-m-b\gamma)^2 + 4am} - \frac{2am}{a + 2m + \frac{2m(b-a)}{m+a}\gamma}\right] \\
&= \lim_{\gamma\to+\infty}\left[\frac{4am\gamma}{\sqrt{(a-m-b\gamma)^2 + 4am} - (a-m-b\gamma)} - \frac{2am\gamma}{a + 2m + \frac{2m(b-a)}{m+a}\gamma}\right] \\
&= \frac{a(m-a)}{b(b-a)}\left(b - \frac{2am}{m-a}\right) > 0.
\end{aligned}$$

Then, as a consequence of Theorem 1.3, we obtain immediately

Corollary 1.5. *Suppose $d_1, d_2, a, m, b, \gamma$ are positive constants with $d_1 = d_2$, $m > a$ and $b > \frac{2am}{m-a}$. Then there exists a positive constant γ_0 depending only on b, a, m such that $\mathbf{E}_*$ is globally asymptotically stable for any $\gamma \geq \gamma_0$.*

The steady-states of system (1.1) satisfy

$$\begin{cases} -d_1\Delta u = au - u^2 - \frac{uv}{m+u}, & x \in \Omega, \\ -d_2\Delta v = bv - \frac{v^2}{\gamma u}, & x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Theorems 1.1–1.3 obviously imply some conditions for the non-existence of positive non-constant solutions of system (1.3), which are independent of the coefficients d_1 and d_2. In [9], Peng and Wang gave some conditions for the non-existence of positive non-constant solutions of system (1.3), which depend on d_1 and d_2, see [9, Theorems 3.1 and 3.5]. For example, they proved that system (1.3) has no positive non-constant solution if d_1 and d_2 are sufficiently large, see [9, Theorems 3.1]. By using a different approach from those in literature (see e.g. [8, 10]), we prove the following result on the non-existence of positive non-constant solutions.

Theorem 1.6. *Suppose $m \geq a$. Then system (1.3) has no positive non-constant solution.*

We point out that the approach used to show Theorem 1.6 can be applied to some interesting models to discuss non-existence of positive non-constant solutions, for instance, the steady-state Sel'kov model (see [12]):

$$\begin{cases} -\theta\Delta u = \lambda(1 - uv^p), & x \in \Omega, \\ -\Delta v = \lambda(uv^p - v), & x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where θ, λ, p are positive constants, which had been studied in [6, 8, 14]. For the case when $0 < p \leq 1$, Peng [8] proved the non-existence of positive non-constant solutions of system (1.3) if θ is sufficiently large. In the present paper, we remove the restriction on θ and obtain

Theorem 1.7. *Suppose θ, λ, p are positive constants. If $0 < p \leq 1$, then system (1.4) has no positive non-constant solution.*

The rest of this paper is organized as follows. In Section 2, we will prove Theorems 1.2 and 1.3 by using Lyapunov function method. In Section 3, we will prove Theorems 1.6 and 1.7 by a novel approach. Finally, our conclusions are given in Section 4.

2 Proofs of Theorems 1.2 and 1.3

We begin with the following lemma.

Lemma 2.1. *The following statements hold.*

(a) $m(m + u_*) > b\gamma u_*$ *if and only if* $m > M_1 = \frac{ab\gamma}{a+b\gamma}$.

(b) $\frac{am}{a+2m} < u_*$ *if and only if* $m > M_2 = \frac{1}{2}\left[(b\gamma - 2a)_+ + \sqrt{b\gamma(b\gamma - 2a)_+}\,\right]$, *where* $s_+ = \max\{0, s\}$.

Proof. As for the conclusion (a), it it clear to see that the case where $m \geq b\gamma$ is trivial. We now suppose $m < b\gamma$. For the case, if $m(m + u_*) > b\gamma u_*$, i.e., $m^2 > (b\gamma - m)u_*$, then

$$2m^2 - (b\gamma - m)(a - m - b\gamma) > (b\gamma - m)\sqrt{(a - m - b\gamma)^2 + 4am}, \tag{2.1}$$

and then taking the square on the two sides of (2.1) yields $m > \frac{ab\gamma}{a+b\gamma}$. Note that the above reasoning process is also inverse since $m > \frac{ab\gamma}{a+b\gamma}$ implies

$$\begin{aligned} 2m^2 - (b\gamma - m)(a - m - b\gamma) &= m^2 + (b\gamma)^2 + ma - ab\gamma \\ &> mb\gamma + ma - ab\gamma \\ &> 0. \end{aligned}$$

Thus the conclusion (a) is valid.

As for the conclusion (b), a simple calculation gives

$$\begin{aligned} (a + 2m)u_* - am > 0 \Leftrightarrow b\gamma &< \frac{2(a+m)^2}{a+2m} \\ &\Leftrightarrow 2m^2 + 2(2a - b\gamma)m + a(2a - b\gamma) > 0. \end{aligned}$$

Solving the latter gives $m > M_2$. This completes the proof of the lemma. □

Proof of Theorem 1.2. Let (u, v) be a positive solution of system (1.1). Adapting the Lyapunov function in [2,3], we define

$$\begin{aligned} V(u,v) &= \int_{u_*}^{u} \frac{\eta - u_*}{\eta g(\eta)} d\eta + \frac{\gamma u_*}{v_*} \int_{v_*}^{v} \frac{\eta - v_*}{\eta} d\eta, \text{ where } g(u) = \frac{u}{m+u}; \\ W(t) &= \int_\Omega V(u(x,t), v(x,t)) dx. \end{aligned} \tag{2.2}$$

Denote $g_1(u,v) = au - u^2 - g(u)v$ and $g_2(u,v) = bv - \frac{v^2}{\gamma u}$. Some calculations give

$$\begin{aligned} \int_\Omega V_u(u,v)u_t dx &= \int_\Omega \frac{u - u_*}{ug(u)} [d_1 \Delta u + g_1(u,v)] dx \\ &= -d_1 \int_\Omega [u_*(g(u) + ug'(u)) - u^2 g'(u)] \frac{|\nabla u|^2}{[ug(u)]^2} dx \\ &\quad + \int_\Omega \frac{u - u_*}{g(u)} \left[u_* - u + \left(\frac{g(u_*)}{u_*} - \frac{g(u)}{u} \right) v_* + \frac{g(u)}{u}(v_* - v) \right] dx \\ &= -d_1 \int_\Omega \frac{(u_* - m)u^2 + 2mu^* u}{(m+u)^2} \frac{|\nabla u|^2}{[ug(u)]^2} dx - \int_\Omega \frac{(u - u_*)(v - v_*)}{u} dx \\ &\quad - \int_\Omega \frac{(u - u_*)^2}{g(u)} \left[1 - \frac{b\gamma u_*}{(m+u)(m+u_*)} \right] dx \quad (\text{note that } v_* = b\gamma u_*), \end{aligned}$$

and

$$\begin{aligned} \int_\Omega V_v(u,v)v_t dx &= \frac{\gamma u_*}{v_*} \int_\Omega \frac{v - v_*}{v} [d_2 \Delta v + g_2(u,v)] dx \\ &= -\gamma d_2 u_* \int_\Omega \frac{|\nabla v|^2}{v^2} dx + \frac{u_*}{v_*} \int_\Omega (v - v_*) \left(\frac{v_*}{u_*} - \frac{v}{u} \right) dx \\ &= -\gamma d_2 u_* \int_\Omega \frac{|\nabla v|^2}{v^2} dx + \frac{u_*}{v_*} \int_\Omega (v - v_*) \left(\frac{v_*}{u_*} - \frac{v_*}{u} + \frac{v_*}{u} - \frac{v}{u} \right) dx \\ &= -\gamma d_2 u_* \int_\Omega \frac{|\nabla v|^2}{v^2} dx + \int_\Omega \frac{(v - v_*)(u - u_*)}{u} dx - \frac{u_*}{v_*} \int_\Omega \frac{(v - v_*)^2}{u} dx. \end{aligned}$$

It follows that

$$
\begin{aligned}
W'(t) = & -d_1 \int_\Omega \frac{(u_* - m)u^2 + 2mu_* u}{(m+u)^2} \frac{|\nabla u|^2}{[ug(u)]^2} dx - \gamma d_2 u_* \int_\Omega \frac{|\nabla v|^2}{v^2} dx \\
& - \int_\Omega \frac{(u-u_*)^2}{g(u)} \left[1 - \frac{b\gamma u_*}{(m+u)(m+u_*)}\right] dx - \frac{u_*}{v_*} \int_\Omega \frac{(v-v_*)^2}{u} dx \\
\le & -d_1 \int_\Omega \frac{(u_* - m)u^2 + 2mu_* u}{(m+u)^2} \frac{|\nabla u|^2}{[ug(u)]^2} dx - \gamma d_2 u_* \int_\Omega \frac{|\nabla v|^2}{v^2} dx \\
& - \left[1 - \frac{b\gamma u_*}{m(m+u_*)}\right] \int_\Omega \frac{(u-u_*)^2}{g(u)} dx - \frac{u_*}{v_*} \int_\Omega \frac{(v-v_*)^2}{u} dx.
\end{aligned}
\tag{2.3}
$$

We now assume that $m > \max\{M_1, M_2\}$. Then, $1 - \frac{b\gamma u_*}{m(m+u_*)} > 0$ by Lemma 2.1(a), and $\frac{am}{a+2m} < u_*$ by Lemma 2.1(b), so there exists a constant $\varepsilon > 0$ such that

$$
\frac{(a+\varepsilon)m}{a+\varepsilon+2m} < u_*. \tag{2.4}
$$

On the other hand, from (1.1), we have

$$
u_t - d_1 \Delta u \le u(a-u), \quad \forall (x,t) \in \Omega \times (0, +\infty).
$$

It follows from the comparison principle that $\limsup_{t\to+\infty} \max_{x\in\overline{\Omega}} u(x,t) \le a$, and hence there exists some $T > 0$, such that

$$
u(x,t) < a + \varepsilon, \quad \forall (x,t) \in \overline{\Omega} \times [T, +\infty). \tag{2.5}
$$

Combining (2.4) and (2.5) gives

$$
\begin{aligned}
(u_* - m)u^2(x,t) + 2mu_* u(x,t) &= u(x,t)[u(x,t) + 2m]\left[u_* - \frac{mu(x,t)}{u(x,t)+2m}\right] \\
&> u(x,t)[u(x,t) + 2m]\left[u_* - \frac{m(a+\varepsilon)}{a+\varepsilon+2m}\right] \\
&> 0, \quad \forall (x,t) \in \overline{\Omega} \times [T, +\infty),
\end{aligned}
$$

therefore, $W'(t) \le 0$ for all $t \ge T$, and equality holds if and only if $(u,v) = \mathbf{E}_*$, so $\mathbf{E}_*$ is globally attractive. Since $m > M_1$ (i.e., $m(a + b\gamma) > ab\gamma$), $\mathbf{E}_*$ is locally asymptotically stable according to Theorem 1.1(a), so is globally asymptotically stable. The proof of the theorem is complete. □

We now are ready to show Theorem 1.3, whose proof is based on the following lemma.

Lemma 2.2. *Suppose $d_1 = d_2$ and $b > a$. Then*

$$
\limsup_{t\to+\infty} \max_{x\in\overline{\Omega}} u(x,t) \le a\left(1 + \frac{b-a}{m+a}\gamma\right)^{-1}.
$$

Proof. Like in [11], we set $\varphi = \frac{v}{u}$. Then a simple calculation gives

$$
\varphi_t = \frac{1}{u} v_t - \frac{v}{u^2} u_t, \qquad \nabla \varphi = \frac{1}{u} \nabla v - \frac{v}{u^2} \nabla u,
$$

and

$$\Delta\varphi = \frac{1}{u}\Delta v - \frac{v}{u^2}\Delta u - \frac{2}{u}\nabla u\cdot\nabla\varphi,$$

so that

$$\begin{aligned}\varphi_t - \frac{2d_1}{u}\nabla u\cdot\nabla\varphi - d_1\Delta\varphi &= \varphi\left(b-a-\frac{1}{\gamma}\varphi + u + \frac{v}{m+u}\right)\\ &\geq \varphi\left(b-a-\frac{1}{\gamma}\varphi\right),\end{aligned}$$

therefore, from the comparison principle, for any $0<\varepsilon\ll 1$ there exists some constant $T_1^\varepsilon \gg 1$ such that

$$\varphi(x,t) \geq (b-a)\gamma - \varepsilon > 0, \quad \forall (x,t)\in\overline{\Omega}\times[T_1^\varepsilon,+\infty). \tag{2.6}$$

By a similar argument to (2.5), there exists some constant $T_2^\varepsilon > T_1^\varepsilon$ such that

$$u(x,t) < a+\varepsilon, \quad \forall (x,t)\in\overline{\Omega}\times[T_2^\varepsilon,+\infty). \tag{2.7}$$

Combining (2.6), (2.7) and $(1.1)_1$, we obtain

$$u_t - d_1\Delta u \leq u\left[a - \left(1+\frac{(b-a)\gamma-\varepsilon}{m+a+\varepsilon}\right)u\right], \quad \forall (x,t)\in\overline{\Omega}\times[T_2^\varepsilon,+\infty).$$

This implies $\limsup_{t\to+\infty}\max_{x\in\overline{\Omega}} u(x,t) \leq \frac{a}{1+\frac{(b-a)\gamma-\varepsilon}{m+a+\varepsilon}}$. Then, letting $\varepsilon\to 0$ gives the desired result. □

Proof of Theorem 1.3. We adapt the same Lyapunov function as that in (2.2).

From Lemma 2.2 and (1.2), there exist some constants $0<\varepsilon\ll 1$ and $T\gg 1$ such that

$$u(x,t) \leq a\left[1+\frac{(b-a)\gamma-\varepsilon}{m+a}\right]^{-1}, \quad \forall (x,t)\in\overline{\Omega}\times[T,\infty), \tag{2.8}$$

and

$$\begin{aligned}2am\left\{a+2m+\frac{2m[(b-a)\gamma-\varepsilon]}{m+a}\right\}^{-1} &< a-m-b\gamma+\sqrt{(a-m-b\gamma)^2+4am}\\ &= 2u_*.\end{aligned} \tag{2.9}$$

Since $F(x) = x/(2m+x)$ is increasing in $[0,\infty)$, it follows from (2.8) and (2.9) that

$$\frac{mu(x,t)}{2m+u(x,t)} \leq \frac{am}{a+2m+\frac{2m[(b-a)\gamma-\varepsilon]}{m+a}} < u_*, \quad \forall (x,t)\in\overline{\Omega}\times[T,+\infty).$$

That is,

$$(u_*-m)u^2 + 2muu_* = u\left[u_*(u+2m) - mu\right] > 0, \quad \forall (x,t)\in\overline{\Omega}\times[T,+\infty).$$

Combining this and (2.3) with $d_1 = d_2$ yields $W'(t)\leq 0$ for all $t\geq T$, and equality holds if and only if $(u,v) = \mathbf{E}_*$, so $\mathbf{E}_*$ is globally attractive. Since $m > M_1$, $\mathbf{E}_*$ is locally asymptotically stable according to Theorem 1.1(a), so is globally asymptotically stable. The proof is complete. □

3 Proofs of Theorems 1.6 and 1.7

We first show Theorem 1.6.

Proof of Theorem 1.6. Assume that (u, v) is a positive solution of system (1.3). Multiplying $(1.1)_1$ by $[(a-u)(m+u)-v]$ and integrating by parts over Ω, we have

$$d_1\int_\Omega \nabla u\cdot\nabla[(a-u)(m+u)-v]dx=\int_\Omega\frac{u}{m+u}[(a-u)(m+u)-v]^2dx,$$

that is,

$$d_1\int_\Omega(a-m-2u)|\nabla u|^2dx-d_1\int_\Omega\nabla u\cdot\nabla v dx=\int_\Omega\frac{u}{m+u}[(a-u)(m+u)-v]^2dx. \tag{3.1}$$

Multiplying $(1.1)_2$ by $(u-\frac{v}{b\gamma})$ and integrating over Ω, we obtain

$$d_2\int_\Omega\nabla u\cdot\nabla v dx-\frac{d_2}{b\gamma}\int_\Omega|\nabla v|^2dx=\int_\Omega\frac{bv}{u}\Big(u-\frac{v}{b\gamma}\Big)^2dx. \tag{3.2}$$

We first multiply (3.2) by d_1/d_2, and then add the resulting equation and (3.1) to get

$$\begin{aligned}&d_1\int_\Omega(a-m-2u)|\nabla u|^2dx-\frac{d_1}{b\gamma}\int_\Omega|\nabla v|^2dx\\&\quad-\int_\Omega\left\{\frac{u}{m+u}[(a-u)(m+u)-v]^2+\frac{d_1bv}{d_2u}\Big(u-\frac{v}{b\gamma}\Big)^2\right\}dx=0.\end{aligned} \tag{3.3}$$

Since $m\ge a$, the first term on the left hand side of (3.3) is non–positive and hence u and v must be constants. The proof is complete. □

Proof of Theorem 1.7. Assume that (u, v) is a positive solution of system (1.4). Multiplying (1.4) by $(\frac{1}{u}-v^p)$ and $(uv^{p-1}-1)$, respectively, and integrating by parts over Ω, we have

$$-\theta\int_\Omega\frac{|\nabla u|^2}{u^2}dx-\theta\int_\Omega\nabla u\cdot\nabla v^p dx=\lambda\int_\Omega u\Big(\frac{1}{u}-v^p\Big)^2dx, \tag{3.4}$$

and

$$(p-1)\int_\Omega uv^{p-2}|\nabla v|^2dx+\frac{1}{p}\int_\Omega\nabla u\cdot\nabla v^p dx=\lambda\int_\Omega v(uv^{p-1}-1)^2dx. \tag{3.5}$$

We first multiply (3.5) by $p\theta$, and then add the resulting equation and (3.4) to obtain

$$\int_\Omega\Big[\theta\frac{|\nabla u|^2}{u^2}+\theta p(1-p)uv^{p-2}|\nabla v|^2\Big]dx+\lambda\int_\Omega\Big[u\Big(\frac{1}{u}-v^p\Big)^2+p\theta v(uv^{p-1}-1)^2\Big]dx=0.$$

Consequently, u and v must be constants if $p\in(0,1]$. The proof is complete. □

Remark 3.1. In [14, Remark 2.1], the authors pointed out that it is difficult to expect the bifurcation of (1.4) near $(u, v) = (1, 1)$ if $0 < p \le 1$ since the constant positive solution $(u, v) = (1, 1)$ is uniformly asymptotically stable for the corresponding reaction–diffusion system to (1.4) for the case. Our Theorem 1.7 shows that no bifurcation will happen for system (1.4) provided that $0 < p \le 1$.

4 Conclusions

In this paper, we prove some new global stability results. In particular, the works by Chen and Shi [1] and Duan, Niu and Wei [2], mentioned above, have been improved. In addition, we derive a non-existence result of the positive non-constant steady-states for system (1.1) by using a different approach from those in literature. By virtue of the approach, we also obtain a complete understanding of the steady-state Sel'kov model for the case when $0 < p \leq 1$.

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Appendix

In this part, we will only prove the global existence of positive solutions of problem (1.1) since the proof to uniqueness is standard. To this end, we will use the regularization method. In what follows, we assume that the initial data u_0 and v_0 are $C^1(\overline{\Omega})$ functions satisfying $u_0, v_0 > 0$ on $\overline{\Omega}$ and $\partial_\nu u_0 = \partial_\nu v_0 = 0$ on $\partial\Omega$.

Let $\varepsilon \in (0,1)$ be a constant. Consider the regularized problem:

$$\begin{cases} u_t - d_1\Delta u = au - u^2 - \dfrac{uv}{m+u}, & x \in \Omega, t > 0, \\ v_t - d_2\Delta v = bv - \dfrac{v^2}{\gamma(u+\varepsilon)}, & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \overline{\Omega}. \end{cases} \tag{$(P)_\varepsilon$}$$

From the standard theory of parabolic equations, system $(P)_\varepsilon$ has a unique nonnegative global solution $(u_\varepsilon, v_\varepsilon)$ for any given $\varepsilon \in (0,1)$.

Let $\overline{u}(t)$ be a solution of the following problem:

$$\begin{cases} \dfrac{d\overline{u}}{dt} = a\overline{u} - \overline{u}^2, \quad t > 0, \\ \overline{u}(0) = \max\limits_{\overline{\Omega}} u_0(x) =: M > 0. \end{cases}$$

It is easy to check that $\overline{u}(t) = \frac{e^{at}}{M^{-1}+\int_0^t e^{as}ds} \leq M_1$ on $[0,+\infty)$ for some constant M_1 independent of ε. Note that u_ε satisfies

$$(u_\varepsilon)_t - d_1\Delta u_\varepsilon \leq u_\varepsilon(a - u_\varepsilon), \qquad x \in \Omega, t > 0.$$

It follows from the comparison principle that $u_\varepsilon(x,t) \leq \overline{u}(t) \leq M_1$ on $\overline{\Omega} \times [0,+\infty)$. Consequently, we have

$$(v_\varepsilon)_t - d_2\Delta v_\varepsilon \leq bv_\varepsilon - \frac{v_\varepsilon^2}{\gamma(M_1+1)}, \qquad x \in \Omega,\ t > 0.$$

Similarly, there exists some constant $M_2 > 0$, independent of ε, such that $v_\varepsilon(x,t) \le M_2$ on $\overline{\Omega} \times [0,+\infty)$. Hence,

$$(u_\varepsilon)_t - d_1\Delta u_\varepsilon \ge -(M_1 + M_2 m^{-1})u_\varepsilon =: -C_1 u_\varepsilon, \qquad x \in \Omega, t > 0.$$

By the comparison principle, $u_\varepsilon(x,t) \ge \underline{u}(t)$ on $\overline{\Omega} \times [0,+\infty)$, where $\underline{u}(t) = (\min_{\overline{\Omega}} u_0)e^{-C_1 t}$ satisfies

$$\begin{cases} \frac{d\underline{u}}{dt} = -C_1\underline{u}, & t > 0, \\ \underline{u}(0) = \min_{\overline{\Omega}} u_0 > 0. \end{cases}$$

It follows that

$$(v_\varepsilon)_t - d_2\Delta v_\varepsilon \ge -\frac{M_2 e^{C_1 t}}{\gamma \min_{\overline{\Omega}} u_0} v_\varepsilon =: -C_2 e^{C_1 t} v_\varepsilon, \qquad x \in \Omega,\ t > 0.$$

Again using the comparison principle, we see that $v_\varepsilon(x,t) \ge \underline{v}(t)$ on $\overline{\Omega} \times [0,+\infty)$, where $\underline{v}(t) = (\min_{\overline{\Omega}} v_0)e^{-C_2\int_0^t e^{C_1 s}ds}$ satisfies

$$\begin{cases} \frac{d\underline{v}}{dt} = -C_2 e^{C_1 t}\underline{v}, & t > 0, \\ \underline{v}(0) = \min_{\overline{\Omega}} v_0 > 0. \end{cases}$$

In summary, we have, for all $\varepsilon \in (0,1)$,

$$M_1 \ge u_\varepsilon(x,t) \ge \underline{u}(t), \qquad M_2 \ge v_\varepsilon(x,t) \ge \underline{v}(t), \qquad \forall (x,t) \in \overline{\Omega} \times [0,+\infty).$$

Then, by a standard compactness argument, one can obtain a positive global solution of system (1.1). This proof is complete.

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Normalized solutions to the Schrödinger systems with double critical growth and weakly attractive potentials

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Abstract. In this paper, we look for solutions to the following critical Schrödinger system

$$\begin{cases} -\Delta u + (V_1 + \lambda_1)u = |u|^{2^*-2}u + |u|^{p_1-2}u + \beta r_1 |u|^{r_1-2}u|v|^{r_2} & \text{in } \mathbb{R}^N, \\ -\Delta v + (V_2 + \lambda_2)v = |v|^{2^*-2}v + |v|^{p_2-2}v + \beta r_2 |u|^{r_1}|v|^{r_2-2}v & \text{in } \mathbb{R}^N, \end{cases}$$

having prescribed mass $\int_{\mathbb{R}^N} u^2 = a_1 > 0$ and $\int_{\mathbb{R}^N} v^2 = a_2 > 0$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ will arise as Lagrange multipliers, $N \geqslant 3$, $2^* = 2N/(N-2)$ is the Sobolev critical exponent, $r_1, r_2 > 1$, $p_1, p_2, r_1 + r_2 \in (2 + 4/N, 2^*)$ and $\beta > 0$ is a coupling constant. Under suitable conditions on the potentials V_1 and V_2, $\beta_* > 0$ exists such that the above Schrödinger system admits a positive radial normalized solution when $\beta \geqslant \beta_*$. The proof is based on comparison argument and minmax method.

Keywords: Schrödinger systems, weakly attractive potentials, normalized solutions, positive solutions.

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1 Introduction and main results

We study the following critical Schrödinger system

$$\begin{cases} -\Delta u + (V_1 + \lambda_1)u = |u|^{2^*-2}u + |u|^{p_1-2}u + \beta r_1 |u|^{r_1-2}u|v|^{r_2} & \text{in } \mathbb{R}^N, \\ -\Delta v + (V_2 + \lambda_2)v = |v|^{2^*-2}v + |v|^{p_2-2}v + \beta r_2 |u|^{r_1}|v|^{r_2-2}v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

with prescribed mass

$$\int_{\mathbb{R}^N} u^2 = a_1 > 0 \quad \text{and} \quad \int_{\mathbb{R}^N} v^2 = a_2 > 0, \tag{1.2}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ will arise as Lagrange multipliers, $N \geqslant 3$, $2^* = 2N/(N-2)$ is the Sobolev critical exponent, $r_1, r_2 > 1$, $p_1, p_2, r_1 + r_2 \in (2 + 4/N, 2^*)$, V_1 and V_2 are the potentials and $\beta > 0$ is a coupling constant. Solutions of (1.1) with prescribed mass (1.2) are called as the normalized solutions in the literature.

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The problem (1.1) comes from the research of solitary waves to the following system

$$\begin{cases} -\Delta\Phi_1 + V_1\Phi_1 - i\frac{\partial}{\partial t}\Phi_1 = |\Phi_1|^{2^*-2}\Phi_1 + |\Phi_1|^{p_1-2}\Phi_1 + \beta r_1|\Phi_1|^{r_1-2}\Phi_1|\Phi_2|^{r_2} \\ -\Delta\Phi_2 + V_2\Phi_2 - i\frac{\partial}{\partial t}\Phi_2 = |\Phi_2|^{2^*-2}\Phi_2 + |\Phi_2|^{p_2-2}\Phi_2 + \beta r_2|\Phi_1|^{r_1}|\Phi_2|^{r_2-2}\Phi_2 \\ \Phi_j = \Phi_j(x,t), (x,t) \in \mathbb{R}^N \times \mathbb{R}_+, \quad j = 1,2, \end{cases} \tag{1.3}$$

where t denotes the time, i is imaginary unit, Φ_j is the wave function of the jth component, β is a coupling constant which describes the scattering length of the attractive and repulsive interaction. If $\beta > 0$, then the interaction is attractive; if $\beta < 0$, then the interaction is repulsive. Set $\Phi_1(x,t) = e^{i\lambda_1 t}u(x)$ and $\Phi_2(x,t) = e^{i\lambda_2 t}v(x)$. It is easy to see that a couple (Φ_1, Φ_2) is the solution of (1.3) if and only if (u,v) is the solution of (1.1). The system (1.3) appears in many physical problems, especially in nonlinear optics and the mean-field models for binary mixtures of Bose-Einstein condensation, see [1,13,14] and reference therein for more physical background. An important, of course well known, feature of (1.3) is conservation of mass:

$$\int_{\mathbb{R}^N} |\Phi_j(x,t)|^2 dx = \int_{\mathbb{R}^N} |\Phi_j(x,0)|^2 dx, \qquad t \in \mathbb{R}_+.$$

Physically, the mass represents the number of particles of each component in Bose–Einstein condensates.

The presence of the mass constraint makes some methods developed to deal with unconstrained problems unavailable, and a new critical exponent appears, the mass critical exponent $2 + 4/N \in (2, 2^*)$. In the mass subcritical case, the Schrödinger equation are usually considered by the minimization arguments, we refer the readers to [8,9,29]. As far as we are aware, the mass supercritical case was first considered by Jeanjean in [21], for the Schrödinger equation. The key idea is to obtain mountain pass solution on S_a by constructing the mountain pass structure on a natural constraint related to the Pohozaev identity. Much work has been done extensively on the normalized solutions to the Schrödinger equation in in the last decades by variational methods. Since numerous contributions flourished within this topic and we just mention, among many possible numerous choices, [23,30,31]. For the nonautonomous Schrödinger equations, we refer the readers to [20,33] when mass subcritical case occurs and [5,12,28] when mass supercritical case occurs.

The existence and multiplicity of normalized solutions to the Schrödinger systems also attracted much attention of researchers in recent decades, see [2–4,6,7,10,17,18,22,25–27] and reference therein. In particular, for the Schrödinger system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1|u|^{p-2}u + \nu_1|u|^{p_1-2}u + \beta r_1|u|^{r_1-2}u|v|^{r_2} & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2|v|^{p-2}v + \nu_2|v|^{p_2-2}v + \beta r_2|u|^{r_1}|v|^{r_2-2}v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

when $N \geqslant 3$, $\nu_1 = \nu_2 = 0$, $p = 4$ and $r_1 = r_2 = 2$, the existence and multiplicity of normalized solutions to (1.4) are studied in [4,6,7]; when $N = 3,4$, $\mu_1 = \mu_2 = 0$, $r_1, r_2 > 1$, $p_1, r_1 + r_2 \in (2, 2^*)$ and $p_2 \in (2, 2^*]$, Li and Zou in [22] studied the geometry of the associated Pohozaev manifold and obtained a normalized solution to (1.4); when $N = 4$, $p = 3$, $p_1, p_2 \in (2,4)$ and $r_1 = r_2 = 2$, the coupling terms are the Sobolev critical case, Luo et al. in [27] considered the existence, nonexistence and asymptotic behavior of normalized solutions to (1.4); when $N = 3,4$, $r_1, r_2 > 1$, $p = 2^*$ and $p_1, p_2, r_1 + r_2 \in (2 + 4/N, 2^*]$, recently, Liu and Fang in [26] obtained the existence and nonexistence of normalized solutions to system (1.4).

To the best of our knowledge, a few studies have addressed the existence of normalized solutions to Schrödinger system with potential. We know only [10,25], in which they considered the mass subcritical case. There is no work concerning normalized solutions to Schrödinger systems with mass supercritical, Sobolev critical and potential. This problem is more complicated and stimulating by the fact that both the potential and the critical term are present, which is the focus of this article. Specifically, in this paper, we consider Schrödinger system (1.1) with weakly attractive potentials, that is,

$$V_i(x) \leqslant \limsup_{|x|\to\infty} V_i(x) < \infty, \qquad i = 1,2,$$

and obtain a positive radial normalized solution. For the weakly repulsive potentials, that is,

$$V_i(x) \geqslant \liminf_{|x|\to\infty} V_i(x) > -\infty, \qquad i = 1,2,$$

does the system (1.1) have a normalized solution? This still is an open problem.

Precisely, $V_i \in C^1(\mathbb{R}^N)$ fulfills

(H$_1$) $\lim_{|x|\to\infty} V_i(x) = \sup_{x\in\mathbb{R}^N} V_i(x) = 0$ and there exists $\tau_i \in [0, 1/2)$ such that $|V_i|_{N/2} \leqslant \tau_i S$, where

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}}; \tag{1.5}$$

(H$_2$) set $W_i(x) := (\nabla V_i(x)\cdot x)/2$, $W_i \in C^1(\mathbb{R}^N)$, $\lim_{|x|\to\infty} W_i(x) = 0$ and there exists $\theta_i \in [0, 1)$ with $(1-\tau_i)/2 - (1+\theta_i)/(\min\{\gamma_{p_1}p_1, \gamma_{p_2}p_2, \gamma_r r\}) > 0$ such that $|W_i|_{N/2} \leqslant \theta_i S$, where $\gamma_q = N(q-2)/(2q)$.

(H$_3$) set $Y_i(x) := \gamma_{p_i}p_i W_i(x) + Z_i(x)$, where $Z_i(x) := \nabla W_i(x)\cdot x$ and $Z_i \in L^s(\mathbb{R}^N)$ for some $s \in [N/2, \infty]$, there exists $\rho_i \in [0, \gamma_{p_i}p_i - 2)$ such that $|Y_{i,+}|_{N/2} \leqslant \rho_i S$ for any $u \in E_i$, where $Y_{i,+} = \max\{Y_i, 0\}$.

An example satisfying the conditions (H$_1$)–H$_3$) is $V_i(x) = -\frac{b}{|x|^c+1}$, $x \in \mathbb{R}^N$ with constant $c > 2$ and suitable small constant b. Obviously, $V = 0$ also satisfies the conditions (H$_1$)–(H$_3$). Hence, the following theorem includes the autonomous case $V = 0$.

Normalized solutions of (1.1) can be found as critical points of the C^1 functional

$$\begin{aligned} I(u,v) = &\frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + V_1u^2 + V_2v^2) - \frac{1}{2^*}\int_{\mathbb{R}^N} (|u|^{2^*} + |v|^{2^*}) \\ &- \frac{1}{p_1}\int_{\mathbb{R}^N} |u|^{p_1} - \frac{1}{p_2}\int_{\mathbb{R}^N} |v|^{p_2} - \beta\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}, \ (u,v) \in E_1\times E_2, \end{aligned}$$

on

$$S_{a_1}\times S_{a_2} := \left\{(u,v) \in E_1\times E_2 : \int_{\mathbb{R}^N} u^2 = a_1, \ \int_{\mathbb{R}^N} v^2 = a_2\right\},$$

with Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$. Here

$$E_i := \left\{u \in H^1_r(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_i u^2 < \infty\right\}, \qquad i = 1,2$$

and $H^1_r(\mathbb{R}^N)$ is the usual radial Sobolev space. The norm of E_i is defined by

$$\|u\|_i = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V_i u^2 + u^2)\right)^{1/2}, \qquad u \in E_i, \ i = 1,2,$$

which is equivalent to the usual norm $\|u\|_{H^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)\right)^{1/2}$ due to the condition (H_1). The solution $(u, v) \in S_{a_1} \times S_{a_2}$ is called a positive radial normalized solution of (1.1) if $u > 0$ and $v > 0$.

Now we state our main results.

Theorem 1.1. *Let $N = 3, 4$, $r_1, r_2 > 1$, $p_1, p_2, r_1 + r_2 \in (2 + 4/N, 2^*)$, $\beta > 0$ and (H_1)–(H_3) hold. Then there exists $\beta_* > 0$ such that the system (1.1) has a positive radial normalized solution $(u, v) \in S_{a_1} \times S_{a_2}$ with $\lambda_1, \lambda_2 > 0$ when $\beta \geqslant \beta_*$.*

Remark 1.2.

(i) This seems to be the first study to consider the existence of normalized solutions to Schrödinger system with critical exponent and weakly attractive potentials;

(ii) To simplify, note that $r := r_1 + r_2$. In the proof of Theorem 1.1, we discuss three cases, that is, $p_1 = \min\{p_1, p_2, r\}$, $p_2 = \min\{p_1, p_2, r\}$ and $r = \min\{p_1, p_2, r\}$.

Since the scalar setting will of course be relevant when dealing with system, it is necessary to study firstly some related results of scalar equations. When $\beta = 0$, (1.1) turns to be the scalar equations

$$-\Delta u + (V_i + \lambda_i)u = |u|^{2^*-2}u + |u|^{p_i-2}u \quad \text{in } \mathbb{R}^N,\ i = 1, 2. \tag{1.6}$$

Normalized solutions of (1.6) can be found as critical points of the C^1 functional

$$J_{V_i}(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V_i u^2) - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*} - \frac{1}{p_i}\int_{\mathbb{R}^N}|u|^{p_i}, \quad u \in E_i,$$

on

$$S_{a_i} := \left\{u \in E_i : \int_{\mathbb{R}^N} u^2 = a_i\right\}.$$

Moreover, u_{a_i} is a ground state normalized solution to (1.6) on S_{a_i} if $J_{V_i}|'_{S_{a_i}}(u_{a_i}) = 0$ and

$$J_{V_i}(u_{a_i}) = \inf\{J_{V_i}(v) : v \in S_{a_i}, J_{V_i}|'_{S_{a_i}}(v) = 0\}.$$

Here comes our second main result.

Theorem 1.3. *Let $N = 3, 4$, $i = 1$ or $i = 2$, $p_i \in (2 + 4/N, 2^*)$ and (H_1)–(H_3) hold. Then the equation (1.6) has a positive radial ground state normalized solution $u_{a_i} \in S_{a_i}$ with $\lambda_i > 0$.*

Remark 1.4. This is probably the first result to consider the existence of normalized solutions to Schrödinger equation with critical exponent and weakly attractive potentials.

To obtain normalized solution of (1.6), as [12,21,23], we introduce the Pohozaev set

$$\mathcal{P}_{a_i,V_i} = \{u \in S_{a_i} : P_{V_i}(u) = 0\},$$

where

$$P_{V_i}(u) = \int_{\mathbb{R}^N}|\nabla u|^2 - \int_{\mathbb{R}^N} W_i u^2 - \int_{\mathbb{R}^N}|u|^{2^*} - \gamma_{p_i}\int_{\mathbb{R}^N}|u|^{p_i}, \qquad u \in E_i.$$

As a matter of fact, the condition $P_{V_i}(u) = 0$ obtained in Lemma 2.1 is the linear combination of Nehari and Pohozaev identities. Furthermore, J is bounded from below on $\mathcal{P}_{a_i,V_i}$, see Lemma 2.5 (iv). Hence, for $a_i > 0$, define

$$m_{V_i}(a_i) := \inf_{\mathcal{P}_{a_i,V_i}} J_{V_i} \tag{1.7}$$

and consider the reachability of $m_{V_i}(a_i)$. Inspired by [12, 33], we need use the comparison arguments between $m_{V_i}(a_i)$and that to the limit equation

$$-\Delta u + \lambda_i u = |u|^{2^*-2}u + |u|^{p_i-2}u \quad \text{in } \mathbb{R}^N. \tag{1.8}$$

The analogue corresponding (1.8) are denoted by J_∞, P_∞, $\mathcal{P}_{a_i,\infty}$ and $m_\infty(a_i)$. Soave in [31, Theorem 1.1 and Section 6] obtained that $m_\infty(a_i) \in (0, S^{N/2}/N)$ can be reached by u_{a_i} when $N = 3, 4, a_i > 0$ and $p_i \in (2 + 4/N, 2^*)$, furthermore, u_{a_i} is a real-valued, positive and radial.

The Gagliardo–Nirenberg inequality is the key point to study the above problems variationally. For $q \in [1, \infty)$, $|u|_q = \left(\int_{\mathbb{R}^N} |u|^q\right)^{1/q}$ stands for the norm in $L^q(\mathbb{R}^N)$.

Proposition 1.5. *Let $N \geqslant 3$ and $u \in H^1(\mathbb{R}^N)$. Then there exists a constant $C(N, q) > 0$ such that, for any $q \in [2, 2^*]$, we have*

$$|u|_q \leqslant C(N, q)|\nabla u|_2^\theta |u|_2^{1-\theta},$$

where $\theta \in [0, 1]$ satisfies $1/q = \theta/2^* + (1-\theta)/2$. In particular, when $q = 2^*$, $C(N, q) = S^{-1/2}$.

In this article, B_R denotes an open ball at 0 with radius of $R > 0$ and $C, C_1, C_2, \ldots$ denote various positive constants whose exact values are irrelevant.

The paper is organized as follows. In Sections 2 and 4, we give some preliminary results about the scalar equation (1.6) and the system (1.1), respectively. The proofs of Theorems 1.3 and 1.1 are given in Sections 3 and 5, respectively.

2 Preliminaries about the scalar equation

In this section, without loss of generality, we may assume that $i = 1$ and the potential V_1 satisfies (H_1)–(H_3).

Lemma 2.1. *If $u \in E_1$ is a weak solution to (1.6), then $P_{V_1}(u) = 0$.*

Proof. Let $u \in E_1$ be a weak solution of (1.6). We see that the following Nehari and Pohozaev identities hold

$$|\nabla u|_2^2 + \int_{\mathbb{R}^N} (V_1 + \lambda_1)u^2 - |u|_{2^*}^{2^*} - |u|_{p_1}^{p_1} = 0, \tag{2.1}$$

$$\frac{N-2}{2}|\nabla u|_2^2 + \frac{N}{2}\int_{\mathbb{R}^N} (V_1 + \lambda_1)u^2 + \int_{\mathbb{R}^N} W_1 u^2 - \frac{N}{2^*}|u|_{2^*}^{2^*} - \frac{N}{p_1}|u|_{p_1}^{p_1} = 0. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain $P_{V_1}(u) = 0$. □

Lemma 2.2. *Assume that $N = 3, 4$ and $u \in E_1$ is a nonnegative solution of (1.6). Then, $u \geqslant 0$ and $u \neq 0$ implies that $\lambda_1 > 0$.*

Proof. Since $u \neq 0$ satisfies

$$-\Delta u = -(V_1 + \lambda_1)u + |u|^{2^*-2}u + |u|^{p_1-2}u \quad \text{in } \mathbb{R}^N,$$

it follows from $u \geqslant 0$ that the right hand side is nonnegative if $\lambda_1 \leqslant 0$, and by [19, Lemma A.2], we obtain $u = 0$, which contradicts to the assumption $u \neq 0$. Hence, $\lambda_1 > 0$. □

For $u \in E_1$ and $t \in \mathbb{R}$, we introduce the transformation $u^t(x) := e^{Nt/2}u(e^t x), x \in \mathbb{R}^N$, it is easy to check that $|u^t|_2 = |u|_2$. We fix $u \neq 0$ and consider the continuous real valued function $f_u : \mathbb{R} \to \mathbb{R}$ with

$$f_u(t) := J_{V_1}(u^t) = \frac{1}{2}e^{2t}|\nabla u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_1(e^{-t}x)u^2 - \frac{1}{2^*}e^{2^*t}|u|_{2^*}^{2^*} - \frac{1}{p_1}e^{\gamma_{p_1}p_1 t}|u|_{p_1}^{p_1},$$

and

$$P_{V_1}(u^t) = e^{2t}|\nabla u|_2^2 - \int_{\mathbb{R}^N} W_1(e^{-t}x)u^2 - e^{2^*t}|u|_{2^*}^{2^*} - \gamma_{p_1}e^{\gamma_{p_1}p_1 t}|u|_{p_1}^{p_1}.$$

By a simple calculation, we see that $P_{V_1}(u^t) = f_u'(t)$.

Lemma 2.3. *Fix $u \in S_{a_1}$. Then $J_{V_1}(u^t) \to 0^+$ as $t \to -\infty$ and $J_{V_1}(u^t) \to -\infty$ as $t \to \infty$.*

Proof. By the condition (H_1), we have

$$J_{V_1}(u^t) \geqslant \frac{1-\tau_1}{2}e^{2t}|\nabla u|_2^2 - \frac{1}{2^*}e^{2^*t}|u|_{2^*}^{2^*} - \frac{1}{p_1}e^{\gamma_{p_1}p_1 t}|u|_{p_1}^{p_1}$$

and

$$J_{V_1}(u^t) \leqslant \frac{1}{2}e^{2t}|\nabla u|_2^2 - \frac{1}{2^*}e^{2^*t}|u|_{2^*}^{2^*} - \frac{1}{p_1}e^{\gamma_{p_1}p_1 t}|u|_{p_1}^{p_1},$$

it is easy to see that the conclusion holds. □

Lemma 2.4. *Let $D_k := \{u \in S_{a_1} : |\nabla u|_2^2 \leqslant k\}$. Then there exists $k_0 > 0$ such that $J_{V_1}(u) > 0$ and $P_{V_1}(u) > 0$ when $u \in D_{k_0}$.*

Proof. By the conditions (H_1) and (H_2), (1.5) and the Gagliardo–Nirenberg inequalities, we have

$$J_{V_1}(u) \geqslant \frac{1-\tau_1}{2}|\nabla u|_2^2 - \frac{1}{2^*}S^{-2^*/2}|\nabla u|_2^{2^*} - \frac{1}{p_1}C(N,p_1)a^{(1-\gamma_{p_1})p_1/2}|\nabla u|_2^{\gamma_{p_1}p_1}$$

and

$$P_{V_1}(u) \geqslant (1-\tau_2)|\nabla u|_2^2 - S^{-2^*/2}|\nabla u|_2^{2^*} - \gamma_{p_1}C(N,p_1)a^{(1-\gamma_{p_1})p_1/2}|\nabla u|_2^{\gamma_{p_1}p_1},$$

it is easy to see that there exists $k_0 > 0$ small enough such that $J_{V_1}(u) > 0$ and $P_{V_1}(u) > 0$ for all $u \in D_{k_0}$. □

Hence, we can define

$$\bar{m}_{V_1}(a_1) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_{V_1}(\gamma(t)) > 0,$$

where $\Gamma = \{\gamma \in C([0,1], S_{a_1}) : \gamma(0) \in D_{k_0}, J_{V_1}(\gamma(1)) \leqslant 0\}$, k_0 is given by Lemma 2.4.

Consider the decomposition of $\mathcal{P}_{a_1,V_1} = \mathcal{P}^+_{a_1,V_1} \cup \mathcal{P}^0_{a_1 V_1} \cup \mathcal{P}^-_{a_1,V_1}$ and

$$\mathcal{P}^+_{a_1,V_1} := \{u \in \mathcal{P}_{a_1,V_1} : f_u''(0) > 0\},$$
$$\mathcal{P}^0_{a_1,V_1} := \{u \in \mathcal{P}_{a_1,V_1} : f_u''(0) = 0\},$$
$$\mathcal{P}^-_{a_1,V_1} := \{u \in \mathcal{P}_{a_1,V_1} : f_u''(0) < 0\}.$$

Lemma 2.5.

(i) $\mathcal{P}_{a_1,V_1} = \mathcal{P}^-_{a_1,V_1}$;

(ii) for any $u \in S_{a_1}$, there exists a unique $t_u := t(u) \in \mathbb{R}$ such that $u^{t_u} \in \mathcal{P}_{a_1,V_1}$, moreover, $J_{V_1}(u^{t_u}) = \max_{t\in\mathbb{R}} J_{V_1}(u^t)$;

(iii) J_{V_1} *is coercive on* $\mathcal{P}_{a_1,V_1}$*, that is,* $J_{V_1}(u) \to \infty$ *for any* $u \in \mathcal{P}_{a_1,V_1}$ *with* $\|u\| \to \infty$*;*

(iv) there exist constants $\delta, \sigma > 0$ *such that* $|\nabla u|_2 \geqslant \delta$ *and* $J_{V_1}(u) \geqslant \sigma$ *for all* $u \in \mathcal{P}_{a_1,V_1}$*.*

Proof. (i) Using $P_{V_1}(u) = 0$ and the conditions (H_2) and (H_3), we have

$$\begin{aligned} f_u''(0) &= 2|\nabla u|_2^2 + \int_{\mathbb{R}^N} Z_1 u^2 - 2^*|u|_{2^*}^{2^*} - \gamma_{p_1}^2 p_1 |u|_{p_1}^{p_1} \\ &= \int_{\mathbb{R}^N} Y_1 u^2 + (2 - \gamma_{p_1} p_1)|\nabla u|_2^2 + (\gamma_{p_1} p_1 - 2^*)|u|_{2^*}^{2^*} \\ &\leqslant (\rho_1 + 2 - \gamma_{p_1} p_1)|\nabla u|_2^2 < 0. \end{aligned}$$

Hence, $\mathcal{P}^+_{a_1,V_1} = \mathcal{P}^0_{a_1,V_1} = \emptyset$, which implies that $\mathcal{P}_{a_1,V_1} = \mathcal{P}^-_{a_1,V_1}$.

(ii) By Lemmas 2.3 and 2.4, we know that $\max_{t\in\mathbb{R}} J_{V_1}(u^t)$ is achieved at $t_u \in \mathbb{R}$ and $J_{V_1}(u^{t_u}) > 0$. In view of $\partial_t J_{V_1}(u^t) = P_{V_1}(u^t)$, we see $P_{V_1}(u^{t_u}) = 0$. Hence, $u^{t_u} \in \mathcal{P}_{a_1,V_1}$. Suppose that there exists another $t_u' \in \mathbb{R}$ such that $u^{t_u'} \in \mathcal{P}_{a_1,V_1}$. Then by Lemma 2.5 (i), we see that t_u and t_u' are strict local maximum points of $f_u(t) := J(u^t)$. Without loss of generality, we assume that $t_u < t_u'$. Hence, there exists $t_u'' \in (t_u, t_u')$ such that $f_u(t_u'') = \min_{t\in[t_u,t_u']} f_u(t)$, and we have $f_u'(t_u'') = 0$ and $f_u''(t_u'') \geqslant 0$. Thus, $u^{t_u''} \in \mathcal{P}^+_{a_1,V_1} \cup \mathcal{P}^0_{a_1,V_1}$, which contradict to (i).

(iii) For $u \in \mathcal{P}_{a_1,V_1}$, by the conditions ($H_1$) and ($H_2$), we have

$$\begin{aligned} J_{V_1}(u) &= J_{V_1}(u) - \frac{1}{\gamma_{p_1} p_1} P_{V_1}(u) \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\gamma_{p_1} p_1}\right)|\nabla u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_1 u^2 + \frac{1}{\gamma_{p_1} p_1}\int_{\mathbb{R}^N} W_1 u^2 \\ &\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_{p_1} p_1}\right)|\nabla u|_2^2. \end{aligned} \tag{2.3}$$

Hence, J_{V_1} is coercive on $\mathcal{P}_{a_1,V_1}$.

(iv) If

$$|\nabla u|_2 < \min\left\{ \left(\frac{1-\theta_1}{3S^{2^*/2}}\right)^{1/(2^*-2)}, \left(\frac{1-\theta_1}{3\gamma_{p_1} C(N,p_1) a^{(1-\gamma_{p_1})p_1/2}}\right)^{1/(\gamma_{p_1} p_1 - 2)} \right\},$$

using the condition (H_2) and Proposition 1.5, we have

$$\begin{aligned} \Psi(u) &:= \int_{\mathbb{R}^N} W_1 u^2 + |u|_{2^*}^{2^*} + \gamma_{p_1}|u|_{p_1}^{p_1} \\ &\leqslant \left(\theta_1 + S^{-2^*/2}|\nabla u|_2^{2^*-2} + \gamma_{p_1} C(N,p_1) a^{(1-\gamma_{p_1})p_1/2}|\nabla u|_2^{\gamma_{p_1} p_1 - 2}\right)|\nabla u|_2^2 \\ &\leqslant \frac{2+\theta_1}{3}|\nabla u|_2^2. \end{aligned}$$

Now, we prove that there exists $\delta > 0$ such that $|\nabla u|_2 \geqslant \delta$ for all $u \in \mathcal{P}_{a_1,V_1}$. On the contrary, there exists $\{u_n\} \subset \mathcal{P}_{a_1,V_1}$ such that $|\nabla u_n|_2 \to 0$, then, for n large enough, we have

$$0 = P_{V_1}(u_n) = |\nabla u_n|_2^2 - \Psi(u_n) \geqslant \frac{1-\theta_1}{3}|\nabla u_n|_2^2 > 0,$$

which is a contradiction. In view of (2.3), we see that there exists $\sigma > 0$ such that $J_{V_1}(u) \geqslant \sigma$ for all $u \in \mathcal{P}_{a_1,V_1}$. □

Lemma 2.6. $m_{V_1}(a_1) = \bar{m}_{V_1}(a_1) > 0$. *Moreover, there exist* $\{v_n\} \subset S_{a_1}$ *such that, as* $n \to \infty$,

$$J_{V_1}(v_n) \to m_{V_1}(a_1), \qquad J_{V_1}|'_{S_{a_1}}(v_n) \to 0, \qquad P_{V_1}(v_n) \to 0, \tag{2.4}$$

and $v_n^- \to 0$ *a.e. in* $\mathbb{R}^N$.

Proof. For any $v \in \mathcal{P}_{a_1,V_1}$, there exist $t_1, t_2 \in \mathbb{R}$ such that $v^{t_1} \in D_{k_0}$ and $J_{V_1}(v^{t_2}) \leqslant 0$. Set

$$\gamma_0(t) := v^{(1-t)t_1+tt_2}, \qquad t \in [0,1],$$

then $\gamma_0 \in \Gamma$ and $\max_{t\in[0,1]} J_{V_1}(\gamma_0(t)) = J_{V_1}(v)$ by Lemma 2.5 (ii), which implies $\bar{m}_{V_1}(a_1) \leqslant m_{V_1}(a_1)$. Now, we prove that any path γ in Γ crosses $\mathcal{P}_{a_1,V_1}$. Using Lemma 2.4, for any $\gamma \in \Gamma$, $P_{V_1}(\gamma(0)) > 0$. On the other hand, by (2.3), $P_{V_1}(\gamma(1)) \leqslant \gamma_{p_1} p_1 J_{V_1}(\gamma(1)) \leqslant 0$. Therefore, there exists $t_0 \in (0,1]$ such that $P_{V_1}(\gamma(t_0)) = 0$, which implies $\bar{m}_{V_1}(a_1) \geqslant m_{V_1}(a_1)$. Thus, $\bar{m}_{V_1}(a_1) = m_{V_1}(a_1)$. In view of Lemma 2.5 (iv), we see that $\bar{m}_{V_1}(a_1) = m_{V_1}(a_1) > 0$.

Now, we recall the stretched functional introduced first in [21]:

$$\tilde{J}_{V_1} : E_1 \times \mathbb{R} \to \mathbb{R}, \qquad (u,t) \mapsto J_{V_1}(u^t)$$

and define

$$\tilde{\Gamma} = \{g \in C([0,1], S_{a_1} \times \mathbb{R}) : g(0) \in D_{k_0} \times \{0\}, g(1) \in J^0 \times \{0\}\},$$

where k is given by Lemma 2.4 and $J^0 := \{u \in E_1 : J_{V_1}(u) \leqslant 0\}$. If $\gamma \in \Gamma$, then $g := (\gamma, 0) \in \tilde{\Gamma}$ and $\tilde{J}_{V_1}(g(t)) = J_{V_1}(\gamma(t)), t \in [0,1]$. And if $g = (g_1, g_2) \in \tilde{\Gamma}$, then $\gamma := g_1^{g_2} \in \Gamma$ and $J_{V_1}(\gamma(t)) = \tilde{J}_{V_1}(g(t)), t \in [0,1]$. Hence, we have

$$\inf_{g\in\tilde{\Gamma}} \max_{t\in[0,1]} \tilde{J}_{V_1}(g(t)) = \bar{m}_{V_1}(a_1) = m_{V_1}(a_1).$$

Thus, using the Ekeland variational principle as in [21, Lemma 2.3], it follows that there exists a sequence $\{(u_n, t_n)\} \subset S_{a_1} \times \mathbb{R}$ such that, as $n \to \infty$,

$$\tilde{J}_{V_1}(u_n, t_n) \to m_{V_1}(a_1), \qquad \tilde{J}_{V_1}|'_{S_{a_1}\times\mathbb{R}}(u_n, t_n) \to 0, \qquad t_n \to 0.$$

Note $v_n := u_n^{t_n}$. For any $w \in \{z \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} v_n z = 0\}$, setting $w_n := w^{-t_n}$, then $(w_n, 0) \in \{(z,t) \in H^1(\mathbb{R}^N) \times \mathbb{R} : \int_{\mathbb{R}^N} u_n z = 0\}$. Hence,

$$J_{V_1}(v_n) \to m_{V_1}(a_1), \qquad \langle J_{V_1}|'_{S_{a_1}}(v_n), w\rangle = \langle \tilde{J}_{V_1}|'_{S_a\times\mathbb{R}}(u_n, t_n), (w_n, 0)\rangle.$$

and by$\|w_n\| \leqslant 2\|w\|$ for n enough large due to $t_n \to 0$, we have $J_{V_1}|'_{S_{a_1}}(v_n) \to 0$. Moreover, by $\langle \tilde{J}_{V_1}|'_{S_{a_1}\times\mathbb{R}}(u_n, t_n), (0,1)\rangle \to 0$, we see $P_{V_1}(v_n) \to 0$. Hence, (2.4) holds. Since $J_{V_1}(v_n) = J_{V_1}(|v_n|)$, $v_n^- \to 0$ a.e. in $\mathbb{R}^N$. □

3 Proof of Theorem 1.3

In this section, the potential $V_1 \neq 0$ and V_1 satisfies (H_1)–(H_3). When $V_1 = 0$, we denote $J_{V_1}, P_{V_1}, \mathcal{P}_{a_1,V_1}$, and $m_{V_1}(a_1)$ by J_∞, $P_\infty, \mathcal{P}_{a_1,\infty}$, and $m_\infty(a_1)$, respectively.

Before proving Theorem 1.1, we first consider the monotonicity of $m_\infty(\cdot)$.

Lemma 3.1. *The map* $m_\infty(\cdot)$ *is decreasing on* $\mathbb{R}_+ \setminus \{0\}$.

Proof. Fix $a > a_1 > 0$. By [31, Theorem 1.1 and Section 6], there exists $u \in \mathcal{P}_{a_1,\infty}$ such that $J_\infty(u) = m_\infty(a_1)$. Set $v := (a_1/a)^{(N-2)/4}u((a_1/a)^{1/2}\cdot)$. Then $|v|_2^2 = a$, and by Lemma 2.5 (ii), there exists $t_v \in \mathbb{R}$ such that $v^{t_v} \in \mathcal{P}_{a,\infty}$. Moreover,

$$|\nabla v^{t_v}|_2^2 = e^{2t_v}|\nabla v|_2^2 = e^{2t_v}|\nabla u|_2^2 = |\nabla u^{t_v}|_2^2,$$
$$|v^{t_v}|_{2^*}^{2^*} = e^{2^* t_v}|v|_{2^*}^{2^*} = e^{2^* t_v}|u|_{2^*}^{2^*} = |u^{t_v}|_{2^*}^{2^*},$$
$$|v^{t_v}|_{p_1}^{p_1} = e^{\gamma_{p_1}p_1 t_v}|v|_{p_1}^{p_1} = e^{\gamma_{p_1}p_1 t_v}(a_1/a)^{p_1(\gamma_{p_1}-1)/2}|u|_{p_1}^{p_1} = (a_1/a)^{p_1(\gamma_{p_1}-1)/2}|u^{t_v}|_{p_1}^{p_1}.$$

Let

$$\Psi(u,t_v) := \frac{1}{p_1}e^{\gamma_{p_1}p_1 t_v}\left(1-(a_1/a)^{p_1(\gamma_{p_1}-1)/2}\right)|u|_{p_1}^{p_1} < 0.$$

Then, we can deduce that

$$m_\infty(a) \leqslant J_\infty(v^{t_v}) = J_\infty(u^{t_v}) + \Psi(u,t_v) < J_\infty(u) = m_\infty(a_1),$$

which indicate $m_\infty(\cdot)$ is decreasing on $\mathbb{R}_+ \setminus \{0\}$. □

Now, we present a key estimate for $m_{V_1}(a_1)$.

Lemma 3.2. *One has that $m_{V_1}(a_1) < m_\infty(a_1)$.*

Proof. By [31, Theorem 1.1 and Section 6], there exists a positive radial $v_{a_1} \in \mathcal{P}_{a_1,\infty}$ such that $J_\infty(v_{a_1}) = m_\infty(a_1)$. Using Lemma 2.5 (ii), there exists $t_{v_{a_1}} := t(v_{a_1}) > 0$ such that $v_{a_1}^{t_{v_{a_1}}} \in \mathcal{P}_{a_1,V_1}$. Since $V_1 \leqslant 0$ and $V_1 \neq 0$, it is easy to check that

$$m_{V_1}(a_1) \leqslant J(v_{a_1}^{t_{v_{a_1}}}) < J_\infty(v_{a_1}^{t_{v_{a_1}}}) \leqslant \max_{t>0} J_\infty(v_{a_1}^t) = J_\infty(v_{a_1}) = m_\infty(a_1).$$ □

Proof of Theorem 1.3. In view of Lemma 2.6, we can obtain a sequence $\{u_n\} \subset S_{a_1}$ satisfying

$$J_{V_1}(u_n) \to m(a_1), \qquad J_{V_1}|'_{S_{a_1}}(u_n) \to 0, \qquad P_{V_1}(u_n) \to 0, \qquad n \to \infty,$$

and $u_n^- \to 0$ a.e. in $\mathbb{R}^N$, and by Lemma 2.5 (iii), it is easy to see that $\{u_n\}$ is bounded in E_1. Up to a subsequence, we assume that $u_n \rightharpoonup u_{a_1}$ in E_1, $u_n \to u_{a_1}$ in $L^s(\mathbb{R}^N)$, $s \in (2, 2^*)$, a.e. in $\mathbb{R}^N$ and $u_{a_1} \geqslant 0$ a.e. in $\mathbb{R}^N$. Moreover, since $J_{V_1}|'_{S_{a_1}}(u_n) \to 0$, by [32, Proposition 5.12], there exists $\lambda_n \in \mathbb{R}$ such that, for any $\varphi \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}[\nabla u_n \cdot \nabla\varphi + (V_1 + \lambda_n)u_n\varphi - |u_n|^{2^*-2}u_n\varphi - |u_n|^{p_1-2}u_n\varphi] = o_n(1)\|\varphi\|. \tag{3.1}$$

Choosing $\varphi = u_n$, we deduce that $\{\lambda_n\}$ is bounded in $\mathbb{R}$, and hence up to a subsequence, $\lambda_n \to \lambda_1 \in \mathbb{R}$. Now, we prove $u_{a_1} \neq 0$. If not, then $u_n \rightharpoonup 0$ in $H_r^1(\mathbb{R}^N)$ and $u_n \to 0$ in $L^s(\mathbb{R}^N)$, $s \in (2, 2^*)$. By Lemma 2.5 (ii), there exists $t_n := t(u_n) \in \mathbb{R}$ such that $P_\infty(u_n^{t_n}) = 0$ and $u_n^{t_n} \in \mathcal{P}_{a_1,\infty}$. By $P_{V_1}(u_n) \to 0$ and $J_{V_1}(u_n) \to m(a_1)$, we see that there exists $\delta > 0$ such that $|\nabla u_n|_2 \geqslant \delta$ for sufficient large n. Using $P_{V_1}(u_n) \to 0$ again, we can assume that $|u_n|_{2^*}^{2^*} \geqslant \delta^2$ for sufficient large n. In view of Lemma 2.5 (iv), we see that $\liminf_{n\to\infty} e^{t_n} > 0$. If $t_n \to \infty$, then,

$$\begin{aligned} 0 &\leqslant e^{-2t_n}J_\infty(u_n^{t_n}) \\ &= \frac{1}{2}|\nabla u_n|_2^2 - \frac{1}{2^*}e^{(2^*-2)t_n}|u_n|_{2^*}^{2^*} - \frac{1}{p_1}e^{(\gamma_{p_1}p_1-2)t_n}|u_n|_{p_1}^{p_1} \\ &\leqslant \frac{1}{2}C - \frac{1}{2^*}e^{(2^*-2)t_n}\delta^2 \to -\infty, \end{aligned} \tag{3.2}$$

which is a contradiction. Hence, $\{t_n\}$ is bounded in $\mathbb{R}$ and we can assume that $t_n \to t_* \in (-\infty, \infty)$. Since $u_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} W_1(x) = 0$, we can obtain that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W_1 u_n^2 = 0,$$

and by $P_{V_1}(u_n) \to 0$, we have

$$\begin{aligned} 0 &= P_\infty(u_n^{t_n}) \\ &= e^{2t_n} \int_{\mathbb{R}^N} W_1 u_n^2 + (e^{2t_n} - e^{2^* t_n})|u_n|_{2^*}^{2^*} + \gamma_{p_1}(e^{2t_n} - e^{\gamma_{p_1} p_1 t_n})|u_n|_{p_1}^{p_1} + o_n(1) \\ &= (e^{2t_n} - e^{2^* t_n})|u_n|_{2^*}^{2^*} + o_n(1) \end{aligned} \tag{3.3}$$

which implies $t_* = 0$. Therefore,

$$m_\infty(a_1) \leqslant J_\infty(u_n^{t_n}) = J_{V_1}(u_n) + o_n(1) = m_{V_1}(a_1) + o_n(1),$$

that is, $m_\infty(a_1) \leqslant m_{V_1}(a_1)$, this is impossible, and thus $u_{a_1} \neq 0$. Moreover, passing to the limit in (3.1) by the weak convergence, we infer that u_{a_1} solves (1.6) with $\lambda = \lambda_1$, and by Lemma 2.2, we see that $\lambda_1 > 0$. Hence, $\langle J'_{V_1}(u_{a_1}), u_{a_1}\rangle + \lambda_1 |u_{a_1}|_2^2 = 0$ and $P_{V_1}(u_{a_1}) = 0$, and by (2.3), we have $J_{V_1}(u_{a_1}) > 0$.

Set $a := |u_{a_1}|_2^2$. We claim that $a = a_1$. If not, then $b := a_1 - a \in (0, a_1)$ due to $a \leqslant a_1$. Let $v_n := u_n - u_{a_1}$, then $v_n \rightharpoonup 0$ in E_1 and $v_n \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$, and by $\lim_{|x|\to\infty} V_1(x) = \lim_{|x|\to\infty} W_1(x) = 0$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V_1 v_n^2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} W_1 v_n^2 = 0.$$

From the Brezis–Lieb lemma and (3.1), one have $|v_n|_2^2 = b + o_n(1)$ and

$$J_\infty(v_n) = J_{V_1}(v_n) + o_n(1) = J_{V_1}(u_n) - J_{V_1}(u_{a_1}) + o_n(1) = m_{V_1}(a_1) - J_{V_1}(u_{a_1}) + o_n(1), \tag{3.4}$$

$$\begin{aligned} \langle J'_\infty(v_n), v_n\rangle &= \langle J'_{V_1}(v_n), v_n\rangle + o_n(1) \\ &= \langle J'_{V_1}(u_n), u_n\rangle - \langle J'_{V_1}(u_{a_1}), u_{a_1}\rangle + o_n(1) \\ &= -\lambda_1 a_1 - \langle J'_{V_1}(u_{a_1}), u_{a_1}\rangle + o_n(1) \\ &= -\lambda_1 a_1 + \lambda_1 a + o_n(1) = -\lambda_1 b + o_n(1), \end{aligned} \tag{3.5}$$

$$P_\infty(v_n) = P_{V_1}(v_n) + o_n(1) = P_{V_1}(u_n) - P_{V_1}(u_{a_1}) + o_n(1) = o_n(1). \tag{3.6}$$

We claim that

$$\liminf_{n\to\infty} |\nabla v_n|_2^2 > 0. \tag{3.7}$$

As a matter of fact, if not, then we may assume that $v_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$ and hence in $L^{2^*}(\mathbb{R}^N)$ by the Sobolev inequality. We also have $|v_n|_{p_1} \to 0$ by the Gagliardo–Nirenberg inequality. Therefore, $\langle J'_\infty(v_n), v_n\rangle \to 0$, and by (3.5), we have $b = 0$, this is a contradiction. Thus, (3.7) holds. Using Lemma 2.5 (ii) again, there exists $t_n := t(v_n) \in \mathbb{R}$ such that $P_\infty(v_n^{t_n}) = 0$ and $v_n^{t_n} \in \mathcal{P}_{|v_n|_2^2,\infty}$. By Lemma 2.5 (iv) and (3.7), it is easy to see that $\liminf_{n\to\infty} e^{t_n} > 0$. Since (3.6) and $P_\infty(v_n^{t_n}) = 0$, by a similar proof as (3.2) and (3.3), we know that $\{t_n\}$ is bounded and $t_n \to 0$. Hence, by (3.4), we have

$$m_\infty(|v_n|_2^2) \leqslant J_\infty(v_n^{t_n}) = J_\infty(v_n) + o_n(1) = m_{V_1}(a_1) - J_{V_1}(u_{a_1}) + o_n(1).$$

Noting that $m_\infty(\cdot)$ is decreasing in $\mathbb{R}_+ \setminus \{0\}$ by Lemma 3.1, we have, for n large enough,

$$m_\infty(a_1) < m_\infty(|v_n|_2^2) \leqslant m_{V_1}(a_1) - J_{V_1}(u_a) + o_n(1) < m_\infty(a_1) - J_{V_1}(u_{a_1}) + o_n(1),$$

which implies that $J_{V_1}(u_{a_1}) \leqslant 0$ contradicting to $J_{V_1}(u_{a_1}) > 0$. Hence, $|u_{a_1}|_2^2 = a = a_1$. Using $u_n \to u_{a_1}$ in $L^2_{\text{loc}}(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} V_1(x) = \lim_{|x|\to\infty} W_1(x) = 0$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V_1 u_n^2 = \int_{\mathbb{R}^N} V_1 u_{a_1}^2, \qquad \lim_{n\to\infty} \int_{\mathbb{R}^N} W_1 u_n^2 = \int_{\mathbb{R}^N} W_1 u_{a_1}^2,$$

and by $P_{V_1}(u_{a_1}) = 0$, we deduce that

$$\begin{aligned}
&J_{V_1}(u_{a_1}) \\
&\quad = J_{V_1}(u_{a_1}) - \frac{1}{\gamma_{p_1} p_1} P_{V_1}(u_{a_1}) \\
&\quad = \left(\frac{1}{2} - \frac{1}{\gamma_{p_1} p_1}\right) |\nabla u_{a_1}|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} V_1 u_{a_1}^2 + \frac{1}{\gamma_{p_1} p_1} \int_{\mathbb{R}^N} W_1 u_{a_1}^2 + \left(\frac{1}{\gamma_{p_1} p_1} - \frac{1}{2^*}\right) |u_{a_1}|_{2^*}^{2^*} \\
&\quad \leqslant \liminf_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{\gamma_{p_1} p_1}\right) |\nabla u_n|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} V_1 u_n^2 + \frac{1}{\gamma_{p_1} p_1} \int_{\mathbb{R}^N} W_1 u_n^2 + \left(\frac{1}{\gamma_{p_1} p_1} - \frac{1}{2^*}\right) |u_n|_{2^*}^{2^*}\right] \\
&\quad = \lim_{n\to\infty} J_{V_1}(u_n) = m(a_1),
\end{aligned}$$

in view of $m_{V_1}(a_1) \leqslant J_{V_1}(u_{a_1})$, consequently, $J_{V_1}(u_{a_1}) = m_{V_1}(a_1)$. Using the strong maximum principle [16, Theorem 8.19], we see that $u_{a_1} > 0$. Therefore, u_{a_1} is a positive radial ground state normalized solution of (1.6). □

4 Preliminaries about the system

In this section, we may assume that the potentials V_i, $i = 1, 2$ satisfy (H_1)–(H_3).

First, we prove the following monotonicity result.

Lemma 4.1. *The map $m_{V_i}(\cdot)$ is nonincreasing on $\mathbb{R}_+ \setminus \{0\}$, where $m_{V_i}(a)$ is defined in (1.7), i=1,2.*

Proof. Here, we only consider the case $i = 1$. The case $i = 2$ is similar to the case $i = 1$. Fix $a > a_1 > 0$. By the definition of $m_{V_1}(a_1)$, there exists $u_0 \in \mathcal{P}_{a_1,V_1}$ such that

$$J_{V_1}(u_0) \leqslant m_{V_1}(a_1) + \varepsilon/3. \tag{4.1}$$

Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be a radial cut off function such that $\phi(x) = 1$ when $x \in B_1$, $\phi(x) = 0$ when $x \in B_2^c$. Set $u_\delta(x) := \phi(\delta x) u_0(x)$, $x \in \mathbb{R}^N, \delta > 0$. Then $u_\delta \in E_1 \setminus \{0\}$ and $u_\delta \to u_0$ in E_1 as $\delta \to 0^+$. It follows from Lemma 2.5 (ii) that, for any $u \in S_{a_1}$, there exists a unique $t_u := t(u) \in \mathbb{R}$ such that $u^{t_u} \in \mathcal{P}_{a_1,V_1}$. Moreover, the map $u \mapsto t_u$ is C^1 by the Implicit Function Theorem. Hence, $t(u_\delta) \to t(u_0) = 0$ in $\mathbb{R}$ and $u_\delta^{t(u_\delta)} \to u_0$ in E_1 as $\delta \to 0^+$. Take a fixed $\delta > 0$ small enough such that

$$J_{V_1}(u_\delta^{t(u_\delta)}) \leqslant J_{V_1}(u_0) + \varepsilon/3 \tag{4.2}$$

and take $\zeta \in C_0^\infty(\mathbb{R}^N)$ such that $\text{supp}(\zeta) \subset B_{1+4/\delta} \setminus B_{4/\delta}$. Set $\bar{\zeta} := (a - |u_\delta|_2^2)/|\zeta|_2^2 \zeta$. Then $|\bar{\zeta}|_2^2 = a - |u_\delta|_2^2$ and $\text{supp}(\bar{\zeta}) \cap \text{supp}(u_\delta) = \emptyset$. For every $s \leqslant 0$, let $w_s := u_\delta + \bar{\zeta}^s$, then $w_s \in S_a$ and there exists $t(w_s) \in \mathbb{R}$ such that $w_s^{t(w_s)} \in \mathcal{P}_{a,V_1}$. We claim that $t(w_s)$ is bounded from above as $s \to -\infty$. Suppose by contradiction that $t(w_s) \to \infty$ as $s \to -\infty$, and by $w_s \to u_\delta \neq 0$ a.e.

in $\mathbb{R}^N$, we deduce that $J_{V_1}(w_s^{t(w_s)}) \to -\infty$ as $s \to -\infty$. However, $J_{V_1}(w_s^{t(w_s)}) > 0$ by Lemma 2.5 (ii). This is absurd. Hence, the claim holds. Since $s + t(w_s) \to -\infty$ as $s \to -\infty$, we have, as $s \to -\infty$,

$$|\nabla \bar{\zeta}^{s+t(w_s)}|_2 \to 0, \qquad \int_{\mathbb{R}^N} V_1(e^{-(s+t(w_s))})\bar{\zeta}^2 \to 0,$$
$$|\bar{\zeta}^{s+t(w_s)}|_{2^*} \to 0, \qquad |\bar{\zeta}^{s+t(w_s)}|_{p_1} \to 0.$$

Consequently, $J_{V_1}(\bar{\zeta}^{s+t(w_s)}) \leqslant \varepsilon/3$ when $s < 0$ small enough. Thus, by (4.2) and (4.1),

$$\begin{aligned} m_{V_1}(a) &\leqslant J_{V_1}(w_s^{t(w_s)}) \\ &= J_{V_1}(u_\delta^{t(w_s)}) + J_{V_1}(\bar{\zeta}^{s+t(w_s)}) \\ &\leqslant J_{V_1}(u_\delta^{t(u_\delta)}) + J_{V_1}(\bar{\zeta}^{s+t(w_s)}) \\ &\leqslant J_{V_1}(u_0) + 2\varepsilon/3 \leqslant m_{V_1}(a_1) + \varepsilon, \end{aligned}$$

which implies $m_{V_1}(a) \leqslant m_{V_1}(a_1)$. Hence, the conclusion holds. □

Lemma 4.2. *Assume that $N = 3, 4$ and $(u, v) \in E_1 \times E_2$ is a nonnegative solution of (1.1). Then, $u \geqslant 0$ and $u \neq 0$ imply that $\lambda_1 > 0$; $v \geqslant 0$ and $v \neq 0$ imply that $\lambda_2 > 0$.*

Proof. Since $u \neq 0$ satisfies

$$-\Delta u = -(V_1 + \lambda_1)u + |u|^{2^*-2}u + |u|^{p_1-2}u + \beta r_1 |u|^{r_1-2}u|v|^{r_2} \quad \text{in } \mathbb{R}^N,$$

it follows from $u \geqslant 0$ that the right hand side is nonnegative if $\lambda_1 \leqslant 0$, and by [19, Lemma A.2], we obtain $u = 0$, which contradicts to the assumption $u \neq 0$. Hence, $\lambda_1 > 0$. Similarly, we also can obtain that $v \geqslant 0$ and $v \neq 0$ implies that $\lambda_2 > 0$. □

The following lemma is a version of the Brezis–Lieb lemma.

Lemma 4.3. *Suppose that $N \geqslant 3$, $r_1, r_2 > 1$ and $r \in (2, 2^*]$. If $(u_n, v_n) \rightharpoonup (u, v)$ in $E_1 \times E_2$, then, up to a subsequence if you need,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (|u_n|^{r_1}|v_n|^{r_2} - |u_n - u|^{r_1}|v_n - v|^{r_2} - |u|^{r_1}|v|^{r_2}) = 0.$$

Proof. See [11, Lemma 2.3] for the proof of the lemma. □

Let $\eta : \mathbb{R} \times E_1 \times E_2 \to E_1 \times E_2$,

$$\eta(t, u, v) := (u^t, v^t) = (e^{Nt/2}u(e^t\cdot), e^{Nt/2}v(e^t\cdot)).$$

Then

$$\begin{aligned} I(\eta(t,u,v)) = \frac{e^{2t}}{2}\left(|\nabla u|_2^2 + |\nabla v|_2^2\right) + \frac{1}{2}\int_{\mathbb{R}^N}\left(V_1(e^{-t}x)u^2 + V_2(e^{-t}x)v^2\right) - \frac{e^{2^*t}}{2^*}\left(|u|_{2^*}^{2^*} + |v|_{2^*}^{2^*}\right) \\ - \frac{e^{\gamma_{p_1}p_1 t}}{p_1}|u|_{p_1}^{p_1} - \frac{e^{\gamma_{p_2}p_2 t}}{p_2}|v|_{p_2}^{p_2} - \beta e^{\gamma_r r t}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}. \end{aligned}$$

Lemma 4.4. *Fix $(u, v) \in S_{a_1} \times S_{a_2}$. Then $I(\eta(t, u, v)) \to 0^+$ as $t \to -\infty$ and $I(\eta(t, u, v)) \to -\infty$ as $t \to \infty$.*

Proof. The proof is standard, therefore it is omitted here. □

Lemma 4.5. *Let $D_k := \{(u,v) \in S_{a_1} \times S_{a_2} : |\nabla u|_2^2 + |\nabla v|_2^2 \leqslant k\}$. Then there exists $k_0 > 0$ sufficiently small such that*

$$0 < \sup_{(u,v)\in D_{k_0}} I < \inf_{(u,v)\in\partial D_{2k_0}} I.$$

Proof. For any $(u,v) \in S_{a_1} \times S_{a_2}$, using the condition (H_1), (1.5), the Gagliardo–Nirenberg and Hölder inequalities, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} (V_1 u^2 + V_2 v^2) &\geqslant -\max\{\tau_1,\tau_2\}\left(|\nabla u|_2^2 + |\nabla v|_2^2\right),\\
\frac{1}{2^*}\left(|u|_{2^*}^{2^*} + |v|_{2^*}^{2^*}\right) &\leqslant \frac{1}{2^* S^{2^*/2}}\left(|\nabla u|_2^2 + |\nabla v|_2^2\right)^{2^*/2},\\
\frac{1}{p_1}|u|_{p_1}^{p_1} \leqslant C_1|\nabla u|_2^{\gamma_{p_1}p_1} &\leqslant C_1\left(|\nabla u|_2^2 + |\nabla v|_2^2\right)^{\gamma_{p_1}p_1/2},\\
\frac{1}{p_2}|v|_{p_2}^{p_2} \leqslant C_2|\nabla v|_2^{\gamma_{p_2}p_2} &\leqslant C_2\left(|\nabla u|_2^2 + |\nabla v|_2^2\right)^{\gamma_{p_2}p_2/2}
\end{aligned}$$

and

$$\beta\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} \leqslant \beta|u|_r^{r_1}|v|_r^{r_2} \leqslant \beta C_3\left(|\nabla u|_2^2 + |\nabla v|_2^2\right)^{\gamma_r r/2}, \tag{4.3}$$

where $C_1 = C(N,p_1,a_1)$, $C_2 = C(N,p_2,a_2)$ and $C_3 = C(N,r_1,r_2,a_1,a_2)$. Set $d := |\nabla u|_2^2 + |\nabla v|_2^2$. Then

$$I(u,v) \geqslant \frac{1}{2}\left(1 - \max\{\tau_1,\tau_2\}\right)d - \frac{1}{2^* S^{2^*/2}}d^{2^*/2} - C_1 d^{\gamma_{p_1}p_1/2} - C_2 d^{\gamma_{p_2}p_2/2} - \beta C_3 d^{\gamma_r r/2}.$$

Since $2^*, \gamma_{p_1}p_1, \gamma_{p_2}p_2, \gamma_r r > 2$, it is easy to see that there exists $k_0 > 0$ small enough such that $I(u,v) > 0$ for all $(u,v) \in D_{2k_0}$. Fixing $(u_1,v_1) \in D_{k_0}$ and $(u_2,v_2) \in \partial D_{2k_0}$, we have

$$\begin{aligned}
&I(u_2,v_2) - I(u_1,v_1)\\
&\geqslant \frac{1}{2}(|\nabla u_2|_2^2 + |\nabla v_2|_2^2) + \frac{1}{2}\int_{\mathbb{R}^N}(V_1 u_2^2 + V_2 v_2^2) - \frac{1}{2^*}(|u_2|_2^{2^*} + |v_2|_2^{2^*})\\
&\quad - \frac{1}{p_1}\int_{\mathbb{R}^N}|u_2|^{p_1} - \frac{1}{p_2}\int_{\mathbb{R}^N}|v_2|^{p_2} - \beta\int_{\mathbb{R}^N}|u_2|^{r_1}|v_2|^{r_2} - \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_1|^2 + |\nabla v_1|^2)\\
&\geqslant \left(\frac{1}{2} - \max\{\tau_1,\tau_2\}\right)k_0 - \frac{1}{2^* S^{2^*/2}}(2k_0)^{2^*/2} - C_1(2k_0)^{\gamma_{p_1}p_1/2} - C_2(2k_0)^{\gamma_{p_2}p_2/2} - \beta C_3(2k_0)^{\gamma_r r/2}\\
&\geqslant \frac{1}{4}\left(\frac{1}{2} - \max\{\tau_1,\tau_2\}\right)k_0,
\end{aligned}$$

for $k_0 > 0$ small enough. Thus, we can choose a sufficient small $k_0 > 0$ to satisfy the desired result. □

Let $\tilde{u} \in S_{a_1}$ be the positive radial ground state normalized solution of (1.6) with $i = 1$ and $\tilde{v} \in S_{a_2}$ be the positive radial ground state normalized solution of (1.6) with $i = 2$. By Lemmas 4.4 and 4.5, there exist $t_1, t_2 \in \mathbb{R}$ with $t_1 < -1 < 1 < t_2$ such that

$$e^{2t_1}\left(|\nabla\tilde{u}|_2^2 + |\nabla\tilde{v}|_2^2\right) < k, \qquad I(\eta(t_1,\tilde{u},\tilde{v})) > 0,$$

and

$$e^{2t_2}\left(|\nabla\tilde{u}|_2^2 + |\nabla\tilde{v}|_2^2\right) > 2k, \qquad I(\eta(t_1,\tilde{u},\tilde{v})) \leqslant 0.$$

Set

$$\Gamma_0 := \{h \in C([0,1], S_{a_1} \times S_{a_2}) : h(0) = \eta(t_1,\tilde{u},\tilde{v}),\ h(1) = \eta(t_2,\tilde{u},\tilde{v})\}.$$

Then $\Gamma_0 \neq \emptyset$. In fact, set $h_0(t) = \eta((1-t)t_1 + tt_2, \tilde{u}, \tilde{v})$, then $h_0 \in \Gamma_0$. Thus, we can define

$$c_\beta(a_1, a_2) := \inf_{h \in \Gamma_0} \max_{t \in [0,1]} I(h(t)).$$

Clearly, $c_\beta(a_1, a_2) > 0$.

Lemma 4.6. $\lim_{\beta \to \infty} c_\beta(a_1, a_2) = 0$.

Proof. Since $h_0 \in \Gamma_0$, we have

$$\begin{aligned} c_\beta(a_1, a_2) &\leqslant \max_{t \in [0,1]} I(h_0(t)) \\ &\leqslant \max_{t \geqslant 0} \left(\frac{1}{2} t^2 (|\nabla \tilde{u}|_2^2 + |\nabla \tilde{v}|_2^2) - \beta t^{\gamma_r r} \int_{\mathbb{R}^N} |\tilde{u}|^{r_1} |\tilde{v}|^{r_2} \right) \\ &= C\beta^{-2/(\gamma_r r - 2)} \to 0, \quad \beta \to \infty, \end{aligned}$$

where C is a positive constant independent of β. □

5 Proof of Theorem 1.1

In order to construct a bounded PS sequence of I at the level $c_\beta(a_1, a_2)$. Adapting the approach from [21], we introduce the C^1-functional $\Phi : E_1 \times E_2 \times \mathbb{R} \to \mathbb{R}$ with $\Phi(u, v, t) := I(\eta(t, u, v))$ and define

$$\tilde{c}_\beta(a_1, a_2) := \inf_{\tilde{h} \in \tilde{\Gamma}_0} \max_{t \in [0,1]} \Phi(\tilde{h}(t)),$$

where $\tilde{\Gamma}_0 = \{\tilde{h} \in C([0,1], S_{a_1} \times S_{a_2} \times \mathbb{R}) : \tilde{h}(0) = (\eta(t_1, \tilde{u}, \tilde{v}), 0), \tilde{h}(1) = (\eta(t_2, \tilde{u}, \tilde{v}), 0)\}$. It is easy to prove that $c_\beta(a_1, a_2) = \tilde{c}_\beta(a_1, a_2)$. The next lemma is special case of [15, Theorem 4.5].

Lemma 5.1. *Let X be a Hilbert manifold, $F \in C^1(X, \mathbb{R})$ be a given functional, $K \subset X$ be compact and consider a subset*

$$\mathcal{D} \subset \{E \subset X : E \text{ is compact}, \ K \subset E\},$$

which is homotopy-stable, that is, it is invariant with respect to deformations leaving K fixed. Assume that

$$\max_{u \in K} F(u) < c := \inf_{E \in \mathcal{D}} \max_{u \in E} F(u) \in \mathbb{R}.$$

Let $\varepsilon_n \in \mathbb{R}$, $\varepsilon_n \to 0$ and $E_n \in \mathcal{D}$ be a sequence such that

$$0 \leqslant \max_{u \in E_n} F(u) - c \leqslant \varepsilon_n.$$

Then there exists a sequence $u_n \in X$ such that, for some constant $C > 0$,

$$|F(u_n) - c| \leqslant \varepsilon_n, \quad \|F|'_X(u_n)\| \leqslant C\sqrt{\varepsilon_n}, \quad \text{dist}(u_n, E_n) \leqslant C\sqrt{\varepsilon_n}.$$

Lemma 5.2. *Let $\{\tilde{h}_n\} \subset \tilde{\Gamma}_0$ be a sequence such that*

$$\max_{t \in [0,1]} \Phi(\tilde{h}_n(t)) \leqslant c_\beta(a_1, a_2) + \frac{1}{n}.$$

Then there exist a sequence $(u_n, v_n, t_n) \in S_{a_1} \times S_{a_2} \times \mathbb{R}$ such that, as $n \to \infty$,

$$\Phi(u_n, v_n, t_n) \to c_\beta(a_1, a_2), \qquad \Phi|'_{S_{a_1} \times S_{a_2} \times \mathbb{R}}(u_n, v_n, t_n) \to 0, \tag{5.1}$$

and

$$\min_{t \in [0,1]} \|(u_n, v_n, t_n) - \tilde{h}_n(t)\|_{H^1(\mathbb{R}^N) \times \mathbb{R}} \to 0. \tag{5.2}$$

Proof. This lemma follows directly from Lemma 5.1 applied to Φ with

$$X := S_{a_1} \times S_{a_2} \times \mathbb{R}, \qquad K := \{(\eta(t_1, \tilde{u}, \tilde{v}), 0), (\eta(t_2, \tilde{u}, \tilde{v}), 0)\},$$

$$\mathcal{D} := \{\tilde{h}([0,1]) : \tilde{h} \in \tilde{\Gamma}_0\}, \qquad E_n := \{\tilde{h}_n(t) : t \in [0,1]\}.$$

Indeed, $c := \inf_{E \in \mathcal{D}} \max_{(u,v,t) \in E} \Phi(u,v,t) = \inf_{E \in \mathcal{D}} \max_{(u,v,t) \in E} I(\eta(t,u,v)) = c_\beta(a_1, a_2)$. On the one hand, for any $h \in \Gamma_0$, $\tilde{h}([0,1]) = (h([0,1]), 0) \in \mathcal{D}$. Hence,

$$c \leqslant \max_{(u,v,t) \in \tilde{h}([0,1])} I(\eta(t,u,v)) = \max_{(u,v) \in h([0,1])} I(u,v) = \max_{t \in [0,1]} I(h(t)).$$

Thus, $c \leqslant c_\beta(a_1, a_2)$. On the other hand, we show that $c_\beta(a_1, a_2) \leqslant c$. Suppose by contradiction that $c < c_\beta(a_1, a_2)$. Then $\max_{(u,v,t) \in E} I(\eta(t,u,v)) < c_\beta(a_1, a_2)$ for some $E \in \mathcal{D}$, hence $\sup_{(u,v,t) \in B_\delta(E)} I(\eta(t,u,v)) < c_\beta(a_1, a_2)$ for some $\delta > 0$, where $B_\delta(E)$ is the δ neighborhood of E. Moreover, $B_\delta(E)$ is open and connected, so it is path connected. Therefore, there exists a path $\tilde{h}_0 \in \tilde{\Gamma}_0$ such that $\max_{t \in [0,1]} \Phi(\tilde{h}_0(t)) < c_\beta(a_1, a_2)$. This is impossible. □

Lemma 5.3. *There exists a bounded sequence $\{(w_n, z_n)\} \subset S_{a_1} \times S_{a_2}$ such that, as $n \to \infty$,*

$$I(w_n, z_n) \to c_\beta(a_1, a_2), \qquad I|'_{S_{a_1} \times S_{a_2}}(w_n, z_n) \to 0, \tag{5.3}$$

$$\begin{aligned} P(w_n, z_n) := |\nabla w_n|_2^2 + |\nabla z_n|_2^2 - \int_{\mathbb{R}^N} (W_1 w_n^2 + W_2 z_n^2) - |w_n|_{2^*}^{2^*} - |z_n|_{2^*}^{2^*} \\ - \gamma_{p_1} |w_n|_{p_1}^{p_1} - \gamma_{p_2} |z_n|_{p_2}^{p_2} - \beta \gamma_r r \int_{\mathbb{R}^N} |w_n|^{r_1} |z_n|^{r_2} \to 0, \end{aligned} \tag{5.4}$$

$w_n^- \to 0$ a.e. in $\mathbb{R}^N$ and $z_n^- \to 0$ a.e. in $\mathbb{R}^N$.

Proof. First, by the definition of $c_\beta(a_1, a_2)$, there exists a sequence $\{h_n\} \subset \Gamma_0$ such that

$$\max_{t \in [0,1]} I(h_n(t)) \leqslant c_\beta(a_1, a_2) + \frac{1}{n}.$$

We observe that, since $I(u,v) = I(|u|, |v|)$ for any $(u,v) \in E_1 \times E_2$, we can take $h_n(t) \geqslant 0$ a.e. in $\mathbb{R}^N$ for every $t \in [0,1]$ and $n \in \mathbb{N}$. Applying Lemma 5.2 to $\tilde{h}_n := (h_n, 0) \in \tilde{\Gamma}_0$, we see that there exists a sequence $\{(u_n, v_n, t_n)\} \subset S_{a_1} \times S_{a_2} \times \mathbb{R}$ such that (5.1) and (5.2) hold. Note $(w_n, z_n) := (u_n^{t_n}, v_n^{t_n})$. By $h_n(t) \geqslant 0$ a.e. in $\mathbb{R}^N$ and (5.2), we see that, up to a subsequence, $u_n^- \to 0$ a.e. and $v_n^- \to 0$ a.e.. Hence, $w_n^- \to 0$ a.e. and $z_n^- \to 0$ a.e.. For any

$$(w_1, w_2) \in \{(u,v) \in E_1 \times E_2 : \int_{\mathbb{R}^N} w_n u = \int_{\mathbb{R}^N} z_n v = 0\},$$

setting $(w_1^n, w_2^n) := (w_1^{-t_n}, w_2^{-t_n})$, then

$$(w_1^n, w_2^n, 0) \in \left\{(u,v,t) \in E_1 \times E_2 \times \mathbb{R} : \int_{\mathbb{R}^N} u_n u = \int_{\mathbb{R}^N} v_n v = 0\right\}.$$

Hence,

$$I(w_n, z_n) \to c_\beta(a_1, a_2), \qquad t_n \to 0$$

and

$$\langle I|'_{S_{a_1} \times S_{a_2}}(w_n, z_n), (w_1, w_2)\rangle = \langle \Phi|'_{S_{a_1} \times S_{a_2} \times \mathbb{R}}(u_n, v_n, t_n), (w_1^n, w_2^n, 0)\rangle.$$

Since $\|(w_1^n, w_2^n)\| \leqslant 4\|(w_1, w_2)\|$ for n enough large, we have $I|'_{S_{a_1}\times S_{a_2}}(w_n, z_n) \to 0$. Therefore, (5.3) hold. Moreover, by $\langle \Phi|'_{S_{a_1}\times S_{a_2}\times\mathbb{R}}(u_n, v_n, t_n), , (0,0,1)\rangle \to 0$, we see $P(w_n, z_n) \to 0$. Hence, (5.4) hold.

Now, we prove that $\{(w_n, z_n)\} \subset S_{a_1} \times S_{a_2}$ is bounded in $E_1 \times E_2$. By (H_1) and (H_2), if $r = \min\{p_1, p_2, r\}$, then, for sufficiently large n,

$$\begin{aligned}
&c_\beta(a_1, a_2) + 1 \\
&\quad\geqslant I(w_n, z_n) - \frac{1}{\gamma_r r}P(w_n, z_n) \\
&\quad\geqslant \left(\frac{1}{2} - \frac{1}{\gamma_r r}\right)(|\nabla w_n|_2^2 + |\nabla z_n|_2^2) + \frac{1}{2}\int_{\mathbb{R}^N}(V_1 w_n^2 + V_2 z_n^2) + \frac{1}{\gamma_r r}\int_{\mathbb{R}^N}(W_1 w_n^2 + W_2 z_n^2) \\
&\quad\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r}\right)|\nabla w_n|_2^2 + \left(\frac{1-\tau_2}{2} - \frac{1+\theta_2}{\gamma_r r}\right)|\nabla z_n|_2^2;
\end{aligned}$$

if $p_1 = \min\{p_1, p_2, r\}$, then, for sufficiently large n,

$$\begin{aligned}
&c_\beta(a_1, a_2) + 1 \\
&\quad\geqslant I(w_n, z_n) - \frac{1}{\gamma_{p_1} p_1}P(w_n, z_n) \\
&\quad\geqslant \left(\frac{1}{2} - \frac{1}{\gamma_{p_1} p_1}\right)(|\nabla w_n|_2^2 + |\nabla z_n|_2^2) + \frac{1}{2}\int_{\mathbb{R}^N}(V_1 w_n^2 + V_2 z_n^2) + \frac{1}{\gamma_{p_1} p_1}\int_{\mathbb{R}^N}(W_1 w_n^2 + W_2 z_n^2) \\
&\quad\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_{p_1} p_1}\right)|\nabla w_n|_2^2 + \left(\frac{1-\tau_2}{2} - \frac{1+\theta_2}{\gamma_{p_1} p_1}\right)|\nabla z_n|_2^2;
\end{aligned}$$

if $p_2 = \min\{p_1, p_2, r\}$, then, for sufficiently large n,

$$\begin{aligned}
&c_\beta(a_1, a_2) + 1 \\
&\quad\geqslant I(w_n, z_n) - \frac{1}{\gamma_{p_2} p_2}P(w_n, z_n) \\
&\quad\geqslant \left(\frac{1}{2} - \frac{1}{\gamma_{p_2} p_2}\right)(|\nabla w_n|_2^2 + |\nabla z_n|_2^2) + \frac{1}{2}\int_{\mathbb{R}^N}(V_1 w_n^2 + V_2 z_n^2) + \frac{1}{\gamma_{p_2} p_2}\int_{\mathbb{R}^N}(W_1 w_n^2 + W_2 z_n^2) \\
&\quad\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_{p_2} p_2}\right)|\nabla w_n|_2^2 + \left(\frac{1-\tau_2}{2} - \frac{1+\theta_2}{\gamma_{p_2} p_2}\right)|\nabla z_n|_2^2.
\end{aligned}$$

In these three cases, we conclude that $\{(w_, z_n)\}$ is bounded in $E_1 \times E_2$. □

It follows from Lemma 5.2 that there exists a nonnegative $(w_0, z_0) \in E_1 \times E_2$ such that, up to a subsequence,

$$\begin{cases}
(w_n, z_n) \rightharpoonup (w_0, z_0) & \text{in } E_1 \times E_2, \\
(w_n, z_n) \rightharpoonup (w_0, z_0) & \text{in } L^{q_1}(\mathbb{R}^N) \times L^{q_2}(\mathbb{R}^N),\ q_1, q_1 \in [2, 2^*], \\
(w_n, z_n) \to (w_0, z_0) & \text{in } L^{q_1}(\mathbb{R}^N) \times L^{q_2}(\mathbb{R}^N),\ q_1, q_1 \in (2, 2^*), \\
(w_n, z_n) \to (w_0, z_0) & \text{a.e. in } \mathbb{R}^N.
\end{cases} \tag{5.5}$$

Since $I|'_{S_{a_1}\times S_{a_2}}(w_n, z_n) \to 0$, by the Lagrange multipliers rule, there exists a sequence $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R} \times \mathbb{R}$ such that

$$I'(w_n, z_n) + \lambda_1^n(w_n, 0) + \lambda_2^n(0, z_n) \to 0, \quad \text{in } (E_1 \times E_2)^*. \tag{5.6}$$

Take $(w_n, 0)$ and $(0, z_n)$ as test functions in (5.6), we see that $\{(\lambda_1^n, \lambda_2^n)\}$ is bounded in $\mathbb{R} \times \mathbb{R}$. Then there exists $(\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}$ such that, up to a subsequence, $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$.

Lemma 5.4. *There exists $\beta_* > 0$ sufficiently large such that $(w_n, z_n) \to (w_0, z_0)$ in $L^{2^*}(\mathbb{R}^N) \times L^{2^*}(\mathbb{R}^N)$ when $\beta \geqslant \beta_*$, moreover, $(w_0, z_0) \neq 0$.*

Proof. We firstly prove that $w_n \to w_0$ in $L^{2^*}(\mathbb{R}^N)$. Using the concentration-compactness principle [24], we see that there exist finite nonnegative measure μ and ν, and a most countable index set Λ such that $|\nabla w_n|^2 \rightharpoonup \mu$ in sense of measure, $|w_n|^{2^*} \rightharpoonup \nu$ in sense of measure and

$$\begin{cases} \mu \geqslant |\nabla w_0|^2 + \sum_{j\in\Lambda} \mu_j \delta_{x_j} & \mu_j \geqslant 0, \\ \nu = |w_0|^{2^*} + \sum_{j\in\Lambda} \nu_j \delta_{x_j} & \nu_j \geqslant 0, \\ \nu_j \leqslant S^{-2^*/2} \mu_j^{2^*/2} & j \in \Lambda, \end{cases} \tag{5.7}$$

where $x_j \in \mathbb{R}^N$ and δ_{x_j} is the Dirac measure at x_j. Let $\chi_R \in C_0^\infty(\mathbb{R}^N)$ be a cut off function satisfying $\chi_R(x) = 1$ in $B_R(x_j)$, $\chi_R(x) = 0$ in $B_{2R}^c(x_j)$ and $|\nabla \chi_R| \leqslant 2/R$. It follows from Lemma 5.2 that $\{\chi_R w_n\}$ is bounded in E_1. Now, take $(\chi_R w_n, 0)$ as a test function in (5.6), then

$$\lim_{n\to\infty} \langle I'(w_n, z_n) + \lambda_1^n (w_n, 0) + \lambda_2^n (0, z_n), (\chi_R w_n, 0)\rangle = 0. \tag{5.8}$$

By (5.5), the absolute continuity of integral and the Hölder inequality, we can deduce that

$$\lim_{R\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} V_1 w_n^2 \chi_R = \lim_{R\to 0} \int_{\mathbb{R}^N} V_1 w_0^2 \chi_R = 0, \tag{5.9}$$

$$\lim_{R\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} \lambda_1^n w_n^2 \chi_R = \lambda_1 \lim_{R\to 0} \int_{\mathbb{R}^N} w_0^2 \chi_R = 0, \tag{5.10}$$

$$\lim_{R\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} w_n \nabla w_n \cdot \nabla \chi_R = \lim_{R\to 0} \int_{\mathbb{R}^N} w_0 \nabla w_0 \cdot \nabla \chi_R = 0, \tag{5.11}$$

$$\lim_{R\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} |w_n|^{p_1} \chi_R = \lim_{R\to 0} \int_{\mathbb{R}^N} |w_0|^{p_1} \chi_R = 0, \tag{5.12}$$

and

$$\lim_{R\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} |w_n|^{r_1} |z_n|^{r_2} \chi_R = \lim_{R\to 0} \int_{\mathbb{R}^N} |w_0|^{r_1} |z_0|^{r_2} \chi_R = 0. \tag{5.13}$$

It follows from (5.8) and (5.9)–(5.13) that

$$\lim_{R\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 \chi_R = \lim_{R\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^N} |w_n|^{2^*} \chi_R,$$

that is,

$$\lim_{R\to 0} \int_{\mathbb{R}^N} \chi_R d\mu = \lim_{R\to 0} \int_{\mathbb{R}^N} \chi_R d\nu. \tag{5.14}$$

Using (5.7) and (5.14), we can obtain $\nu_j \geqslant \mu_j$, furthermore, either $\mu_j = 0$ or $\mu_j \geqslant S^{N/2}$ for $j \in \Lambda$. Observe that, for any $j \in \Lambda$, $\mu_j = 0$ if and only if $\nu_j = 0$. If $\mu_j = 0$, then $\nu_j = 0$ and $|w_n|_{2^*}^{2^*} \to |w_0|_{2^*}^{2^*}$ by (5.7), combining $w_n \rightharpoonup w_0$ in $L^{2^*}(\mathbb{R}^N)$, we conclude that $w_n \to w_0$ in $L^{2^*}(\mathbb{R}^N)$. If $\mu_j \geqslant S^{N/2}$, then we split three cases.

If $r = \min\{r, p_1, p_2\}$, then, by Lemma 4.6, there exists $\beta_1 > 0$ sufficiently large such that, for $\beta \geqslant \beta_1$,

$$c_\beta(a_1, a_2) < \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r} \right) S^{N/2}. \tag{5.15}$$

It follows from (5.7) that

$$\begin{aligned} c_\beta(a_1,a_2) &= \lim_{n\to\infty} I(w_n,z_n) - \frac{1}{\gamma_r r}P(w_n,z_n) \\ &\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r}\right)\int_{\mathbb{R}^N} |\nabla w_n|^2\chi_R dx \\ &= \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r}\right)\int_{\mathbb{R}^N} \chi_R d\mu \\ &\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r}\right)\mu_j \geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r}\right)S^{N/2}, \end{aligned}$$

which contradicts to (5.15). If $p_1 = \min\{r,p_1,p_2\}$ or $p_2 = \min\{r,p_1,p_2\}$, similarly as the case $r = \min\{r,p_1,p_2\}$, them also yields a contradiction.

In summary, going if necessary to replace a larger β_*, we obtain $\mu_j = \nu_j = 0$ for all $j \in \Lambda$ and $\beta \geqslant \beta_*$. Consequently, $w_n \to w_0$ in $L^{2^*}(\mathbb{R}^N)$ when $\beta \geqslant \beta_*$. $z_n \to z_0$ in $L^{2^*}(\mathbb{R}^N)$ can be obtained in the similar way.

By Lemma 5.3, we know that (w_0,z_0) is a nonnegative solution of (1.1). Suppose that by contradiction $(w_0,z_0) = 0$, and by (4.3), $\int_{\mathbb{R}^N} W_1 w_n^2 \to 0$, $\int_{\mathbb{R}^N} W_2 z_n^2 \to 0$, the strong convergence of $L^{2^*}, L^{p_1}, L^{p_2}, L^r$ and $P(w_n,z_n) \to 0$, we see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla w_n|^2 + |\nabla z_n|^2) = 0.$$

Hence, by $\int_{\mathbb{R}^N} V_1 w_n^2 \to 0$, $\int_{\mathbb{R}^N} V_2 z_n^2 \to 0$, we have $c_\beta(a_1,a_2) = \lim_{n\to\infty} I(w_n,z_n) = 0$, which contradicts to $c_\beta(a_1,a_2) > 0$. Hence, $(w_0,z_0) \neq 0$. □

Lemma 5.5. *If $c_\beta(a_1,a_2) < \min\{m_{V_1}(a_1), m_{V_2}(a_2)\}$, then $(w_n,z_n) \to (w_0,z_0)$ in $E_1 \times E_2$. Moreover, $(u_0,v_0) \in S_{a_1} \times S_{a_2}$ is a positive radial normalized solution of (1.1) with $\lambda_1 > 0$ and $\lambda_2 > 0$.*

Proof. We know from Lemmas 5.3 and 5.4 that (w_0,z_0) is nonnegative and $(w_0,z_0) \neq 0$.

If $w_0 \neq 0$ and $z_0 = 0$, then w_0 is a nontrivial radial solutions of (1.6) with $i = 1$ and $w_0 > 0$ by the maximum principle, where $|w_0|_2^2 = a \leqslant a_1$. By Lemma 4.1 and Theorem 1.3, we see that $m_{V_1}(a_1) \leqslant m_{V_1}(a) \leqslant J_{V_1}(w_0) = I(w_0,0)$. It follows from the conditions (H$_1$) and (H$_2$) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V_1\left[w_n^2 - (w_n-w_0)^2 - w_0^2\right] = 0, \qquad \lim_{n\to\infty}\int_{\mathbb{R}^N} V_1(w_n-w_0)^2 = 0 \tag{5.16}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} W_1\left[w_n^2 - (w_n-w_0)^2 - w_0^2\right] = 0, \qquad \lim_{n\to\infty}\int_{\mathbb{R}^N} W_1(w_n-w_0)^2 = 0. \tag{5.17}$$

Applying the Brezis–Lieb lemma, Lemma 4.3, (5.17), (5.16) and the $L^{p_1}, L^{p_2}, L^{2^*}, L^r$ strong convergence, we deduce that

$$\begin{aligned} o_n(1) &= P(w_n,z_n) \\ &= P(w_n-w_0,z_n) + P(w_0,0) + o_n(1) \\ &= \int_{\mathbb{R}^N}(|\nabla(w_n-w_0)|^2 + |\nabla z_n|^2) + o_n(1) \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} c_\beta(a_1,a_2) &= \lim_{n\to\infty} I(w_n,z_n) \\ &= \lim_{n\to\infty} I(w_n-w_0,z_n)+I(w_0,0)+o_n(1) \\ &\geqslant \frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla(w_n-w_0)|^2+|\nabla z_n|^2)+m_{V_1}(a_1)\geqslant m_{V_1}(a_1), \end{aligned} \tag{5.19}$$

which contradicts to $c_\beta(a_1,a_2)<m_{V_1}(a_1)$.

If $w_0=0$ and $z_0\neq 0$, then z_0 is a nontrivial radial solutions of (1.6) with $i=2$ and $z_0>0$ by the maximum principle, where $b=|z_0|_2^2\leqslant a_2$ and $m_{V_2}(a_2)\leqslant m_{V_2}(b)\leqslant J_{V_2}(z_0)=I(0,z_0)$. Similarly as (5.18) and (5.19), we also can derive a contradiction.

Hence, (w_0,z_0) is nonnegative, $w_0\neq 0$ and $z_0\neq 0$, and by Lemma 4.2, we can obtain $\lambda_1>0$ and $\lambda_2>0$. By the Pohozaev and Nehari identities, it is easy to see that

$$\begin{aligned} \lambda_1|w_0|_2^2+\lambda_2|z_0|_2^2 = &-\int_{\mathbb{R}^N}(V_1w_0^2+V_2z_0^2)-\int_{\mathbb{R}^N}(W_1w_0^2+W_2z_0^2) \\ &+(1-\gamma_{p_1})|w_0|_{p_1}^{p_1}+(1-\gamma_{p_2})|z_0|_{p_2}^{p_2}+\beta r(1-\gamma_r)\int_{\mathbb{R}^N}|w_0|^{r_1}|z_0|^{r_2}, \end{aligned}$$

and combining $P(w_n,z_n)\to 0$, we have

$$\begin{aligned} \lambda_1a_1+\lambda_2a_2 &= \lim_{n\to\infty}\left(\lambda_1^n|w_n|_2^2+\lambda_2^n|z_n|_2^2\right) \\ &= \lim_{n\to\infty}\Big[-\int_{\mathbb{R}^N}(V_1w_n^2+V_2z_n^2)-\int_{\mathbb{R}^N}(W_1w_n^2+W_2z_n^2) \\ &\quad+(1-\gamma_{p_1})|w_n|_{p_1}^{p_1}+(1-\gamma_{p_2})|z_n|_{p_2}^{p_2}+\beta r(1-\gamma_r)\int_{\mathbb{R}^N}|w_n|^{r_1}|z_n|^{r_2}\Big] \\ &= -\int_{\mathbb{R}^N}(V_1w_0^2+V_2z_0^2)-\int_{\mathbb{R}^N}(W_1w_0^2+W_2z_0^2) \\ &\quad+(1-\gamma_{p_1})|w_0|_{p_1}^{p_1}+(1-\gamma_{p_2})|z_0|_{p_2}^{p_2}+\beta r(1-\gamma_r)\int_{\mathbb{R}^N}|w_0|^{r_1}|z_0|^{r_2} \\ &= \lambda_1|w_0|_2^2+\lambda_2|z_0|_2^2, \end{aligned}$$

which implies that $|w_0|_2^2=a_1$ and $|z_0|_2^2=a_2$, that is, $(w_n,z_n)\to(w_0,z_0)$ in $L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$. Therefore, from (5.5), (5.6) and Lemma 5.4, we know that

$$\begin{aligned} \lim_{n\to\infty}(|\nabla w_n|_2^2+\lambda_1|w_n|_2^2) &= \lim_{n\to\infty}\left(-\int_{\mathbb{R}^N}V_1w_n^2+|w_n|_{2^*}^{2^*}+|w_n|_{p_1}^{p_1}+\beta r_1\int_{\mathbb{R}^N}|w_n|^{r_1}|z_n|^{r_2}\right) \\ &= -\int_{\mathbb{R}^N}V_1w_0^2+|w_0|_{2^*}^{2^*}+|w_0|_{p_1}^{p_1}+\beta r_1\int_{\mathbb{R}^N}|w_0|^{r_1}|z_0|^{r_2} \\ &= |\nabla w_0|_2^2+\lambda_1|w_0|_2^2, \end{aligned}$$

that is $\|w_n\|_1\to\|w_0\|_1$ as $n\to\infty$. Similarly, we also have $\|z_n\|_2\to\|z_0\|_2$. Hence, it is easy to see that $(w_n,z_n)\to(w_0,z_0)$ in $E_1\times E_2$. This completes the proof. □

Proof of Theorem 1.1. By Lemmas 5.3, 5.4 and 5.5, we complete the proof of Theorem 1.1. □

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Addendum to "Ulam–Hyers stability and exponentially dichotomic equations in Banach spaces" [*Electron. J. Qual. Theory Differ. Equ.* 2023, No. 8, 1–10]

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We add relevant references about which we learned after the completion of the initial work. We mainly show how the concept of exponential trichotomy can successfully replace the one of exponential dichotomy in some results from the paper in the title.

Keywords: Ulam–Hyers stability, exponential dichotomy, exponential trichotomy.

2020 Mathematics Subject Classification: 34G10, 34G20, 34D09.

After the completion of [2] we learned about a few works of L. Backes, D. Dragičević et al. In particular, they work with the notion of *Lipschitz shadowing property* that coincides with the notion of *Ulam–Hyers stability* used in [2]. Theorem 4.4 in [2] is a particular case of Theorem 6 in [1], where the linear part admits an exponential trichotomy. Also, note that the hypothesis that the evolution family is exponentially bounded is not used in the proofs of Theorems 4.2 and 4.4 in [2].

We also learned about the paper [3] by S. Elaydi and O. Hájek whose Theorem 5.3 can be used to generalize Theorem 3.5 in [2]. The statement of the new result is mainly obtained by replacing "Ulam–Hyers stability with uniqueness" and "exponential dichotomy" in Theorem 3.5 [2] with "Ulam–Hyers stability" and "exponential trichotomy", respectively.

We apologize for any inconvenience caused by our omissions.

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Complex dynamics of the system of nonlinear difference equations in the Hilbert space

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Abstract. In the given article the necessary and sufficient conditions of the existence of solutions of boundary value problem for the nonlinear system in the Hilbert spaces are obtained. Examples of such systems like a Lotka–Volterra are considered. Bifurcation and branching conditions of solutions are obtained.

Keywords: Lotka–Volterra models, population dynamics, Moore–Penrose pseudo-inverse operators, Fibonacci numbers.

2020 Mathematics Subject Classification: 39A70, 47B39, 39A22.

1 Introduction

The system of difference equations is the subject of numerous publications, and it is impossible to analyse all of them in detail. In this article we develop constructive methods of analysis of linear and weakly nonlinear boundary-value problems for difference equations, which occupy a central place in the qualitative theory of dynamical systems. We consider such problems that the operator of the linear part of the equation does not have an inverse. Such problems include the so called critical (or resonance) problems (when the considered problem can have non unique solution and not for any right-hand sides). We use the well-known technique of generalised inverse operators [4] and the notion of a strong generalised solution of an operator equation developed in [20]. In such way, one can prove the existence of solutions of different types for the system of operator equations in the Hilbert spaces. There exist three possible types of solutions: classical solutions, strong generalised solutions, and strong pseudo solutions [32]. For the analysis of a weakly nonlinear system, we develop the well-known Lyapunov–Schmidt method. This approach gives possibility to investigate a lot of problems in difference equations and mathematical biology from a single point of view.

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2 Statement of the problem

Consider the following boundary-value problem

$$x(n+1,\varepsilon) = a(n)x(n,\varepsilon) + b(n)y(n,\varepsilon) + \varepsilon Z_1(x(n,\varepsilon), y(n,\varepsilon), n, \varepsilon) + f_1(n); \tag{2.1}$$

$$y(n+1,\varepsilon) = c(n)x(n,\varepsilon) + d(n)y(n,\varepsilon) + \varepsilon Z_2(x(n,\varepsilon), y(n,\varepsilon), n, \varepsilon) + f_2(n); \tag{2.2}$$

$$l\begin{pmatrix} x(\cdot,\varepsilon) \\ y(\cdot,\varepsilon) \end{pmatrix} = \alpha, \tag{2.3}$$

where operators $\{a(n), b(n), c(n), d(n) \in \mathcal{L}(\mathcal{H}), n \in J \subset \mathbb{Z}\}$, $\mathcal{L}(\mathcal{H})$ is the space of linear and bounded operators which acts from $\mathcal{H}$ into itself, vector-functions $f_1(n), f_2(n) \in l_\infty(J,\mathcal{H})$,

$$l_\infty(J,\mathcal{H}) = \left\{ f : J \to \mathcal{H}, \|f\|_{l_\infty} = \sup_{n\in J} \|f(n)\|_{\mathcal{H}} < \infty \right\},$$

Z_1, Z_2 are smooth nonlinearities; a linear and bounded operator l translates solutions of (2.1), (2.2) into the Hilbert space $\mathcal{H}_1$, α is an element of the space $\mathcal{H}_1$, $\alpha \in \mathcal{H}_1$ (instead of $l_\infty(J,\mathcal{H})$ we can consider another functional space $\mathcal{T}(J,\mathcal{L}(\mathcal{H}))$).

We find solutions of the boundary-value problem (2.1)–(2.3) which for $\varepsilon = 0$ turns in one of solutions of generating boundary-value problem

$$x_0(n+1) = a(n)x_0(n) + b(n)y_0(n) + f_1(n); \tag{2.4}$$

$$y_0(n+1) = c(n)x_0(n) + d(n)y_0(n) + f_2(n); \tag{2.5}$$

$$l\begin{pmatrix} x_0(\cdot) \\ y_0(\cdot) \end{pmatrix} = \alpha. \tag{2.6}$$

3 Results

3.1 Linear case

Consider the following vector $z_0(n) = (x_0(n), y_0(n))$, sequence of operator matrices

$$A_n = \begin{pmatrix} a(n) & b(n) \\ c(n) & d(n) \end{pmatrix},$$

and sequence of vector–functions $f(n) = (f_1(n), f_2(n))$. Then we can rewrite the generating boundary-value problem (2.4)–(2.6) in the following form

$$z_0(n+1) = A_n z_0(n) + f(n), \tag{3.1}$$

$$l z_0(\cdot) = \alpha. \tag{3.2}$$

Define an operator $\Phi(m,n) = A_{m+1}A_m \ldots A_{n+1}, m > n$, $\Phi(m,m) = I$. The operator $U(m) = \Phi(m,0)$ is an evolution operator [6]. General solution $z_0(n)$ of (3.1) can be represented in the following form

$$z_0(n) = \Phi(n,0)z_0 + g(n), \tag{3.3}$$

where

$$g(n) = \sum_{i=0}^{n} \Phi(n,i) f(i).$$

Remark 3.1. It should be noted that if the sequence of operator matrices A_n each has bounded inverse $A_n^{-1} \in \mathcal{L}(\mathcal{H})$, then general solution of (3.1) can be represented in the following form

$$z_0(n) = U(n)z_0 + \sum_{i=0}^{n} U(n)U^{-1}(i)f(i).$$

Substituting representation (3.3) in the boundary condition (3.2) we obtain the following operator equation

$$Qz_0 = h, \tag{3.4}$$

where the operator Q and the element h have the following form

$$Q = l\Phi(\cdot,0), \qquad Q : \mathcal{H} \to \mathcal{H}_1, \qquad h = \alpha - lg(\cdot).$$

According to the theory of generalised solutions which was represented in [2] and theory of Moore–Penrose pseudo invertible operators [4] for the equation (3.4) we have the following variants:

1) Suppose that $R(Q) = \overline{R(Q)}$ ($R(Q)$ is the image of the operator Q). In this case we have that the equation (3.4) is solvable if and only if the following condition is hold [4]:

$$P_Y h = 0, \qquad \mathcal{H}_1 = R(Q) \oplus Y. \tag{3.5}$$

Here P_Y is an orthoprojector onto subspace Y. Under condition (3.5) the set of solutions of (3.4) has the following form:

$$z_0 = Q^+ h + P_{N(Q)}c, \qquad \forall c \in \mathcal{H},$$

where Q^+ is Moore–Penrose pseudo inverse [4,24,29] to the operator Q, $P_{N(Q)}$ is orthoprojector onto the kernel of the operator Q.

2) Consider the case when $R(Q) \neq \overline{R(Q)}$. In this case there is strong Moore–Penrose pseudo inverse $\overline{Q}^+$ [2] to the operator Q ($\overline{Q} : \overline{\mathcal{H}} \to \mathcal{H}_1$ is extension of the operator Q onto extended space $\mathcal{H} \subset \overline{\mathcal{H}}$ [2]). Condition of generalised solvability has the following form:

$$P_Y h = 0, \qquad \mathcal{H}_1 = \overline{R(Q)} \oplus Y. \tag{3.6}$$

Condition (3.6) guarantees only that $h \in \overline{R(Q)}$. Under condition (3.6) the set of strong generalised solutions of the equation (3.4) has the following form:

$$z_0 = \overline{Q}^+ h + P_{N(\overline{Q})}c, \qquad \forall c \in \mathcal{H}. \tag{3.7}$$

If $h \in R(Q)$ then strong generalised solutions are classical.

3) Suppose that $R(Q) \neq \overline{R(Q)}$ and $h \notin \overline{R(Q)}$. It means that the following condition is hold

$$P_Y h \neq 0. \tag{3.8}$$

Under condition (3.8) the set of strong generalised quasisolutions [2,4] has the following form:

$$z_0 = \overline{Q}^+ h + P_{N(\overline{Q})}c, \qquad \forall c \in \mathcal{H}.$$

Using the notion presented above, we obtain the following theorem.

Theorem 3.2. *Boundary value problem* (3.1), (3.2) *is solvable.*

a1) There are strong generalised solutions of (3.1), (3.2) *if and only if*

$$P_Y\left\{\alpha - l\sum_{i=0}^{\cdot}\Phi(\cdot,i)f(i)\right\} = 0, \tag{3.9}$$

if the element $(\alpha - l\sum_{i=0}^{\cdot}\Phi(\cdot,i)f(i)) \in R(Q)$ *then solutions are classical;*

b1) under condition (3.9) *the set of generalised solutions of the boundary-value problem* (3.1), (3.2) *has the following form*

$$z_0(n,c) = \overline{G[f,\alpha]}(n) + P_{N(\overline{Q})}c, \qquad \forall c \in \mathcal{H},$$

where the generalised Green operator has the form

$$\overline{G[f,\alpha]}(n) = \Phi(n,0)\overline{Q}^+\left\{\alpha - l\sum_{i=0}^{\cdot}\Phi(\cdot,i)f(i)\right\};$$

a2) There are strong quasisolutions of (3.1), (3.2) *if and only if the following condition is hold*

$$P_Y\left\{\alpha - l\sum_{i=0}^{\cdot}\Phi(\cdot,i)f(i)\right\} \neq 0; \tag{3.10}$$

b2) Under condition (3.10) *the set of strong quasisolutions of the boundary-value problem* (3.1), (3.2) *has the following form*

$$z_0(n,c) = \overline{G[f,\alpha]}(n) + P_{N(\overline{Q})}c, \qquad \forall c \in \mathcal{H}.$$

3.2 Nonlinear case

Consider the nonlinear boundary-value problem (2.1)–(2.3). Using the introduced notations we can rewrite this problem in the following form

$$z(n+1,\varepsilon) = A_n z(n,\varepsilon) + \varepsilon Z(z(n,\varepsilon),n,\varepsilon), \tag{3.11}$$

$$lz(\cdot,\varepsilon) = \alpha. \tag{3.12}$$

Theorem 3.3 (Necessary condition)**.** *Suppose that the boundary value problem* (3.11), (3.12) *has solution* $z(n,\varepsilon)$ *which for* $\varepsilon = 0$ *turns in one of solutions* $z_0(n,c)$ *with element* $c \in \mathcal{H}$ $(z(n,0) = z_0(n,c))$*. Then* c *satisfies the following operator equation for generating elements*

$$F(c) = P_Y l\sum_{i=0}^{\cdot}\Phi(\cdot,i)Z(z_0(i,c),i,0) \tag{3.13}$$

$$= P_Y l\sum_{i=0}^{\cdot}\Phi(\cdot,i)Z(\overline{G[f,\alpha]}(i) + P_{N(\overline{Q})}c,\cdot,0) = 0. \tag{3.14}$$

Proof. According to Theorem 3.3, the boundary value problem (3.11), (3.12) has solution if and only if the following condition is true:

$$P_Y\left\{\alpha - l\sum_{i=0}^{\cdot}\Phi(\cdot,i)(f(i) + \varepsilon Z(z(i,\varepsilon),i,\varepsilon))\right\} = 0. \tag{3.15}$$

From the condition (3.15) follows condition (3.13). □

Remark 3.4. It should be noted that theorem 3.2 is hold when the nonlinearities Z_1, Z_2 are continuous in the neighborhood of generating solution $z_0(n, c_0)$.

Now, we propose the following change of variables:

$$z(n,\varepsilon) = z_0(n, c_0) + u(n,\varepsilon),$$

where the element c_0 satisfies the operator equation (3.13). Then we can rewrite the boundary value problem (3.11), (3.12) in the following form

$$u(n+1,\varepsilon) = A_n u(n,\varepsilon) + \varepsilon\{Z(z_0(n,c^0), n, 0) + Z'_u(z_0(n,c^0), n, 0)u(n,\varepsilon) + \mathcal{R}(u(n,\varepsilon), n, \varepsilon)\}, \tag{3.16}$$

$$lu(\cdot,\varepsilon) = 0. \tag{3.17}$$

Here Z'_u is the Fréchet derivative,

$$\mathcal{R}(0,0,0) = \mathcal{R}'_u(0,0,0) = 0.$$

Boundary value problem (3.16), (3.17) has solutions if and only if the following condition is true:

$$P_{Y} l \sum_{i=0}^{\cdot} \Phi(\cdot, i)(Z(z_0(i,c^0), i, 0) + Z'_u(z_0(i,c^0), i, 0)u(i,\varepsilon) + \mathcal{R}(u(i,\varepsilon), i, \varepsilon)) = 0. \tag{3.18}$$

Under this condition the set of solutions of boundary value problem (3.16), (3.17) has the following form

$$u(n,\varepsilon) = P_{N(\overline{Q})} c + \overline{u}(n,\varepsilon), \tag{3.19}$$

where

$$\overline{u}(n,\varepsilon) = \varepsilon \overline{G[Z(z_0(\cdot,c^0),\cdot,0) + Z'_u(z_0(\cdot,c^0),\cdot,0)u(\cdot,\varepsilon) + \mathcal{R}(u(\cdot,\varepsilon),\cdot,\varepsilon),0]}(n). \tag{3.20}$$

Substituting (3.19) in (3.18) we obtain the following operator equation

$$B_0 c = r, \tag{3.21}$$

where the operator

$$B_0 = -P_{Y} l \sum_{i=0}^{\cdot} \Phi(\cdot, i) Z'_u(z_0(i,c^0), i, 0) P_{N(\overline{Q})},$$

$$r = P_{Y} l \sum_{i=0}^{\cdot} \Phi(\cdot, i)(Z'_u(z_0(i,c^0), i, 0)\overline{u}(i,\varepsilon) + \mathcal{R}(u(i,\varepsilon), i, \varepsilon)).$$

Condition $P_{N(B_0^*)} P_Y = 0$ guarantees that equation (3.21) is solvable and has at least one generalized solution in the following form $c = \overline{B}_0^+ r$. For a small enough ε considered operator system (3.19)–(3.21) has a contracting operator in the right-hand side and using contraction mapping principle [2] we have the following assertion.

Theorem 3.5 (Sufficient condition)**.** *Suppose that the following condition is true: $P_{N(\overline{B}_0^*)} P_Y = 0$. ($P_{N(\overline{B}_0^*)}$ is an orthoprojector onto the kernel of adjoint to the operator B_0). Then the boundary value problem* (3.11), (3.12) *has generalised solutions which can be found with using of iterative processes:*

$$u_{k+1}(n,\varepsilon) = P_{N(\overline{Q})} c_k + \overline{u}_k(n,\varepsilon),$$

$$c_{k+1} = \overline{B}_0^+ P_Y l \sum_{i=0}^{\cdot} \Phi(\cdot, i)(Z'_u(z_0(i,c^0), i, 0)\overline{u}_k(i,\varepsilon) + \mathcal{R}(u_k(i,\varepsilon), i, \varepsilon)),$$

$$\overline{u}_{k+1}(n,\varepsilon) = \overline{\varepsilon G[Z(z_0(\cdot,c^0),\cdot,0) + Z'_u(z_0(\cdot,c^0),\cdot,0)\overline{u}_k(\cdot,\varepsilon) + \mathcal{R}(u_k(\cdot,\varepsilon),\cdot,\varepsilon),0]}(n),$$

where

$$\mathcal{R}(u_k(n,\varepsilon),n,\varepsilon) = Z(z_0(n,c^0) + u_k(n,\varepsilon),n,\varepsilon) - Z(z_0(n,c^0),n,0) - Z'_u(z_0(n,c^0),n,0)u_k(n,\varepsilon),$$

$$u_0 = c_0 = \overline{y}_0 = 0.$$

4 Applications

It is well-known that systems like a Lotka–Volterra [34, 35] plays an important role in the dynamics of population [26, 27] (mathematical biology). There exist many papers which are dedicated to investigation of such problems in continuous and discrete cases (see for example the recent works [1,5,7,9–19,21–23,25,28,30,31,33]). As a rule such problems are regular. We consider some examples of systems with different type of boundary conditions in the critical case. We show that the operator which generates considering problem can be Fredholm. We find bifurcation conditions of solutions with using of the equation for generating constants [3]. It should be noted that the proposed method also works in the case of boundary-value problems with fractional derivative [8].

4.1 Examples

4.1.1 Example 1

Consider the following periodic boundary-value problem in the finite dimensional case:

$$x_i(n+1,\varepsilon) = a_i(n)x_i(n,\varepsilon) + b_i(n)y_i(n,\varepsilon) + \varepsilon g_i^1(n)x_i(n,\varepsilon)\left(1 - \sum_{j=1}^{t} a_{ij}(n)y_j(n,\varepsilon)\right) + f_1^i(n), \quad (4.1)$$

$$y_i(n+1,\varepsilon) = c_i(n)x_i(n,\varepsilon) + d_i(n)y_i(n,\varepsilon) + \varepsilon g_i^2(n)y_i(n,\varepsilon)\left(1 - \sum_{j=1}^{t} b_{ij}(n)x_j(n,\varepsilon)\right) + f_2^i(n), \quad (4.2)$$

$$x_i(0,\varepsilon) = x_i(m,\varepsilon), \quad (4.3)$$

$$y_i(0,\varepsilon) = y_i(m,\varepsilon), \qquad i = \overline{1,p}. \quad (4.4)$$

Here $x_i(n,\varepsilon), y_i(n,\varepsilon), a_i(n), b_i(n), c_i(n), d_i(n), g_i^1(n), g_i^2(n), a_{ij}(n), b_{ij}(n) \in \mathbb{R},\ i = \overline{1,p},\ j = \overline{1,t}.$

For $\varepsilon = 0$ we obtain the following generating boundary-value problem

$$x_i^0(n+1) = a_i(n)x_i^0(n) + b_i(n)y_i^0(n) + f_1^i(n), \quad (4.5)$$

$$y_i^0(n+1) = c_i(n)x_i^0(n) + d_i(n)y_i^0(n) + f_2^i(n), \quad (4.6)$$

$$x_i^0(0) = x_i^0(m), \quad (4.7)$$

$$y_i^0(0) = y_i^0(m). \quad (4.8)$$

$$l\begin{pmatrix} x_0(\cdot) \\ y_0(\cdot) \end{pmatrix} = \begin{pmatrix} x_i^0(m) - x_i^0(0) \\ y_i^0(m) - y_i^0(0) \end{pmatrix}_{i=\overline{1,p}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For the vector $z_i^0(n) = (x_i^0(n), y_i^0(n))$ we can write the following assertion.

Corollary 4.1. *The boundary value problem (4.5)–(4.8) has periodic solutions if and only if*

$$P_{Y_d}\sum_{k=0}^{m}\Phi(m,k)f(k)=0, \tag{4.9}$$

where $Q=\Phi(m,0)-I$, d is a number of linearly independent columns of Q; under condition (4.9) the set of solutions has the form

$$z_i^0(n,c_r)=(G[f,0])(n)+P_{Q_r}c_r,\ \ c_r\in\mathbb{R}^r, \tag{4.10}$$

where the generalised Green's operator $(G[f,0])(n)$ has the following form

$$(G[f,0])(n)=-\Phi(n,0)Q^+\sum_{k=0}^{m}\Phi(m,k)f(k),$$

r is a number of linearly independent rows of Q (P_{Q_r} is an orthoprojector onto the kernel of matrix Q).

Remark 4.2. It should be noted that in the considered above case index of an operator $\mathcal{S}$ can be calculated in the following way

$$\operatorname{ind}\mathcal{S}=r-d,$$

where the operator $\mathcal{S}$ with boundary conditions has the following form

$$\mathcal{S}\begin{pmatrix} x_i^0(n)\\ y_i^0(n)\end{pmatrix}:=\begin{pmatrix} x_i^0(n+1)-a_i(n)x_i^0(n)-b_i(n)y_i^0(n)\\ y_i^0(n+1)-c_i(n)x_i^0(n)-d_i(n)y_i^0(n)\end{pmatrix}.$$

It means that the operator $\mathcal{S}$ is Fredholm [4].

For the nonlinear boundary value problem (4.1)–(4.4) we obtain the following assertions.

Corollary 4.3 (Necessary condition)**.** *If the boundary value problem (4.1)–(4.4) has solution, then the element $c_r=c_r^0$ satisfies the following equation for generating constants:*

$$F(c_r)=P_{Y_d}\sum_{i=0}^{m}\Phi(m,i)Z(z_0(i,c_r),i,0)=0,$$

where

$$Z(z_0(n,c_r),n,0)=\begin{pmatrix} g_i^1(n)x_i^0(n,c_r)(1-\sum_{j=1}^{t}a_{ij}(n)y_j^0(n,c_r))\\ g_i^2(n)y_i^0(n,c_r)(1-\sum_{j=1}^{t}b_{ij}(n)x_j^0(n,c_r))\end{pmatrix}.$$

Corollary 4.4 (Sufficient condition)**.** *Suppose that the following condition is true:*

$$P_{N(B_0^*)}P_{Q_d^*}=0.$$

Then the boundary value problem (4.1)–(4.4) has generalized solutions which can be found using of iterative processes:

$$u_{k+1}(n,\varepsilon)=P_{N(Q)_r}c_k+\overline{u}_k(n,\varepsilon),$$

$$c_{k+1}=B_0^+P_{N(Q^*)}\sum_{i=0}^{m}\Phi(m,i)(Z_u'(z_0(i,c^0),i,0)\overline{u}_k(i,\varepsilon)+\mathcal{R}(u_k(i,\varepsilon),i,\varepsilon)),$$

$$\overline{u}_{k+1}(n,\varepsilon)=\varepsilon G[Z(z_0(\cdot,c^0),\cdot,0)+Z_u'(z_0(\cdot,c^0),\cdot,0)\overline{u}_k(\cdot,\varepsilon)+\mathcal{R}(u_k(\cdot,\varepsilon),\cdot,\varepsilon),0](n),$$

where

$$\mathcal{R}(u_k(n,\varepsilon),n,\varepsilon) = Z(z_0(n,c^0)+u_k(n,\varepsilon),n,\varepsilon) - Z(z_0(n,c^0),n,0) - Z'_u(z_0(n,c^0),n,0)u_k(n,\varepsilon),$$

$$u_0 = c_0 = \overline{y}_0 = 0,$$

$$Z'_u(z_0(n,c^0),n,0)u_k(n,\varepsilon)$$
$$= \begin{pmatrix} g_i^1(n)x_i^0(n,c_r^0)(1-\sum_{j=1}^t a_{ij}(n)u_{jk}^2(n)) + g_i^1(n)u_{ik}^1(n)(1-\sum_{j=1}^t a_{ij}(n)y_j^0(n,c_r^0)) \\ g_i^2(n)y_i^0(n,c_r^0)(1-\sum_{j=1}^t b_{ij}(n)u_{jk}^1(n)) + g_i^2(n)u_{ik}^2(n)(1-\sum_{j=1}^t b_{ij}(n)x_j^0(n,c_r^0)) \end{pmatrix}.$$

4.1.2 Example 2

Suppose that $a_i(n) = b_i(n) = c_i(n) = g_i^1(n) = g_i^2(n) = a_{ij}(n) = b_{ij}(n) = 1, d_i(n) = 0$. In this case

$$A_n = A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad n \in \mathbb{N}.$$

Then for the linear boundary value problem (4.5)–(4.8) we obtain that the evolution operator $\Phi(m,n)$ has the following form

$$\Phi(m,n) = A^{m-n+1} = \begin{pmatrix} F_{m-n+2} & F_{m-n+1} \\ F_{m-n+1} & F_{m-n} \end{pmatrix}.$$

Here $F_0 = 1, F_1 = 1$, $F_{n+2} = F_n + F_{n+1}, n \geq 0$ are Fibonacci numbers. In this case the matrix Q is nondegenerate ($Q^+ = Q^{-1}$, $P_{N(Q)} = I, P_Y = I$, I is an identity matrix) and we obtain the following corollary.

Corollary 4.5. *The boundary value problem* (4.1)–(4.4) *has periodic solution if and only if*

$$\sum_{k=0}^{m} A^{m-k+1} f(k) = \sum_{k=0}^{m} \begin{pmatrix} F_{m-k+2} & F_{m-k+1} \\ F_{m-k+1} & F_{m-k} \end{pmatrix} \begin{pmatrix} f_1^i(k) \\ f_2^i(k) \end{pmatrix} = 0; \tag{4.11}$$

under condition (4.11) *the solution of the boundary value problem* (4.1)–(4.4) *has the form*

$$z_i^0(n) = (G[f,0])(n) = -A^{n+1}Q^{-1}\sum_{k=0}^{m} A^{m-k+1}f(k)$$
$$= -\frac{1}{\Delta(m)}\sum_{k=0}^{m}\begin{pmatrix} a_{11}(n,m,k)f_1^i(k) + a_{12}(n,m,k)f_2^i(k) \\ a_{21}(n,m,k)f_1^i(k) + a_{22}(n,m,k)f_2^i(k) \end{pmatrix},$$

where

$$\Delta(m) = (F_{m+2}-1)(F_m-1) - F_{m+1}^2;$$

$$a_{11}(n,m,k) = F_{n+2}(F_mF_{m-k+2} - F_{m+1}F_{m-k+1}) - (F_{n+2}F_{m-k+2} + F_{n+1}F_{m-k+1}) + F_{n+1}(F_{m+2}F_{m-k+1} - F_{m+1}F_{m-k+2});$$

$$a_{12}(n,m,k) = F_{n+2}(F_mF_{m-k+1} - F_{m+1}F_{m-k}) - (F_{n+2}F_{m-k+1} + F_{n+1}F_{m-k}) + F_{n+1}(F_{m+2}F_{m-k} - F_{m+1}F_{m-k+1});$$

$$a_{21}(n,m,k) = F_{n+1}(F_mF_{m-k+2} - F_{m+1}F_{m-k+1}) - (F_{n+1}F_{m-k+2} + F_{n+1}F_{m-k+1}) + F_n(F_{m+2}F_{m-k+1} - F_{m+1}F_{m-k+2});$$

$$a_{22}(n,m,k) = F_{n+1}(F_mF_{m-k+1} - F_{m+1}F_{m-k}) - (F_{n+2}F_{m-k+1} + F_{n+1}F_{m-k}) + F_{n+1}(F_{m+2}F_{m-k} - F_{m+1}F_{m-k+1}).$$

In this case the necessary condition of solvability for the nonlinear boundary-value problem (4.1)–(4.4) has the form

$$\begin{pmatrix} \sum_{i=0}^{m} F_{m-i+2}x_i^0(n)(1-\sum_{j=1}^{n} y_j^0(n)) + F_{m-i+1}y_i^0(n)(1-\sum_{j=1}^{n} x_j^0(n)) \\ \sum_{i=0}^{m} F_{m-i+1}x_i^0(n)(1-\sum_{j=1}^{n} y_j^0(n)) + F_{m-i}y_i^0(n)(1-\sum_{j=1}^{n} x_j^0(n)) \end{pmatrix} = 0.$$

Fréchet derivate Z'_u has the following form

$$Z'_u(z_0(n), n, 0)u_k(n,\varepsilon) = \begin{pmatrix} x_i^0(n)(1-\sum_{j=1}^{n} u_{jk}^2(n,\varepsilon)) + u_{ik}^1(n,\varepsilon)(1-\sum_{j=1}^{n} y_j^0(n)) \\ y_i^0(n)(1-\sum_{j=1}^{n} u_{jk}^1(n,\varepsilon)) + u_{ik}^2(n,\varepsilon)(1-\sum_{j=1}^{n} x_j^0(n)) \end{pmatrix}.$$

4.1.3 Example 3

Consider the following boundary value problem

$$\begin{aligned} x_i(n+1,\varepsilon) &= a_i(n)x_i(n,\varepsilon) + b_i(n)y_i(n,\varepsilon) \\ &\quad + \varepsilon g_i^1(n)x_i(n,\varepsilon)\left(1-\sum_{j=1}^{n} a_{ij}(n)y_j(n,\varepsilon)\right) + f_1^i(n), \qquad (4.12) \\ y_i(n+1,\varepsilon) &= c_i(n)x_i(n,\varepsilon) + d_i(n)y_i(n,\varepsilon) \\ &\quad + \varepsilon g_i^2(n)y_i(n,\varepsilon)\left(1-\sum_{j=1}^{n} b_{ij}(n)x_j(n,\varepsilon)\right) + f_2^i(n), \qquad (4.13) \end{aligned}$$

with the following boundary conditions

$$l\begin{pmatrix} x_i(\cdot,\varepsilon) \\ y_i(\cdot,\varepsilon) \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{p_1} x_i(n_k,\varepsilon) \\ \sum_{l=0}^{p_2} y_i(n_l,\varepsilon) \end{pmatrix}_{i=\overline{1,p}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}. \qquad (4.14)$$

Here $n_k, k = \overline{0,p_1}$, $n_l, l = \overline{0,p_2}$ are finite sequences of integer numbers. In this case we obtain the multi-point boundary-value problem.

4.1.4 Example 4

Suppose that $x_i(n), y_i(n) \geq 0$ and boundary condition has the following form

$$l\begin{pmatrix} x_i(\cdot,\varepsilon) \\ y_i(\cdot,\varepsilon) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{p} x_i(0,\varepsilon) \\ \sum_{i=0}^{p} y_i(0,\varepsilon) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \qquad (4.15)$$

Such condition has practical meaning. It means the population distribution at the initial time (the proportion of the population in species).

5 Conclusion

Proposed in the given article approach gives possibility to investigate a lot of biological problems.

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Multiplicity of solutions of Kirchhoff-type fractional Laplacian problems with critical and singular nonlinearities

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Abstract. In this article, the following Kirchhoff-type fractional Laplacian problem with singular and critical nonlinearities is studied:

$$\begin{cases} \left(a+b\|u\|^{2\mu-2}\right)(-\Delta)^s u = \lambda l(x) u^{2_s^*-1} + h(x) u^{-\gamma}, & \text{in } \Omega, \\ u>0, & \text{in } \Omega, \\ u=0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $N > 2s$, $(-\Delta)^s$ is the fractional Laplace operator, $2_s^* = 2N/(N-2s)$ is the critical Sobolev exponent, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $l \in L^\infty(\Omega)$ is a non-negative function and $\max\{l(x), 0\} \not\equiv 0$, $h \in L^{\frac{2_s^*}{2_s^*+\gamma-1}}(\Omega)$ is positive almost everywhere in Ω, $\gamma \in (0,1)$, $a > 0, b > 0$, $\mu \in [1, 2_s^*/2)$ and parameter λ is a positive constant. Here we utilize a special method to recover the lack of compactness due to the appearance of the critical exponent. By imposing appropriate constraint on λ, we obtain two positive solutions to the above problem based on the Ekeland variational principle and Nehari manifold technique.

Keywords: fractional Laplacian problem, singular, critical nonlinearity, Kirchhoff-type problem.

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1 Introduction

This paper is concerned with the existence and multiplicity of positive solutions for the following Kirchhoff-type problem with singular nonlinearity and critical exponent driven by

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fractional Laplacian operator:

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u = \lambda l(x)u^{2_s^*-1} + h(x)u^{-\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $0 < s < 1$, $N > 2s$, $2_s^* = 2N/(N-2s)$ is the fractional critical Sobolev exponent, $M(t) = a + bt^{\mu-1}$, $a > 0, b > 0$, $\mu \in [1, 2_s^*/2)$, $l(x)$ is non-negative and $l(x) \in L^\infty(\Omega)$ satisfies $l(x) \not\equiv 0$ in Ω, $0 < \gamma < 1$ and $h \in L^{\frac{2_s^*}{2_s^*+\gamma-1}}(\Omega)$ is positive almost everywhere in Ω, parameter $\lambda > 0$ and $(-\Delta)^s$ is the fractional Laplace operator which defined up to normalization factors as

$$(-\Delta)^s \Psi(x) = 2 \lim_{\tau \to 0^+} \int_{\mathbb{R}^N \backslash B_\tau(x)} \frac{\Psi(x) - \Psi(y)}{|x-y|^{N+2s}} dy, \qquad x \in \mathbb{R}^N, \tag{1.2}$$

for any $\Psi \in C_0^\infty(\mathbb{R}^N)$, where $B_\tau(x)$ is the ball with radius τ and center $x \in \mathbb{R}^N$. For more details, we can refer to [25] and references therein. The fractional elliptic problem appeared in many different practical applications and phenomena, such as resilience, phase transformation and minimal surface problems, etc. For more related introduction, see [1,2,6,20,29].

Above all, let us review the relevant progress on Kirchhoff-type equation. The Kirchhoff-type equation is a generalization of the classical D'Alembert wave equation, which was raised by Kirchhoff to describe the lateral vibration of stretched strings in [17]. The basic model for problem (1.1) can be summarized as follows:

$$\rho u_{tt} - M\left(\int_0^L u_x^2 dx\right) u_{xx} = 0,$$

where ρ, a, b, L are constants, $M(\int_0^L u_x^2 dx) := a + b(\int_0^L u_x^2 dx)^{\mu-1}$ describes the tension changes arise from changes in string length during the vibrations. Concerning the Kirchhoff term M, we consider a specific version of M,

$$M(t) = a + bt^{\mu-1}, \quad a, b > 0, \quad 1 \le \mu < 2_s^*/2. \tag{1.3}$$

Where, a represents the initial tension while b is related to the inherent properties of string (such as Young's modulus). In particular, in the case of $M(0) = 0$ while $M(t) > 0$ for all $t \in \mathbb{R}^+$, Kirchhoff-type equation is often referred as degenerate. If $M(t) \ge c > 0$ for all $t \in \mathbb{R}_0^+$ and some constant c, equation is commonly known as non-degenerate. For some advance of degenerate Kirchhoff-type problems, see for instance [3,7,32]. In addition, we refer to [8,10,13,31,33] about some existence results of non-degenerate Kirchhoff-type problems.

Next, let us present some progress of Laplacian equations involving singular terms. A general version of this type of problem can be formed as follows:

$$\begin{cases} -\Delta u = \lambda m(x)u^{-\gamma} + h(x)u^q, & \text{in} \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{in } \partial\Omega. \end{cases} \tag{1.4}$$

In the early days, when $\lambda \equiv 1$ and $h(x) = 0$, the existence and regularity results of the solutions to problem (1.4) were studied by Boccardo et al. in [5]. The difference in results depends on the summability of m in some Lebesgue function spaces and on the value range of γ (which can be smaller, equal or larger than 1). When $0 < \gamma < 1$, in the case of $m(x) \equiv 1$ and $h(x) \equiv 0$, Crandall et al. solved problem (1.4) in [9] and learned that it has a unique weak solution. Subsequently, the multiplicity of solutions to such problems was obtained by Sun et al. in [28]. Moreover, Liu and Sun solved the Kirchhoff equation involving singular terms and Hardy potential in [23]. For equations involving the critical case, we may refer to [12, 15, 16]. To be specific, the author in [12] solved the Kirchhoff equation involving the critical exponent and obtained two different solutions. In [15], when $m(x), h(x) \equiv 1$, the authors obtained that if λ is less than a positive constant, then problem (1.4) has two positive solutions. Furthermore, Yang in [16] studied the multiplicity and asymptotic behavior of positive solutions to problem (1.4), where $0 < \gamma < 1 < q \leq (N+2)/(N-2)$. By applying variational method and sub-supersolution technique, the author learned what happens to the number and properties of solutions for the equation with different values of λ. In the setting of $\gamma = 1$, minimization theory is used by the authors in [31] to obtain a unique positive solution in the subcritical case. Regarding $\gamma > 1$, in the case of $h(x) \equiv 0$ and $\lambda \equiv 1$, the authors in [18] also got the unique solution. It is worth mentioning that Wang et al. used Ekeland's variational principle and the Nehari method to prove the existence of a unique positive solution for a Kirchhoff equation involving strong singularity in [30]. Besides, there are equations for (1.4) with Kirchhoff terms that we can refer to [19, 21, 22]. In [19], the authors obtained two different positive solutions through the variational and perturbation methods. Liao et al. in [22] studied the solutions of equation (1.4) in the weak singular case under different constraints. On the basis of [22], they solved the critical case in [21] and got the unique positive solution.

In the above context, the following class of singular Kirchhoff problem with fractional Laplace operators has been extensively studied:

$$\begin{cases} M(\int_\Omega |\nabla u(x)|^2)(-\Delta)^s u = \lambda f(x)u^{-\gamma} + g(x)u^{2_s^*-1}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.5}$$

in the setting of $M \equiv 1$. Mukherjee and Sreenadh in [24] studied a singular problem with critical growth and obtained two solutions, where γ can be equal to 1. In the case of $\gamma > 0$, Barrios et al. discussed the existence of solutions to the equation 1.5 in two cases: $g(x) = 0$ and $g(x) = 1$. Besides, the authors in [14] solved a variant of problem (1.5) in which λ is multiplied to the critical term. Through the variational method, they learned about the existence and multiplicity of solutions to the equation when λ takes different values. For such problems with different Kirchhoff terms, we may consult [11–13] and the references therein. Equation (1.5) was discussed in [12], where there is no weight function and the Kirchhoff term may be degenerate, the variational method and appropriate truncation theory were used to obtain two solutions. In [13], Fiscella et al. proved that equation (1.5) of the non-degenerate type has two distinct solutions by using the Nehari method. At last, the authors in [11] considered a critical degenerate Kirchhoff problem with strong singularity, and the only positive solution was obtained.

In view of the aforementioned works, in particular, according to [11–13,30], we are inspired to investigate the existence and multiplicity of solutions to problem (1.1) under appropriate assumptions. The most significant difficulty lies in the lack of compactness caused by the presence of critical term. For this, we use the method of [13] to recover compactness. Especially, we are interested in a natural problem: whether problem (1.1) can be solved in the strong singular case? We will try our best to study this situation in the future.

Here is the main result we obtain.

Theorem 1.1. *Let $s \in (0,1)$, $N > 2s$, $0 < \gamma < 1$, a be small enough and $h \in L^{\frac{2_s^*}{2_s^*+\gamma-1}}(\Omega)$ be positive a.e. in Ω. Then there exists $\Gamma_0 > 0$, when $0 < \lambda < \Gamma_0$, then problem (1.1) has at least two positive solutions with negative energies.*

Remark 1.2. Compared to the fundamental conclusion in [11], there are three main differences: (i) The range of $M(t)$ is different, in this paper we only consider non-degenerate case. (ii) We utilize a different method to recover the lack of compactness caused by the critical term. (iii) By controlling the range of λ in the weak singular case, we obtain two positive solutions.

Remark 1.3. Compared to [13], our result refines and improves the main result of [13] from the following aspects: (i) We do not need to control b to be as small as possible but we need to control $a > 0$ small enough to ensure that λ is positive in order to obtain the second positive solution. (ii) Our nonlinearities do not involve sign changing functions and we don't need to control $l(x) = \|l\|_\infty$ in $B_{\rho_0}(0)$ for some $\rho_0 > 0$.

2 Variational setting

Regarding problem (1.1), we mainly solve it in fractional Sobolev space, which is specifically defined by

$$\mathbb{M} = \left\{ \Phi \mid \Phi : \mathbb{R}^N \to \mathbb{R} \text{ is measureable, } \Phi|_\Omega \in L^2(\Omega), \ \frac{\Phi(x) - \Phi(y)}{|x-y|^{\frac{N+2s}{2}}} \in L^2(G) \right\}, \tag{2.1}$$

where $G = \mathbb{R}^{2N} \backslash (\Omega^c \times \Omega^c)$, with $\Omega^c = \mathbb{R}^N \backslash \Omega$. Moreover, $\mathbb{M}_0$ is defined as the linear subspace of $\mathbb{M}$, which is

$$\mathbb{M}_0 := \left\{ \Phi \in \mathbb{M} : \Phi = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\}.$$

As for the norm of two spaces, the norm of the space $\mathbb{M}$ is given as shown below:

$$\|\Phi\|_{\mathbb{M}} = \|\Phi\|_{L^2(\Omega)} + \left(\iint_G \frac{|\Phi(x) - \Phi(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}. \tag{2.2}$$

Besides, we confirm the following norm on $\mathbb{M}_0$:

$$\|\Phi\|_{\mathbb{M}_0} := \left(\iint_G \frac{|\Phi(x) - \Phi(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}. \tag{2.3}$$

According to Lemma 6 in [26], it is easy to know that (2.2) and (2.3) are equivalent. In addition, it is standard to verify that $(\mathbb{M}_0, \|.\|_{\mathbb{M}_0})$ is a Hilbert space and the form of scalar product in $\mathbb{M}_0$ is as follows:

$$\langle \imath, \jmath \rangle : \langle \imath, \jmath \rangle_{\mathbb{M}_0} = \iint_G \frac{(\imath(x) - \imath(y))(\jmath(x) - \jmath(y))}{|x-y|^{N+2s}} dxdy, \quad \text{for } \imath, \jmath \in \mathbb{M}_0, \tag{2.4}$$

see for example Lemma 7 in [26]. The embedding $\mathbb{M}_0 \hookrightarrow L^\eta(\Omega)$ is compact and continuous for $2 \le \eta < 2_s^*$, see [26, Lemma 8]). Then, an appropriate selection linked to the best Sobolev constant can be defined as

$$S_s = \inf_{\Phi \in \mathbb{M}_0 \backslash \{0\}} S_s(\Phi) = \frac{\iint_G \frac{|\Phi(x) - \Phi(y)|^2}{|x-y|^{N+2s}} dxdy}{\left(\int_\Omega |\Phi(x)|^{2_s^*} dx \right)^{2/2_s^*}}. \tag{2.5}$$

In what follows, for the sake of simplicity of notations, we shall denote $\|\cdot\|_{\mathbb{M}_0}$ and $\|\cdot\|_{L^\eta(\Omega)}$ by $\|\cdot\|$ and $\|\cdot\|_\eta$ for any $\eta \in [2,\infty]$.

In the process of obtaining multiple solutions, we will use Nehari manifold method and fibering maps. Before this, let us first introduce the definition of weak solutions to problem (1.1).

Definition 2.1. $u \in \mathbb{M}_0$ is a weak solution of problem (1.1) if for all $\ell \in \mathbb{M}_0$ the following weak formulation is satisfied:

$$a\langle u,\ell\rangle + b\|u\|^{2\mu-2}\langle u,\ell\rangle - \lambda\int_\Omega l(x)|u|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u|^{-\gamma}\ell dx = 0.$$

The energy functional associated to problem (1.1): $\mathcal{I} : \mathbb{M}_0 \to \mathbb{R}$ is defined as

$$\mathcal{I}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{2\mu}\|u\|^{2\mu} - \frac{\lambda}{2_s^*}\int_\Omega l(x)|u|^{2_s^*}dx - \frac{1}{1-\gamma}\int_\Omega h(x)|u|^{1-\gamma}dx. \tag{2.6}$$

3 Fibering maps analysis

For any $u \in \mathbb{M}_0$, we first introduce the fibering map: $\phi_u(t) : (0,\infty) \to \mathbb{R}$, defined as

$$\phi_u(t) = \mathcal{I}(tu) = \frac{a}{2}t^2\|u\|^2 + \frac{b}{2\mu}t^{2\mu}\|u\|^{2\mu} - \lambda\frac{t^{2_s^*}}{2_s^*}\int_\Omega l(x)|u|^{2_s^*}dx - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega h(x)|u|^{1-\gamma}dx.$$

Through simple calculation, we get

$$\phi_u'(t) = at\|u\|^2 + bt^{2\mu-1}\|u\|^{2\mu} - \lambda t^{2_s^*-1}\int_\Omega l(x)|u|^{2_s^*}dx - t^{-\gamma}\int_\Omega h(x)|u|^{1-\gamma}dx,$$

where in particular

$$\phi_u'(1) = a\|u\|^2 + b\|u\|^{2\mu} - \lambda\int_\Omega l(x)|u|^{2_s^*}dx - \int_\Omega h(x)|u|^{1-\gamma}dx. \tag{3.1}$$

From this, we may define the constrained set as

$$\mathbb{X} = \left\{u \in \mathbb{M}_0 : a\|u\|^2 + b\|u\|^{2\mu} - \lambda\int_\Omega l(x)|u|^{2_s^*}dx - \int_\Omega h(x)|u|^{1-\gamma}dx = 0\right\}. \tag{3.2}$$

Furthermore,

$$\phi_u''(t) = a\|u\|^2 + (2\mu-1)bt^{2\mu-2}\|u\|^{2\mu} - \lambda(2_s^*-1)t^{2_s^*-2}\int_\Omega l(x)|u|^{2_s^*}dx + \gamma t^{-\gamma-1}\int_\Omega h(x)|u|^{1-\gamma}dx.$$

Apparently,

$$\phi_u''(1) = a\|u\|^2 + (2\mu-1)b\|u\|^{2\mu} - \lambda(2_s^*-1)\int_\Omega l(x)|u|^{2_s^*}dx + \gamma\int_\Omega h(x)|u|^{1-\gamma}dx. \tag{3.3}$$

As a matter of fact, the two weak solutions we want are in $\mathbb{X}$. In order to better explore the existence of solutions, $\mathbb{X}$ can be further decomposed into $\mathbb{X}^+$, $\mathbb{X}^-$ and $\mathbb{X}^0$:

$$\mathbb{X}^+ = \left\{u\in\mathbb{X} : a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu} - \lambda(2_s^*-1+\gamma)\int_\Omega l(x)|u|^{2_s^*}dx > 0\right\}, \tag{3.4}$$

$$\mathbb{X}^- = \left\{u\in\mathbb{X} : a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu} - \lambda(2_s^*-1+\gamma)\int_\Omega l(x)|u|^{2_s^*}dx < 0\right\}, \tag{3.5}$$

$$\mathbb{X}^0 = \left\{u\in\mathbb{X} : a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu} - \lambda(2_s^*-1+\gamma)\int_\Omega l(x)|u|^{2_s^*}dx = 0\right\}. \tag{3.6}$$

4 Technical lemmas

In this section, we shall present several relevant lemmas in this section, which will be helpful for the proof of Theorem 1.1.

Lemma 4.1. *When* $0 < \lambda < \Gamma_1$ *hold, where*

$$\Gamma_1 = \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{a(2_s^*-2)}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} S_s^{\frac{2_s^*-1+\gamma}{1+\gamma}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}^{\frac{2-2_s^*}{1+\gamma}} \|l\|_\infty^{-1},$$

there exist unique $t_0 = t_0(u) > 0$, $t_- = t_-(u) > 0$, $t_+ = t_+(u) > 0$, *with* $t_- < t_0 < t_+$, *such that* $t_+u \in \mathbb{X}^+$, $t_-u \in \mathbb{X}^-$.

Proof. For any $u \in \mathbb{M}_0$, we may write $\psi_u(t)$ in the form

$$\psi_u(t) = at^{2-2_s^*}\|u\|^2 + bt^{2\mu-2_s^*}\|u\|^{2\mu} - t^{1-\gamma-2_s^*}\int_\Omega h(x)|u|^{1-\gamma}dx, \qquad t>0. \tag{4.1}$$

It is noticeable that if $\psi_u(t) = \lambda \int_\Omega l(x)|u|^{2_s^*}dx$, that is

$$at^{2-2_s^*}\|u\|^2 + bt^{2\mu-2_s^*}\|u\|^{2\mu} - t^{1-\gamma-2_s^*}\int_\Omega h(x)|u|^{1-\gamma}dx = \lambda\int_\Omega l(x)|u|^{2_s^*}dx, \tag{4.2}$$

multiplying $t^{2_s^*}$ on the both sides of the equation, one has

$$a\|tu\|^2 + b\|tu\|^{2\mu} - \int_\Omega h(x)|tu|^{1-\gamma}dx = \lambda\int_\Omega l(x)|tu|^{2_s^*}dx, \tag{4.3}$$

then we can deduce that $tu \in \mathbb{X}$.

We can easily infer from (4.1) that $\lim_{t\to 0^+}\psi_u(t) = -\infty$ and $\lim_{t\to\infty}\psi_u(t) = 0$. Furthermore, one step derivative calculation can get

$$\begin{aligned}\psi_u'(t) &= a(2-2_s^*)t^{1-2_s^*}\|u\|^2 + b(2\mu-2_s^*)t^{2\mu-1-2_s^*}\|u\|^{2\mu}\\ &\quad + (2_s^*+\gamma-1)t^{-\gamma-2_s^*}\int_\Omega h(x)|u|^{1-\gamma}dx.\end{aligned} \tag{4.4}$$

Based on the fact that $1 < 2\mu < 2_s^*$ and $0<\gamma<1$, one can obtain that $\lim_{t\to 0^+}\psi_u'(t) > 0$ and $\lim_{t\to\infty}\psi_u'(t) < 0$.

Rewrite $\psi_u'(t) = t^{2\mu-1-2_s^*}g_u(t)$, where

$$g_u(t) = a(2-2_s^*)t^{2-2\mu}\|u\|^2 + b(2\mu-2_s^*)\|u\|^{2\mu} + (2_s^*+\gamma-1)t^{1-\gamma-2\mu}\int_\Omega h(x)|u|^{1-\gamma}dx.$$

If

$$g_u'(t) = a(2-2_s^*)(2-2\mu)t^{1-2\mu}\|u\|^2 - (1-\gamma-2_s^*)(1-\gamma-2\mu)t^{-\gamma-2\mu}\int_\Omega h(x)|u|^{1-\gamma}dx = 0,$$

then it could be seen that there exists a unique

$$t_1 = \left(\frac{(2_s^*-1+\gamma)(2\mu-1+\gamma)\int_\Omega h(x)|u|^{-\gamma}dx}{a(2_s^*-2)(2\mu-2)\|u\|^2}\right)^{\frac{1}{1+\gamma}} > 0$$

such that $g_u'(t_1) = 0$. Similarly, since $1<2\mu<2_s^*$ and $0<\gamma<1$, we have $\lim_{t\to 0^+} g_u(t) = +\infty$ and $\lim_{t\to+\infty} g_u(t) = b(2\mu-2_s^*)\|u\|^{2\mu} < 0$. Also, $\lim_{t\to 0^+} g_u'(t) < 0$ and $\lim_{t\to+\infty} g_u'(t) > 0$.

Subsequently, we infer that there is only one $t_0 > 0$ that satisfies $g_u(t_0) = 0$. Actually, it follows from $\psi_u'(t) = t^{2\mu-1-2_s^*} g_u(t)$ that t_0 is a unique critical point of $\psi_u(t)$, which is the global maximum point. In another word, this means that when $0 < t < t_0$, $\psi_u(t)$ is increasing. $\psi_u(t)$ is decreasing in the range greater than t_0 and $\psi_u'(t_0) = 0$. We define

$$\psi_u(t_0) = \max_{t>0} \psi_u(t) = \max_{t>0}(b\|u\|^{2\mu} t^{2\mu-2_s^*} + \varphi_u(t)) \geq \max_{t>0} \varphi_u(t), \tag{4.5}$$

where

$$\varphi_u(t) = at^{2-2_s^*}\|u\|^2 - t^{1-\gamma-2_s^*} \int_\Omega h(x)|u|^{1-\gamma} dx.$$

With respect to $\varphi_u(t)$, there holds

$$\varphi_u'(t) = a(2-2_s^*)t^{1-2_s^*}\|u\|^2 - (1-\gamma-2_s^*)t^{-\gamma-2_s^*} \int_\Omega h(x)|u|^{1-\gamma} dx,$$

we observe that $\lim_{t\to 0^+} \varphi_u'(t) > 0$ and $\lim_{t\to+\infty} \varphi_u'(t) < 0$,

$$\max_{t>0} \varphi_u(t) = \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{2_s^*-2}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} \frac{(a\|u\|^2)^{\frac{2_s^*+\gamma-1}{1+\gamma}}}{\left(\int_\Omega h(x)|u|^{1-\gamma}dx\right)^{\frac{2_s^*-2}{1+\gamma}}}. \tag{4.6}$$

Hence by (4.5) and (4.6), we obtain

$$\begin{aligned}
&\psi_u(t_0) - \lambda \int_\Omega l(x)|u|^{2_s^*} dx \\
&\geq \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{2_s^*-2}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} \frac{(a\|u\|^2)^{\frac{2_s^*+\gamma-1}{1+\gamma}}}{\left(\int_\Omega h(x)|u|^{1-\gamma}dx\right)^{\frac{2_s^*-2}{1+\gamma}}} - \lambda \int_\Omega l(x)|u|^{2_s^*} dx \\
&\geq \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{a(2_s^*-2)}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} \frac{\|u\|^{\frac{22_s^*+2\gamma-2}{1+\gamma}}}{\left[\left(\int_\Omega h(x)^{\frac{2_s^*}{2_s^*-1+\gamma}}\right)^{\frac{2_s^*-1+\gamma}{2_s^*}} \left(\int_\Omega |u|^{2_s^*} dx\right)^{\frac{1-\gamma}{2_s^*}}\right]^{\frac{2_s^*-2}{1+\gamma}}} \\
&\quad - \lambda \|l\|_\infty S_s^{-\frac{2_s^*}{2}} \|u\|^{2_s^*} \\
&\geq \|u\|^{2_s^*} \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{a(2_s^*-2)}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} S_s^{\frac{(1-\gamma)(2_s^*-2)}{2(1+\gamma)}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}^{\frac{2-2_s^*}{1+\gamma}} - \lambda \|l\|_\infty S_s^{-\frac{2_s^*}{2}} \|u\|^{2_s^*} \\
&> 0,
\end{aligned} \tag{4.7}$$

for all $0 < \lambda < \left(\frac{1+\gamma}{2_s^*-2}\right)\left(\frac{a(2_s^*-2)}{2_s^*+\gamma-1}\right)^{\frac{2_s^*+\gamma-1}{1+\gamma}} S_s^{\frac{2_s^*-1+\gamma}{1+\gamma}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}^{\frac{2-2_s^*}{1+\gamma}} \|l\|_\infty^{-1} = \Gamma_1$. From (4.7), we can observe that there are unique $t_+ = t_+(u) < t_0$ and $t_- = t_-(u) > t_0$ satisfying

$$\psi_u(t_+) = \lambda \int_\Omega l(x)|u|^{2_s^*} dx = \psi_u(t_-).$$

Similar to (4.2) and (4.3), we confirm that $t_+u \in \mathbb{X}$ and $t_-u \in \mathbb{X}$. Since $\psi_u'(t_+) > 0$ and $\psi_u'(t_-) < 0$, we can get $t_+u \in \mathbb{X}^+$ and $t_-u \in \mathbb{X}^-$. Specifically,

$$\psi_u'(t_+) = a\|u\|^2(2-2_s^*)t_+^{1-2_s^*} + b\|u\|^{2\mu}(2\mu-2_s^*)t_+^{2\mu-1-2_s^*} - (1-\gamma-2_s^*)t_+^{-\gamma-2_s^*} \int_\Omega h(x)|u|^{1-\gamma} dx > 0$$

multiplying $t_+^{2_s^*+1}$ on the both sides of the inequation, one has

$$a\|t_+u\|^2(2-2_s^*)+b\|t_+u\|^{2\mu}(2\mu-2_s^*)-(1-\gamma-2_s^*)\int_\Omega h(x)|t_+u|^{1-\gamma}dx>0. \tag{4.8}$$

As to the definition of $\mathbb{X}^+$, the prerequisite is $u\in\mathbb{X}$,

$$\begin{aligned}\phi_t''(1)&=a\|u\|^2+(2\mu-1)b\|u\|^{2\mu}-\lambda(2_s^*-1)\int_\Omega l(x)|u|^{2_s^*}dx+\gamma\int_\Omega h(x)|u|^{1-\gamma}dx\\&=a\|u\|^2+(2\mu-1)b\|u\|^{2\mu}-(2_s^*-1)\big(a\|u\|^2+b\|u\|^{2\mu}\\&\quad-\int_\Omega h(x)|u|^{1-\gamma}dx\big)+\gamma\int_\Omega h(x)|u|^{1-\gamma}dx\\&=a\|u\|^2(2-2_s^*)+b(2\mu-2_s^*)\|u\|^{2\mu}-(1-\gamma-2_s^*)\int_\Omega h(x)|u|^{1-\gamma}dx.\end{aligned}$$

Therefore, another expression of $\mathbb{X}^+$ can be written as

$$\mathbb{X}^+=\left\{u\in\mathbb{X}:a\|u\|^2(2-2_s^*)+b(2\mu-2_s^*)\|u\|^{2\mu}-(1-\gamma-2_s^*)\int_\Omega h(x)|u|^{1-\gamma}dx>0\right\}. \tag{4.9}$$

Similarly,

$$\mathbb{X}^-=\left\{u\in\mathbb{X}:a\|u\|^2(2-2_s^*)+b(2\mu-2_s^*)\|u\|^{2\mu}-(1-\gamma-2_s^*)\int_\Omega h(x)|u|^{1-\gamma}dx<0\right\}, \tag{4.10}$$

$$\mathbb{X}^0=\left\{u\in\mathbb{X}:a\|u\|^2(2-2_s^*)+b(2\mu-2_s^*)\|u\|^{2\mu}-(1-\gamma-2_s^*)\int_\Omega h(x)|u|^{1-\gamma}dx=0\right\}. \tag{4.11}$$

Because (4.8) is established, we know that $t_+u\in\mathbb{X}^+$. At the same time, $t_-u\in\mathbb{X}^-$ can be obtained using the same method. □

Lemma 4.2. *There is $\Gamma_2>0$ satisfies $\mathbb{X}^0=\{0\}$ for all $0<\lambda<\Gamma_2$, where*

$$\Gamma_2=\frac{2\big[(1+\gamma)(2\mu-1+\gamma)ab\big]^{\frac12}}{(2_s^*+\gamma-1)\|l\|_\infty}\frac{S_s^{\frac{(\mu+1)(2_s^*+\gamma-1)}{2(\mu+\gamma)}}}{\left[\frac{(2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}}}{2\left[(2_s^*-2)(2_s^*-2\mu)ab\right]^{\frac12}}\right]^{\frac{2_s^*-\mu-1}{\mu+\gamma}}}.$$

Proof. We can prove it in two cases.

Case 1: $u\in\mathbb{X}\backslash\{0\}$ and $\int_\Omega l(x)|u|^{2_s^*}dx=0$.

According to the definition of $\mathbb{X}$, it follows that (3.2) that

$$a\|u\|^2+b\|u\|^{2\mu}-\int_\Omega h(x)|u|^{1-\gamma}dx=0.$$

On account of $0<\gamma<1$, we extrapolate that

$$\begin{aligned}\phi_u''(1)&=a\|u\|^2+b(2\mu-1)\|u\|^{2\mu}+\gamma\int_\Omega h(x)|u|^{1-\gamma}dx\\&=a\|u\|^2+b(2\mu-1)\|u\|^{2\mu}+\gamma(a\|u\|^2+b\|u\|^{2\mu})\\&=a(1+\gamma)\|u\|^2+(2\mu-1+\gamma)b\|u\|^{2\mu}>0.\end{aligned} \tag{4.12}$$

From this, we can learn that $u\notin\mathbb{X}^0$.

Case 2: $u \in \mathbb{X}\setminus\{0\}$ and $\int_\Omega l(x)|u|^{2_s^*}dx \neq 0$.
We may paradoxically assume there exists $u \in \mathbb{X}^0$ and $u \not\equiv 0$. On the basis of (3.2) and (3.3), we obtain

$$a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu} - (2_s^*-1+\gamma)\lambda\int_\Omega l(x)|u|^{2_s^*}dx = 0 \tag{4.13}$$

and

$$a(2-2_s^*)\|u\|^2 + b(2\mu-2_s^*)\|u\|^{2\mu} - (1-\gamma-2_s^*)\int_\Omega h(x)|u|^{1-\gamma}dx = 0. \tag{4.14}$$

Inspired by (4.13), we may define $H : \mathbb{X} \to \mathbb{R}$ as

$$H(u) = \frac{a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu}}{(2_s^*+\gamma-1)\lambda} - \int_\Omega l(x)|u|^{2_s^*}dx.$$

Obviously, if $u \in \mathbb{X}^0$, then $H(u) = 0$. Using (2.5) and the basic inequality $(\varrho+\kappa) \geq 2(\varrho\kappa)^{\frac{1}{2}}$, for any $\varrho, \kappa \geq 0$, we conclude that

$$\begin{aligned} H(u) &\geq \frac{2\left[(1+\gamma)(2\mu-1+\gamma)ab\right]^{\frac{1}{2}}}{(2_s^*+\gamma-1)\lambda}\|u\|^{\mu+1} - \|l\|_\infty S_s^{-\frac{2_s^*}{2}}\|u\|^{2_s^*} \\ &\geq \|u\|^{2_s^*}\left(\frac{2\left[(1+\gamma)(2\mu-1+\gamma)ab\right]^{\frac{1}{2}}}{(2_s^*+\gamma-1)\lambda}\frac{1}{\|u\|^{2_s^*-\mu-1}} - \|l\|_\infty S_s^{-\frac{2_s^*}{2}}\right). \end{aligned}$$

Besides, by (2.5), (4.14) and the Hölder inequality, we know

$$\begin{aligned} 2\left[(2_s^*-2\mu)(2_s^*-2)ab\right]^{\frac{1}{2}}\|u\|^{\mu+1} &\leq (2_s^*+\gamma-1)\left(\int_\Omega h(x)^{\frac{2_s^*}{2_s^*-1+\gamma}}dx\right)^{\frac{2_s^*-1+\gamma}{2_s^*}}\left(\int_\Omega |u|^{2_s^*}dx\right)^{\frac{1-\gamma}{2_s^*}} \\ &\leq (2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}}S_s^{\frac{\gamma-1}{2}}\|u\|^{1-\gamma}. \end{aligned}$$

Therefore,

$$\|u\| \leq \left[\frac{(2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}}S_s^{\frac{\gamma-1}{2}}}{2\left[(2_s^*-2\mu)(2_s^*-2)ab\right]^{\frac{1}{2}}}\right]^{\frac{1}{\mu+\gamma}}.$$

We control

$$\frac{2\left[(1+\gamma)(2\mu-1+\gamma)ab\right]^{\frac{1}{2}}}{(2_s^*+\gamma-1)\lambda}\frac{1}{\|u\|^{2_s^*-\mu-1}} - \|l\|_\infty S_s^{-\frac{2_s^*}{2}} > 0,$$

which leads to the following conclusion

$$\begin{aligned} \lambda &< \frac{2\left[(1+\gamma)(2\mu-1+\gamma)ab\right]^{\frac{1}{2}}}{(2_s^*+\gamma-1)\|l\|_\infty}\frac{S_s^{\frac{2_s^*}{2}}}{\left[\frac{(2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}}S_s^{\frac{\gamma-1}{2}}}{2\left[(2_s^*-2)(2_s^*-2\mu)ab\right]^{\frac{1}{2}}}\right]^{\frac{2_s^*-\mu-1}{\mu+\gamma}}} \\ &= \frac{2\left[(1+\gamma)(2\mu-1+\gamma)ab\right]^{\frac{1}{2}}}{(2_s^*+\gamma-1)\|l\|_\infty}\frac{S_s^{\frac{(\mu+1)(2_s^*+\gamma-1)}{2(\mu+\gamma)}}}{\left[\frac{(2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}}}{2\left[(2_s^*-2)(2_s^*-2\mu)ab\right]^{\frac{1}{2}}}\right]^{\frac{2_s^*-\mu-1}{\mu+\gamma}}} = \Gamma_2. \end{aligned}$$

Since $2_s^* > 2\mu$, $H(u) > 0$ for all $u \in \mathbb{X}^0 \setminus \{0\}$ can be confirmed. This causes the desired contradiction. □

Lemma 4.3. *$\mathcal{I}$, in addition to being coercive, is bounded from below on $\mathbb{X}$.*

Proof. For all $u \in \mathbb{X}$, we can deduce that

$$\begin{aligned}\mathcal{I}(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{2\mu}\|u\|^{2\mu} - \frac{1}{1-\gamma}\int_\Omega h(x)|u|^{1-\gamma}dx - \frac{1}{2_s^*}\left(a\|u\|^2 + b\|u\|^{2\mu} - \int_\Omega h(x)|u|^{1-\gamma}dx\right)\\ &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right)a\|u\|^2 + \left(\frac{1}{2\mu} - \frac{1}{2_s^*}\right)b\|u\|^{2\mu} - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right)\int_\Omega h(x)|u|^{1-\gamma}dx\\ &\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right)a\|u\|^2 - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}\|u\|^{1-\gamma}\end{aligned}$$

from the condition that $2\mu < 2_s^*$ and (2.5). Based on the fact of $1-\gamma < 2$, it can be determined that $\mathcal{I}$ is coercive. In addition, we may define

$$G_a(q) = \left(\frac{1}{2} - \frac{1}{2_s^*}\right)aq^2 - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}} q^{1-\gamma},$$

then

$$\begin{aligned}G_a'(q) &= \frac{2_s^*-2}{2_s^*}aq - \frac{2_s^*-1+\gamma}{2_s^*}\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}} q^{-\gamma},\\ G_a''(q) &= \frac{2_s^*-2}{2_s^*}a + \frac{2_s^*-1+\gamma}{2_s^*}\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}} \gamma q^{-\gamma-1}.\end{aligned}$$

We can obtain a unique stationary point q_{min}, where

$$q_{min} = \left(\frac{(2_s^*-1+\gamma)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}}{(2_s^*-2)a}\right)^{\frac{1}{1+\gamma}},$$

and

$$G_a''(q_{min}) = \frac{a(2_s^*-2)(1+\gamma)}{2_s^*} > 0.$$

Then $G_a(q)$ attains its minimum at q_{min}. Accordingly,

$$\begin{aligned}\mathcal{I}(u) \geq\ & \frac{\left((2_s^*-2)a\right)^{\frac{\gamma-1}{\gamma+1}}}{22_s^*}\left((2_s^*-1+\gamma)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2}{1+\gamma}}\\ & - \frac{1}{2_s^*(1-\gamma)}\left((2_s^*-2)a\right)^{\frac{\gamma-1}{\gamma+1}}\left((2_s^*-1+\gamma)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2}{1+\gamma}}\\ =\ & \frac{\gamma+1}{22_s^*(\gamma-1)}\left((2_s^*-2)a\right)^{\frac{\gamma-1}{\gamma+1}}\left((2_s^*-1+\gamma)\|h\|_{\frac{2_s^*}{2_s^*-1+\gamma}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2}{1+\gamma}} > -C_0\end{aligned}$$

for some constant $C_0 > 0$. This proof is completed. □

Lemma 4.4. *Let $\lambda \in (0, \Gamma_2)$, assume that $\gamma \in (0,1)$, then $\|u\| > \rho$ for all $u \in \mathbb{X}^-$, where*

$$\rho = \left(\frac{2\sqrt{(1+\gamma)(2\mu-1+\gamma)ab}}{(2_s^*+\gamma-1)\lambda\|l\|_\infty S_s^{-\frac{2_s^*}{2}}}\right)^{\frac{1}{2_s^*-\mu-1}}.$$

Proof. If $u \in \mathbb{X}^- \subset \mathbb{X}$, from (3.3), then we are sure that

$$a\|u\|^2 + b\|u\|^{2\mu} - \lambda \int_\Omega l(x)|u|^{2_s^*}dx - \int_\Omega h(x)|u|^{1-\gamma}dx = 0$$

and

$$a\|u\|^2 + (2\mu - 1)b\|u\|^{2\mu} - (2_s^* - 1)\lambda \int_\Omega l(x)|u|^{2_s^*}dx + \gamma \int_\Omega h(x)|u|^{1-\gamma}dx < 0,$$

which yields

$$\begin{aligned} &a(1+\gamma)\|u\|^2 + (2\mu - 1 + \gamma)b\|u\|^{2\mu} \\ &\qquad < (2_s^* + \gamma - 1)\lambda \int_\Omega l(x)|u|^{2_s^*}dx \le (2_s^* + \gamma - 1)\lambda\|l\|_\infty S_s^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}. \end{aligned}$$

Hence, we infer that $\rho < \|u\|$. □

Lemma 4.5. *Assume that $u_n \to u$ in $\mathbb{M}_0$, then*

$$\lim_{n\to+\infty} \int_\Omega l(x)|u_n|^{2_s^*}dx = \int_\Omega l(x)|u|^{2_s^*}dx, \tag{4.15}$$

and

$$\lim_{n\to+\infty} \int_\Omega h(x)|u_n|^{1-\gamma}dx = \int_\Omega h(x)|u|^{1-\gamma}dx. \tag{4.16}$$

Proof. Let $\{u_n\} \subset \mathbb{M}_0$ and $u_n \to u$ in $\mathbb{M}_0$. Due to $l \in L^\infty(\Omega)$ and $u_n \to u$, we deduce that there must be $C_1 > 0$ and $C_2 > 0$ satisfying $\|u_n\| \le C_1$ and $|l(x)| \le C_2$ a.e. in Ω. Set $k_n(x) = l(x)^{\frac{1}{2_s^*}}u_n$, $k(x) = l(x)^{\frac{1}{2_s^*}}u$, then

$$\left(\int_\Omega |k_n(x)|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} = \left(\int_\Omega l(x)|u_n|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} \le C_2^{\frac{1}{2_s^*}}\left(\int_\Omega |u_n|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} \le C_2^{\frac{1}{2_s^*}}S_s^{-\frac{1}{2}}C_1. \tag{4.17}$$

It can be clearly determined from this that

$$\begin{cases} \{k_n\} \text{ is bounded in } L^{2_s^*}(\Omega), \\ k_n \to k \text{ a.e. in } \Omega. \end{cases}$$

Moreover,

$$\begin{aligned} \int_\Omega |k_n(x) - k(x)|^{2_s^*}dx = \int_\Omega l(x)|u_n - u|^{2_s^*}dx &\le C_2 \int_\Omega |u_n - u|^{2_s^*}dx \\ &\le C_2\|u_n - u\|_{2_s^*}^{2_s^*} \to 0, \end{aligned} \tag{4.18}$$

for n large enough. All prerequisites have been met, and the Brézis–Lieb lemma can be used to obtain

$$\lim_{n\to+\infty}\left(\int_\Omega |k_n(x)|^{2_s^*}dx\right)^{\frac{1}{2_s^*}} = \left(\int_\Omega |k(x)|^{2_s^*}dx\right)^{2_s^*} + \lim_{n\to+\infty}\left(\int_\Omega |k_n(x) - k(x)|^{2_s^*}dx\right)^{\frac{1}{2_s^*}}.$$

On account of (4.18), we obtain (4.15). Using the same method, we can prove that (4.16) is valid. □

Lemma 4.6. *For all $0 < \lambda < \Gamma_2$, $\mathbb{X}^+ \cup \mathbb{X}^0$ and $\mathbb{X}^-$ are closed sets in G_0-topology.*

Proof. We prove this lemma in two parts. Let us first prove that $\mathbb{X}^+ \cup \mathbb{X}^0$ is a closed set.

Part 1: Suppose $\{u_n\} \subset \mathbb{X}^+ \cup \mathbb{X}^0$ and $u_n \to u_0$ in $\mathbb{M}_0$, we need to prove that $u_0 \in \mathbb{X}^+ \cup \mathbb{X}^0$. Since $\{u_n\} \subset \mathbb{X}^+ \cup \mathbb{X}^0$, we get

$$a(1+\gamma)\|u_n\|^2 + b(2\mu - 1 + \gamma)\|u_n\|^{2\mu} - (2_s^* + \gamma - 1)\lambda \int_\Omega l(x)|u_n|^{2_s^*} dx \geq 0.$$

Since $|\|u_n\| - \|u_0\|| \leq \|u_n - u_0\| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|u_n\|^2 = \|u_0\|^2, \qquad \lim_{n\to\infty} \|u_n\|^{2\mu} = \|u_0\|^{2\mu}.$$

Then, letting $n \to \infty$, it follows from Lemma 4.5 that

$$a(1+\gamma)\|u_0\|^2 + b(2\mu - 1 + \gamma)\|u_0\|^{2\mu} - (2_s^* + \gamma - 1)\lambda \int_\Omega l(x)|u_0|^{2_s^*} dx \geq 0.$$

Therefore, $\mathbb{X}^+ \cup \mathbb{X}^0$ is a closed set.

Part 2: Suppose that $\{u_n\} \subset \mathbb{X}^-$ such that $u_n \to u_0$ in $\mathbb{M}_0$. We infer that $u_0 \in \overline{\mathbb{X}^-} = \mathbb{X}^- \cup \{0\}$. By using Lemma 4.4, we have

$$\|u_0\| = \lim_{n\to\infty} \|u_n\| \geq \rho > 0. \tag{4.19}$$

Therefore $u_0 \neq 0$, which implies $u_0 \in \mathbb{X}^-$. This proof in completed. □

Lemma 4.7. *Let $u \in \mathbb{X}^+$(respectively $\mathbb{X}^-$) with $u \geq 0$, $0 < \gamma < 1$ and $h \in L^{\frac{2_s^*}{2_s^*+\gamma-1}}(\Omega)$. Subsequently, there exist $\varepsilon > 0$ and the continuous function $\varsigma : B_\varepsilon(0) \to \mathbb{R}^+$ satisfying*

$$\varsigma(z) > 0, \qquad \varsigma(0) = 1, \qquad \varsigma(z)(u+z) \in \mathbb{X}^\pm$$

for any $z \in B_\varepsilon(0)$, where $B_\varepsilon(0) = \{z \in \mathbb{M}_0 : \|z\| < \varepsilon\}$.

Proof. With regard to any $u \in \mathbb{X}^+ \subset \mathbb{X}$, define $Q : \mathbb{M}_0 \times \mathbb{R}^+ \to \mathbb{R}$ as follows

$$\begin{aligned} Q(z,\omega) &= \omega^{1+\gamma} a\|u+z\|^2 + \omega^{2\mu-1+\gamma} b\|u+z\|^{2\mu} \\ &\quad - \omega^{2_s^*-1+\gamma}\lambda \int_\Omega l(x)|u+z|^{2_s^*} dx - \int_\Omega h(x)|u+z|^{1-\gamma} dx. \end{aligned}$$

Differentiating the above equation, we determine that

$$\begin{aligned} \frac{\partial Q}{\partial \omega} &= a(1+\gamma)\omega^\gamma \|u+z\|^2 + b(2\mu-1+\gamma)\omega^{2\mu-2+\gamma}\|u+z\|^{2\mu} \\ &\quad - (2_s^* - 1 + \gamma)\lambda \omega^{2_s^*-2+\gamma} \int_\Omega l(x)|u+z|^{2_s^*} dx. \end{aligned}$$

Due to $u \in \mathbb{X}^+ \subset \mathbb{X}$, it is clear that

$$Q(0,1) = a\|u\|^2 + b\|u\|^{2\mu} - \int_\Omega h(x)|u|^{1-\gamma} dx - \lambda \int_\Omega l(x)|u|^{2_s^*} dx = 0 \tag{4.20}$$

and

$$\frac{\partial Q}{\partial \omega}(0,1) = a(1+\gamma)\|u\|^2 + b(2\mu-1+\gamma)\|u\|^{2\mu} - (2_s^* - 1 + \gamma)\lambda \int_\Omega l(x)|u|^{2_s^*} dx > 0. \tag{4.21}$$

The implicit function theorem applies to Q at the point $(0,1)$. A continuous function $\omega = \varsigma(z) > 0$ can be got, it can be seen that $\varsigma(0) = 1$ from (4.20). There is $\varepsilon^* > 0$. So $Q(z, \varsigma(z)) = 0$ for any $z \in \mathbb{M}_0$ with $\|z\| < \varepsilon^*$.

$$\begin{aligned} Q(z,w) &= Q(z,\varsigma(z)) \\ &= \varsigma^{1+\gamma}(z)a\|u+z\|^2 + \varsigma^{2\mu-1+\gamma}(z)b\|u+z\|^{2\mu} \\ &\quad -\varsigma^{2_s^*-1+\gamma}(z)\lambda\int_\Omega l(x)|u+z|^{2_s^*}dx - \int_\Omega h(x)|u+z|^{1-\gamma}dx \\ &= [a\|\varsigma(z)(u+z)\|^2 + b\|\varsigma(z)(u+z)\|^{2\mu} \\ &\quad -\lambda\int_\Omega l(x)|\varsigma(z)(u+z)|^{2_s^*}dx - \int_\Omega h(x)|\varsigma(z)(u+z)|^{1-\gamma}dx]/\varsigma^{1-\gamma}(z) \\ &= 0, \end{aligned} \tag{4.22}$$

that is $\varsigma(z)(u+z) \in \mathbb{X}$ for any $z \in \mathbb{M}_0$ with $\|z\| < \varepsilon^*$.

$$\begin{aligned} &\frac{\partial Q}{\partial \omega}(z,\varsigma(z)) \\ &= \frac{a(\gamma+1)\|\varsigma(z)(u+z)\|^2 + b(2\mu-1+\gamma)\|\varsigma(z)(u+z)\|^{2\mu} - (2_s^*-1+\gamma)\lambda\int_\Omega l(x)|\varsigma(z)(u+z)|^{2_s^*}dx}{\varsigma^{2-\gamma}(z)}. \end{aligned}$$

Taking sufficiently small $\varepsilon > 0$ so that $\varepsilon < \varepsilon^*$, we determine that

$$\varsigma(z)(u+z) \in \mathbb{X}^+, \qquad \forall z \in \mathbb{M}_0,\ \|z\| < \varepsilon.$$

As for $u \in \mathbb{X}^-$, we can proceed similarly to arrive at the same conclusion. □

5 Proof of Theorem 1.1

At present, let us show that problem (1.1) has a positive solution on each of $\mathbb{X}^+$ and $\mathbb{X}^-$, respectively. From Lemma 4.1, when $0 < \lambda < \Gamma_1$, one has $\mathbb{X}^\pm \neq \emptyset$. We complete this proof in two steps.

Step 1: We analyze problem (1.1) on $\mathbb{X}^+ \cup \mathbb{X}^0$.
According to Lemma 4.6, for $0 < \lambda < \Gamma_2$, we know $\mathbb{X}^+ \cup \mathbb{X}^0$ must be a closed set in $\mathbb{M}_0$. In the light of Lemma 4.3, $\mathcal{I}$ can be determined to be coercive and bounded below, $c^+ = \inf_{u\in\mathbb{X}^+\cup\mathbb{X}^0}\mathcal{I}$ can be clearly defined. Then, this minimization problem can be handled by Ekeland's variational principle. Then, a sequence $\{u_k\} \subset \mathbb{X}^+ \cup \mathbb{X}^0$ exists and satisfies the following properties:

$$(i)\ \mathcal{I}(u_k) < \inf_{u\in\mathbb{X}^+\cup\mathbb{X}^0}\mathcal{I}(u) + \frac{1}{k}, \qquad (ii)\ \mathcal{I}(u_k) \le \mathcal{I}(u) + \frac{1}{k}\|u_k - u\|, \qquad \forall u \in \mathbb{X}^+ \cup \mathbb{X}^0. \tag{5.1}$$

By means of $\mathcal{I}(u) = \mathcal{I}(|u|)$, we know that $u_k(x) \ge 0$ almost everywhere in Ω. Significantly, $\{u_k\}$ must be bounded in $\mathbb{M}_0$, going to a subsequence if necessary, let us represent the subsequence in terms of $\{u_n\}$. There exists u_0 satisfies

$$\begin{aligned} &u_n \rightharpoonup u_0 \quad \text{in } \mathbb{M}_0, \\ &u_n \to u_0 \quad \text{a.e. in } \Omega, \\ &u_n \rightharpoonup u_0 \quad \text{in } L^{2_s^*}, \\ &u_n \to u_0 \quad \text{in } L^r(\Omega) \text{ for } 2 \le r < 2_s^*, \end{aligned} \tag{5.2}$$

as $n \to \infty$. For all $u \in \mathbb{X}^+$, it follows from (3.4) and $2\mu < 2_s^*$ that

$$\begin{aligned}
\mathcal{I} &= \frac{a}{2}\|u\|^2 + \frac{b}{2\mu}\|u\|^{2\mu} - \frac{1}{2_s^*}\lambda \int_\Omega l(x)|u|^{2_s^*}dx - \frac{1}{1-\gamma}\int_\Omega h(x)|u|^{1-\gamma}dx \\
&= -\frac{a(1+\gamma)}{2(1-\gamma)}\|u\|^2 - \frac{2\mu+\gamma-1}{2\mu(1-\gamma)}b\|u\|^{2\mu} + \frac{\gamma-1+2_s^*}{2_s^*(1-\gamma)}\lambda\int_\Omega l(x)|u|^{2_s^*}dx \\
&< -\frac{a(1+\gamma)}{2(1-\gamma)}\|u\|^2 - \frac{2\mu+\gamma-1}{2\mu(1-\gamma)}b\|u\|^{2\mu} + \frac{a(1+\gamma)}{2_s^*(1-\gamma)}\|u\|^2 + \frac{2\mu+\gamma-1}{2_s^*(1-\gamma)}b\|u\|^{2\mu} \\
&< -\frac{a(1+\gamma)}{(1-\gamma)}\frac{(2_s^*-2)}{22_s^*}\|u\|^2 \\
&< 0.
\end{aligned} \tag{5.3}$$

So we are sure that $\inf_{u\in\mathbb{X}^+}\mathcal{I}(u) < 0$. Thus, $c^+ = \inf_{u\in\mathbb{X}^+}\mathcal{I}(u) < 0$, which in particular implies we might as well consider a subsequence $\{u_n\} \subset \mathbb{X}^+$. As to this fact, in terms of Lemma 4.7 with $u = u_n$, a series of functions ς_n satisfying $\varsigma_n(0) = 1$ can be obtained. Meanwhile, for $\varphi \in \mathbb{M}_0$ with $\varphi \geq 0$ and $\wp > 0$ sufficiently small, the fact that $\varsigma_n(\wp\varphi)(u_n + \wp\varphi) \in \mathbb{X}^+$ holds can be established. With these basic facts in mind, it is easy to know

$$a\|u_n\|^2 + b\|u_n\|^{2\mu} - \lambda\int_\Omega l(x)|u_n|^{2_s^*}dx - \int_\Omega h(x)|u_n|^{1-\gamma}dx = 0 \tag{5.4}$$

and

$$\begin{aligned}
&a\varsigma_n^2(\wp\varphi)\|u_n+\wp\varphi\|^2 + b\varsigma_n^{2\mu}(\wp\varphi)\|u_n+\wp\varphi\|^{2\mu} \\
&\quad - \varsigma_n^{2_s^*}(\wp\varphi)\lambda\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}dx - \varsigma_n^{1-\gamma}(\wp\varphi)\int_\Omega h(x)|u_n+\wp\varphi|^{1-\gamma}dx = 0.
\end{aligned} \tag{5.5}$$

It should be noted that $\varsigma_n'(0)$ is treated by us as the derivative of ς_n at zero, and its specific representation is as follows:

$$\varsigma_n'(0) = (\varsigma_n', \varphi) := \lim_{\wp\to 0}\frac{\varsigma_n(\wp\varphi)-1}{\wp} \in [-\infty, +\infty],$$

for all $\varphi \in \mathbb{M}_0$. Now let us prove when $\lambda < \Gamma_1$, $\{u_n\} \subset \mathbb{X}^\pm$ satisfies (5.1). Then, (ς_n', φ) must be uniformly bounded for all $\varphi \in \mathbb{M}_0$ with $\varphi \geq 0$. In particular, here we just consider $\{u_n\} \subset \mathbb{X}^+$, the case on $\mathbb{X}^-$ can be proved in the same way.

It follows from (5.4) and (5.5) that

$$\begin{aligned}
0 &= a[(\varsigma_n^2(\wp\varphi)-1)\|u_n+\wp\varphi\|^2 + \|u_n+\wp\varphi\|^2 - \|u_n\|^2] \\
&\quad + b[(\varsigma_n^{2\mu}(\wp\varphi)-1)\|u_n+\wp\varphi\|^{2\mu} + \|u_n+\wp\varphi\|^{2\mu} - \|u_n\|^{2\mu}] \\
&\quad - (\varsigma_n^{2_s^*}(\wp\varphi)-1)\lambda\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}dx - \lambda\int_\Omega l(x)(|u_n+\wp\varphi|^{2_s^*} - |u_n|^{2_s^*})dx \\
&\quad - (\varsigma_n^{1-\gamma}(\wp\varphi)-1)\int_\Omega h(x)|u_n+\wp\varphi|^{1-\gamma} - \int_\Omega h(x)(|u_n+\wp\varphi|^{1-\gamma} - |u_n|^{1-\gamma})dx \\
&\leq a(\varsigma_n^2(\wp\varphi)-1)\|u_n+\wp\varphi\|^2 + a(\|u_n+\wp\varphi\|^2 - \|u_n\|^2) \\
&\quad + b(\varsigma_n^{2\mu}(\wp\varphi)-1)\|u_n+\wp\varphi\|^{2\mu} + b(\|u_n+\wp\varphi\|^{2\mu} - \|u_n\|^{2\mu}) \\
&\quad - (\varsigma_n^{2_s^*}(\wp\varphi)-1)\lambda\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}dx - \lambda\int_\Omega l(x)(|u_n+\wp\varphi|^{2_s^*} - |u_n|^{2_s^*})dx \\
&\quad - (\varsigma_n^{1-\gamma}(\wp\varphi)-1)\int_\Omega h(x)|u_k+\wp\varphi|^{1-\gamma}.
\end{aligned}$$

Afterwards, dividing the above inequation by $\wp > 0$, we get

$$\begin{aligned}&\frac{\varsigma_n(\wp\varphi)-1}{\wp}\Big[a(\varsigma_n(\wp\varphi)+1)\|u_n+\wp\varphi\|^2+b\frac{\varsigma_n^{2\mu}(\wp\varphi)-1}{\varsigma_n(\wp\varphi)-1}\|u_n+\wp\varphi\|^{2\mu}\\&-\frac{\varsigma_n^{2_s^*}(\wp\varphi)-1}{\varsigma_n(\wp\varphi)-1}\lambda\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}dx-\frac{\varsigma_n^{1-\gamma}(\wp\varphi)-1}{\varsigma_n(\wp\varphi)-1}\int_\Omega h(x)|u_n+\wp\varphi|^{1-\gamma}dx\Big]\\&+a\frac{\|u_n+\wp\varphi\|^2-\|u_n\|^2}{\wp}+b\frac{\|u_n+\wp\varphi\|^{2\mu}-\|u_n\|^{2\mu}}{\wp}-\lambda\int_\Omega l(x)\frac{|u_n+\wp\varphi|^{2_s^*}-|u_n|^{2_s^*}}{\wp}dx\geq 0.\end{aligned}$$

Letting $\wp \to 0$, we extrapolate that

$$\begin{aligned}&(\varsigma_n',\varphi)\Big[2a\|u_n\|^2+2\mu b\|u_n\|^{2\mu}-2_s^*\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx-(1-\gamma)\int_\Omega h(x)|u_n|^{1-\gamma}dx\Big]\\&+2a\langle u_n,\varphi\rangle+2\mu b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle-2_s^*\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\geq 0.\end{aligned}\tag{5.6}$$

According to $\{u_n\} \in \mathbb{X}$, using (5.4) in (5.6), we have

$$\begin{aligned}&(\varsigma_n',\varphi)\Big[a(1+\gamma)\|u_n\|^2+(2\mu-1+\gamma)b\|u_n\|^{2\mu}-(2_s^*-1+\gamma)\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\Big]\\&+2a\langle u_n,\varphi\rangle+2\mu b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle-2_s^*\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\geq 0,\end{aligned}\tag{5.7}$$

that is

$$(\varsigma_n',\varphi)\geq\frac{-(2a\langle u_n,\varphi\rangle+2\mu b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle-2_s^*\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx)}{a(1+\gamma)\|u_n\|^2+(2\mu-1+\gamma)b\|u_n\|^{2\mu}-(2_s^*-1+\gamma)\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx}.$$

Since $\{u_n\}$ is bounded in $\mathbb{M}_0$, the above inequality means that (ς_n',φ) is bounded from below uniformly for any $\varphi\in\mathbb{M}_0$ with $\varphi\geq 0$, that is $(\varsigma_n',\varphi)\neq-\infty$.

Now we have to prove that (ς_n',φ) is bounded from above. By (5.1)-(ii), we have

$$\frac{\|u_n-\varsigma_n(\wp\varphi)(u_n+\wp\varphi)\|}{n}\geq\mathcal{I}(u_n)-\mathcal{I}[\varsigma_n(\wp\varphi)(u_n+\wp\varphi)]\tag{5.8}$$

and

$$\begin{aligned}\frac{\|u_n-\varsigma_n(\wp\varphi)(u_n+\wp\varphi)\|}{n}&=\frac{\|(1-\varsigma_n(\wp\varphi))u_n-\varsigma_n(\wp\varphi)\wp\varphi\|}{n}\\&\leq\frac{\|(1-\varsigma_n(\wp\varphi))u_n\|}{n}+\frac{\|-\varsigma_n(\wp\varphi)\wp\varphi\|}{n}\\&\leq|\varsigma_n(\wp\varphi)-1|\frac{\|u_n\|}{n}+\wp\zeta_n(\wp\varphi)\frac{\|\varphi\|}{n},\end{aligned}\tag{5.9}$$

which implies

$$\begin{aligned}&\Big|\varsigma_n(\wp\varphi)-1\Big|\frac{\|u_n\|}{n}+\wp\varsigma_n(\wp\varphi)\frac{\|\varphi\|}{n}\geq\mathcal{I}(u_n)-\mathcal{I}[\varsigma_n(\wp\varphi)(u_n+\wp\varphi)]\\&=\frac{a(1+\gamma)}{2(1-\gamma)}\Big[(\varsigma_n^2(\wp\varphi)-1)\|u_n+\wp\varphi\|^2+(\|u_n+\wp\varphi\|^2-\|u_n\|^2)\Big]\\&\quad+\frac{b(2\mu-1+\gamma)}{2\mu(1-\gamma)}\Big[(\varsigma_n^{2\mu}(\wp\varphi)-1)\|u_n+\wp\varphi\|^{2\mu}+(\|u_n+\wp\varphi\|^{2\mu}-\|u_n\|^{2\mu})\Big]\\&\quad-\frac{2_s^*-1+\gamma}{2_s^*(1-\gamma)}\lambda\Big[(\varsigma_n^{2_s^*}(\wp\varphi)-1)\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}+\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}-l(x)|u_n|^{2_s^*}dx\Big].\end{aligned}\tag{5.10}$$

Dividing (5.10) by $\wp > 0$, and letting $\wp \to 0$, we deduce

$$\begin{aligned}
&\left|(\varsigma_n', \varphi)\right| \frac{\|u_n\|}{n} + \lim_{\wp\to 0} \varsigma_n(\wp\varphi)\frac{\|\varphi\|}{n} \\
&\geq a\frac{1+\gamma}{1-\gamma}\left[(\varsigma_n', \varphi)\|u_n\|^2 + \langle u_n, \varphi\rangle\right] \\
&\quad + b\frac{2\mu-1+\gamma}{1-\gamma}\left[(\varsigma_n', \varphi)\|u_n\|^{2\mu} + \|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle\right] \\
&\quad - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\left[(\varsigma_n', \varphi)\int_\Omega l(x)|u_n|^{2_s^*}dx + \int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right].
\end{aligned}$$

So

$$\begin{aligned}
&\left|(\varsigma_n', \varphi)\right| \frac{\|u_n\|}{n} + \frac{\|\varphi\|}{n} \\
&\geq (\varsigma_n', \varphi)\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] \\
&\quad + a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle \\
&\quad - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx.
\end{aligned}$$

If $(\varsigma_n', \varphi) \geq 0$, then

$$\begin{aligned}
(\varsigma_n', \varphi) &\leq \frac{\frac{\|\varphi\|}{n} - \left(a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right)}{\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} + \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] - \frac{\|u_n\|}{n}} \\
&\leq \frac{\frac{\|\varphi\|}{n} + \left|a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right|}{\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} + \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] - \frac{\|u_n\|}{n}}.
\end{aligned} \tag{5.11}$$

If $(\varsigma_n', \varphi) < 0$, that is

$$\begin{aligned}
(\varsigma_n', \varphi) &\leq \frac{\frac{\|\varphi\|}{n} - \left(a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right)}{\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} + \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] + \frac{\|u_n\|}{n}} \\
&\leq \frac{\frac{\|\varphi\|}{n} + \left|a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right|}{\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} + \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] + \frac{\|u_n\|}{n}}.
\end{aligned} \tag{5.12}$$

Combining (5.11) and (5.12), we deduce that

$$(\varsigma_n', \varphi) \leq \frac{\frac{\|\varphi\|}{n} + \left|a\frac{1+\gamma}{1-\gamma}\langle u_n, \varphi\rangle + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu-2}\langle u_n, \varphi\rangle - \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx\right|}{\left[a\frac{1+\gamma}{1-\gamma}\|u_n\|^2 + b\frac{2\mu-1+\gamma}{1-\gamma}\|u_n\|^{2\mu} + \frac{2_s^*-1+\gamma}{1-\gamma}\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx\right] - \frac{\|u_n\|}{n}}.$$

By the boundedness of $\{u_n\}$, the above inequality can already explain $(\varsigma_n', \varphi) \neq +\infty$. In summary, there is a positive constant C_3 such that $|(\varsigma_n', \varphi)| \leq C_3$.

At present, again we use (5.8) and (5.9) in combination and divide by $\wp > 0$.

$$\begin{aligned}
&\Big|\frac{\varsigma_n(\wp\varphi)-1}{\wp}\Big|\frac{\|u_n\|}{n}+\varsigma_n(\wp\varphi)\frac{\|\varphi\|}{n} \\
&\geq \frac{\mathcal{I}(u_n)-\mathcal{I}[\varsigma_n(\wp\varphi)(u_n+\wp\varphi)]}{\wp} \\
&= -\frac{\varsigma_n(\wp\varphi)-1}{\wp}\Bigg[\frac{a(\varsigma_n(\wp\varphi)+1)}{2}\|u_n+\wp\varphi\|^2+\frac{b(\varsigma_n^{2\mu}(\wp\varphi)-1)}{2\mu(\varsigma_n(\wp\varphi)-1)}\|u_n+\wp\varphi\|^{2\mu} \\
&\quad -\frac{\varsigma_n^{2_s^*}(\wp\varphi)-1}{2_s^*(\varsigma_n(\wp\varphi)-1)}\lambda\int_\Omega l(x)|u_n+\wp\varphi|^{2_s^*}dx-\frac{\varsigma_n^{1-\gamma}(\wp\varphi)-1}{(1-\gamma)(\varsigma_n(\wp\varphi)-1)}\int_\Omega h(x)|u_k+\wp\varphi|^{1-\gamma}dx\Bigg] \\
&\quad -\Bigg[\frac{a(\|u_n+\wp\varphi\|^2-\|u_n\|^2)}{2\wp}+\frac{b(\|u_n+\wp\varphi\|^{2\mu}-\|u_n\|^{2\mu})}{2\mu\wp} \\
&\quad -\lambda\int_\Omega\frac{l(x)|u_n+\wp\varphi|^{2_s^*}-l(x)|u_n|^{2_s^*}}{2_s^*\wp}-\int_\Omega\frac{h(x)|u_n+\wp\varphi|^{1-\gamma}-h(x)|u_n|^{1-\gamma}}{(1-\gamma)\wp}dx\Bigg].
\end{aligned} \tag{5.13}$$

Based on the above inequality, now we let $\wp \to 0$. With the help of Fatou's lemma, we infer

$$\begin{aligned}
&|(\varsigma_n',\varphi)|\frac{\|u_n\|}{n}+\lim_{\wp\to0}\varsigma_n(\tau\varphi)\frac{\|\varphi\|}{n}=|(\varsigma_n',\varphi)|\frac{\|u_n\|}{n}+\frac{\|\varphi\|}{n} \\
&\geq -(\varsigma_n,\varphi)\Big[a\|u_n\|^2+b\|u_n\|^{2\mu}-\lambda\int_\Omega l(x)|u_n|^{2_s^*}dx-\int_\Omega h(x)|u_n|^{1-\gamma}dx\Big] \\
&\quad -\Big[a\langle u_n,\varphi\rangle+b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle-\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx \\
&\quad -\int_\Omega\liminf_{\wp\to0}\frac{h(x)|u_n+\wp\varphi|^{1-\gamma}-h(x)|u_n|^{1-\gamma}}{(1-\gamma)\wp}dx\Big] \\
&= -\Big[a\langle u_n,\varphi\rangle+b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle-\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx-\int_\Omega h(x)|u_n|^{-\gamma}\varphi dx\Big],
\end{aligned}$$

owing to $u_n \in \mathbb{X}$ and $|(\varsigma_n', \varphi)| \leq C_3$ uniformly for large n. Consequently,

$$\begin{aligned}
&(a+b\|u_n\|^{2\mu-2})\langle u_n,\varphi\rangle-\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx+\frac{|(\varsigma_n',\varphi)\|u_n\|+\|\varphi\||}{n} \\
&\geq \int_\Omega h(x)|u_n|^{-\gamma}\varphi dx,
\end{aligned} \tag{5.14}$$

which implies that as $n \to \infty$

$$a\langle u_n,\varphi\rangle+b\|u_n\|^{2\mu-2}\langle u_n,\varphi\rangle-\lambda\int_\Omega l(x)|u_n|^{2_s^*-1}\varphi dx-\int_\Omega h(x)|u_n|^{-\gamma}\varphi dx\geq o_n(1), \tag{5.15}$$

for any $\varphi \in \mathbb{M}_0$ with $\varphi \geq 0$.

After that, our purpose is to prove that (5.15) applicable to any arbitrary $\ell \in \mathbb{M}_0$. We set $\psi_\varepsilon = u_n + \varepsilon\ell$ with $\varepsilon > 0$ and $\ell \in \mathbb{M}_0$. Denoting $\Omega_\varepsilon = \{x \in \mathbb{R}^N : \psi_\varepsilon(x) \leq 0\}$. Afterwards, letting $\varphi = \psi_\varepsilon^+$ in (5.15), we confirm

$$\begin{aligned}
o_n(1) &\le (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^+\rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}\psi_\varepsilon^+ dx - \int_\Omega h(x)|u_n|^{-\gamma}\psi_\varepsilon^+ dx \\
&= (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon + \psi_\varepsilon^-\rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}(\psi_\varepsilon + \psi_\varepsilon^-)dx \\
&\quad - \int_\Omega h(x)|u_n|^{-\gamma}(\psi_\varepsilon + \psi_\varepsilon^-)dx \\
&= (a+b\|u_n\|^{2\mu-2})\langle u_n, u_n + \varepsilon\ell\rangle + (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^-\rangle \\
&\quad - \Big(\int_\Omega - \int_{\Omega_\varepsilon}\Big)\big[\lambda l(x)|u_n|^{2_s^*-1}(u_n+\varepsilon\ell) + h(x)|u_n|^{-\gamma}(u_n+\varepsilon\ell)\big]dx \\
&= \Big[a\|u_n\|^2 + b\|u_n\|^{2\mu} - \lambda \int_\Omega l(x)|u_n|^{2_s^*}dx - \int_\Omega h(x)|u_n|^{1-\gamma}dx\Big] \\
&\quad + \varepsilon\Big[(a+b\|u_n\|^{2\mu-2})\langle u_n, \ell\rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx\Big] \\
&\quad + (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^-\rangle + \int_{\Omega_\varepsilon} [\lambda l(x)|u_n|^{2_s^*-1}(u_n+\varepsilon\ell) + h(x)|u_n|^{-\gamma}(u_n+\varepsilon\ell)]dx \\
&= \Big[a\|u_n\|^2 + b\|u_n\|^{2\mu} - \lambda \int_\Omega l(x)|u_n|^{2_s^*}dx - \int_\Omega h(x)|u_n|^{1-\gamma}dx\Big] \\
&\quad + \varepsilon\Big[(a+b\|u_n\|^{2\mu-2})\langle u_n, \ell\rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx\Big] \\
&\quad + (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^-\rangle + \lambda \int_{\Omega_\epsilon} l(x)|u_n|^{2_s^*-1}(u_n+\varepsilon\ell)dx \\
&\quad + \int_{\Omega_\epsilon} h(x)|u_n|^{-\gamma}(u_n+\varepsilon\ell)dx.
\end{aligned}$$

Note that $u_n \in \mathbb{X}$ and $u_n + \varepsilon\ell \le 0$ in Ω_ε, thus

$$\int_{\Omega_\varepsilon} h(x)|u_n|^{-\gamma}(u_n+\varepsilon\ell)dx < 0.$$

Considering these facts, we deduce that

$$\begin{aligned}
o_n(1) &\le \varepsilon\Big[(a+b\|u_n\|^{2\mu-2})\langle u_n, \ell\rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx\Big] \\
&\quad + (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^-\rangle + \lambda \int_{\Omega_\varepsilon} l(x)|u_n|^{2_s^*-1}(u_n+\varepsilon\ell)dx \\
&\quad + \int_{\Omega_\varepsilon} h(x)|u_n|^{-\gamma}(u_n+\varepsilon\ell)dx \qquad (5.16)\\
&\le \varepsilon\Big[(a+b\|u_n\|^{2\mu-2})\langle u_n, \ell\rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx\Big] \\
&\quad + (a+b\|u_n\|^{2\mu-2})\langle u_n, \psi_\varepsilon^-\rangle + \lambda \int_{\Omega_\varepsilon} l(x)|u_n|^{2_s^*-1}(u_n+\varepsilon\ell)dx.
\end{aligned}$$

Then, denote

$$\Im_\varepsilon(x,y) = \frac{(u_n(x)-u_n(y))(\psi_\varepsilon^-(x) - \psi_\varepsilon^-(y))}{|x-y|^{N+2s}}$$

and

$$\Im(x,y) = \frac{(u_n(x)-u_n(y))(\ell(x)-\ell(y))}{|x-y|^{N+2s}}.$$

The definition of scalar products and the symmetry of the fraction kernel can be used here. Therefore, we may write

$$\begin{aligned}\langle u_n, \psi_\varepsilon^- \rangle &= \iint_G \frac{(u_n(x)-u_n(y))(\psi_\varepsilon^-(x)-\psi_\varepsilon^-(y))}{|x-y|^{N+2s}}dxdy \\ &= \iint_{(\mathbb{R}^N\times\mathbb{R}^N)\setminus(\Omega^c\times\Omega^c)} \Im_\varepsilon(x,y)dxdy \\ &= \left(\iint_{\Omega\times\Omega} + \iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)} + \iint_{(\mathbb{R}^N\setminus\Omega)\times\Omega}\right)\Im_\varepsilon(x,y)dxdy \\ &= \iint_{\Omega\times\Omega}\Im_\varepsilon(x,y)dxdy + 2\iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)}\Im_\varepsilon(x,y)dxdy.\end{aligned}$$

It is worth noting that $\psi_\varepsilon^- = 0$ in the case of ψ_ε is not in Ω_ε. From this, we

$$\begin{aligned}&\iint_{\Omega\times\Omega}\Im_\varepsilon(x,y)dxdy + 2\iint_{\Omega\times(\mathbb{R}^N\setminus\Omega)}\Im_\varepsilon(x,y)dxdy \\ &\quad= \left(\iint_{\Omega_\varepsilon\times\Omega_\varepsilon} +2\iint_{\Omega_\varepsilon\times(\Omega\setminus\Omega_\varepsilon)} +2\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega)}\right)\Im_\varepsilon(x,y)dxdy \\ &\quad= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\Im_\varepsilon(x,y)dxdy + 2\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)}\Im_\varepsilon(x,y)dxdy.\end{aligned}$$

Next,

$$\begin{aligned}&\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(u_n(x)-u_n(y))(\psi_\varepsilon^-(x)-\psi_\varepsilon^-(y))}{|x-y|^{N+2s}}dxdy \\ &\quad= \iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(u_n(x)-u_n(y))(\psi_\varepsilon^+(x)-\psi_\varepsilon^+(y))}{|x-y|^{N+2s}}dxdy \\ &\qquad- \iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(u_n(x)-u_n(y))(\psi_\varepsilon(x)-\psi_\varepsilon(y))}{|x-y|^{N+2s}}dxdy \\ &\quad= -\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}dxdy - \varepsilon\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\frac{(u_n(x)-u_n(y))(\ell(x)-\ell(y))}{|x-y|^{N+2s}}dxdy \\ &\quad\le -\varepsilon\iint_{\Omega_\varepsilon\times\Omega_\varepsilon}\Im(x,y)dxdy.\end{aligned}$$

In the same way,

$$2\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)}\Im_\varepsilon(x,y)dxdy \le -2\varepsilon\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)}\Im(x,y)dxdy.$$

In combination with the above, we can obtain

$$\langle u_n, \psi_\varepsilon^- \rangle \le -\varepsilon\left(\iint_{\Omega_\varepsilon\times\Omega_\varepsilon} +2\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus\Omega_\varepsilon)}\right)\Im(x,y)dxdy \le 2\varepsilon\iint_{\Omega_\varepsilon\times\mathbb{R}^N}|\Im(x,y)|dxdy. \tag{5.17}$$

Apparently, $\Im(x,y) \in L^1(\mathbb{R}^N\times\mathbb{R}^N)$. Besides, for any $\sigma > 0$, there exists R_σ sufficiently large.

From the basic definition of Ω_ε, we infer $\Omega_\varepsilon \subset \operatorname{supp}\ell$.

Since

$$\iint_{\Omega_\varepsilon\times\mathbb{R}^N}|\Im(x,y)|dxdy = \iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus B_{R_\sigma})}|\Im(x,y)|dxdy + \iint_{\Omega_\varepsilon\times B_{R_\sigma}}|\Im(x,y)|dxdy,$$

for the first item, we may obtain

$$\iint_{\Omega_\varepsilon\times(\mathbb{R}^N\setminus B_{R_\sigma})}|\Im(x,y)|dxdy < \iint_{(\operatorname{supp}\ell)\times(\mathbb{R}^N\setminus B_{R_\sigma})}|\Im(x,y)|dxdy \quad < \frac{\sigma}{2}.$$

Also, we know that $|\Omega_\varepsilon \times B_{R_\sigma}| \to 0$ as $\varepsilon \to 0^+$. The absolute continuity of the integral can be used, so there exists δ_σ and ε_σ such that for any $\varepsilon \in (0, \varepsilon_\sigma]$,

$$|\Omega_\varepsilon \times B_{R_\sigma}| < \delta_\sigma, \text{and} \iint_{\Omega_\varepsilon \times B_{R_\sigma}} |\Im(x,y)| dxdy < \frac{\sigma}{2}.$$

Accordingly, for all $\varepsilon \in (0, \varepsilon_\sigma]$,

$$\iint_{\Omega_\varepsilon \times \mathbb{R}^N} |\Im(x,y)| dxdy < \sigma,$$

$$\lim_{\varepsilon \to 0^+} \iint_{\Omega_\varepsilon \times \mathbb{R}^N} |\Im(x,y)| dxdy = 0.$$

Thus, according to (5.17), we get

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \langle u_n, \psi_\varepsilon^- \rangle = 0. \tag{5.18}$$

With respect to $\int_{\Omega_\varepsilon} l(x)|u_n|^{2_s^*-1}(u_n + \varepsilon\ell)dx$, since $\{u_n + \varepsilon\ell \le 0\} \to 0$ as $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} l(x)|u_n|^{2_s^*-1}(u_n + \varepsilon\phi)dx = 0. \tag{5.19}$$

Finally, dividing by ε and letting $\varepsilon \to 0$ in (5.16), for n large enough, we get from (5.18) and (5.19) that

$$a\langle u_n, \ell \rangle + b||u_n||^{2\mu-2}\langle u_n, \ell \rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx \ge o_n(1). \tag{5.20}$$

Replace ℓ in (5.20) with $-\ell$, and the inequality is also true. Thus it can be seen that

$$a\langle u_n, \ell \rangle + b\|u_n\|^{2\mu-2}\langle u_n, \ell \rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_n|^{-\gamma}\ell dx = o_n(1) \tag{5.21}$$

as $n \to \infty$.

Step 2: We analyze problem (1.1) on $\mathbb{X}^-$.

We have learned that $\mathcal{I}$ is bounded from below on $\mathbb{X}^-$ and coercive from Lemma 4.3. And it turns out that $\mathbb{X}^-$ is a closed set. Ekeland's variational principle can also be used. We may define $c^- = \inf_{u \in \mathbb{X}^-} \mathcal{I}$ accordingly. There exists a $\xi \in \mathbb{X}^-$, we deduce that

$$a(1+\gamma)\|\xi\|^2 + b(2\mu - 1 + \gamma)\|\xi\|^{2\mu} - \lambda(2_s^* - 1 + \gamma)\int_\Omega l(x)|\xi|^{2_s^*}dx < 0.$$

From (3.5) it follows that

$$b\|\xi\|^{2\mu} < \lambda \frac{2_s^* - 1 + \gamma}{2\mu - 1 + \gamma}\int_\Omega l(x)|\xi|^{2_s^*}dx - a\frac{1+\gamma}{2\mu - 1 + \gamma}\|\xi\|^2.$$

$$\begin{aligned}
\mathcal{I}(\xi) &= \frac{2_s^* - 2}{22_s^*}a\|\xi\|^2 + \frac{2_s^* - 2\mu}{2\mu 2_s^*}b\|\xi\|^{2\mu} - \frac{2_s^* + \gamma - 1}{2_s^*(1-\gamma)}\int_\Omega h(x)|\xi|^{1-\gamma}dx \\
&< \frac{2_s^* - 2}{22_s^*}a\|\xi\|^2 + \frac{2_s^* - 2\mu}{2\mu 2_s^*}\left[\lambda\frac{2_s^* - 1 + \gamma}{2\mu - 1 + \gamma}\int_\Omega l(x)|\xi|^{2_s^*}dx - a\frac{1+\gamma}{2\mu - 1 + \gamma}\|\xi\|^2\right] \\
&\quad - \frac{2_s^* + \gamma - 1}{2_s^*(1-\gamma)}\int_\Omega h(x)|\xi|^{1-\gamma}dx \\
&< \frac{\mu}{2\mu - 1 + \gamma}a\|\xi\|^2 + \lambda\left(\frac{2_s^* - 2\mu}{2\mu 2_s^*}\right)\left(\frac{2_s^* - 1 + \gamma}{2\mu - 1 + \gamma}\right)\int_\Omega l(x)|\xi|^{2_s^*}dx \\
&\quad - \frac{2_s^* + \gamma - 1}{2_s^*(1-\gamma)}\int_\Omega h(x)|\xi|^{1-\gamma}dx.
\end{aligned}$$

Let

$$\frac{\mu}{2\mu-1+\gamma}a\|\xi\|^2+\lambda\left(\frac{2_s^*-2\mu}{2\mu 2_s^*}\right)\left(\frac{2_s^*-1+\gamma}{2\mu-1+\gamma}\right)\int_\Omega l(x)|\xi|^{2_s^*}dx - \frac{2_s^*+\gamma-1}{2_s^*(1-\gamma)}\int_\Omega h(x)|\xi|^{1-\gamma}dx<0. \tag{5.22}$$

Then (5.22) implies that

$$0<\lambda<\left(\frac{2_s^*+\gamma-1}{2_s^*(1-\gamma)}\int_\Omega h(x)|\xi|^{1-\gamma}dx-\frac{\mu}{2\mu-1+\gamma}a\|\xi\|^2\right)\frac{2\mu 2_s^*(2\mu-1+\gamma)}{(2_s^*-2\mu)(2_s^*-1+\gamma)\int_\Omega l(x)|\xi|^{2_s^*}dx} = \Gamma_*.$$

In order to guarantee that λ is positive, we have to make a sufficiently small. Hence, when $0<\lambda<\Gamma_*$, $\mathcal{I}(\xi)<0$. Furthermore, we know that $c^-<0$.

Based on what is explained above, there exists a sequence $\{v_k\}\subset\mathbb{X}^-$ satisfies the following properties

$$(i)\ \mathcal{I}(v_k)<c^-+\frac{1}{k},\qquad (ii)\ \mathcal{I}(v_k)\le\mathcal{I}(v)+\frac{1}{k}\|v_k-v\|,\qquad \forall v\in\mathbb{X}^-. \tag{5.23}$$

Similarly, let us assume that $v_k(x)\ge 0$ for all $x\in\Omega$. Because $\mathbb{X}^-$ does not contain $\{0\}$. So $v_k(x)>0$ for all $x\in\Omega$. Apparently, $\{v_k\}$ is bounded in $\mathbb{M}_0$, we use $\{v_n\}$ to represent its subsequence, so there will be $v_0>0$ such that

$$\begin{aligned} v_n &\rightharpoonup v_0 && \text{in } \mathbb{M}_0,\\ v_n &\rightharpoonup v_0 && \text{in } L^{2_s^*},\\ v_n &\to v_0 && \text{a.e. in } \Omega,\\ v_n &\to v_0 && \text{in } L^\eta(\Omega) \text{ for } 2\le\eta<2_s^*, \end{aligned} \tag{5.24}$$

owing to $\mathbb{X}^-$ is a closed set. Applying Lemma 4.7 with $u=v_n$, $\varphi\in\mathbb{M}_0$, $\varphi\ge 0$ and $\wp>0$ small enough. A series of continuous functions satisfying $\varsigma_n(0)=1$ and $\varsigma_n(\wp\varphi)(v_n+\wp\varphi)\in\mathbb{X}^-$ can certainly be obtained. The proof procedure in **Step 1** can be used again to obtain

$$a\langle v_n,\ell\rangle+b||v_n||^{2\mu-2}\langle v_n,\ell\rangle-\lambda\int_\Omega l(x)|v_n|^{2_s^*-1}\ell dx-\int_\Omega h(x)|v_n|^{-\gamma}\ell dx\ge o_n(1) \tag{5.25}$$

as $n\to\infty$.

Lemma 5.1. *For $0<\lambda<\Gamma_1$, let $\{u_k\}\subset\mathbb{X}^+$ in **Step 1** and $\{v_k\}\subset\mathbb{X}^-$ in **Step 2** respectively satisfying (5.1) and (5.23) and simultaneously satisfying $\mathcal{I}\to c<C_\lambda$ as $k\to\infty$, where*

$$C_\lambda=\frac{s}{N}S_s^{\frac{N}{2s}}\left(\frac{a^{\frac{2_s^*}{2}}}{\|l\|_\infty}\right)^{\frac{2}{2_s^*-2}}\lambda^{\frac{2}{2-2_s^*}}-\frac{2\mu-1+\gamma}{2_s^*(1-\gamma)2\mu}\frac{\left((2_s^*+\gamma-1)\|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2\mu}{2\mu-1+\gamma}}}{[b(2_s^*-2\mu)]^{\frac{1-\gamma}{2\mu-1+\gamma}}}.$$

Then, both $\{u_k\}$ and $\{v_k\}$ have strongly convergent subsequences in $\mathbb{M}_0$.

Proof. Let us just think about $\{u_k\}\subset\mathbb{X}^+$, the case $\{v_k\}\subset\mathbb{X}^-$ can be obtained in the same way. Note that $\{u_k\}$ is bounded in $\mathbb{M}_0$ and $u_k\ge 0$. Furthermore, there is a subsequence $\{u_n\}$

that satisfies

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{in } \mathbb{M}_0, \\ u_n &\rightharpoonup u_0 \quad \text{in } L^{2_s^*}, \\ u_n &\to u_0 \quad \text{a.e. in } \ \Omega, \\ u_n &\le \hbar \quad \text{a.e. in } \ \Omega, \\ u_n &\to u_0 \quad \text{in } L^r(\Omega) \text{ for } 2 \le r < 2_s^*, \\ \|u_n\| &\to \zeta, \end{aligned} \tag{5.26}$$

as $n \to \infty$, with $\hbar \in L^r(\Omega)$ for a fixed $r \in [1, 2_s^*)$ and $u_0 \ge 0$. If $\zeta = 0$, that is $\|u_n - 0\| \to 0$ as $n \to 0$, which implies $u_n \to 0$ in $\mathbb{M}_0$ as $n \to \infty$. The situation of $\zeta > 0$ will be considered below.

According to (5.26), we get

$$\begin{aligned} 2\langle u_n, u_0 \rangle &= 2\langle u_0, u_0 \rangle + o(1) \\ &= \langle u_n, u_n \rangle - \langle u_n, u_n \rangle + 2\langle u_0, u_0 \rangle + o(1), \end{aligned}$$

as $n \to \infty$, which implies

$$\|u_n\|^2 = \|u_n - u_0\|^2 + \|u_0\|^2 + o(1) \tag{5.27}$$

as $n \to \infty$. Applying the Brézis–Lieb lemma and the process in Lemma 4.5, we have

$$\int_\Omega l(x)|u_n|^{2_s^*} dx = \int_\Omega l(x)|u_n - u_0|^{2_s^*} dx + \int_\Omega l(x)|u_0|^{2_s^*} dx + o(1) \tag{5.28}$$

as $n \to \infty$. We infer from (5.21), (5.27) and (5.28) that, as $n \to \infty$

$$\begin{aligned} o(1) &= (a + b\|u_n\|^{2\mu-2})\langle u_n, u_n - u_0 \rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}(u_n - u_0)dx \\ &\quad - \int_\Omega h(x)|u_n|^{-\gamma}(u_n - u_0)dx \\ &= (a + b\zeta^{2\mu-2})(\zeta^2 - \|u_0\|^2) - \lambda \int_\Omega l(x)|u_n|^{2_s^*} dx \\ &\quad + \lambda \int_\Omega l(x)|u_0|^{2_s^*} dx - \int_\Omega h(x)|u_n|^{-\gamma}(u_n - u_0)dx + o(1) \\ &= (a + b\zeta^{2\mu-2})\|u_n - u_0\|^2 - \lambda \int_\Omega l(x)|u_n - u_0|^{2_s^*} dx - \int_\Omega h(x)|u_n|^{-\gamma}(u_n - u_0)dx + o(1). \end{aligned}$$

Consequently,

$$\begin{aligned} &(a + b\zeta^{2\mu-2}) \lim_{n\to\infty} \|u_n - u_0\|^2 \\ &\quad = \lim_{n\to\infty} \lambda \int_\Omega l(x)|u_n - u_0|^{2_s^*} dx + \lim_{n\to\infty} \int_\Omega h(x)|u_n|^{1-\gamma} dx - h(x)|u_n|^{-\gamma} u_0 dx. \end{aligned} \tag{5.29}$$

By (5.26), we have $u_n^{1-\gamma} \le \hbar^{1-\gamma}$ a.e. in Ω. Then, the dominated convergence theorem can be used, that is

$$\lim_{n\to\infty} \int_\Omega h(x)|u_n|^{1-\gamma} dx = \int_\Omega h(x)|u_0|^{1-\gamma} dx. \tag{5.30}$$

In the light of (5.20), we know that $h(x)u_n^{-\gamma} u_0 dx \in L^1(\Omega)$. Then Fatou's lemma yields

$$\liminf_{n\to\infty} \int_\Omega h(x)u_n^{-\gamma} u_0 dx \ge \int_\Omega h(x)u_0^{1-\gamma} dx. \tag{5.31}$$

For convenience, we define

$$\aleph^{2_s^*} = \lim_{n\to\infty} \int_\Omega l(x)|u_n - u_0|^{2_s^*} dx. \tag{5.32}$$

Combining (5.29), (5.30) and (5.31), we get

$$(a + b\zeta^{2\mu-2}) \lim_{n\to\infty} \|u_n - u_0\|^2 \leq \lambda\aleph^{2_s^*}, \tag{5.33}$$

which means that $0 \leq \aleph$. If $\aleph = 0$, we can immediately infer that $u_n \to u_0$ in $\mathbb{M}_0$ owing to $\zeta > 0$. Let us paradoxically say that $\aleph > 0$ to complete this proof. From (5.32), as $n \to \infty$, we obtain

$$\aleph^{2_s^*} \leq \|l\|_\infty S_s^{-\frac{2_s^*}{2}} \lim_{n\to\infty} \|u_n - u_0\|^{2_s^*},$$

which implies that

$$\|l\|_\infty^{-\frac{2}{2_s^*}} \aleph^2 S_s \leq \lim_{n\to\infty} \|u_n - u_0\|^2. \tag{5.34}$$

Combining (5.33) and (5.34), we have

$$\aleph^{2_s^*-2} \geq (a + b\zeta^{2\mu-2})\|l\|_\infty^{-\frac{2}{2_s^*}} S_s\lambda^{-1}. \tag{5.35}$$

As $n \to \infty$, it is easy to get

$$\aleph^{2_s^*} \geq (a + b\zeta^{2\mu-2})(\zeta^2 - \|u_0\|^2)\lambda^{-1} \tag{5.36}$$

from (5.33). Using (5.34) and (5.35) in (5.36), we have

$$\begin{aligned}
(\aleph^{2_s^*})^{\frac{2_s^*-2}{2}} &\geq (a + b\zeta^{2\mu-2})^{\frac{2_s^*-2}{2}} (\zeta^2 - \|u_0\|^2)^{\frac{2_s^*-2}{2}} \lambda^{\frac{2-2_s^*}{2}} \\
&= (a + b\zeta^{2\mu-2})^{\frac{2_s^*-2}{2}} (\lim_{n\to\infty} \|u_n - u_0\|^2)^{\frac{2_s^*-2}{2}} \lambda^{\frac{2-2_s^*}{2}} \\
&\geq (a + b\zeta^{2\mu-2})^{\frac{2_s^*-2}{2}} (\|l\|_\infty^{-\frac{2}{2_s^*}} S_s)^{\frac{2_s^*-2}{2}} \aleph^{2_s^*-2} \lambda^{\frac{2-2_s^*}{2}} \\
&\geq (a + b\zeta^{2\mu-2})^{\frac{2_s^*}{2}} S_s^{\frac{2_s^*}{2}} \|l\|_\infty^{-1} \lambda^{-\frac{2_s^*}{2}}.
\end{aligned} \tag{5.37}$$

At the same time, according to (5.34) and (5.35), we get

$$\begin{aligned}
(\zeta^2 - \|u_0\|^2)^{\frac{2_s^*-2}{2}} &\geq \|l\|_\infty^{-\frac{2_s^*-2}{2_s^*}} \aleph^{2_s^*-2} S_s^{\frac{2_s^*-2}{2}} \\
&\geq (a + b\zeta^{2\mu-2})\|l\|_\infty^{-1} S_s^{\frac{2_s^*}{2}} \lambda^{-1}
\end{aligned} \tag{5.38}$$

as $n \to \infty$. Consequently, we have

$$(\zeta^2)^{\frac{2_s^*-2}{2}} \geq (\zeta^2 - \|u_0\|^2)^{\frac{2_s^*-2}{2}} \geq (a + b\zeta^{2\mu-2})\|l\|_\infty^{-1} S_s^{\frac{2_s^*}{2}} \lambda^{-1}, \tag{5.39}$$

that is

$$\zeta^2 \geq S_s^{\frac{N}{2s}} \|l\|_\infty^{-\frac{2}{2_s^*-2}} (a + b\zeta^{2\mu-2})^{\frac{2}{2_s^*-2}} \lambda^{\frac{2}{2-2_s^*}} \geq S_s^{\frac{N}{2s}} \|l\|_\infty^{-\frac{2}{2_s^*-2}} a^{\frac{2}{2_s^*-2}} \lambda^{\frac{2}{2-2_s^*}}. \tag{5.40}$$

We define

$$F(u_n, \phi) = (a + b\|u_n\|^{2\mu-2})\langle u_n, \phi\rangle - \lambda \int_\Omega l(x)|u_n|^{2_s^*-1}\phi dx - \int_\Omega h(x)|u_n|^{-\gamma}\phi dx \tag{5.41}$$

for any $\phi \in \mathbb{M}_0$. Subsequently,

$$\begin{aligned} &\mathcal{I}(u_n) - \frac{1}{2_s^*}F(u_n,u_n) \\ &= \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|u_n\|^2 + \left(\frac{1}{2\mu} - \frac{1}{2_s^*}\right) b\|u_n\|^{2\mu} - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right) \int_\Omega h(x)|u_n|^{1-\gamma}dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|u_n\|^2 + \left(\frac{1}{2\mu} - \frac{1}{2_s^*}\right) b\|u_n\|^{2\mu} \\ &\quad - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right) S_s^{\frac{\gamma-1}{2}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} \|u_n\|^{1-\gamma}. \end{aligned} \tag{5.42}$$

Define

$$P(t) = \left(\frac{1}{2\mu} - \frac{1}{2_s^*}\right) bt^{2\mu} - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right) S_s^{\frac{\gamma-1}{2}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} t^{1-\gamma},$$

$$P'(t) = \left(\frac{1}{2\mu} - \frac{1}{2_s^*}\right) 2\mu bt^{2\mu-1} - \left(\frac{1}{1-\gamma} - \frac{1}{2_s^*}\right) S_s^{\frac{\gamma-1}{2}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} (1-\gamma)t^{-\gamma}.$$

When $p'(t) = 0$, we can get

$$t = \left[\frac{2_s^* - 1 + \gamma}{b(2_s^* - 2\mu)}\right]^{\frac{1}{2\mu-1+\gamma}} S_s^{\frac{\gamma-1}{2(2\mu-1+\gamma)}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}^{\frac{1}{2\mu-1+\gamma}}.$$

Hence we have

$$\begin{aligned} P(t) &\geq \frac{2_s^* - 2\mu}{2\mu 2_s^*} b \left[\frac{2_s^* - 1 + \gamma}{b(2_s^* - 2\mu)}\right]^{\frac{2\mu}{2\mu-1+\gamma}} S_s^{\frac{(\gamma-1)\mu}{2\mu-1+\gamma}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}^{\frac{2\mu}{2\mu-1+\gamma}} \\ &\quad - \frac{2_s^* - 1 + \gamma}{(1-\gamma)2_s^*} S_s^{\frac{\gamma-1}{2}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} \left[\frac{2_s^* - 1 + \gamma}{b(2_s^* - 2\mu)}\right]^{\frac{1-\gamma}{2\mu-1+\gamma}} S_s^{\frac{-\gamma^2+2\gamma-1}{2(2\mu-1+\gamma)}} \|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}^{\frac{1-\gamma}{2\mu-1+\gamma}} \\ &= -\frac{2\mu - 1 + \gamma}{2_s^*(1-\gamma)2\mu} \frac{\left((2_s^* + \gamma - 1)\|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2\mu}{2\mu-1+\gamma}}}{[b(2_s^* - 2\mu)]^{\frac{1-\gamma}{2\mu-1+\gamma}}}. \end{aligned} \tag{5.43}$$

Then, from (5.42), we have

$$\begin{aligned} \mathcal{I}(u_n) - \frac{1}{2_s^*}F(u_n,u_n) &\geq \left(\frac{1}{2} - \frac{1}{2_s^*}\right) a\|u_n\|^2 \\ &\quad - \frac{2\mu - 1 + \gamma}{2_s^*(1-\gamma)2\mu} \frac{\left((2_s^* + \gamma - 1)\|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2\mu}{2\mu-1+\gamma}}}{[b(2_s^* - 2\mu)]^{\frac{1-\gamma}{2\mu-1+\gamma}}}. \end{aligned} \tag{5.44}$$

Letting $n \to \infty$, we get

$$c \geq \frac{s}{N} S_s^{\frac{N}{2s}} \left(\frac{a^{\frac{2_s^*}{2}}}{\|l\|_\infty}\right)^{\frac{2}{2_s^*-2}} \lambda^{\frac{2}{2-2_s^*}} - \frac{2\mu - 1 + \gamma}{2_s^*(1-\gamma)2\mu} \frac{\left((2_s^* + \gamma - 1)\|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}} S_s^{\frac{\gamma-1}{2}}\right)^{\frac{2\mu}{2\mu-1+\gamma}}}{[b(2_s^* - 2\mu)]^{\frac{1-\gamma}{2\mu-1+\gamma}}},$$

which contradicts the assumption $c < C_\lambda$. This proof is completed. □

Let us fix $\lambda < \Gamma_0 = \min\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_*\}$, with Γ_1 and Γ_2 given respectively in Lemma 4.1 and Lemma 4.2, and

$$\Gamma_3 = \frac{a^{\frac{2_s^*}{2}}}{\|l\|_\infty}\left(\frac{s}{N}S_s^{\frac{N}{2s}}\frac{2_s^*(1-\gamma)2\mu}{2\mu-1+\gamma}\right)^{\frac{2_s^*-2}{2}}\left[\frac{[b(2_s^*-2\mu)]^{\frac{1-\gamma}{2\mu-1+\gamma}}}{(2_s^*-1+\gamma)\|h\|_{\frac{2_s^*}{2_s^*+\gamma-1}}S_s^{\frac{\gamma-1}{2}}}\right]^{\frac{2_s^*-2}{2}},$$

which shows that $C_\lambda > 0$. From (5.3) and (5.22), we know that $c^+ < 0 < C_\lambda$ and $c^- < 0 < C_\lambda$. Applying the Lemma 5.1, the minimization sequence $\{u_n\}$ will satisfy $u_n \to u_0$ in $\mathbb{M}_0$, and the minimization sequence $\{v_n\}$ will satisfy $v_n \to v_0$ in $\mathbb{M}_0$. According to (5.20) and (5.25), we can separately obtain

$$a\langle u_0, \ell\rangle + b\|u_0\|^{2\mu-2}\langle u_0, \ell\rangle - \lambda\int_\Omega l(x)|u_0|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_0|^{-\gamma}\ell dx \geq 0,$$

and

$$a\langle v_0, \ell\rangle + b||v_0||^{2\mu-2}\langle v_0, \ell\rangle - \lambda\int_\Omega l(x)|v_0|^{2_s^*-1}\ell dx - \int_\Omega h(x)|v_0|^{-\gamma}\ell dx \geq 0$$

for any $\ell \in \mathbb{M}_0$. From the two inequalities above, we know $h(x)|u_0|^{-\gamma}\ell$ and $h(x)|v_0|^{-\gamma}\ell$ are integrable, which imply that $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$ in Ω, then the strong maximum principle (see Proposition 2.2.8 in [27]) yields that $u_0 > 0$ and $v_0 > 0$ in Ω. According to the arbitrariness of ℓ, we know that (5.20) fits any $\ell \in \mathbb{M}_0$. It follows that

$$a\langle u_0, \ell\rangle + b\|u_0\|^{2\mu-2}\langle u_0, \ell\rangle - \lambda\int_\Omega l(x)|u_0|^{2_s^*-1}\ell dx - \int_\Omega h(x)|u_0|^{-\gamma}\ell dx = 0,$$

and

$$a\langle v_0, \ell\rangle + b||v_0||^{2\mu-2}\langle v_0, \ell\rangle - \lambda\int_\Omega l(x)|v_0|^{2_s^*-1}\ell dx - \int_\Omega h(x)|v_0|^{-\gamma}\ell dx = 0$$

as $n \to \infty$. This indicates that problem (1.1) has a positive solution on both $\mathbb{X}^+$ and $\mathbb{X}^-$, respectively.

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Some new results for the smoothness of topological equivalence in uniformly asymptotically stable systems

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Abstract. In this article we revisit a method of topological linearization for nonautonomous and uniformly asymptotically stable ordinary differential equations developed by Kenneth J. Palmer and Faxing Lin. In particular, sufficient conditions are obtained ensuring the smoothness of the above mentioned topological linearization.

Keywords: nonautonomous ordinary differential equations, uniform asymptotic stability, topological equivalence, diffeomorphisms.

2020 Mathematics Subject Classification: 34A30, 34C41, 34D10, 37B55.

1 Introduction

The smoothness of the topological equivalence or topological conjugacy is a classical topic on autonomous dynamical systems and we refer the reader to [16] for an overview on the latest advances. Nevertheless, the nonautonomous case is considerably less developed than the autonomous one; and the first results go back to the last decade. In this note, we will continue this study in the nonautonomous case.

More specifically, we obtain sufficient conditions ensuring the differentiability of the topological equivalence for certain families of nonautonomous systems

$$\dot{x} = F_1(t,x) \quad \text{for any } t \in J, \tag{1.1}$$

and

$$\dot{y} = F_2(t,y) \quad \text{for any } t \in J, \tag{1.2}$$

where $J \subseteq \mathbb{R}$ is an upperly unbounded interval while the functions $F_i\colon J \times \mathbb{R}^n \to \mathbb{R}^n$ are such that the existence and uniqueness of the solutions on J is ensured. In addition, the solutions of (1.1) and (1.2) passing through x_0 and y_0 at $t = \tau \in J$ will be denoted respectively by $t \mapsto x(t,\tau,x_0)$ and $t \mapsto y(t,\tau,y_0)$.

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The above systems are J-topologically equivalent when there exists a family of homeomorphisms parametrized by J mapping solutions of a system into solutions of the other one and viceversa; this property is described formally as follows:

Definition 1.1 ([21])**.** The systems (1.1) and (1.2) are J-topologically equivalent if there exists a function $H\colon J \times \mathbb{R}^n \to \mathbb{R}^n$ such that:

i) For any fixed $\tau \in J$, $x_0 \mapsto H_\tau(x_0) := H(\tau, x_0)$ is a homeomorphism of $\mathbb{R}^n$, whose inverse is denoted by $y_0 \mapsto G_\tau(y_0) := G(\tau, y_0)$.

ii) If $t \mapsto x(t,\tau,x_0)$ is a solution of (1.1) then $t \mapsto H(t, x(t,\tau,x_0))$ is a solution of (1.2). Similarly, if $t \mapsto y(t,\tau,y_0)$ is a solution of (1.2), then $t \mapsto G(t, y(t,\tau,y_0))$ is a solution of (1.1). That is, for any $t,\tau \in J$ it follows:

$$\begin{cases} H(t,x(t,\tau,x_0)) = y(t,\tau,H(\tau,x_0)) \\ G(t,y(t,\tau,y_0)) \;= x(t,\tau,G(\tau,y_0)). \end{cases} \tag{1.3}$$

iii) For any fixed $\tau \in J$, it is verified that the norms

$$\|H(\tau,x_0)\| \to +\infty \quad \text{and} \quad \|G(\tau,y_0)\| \to +\infty \quad \text{as} \quad \|x_0\|, \|y_0\| \to +\infty.$$

Altough there is not a universally accepted definition of topological equivalence, the statements i) and ii) hold consistently in the specialized literature, while the asymptotic property iii) can be replaced by other types of conditions; see *e.g.* [18, p. 12], [19, p. 357] and [21] for details.

In the nonautonomous framework, the problem of search sufficient conditions ensuring the differentiability properties of a topological equivalence is relatively recent. In addition, there are diverse approaches to construct the homeomorphisms stated in Definition 1.1. Having this in mind, in Section 2 we describe the two main strategies: the use of the Green's function and the crossing time function.

The rest of the article is organized as follows: Sections 3 and 4 focus on deducting the differentiability for the $\mathbb{R}$-topological equivalence between uniformly asymptotically stable systems (1.1)–(1.2), obtained by F. Lin [14] and K. J. Palmer [18] by using a crossing time approach. Section 5 addresses the higher order differentiability. Section 6 provides an additional result and compares it with the current literature.

Last but not least, we point out that, to the best of our knowledge, there are no smoothness results for the crossing time approach in the nonautonomous context, which is the main novelty and contribution of this article.

2 The topological equivalence problem

The *topological equivalence problem* can be understood as the research of sufficient conditions on the vector fields F_i with $i = 1,2$ such that (1.1) and (1.2) are J-topologically equivalent. In this article, we will distinguish some approaches carried out to cope with this problem: the **Green's function approach** and the **crossing time approach**. Nevertheless, it is important to emphasize that this distinction is not exhaustive.

From now on, the symbol $\|\cdot\|$ denotes either the euclidean vector norm or its induced matrix norm. The particular context of its appearance will indicate what its meaning is. In addition, $u = o(v)$ is the classical Landau's little-o notation.

2.1 The Green's function approach

The topological equivalence problem is studied for the particular case of systems (1.1)–(1.2), described by the linear system

$$\dot{x} = A(t)x \quad \text{for any } t \in J, \tag{2.1}$$

and a family of quasilinear perturbations, such as:

$$\dot{y} = A(t)y + f(t,y) \quad \text{for any } t \in J, \tag{2.2}$$

which is to say $F_1(t,x) = A(t)x$ and $F_2(t,y) = A(t)y + f(t,y)$ with $J = \mathbb{R}$ or $J = [0,+\infty)$.

A pivotal assumption of this approach is that (2.1) has a *dichotomy* property on J, which is defined as follows:

Definition 2.1. The system (2.1) has a dichotomy on J if there exist a projector $t \mapsto P(t) \in M_n(\mathbb{R})$, positive constants K, α and two functions $h, \mu \colon J \to [1,+\infty)$ continuous, increasing and verifying $\mu = o(h^\alpha)$ such that any fundamental matrix $t \mapsto \Phi(t)$ of (2.1) verifies:

$$P(t)\Phi(t,s) = \Phi(t,s)P(s) \quad \text{for any } t,s \in J$$

and

$$\begin{cases} \|\Phi(t,s)P(s)\| \leq K\mu(|s|)\left(\frac{h(t)}{h(s)}\right)^{-\alpha} & \text{for any } t \geq s \text{ with } t,s \in J, \\ \|\Phi(t,s)Q(s)\| \leq K\mu(|s|)\left(\frac{h(s)}{h(t)}\right)^{-\alpha} & \text{for any } s \geq t \text{ with } t,s \in J, \end{cases}$$

where $\Phi(t,s) := \Phi(t)\Phi^{-1}(s)$ and $Q(t) = I - P(t)$.

Note that any nontrivial solution $t \mapsto x(t,\tau,x_0) = \Phi(t,\tau)x_0$ of (2.1) can be splitted as

$$x(t,\tau,x_0) = \underbrace{\Phi(t,\tau)P(\tau)x_0}_{=x^+(t,\tau,x_0)} + \underbrace{\Phi(t,\tau)Q(\tau)x_0}_{=x^-(t,\tau,x_0)},$$

where $t \mapsto x^+(t,\tau,x_0) := x^+(t)$ and $t \mapsto x^-(t,\tau,x_0) := x^-(t)$ verify

$$\|x^+(t)\| \leq K\|P(\tau)x_0\|\mu(|\tau|)\left(\frac{h(t)}{h(\tau)}\right)^{-\alpha} \quad \text{and} \quad \left(\frac{h(t)}{h(\tau)}\right)^{\alpha}\frac{\|Q(\tau)x_0\|}{K\mu(|t|)} \leq \|x^-(t)\|,$$

for any $t \geq \tau$.

The above mentioned properties of μ,h and α allow to deduce that $x^+(t)$ is a forward contraction and $x^-(t)$ is a forward expansion. This splitting and its *dichotomic* asymptotic behavior motivate the use of the name dichotomy.

There exist several kinds of dichotomies describing the contractions and expansions at a specific rate; we refer the reader to the Table 1 from [23] and references therein for a detailed description.

The Green's function associated to the above mentioned dichotomy property is

$$\mathcal{G}(t,s) = \begin{cases} \Phi(t,s)P(s) & \text{if } t \geq s, \\ -\Phi(t,s)Q(s) & \text{if } t < s, \end{cases}$$

and allows an explicit construction of the homeomorphisms H_t and their inverses G_t mentioned on Definition 1.1.

The first homeomorphism was established by K. J. Palmer [17], which was constructed under the following assumptions: (2.1) has an exponential dichotomy on $\mathbb{R}$, namely $h(t) = e^t$ and $\mu(t) = 1$, the function f is uniformly bounded on $\mathbb{R} \times \mathbb{R}^n$ and $x \mapsto f(t,x)$ is uniformly Lipschitz with respect to t.

The first improvement of Palmer´s result was done by J. Shi and K. Xiong in [21], who demonstrated that the maps $\xi \mapsto H_t(\xi)$ and $\xi \mapsto G_t(\xi)$ are uniformly continuous with respect to t.

There exist a vast corpus of literature devoted to the topological equivalence problem by following this approach. In general, the problem is addressed by considering dichotomies more general than the exponential one; and, at the same time, imposing more restrictive assumptions on the perturbation f. In this context, we highlight the work of L. Barreira and C. Valls [2], which assumes that (2.1) has a nonuniform exponential dichotomy on $\mathbb{R}$, that is, $h(t) = e^t$ and $\mu(t) = e^{\varepsilon|t|}$. We refer the reader again to the Table 1 from [23] and [19] for more results.

2.2 The crossing time approach

In the work [18] of K. J. Palmer, the topological equivalence problem is considered for systems (1.1)–(1.2), where the maps $F_1 \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $F_2 \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following properties:

(P1) The origin is an equilibrium for any $t \in \mathbb{R}$, that is

$$F_1(t,0) = F_2(t,0) = 0 \quad \text{for any } t \in \mathbb{R},$$

(P2) There exists $L > 0$ such that, given any $t \in \mathbb{R}$ and $x, \tilde{x} \in \mathbb{R}^n$,

$$\|F_1(t,x) - F_1(t,\tilde{x})\| \leq L\|x - \tilde{x}\| \quad \text{and} \quad \|F_2(t,x) - F_2(t,\tilde{x})\| \leq L\|x - \tilde{x}\|.$$

(P3) There exists a continuous function $V \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and positive constants C_1, C_2 and β such that

$$C_1\|x\|^\beta \leq V(t,x) \leq C_2\|x\|^\beta \quad \text{for any } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n.$$

(P4) There exists $\eta > 0$ such that any solution $t \mapsto \phi(t)$ either of (1.1) or (1.2) verifies

$$DV_-(t,\phi(t)) := \liminf_{h \to 0^+} \frac{V(t,\phi(t)) - V(t-h,\phi(t-h))}{h} \leq -\eta\|\phi(t)\|^\beta.$$

A consequence of **(P3)** and **(P4)** is that V is a Lyapunov function for the systems (1.1) and (1.2). Then, classical results of Lyapunov's stability [13, Theorem 4.9], [20, Chapter 1] imply that the origin is a globally uniformly asymptotically stable equilibrium of (1.1) and (1.2), which also implies the existence and uniqueness of the *crossing times* $T := T(\tau, x_0)$ and $S := S(\tau, y_0)$, namely, the unique times such that

$$V(T, x(T,\tau,x_0)) = V(S, y(S,\tau,y_0)) = 1. \tag{2.3}$$

We now state the following result obtained by K. J. Palmer in [18, Lemma]:

Proposition 2.2. *If the systems* (1.1) *and* (1.2) *verify* **(P1)–(P4)**, *then,* (1.1) *and* (1.2) *are* $\mathbb{R}$*-topologically equivalent with H and G defined by:*

$$H(\tau,x_0)=\begin{cases} y(\tau,T(\tau,x_0),x(T(\tau,x_0),\tau,x_0)) & x_0\neq 0,\\ 0 & x_0=0,\end{cases} \tag{2.4}$$

and

$$G(\tau,y_0)=\begin{cases} x(\tau,S(\tau,y_0),y(S(\tau,y_0),\tau,y_0)) & y_0\neq 0,\\ 0 & y_0=0.\end{cases} \tag{2.5}$$

From now on, for each $\tau\in\mathbb{R}$, the maps H_τ and G_τ will be called as the **Palmer's homeomorphism**.

A strong assumption of Palmer's result is that (1.1) and (1.2) must have the same Lyapunov's function; a particular example of this result is studied by F. Lin [14], which considers the linear diagonal system

$$\dot{x}=-\frac{\delta}{2}x, \tag{2.6}$$

and also the quasilinear system

$$\dot{y}=C(t)y+B(t)y+g(t,y), \tag{2.7}$$

such that $x,y\in\mathbb{R}^n$, $\delta>0$ while the functions $C\colon\mathbb{R}\to M_n(\mathbb{R})$, $B\colon\mathbb{R}\to M_n(\mathbb{R})$ and $g\colon\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous and also verify:

- **(L1)** The function $t\mapsto C(t)=\{c_{ij}(t)\}_{i,j=1}^{n}$ is bounded in $\mathbb{R}$ and $C(t)$ is a diagonal matrix with $c_{ii}(t)\leq-\delta$ for any $t\in\mathbb{R}$,
- **(L2)** For any $t\in\mathbb{R}$, it follows that $\|B(t)\|\leq\frac{\delta}{4}$,
- **(L3)** For any $t\in\mathbb{R}$ and any couple $y,\tilde{y}\in\mathbb{R}^n$, it is satisfied that

$$\|g(t,y)-g(t,\tilde{y})\|\leq\frac{\delta}{4}\|y-\tilde{y}\| \quad\text{and}\quad g(t,0)=0.$$

A careful reading of [14, Proposition 7, p. 41] allows us to deduce that a consequence of **(L1)–(L3)** is that the systems (2.6) and (2.7) have the same Lyapunov function and, consequently, the origin is a uniformly asymptotically stable equilibrium, emulating the properties **(P3)** and **(P4)** considered by Palmer.

The following result is obtained by F. Lin in [14, Lemma 1]:

Proposition 2.3. *If the systems* (2.6)–(2.7) *verify* **(L1)–(L3)**, *then,* (2.6)–(2.7) *are* $\mathbb{R}$*-topologically equivalent with H and G defined by:*

$$H(\tau,x_0)=\begin{cases} y\left(\tau,T(\tau,x_0),e^{-\frac{\delta}{2}(T(\tau,x_0)-\tau)}x_0\right) & x_0\neq 0,\\ 0 & x_0=0,\end{cases} \tag{2.8}$$

and

$$G(\tau,y_0)=\begin{cases} y(S(\tau,y_0),\tau,y_0)e^{-\frac{\delta}{2}(\tau-S(\tau,y_0))} & y_0\neq 0,\\ 0 & y_0=0,\end{cases} \tag{2.9}$$

where $T:=T(\tau,x_0)$ *and* $S:=S(\tau,y_0)$ *are the unique times such that the euclidean norm of its solutions verify*

$$\|x(T(\tau,x_0),\tau,x_0)\|^2=\|y(S(\tau,y_0),\tau,y_0)\|^2=1. \tag{2.10}$$

From now on, for each $\tau \in \mathbb{R}$, the maps H_τ and G_τ will be called as the **Lin's homeomorphism**.

Remark 2.4. If $x_0 \neq 0$ and $y_0 \neq 0$, the identity $x(t,t_0,x_0) = e^{-\frac{\delta}{2}(t-t_0)}x_0$ implies that $H(\tau,x_0)$ and $G(\tau,y_0)$ have the alternative characterizations:

$$H(\tau,x_0) = y(\tau,T(\tau,x_0),x(T(\tau,x_0),\tau,x_0)),$$

and

$$G(\tau,y_0) = x(\tau,S(\tau,y_0),y(S(\tau,y_0),\tau,y_0)),$$

which coincide with (2.4)–(2.5) and also implies the identities

$$y(T(\tau,x_0),\tau,H(\tau,x_0)) = x(T(\tau,x_0),\tau,x_0),$$

and

$$x(S(\tau,y_0),\tau,G(\tau,y_0)) = y(S(\tau,y_0),\tau,y_0).$$

It is important to emphasize that the literature devoted to the crossing time based homeomorphisms is considerably less developed in comparison with the Green's function approach. In fact, while the topological linearization via the Green's function has become an interesting topic in itself, the linearization via crossing time has been used as a technical step inside more general results. For example, in [18] the crossing time is used to relate topological equivalence with exponential dichotomy; furthermore, in [14] is a tool employed to obtain a topological equivalence result for a more general family of systems that can be reduced to (2.6)–(2.7).

2.3 The smoothness of the topological equivalence and the main novelty of this work

While the topological equivalence problem goes back to the 70's, the study of the differentiability properties of the homeomorphisms H_t and G_t of Definition 1.1 started in the 2010's decade and, obviously, is considerably less studied.

The first results on the smoothness of the maps H_t and G_t were based on the Green's function approach and were obtained for the contractive case in [4–7], while the contractive/expansive case is treated later in [11] under strong assumptions on the quasilinear perturbation.

It is important to stress that less restrictive smoothness results have recently been obtained in the contractive/expansive case by Cuong *et al.* and Dragičević *et al.* both cases are inspired by the ideas developed by Sternberg's, and considering resonance conditions described in terms of the spectra associated to the uniform exponential dichotomy [8] and the nonuniform exponential dichotomy [9, 10]. In this context, we also highlight the noticeable contributions of Backes & Dragičević in [1], Barreira & Valls in [3] and Lu *et al.* in [15].

Surprisingly, and to the best of our knowledge; there are no studies about the smoothness properties of homeomorphisms H_t and G_t when considering the crossing time approach and this work can be seen as a contribution on this subject.

3 Smoothness of Lin's homeomorphism

Throughout this section, we will assume that the conditions **(L1)–(L3)** of the Proposition 2.3 are verified and, in consequence, the systems (2.6) and (2.7) are $\mathbb{R}$-topologically equivalent with maps H and G described respectively by (2.8) and (2.9). Moreover, as a convenient shorthand, we will refer to $\mathbb{R}_0^n$ rather than $\mathbb{R}^n \setminus \{0\}$ $(n \geq 1)$ in all that follows.

Firstly, we will study the smoothness properties of the crossing time function S stated in (2.9). In order to do that, it will be useful to introduce the map $\mathcal{F} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\mathcal{F}(s,y) = C(s)y + B(s)y + g(s,y), \tag{3.1}$$

and, if $y \mapsto \mathcal{F}(s,y)$ is derivable on $\mathbb{R}^n$, its jacobian matrix for any fixed $s \in \mathbb{R}$ will be denoted by $D_y\mathcal{F}(s,y)$.

Lemma 3.1. *If the system (2.7) satisfies* **(L1)–(L3)** *and the maps $(t,y) \mapsto g(t,y)$, $(t,y) \mapsto C(t)y$ and $(t,y) \mapsto B(t)y$ belong to $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ then the crossing time $S : \mathbb{R} \times \mathbb{R}_0^n \to \mathbb{R}$ is continuously differentiable on its domain of definition.*

Moreover, for any fixed $\tau \in \mathbb{R}$, the partial derivative of S with respect to ξ is given explicitly by

$$D_\xi S(\tau,\xi) = -\frac{D_\xi y(S(\tau,\xi),\tau,\xi) y(S(\tau,\xi),\tau,\xi)}{\mathcal{F}(\tau, y(S(\tau,\xi),\tau,\xi)) \cdot y(S(\tau,\xi),\tau,\xi)}. \tag{3.2}$$

Proof. As a first step, let us define the auxiliary map $\psi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by:

$$\psi(\tau,\xi,t) = \|y(t,\tau,\xi)\|^2 - 1.$$

The above assumptions imply that $\mathcal{F} \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Therefore, the differentiability of the solutions of (2.7) with respect to the initial conditions [22, Theorem 6.1, p. 89] states that $(t,\tau,\xi) \mapsto y(t,\tau,\xi) \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $D_\xi y(t,\tau,\xi)$ is solution of the linear variational equation

$$Y' = D_y\mathcal{F}(t, y(t,\tau,\xi))Y \quad \text{with} \quad Y(\tau) = I,$$

which leads to

$$D_\xi \psi(\tau,\xi,t) = 2D_\xi y(t,\tau,\xi)\, y(t,\tau,\xi). \tag{3.3}$$

Moreover, by (2.7) it is straightforward to verify that

$$D_t \psi(\tau,\xi,t) = 2\mathcal{F}(t, y(t,\tau,\xi)) \cdot y(t,\tau,\xi), \tag{3.4}$$

where $\mathcal{F}$ is defined in (3.1). By gathering the above derivatives and recalling the assumptions, we have that $\psi \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$.

As a second step, let us consider the Banach spaces $X = (\mathbb{R} \times \mathbb{R}^n, |\cdot|_X)$ and $Y = Z = (\mathbb{R}, |\cdot|)$, where $|(t,x)|_X = \|x\| + |t|$.

Given the open set $\mathcal{O} = \mathbb{R} \times \mathbb{R}_0^n \times \mathbb{R} \subseteq X \times Y$, we define F as the restriction of ψ into the set $\mathcal{O}$, namely, $F : \mathcal{O} \to \mathbb{R}$ is defined by

$$F(\tau,\xi,t) = \psi(\tau,\xi,t) = \|y(t,\tau,\xi)\|^2 - 1,$$

which belongs to $\mathscr{C}^1(\mathcal{O}, \mathbb{R})$. Moreover, by (2.10), it follows that

$$F(\tau_0, \xi_0, S(\tau_0,\xi_0)) = 0 \quad \text{for any } \xi_0 \neq 0 \text{ and } \tau_0 \in \mathbb{R}. \tag{3.5}$$

The next step applies a Lin's estimation, namely, (2.10) and the proof of [14, Proposition 7, p. 41–42], which allow us to deduce that

$$D_tF(\tau_0, \xi_0, S(\tau_0,\xi_0)) \leq -\delta \underbrace{\|y(S(\tau_0,\xi_0),\tau_0,\xi_0)\|^2}_{=1} = -\delta < 0. \tag{3.6}$$

By using (3.5) combined with the implicit function theorem [22, Theorem 5.7, p. 82] applied to F, we can prove the existence of $\varphi \in \mathscr{C}^1(U,W)$, where U is a neighborhood of (τ_0,ξ_0) while W is one of $S(\tau_0,\xi_0)$, such that $\varphi(\tau_0,\xi_0) = S(\tau_0,\xi_0)$ with $U \times W \subseteq \mathcal{O}$ and

$$F(\tau,\xi,\varphi(\tau,\xi)) = 0 \quad \text{for any } (\tau,\xi) \in U,$$

which is equivalent to

$$\|y(\varphi(\tau,\xi),\tau,\xi)\|^2 = 1 = \|y(S(\tau,\xi),\tau,\xi)\|^2 \quad \text{for any} \quad (\tau,\xi) \in U,$$

then, the uniqueness of S implies $S(\tau,\xi) = \varphi(\tau,\xi)$ on U and $S \in \mathscr{C}^1(U,W)$.

In particular, we have that S is continuously differentiable on each $(\tau_0,\xi_0) \in \mathbb{R} \times \mathbb{R}^n_0$ and it follows that $S \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n_0, \mathbb{R})$. In addition, the partial derivative can be explicitly computed as

$$D_\xi S(\tau,\xi) = -\left[D_t F(\tau,\xi,S(\tau,\xi))\right]^{-1} D_\xi F(\tau,\xi,S(\tau,\xi)).$$

As a final step, the identity $F = \psi$ on $\mathcal{O}$ combined with (3.1) imply that the above partial derivatives coincides with those described by (3.3)–(3.4), then the above identity becomes (3.2) and the result follows. □

Corollary 3.2. *The crossing time T corresponding to the solutions of* (2.6) *verifies $T \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n_0, \mathbb{R})$ and the partial derivative of T with respect to ξ satisfies*

$$D_\xi T(\tau,\xi) = \frac{2}{\delta}\frac{\xi}{\|\xi\|^2} \tag{3.7}$$

for any $\xi \neq 0$ and any fixed $\tau \in \mathbb{R}$.

Proof. Let us define $C_0, B_0 : \mathbb{R} \to M_n(\mathbb{R})$ and $g_0 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$C_0(t) = -\delta I, \quad B_0(t) = \frac{\delta}{4} I \quad \text{and} \quad g_0(t,x) = \frac{\delta}{4} x.$$

For any $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, we have

$$C_0(t)x + B_0(t)x + g_0(t,x) = -\frac{\delta}{2} x,$$

and the matrix functions C_0, B_0 verify **(L1)** and **(L2)**, while g_0 verifies **(L3)** of the Proposition 2.3. Moreover, we have that $(t,x) \mapsto C_0(t)x, B_0(t)x$ and $(t,x) \mapsto g_0(t,x)$ are maps of class $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. Then, Lemma 3.1 implies that the crossing time T is a function of class $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n_0, \mathbb{R})$.

In order to verify (3.7), let us remember that

$$x(t,\tau,\xi) = e^{-\frac{\delta}{2}(t-\tau)}\xi,$$

and we can obtain an explicit description for the crossing time:

$$\|x(T(\tau,\xi),\tau,\xi)\|^2 = 1 \iff T(\tau,\xi) = \frac{2}{\delta}\ln(\|\xi\|) + \tau \tag{3.8}$$

and (3.7) follows by calculating the derivative of $(\tau,\xi) \mapsto \frac{2}{\delta}\ln(\|\xi\|) + \tau$ with respect to ξ. □

The results of continuous differentiability for the crossing time functions will be useful to achieve the following result:

Theorem 3.3. *If the systems* (2.6)–(2.7) *satisfy* **(L1)–(L3)** *and the maps* $(t,y) \mapsto g(t,y)$, $(t,y) \mapsto C(t)y$ *and* $(t,y) \mapsto B(t)y$ *belong to* $\mathscr{C}^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$*, the Lin's homeomorphism* H_τ *is a diffeomorphism of class* $\mathscr{C}^1(\mathbb{R}^n_0,\mathbb{R}^n_0)$ *for any fixed* $\tau\in\mathbb{R}$.

Moreover, the derivative of $H_\tau := H(\tau,\cdot)$ *with respect to* ξ *is given by*

$$D_\xi H(\tau,\xi) = \frac{2}{\delta}\left\{D_T\, y_i\left(\tau, T(\tau,\xi), \frac{\xi}{\|\xi\|}\right)\frac{\xi_j}{\|\xi\|^2}\right\}_{i,j=1}^{n} + D_\xi y\left(\tau, T(\tau,\xi), \frac{\xi}{\|\xi\|}\right) D_\xi\left[\frac{\xi}{\|\xi\|}\right], \tag{3.9}$$

while the derivative of $G(\tau,\cdot) := H_\tau^{-1}$ *with respect to* ξ *is*

$$D_\xi G(\tau,\xi) = e^{\frac{\delta}{2}(S(\tau,\xi)-\tau)}\left[\frac{\delta}{2}\mathcal{G}(\tau,\xi) + \mathcal{V}(\tau,\xi) + D_\xi y(S(\tau,\xi),\tau,\xi)\right] \tag{3.10}$$

where $\mathcal{G}$ *and* $\mathcal{V}$ *are* n*-th order matrices with*

$$\mathcal{G}_{i,j}(\tau,\xi) := \frac{\partial S(\tau,\xi)}{\partial \xi_j} y_i(S(\tau,\xi),\tau,\xi) \quad \text{and} \quad \mathcal{V}_{i,j}(\tau,\xi) := \mathcal{F}_i(S, y(S,\tau,\xi))\frac{\partial S(\tau,\xi)}{\partial \xi_j}$$

for any $i,j\in\{1,\dots,n\}$ *and* $\mathcal{F}_i$ *is the* i*-th coordinate of* $\mathcal{F}$ *defined in* (3.1).

Proof. By Lemma 3.1 and Corollary 3.2, the crossing time functions S and T are of class $\mathscr{C}^1(\mathbb{R}\times\mathbb{R}^n_0,\mathbb{R})$, whose proofs used that the map $(t,\tau,\xi)\mapsto y(t,\tau,\xi)$ is of class $\mathscr{C}^1(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$.

The above facts combined with (2.8) imply that, for any fixed $\tau\in\mathbb{R}$, the map $0\neq\xi\mapsto H_\tau(\xi) = y(\tau, T(\tau,\xi), e^{-\frac{\delta}{2}(T(\tau,\xi)-\tau)}\xi)$ can be seen as a composition of functions of class $\mathscr{C}^1$, which leads to $H_\tau\in\mathscr{C}^1(\mathbb{R}^n_0,\mathbb{R}^n_0)$.

Similarly, by using (2.9), the map $0\neq\xi\mapsto G_\tau(\xi) = y(\tau,S(\tau,\xi),\xi)e^{-\frac{\delta}{2}(\tau-S(\tau,\xi))}$ is a composition and product of functions of class $\mathscr{C}^1$. In consequence, we have $G_\tau\in\mathscr{C}^1(\mathbb{R}^n_0,\mathbb{R}^n_0)$ and H_τ is a diffeomorphism of class $\mathscr{C}^1(\mathbb{R}^n_0,\mathbb{R}^n_0)$.

In order to verify (3.9), the explicit characterization of the crossing T given by (3.8) allows to obtain a simpler expression

$$H(\tau,\xi) = y\left(\tau, T(\tau,\xi), e^{-\frac{\delta}{2}(T(\tau,\xi)-\tau)}\xi\right) = y\left(\tau, T(\tau,\xi), \frac{\xi}{\|\xi\|}\right).$$

Let $H_i(\tau,\xi)$ be the i-th coordinate of the map $H(\tau,\cdot)$. By using $\frac{\partial\tau}{\partial\xi_j}=0$ combined with the chain rule and (3.7), we can deduce that the partial derivatives are

$$\begin{aligned}
\frac{\partial H_i(\tau,\xi)}{\partial \xi_j} &= \frac{\partial}{\partial \xi_j}\left\{y_i\left(\tau, T(\tau,\xi), \frac{\xi}{\|\xi\|}\right)\right\} \\
&= \frac{\partial y_i\left(\tau, T(\tau,\xi), \frac{\xi}{\|\xi\|}\right)}{\partial \tau}\underbrace{\frac{\partial \tau}{\partial \xi_j}}_{=0} + \frac{\partial y_i\left(\tau, T(\tau,\xi), \frac{\xi}{\|\xi\|}\right)}{\partial T(\tau,\xi)}\frac{\partial T(\tau,\xi)}{\partial \xi_j} \\
&\quad + \sum_{k=1}^{n}\frac{\partial y_i\left(\tau, T(\tau,\xi), \frac{\xi}{\|\xi\|}\right)}{\partial \xi_k}\frac{\partial}{\partial \xi_j}\left\{\frac{\xi_k}{\|\xi\|}\right\} \\
&= \frac{2}{\delta}\frac{\partial y_i\left(\tau, T(\tau,\xi), \frac{\xi}{\|\xi\|}\right)}{\partial T}\frac{\xi_j}{\|\xi\|^2} + \sum_{k=1}^{n}\frac{\partial y_i\left(\tau, T(\tau,\xi), \frac{\xi}{\|\xi\|}\right)}{\partial \xi_k}\frac{\partial}{\partial \xi_j}\left\{\frac{\xi_k}{\|\xi\|}\right\},
\end{aligned}$$

which corresponds to the (i,j)-coefficient of (3.9).

In order to verify the identity (3.10), if $G_i(\tau,\xi)$ is the i-th coordinate of $G(\tau,\cdot)$, notice that

$$\begin{aligned}
\frac{\partial G_i(\tau,\xi)}{\partial \xi_j} &= \frac{\partial}{\partial \xi_j}\left[y_i(S(\tau,\xi),\tau,\xi)e^{-\frac{\delta}{2}(\tau-S(\tau,\xi))}\right] \\
&= \frac{\delta}{2}e^{\frac{\delta}{2}(S(\tau,\xi)-\tau)}\frac{\partial S(\tau,\xi)}{\partial \xi_j}y_i(S(\tau,\xi),\tau,\xi) \\
&\quad + e^{\frac{\delta}{2}(S(\tau,\xi)-\tau)}\left(\underbrace{\frac{\partial y_i(S(\tau,\xi),\tau,\xi)}{\partial S}}_{=\mathcal{F}_i(S,y(S,\tau,\xi))}\frac{\partial S(\tau,\xi)}{\partial \xi_j} + \frac{\partial y_i(S(\tau,\xi),\tau,\xi)}{\partial \xi_j}\right) \\
&= e^{\frac{\delta}{2}(S(\tau,\xi)-\tau)}\left\{\frac{\delta}{2}\mathcal{G}_{i,j}(\tau,\xi) + \mathcal{V}_{i,j}(\tau,\xi) + \frac{\partial y_i(S(\tau,\xi),\tau,\xi)}{\partial \xi_j}\right\}
\end{aligned}$$

and similarly, we can verify that it corresponds to the (i,j)-coefficient of (3.10), and the Theorem follows. □

Corollary 3.4. *Under the assumptions of Theorem 3.3, the Lin's homeomorphism $G_\tau : \mathbb{R}^n_0 \to \mathbb{R}^n_0$ is a preserving orientation diffeomorphism for $n \geq 2$.*

Proof. Let $\tau \in \mathbb{R}$ fixed. Firstly, as H_τ is a bijective map with inverse G_τ, it follows that

$$H_\tau(G_\tau(\xi)) = \xi \quad \text{for any } \xi \in \mathbb{R}^n_0.$$

Then, by the Theorem 3.3, we have that H_τ is a diffeomorphism of $\mathbb{R}^n_0$ on itself and

$$D_G H_\tau(G_\tau(\xi))\, D_\xi G_\tau(\xi) = I \implies \det[D_G H_\tau(G_\tau(\xi))]\, \det[D_\xi G_\tau(\xi)] = 1,$$

where $\det[D_\xi G_\tau(\xi)] \neq 0$ for any $\xi \in \mathbb{R}^n_0$.

Therefore, we only have to verify that $\det[D_\xi G_\tau(\xi)] > 0$ for any $\xi \in \mathbb{R}^n_0$. In order to prove this property, we construct the function $\Gamma : \mathbb{R}^n_0 \to \mathbb{R}\setminus\{0\}$ defined by $\Gamma(\xi) = \det[D_\xi G_\tau(\xi)]$. Note that Γ can be seen as a composition of continuous maps described by:

$$\begin{aligned}
\mathbb{R}^n_0 &\xrightarrow{D_\xi G_\tau} GL_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}\setminus\{0\} \\
\xi &\longmapsto D_\xi G_\tau(\xi) \longmapsto \det[D_\xi G_\tau(\xi)] = \Gamma(\xi).
\end{aligned}$$

By the continuity of Γ on the connected set $\mathbb{R}^n_0$ for $n \geq 2$, we have that $\Gamma(\mathbb{R}^n_0)$ is connected, then we have that

$$\text{either} \quad \Gamma(\mathbb{R}^n_0) \subseteq \,]-\infty,0[\quad \text{or} \quad \Gamma(\mathbb{R}^n_0) \subseteq \,]0,+\infty[\,. \tag{3.11}$$

Hence, in order to prove that $\det D_\xi G(\tau,\xi) > 0$ for any $\xi \in \mathbb{R}^n_0$, we have to show that $\Gamma(\mathbb{R}^n_0) \subseteq \,]0,+\infty[$. By the above paragraph, we only need to show that $\det D_\xi G_\tau(\xi) > 0$ for some specific $\xi \in \mathbb{R}^n_0$. Indeed, we will verify this property for $\xi = (1,0,\ldots,0) \in \mathbb{R}^n_0$.

Now, as $\|\xi\| = 1$, we have that

$$\|y(\tau,\tau,\xi)\|^2 = \|\xi\|^2 = 1 = \|y(S(\tau,\xi),\tau,\xi)\|^2,$$

which implies that $S(\tau,\xi) = \tau$ by the uniqueness of S. In addition, by [22, Theorem 6.1, p. 189], we have that $t \mapsto D_\xi y(t,\tau,\xi)$ is solution of the linear variational equation

$$\frac{dY}{dt} = D_y\mathcal{F}(t,y(t,\tau,\xi))Y \quad \text{with} \quad Y(\tau) = I,$$

and (3.2) combined with $S(\tau,\xi)=\tau$ imply that

$$D_\xi\, S(\tau,\xi) = -\frac{D_\xi\, y(S(\tau,\xi),\tau,\xi)\, y(S(\tau,\xi),\tau,\xi)}{\mathcal{F}(\tau,y(S(\tau,\xi),\tau,\xi))\cdot y(S(\tau,\xi),\tau,\xi)} = -\frac{\xi}{\mathcal{F}(\tau,\xi)\cdot\xi}$$

then we have

$$\frac{\partial S(\tau,\xi)}{\partial \xi_j} = -\frac{\xi_j}{\mathcal{F}(\tau,\xi)\cdot\xi} \quad \text{for any } 1\leq j\leq n. \tag{3.12}$$

On the other hand, we have that $\frac{\partial y_i(\tau,\tau,\xi)}{\partial \xi_j}=\delta_{ij}$ where δ_{ij} is the Kronecker delta. Then, (3.10), (3.12) and $S(\tau,\xi)=\tau$ imply that the i,j-coordinate of $D_\xi G_\tau(\xi)$ is

$$\begin{aligned}
\frac{\partial G_i(\tau,\xi)}{\partial \xi_j} &= \frac{\delta}{2}e^{\frac{\delta}{2}(S(\tau,\xi)-\tau)}\frac{\partial S(\tau,\xi)}{\partial \xi_j}y_i(S(\tau,\xi),\tau,\xi) \\
&\quad + e^{\frac{\delta}{2}(S(\tau,\xi)-\tau)}\left\{\mathcal{F}_i(S(\tau,\xi),y(S(\tau,\xi),\tau,\xi))\frac{\partial S(\tau,\xi)}{\partial \xi_j}+\frac{\partial y_i(S(\tau,\xi),\tau,\xi)}{\partial \xi_j}\right\} \\
&= \frac{\partial S(\tau,\xi)}{\partial \xi_j}\left\{\frac{\delta}{2}y_i(S(\tau,\xi),\tau,\xi)+\mathcal{F}_i(S(\tau,\xi),y(S(\tau,\xi),\tau,\xi))\right\}+\frac{\partial y_i(S(\tau,\xi),\tau,\xi)}{\partial \xi_j} \\
&= -\frac{\xi_j}{\mathcal{F}(\tau,\xi)\cdot\xi}\left\{\frac{\delta}{2}\xi_i+\mathcal{F}_i(\tau,\xi)\right\}+\delta_{ij}.
\end{aligned}$$

By considering $\xi=(1,0,0\ldots,0)=(\xi_1,\xi_2,\ldots,\xi_n)$, we can deduce that

$$\frac{\partial G_i(\tau,\xi)}{\partial \xi_j} = \begin{cases} -\dfrac{\mathcal{F}_i(\tau,\xi)}{\mathcal{F}_1(\tau,\xi)} & j=1\neq i, \\ 0 & j\neq i, j\neq 1, \\ -\dfrac{\delta}{2\mathcal{F}_1(\tau,\xi)} & i=j=1 \\ 1 & i=j, i\neq 1, \end{cases}$$

and the derivative $D_\xi G_\tau(\xi)$ is described by the block matrix

$$D_\xi G_\tau(\xi) = \left[\begin{array}{c|c} -\frac{\delta}{2\mathcal{F}_1(\tau,\xi)} & 0_{1\times n} \\ \hline \mathcal{D} & I_{n-1}\end{array}\right] \quad \text{where} \quad \mathcal{D} = \begin{bmatrix} -\frac{\mathcal{F}_2(\tau,\xi)}{\mathcal{F}_1(\tau,\xi)} \\ -\frac{\mathcal{F}_3(\tau,\xi)}{\mathcal{F}_1(\tau,\xi)} \\ \vdots \\ -\frac{\mathcal{F}_n(\tau,\xi)}{\mathcal{F}_1(\tau,\xi)} \end{bmatrix},$$

and $I_{n-1}\in M_{n-1}(\mathbb{R})$ is the identity matrix. That is, $D_\xi G_\tau(\xi)$ is a lower triangular matrix where its diagonal terms are

$$\frac{\partial G_i(\tau,\xi)}{\partial \xi_i} = \begin{cases} -\dfrac{\delta}{2\mathcal{F}_1(\tau,\xi)} & i=1, \\ 1 & i\neq 1, \end{cases}$$

and we can explicitely see that $\det[D_\xi G_\tau(\xi)] = -\frac{\delta}{2\mathcal{F}_1(\tau,\xi)}$.

Let us recall that, in the proof of the Lemma 3.1, we constructed the function $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(\tau,\xi,t) = \|y(t,\tau,\xi)\|^2 - 1 \quad \text{for any } (\tau,\xi,t) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R},$$

which satisfies (3.6). This property combined with $S(\tau,\xi) = \tau$ lead to

$$\begin{aligned}\mathcal{F}_1(\tau,\xi) &= \sum_{i=1}^{n} \mathcal{F}_i(\tau,\xi)\xi_i = \mathcal{F}(\tau,\xi)\cdot\xi \\ &= \mathcal{F}(S(\tau,\xi), y(S(\tau,\xi),\tau,\xi)) \cdot y(S(\tau,\xi),\tau,\xi) \\ &= \frac{1}{2} D_t F(\tau,\xi,S(\tau,\xi)) \leq -\frac{\delta}{2} < 0,\end{aligned}$$

then we have that $\mathcal{F}_1(\tau,\xi) < 0$ and consequently $\det[D_\xi G_\tau(\xi)] = -\frac{\delta}{2\mathcal{F}_1(\tau,\xi)} > 0$.

We have verified the existence of $\xi \in \mathbb{R}_0^n$ such that $\det[D_\xi G_\tau(\xi)] = \Gamma(\xi) > 0$. The connectedness of $\Gamma(\mathbb{R}_0^n)$ and (3.11) imply that $\Gamma(\mathbb{R}_0^n) \subseteq]0,+\infty[$, or equivalently, $\det[D_\xi G_\tau(\tilde{\xi})] > 0$ for every $\tilde{\xi} \in \mathbb{R}_0^n$ and we have proved that G_τ is a preserving orientation diffeomorphism of $\mathbb{R}_0^n$ on itself for any $n \geq 2$. □

The connectedness of $\mathbb{R}_0^n$ with $n \geq 2$ played a key role in the above proof. Nevertheless, we will make minor adaptations to cope with the case $n = 1$.

Corollary 3.5. *Under the assumptions of Theorem 3.3, the Lin's homeomorphism $G_\tau : \mathbb{R}_0^n \to \mathbb{R}_0^n$ is also a preserving orientation diffeomorphism for $n = 1$.*

Proof. Let be $\Gamma : \mathbb{R}_0 \to \mathbb{R}_0$ as in the previous proof, where $\mathbb{R}_0$ is the disconnected set

$$\mathbb{R}_0 := \mathbb{R} \setminus \{0\} =]-\infty,0[\cup]0,+\infty[.$$

However, we will emulate the proof of the Corollary (3.4) in the connected components of $\mathbb{R}_0$, namely, $\mathcal{C}^+ :=]0,+\infty[$ and $\mathcal{C}^- :=]-\infty,0[$.

Let us consider $\xi^+ := 1 \in \mathcal{C}^+$ and $\xi^- := -1 \in \mathcal{C}^-$. Then we have $|\xi^-| = |\xi^+| = 1$, which leads to

$$S(\tau,\xi^-) = \tau = S(\tau,\xi^+).$$

By replying the proof of the Corollary 3.4, we have that

$$\frac{\partial S(\tau,\xi^+)}{\partial \xi} = \frac{-1}{\mathcal{F}(\tau,1)} \quad \text{and} \quad \det[D_\xi G_\tau(\xi^+)] = D_\xi G_\tau(\xi^+) = -\frac{\delta}{2\mathcal{F}(\tau,1)},$$

and we verify, similarly as in the previous result, that $\det D_\xi[G_\tau(\xi^+)] = \Gamma(\xi^+) > 0$, which implies $\Gamma(\mathcal{C}^+) \in]0,+\infty[$ and it follows that $\det[D_\xi G_\tau(\xi)] = \Gamma(\xi) > 0$ for any $\xi \in \mathcal{C}^+$.

By proceeding analogously, we also can verify that

$$\frac{\partial S(\tau,\xi^-)}{\partial \xi} = \frac{-1}{\mathcal{F}(\tau,-1)} \quad \text{and} \quad D_\xi G_\tau(\xi^-) = \frac{\delta}{2\mathcal{F}(\tau,-1)}.$$

Now, let us recall that $\mathcal{F}(t,\xi) = C(t)\xi + B(t)\xi + g(t,\xi)$ where $C, B : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the properties **(L1)–(L3)** stated in Section 2 with $n = 1$, then, we have

$$C(\tau)\xi^- = -C(\tau) \geq \delta, \quad B(\tau)\xi^- = -B(\tau) \geq -\frac{\delta}{4}, \quad \text{and} \quad g(\tau,\xi^-) \geq -\frac{\delta}{4},$$

which implies that $\mathcal{F}(\tau,\xi^-) = -C(\tau) - B(\tau) + g(\tau,\xi^-) \geq \frac{\delta}{4} > 0$ and we have that

$$\Gamma(\xi^-) = \det[D_\xi G_\tau(\xi^-)] = D_\xi G_\tau(\xi^-) = \frac{\delta}{2\mathcal{F}(\tau,\xi^-)} > 0.$$

By the connectedness of $\Gamma(\mathcal{C}^-)$, we have that $\Gamma(\mathcal{C}^-) \subseteq]0,+\infty[$ and $\det[D_\xi G_\tau(\xi)] = \Gamma(\xi) > 0$ for any $\xi \in \mathcal{C}^-$ and the result holds. □

4 Smoothness of Palmer's homeomorphism

This section studies the differentiability properties of the homeomorphisms H_t and G_t defined by (2.4) and (2.5) in the Proposition 2.2.

Lemma 4.1. *If the systems* (1.1) *and* (1.2) *verify* **(P1)–(P4)** *while the functions* $F_1, F_2 \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *are of class* $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ *and the Lyapunov function* V *is of class* $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$*, the crossing times* T *and* S *of* (2.3) *are of class* $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$.

Moreover, the derivative of $T := T(\tau,\cdot)$ *with respect to* $\xi \in \mathbb{R}_0^n$ *verifies*

$$D_\xi T(\tau,\xi) = -\frac{D_\xi x(T,\tau,\xi)\, D_\xi V(T,x(T,\tau,\xi))}{D_T V(T,x(T,\tau,\xi)) + D_\xi V(T,x(T,\tau,\xi)) \cdot F_1(T,x(T,\tau,\xi))}. \tag{4.1}$$

while, the derivative of $S := S(\tau,\cdot)$ *with respect to* $\xi \in \mathbb{R}_0^n$ *is given by*

$$D_\xi S(\tau,\xi) = -\frac{D_\xi y(S,\tau,\xi)\, D_\xi V(S,y(S,\tau,\xi))}{D_S V(S,y(S,\tau,\xi)) + D_\xi V(S,y(S,\tau,\xi)) \cdot F_2(S,y(S,\tau,\xi))}. \tag{4.2}$$

Proof. We will work with the same Banach spaces X, Y and Z and the same open set $\mathcal{O} \subseteq X \times Y$ of the proof of the Lemma 3.1.

Moreover, we will construct a $\mathscr{C}^1(\mathcal{O}, Z)$-map ϕ verifying $\phi(\tau,\xi,T(\tau,\xi)) = 0$ in order to apply the implicit function theorem for Banach spaces [22, Theorem 5.7, p. 82].

Firstly, let us define the auxiliary map $\nu : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as follows:

$$\nu(\tau,\xi,t) = V(t,x(t,\tau,\xi)) - 1.$$

As $F_1 \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, the differentiability of the solutions of (1.1) with respect to the initial conditions states that $(t,\tau,\xi) \mapsto x(t,\tau,\xi) \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $t \mapsto D_\xi\, x(t,\tau,\xi)$ is solution of the linear variational equation

$$X' = D_x F_1(t,x(t,\tau,\xi))X \quad \text{with} \quad X(\tau) = I.$$

By hypothesis, $(t,x) \mapsto V(t,x)$ is of class $\mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, which leads to

$$D_\xi\, \nu(\tau,\xi,t) \quad = \quad D_x V(t,x(t,\tau,\xi)) D_\xi\, x(t,\tau,\xi). \tag{4.3}$$

Moreover, by (1.1) it is straightforward to verify that

$$D_t\, \nu(\tau,\xi,t) = D_t V(t,x(t,\tau,\xi)) + D_x V(t,x(t,\tau,\xi)) F_1(t,x(t,\tau,\xi)), \tag{4.4}$$

and, by recalling the above assumptions, we can see that $\nu \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$.

Now, let us construct the restriction on ν into $\mathcal{O}$, namely, $\phi : \mathcal{O} \to \mathbb{R}$ defined by

$$\phi(\tau,\xi,t) = \nu(\tau,\xi,t) = V(t,x(t,\tau,\xi)) - 1,$$

which clearly belongs to $\mathscr{C}^1(\mathcal{O},\mathbb{R})$. In addition, (2.3) implies that

$$\phi(\tau_0,\xi_0,T(\tau_0,\xi_0)) = 0 \quad \text{for any } \xi_0 \neq 0 \text{ and } \tau_0 \in \mathbb{R}. \tag{4.5}$$

By the assumptions **(P3)–(P4)** of the Proposition 2.2 we can deduce that

$$\begin{aligned} D_t\phi(\tau,\xi,t) = D_t\nu(\tau,\xi,t) &= D_t\{V(t,x(t,\tau,\xi))-1\} \\ &= D_tV(t,x(t,\tau,\xi)) \leq -\eta\|x(t,\tau,\xi)\|^{\beta}, \end{aligned}$$

for all $(\tau,\xi,t)\in\mathcal{O}$. Moreover, by **(P1)**, **(P2)** and [14, Prop 2, p. 40], we have the inequality $\|\xi\|e^{-L|t-\tau|} \leq \|x(t,\tau,\xi)\|$, obtaining the sharper estimation:

$$D_t\phi(\tau_0,\xi_0,T(\tau_0,\xi_0)) \leq -\eta e^{-L\beta|T(\tau_0,\xi_0)-\tau_0|}\|\xi_0\|^{\beta} < 0, \tag{4.6}$$

then, the implicit function theorem and (4.5) establish the existence of $\mathscr{C}^1(U,W)$-map φ, where U is a neighborhood of (τ_0,ξ_0) while W is one of $T(\tau_0,\xi_0)$, which verifies

$$\varphi(\tau_0,\xi_0) = T(\tau_0,\xi_0) \quad \text{with} \quad U\times W\subseteq\mathcal{O} \text{ and } \phi(\tau,\xi,\varphi(\tau,\xi)) = 0 \quad \text{for any } (\tau,\xi)\in U,$$

which also can be written as

$$V(\varphi(\tau,\xi),x(\varphi(\tau,\xi),\tau,\xi)) = 1 = V(T(\tau,\xi),x(T(\tau,\xi),\tau,\xi)) \quad \text{on } U,$$

which leads to $T(\tau,\xi)=\varphi(\tau,\xi)$ for any $(\tau,\xi)\in U$ by the uniqueness of T, which also implies that $T\in\mathscr{C}^1(U,W)$.

In particular, we have the continuous differentiability of T on each arbitrary $(\tau_0,\xi_0)\in\mathbb{R}\times\mathbb{R}_0^n$, this implies that $T\in\mathscr{C}^1(\mathbb{R}\times\mathbb{R}_0^n,\mathbb{R})$. The partial derivative is obtained explicitly as

$$D_\xi T(\tau,\xi) = -\left[D_t\phi(\tau,\xi,T(\tau,\xi))\right]^{-1} D_\xi\,\phi(\tau,\xi,T(\tau,\xi)).$$

By using that $\nu\,|_{\mathcal{O}}=\phi$, the above partial derivatives coincides with (4.3)–(4.4) and the above identity becomes (4.1).

Finally, in order to show (4.2) and $S\in\mathscr{C}^1$, we can make an identical proof to the previous one by using the maps F_2 and $t\mapsto y(t,\tau,\xi)$ instead of F_1 and $t\mapsto x(t,\tau,\xi)$, respectively, and the result follows. □

The above result of continuous differentiability for the crossing times will be useful to achieve the continuous differentiability of the Palmer's homeomorphisms H_τ and G_τ.

Theorem 4.2. *Under the assumptions of the Lemma 4.1, the Palmer's homeomorphism H_τ described in (2.4) is a diffeomorphism of class $\mathscr{C}^1(\mathbb{R}_0^n,\mathbb{R}_0^n)$ for any fixed $\tau\in\mathbb{R}$.*

Moreover, if we use the notation $S:=S(\tau,\xi)$ and $T:=T(\tau,\xi)$, the derivative of $H_\tau:=H(\tau,\cdot)$, with respect to ξ is given by

$$D_\xi H(\tau,\xi) \;=\; \mathcal{V}(\tau,\xi)+D_\xi y(\tau,T,x(T,\tau,\xi))[\mathcal{A}(\tau,\xi)+D_\xi x(T,\tau,\xi)]. \tag{4.7}$$

where the (i,j)-coordinates of $\mathcal{V}(\tau,\xi)$ and $\mathcal{A}(\tau,\xi)$ are given by

$$\mathcal{V}_{i,j}(\tau,\xi) = D_T\,y_i(\tau,T,x(T,\tau,\xi))\,D_{\xi_j}T(\tau,\xi), \quad \mathcal{A}_{i,j}(\tau,\xi) = D_Tx_i(T,\tau,\xi)D_{\xi_j}T(\tau,\xi),$$

while the derivative of $G(\tau,\cdot):=H_\tau^{-1}$ with respect to ξ is

$$D_\xi G(\tau,\xi) = \mathcal{W}(\tau,\xi)+D_\xi x(\tau,S,x(S,\tau,\xi))[\mathcal{B}(\tau,\xi)+D_\xi y(S,\tau,\xi)] \tag{4.8}$$

where the (i,j)-coordinates of $\mathcal{W}(\tau,\xi)$ and $\mathcal{B}(\tau,\xi)$ are given by

$$\mathcal{W}_{i,j}(\tau,\xi) = D_S\,x_i(\tau,S,x(S,\tau,\xi))\,D_{\xi_j}S(\tau,\xi), \quad \mathcal{B}_{i,j}(\tau,\xi) = D_Sy_i(S,\tau,\xi)D_{\xi_j}S(\tau,\xi).$$

Proof. Note that $S, T \in \mathscr{C}^1(\mathbb{R} \times \mathbb{R}^n_0, \mathbb{R})$ by Lemma 4.1, whose proof used that the maps $(t,\tau,\xi) \mapsto y(t,\tau,\xi)$ and $(t,\tau,\xi) \mapsto x(t,\tau,\xi)$ are of class $\mathscr{C}^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

The above paragraph together with (2.4) imply that the map

$$0 \neq \xi \mapsto H_\tau(\xi) = y(\tau, T(\tau,\xi), x(T(\tau,\xi),\tau,\xi)),$$

is a composition of $\mathscr{C}^1$ maps, and we have that $H_\tau \in \mathscr{C}^1(\mathbb{R}^n_0, \mathbb{R}^n_0)$ for any fixed $\tau \in \mathbb{R}$.

To verify (4.7), let $H_i(\tau,\xi)$ be the i-th coordinate of the map $H(\tau,\cdot)$. By using (2.4) and the chain's rule, we can deduce that the partial derivatives are

$$\begin{aligned}
\frac{\partial H_i(\tau,\xi)}{\partial \xi_j} &= \frac{\partial}{\partial \xi_j}\{y_i(\tau, T(\tau,\xi), x(T(\tau,\xi),\tau,\xi))\} \\
&= \frac{\partial y_i(\tau, T(\tau,\xi), x(T(\tau,\xi),\tau,\xi))}{\partial T}\frac{\partial T(\tau,\xi)}{\partial \xi_j} \\
&\quad + \sum_{k=1}^{n} \frac{\partial y_i(\tau, T(\tau,\xi), x(T(\tau,\xi),\tau,\xi))}{\partial x_k}\frac{\partial x_k(T(\tau,\xi),\tau,\xi)}{\partial T}\frac{\partial T(\tau,\xi)}{\partial \xi_j} \\
&\quad + \sum_{k=1}^{n} \frac{\partial y_i(\tau, T(\tau,\xi), x(T(\tau,\xi),\tau,\xi))}{\partial x_k}\frac{\partial x_k(T(\tau,\xi),\tau,\xi)}{\partial \xi_j} \\
&= \mathcal{V}_{i,j}(\tau,\xi) + \sum_{k=1}^{n} \frac{\partial y_i(\tau, T, x(T,\tau,\xi))}{\partial x_k}\left\{\mathcal{A}_{k,j}(\tau,\xi) + \frac{\partial x_k(T,\tau,\xi)}{\partial \xi_j}\right\},
\end{aligned}$$

where the last equation is due by using $T = T(\tau,\xi)$. Then, we can verify that it corresponds to the (i,j)-coefficient of (4.7).

In a similar way, by (2.5), the map $0 \neq \xi \mapsto G_\tau(\xi) = x(\tau, S(\tau,\xi), y(S(\tau,\xi),\tau,\xi))$ is a composition of continuously differentiable functions. In consequence, G_τ is also a continuously differentiable map of $\mathbb{R}^n_0$ on itself. Therefore, H_τ is a diffeomorphism of class $\mathscr{C}^1(\mathbb{R}^n_0, \mathbb{R}^n_0)$.

The proof of the identity (4.8) is similar to (4.7) by using $(t,\tau,\xi) \mapsto y(t,\tau,\xi)$ and $(\tau,\xi) \mapsto S(\tau,\xi)$ instead of $(t,\tau,\xi) \mapsto x(t,\tau,\xi)$ and $(\tau,\xi) \mapsto T(\tau,\xi)$, respectively. Therefore, we can prove that it corresponds to the (i,j)-coefficient of the identity (4.8), and the Theorem follows. □

5 The smoothness of class $\mathscr{C}^k$ for the Palmer's and Lin's homeomorphism

Throughout this section, we will see that; provided some additional properties; the Palmer's and Lin's homeomorphisms via crossing times are diffeomorphisms of class $\mathscr{C}^k$ for any fixed $\tau \in \mathbb{R}$ and $k \geq 2$.

We will start by studying the Palmer's homeomorphism:

Lemma 5.1. *If the systems (1.1) and (1.2) verify* **(P1)–(P4)**, *F_1 and F_2 are in $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ while the Lyapunov function V verifies $V \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, then, the crossing times S and T of (2.3) verify $T, S \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n_0, \mathbb{R})$.*

Proof. We will consider the Banach spaces X, Y and Z together with the open set $\mathcal{O} \subseteq X \times Y$ and the functions ν and $\phi := \nu|_{\mathcal{O}}$ of the proof of Lemma 4.1.

By [22, Cor. 6.1, p. 92], $(t, \tau, \xi) \mapsto x(t, \tau, \xi)$ is of class $\mathscr{C}^k(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$. In addition, ν is a composition of $\mathscr{C}^k$-maps, which implies that $\nu \in \mathscr{C}^k(X \times Y, Z)$, and we conclude that $\phi \in \mathscr{C}^k(\mathcal{O}, Z)$.

By (4.5), we have that $\phi(\tau_0, \xi_0, T(\tau_0, \xi_0)) = 0$ while $D_t\phi(\tau_0, \xi_0, T(\tau_0, \xi_0)) \neq 0$ by (4.6). Hence, the implicit function theorem for maps in $\mathscr{C}^k$ [22, Cor. 5.1, p. 84] implies the existence of $\varphi : U_0 \to V_0$ of class $\mathscr{C}^k$, where U_0 and V_0 are neighborhoods of (τ_0, ξ_0) and $T(\tau_0, \xi_0)$, respectively, and ϕ verifies

$$\phi(\tau, \xi, \varphi(\tau, \xi)) = 0 \quad \text{for any } (\tau, \xi) \in U_0.$$

Moreover, the definition of T and the above property of φ establish that

$$V(\varphi(\tau, \xi), x(\varphi(\tau, \xi), \tau, \xi)) = 1 = V(T(\tau, \xi), x(T(\tau, \xi), \tau, \xi)),$$

and the uniqueness of T implies that $\varphi(\tau, \xi) = T(\tau, \xi)$ for any $(\tau, \xi) \in U_0$. Then, T has continuous k-th derivatives on any $(\tau_0, \xi_0) \in \mathbb{R} \times \mathbb{R}_0^n$, which implies that $T \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$.

By using $(t, \tau, \xi) \mapsto y(t, \tau, \xi)$, F_2 and S instead of $(t, \tau, \xi) \mapsto x(t, \tau, \xi)$, F_1 and T respectively, we have that $S \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$. □

Theorem 5.2. *If the systems (1.1) and (1.2) verify* **(P1)–(P4)**, *F_1, F_2 are in $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and the Lyapunov function also verifies that $V \in \mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, then the Palmer's homeomorphism H_τ is a diffeomorphism of class $\mathscr{C}^k(\mathbb{R}_0^n, \mathbb{R}_0^n)$ for any fixed $\tau \in \mathbb{R}$.*

Proof. The crossing time functions S and T are of class $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$ by Lemma 5.1, whose proof used that $(t, \tau, \xi) \mapsto y(t, \tau, \xi)$ and $(t, \tau, \xi) \mapsto x(t, \tau, \xi)$ are of class $\mathscr{C}^k(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

A byproduct of the above facts combined with (2.4) is that, for any fixed $\tau \in \mathbb{R}$, the map $0 \neq \xi \mapsto H_\tau(\xi) = y(\tau, T(\tau, \xi), x(T(\tau, \xi), \tau, \xi))$ can be seen as a composition of function of class $\mathscr{C}^k$, which leads to $H_\tau := H(\tau, \cdot) \in \mathscr{C}^k(\mathbb{R}_0^n, \mathbb{R}_0^n)$.

Similarly, we have that $0 \neq \xi \mapsto G_\tau(\xi) = x(\tau, S(\tau, \xi), y(S(\tau, \xi), \tau, \xi))$ is a composition of $\mathscr{C}^k$-functions, leading to $G_\tau := G(\tau, \cdot) \in \mathscr{C}^k(\mathbb{R}_0^n, \mathbb{R}_0^n)$. This implies that each H_τ is a diffeomorphism of class $\mathscr{C}^k(\mathbb{R}_0^n, \mathbb{R}_0^n)$. □

As a byproduct of the above result, we can study the smoothness properties for the Lin's homeomorphism:

Corollary 5.3. *Under the assumptions of Lemma 3.1, if the maps $(t, \xi) \mapsto C(t)\xi$, $(t, \xi) \mapsto B(t)\xi$ and g are of class $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, the crossing times T and S of (2.10) are of class $\mathscr{C}^k(\mathbb{R} \times \mathbb{R}_0^n, \mathbb{R})$.*

Moreover, for all $\tau \in \mathbb{R}$ fixed, the Lin's homeomorphism H_τ is a diffeomorphism of class $\mathscr{C}^k(\mathbb{R}_0^n, \mathbb{R}_0^n)$.

Proof. Let us define the functions $F_1, F_2 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$F_1(t, \xi) = C(t)\xi + B(t)\xi + g(t, \xi), \quad \text{and} \quad F_2(t, \xi) = -\frac{\delta}{2}I\xi \quad \text{for any } (t, \xi) \in \mathbb{R} \times \mathbb{R}^n,$$

and we will verify the properties **(P1)–(P4)**.

Firstly, note that $F_1(t, 0) = F_2(t, 0) = 0$ for any $t \in \mathbb{R}$. In fact, one identity is trivial while the other one is by **(L3)**, then, **(P1)** follows.

To verify **(P2)**, note that $t \mapsto C(t)$ is a bounded matrix function by **(L1)**, then, we can define $M = \sup_{t\in\mathbb{R}} \|C(t)\|$ and $L := M + \frac{\delta}{2}$. Now, by **(L1)–(L3)**, we can deduce that for any $\xi_1, \xi_2 \in \mathbb{R}^n$ and $t \in \mathbb{R}$

$$\|F_1(t,\xi_1) - F_1(t,\xi_2)\| \leq L\|\xi_1 - \xi_2\| \quad \text{and} \quad \|F_2(t,\xi_1) - F_2(t,\xi_2)\| \leq L\|\xi_1 - \xi_2\|$$

and we have verified **(P2)**.

We will see that **(P3)** and **(P4)** are verified if we consider $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$V(t,\xi) := \|\xi\|^2 = \sum_{i=1}^{n} \xi_i^2.$$

In fact, if $C_1 = C_2 = 1$ and $\beta = 2$, we have that

$$C_1\|\xi\|^\beta \leq \|\xi\|^2 = V(t,\xi) \leq C_2\|\xi\|^\beta$$

for any $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, this proves **(P3)**.

The last step consists in verify the existence of $\eta > 0$ such that

$$DV(t,\gamma(t)) \leq -\eta\|\gamma(t)\|^\beta,$$

for any solution $t \mapsto \gamma(t)$ either of (2.6) or (2.7). But, let us define $\eta = \delta > 0$. By the proof of [14, Proposition 7, p. 41], any solution $t \mapsto y(t,\tau,\xi)$ of (2.7) satisfies

$$DV(t,y(t,\tau,\xi)) = \frac{d}{dt}\left(\|y(t,\tau,\xi)\|^2\right) \leq -\delta\|y(t,\tau,\xi)\|^2,$$

and any solution $t \mapsto x(t,\tau,\xi)$ can be written as a solution of (2.7) with the maps

$$C_0(t) = -\delta I, \quad B_0(t) = \frac{\delta}{4}I \quad \text{and} \quad g_0(t,\xi) = \frac{\delta}{4}I\xi \quad \text{for any } (t,\xi) \in \mathbb{R}\times\mathbb{R}^n,$$

then

$$DV(t,x(t,\tau,\xi)) \leq -\delta\|x(t,\tau,\xi)\|^2,$$

and we proved **(P4)**.

Now, the function F_1 is a sum of functions of class $\mathscr{C}^k(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ which implies that $F_1 \in \mathscr{C}^k(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$.

On the order hand, we have that $F_2 = -\frac{\delta}{2}I\xi$ belongs to $\mathscr{C}^\infty(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ which leads to $F_2 \in \mathscr{C}^k(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$.

In addition, we have that V is a quadratic polinomial map of n variables, then $V \in \mathscr{C}^k(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$ and we have that T and S are in $\mathscr{C}^k(\mathbb{R}\times\mathbb{R}_0^n,\mathbb{R})$ by Lemma 5.1.

Finally, the Lin's homeomorphism can be seen as a Palmer homeomorphism's between the systems (2.6) and (2.7) by Remark 2.4, then the Lin's homeomorphism is a diffeomorphism of class $\mathscr{C}^k(\mathbb{R}_0^n,\mathbb{R}_0^n)$ by the Theorem 5.2. □

6 A generalization of Theorem 3.3

Under the assumption that the properties **(L1)** and **(L2)** are verified, let us consider the diagonal dominant linear system:

$$\dot{y} = [C(t) + B(t)]y. \tag{6.1}$$

By the variation of parameters method, we have that any solution $t \mapsto x(t,t_0,\xi)$ of (6.1) passing by ξ at $t = t_0$ verifies

$$x(t,t_0,\xi) = \Phi_C(t,t_0)\xi + \int_{t_0}^{t} \Phi_C(t,s)B(s)x(s,t_0,\xi)\,ds,$$

where $t \mapsto \Phi_C(t,s)$ is a transition matrix of $z' = C(t)z$.

On the other hand, we have that $x(t,t_0,\xi) = \Phi_{C+B}(t,t_0)\xi$ where $\Phi_{B+C}(t,t_0)$ is a transition matrix of (6.1). In addition, by considering

$$\Phi_C(t,s) = \text{Diag}\left\{e^{\int_s^t c_{ii}(\tau)\,d\tau}\right\}_{i=1}^{n},$$

and **(L1)**, we can deduce that $\|\Phi_C(t,s)\| \leq e^{-\delta(t-s)}$ for any $t \geq s$. The estimate of Φ_C combined with **(L2)** imply that

$$\begin{aligned} e^{\delta t}\|\Phi_{B+C}(t,t_0)\xi\| = e^{\delta t}\|x(t,t_0,\xi)\| &\leq e^{\delta t_0}\|\xi\| + \int_{t_0}^{t} \frac{\delta}{4} e^{\delta s}\|x(s,t_0,\xi)\|\,ds \\ &= e^{\delta t_0}\|\xi\| + \int_{t_0}^{t} \frac{\delta}{4} e^{\delta s}\|\Phi_{C+B}(s,t_0)\xi\|\,ds \end{aligned}$$

then, by using the classical Gronwall's inequality, it is straightforward to infer that

$$\|\Phi_{B+C}(t,t_0)\| \leq e^{-\frac{\delta}{2}(t-t_0)} \quad \text{for any } t \geq t_0, \tag{6.2}$$

namely, the linear system (6.1) is $\mathbb{R}$-uniformly exponentially stable.

Similarly, as it was stated in the subsection 2.2, the properties **(L1)–(L3)** imply that any solution $t \mapsto y(t,t_0,y_0)$ of (2.7) verifies

$$\|y(t,t_0,y_0)\| \leq \|y_0\|e^{-\frac{\delta}{2}(t-t_0)} \quad \text{for any } t \geq t_0.$$

A nice consequence of the Lin's homeomorphism is that the linear system (6.1) is topologically equivalent to its quasilinear perturbation (2.7) by the proof of [14, Lemma 2].

A direct byproduct of our previous results is the smoothness of the above mentioned topological equivalence

Theorem 6.1. *If* **(L1)–(L3)** *are satisfied and the maps $(t,y) \mapsto g(t,y)$, $(t,y) \mapsto C(t)y$ and $(t,y) \mapsto B(t)y$ belong to $\mathscr{C}^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$, then the linear system (6.1) and its quasilinear perturbation (2.7) are $\mathbb{R}$-topologically equivalent via a function $P\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ which, for any fixed t, is a preserving orientation diffeomorphism of class $\mathscr{C}^1$ on $\mathbb{R}_0^n$.*

Proof. By Theorem 3.3 and Corollary 3.4, the quasilinear system (2.7) and the diagonal autonomous system (2.6) are $\mathbb{R}$-topologically equivalent via the function $H\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ described by (2.8), which is a preserving orientation diffeomorphism of class $\mathscr{C}^1$ on $\mathbb{R}_0^n$ for any fixed t.

On the other hand, as pointed out by Lin in [14], the function $(t,x) \mapsto g(t,x) \equiv 0$ satisfies **(L3)** and belongs to $\mathscr{C}^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$, then, the linear system (6.1), and the diagonal autonomous system (2.6), are $\mathbb{R}$-topologically equivalent via a function $Q\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ which for any fixed t, is a preserving orientation diffeomorphism of class $\mathscr{C}^1$ on $\mathbb{R}_0^n$.

Finally, as the $\mathbb{R}$-topological equivalence is an equivalence relation; it follows that the linear system (6.1) and its quasilineal perturbation (2.7), are $\mathbb{R}$-topologically equivalent via a function $P = H \circ Q\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ which, for any fixed t, is a composition of two preserving orientation diffeomorphisms of class $\mathscr{C}^1$ on $\mathbb{R}_0^n$. □

It is of interest to point out that the Theorem 6.1 has a similar structure that a result obtained in [6] by following a Green's function approach. Both results are about the smoothness of a $\mathbb{R}$-topological equivalence between a uniformly exponentially stable linear system and a quasilinear perturbation. Now, it is interesting for us to describe the advantages and drawbacks of our result compared with the one obtained in [6].

A difference at first glance is that the Theorem 6.1 does not assumes that the quasilinear perturbation g is bounded on $\mathbb{R} \times \mathbb{R}^n$. For example, a linear perturbation $g(t,x) = H(t)x$ with $\|H(t)\| \leq \frac{\delta}{4}$ is covered by our result when $t \mapsto H(t)$ is continuously differentiable. On the other hand, in [6] the global boundedness of the perturbation is an essential assumption to the construction of the homeomorphism; nevertheless, Theorem 6.1 assumes that $g(t,0) = 0$ for any $t \in \mathbb{R}$, which is not necessary in [6].

A second difference is that Theorem 6.1 allows an easier generalization to derivatives of higher order, which is not the case in [6]. From this perspective, our approach has a clear advantage.

Finally, the result of [6] only assumes that the linear part is a uniformly asymptotically stable linear system, while in Theorem 6.1 this assumption is restricted for the special case of linear systems with diagonal dominance, making our result a more restrictive one.

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Rational limit cycles of Abel differential equations

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Abstract. We study the number of rational limit cycles of the Abel equation $x' = A(t)x^3 + B(t)x^2$, where $A(t)$ and $B(t)$ are real trigonometric polynomials. We show that this number is at most the degree of $A(t)$ plus one.

Keywords: periodic solution, limit cycle, Abel equation.

2020 Mathematics Subject Classification: 34C25.

1 Introduction

The Abel differential equation

$$x' = A(t)x^3 + B(t)x^2 + C(t)x,$$

where $A(t), B(t)$ and $C(t)$ are trigonometric polynomials has been studied by many authors, either for its relation to higher degree phenomena (see e.g. [11]), for applications to real-world models (see e.g. [2]), or for its own intrinsic interest (see e.g. [7]).

In this paper we consider Abel differential equations without linear term, that is,

$$x' = A(t)x^3 + B(t)x^2, \tag{1.1}$$

with $A(t)$ and $B(t)$ being real trigonometric polynomials.

Among the main problems related to this equation, we could name the Smale–Pugh [16] problem, which is considered as a particular case of the 16th Hilbert problem. The problem consists in bounding the number of limit cycles of (1.1), that is, the number of isolated periodic solutions in the set of periodic solutions of the equation. In connection with this problem, Lins Neto [11] proved that there is no upper bound on the number of limit cycles of (1.1).

Another important problem often mentioned in the literature is the Poincaré center-focus problem applied to this setting. Trivially, $x(t) = 0$ is a solution of the equation. The problem asks when the equation 1.1 has a center at $x(t) = 0$, i.e., all solutions in a neighborhood of the solution $x(t) = 0$ are closed. This problem for (1.1) was proposed by Briskin, Françoise, and

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Yondim [4, 5]. There are some progresses in solving this problem, for example, it has been proved that the composition condition determine certain parametric centers when the coefficients are polynomials [13], but the problem is open for trigonometric coefficients. Moreover, it is conjectured that the converse is true when $A(t)$ and $B(t)$ are polynomials, with some relevant evidences that support it. For more more details, the reader may refer to [7, 10].

When faced with the Smale–Pugh problem, one of the most common strategies for obtaining upper bounds on the number of limit cycles is to impose some additional restrictions on $A(t)$ and $B(t)$. For example, it has been shown that if $A(t) \neq 0$ or $B(t) \neq 0$ does not change sign, or if there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha A(t) + \beta B(t) \neq 0$ does not change sign, then the equation has at most three limit cycles [1, 8]. See [7] for more information.

Another strategy is to focus on the problem for limit cycles of a certain form or with certain properties that may be of particular interest. An example of such a result is that the generalized Abel equation with polynomial coefficients and degree n has at most n polynomial limit cycles, see [9].

Also, the number of rational solutions of (1.1) when $A(t), B(t)$ are polynomials has been profusely studied. For instance, in [15] the authors obtain upper bounds on the number of rational periodic solutions of (1.1) under certain conditions on the degrees of $A(t), B(t)$, and in [3], a general, non-optimal upper bound has been obtained. It has been also studied for $A(t), B(t)$ trigonometric polynomials, in [17]. Note that the rational solutions are not necessarily limit cycles, as they may be part of a centre.

In this paper we obtain an upper bound on the number of rational limit cycles of the equation (1.1), i.e., limit cycles of the form $x(t) = Q(t)/P(t)$, where $P(t)$ and $Q(t)$ are real trigonometric polynomials and $P(t) \neq 0$ for all $t \in \mathbb{R}$. Recall that a real trigonometric polynomial of degree n is an expression of the form

$$\sum_{k=0}^{n} a_k \cos(kt) + b_k \sin(kt), \quad a_k, b_k \in \mathbb{R}$$

with $a_n \cdot b_n \neq 0$. As usual, we write $\mathbb{R}[\cos(t), \sin(t)]$ for the ring of real trigonometric polynomials.

Using this notation, our main result is as follows.

Theorem 1.1. *Let $A(t), B(t) \in \mathbb{R}[\cos(t), \sin(t)]$. If the degree of $A(t)$ is odd or less than twice the degree of $B(t)$, then* (1.1) *has at most two non-trivial rational limit cycles. Otherwise, the number of non-trivial rational limit cycles of equation* (1.1) *is at most the degree of $A(t)$ plus one.*

To prove this result, we consider each rational limit cycle $x(t) = Q(t)/P(t)$ as an invariant trigonometric algebraic curve of degree one in x with real trigonometric coefficients (Proposition 2.2), that is, an invariant curve of (1.1) of the form $Q(t) - P(t)x = 0$, where $P(t)$ and $Q(t)$ are real trigonometric polynomials and $P(t) \neq 0$ for all $t \in \mathbb{R}$. Therefore, to bound the number of limit rational cycles, we bound the number of invariant algebraic curves of degree one in x with real trigonometric coefficients of (1.1) such that (1.1) has no center at the origin. In particular, to prove the second part of Theorem 1.1, we bound the maximum number of invariant curves of this type such that (1.1) does not have a Darboux first integral.

The study of invariant curves of degree one in x is interesting in itself, regardless of whether they correspond to rational limit cycles or not. Therefore, we include a method to parameterize the Abel equations 1.1 that have at least two non-trivial invariant algebraic curves of degree one in x with real trigonometric coefficients.

The structure of the paper is as follows. In the first section we characterize the invariant algebraic curves of degree one in x with real trigonometric coefficients such that the equation (1.1) has and we show some of their properties. Then we prove the first part of Theorem 1.1 and give the parameterization mentioned above. In the second section, we apply Darboux's integrability theory to this situation, which allows us to complete the proof of the main result of the paper.

Due to the fact that $\mathbb{R}[\cos(t), \sin(t)]$ is not a unique factorization domain, some necessary facts and results about factorization in these rings are collected in Appendix A.

2 Real trigonometric invariant algebraic curves of degree one in x

Consider the Abel equation (1.1), set $g(t,x) := A(t)x^3 + B(t)x^2$ and denote the associated vector field by $\mathcal{X}$, that is,

$$\mathcal{X} = \frac{\partial}{\partial t} + g\frac{\partial}{\partial x}.$$

Fixed $f \in \mathcal{C}^1$, the curve $f(t,x) = 0$ is said to be an invariant curve of (1.1) if there exists $K \in \mathcal{C}^0$, called the cofactor of $f(t,x)$, such that

$$(\mathcal{X}f)(t,x) = \left(\frac{\partial f}{\partial t} + g\frac{\partial f}{\partial x}\right)(t,x) = K(t,x)f(t,x).$$

Note that $f_0(t,x) := x = 0$ is always an invariant curve of (1.1) with the cofactor $K_0(t,x) = A(t)x^2 + B(t)x$.

If $f(t,x) = 0$ is invariant and $x(t)$ is a solution of (1.1), then, for any t_0 in the domain of the solution,

$$f(t,x(t)) = f(t_0,x(t_0))\exp\left(\int_{t_0}^{t} K(s,x(s))ds\right).$$

Therefore, if $f(t_0,x(t_0)) = 0$, then $f(t,x(t)) = 0$ for all t. Consequently $f(t,x) = 0$ consists of trajectories of solutions of the equation.

From now on, unless otherwise stated, $A(t)$ and $B(t)$ are real trigonometric polynomials, that is, $A(t), B(t) \in \mathbb{R}[\cos(t), \sin(t)]$.

The first objective of this section is to characterize the invariant algebraic curves $f(t,x) = 0$ of (1.1) such that $f(t,x) = Q(t) - P(t)x \in \mathbb{R}[\cos(t), \sin(t)][x]$ with $P(t) \neq 0$, for all $t \in \mathbb{R}$.

Let $P(t)$ and $Q(t)$ be real trigonometric polynomials with $P(t) \neq 0$, for all $t \in \mathbb{R}$. If $R(t) \in \mathbb{R}[\cos(t), \sin(t)]$ is a common factor of $P(t)$ and $Q(t)$, meaning that there exist factorizations of $P(t)$ and $Q(t)$ in which $R(t)$ appears (see Appendix A for details), we have that $Q(t) - P(t)x = 0$ is an invariant curve of (1.1) if and only if $Q(t)/R(t) - (P(t)/R(t))x = 0$ is an invariant curve of (1.1). Thus, in what follows, we always assume that $P(t)$ and $Q(t)$ have no common factors, that is, $Q(t) - P(t)x$ is irreducible in $\mathbb{R}[\cos(t), \sin(t)][x]$ and, by Corollary A.4, in $\mathbb{C}[\cos(t), \sin(t)][x]$.

Remark 2.1. To simplify the exposition from now on, we simply say invariant curves of (1.1) to refer to invariant curves of (1.1) of the form $Q(t) - P(t)x = 0$, where $P(t) \neq 0$, for all $t \in \mathbb{R}$, and $Q(t) - P(t)x$ is irreducible.

The following result establishes the relationship between rational limit cycles $x(t) = Q(t)/P(t)$ and invariant curves of (1.1).

Proposition 2.2. *Let $P(t)$ and $Q(t)$ be real trigonometric polynomials. If $P(t) \neq 0$ for all $t \in \mathbb{R}$, then $x(t) = Q(t)/P(t)$ is a solution of* (1.1) *if and only if $Q(t) - P(t)x = 0$ is an invariant curve of* (1.1).

Proof. Since $\mathbb{R}[\cos(t), \sin(t)]$ is a domain (see [14, Theorem 3.1]) its field of fractions, Σ, is well-defined. Thus, we can perform the Euclidean division of $(\mathcal{X}f)(t,x) = Q'(t) - P'(t)x - P(t)\Big(A(t)x^3 + B(t)x^2\Big)$ by $f(t,x) := Q(t) - P(t)x$ in $\Sigma[x]$, so that there exist unique $K(t,x)$ and $Z(t,x) \in \Sigma[x]$ such that

$$(\mathcal{X}f)(t,x) = K(t,x)(Q(t) - P(t)x) + Z(t).$$

Concretely,

$$Z(t) = P(t)\left(\left(\frac{Q(t)}{P(t)}\right)' - A(t)\left(\frac{Q(t)}{P(t)}\right)^3 - B(t)\left(\frac{Q(t)}{P(t)}\right)^2\right)$$

and

$$\begin{aligned} K(t,x) = A(t)x^2 + \left(A(t)\left(\frac{Q(t)}{P(t)}\right) + B(t)\right)x \\ + A(t)\left(\frac{Q(t)}{P(t)}\right)^2 + B(t)\left(\frac{Q(t)}{P(t)}\right) + \frac{P'(t)}{P(t)}. \end{aligned}$$

Note that $K(t,x) \in \mathcal{C}^0$ because $P(t) \neq 0$ for all $t \in \mathbb{R}$. Therefore, we conclude that the necessary and sufficient condition for $Q(t) - P(t)x = 0$ to be an invariant curve of (1.1) is $Z(t) = 0$, which, given the expression of $Z(t)$, is equivalent to $x(t) = Q(t)/P(t)$ being a solution of (1.1). □

Now, our approach is as follows: instead of directly bounding the number of rational limit cycles of (1.1), we bound the number of invariant curves of degree one in x such that (1.1) has not a center. According to Proposition 2.2, this will be an upper bound on the number of rational limit cycles.

Next we give a condition so that $Q(t) - P(t)x = 0$ is an invariant curve of (1.1). But first we need a lemma.

Lemma 2.3. *Let $P(t)$ and $Q(t)$ be real trigonometric polynomials with $P(t) \neq 0$ for all $t \in \mathbb{R}$. If $Q(t) - P(t)x = 0$ is an invariant curve of* (1.1)*, then the corresponding cofactor is a polynomial in x with real trigonometric polynomial coefficients.*

Proof. Let $f(t,x) := Q(t) - P(t)x$. Arguing as in the proof of Proposition 2.2, we obtain that there exists $\tilde{K}(t,x) \in \mathbb{R}[\cos(t), \sin(t)][x]$ such that $P(t)^2(\mathcal{X}f)(t,x) = \tilde{K}(t,x)f(t,x)$.

By Corollary A.4, $f(t,x)$ is irreducible in $\mathbb{C}[\cos(t), \sin(t)][x]$. Moreover, as $\mathbb{C}[\cos(t), \sin(t)]$ is an Euclidean domain (see [14, Theorem 2.1]), it is an unique factorization domain and therefore $\mathbb{C}[\cos(t), \sin(t)][x]$ is also an unique factorization domain. Thus, we have that $P(t)^2$ or $(\mathcal{X}f)(t,x)$ are divisible by $f(t,x)$ which necessarily implies that there exists $H(t,x) \in \mathbb{C}[\cos(t), \sin(t)][x]$ such that $(\mathcal{X}f)(t,x) = H(t,x)f(t,x)$ for degree reasons.

Finally, since both $(\mathcal{X}f)(t,x)$ and $f(t,x)$ are polynomials in x with real trigonometric polynomial coefficients, we conclude that $H(t,x)$ is also a polynomial in x with real trigonometric polynomial coefficients. □

The next result gives the condition for $Q(t) - P(t)x = 0$ to be an invariant curve of (1.1). This result is proved in [12] for the polynomial case. Moreover, since rational solutions are equivalent to the invariant curves of degree one in x, as mentioned in the introduction, the following result can also be obtained from those of [17] for the trigonometric case. Here, we provide a simplified proof.

Proposition 2.4. *The curve* $Q(t) - P(t)x = 0$ *is an invariant curve of* (1.1) *if and only if* $Q(t) = c \in \mathbb{R}$ *and there exists a trigonometric polynomial* $R(t)$ *such that*

$$A(t) = (P(t)/c)R(t), \quad B(t) = -P'(t)/c - R(t).$$

In this case, the corresponding cofactor is equal to $A(t)x^2 - (P'(t)/c)x$.

Proof. Let $f(t,x) := Q(t) + P(t)x = 0$ be an invariant curve of (1.1). Arguing as in the proof of Proposition 2.2 and taking advantage of the fact that $x(t) = Q(t)/P(t)$ is a solution of (1.1), we have that the corresponding cofactor can be written as

$$K(t,x) = A(t)x^2 - \left(\frac{P(t)}{Q(t)}\right)' x + \frac{Q'(t)}{Q(t)}.$$

Since, by Lemma 2.3, $K(t,x) \in \mathbb{R}[\cos(t), \sin(t)][x]$, we have in particular that $Q'(t)$ is equal to $K_0(t)Q(t)$ for some $K_0(t) \in \mathbb{R}[\cos(t), \sin(t)]$. Comparing degrees, either $Q(t) = 0$ or K_0 is constant. In the latter case,

$$|Q(t)| = e^{\int K_0 dt}.$$

Thus we conclude that $K_0 = 0$, and $Q(t) = c \in \mathbb{R}$. Note that, in this case, $K(t,x) = A(t)x^2 - (P'(t)/c)x$.

Finally, noting that

$$-\frac{P'(t)}{c} = A(t)\left(\frac{c}{P(t)}\right) + B(t)$$

we conclude that $B(t) = -P'(t)/c - R(t)$ where $R(t) = cA(t)/P(t) \in \mathbb{R}[\cos(t), \sin(t)]$.

The converse follows by direct checking. □

As mentioned in the proof of Proposition 2.4, the curve $c - P(t)x = 0$ is an invariant curve of the equation (1.1) if and only if

$$-\frac{P'(t)}{c} = A(t)\left(\frac{c}{P(t)}\right) + B(t).$$

Without loss of generality, we can assume $c = 1$, so that

$$P(t)P'(t) + P(t)B(t) + A(t) = 0. \tag{2.1}$$

Note that if equation (1.1) has an invariant curve of the form $1 - Kx = 0$ with K a non-zero constant, then the Abel equation becomes the separated variable equation $x' = B(t)x^2(-Kx + 1)$ with constant solutions 0 and $1/K$. If $\int_0^{2\pi} B(t)\,dt \neq 0$ these constant solutions are the unique limit cycles, while if $\int_0^{2\pi} B(t)\,dt = 0$ every bounded solution is periodic, so it has no limit cycles. Hence, we consider only the case $\deg(P) \geq 1$.

We will say that an invariant curve $1 - P(t)x = 0$ has degree n if $\deg(P) = n$. Next we prove that the sum of the degrees of two invariant curves is the degree of A.

Proposition 2.5. *If* $1 - P_1(t)x = 0$ *and* $1 - P_2(t)x = 0$ *are two different invariant curves of* (1.1), *then* $\deg(P_1) + \deg(P_2) = \deg(A)$. *Consequently, if* $\deg(P_1) = \deg(P_2)$, *then* $\deg(P_1) = \deg(P_2) = \deg(A)/2$.

Proof. By Proposition 2.4, there exist trigonometric polynomials $R_1(t)$ and $R_2(t)$ such that $P_1(t)R_1(t) = A(t) = P_2(t)R_2(t)$ and $-P_1'(t) - R_1(t) = B(t) = -P_2'(t) - R_2(t)$. Thus,

$$P_1(t)(P_2'(t) + R_2(t) - P_1'(t)) = A(t) = P_2(t)R_2(t).$$

Therefore

$$P_1(t)(P_2(t) - P_1(t))' = P_1(t)(P_2'(t) - P_1'(t)) = R_2(t)(P_2(t) - P_1(t)).$$

Now, since $\deg(P_2(t) - P_1(t)) = \deg((P_2(t) - P_1(t))')$, we conclude that $\deg(P_1) = \deg(R_2) = \deg(A) - \deg(P_2)$, from which our claim follows. □

The following example shows that (1.1) can have two limit cycles of different degrees.

Example 2.6. Let $P_1(t) = 2(\cos(t)+2)(\sin(t)+2)$ and $P_2(t) = (\cos(t)+2)(\sin(t)+2)(\sin(t)+4)$. By Proposition 2.4, we have that $1 - P_i(t)x,\ i = 1, 2$ are invariant curves of (1.1) for

$$A(t) = (\cos(t) + 2)(\sin(t) + 2)(\sin(t) + 4)(3\cos(2t) + 8\cos(t) - 4\sin(t) + 1)$$

and

$$B(t) = -3/4\sin(3t) - 9\cos(2t) - 2\sin(2t) - 20\cos(t) + 49/4\sin(t) - 1.$$

Note that $5 = \deg(A) = \deg(P_1) + \deg(P_2) = 2 + 3$. Moreover, since $\int_0^{2\pi} B(t)\,dt = -2\pi \neq 0$, (1.1) does not have a center (see, for instance, [1, Lemma 7]), so the solutions $x(t) = 1/P_1(t)$ and $x(t) = 1/P_2(t)$ are limit cycles.

Remember that $x(t) = 0$ is always an invariant curve of (1.1). It corresponds to the case $Q(t) = 0$ and we call it a trivial invariant curve.

Corollary 2.7. *If equation* (1.1) *has three or more non-trivial invariant curves, then they all have degree* $\deg(A)/2$.

Proof. Suppose that equation (1.1) has three invariant curves $1 - P_1(t)x = 0$, $1 - P_2(t)x = 0$ and $1 - P_3(t)x = 0$. Then, by Proposition 2.5, $\deg(P_1) + \deg(P_2) = \deg(P_1) + \deg(P_3) = \deg(P_2) + \deg(P_3) = \deg(A)$, which implies $\deg(P_1) = \deg(P_2) = \deg(P_3) = \deg(A)/2$. □

Now, it is easy to give two conditions for equation (1.1) to have at most two non-trivial invariant curves, which proves the first part of the main theorem (Theorem 1.1).

Corollary 2.8. *If* $\deg(A)$ *is odd or if* $\deg(B) > \deg(A)/2$, *then* (1.1) *has has at most two non-trivial invariant curves.*

Proof. If $\deg(A)$ is odd the claim follows directly from Corollary 2.7.

So, suppose that $\deg(A)$ is even and $\deg(B) > \deg(A)/2$. If $1 - P(t)x = 0$ is an invariant curve of (1.1), then by Proposition 2.4 there exists $R(t) \in \mathbb{R}[\cos(t), \sin(t)]$ such that $A(t) = P(t)R(t)$ and $B(t) = -P'(t) - R(t)$, then $\deg(B) \leq \max\{\deg(P) = \deg(P'), \deg(A) - \deg(P)\}$. Thus if $\deg(P) = \deg(A)/2$, then $\deg(B) \leq \deg(A)/2$, contradicting the hypothesis. This fact, together with Proposition 2.5 and Corollary 2.7, completes the proof of the claim. □

We can now write this last result in terms of rational limit cycles.

Corollary 2.9. *If* $\deg(A)$ *is odd or if* $\deg(B) > \deg(A)/2$*, then* (1.1) *has has at most two non-trivial rational limit cycles.*

In [3] a parameterization is given for all cases of equations $x' = A(t)x^3 + B(t)x^2$, with $A(t), B(t) \in \mathbb{C}[t]$ which have at least two non-trivial polynomial polynomial invariant curves. Before we finish proving the main result of the paper in the next section, let us see that a similar parametrization of the rational limit cycles can be obtained in this case.

Proposition 2.10. *Equation* (1.1) *has two different non-trivial invariant curves if and only if there exist* $G(t), \hat{G}(t), S_1(t) \in \mathbb{R}[\cos(t), \sin(t)]$ *and* $k \in \mathbb{R} \setminus \{0\}$*, such that* $G(t), S_1(t), S_1(t) + k\hat{G}(t) \neq 0$ *for all* $t \in \mathbb{R}$*, every irreducible factor of* $\hat{G}(t)$ *divides* $G(t)$*, and the functions* A, B *satisfy*

$$A(t) = G(t)S_1(t)(S_1(t) + k\hat{G}(t)) \left(G'(t) + \frac{G(t)\hat{G}'(t)}{\hat{G}(t)} \right),$$
$$B(t) = -(G(t)S_1(t))' - (S_1(t) + k\hat{G}(t)) \left(G'(t) + \frac{G(t)\hat{G}'(t)}{\hat{G}(t)} \right),$$

Furthermore, in this case the two different invariant curves are

$$1 - G(t)S_1(t)x = 0 \quad \text{and} \quad 1 - G(t)(S_1(t) + k\hat{G}(t))x = 0.$$

Proof. Assume $1 - P_1(t)x = 0$, $1 - P_2(t)x = 0$ are two different non-trivial invariant curves of (1.1). Recall that by Remark 2.1, $P_1(t) \neq 0$ and $P_2(t) \neq 0$, for all $t \in \mathbb{R}$, so they have unique decomposition (see Corollary A.2). Thus there exists their greatest common divisor. Set $G(t) := \gcd(P_1(t), P_2(t))$, $S_1(t) := P_1(t)/G(t)$ and $S_2(t) := P_2(t)/G(t)$. Moreover, since, by Proposition 2.4, $P_1(t)$ and $P_2(t)$ divide $A(t)$, there exists $S(t) \in \mathbb{R}[\cos(t), \sin(t)]$ such that

$$A(t) = G(t)S_1(t)S_2(t)S(t)$$

and, by Proposition 2.4,

$$B(t) = -(G(t)S_1(t))' - S_2(t)S(t) = -(G(t)S_2(t))' - S_1(t)S(t).$$

Thus,

$$\Big(G(t)(S_2(t) - S_1(t))\Big)' = (S_2(t) - S_1(t))S(t).$$

Let $\hat{G}(t)$ be the product of all the factors of $S_2(t) - S_1(t)$ that divide $G(t)$; note that $\hat{G}(t)$ is well-defined by Proposition A.1 because $G(t) \neq 0$ for all $t \in \mathbb{R}$. Set $H(t) := (S_2(t) - S_1(t))/\hat{G}(t)$.$H(t)$ does not necessarily have a unique decomposition; however, by construction, no irreducible factor of $H(t)$ (in any of its factorizations) can divide $G(t)$. Now, from

$$\begin{aligned} G'(t)H(t)\hat{G}(t) + G(t)(H(t)\hat{G}(t))' &= (G(t)(S_2(t) - S_1(t))) \\ &= (S_2(t) - S_1(t))S(t) \\ &= H(t)\hat{G}(t)S(t), \end{aligned}$$

it follows that $G(t)H'(t)\hat{G}(t) + G(t)H(t)\hat{G}'(t) = H(t)\hat{G}(t)(S(t) - G'(t))$. So,

$$G(t)H'(t)\hat{G}(t) = H(t)\Big(\hat{G}(t)(S(t) - G'(t)) - G(t)\hat{G}'(t)\Big). \tag{2.2}$$

Therefore, since $G(t)\hat{G}(t)$ have no real zeros and no common irreducible factors with $H(t)$, by Corollary A.3, $G(t)\hat{G}(t)$ divides $R(t) := \hat{G}(t)(S(t) - G'(t)) - G(t)\hat{G}'(t)$. Moreover, noticing that $\deg(R(t)) \le \deg(\hat{G}(t)G'(t)) = \deg(\hat{G}(t)G(t))$, we have that $R(t)/(G(t)\hat{G}(t)) = k_0 \in \mathbb{R}[\cos(t), \sin(t)]$. Therefore, $H'(t) = H(t)R(t)/(\hat{G}(t)G(t)) = H(t)k_0$ and we conclude that $H(t) = k$, for some $k \in \mathbb{R}$, and that $S_2(t) = S_1(t) + k\hat{G}(t)$. Now, since $P_1(t) \neq P_2(t)$, we have that $k \neq 0$. So, replacing $H(t)$ by k in (2.2), we obtain that $S(t) = G'(t) + (G(t)\hat{G}'(t))/\hat{G}(t)$ as claimed.

Finally, since $S_1(t) \neq S_2(t)$, The opposite is deduced by direct verification using Proposition 2.4. □

Example 2.11. In order to obtain Example 2.6, it suffices to apply Proposition 2.10 with $G(t) = (\cos(t) + 2)(\sin(t) + 2)$, $\hat{G}(t) = \sin(t) + 2$, $S_1(t) = \sin(t) + 4$, and $k = -1$.

3 Darboux first integrals and proof of the main result

In this section, we use Darboux integrability theory to bound the maximum number of invariant curves that (1.1) can have without forcing the existence of a center.

We say that $f(t, x)$, smooth enough and not identically constant, is a first integral of (1.1) if $\mathcal{X}f = 0$, that is, $f(t, x) = 0$ is an invariant curve of (1.1) with zero cofactor. Equivalently, $f(t, x(t)) = 0$ is constant if $x(t)$ is a solution of the equation.

We say that a first integral f of (1.1) is of Darboux type if

$$f(t, x) = \prod_{i=0}^{r} f_i(t, x)^{\alpha_i},$$

where $f_i(t, x) = 0$ are invariant curves of the equation and $\alpha_i \in \mathbb{C}$.

First, we present Darboux's general result that relates the existence of a first Darboux integral with the linear dependence of the cofactors of the invariant curves. We have adapted its statement to our situation

Theorem 3.1 (Darboux's Theorem, [6]). *Let $f_0(t, x) = 0, \ldots, f_r(t, x) = 0$ be invariant curves of (1.1) with cofactors $K_0(t, x), \ldots, K_r(t, x)$, respectively. If there exist $\alpha_0, \ldots, \alpha_r \in \mathbb{C}$ such that $\sum_{i=0}^{r} \alpha_i K_i(t, x) = 0$ then $f(t, x) = \prod_{i=0}^{r} f_i(t, x)^{\alpha_i}$ is a first integral of (1.1).*

The following result is a direct application of Theorem 3.1 for the case where the invariant curves $f_0(t, x) = 0, \ldots, f_r(t, x) = 0$ are all non-trivial and have the form described in Remark 2.1.

Proposition 3.2. *Let $\alpha_i \in \mathbb{R}, i = 1, \ldots, r$, and $\alpha_0 := -\sum_{i=1}^{r} \alpha_i$. If $1 - P_i(t)x = 0, i = 1, \ldots, r$ are invariant curves of (1.1), then $f(t, x) := x^{\alpha_0} \prod_{i=1}^{r} (1 - P_i(t)x)^{\alpha_i}$ is a first integral of (1.1) if and only if*

$$\sum_{i=1}^{r} \alpha_i \frac{A(t)}{P_i(t)} = 0. \tag{3.1}$$

Proof. First, we recall that $f_0(t, x) = x = 0$ is always an invariant curve of (1.1) with cofactor $K_0(t, x) = A(t)x^2 + B(t)x$. Furthermore, by Proposition 2.4, we have the cofactor of $1 - P_i(t)x = 0$ is $K_i(t, x) = A(t)x^2 - P_i'(t)x$ for each $i = 1, \ldots, r$. Therefore,

$$\begin{aligned}(\mathcal{X}f)(t, x) &= \left(\alpha_0 A(t)x^2 + \alpha_0 B(t)x + \sum_{i=1}^{r} \alpha_i \Big(A(t)x^2 - P_i'(t)x\Big)\right) f(t, x) \\ &= -\left(\sum_{i=1}^{r} \alpha_i \Big(B(t) + P_i'(t)\Big) x\right) f(t, x).\end{aligned}$$

Moreover, by Proposition 2.4, $B(t) = -P_i'(t) - \frac{A(t)}{P_i(t)}, i = 1, \ldots, r$. So, we conclude that

$$(\mathcal{X}f)(t,x) = \left(\sum_{i=1}^{r} \alpha_i \frac{A(t)}{P_i(t)} x \right) f(t,x)$$

Now, by the definition of first integral and Theorem 3.1, our claim follows. □

Note that, as we have seen in the proof of the previous result, (3.1) is a necessary and sufficient condition for the cofactors to be linearly dependent.

We can now complete the proof of the main result (Theorem 1.1).

Proof of Theorem 1.1. If (1.1) has less than three non-trivial rational limit cycles, then the number of rational limit cycles is bounded by $\deg(A) + 1$ because $\deg(A) \geq 1$. Thus, we assume that (1.1) has $r \geq 3$ non-trivial rational limit cycles, corresponding to the invariant curves $1 - P_1(t)x = 0, \ldots, 1 - P_r(t)x = 0$ of (1.1) by Proposition 2.2.

Since $r \geq 3$, by Corollary 2.7 we know that $\deg(P_i) = \deg(A)/2, i = 1, \ldots, r$, and consequently $\deg(A/P_i) = \deg(A)/2$ for all i by Proposition 2.4. Moreover, by Proposition 3.2, the trigonometric polynomials $A/P_i, i = 1, \ldots, r$, are linearly independent. Otherwise, there would be a Darboux first integral and thus a center.

Finally, since the $\mathbb{R}-$vector space of trigonometric polynomials of degree $\deg(A)/2$ has dimension $\deg(A) + 1$, we conclude that $r \leq \deg(A) + 1$. □

A On factorization issues in the ring of real trigonometric polynomials

It is well known that the ring of real trigonometric polynomials is not a unique factorization domain. However, it is a Dedekind half-factorial domain ([14, Theorem 3.1]). Therefore, every non-zero non-unit is a finite product of irreducible elements, and any two factorizations into irreducibles of an element in $\mathbb{R}[\cos(t), \sin(t)]$ have the same number of irreducible factors. This allows us to consider the irreducible factors of a given real trigonometric polynomial or to use expressions such us "$P(t)$ and $Q(t)$ have no common irreducible factors", regardless of the fact that the greatest common divisor is not defined in half-factorial domains in general.

Recall that, given a non-zero real trigonometric polynomial

$$P(t) = \sum_{k=0}^{n} a_0 \cos(kt) + b_0 \sin(kt), \quad a_k, b_k \in \mathbb{R}$$

the degree of $P(t), \deg(P)$, is the biggest k such that $a_k \cdot b_k \neq 0$. Note that $\deg(PQ) = \deg(P) + \deg(Q)$ and, if $\deg(P) > 0$, then $\deg(P') = \deg(P)$. In particular, $P' = 0$ if and only if $P \in \mathbb{R}$.

Furthermore, since the irreducible elements of $\mathbb{R}[\cos(t), \sin(t)]$ are those of the form

$$a\cos(t) + b\sin(t) + c, \quad a, b, c \in \mathbb{R}, (a,b) \neq (0,0)$$

by [14, Theorem 3.4], we have that the degree of a non-zero non-unit element of the ring $\mathbb{R}[\cos(t), \sin(t)]$ is the number of its irreducible factors.

Given $z \in \mathbb{R}[\cos(t), \sin(t)]$, in the following we write $\langle z \rangle$ for the principal ideal of the ring $\mathbb{R}[\cos(t), \sin(t)]$ generated by z.

The irreducible factors of real trigonometric polynomials without real zeros are characterized by the following proposition.

Proposition A.1. *Let $P(t) \in \mathbb{R}[\cos(t), \sin(t)]$ be non-zero and non-unit. The following statements are equivalent.*

1. *$P(t) \neq 0$ for all $t \in \mathbb{R}$.*

2. *Any irreducible factor of $P(t)$ can be written (up to units) in the form $a\cos(t) + b\sin(t) + c$ where $a, b, c \in \mathbb{R}$, $(a, b) \neq (0, 0)$ and $c^2 > a^2 + b^2$.*

3. *If z is an irreducible factor of $P(t)$, then $\langle z\rangle$ is a maximal ideal of $\mathbb{R}[\cos(t), \sin(t)]$.*

4. *Every maximal ideal of $\mathbb{R}[\cos(t), \sin(t)]$ containing $\langle P(t)\rangle$ is principal.*

Proof. (1) $\iff$ (2). Let $z_i \in \mathbb{R}[\cos(t), \sin(t)]$, $i = 1, \ldots, n$, be irreducible elements such that $P = u\, z_1 \cdots z_n$ for some $u \in \mathbb{R}$. Obviously, $P(t) \neq 0$ for all $t \in \mathbb{R}$ if and only if $z_i(t) \neq 0$ for all $t \in \mathbb{R}$ and all $i \in \{1, \ldots, n\}$. Since irreducible elements in $\mathbb{R}[\cos(t), \sin(t)]$ have the form $a\cos(t) + b\sin(t) + c$ with $a, b, c \in \mathbb{R}$ and $(a, b) \neq (0, 0)$, and $a\cos(t) + b\sin(t) + c$ has no real zeros if and only if $c^2 > a^2 + b^2$, we are done.
(2) $\iff$ (3). Let $z = a\cos(t) + b\sin(t) + c \in \mathbb{R}[\cos(t), \sin(t)]$ with $(a, b) \neq (0, 0)$. Since, by [14, Theorem 3.8], $\langle z\rangle$ is a maximal ideal if and only if $c^2 > a^2 + b^2$, we have the desired equivalence.
(3) $\implies$ (4). Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{R}[\cos(t), \sin(t)]$ such that $P(t) \in \mathfrak{m}$. Since $\mathfrak{m}$ is a prime ideal, there exists an irreducible factor z of $P(t)$ such that $z \in \mathfrak{m}$; equivalently, $\langle z\rangle \subseteq \mathfrak{m}$. From the maximality of $\langle z\rangle$ follows that $\mathfrak{m} = \langle z\rangle$.
(4) $\implies$ (3). Let z be an irreducible factor of $P(t)$ and let $\mathfrak{m}$ be a maximal ideal of the ring $\mathbb{R}[\cos(t), \sin(t)]$ containing $\langle z\rangle$. Since $\mathfrak{m}$ is principal, there exists $w \in \mathbb{R}[\cos(t), \sin(t)]$ such that $\mathfrak{m} = \langle w\rangle$; in particular, w divides z and the irreducibility of z implies $\langle z\rangle = \mathfrak{m}$. □

Corollary A.2. *Let $P(t) \in \mathbb{R}[\cos(t), \sin(t)]$ be non-zero and non-unit. If $P(t) \neq 0$ for all $t \in \mathbb{R}$, then $P(t)$ has a unique factorization except for order of factors or product by units.*

Proof. Let $u\, z_1 \cdots z_n = v\, w_1 \cdots w_n$ be two factorizations of P into irreducibles, z_i, w_i, $i = 1, \ldots, n$, for some $u, v \in \mathbb{R}$. Since, by Proposition A.1, $\langle w_1\rangle$ is maximal and $u\, z_1 \cdots z_n = P(t) \in \langle w_1\rangle$, we have that there exists j such that $z_j \in \langle w_1\rangle$. So it follows from the irreducibility of z_j that $z_j = u_1 w_1$ for some $u_1 \in \mathbb{R}$. Now it is sufficient to repeat the same argument with $u u_1\, z_1 \cdots z_{j-1} z_{j+1} \cdots z_n = v\, w_2 \cdots w_n$, and so on, to get the desired result. □

Clearly, the converse of the previous corollary is not true, since there are many real irreducible trigonometric polynomials with real zeros.

Corollary A.3. *Let $P(t), H(t)$ and $R(t) \in \mathbb{R}[\cos(t), \sin(t)]$ be non-zero and non-units. If $P(t) \neq 0$, for every $t \in \mathbb{R}$, $P(t) = H(t)R(t)$ and no irreducible factor of $P(t)$ divides $H(t)$, then $P(t)$ divides $R(t)$.*

Proof. By Corollary A.2, there exist unique irreducible real trigonometric polynomials $z_1, \ldots, z_n$ such that $P(t) = u\, z_1 \cdots z_n$ for some $u \in \mathbb{R}$. If z_1 is an irreducible factor of $P(t)$, then $H(t)R(t) = P(t) \in \langle z_1\rangle$. By Proposition A.1, $\langle z_1\rangle$ is maximal. Therefore, $H(t) \in \langle z_1\rangle$ or $R(t) \in \langle z_1\rangle$. However, since no irreducible factor of $P(t)$ divides $H(t)$, we conclude that $R(t) \in \langle z_1\rangle$ and therefore $R(t) = \tilde{R}(t) z_1$ for some $\tilde{R}(t) \in \mathbb{R}[\cos(t), \sin(t)]$. Now, if we repeat the same argument with $\tilde{P}(t) = u\, z_2 \cdots z_n$, $\tilde{R}(t)$ and z_2, and so on, we get the desired result. □

Now, it is convenient to recall that $\mathbb{C}[\cos(t), \sin(t)]$ is an Euclidean domain (see [14, Theorem 2.1]. In particular, it is a unique factorization domain.

Corollary A.4. *Let $P(t)$ and $Q(t) \in \mathbb{R}[\cos(t), \sin(t)]$ be non-zero and non-units. If $P(t) \neq 0$, for all $t \in \mathbb{R}$, then $P(t)$ and $Q(t)$ are coprime in $\mathbb{C}[\cos(t), \sin(t)]$ if and only if they have no common irreducible factors in $\mathbb{R}[\cos(t), \sin(t)]$.*

Proof. If $P(t)$ and $Q(t)$ have common irreducible factors in $\mathbb{R}[\cos(t), \sin(t)]$, then they have common irreducible factors in $\mathbb{C}[\cos(t), \sin(t)]$.

Conversely, let us suppose that $P(t)$ and $Q(t)$ have no common irreducible factors in $\mathbb{R}[\cos(t), \sin(t)]$. If $z \in \mathbb{C}[\cos(t), \sin(t)]$ is an irreducible factor of $P(t)$ and $Q(t)$, then $P(t)$ and $Q(t)$ belong to $\langle z \rangle \cap \mathbb{R}[\cos(t), \sin(t)]$. Thus there exists a maximal ideal $\mathfrak{m}$ of $\mathbb{R}[\cos(t), \sin(t)]$ such that $P(t) \in \mathfrak{m}$ and $Q(t) \in \mathfrak{m}$. Since, by Proposition A.1, $\mathfrak{m}$ is principal, we conclude that, contrary to the hypothesis, $P(t)$ y $Q(t)$ have a real common factor. □

Note that for the above corollary to hold, the condition $P(t) \neq 0$, for all $t \in \mathbb{R}$, is mandatory.

Example A.5. The trigonometric polynomials $P(t) = \sqrt{2}\sin(t) - 1$ and $Q(t) = -\sqrt{2}\cos(t) + 1$ are irreducible in $\mathbb{R}[\cos(t), \sin(t)]$ and their respective factorizations in $\mathbb{C}[\cos(t), \sin(t)]$ are

$$\left(\frac{(i-1)\sin(t) + (i-1)\cos(t) - \sqrt{2}i}{2}\right)\left(i\sin(t) + \cos(t) - \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

and

$$-\left(\frac{(i+1)\sin(t) + (i-1)\cos(t) + \sqrt{2}}{2}\right)\left(i\sin(t) + \cos(t) - \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right).$$

Note that they both have the same last irreducible complex factor.

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Existence of nontrivial solutions for a quasilinear Schrödinger–Poisson system in $\mathbb{R}^3$ with periodic potentials

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Abstract. In this paper, we study the following quasilinear Schrödinger–Poisson system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4\Delta_4\phi = \lambda u^2, & x \in \mathbb{R}^3, \end{cases}$$

where λ and ε are positive parameters, $\Delta_4 u = \operatorname{div}(|\nabla u|^2\nabla u)$, V is a continuous and periodic potential function with positive infimum, $f(x,t) \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ is periodic with respect to x and only needs to satisfy some superquadratic growth conditions with respect to t. One nontrivial solution is obtained for λ small enough and ε fixed by a combination of variational methods and truncation technique.

Keywords: quasilinear Schrödinger–Poisson system, periodic potential, variational methods, truncation technique, nontrivial solution.

2020 Mathematics Subject Classification: 35B38, 35D30, 35J50.

1 Introduction and main result

In this paper, we consider the following system

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4\Delta_4\phi = \lambda u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where λ and ε are positive parameters, $\Delta_4 u = \operatorname{div}(|\nabla u|^2\nabla u)$, V is a continuous and periodic potential function with positive infimum, f is a continuous function defined on $\mathbb{R}^3 \times \mathbb{R}$ which is periodic with respect to the first variable and satisfies some superquadratic growth conditions with respect to the second variable. Precisely, we assume that

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ with $\inf_{x\in\mathbb{R}^3} V(x) = V_0 > 0$ and it is a 1-periodic potential function, that is,

$$V(x+y) = V(x), \quad \text{for every } x \in \mathbb{R}^3 \text{ and } y \in \mathbb{Z}^3.$$

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(f_1) f is 1-periodic with respect to x. There exist positive constants C and $p \in (2,6)$ such that
$$|f(x,t)| \leq C(1+|t|^{p-1}), \quad \text{for } (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

(f_2) $\lim_{|t|\to 0} \frac{f(x,t)}{t} = 0$, uniformly for $x \in \mathbb{R}^3$.

(f_3) There exist $\alpha \in (2,6)$ and $R > 0$ such that
$$\inf_{x\in\mathbb{R}^3,\ |t|\geq R} F(x,t) > 0,\ f(x,t)t \geq \alpha F(x,t), \quad \text{for } (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$
where $F(x,t) = \int_0^t f(x,s)ds$.

This class of system appears by studying a quantum mechanical model of extremely small devices in semiconductor nanostructures taking into account quantum structure and the longitudinal field oscillations during the beam propagation, for more details on the physical background of this class of system see [19]. Although this class of system has been well-known among the physicists, it has never been considered before [12,13] in the mathematical literature. One of them is something of type
$$\begin{cases} -\Delta u + \omega u + (\phi + \widetilde{\phi})u = 0, \\ -\Delta\phi - \varepsilon^4 \Delta_4 \phi = u^2 - n^*, \end{cases} \tag{1.2}$$
where u and ϕ represent the modulus of the wave function and the electrostatic potential respectively, n^* and $\widetilde{\phi}$ are given data of the problem which represent respectively the dopant density and the effective external potential. System (1.2) with some periodicity conditions was studied in [12] by the Krasnoselskii genus. Under minimal summability conditions on the data n^* and $\widetilde{\phi}$, existence of ground state solutions for system (1.2) was proved in [3] by means of minimization techniques, and the behaviour of these solutions whenever $\varepsilon \to 0^+$ was studied: these solutions converge to a ground state solution of Schrödinger–Poisson system associated with $\varepsilon = 0$ in system (1.2). A quasilinear Schrödinger–Poisson system in the unitary cube under periodic boundary conditions was studied in [13], the global existence and uniqueness of solution was obtained by using Galerkin scheme. There are also some studies on quasilinear Schrödinger–Poisson system with nonlinearities by variational methods. In [8], a class of quasilinear Schrödinger–Poisson system with an asymptotically linear term was studied, the existence and behaviour of ground state solutions as $\varepsilon \to 0^+$ were given. Recently, [11] studied the existence and asymptotic behaviour of solutions for a class of quasilinear Schrödinger–Poisson system with a critical nonlinearity combining with a 4-suplinear nonlinearity. Similar results were obtained in [10] in the two-dimensional case. In [21], we also got the existence and asymptotic behaviour of solutions for a class of quasilinear Schrödinger–Poisson system with coercive potential by variational methods and a truncation technique.

Formally, system (1.1) is the well-known Schrödinger–Poisson system if $\varepsilon = 0$ which has been given extensive attention and research in the last few decades. We mention that a reduction procedure for this class of system was proposed in [2] and an eigenvalue problem in bounded domains was considered. Schrödinger–Poisson system with general nonlinearity was first studied in [6] and later studied in many literatures, see for example [1,5,7,17,20,23] and the references therein. More recently, [9] studied the following quasilinear elliptic system by variational methods
$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \lambda\phi|u|^{p-2}u = |u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = |u|^p, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $1 < p < 3$, $p < q < \frac{3p}{3-p}$, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ and $\lambda > 0$ is a parameter. The existence of nontrivial solutions for system (1.3) is obtained by the Mountain Pass theorem. According to the range of q, the scaling technique [14] and the truncation technique [15] were used to obtain a bounded Palais–Smale sequence respectively in [9].

From a mathematical point of view, on the one hand, the main novelty of system (1.1) is that the equation of the electrostatic potential in the system is not linear, that is, it is not the classical Poisson equation. Contrast to the classical Poisson equation or the second equation in system (1.3), the solution of the second equation in system (1.1) has neither an explicit formula nor homogeneity properties. It leads to that the scaling technique [14] is no longer applicable. It is natural to ask whether the truncation technique [15] can be used to deal with system (1.1), especially for the case $\alpha \in (2, 4]$. One the other hand, since under our assumptions there is no compact embedding between the main working spaces, we can not prove that the variational functional associated to system (1.1) satisfies (PS) condition directly. Lions vanishing lemma[18] will be applied to prove that system (1.1) enjoys at least one nontrivial solutions whenever the positive parameter λ is small enough. In this process, the weak convergence property of the solutions for the second equation in system (1.1) plays an important role. However, due to the "bad" properties of those solutions, this weak convergence property of them is not apparent. We will follow the arguments of [4, 9], together with the uniqueness of the solution for the second equation in system (1.1), to solve this key technique problem, for more details, see Lemma 2.2.

Before stating our main result, we give several notations. For any $q \in [1, +\infty]$, we denote by $|\cdot|_q$ the norm of the Lebesgue space $L^q(\mathbb{R}^3)$. $D^{1,2}(\mathbb{R}^3)$ is the Hilbert space defined as the completion of the test functions $C_0^\infty(\mathbb{R}^3)$ with respect to the L^2 norm of the gradient. We denote by X the completion of the functions $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $|\nabla \cdot|_2 + |\nabla \cdot|_4$, which is a reflexive Banach space. Under assumption (V), let $H_V^1(\mathbb{R}^3)$ be $H^1(\mathbb{R}^3)$ equipped with the following norm and inner product

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\, dx \right)^{\frac{1}{2}}, \qquad (u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv) dx.$$

Assumption (V) also guarantees the continuous embedding from $H_V^1(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$, $q \in [2, 6]$ and local compact embedding from $H_V^1(\mathbb{R}^3)$ to $L^q_{loc}(\mathbb{R}^3)$, $q \in [1, 6)$.

As usual, a weak solution for system (1.1) is a pair $(u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}) \in H_V^1(\mathbb{R}^3) \times X$ such that

$$\begin{cases} \int_{\mathbb{R}^3} (\nabla u_{\lambda,\varepsilon} \nabla v + V(x) u_{\lambda,\varepsilon} v + \lambda \phi_{\lambda,\varepsilon} u_{\lambda,\varepsilon} v) dx = \int_{\mathbb{R}^3} f(x, u_{\lambda,\varepsilon}) v dx, & v \in H_V^1(\mathbb{R}^3), \\ \int_{\mathbb{R}^3} (\nabla \phi_{\lambda,\varepsilon} + \varepsilon^4 |\nabla \phi_{\lambda,\varepsilon}|^2 \nabla \phi_{\lambda,\varepsilon}) \nabla \varphi dx = \lambda \int_{\mathbb{R}^3} u_{\lambda,\varepsilon}^2 \varphi dx, & \varphi \in X. \end{cases}$$

Our main result is as follows.

Theorem 1.1. *Under the assumptions (V) and (f_1)–(f_3), there exists $\lambda_0 > 0$ such that system (1.1) has at least one nontrivial solution $(u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}) \in H_V^1(\mathbb{R}^3) \times X$ for all $(\lambda, \varepsilon) \in (0, \lambda_0) \times (0, \infty)$. Moreover, $\phi_{\lambda,\varepsilon}$ is nonnegative.*

Remark 1.2. Compared with our last result in [21], the main difficulty here is the lack of compactness. In particular, the weak convergence property of the solutions for the second equation in system (1.1) is the key to obtaining a nontrivial solution for system (1.1).

Remark 1.3. The constraint on λ is mainly used to guarantee the variational functional associated to system (1.1) enjoys a (PS) sequence with a prior bound. If $\alpha \in (4, 6)$, due to (i)

of Lemma 2.2, it is easy to obtain a bounded (PS) sequence of the variational functional associated to system (1.1) with $(\lambda, \varepsilon) \in (0, \infty) \times (0, \infty)$ by using standard methods. Thus, the constraint that $\lambda < \lambda_0$ can be got rid of in this case. We leave details of the proof to the interested readers.

Throughout the paper, we denote C_q the constant of Sobolev imbedding from $H^1_V(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ for $q \in [2,6]$. $S = \inf_{\varphi \in D^{1,2}(\mathbb{R}^3))\setminus\{0\}} \frac{|\nabla \varphi|_2^2}{|\varphi|_6^2}$ is the optimal constant in the Sobolev inequality. The rest of the paper is organized as follows. We give some preliminaries in Section 2. The proof of Theorem 1.1 is given in Section 3.

2 Preliminaries

First, under our assumptions, system (1.1) has a variational structure. Formally, its corresponding functional is defined by

$$\mathcal{J}_{\lambda,\varepsilon}(u,\phi) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{2}\int_{\mathbb{R}^3} \phi u^2 dx - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla \phi|^2 dx - \frac{\varepsilon^4}{8}\int_{\mathbb{R}^3} |\nabla \phi|^4 dx - \int_{\mathbb{R}^3} F(x,u) dx.$$

It is not difficult to see that the critical points of $\mathcal{J}_{\lambda,\varepsilon}$ are the weak solutions of system (1.1). Since the functional $\mathcal{J}_{\lambda,\varepsilon}$ is strongly indefinite, the reduction procedure which is successfully used to study the classical Schrödinger–Poisson system will be applied to deal with system (1.1). Similar to Lemma 2.1 of [21] or Lemma 2.2 of [8], we have the following result.

Lemma 2.1. *For any $u \in H^1_V(\mathbb{R}^3)$ and $\lambda, \varepsilon > 0$, there exists a unique nonnegative weak solution $\phi_{\lambda,\varepsilon}(u) \in X$ for*

$$-\Delta \phi - \varepsilon^4 \Delta_4 \phi = \lambda u^2, \qquad x \in \mathbb{R}^3. \tag{2.1}$$

That is, for any $\varphi \in X$, we have

$$\int_{\mathbb{R}^3} (\nabla \phi_{\lambda,\varepsilon}(u) + \varepsilon^4 |\nabla \phi_{\lambda,\varepsilon}(u)|^2 \nabla \phi_{\lambda,\varepsilon}(u)) \nabla \varphi dx = \lambda \int_{\mathbb{R}^3} u^2 \varphi dx.$$

Next, we give some properties of the weak solution $\phi_{\lambda,\varepsilon}(u)$ for equation (2.1).

Lemma 2.2. *For every $\lambda, \varepsilon > 0$, $\phi_{\lambda,\varepsilon}(u)$ enjoys the following properties.*

(i) For every $u \in H^1_V(\mathbb{R}^3)$,

$$|\nabla \phi_{\lambda,\varepsilon}(u)|_2^2 + \varepsilon^4 |\nabla \phi_{\lambda,\varepsilon}(u)|_4^4 = \lambda \int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(u) u^2 dx \le \lambda^2 S^{-1} C^4_{\frac{12}{5}} \|u\|^4;$$

(ii) if $\{u_n\}$ is bounded in $H^1_V(\mathbb{R}^3)$, then there exist a subsequence still denoted by $\{u_n\}$ and $u \in H^1_V(\mathbb{R}^3)$ such that

$$\phi_{\lambda,\varepsilon}(u_n) \rightharpoonup \phi_{\lambda,\varepsilon}(u) \text{ in } X, \int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(u_n) u_n v dx \to \int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(u) u v dx, \quad \text{for } v \in H^1_V(\mathbb{R}^3);$$

(iii) $\phi_{\lambda,\varepsilon}(u_y)(\cdot) = \phi_{\lambda,\varepsilon}(u)(\cdot + y)$, for every $y \in \mathbb{R}^3$, where $u_y(\cdot) = u(\cdot + y)$.

Proof. (1) By the definition of $\phi_{\lambda,\varepsilon}(u)$, the first equality in (i) is true. Then by the Hölder inequality and the Sobolev embedding theorem, we can get that the first conclusion is true.

(2) Since $\{u_n\}$ is bounded in $H^1_V(\mathbb{R}^3)$, going if necessary to a subsequence, there exists $u \in H^1_V(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H^1_V(\mathbb{R}^3)$. By the Sobolev embedding theorem and the local compact embedding theorem, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u && \text{in } L^p(\mathbb{R}^3),\ p \in [2,6];\\ u_n &\to u && \text{in } L^p_{loc}(\mathbb{R}^3),\ p \in [1,6);\\ u_n(x) &\to u(x), && a.e.\ x \in \mathbb{R}^3. \end{aligned}$$

Since (i) leads to that $\{\phi_{\lambda,\varepsilon}(u_n)\}$ is bounded in X, going if necessary to a subsequence, there exists $\phi_{\lambda,\varepsilon} \in X$ such that

$$\phi_{\lambda,\varepsilon}(u_n) \rightharpoonup \phi_{\lambda,\varepsilon} \quad \text{in } X \quad (\text{which is also valid in } D^{1,2}(\mathbb{R}^3)).$$

Furthermore, we can also assume that

$$\begin{aligned} \phi_{\lambda,\varepsilon}(u_n) &\rightharpoonup \phi_{\lambda,\varepsilon} && \text{in } L^6(\mathbb{R}^3);\\ \phi_{\lambda,\varepsilon}(u_n) &\to \phi_{\lambda,\varepsilon} && \text{in } L^p_{loc}(\mathbb{R}^3),\ p \in [1,6);\\ \phi_{\lambda,\varepsilon}(u_n)(x) &\to \phi_{\lambda,\varepsilon}(x), && a.e.\ x \in \mathbb{R}^3. \end{aligned}$$

On the one hand, by Lemma 2.1, we have

$$\int_{\mathbb{R}^3} (\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\varphi + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\varphi)dx = \lambda\int_{\mathbb{R}^3} u_n^2\varphi dx, \tag{2.2}$$

and

$$\int_{\mathbb{R}^3} (\nabla\phi_{\lambda,\varepsilon}(u)\nabla\varphi + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u)|^2\nabla\phi_{\lambda,\varepsilon}(u)\nabla\varphi)dx = \lambda\int_{\mathbb{R}^3} u^2\varphi dx, \quad \text{for } \varphi \in X. \tag{2.3}$$

Set $\varphi = (\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R$ in (2.2), where $\psi_R \in C_0^\infty(\mathbb{R}^3,\ [0,1])$ is a cut-off function such that $\psi_R|_{B_R(0)} = 1$, $\operatorname{supp}\psi_R \subset B_{2R}(0)$ and $|\nabla\psi_R| \le \frac{2}{R}$, we can get

$$\begin{aligned} 0 = &\int_{\mathbb{R}^3} (1+\varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2)\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx\\ &+ \int_{\mathbb{R}^3} (1+\varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2)\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx\\ &- \lambda\int_{\mathbb{R}^3} u_n^2(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx. \end{aligned} \tag{2.4}$$

On the other hand, by the definition of weak convergence in X, we have

$$\int_{\mathbb{R}^3} (1+\varepsilon^4|\nabla\phi_{\lambda,\varepsilon}|^2)\nabla\phi_{\lambda,\varepsilon}\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx \to 0, \quad \text{as } n\to\infty.$$

The local compact embedding theorem implies that

$$\int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0, \qquad \int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}|^2\nabla\phi_{\lambda,\varepsilon}\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0,$$

as $n\to\infty$. Then

$$\begin{aligned} o_n(1) = &\int_{\mathbb{R}^3} (1+\varepsilon^4|\nabla\phi_{\lambda,\varepsilon}|^2)\nabla\phi_{\lambda,\varepsilon}\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx\\ &+ \int_{\mathbb{R}^3} (1+\varepsilon^4|\nabla\phi_{\lambda,\varepsilon}|^2)\nabla\phi_{\lambda,\varepsilon}\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx. \end{aligned} \tag{2.5}$$

By calculating (2.4) minus (2.5), we deduce that

$$\begin{aligned} o_n(1) = & \int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u_n) - \nabla\phi_{\lambda,\varepsilon}|^2\psi_R dx \\ & + \varepsilon^4 \int_{\mathbb{R}^3} (|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n) - |\nabla\phi_{\lambda,\varepsilon}|^2\nabla\phi_{\lambda,\varepsilon})\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx \\ & + \int_{\mathbb{R}^3} (\nabla\phi_{\lambda,\varepsilon}(u_n) - \nabla\phi_{\lambda,\varepsilon})\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \\ & + \varepsilon^4 \int_{\mathbb{R}^3} (|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n) - |\nabla\phi_{\lambda,\varepsilon}|^2\nabla\phi_{\lambda,\varepsilon})\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \\ & - \lambda \int_{\mathbb{R}^3} u_n^2(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx. \end{aligned} \tag{2.6}$$

Since $\phi_{\lambda,\varepsilon}(u_n) \to \phi_{\lambda,\varepsilon}$ in $L^p_{loc}(\mathbb{R}^3),\ p \in [1,6)$, by the Hölder inequality and the definition of ψ_R, we can get that

$$\int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0, \quad \text{as } n \to \infty, \tag{2.7}$$

$$\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0, \quad \text{as } n \to \infty. \tag{2.8}$$

In fact, by the Hölder inequality, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \right| & \leq C|\nabla\phi_{\lambda,\varepsilon}(u_n)|_2 \left(\int_{B_{2R}(0)} |\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon}|^2 dx \right)^{\frac{1}{2}} \\ & \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \right| \\ & \leq C|\nabla\phi_{\lambda,\varepsilon}(u_n)|_4^3 \left(\int_{B_{2R}(0)} |\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon}|^4 dx \right)^{\frac{1}{4}} \\ & \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

Similarly, we can also get that

$$\int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0, \ \int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}|^2\nabla\phi_{\lambda,\varepsilon}\nabla\psi_R(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})dx \to 0,$$

and

$$\int_{\mathbb{R}^3} u_n^2(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})\psi_R dx \to 0, \quad \text{as } n \to \infty.$$

Thus, it follows from (2.6)–(2.8) that

$$\begin{aligned} o_n(1) = & \int_{\mathbb{R}^3} [\varepsilon^4(|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2\nabla\phi_{\lambda,\varepsilon}(u_n) - |\nabla\phi_{\lambda,\varepsilon}|^2\nabla\phi_{\lambda,\varepsilon})\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon}) \\ & + |\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})|^2]\psi_R dx. \end{aligned}$$

Then the Simon inequality leads to that

$$\int_{\mathbb{R}^3} (|\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})|^2 + \varepsilon^4|\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})|^4)\psi_R dx \to 0, \quad \text{as } n \to \infty.$$

Thus,

$$\int_{B_R(0)} |\nabla(\phi_{\lambda,\varepsilon}(u_n) - \phi_{\lambda,\varepsilon})|^2 dx \to 0, \quad \text{as } n \to \infty.$$

Up to a subsequence, we have

$$\nabla\phi_{\lambda,\varepsilon}(u_n)(x) \to \nabla\phi_{\lambda,\varepsilon}(x), \quad \text{a.e. } x \in B_R(0), \text{ as } n \to \infty.$$

The arbitrariness of R implies that, going to a subsequence,

$$\nabla\phi_{\lambda,\varepsilon}(u_n)(x) \to \nabla\phi_{\lambda,\varepsilon}(x), \quad \text{a.e. } x \in \mathbb{R}^3, \text{ as } n \to \infty.$$

The boundedness of $\{|\nabla\phi_{\lambda,\varepsilon}(u_n)|\}$ in $L^4(\mathbb{R}^3)$ ensures that $\{|\nabla\phi_{\lambda,\varepsilon}(u_n)|^3\}$ is also bounded in $L^{\frac{4}{3}}(\mathbb{R}^3)$. Thus, it follows from [22, Proposition 5.4.7] that

$$|\nabla\phi_{\lambda,\varepsilon}(u_n)|^2 D_i\phi_{\lambda,\varepsilon}(u_n) \rightharpoonup |\nabla\phi_{\lambda,\varepsilon}|^2 D_i\phi_{\lambda,\varepsilon} \quad \text{in } L^{\frac{4}{3}}(\mathbb{R}^3),\ i = 1,2,3.$$

Therefore, for every $\varphi \in X$,

$$\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u_n)|^2 D_i\phi_{\lambda,\varepsilon}(u_n) D_i\varphi dx \to \int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}|^2 D_i\phi_{\lambda,\varepsilon} D_i\varphi dx, \qquad i = 1,2,3.$$

Then

$$\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u_n)|^2 \nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\varphi dx \to \int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}|^2 \nabla\phi_{\lambda,\varepsilon}\nabla\varphi dx, \quad \text{as } n \to \infty.$$

It follows from $\phi_{\lambda,\varepsilon}(u_n) \rightharpoonup \phi_{\lambda,\varepsilon}$ in $D^{1,2}(\mathbb{R}^3)$ that

$$\int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}(u_n)\nabla\varphi dx \to \int_{\mathbb{R}^3} \nabla\phi_{\lambda,\varepsilon}\nabla\varphi dx, \quad \text{as } n \to \infty.$$

Since $\varphi \in L^6(\mathbb{R}^3)$ and $u_n^2 \rightharpoonup u^2$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$ by [22, Proposition 5.4.7], we have

$$\int_{\mathbb{R}^3} u_n^2\varphi dx \to \int_{\mathbb{R}^3} u^2\varphi dx, \quad \text{as } n \to \infty.$$

Therefore, by taking limits as $n \to \infty$ on both sides of (2.2), we can obtain that

$$\int_{\mathbb{R}^3} (\nabla\phi_{\lambda,\varepsilon}\nabla\varphi + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}|^2\nabla\phi_{\lambda,\varepsilon}\nabla\varphi)dx = \lambda\int_{\mathbb{R}^3} u^2\varphi dx, \quad \text{for } \varphi \in X.$$

The uniqueness of solution for equation (2.1) with given u and (2.3) result that $\phi_{\lambda,\varepsilon} = \phi_{\lambda,\varepsilon}(u)$.

By [22, Proposition 5.4.7] again, we can get that $\phi_{\lambda,\varepsilon}(u_n)u_n \rightharpoonup \phi_{\lambda,\varepsilon}(u)u$ in $L^{\frac{3}{2}}(\mathbb{R}^3)$. Then for every $v \in H_V^1(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(u_n)u_n v dx \to \int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(u)uv dx, \quad \text{as } n \to \infty.$$

(3) The uniqueness of solution for equation (2.1) and the translation invariance of Lebesgue integral on $\mathbb{R}^3$ also guarantee that (iii) is true. In fact, for every $\varphi \in X$ and $y \in \mathbb{R}^3$,

$$\int_{\mathbb{R}^3} (1 + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u)(x)|^2)\nabla\phi_{\lambda,\varepsilon}(u)(x)\nabla\varphi(x-y)dx = \lambda\int_{\mathbb{R}^3} u^2(x)\varphi(x-y)dx.$$

By the translation invariance of Lebesgue integral on $\mathbb{R}^3$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} (1 + \varepsilon^4|\nabla\phi_{\lambda,\varepsilon}(u)(x+y)|^2)\nabla\phi_{\lambda,\varepsilon}(u)(x+y)\nabla\varphi(x)dx \\
&\quad = \lambda\int_{\mathbb{R}^3} u^2(x+y)\varphi(x)dx \\
&\quad = \lambda\int_{\mathbb{R}^3} u_y^2(x)\varphi(x)dx.
\end{aligned}$$

The uniqueness of solution for equation (2.1) leads to $\phi_{\lambda,\varepsilon}(u_y)(\cdot) = \phi_{\lambda,\varepsilon}(u)(\cdot + y)$. □

As shown in [11], the functional

$$\begin{aligned} J_{\lambda,\varepsilon}(u) &:= \mathcal{J}_{\lambda,\varepsilon}(u, \phi_{\lambda,\varepsilon}(u)) \\ &= \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u)|^2 dx + \frac{3\varepsilon^4}{8}\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u)|^4 dx - \int_{\mathbb{R}^3} F(x,u)dx, \quad u \in H^1_V(\mathbb{R}^3) \end{aligned}$$

is of class C^1. Its Fréchet derivative at $u \in H^1_V(\mathbb{R}^3)$ is given by

$$\begin{aligned} \langle J'_{\lambda,\varepsilon}(u), v\rangle &= \langle \partial_u \mathcal{J}_{\lambda,\varepsilon}(u, \phi_{\lambda,\varepsilon}(u)), v\rangle \\ &= \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv + \lambda\phi_{\lambda,\varepsilon}(u)uv)dx - \int_{\mathbb{R}^3} f(x,u)vdx. \end{aligned}$$

Lemma 2.3 ([11, Lemma 4])**.** *Let $\lambda, \varepsilon > 0$ be fixed, the following statements are equivalent:*

(i) the pair $(u_{\lambda,\varepsilon},\ \phi_{\lambda,\varepsilon}) \in H^1_V(\mathbb{R}^3) \times X$ is a critical point of $\mathcal{J}_{\lambda,\varepsilon}$;

(ii) $u_{\lambda,\varepsilon} \in H^1_V(\mathbb{R}^3)$ is a critical point of $J_{\lambda,\varepsilon}$ and $\phi_{\lambda,\varepsilon} = \phi_{\lambda,\varepsilon}(u_{\lambda,\varepsilon})$.

For convenience, we set the functional

$$I_{\lambda,\varepsilon}(u) = \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u)|^2 dx + \frac{3\varepsilon^4}{8}\int_{\mathbb{R}^3} |\nabla\phi_{\lambda,\varepsilon}(u)|^4 dx, \qquad u \in H^1_V(\mathbb{R}^3).$$

It follows from [3, Proposition 4.1] that $I_{\lambda,\varepsilon} \in C^1(H^1_V(\mathbb{R}^3), \mathbb{R})$ and

$$\langle I'_{\lambda,\varepsilon}(u), v\rangle = \lambda \int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(u)uvdx, \quad \text{for } v \in H^1_V(\mathbb{R}^3).$$

In such a way, $J_{\lambda,\varepsilon}$ can be rewritten as

$$J_{\lambda,\varepsilon}(u) = \frac{1}{2}\|u\|^2 + I_{\lambda,\varepsilon}(u) - \int_{\mathbb{R}^3} F(x,u)dx.$$

In view of the above facts, in order to obtain a weak solution for system (1.1), it is sufficient to find a critical point of the functional $J_{\lambda,\varepsilon}$ in $H^1_V(\mathbb{R}^3)$.

3 Proof of our main result

In this section, we complete the proof of our main result. It is a difficult task to get a bounded Palais–Smale sequence for the functional $J_{\lambda,\varepsilon}$ directly due to the presence of nonlocal term for the case $\alpha \in (2,4]$ in (f_3). We use a truncation method which has been widely used [1,9,15–17] to deal with it. Precisely, we define a truncation for the functional $J_{\lambda,\varepsilon}$ in the following way. Let $\chi \in C^\infty([0,+\infty),[0,1])$ satisfy

$$\begin{cases} \chi(s) = 1,\ s \in [0,1], \\ 0 \le \chi(s) \le 1,\ s \in (1,2), \\ \chi(s) = 0,\ s \in [2,+\infty), \\ -2 \le \chi'(s) \le 0. \end{cases}$$

For each $T > 0$, we define $h_T(u) = \chi(\frac{\|u\|^2}{T^2})$ for $u \in H^1_V(\mathbb{R}^3)$ and the truncated functional

$$J^T_{\lambda,\varepsilon}(u) = \frac{1}{2}\|u\|^2 + h_T(u)I_{\lambda,\varepsilon}(u) - \int_{\mathbb{R}^3} F(x,u)dx. \tag{3.1}$$

The functional $J^T_{\lambda,\varepsilon} \in C^1(H^1_V(\mathbb{R}^3), \mathbb{R})$ with Fréchet derivative at u given by

$$\langle {J^T_{\lambda,\varepsilon}}'(u), v\rangle = \left(1 + \frac{2}{T^2}\chi'(\frac{\|u\|^2}{T^2})I_{\lambda,\varepsilon}(u)\right)\int_{\mathbb{R}^3}(\nabla u\nabla v + V(x)uv)dx + \lambda h_T(u)\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u)uvdx - \int_{\mathbb{R}^3} f(x,u)vdx,\ v \in H^1_V(\mathbb{R}^3).$$

Then $u_{\lambda,\varepsilon} \in H^1_V(\mathbb{R}^3)$ is a critical point of $J^T_{\lambda,\varepsilon}$ if and only if $(u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}(u_{\lambda,\varepsilon})) \in H^1_V(\mathbb{R}^3) \times X$ is a weak solution of

$$\begin{cases} \left(1 + \frac{2}{T^2}\chi'(\frac{\|u\|^2}{T^2})I_{\lambda,\varepsilon}(u)\right)(-\Delta u + V(x)u) + \lambda h_T(u)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi - \varepsilon^4\Delta_4\phi = \lambda u^2, & x \in \mathbb{R}^3. \end{cases}$$

From the definition of χ, for given T, we have

$$J^T_{\lambda,\varepsilon}(u) = J_{\lambda,\varepsilon}(u) \quad \text{and} \quad {J^T_{\lambda,\varepsilon}}'(u) = J'_{\lambda,\varepsilon}(u), \quad \text{if } \|u\| \le T.$$

Thus, if $\{u_n\}$ is a (PS) sequence of $J^T_{\lambda,\varepsilon}$ with $\|u_n\| \le T$, then it is also a bounded (PS) sequence of $J_{\lambda,\varepsilon}$.

We firstly prove that the truncated functional $J^T_{\lambda,\varepsilon}$ enjoys the mountain pass geometry structure.

Lemma 3.1. *For every fixed $(\lambda,\varepsilon) \in (0,\infty)\times(0,\infty)$, there exist $\rho > 0$ and $e_T \in H^1_V(\mathbb{R}^3)$ such that $\|e_T\| > \rho$ and*

$$\inf_{u\in H^1_V(\mathbb{R}^3),\ \|u\|=\rho} J^T_{\lambda,\varepsilon}(u) > J^T_{\lambda,\varepsilon}(0) = 0 > J^T_{\lambda,\varepsilon}(e_T).$$

Proof. On the one hand, it follows from (f_1) and (f_2) that there exists $a_1 > 0$ such that

$$|f(x,t)| \le \frac{V_0}{2}|t| + a_1|t|^{p-1},\ |F(x,t)| \le \frac{V_0}{4}t^2 + \frac{a_1}{p}|t|^p, \quad \text{for } (x,t) \in \mathbb{R}^3\times\mathbb{R}. \tag{3.2}$$

Then (3.1) and (3.2) imply that

$$\begin{aligned} J^T_{\lambda,\varepsilon}(u) &= \frac{1}{2}\|u\|^2 + h_T(u)I_{\lambda,\varepsilon}(u) - \int_{\mathbb{R}^3}F(x,u)dx \\ &\ge \frac{1}{4}\|u\|^2 - \frac{a_1}{p}\int_{\mathbb{R}^3}|u|^p dx \\ &\ge \frac{1}{4}\|u\|^2 - \frac{a_1}{p}C_p^p\|u\|^p. \end{aligned}$$

We conclude that there exists $\rho > 0$ small enough such that for any $u \in H^1_V(\mathbb{R}^3)$ with $0 < \|u\| \le \rho$, it results that $J^T_{\lambda,\varepsilon}(u) > 0$. In particular, we have

$$J^T_{\lambda,\varepsilon}(u) \ge \frac{1}{4}\rho^2 - \frac{a_1}{p}C_p^p\rho^p > 0, \quad \text{for } u \in H^1_V(\mathbb{R}^3) \text{ with } \|u\| = \rho.$$

On the other hand, by (f_1)–(f_3), there exist $a_2, a_3 > 0$ such that

$$F(x,t) \ge a_2|t|^\alpha - a_3t^2, \quad \text{for } (x,t) \in \mathbb{R}^3\times \in \mathbb{R}. \tag{3.3}$$

Then for $\overline{u} \in H_V^1(\mathbb{R}^3)$ with $\|\overline{u}\| = 1$ fixed and $s > \sqrt{2}T$, by (3.3) and the definition of h_T, we have

$$\begin{aligned} J_{\lambda,\varepsilon}^T(s\overline{u}) &= \frac{s^2}{2}\|\overline{u}\|^2 + h_T(s\overline{u})I_{\lambda,\varepsilon}(s\overline{u}) - \int_{\mathbb{R}^3} F(x, s\overline{u})dx \\ &\leq \left(\frac{1}{2} + a_3\right) s^2\|\overline{u}\|^2 - a_2|s|^\alpha \int_{\mathbb{R}^3} |\overline{u}|^\alpha dx \\ &\to -\infty, \qquad s \to +\infty. \end{aligned}$$

Thus, by choosing $s_T > \max\{\rho, \sqrt{2}T\}$ large enough, we can get $J_{\lambda,\varepsilon}^T(s_T\overline{u}) < 0$. So we can set $e_T = s_T\overline{u}$. □

Then it follows from Lemma 3.1 and the Mountain Pass lemma that there exists a $(\mathrm{PS})_{c_T}$ sequence $\{u_n\}$ for $J_{\lambda,\varepsilon}^T$ in $H_V^1(\mathbb{R}^3)$, where

$$c_T = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_{\lambda,\varepsilon}^T(\gamma(t)),$$

with

$$\Gamma := \{\gamma \in C([0,1], H_V^1(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = e_T\}.$$

From the proof of Lemma 3.1, we can also get that $c_T > 0$, for every $T > 0$.

Second, we study the boundedness of the $(\mathrm{PS})_{c_T}$ sequence $\{u_n\}$ of $J_{\lambda,\varepsilon}^T$ which has been obtained by the Mountain Pass lemma. In this process, the truncation of the nonlocal term plays an important role.

Lemma 3.2. *For $T > 0$ sufficiently large, there exists $\lambda_T > 0$ such that for any $\lambda \in (0, \lambda_T)$ and $\varepsilon > 0$,*

$$\limsup_{n\to\infty} \|u_n\| < T$$

holds, where $\{u_n\}$ is the $(PS)_{c_T}$ sequence of $J_{\lambda,\varepsilon}^T$ obtained above.

Proof. If $\|u_n\| \to \infty$, as $n \to \infty$, then $h_T(u_n) = \chi(\frac{\|u_n\|^2}{T^2}) \to 0$, as $n \to \infty$. Thus, for all $n \in \mathbb{N}$ large enough

$$J_{\lambda,\varepsilon}^T(u_n) = \frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^3} F(x, u_n)dx, \ \langle J_{\lambda,\varepsilon}^{T\,\prime}(u_n), u_n\rangle = \|u\|^2 - \int_{\mathbb{R}^3} f(x, u_n)u_n dx.$$

Then, by (f_3), for $n \in \mathbb{N}$ large enough

$$\begin{aligned} c_T + 1 + \|u_n\| &\geq J_{\lambda,\varepsilon}^T(u_n) - \frac{1}{\alpha}\langle J_{\lambda,\varepsilon}^{T\,\prime}(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|^2 - \int_{\mathbb{R}^3}\left(F(x, u_n) - \frac{1}{\alpha}f(x, u_n)u_n\right)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|^2, \end{aligned}$$

which is impossible, since $\|u_n\| \to \infty$, $n \to \infty$. Therefore, $\{u_n\}$ is bounded in $H_V^1(\mathbb{R}^3)$ which may be dependent on T.

On the contrary, we assume that $\limsup_{n\to\infty}\|u_n\| \geq T$. Up to a subsequence and still denoted by $\{u_n\}$, we have $\lim_{n\to\infty}\|u_n\| \geq T$. By (f_3), we obtain that

$$\begin{aligned} J^T_{\lambda,\varepsilon}(u_n) - \frac{1}{\alpha}\langle {J^T_{\lambda,\varepsilon}}'(u_n), u_n\rangle &= \left[\frac{1}{2} - \frac{1}{\alpha}\left(1 + \frac{2}{T}\chi'(\frac{\|u_n\|^2}{T^2})I_{\lambda,\varepsilon}(u_n)\right)\right]\|u_n\|^2 \\ &\quad + h_T(u_n)\left(I_{\lambda,\varepsilon}(u_n) - \frac{\lambda}{\alpha}\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u_n)u_n^2dx\right) \\ &\quad - \int_{\mathbb{R}^3}\left(F(x,u_n) - \frac{1}{\alpha}f(x,u_n)u_n\right)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|^2 - \frac{\lambda}{\alpha}h_T(u_n)\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u_n)u_n^2dx. \end{aligned}$$

Then

$$\left(\frac{1}{2} - \frac{1}{\alpha}\right)\|u_n\|^2 + \frac{1}{\alpha}\langle {J^T_{\lambda,\varepsilon}}'(u_n), u_n\rangle \leq J^T_{\lambda,\varepsilon}(u_n) + \frac{\lambda}{\alpha}h_T(u_n)\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u_n)u_n^2dx. \tag{3.4}$$

By the definition of $I_{\lambda,\varepsilon}$ and (i) of Lemma 2.2,

$$0 \leq I_{\lambda,\varepsilon}(v) \leq \frac{3}{8}\lambda^2 S^{-1}C^4_{\frac{12}{5}}\|v\|^4, \qquad v \in H^1_V(\mathbb{R}^3). \tag{3.5}$$

By (3.3), (3.5) and the definitions of c_T and e_T, we have

$$\begin{aligned} c_T &\leq \max_{t\in[0,1]} J^T_{\lambda,\varepsilon}(te_T) \\ &\leq \max_{t\in[0,1]}\left(\frac{t^2}{2}\|e_T\|^2 - \int_{\mathbb{R}^N}F(x,te_T)dx\right) + \max_{t\in[0,1]} h_T(te_T)I_{\lambda,\varepsilon}(te_T) \\ &\leq \max_{t\in[0,1]}\left(\frac{(s_Tt)^2}{2}(1+2a_3C_2^2)\|\overline{u}\|^2 - a_2|\overline{u}|^\alpha_\alpha(s_Tt)^\alpha\right) + \frac{3}{2}\lambda^2S^{-1}C^4_{\frac{12}{5}}T^4 \\ &\leq \max_{t\in[0,\infty)}\left(\frac{t^2}{2}(1+2a_3C_2^2)\|\overline{u}\|^2 - a_2|\overline{u}|^\alpha_\alpha t^\alpha\right) + \frac{3}{2}\lambda^2S^{-1}C^4_{\frac{12}{5}}T^4 \\ &=: c_* + \frac{3}{2}\lambda^2S^{-1}C^4_{\frac{12}{5}}T^4. \end{aligned} \tag{3.6}$$

It should be pointed out that $c_* > 0$ is independent of T and λ. It follows from the definition of h_T and (i) of Lemma 2.2 that

$$h_T(u_n)\int_{\mathbb{R}^3}\phi_{\lambda,\varepsilon}(u_n)u_n^2dx \leq 4\lambda S^{-1}C^4_{\frac{12}{5}}T^4. \tag{3.7}$$

By taking upper limits as $n \to \infty$ on both sides of (3.4), (3.6) and (3.7) lead to

$$\left(\frac{1}{2} - \frac{1}{\alpha}\right)T^2 \leq c_* + \left(\frac{3}{2} + \frac{4}{\alpha}\right)\lambda^2S^{-1}C^4_{\frac{12}{5}}T^4.$$

For every T large enough such that $\left(\frac{1}{2} - \frac{1}{\alpha}\right)T^2 > c_* + 1$, we can obtain $\lambda_T > 0$ small such that $\left(\frac{3}{2} + \frac{4}{\alpha}\right)\lambda_T{}^2S^{-1}C^4_{\frac{12}{5}}T^4 \leq 1$. Therefore, we can get a contradiction for every $\lambda \in (0, \lambda_T)$. □

It follows from Lemma 3.2 that there exists a $(PS)_{c_T}$ sequence of $J^T_{\lambda,\varepsilon}$ still denoted by $\{u_n\}$ with $\|u_n\| \leq T$ for every $T > \sqrt{\frac{2\alpha(c_*+1)}{\alpha-2}}$ and $\lambda \in (0, \lambda_T)$. By the definition of h_T again, we can get that

$$J_{\lambda,\varepsilon}(u_n) = J^T_{\lambda,\varepsilon}(u_n) \to c_T, \qquad J'_{\lambda,\varepsilon}(u_n) = {J^T_{\lambda,\varepsilon}}'(u_n) \to 0, \quad \text{as } n \to \infty.$$

That is, for every fixed $T > \sqrt{\frac{2\alpha(c_*+1)}{\alpha-2}}$, $\{u_n\}$ is also a bounded $(PS)_{c_T}$ of $J_{\lambda,\varepsilon}$ for $\lambda \in (0, \lambda_T)$.

By using Lemma 2.2, we can obtain the following lemma which plays a crucial role in finding a nontrivial solution of system (1.1).

Lemma 3.3. *Let $\{u_n\}$ be a bounded $(PS)_c$ sequence of $J_{\lambda,\varepsilon}$ with $c > 0$, then there exists $\widetilde{u} \in H^1_V(\mathbb{R}^3) \setminus \{0\}$ such that $J'_{\lambda,\varepsilon}(\widetilde{u}) = 0$.*

Proof. Let $\{u_n\}$ be a bounded $(PS)_c$ sequence of $J_{\lambda,\varepsilon}$. That is,

$$J_{\lambda,\varepsilon}(u_n) \to c > 0, \qquad J'_{\lambda,\varepsilon}(u_n) \to 0 \quad \text{in } H^{-1}_V(\mathbb{R}^3), \text{ as } n \to \infty. \tag{3.8}$$

It is clear that $\{u_n\}$ is either

(*i*) *vanishing*: for each $r > 0$, $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} u_n^2 dx = 0$, or

(*ii*) *non-vanishing*: there exist $r, \eta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\limsup_{n\to\infty} \int_{B_r(y_n)} u_n^2 dx \geq \eta.$$

If $\{u_n\}$ is vanishing, then it follows from Lemma I.1 in [18] that $u_n \to 0$ in $L^s(\mathbb{R}^3)$ whenever $2 < s < 6$. By [11, Lemma 2], we have

$$\int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(u_n) u_n^2 dx \to 0, \qquad n \to \infty. \tag{3.9}$$

It follows from (f_1) and (f_2) that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|f(x,t)| \leq \epsilon|t| + C_\epsilon |t|^{p-1}, \quad \text{for } (x,t) \in \mathbb{R}^3 \times \in \mathbb{R}. \tag{3.10}$$

Then

$$\left| \int_{\mathbb{R}^3} f(x,u_n) u_n dx \right| \leq \int_{\mathbb{R}^3} (\epsilon u_n^2 + C_\epsilon |u_n|^p) dx.$$

By the arbitrariness of ϵ and $u_n \to 0$ in $L^p(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} f(x,u_n) u_n dx \to 0, \qquad n \to \infty. \tag{3.11}$$

It follows from (3.9), (3.11) and $\langle J'_{\lambda,\varepsilon}(u_n), u_n \rangle \to 0$ that $u_n \to 0$ in $H^1_V(\mathbb{R}^3)$. Then $J_{\lambda,\varepsilon}(u_n) \to 0$, which is a contradiction with the fact that $c > 0$ in (3.8). Therefore, $\{u_n\}$ must be non-vanishing. Furthermore, we can assume that $\{y_n\} \subset \mathbb{Z}^3$ since $B_r(y_n) \subset B_{r+1}(z_n)$ for some $z_n \in \mathbb{Z}^3$.

Let $\widetilde{u}_n(x) := u_n(x + y_n)$. (*iii*) of Lemma 2.2 and the periodic assumptions of V and f guarantee that $\|\widetilde{u}_n\| = \|u_n\|$ and $\|J'_{\lambda,\varepsilon}(\widetilde{u}_n)\| = \|J'_{\lambda,\varepsilon}(u_n)\|$. Since $\{\widetilde{u}_n\}$ is bounded in $H^1_V(\mathbb{R}^3)$, there exists $\widetilde{u} \in H^1_V(\mathbb{R}^3)$, which is nonzero due to the fact that $\limsup_{n\to\infty} \int_{B_r(0)} \widetilde{u}_n^2 dx \geq \eta$, such that $\widetilde{u}_n \rightharpoonup \widetilde{u}$ in $H^1_V(\mathbb{R}^3)$ after passing to a subsequence. A direct computation shows that $J'_{\lambda,\varepsilon}(\widetilde{u}) = 0$. In fact, for every $v \in H^1_V(\mathbb{R}^3)$,

$$o_n(1) = \langle J'_{\lambda,\varepsilon}(\widetilde{u}_n), v \rangle = \int_{\mathbb{R}^3} (\nabla \widetilde{u}_n \nabla v + V(x) \widetilde{u}_n v + \phi_{\lambda,\varepsilon}(\widetilde{u}_n) \widetilde{u}_n v - f(x, \widetilde{u}_n) v) dx.$$

The weak convergence in $H^1_V(\mathbb{R}^3)$ leads to

$$\int_{\mathbb{R}^3} (\nabla \widetilde{u}_n \nabla v + V(x) \widetilde{u}_n v) dx \to \int_{\mathbb{R}^3} (\nabla \widetilde{u} \nabla v + V(x) \widetilde{u} v) dx, \quad \text{as } n \to \infty.$$

By (*ii*) of Lemma 2.2, we can get that $\phi_{\lambda,\varepsilon}(\widetilde{u}_n) \rightharpoonup \phi_{\lambda,\varepsilon}(\widetilde{u})$ in X and

$$\int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(\widetilde{u}_n)\widetilde{u}_n v dx \to \int_{\mathbb{R}^3} \phi_{\lambda,\varepsilon}(\widetilde{u})\widetilde{u} v dx, \quad \text{as } n \to \infty.$$

It follows from (3.10) that

$$|f(x,\widetilde{u}_n)v| \le |\widetilde{u}_n||v| + C|\widetilde{u}_n|^{p-1}|v|, \quad \text{for some } C > 0.$$

By the definitions of weak convergence in $L^2(\mathbb{R}^3)$ and $L^{\frac{p}{p-1}}(\mathbb{R}^3)$, we can get that

$$\int_{\mathbb{R}^3}(|\widetilde{u}_n||v| + C|\widetilde{u}_n|^{p-1}|v|)dx \to \int_{\mathbb{R}^3}(|\widetilde{u}||v| + C|\widetilde{u}|^{p-1}|v|)dx, \quad \text{for } v \in H^1_V(\mathbb{R}^3).$$

Then, by applying the Fatou lemma twice, we have

$$\int_{\mathbb{R}^3} f(x,\widetilde{u}_n)v dx \to \int_{\mathbb{R}^3} f(x,\widetilde{u})v dx, \qquad n \to \infty.$$

Thus, $\langle J'_{\lambda,\varepsilon}(\widetilde{u}), v\rangle = 0$. That is, $\widetilde{u}$ is a nontrivial critical point of $J_{\lambda,\varepsilon}$. □

Proof of Theorem 1.1. Let $T_0 > \sqrt{\frac{2\alpha(c_*+1)}{\alpha-2}}$ and $\lambda_0 := \lambda_{T_0}$ be chosen as in Lemma 3.2. By Lemma 3.2 and Lemma 3.3, for every $\lambda \in (0,\lambda_0)$ and $\varepsilon > 0$, $J_{\lambda,\varepsilon}$ has at least one nontrivial critical point $u_{\lambda,\varepsilon} \in H^1_V(\mathbb{R}^3)$. Lemma 2.3 indicates that $(u_{\lambda,\varepsilon}, \phi_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}))$ is a nontrivial solution of system (1.1). The proof is completed. □

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Limit cycles in piecewise smooth perturbations of a class of cubic differential systems

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Abstract. In this paper, we study the bifurcation of limit cycles from a class of cubic integrable non-Hamiltonian systems under arbitrarily small piecewise smooth perturbations of degree n. By using the averaging theory and complex method, the lower and upper bounds for the maximum number of limit cycles bifurcating from the period annulus of the unperturbed systems are given at first order in ε. It is also shown that in this case, the maximum number of limit cycles produced by piecewise smooth perturbations is almost twice the upper bound of the maximum number of limit cycles produced by smooth perturbations for the considered systems.

Keywords: bifurcation of limit cycles, piecewise smooth perturbation, cubic differential system, averaging theory, complex method.

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1 Introduction

Non-smooth phenomena widely exists in real world and scientific fields, such as dynamic compensation of inertial element error in autonomous navigation of high dynamic aircraft, the non-smooth switching between modules in multi-source information fusion, electronic relays, mechanical impact, neuronal networks and etc., see for instance [1, 14, 16, 27]. Generally, it can be modeled by non-smooth differential systems. Piecewise smooth differential systems, served as one of the most important non-smooth dynamical systems, attracts many researcher's interest. In recent years, more attention focuses on studying dynamical behaviors, especially the bifurcation theory of limit cycles in piecewise smooth systems, see [5, 7, 11, 17, 18, 34, 39, 42, 43]. There are quite a few innovative methods which have been proposed and some theoretical results were established. For example, the conjecture that a class of piecewise Liénard equations with $n + 1$ intervals has up to $2n$ limit cycles was proved in

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[44]. Through analyzing the Lyapunov constants, Hopf bifurcation of non-smooth systems was presented in [10,12,23]. The Melnikov method for Hopf and homoclinic bifurcations was applied to non-smooth systems [2,19,25,31,32]. In addition, the first order Melnikov function for planar piecewise smooth Hamiltonian systems was derived to study Poincaré bifurcation [33], while the averaging theory of discontinuous dynamical systems was developed to find limit cycles of piecewise continuous dynamical systems [35].

It is well known that the simplest piecewise smooth systems are the piecewise linear ones with two zones separated by a straight line. Lum and Chua [40, 41] conjectured that such a continuous piecewise linear differential system in the plane has at most one limit cycle, which was proved by Freire et al. [21]. While for the planar discontinuous piecewise linear differential systems with two zones separated by a straight line, Han and Zhang [25] showed that such systems may have two limit cycles. Huan and Yang [26] provided a numerical example which possesses three limit cycles. Llibre and Ponce [38] presented three nested limit cycles in discontinuous piecewise linear differential systems with two zones. Some results on other discontinuous piecewise linear differential systems with two zones separated by a straight line exhibiting three limit cycles can also be seen in [7, 8, 28] etc. There are also some works concerning the limit cycles bifurcation from a linear center under piecewise smooth perturbations. For example, the nth degree piecewise polynomial perturbations of a linear center were considered in [6], and an upper bound of no more than $Nn-1$ limit cycles appearing up to a study of order N was presented. Cen et al. [9] studied quadratic isochronous centers $S1, S2, S3$ and $S4$ under the piecewise polynomial perturbations of degree n by applying the first order averaging theory, and found the sharp upper bound for the first three isochronous centers and an upper bound for the last center. More results on this topic can be found in [15,20,22,24,29,30,36,37] and the references therein.

In the present paper, we focus our attention on the study of limit cycle bifurcation from a class of planar cubic integrable non-Hamilton differential system.

$$(\dot{x},\dot{y}) = (-y(x+a)(y+b), x(x+a)(y+b)), \tag{1.1}$$

which has

$$H(x,y) = x^2 + y^2 = h, \quad h \in (0, \min\{a^2, b^2\})$$

as its first integral with the integrating factor $\mu = 2/((x+a)(y+b))$, and $(0,0)$ is the unique center.

Consider arbitrarily small piecewise smooth perturbations of system (1.1)

$$(\dot{x},\dot{y}) = \begin{cases} (-y(x+a)(y+b) + \varepsilon f^+(x,y), x(x+a)(y+b) + \varepsilon g^+(x,y)), & x > 0, \\ (-y(x+a)(y+b) + \varepsilon f^-(x,y), x(x+a)(y+b) + \varepsilon g^-(x,y)), & x < 0, \end{cases} \tag{1.2}$$

where the polynomials $f^\pm(x,y), g^\pm(x,y), i = 1,2$ are given by

$$f^+(x,y) = \sum_{i+j=0}^{n} a_{i,j}x^iy^j, \quad g^+(x,y) = \sum_{i+j=0}^{n} b_{i,j}x^iy^j,$$
$$f^-(x,y) = \sum_{i+j=0}^{n} c_{i,j}x^iy^j, \quad g^-(x,y) = \sum_{i+j=0}^{n} d_{i,j}x^iy^j,$$

with any real coefficients $a_{i,j}, b_{i,j}, c_{i,j}$ and $d_{i,j}$, and $|\varepsilon| \neq 0$ is a small parameter. By using the first order averaging theory for discontinuous systems and complex method, we study the maximum number, denoted by $H(n)$, of limit cycles of system (1.2) bifurcating from the period annulus around the center of system (1.1). The main results are summarized as follows.

Theorem 1.1. *For system (1.2) with* $|\varepsilon| \neq 0$ *sufficiently small, we have*

(i) If $|a| > |b| \neq 0$, *then* $2[\frac{n}{2}] + 2n + 3 \leq H(n) \leq 2[\frac{n}{2}] + 4n + 14$;

(ii) If $|b| > |a| \neq 0$, *then* $2[\frac{n}{2}] + 2n + 3 \leq H(n) \leq 4[\frac{n}{2}] + 3n + 14$;

(iii) If $|a| = |b| \neq 0$, *then* $[\frac{n}{2}] + 2n + 3 \leq H(n) \leq 3n + 6$;

(iv) If $b = 0, a \neq 0$, *then* $H(n) = 2[\frac{n}{2}] + n + 1$,

where $[\cdot]$ *is the integer function, and* $H(n)$ *denotes the maximum number of limit cycles of system (1.2) bifurcating from the period annulus of the unperturbed system (1.1) at first order in* ε.

Remark 1.2. It is noted that the limit cycle bifurcation from the unperturbed system (1.1) with $a, b \in R\backslash\{0\}$ under arbitrarily small smooth polynomial perturbations of degree n is studied in [4], which shows that $3[(n-1)/2] + 4$ if $a \neq b$ and, respectively, $2[(n-1)/2] + 2$ if $a = b$, up to first order in ε, are upper bounds for the number of the limit cycles bifurcating from the period annulus of the cubic center (1.1). Comparing Theorem 1.1 with the results in [4], we obtained that at first order in ε, the lower bound of the maximum number of limit cycles produced by piecewise smooth perturbations is almost twice the upper bound of the maximum number of limit cycles produced by smooth perturbations. Hence, for one differential system, piecewise smooth perturbations generally produce more limit cycles than smooth ones.

The organization of this paper is as follows. In Section 2, we present some preliminary results, including the first order averaging theory for discontinuous systems and the method estimating the number of zeros of some functions. The explicit expression and properties of the averaged function are derived in Section 3. Sections 4-6 are dedicated to the investigation of the lower and upper bounds for the maximum number of the zeros of the averaged function, respectively. Finally we prove Theorem 1.1 in Section 7.

2 Preliminary results

In this section, we briefly introduce the first order averaging theory for discontinuous systems and the method concerning the estimate of the number of zeros of some functions, which will be used in the proof of our main results.

Lemma 2.1 ([34])**.** *Consider the following discontinuous differential systems*

$$\frac{dr}{d\theta} = \varepsilon F(\theta, r) + \varepsilon^2 R(\theta, r, \varepsilon), \tag{2.1}$$

with

$$\begin{aligned} F(\theta, r) &= F_1(\theta, r) + \operatorname{sign}(h(\theta, r))F_2(\theta, r), \\ R(\theta, r, \varepsilon) &= R_1(\theta, r, \varepsilon) + \operatorname{sign}(h(\theta, r))R_2(\theta, r, \varepsilon), \end{aligned}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ *and* $h : \mathbb{R} \times D \to \mathbb{R}$ *are continuous functions, T-periodic in the first variable* θ *and* D *is an open subset of* $\mathbb{R}^n$. *We also suppose that* h *is a* C^1 *function having zero as a regular value, and the sign function* $\operatorname{sign}(u)$ *is given by*

$$\operatorname{sign}(u) = \begin{cases} 1, & u > 0, \\ 0, & u = 0, \\ -1, & u < 0. \end{cases}$$

Define the averaged function $f : D \to \mathbb{R}^n$ *as*

$$f(r) = \int_0^T F(\theta, r)\, d\theta. \tag{2.2}$$

Assume that the following hypotheses (i), (ii) *and* (iii) *hold.*

(i) F_1, F_2, R_1, R_2 *and* h *are locally Lipschitz with respect to* r.

(ii) There exists an open bounded subset $C \subset D$ *such that for the sufficiently small* $|\varepsilon| > 0$, *every orbit starting in* $\overline{C}$ *reaches the set of discontinuity only at its crossing regions.*

(iii) For $a \in C$ *with* $f(a) = 0$, *there exists a neighborhood* V *of* a *such that* $f(z) \neq 0$ *for all* $z \in \overline{V} \setminus \{a\}$ *and the Brouwer degree function* $d_B(f, V, a) \neq 0$.

Then, for the sufficiently small $|\varepsilon| > 0$ *there exists a T-periodic solution* $r(\theta, \varepsilon)$ *of system* (2.1) *such that* $r(0, \varepsilon) \to a$ *as* $\varepsilon \to 0$.

The result from [3] is often used to replace the condition (iii) in Lemma 2.1, which is stated as follows.

Remark 2.2. $(\widetilde{iii})$ Let $f : D \to \mathbb{R}$ be a C^1 function with $f(a) = 0$, where D is an open subset of $\mathbb{R}$ and $a \in D$. Whenever the Jacobian determinant $J_f(a) \neq 0$, there exists a neighborhood V of a such that $f(r) \neq 0$ for all $r \in \overline{V} \setminus \{a\}$. Then $d_B(f, V, 0) \neq 0$.

To estimate the number of zeros of some functions, we recall an important result from [13].

Lemma 2.3. *Consider* $p+1$ *linearly independent analytical functions* $f_i : U \to \mathbb{R}, i = 0, 1, \ldots, p$, *where* $U \subset \mathbb{R}$ *is an interval. Suppose that there exists* $j \in \{0, 1, \ldots, p\}$ *such that* f_j *has constant sign. Then there exists* $p+1$ *constants* $C_i, i = 0, 1, \ldots, p$ *such that* $f(x) = \sum_{i=0}^{p} C_i f_i(x)$ *has at least* p *simple zeros in* U.

3 Explicit expression of averaged function

This section is devoted to the derivation and simplification of the expression for the averaged function.

After making the polar coordinate transformations $x = r\cos\theta$ and $y = r\sin\theta$, system (1.2) becomes the following

$$\frac{dr}{d\theta} = \begin{cases} \varepsilon X^+(\theta, r) + \varepsilon^2 Y^+(\theta, r, \varepsilon), & \cos\theta > 0, \\ \varepsilon X^-(\theta, r) + \varepsilon^2 Y^-(\theta, r, \varepsilon), & \cos\theta < 0, \end{cases} \tag{3.1}$$

where

$$X^+(\theta, r) = \frac{P^+(\theta, r)}{(r\cos\theta + a)(r\sin\theta + b)}, \qquad X^-(\theta, r) = \frac{P^-(\theta, r)}{(r\cos\theta + a)(r\sin\theta + b)},$$
$$Y^+(\theta, r, \varepsilon) = -\frac{X^+(\theta, r) Q^+(\theta, r)}{r(r\cos\theta + a)(r\sin\theta + b) + \varepsilon Q^+(\theta, r)},$$
$$Y^-(\theta, r, \varepsilon) = -\frac{X^-(\theta, r) Q^-(\theta, r)}{r(r\cos\theta + a)(r\sin\theta + b) + \varepsilon Q^-(\theta, r)},$$

with

$$P^\pm(\theta, r) = \cos\theta f^\pm(r\cos\theta, r\sin\theta) + \sin\theta g^\pm(r\cos\theta, r\sin\theta),$$

$$Q^{\pm}(\theta, r) = \cos\theta g^{\pm}(r\cos\theta, r\sin\theta) - \sin\theta f^{\pm}(r\cos\theta, r\sin\theta).$$

Denote

$$r_1 = \begin{cases} -a, & a < 0, \\ +\infty, & a > 0, \end{cases} \qquad r_2 = \begin{cases} a, & a > 0, \\ +\infty, & a < 0, \end{cases} \qquad r_3 = \begin{cases} b, & b > 0, \\ -b, & b < 0, \end{cases}$$

then the functions $X^+(\theta, r)$ and $Y^+(\theta, r, \varepsilon)$ ($X^-(\theta, r)$ and $Y^-(\theta, r, \varepsilon)$, resp.) are well defined in $(0, r_1) \cap (0, r_3)$ ($(0, r_2) \cap (0, r_3)$, resp.) for $ab \neq 0$, while $X^+(\theta, r)$ and $Y^+(\theta, r, \varepsilon)$ ($X^-(\theta, r)$ and $Y^-(\theta, r, \varepsilon)$, resp.) are well defined in $(0, r_1)$ ($(0, r_2)$, resp.) for $b = 0,\ a \neq 0$.

We rewrite system (3.1) as the form

$$\frac{dr}{d\theta} = \varepsilon F(\theta, r) + \varepsilon^2 R(\theta, r, \varepsilon), \tag{3.2}$$

where

$$F(\theta, r) = \begin{cases} X^+(\theta, r), & \cos\theta > 0, \\ X^-(\theta, r), & \cos\theta < 0, \end{cases}$$
$$R(\theta, r, \varepsilon) = \begin{cases} Y^+(\theta, r, \varepsilon), & \cos\theta > 0, \\ Y^-(\theta, r, \varepsilon), & \cos\theta < 0. \end{cases}$$

By Lemma 2.1, the averaged function of system (3.2) can be expressed as

$$\begin{aligned} f(r) &= \int_0^{2\pi} F(\theta, r)\, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\theta, r)\, d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} F(\theta, r)\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} X^+(\theta, r)\, d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} X^-(\theta, r)\, d\theta \\ &= \sum_{i+j=1}^{n+1} \omega_{i,j} r^{i+j-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)(r\sin\theta + b)} d\theta \\ &\quad + \sum_{i+j=1}^{n+1} \tau_{i,j} r^{i+j-1} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)(r\sin\theta + b)} d\theta, \end{aligned} \tag{3.3}$$

where $\omega_{i,j} = a_{i-1,j} + b_{i,j-1}$, $\tau_{i,j} = c_{i-1,j} + d_{i,j-1}$, and $\omega_{0,0} = \tau_{0,0} = 0$ provided that $a_{-1,j} = b_{i,-1} = c_{-1,j} = d_{i,-1} = 0$.

Define

$$\begin{aligned} I_{i,j}(r) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)(r\sin\theta + b)} d\theta, \\ J_{i,j}(r) &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)(r\sin\theta + b)} d\theta \end{aligned} \tag{3.4}$$

for $i, j \geq 0$.

Remark 3.1. Note that in the interval $\cap_{i=1}^{3}(0, r_i) = (0, \min\{|a|, |b|\})$ ($(0, |a|)$, resp.) for $ab \neq 0$ ($b = 0, a \neq 0$, resp.), the zeros of the function $f(r)$ coincide with the non-zero zeros of $F(r) = rf(r)$. To make the calculation easier, we investigate the zeros of the function $F(r)$ instead of $f(r)$ in the subsequent sections.

Lemma 3.2. *The function $F(r) = rf(r)$ defined above can be expressed as*

$$F(r) = rf(r) = F_1(r) + F_2(r), \tag{3.5}$$

where

$$F_1(r) = \sum_{i=0}^{n+1} r^i I_{i,0}(r) \sum_{j=0}^{[\frac{n+1-i}{2}]} P_{i,j} r^{2j} + \sum_{i=0}^{n} r^{i+1} I_{i,1}(r) \sum_{j=0}^{[\frac{n-i}{2}]} Q_{i,j} r^{2j},$$
$$F_2(r) = \sum_{i=0}^{n+1} r^i J_{i,0}(r) \sum_{j=0}^{[\frac{n+1-i}{2}]} \widetilde{P}_{i,j} r^{2j} + \sum_{i=0}^{n} r^{i+1} J_{i,1}(r) \sum_{j=0}^{[\frac{n-i}{2}]} \widetilde{Q}_{i,j} r^{2j},$$

with

$$P_{i,j} = \sum_{k=0}^{[\frac{i}{2}]} (-1)^k C_{k+j}^k \omega_{i-2k,2k+2j}, \qquad Q_{i,j} = \sum_{k=0}^{[\frac{i}{2}]} (-1)^k C_{k+j}^k \omega_{i-2k,2k+2j+1},$$
$$\widetilde{P}_{i,j} = \sum_{k=0}^{[\frac{i}{2}]} (-1)^k C_{k+j}^k \tau_{i-2k,2k+2j}, \qquad \widetilde{Q}_{i,j} = \sum_{k=0}^{[\frac{i}{2}]} (-1)^k C_{k+j}^k \tau_{i-2k,2k+2j+1}.$$

Moreover, $P_{0,0} = \widetilde{P}_{0,0} = 0$, C_{k+j}^k is the combinatorial number, and the other coefficients $P_{i,j}$, $Q_{i,j}$, $\widetilde{P}_{i,j}$ and $\widetilde{Q}_{i,j}$ are independent.

Proof. From (3.3) and (3.4), we have

$$\begin{aligned}
F(r) &= \sum_{i+j=1}^{n+1} \omega_{i,j} r^{i+j} I_{i,j}(r) + \sum_{i+j=1}^{n+1} \tau_{i,j} r^{i+j} J_{i,j}(r) \\
&= \sum_{i=1}^{n+1} r^i \sum_{j=0}^{i} \omega_{i-j,j} I_{i-j,j}(r) + \sum_{i=1}^{n+1} r^i \sum_{j=0}^{i} \tau_{i-j,j} J_{i-j,j}(r) \\
&= \sum_{i=1}^{n+1} r^i \sum_{j=0}^{[\frac{i}{2}]} \omega_{i-2j,2j} I_{i-2j,2j}(r) + \sum_{i=1}^{n+1} r^i \sum_{j=0}^{[\frac{i-1}{2}]} \omega_{i-2j-1,2j+1} I_{i-2j-1,2j+1}(r) \\
&\quad + \sum_{i=1}^{n+1} r^i \sum_{j=0}^{[\frac{i}{2}]} \tau_{i-2j,2j} J_{i-2j,2j}(r) + \sum_{i=1}^{n+1} r^i \sum_{j=0}^{[\frac{i-1}{2}]} \tau_{i-2j-1,2j+1} J_{i-2j-1,2j+1}(r).
\end{aligned} \tag{3.6}$$

On the other hand, some computations shows that

$$\begin{aligned}
I_{i,2j}(r) &= \sum_{k=0}^{j} (-1)^k C_j^k I_{i+2k,0}(r), \qquad I_{i,2j+1}(r) = \sum_{k=0}^{j} (-1)^k C_j^k I_{i+2k,1}(r), \\
J_{i,2j}(r) &= \sum_{k=0}^{j} (-1)^k C_j^k J_{i+2k,0}(r), \qquad J_{i,2j+1}(r) = \sum_{k=0}^{j} (-1)^k C_j^k J_{i+2k,1}(r).
\end{aligned} \tag{3.7}$$

Putting (3.7) into (3.6), we can get (3.5). The independence of $P_{i,j}$, $Q_{i,j}$, $\widetilde{P}_{i,j}$ and $\widetilde{Q}_{i,j}$ follows from their definitions.

This completes the proof of Lemma 3.2. □

Define

$$\begin{aligned}
Y_{i,j}(r) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{a + r\cos\theta} d\theta, \qquad Z_{i,j}(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{b + r\sin\theta} d\theta, \\
\widetilde{Y}_{i,j}(r) &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{a + r\cos\theta} d\theta, \qquad \widetilde{Z}_{i,j}(r) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{b + r\sin\theta} d\theta,
\end{aligned} \tag{3.8}$$

for $i, j \geq 0$, then a straightforward computation yields Lemma 3.3 below.

Lemma 3.3. *For* (3.4) *and* (3.8), *the following equalities hold.*

(i) $I_{i,0}(r) = \frac{1}{r}[Z_{i-1,0}(r) - aI_{i-1,0}(r)]$ *for* $i \geq 1$.

(ii) $Z_{i,0}(r) = -\frac{1}{r^2}[(b^2 - r^2)Z_{i-2,0}(r) - b^* m_{i-2}]$ *for* $i \geq 2$.

(iii) $I_{i,1}(r) = \frac{1}{r}[Y_{i,0}(r) - bI_{i,0}(r)]$ *for* $i \geq 0$.

(iv) $Y_{i,0}(r) = \frac{1}{r}[c^* m_{i-1} - aY_{i-1,0}(r)]$ *for* $i \geq 1$.

(v) $J_{i,0}(r) = \frac{1}{r}[\widetilde{Z}_{i-1,0}(r) - aJ_{i-1,0}(r)]$ *for* $i \geq 1$.

(vi) $\widetilde{Z}_{i,0}(r) = -\frac{1}{r^2}[(b^2 - r^2)\widetilde{Z}_{i-2,0}(r) - \widetilde{b}^* m_{i-2}]$ *for* $i \geq 2$.

(vii) $J_{i,1}(r) = \frac{1}{r}[\widetilde{Y}_{i,0}(r) - bJ_{i,0}(r)]$ *for* $i \geq 0$.

(viii) $\widetilde{Y}_{i,0}(r) = \frac{1}{r}[\widetilde{c}^* m_{i-1} - a\widetilde{Y}_{i-1,0}(r)]$ *for* $i \geq 1$,

where $m_i = \frac{(i-1)!!}{i!!}$, $m_0 = m_1 = 1$, *and*

$$b^* = \begin{cases} \pi b, & i \text{ is even}, \\ 2b, & i \text{ is odd}, \end{cases} \qquad c^* = \begin{cases} \pi, & i \text{ is odd}, \\ 2, & i \text{ is even}, \end{cases}$$
$$\widetilde{b}^* = \begin{cases} \pi b, & i \text{ is even}, \\ -2b, & i \text{ is odd}, \end{cases} \qquad \widetilde{c}^* = \begin{cases} \pi, & i \text{ is odd}, \\ -2, & i \text{ is even}. \end{cases}$$

Moreover, we have

$$\Big(-a^2 - b^2 + r^2\Big) I_{0,0}(r) = -bY_{0,0}(r) + rZ_{1,0}(r) - aZ_{0,0}(r),$$
$$\Big(-a^2 - b^2 + r^2\Big) J_{0,0}(r) = -b\widetilde{Y}_{0,0}(r) - rZ_{1,0}(r) - aZ_{0,0}(r).$$

Now, we start with simplifying $F(r)$. Firstly, substituting Lemma 3.3 into (3.5), we get

$$\begin{aligned} F_1(r) = {} & \sum_{i=0}^{[\frac{n+1}{2}]} W_{0,i} r^{2i} I_{0,0}(r) + \sum_{i=0}^{[\frac{n}{2}]} Z_{2i,0}(r) r^{2i} \sum_{j=0}^{[\frac{n}{2}]-i} W_{2i+1,j} r^{2j} + \sum_{i=0}^{[\frac{n-1}{2}]} Z_{2i+1,0}(r) r^{2i+1} \sum_{j=0}^{[\frac{n-1}{2}]-i} W_{2i+2,j} r^{2j} \\ & - b \sum_{i=0}^{[\frac{n}{2}]} T_{0,i} r^{2i} I_{0,0}(r) - b \sum_{i=0}^{[\frac{n-1}{2}]} Z_{2i,0}(r) r^{2i} \sum_{j=0}^{[\frac{n-1}{2}]-i} T_{2i+1,j} r^{2j} \\ & - b \sum_{i=0}^{[\frac{n}{2}]-1} Z_{2i+1,0}(r) r^{2i+1} \sum_{j=0}^{[\frac{n}{2}]-1-i} T_{2i+2,j} r^{2j} + \sum_{i=0}^{[\frac{n}{2}]} T_{0,i} r^{2i} Y_{0,0}(r) + \sum_{i=0}^{[\frac{n-1}{2}]} m_{2i} c^* r^{2i} \sum_{j=0}^{[\frac{n-1}{2}]-i} T_{2i+1,j} r^{2j} \\ & + \sum_{i=0}^{[\frac{n}{2}]-1} m_{2i+1} c^* r^{2i+1} \sum_{j=0}^{[\frac{n}{2}]-1-i} T_{2i+2,j} r^{2j}, \end{aligned} \tag{3.9}$$

where

$$W_{i,j} = \sum_{k=i}^{n+1-2j} P_{k,j}(-a)^{k-i}, \qquad T_{i,j} = \sum_{k=i}^{n-2j} Q_{k,j}(-a)^{k-i}.$$

And $F_2(r)$ can be similarly expressed as

$$\begin{aligned}F_2(r) =& \sum_{i=0}^{[\frac{n+1}{2}]} \widetilde{W}_{0,i} r^{2i} J_{0,0}(r) + \sum_{i=0}^{[\frac{n}{2}]} \widetilde{Z}_{2i,0}(r) r^{2i} \sum_{j=0}^{[\frac{n}{2}]-i} \widetilde{W}_{2i+1,j} r^{2j} + \sum_{i=0}^{[\frac{n-1}{2}]} \widetilde{Z}_{2i+1,0}(r) r^{2i+1} \sum_{j=0}^{[\frac{n-1}{2}]-i} \widetilde{W}_{2i+2,j} r^{2j} \\ & - b \sum_{i=0}^{[\frac{n}{2}]} \widetilde{T}_{0,i} r^{2i} J_{0,0}(r) - b \sum_{i=0}^{[\frac{n-1}{2}]} \widetilde{Z}_{2i,0}(r) r^{2i} \sum_{j=0}^{[\frac{n-1}{2}]-i} \widetilde{T}_{2i+1,j} r^{2j} \\ & - b \sum_{i=0}^{[\frac{n}{2}]-1} \widetilde{Z}_{2i+1,0}(r) r^{2i+1} \sum_{j=0}^{[\frac{n}{2}]-1-i} \widetilde{T}_{2i+2,j} r^{2j} + \sum_{i=0}^{[\frac{n}{2}]} \widetilde{T}_{0,i} r^{2i} \widetilde{Y}_{0,0}(r) + \sum_{i=0}^{[\frac{n-1}{2}]} m_{2i} \widetilde{c}^* r^{2i} \sum_{j=0}^{[\frac{n-1}{2}]-i} \widetilde{T}_{2i+1,j} r^{2j} \\ & + \sum_{i=0}^{[\frac{n}{2}]-1} m_{2i+1} \widetilde{c}^* r^{2i+1} \sum_{j=0}^{[\frac{n}{2}]-1-i} \widetilde{T}_{2i+2,j} r^{2j},\end{aligned}$$

where

$$\widetilde{W}_{i,j} = \sum_{k=i}^{n+1-2j} \widetilde{P}_{k,j} (-a)^{k-i}, \qquad \widetilde{T}_{i,j} = \sum_{k=i}^{n-2j} \widetilde{Q}_{k,j} (-a)^{k-i}.$$

Obviously, $W_{i,j}, T_{i,j}, \widetilde{W}_{i,j}$ and $\widetilde{T}_{i,j}$ are independent.

From Lemma 3.3, we have when $k \geq 1$

$$\begin{aligned}r^{2k} I_{0,0}(r) =& \left(a^2+b^2\right)^k I_{0,0}(r) - b \sum_{i=0}^{k-1} \left(a^2+b^2\right)^{k-1-i} r^{2i} Y_{0,0}(r) \\ & + \sum_{i=0}^{k-1} \left(a^2+b^2\right)^{k-1-i} r^{2i+1} Z_{1,0}(r) - a \sum_{i=0}^{k-1} \left(a^2+b^2\right)^{k-1-i} r^{2i} Z_{0,0}(r).\end{aligned} \tag{3.10}$$

Noting $W_{0,0} = -aW_{1,0}$ and substituting (3.10) into (3.9), we have

$$\begin{aligned}F_1(r) =& \left[\sum_{i=0}^{[\frac{n-1}{2}]} T_{2i+1,0}(-b^2)^i - \frac{1}{b} \sum_{i=1}^{[\frac{n}{2}]} W_{2i+1,0}(-b^2)^i \right] (\pi - bZ_{0,0}(r)) \\ & + \left[T_{0,0} - b \sum_{j=1}^{[\frac{n+1}{2}]} (a^2+b^2)^{j-1} W_{0,j} + b^2 \sum_{j=1}^{[\frac{n}{2}]} T_{0,j} (a^2+b^2)^{j-1} \right] (Y_{0,0}(r) - bI_{0,0}(r)) \\ & + \left[W_{1,0} - a \sum_{j=1}^{[\frac{n+1}{2}]} (a^2+b^2)^{j-1} W_{0,j} + ab \sum_{j=1}^{[\frac{n}{2}]} T_{0,j} (a^2+b^2)^{j-1} \right] (Z_{0,0}(r) - aI_{0,0}(r)) \\ & + \left[\sum_{i=1}^{[\frac{n}{2}]} T_{0,i} - b \sum_{i=1}^{[\frac{n+1}{2}]-1} \sum_{j=i+1}^{[\frac{n+1}{2}]} (a^2+b^2)^{j-i-1} W_{0,j} + b^2 \sum_{i=1}^{[\frac{n}{2}]-1} \sum_{j=i+1}^{[\frac{n}{2}]} (a^2+b^2)^{j-i-1} T_{0,j} \right] r^{2i} Y_{0,0}(r) \\ & + \left[\sum_{i=1}^{[\frac{n}{2}]} \sum_{t=0}^{[\frac{n}{2}]-i} \sum_{k+j=i} W_{2k+2t+1,j} C_{k+t}^t (-b^2)^t - b \sum_{i=1}^{[\frac{n-1}{2}]} \sum_{t=0}^{[\frac{n-1}{2}]-i} \sum_{k+j=i} T_{2k+2t+1,j} C_{k+t}^t (-b^2)^t \right. \\ & \left. - a \sum_{i=1}^{[\frac{n+1}{2}]-1} \sum_{j=i+1}^{[\frac{n+1}{2}]} (a^2+b^2)^{j-i-1} W_{0,j} + ab \sum_{i=1}^{[\frac{n}{2}]-1} \sum_{j=i+1}^{[\frac{n}{2}]} (a^2+b^2)^{j-i-1} T_{0,j} \right] r^{2i} Z_{0,0}(r)\end{aligned}$$

$$
\begin{aligned}
&+\left[\sum_{i=0}^{[\frac{n-1}{2}]}\sum_{t=0}^{[\frac{n-1}{2}]-i}\sum_{k+j=i}W_{2k+2t+2,j}C_{k+t}^{t}(-b^2)^t-b\sum_{i=0}^{[\frac{n}{2}]-1}\sum_{t=0}^{[\frac{n}{2}]-1-i}\sum_{k+j=i}T_{2k+2t+2,j}C_{k+t}^{t}(-b^2)^t\right.\\
&\left.+\sum_{i=0}^{[\frac{n+1}{2}]-1}\sum_{j=i+1}^{[\frac{n+1}{2}]}(a^2+b^2)^{j-i-1}W_{0,j}-b\sum_{i=0}^{[\frac{n}{2}]-1}\sum_{j=i+1}^{[\frac{n}{2}]}(a^2+b^2)^{j-i-1}T_{0,j}\right]r^{2i+1}Z_{1,0}(r)\\
&+\left[\sum_{i=1}^{[\frac{n-1}{2}]}\sum_{k+j=i}T_{2k+1,j}c^*m_{2k}+\sum_{i=1}^{[\frac{n}{2}]-1}\sum_{h=1}^{[\frac{n}{2}]-i}\sum_{k+j=i}W_{2k+2h+1,j}(-b^2)^{h-1}b^*\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t-1)}\right.\\
&\left.-b\sum_{i=1}^{[\frac{n-1}{2}]-1}\sum_{h=1}^{[\frac{n-1}{2}]-i}\sum_{k+j=i}T_{2k+2h+1,j}(-b^2)^{h-1}b^*\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t-1)}\right]r^{2i}\\
&+\left[\sum_{i=0}^{[\frac{n}{2}]-1}\sum_{k+j=i}T_{2k+2,j}c^*m_{2k+1}\right.\\
&+\sum_{i=0}^{[\frac{n-1}{2}]-1}\sum_{h=1}^{[\frac{n-1}{2}]-i}\sum_{k+j=i}W_{2k+2h+2,j}(-b^2)^{h-1}b^*\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t)-1}\\
&\left.-b\sum_{i=0}^{[\frac{n}{2}]-2}\sum_{h=1}^{[\frac{n}{2}]-1-i}\sum_{k+j=i}T_{2k+2h+2,j}(-b^2)^{h-1}b^*\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t)-1}\right]r^{2i+1}.
\end{aligned}
$$

Moreover, $F_1(r)$ can be expressed as

$$
\begin{aligned}
F_1(r)=&A_1^{(1)}\Big(\pi-bZ_{0,0}(r)\Big)+A_2\Big(Z_{0,0}(r)-aI_{0,0}(r)\Big)+A_3\Big(Y_{0,0}(r)-bI_{0,0}(r)\Big)+\sum_{i=1}^{[\frac{n}{2}]}B_ir^{2i}Y_{0,0}(r)\\
&+\sum_{i=1}^{[\frac{n}{2}]}D_i^{(1)}r^{2i}Z_{0,0}(r)+\sum_{i=0}^{[\frac{n-1}{2}]}E_i^{(1)}r^{2i+1}Z_{1,0}(r)+\sum_{i=1}^{[\frac{n-1}{2}]}F_i^{(1)}r^{2i}+\sum_{i=0}^{[\frac{n}{2}]-1}G_i^{(1)}r^{2i+1}.
\end{aligned}
$$

Similarly, due to $\widetilde{W}_{0,0}=-a\widetilde{W}_{1,0}$, $F_2(r)$ takes the form

$$
\begin{aligned}
F_2(r)=&A_1^{(2)}\Big(\pi-b\widetilde{Z}_{0,0}(r)\Big)+A_4\Big(\widetilde{Z}_{0,0}(r)-aJ_{0,0}(r)\Big)+A_5\Big(\widetilde{Y}_{0,0}(r)-bJ_{0,0}(r)\Big)+\sum_{i=1}^{[\frac{n}{2}]}C_ir^{2i}\widetilde{Y}_{0,0}(r)\\
&+\sum_{i=1}^{[\frac{n}{2}]}D_i^{(2)}r^{2i}\widetilde{Z}_{0,0}(r)+\sum_{i=0}^{[\frac{n-1}{2}]}E_i^{(2)}r^{2i+1}\widetilde{Z}_{1,0}(r)+\sum_{i=1}^{[\frac{n-1}{2}]}F_i^{(2)}r^{2i}+\sum_{i=0}^{[\frac{n}{2}]-1}G_i^{(2)}r^{2i+1}.
\end{aligned}
$$

Recalling that $Z_{0,0}=\widetilde{Z}_{0,0}, Z_{1,0}=-\widetilde{Z}_{1,0}$, we get

$$
\begin{aligned}
F(r)=&A_1\Big(\pi-bZ_{0,0}(r)\Big)+A_2\Big(Z_{0,0}(r)-aI_{0,0}(r)\Big)\\
&+A_3\Big(Y_{0,0}(r)-bI_{0,0}(r)\Big)+A_4\Big(Z_{0,0}(r)-aJ_{0,0}(r)\Big)\\
&+A_5\Big(\widetilde{Y}_{0,0}(r)-bJ_{0,0}(r)\Big)+\sum_{i=1}^{[\frac{n}{2}]}B_ir^{2i}Y_{0,0}(r)+\sum_{i=1}^{[\frac{n}{2}]}C_ir^{2i}\widetilde{Y}_{0,0}(r)+\sum_{i=1}^{[\frac{n}{2}]}D_ir^{2i}Z_{0,0}(r)\\
&+\sum_{i=0}^{[\frac{n-1}{2}]}E_ir^{2i+1}Z_{1,0}(r)+\sum_{i=1}^{[\frac{n-1}{2}]}F_ir^{2i}+\sum_{i=0}^{[\frac{n}{2}]-1}G_ir^{2i+1}.
\end{aligned}
\tag{3.11}
$$

where the coefficients $A_i (i = 1, 2 \dots, 5)$, $B_i, C_i, D_i \left(i = 1, \dots, [\frac{n}{2}]\right)$, $E_i \left(i = 0, 1, \dots, [\frac{n-1}{2}]\right)$, $F_i \left(i = 1, \dots, [\frac{n-1}{2}]\right)$ and $G_i \left(i = 0, 1, \dots, [\frac{n}{2}] - 1\right)$ are listed in Appendix.

In particular, for $b = 0, a \neq 0$, recall $I_{i,0}(r) = J_{i,0}(r) = 0$, $I_{i,1}(r) = \frac{1}{r} Y_{i,0}(r)$, and $J_{i,1}(r) = \frac{1}{r} \widetilde{Y}_{i,0}(r)$ for $i \geq 0$, then $F(r)$ takes the form

$$F(r) = \sum_{i=0}^{[\frac{n}{2}]} B_i r^{2i} Y_{0,0}(r) + \sum_{i=0}^{[\frac{n}{2}]} C_i r^{2i} \widetilde{Y}_{0,0}(r) + \sum_{i=0}^{[\frac{n-1}{2}]} F_i r^{2i} + \sum_{i=0}^{[\frac{n}{2}]-1} G_i r^{2i+1}, \tag{3.12}$$

where B_i, C_i, F_i and G_i are given by taking $b = 0$ in the corresponding coefficient formulas of (3.11).

Choosing the appropriate parameter variables from (3.11) and (3.12), respectively, we can get the following Jacobian determinants

$$\left| \frac{\partial\left(A_4, A_3, B_1, \dots, B_{[\frac{n}{2}]}, A_5, C_1, \dots, C_{[\frac{n}{2}]}, A_2, D_1, \dots, D_{[\frac{n}{2}]}, E_0, \dots, E_{[\frac{n-1}{2}]}, A_1, F_1, \dots, F_{[\frac{n-1}{2}]}, G_0, \dots, G_{[\frac{n}{2}]-1}\right)}{\partial\left(\widetilde{W}_{1,0}, T_{0,0}, T_{0,1} \dots, T_{0,[\frac{n}{2}]}, \widetilde{T}_{0,0},, \widetilde{T}_{0,1} \dots, \widetilde{T}_{0,[\frac{n}{2}]}, W_{1,0}, W_{1,1}, \dots, W_{1,[\frac{n}{2}]}, W_{2,0}, \dots, W_{2,[\frac{n-1}{2}]}, T_{1,0}, T_{1,1}, \dots, T_{1,[\frac{n-1}{2}]}, T_{2,0}, \dots, T_{2,[\frac{n}{2}]-1}\right)} \right|$$
$$= (c^*)^{n-1} \neq 0,$$

and

$$\left| \frac{\partial\left(B_0, B_1, \dots, B_{[\frac{n}{2}]}, C_0, C_1, \dots, C_{[\frac{n}{2}]}, F_0, F_1, \dots, F_{[\frac{n-1}{2}]}, G_0, G_1, \dots, G_{[\frac{n}{2}]-1}\right)}{\partial\left(T_{0,0}, T_{0,1} \dots, T_{0,[\frac{n}{2}]}, \widetilde{T}_{0,0},, \widetilde{T}_{0,1} \dots, \widetilde{T}_{0,[\frac{n}{2}]}, T_{1,0}, T_{1,1}, \dots, T_{1,[\frac{n-1}{2}]}, T_{2,0}, \dots, T_{2,[\frac{n}{2}]-1}\right)} \right| = (c^*)^n \neq 0,$$

which imply Lemma 3.4 below.

Lemma 3.4. *For the functions $F(r)$ in (3.11) and (3.12), their coefficients are independent, respectively.*

4 Properties of some integrals

In this section, we study the properties of some integrals defined in Section 3, which play the important roles in the proof of Theorem 1.1.

A straightforward computation yields Lemma 4.1 below.

Lemma 4.1. *For the integrals $I_{0,0}(r), J_{0,0}(r), Y_{0,0}(r)$ and $\widetilde{Y}_{0,0}(r)$ with $ab \neq 0$, the following equalities hold.*

$$I_{0,0}(r) = \begin{cases} \dfrac{-4b\sqrt{b^2-r^2}\arctan\sqrt{\frac{a-r}{a+r}} + \sqrt{a^2-r^2}\sqrt{b^2-r^2}\ln\frac{b+r}{b-r} - a\pi\sqrt{a^2-r^2}}{(-a^2-b^2+r^2)\sqrt{a^2-r^2}\sqrt{b^2-r^2}}, & a > 0, r \in (0, a), \\ \dfrac{-2b\sqrt{b^2-r^2}\ln\frac{r+\sqrt{r^2-a^2}}{a} + \sqrt{r^2-a^2}\sqrt{b^2-r^2}\ln\frac{b+r}{b-r} - a\pi\sqrt{r^2-a^2}}{(-a^2-b^2+r^2)\sqrt{r^2-a^2}\sqrt{b^2-r^2}}, & a > 0, r \in (a, +\infty), \\ \dfrac{2}{ab} - \dfrac{1}{b^2}\ln\dfrac{a+b}{b-a} + \dfrac{a\pi}{b^2\sqrt{b^2-a^2}}, & a > 0, r = a, \\ \dfrac{4b\sqrt{b^2-r^2}\arctan\sqrt{\frac{a-r}{a+r}} + \sqrt{a^2-r^2}\sqrt{b^2-r^2}\ln\frac{b+r}{b-r} - a\pi\sqrt{a^2-r^2}}{(-a^2-b^2+r^2)\sqrt{a^2-r^2}\sqrt{b^2-r^2}}, & a < 0, r \in (0, -a). \end{cases}$$

$$J_{0,0}(r) = \begin{cases} \dfrac{-4b\sqrt{b^2-r^2}\arctan\sqrt{\frac{a+r}{a-r}} + \sqrt{a^2-r^2}\sqrt{b^2-r^2}\ln\frac{b-r}{b+r} - a\pi\sqrt{a^2-r^2}}{(-a^2-b^2+r^2)\sqrt{a^2-r^2}\sqrt{b^2-r^2}}, & a > 0, r \in (0, a), \\ \dfrac{2b\sqrt{b^2-r^2}\ln\frac{r+\sqrt{r^2-a^2}}{-a} + \sqrt{r^2-a^2}\sqrt{b^2-r^2}\ln\frac{b-r}{b+r} - a\pi\sqrt{r^2-a^2}}{(-a^2-b^2+r^2)\sqrt{r^2-a^2}\sqrt{b^2-r^2}}, & a < 0, r \in (-a, +\infty), \\ \dfrac{2}{ab} - \dfrac{1}{b^2}\ln\dfrac{a+b}{b-a} + \dfrac{a\pi}{b^2\sqrt{b^2-a^2}}, & a < 0, r = -a, \\ \dfrac{4b\sqrt{b^2-r^2}\arctan\sqrt{\frac{a+r}{a-r}} + \sqrt{a^2-r^2}\sqrt{b^2-r^2}\ln\frac{b-r}{b+r} - a\pi\sqrt{a^2-r^2}}{(-a^2-b^2+r^2)\sqrt{a^2-r^2}\sqrt{b^2-r^2}}, & a < 0, r \in (0, -a). \end{cases}$$

$$Y_{0,0}(r) = \begin{cases} \frac{-4}{\sqrt{a^2-r^2}} \arctan \sqrt{\frac{a-r}{a+r}}, & a<0, r\in(0,-a), \\ \frac{2}{\sqrt{r^2-a^2}} \ln \left(\frac{r+\sqrt{r^2-a^2}}{a} \right), & a>0, r\in(a,+\infty), \\ \frac{2}{a}, & a>0, r=a, \\ \frac{4}{\sqrt{a^2-r^2}} \arctan \sqrt{\frac{a-r}{a+r}}, & a>0, r\in(0,a). \end{cases}$$

$$\widetilde{Y}_{0,0}(r) = \begin{cases} \frac{4}{\sqrt{a^2-r^2}} \arctan \sqrt{\frac{a+r}{a-r}}, & a>0, r\in(0,a), \\ \frac{-2}{\sqrt{r^2-a^2}} \ln \left(\frac{r+\sqrt{r^2-a^2}}{-a} \right), & a<0, r\in(-a,+\infty), \\ \frac{2}{a}, & a<0, r=-a, \\ \frac{-4}{\sqrt{a^2-r^2}} \arctan \sqrt{\frac{a+r}{a-r}}, & a<0, r\in(0,-a). \end{cases}$$

$$Z_{0,0}(r) = \widetilde{Z}_{0,0}(r) = \frac{\pi}{\sqrt{b^2-r^2}}, \qquad b>0, r\in(0,b) \quad \textit{or} \quad b<0, r\in(0,-b).$$

$$Z_{1,0}(r) = -\widetilde{Z}_{1,0}(r) = \frac{1}{r}\ln\frac{b+r}{b-r}, \qquad b>0, r\in(0,b) \quad \textit{or} \quad b<0, r\in(0,-b).$$

Moreover, we have

Lemma 4.2. *For the integrals $Y_{0,0}(r), \widetilde{Y}_{0,0}(r), Z_{0,0}(r)$ and $Z_{1,0}(r)$, the statements given below are true.*

1. *If $a>0$, then $Y_{0,0}(r)$ can be analytically extended to the interval $(-a,+\infty)$. Furthermore, when $r\to(-a)^+, Y_{0,0}(r) \sim \frac{\sqrt{2}\pi}{\sqrt{a(a+r)}}$; when $r\to+\infty, Y_{0,0}(r)\sim \frac{2\ln(r)}{r}$.*

2. *If $a>0$, then $\widetilde{Y}_{0,0}(r)$ can be analytically extended to the interval $(-\infty,a)$. Furthermore, when $r\to a^-, \widetilde{Y}_{0,0}(r)\sim\frac{\sqrt{2}\pi}{\sqrt{a(a-r)}}$; when $r\to-\infty, \widetilde{Y}_{0,0}(r)\sim-\frac{2\ln(-r)}{r}$.*

3. *If $b>0$, then $Z_{0,0}(r)$ and $Z_{1,0}(r)$ can be analytically extended to the interval $(-b,b)$. Furthermore, when $r\to b^-$, $Z_{0,0}(r)\sim\frac{\pi}{\sqrt{2b(b-r)}}$, $Z_{1,0}(r)\sim\frac{1}{b}\ln\frac{2b}{b-r}$; when $r\to(-b)^+, Z_{0,0}(r)\sim \frac{\pi}{\sqrt{2b(b+r)}}$, $Z_{1,0}(r)\sim\frac{1}{b}\ln\frac{2b}{b+r}$.*

And other cases can be discussed similarly.

Proof. For $a>0$, when $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$, the function $\frac{1}{a+r\cos\theta}$ is an analytic function of r in $(-a,+\infty)$, so $Y_{0,0}(r)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{a+r\cos\theta}d\theta$ is also analytic.

For $r\in(-a,0), Y_{0,0}(r)$ takes the same form as given for $r\in(0,a)$. Thus we have that when $r\to(-a)^+$,

$$Y_{0,0}(r)\sim\frac{\sqrt{2}\pi}{\sqrt{a(a+r)}},$$

when $r\to+\infty$,

$$Y_{0,0}(r)\sim\frac{2\ln r}{r}.$$

Similarly, we can prove other results. □

Lemma 4.3.

1. *If* $a > 0$, *then* $Y_{0,0}(r)$ *can be analytically extended to the complex domain* $D_1 = \mathbb{C} \setminus \{r \in \mathbb{R} \mid r \leq -a\}$.
2. *If* $a > 0$, *then* $\widetilde{Y}_{0,0}(r)$ *can be analytically extended to the complex domain* $D_2 = \mathbb{C} \setminus \{r \in \mathbb{R} \mid r \geq a\}$.
3. *If* $b > 0$, *then* $Z_{0,0}(r)$ *and* $Z_{1,0}(r)$ *can be analytically extended to the complex domain* $D_3 = \mathbb{C} \setminus \{r \in \mathbb{R} \mid r \leq -b, r \geq b\}$.

And other cases can be discussed similarly.

Proof. It is not difficult to know that $Y_{0,0}(r)$ takes the form

$$Y_{0,0}(r) = \frac{1}{\sqrt{a^2 - r^2}} \left(\pi - \int_0^r \frac{2}{\sqrt{a^2 - z^2}} dz \right).$$

The function $\sqrt{a^2 - r^2}$ is analytic in the domain $D^* = \mathbb{C} \setminus \{r \in \mathbb{R} \mid r^2 - a^2 \geq 0\}$, so $Y_{0,0}(r)$ is analytic in the domain D^*. Together with Lemma 4.2, $Y_{0,0}(r)$ is analytic in $D_1 = \mathbb{C} \setminus \{r \in \mathbb{R} \mid r \leq -a\}$.

The other results can be obtained in a similar way. □

For $r \leq -a$ with $a > 0$, denote by $Y^+_{0,0}(r)$ ($Y^-_{0,0}(r)$, resp.) the analytic continuation of $Y_{0,0}(\bar{r})$ along an arc such that $Im(\bar{r}) > 0$ ($Im(\bar{r}) < 0$, resp.) for $\bar{r} \in D_1$. Similarly, we can define $\widetilde{Y}^{\pm}_{0,0}(r), Z^{\pm}_{0,0}(r)$ and $Z^{\pm}_{1,0}(r)$ for $r \in \mathbb{C} \setminus D_2$ and $r \in \mathbb{C} \setminus D_3$, respectively.

Lemma 4.4.

1. *If* $a > 0$, *then the functions* $Y^{\pm}_{0,0}(r)$ *defined in* $(-\infty, -a)$ *satisfy*

$$Y^+_{0,0}(r) - Y^-_{0,0}(r) = \frac{c_1 i}{\sqrt{r^2 - a^2}},$$

2. *If* $a > 0$, *then the functions* $\widetilde{Y}^{\pm}_{0,0}(r)$ *defined in* $(a, +\infty)$ *satisfy*

$$\widetilde{Y}^+_{0,0}(r) - \widetilde{Y}^-_{0,0}(r) = \frac{c_2 i}{\sqrt{r^2 - a^2}},$$

3. *If* $b > 0$, *then the functions* $Z^{\pm}_{0,0}(r)$ *defined in* $(-\infty, -b) \cup (b, +\infty)$ *satisfies*

$$Z^+_{0,0}(r) - Z^-_{0,0}(r) = \frac{2i\pi}{\sqrt{r^2 - b^2}}, \qquad r \in (b, +\infty),$$
$$Z^+_{0,0}(r) - Z^-_{0,0}(r) = \frac{-2i\pi}{\sqrt{r^2 - b^2}}, \qquad r \in (-\infty, -b).$$

4. *If* $b > 0$, *then the functions* $Z^{\pm}_{1,0}(r)$ *defined in* $(-\infty, -b) \cup (b, +\infty)$ *satisfy*

$$Z^+_{1,0}(r) - Z^-_{1,0}(r) = \frac{2i\pi}{r},$$

where c_1 *and* c_2 *are all nonzero real numbers and* $i^2 = -1$.

Proof. 1. A straightforward computation shows

$$\left(a^2 - r^2\right) Y_{0,0}'(r) = rY_{0,0}(r) - 2.$$

Noting that $Y_{0,0}^{\pm}(r)$ are the analytic continuation of $Y_{0,0}(r)$ for $r \in (-\infty, -a)$, we have

$$\left(a^2 - r^2\right) Y_{0,0}^{\pm\,'}(r) = rY_{0,0}^{\pm}(r) - 2,$$

which leads to

$$\left(a^2 - r^2\right)\left(Y_{0,0}^{+}(r) - Y_{0,0}^{-}(r)\right)' = r\left(Y_{0,0}^{+}(r) - Y_{0,0}^{-}(r)\right).$$

Solving the equation, we get that

$$Y_{0,0}^{+}(r) - Y_{0,0}^{-}(r) = \frac{c_1^*}{\sqrt{r^2 - a^2}},$$

where $c_1^* = c_1 i, c_1$ is a nonzero real number. Otherwise, $-a$ is the analytic point or the pole of $Y_{0,0}(r)$, which contradicts with the fact

$$Y_{0,0}(r) \sim \frac{\sqrt{2}\pi}{\sqrt{a(a+r)}}$$

as $r \to (-a)^+$.

The second result can be proved in a similar way.

2. Note that $Z_{0,0}^{\pm}(r)$ are both analytic continuation of $Z_{0,0}(r)$ for $r \in (b, +\infty)$, where b is not an analytic point, so we have

$$\begin{aligned} Z_{0,0}^{+}(r) - Z_{0,0}^{-}(r) &= \frac{\pi}{\sqrt{b+r}} |b-r|^{-\frac{1}{2}} e^{i\frac{\pi}{2}} - \frac{\pi}{\sqrt{b+r}} |b-r|^{-\frac{1}{2}} e^{-i\frac{\pi}{2}} \\ &= \frac{2i\pi}{\sqrt{r^2 - b^2}}. \end{aligned}$$

In a similar way, we can get the other results.

Hence we complete the proof of Lemma 4.4. □

5 Lower bound for the maximum number of zeros of averaged function

In this section we firstly prove the linearly independence of the generating functions of $F(r)$, then give the estimate on the lower bound for the maximum number of zeros of the averaged function $f(r)$.

Lemma 5.1. *For the function $F(r)$, we have*

1. *If $ab \neq 0$ and $|a| \neq |b|$, then the generating functions of the function $F(r)$ in (3.11) are the following $4[\frac{n}{2}] + 2[\frac{n-1}{2}] + 6$ linearly independent functions:*

$$\begin{aligned} &\pi - bZ_{0,0}(r),\ Z_{0,0}(r) - aI_{0,0}(r),\ Z_{0,0}(r) - aJ_{0,0}(r),\ Y_{0,0}(r) - bI_{0,0}(r),\ \widetilde{Y}_{0,0}(r) - bJ_{0,0}(r), \\ &r^2 Y_{0,0}(r),\ r^4 Y_{0,0}(r), \dots, r^{2[\frac{n}{2}]} Y_{0,0}(r),\ r^2 \widetilde{Y}_{0,0}(r),\ r^4 \widetilde{Y}_{0,0}(r), \dots, r^{2[\frac{n}{2}]} \widetilde{Y}_{0,0}(r), \\ &r^2 Z_{0,0}(r),\ r^4 Z_{0,0}(r), \dots, r^{2[\frac{n}{2}]} Z_{0,0}(r),\ rZ_{1,0}(r),\ r^3 Z_{1,0}(r),\ r^5 Z_{1,0}(r), \dots,\ r^{2[\frac{n-1}{2}]+1} Z_{1,0}(r), \\ &r^2, \dots, r^{2[\frac{n-1}{2}]},\ r, r^3, \dots, r^{2[\frac{n}{2}]-1}. \end{aligned} \tag{5.1}$$

2. *If $|a| = |b| \neq 0$, then the generating functions of the function $F(r)$ in (3.11) are the following $3[\frac{n}{2}] + 2[\frac{n-1}{2}] + 6$ linearly independent functions:*

$$\pi - bZ_{0,0}(r),\ Z_{0,0}(r) - aI_{0,0}(r),\ Z_{0,0}(r) - aJ_{0,0}(r),\ Y_{0,0}(r) - bI_{0,0}(r),\ \widetilde{Y}_{0,0}(r) - bJ_{0,0}(r),$$
$$r^2Y_{0,0}(r),\ r^4Y_{0,0}(r),\dots,r^{2[\frac{n}{2}]}Y_{0,0}(r),\ r^2Z_{0,0}(r),\ r^4Z_{0,0}(r),\dots,r^{2[\frac{n}{2}]}Z_{0,0}(r),$$
$$rZ_{1,0}(r),\ r^3Z_{1,0}(r),\ r^5Z_{1,0}(r),\dots,r^{2[\frac{n-1}{2}]+1}Z_{1,0}(r),$$
$$r^2,\dots,r^{2[\frac{n-1}{2}]},\ r, r^3,\dots,r^{2[\frac{n}{2}]-1}.$$

3. *If $b = 0, a \neq 0$, then the generating functions of the function $F(r)$ in (3.12) are the following $3[\frac{n}{2}] + [\frac{n-1}{2}] + 3$ linearly independent functions:*

$$Y_{0,0}(r),\ r^2Y_{0,0}(r),\ r^4Y_{0,0}(r),\dots,r^{2[\frac{n}{2}]}Y_{0,0}(r),$$
$$\widetilde{Y}_{0,0}(r),\ r^2\widetilde{Y}_{0,0}(r),\ r^4\widetilde{Y}_{0,0}(r),\dots,r^{2[\frac{n}{2}]}\widetilde{Y}_{0,0}(r),$$
$$1, r^2,\dots,r^{2[\frac{n-1}{2}]},\ r,\ r^3,\dots,r^{2[\frac{n}{2}]-1}.$$

where $Y_{0,0}(r), \widetilde{Y}_{0,0}(r), Z_{0,0}(r)$ and $Z_{1,0}(r)$ are given by (3.8).

Proof. We only prove the first result, the other ones are similar.

From (3.11), we can analytically extend the domain of $F(r)$ to the complex plane $\mathbb{C} \setminus \{r \in \mathbb{R} \mid |r| \geq \min\{|a|, |b|\}\}$ for $ab \neq 0$, and suppose that there exist some coefficients such that $F(r) \equiv 0$, that is

$$\begin{aligned} F(r) = {} & a_1\Big(\pi - bZ_{0,0}(r)\Big) + a_2\Big(Z_{0,0}(r) - aI_{0,0}(r)\Big) \\ & + a_3\Big(Y_{0,0}(r) - bI_{0,0}(r)\Big) + a_4\Big(Z_{0,0}(r) - aJ_{0,0}(r)\Big) \\ & + a_5\Big(\widetilde{Y}_{0,0}(r) - bJ_{0,0}(r)\Big) + \sum_{i=1}^{[\frac{n}{2}]} b_i r^{2i} Y_{0,0}(r) + \sum_{i=1}^{[\frac{n}{2}]} c_i r^{2i}\widetilde{Y}_{0,0}(r) + \sum_{i=1}^{[\frac{n}{2}]} d_i r^{2i} Z_{0,0}(r) \\ & + \sum_{i=0}^{[\frac{n-1}{2}]} e_i r^{2i+1} Z_{1,0}(r) + \sum_{i=1}^{[\frac{n-1}{2}]} f_i r^{2i} + \sum_{i=0}^{[\frac{n}{2}]-1} g_i r^{2i+1} \equiv 0, \end{aligned} \tag{5.2}$$

we just need to prove $a_i = 0 \ \ (i = 1, 2\dots, 5), b_i = c_i = d_i = 0 \ \ \big(i = 1,\dots,[\frac{n}{2}]\big)$, $e_i = 0 \ \ \big(i = 0, 1,\dots,[\frac{n-1}{2}]\big)$, $f_i = 0 \ \ \big(i = 1,\dots,[\frac{n-1}{2}]\big)$, and $g_i = 0 \ \ \big(i = 0, 1,\dots,[\frac{n}{2}] - 1\big)$.

Obviously, $F(r) \equiv 0$ is equivalent to

$$V_*(r) := (-a^2 - b^2 + r^2)F(r) \equiv 0$$

for $r \in \mathbb{C} \setminus \{r \in \mathbb{R} \| r| \geq min\{|a|, |b|\}\}$ and $ab \neq 0$.

From (5.2), we have

$$\begin{aligned} V_*(r) = {} & \sum_{i=0}^{[\frac{n+2}{2}]} \overline{A}_i r^{2i} Y_{0,0}(r) + \sum_{i=0}^{[\frac{n+2}{2}]} \overline{B}_i r^{2i}\widetilde{Y}_{0,0}(r) + \sum_{i=0}^{[\frac{n+2}{2}]} \overline{C}_i r^{2i} Z_{0,0}(r) \\ & + \sum_{i=0}^{[\frac{n+1}{2}]} \overline{D}_i r^{2i+1} Z_{1,0}(r) + \sum_{i=0}^{[\frac{n-1}{2}]+1} \overline{E}_i r^{2i} + \sum_{i=0}^{[\frac{n}{2}]} \overline{F}_i r^{2i+1}, \end{aligned}$$

where

$$\overline{A}_0 = aba_2 - a^2a_3, \quad \overline{A}_1 = a_3 - (a^2+b^2)b_1, \quad \overline{A}_i = b_{i-1} - (a^2+b^2)b_i, \quad \overline{A}_{[\frac{n+2}{2}]} = b_{[\frac{n}{2}]},$$
$$\overline{B}_0 = aba_4 - a^2a_5, \quad \overline{B}_1 = a_5 - (a^2+b^2)c_1, \quad \overline{B}_i = c_{i-1} - (a^2+b^2)c_i, \quad \overline{B}_{[\frac{n+2}{2}]} = c_{[\frac{n}{2}]},$$
$$\overline{C}_0 = b(a^2+b^2)a_1 - b^2(a_2+a_4) + ab(a_3+a_5), \quad \overline{C}_1 = -ba_1 + a_2 + a_4 - (a^2+b^2)d_1,$$
$$\overline{C}_i = d_{i-1} - (a^2+b^2)d_i, \quad \overline{C}_{[\frac{n+2}{2}]} = d_{[\frac{n}{2}]},$$
$$\overline{D}_0 = a(a_4-a_2) + b(a_5-a_3) - (a^2+b^2)e_0, \quad \overline{D}_i = e_{i-1} - (a^2+b^2)e_i, \quad \overline{D}_{[\frac{n+1}{2}]} = e_{[\frac{n-1}{2}]},$$
$$\overline{E}_0 = (-a^2-b^2)a_1\pi, \quad \overline{E}_1 = a_1\pi - (a^2+b^2)f_1, \quad \overline{E}_i = f_{i-1} - (a^2+b^2)f_i, \quad \overline{E}_{[\frac{n-1}{2}]+1} = f_{[\frac{n-1}{2}]},$$
$$\overline{F}_0 = (-a^2-b^2)g_0, \quad \overline{F}_i = g_{i-1} - (a^2+b^2)g_i, \quad \overline{F}_{[\frac{n}{2}]} = g_{[\frac{n}{2}]-1}. \tag{5.3}$$

If $a > b > 0$, then $V_*(r)$ are analytic in $D = D_1 \cap D_2 \cap D_3 = \mathbb{C} \setminus \{r \in \mathbb{R} | r \geq b,\ r \leq -b\}$.

Similarly, we can define the functions $V_*^{\pm}(r)$ as the analytic continuation of $V_*(r)$ to $(-\infty, -b) \cup (b, +\infty)$ from the upper and lower half planes, respectively.

By Lemma 4.4, when $r \in (b, a)$, we have

$$V_*^+(r) - V_*^-(r) = \frac{2i\pi}{\sqrt{r^2-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} \overline{C}_i r^{2i} + \frac{2i\pi}{r} \sum_{i=0}^{[\frac{n+1}{2}]} \overline{D}_i r^{2i+1}.$$

Thus $V_*^+(r) - V_*^-(r) \equiv 0$ yields

$$\frac{1}{\sqrt{r^2-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} \overline{C}_i r^{2i} + \sum_{i=0}^{[\frac{n+1}{2}]} \overline{D}_i r^{2i} \equiv 0,$$

which implies that $\overline{C}_i = 0 \ \left(i = 0,1,2,\ldots,[\frac{n+2}{2}]\right), \overline{D}_i = 0 \ \left(i = 0,1,2,\ldots,[\frac{n+1}{2}]\right)$.

Similarly, when $r \in (a, +\infty)$,

$$V_*^+(r) - V_*^-(r) = \frac{c_2^*}{\sqrt{r^2-a^2}} \sum_{i=0}^{[\frac{n+2}{2}]} \overline{B}_i r^{2i} \equiv 0$$

if and only if $\overline{B}_i = 0 \ \left(i = 0,1,2,\ldots,[\frac{n+2}{2}]\right)$.

And when $r \in (-\infty, -a)$,

$$V_*^+(r) - V_*^-(r) = \frac{c_1^*}{\sqrt{r^2-a^2}} \sum_{i=0}^{[\frac{n+2}{2}]} \overline{A}_i r^{2i} \equiv 0$$

if and only if $\overline{A}_i = 0 \ \left(i = 0,1,2,\ldots,[\frac{n+2}{2}]\right)$.

Consequently, $V_*(r) \equiv 0$ becomes

$$\sum_{i=0}^{[\frac{n-1}{2}]+1} \overline{E}_i r^{2i} + \sum_{i=0}^{[\frac{n}{2}]} \overline{F}_i r^{2i+1} \equiv 0,$$

so we get $\overline{E}_i = 0 \ \left(i = 0,1,2,\ldots,[\frac{n-1}{2}]+1\right)$ and $\overline{F}_i = 0 \ \left(i = 0,1,2,\ldots,[\frac{n}{2}]\right)$.

The above results lead to $a_i = 0 \ (i = 1,2,\ldots,5), b_i = c_i = d_i = 0 \ \left(i = 1,\ldots,[\frac{n}{2}]\right)$, $e_i = 0 \ \left(i = 0,1,\ldots,[\frac{n-1}{2}]\right)$, $f_i = 0 \ \left(i = 1,\ldots,[\frac{n-1}{2}]\right)$, and $g_i = 0 \ \left(i = 0,1,\ldots,[\frac{n}{2}]-1\right)$.

Hence, for the case $a > b > 0$, the functions listed in (5.1) are $4[\frac{n}{2}] + 2[\frac{n-1}{2}] + 6$ linearly independent generating ones of $F(r)$.

The other cases for $ab \neq 0, |a| \neq |b|$ can be investigated in the similar way.

This completes the proof of the first result in Lemma 5.1. □

Recall that $[\frac{n}{2}] + [\frac{n-1}{2}] = n - 1$, then Lemma 5.2 follows from Remark 3.1, Lemmas 2.3 and 5.1.

Lemma 5.2. *Denoting by $N(f)$ the maximum number of simple zeros of the averaged function $f(r)$ in $r \in (0, min\{|a|, |b|\})$ for $ab \neq 0$ or $r \in (0, |a|)$ for $b = 0, a \neq 0$, we have*

1. *when $ab \neq 0$ and $|a| \neq |b|$, $N(f) \geq 2\left[\frac{n}{2}\right] + 2n + 3$;*
2. *when $|a| = |b| \neq 0$, $N(f) \geq \left[\frac{n}{2}\right] + 2n + 3$;*
3. *when $b = 0, a \neq 0, N(f) \geq 2\left[\frac{n}{2}\right] + n + 1$,*

where $[\cdot]$ denotes the integer function.

6 Upper bound for the number of zeros of averaged function

In this section, we extend the variable r to the complex plane to investigate the upper bound for the number of zeros of the function $F(r)$, which is closely related to that of the averaged function $f(r)$.

Here, we look r as the complex number.

Lemma 6.1. *For the complex variable r, we have*

1. *If $a > 0$, then when $r \to (-a)^+, Y_{0,0}(r) \sim \frac{\sqrt{2}\pi}{\sqrt{a(a+r)}}$; when $r \to \infty, Y_{0,0}(r) \sim \frac{2\ln(r)}{r}$.*
2. *If $a > 0$, then when $r \to a^-, \widetilde{Y}_{0,0}(r) \sim \frac{\sqrt{2}\pi}{\sqrt{a(a-r)}}$; when $r \to \infty, \widetilde{Y}_{0,0}(r) \sim -\frac{2\ln(-r)}{r}$.*
3. *If $b > 0$, then when $r \to b^-, Z_{0,0}(r) \sim \frac{\pi}{\sqrt{2b(b-r)}}$, $Z_{1,0}(r) \sim \frac{1}{b}\ln\frac{2b}{b-r}$; when $r \to (-b)^+$, $Z_{0,0}(r) \sim \frac{\pi}{\sqrt{2b(b+r)}}$, $Z_{1,0}(r) \sim \frac{1}{b}\ln\frac{2b}{b+r}$.*

And other cases can be discussed similarly.

Proof. For $Y_{0,0}(r)$, we have

$$\begin{aligned} \left(a^2 - r^2\right) Y_{0,0}'(r) &= rY_{0,0}(r) - 2, \\ \left(a^2 - r^2\right) Y_{0,0}''(r) - 3rY_{0,0}'(r) - Y_{0,0}(r) &= 0. \end{aligned} \tag{6.1}$$

It is easy to check that ∞ and $-a$ are the regular singular points (see [27]) of the second equation in (6.1), so when the complex variable $r \to -a$ and ∞, the solution $Y_{0,0}(r)$ has the same properties as for the real number r. This fact together with Lemma 4.2 yields the first results.

Similarly, the other results can be proved. □

The results concerning the number of zeros of $F(r)$ are given as follows.

Lemma 6.2. *When $|a| > |b| \neq 0$, we have*

$$N^*(F) \leq 2\left[\frac{n}{2}\right] + 4n + 14,$$

where $N^*(F)$ denotes the maximum number of non-zero simple zeros of $F(r)$ in $D = D_1 \cap D_2 \cap D_3 = \mathbb{C} \setminus \{r \in \mathbb{R} \mid |r| \geq |b|\}$, and $[\cdot]$ is the integer function.

Proof. We only give the detailed proof for the case $a > b > 0$, other results can be proved in a similar way.

Let $0 < \varepsilon \ll 1 \ll R$ and $D_{\varepsilon,R}$ be the domain obtained by removing four small discs

$$C_{b,\varepsilon} = \{r \in \mathbb{C} \mid |r-b| \le \varepsilon\}, \qquad C_{a,\varepsilon} = \{r \in \mathbb{C} \mid |r-a| \le \varepsilon\},$$
$$C_{-b,\varepsilon} = \{r \in \mathbb{C} \mid |r+b| \le \varepsilon\}, \qquad C_{-a,\varepsilon} = \{r \in \mathbb{C} \mid |r+a| \le \varepsilon\}$$

and four real intervals

$$L_1 = [b+\varepsilon, a-\varepsilon], \qquad L_2 = [a+\varepsilon, R],$$
$$L_3 = [-a+\varepsilon, -b-\varepsilon], \qquad L_4 = [-R, -a-\varepsilon]$$

from $C_R = \{r \in \mathbb{C} \mid |r| \le R\}$, $L_i^{\pm}$ be the upper and lower bounds of L_i for $i = 1,2,3,4$, see Figure 6.1.

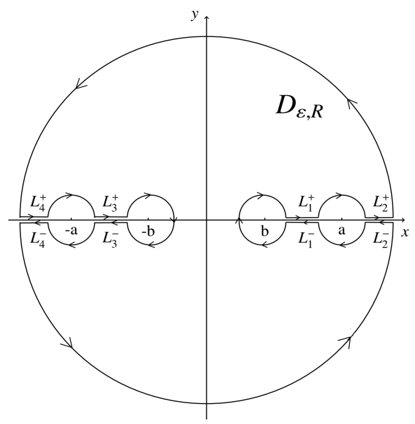

Figure 6.1: The domain $D_{\varepsilon,R}$

Recalling $F(r)$ in (3.11), we define

$$\begin{aligned} V(r) &:= \left(r^2 - a^2 - b^2\right) F(r) \\ &= \sum_{i=0}^{[\frac{n+2}{2}]} \widetilde{A}_i r^{2i} Y_{0,0}(r) + \sum_{i=0}^{[\frac{n+2}{2}]} \widetilde{B}_i r^{2i} \widetilde{Y}_{0,0}(r) + \sum_{i=0}^{[\frac{n+2}{2}]} \widetilde{C}_i r^{2i} Z_{0,0}(r) \\ &\quad + \sum_{i=0}^{[\frac{n+1}{2}]} \widetilde{D}_i r^{2i+1} Z_{1,0}(r) + \sum_{i=0}^{[\frac{n-1}{2}]+1} \widetilde{E}_i r^{2i} + \sum_{i=0}^{[\frac{n}{2}]} \widetilde{F}_i r^{2i+1}, \end{aligned}$$

where $\widetilde{A}_i, \widetilde{B}_i, \widetilde{C}_i, \widetilde{D}_i, \widetilde{E}_i$ and $\widetilde{F}_i$ are similar to the coefficients of (5.3), with a_i $(i = 1,2\ldots,5)$, b_i, c_i, d_i $\left(i = 1,\ldots,[\frac{n}{2}]\right)$, e_i $\left(i = 0,1,\ldots,[\frac{n-1}{2}]\right)$, f_i $\left(i = 1,\ldots,[\frac{n-1}{2}]\right)$ and g_i $\left(i = 0,1,\ldots,[\frac{n}{2}]-1\right)$ being replaced by A_i $(i = 1,2\ldots,5)$, B_i, C_i, D_i $\left(i = 1,\ldots,[\frac{n}{2}]\right)$, E_i $\left(i = 0,1,\ldots,[\frac{n-1}{2}]\right)$, F_i $\left(i = 1,\ldots,[\frac{n-1}{2}]\right)$ and G_i $\left(i = 0,1,\ldots,[\frac{n}{2}]-1\right)$ in (3.11), respectively.

Since the zeros of $F(r)$ coincide with those of $V(r)$ for $r \in D$, we investigate the latter instead of the former. Now consider the change of the argument of $V(r)$ along the boundary of the domain $D_{\varepsilon,R}$.

On $\partial C_{b,\varepsilon}$, $V(r) \sim \frac{C^*}{\sqrt{b-r}}$, where C^* is a constant. This means that the argument of $V(r)$ increases by $\pi + o(1)$. Since $V(b-\varepsilon)$ is a real number, it intersects the real axis only once.

On L_1^+, $V(r)$ is real if and only if $\operatorname{Im} V(r) = 0$, that is

$$0 = V^+(r) - V^-(r) = \frac{2i\pi}{\sqrt{r^2-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} C_i r^{2i} + \frac{2i\pi}{r} \sum_{i=0}^{[\frac{n+1}{2}]} D_i r^{2i+1},$$

where $V^{\pm}(r)$ denote the analytic continuation of $V(r)$ to $(-\infty,-b)\cup(b,+\infty)$ from the upper and lower half planes, respectively. Then letting $u = r^2$, we can get that

$$\sum_{i=0}^{[\frac{n+2}{2}]} C_i u^i + \sqrt{u-b^2} \sum_{i=0}^{[\frac{n+1}{2}]} D_i u^i = 0.$$

Define

$$V_1(u) = \sum_{i=0}^{[\frac{n+2}{2}]} C_i u^i + \sqrt{u-b^2} \sum_{i=0}^{[\frac{n+1}{2}]} D_i u^i = 0,$$

and denote by n_1 the number of the root of the function $V_1(u)$. Then $V(u)$ intersects the real axis exactly n_1 times, which holds for L_1^-. So the argument of $V(r)$ increases by at most $2(n_1+1)\pi + o(1)$ on $\partial C_{b,\varepsilon} \cup L_1^{\pm}$.

Similarly, the argument of $V(r)$ increases by at most $2(n_2+1)\pi + o(1)$ on $\partial C_{-b,\varepsilon} \cup L_3^{\pm}$, where n_2 is the number of zeros of the function $V_2(u)$ defined by

$$V_2(u) = -\sum_{i=0}^{[\frac{n+2}{2}]} C_i u^i + \sqrt{u-b^2} \sum_{i=0}^{[\frac{n+1}{2}]} D_i u^i.$$

On $\partial C_{a,\varepsilon}$, $V(r) \sim \frac{C^{**}}{\sqrt{a-r}}$ for some constant C^{**}, which implies that the argument of $V(r)$ increases by at most $\pi + o(1)$.

On L_2^+, $V(r)$ is real if and only if $\operatorname{Im} V(r) = 0$, that is

$$\begin{aligned} 0 = \frac{V^+(r)-V^-(r)}{i} &= \frac{c_2}{\sqrt{r^2-a^2}} \sum_{i=0}^{[\frac{n+2}{2}]} B_i r^{2i} + \frac{2\pi}{\sqrt{r^2-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} C_i r^{2i} + \frac{2\pi}{r} \sum_{i=0}^{[\frac{n+1}{2}]} D_i r^{2i+1} \\ &= \frac{1}{\sqrt{r^2-a^2}} \sum_{i=0}^{[\frac{n+2}{2}]} B_i^* r^{2i} + \frac{1}{\sqrt{r^2-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} C_i^* r^{2i} + \sum_{i=0}^{[\frac{n+1}{2}]} D_i^* r^{2i}, \end{aligned} \tag{6.2}$$

where $B_i^* = c_2 B_i, C_i^* = 2\pi C_i, D_i^* = 2\pi D_i$. Let $u = r^2$, then the function (6.2) becomes

$$\frac{1}{\sqrt{u-a^2}} \sum_{i=0}^{[\frac{n+2}{2}]} B_i^* u^i + \frac{1}{\sqrt{u-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} C_i^* u^i + \sum_{i=0}^{[\frac{n+1}{2}]} D_i^* u^i = 0. \tag{6.3}$$

By the Argument Principle, the number of roots of (6.3) is not greater than $n + [\frac{n}{2}] + 4$ on L_2^+. So $V(r)$ intersects the real axis at most $n + [\frac{n}{2}] + 4$ times, which is true for L_2^-. Then the argument of $V(r)$ totally increases by at most $2\left(n + [\frac{n}{2}] + 5\right)\pi + o(1)$ on $C_{a,\varepsilon} \cup L_2^{\pm}$.

In a similar way, we get that the argument of $V(r)$ increases by at most $2\left(n + [\frac{n}{2}] + 5\right)\pi + o(1)$ on $C_{-a,\varepsilon} \cup L_4^{\pm}$.

On C_R, we have

$$\begin{aligned} &r^{2[\frac{n+2}{2}]} Y_{0,0}(r) \sim r^{2[\frac{n+2}{2}]-1} \ln r, && r^{2[\frac{n+2}{2}]} \widetilde{Y}_{0,0}(r) \sim r^{2[\frac{n+2}{2}]-1} \ln(-r), \\ &r^{2[\frac{n+2}{2}]} Z_{0,0}(r) \sim r^{2[\frac{n+2}{2}]-1}, && r^{2[\frac{n+1}{2}]+1} Z_{1,0}(r) \sim r^{2[\frac{n+1}{2}]}, \end{aligned}$$

then the corresponding arguments of these terms increase by $2\left(2[\frac{n+2}{2}]-1\right)\pi+o(1)$, $2\left(2[\frac{n+2}{2}]-1\right)\pi+o(1)$, $2\left(2[\frac{n+2}{2}]-1\right)\pi$, $2\left(2[\frac{n+1}{2}]\right)\pi$, respectively. And the argument of r^{n+1} increases by $2(n+1)\pi$. Since $2[\frac{n+2}{2}]-1\leq n+1, 2[\frac{n+1}{2}]\leq n+1$, the argument of $V(r)$ increases by at most $2(n+1)\pi+o(1)$.

So along the boundary of $D_{\varepsilon,R}$, the arguments of $V(r)$ increase by at most

$$\begin{aligned}&2(n_1+1)\pi+2\left(n+\left[\frac{n}{2}\right]+5\right)\pi+2(n_2+1)\pi+2\left(n+\left[\frac{n}{2}\right]+5\right)\pi+2(n+1)\pi+o(1)\\&\quad=2\left(3n+n_1+n_2+2\left[\frac{n}{2}\right]+13\right)\pi+o(1).\end{aligned}$$

On the other hand, a straightforward computation yields

$$\begin{aligned}V_1(u)\cdot V_2(u)&=\left(\sum_{i=0}^{[\frac{n+2}{2}]}C_iu^i+\sqrt{u-b^2}\sum_{i=0}^{[\frac{n+1}{2}]}D_iu^i\right)\cdot\left(-\sum_{i=0}^{[\frac{n+2}{2}]}C_iu^i+\sqrt{u-b^2}\sum_{i=0}^{[\frac{n+1}{2}]}D_iu^i\right)\\&=(u-b^2)\sum_{i=0}^{2[\frac{n+1}{2}]}\widetilde{D}_iu^i-\sum_{i=0}^{2[\frac{n+2}{2}]}\widetilde{C}_iu^i,\end{aligned}$$

which implies that $V_1(u)\cdot V_2(u)=0$ has at most $n+2$ zeros, taking into account the multiplicities. Thus we have $n_1+n_2\leq n+2$. By the Argument Principle, $V(r)$ has at most $4n+2[\frac{n}{2}]+15$ zeros in $D_{\varepsilon,R}$.

Let $\varepsilon\to 0$ and $R\to+\infty$, then we have that $V(r)$ has at most $4n+2\left[\frac{n}{2}\right]+15$ zeros in $D=D_1\cap D_2\cap D_3$ obtained by removing two real intervals $(-\infty,-b]$ and $[b,+\infty)$ from $\mathbb{C}$. Since $V(0)=0$, we get that

$$N^*(V)\leq 2\left[\frac{n}{2}\right]+4n+14,$$

where $N^*(V)$ denotes the maximum number of the non-zero simple zeros of $V(r)$. Thus we also have

$$N^*(F)\leq 2\left[\frac{n}{2}\right]+4n+14.$$

This completes the proof of Lemma 6.2 for the case $a>b>0$. □

Similarly, we have

Lemma 6.3.

1. *When* $|b|>|a|\neq 0$, $N^*(F)\leq 4[\frac{n}{2}]+3n+14$.
2. *When* $|a|=|b|\neq 0$, $N^*(F)\leq 3n+6$.
3. *When* $b=0, a\neq 0$, $N^*(F)\leq 2[\frac{n}{2}]+n+1$, *where* $N^*(F)$ *denotes the maximum number of non-zero simple zeros of* $F(r)$ *in* $r\in(0,min\{|a|,|b|\})$ *for* $ab\neq 0$ *or* $r\in(0,|a|)$ *for* $b=0, a\neq 0$.

Based on Remark 3.1, we get the upper bound of zeros of the averaged function $f(r)$, which is stated as follows.

Lemma 6.4. *Let* $N(f)$ *be the maximum number of simple zeros of* $f(r)$ *in* $r\in(0,min\{|a|,|b|\})$ *for* $ab\neq 0$ *or* $r\in(0,|a|)$ *for* $b=0, a\neq 0$, *then the following statements hold.*

1. *When* $ab\neq 0$ *and* $|a|>|b|$, $N(f)\leq 2[\frac{n}{2}]+4n+14$;
2. *When* $ab\neq 0$ *and* $|a|<|b|$, $N(f)\leq 4[\frac{n}{2}]+3n+14$;

3. *When* $|a| = |b| \neq 0,\ N(f) \leq 3n + 6$;
4. *When* $b = 0, a \neq 0,\ N(f) \leq 2[\frac{n}{2}] + n + 1$,

where $[\cdot]$ *is the integer function.*

7 Proof of Theorem 1.1

Following Lemmas 5.2 and 6.4, we obtain Lemma 7.1 below.

Lemma 7.1. *The following statements are true.*

1. *When* $|a| > |b| \neq 0,\ 2[\frac{n}{2}] + 2n + 3 \leq N(f) \leq 2[\frac{n}{2}] + 4n + 14$;
2. *When* $|b| > |a| \neq 0,\ 2[\frac{n}{2}] + 2n + 3 \leq N(f) \leq 4[\frac{n}{2}] + 3n + 14$;
3. *When* $|a| = |b| \neq 0,\ [\frac{n}{2}] + 2n + 3 \leq N(f) \leq 3n + 6$;
4. *When* $b = 0, a \neq 0,\ N(f) = 2[\frac{n}{2}] + n + 1$,

where $N(f)$ *is the same as defined by Lemma 6.4.*

Proof of Theorem 1.1. By the first order averaging theory, the number of non-zero simple zeros of the averaged function $f(r)$ corresponds to that of limit cycles bifurcating from the period annulus around the center of the unperturbed systems (1.1). Then Theorem 1.1 follows from Lemmas 7.1 and 2.1.

This completes the proof of Theorem 1.1. □

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Appendix. Coefficients of the function $F(r)$ in (3.11)

For the odd number n, the coefficients in (3.11) take the form

$$A_1 = \sum_{i=0}^{\frac{n-1}{2}} \Big(T_{2i+1,0} + \widetilde{T}_{2i+1,0}\Big)\Big(-b^2\Big)^i - \frac{1}{b}\sum_{i=1}^{\frac{n-1}{2}} \Big(W_{2i+1,0} + \widetilde{W}_{2i+1,0}\Big)\Big(-b^2\Big)^i,$$

$$A_2 = W_{1,0} - a\sum_{j=1}^{\frac{n+1}{2}} W_{0,j}\Big(a^2+b^2\Big)^{j-1} + ab\sum_{j=1}^{\frac{n-1}{2}} T_{0,j}\Big(a^2+b^2\Big)^{j-1},$$

$$A_3 = T_{0,0} - b\sum_{j=1}^{\frac{n+1}{2}} W_{0,j}\Big(a^2+b^2\Big)^{j-1} + b^2\sum_{j=1}^{\frac{n-1}{2}} T_{0,j}\Big(a^2+b^2\Big)^{j-1},$$

$$A_4 = \widetilde{W}_{1,0} - a\sum_{j=1}^{\frac{n+1}{2}} \widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-1} + ab\sum_{j=1}^{\frac{n-1}{2}} \widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-1},$$

$$A_5 = \widetilde{T}_{0,0} - b\sum_{j=1}^{\frac{n+1}{2}} \widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-1} + b^2\sum_{j=1}^{\frac{n-1}{2}} \widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-1},$$

$$B_i = T_{0,i} - b\sum_{j=i+1}^{\frac{n+1}{2}} W_{0,j}\Big(a^2+b^2\Big)^{j-i-1} + b^2\sum_{j=i+1}^{\frac{n-1}{2}} T_{0,j}\Big(a^2+b^2\Big)^{j-i-1},$$

$$B_{\left[\frac{n}{2}\right]} = T_{0,\frac{n-1}{2}} - bW_{0,\frac{n+1}{2}},$$

$$C_i = \widetilde{T}_{0,i} - b\sum_{j=i+1}^{\frac{n+1}{2}} \widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-i-1} + b^2\sum_{j=i+1}^{\frac{n-1}{2}} \widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-i-1},$$

$$C_{\left[\frac{n}{2}\right]} = \widetilde{T}_{0,\frac{n-1}{2}} - b\widetilde{W}_{0,\frac{n+1}{2}},$$

$$\begin{aligned} D_i = &\sum_{t=0}^{\frac{n-1}{2}-i}\sum_{k+j=i}\Big(W_{2k+2t+1,j}+\widetilde{W}_{2k+2t+1,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t - a\sum_{j=i+1}^{\frac{n+1}{2}}\Big(W_{0,j}+\widetilde{W}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1} \\ &- b\sum_{t=0}^{\frac{n-1}{2}-i}\sum_{k+j=i}\Big(T_{2k+2t+1,j}+\widetilde{T}_{2k+2t+1,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t + ab\sum_{j=i+1}^{\frac{n-1}{2}}\Big(T_{0,j}+\widetilde{T}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1}, \end{aligned}$$

$$D_{\left[\frac{n}{2}\right]} = \sum_{k+j=\frac{n-1}{2}}\Big(W_{2k+1,j}+\widetilde{W}_{2k+1,j}\Big) - b\sum_{k+j=\frac{n-1}{2}}\Big(T_{2k+1,j}+\widetilde{T}_{2k+1,j}\Big) - a\Big(W_{0,\frac{n+1}{2}}+\widetilde{W}_{0,\frac{n+1}{2}}\Big),$$

$$\begin{aligned} E_i = &\sum_{t=0}^{\frac{n-1}{2}-i}\sum_{k+j=i}\Big(W_{2k+2t+2,j}-\widetilde{W}_{2k+2t+2,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t + \sum_{j=i+1}^{\frac{n+1}{2}}\Big(W_{0,j}-\widetilde{W}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1} \\ &- b\sum_{t=0}^{\frac{n-1}{2}-i-1}\sum_{k+j=i}\Big(T_{2k+2t+2,j}-\widetilde{T}_{2k+2t+2,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t - b\sum_{j=i+1}^{\frac{n-1}{2}}\Big(T_{0,j}-\widetilde{T}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1}, \end{aligned}$$

$$E_{\left[\frac{n-1}{2}\right]} = \sum_{k+j=\frac{n-1}{2}}\Big(W_{2k+2,j}-\widetilde{W}_{2k+2,j}\Big) + \Big(W_{0,\frac{n+1}{2}}-\widetilde{W}_{0,\frac{n+1}{2}}\Big),$$

$$\begin{aligned} F_i = &\sum_{k+j=i}\Big(T_{2k+1,j}c^* + \widetilde{T}_{2k+1,j}\widetilde{c}^*\Big)m_{2k} \\ &+ \sum_{h=1}^{\frac{n-1}{2}-i}\sum_{k+j=i}(-b^2)^{h-1}\Big(W_{2k+2h+1,j}b^* + \widetilde{W}_{2k+2h+1,j}\widetilde{b}^*\Big)\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t-1)} \\ &- b\sum_{h=1}^{\frac{n-1}{2}-i}\sum_{k+j=i}\Big(T_{2k+2h+1,j}b^* + \widetilde{T}_{2k+2h+1,j}\widetilde{b}^*\Big)\Big(-b^2\Big)^{h-1}\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t-1)}, \end{aligned}$$

$$F_{\left[\frac{n-1}{2}\right]} = \sum_{k+j=\frac{n-1}{2}}\Big(T_{2k+1,j}c^* + \widetilde{T}_{2k+1,j}\widetilde{c}^*\Big)m_{2k},$$

$$\begin{aligned} G_i = &\sum_{k+j=i}\Big(T_{2k+2,j}c^* + \widetilde{T}_{2k+2,j}\widetilde{c}^*\Big)m_{2k+1} \\ &+ \sum_{h=1}^{\frac{n-1}{2}-i}\sum_{k+j=i}(-b^2)^{h-1}\Big(W_{2k+2h+2,j}b^* + \widetilde{W}_{2k+2h+2,j}\widetilde{b}^*\Big)\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t)-1} \\ &- b\sum_{h=1}^{\frac{n-3}{2}-i}\sum_{k+j=i}(-b^2)^{h-1}\Big(T_{2k+2h+2,j}b^* + \widetilde{T}_{2k+2h+2,j}\widetilde{b}^*\Big)\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t)-1}, \end{aligned}$$

$$G_{\left[\frac{n}{2}\right]-1} = \sum_{k+j=\frac{n-3}{2}} \Big(T_{2k+2,j}c^* + \widetilde{T}_{2k+2,j}\widetilde{c^*}\Big)m_{2k+1} + \sum_{k+j=\frac{n-3}{2}} \Big(W_{2k+4,j}b^* + \widetilde{W}_{2k+4,j}\widetilde{b}^*\Big) \sum_{t=0}^{k} m_{2(k-t)+1}.$$

For the even number n, the coefficients in (3.11) take the form

$$\begin{aligned}
A_1 &= \sum_{i=0}^{\frac{n}{2}-1} \Big(T_{2i+1,0} + \widetilde{T}_{2i+1,0}\Big)\Big(-b^2\Big)^i - \frac{1}{b}\sum_{i=1}^{\frac{n}{2}} \Big(W_{2i+1,0} + \widetilde{W}_{2i+1,0}\Big)\Big(-b^2\Big)^i,\\
A_2 &= W_{1,0} - a\sum_{j=1}^{\frac{n}{2}} W_{0,j}\Big(a^2+b^2\Big)^{j-1} + ab\sum_{j=1}^{\frac{n}{2}} T_{0,j}\Big(a^2+b^2\Big)^{j-1},\\
A_3 &= T_{0,0} - b\sum_{j=1}^{\frac{n}{2}} W_{0,j}\Big(a^2+b^2\Big)^{j-1} + b^2\sum_{j=1}^{\frac{n}{2}} T_{0,j}\Big(a^2+b^2\Big)^{j-1},\\
A_4 &= \widetilde{W}_{1,0} - a\sum_{j=1}^{\frac{n}{2}} \widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-1} + ab\sum_{j=1}^{\frac{n}{2}} \widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-1},\\
A_5 &= \widetilde{T}_{0,0} - b\sum_{j=1}^{\frac{n}{2}} \widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-1} + b^2\sum_{j=1}^{\frac{n}{2}} \widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-1},\\
B_i &= T_{0,i} - b\sum_{j=i+1}^{\frac{n}{2}} W_{0,j}\Big(a^2+b^2\Big)^{j-i-1} + b^2\sum_{j=i+1}^{\frac{n}{2}} T_{0,j}\Big(a^2+b^2\Big)^{j-i-1},\\
B_{\left[\frac{n}{2}\right]} &= T_{0,\frac{n}{2}},\\
C_i &= \widetilde{T}_{0,i} - b\sum_{j=i+1}^{\frac{n}{2}} \widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-i-1} + b^2\sum_{j=i+1}^{\frac{n}{2}} \widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-i-1},\\
C_{\left[\frac{n}{2}\right]} &= \widetilde{T}_{0,\frac{n}{2}},
\end{aligned}$$

$$\begin{aligned}
D_i = &\sum_{t=0}^{\frac{n}{2}-i}\sum_{k+j=i} \Big(W_{2k+2t+1,j} + \widetilde{W}_{2k+2t+1,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t - a\sum_{j=i+1}^{\frac{n}{2}} \Big(W_{0,j}+\widetilde{W}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1}\\
&- b\sum_{t=0}^{\frac{n}{2}-i-1}\sum_{k+j=i} \Big(T_{2k+2t+1,j} + \widetilde{T}_{2k+2t+1,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t + ab\sum_{j=i+1}^{\frac{n}{2}} \Big(T_{0,j}+\widetilde{T}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1},\\
D_{[\frac{n}{2}]} = &\sum_{k+j=\frac{n}{2}} \Big(W_{2k+1,j} + \widetilde{W}_{2k+1,j}\Big),\\
E_i = &\sum_{t=0}^{\frac{n}{2}-1-i}\sum_{k+j=i} \Big(W_{2k+2t+2,j} - \widetilde{W}_{2k+2t+2,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t + \sum_{j=i+1}^{\frac{n}{2}} \Big(a^2+b^2\Big)^{j-i-1}\Big(W_{0,j}-\widetilde{W}_{0,j}\Big)\\
&- b\sum_{t=0}^{\frac{n}{2}-i-1}\sum_{k+j=i} \Big(T_{2k+2t+2,j} - \widetilde{T}_{2k+2t+2,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t - b\sum_{j=i+1}^{\frac{n}{2}} \Big(a^2+b^2\Big)^{j-i-1}\Big(T_{0,j}-\widetilde{T}_{0,j}\Big),\\
E_{\left[\frac{n-1}{2}\right]} = &\sum_{k+j=\frac{n}{2}-1} \Big(W_{2k+2,j} - \widetilde{W}_{2k+2,j}\Big)\\
&- b\sum_{k+j=\frac{n}{2}-1} \Big(T_{2k+2,j} - \widetilde{T}_{2k+2,j}\Big) + \Big(W_{0,\frac{n}{2}} - \widetilde{W}_{0,\frac{n}{2}}\Big) - b\Big(T_{0,\frac{n}{2}} - \widetilde{T}_{0,\frac{n}{2}}\Big),
\end{aligned}$$

$$
\begin{aligned}
F_i = & \sum_{k+j=i} \Big(T_{2k+1,j} c^* + \widetilde{T}_{2k+1,j} \widetilde{c}^* \Big) m_{2k} \\
& + \sum_{h=1}^{\frac{n}{2}-i} \Big(-b^2 \Big)^{h-1} \sum_{k+j=i} \Big(W_{2k+2h+1,j} b^* + \widetilde{W}_{2k+2h+1,j} \widetilde{b}^* \Big) \sum_{t=h-1}^{k+h-1} C_t^{t-h+1} m_{2(k+h-t-1)} \\
& - b \sum_{h=1}^{\frac{n}{2}-1-i} \sum_{k+j=i} \Big(T_{2k+2h+1,j} b^* + \widetilde{T}_{2k+2h+1,j} \widetilde{b}^* \Big) \Big(-b^2 \Big)^{h-1} \sum_{t=h-1}^{k+h-1} C_t^{t-h+1} m_{2(k+h-t-1)},
\end{aligned}
$$

$$
F_{\left[\frac{n-1}{2}\right]} = \sum_{k+j=\frac{n}{2}-1} \Big(T_{2k+1,j} c^* + \widetilde{T}_{2k+1,j} \widetilde{c}^* \Big) m_{2k} + \sum_{k+j=\frac{n}{2}-1} \Big(W_{2k+3,j} b^* + \widetilde{T}_{2k+3,j} \widetilde{b}^* \Big) \sum_{t=0}^{k} m_{2(k-t)},
$$

$$
\begin{aligned}
G_i = & \sum_{k+j=i} \Big(T_{2k+2,j} c^* + \widetilde{T}_{2k+2,j} \widetilde{c}^* \Big) m_{2k+1} \\
& + \sum_{h=1}^{\frac{n}{2}-1-i} \Big(-b^2 \Big)^{h-1} \sum_{k+j=i} \Big(W_{2k+2h+2,j} b^* + \widetilde{W}_{2k+2h+2,j} \widetilde{b}^* \Big) \sum_{t=h-1}^{k+h-1} C_t^{t-h+1} m_{2(k+h-t)-1} \\
& - b \sum_{h=1}^{\frac{n}{2}-1-i} \Big(-b^2 \Big)^{h-1} \sum_{k+j=i} \Big(T_{2k+2h+2,j} b^* + \widetilde{T}_{2k+2h+2,j} \widetilde{b}^* \Big) \sum_{t=h-1}^{k+h-1} C_t^{t-h+1} m_{2(k+h-t)-1},
\end{aligned}
$$

$$
G_{\left[\frac{n}{2}\right]-1} = \sum_{k+j=\frac{n}{2}-1} \Big(T_{2k+2,j} c^* + \widetilde{T}_{2k+2,j} \widetilde{c}^* \Big) m_{2k+1}.
$$

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A class of nonlinear oscillators with non-autonomous first integrals and algebraic limit cycles

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Abstract. In this paper, we present a class of autonomous nonlinear oscillators with non-autonomous first integral. We prove explicitly the existence of a global sink which is, under some conditions, an algebraic limit cycle. For that class, we draw the possible phase portraits in the Poincaré disk.

Keywords: algebraic limit cycles, global sink, non-autonomous first integrals.

2020 Mathematics Subject Classification: 34A05, 34C05, 34C07, 34C10, 34C25.

1 Introduction and the main result

In this paper, we consider the class of second-order nonlinear ordinary differential equations of the form

$$x_{tt} + f_3(x)x_t^3 + f_2(x)x_t^2 + f_1(x)x_t + f_0(x) = 0, \tag{1.1}$$

where $f_i \neq 0, i = 0,1,2,3$ are smooth real functions of the variable $x = x(t)$. In the $(x, \dot{x})$ phase plane, equation (1.1) is equivalent to

$$\begin{cases} \dot{x} = y, \\ \dot{y} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x), \end{cases} \tag{1.2}$$

where the dot is the derivative with respect to the independent variable t. This kind of oscillator arises in modeling physical, chemical or electronic processes [3, 13]. The qualitative behavior of the solutions of such oscillators is very important and complicated. Various methods have been proposed in the literature to examine the global dynamics of these solutions. Analytical methods, such as the integrability method, attempt to transform the differential system (1.2) into a known differential equation (linear, Bernoulli, Riccati, Abel). This method is used to obtain the solutions explicitly. However, this method may not be sufficient to characterize all the features of the system, especially when the solutions are not analytically known.

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On the other hand, some mathematicians have introduced new tools allowing to obtain the maximum qualitative information about the dynamics of planar differential systems in general. The tool relies on geometric characteristics is called the classification of phase portraits in the Poincaré disk.

A significant number of papers regarding limit cycles, first integrals and invariants curves [9, 16, 18, 20, 22, 24, 31] has been published, where the main goal was studying the qualitative behavior of these solutions.

Starting with [7], Chandrasekar et al. investigated the integrability of a class of oscillators, described by the generalized second-order nonlinear ordinary differential equation

$$\ddot{x} + (k_1 x^q + k_2)\dot{x} + k_3 x^{2q+1} + k_4 x^{q+1} + \lambda_1 x = 0, \tag{1.3}$$

where the parameters λ_1, q and $k_i, i = 1, 2, 3, 4$ are real. Using the extended Prelle–Singer method, the authors were able to determine the first integrals and general solutions for the integrable cases.

In [27], Sinelshchikov proved that two subfamilies of the following family of oscillators

$$y_{zz} + k(y)y_z^3 + h(y)y_z^2 + f(y)y_z + g(y) = 0, \tag{1.4}$$

with k, h, f and $g \neq 0$ are arbitrary sufficiently smooth functions, are integrable and each subfamily possesses an autonomous parametric first integral and two autonomous invariant curves.

In [28], the same author, along with Guha and Choudhury, studied a family of non-autonomous second-order differential equations of the type

$$y_{zz} + a_3(z, y)y_z^3 + a_2(z, y)y_z^2 + a_1(z, y)y_z + a_0(z, y) = 0, \tag{1.5}$$

where $a_i, i = 0, 1, 2, 3$ are smooth functions such that $a_3 \neq 0$ and $|a_2|^2 + |a_1|^2 + |a_0|^2 \neq 0$. The authors showed that equations from (1.5) with a Lax representation admit a quadratic rational first integral.

As a continuation of [7], Jibin and Han [17] showed that the oscillator considered in [7] has a unique and stable limit cycle and they gave its exact parametric representation. In fact, this limit cycle was obtained explicitly a long time before (see [1, 4] and references therein).

Naturally, the following question arises: is there an integrable polynomial planar oscillator of the form (1.1) with an explicit algebraic limit cycle? To the best of our knowledge, we have not encountered such an example in the literature. In this paper, we provide the answer to this question. Moreover, since autonomous rational first integrals and limit cycles are incompatible, in the sense that a planar vector field may have at most one of them. We think that it is interesting to provide an example in which algebraic limit cycles and non-autonomous first integrals can coexist. We consider the class of autonomous oscillators of the form (1.2), where

$$\begin{aligned}
f_3(x) &= -\frac{w}{2h}, \\
f_2(x) &= -\frac{3w^2(x^3 - hx)}{4h^2}, \\
f_1(x) &= -\frac{3}{8}\left(\frac{w}{h}\right)^3 x^6 + \frac{3}{4}\frac{w^3}{h^2}x^4 - \frac{1}{8}\frac{w}{h}(3w^2 + 20)x^2 + w, \\
f_0(x) &= -\frac{x}{16h^4}h_1(x)h_2(x),
\end{aligned}$$

with

$$h_1(x) = w^2x^4 - hw^2x^2 + 4h^2,$$
$$h_2(x) = w^2x^4 - 2hw^2x^2 + h^2w^2 + 4h^2,$$

are smooth real functions of the variable $x \in \mathbb{R}$, while $h \in \mathbb{R}^*$ and $w \in \mathbb{R}$ are parameters.

Our main result is the following.

Theorem 1.1. *Let X be the vector field given by (1.2) and let Γ be the set*

$$\left\{(x,y) \in \mathbb{R}^2 \colon x^2 + \left(y + \frac{w}{2h}(x^2 - h)x\right)^2 = h\right\}.$$

Then the following statements hold:

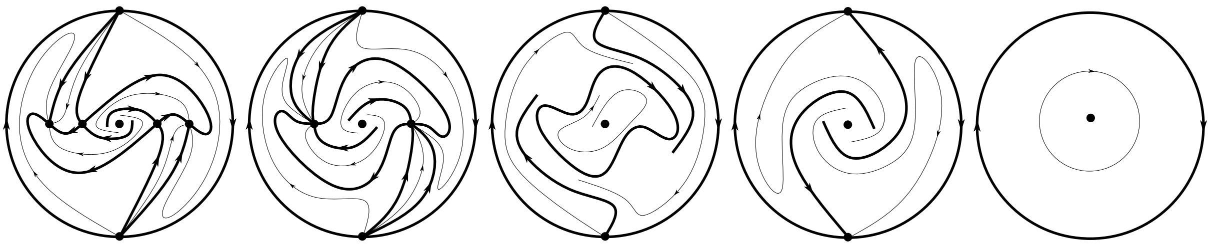
(a) X has a non-autonomous first integral given by

$$H(x,y,t) = \frac{x^2+\left(y+\frac{w}{2h}(w^2-h)x\right)^2-h}{x^2+\left(y+\frac{w}{2h}(w^2-h)x\right)^2}e^{wt}, \qquad \forall (x,y,t) \in \mathbb{R}^3.$$

(b) If $h > 0$ and $w > 0$ (resp. $w < 0$), then Γ is a global sink (resp. source) of X.

(c) If $h > 0$ and $0 < w < 4$ (resp. $-4 < w < 0$), then Γ is a hyperbolic stable (resp. unstable) algebraic limit cycle of X. Moreover, Γ is the unique limit cycle of X and is a global sink (resp. source) of X.

Moreover, the phase portraits of X in the Poincaré disk are topologically equivalent to those given in Figure 1.1.



$h > 0$	$h > 0$	$h > 0$	$h < 0$	$h \neq 0$
$\lvert w\rvert > 4.$	$\lvert w\rvert = 4.$	$0 < \lvert w\rvert < 4.$	$w \neq 0.$	$w = 0.$

Figure 1.1: The topological distinct phase portraits of X.

The paper is organized as follows. In Section 2, we introduce some preliminary results. Theorem 1.1 is proved in Section 3.

2 Preliminary results

2.1 First integrals and invariant algebraic curves

Let $X = (P, Q)$ be a polynomial vector field. We say that X is integrable if and only if there exists a non-constant C^1 function $H \colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$P(x,y)\frac{\partial H}{\partial x}(x,y) + Q(x,y)\frac{\partial H}{\partial y}(x,y) = 0, \tag{2.1}$$

for all $(x,y) \in \mathbb{R}^2$. Therefore the function H is constant along the trajectories $(x(t), y(t))$ of X, i.e., if $I \subset \mathbb{R}$ is an interval, then there exists $c \in \mathbb{R}$: $H(x(t), y(t)) = c$, for all $t \in I$. In such a case the function H is called *first integral* and the trajectories of X are contained in the level sets of H. If the first integral depends on the time t, i.e., $H = H(x,y,t)$, thus we say that H is a *non-autonomous first integral* of X if

$$P(x,y)\frac{\partial H}{\partial x}(x,y,t) + Q(x,y)\frac{\partial H}{\partial y}(x,y,t) + \frac{\partial H}{\partial t}(x,y,t) = 0, \tag{2.2}$$

for all $(x,y,t) \in \mathbb{R}^2 \times I$. Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a real polynomial. We say that F is an *invariant* for X if it satisfies

$$P(x,y)\frac{\partial F}{\partial x}(x,y) + Q(x,y)\frac{\partial F}{\partial y}(x,y) = K(x,y)F(x,y), \tag{2.3}$$

for all $(x,y) \in \mathbb{R}^2$. Here, $K\colon \mathbb{R}^2 \to \mathbb{R}$, which is the *cofactor* of F, is a real polynomial and its degree is at most $n-1$, where n represents the maximum of the degrees of P and Q. It can be observed that the set defined by the equation $F(x,y) = 0$ is invariant under the flow of X. In this case, this set may contain ovals, which can be *algebraic limit cycles*. For more details about first integrals, invariant algebraic curves and algebraic limit cycles, see [4,6,8,10,21,23,32] and Chapter 8 of [11] and the references therein.

2.2 Singular points

Let $X = (P,Q)$ be a polynomial vector field. We say $q \in \mathbb{R}^2$ is a *singularity* of X if $P(q) = Q(q) = 0$. The Jacobian matrix J of the vector field X at q is given by

$$J(q) = \begin{pmatrix} \frac{\partial P}{\partial x}(q) & \frac{\partial P}{\partial y}(q) \\ \frac{\partial Q}{\partial x}(q) & \frac{\partial Q}{\partial y}(q) \end{pmatrix}. \tag{2.4}$$

Let $D(q) = \lambda_1\lambda_2$ be the determinant and $T(q) = \lambda_1 + \lambda_2$ the trace of $J(q)$, where λ_1 and λ_2 are the eigenvalues of $J(q)$, that are the solutions of the characteristic equation

$$\lambda^2 - T(q)\lambda + D(q) = 0.$$

The singularity q is:

(a) *Hyperbolic* if both eigenvalues have real parts different from zero. Here, we distinguish:

  (i) If $D(q) < 0$, then q is a saddle.
  (ii) If $D(q) > 0$ and $T(q) > 0$, then q is an unstable focus/node.
  (iii) If $D(q) > 0$ and $T(q) < 0$, then q is a stable focus/node.

(b) *Degenerate monodromic* if $D(q) > 0$ and $T(q) = 0$. In this case, q is a weak focus or a center.

(c) *Semi-hyperbolic* if $D(q) = 0$ and $T(q) \neq 0$.

(d) *Nilpotent* if $D(q) = T(q) = 0$ and $J(q)$ is not identically zero.

(e) *Degenerate* if $D(q) = T(q) = 0$ and $J(q)$ is identically zero.

We characterize the local phase portraits at hyperbolic, semi-hyperbolic and nilpotent singular points using Theorems 2.15, 2.19 and 3.5 of [11], respectively. For the degenerate singularities, we employ the blow-up technique, see [2] for details.

2.3 The blow-up technique

Consider X a planar polynomial vector field with an isolated singularity at the origin, then we can apply the change of coordinates $\phi : \mathbb{S}^1 \times \mathbb{R}_+ \to \mathbb{R}^2$ given by $\phi(\theta, r) = (r\cos\theta, r\sin\theta) = (x, y)$, where $\mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$. Consequently, we can induce the vector field X_0 in $\mathbb{S}^1 \times \mathbb{R}_+$ by pullback, i.e., $X_0 = D\phi^{-1}X$. One can see that if the k-jet of X (i.e., the Taylor expansion of order k of X, denoted by j_k) is zero at the origin, then the k-jet of X_0 is also zero at every point in $\mathbb{S}^1 \times \{0\}$. Thus, taking the first $k \in \mathbb{N}$ satisfying $j_k(X(0,0)) = 0$ and $j_{k+1}(X(0,0)) \neq 0$, we can define the vector field $\hat{X} = \frac{1}{r^k}X_0$. Therefore, it follows that the behavior of $\hat{X}$ near $\mathbb{S}^1$ is the same as that of X near the origin. One can also see that $\mathbb{S}^1$ is invariant under the flow of $\hat{X}$. For a more detailed study of this technique, see [2] or Chapter 3 of [11]. The vector field $\hat{X}$ can be also expressed as

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r^{k+1}}, \qquad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^{k+2}}.$$

The blow-up technique has a generalization called the *quasihomogeneous blow-up*. In this case, we consider the change of coordinates $\psi(\theta, r) = (r^\alpha\cos\theta, r^\beta\sin\theta) = (x, y)$ for $(\alpha, \beta) \in \mathbb{N}^2$. In a similar way, we can induce the vector field X_0 in $\mathbb{S}^1 \times \mathbb{R}_+$. For some $k \in \mathbb{N}$ maximal, one can define $X_{\alpha,\beta} = \frac{1}{r^k}X_0$ and such a vector field is given by

$$\dot{r} = \xi(\theta)\frac{\cos\theta\, r^\beta\dot{x} + \sin\theta\, r^\alpha\dot{y}}{r^{\alpha+\beta+k-1}}, \qquad \dot{\theta} = \xi(\theta)\frac{\alpha\cos\theta\, r^\alpha\dot{y} - \beta\sin\theta\, r^\beta\dot{x}}{r^{\alpha+\beta+k}},$$

where $\xi(\theta) = (\beta\sin^2\theta + \alpha\cos^2\theta)^{-1}$. Since $\xi(\theta) > 0$ for all $\theta \in \mathbb{S}^1$, therefore it can be eliminated by a change in the time variable. Thus, it follows then

$$\dot{r} = \frac{\cos\theta\, r^\beta\dot{x} + \sin\theta\, r^\alpha\dot{y}}{r^{\alpha+\beta+k-1}}, \qquad \dot{\theta} = \frac{\alpha\cos\theta\, r^\alpha\dot{y} - \beta\sin\theta\, r^\beta\dot{x}}{r^{\alpha+\beta+k}},$$

As in the previous technique, the behavior of $X_{\alpha,\beta}$ near $\mathbb{S}^1$ (which is invariant) is similar to the behavior of X near the origin.

2.4 The Poincaré compactification

To study the behavior of the trajectories of a planar vector field near infinity, we will employ the *Poincaré compactification* (for more details, see [30] or Chapter 5 of [11]).

Let $X = (P, Q)$ be a planar polynomial vector field of degree $n \in \mathbb{N}$. We identify $\mathbb{R}^2$ with the plane $(x_1, x_2, 1)$ in $\mathbb{R}^3$ and define the *Poincaré sphere* as $\mathbb{S}^2 = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$. We denote the *northern hemisphere*, the *southern hemisphere* and the *equator* by $H_+ = \{y \in \mathbb{S}^2 : y_3 > 0\}$, $H_- = \{y \in \mathbb{S}^2 : y_3 < 0\}$ and $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$, respectively. The *Poincaré compactified vector field* $p(X)$ associated with X is an analytic vector field generated on $\mathbb{S}^2$ by the central projections $f_\pm : \mathbb{R}^2 \to H_\pm$, given by $f_\pm(x_1, x_2) = \pm\Delta(x_1, x_2)(x_1, x_2, 1)$, where $\Delta(x_1, x_2) = (x_1^2 + x_2^2 + 1)^{-\frac{1}{2}}$. These two maps define two symmetric copies of X, one copy X^+ in H_+ and the other copy X^- in H_-. In brief, we obtain the vector field $X' = X^+ \cup X^-$ defined on $\mathbb{S}^2\backslash\mathbb{S}^1$. Note that the equator $\mathbb{S}^1$ of the sphere $\mathbb{S}^2$ corresponds with the *infinity* of $\mathbb{R}^2$. The analytic extension of X' from $\mathbb{S}^2\backslash\mathbb{S}^1$ to $\mathbb{S}^2$, given by $y_3^{n-1}X'$, is the Poincaré compactified vector field $p(X)$. The *Poincaré disk* $\mathbb{D}$ is the projection of the closed northern hemisphere on $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ (the vector field given by this projection will also be denoted by $p(X)$). The behavior of $p(X)$ near $\mathbb{S}^1$ is the same as the behavior of X near infinity of $\mathbb{R}^2$. We define the local charts of $\mathbb{S}^2$ by $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ for $i \in \{1,2,3\}$ and their corresponding local maps by $\phi_i : U_i \to \mathbb{R}^2$, $\psi_i : V_i \to \mathbb{R}^2$ with $\phi_i(y_1, y_2, y_3) = -\psi_i(y_1, y_2, y_3) = (\frac{y_m}{y_i}, \frac{y_n}{y_i})$, where $m \neq i$, $n \neq i$ and $m < n$. Denoting by (u, v) the image of ϕ_i and ψ_i, for $i = 1, 2$, in each chart. The expression of $p(X)$ in the local chart U_1 is

$$\dot{u} = v^n \left[Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right) \right], \quad \dot{v} = -v^{n+1} P\left(\frac{1}{v}, \frac{u}{v}\right),$$

and in the local chart U_2, it is given by

$$\dot{u} = v^n \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \quad \dot{v} = -v^{n+1} Q\left(\frac{u}{v}, \frac{1}{v}\right).$$

The expression of $p(X)$ in V_1 and V_2 is the same as that for U_1 and U_2, except by a multiplicative factor of $(-1)^{n-1}$. In these local charts for $i \in \{1, 2\}$, the coordinate $v = 0$ represents the points of $\mathbb{S}^1$. Thus, the singularities at infinity of $\mathbb{R}^2$. Note that $\mathbb{S}^1$ is invariant under the flow of $p(X)$.

2.5 The Markus–Neumann theorem

Let X be a polynomial vector field and $p(X)$ be its compactification defined on $\mathbb{D}$. Consider ϕ the flow associated to $p(X)$. The separatrices of $p(X)$ are orbits, which can be:

1. All the orbits contained in $\mathbb{S}^1$, i.e., at infinity;
2. All the singular points;
3. All the trajectories that are located on the boundaries of the hyperbolic sectors of the finite and infinite singular points;
4. All the limit cycles of X.

The set of all separatrices, denoted by $\mathcal{S}$ is closed. Each connected component of $\mathbb{D} \backslash \mathcal{S}$ is called a *canonical region* of the flow $(\mathbb{D}, \phi)$.

The *separatrix configuration* $\mathcal{S}_c$ of the flow $(\mathbb{D}, \phi)$, is the union of all the separatrices $\mathcal{S}$ of the flow, together with one orbit from each canonical region.

Two separatrix configurations $\mathcal{S}_c$ and $\mathcal{S}_c^*$ of the flow $(\mathbb{D}, \phi)$ are *topologically equivalent* if there exists a homeomorphism from $\mathbb{D}$ to $\mathbb{D}$ that transforms orbits of $\mathcal{S}_c$ into those of $\mathcal{S}_c^*$ while preserving or reversing the orientation of all these orbits.

Theorem 2.1 (Markus–Neumann)**.** *Let $p(X)$ and $p(Y)$ be two Poincaré compactifications in the Poincaré disk $\mathbb{D}$ of two polynomial vector fields X and Y, with finitely many singularities. Then the phase portraits of $p(X)$ and $p(Y)$ are topologically equivalent if and only if their separatrix configurations are topologically equivalent.*

Proof. See [5,25,26] and Section 1.9 of [11]. □

2.6 The solutions of the quartic algebraic equation of degree four

It is well known that the quartic equation

$$ax^4 + bx^3 + cx^2 + d = 0, \tag{2.5}$$

where $a \neq 0$, can be transformed via the change of variable $x \mapsto x - \frac{b}{4a}$ into the equation

$$x^4 + px^2 + qx + r = 0. \tag{2.6}$$

The discriminant of equation (2.6) is given by

$$\Delta = 16p^4r - 4p^3q^2 - 128p^2r^2 + 144pq^2r - 27q^4 + 256r^3.$$

Suppose $\Delta > 0$. Then the following statements hold (see [19] or Chapter 12 of [12]).

(i) If $p < 0$ and $4r < p^2$, then all roots of (2.6) are simple and real.

(ii) If $p \geqslant 0$ or $4r \geqslant p^2$, then all roots of (2.6) are simple and complex.

Moreover, we observe that if $q = 0$, then $\Delta = 16(p^2 - 4r)^2 r$.

3 Proof of Theorem 1.1

Let us look at statement (a). To see that

$$H(x,y,t) = \frac{x^2 + \left(y + \frac{w}{2h}(w^2 - h)x\right)^2 - h}{x^2 + \left(y + \frac{w}{2h}(w^2 - h)x\right)^2} e^{wt}, \tag{3.1}$$

is a non-autonomous first integral of X, it is sufficient to observe that the equation

$$P(x,y)\frac{\partial H}{\partial x}(x,y,t) + Q(x,y)\frac{\partial H}{\partial y}(x,y,t) + \frac{\partial H}{\partial t}(x,y,t) = 0,$$

is satisfied. We now look at statement (b). Suppose $w > 0$ and let

$$H_1(x,y) = \frac{x^2 + \left(y + \frac{w}{2h}(w^2 - h)x\right)^2 - h}{x^2 + \left(y + \frac{w}{2h}(w^2 - h)x\right)^2}.$$

We want to prove that

$$\Gamma = \left\{(x,y) \in \mathbb{R}^2 \colon x^2 + \left(y + \frac{w}{2h}(x^2 - h)x\right)^2 = h\right\},$$

is a global sink of X. It follows from (3.1) that $H(x,y,t) = H_1(x,y)e^{wt}$.

Let $(x(t), y(t)) \in \mathbb{R}^2$ be an orbit of X. Since $w > 0$, observe that if $t \to +\infty$, then $e^{wt} \to +\infty$. However, it follows from statement (a) that $H(x(t), y(t), t)$ is constant, for every $t \in \mathbb{R}$. Therefore, we have $H_1(x(t), y(t)) \to 0$ as $t \to +\infty$. The statement now follows from the fact that Γ coincides with the set $\{(x,y) \in \mathbb{R}^2 \colon H_1(x,y) = 0\}$. For $w < 0$, the proof follows straightforwardly from the fact that X is invariant under the change of variables and parameters $(x, t, w) \mapsto (-x, -t, -w)$.

Let us look at statement (c). First, let

$$F(x,y) = x^2 + \left(y + \frac{w}{2h}(x^2 - h)x\right)^2 - h. \tag{3.2}$$

Notice that if $h > 0$, then

$$P(x,y)\frac{\partial F}{\partial x}(x,y) + Q(x,y)\frac{\partial F}{\partial y}(x,y) = K(x,y)F(x,y), \tag{3.3}$$

where

$$K(x,y) = -\frac{w}{4h^3}(w^2x^6 - 2hw^2x^4 + 4hwx^3y + h^2(w^2+4)x^2 - 4h^2wxy + 4h^2y^2). \tag{3.4}$$

Therefore, if $h > 0$, the curve $F = 0$ is an invariant algebraic curve of X.

We claim that if $0 < |w| < 4$, then the origin is the unique finite singularity of X. Indeed, it follows from (1.2) that the finite singularities of X, other then the origin, are of the form $(x_i, 0)$, $i \in \{1,2,3,4\}$ where x_i are the real solutions of $h_1(x)h_2(x) = 0$, with

$$h_1(x) = w^2x^4 - hw^2x^2 + 4h^2, \qquad h_2(x) = w^2x^4 - 2hw^2x^2 + h^2w^2 + 4h^2.$$

Let Δ_i denote the discriminant of h_i, $i \in \{1,2\}$. It follows from Subsection 2.6 that

$$\Delta_1 = 64h^6w^6(w^2-16)^2, \qquad \Delta_2 = 4096h^6w^6(w^2+4).$$

Therefore, we conclude that if $w \neq 0$ and $h \neq 0$, then h_2 always has a positive discriminant. Hence, all its singularities are either real or complex. Since h_2 satisfies statement (ii) of Subsection 2.6, we conclude that h_2 never has real solutions.

Similarly, it can be seen that if $h > 0$ and $0 < |w| < 4$, then h_1 also does not have real solutions. Thus, if $h > 0$ and $0 < |w| < 4$, then the origin is the unique finite singularity of X. Since it does not lie on the curve $F^{-1}(0) = \Gamma$, we conclude that Γ is an algebraic limit cycle. Moreover, the limit cycle Γ is hyperbolic (for more details, see [14]) if only if

$$I(\Gamma) = \int_0^T K(\gamma(t))dt \neq 0,$$

where $T > 0$ is the period of Γ, $\gamma(t)$ is the parameterization of Γ and the cofactor K is given by (3.3) and (3.4), hence

- if $I(\Gamma) < 0$, Γ is a stable limit cycle;
- if $I(\Gamma) > 0$, Γ is an unstable limit cycle.

It follows from (3.4) that $K(x,y) < 0$ (resp. $K(x,y) > 0$) if $w > 0$ (resp. $w < 0$). Consequently, Γ is a hyperbolic limit cycle. In particular, it follows from statement (b) that Γ is the unique limit cycle of X and that it is stable if $w > 0$ and unstable if $w < 0$.

We now look to the phase portraits of X. If $w = 0$ then

$$\dot{x} = y, \quad \dot{y} = -x,$$

and thus X has a global center. In the sequel, we assume $w \neq 0$. Since X is invariant under the change of variables $(x,t,w) \mapsto (-x,-t,-w)$, it is enough to assume $w > 0$. Similarly to the previous analysis on the roots of h_1 and h_2, one can see that:

(a) All the roots of h_2 are complex;

(b) If $h < 0$, then all roots of h_1 are complex;

(c) If $h > 0$ and $0 < w < 4$, then all the roots of h_1 are complex;

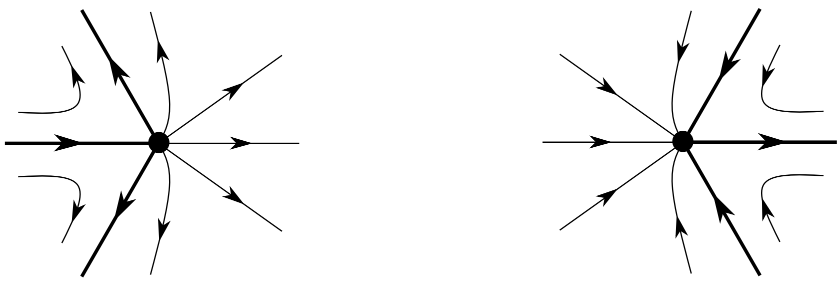
(d) If $h > 0$ and $w = 4$, then h_1 has two real solutions of multiplicity two, given by $x^{\pm} = \pm\sqrt{\frac{h}{2}}$;

(e) If $h > 0$ and $w > 4$, then h_1 has four distinct real solutions, given by

$$x_1 = -\frac{1}{\sqrt{2}}\sqrt{h\left(1+\frac{\sqrt{w^2-16}}{w^2}\right)}, \qquad x_2 = -\frac{1}{\sqrt{2}}\sqrt{h\left(1-\frac{\sqrt{w^2-16}}{w^2}\right)},$$
$$x_3 = \frac{1}{\sqrt{2}}\sqrt{h\left(1-\frac{\sqrt{w^2-16}}{w^2}\right)}, \qquad x_4 = \frac{1}{\sqrt{2}}\sqrt{h\left(1+\frac{\sqrt{w^2-16}}{w^2}\right)}.$$

Let $p_i = (x_i, 0)$, $i \in \{1,2,3,4\}$ be the singularities associated to x_i and let $\mathcal{O}$ denote the origin. Calculations show that the origin is always a hyperbolic unstable focus. Moreover, if $w > 4$ then p_1 and p_4 are hyperbolic stable nodes, while p_2 and p_3 are hyperbolic saddles. Furthermore, if $w = 0$ then $p_1 = p_2$ and $p_3 = p_4$ are semi-hyperbolic saddle-nodes.

We now look at the infinity. The unique singularity at infinity is the origin of the second chart of the Poincaré compactification. In this case, after performing two quasihomogeneous blow-ups, with weights $(\alpha_1, \beta_1) = (2,3)$ and $(\alpha_2, \beta_2) = (2,1)$ respectively, we obtain the local phase portraits as illustrated in Figure 3.1.



$h > 0$. $h < 0$.

Figure 3.1: Local phase portrait at the origin of the second chart of the Poincaré compactification.

We now study the phase portrait for the case $w > 4$. In this case, the local phase portrait is shown in Figure 3.2.

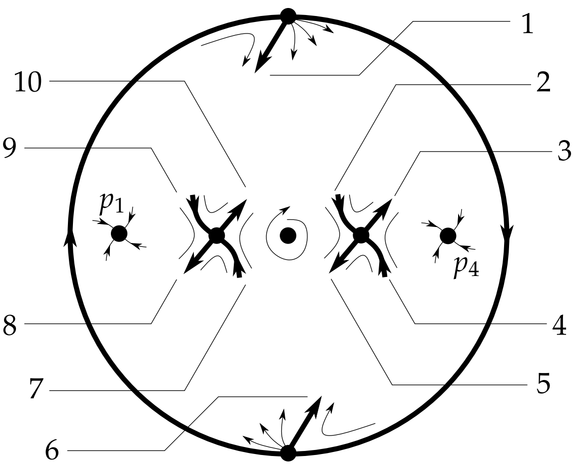


Figure 3.2: Uncompleted phase portrait for $w > 4$.

Observe that the invariant algebraic curve $F(x, y) = 0$ is given by the union of the curves

$$y^{\pm} = \frac{w}{2h}(h - x^2)x \pm \sqrt{h - x^2}, \tag{3.5}$$

for $|x| < \sqrt{h}$. It follows that separatrix 10 goes to the stable node p_4, while separatrix 8 goes to the stable node p_1. Since X is invariant under the change of variables $(x, y) \mapsto (-x, -y)$, it follows that separatrix 5 goes to p_1 and separatrix 3 goes to p_4. Separatrices 7 and 2 are now enclosed in the bounded region delimited by Γ and thus have no other option than to be generated at the origin. See Figure 3.3.

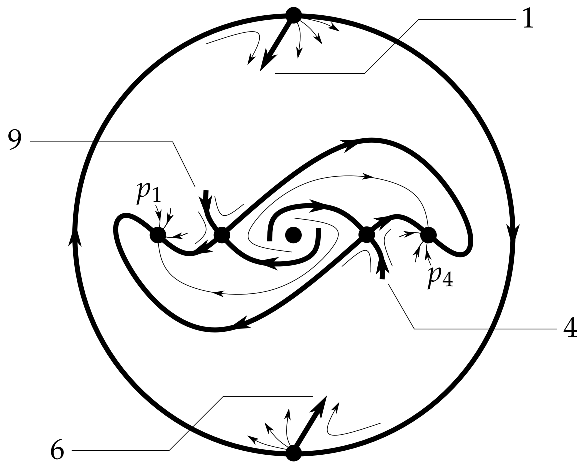


Figure 3.3: Uncompleted phase portrait for $w > 4$.

We now have numerical evidence, according to software $P4$ (see Chapters 9 and 10 of [11]), that separatrix 1 goes to p_1. Therefore, it follows from the invariance of X under the change of variables $(x, y) \mapsto (-x, -y)$ that separatrix 6 goes to p_4. Hence, separatrix 9 must be born at the north pole, while separatrix 4 must be born at the south pole. The other phase portraits can be obtained in a similar manner.

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Declarations

Conflicts of interest The authors declare that they have no conflicts of interest.

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A minimization problem related to the principal frequency of the p-Bilaplacian with coupled Dirichlet–Neumann boundary conditions

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Abstract. For each fixed integer $N \geq 2$ let $\Omega \subset \mathbb{R}^N$ be an open, bounded and convex set with smooth boundary. For each real number $p \in (1, \infty)$ define

$$M(p;\Omega) = \inf_{u \in \mathcal{W}_C^{2,\infty}(\Omega)\setminus\{0\}} \frac{\int_\Omega (\exp(|\Delta u|^p) - 1)\, dx}{\int_\Omega (\exp(|u|^p) - 1)\, dx},$$

where $\mathcal{W}_C^{2,\infty}(\Omega) := \cap_{1<p<\infty}\{u \in W_0^{2,p}(\Omega) :\ \Delta u \in L^\infty(\Omega)\}$. We show that if the radius of the largest ball which can be inscribed in Ω is strictly larger than a constant which depends on N then $M(p;\Omega)$ vanishes while if the radius of the largest ball which can be inscribed in Ω is strictly less than 1 then $M(p;\Omega)$ is a positive real number. Moreover, in the latter case when p is large enough we can identify the value of $M(p;\Omega)$ as being the principal frequency of the p-Bilaplacian on Ω with coupled Dirichlet–Neumann boundary conditions.

Keywords: p-Bilaplacian, principal frequency, Dirichlet–Neumann boundary conditions.

2020 Mathematics Subject Classification: 35P30, 47J05, 47J20, 49J40, 49S05.

1 Introduction

1.1 Notations

For each integer $N \geq 1$ we denote by $\mathbb{R}^N$ the N-dimensional Euclidean space. Let $|\cdot|$ denote the modulus on $\mathbb{R}$ and for each integer $N \geq 2$ let $|\cdot|_N$ denote the Euclidean norm on $\mathbb{R}^N$. For

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each open and bounded subset Ω of $\mathbb{R}^N$ denote by R_Ω the inradius of Ω (that is the radius of the largest ball which can be inscribed in Ω). Finally, for each integer $N \geq 1$ define

$$\mathbb{P}^N := \{\Omega \subset \mathbb{R}^N : \Omega \text{ is an open, bounded, convex set with smooth boundary } \partial\Omega\}.$$

1.2 Statement of the problem

For each $\Omega \in \mathbb{P}^N$ and each real number $p \in (1, \infty)$ we define

$$M(p;\Omega) := \inf_{u \in \mathcal{W}_C^{2,\infty}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (\exp(|\Delta u|^p) - 1)dx}{\displaystyle\int_\Omega (\exp(|u|^p) - 1)dx} \tag{1.1}$$

where $\mathcal{W}_C^{2,\infty}(\Omega) := \cap_{1<p<\infty}\{u \in W_0^{2,p}(\Omega) : \Delta u \in L^\infty(\Omega)\}$. The goal of this paper is to emphasize the following phenomena which appear in relation with the minimization problem (1.1): if R_Ω is large enough then $M(p;\Omega) = 0$ for each $p \in (1, \infty)$ while if R_Ω is small enough then $M(p;\Omega) > 0$ for each $p \in (1, \infty)$. Moreover, in the latter case we can identify the value of $M(p;\Omega)$ for each p large enough as being equal with the following quantity

$$\Lambda_C(p;\Omega) := \inf_{u \in W_0^{2,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta u|^p dx}{\displaystyle\int_\Omega |u|^p dx}, \tag{1.2}$$

(see Theorem 1.1 for the precise result on problem (1.1)). Regarding $\Lambda_C(p;\Omega)$ we recall the well-known fact that it represents the principal eigenvalue of the p-Bilaplacian with coupled Dirichlet–Neumann boundary conditions (see, e.g., N. Katzourakis & E. Parini [5, relation (1.6)]). In other words, $\Lambda_C(p;\Omega)$ is the smallest real number Λ for which the following equation has a nontrivial solution

$$\begin{cases} \Delta_p^2 u = \Lambda |u|^{p-2}u, & \text{in } \Omega, \\ u = |\nabla u|_N = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ stands for the p-Bilaplacian. At this point we consider important to recall the fact that problem (1.3) with $p = 2$ represents the famous "*clamped plate*" problem, which was initially studied by Lord J. W. S. Rayleigh in his famous book *The Theory of Sound* (1877), and subsequently deeply investigated by G. Szegö (1950), G. Talenti (1981), M. Ashbaugh & R. Benguria (1995) and N. Nadirashvili (1995) from an isoperimetric point of view.

1.3 Motivation

For each $\Omega \in \mathbb{P}^N$ and each real number $p \in (1, \infty)$ we recall the eigenvalue problem for the p-Laplacian under homogeneous Dirichlet boundary conditions

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where λ is a real parameter and $\Delta_p u := \operatorname{div}(|\nabla u|_N^{p-2}\nabla u)$ is the p-Laplace operator. It is well-known (see, e.g., P. Lindqvist [7]) that the first eigenvalue of problem (1.4) has the following variational characterization

$$\lambda_1(p;\Omega) := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^p dx}{\displaystyle\int_\Omega |u|^p dx}.$$

Defining

$$\Lambda_1(p;\Omega) := \inf_{u\in X_0(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (\exp(|\nabla u|_N^p) - 1)\, dx}{\displaystyle\int_\Omega (\exp(|u|^p) - 1)\, dx}, \tag{1.5}$$

where $X_0(\Omega) := W^{1,\infty}(\Omega) \cap \big(\cap_{1<p<\infty} W_0^{1,p}(\Omega)\big)$, we recall that by [2, Theorem 2] (see also [1] for similar results) we know that $\Lambda_1(p;\Omega) = 0$ if $R_\Omega > 1$ while $\Lambda_1(p;\Omega) > 0$ if $R_\Omega \leq 1$. Moreover, there exists a constant $M \in [e^{-1}, 1]$ such that if $R_\Omega \leq M$ we have $\Lambda_1(p;\Omega) = \lambda_1(p;\Omega)$, for all $p \in (1,\infty)$. Furthermore, by [1, Theorem 2] we have that if $R_\Omega < 1$ then there exists a constant $P \in (1,\infty)$ such that $\Lambda_1(p;\Omega) = \lambda_1(p;\Omega)$, for all $p \in [P,\infty)$.

Motivated by these results regarding $\Lambda_1(p;\Omega)$ and $\lambda_1(p;\Omega)$ in this paper we show that we can arrive to a similar conclusion in relation with $M(p;\Omega)$ and $\Lambda_C(p;\Omega)$.

1.4 Main result

The main result of this paper is given by the following theorem.

Theorem 1.1. *Assume $N \geq 2$ is a given integer and let C_N be the constant given by*

$$C_N := \begin{cases} \dfrac{4}{\ln 2}, & \text{if } N = 2, \\[2mm] \dfrac{2^{\frac{2}{N}}(N-2)}{1 - 2^{\frac{2}{N}-1}}, & \text{if } N \geq 3. \end{cases} \tag{1.6}$$

Then for each $\Omega \in \mathbb{P}^N$ and each $p \in (1,\infty)$ we have that $M(p;\Omega) > 0$, if $R_\Omega < 1$ and $M(p;\Omega) = 0$ if $R_\Omega > C_N^{1/2}$. Moreover, if $\Omega \in \mathbb{P}^N$ with $R_\Omega < 1$ then there exists a constant $P^\star > 1$ such that $M(p;\Omega) = \Lambda_C(p;\Omega)$ for all $p \in [P^\star, \infty)$.

Actually, a careful look at the proof of Theorem 1.1 (more precisely, observing the fact that relation (3.1) holds true for a ball with the radius strictly smaller that $C_N^{1/2}$) shows that it can be improved in the particular case when Ω is a ball, in the following sense.

Corollary 1.2. *Assume $N \geq 2$ is a given integer and let B_R be a ball of radius R from $\mathbb{R}^N$ centered at the origin. Then for each $p \in (1,\infty)$ we have that $M(p;B_R) > 0$, if $R < C_N^{1/2}$ and $M(p;B_R) = 0$ if $R > C_N^{1/2}$. Moreover, if $R < C_N^{1/2}$ then there exists a constant $P^\star > 1$ such that $M(p;B_R) = \Lambda_C(p;B_R)$ for all $p \in [P^\star, \infty)$.*

Note that, unfortunately, our proof of Theorem 1.1 cannot fill the gap which occurs when $R_\Omega \in [1, C_N^{1/2}]$. In the case of Corollary 1.2 this gap reduces to an uncovered case when $R = C_N^{1/2}$.

The rest of the paper comprises two more sections offering the following pieces of information: in Section 2 we recall the asymptotic behaviour of $\Lambda_C(p;\Omega)^{1/p}$, as $p \to \infty$, and we give a lower bound for $\Lambda_C(p;\Omega)$; Section 3 is devoted to the proof of the main result.

2 Auxiliary results on $\Lambda_C(p;\Omega)$

2.1 The asymptotic behaviour of $\Lambda_C(p;\Omega)^{1/p}$, as $p \to \infty$

Define

$$\Lambda_\infty^C(\Omega) := \inf_{u \in W_C^{2,\infty}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\Omega)}} . \tag{2.1}$$

By [5, Theorem 1.1] we know that

$$\lim_{p\to\infty} \Lambda_C(p;\Omega)^{1/p} = \Lambda_\infty^C(\Omega) . \tag{2.2}$$

Note that in general an explicit expression of $\Lambda_\infty^C(\Omega)$ is not available in the literature but when $\Omega = B_R$, where B_R stands for a ball of radius R from $\mathbb{R}^N$ centered at the origin, we have (by [5, Proposition 3.5]) that $\Lambda_\infty^C(B_R) = C_N R^{-2}$, where C_N is given by relation (1.6). Moreover, by [5, Proposition 3.5] we have that the minimizer realising the infimum in the definition of $\Lambda_\infty^C(B_R)$ is the positive, radially symmetric function $u_0(x) := w_1\left(\frac{x}{R}\right)$ with w_1 being the solution of the problem

$$\begin{cases} -\Delta w_1(x) = f(x), & \text{for } x \in B_1, \\ w_1(x) = 0, & \text{for } x \in \partial B_1, \end{cases}$$

where

$$f(x) := \begin{cases} 1, & \text{if } |x|_N \le 2^{-\frac{1}{N}}, \\ -1, & \text{if } 2^{-\frac{1}{N}} < |x|_N < 1 . \end{cases}$$

Actually, by [5, Lemma 3.3]) we know that for $N = 2$ we have

$$w_1(x) = \begin{cases} \dfrac{\ln 2}{4} - \dfrac{|x|_2^2}{4}, & \text{for } |x|_2 \le 2^{-\frac{1}{2}}, \\ \dfrac{|x|_2^2}{4} - \dfrac{\ln(|x|_2)}{2} - \dfrac{1}{4}, & \text{for } 2^{-\frac{1}{2}} < |x|_2 < 1, \end{cases}$$

while for $N \ge 3$ we have

$$w_1(x) = \begin{cases} \dfrac{2^{-\frac{2}{N}}}{N} - \dfrac{1}{2N} - \dfrac{1}{N(N-2)} + \dfrac{2^{1-\frac{2}{N}}}{N(N-2)} - \dfrac{|x|_N^2}{2N}, & \text{for } |x|_N \le 2^{-\frac{1}{N}}, \\ \dfrac{|x|_N^2}{2N} + \dfrac{|x|_N^{2-N}}{N(N-2)} - \dfrac{1}{2N} - \dfrac{1}{N(N-2)}, & \text{for } 2^{-\frac{1}{N}} < |x|_N < 1 . \end{cases}$$

Consequently, we have that the function $u_0 : B_R \to \mathbb{R}$, given by $u_0(x) := w_1\left(\frac{x}{R}\right)$, has the following expressions:

- if $N = 2$ then

$$u_0(x) = \begin{cases} \dfrac{\ln 2}{4} - \dfrac{|x|_2^2}{4R^2}, & \text{for } |x|_2 \le 2^{-\frac{1}{2}}R, \\ \dfrac{|x|_2^2}{4R^2} - \dfrac{\ln(|x|_2) - \ln(R)}{2} - \dfrac{1}{4}, & \text{for } 2^{-\frac{1}{2}}R < |x|_2 < R. \end{cases}$$

- if $N \ge 3$ then

$$u_0(x) = \begin{cases} \dfrac{1 - 2^{\frac{2}{N}-1}}{2^{\frac{2}{N}}(N-2)} - \dfrac{|x|_N^2}{2NR^2}, & \text{for } |x|_N \le 2^{-\frac{1}{N}}R, \\ \dfrac{|x|_N^2}{2NR^2} + \dfrac{|x|_N^{2-N}}{N(N-2)R^{2-N}} - \dfrac{1}{2N} - \dfrac{1}{N(N-2)}, & \text{for } 2^{-\frac{1}{N}}R < |x|_N < R. \end{cases}$$

Remark 2.1. Simple computations show that when $N = 2$ the function u_0 satisfies $\|u_0\|_{L^\infty(B_R)} = \frac{\ln 2}{4}$ and $\|\Delta u_0\|_{L^\infty(B_R)} = R^{-2}$. Similarly, when $N \geq 3$ the function u_0 verifies $\|u_0\|_{L^\infty(B_R)} = \frac{1-2^{\frac{2}{N}-1}}{2^{\frac{2}{N}}(N-2)}$ and $\|\Delta u_0\|_{L^\infty(B_R)} = R^{-2}$. Consequently, in both cases u_0 is a minimizer for $\Lambda_\infty^C(B_R)$ with $\|u_0\|_{L^\infty(B_R)} = C_N^{-1}$, where C_N is given by relation (1.6).

2.2 A lower bound for $\Lambda_C(p;\Omega)$

The goal of this section is to prove the following result:

Proposition 2.2. *Let $N \geq 2$ be an integer and $\Omega \in \mathbb{P}^N$ be a set. Then we have*

$$\Lambda_C(p;\Omega) \geq p^{-1} R_\Omega^{-2p}, \qquad \forall\, p \in (1,\infty)\,.$$

The main ingredient in proving Proposition 2.2 is a Hardy-type inequality due to E. Mitidieri [8, Corollary 2.2]. We recall this inequality below.

Theorem 2.3. *If $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $\phi : \Omega \to (0,\infty)$ is a superharmonic function such that $\phi \in C^2(\overline{\Omega})$ and it satisfies $-\Delta\phi \geq a|\nabla\phi|_N^2 \phi^{-1}$, in Ω, for some constant $a > 0$ then for each real number $p \in (1,\infty)$ the following inequality holds true*

$$\frac{(p-1)a+p}{p^2}\int_\Omega |\Delta\phi||u|^p dx \leq \int_\Omega \phi^p |\Delta\phi|^{1-p}|\Delta u|^p dx, \qquad \forall\, u \in C_0^\infty(\Omega)\,. \tag{2.3}$$

2.2.1 Proof of Proposition 2.2.

For each $\Omega \in \mathbb{P}^N$ let v be the unique function satisfying

$$\begin{cases} -\Delta v = 1, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

In particular, we have that $v \in C^2(\overline{\Omega})$. Letting $M_2(\Omega) := \max_{x\in\overline{\Omega}} v(x)$, we have by [4, Theorem 1.2 with $p = q = 2$] that

$$M_2(\Omega) \leq \frac{R_\Omega^2}{2}\,.$$

On the other hand, by [4, Theorem 3.2] (with $p = 2$ and F being the Euclidean norm on $\mathbb{R}^N$) we know that

$$2^{-1}|\nabla v(x)|_N^2 + v(x) \leq M_2(\Omega), \qquad \forall\, x \in \Omega\,.$$

Thus, defining $\phi : \Omega \to (0,\infty)$ by

$$\phi(x) := v(x) + M_2(\Omega), \qquad \forall\, x \in \Omega\,,$$

we have that $\phi \in C^2(\overline{\Omega})$ and since $-\Delta\phi(x) = -\Delta v(x) = 1$ for all $x \in \Omega$, by the above estimate we deduce that

$$2^{-1}\phi^{-1}(x)|\nabla\phi(x)|_N^2 \leq -\Delta\phi(x), \qquad \forall\, x \in \Omega\,.$$

In other words, ϕ given above satisfies the hypothesis from Theorem 2.3 with $a = 2^{-1}$ and, consequently, the following inequality holds true

$$\frac{3p-1}{2p^2}\int_\Omega |u|^p\, dx \leq \int_\Omega (v + M_2(\Omega))^p |\Delta u|^p\, dx, \qquad \forall\, u \in C_0^\infty(\Omega)\,. \tag{2.4}$$

Since $v(x) \leq M_2(\Omega) \leq 2^{-1}R_\Omega^2$ for each $x \in \Omega$ inequality (2.4) implies the conclusion of Proposition 2.2.

3 Proof of the main result

We start by establishing three lemmas which will be helpful in the proof of our main result.

Lemma 3.1. *Assume $N \geq 2$ is an integer. For each $\Omega \in \mathbb{P}^N$ and each $p \in (1,\infty)$ we have $M(p;\Omega) \leq \Lambda_C(p;\Omega)$.*

Proof. Assume $p \in (1,\infty)$ is arbitrary but fixed. Taking into account relation (1.1) for any $u \in C_0^\infty(\Omega) \setminus \{0\} \subset \mathcal{W}_C^{2,\infty}(\Omega) \setminus \{0\}$ and $t \in (0,1)$ we have

$$M(p;\Omega) \leq \frac{\displaystyle\int_\Omega (\exp(|\Delta(tu)|^p) - 1)dx}{\displaystyle\int_\Omega (\exp(|tu|^p) - 1)dx} = \frac{\displaystyle\int_\Omega |\Delta u|^p dx + \sum_{k=2}^\infty t^{(k-1)p} \int_\Omega \frac{|\Delta u|^{kp}}{k!} dx}{\displaystyle\int_\Omega |u|^p dx + \sum_{k=2}^\infty t^{(k-1)p} \int_\Omega \frac{|u|^{kp}}{k!} dx}.$$

Letting $t \to 0^+$ in the above inequality we get

$$M(p;\Omega) \leq \frac{\displaystyle\int_\Omega |\Delta u|^p\, dx}{\displaystyle\int_\Omega |u|^p\, dx}, \qquad \forall\, u \in C_0^\infty(\Omega) \setminus \{0\}.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{2,p}(\Omega)$ and $\Lambda_C(p;\Omega)$ is defined by relation (1.2) we deduce that the conclusion of Lemma 3.1 holds true. □

Lemma 3.2. *Assume $N \geq 2$ is an integer. For each $\Omega \in \mathbb{P}^N$ and each $p \in (1,\infty)$ we have $M(p;\Omega) \geq \inf_{k \in \mathbb{N}\setminus\{0\}} \Lambda_C(kp;\Omega)$.*

Proof. Assume $p \in (1,\infty)$ is arbitrary but fixed. Using the definition of $\Lambda_C(p;\Omega)$ given by relation (1.2) we deduce that for each $u \in \mathcal{W}_C^{2,\infty}(\Omega) \setminus \{0\}$ (which, in particular, ensures that $u \in W_0^{2,q}(\Omega) \setminus \{0\}$ for any $q > 1$), we have

$$\frac{\displaystyle\int_\Omega (\exp(|\Delta u|^p) - 1)\, dx}{\displaystyle\int_\Omega (\exp(|u|^p) - 1)\, dx} \geq \frac{\displaystyle\sum_{k=1}^\infty \frac{\Lambda_C(kp;\Omega)}{k!} \int_\Omega |u|^{kp}\, dx}{\displaystyle\sum_{k=1}^\infty \frac{1}{k!} \int_\Omega |u|^{kp}\, dx} \geq \inf_{k \in \mathbb{N}\setminus\{0\}} \Lambda_C(kp;\Omega).$$

Passing above to the infimum over all $u \in \mathcal{W}_C^{2,\infty}(\Omega) \setminus \{0\}$, we arrive at the conclusion of Lemma 3.2. □

Lemma 3.3. *Assume that $\Omega \in \mathbb{P}^N$ satisfies $\Lambda_\infty^C(\Omega) > 1$. Define*

$$\mathcal{O} := \{p \in (1,\infty) :\ \Lambda_C(p;\Omega) \leq \Lambda_C(kp;\Omega),\ \forall\, k \geq 1\}.$$

Then there exists an integer $L \geq 1$ such that $(L,\infty) \subset \mathcal{O}$.

Proof. The proof of this lemma follows the ideas used in the proof of Step 5 from the proof of Theorem 2 in [1, p. 10]. We recall it just for the reader's convenience.

We argue by contradiction. Indeed, assume that for each integer $m \geq 1$ there exists a real number $p_m \geq m$ and an integer $k_m \geq 2$ such that $\Lambda_C(p_m;\Omega) > \Lambda_C(k_m p_m;\Omega)$. Since $\Lambda_\infty^C(\Omega) > 1$ it follows that $\Lambda_\infty^C(\Omega) - \sqrt{\Lambda_\infty^C(\Omega)} > 0$. Let us now fix $\varepsilon \in (0, \Lambda_\infty^C(\Omega) - \sqrt{\Lambda_\infty^C(\Omega)})$. It is clear that $(\Lambda_\infty^C(\Omega)-\varepsilon)^2 > \Lambda_\infty^C(\Omega)$. On the other hand, by (2.2), $\lim_{q\to\infty}\sqrt[q]{\Lambda_C(q;\Omega)} = \Lambda_\infty^C(\Omega)$, and thus there exists a positive integer A_ε such that $1 < \Lambda_\infty^C(\Omega)-\varepsilon < \sqrt[q]{\Lambda_C(q;\Omega)}$, for all $q \geq A_\varepsilon$. Then,

$$(\Lambda_\infty^C(\Omega)-\varepsilon)^{2p_m} \leq (\Lambda_\infty^C(\Omega)-\varepsilon)^{k_m p_m} < \Lambda_C(k_m p_m;\Omega) < \Lambda_C(p_m;\Omega), \qquad \forall\, m > A_\varepsilon\,.$$

Hence, using again (2.2), we conclude that

$$(\Lambda_\infty^C(\Omega)-\varepsilon)^2 \leq \lim_{m\to\infty} \sqrt[p_m]{\Lambda_C(p_m;\Omega)} = \Lambda_\infty^C(\Omega)\,,$$

which is a contradiction. The proof of Lemma 3.3 is complete. □

Proof of Theorem 1.1.

• *Step 1.* We show that $M(p;\Omega)=0$, for each $\Omega\in\mathbb{P}^N$ with $R_\Omega > C_N^{1/2}$ and each $p\in(1,\infty)$.

Assume that $p\in(1,\infty)$ is arbitrary but fixed. Firstly, note that for each $\Omega\in\mathbb{P}^N$ we may assume without loss of generality, by a translation of the domain, that $0\in\Omega$ is exactly the center of the largest ball which can be inscribed in Ω, in other words $B_{R_\Omega}\subset\Omega$. Next, let u_0 be a minimizer for $\Lambda_\infty^C(B_{R_\Omega})$ with $\|u_0\|_{L^\infty(B_{R_\Omega})} = C_N^{-1}$, where C_N is given by relation (1.6), and $\|\Delta u_0\|_{L^\infty(B_{R_\Omega})} = R_\Omega^{-2}$ (see Remark 2.1 for details). Then we can define $U_0:\Omega\to\mathbb{R}$ by

$$U_0(x) := \begin{cases} u_0(x), & \text{if } x\in B_{R_\Omega},\\ 0, & \text{if } x\in\Omega\setminus B_{R_\Omega}.\end{cases}$$

Since $u_0\in\mathcal{W}_C^{2,\infty}(B_{R_\Omega})$ it follows that $u_0\in W_0^{2,q}(B_{R_\Omega})$ for each $q\in(1,\infty)$ and by [6, Lemma 5.2.5 & Theorem 5.4.4 & Section 5.5] we deduce that $U_0\in W_0^{2,q}(\Omega)$ for each $q\in(1,\infty)$. It follows that, actually, we have $nU_0\in\mathcal{W}_C^{2,\infty}(\Omega)\setminus\{0\}$, for each positive integer n. Testing with nU_0 in the definition of $M(p;\Omega)$, and taking into account that $|\Delta U_0(x)|\leq R_\Omega^{-2}$, for a.a. $x\in B_{R_\Omega}$, we get

$$M(p;\Omega) \leq \frac{\displaystyle\int_\Omega [\exp(|\Delta(nU_0(x))|^p)-1]\,dx}{\displaystyle\int_\Omega [\exp(|nU_0(x)|^p)-1]\,dx} \leq \frac{\displaystyle\int_{B_{R_\Omega}} [\exp(|nR_\Omega^{-2}|^p)-1]\,dx}{\displaystyle\int_{B_{R_\Omega}} [\exp(n^p|u_0(x)|^p)-1]\,dx}\,.$$

On the other hand, we recall that by Remark 2.1 we know that $\|u_0\|_{L^\infty(B_{R_\Omega})} = C_N^{-1}$, where C_N is given by relation (1.6). We deduce that if we assume $R_\Omega > C_N^{1/2}$, then letting $\epsilon_0>0$ be such that $\epsilon_0 + R_\Omega^{-2} < C_N^{-1}$, we get that there exists a subset $\omega\subset B_{R_\Omega}$ with $|\omega|>0$ such that $|u_0(x)| > \epsilon_0 + R_\Omega^{-2}$, for all $x\in\omega$. It follows that, for each positive integer n we have

$$M(p;\Omega) \leq \frac{|B_{R_\Omega}|[\exp(|nR_\Omega^{-2}|^p)-1]}{\displaystyle\int_\omega [\exp(n^p|u_0(x)|^p)-1]\,dx} \leq \frac{|B_{R_\Omega}|[\exp(|nR_\Omega^{-2}|^p)-1]}{|\omega|\left[\exp\left[n^p\left(\epsilon_0+R_\Omega^{-2}\right)^p\right]-1\right]}\,.$$

Letting $n\to\infty$ we find $M(p;\Omega)=0$.

• *Step 2.* We show that $M(p;\Omega)>0$, for each $\Omega\in\mathbb{P}^N$ with $R_\Omega<1$ and each $p\in(1,\infty)$. Moreover, there exists $P^\star>1$ such that $M(p;\Omega)=\Lambda_C(p;\Omega)$ for all $p\geq P^\star$.

Let $\Omega\in\mathbb{P}^N$ with $R_\Omega<1$ and $p\in(1,\infty)$ be arbitrary but fixed. By Proposition 2.2 we know that

$$\Lambda_C(q;\Omega) \geq q^{-1}R_\Omega^{-2q}, \qquad \forall\, q\in(1,\infty)\,.$$

That fact and relation (2.2) yield

$$\Lambda_\infty^C(\Omega) = \lim_{q\to\infty} \Lambda_C(q;\Omega)^{1/q} \geq \lim_{q\to\infty} \sqrt[q]{q^{-1}R_\Omega^{-2q}} = R_\Omega^{-2} > 1\,. \tag{3.1}$$

Since $\Lambda_\infty^C(\Omega) > 1$ the hypothesis of Lemma 3.3 is fulfilled. Let $L \geq 1$ be the smallest integer number for which Lemma 3.3 holds true. It follows that

$$\Lambda_C(q;\Omega) \leq \Lambda_C(kq;\Omega), \qquad \forall\, k \geq 1,\ \forall\, q > L\,.$$

Taking $k_0 := \left[Lp^{-1}\right] + 2$ we get $k_0 p > L$ and consequently, by the above inequality we find that

$$\Lambda_C(k_0 p;\Omega) \leq \Lambda_C(kp;\Omega)\,,$$

for each integer $k \geq k_0$. Thus,

$$\Lambda_C(k_0 p;\Omega) \leq \inf_{k\geq k_0} \Lambda_C(kp;\Omega)\,.$$

On the other hand, by Lemma 3.2 we know that

$$M(p;\Omega) \geq \inf_{k\in\mathbb{N}\setminus\{0\}} \Lambda_C(kp;\Omega)\,.$$

All the above pieces of information imply that

$$M(p;\Omega) \geq \inf_{k\in\{1,2,\ldots,k_0\}} \Lambda_C(kp;\Omega) > 0\,.$$

Finally, if we assume, in addition, that $p > L$ then similar arguments as above yield $M(p;\Omega) \geq \Lambda_C(p;\Omega)$. On the other hand, by Lemma 3.1 we have $M(p;\Omega) \leq \Lambda_C(p;\Omega)$, and, consequently, we conclude that $M(p;\Omega) = \Lambda_C(p;\Omega)$, for all $p \geq P^\star := L+1$. The proof of Theorem 1.1 is now complete. □

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Linearized instability for differential equations with dependence on the past derivative

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Abstract. We provide a criterion for instability of equilibria of equations in the form $\dot{x}(t) = g(x'_t, x_t)$, which includes neutral delay equations with state-dependent delay. The criterion is based on a lower bound $\Delta > 0$ for the delay in the neutral terms, on regularity assumptions of the functions in the equation, and on spectral assumptions on a semigroup used for approximation. The spectral conditions can be verified studying the associated characteristic equation. Estimates in the C^1-norm, a manifold containing the state space X_2 of the equation and another manifold contained in X_2, and an invariant cone method are used for the proof. We also give mostly self-contained proofs for the necessary prerequisites from the constant delay case, and conclude with an application to a mechanical example.

Keywords: neutral delay equations, dependence on past derivative, state-dependent delay, linearized instability.

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1 Introduction

Functional differential equations with constant delays, distributed delays, time-dependent delays, and state-dependent delays are all special cases of a dependence of the present derivative $\dot{x}(t)$ on the past history $\dot{x}|_{(-\infty,t]}$. (Some models also include dependence on the future.) A basic theory for equations with a general past dependence, following a generally familiar dynamical systems framework, is still in development, see e.g. the work of Nishiguchi [41]. The present paper is a contribution in this sense. We consider neutral equations $\dot{x}(t) = g(\dot{x}_t, x_t)$ with dependence on a bounded past interval, and with a lower bound of the delay in the derivative terms on the rght hand side. This includes neutral equations with a state-dependent point delay.

Equations with state-dependent retarded and advanced terms appear already in work of Poisson [44] from 1806. Papers by Driver [10] going back to the 1960s (on the particularly

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difficult case of the two-body problem of electrodynamics), or by Grimm [18] from 1971 are among the earliest that consider models with state-dependent time shift. But it seems that a systematic treatment by a larger number of authors started not earlier than in the late 1980s, for example, Jackiewicz 1987 [31], Mallet-Paret, Nussbaum and Paraskevopoulos [40], Jackiewicz 1995 [32], Hartung, Herdman and Turi [24], Krisztin [36] and Walther [49]. The article [25] gives an impression of the history of the subject. In models for real-world phenomena, state-dependent time shifts arise from position-dependent signal (or force, in the electrodynamics problem) propagation times, or from threshold conditions in mathematical biology. The resulting time shifts are sometimes implicitly defined via properties of the system state, and then a solution theory has to take the solvability of these implicit equations into account.

Neutral differential equations (i.e., the time derivative of the solution appears also on the right hand side of the equation $\dot{x}(t) = \dots$) arise in the famous two-body problem of electrodynamics, as well as in models of biological and mechanical systems, see for example [37], Chapter 9, [52] and [38]. Constant delays in such models certainly result from simplification, so it seems desirable to have a basic theory that covers also state-dependent variable delays, or, more general forms of dependence on the past derivative.

We introduce some notational conventions: Let $n \in \mathbb{N}$ and $h > 0$ be given. We assume that all delays are bounded above by h, so that the system state at time t is given by the segment $x_t \in (C^0[-h,0], \mathbb{R}^n), x_t(s) = x(t+s), s \in [-h,0]$. By C^0 we briefly denote the Banach space of continuous functions $C^0([-h,0],\mathbb{R}^n)$ with the norm given by $|\phi|_{C^0} = \max_{-h\le t\le 0} |\phi(t)|$, here $|\ |$ is the 1-norm given by $|z| = \max_{j=1,\dots,n} |z_j|$ on $\mathbb{C}^n$, which also induces the 1-norm on $\mathbb{R}^n$. More generally, C^k denotes the Banach space of k-times continuously differentiable functions $\phi\colon [-h,0] \to \mathbb{R}^n$, with the C^k-norm given by $|\phi|_{C^k} = |\phi|_{C^0} + |\phi'|_{C^0} + \dots + |\phi^{(k)}|_{C^0}$. We write $C^k_{\mathbb{C}}$ for the complexified spaces, which we identify with $C^k([-h,0],\mathbb{C}^n)$. For functions defined on a different domain, e.g., an interval of the form $[-h,-\Delta]$, the corresponding notation is used. Sometimes balls are indexed with the intended norm, for example $B_{|\ |_{C^2}}(0,\delta) = \left\{\psi \in C^2 \ \middle|\ |\psi|_{C^2} < \delta\right\}$. We also use the index $\mathbb{C}$ for canonical complexifications of linear operators, in particular, for semigroups and their generators.

In the present paper we adopt the framework of equations of the form

$$\dot{x}(t) = g(x'_t, x_t) \tag{1.1}$$

introduced by Walther in [55]. We use the notation ψ', ψ'' etc. (instead of $\partial\psi, \partial\partial\psi$ etc. as in [55]), and we write a dot for derivatives at specific times. Note that $(x')_t = (x_t)'$ if $x|_{[t-h,t]}$ is of class C^1.

In eq. (1.1), the functional $g : W \subset C^0 \times C^1 \to \mathbb{R}^n$ is continuous on an open neighborhood W of zero in the product space $C^0 \times C^1$ (with $|\ |_{C^0}$ in the first and $|\ |_{C^1}$ in the second factor), and with an equilibrium at zero: $g(0,0) = 0$. For real numbers t_0, T with $t_0 < T$, a function $x : [t_0 - h, T) \to \mathbb{R}^n$ is a solution of equation (1.1) if it is of class C^1, satisfies $(x'_t, x_t) \in W$ for $t \in [t_0, T)$, and (1.1) is true for $t \in [t_0, T)$.

This setting includes state dependent point delays of the form $\tau(x_t)$ as a special case. One main assumption is that the dependence of g on the first argument (the derivative history) has a minimal delay $\Delta > 0$, meaning that one stays in safe distance to implicit differential equations. Similar conditions were used in [31](p. 10, before Section 2), [45](condition (H), Section 4, p. 3980), and [23](condition (A4), p. 6), but, for example, not in [32]. This property (and also the presence of delayed, but not advanced terms) excludes the classical electrodynamics problem, as also remarked at the end of the introduction to [55]. A typical example class that does fit our framework is the $\mathbb{R}^n$ valued version

$$\dot{x}(t) = A[x'(t - d(x(t)))] + f[x(t - r(x(t)))], \tag{1.2}$$

of the example class from [54] (details in Section 2).

The purpose of the present work is to complement the linearized stability results from the papers [53] and [54], and also those from [2] and [22], with a linearized instability result. As in [54], one difficulty lies in the fact that the 'obvious candidate' for linearization at the zero solution, given by a semigroup $\{S^0(t)\}_{t\geq 0}$ of linear operators, does not have the usual quality of approximation for the full nonlinear equation.

The further organization of the paper is as follows: Section 2 lists the essential assumptions and gives a class of typical examples where they hold. Then Section 3 studies the linearization equation, associated semigroups on spaces of C^0 and C^1 functions, and spectral properties of generators and semigroups. The second part of Section 3 then prepares the study of the nonlinear equation, in particular, by a variation of constants formula from [53]. Properties related to the minimal value Δ of the delay in the derivative become important here. In Section 4, we introduce the nonlinear semiflow and preparatory estimates for solutions in the C^1-norm. The state space X_2 of the semiflow is contained in a manifold $\mathcal{M}_2$, which is tangent at zero to the state space of a semigroup obtained from linearization.

In Section 5 we use the manifold $\mathcal{M}_2$ to obtain a splitting of solutions into three terms, the first of which is given by a linear semigroup on a space of C^1 functions (namely, the so-called extended tangent space of $\mathcal{M}_2$ at zero), the second corresponds to the deviation between $\mathcal{M}_2$ and its tangent space, and the third to the nonlinear part of the equation.

For each of the three terms, C^1-estimates are possible for short time. In Lemma 5.4 we obtain the decisive estimate that expresses smallness of the nonlinear effects w.r. to the C^1-norm. In the following part of Section 5 we employ the additional smoothness condition $(\mathbf{D^2 g_2})$ to construct a manifold $\mathcal{M}_4$ contained in X_2, and describe its tangent space at to zero. A part of this manifold is then used to provide initial functions which will have an unstable evolution under the semiflow.

Section 6 contains the main theorem. Based on spectral assumptions, the estimate on the deviation between linear approximation and 'remainder terms', and the presence of suitable initial functions, an appropriate invariant cone method allows to prove the 'linearized instability' result.

Finally, in Section 7, we consider an example from [38] which models mechanical systems coupled to computer simulations. We show that generalizations of the equations considered in [38], in the sense of equations with state-dependent delay and nonlinear dependence of the delayed derivative, fit in our framework. The linearization at zero and its characteristic equation remain unchanged for these generalizations. Compared to [38] we give some additional analysis of the characteristic equation, and obtain an instability result for suitable values of the parameters, in particular, large enough values of the delay functional at zero.

2 Assumptions and typical examples

We adopt the general setting from [53–55]; in particular, we now list a number of hypotheses from these papers with the same numbering as in [53, 55], but in some cases described in slightly different notation. Conditions $(\widetilde{\mathbf{g1}})$ and $(\widetilde{\mathbf{g8}})$ are stronger versions of $(\mathbf{g1})$ and $(\mathbf{g8})$ from [54]; we comment on the assumptions in detail below.

Consider eq. (1.1), and define $U_1 := \left\{\psi \in C^1 \mid (\psi', \psi) \in W\right\}$; this is an open subset of C^1. We shall use the term **'bounding function'** for any nondecreasing function $\varphi\colon [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0 = \lim_{t\to 0} \varphi(t)$. Such bounding functions appear in several assumptions.

(g0) g is continuous (w.r. to $|\ |_{C^0}$ in the first and $|\ |_{C^1}$ in the second argument).

$(\widetilde{\text{g1}})$ **(The delay in the neutral term of (1.1) has a lower bound.)** There exists $\Delta \in (0,h)$ such that for $(\phi_1,\psi), (\phi_2,\psi) \in W \subset C^0 \times C^1$, one has the implication

$$\forall t \in [-h,-\Delta]: \ \phi_1(t) = \phi_2(t) \Longrightarrow g(\phi_1,\psi) = g(\phi_2,\psi). \tag{2.1}$$

(g2) For every $\psi \in U_1 \subset C^1$, there exists $L_2 \geq 0$ and a neighborhood $N \subset W$ of (ψ',ψ) in $C^0 \times C^1$ such that for all $(\phi_1,\psi_1), (\phi_2,\psi_2)$ in N, with ϕ_2 Lipschitz continuous and best possible Lipschitz constant $\mathrm{Lip}(\phi_2)$, we have:

$$|g(\phi_2,\psi_2) - g(\phi_1,\psi_1)| \leq L_2(|\phi_2-\phi_1|_{C^0} + (\mathrm{Lip}(\phi_2)+1)|\psi_2-\psi_1|_{C^0}).$$

(g3) The restriction g_1 of g to the open subset $W_1 = W \cap (C^1 \times C^1)$ of the space $C^1 \times C^1$ is continuously differentiable, and hence also has continuous partial derivatives $D_1g_1, D_2g_1 : W_1 \to L_c(C^1,\mathbb{R}^n)$. Every derivative $Dg_1(\phi,\psi) : C^1 \times C^1 \to \mathbb{R}^n$, $(\phi,\psi) \in W_1$, has a continuous linear extension: $D_eg_1(\phi,\psi) \in L_c(C^0 \times C^0, \mathbb{R}^n)$, and the map

$$W_1 \times C^0 \times C^0 \ni (\phi,\psi,\chi,\rho) \mapsto D_eg_1(\phi,\psi)(\chi,\rho) \in \mathbb{R}^n$$

is continuous. The corresponding properties then hold for the partial derivatives and their extensions $D_{1,e}g_1, D_{2,e}g_1 : W_1 \to L_c(C^0,\mathbb{R}^n)$.

(g4) **('Linear' case.)** This condition was used in [53] and essentially requires g to be linear in the first argument; we do not use this assumption.

(g5) (is an additional condition on $D_eg_1(\phi,\psi)$ which we do not use.)

(g6) (Recall that $(0,0) \in W_1$, $g(0,0) = 0$). The map

$$C^1 \times C^1 \supset W_1 \ni (\phi,\psi) \mapsto \|D_{1,e}g_1(\phi,\psi)\|_{L_c(C^0,\mathbb{R}^n)} \in \mathbb{R}$$

(see **(g3)**) is upper semicontinuous at $(0,0)$.

(g7) There exist $c_7 > 0$ and a bounding function ζ_7 so that for every $(\phi,\psi) \in W_1$ with $\max\{|\phi|_{C^0}, |\psi|_{C^0}\} \leq 1$ and for all $\rho \in C^1$, we have

$$|[D_2g_1(\phi,\psi) - D_2g_1(0,0)]\rho| \leq \zeta_7(|\phi|_{C^1} + |\psi|_{C^1})|\rho|_{C^0} + c_7 \cdot |\rho|_{C^1}|\psi|_{C^0}.$$

$(\widetilde{\text{g8}})$ **('Nonlinear' case.)** There exist a constant $c_8 > 0$, and a bounding function α such that, with W_1 as in **(g3)** and Δ from **(g1)**, one has for $\phi,\psi \in W_1$ with $\max\{|\phi|_{C^0}, |\psi|_{C^0}\} \leq 1$ and $\chi \in C^1$:

$$|[D_1g_1(\phi,\psi) - D_1g_1(0,0)]\chi| \leq c_8|\chi'|_{C^0} \cdot |\psi|_{C^0} + \alpha(|\phi|_{C^0}) \cdot |\chi|_{[-h,-\Delta]}|_{C^0}.$$

(g9) There exist a convex neighborhood $U_2 \subset U_1 \cap C^2$ of 0 in C^2, a constant $c_9 > 0$ and a bounding function ζ_9 such that for $\psi \in U_2$ one has

$$\max_{0 \le s \le 1} |[D_2 g_1(s\psi', s\psi) - D_2 g_1(0,0)]\psi| \le \zeta_9(|\psi|_{C^2})|\psi|_{C^0} + c_9|\psi|_{C^1}|\psi|_{C^0}.$$

Comments on the above hypotheses:

1) Define

$$X_1 := \Big\{ \psi \in U_1 \mid \dot\psi(0) = g(\psi', \psi) \Big\}, \text{ and}$$
$$X_{1+} := \Big\{ \psi \in X_1 \mid \psi' \text{ is Lipschitz continuous} \Big\}.$$

Note that the condition defining X_1 is satisfied by any segment $\psi = x_t$ of a solution x of equation (1.1), if $x|_{[t-h,t]}$ is of class C^1. Under assumptions **(g0)**, $\widetilde{\textbf{(g1)}}$, **(g2)**, equation (1.1) defines a (local, in time) semiflow on the set X_{1+} which is continuous with respect to the topology from $\mathbb{R}_0^+ \times C^1$ (see [55], Section 4, in particular, Corollary 4.6). Semiflows on smaller sets, with additional smoothness properties, are restrictions of this one.

2) Condition $\widetilde{\textbf{(g1)}}$ expresses that the values of $g(x_t', x_t)$ do not depend on the 'recent past' of $\dot x$, namely, on the values of $\dot x$ on $[t-\Delta, t]$. (Our assumption is apparently stronger than the corresponding assumption **(g1)** from [54, 55], since we assume Δ to exist uniformly for W. It was, however, shown in Proposition 2.7 of [55] that Δ can be chosen locally uniformly, so that the difference is actually minimal.)

This condition excludes, in particular, implicit differential equations. This restriction and also the upper bound h on the delay exclude, for example, the famous two-body-problem of electrodynamics, as considered by Driver e.g. in [10, 11], from the framework chosen here.

3) The extension property **(g3)** can be seen as saying that $Dg_1(\phi, \psi)(\chi, \rho)$ does not depend on χ' and ρ'. Such conditions in the context of state-dependent delay equations were employed, e.g., in [36, 49, 53], and seem to go back to Definition 3.2 in [40]. There a corresponding property was called 'almost Fréchet differentiable' and defined as differentiability from a subspace with stronger norm to an ambient space with weaker norm. Extensibility of the derivative to a linear map continuous w.r. to the weaker norm was not part of the definition in [40], but was present in the applications there.

4) With X_1 from above, define $\mathcal{M}_2 := X_1 \cap C^2$; this set is called X_2 in [55]. It is shown in Proposition 5.1 of that reference that if g satisfies **(g0), (g1), (g3)**, then $\mathcal{M}_2$ is a C^1-submanifold of C^2 with codimension n; its tangent spaces are given by

$$T_\psi \mathcal{M}_2 = \Big\{ \chi \in C^2 \mid \chi'(0) = Dg_1(\psi', \psi)(\chi', \chi) \Big\}.$$

Note that the condition determining these tangent spaces involves only the first derivative of χ, and using the extension property **(g1)**, one can define the so-called extended tangent spaces

$$T_{e,\psi} \mathcal{M}_2 = \Big\{ \chi \in C^1 \mid \chi'(0) = D_e g_1(\psi', \psi)(\chi', \chi) \Big\}. \tag{2.2}$$

The set $\mathcal{M}_2$ is not invariant under the semiflow on X_1, because the property of being C^2 is not, but the following subset of $\mathcal{M}_2$, which is characterized by a second order compatibility condition, (called X_{2*} in [55]) is invariant:

$$X_2 := \Big\{ \psi \in \mathcal{M}_2 \mid \psi' \in T_{e,\psi} \mathcal{M}_2 \Big\}.$$

Combining the definitions, one gets the following explicit description of X_2:

$$X_2 = \Big\{\psi \in C^2 \;\big|\; \text{(i) } \dot\psi(0) = g(\psi', \psi); \quad \text{(ii) } \ddot\psi(0) = D_e g_1(\psi', \psi)(\psi'', \psi')\Big\} \tag{2.3}$$

Under assumptions **(g0)–(g3)**, the semiflow induced on X_2 is continuous w.r. to the topology from $\mathbb{R}_0^+ \times C^2$, as shown in Section 6 of [55]. It has differentiability properties under the linearity assumption **(g4)** (in brief: differentiation w.r. to initial values is possible w.r. to directions tangent to X_2, is given by a variational equation, and has continuity properties under an additional assumption **(g5)**). Condition **(g5)** is not required for the stability results in the papers [53] and [54], and **(g4)** is assumed in [53], but not in [54]. For the instability result of the present paper we assume neither of these two conditions.

The set $X_2 \subset X_{1,+}$ is invariant under the semiflow, but no smooth submanifold of C^2, and we prove in Section 4 that $\mathcal{M}_4 := X_2 \cap C^4$ is a C^1-submanifold of C^4, under an additional condition on g. We employ that manifold in order to get initial values exhibiting instability.

5) Assumptions **(g6)** and **(g7)** are from the paper [53] on linearized stability – except that here ζ_7 is required to be nondecreasing (clearly, if this would not hold, it could be achieved achieved replacing ζ_7 with $\tilde\zeta_7(s) := \sup\Big\{\zeta_7(t) \;\big|\; t \in [0, s]\Big\}$), and that the statement here uses the partial derivative. These conditions are used, in particular, to estimate the 'nonlinear part' $r_g(\psi) = g_1(\psi', \psi) - Dg_1(0, 0)(\psi', \psi)$ of equation (1.1).

Condition **(g7)** is slightly stronger than **(g9)** from the paper [54](because the arguments ϕ, ψ, ρ are independent in **(g7)**), but we keep this condition in the present paper (see Prop. 2.1 below).

6) Our condition **$\widetilde{\text{(g8)}}$** is easily seen to imply condition **(g8)** from the 'nonlinear' paper [54], by specialization to the case $\phi := s\psi', \psi := s\psi$, where $s \in [0, 1]$, and $\chi := \psi'$. On the other hand, the equations of the primary example class from [54] also satisfy **$\widetilde{\text{(g8)}}$**, as we prove below.

7) Condition **(g9)** above is easily seen to be equivalent with condition **(g9)** from [54]: The $\max\limits_{0\le s\le 1}$ from [54] disappears in our case since we assume that ζ_9 is a bounding function, and thereby nondecreasing. Therefore we use the same symbol for 'our' condition **(g9)**.

8) One concrete type of 'linear' equation (meaning linear in the delayed derivative) which was shown to satisfy **(g1)–(g7)** in [55] is the scalar equation

$$\dot x(t) = a\dot x(t - d(x(t))) + f[x(t) - r(x(t))], \tag{2.4}$$

if $a \in \mathbb{R}$ and, for example, $d \in C^2(\mathbb{R}, [\Delta, h])$, $r \in C^2(\mathbb{R}, [0, h])$, and $f \in C^2(\mathbb{R}, \mathbb{R})$, $f(0) = 0$. (Note only that in the notation of the present paper the delays appear with a minus sign.) Correspondingly, the example class from [54] is

$$\dot x(t) = A[\dot x(t - d(x(t))] + f[x(t - r(x(t)))], \tag{2.5}$$

with a nonlinear C^2 function A and $d \in C^2(\mathbb{R}, [\Delta, h])$, $r \in C^2(\mathbb{R}, [0, h])$, and $f \in C^2(\mathbb{R}, \mathbb{R})$, and $A(0) = f(0) = 0$.

We introduce the additional hypothesis on g, mentioned in point 4) above:

($\mathbf{D^2 g_2}$) The map $g_2 := g_1|_{W_1 \cap (C^2 \times C^2)} : W_1 \cap (C^2 \times C^2) \to \mathbb{R}^n$ induced by g_1 is C^2 on $C^2 \times C^2$, and for $(\psi, \phi) \in W_1 \cap (C^2 \times C^2)$, the continuous bilinear form $D^2 g_2(\psi, \phi) : C^2 \times C^2 \to \mathbb{R}^n$ has a continuous extension $D_e^2 g_2(\psi, \phi)$ to $C^1 \times C^1$.

Proposition 2.1.

a) Under conditions **(g0) – (g3)**, **(g6)**, and **(g7)**, $\widetilde{\textbf{(g8)}}$ *(instead of* **(g8)**, **(g9)** *from [54]), the results from [54] remain valid.*

b) If $\Delta \in (0,h]$ *and* $A \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, $d \in C^2(\mathbb{R}^n, [-h,-\Delta])$, $r \in C^2(\mathbb{R}^n, [-h,0])$, *and* $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, *then equation (2.5) (written in the form (1.1)) satisfies conditions* **(g0)–(g3)**, **(g6)**, **(g7)**, $\widetilde{\textbf{(g8)}}$ *from above, and also condition* $(\mathbf{D^2 g_2})$.

Proof. a) We show that **(g7)** implies **(g9)** from the present paper, and hence also **(g9)** from [54]: There exists a convex neighborhhod $U_2 \subset U_1$ of 0 in C^2 such for $\psi \in U_2$ and $s \in [0,1]$ one has $(s\psi', s\psi) \in W^1$, and $\max\{|s\psi'|_{C^0}, |s\psi|_{C^0}\} \le 1$ for $s \in [0,1]$. Then **(g7)** gives

$$\max_{0\le s\le 1} |[D_2 g_1(s\psi', s\psi) - D_2 g_1(0,0)]\psi| \le \max_{0\le s\le 1} [\zeta_7(|s\psi'|_{C^1} + |s\psi|_{C^1}) \cdot |\psi|_{C^0} + c_7 |\psi|_{C^1} |s\psi|_{C^0}].$$

Using that ζ_7 is nondecreasing, we can estimate the last expression by

$$\zeta_7(2|\psi|_{C^2}) \cdot |\psi|_{C^0} + c_7|\psi|_{C^1}|\psi|_{C^0} = \zeta_9(|\psi|_{C^2}) \cdot |\psi|_{C^0} + c_9|\psi|_{C^1}|\psi|_{C^0},$$

where $c_9 := c_7$ and $\zeta_9(r) := \zeta_7(2r)$. This estimate has the form required in **(g9)**.

It was already remarked that $\widetilde{\textbf{(g8)}}$ implies **(g8)**, so the assertion of a) follows.

Ad b): We can set $W = C^0 \times C^1$ and then have

$$g(\phi,\psi) = A[\phi(-d(\psi(0)))] + f[\psi(-r(\psi(0)))] \text{ for } (\phi,\psi) \in W = C^0 \times C^1. \tag{2.6}$$

The calculation from p. 321 and formula (2.1) on p. 322 from [54] carry over to the n-dimensional case to show that **(g0)** is satisfied, that the restriction g_1 is of class C^1 on $W_1 = C^1 \times C^1$, and that for $(\phi,\psi) \in W_1$, $\chi, \rho \in C^1$ one has

$$\begin{aligned} Dg_1(\phi,\psi)(\chi,\rho) = DA[\phi(-d(\psi(0)))]\,[-\dot\phi(-d(\psi(0)))Dd(\psi(0))\rho(0) + \chi(-d(\psi(0)))] \\ + Df[\psi(-r(\psi(0)))]\,[-\dot\psi(-r(\psi(0)))Dr(\psi(0)))\rho(0) + \rho(-r(\psi(0)))]. \end{aligned} \tag{2.7}$$

In particular,

$$D_1 g_1(\phi,\psi)\chi = DA[\phi(-d(\psi(0)))]\chi(-d(\psi(0))), \tag{2.8}$$

$$\begin{aligned} D_2 g_1(\phi,\psi)\rho = DA[\phi(-d(\psi(0)))]\,[-\dot\phi(-d(\psi(0)))Dd(\psi(0))\rho(0)] \\ + Df[\psi(-r(\psi(0)))]\,[-\dot\psi(-r(\psi(0)))Dr(\psi(0))\rho(0) + \rho(-r(\psi(0)))]. \end{aligned} \tag{2.9}$$

Property $\widetilde{\textbf{(g1)}}$ is a direct consequence of formula (2.6) and the assumption that $d(v) \in [\Delta, h]$ for all $v \in \mathbb{R}^n$. The proof of **(g2)** is analogous to the corresponding proof in Proposition 2.1, p. 322 of [54], with one-dimensional balls replaced by n-dimensional balls.

The extension property from **(g3)** and the associated continuity property for $D_e g_1$ are seen from (2.7), mainly since no derivatives of χ and ρ are used. As in [54], p. 323, property **(g6)** is true since, in view of (2.8), for $(\phi,\psi) \in W_1$

$$\begin{aligned} \|D_{1,e} g_1(\phi,\psi)\|_{L_c(C^0,\mathbb{R}^n)} &= \sup_{|\chi|_{C^0}\le 1} |DA[\phi(-d(\psi(0)))]\chi(-d(\psi(0)))| \\ &\le \|DA[\phi(-d(\psi(0)))]\|_{L_c(\mathbb{R}^n,\mathbb{R}^n)}, \end{aligned}$$

with equality for appropriately chosen χ, so that the map mentioned in **(g6)** is even continuous.

Proof of **(g7)**: (We use the notation $\|f\|_{\infty,M}$ for $\sup_{x\in M}\|f(x)\|$ for several functions f with values in normed spaces.) For ϕ, ψ and ρ as in **(g7)**, we have from (2.9)

$$\begin{aligned}
&|[D_2g_1(\phi,\psi) - D_2g_1(0,0)]\rho| \\
&\quad = |DA[\phi(-d(\psi(0)))]\,[-\dot\phi(-d(\psi(0)))Dd(\psi(0))\rho(0)] \\
&\qquad + Df[\psi(-r(\psi(0)))]\,[-\dot\psi(-r(\psi(0)))Dr(\psi(0))\rho(0) + \rho(-r(\psi(0)))] \\
&\qquad - DA(0)\cdot 0 - Df(0)\rho(-r(0))| \\
&\quad \le \|DA\|_{\infty,B(0,1)}\cdot\|Dd\|_{\infty,B(0,1)}\cdot|\phi|_{C^1}\cdot|\rho|_{C^0} \\
&\qquad + \|Df\|_{\infty,B(0,1)}\cdot\|Dr\|_{\infty,B(0,1)}\cdot|\psi|_{C^1}\cdot|\rho|_{C^0} \\
&\qquad + |Df[\psi(-r(\psi(0)))]\rho(-r(\psi(0))) - Df(0)\rho(-r(0))|.
\end{aligned}$$

The last term can be estimated by

$$\begin{aligned}
&|Df[\psi(-r(\psi(0))) - Df(0)]\rho(-r(\psi(0)))| + |Df(0)[\rho(-r(\psi(0))) - \rho(-r(0))]| \\
&\quad \le \sup_{|v|\le|\psi|_{C^0}} |Df(v) - Df(0)|\cdot|\rho|_{C^0} + |Df(0)|\cdot|\rho|_{C^1}\cdot\|Dr\|_{\infty,B(0,1)}\cdot|\psi|_{C^0} \\
&\quad \le \sup_{|v|\le|\psi|_{C^1}} |Df(v) - Df(0)|\cdot|\rho|_{C^0} + |Df(0)|\cdot\|Dr\|_{\infty,B(0,1)}|\rho|_{C^1}\cdot|\psi|_{C^0}.
\end{aligned}$$

Dropping the index $B(0,1)$ now, we have with $c_7 := |Df(0)|\cdot\|Dr\|_\infty$ and

$$\zeta_7(u) := \max\{\|DA\|_\infty\cdot\|Dd\|_\infty, \|Df\|_\infty\cdot\|Dr\|_\infty\}\cdot u + \sup_{|v|\le u}|Df(v) - Df(0)|$$

that $|[D_2g_1(\phi,\psi) - D_2g_1(0,0)]\rho| \le \zeta_7(|\phi|_{C^1} + |\psi|_{C^1})\cdot|\rho|_{C^0} + c_7\cdot|\rho|_{C^1}\cdot|\psi|_{C^0}$.

Proof of $\widetilde{\textbf{(g8)}}$: For ϕ, ψ and χ as in $\widetilde{\textbf{(g8)}}$ we obtain from (2.8), using that d has values in $[\Delta, h]$:

$$\begin{aligned}
&|[D_1g_1(\phi,\psi) - D_1g_1(0,0)]\chi| = |DA[\phi(-d(\psi(0)))]\chi(-d(\psi(0))) - DA(0)\chi(-d(0))| \\
&\quad \le |DA[\phi(-d(\psi(0)))]\,[\chi(-d(\psi(0))) - \chi(-d(0))]| \\
&\qquad + |\{DA[\phi(-d(\psi(0)))] - DA(0)\}\chi(-d(0))| \\
&\quad \le \|DA\|_{\infty,B(0,1)}\cdot|\chi'|_{C^0}\cdot\|Dd\|_{\infty,B(0,1)}|\psi|_{C^0} + \underbrace{\sup_{|v|\le|\phi|_{C^0}}|DA(v) - DA(0)|}_{=:\,\alpha(|\phi|_{C^0})}\cdot|\chi|_{[-h,-\Delta]}|_{C^0} \\
&\quad = c_8\cdot|\chi'|_{C^0}\cdot|\psi|_{C^0} + \alpha(|\phi|_{C^0})\cdot|\chi|_{[-h,-\Delta]}|_{C^0},
\end{aligned}$$

with $c_8 := \|DA\|_{\infty,B(0,1)}\cdot\|Dd\|_{\infty,B(0,1)}$ and the indicated bounding function α.

Proof of $(\mathbf{D^2g_2})$: The evaluation map $\mathrm{ev} : (t,\psi) \mapsto \psi(t)$ is of class C^2 on $[-h,0]\times C^2$. Denoting partial derivatives w.r. to the scalar argument t by ∂_1 and identifying them with vectors, one has for $t\in[-h,0], \psi,\chi \in C^2$

$$\partial_1^2\mathrm{ev}(t,\psi) = \ddot\psi(t),\ D_2^2\mathrm{ev}(t,\psi) = 0,\ \partial_1 D_2\mathrm{ev}(t,\psi)\chi = \dot\chi(t).$$

With the evaluation at zero ev_0 and the canonical projections, g can be represented as on p. 321 of [54]:

$$g = A\circ\mathrm{ev}\circ((-d\circ\mathrm{ev}_0\circ\mathrm{pr}_2)\times\mathrm{pr}_1) + f\circ\mathrm{ev}\circ((-r\circ\mathrm{ev}_0\circ\mathrm{pr}_2)\times\mathrm{pr}_2),$$

which shows that under our assumptions the induced map g_2 is C^2 on $W_1\cap(C^2\times C^2)$, as composition of C^2 maps. To prove the extension property, we first compute an expression for

D^2g_2, based on (2.7). (Below, vectors in $\mathbb{R}^n$ are sometimes also multiplied by numbers from the right.) For $(\phi,\psi) \in W_1 \cap (C^2 \times C^2)$ and $(\chi_1,\rho_1),(\chi_2,\rho_2) \in C^2 \times C^2$,

$$
\begin{aligned}
D^2g_2(\phi,\psi)[(\chi_1,\rho_1),(\chi_2,\rho_2)] &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[Dg_2(\phi+\varepsilon\chi_2,\psi+\varepsilon\rho_2) - Dg_2(\phi,\psi)](\chi_1,\rho_1)\\
= D^2A(\phi(-d(\psi(0))))\big[&-\dot\phi(-d(\psi(0)))Dd(\psi(0))\rho_1(0)+\chi_1(-d(\psi(0))),\\
&-\dot\phi(-d(\psi(0)))Dd(\psi(0))\rho_2(0)+\chi_2(-d(\psi(0)))\big]\\
+DA(\phi(-d(\psi(0))))\{&-\dot\chi_2(-d(\psi(0))Dd(\psi(0))\rho_1(0)\\
&+\ddot\phi(-d(\psi(0)))[Dd(\psi(0))\rho_2(0)]\cdot[Dd\psi(0)\rho_1(0)]\\
&-\dot\phi(-d(\psi(0)))D^2d(\psi(0))[\rho_1(0),\rho_2(0)]\\
&-\dot\chi_1(-d(\psi(0)))Dd(\psi(0))\rho_2(0)\}\\
+\ &\text{(a similar expression involving } f \text{ and } r\text{, namely:)}\\
D^2f(\psi(-r(\psi(0))))\big[&-\dot\psi(-r(\psi(0)))Dr(\psi(0))\rho_1(0)+\rho_1(-r(\psi(0))),\\
&-\dot\psi(-r(\psi(0)))Dr(\psi(0))\rho_2(0)+\rho_2(-r(\psi(0)))\big]\\
+Df(\psi(-r(\psi(0))))\{&-\dot\rho_2(-r(\psi(0))Dr(\psi(0))\rho_1(0)\\
&+\ddot\psi(-r(\psi(0)))[Dr(\psi(0))\rho_2(0)]\cdot[Dr(\psi(0))\rho_1(0)]\\
&-\dot\psi(-r(\psi(0)))D^2r(\psi(0))[\rho_1(0),\rho_2(0)]\\
&-\dot\rho_1(-r(\psi(0)))Dr(\psi(0))\rho_2(0)\}.
\end{aligned}
$$

One sees from the above expressions that $D^2g_2(\phi,\psi)$ has a continuous extension to $C^1 \times C^1$, mainly because no second derivatives of $\chi_1,\rho_1,\chi_2,\rho_2$ appear. □

The formal linearization of equation (1.1) at zero, using the extension property **(g3)**, is $\dot y(t) = D_eg_1(0,0)(y'_t,y_t)$, which can also be written as

$$\dot y(t) = D_{1,e}g_1(0,0)y'_t + D_{2,e}g_1(0,0)y_t, \tag{2.10}$$

Remark 2.2. In the special case of equation (2.5) (but n-dimensional, as in Prop. 2.1 b)), the formal linearization in the sense of equation (2.10) is given by

$$\dot y(t) = DA(0)\dot y(t-d(0)) + Df(0)y(t-r(0)), \tag{2.11}$$

i.e., by the 'frozen delay principle' (linearizing in the same way as if the delays were constant, with the values at equilibrium).

Proof. From the expressions for the partial derivatives in (2.8) and (2.9) we see that for $\chi,\rho \in C^0$ one has $D_{1,e}g_1(0,0)\chi = DA(0)\chi(-d(0))$ and

$$D_{2,e}g_1(0,0)\rho = DA(0)[0] + Df(0)[0+\rho(-r(0)] = Df(0)\rho(-r(0)).$$

Applying this with $\chi = y'_t$ and $\rho = y_t$ shows that in this case (2.10) and (2.11) are equivalent. (See also the remarks at the beginning of Section 3.4, p. 472 in [25], which however refer to the non-neutral case.) □

3 Semigroups, spectra, growth estimates and fundamental matrix

We define $L := D_{1,e}g_1(0,0) \in L_c(C^0,\mathbb{R}^n)$, $R := D_{2,e}g_1(0,0) \in L_c(C^0,\mathbb{R}^n)$, and then rewrite the linearization equation (2.10), shifting the derivative-dependent part to the left-hand side, and writing Y for the phase curve $t \mapsto y_t \in C^0$:

$$\frac{d}{dt}(y - L \circ Y)(t) = Ry_t. \tag{3.1}$$

Complexifications $L_\mathbb{C}, R_\mathbb{C} \in L_c(C^0_\mathbb{C}, \mathbb{C}^n)$ are obtained in the obvious way. It turns out that the latter equation generates a semigroup even on the space C^0, and that this semigroup serves as a kind of linearized approximation of the nonlinear semiflow, but in a sense that must be treated with caution: The domains of the semigroup and of the semiflow are different, and for the error made in approximation by the semigroup to be small in the C^1-norm, the trajectory has to stay in small ball w.r. to the C^2-norm. By a solution of (3.1) we mean a continuous function y on, e.g., $[-h, T)$ such that, with the corresponding phase curve Y, the function $(y - L \circ Y) : [0, T) \to \mathbb{R}^n$ is of class C^1 (meaning the right-hand derivative at $t = 0$), and satisfies the equation. Such a solution of equation (3.1) is in general not necessarily differentiable, only the difference $y - L \circ Y$ is.

Lemma 3.1 (**The semigroup** S^0, see [53, Corollary 6.2, p. 457]). *For every $\chi \in C^0$, there is a uniquely determined continuous solution $y^\chi : [-h, \infty) \to \mathbb{R}^n$ of (3.1) with $y^\chi_0 = \chi$. Each linear map $S^0(t) : C^0 \ni \chi \mapsto y^\chi_t \in C^0$, $t \geq 0$ is continuous, and the operators $S^0(t), t \geq 0$, form a strongly continuous semigroup $\{S^0(t)\}_{t\geq 0}$ of operators in $L_c(C^0, C^0)$.*

A large part of this section is concerned with deriving growth estimates for the semigroup S^0 from spectral assumptions on its generator A^0. As always for translation semigroups,. $A^0\varphi = \varphi'$ for $\varphi \in D(A^0)$. Since equation (3.1) is a neutral equation without state-dependent delay, this is not a new topic, but we found that the treatment in the literature does not always provide comfortable reading, and try to improve this in the present paper.

For a complex valued function f, we use the notation $\mathcal{Z}(f)$ for the zero set of f.

Lemma 3.2 (**Spectrum of the generator** A^0 **of** S^0). *The spectrum of its infinitesimal generator A^0 (that is, the spectrum of the complexification $A^0_\mathbb{C}$) consists only of isolated eigenvalues of finite multiplicity. These eigenvalues coincide with the solutions of the characteristic equation*

$$\chi(\lambda) := \det(\Delta(\lambda)) = 0$$

where $\Delta(\lambda) \in \mathbb{C}^{n\times n}$ ($\lambda \in \mathbb{C}$) is a so-called characteristic matrix, obtained from the exponential ansatz $y(t) = \exp(\lambda t) \cdot y(0)$ for solutions of equation (3.1). Thus we have

$$\sigma(A^0) = \mathcal{Z}(\chi).$$

Proof. We can write equation (3.1) as $\frac{d}{dt}[(\mathrm{ev}_0 - L)y_t] = Ry_t$, where ev_0 denotes the evaluation at zero. The operator $\mathrm{ev}_0 - L$ corresponds to the operator M in formula (3.1) on p. 510 of [33], and also to the operator M in formula (6.3) on p. 396 of [34]. It satisfies the condition (3.4) on p. 511 from [33] (in the language of [21], p. 6: 'μ uniformly non-atomic at 0'), and the corresponding condition (6.3) from [34]: In our case L and R are given (in the sense of the Riesz representation theorem) by Riemann–Stieltjes integrals of the form

$$L\varphi = \int_{-h}^{0} d\mu(\theta) \cdot \varphi(\theta), \quad R\varphi = \int_{-h}^{0} d\eta(\theta) \cdot \varphi(\theta),$$

with matrix valued functions μ, η having entries of bounded variation, and defining Borel measures on $[-h, 0]$ (see also [46], Theorem 2.14, p. 40). In our situation, μ is constant (in particular, continuous) on $[-\Delta, 0]$ (compare here part b) of Lemma 3.18 below). The assertions now follow from Corollary 3.3 on p. 512 of [33], together with the definition of the characteristic matrix $\Delta(\lambda)$ as introduced in Theorem 3.2 of that reference.

Alternatively, the statements of the present Lemma follow from Theorem 2.1, p. 109 of [26], or from Theorem 1 on p. 17, Section III of [21]. In the latter reference the corresponding condition on the behavior of μ at zero is found in (5) on p. 6, as mentioned above. □

Note that with the representations of L and R as in the above proof, one has

$$\begin{aligned}\Delta(\lambda) &= \lambda I - \lambda \int_{-h}^{0} d\mu(\theta)\exp(\lambda\theta) - \int_{-h}^{0} d\eta(\theta)\exp(\lambda\theta)\\ &= \lambda\cdot\left[I - \int_{-h}^{0} d\mu(\theta)\exp(\lambda\theta)\right] - \int_{-h}^{0} d\eta(\theta)\exp(\lambda\theta).\end{aligned} \tag{3.2}$$

Remark 3.3. In the reference [33], which was employed in the above proof, the resolvent set $\rho(A)$ of a closed operator $A : X \supset D(A) \to X$ with domain $D(A)$ in a complex Banach space X is described in Section I.1.1 on p. 482 as '...the set of complex numbers λ for which the *resolvent* $R(\lambda, A) = (\lambda - A)^{-1}$ exists.' – taken literally, this would include cases where for example the range of $\lambda - A$ is a closed proper subspace $U \subsetneq X$. It is however obvious from the subsequent text on p. 482 of [33] that the existence of the resolvent is understood as an operator in $L_c(X, X)$, i.e., $\lambda \in \rho(A)$ if and only if $\lambda - A : D(A) \to X$ is an isomorphism with continuous inverse. For a closed operator A in a Banach space, this is equivalent to demanding that $\lambda - A$ is bijective onto its image, with a continuous inverse, and that the image is dense in X (it is then automatically all of X).

Lemma 3.4 (The semigroup S^1). *The solutions of equation (3.1) also induce a C^0-semigroup $\{S^1(t)\}_{t\geq 0}$ of linear operators on the space*

$$T^1 = \left\{\chi \in C^1 \mid \chi'(0) = D_e g_1(0,0)(\chi', \chi)\right\}$$

with the $|\ |_{C^1}$-norm (this is the domain of the generator A^0 of the semigroup $\{S^0(t)\}_{t\geq 0}$ on C^0, with the graph norm). The space T^1 coincides with the extended tangent space $T_{e,0}\mathcal{M}_2$ of $\mathcal{M}_2$ at zero (see (2.2)). The infinitesimal generators A^0 (of the semigroup S^0) and A^1 (of the semigroup S^1) have the same spectra (again, these are the spectra of $A^0_{\mathbb{C}}, A^1_{\mathbb{C}}$). For both operators, these consist only of eigenvalues λ of finite type, which are obtained from the exponential ansatz as in Lemma 3.2. The corresponding (finite dimensional) generalized eigenspaces G_λ of $A^0_{\mathbb{C}}$ and $A^1_{\mathbb{C}}$ coincide.

Proof. From [53], especially the remark on p. 442 preceding condition **(g4)** there, and from [25], Proposition 3.4.1, p. 473, one sees that T^1 coincides with the domain of the 'real' generator A^0, that the restriction of S^0 to T^1 defines a C^0-semigroup with respect to the C^1-topology, and that the spectra/ resolvent sets of the infinitesimal generator A^1 of $\{S^1(t)\}$ and A^0 of $\{S^0(t)\}$ (here we mean the complexified versions) satisfy

$$\sigma(A_1) \subset \sigma(A^0), \text{ and } \rho(A^1) \subset \rho(A^0) \cup \left\{\lambda \in \mathbb{C} \mid \lambda - A^0 \text{ is injective, not surjective}\right\}. \tag{3.3}$$

It is clear from (2.2) that $T^1 = T_{e,0}\mathcal{M}_2$. Now, from Lemma 3.2, the spectrum of A^0 consists only of eigenvalues of finite type, which are obtained from the exponential ansatz. This shows that the set in brackets in (3.3) above is empty, so $\rho(A_1) \subset \rho(A^0)$. Together with the first inclusion in (3.3) we conclude $\sigma(A_1) = \sigma(A^0)$. The assertion on the eigenspaces follows from part (ii) of Proposition 3.4.1, p. 474 in [25]. □

We shall need results expressing how spectral properties of the generator influence growth properties of the semigroup, in particular, to have a separation between different growth rates on complementary subspaces. The difficulty here lies, in principle, in the nontrivial relation between the so-called spectral bound

$$s(A) := \sup \operatorname{Re}(\sigma(A))$$

of the generator A and the growth bound

$$\omega_0(T) := \inf \left\{ \beta \in \mathbb{R} \mid \exists M > 0 : \forall t \geq 0 : \|T(t)\| \leq Me^{\beta t} \right\}$$

of a C^0-semigroup $\{T(t)\}_{t\geq 0}$ of operators in $L_c(X, X)$, where X is a Banach space (see, e.g., [13], Section 2 of Chapter IV). In general, one only has $s(A) \leq \omega_0(T)$ instead of equality, and for the spectral radius $r(T(t))$ (which is again defined as $r(T(t)_{\mathbb{C}})$, if necessary):

$$\forall t \geq 0 : \; r(T(t)) = \exp(\omega_0(T)t) \tag{3.4}$$

(see e.g. [13], Chapter IV, Prop. 2.2, p. 251, and the counterexample 2.7 on p. 253, where $s(A) = -1$ and $\omega_0(T) = 0$). A frequently quoted example (in the Hilbert space ℓ^2) is given in the paper by Zabczyk, [57], which mentions the earlier result by Foiaș [14]. See also Lemma 4.2 from Section 74, p. 180 in [20], where the growth bound is called the order of $\{T(t)\}$. Thus, the growth bound for a semigroup is controlled by the spectral radius of one particular $T(t_0)$ with $t_0 > 0$:

Proposition 3.5. *If $\omega \in \mathbb{R}$ satisfies $r(T(t_0)) < \exp(\omega t_0)$ for some $t_0 > 0$ then there exists $M \geq 1$ with*

$$\|T(t)\|_{L_c(X,X)} \leq M \exp(\omega t) \quad \textit{for all } t \geq 0.$$

Proof. Since $r(T(t)) \leq \|T(t)\|_{L_c(X,X)}$ (see e.g. Cor. 1.4, p. 241 of [13]), such an ω must be larger than the growth bound $\omega_0(T)$, in view of (3.4). The estimate then follows from the definition of $\omega_0(T)$. □

We employ the usual notation $\rho(\dots)$ for the resolvent set and $P\sigma(\dots), C\sigma(\dots), R\sigma(\dots)$ for the point spectrum, i.e., the continuous spectrum and the residual spectrum of an operator, compare e.g. [28], Definition 2.16.1, p. 54. The problem of controlling the spectral radius $r(T(t))$ in turn by the spectrum of the generator A stems from the fact that in general one has

$$P\sigma(T(t)) \subset \{0\} \cup \exp[t \cdot P\sigma(A)] \text{ and } R\sigma(T(t)) \subset \{0\} \cup \exp[t \cdot R\sigma(A)]$$

(spectral mapping theorems for the point and residual spectrum, see [13], Theorem 3.7, p. 277), but no corresponding control over possible continuous spectrum $C\sigma(T(t))$. See [28], p. 54 for this subdivision of the spectrum $\sigma(T(t))$, and also Theorems 16.7.2 and 16.7.2 on pages 467 and 469 of [28].

The idea of controlling the spectrum of the semigroup $S^0(t)$ (which is if interest for us) as used in [20], in [26], and also in [17], is to treat $S^0(t)$ as a compact perturbation of a 'simpler' semigroup. For this 'simpler' semigroup an appropriate spectral mapping theorem is known, and then a result for compact perturbations can be used. We start carrying out this approach now.

The operator $L = D_{1,e}g_1(0,0) \in L_c(C^0, \mathbb{R}^n)$ from equation (3.1) has a representation as $L\varphi = \int_{-h}^{0} d\mu(\theta)\varphi(\theta)$ in the sense of the Riesz representation theorem. For simplicity, we introduce assumptions on L which are slightly stronger than needed.

Assumption on L: There exist $k \in \mathbb{N}$ and $A_j \in \mathbb{R}^{n\times n}, \tau_j \in (0,h], j = 1,\dots,k$, and an L^1 function $A : [-h,0] \to \mathbb{R}^{n\times n}$ such that L has the form

$$L\varphi = \sum_{j=1}^{k} A_j\varphi(-\tau_j) + \int_{-h}^{0} A(\theta)\varphi(\theta)d\theta \quad (\varphi \in C^0). \tag{3.5}$$

This condition implies the 'non-atomic at zero' and 'no singular part' assumptions made in [21] (Section IV there) and in [26] (assumptions (i) and (ii) on p. 108 there). (In combination with the presently not needed condition $\widetilde{\textbf{(g1)}}$, Lemma 3.18 below shows that even $A(\theta) = 0$ for almost all θ in $[-\Delta, 0]$.) With the difference operator $D_0 : C^0 \to C^0$ defined by

$$D_0\varphi := \varphi(0) - \sum_{j=1}^{k} A_j\varphi(-\tau_j), \tag{3.6}$$

equation (3.1) takes the form $\frac{d}{dt}[D_0y_t - \int_{-h}^{0} A(\theta)y_t(\theta)d\theta] = Ry_t$, and is therefore (remotely) related to the difference equation $D_0y_t = 0$, as we see below. In order to use this relation, the next result will be important. It is known, and we include a proof for completeness.

Proposition 3.6. *For fixed $t \geq 0$, consider the continuous linear operator*

$$K(t) : C^0 \to C^0([0,t],\mathbb{R}^n),\ \phi \mapsto H(\cdot,\phi),\ \textit{where}$$

$$H(s,\phi) := \int_{-h}^{0} A(\theta)y^{\phi}(s+\theta)d\theta - \int_{-h}^{0} A(\theta)\phi(\theta)d\theta + \int_{0}^{s} R(y^{\phi}_{\sigma})\,d\sigma$$

and $y^{\phi}_s := S^0(s)\phi$. This operator is compact.

Proof. Continuity of $K(t)$ is obvious. The middle term in the formula for H just defines a continuous linear functional into $\mathbb{R}^n$, and hence certainly a compact operator. The C^0-semigroup S^0 satisfies an exponential growth estimate of the form $|S^0(t)\phi|_{C^0} \leq c(t)|\phi|_{C^0}$, with c nondecreasing. Using this for the last term in the definition of H shows that this part even produces functions bounded in C^1, if $|\phi|_{C^0} \leq 1$. It follows from the Arzelà–Ascoli theorem that this third part defines a compact operator.

Abbreviating the first term with $U(s,\phi)$, considering $s \in [0,t]$ and $\tau \in [0,h]$ with $s+\tau \in [0,t]$, and extending A to an L^1 function on all of $\mathbb{R}$ (by zero), we obtain for ϕ with $|\phi|_{C^0} \leq 1$:

$$\begin{aligned}
|U(s+\tau,\phi) - U(s,\phi)| &= \left|\int_{-h}^{0} A(\theta)[y^{\phi}(s+\tau+\theta) - y^{\phi}(s+\theta)]d\theta\right| \\
&= \left|\int_{-h+\tau}^{\tau} A(\theta-\tau)y^{\phi}(s+\theta)\,d\theta - \int_{-h}^{0} A(\theta)y^{\phi}(s+\theta)]d\theta\right| \\
&\leq \int_{-h+\tau}^{\tau} |A(\theta-\tau) - A(\theta)|\cdot|y^{\phi}(s+\theta)|\,d\theta \\
&\leq c(t)\cdot|\phi|_{C^0}\int_{-h+\tau}^{\tau} |A(\theta-\tau) - A(\theta)|\,d\theta \leq c(t)\cdot\int_{\mathbb{R}} |A(\theta-\tau) - A(\theta)|\,d\theta.
\end{aligned}$$

Now translation $\mathbb{R} \ni \tau \mapsto A(\cdot+\tau) \in L^1(\mathbb{R},\mathbb{R}^{n\times n})$ is continuous, as follows from approximation by continuous functions with compact support and the Lebesgue convergence theorem. Thus the last term goes to zero as $|\tau| \to 0$. This proves that the functions $U(\cdot,\phi)$, $|\phi|_{C^0} \leq 1$ are a (bounded) and equicontinuous set in $C^0([0,t],\mathbb{R}^n)$, and hence the compactness of also the first part of $K(t)$ follows again from the Arzelà–Ascoli theorem. □

We turn to the analysis of characteristic functions now. For L as in (3.5), the characteristic matrix (compare (3.2)) takes the form

$$\Delta(\lambda) = \lambda \cdot \left[I - \sum_{j=1}^{k} A_j \exp(-\lambda\tau_j) - \int_{-h}^{0} A(\theta)\exp(\lambda\theta)d\theta \right] - \int_{-h}^{0} d\eta(\theta)\exp(\lambda\theta). \tag{3.7}$$

In this context, the following functions and sets are important:

Define $\Delta_0(\lambda) := I - \sum_{j=1}^{k} A_j \exp(-\lambda\tau_j)$ and

$$\chi_0(\lambda) := \det \Delta_0(\lambda), \tag{3.8}$$

with the zero set $\mathcal{Z}(\chi_0) = \left\{ \lambda \in \mathbb{C} \mid \det \Delta_0(\lambda) = 0 \right\}$, and

$$Z_0 := \operatorname{Re}(\mathcal{Z}(\chi_0)) = \left\{ \operatorname{Re}(\lambda) \mid \det \Delta_0(\lambda) = 0 \right\}. \tag{3.9}$$

With $M_1(\lambda) := \int_{-h}^{0} A(\theta)\exp(\lambda\theta)d\theta$, $M_2(\lambda) := \int_{-h}^{0} d\eta(\theta)\exp(\lambda\theta)$ one has $\Delta(\lambda) = \lambda \cdot [\Delta_0(\lambda) - M_1(\lambda)] - M_2(\lambda)$, so for $\lambda \neq 0$: $\Delta(\lambda) = \lambda \cdot [\Delta_0(\lambda) - M_1(\lambda) - \frac{1}{\lambda}M_2(\lambda)]$, and

$$\chi(\lambda) = \det(\Delta(\lambda)) = \lambda^n \cdot \det \left[\Delta_0(\lambda) - M_1(\lambda) - \frac{1}{\lambda} M_2(\lambda) \right].$$

It follows that the function defined by $\tilde{\chi}(\lambda) := \chi(\lambda)/\lambda^n$ for $\lambda \neq 0$ satisfies

$$\tilde{\chi}(\lambda) = \det \left[\Delta_0(\lambda) - M_1(\lambda) - \frac{1}{\lambda} M_2(\lambda) \right]. \tag{3.10}$$

For intervals $I \subset \mathbb{R}$, we shall consider $\chi, \tilde{\chi}$ and χ_0 in vertical strips of the form

$$\mathbf{S}_I := \left\{ r + is \mid r \in I,\, s \in \mathbb{R} \right\}.$$

Note that if $I = (\alpha, \beta)$ then the function χ_0 is holomorphic and almost periodic in the strip $\mathbf{S}_I$. (The almost periodicity in the vertical direction corresponds to the definition of H. Bohr in [7], section **104**, p. 86, which includes uniformity w.r. to the real part. Sometimes it is also defined correspondingly for horizontal strips, see e.g. formula (6.09), p. 266 in [39].) The approach in the subsequent results, based on almost periodicity, is essentially contained in [26] and also in sections 12.3 and 12.10 of [20], but we include proofs for completeness.

Lemma 3.7. *Consider a vertical strip $\mathbf{S}_{(\alpha,\beta)}$, where $\alpha, \beta \in \mathbb{R}$, and holomorphic functions f_0, f_1 : $\mathbf{S}_{(\alpha,\beta)} \to \mathbb{C}$, with f_0 almost periodic, and such that $f_1(r+is) \to f_0(r+is)$ as $s \to \infty$, uniformly w.r. to $r \in (\alpha, \beta)$.*

- *a) If $z_0 = r_0 + is_0$ is a zero of f_0 in $\mathbf{S}_{(\alpha,\beta)}$ then there exists a sequence $(z_j) = (r_j + is_j)$ in $\mathcal{Z}(f_0)$ with $r_j \to r_0$ and $s_j \to \infty$ as $j \to \infty$.*
- *b) If z_0 and the sequence $(z_j) \subset \mathcal{Z}(f_0)$ are as in a) then there exists a subsequence $(z_{\psi(j)}) \subset (z_j)$ and a sequence (ζ_j) of zeroes of f_1 such that $|\zeta_j - z_{\psi(j)}| \to 0$ as $j \to \infty$, in particular, $\operatorname{Re}(\zeta_j) \to r_0$.*

Proof. Ad a): Assume the opposite; then there exist $\delta > 0$ and $S > 0$ such that the set $\left\{ z = r + is \mid |r - r_0| \leq \delta, s \geq S \right\}$ is contained in $\mathbf{S}_{(\alpha,\beta)}$, and disjoint to $\mathcal{Z}(f_0)$. Then all points of the form $r_0 + is$ with $s \geq S + \delta$ would be at least a distance δ away from $\mathcal{Z}(f_0)$. It follows now from the fact that f_0 is both almost periodic and holomorphic that there exists a number $m(\delta) > 0$ such that

$$\forall s \geq S + \delta : \ |f_0(r_0 + is)| \geq m(\delta).$$

[See Lemma 3.1, part (ii) on p. 111 of [26], and Lemma 1 in Section 2 of Chapter VI [39], p. 268. The proof there (stated for horizontal strips) employs the characterization of almost periodic functions (due to Bochner) by the fact that the set of translates of such a function is relatively compact with respect to uniform convergence, see [39], p. 266, and [6], Satz XII on p. 143, where this property is called „Normaleigenschaft".]

Now associated to $\varepsilon := m(\delta)/2$ there exists an ε-almost period $T > 0$ of $f_0(r_0 + i\cdot)$ which satisfies $s_0 + T > S + \delta$, and hence with $s := s_0 + T$ one has

$$\begin{aligned} m(\delta) &\leq |f_0(r_0 + is)| = |f_0(r_0 + i(s_0 + T))| \\ &\leq \underbrace{|f_0(r_0 + is_0)|}_{=0} + \underbrace{|f_0(r_0 + is_0) - f_0(r_0 + i(s_0 + T))|}_{\leq \varepsilon} \leq \varepsilon = m(\delta)/2, \end{aligned}$$

a contradiction.

Ad b): Assume that the sequence $(z_j) = (r_j + is_j)$ is as in a). Since $r_j \to r_0$, we can pick $\delta_0 > 0$ is such that $\forall j \in \mathbb{N} : B(z_j, \delta_0) \subset \mathbf{S}_{(\alpha,\beta)}$. Next, for $j, k \in \mathbb{N}$, the circular rings $R_{j,k} := \left\{ z \in \mathbb{C} \mid \delta_0/(k+1) < |z - z_j| < \delta_0/k \right\}$ are also contained in $\mathbf{S}_{(\alpha,\beta)}$.

Claim: There exists $k_0 \in \mathbb{N}$ such that $\forall j_0 \in \mathbb{N}\ \exists j \geq j_0 : R_{j,k_0} \cap \mathcal{Z}(f_0) = \emptyset$.

Proof: The opposite of the claim is

$$\forall k_0 \in \mathbb{N}\ \exists j_0 \in \mathbb{N}\ \forall j \geq j_0 : R_{j,k} \cap \mathcal{Z}(f_0) \neq \emptyset. \tag{$*$}$$

Assume that ($*$) holds, and fix $N \in \mathbb{N}$, and take numbers $j_0(1), \ldots, j_0(N)$ corresponding to $k_0 = 1, 2, \ldots, N$ according to ($*$), and set $j^* := \max\{j_0(1), \ldots, j_0(N)\}$. Then one has $R_{j^*,k} \cap \mathcal{Z}(f_0) \neq \emptyset$ for $k = 1, \ldots, N$, i.e., f_0 has at least N zeroes in $B(z_{j*}, \delta_0)$. This argument works for every $N \in \mathbb{N}$, which contradicts the fact that there exists $N^* \in \mathbb{N}$ such that in each rectangle of the form $\left\{ z = r + is \in \mathbb{C} \mid r \in (\alpha, \beta),\, s \in (t, t+1] \right\}$ (where $t \in \mathbb{R}$), the almost periodic holomorphic function f_0 has at most N^* zeroes. (See [39], Lemma 2, p. 269; the proof again uses Bochner's compactness theorem. See also [26], p. 111, Lemma 3.1, part (i).) The claim is proved.

The above claim allows us to choose a subsequence $(z_{\varphi(j)}) \subset (z_j)$ such that

$$\forall j \in \mathbb{N} : \ R_{\varphi(j),k_0} \cap \mathcal{Z}(f_0) = \emptyset.$$

With $\delta^* := \frac{1}{2}(\delta_0/(k_0+1) + \delta_0/k_0)$, the central circular lines $\partial B(z_{\varphi(j)}, \delta^*)$ of the rings $R_{\varphi(j),k_0}$ all have distance at least $\delta^* - \delta_0/(k_0+1) > 0$ from $\mathcal{Z}(f_0)$. As above, it follows from Lemma 1 on p. 268 of [39] that there exists a number $m > 0$ such that $|f_0| \geq m$ on $\partial B(z_{\varphi(j)}, \delta^*)$ for every $j \in \mathbb{N}$. The assumed uniform convergence of $f_1(r + is)$ to $f_0(r + is)$ as $s \to \infty$ implies that for all large enough j, $|f_1 - f_0| < m$ on $\partial B(z_{\varphi(j)}, \delta^*)$. Then the Rouché theorem implies that f_1 and f_0 have the same number of zeroes in $B(z_{\varphi(j)}, \delta^*)$, in particular, f_1 has a zero in this set.

Together we have proved that, with δ_0 as above, there exists a subsequence $(z_{\varphi(j)}) \subset (z_j)$ and $j_0 \in \mathbb{N}$ such that

$$\forall j \geq j_0 : \mathcal{Z}(f_1) \cap B(z_{\varphi(j)}, \delta_0) \neq \emptyset.$$

In particular, there exist arbitrarily large $j \in \mathbb{N}$ with $\mathcal{Z}(f_1) \cap B(z_{\varphi(j)}, \delta_0) \neq \emptyset$.

Since this argument also works for every smaller positive value of δ_0, we can pick a sequence of numbers $\delta_j > 0$ with $\delta_0 \geq \delta_j \to 0$ and associated indices $\psi(j)$ with $\psi(j+1) > \psi(j)$ (so that $(z_{\psi(j)})$ is a subsequence of (z_j)) such that $\mathcal{Z}(f_1) \cap B(z_{\psi(j)}, \delta_j) \neq \emptyset$ for all $j \in \mathbb{N}$. Choosing ζ_j from the last set for every j, we obtain $(\zeta_j) \subset \mathcal{Z}(f_1)$ and $|\zeta_j - z_{\psi(j)}| < \delta_j \to 0$ as $j \to \infty$. □

We have the following relation between χ_0 and $\tilde{\chi}$, which will allow us to apply the last proposition:

Proposition 3.8. *In a vertical strip $\mathbf{S}_{(\alpha,\beta)}$, where $\alpha, \beta \in \mathbb{R}$, $\tilde{\chi}(r+is) \to \chi_0(r+is)$ as $s \to \infty$, uniformly w.r. to $r \in (\alpha, \beta)$.*

Proof. Recall from (3.8) and (3.10) that

$$\chi_0(\lambda) = \det \Delta_0(\lambda) \quad \text{and} \quad \tilde{\chi}(\lambda) = \det[\Delta_0(\lambda) - M_1(\lambda) - \frac{1}{\lambda} M_2(\lambda)].$$

We claim that $M_1(r+is) \to 0$ as $s \to \infty$, uniformly w.r. to $r \in (\alpha, \beta)$. The proof is mainly a Riemann–Lebesgue-type argument which we include for completeness. (We choose a matrix norm $\| \ \|$ on $\mathbb{C}^{n\times n}$, and use the corresponding C^0- and L^1-norms.) Set $\mu := \max\{|\alpha|, |\beta|\}$ and let $\varepsilon > 0$ be given. Choose a matrix-valued function $\tilde{A} \in C^1([-h, 0], \mathbb{R}^{n\times n})$ with $e^{\mu h} \cdot \|\tilde{A} - A\|_{L^1([-h,0])} \leq \varepsilon/2$. Then one has for $r \in (\alpha, \beta)$

$$M_1(r+is) = \int_{-h}^{0} A(\theta) e^{(r+is)\theta}\, d\theta = \int_{-h}^{0} \tilde{A}(\theta) e^{(r+is)\theta}\, d\theta + \int_{-h}^{0} [A(\theta) - \tilde{A}(\theta)] e^{(r+is)\theta}\, d\theta.$$

The second term can be estimated by $e^{\mu h} \|\tilde{A} - A\|_{L^1([-h,0])} < \varepsilon/2$.

The first term equals, by partial integration,

$$\left[\tilde{A}(\theta) \frac{1}{r+is} e^{(r+is)\theta}\right]_{\theta=-h}^{\theta=0} - \frac{1}{r+is} \int_{-h}^{0} \tilde{A}'(\theta) \cdot e^{(r+is)\theta} d\theta,$$

which for $s > 0$ can be estimated by $\frac{1}{s}[2e^{\mu h} \|\tilde{A}\|_{C^0} + e^{\mu h} \|\tilde{A}'\|_{L^1([-h,0])}]$, and the latter is less than $\varepsilon/2$ for all large enough s. This proves the asserted uniform convergence.

Since $M_2(\lambda)$ is uniformly bounded for $\lambda \in \mathbf{S}_{(\alpha,\beta)}$, we also have $\frac{1}{r+is} M_2(r+is) \to 0$ as $s \to \infty$, uniformly w.r. to $r \in (\alpha, \beta)$.

Note that the matrix $\Delta_0(\lambda)$ is bounded for $\lambda \in \mathbf{S}_{(\alpha,\beta)}$. It follows from the convergence of $M_1(\lambda)$ and $\frac{1}{\lambda} M_2(\lambda)$ to zero as $s \to \infty$ ($\lambda = r + is, r \in (\alpha, \beta)$) that all matrices $\Delta_0(\lambda)$ and $\Delta_0(\lambda) - M_1(\lambda) - \frac{1}{\lambda} M_2(\lambda)$ for $s \geq 1$ are contained in a ball in $\mathbb{C}^{n\times n}$, on which the determinant function det is uniformly continuous, so that $|\det(A) - \det(B)| \leq \rho_{\det}(\|A - B\|)$ for A, B in that ball, with a bounding function $\rho_{\det}$. It follows then that for $\lambda = r + is \in \mathbf{S}_{(\alpha,\beta)}$ with $s \geq 1$ one has

$$|\chi_0(r+is) - \tilde{\chi}(r+is)| \leq \rho_{\det}\left(\left\| M_1(r+is) + \frac{1}{r+is} M_2(r+is) \right\|\right) \to 0 \qquad (s \to \infty),$$

uniformly w.r. to $r \in (\alpha, \beta)$. □

Corollary 3.9.

a) If $z_0 = r_0 + is_0$ is a zero of χ_0 then there exist sequences $(z_j = r_j + is_j)$ in $\mathcal{Z}(\chi_0)$ and $(\zeta_j = \rho_j + i\sigma_j)$ in $\mathcal{Z}(\chi)$ such that $r_j \to r_0$, $s_j \to \infty$, and $|\zeta_j - z_j| \to 0$ as $j \to \infty$.

b) *For the real parts of the zero sets* $Z_0 := \operatorname{Re}(\mathcal{Z}(\chi_0))$ *and* $Z := \operatorname{Re}(\mathcal{Z}(\chi))$ *we have*

$$Z_0 \subset \overline{Z}.$$

Proof. Part a) with $\tilde{\chi}$ in place of χ follows directly from Proposition 3.8 and from Lemma 3.7, since χ_0 is holomorphic and almost periodic in very strip $\mathbf{S}_{(\alpha,\beta)}$ containing z_0. We may assume that all ζ_j of $\tilde{\chi}$ are different from 0, and then they are also zeroes of χ, which proves part a). Part b) is a direct consequence of a), since we also have $\rho_j \to r_0$. □

For a subset $I \subset \mathbb{R}$ we define the corresponding 'circular ring' in $\mathbb{C}$ by

$$\mathbf{R}_I := \Big\{ z \in \mathbb{C} \;\Big|\; |z| \in \exp(I) \Big\},$$

so $\mathbf{R}_I$ is the image of the 'vertical strip' $\mathbf{S}_I = I + \mathbb{R} \cdot i$ under the exponential map. The next auxiliary result has an early precursor in Lemma 5.2 on p. 16 of [19].

Lemma 3.10.

a) *With* D_0 *from* (3.6), *the corresponding difference equation* $D_0 x_t = 0$ *or* $x(t) = \sum_{j=1}^{k} A_j x(t - \tau_j)$ *generates a* C^0*-semigroup* $\{T_{D_0}(t)\}_{t \geq 0}$ *on the kernel of* D_0 *(a subspace of* C^0 *with finite codimension).*

b) *With* $Z_0 = \operatorname{Re}(\mathcal{Z}(\chi_0))$ *from Corollary* 3.9 *we have for the spectra:*

$$\sigma(T_{D_0}(t)) \subset \{0\} \cup \mathbf{R}_{\overline{Z_0}t} \qquad (t \geq 0).$$

c) *If* $\operatorname{pr}_{D_0} \in L_c(C^0, C^0)$ *is a projection onto* $\ker(D_0)$, *the semigroup* $\{S^0(t)\}$ *generated by equation* (3.1) *can be written as*

$$S^0(t) = T_{D_0}(t) \circ \operatorname{pr}_{D_0} + K(t),$$

with compact operators $K(t) \in L_c(C^0, C^0)$. *(Here, formally,* $T_{D_0}(t) \in L_c(\ker(D_0), \ker(D_0))$ *should be followed by the inclusion map from* $\ker(D_0)$ *to* C^0*, which we omit.)*

Proof. Part a) is stated (not proved) in [26], Section 3, p. 110, and follows from much more general existence results in Chapter 12 of [20]. For D_0 as considered here, and assuming that τ_1 is minimal among the discrete delays τ_j ($j = 1, \ldots, k$), the forward solution of $D_0 x_t = 0$ (given $x_0 \in \ker(D_0)$) can be directly obtained by stepwise forward definition: $x(t) := \sum_{j=1}^{k} A_j x(t - \tau_j)$ on $[0, \tau_1]$, then by the same formula on $[\tau_1, \tau_1 + 2\tau_1]$, etc.

Part b) is proved in [26], Theorem 3.2, p. 114, based on exponential estimates for $\|T_{D_0}(t)\|$ obtained by Laplace transform methods. The proof in [26] (Lemma 3.4, p. 111) quotes reference [12] of that paper, which apparently was never published. Another proof is given in [17], Theorem 2.1, p. 209. Both proofs use Laplace transform methods and a result due to Cameron and Pitt [8, 43] on exponential expansion $1/h(z)$, if h is almost periodic and holomorphic.

Part c) is proved in [26], Lemma 4.1, p. 116. We sketch the idea: For $\phi \in C^0$ and $\phi_0 := \operatorname{pr}_{D_0}\phi$, $y_t := S^0(t)\phi$ it follows from equations (3.1), (3.5) and (3.6) that

$$D_0 y_t - \int_{-h}^{0} A(\theta) y_t(\theta) d\theta - \left[D_0 \phi - \int_{-h}^{0} A(\theta)\phi(\theta) d\theta \right] = \int_0^t R y_s \, ds,$$

and hence, with H from Proposition 3.6,

$$D_0(y_t - \phi) = \int_{-h}^{0} A(\theta) y_t(\theta) d\theta - \int_{-h}^{0} A(\theta)\phi(\theta) d\theta + \int_0^t R y_s \, ds = H(t, \phi).$$

Thus, setting $\phi_1 := \phi - \phi_0 = (\mathrm{id} - \mathrm{pr}_{D_0})\phi$, $D_0(y_t - \phi) = D_0(y_t - \phi_1)$, and defining $z : [-h, \infty) \to \mathbb{R}^n$ by $z_t := y_t - \phi_1$, we have $z_0 = \phi_0$, so $D_0 z_0 = 0$, and z solves $D_0 z_t = H(t, \phi)$ $(t \geq 0)$, an inhomogeneous version of the equation generating T_{D_0}. The solution theory for this equation implies that, for fixed $t \geq 0$, $z_t = T_{D_0}(t) z_0 + \mathcal{K}(t) H(\cdot, \phi)|_{[0,t]}$, with a continuous and linear operator $\mathcal{K}(t) : C^0([0,t], \mathbb{R}^n) \to C^0$. It follows that

$$S^0(t)\phi = y_t = z_t + \phi_1 = T_{D_0}(t)\mathrm{pr}_{D_0}\phi + \mathcal{K}(t) H(\cdot, \phi)|_{[0,t]},$$

and from Proposition 3.6 we know that the operator $C^0 \ni \phi \mapsto H(\cdot, \phi)|_{[0,t]} \in C^0([0,t], \mathbb{R}^n)$ is compact. The assertion of c) follows. □

We need some functional analytic results of general nature, in particular, a 'compact perturbation' result. The version below suffices for our purposes. (As above, we write $\rho(\dots)$ for the resolvent set and $P\sigma(\dots)$ for the point spectrum, i.e., the eigenvalues of an operator.)

Lemma 3.11. *Assume that X is a real or complex Banach space, and $U, K \in L_c(X, X)$, with K compact.*

(i) *If $G \subset \mathbb{C}$ satisfies*

$$(1) \quad G \subset \rho(U) \quad \textit{and} \quad (2) \quad G \cap P\sigma(U + K) = \emptyset, \tag{3.11}$$

then also $G \subset \rho(U + K)$.

(ii) *If $\mu \in \rho(U) \cap \sigma(U + K)$ is an isolated spectral value, then it is an eigenvalue of $U + K$ of finite multiplicity (in the sense that the spectral subspace of $X_{\mathbb{C}}$ associated to μ is finite-dimensional).*

Proof. We can assume that X is a $\mathbb{C}$-Banach space, otherwise we would have to consider the complexifications of spaces and operators, We write $\mathrm{GL}(X, X)$ for the topological linear isomorphisms of X.

Ad (i): For $\lambda \in G$, condition (1) gives that $\lambda - U \in \mathrm{GL}(X, X)$, and one has

$$\lambda - (U + K) = (\lambda - U) \circ \underbrace{[\mathrm{id}_X - (\lambda - U)^{-1} K)]}_{=: F_\lambda} = (\lambda - U) \circ F_\lambda. \tag{3.12}$$

The operators F_λ are of the form $\mathrm{id}_X - K_\lambda$ with compact operators K_λ, and hence Fredholm operators of index zero (see [29], Korollar 25.3., p. 109). This property implies that $F_\lambda \in \mathrm{GL}(X, X)$ if and only if $\ker F_\lambda = \{0\}$. Now assumption (2) shows that the operators $\lambda - (U + K)$ are injective for all $\lambda \in G$, and hence also F_λ is injective, and thereby in $\mathrm{GL}(X, X)$ for $\lambda \in G$. It follows that for these λ also $\lambda - (U + K) \in \mathrm{GL}(X, X)$, so $\lambda \in \rho(U + K)$.

Ad (ii): (The proof here follows the proof of Lemma 5.2, p. 22 in [15].) Assume that μ is as in (ii). Part (i) applied to $G := \rho(U) \setminus P\sigma(U + K)$ shows that we must have $\mu \in P\sigma(U + K)$, so μ is an eigenvalue of $U + K$. For λ close enough to μ, but different from μ, we have $\lambda \in \rho(U + K) \cap \rho(U)$. Thus, for small enough $r > 0$, the spectral projection associated to the spectral set $\{\mu\}$ of $U + K$ is given by $\mathrm{pr}_\mu = \frac{1}{2\pi i} \oint_{|\lambda - \mu| = r} (\lambda - (U + K))^{-1} \, d\lambda$, and from (3.12) we see that $(\lambda - (U + K))^{-1} = F_\lambda^{-1} \circ (\lambda - U)^{-1}$, if $\lambda \in \rho(U + K) \cap \rho(U)$. Switching to resolvent notation we obtain $F_\lambda \circ R(\lambda; U + K) = R(\lambda; U)$ and hence, using the definition of $F_\lambda = \mathrm{id}_X - R(\lambda; U) K$,

$$R(\lambda; U + K) = R(\lambda; U) \circ K \circ R(\lambda; U + K) + R(\lambda; U).$$

Using this in the spectral projection formula we obtain

$$\mathrm{pr}_\mu = \frac{1}{2\pi i} \oint\limits_{|\lambda-\mu|=r} \{R(\lambda, U) \circ K \circ R(\lambda; U+K) + R(\lambda; U)\}\, d\lambda.$$

The second term under the integral is holomorphic in the neighborhood of μ and hence contributes zero; the first term consists of compact operators, and so we conclude that pr_μ is compact, which (for a projection) means it has finite-dimensional range. (From which it follows again that μ must be an eigenvalue, since $U+K$ induces a finite dimensional endomorphism of $\mathrm{image}(\mathrm{pr}_\mu)$ with spectrum $\{\mu\}$.) □

Remark 3.12. Assume that μ is an isolated eigenvalues with finite-dimensional spectral subspace of the operator $T \in L_c(X, X)$, where X is a complex Banach space (in particular $T := U+K$ and μ as above). Then the space of generalized eigenvectors $\mathcal{G}_\mu := \bigcup_{j=1}^\infty \ker\,(\mu - T)^j$ equals the image of the spectral projection pr_μ, and with its dimension $\nu(\mu)$ one has $\mathcal{G}_{\mu,T} = \ker\,(\mu - T)^{\nu(\mu)}$, and the direct sum decomposition

$$X = \ker(\mu - T)^{\nu(\mu)} \oplus \mathrm{image}(\mu - T)^{\nu(\mu)} = \mathrm{image}(\mathrm{pr}_\mu) \oplus \ker(\mathrm{pr}_\mu), \tag{3.13}$$

with both decompositions coinciding.

Proof. For the first decomposition and the identity $\mathrm{image}(\mathrm{pr}_\mu) = \ker(\mu - T)^{\nu(\mu)}$, see [15], Theorem 2.1, p. 9, and the passage preceding it. Note that these results are independent of the Hilbert space setting of [15], like many results of Chapter I of that reference, see also the first sentence on p. 1 there. The existence of the second decomposition is clear, since pr_μ is a projection, and it remains to prove equality of the spaces to the right of the $\oplus$-signs. If $v \in \mathrm{image}(\mu - T)^{\nu(\mu)}$, there exists $w \in X$ with $v = (\mu - T)^{\nu(\mu)} w$, and then for small $r > 0$

$$\mathrm{pr}_\mu v = \frac{1}{2\pi i} \oint\limits_{|\lambda-\mu|=r} R(\lambda; T) v\, d\lambda = \frac{1}{2\pi i} \oint\limits_{|\lambda-\mu|=r} R(\lambda; T)(\mu - T)^{\nu(\mu)} w\, d\lambda = 0,$$

since $\nu(\mu)$ equals the pole order of $R(\cdot; T)$ at μ (see e.g. Theorem 10.1, p. 330 in [47] or formula (2.3) on p. 9 of [15]), and hence the integrand has a holomorphic extension at μ. This shows that $\mathrm{image}(\mu - T)^{\nu(\mu)} \subset \ker(\mathrm{pr}_\mu)$. Since both are direct complements of the same space, equality follows. □

We will need another result of general nature. In the lemma below, the restriction $T|_Y$ of an operator T to an invariant subspace Y of T is meant as simultaneous restriction in the domain and the image space.

Lemma 3.13.

a) *Let X be a complex Banach space and A a closed operator with domain $D(A)$ and range in X, and let Σ be a bounded spectral subset of A. Then the associated spectral subspace X_Σ (with the corresponding projection given by a contour integral with contour enclosing Σ) satisfies $X_\Sigma \subset D(A)$, and $A|_{X_\Sigma}$ is bounded.*

b) *Assume that $\{S(t)\}_{t\geq 0}$ is a C^0-semigroup of operators in $L_c(X, X)$, with generator A, and let Σ be a bounded subset of the spectrum $\sigma(A)$, with complement $\Sigma' := \sigma_e(A) \setminus \Sigma$, where $\sigma_e(A)$ denotes the extended spectrum of A (i.e., $\sigma(A) \cup \{\infty\}$ in case A is unbounded). Then the corresponding spectral decomposition $X = X_{\Sigma'} \oplus X_\Sigma$ is invariant under all $S(t)$ $(t \geq 0)$.*

c) The spectral projections $\mathrm{pr}_{M,S(t)}$*, associated to the operator* $S(t)$ *and some spectral subset* M *of* $S(t)$ *for some* $t \geq 0$*, and* $\mathrm{pr}_{\Sigma,A}$*, associated to* A *and* Σ*, commute.*

d) If for some $t > 0$ *one has*

$$\exp(t\Sigma) \cap \sigma(S(t)|_{X_{\Sigma'}}) = \emptyset, \tag{3.14}$$

then the disjoint sets in (3.14) are spectral sets for the operator $S(t)$*. In this case the spectral projection for* $S(t)$ *corresponding to the set* $\exp(t\Sigma)$ *coincides with the spectral projection for* A *corresponding to* Σ*:*

$$\mathrm{pr}_{\exp(t\Sigma),S(t)} = \mathrm{pr}_{\Sigma,A}.$$

Proof. For a), see [47], Theorem 9.2, p. 322.

Ad b): For $t \geq 0$, one has $S(t) \circ A = A \circ S(t)$ on $D(A)$ (see [42], Theorem 2.4 c, p. 5), which implies $(\lambda - A)S(t) = S(t)(\lambda - A)$ on $D(A)$ for all $\lambda \in \mathbb{C}$, and hence for $\lambda \in \rho(A)$: $S(t) = R(\lambda; A)S(t)(\lambda - A)$ and finally $S(t)R(\lambda; A) = R(\lambda; A)S(t)$ on the dense subspace $D(A)$, hence on all of X. Since the spectral projection $\mathrm{pr}_{\Sigma,A}$ associated to A and Σ is given by a contour integral over of $R(\lambda; A)$, it follows that also $\mathrm{pr}_{\Sigma,A}$ and $S(t)$ commute, which proves the invariance.

Ad c): For $z \in \rho(S(t))$ and $w \in \rho(A)$, the fact that $R(w, A)$ and $S(t)$ commute implies

$$R(w; A) = R(w; A)(z - S(t))R(z; S(t)) = (z - S(t))R(w; A)R(z; S(t)),$$

and hence $R(z; S(t)R(w; A) = R(w; A)R(z; S(t))$ (the resolvents commute). Now

$$\mathrm{pr}_{M,S(t)} = \frac{1}{2\pi i}\int_{\Gamma_M} R(z; S(t))\, dz \text{ and } \mathrm{pr}_{\Sigma,A} = \frac{1}{2\pi i}\int_{\Gamma_\Sigma} R(w; A)\, dw$$

with appropriate cycles Γ_M and Γ_Σ, and the fact that both projections commute is obtained from Fubini's theorem and the commuting property of the resolvents.

Ad d): Assume condition (3.14) for some $t > 0$. Since $A|_{X_\Sigma}$ is bounded and obviously the generator of the semigroup $\{S(t)|_{X_\Sigma}\}$, the latter semigroup is uniformly continuous (i.e., continuous with respect to $\| \ \|_{L_c(X,X)}$), see [42], Theorem 1.2, p. 2. The spectral mapping theorem for uniformly continuous semigroups ([13], Lemma 3.1.3, p. 19) shows that $\exp(t\Sigma) = \sigma(S(t)|_{X_\Sigma})$. This set is compact and, in view of (3.14), disjoint to the as well compact set $\sigma(S(t)|_{X_{\Sigma'}})$, but since (in view of b)) $S(t)$ is completely reduced by the subspaces X_Σ and $X_{\Sigma'}$, the union of both sets gives $\sigma(S(t))$ (see [47], Theorem 5.4, p. 289), so both are spectral sets for $S(t)$. Hence the spectral projection $\mathrm{pr}_{\exp(t\Sigma),S(t)}$ is well- defined. We briefly write pr_A for $\mathrm{pr}_{\Sigma,A}$ and pr_S for $\mathrm{pr}_{\exp(t\Sigma),S(t)}$. As above, both are given by appropriate contour integrals over cycles Γ_Σ and $\Gamma_{\exp(t\Sigma)}$ enclosing the respective sets. Note that $\mathrm{pr}_S|_{X_\Sigma} = \frac{1}{2\pi i}\int_{\Gamma_{\exp(t\Sigma)}} R(z; S(t)|_{X_\Sigma})\, dz = \mathrm{id}_{X_\Sigma}$, since $S(t)|_{X_\Sigma}$ has only spectrum in the interior of $\Gamma_{\exp(t\Sigma)}$ (namely, the set $\exp(t\Sigma)$). Hence $X_\Sigma \subset \mathrm{image}(\mathrm{pr}_S)$. Further, $R(\cdot; S(t)|_{X_{\Sigma'}})$ is a holomorphic function in the interior of $\Gamma_{\exp(t\Sigma)}$, due to condition (3.14). It follows that $\mathrm{pr}_S|_{X_{\Sigma'}} = 0$, or $X_{\Sigma'} \subset \ker(\mathrm{pr}_S)$. Now from

$$\begin{aligned} X &= \ker(\mathrm{pr}_A) \oplus \mathrm{image}(\mathrm{pr}_A) = X_{\Sigma'} \oplus X_\Sigma; \\ X &= \ker(\mathrm{pr}_S) \oplus \mathrm{image}(\mathrm{pr}_S) \end{aligned}$$

and the inclusions between the subspaces of both decompositions, equality follows, and hence $\mathrm{pr}_S = \mathrm{pr}_A$. □

We want to obtain an exponential dichotomy (or pseudo-hyperbolicity) result for the semigroup $\{S^0(t)\}_{t\geq 0}$ from assumptions on the spectrum of the generator A_0, i.e., on $\sigma(A_0) = \mathcal{Z}(\chi)$, see Lemma 3.2. The lemma below is a main step.

Lemma 3.14. *Assume L is as in (3.5), and that there exist real numbers α, β with $\alpha < \beta$ such that the spectrum of A^0 decomposes as*

$$\sigma(A^0) = \sigma(A^0_{\mathbb{C}}) = \underbrace{\Big(\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(-\infty,\alpha]}\Big)}_{=:\Sigma'} \cup \underbrace{\Big(\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}\Big)}_{=:\Sigma},$$

with the set Σ nonempty and finite. Then

(i) *For $t \geq 0$, $\sigma(S^0(t)_{\mathbb{C}}) \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]} \cup \mathbf{R}_{(t\beta,\infty)}$.*

(ii) *For $t > 0$, $\sigma(S^0(t)_{\mathbb{C}}) \cap \mathbf{R}_{(t\beta,\infty)} = \Big\{\exp(t\lambda) \mid \lambda \in \sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}\Big\}$, and all of these finitely many numbers are eigenvalues of finite multiplicity for $S^0(t)$.*

(iii) *If μ is one of the finitely many spectral values of $S^0(t)_{\mathbb{C}}$ in $\mathbf{R}_{(t\beta,\infty)}$, then the associated spectral space of $S^0(t)$ is given by*

$$\mathcal{G}_{\mu,S^0(t)_{\mathbb{C}}} = \bigoplus_{\substack{\lambda \in \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)},\\ \exp(t\lambda) = \mu}} \mathcal{G}_{\lambda,A^0_{\mathbb{C}}}, \tag{3.15}$$

where $\mathcal{G}_{\lambda,A^0_{\mathbb{C}}}$ denotes the associated finite-dimensional spectral space of $A^0_{\mathbb{C}}$ associated with λ.

(iv) *For every $t > 0$, condition (3.14) from Lemma 3.13 is satisfied with Σ from above.*

Proof. In the proof, we omit the subscript $\mathbb{C}$. From the assumptions, $\sigma(A^0) \cap \mathbf{S}_{(\alpha,\infty)} = \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)}$ is finite. Since $\sigma(A^0) = \mathcal{Z}(\chi)$ (Lemma 3.2), we see from Corollary 3.9 that $\mathcal{Z}(\chi_0) \cap \mathbf{S}_{(\alpha,\infty)} = \emptyset$, since any number in this set would imply the existence of infinitely many numbers in $\mathcal{Z}(\chi) \cap \mathbf{S}_{(\alpha,\infty)}$. It follows that, with the notation from Corollary 3.9, $Z_0 \subset (-\infty, \alpha]$ and hence also $\overline{Z_0} \subset (-\infty, \alpha]$. We obtain for $t \geq 0$ that $t \cdot \overline{Z_0} \subset (-\infty, t\alpha]$, and Lemma 3.10 b) shows that $\sigma(T_{D_0}(t)) \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]}$. Since $\sigma[T_{D_0}(t) \circ \mathrm{pr}_{D_0}] = \{0\} \cup \sigma(T_{D_0}(t))$, we conclude that

$$\forall t \geq 0: \ \sigma[T_{D_0}(t) \circ \mathrm{pr}_{D_0}] \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]}. \tag{3.16}$$

For $t \geq 0$, Lemma 3.10 c) allows us to apply Lemma 3.11 with $U := T_{D_0}(t) \circ \mathrm{pr}_{D_0}$ and $K := K(t)$ (so that $U + K = S^0(t)$), and with $G := \mathbf{R}_{(t\alpha,\infty)} \setminus P\sigma(U + K)$, which obviously satisfies the second condition in (3.11). We see from (3.16) that the first condition in (3.11) also holds, and so we can conclude that $G \subset \rho(S^0(t))$. It follows that

$$\sigma(S^0(t)) \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]} \cup P\sigma(S^0(t)). \tag{3.17}$$

Now the spectral mapping theorem for the point spectrum (see Theorem 16.7.2 on page 467 of [28], or Theorem 3.7 on p. 277 of [13]) gives $P\sigma(S^0(t)) \setminus \{0\} = \exp(t \cdot P\sigma(A^0))$. Since $P\sigma(A^0) = \sigma(A^0)$ and A^0 has no spectrum in $\mathbf{S}_{(\alpha,\beta]}$, we conclude that $\sigma(S^0(t)) \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]} \cup \mathbf{R}_{(t\beta,\infty)}$, and assertion (i) is proved.

Ad (ii): Assume $t > 0$. We also see from (3.17) and the spectral mapping theorem for the point spectrum that

$$\begin{aligned}\sigma(S^0(t)) \cap \mathbf{R}_{(t\beta,\infty)} &= P\sigma(S^0(t)) \cap \mathbf{R}_{(t\beta,\infty)} \\ &= \exp(t \cdot P\sigma(A^0)) \cap \mathbf{R}_{(t\beta,\infty)} = \exp[t \cdot (P\sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)})].\end{aligned}$$

(Note that $\exp(t\lambda) \in \mathbf{R}_{(t\beta,\infty)}$ if and only if $\mathrm{Re}(t\lambda) > t\beta$, which means $\mathrm{Re}(\lambda) > \beta$.) Thus we obtain that $\sigma(S^0(t)) \cap \mathbf{R}_{(t\beta,\infty)} = \left\{\exp(t\lambda) \mid \lambda \in \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)}\right\}$, and that these numbers are all eigenvalues of $S^0(t)$. Finite multiplicity can be seen as follows: Consider a spectral value $\mu = \exp(\lambda t)$ of $S^0(t)$, where $\lambda \in \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)}$. Then, with U and K defined as above, μ is obviously an isolated spectral value of $U + K$ and (3.16) shows that $\mu \in \rho(U)$. Finite multiplicity now follows from part (ii) of Lemma 3.11.

Ad (iii): We briefly write $\mathcal{G}$ for $\mathcal{G}_{\mu,S^0(t)}$. Remark 3.12 shows that $\mathcal{G} = \ker\,(\mu - S^0(t))^{\nu(\mu)}$, and this space is invariant under all $S^0(s)$, since the null space of an operator T_1 is invariant under a second operator T_2 that commutes with T_1. Hence $\{S^0(s)|_{\mathcal{G}}\}_{s\geq 0}$ is a semigroup on the finite dimensional space $\mathcal{G}$. Its generator is defined on all of $\mathcal{G}$ and coincides with $A^0|_{\mathcal{G}}$. Since the spectrum is natural with respect to restriction to spectral subspaces (see [47], Theorem 9.2 and Corollary 9.3, pp. 322-323), we have $\sigma(S^0(t)|_{\mathcal{G}}) = \{\mu\}$, and the finite-dimensional spectral mapping theorem for $\exp(t\cdot)$ (which follows easily from the Jordan canonical form theorem, but also e.g. from Lemma 3.13 on p. 19 of [13]) gives

$$\sigma(S^0(t)|_{\mathcal{G}}) = \exp[t \cdot \sigma(A^0|_{\mathcal{G}})].$$

It follows that $\sigma(A^0|_{\mathcal{G}}) \subset \left\{\lambda \in \mathbb{C} \mid \exp(t\lambda) = \mu\right\}$, and since $\mu \in \mathbf{R}_{(t\beta,\infty)}$, we conclude $\sigma(A^0|_{\mathcal{G}}) \subset \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)} \cap \left\{\lambda \in \mathbb{C} \mid \exp(t\lambda) = \mu\right\}$. It follows now from the Jordan canonical form theorem, with the obvious notation for the generalized eigenspaces of $A^0|_{\mathcal{G}}$, that

$$\mathcal{G} = \bigoplus_{\lambda\in\sigma(A^0|_{\mathcal{G}})} \mathcal{G}_{\lambda,A^0|_{\mathcal{G}}} = \bigoplus_{\substack{\lambda \in \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)},\\ \exp(t\lambda) = \mu}} \mathcal{G}_{\lambda,A^0|_{\mathcal{G}}} \subset \bigoplus_{\substack{\lambda \in \sigma(A^0) \cap \mathbf{S}_{(\beta,\infty)},\\ \exp(t\lambda) = \mu}} \mathcal{G}_{\lambda,A^0},$$

where the last inclusion is obvious. This proves the inclusion '$\subset$' in (3.15). To prove the inclusion '$\supset$', it suffices to prove that for λ as on the right hand side one has $\mathcal{G}_{\lambda,A^0} \subset \mathcal{G}_{\mu,S^0(t)}$. For such λ, the space $\mathcal{G}_{\lambda,A^0}$ is finite dimensional, contained in the domain of A^0 and invariant under A^0, and $A_0|_{\mathcal{G}_{\lambda,A^0}} = \lambda + N_\lambda$, with a nilpotent operator N_λ. Further,

$$\begin{aligned} S^0(t)|_{\mathcal{G}_{\lambda,A^0}} &= \exp[tA_0|_{\mathcal{G}_{\lambda,A^0}}] = \exp[t(\lambda + N_\lambda)] = \exp(t\lambda) \circ \exp(tN_\lambda) = \mu \circ [\mathrm{id}_{\mathcal{G}_{\lambda,A^0}} + \tilde{N}] \\ &= \mu + \hat{N}, \end{aligned}$$

where $\hat{N}$ is also a nilpotent endomorphism of $\mathcal{G}_{\lambda,A^0}$. It follows that $(\mu - S^0(t))^k|_{\mathcal{G}_{\lambda,A^0}} = 0$ for some $k \in \mathbb{N}$ (certainly for $k = \dim \mathcal{G}_{\lambda,A^0}$), and hence $\mathcal{G}_{\lambda,A^0} \subset \bigcup_{j=1}^{\infty} \ker\,(\mu - S^0(t))^j = \mathcal{G}_{\mu,S^0(t)}$, see Remark 3.12.

Ad (iv): Consider $\mu \in \exp(t\Sigma)$. To the isolated eigenvalue μ of $S^0(t)$ corresponds a spectral projection pr_μ, and from Lemma 3.13 c) we see that it commutes with pr_{Σ,A^0}. From (3.15) we see that $\mathcal{G}_{\mu,S^0(t)} \subset C^0_\Sigma$ (where the last symbol denotes the spectral subspace of A^0 corresponding to Σ), which implies that

$$\mathrm{pr}_{\Sigma,A^0} \circ \mathrm{pr}_\mu = \mathrm{pr}_\mu. \tag{3.18}$$

The operator $\mu - S^0(t)$ induces an isomorphism on $\ker(\mathrm{pr}_\mu)$, since $\mu \notin \sigma(S^0(t)|_{\ker(\mathrm{pr}_\mu)})$. We show that $C^0_{\Sigma'} \subset \ker(\mathrm{pr}_\mu)$: Since $C^0_{\Sigma'} = \ker(\mathrm{pr}_{\Sigma,A^0})$, we obtain using the commuting property and (3.18):

$$\mathrm{pr}_\mu|_{C^0_{\Sigma'}} = \mathrm{pr}_\mu(\mathrm{id} - \mathrm{pr}_{\Sigma,A^0})|_{C^0_{\Sigma'}} = (\mathrm{pr}_\mu - \mathrm{pr}_{\Sigma,A^0}\mathrm{pr}_\mu)|_{C^0_{\Sigma'}} = 0.$$

Thus, $\mu - S^0(t)$ also induces an isomorphism on the space $C^0_{\Sigma'}$ (which, as we know from Lemma 3.13b), is invariant under $S^0(t)$), and we conclude $\mu \notin \sigma(S^0(t)|_{C^0_{\Sigma'}})$, so condition (3.14) holds. □

Putting together the above results on spectra and characteristic functions, we arrive at the theorem below.

Theorem 3.15 (Exponential separation for S^0). *Assume L is as in (3.5), and that there exist real numbers α, β with $\alpha < \beta$ such that the spectrum $\sigma(A^0)$ of the generator of S^0 can be split as in Lemma 3.14:*

$$\sigma(A^0_{\mathbb{C}}) = \Big(\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}\Big) \cup \Big(\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(-\infty,\alpha]}\Big)\,, \tag{3.19}$$

and $\big(\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}\big)$ is a nonempty finite set. Then the following hold:

a) *For $t > 0$, the decomposition $C^0_{\mathbb{C}} = E^+_{\mathbb{C}} \oplus E^-_{\mathbb{C}}$ into spectral subspaces of $A^0_{\mathbb{C}}$ according to (3.19) is invariant under $S^0(t)_{\mathbb{C}}$. These spaces coincide with the spectral subspaces of $S^0(t)_{\mathbb{C}}$ coming from the spectral sets $\sigma(S^0(t)_{\mathbb{C}}) \cap \mathbf{R}_{(t\beta,\infty)}$ and $\sigma(S^0(t)_{\mathbb{C}}) \cap \big(\{0\} \cup \mathbf{R}_{(-\infty,t\alpha]}\big)$ (see Lemma 3.14 (i)).*

b) *Analogous to a), setting $E^\pm :=$ Re$(E^\pm_{\mathbb{C}})$, the 'real' decomposition $C^0 = E^+ \oplus E^-$ is invariant under the 'real' operator family $\{S^0(t)\}_{t\geq 0}$.*

c) *There exists a constant $K > 0$ such that for all $t \geq 0$*

$$|S^0(t)\varphi|_{C^0} \geq K^{-1}\exp(\beta t)|\varphi|_{C^0} \quad \textit{for } \varphi \in E^+. \tag{3.20}$$

d) *If $\tilde{\alpha} \in (\alpha, \beta)$, then there exists $\tilde{K} > 0$ such that for all $t \geq 0$*

$$|S^0(t)\varphi|_{C^0} \leq \tilde{K}\exp(\tilde{\alpha} t)|\varphi|_{C^0} \quad \textit{for } \varphi \in E^-. \tag{3.21}$$

Proof. Ad a): Fix $t > 0$. Lemma 3.13 b) applied to the semigroup $S^0(\cdot)_{\mathbb{C}}$ and with $\Sigma := \sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}$ and Σ' as in Lemma 3.14 gives the invariance of the spaces $E^\pm_{\mathbb{C}}$ under $S^0(t)_{\mathbb{C}}$. Further, we see from Lemma 3.14(ii) that $\exp(\Sigma t) = \sigma(S^0(t)_{\mathbb{C}} \cap \mathbf{R}_{(t\beta,\infty)}$ and from part (iv) of the same lemma that condition (3.14) holds. Hence part d) of Lemma 3.13 gives that the spectral projections $\mathrm{pr}_{\exp(t\Sigma),S^0(t)_{\mathbb{C}}}$ and $\mathrm{pr}_{\Sigma,A^0_{\mathbb{C}}}$ onto $E^+_{\mathbb{C}}$ coincide, and hence the complementary spectral subspaces (corresponding to the set $\{0\} \cup \mathbf{R}_{(-\infty,t\alpha]}$ for $S^0(t)_{\mathbb{C}}$ and to Σ' for $A^0_{\mathbb{C}}$) also coincide, namely, with $E^-_{\mathbb{C}}$.

b) Follows from a) taking real parts of the involved spaces; noting that $S^0(t)$ is the restriction of $S^0(t)_{\mathbb{C}}$ to C^0, and the analogous property for the spaces $E^\pm_{\mathbb{C}}$ and $E^\pm$.

Ad c): E^+ is finite-dimensional, with

$$\min\Big\{\ \mathrm{Re}(\lambda)\ \Big|\ \lambda \text{ is eigenvalue of } A^0_{\mathbb{C}}|_{E^+_{\mathbb{C}}}\Big\} > \beta.$$

Hence estimate (3.20) for $S^0(t)|_{E^+}$ with respect to the C^0-norm is obtained in a standard way, as for ordinary differential equations (even in the case of possible multiple eigenvalues, since their minimal real part is larger than β). Alternatively, one can also use that $\sigma(S^0(t)_{\mathbb{C}}|_{E^+_{\mathbb{C}}}) \subset \mathbf{R}_{(t\beta,\infty)}$ for $t > 0$ implies $\sigma([S^0(t)_{\mathbb{C}}|_{E^+_{\mathbb{C}}}]^{-1}) \subset \Big\{z \in \mathbb{C}\ \Big|\ |z| < \exp(-\beta t)\Big\}$ and then use Proposition 3.5 for the semigroup $t \mapsto \sigma([S^0(t)_{\mathbb{C}}|_{E^+_{\mathbb{C}}}]^{-1})$ of inverse operators. The analogous 'real' estimate is then obtained by restriction.

Ad d): From a), we have $\sigma(S^0(t)_{\mathbb{C}}|_{E^-_{\mathbb{C}}}) \subset \{0\} \cup \mathbf{R}_{(-\infty,t\alpha]}$, which for $\tilde{\alpha} \in (\alpha, \beta)$ implies that the spectral radius satisfies $r(S^0(t)_{\mathbb{C}}|_{E^-_{\mathbb{C}}}) < \exp(\tilde{\alpha}t)$. Proposition 3.5 applied to the semigroup $\{S^0(s)_{\mathbb{C}}|_{E^-_{\mathbb{C}}}\}_{s\geq 0}$ gives estimate (3.21) first for the complexification, and the real version follows. □

We can now easily obtain a result corresponding to the above theorem for the semigroup S^1 on the space $T^1 = D(A^0)$ from Lemma 3.4, which is what we actually need later.

Corollary 3.16 (Exponential separation for S^1). *Under the assumptions and with the notation of Theorem 3.15, one has $E^+ \subset T^1$ and the S^1-invariant decomposition*

$$T^1 = E^+ \oplus (T^1 \cap E^-) \tag{3.22}$$

With respect to the C^1-norm, the semigroup S^1 satisfies estimates analogous to (3.20) and (3.21) on these spaces.

Proof. 1. From Lemma 3.13, applied with Σ as in Lemma 3.14, we see that $E^+_{\mathbb{C}} \subset D(A^0_{\mathbb{C}})$, which implies $E^+ = \operatorname{Re}(E^+_{\mathbb{C}}) \subset \operatorname{Re}(D(A^0_{\mathbb{C}})) = D(A^0) = T^1$. The complex spectral projection $\mathrm{pr}_{\Sigma,A^0_{\mathbb{C}}} \in L_{\mathbb{C}}(C^0_{\mathbb{C}}, C^0_{\mathbb{C}})$ onto $E^+_{\mathbb{C}}$ induces a projection $\mathrm{pr}_{\Sigma,A^0} \in L_{\mathbb{C}}(C^0, C^0)$ onto E^+ which corresponds to the decomposition in Theorem 3.15b). For $\varphi \in T^1$ one has also $\varphi - \mathrm{pr}_{\Sigma,A^0}\varphi \in T^1$ (and certainly $\varphi - \mathrm{pr}_{\Sigma,A^0}\varphi \in E^-$), so we have the decomposition $T^1 = E^+ \oplus (T^1 \cap E^-)$. It is invariant under all $S^1(t)$ $(t \geq 0)$, since the spaces $E^\pm$ are invariant under $S^0(t)$, of which $S^1(t)$ is a restriction.

2. On the finite-dimensional space E^+ all norms are equivalent, hence it is clear that an estimate analogous to (3.20) also holds w.r. to the C^1-norm, and hence for $S^1(t)$ restricted to this space.

3. Since $T^1 = D(A^0)$ and since $S^0(t)$ and A^0 commute on $D(A^0)$ ([42], Theorem 2.4 c, p. 5), we have for $\varphi \in T^1$ and $t \geq 0$ in view of (3.21):

$$|(S^1(t)\varphi)'|_{C^0} = |(S^0(t)\varphi)'|_{C^0} = |A^0S^0(t)\varphi|_{C^0} = |S^0(t)A^0\varphi|_{C^0} = |S^0(t)\varphi'|_{C^0} \leq \tilde{K}\exp(\tilde{\alpha}t)|\varphi'|_{C^0}.$$

In combination with estimate (3.21) for the C^0-norm, it is now obvious that we obtain an analogous estimate for the C^1-norm:

$$|S^1(t)\varphi|_{C^1} = |S^1(t)\varphi|_{C^0} + |(S^1(t)\varphi)'|_{C^0} \leq \tilde{K}\exp(\tilde{\alpha}t)|\varphi|_{C^0} + \tilde{K}\exp(\tilde{\alpha}t)|\varphi'|_{C^0} = \tilde{K}\exp(\tilde{\alpha}t)|\varphi|_{C^1}.$$

□

The remark below may explain why we decided to give proofs for Theorem 3.15 and its prerequisites, although a number of related references exist.

Remark 3.17 (on related literature). a) Recall the sets Z_0 and Z from Corollary 3.9 (in [26], Theorem 4.1, p. 117, the set Z_0 is named Z). In Theorem 4.2 of [26], which essentially describes consequences of a splitting of the spectrum at real part = $\alpha \in \mathbb{R}$, there is no assumption like $\alpha \notin \overline{Z}$ (in the notation from that paper), which one would expect, in view of the preparations leading to that theorem. The proof of Theorem 4.2 in [26] uses Theorem 4.1 of the same reference, and that does have assumptions on Z, so their absence from the hypotheses of Theorem 4.2 is surprising. This can be explained using ideas sketched in the first remark on p. 18 of [26], but such an explanation is not given in [26]. We tried to carry this out in the proof of Lemma 3.14 above.

b) Some results of [26] take reference to the paper of the same author titled 'Adjoint theory and boundary value problems for neutral linear FDEs', which apparently was never published.

c) Contrary to Theorem 4.2 in [26], the somewhat analogous Theorem 6.1 of [34] contains assumptions on both the zeroes of $\lambda \mapsto \det \Delta(\lambda)$ and $\lambda \mapsto \det \Delta_0(\lambda)$ – this is apparently due to the more general form of the operator L considered in [34]; compare the remark after condition (J) on p. 397 of [34].

d) Theorem 6.4 from the section with application to neutral delay equations from [34] would allow to transfer the hyperbolic splitting from Theorem 6.1 of the same paper to a splitting by some growth rate $\exp(\alpha t)$ for nonzero α, but the proof contains an unclear point: It uses a rescaling argument familiar in semigroup theory (see e.g. Section 2 of Chapter II in [13]). But the rescaled semigroup and its generator are not necessarily obtained from a neutral delay equation as the original ones.

e) Above we proved and used the 'compact perturbation' result Lemma 3.11. It is a simpler form of Lemma 5.2 from p. 22 of [15], which however is stated in a Hilbert space context, and also a simpler form of Lemma 4.2 from p. 117 of [26], where it is claimed that the proof can be obtained by modification of the proof from [15]. In the corresponding passage of [17] (Lemma 2.4 on p. 211), a reference from the well-known book of Kato [35] is quoted with a misleading number, and the result of Theorem 5.26 from Chapter IV of that book (which was possibly meant) does not seem to fit well. In the book [20], the reader is referred to Section 12.12 of [20] for references concerning the 'compact perturbation' result, (Lemma 3.4 of Section 12.3, p. 285), but Section 12.12 does not seem to contain such references.

The last part of this section prepares the treatment of nonlinear equations in Section 3. It will be important later that for particular solutions y of equation 3.1 and short time intervals, on which the phase curve Y satisfies $L \circ Y = 0$, y will still be C^1 on a short interval to the right of zero.

For $\nu \in (0, h)$ we define the space

$$\mathcal{N}_\nu := \left\{ \varphi \in C^0 \mid \varphi = 0 \text{ on } [-h, -\nu] \right\}.$$

Recall the set W_1 from condition **(g3)**. Condition **$(\widetilde{\mathbf{g1}})$** implies the following property:

Lemma 3.18. *Assume $(\psi, \phi) \in W_1$, further that $\chi_1, \chi_2, \in C^0$, that $\hat{\chi} \in \mathcal{N}_\Delta$ and $\hat{\psi} \in C^1 \cap \mathcal{N}_\Delta$, and that also $(\psi + \hat{\psi}, \phi) \in W_1$.*

a) Then $D_e g_1(\psi + \hat{\psi}, \phi)(\chi_1 + \hat{\chi}, \chi_2) = D_e g_1(\psi, \phi)(\chi_1, \chi_2)$.

b) In particular, $D_{1,e} g_1(0, 0) = 0$ on $\mathcal{N}_\Delta$, and hence also $L = 0$ on $\mathcal{N}_\Delta$.

Proof. Property **$(\widetilde{\mathbf{g1}})$** implies

$$g(\tilde{\psi} + \hat{\psi}, \tilde{\phi}) = g(\tilde{\psi}, \tilde{\phi})$$

for $(\tilde{\psi}, \tilde{\phi})$ in a neighborhood of (ψ, ϕ) in $C^1 \times C^1$. Hence,

$$Dg_1(\psi + \hat{\psi}, \phi) = Dg_1(\psi, \phi), \text{ and consequently } D_e g_1(\psi + \hat{\psi}, \phi) = D_e g_1(\psi, \phi).$$

(Note that the extensions to $C^0 \times C^0$ are unique, due to density of C^1 in $(C^0, |\ |_{C^0})$.)

It follows that

$$
\begin{aligned}
D_e g_1(\psi+\hat{\psi},\phi)(\chi_1+\hat{\chi},\chi_2) &= D_e g_1(\psi,\phi)(\chi_1+\hat{\chi},\chi_2) \\
&= D_e g_1(\psi,\phi)(\chi_1,\chi_2) + D_e g_1(\psi,\phi)(\hat{\chi},0).
\end{aligned} \tag{3.23}
$$

In case $\hat{\chi} \in C^1 \cap \mathcal{N}_\Delta$, the last term equals $\lim_{s\to 0} \frac{1}{s}[g_1(\psi+s\hat{\chi},\phi) - g_1(\psi,\phi)] = 0$, since $g_1(\psi + s\hat{\chi},\phi) - g_1(\psi,\phi) = 0$ for s sufficiently small. By density, it follows that $D_e g_1(\psi,\phi)(\hat{\chi},0) = 0$ for $\hat{\chi} \in \mathcal{N}_\Delta$. Assertion a) then follows from (3.23).

Proof of b): Assertion a), specialized to the case $\psi = \hat{\psi} = \phi = \chi_1 = \chi_2 = 0$, gives $D_{1,e} g_1(0,0)\hat{\chi} = D_e g_1(0,0)(\hat{\chi},0) = D_e g_1(0,0)(0,0) = 0$ if $\hat{\chi} \in \mathcal{N}_\Delta$, which shows b). □

Associated to equation (3.1), there are not only the semigroups $\{S^0(t)\}_{t\geq 0}$ and $\{S^1(t)\}_{t\geq 0}$, but also the so-called fundamental solution $\mathbf{X} : [-h,\infty) \to \mathbb{C}^{n\times n}$; the column functions $t \mapsto \mathbf{X}_j(t)$ $(j = 1,\dots,n)$ are zero on $[-h,0)$, equal to the j-th unit vector e_j at $t = 0$, continuous on $[0,\infty)$, and solve equation (3.1) on $[0,\infty)$ in the 'integral' sense of formula (3.24) explained below (see Section 6 of [53]): The description of the operators L and R in $L_c(C^0,\mathbb{R}^n)$ by integrals can be naturally extended from continuous functions to bounded Borel-measurable functions, leading to extended operators $\hat{L}$ and $\hat{R}$. The X_j then satisfy

$$
\mathbf{X}_j(t) - \hat{L}\mathbf{X}_{j,t} = e_j + \int_0^t \hat{R}\mathbf{X}_{j,s}\, ds \quad (t \geq 0), \tag{3.24}
$$

where $\mathbf{X}_{j,t} = \mathbf{X}_j(t+\cdot)|_{[-h,0]}$ denotes the segment of $\mathbf{X}_j$ at time t, and the integral is a Lebesgue integral. Compare Prop. 6.7, p. 459 in [53]. In this sense the fundamental solution can be seen as an extension of the solution operators to discontinuous initial segments (which are zero on $[-h,0)$), and it is helpful for the description of solutions to inhomogeneous equations.

Lemma 3.19 ([53, Corollary 6.8])**.** *Let $c \geq 1$, $\omega \in \mathbb{R}$ be given with*

$$
|S^0(t)\chi|_{C^0} \leq c e^{\omega t}|\chi|_{C^0}, \quad \text{for all } \ t \geq 0, \chi \in C^0. \tag{3.25}
$$

Then the columns $\mathbf{X}_j$ of the fundamental matrix satisfy

$$
|\mathbf{X}_j(t)| \leq c e^{\omega t} \quad \text{for all } j \in \{1,\dots,n\} \text{ and } t \geq 0. \tag{3.26}
$$

It will be important how the semigroup S^0 acts on functions in the space $\mathcal{N}_{\Delta/2}$, because such functions span an n-dimensional complement of $T^1 = T_{e,0}\mathcal{M}_2$ in C^2. Although we cannot expect a general solution of the linear equation (3.1) to be of class C^1, it will be important that such solutions for special initial functions (namely, in the space $\mathcal{N}_{\Delta/2}$) are C^1 when restricted to the time interval $[0,\Delta/2]$. We shall see that a similar property holds for the additional term present in solutions of the nonlinear equation 1.1, which term involves the fundamental matrix.

Lemma 3.20. *Assume $\psi \in \mathcal{N}_{\Delta/2}$.*

a) The restriction of the solution y of equation (3.1) with $y_0 = \psi$ (i.e., $y_t = S^0(t)\psi$) to $[0,\Delta/2]$ is of class C^1 (at $t = 0$, this refers to the right hand derivative), and satisfies $\dot{y}(0+) = R\psi$, and there exists a constant $M_1 \geq 1$ such that for all such ψ and $t \in [0,\Delta/2]$, one has

$$
\max\{|\dot{y}(t)|, |y(t)|\} \leq M_1 \cdot |\psi|_{C^0}.
$$

b) The fundamental matrix $\mathbf{X}$ *is absolutely continuous on* $[0, \Delta/2]$ *(right-continuous at* 0*), hence differentiable Lebesgue-almost everywhere on* $[0, \Delta/2]$*, and satisfies* $|\mathbf{X}(t)| \leq \tilde{M}_1$ *for* $t \in [0, \Delta/2]$*, with an appropriate* $\tilde{M}_1 \geq 1$*.*

Proof. Solutions y of equation (3.1) with $y_0 = \psi \in \mathcal{N}_{\Delta/2}$ actually follow the non-neutral retarded equation $\dot{y}(t) = Ry_t$ on $[0, \Delta/2]$, since the segments y_t satisfy $y_t \in \mathcal{N}_\Delta \subset \ker(L)$ for $t \in [0, \Delta/2]$. For the semigroup S^0, there exist constants $M \geq 1$ and $\Omega > 0$ such that for all $t \geq 0$ one has $\|S^0(t)\|_{L_c(C^0,C^0)} \leq M \cdot \exp(\Omega t)$ (see [42], Theorem 2.2, p. 4, or Proposition 3.5 of the present paper). Writing $\|R\|$ for $\|R\|_{L_c(C^0,\mathbb{R}^n)}$, it follows that for such y and $t \in [0, \Delta/2]$ one has

$$|\dot{y}(t)| = |Ry_t| \leq \|R\| \cdot |y_t|_{C^0} \leq \|R\| \cdot M \cdot \exp(\Omega t) \cdot |\psi|_{C^0} \leq \|R\| \cdot M \cdot \exp(\Omega\Delta/2) \cdot |\psi|_{C^0},$$

and clearly $|y(t)| \leq |y_t|_{C^0} \leq M \cdot \exp(\Omega\Delta/2)|\psi|_{C^0}$. Set $M_1 := \max\{\|R\|, 1\} M \cdot \exp(\Omega\Delta/2)$.

Ad b): Continuity on $[0, \infty)$ was already remarked above, and the estimate with $\tilde{M}_1 := M \cdot \exp(\Omega\Delta/2)$ follows from Lemma 3.19. Further, the segments $\mathbf{X}_{j,t}$ are zero on $[-h, -\Delta]$ (even on $[-h, -\Delta/2)$) for $t \in [0, \Delta/2]$, and the operator $\hat{L}$ is zero on such segments, as extension of L which is zero on $\mathcal{N}_\Delta$ (see also [56], Prop. 5.3). In view of equation (3.24), we see that the $\mathbf{X}_j$ actually satisfy

$$\mathbf{X}_j(t) = e_j + \int_0^t \hat{R}\mathbf{X}_{j,s}\, ds \qquad (t \in [0, T])$$

(compare also formula (6.2) in [56]). The integrand here is of class L^1, and it follows that $\mathbf{X}_j$ is absolutely continuous, with derivative $\hat{R}\mathbf{X}_{j,t}$ for Lebesgue-almost every $t \in [0, \Delta/2]$ (see [27], Satz 131.2, p. 113, and [16], Theorem 29, Chap. X, p. 208). □

We turn to inhomogeneous equations now.

Lemma 3.21 (**Variation of constants,** Corollary 6.12, p. 460 in [53]**).** *For every* $\phi \in C^0([-h, 0], \mathbb{C}^n)$ *and every continuous function* $f : [0, \infty) \to \mathbb{C}^n$ *there is a unique continuous solution of the inhomogeneous equation*

$$\frac{d}{dt}(y - L_{\mathbb{C}} \circ Y)(t) = R_{\mathbb{C}} y_t + f(t), \qquad t > 0, \tag{3.27}$$

with $y_0 = \phi$*. (The notion of a solution here is analogous to the case* $f = 0$*.) For all* $t \geq 0$*, one has with* $v_t := S^0(t)_{\mathbb{C}}\phi$ *(i.e.,* v *is the solution of the homogeneous equation* (3.1) *with* $v_0 = \phi$*) the representation*

$$y(t) = v(t) + \int_0^t \mathbf{X}(t - s) f(s)\, ds.$$

4 Solutions of the nonlinear equation

Recall the set X_2 from the introduction, described in (2.3).

Theorem 4.1 (**Semiflow on** $\mathbf{X_2}$**).** *Assume* **(g0)–(g3)**. For each $\varphi \in X_2$, the corresponding solution x^φ of (1.1) is twice continuously differentiable, and for all t in the maximal existence interval $[0, t_\varphi)$, one has $x_t^\varphi \in X_2$. These solutions define a semiflow Φ on X_2 by setting $\Phi(t, \varphi) := x_t^\varphi$ (a restriction of the semiflow on X_{1+}), which is continuous w.r.t. to the obvious topology on $[0, \infty) \times C^2$ induced by $|\ |_{C^2}$ on C^2.

Proof. See Propositions 6.1 and 6.2, and the passage before condition **(g4)** in [55]. □

Recall the set W_1 from condition **(g3)**. As indicated in point 5) of the comments on the hypotheses, we define the map

$$r_g : \{\psi \in C^2 : (\psi', \psi) \in W_1\} \ni \psi \mapsto g_1(\psi', \psi) - Dg_1(0,0)(\psi', \psi) \in \mathbb{R}^n. \tag{4.1}$$

This map is continuously differentiable w.r.t. to $|\ |_{C^2}$ on its domain (and the ordinary topology on $\mathbb{R}^n$).

Lemma 4.2 ([53, Proposition 3.3]). *The twice continuously differentiable solutions $y : [-h, t_\varphi) \to \mathbb{R}^n$ of (1.1) with $y_0 = \varphi \in X_2$ as in Theorem 4.1 are also solutions of the inhomogeneous equation*

$$\frac{d}{dt}(y - L \circ Y)(t) = Ry_t + r_g(y_t). \tag{4.2}$$

Corollary 4.3. *If $t_0 > 0$ and $x : [-h, t_0] \to \mathbb{R}^n$ is a C^2 solution of (1.1) as in Lemma 4.2, then for $t \in [0, t_0]$ one has*

$$x_t = S^0(t)x_0 + N_t, \tag{4.3}$$

where $N(t) = 0$ for $t \in [-h, 0]$ and $N(t) = \int_0^t \mathbf{X}(t-s) r_g(x_s)\, ds$ for $t \in [0, t_0]$.

Proof. The proof follows from Lemma 4.2 and Lemma 3.21. □

For the term N in formula (4.3), we have a result similar to part a) of Lemma 3.20.

Lemma 4.4. *With x and N as in Corollary 4.3, and $T \in (0, \Delta] \cap (0, t_0]$, the function N restricted to $[0, T]$ is of class C^1, satisfies $\dot{N}(t) = RN_t + r_g(x_t)$ for $t \in [0, T]$, and in particular $\dot{N}(0+) = r_g(x_0)$.*

Proof. N is continuous, and one sees from formula (6.2) in Prop. 6.2 of [56] that N satisfies the integral equation

$$N(t) - LN_t = \int_0^t RN_s\, ds + \int_0^t r_g(x_s)\, ds.$$

Now for $t \in [0, T]$, the segments N_t are in $\mathcal{N}_\Delta$, so that $LN_t = 0$, and N actually satisfies

$$N(t) = \int_0^t RN_s\, ds + \int_0^t r_g(x_s)\, ds \qquad (t \in [0, T]).$$

Both integrands here are continuous, since $s \mapsto N_s$ is continuous, and since the map $s \mapsto (x_s', x_s) \in C^1 \times C^1$ is continuous (compare formula (4.1)). Observe here that x is C^2, which (using locally uniform continuity of $\ddot{x}$) implies that, in particular, $s \mapsto (x_s)'' \in C^0$ is continuous, and hence $s \mapsto (x_s)' \in C^1$ is continuous. It follows from the fundamental theorem of calculus that N is C^1 on $[0, T]$, with $\dot{N}(t) = RN_t + r_g(x_t)$, in particular, $\dot{N}(0+) = 0 + r_g(x_0)$. □

We turn to an estimate for the nonlinear term r_g in equation (4.2) now.

Lemma 4.5. *Assume* **$(\widetilde{\text{g1}})$–(g3), (g6), (g7)** *and* **$(\widetilde{\text{g8}})$**. *Then there exists a neighborhood U_2 in C^2 of 0 and a bounding function $\tilde{\zeta}$ such that for $\psi \in U_2$ the following estimate holds:*

$$|r_g(\psi)| \le \tilde{\zeta}(|\psi|_{C^2}) \cdot |\psi|_{C^0} + \alpha(|\psi|_{C^1}) |\psi'|_{[-h,-\Delta]}|_{C^0}.$$

Proof. (The proof follows the proof of part (v) of Proposition 3.1, p. 324 in [54].) With W_1 from assumption **(g3)**, there exists a neighborhood U_2 in C^2 of 0 such that for $\psi \in U_2$ one has $(\psi', \psi) \in W_1$. For such ψ one has

$$
\begin{aligned}
r_g(\psi) &= \int_0^1 [Dg_1(s\psi', s\psi) - Dg_1(0,0)](\psi', \psi))\, ds = \\
&= \int_0^1 [D_1 g_1(s\psi', s\psi) - D_1 g_1(0,0)]\psi'\, ds + \int_0^1 [D_2 g_1(s\psi', s\psi) - D_2 g_1(0,0)]\psi\, ds.
\end{aligned}
$$

Using $\widetilde{\textbf{(g8)}}$ and that α is nondecreasing, the first term can be estimated by

$$
\max_{0\le s\le 1} c_8 |\psi''|_{C^0} |s\psi|_{C^0} + \alpha(|s\psi'|_{C^0}) \cdot |\psi'|_{[-h,-\Delta]}|_{C^0} \le c_8 |\psi''|_{C^0} |\psi|_{C^0} + \alpha(|\psi'|_{C^0}) \cdot |\psi'|_{[-h,-\Delta]}|_{C^0}.
$$

In a similar way, using **(g7)**, the second term is estimated by

$$
\zeta_7(|\psi'|_{C^1} + |\psi|_{C^1}) \cdot |\psi|_{C^0} + c_7 |\psi|_{C^1} |\psi|_{C^0}.
$$

Adding both estimates gives

$$
\begin{aligned}
|r_g(\psi)| &\le \left[c_8 |\psi''|_{C^0} + \zeta_7(|\psi'|_{C^1} + |\psi|_{C^1}) + c_7 |\psi|_{C^1}\right] \cdot |\psi|_{C^0} + \alpha(|\psi'|_{C^0}) \cdot |\psi'|_{[-h,-\Delta]}|_{C^0} \\
&\le [c_8 |\psi|_{C^2} + \zeta_7(2|\psi|_{C^2}) + c_7 |\psi|_{C^2}] \cdot |\psi|_{C^0} + \alpha(|\psi|_{C^1}) \cdot |\psi'|_{[-h,-\Delta]}|_{C^0}.
\end{aligned}
$$

The assertion follows by defining $\tilde{\zeta}(s) := (c_8 + c_7)s + \zeta_7(2s)$ for $s \in [0, \infty)$. ☐

The following coarser estimate for r_g will be convenient to use.

Corollary 4.6. *Under the assumptions of Lemma 4.5, there exists a bounding function ρ_g such that*

$$
\forall \psi \in U_2 : \; |r_g(\psi)| \le \rho_g(|\psi|_{C^2}) \cdot |\psi|_{C^1}.
$$

(The proof is obvious, setting $\rho_g(|\psi|_{C^2}) := \tilde{\zeta}(|\psi|_{C^2}) + \alpha(|\psi|_{C^2})$.)

Under more restrictive assumptions than in the present paper (in particular, the linearity condition **(g4)**), it is possible to show that the nonlinearity r_g satisfies an estimate of the form $|r_g(\psi)| \le \text{const} \cdot |\psi|_{C^2} \cdot |\psi|_{C^0}$, see Proposition 3.2, p. 448 in [53]. In the present work (in view of the estimate in Corollary 4.6) we have to work with the C^1-norm. For this purpose we will replace the decomposition in formula (4.3) by a more suitable one, using that $X_2 \subset \mathcal{M}_2$, and the local graph representation of $\mathcal{M}_2$ at zero.

As another consequence of condition $\widetilde{\textbf{(g1)}}$ (together with continuity properties of Dg_1) we shall next obtain an estimate of $|x_t|_{C^1}$ in terms of $|x_0|_{C^1}$ for solutions of (1.1), if $t \in [0, \Delta]$. It follows from assumptions **(g6)** and **(g7)** that there exists a ball B_2 around zero in C^2 such that $B_2 \subset U_2$, and

$$
\begin{aligned}
\|D_{1,e} g_1\|_{\infty, B_2} &:= \sup\left\{ \|D_{1,e} g_1(\psi)\|_{L_c(C^0, \mathbb{R}^n)} \;\middle|\; \psi \in B_2 \right\} < \infty, \\
\exists \tilde{D}_2 > 0\; \forall \psi \in B_2\; \forall s \in [0,1] &: |D_2 g_1(s\psi', s\psi)\psi| < \tilde{D}_2 |\psi|_{C^0}, \text{ and hence} \\
\bar{D} &:= \max\{\|D_{1,e} g_1\|_{\infty, B_2}, \tilde{D}_2, 1\} < \infty.
\end{aligned}
$$

(Note that the property concerning $\|D_{1,e} g_1\|_{\infty, B_2}$ would even hold on a ball in C^1 around zero.)

Lemma 4.7. *Assume that $B_2 \subset C^2$ and $\bar{D}$ are as described above, and that $x : [-h, t_1] \to \mathbb{R}^n$ is a solution of (1.1) with segments $x_t \in X_2 \cap B_2$, and with $t_1 \in (0, \Delta]$.*

Then, setting $C_1 := \bar{D}[1 + (1 + \Delta \cdot \bar{D}) \cdot \exp(\bar{D}\Delta)]$, one has

$$\forall t \in [0, t_1] : \ |x_t|_{C^1} \le C_1 |x_0|_{C^1}.$$

Proof. Set $\varphi := x_0 \in X_2$, and (as in [55], Proposition 2.3) define the affine-linear C^1 extension $\varphi^d : [-h, \Delta] \to \mathbb{R}^n$ by $\varphi^d(t) := \varphi(0) + t\dot{\varphi}(0)\ (t \in [0, \Delta])$. For $t \in [0, t_1]$ we have $\dot{x}(t + \theta) = \dot{\varphi}(t + \theta) = \dot{\varphi}^d(t + \theta)$ if $\theta \in [-h, -\Delta]$ and hence

$$x_t' - (\varphi^d)_t' \in \mathcal{N}_\Delta.$$

Using Lemma 3.18 a) we obtain for these t

$$\begin{aligned}\dot{x}(t) = g(x_t', x_t) &= \int_0^1 [Dg_1(sx_t', sx_t)(x_t', x_t)\, ds = \int_0^1 [D_e g_1(sx_t', sx_t)(x_t', x_t)\, ds \\ &= \int_0^1 D_e g_1(sx_t', sx_t)((\varphi^d)_t', x_t)\, ds = \int_0^1 [D_{1,e} g_1(sx_t', sx_t)(\varphi^d)_t' + D_2 g_1(sx_t', sx_t) x_t]\, ds,\end{aligned}$$

and hence

$$\begin{aligned}|\dot{x}(t)| &\le \|D_{1,e} g_1\|_{\infty, B_2} |(\varphi^d)_t'|_{C^0} + \tilde{D}_2 |x_t|_{C^0} \\ &\le \|D_{1,e} g_1\|_{\infty, B_2} |\varphi'|_{C^0} + \tilde{D}_2 |x_t|_{C^0} \le \bar{D}(|\varphi|_{C^1} + |x_t|_{C^0}).\end{aligned} \tag{4.4}$$

It follows that for $t \in (0, t_1] \subset [0, \Delta]$

$$|x(t)| \le |\varphi(0)| + \int_0^t |\dot{x}(s)|\, ds \le |\varphi(0)| + \Delta \cdot \bar{D} |\varphi|_{C^1} + \bar{D} \int_0^t |x_s|_{C^0}\, ds.$$

Setting $\mu(t) := \max_{s \in [-h, t]} |x(s)|$ for $t \in [0, \Delta]$, we see that

$$\mu(t) \le (1 + \Delta \cdot \bar{D}) |\varphi|_{C^1} + \bar{D} \int_0^t \mu(s)\, ds,$$

and Gronwall's lemma gives

$$\mu(t) \le (1 + \Delta \cdot \bar{D}) |\varphi|_{C^1} \exp[\bar{D}t] \le [1 + \Delta \cdot \bar{D}] \cdot \exp[\bar{D}\Delta] \cdot |\varphi|_{C^1}.$$

Since $\bar{D} \ge 1$, the last estimate and the definition of C_1 imply

$$|x_t|_{C^0} \le [1 + \Delta \cdot \bar{D}] \cdot \exp[\bar{D}\Delta] \cdot |\varphi|_{C^1} \le C_1 |\varphi|_{C^1} \text{ for } t \in [0, t_1]. \tag{4.5}$$

Combining the first inequality in (4.5) with (4.4) we conclude

$$\begin{aligned}|\dot{x}(t)| \le \bar{D}[|\varphi|_{C^1} + (1 + \Delta \cdot \bar{D}) \cdot \exp(\bar{D}\Delta) \cdot |\varphi|_{C^1}] &= \bar{D}[1 + (1 + \Delta \cdot \bar{D}) \cdot \exp(\bar{D}\Delta)] \cdot |\varphi|_{C^1} \\ &= C_1 |\varphi|_{C^1}.\end{aligned}$$

This estimate together with the second inequality in (4.5) gives the result. □

5 Manifolds, graph representation, and decomposition of solutions.

Recall the set U_1 from the beginning of Section 2, and consider the map $F_2 : U_1 \cap C^2 \to \mathbb{R}^n$, $F_2(\psi) := \dot{\psi}(0) - g_1(\psi', \psi)$. This map is of class C^1 (when considered with $|\ |_{C^2}$ on its domain). It is shown in [55, Proposition 5.1] that, if g satisfies **$(\widetilde{\mathbf{g1}})$** and **(g3)**, then $\mathcal{M}_2 := F_2^{-1}(0)$ (called X_2 in the mentioned reference) is a submanifold of class C^1 of the space C^2. The proof in [55] is based on the fact that the differential $DF_2(\psi)$ at every point of $\mathcal{M}_2$ is surjective. In particular, $DF_2(0)$ is surjective, and given by

$$DF_2(0)\chi = \dot{\chi}(0) - Dg_1(0,0)(\chi', \chi) = \dot{\chi}(0) - L\chi' - R\chi.$$

It is also shown in part 2 of the proof of [55, Proposition 5.1] that even $DF_2(0)|_{\mathcal{N}_\Delta}$ is surjective, and hence there exist functions $\psi_1, \dots, \psi_n \in \mathcal{N}_\Delta \cap C^2$ with

$$DF_2(0)\psi_j = e_j \text{ (the } j\text{-th unit vector)}, \quad j = 1, \dots, n. \tag{5.1}$$

It is clear that one can also choose the ψ_j such that $\psi_j \in \mathcal{N}_{\Delta/2}$, which we assume from now on. Since $L = 0$ on $\mathcal{N}_\Delta$, we have $\dot{\psi}_j(0) - R\psi_j = e_j$, $j = 1, \dots, n$. The tangent space $T_0\mathcal{M}_2$ satisfies $T_0\mathcal{M}_2 = \ker DF_2(0) = \left\{\chi \in C^2 \mid \chi'(0) = Dg_1(0,0)(\chi', \chi)\right\}$, and the so-called extended tangent space to $\mathcal{M}_2$ at zero is

$$T_{e,0}\mathcal{M}_2 = \left\{\chi \in C^1 \mid \chi'(0) = D_e g_1(0,0)(\chi', \chi)\right\} = T^1$$

(see Lemma 3.4). We have the decompositions

$$C^2 = \ker DF_2(0) \oplus \bigoplus_{j=1}^{n} \mathbb{R} \cdot \psi_j, \tag{5.2}$$

and correspondingly also

$$C^1 = T^1 \oplus \bigoplus_{j=1}^{n} \mathbb{R} \cdot \psi_j, \tag{5.3}$$

since T^1 is the kernel of the continuous linear functional $C^1 \ni \chi \mapsto \chi'(0) - D_e g_1(0,0)(\chi', \chi)$, and the ψ_j also span its n-dimensional complement in C^1. Recall also that T^1 is the domain of the generator of the semigroup S^0, which induces the C^0-semigroup S^1 on $(T^1, |\ |_{C^1})$ (see Lemma 3.4).

Segments $\varphi \in \mathcal{M}_2$ close to zero w.r. to $|\ |_{C^2}$, say, with $|\varphi|_{C^2} < \delta_2$, have a graph representation $\varphi = \bar{\varphi} + \sum_{j=1}^{n} m_j(\bar{\varphi}) \cdot \psi_j$, with the projection $\bar{\varphi} \in T_0\mathcal{M}_2 \subset T_{e,0}\mathcal{M}_2 = T^1$ of φ to $T_0\mathcal{M}_2$ according to the decomposition (5.2), and with real-valued functions m_j defined on a neighborhood of zero in $T_0\mathcal{M}_2$ and of class C^1 w.r. to the C^2-topology, satisfying $m_j(0) = 0, Dm_j(0) = 0$. Clearly we can choose $\delta_2 > 0$ such that $B_{|\ |_{C^2}}(0, \delta_2) \subset B_2 \subset U_2$, with B_2 as in Lemma 4.7.

For segments in the state space X_2 of the semiflow from Theorem 4.1(recall $X_2 \subset \mathcal{M}_2$) we have the following close relation between the functions m_j and the components r_j of the nonlinear term r_g in equation (4.2):

Lemma 5.1. *For $\varphi \in X_2 \subset \mathcal{M}_2$ with $|\varphi|_{C^2} < \delta_2$ one has*

$$\varphi = \underbrace{\bar{\varphi}}_{\in T^1} + \sum_{j=1}^{n} m_j(\bar{\varphi}) \cdot \psi_j = \bar{\varphi} + \underbrace{\sum_{j=1}^{n} r_j(\varphi) \cdot \psi_j}_{=:\varphi_*} = \bar{\varphi} + \varphi_*, \tag{5.4}$$

and $|\varphi_*|_{C^1} \leq \rho_*(|\varphi|_{C^2}) \cdot |\varphi|_{C^1}$, with a bounding function ρ_*.

Proof. For φ as in the statement of the lemma, the first equality is clear from the above remarks. Further,

$$\begin{aligned} 0 = F_2(\varphi) &= \dot\varphi(0) - g_1(\varphi', \varphi) = \dot\varphi(0) - Dg_1(0,0)(\varphi', \varphi) - r_g(\varphi) \\ &= DF_2(0)\varphi - r_g(\varphi) = DF_2(0)[\bar\varphi + \sum_{j=1}^{n} m_j(\bar\varphi)\cdot\psi_j] - r_g(\varphi) \\ &= \underbrace{DF_2(0)\bar\varphi}_{=0} + \sum_{j=1}^{n} m_j(\bar\varphi)\cdot \underbrace{DF_2(0)\psi_j}_{=e_j} - r_g(\varphi) \\ &= \sum_{j=1}^{n}[m_j(\bar\varphi) - r_j(\varphi)]\cdot e_j, \end{aligned}$$

so $r_j(\varphi) = m_j(\bar\varphi)), j = 1, \dots, n$, which shows the second equality in (5.4). Finally (recall that we use the 1-norm on $\mathbb{R}^n$), from Corollary 4.6 we get

$$|\varphi_*|_{C^1} \le \sum_{j=1}^{n} |r_j(\varphi)|\cdot|\psi_j|_{C^1} \le |r_g(\varphi)|\cdot\max_j|\psi_j|_{C^1} \le \underbrace{\rho_g(|\varphi|_{C^2})\cdot\max_j|\psi_j|_{C^1}}_{=:\rho_*(|\varphi|_{C^2})} \cdot|\varphi|_{C^1},$$

so the stated estimate follows with the indicated definition of ρ_*. □

The decomposition of solutions from Corollary 4.3 is now replaced by the subsequent one (recall the semiflow Φ from Theorem 4.1).

Corollary 5.2. *For φ and $\bar\varphi$ as in Lemma 5.1 and $t \ge 0$ such that $\Phi(t, \varphi)$ is defined, one has for the corresponding solution x^φ of eq.* (1.1)

$$x_t^\varphi = \Phi(t, \varphi) = S^1(t)\bar\varphi + \sum_{j=1}^{n} r_j(\varphi)\cdot S^0(t)\psi_j + N_t. \tag{5.5}$$

Proof. Using Corollary 4.3 and Lemma 5.1 one gets

$$x_t^\varphi = S^0(t)[\bar\varphi + \sum_{j=1}^{n} r_j(\varphi)\cdot\psi_j] + N_t = S^1(t)\bar\varphi + \sum_{j=1}^{n} r_j(\varphi)\cdot S^0(t)\psi_j + N_t. \qquad \square$$

Remark 5.3. Note that in this decomposition, since x_t^φ is of class C^1 (even C^2), and $S^1(t)\bar\varphi$ is of class C^1, the remaining sum of two terms $\sum_{j=1}^{n} r_j(\varphi)\cdot S^0(t)\psi_j + N_t$ is also of class C^1. This is not true for the parts $\sum_{j=1}^{n} r_j(\varphi)\cdot S^0(t)\psi_j$ and N_t as defined in Corollary 4.3, but as long as $t < \Delta/2$, we have the 'partial smoothness' results from Lemmas 3.20 and 4.4, since the ψ_j are in $\mathcal{N}_{\Delta/2}$. It is also instructive to see how the jump discontinuities at 0 in the derivatives of the middle and the last term in (5.5) cancel (as it must be, since the sum of both terms is C^1 on $[-h, t]$): We have $\dot N(0+) = r_g(\varphi)$, $\dot N(0-) = 0$, hence $\dot N(0+) - \dot N(0-) = r_g(\varphi)$. For the middle term one has, setting $\mu(t) := [\sum_{j=1}^{n} r_j(\varphi)\cdot S^0(t)\psi_j](0) \in \mathbb{R}^n$,

$$\dot\mu(0-) = \sum_{j=1}^{n} r_j(\varphi)\dot\psi_j(0) = \sum_{j=1}^{n} r_j(\varphi)(R\psi_j + e_j) = [\sum_{j=1}^{n} r_j(\varphi)\cdot R\psi_j] + r_g(\varphi), \text{ while}$$

$$\dot\mu(0+) = (L+R)[\sum_{j=1}^{n} r_j(\varphi)\cdot\psi_j] = R[\sum_{j=1}^{n} r_j(\varphi)\cdot\psi_j] = \sum_{j=1}^{n} r_j(\varphi)\cdot R\psi_j,$$

so that $\dot\mu(0+) - \dot\mu(0-) = -r_g(\varphi)$, which just cancels the jump of $\dot N$ at 0.

Lemma 5.4. *There exists a bounding function ρ such that for every $\delta \in (0, \delta_2]$ and every solution $x = x^\varphi$ with segments $x_t \in X_2 \cap B_{|\,|_{C^2}}(0;\delta)$ for $t \in [0, \Delta/2]$ and $\bar{\varphi}$ as in Lemma 5.1 , one has*

$$\forall t \in [0, \Delta/2] : \; |x_t^\varphi - S^1(t)\bar{\varphi}|_{C^1} \leq \rho(\delta) \cdot |\varphi|_{C^1}.$$

Proof. For $\delta \in (0, \delta_2]$ (with δ_2 as in Lemma 5.1), the conclusion of Lemma 4.7 holds with an appropriate number C_1 for solutions with segments in $X_2 \cap B_{|\,|_{C^2}}(0, \delta) \subset X_2 \cap B_2$. Corollary 4.6 and the choice of δ_2 show that

$$\forall \varphi \in B_{|\,|_{C^2}}(0, \delta_2) : \; |r_g(\varphi)| \leq \rho_g(|\varphi|_{C^2}) \cdot |\varphi|_{C^1}. \tag{5.6}$$

Set $M^* := \max_{j=1,\ldots,n} |\psi_j|_{C^1}$, and with $M_1, \tilde{M}_1$ as in Lemma 3.20, and set

$$C := \max\left\{ M_1 M^*, C_1 \tilde{M}_1 \cdot (\Delta/2), \|R\|_{L_c(C^0,\mathbb{R}^n)} \cdot C_1 \tilde{M}_1 \cdot (\Delta/2) + C_1 \right\}.$$

Consider now a solution x as in the assertion, which then has a decomposition according to formula (5.5). From Lemma 3.20 a) and Lemma 4.4 we see that the functions y_j defined by $(y_j)_t := [S^0(t)\psi_j]$, $j = 1, \ldots, n$, as well as the function N, are C^1 when restricted to $[0, \Delta/2]$ and to $[-h, 0]$, with a jump discontinuity of the first derivative at $t = 0$. With $z(t) := \sum_{j=1}^n r_j(\varphi) \cdot y_j(t)$ for $t \in [-h, \Delta/2]$ we have from formula (5.5)

$$x^\varphi(t) - (S^1(t)\bar{\varphi})(0) = z(t) + N(t) \qquad (t \in [0, \Delta/2]),$$

and $x_0^\varphi - \bar{\varphi} = \varphi - \bar{\varphi} = \sum_{j=1}^n r_j(\varphi) \cdot \psi_j = z_0$.

For $t \in [0, \Delta/2]$ we see from Lemma 3.20 that $\max\{|\dot{y}_j(t)|, |y_j(t)|\} \leq M_1 \cdot |\psi_j|_{C^0}$, $j = 1, \ldots, n$. Since we use the 1-norm on $\mathbb{R}^n$, it follows from (5.6) and the definition of M^* that for these t

$$\begin{aligned} \max\{|\dot{z}(t)|, |z(t)|\} &\leq \left(\sum_{j=1}^n |r_j(\varphi)| \right) \cdot M_1 \cdot \max_j |\psi_j|_{C^0} = |r_g(\varphi)| M_1 \max_j |\psi_j|_{C^0} \\ &\leq \rho_g(|\varphi|_{C^2}) \cdot M_1 M^* |\varphi|_{C^1} \leq C \cdot \rho_g(\delta) |\varphi|_{C^1}. \end{aligned}$$

For $t \in [-h, 0]$ we have $\max\{|\dot{z}(t)|, |z(t)|\} \leq |r_g(\varphi)| M^* \leq \rho_g(|\varphi|_{C^2}) M^* |\varphi|_{C^1}$, so that the last estimate holds also for these t, since $M_1 \geq 1$.

Now $N = 0$ on $[-h, 0]$, and for $t \in [0, \Delta/2]$ we obtain, using Lemma 3.20 b), again estimate (5.6), and Lemma 4.7:

$$\begin{aligned} |N(t)| &= \left| \int_0^t \mathbf{X}(t-s) r_g(x_s)\, ds \right| \leq \tilde{M}_1 \int_0^t \rho_g(|x_s|_{C^2}) |x_s|_{C^1}\, ds \\ &\leq \tilde{M}_1 \rho_g(\delta) \int_0^t C_1 |\varphi|_{C^1}\, ds \leq C_1 \tilde{M}_1 \cdot (\Delta/2) \rho_g(\delta) |\varphi|_{C^1} \\ &\leq C \rho_g(\delta) |\varphi|_{C^1}. \end{aligned}$$

Further, for $t \in [0, \Delta/2]$, Lemma 4.4, the second last inequality in the last estimate, and Lemma 4.7 (again) give

$$\begin{aligned} |\dot{N}(t)| &\leq \|R\|_{L_c(C^0,\mathbb{R}^n)} |N_t|_{C^0} + |r_g(x_t)| \leq \|R\|_{L_c(C^0,\mathbb{R}^n)} C_1 \tilde{M}_1 (\Delta/2) \rho_g(\delta) |\varphi|_{C^1} + \rho_g(\delta) C_1 |\varphi|_{C^1} \\ &= \left[\|R\|_{L_c(C^0,\mathbb{R}^n)} C_1 \tilde{M}_1 (\Delta/2) + C_1 \right] \cdot \rho_g(\delta) |\varphi|_{C^1} \leq C \rho_g(\delta) |\varphi|_{C^1}. \end{aligned}$$

Combining the above estimates for $z, \dot{z}, N$, and $\dot{N}$ we see that for $t \in [0, \Delta/2]$

$$|x_t^{\varphi} - S^1(t)\bar{\varphi}|_{C^1} \le \max_{t\in[-h,0]\cup[0,\Delta/2]} \{|z(t)| + |N(t)| + |\dot{z}(t)| + |\dot{N}(t)|\} \le 4C \cdot \rho_g(\delta)|\varphi|_{C^1},$$

which proves the assertion with $\rho(\delta) := 4C \cdot \rho_g(\delta)$. □

We now use condition $(\mathbf{D^2g_2})$ from Section 2 to obtain a manifold containing initial values with unstable behavior.

The proof of the lemma below is methodically similar to the proof that $\mathcal{M}_2$ (called X_2 in [55]) is a submanifold of C^2 in Proposition 5.1 of [55].

Lemma 5.5. *Under the additional assumption* $(\mathbf{D^2g_2})$, *the set* $\mathcal{M}_4 := X_2 \cap C^4$ *is a* C^1*-submanifold of* C^4. *The tangent space at* $0 \in C^4$ *to* $\mathcal{M}_4$ *satisfies*

$$T_0\mathcal{M}_4 = \Big\{\chi \in C^4 \;\Big|\; \dot{\chi}(0) = Dg_1(0,0)(\chi', \chi), \\ \ddot{\chi}(0) = Dg_1(0,0)(\chi'', \chi')\Big\}.$$

Proof. From (2.3), we have

$$\mathcal{M}_4 = \Big\{\psi \in U_1 \cap C^4 \;\Big|\; \text{(i) } \dot{\psi}(0) = g(\psi', \psi); \\ \text{(ii) } \ddot{\psi}(0) = D_e g_1(\psi', \psi)(\psi'', \psi')\Big\}$$

Note that for $\psi \in C^4$, we can write g_1 instead of g and Dg_2 instead of $D_e g_1$ in the definition of X_2 and the description of $\mathcal{M}_4$. Thus, with $F_4 : U_1 \cap C^4 \to \mathbb{R}^n \times \mathbb{R}^n$ given by

$$F_4(\psi) := [\dot{\psi}(0) - g_1(\psi', \psi), \ddot{\psi}(0) - Dg_2(\psi', \psi)(\psi'', \psi')],$$

we have $\mathcal{M}_4 = F_4^{-1}\{(0,0)\}$. F_4 is of class C^1, because the maps

$$C^4 \ni \psi \mapsto (\psi'', \psi') \in C^2 \times C^3 \subset C^2 \times C^2$$

and $C^4 \ni \psi \mapsto (\psi', \psi) \in C^3 \times C^4 \subset C^2 \times C^2$ are linear and continuous, the latter maps $U_1 \cap C^4$ into W_1, and Dg_2 is C^1 on $W_1 \cap (C^2 \times C^2)$, due to assumption $(\mathbf{D^2g_2})$.

Further, for $\psi \in U_1 \cap C^4$ and $\chi \in C^4$, we calculate

$$DF_4(\psi)\chi = [\dot{\chi}(0) - Dg_1(\psi', \psi)(\chi', \chi), \ddot{\chi}(0) \\ - Dg_2(\psi', \psi)(\chi'', \chi') - D^2g_2(\psi', \psi)[(\psi'', \psi'), (\chi', \chi)]]. \tag{5.7}$$

(Note that in the term with Dg_2 of the last formula, it makes no difference if we use Dg_1 or Dg_2.) If $\chi \in C^4 \cap \mathcal{N}_\Delta$ (then also $\chi' \in \mathcal{N}_\Delta$ and $\chi'' \in \mathcal{N}_\Delta$), Lemma 3.18 implies

$$Dg_2(\psi', \psi)(\chi', \chi) = Dg_1(\psi', \psi)(\chi', \chi) = Dg_1(\psi', \psi)(0, \chi) = D_e g_1(\psi', \psi)(0, \chi),$$

and

$$Dg_1(\psi', \psi)(\chi'', \chi') = Dg_1(\psi', \psi)(0, \chi') = D_e g_1(\psi', \psi)(0, \chi').$$

Further, using $\chi' \in \mathcal{N}_\Delta$ and Lemma 3.18 again, one obtains

$$\begin{aligned} D^2 g_2(\psi',\psi)[(\psi'',\psi'),(\chi',\chi)] &= \lim_{s\to 0}\frac{1}{s}Dg_2(\psi'+s\chi',\psi+s\chi)(\psi'',\psi') \\ &= \frac{1}{s}Dg_1(\psi'+s\chi',\psi+s\chi)(\psi'',\psi') \\ &= \lim_{s\to 0}\frac{1}{s}Dg_1(\psi',\psi+s\chi)(\psi'',\psi') \\ &= D^2 g_2(\psi',\psi)[(\psi'',\psi'),(0,\chi)] \\ &= D_e^2 g_2(\psi',\psi)[(\psi'',\psi'),(0,\chi)]. \end{aligned}$$

Thus, for $\chi \in C^4 \cap \mathcal{N}_\Delta$, we obtain

$$\begin{aligned} DF_4(\psi)\chi = &\big[\dot\chi(0) - D_e g_1(\psi',\psi)(0,\chi), \ddot\chi(0) - D_e g_1(\psi',\psi)(0,\chi') \\ &- D_e^2 g_2(\psi',\psi)[(\psi'',\psi'),(0,\chi)]\big]. \end{aligned} \tag{5.8}$$

Take now $j \in \{1,\ldots,n\}$. Slightly modifying the argument from the proof of [55, Proposition 5.1] to C^4-smoothness, one can find a sequence $(\chi_m^{(j)})_{m\in\mathbb{N}} \subset C^4 \cap \mathcal{N}_\Delta$ with $\dot\chi_m^{(j)}(0) = e_j$ (the j-th unit vector in $\mathbb{R}^n$) and

$$\dot\chi_m^{(j)}(0) - D_e g_1(\psi',\psi)(0,\chi_m^{(j)}) \to e_j \quad (m\to\infty).$$

(Here $|\chi_m^{(j)}|_{C^0} \to 0$ as $m \to \infty$, which together with the continuity property of $D_e g_1$ implies $D_e g_1(\psi',\psi)(0,\chi_m^{(j)}) \to 0$. This is why we use the notation $D_e g_1(\psi',\psi)$ here, although $\chi_m^{(j)} \in C^3 \subset C^1$.)

$(\chi_m^{(j)})$ can be also chosen such that the sequences $(\chi_m^{(j)}) \subset C^1$ and $(\ddot\chi_m^{(j)}(0)) \subset \mathbb{R}^n$ are bounded, so that the sequence

$$\ddot\chi_m^{(j)}(0) - D_e g_1(\psi',\psi)(0,(\chi_m^{(j)})') - D_e^2 g_2(\psi',\psi)[(\psi'',\psi'),(0,\chi_m^{(j)})]$$

is bounded in $\mathbb{R}^n$. Hence we can assume that this sequence converges to a vector $f_j \in \mathbb{R}^n$.

Together we obtain

$$DF_4(\psi)\chi_m^{(j)} \to (e_j, f_j) \ \text{ as } \ m \to \infty. \tag{5.9}$$

Next, we find a sequence $(\zeta_m^{(j)}) \subset C^4 \cap \mathcal{N}_\Delta$ such that $\dot\zeta_m^{(j)}(0) = 0$, $|\zeta_m^{(j)}|_{C^1} \to 0$, $\ddot\zeta_m^{(j)}(0) = e_j$. With the 'minimal delay' Δ, it suffices to define $\zeta_m^{(j)}$ for m with $1/m < \Delta$. Take for example

$$\zeta_m^{(j)}(t) = \begin{cases} 0, & -h \le t \le -\dfrac{1}{m} \\ \dfrac{m^5}{2}\cdot t^2\left(t+\dfrac{1}{m}\right)^5 \cdot e_j, & -\dfrac{1}{m} \le t \le 0. \end{cases}$$

Then

$$\begin{aligned} |\zeta_m^{(j)}|_{C^0} &\le \frac{m^5}{2}\cdot\frac{1}{m^2}\cdot\frac{1}{m^5}|e_j| \to 0 \ (m\to\infty), \\ |(\zeta_m^{(j)})'|_{C^0} &\le \frac{m^5}{2}\cdot\left[\frac{2}{m}\left(\frac{1}{m}\right)^5 + \frac{1}{m^2}\cdot\frac{5}{m^4}\right]\cdot|e_j| \to 0 \qquad (m\to\infty), \\ \ddot\zeta_m^{(j)}(0) &= \frac{m^5}{2}\cdot 2\cdot\frac{1}{m^5}\cdot e_j = e_j. \end{aligned}$$

Therefore, we have for the first part in expression (5.8) for $DF_4(\psi)\zeta_m^{(j)}$:

$$\dot{\zeta}_m^{(j)}(0) - D_e g_1(\psi',\psi)(0,\zeta_m^{(j)}) \to 0 \quad \text{as } m \to \infty,$$

and for the second part

$$\begin{aligned}&\ddot{\zeta}_m^{(j)}(0) - D_e g_1(\psi',\psi)(0,(\zeta_m^{(j)})') - D_e^2 g_2(\psi',\psi)[(\psi'',\psi'),(0,\zeta_m^{(j)})]\\ &\quad = e_j - D_e g_1(\psi',\psi)(0,(\zeta_m^{(j)})') - D_e^2 g_2(\psi',\psi)[(\psi'',\psi'),(0,\zeta_m^{(j)})] \to e_j \quad \text{as } m \to \infty,\end{aligned}$$

since $(\zeta_m^{(j)})' \to 0$ in C^0 and $|\zeta_m^{(j)}|_{C^1} \to 0$ $(m \to \infty)$), in view of the continuity properties of $D_e g_1(\psi',\psi)$ and $D_e^2 g_2(\psi',\psi)$. We see now that

$$DF_4(\psi)\zeta_m^{(j)} \to (0,e_j) \quad \text{as } m \to \infty. \tag{5.10}$$

From (5.9) and (5.10), one sees that the $2n$ vectors (e_j, f_j) and $(0, e_j)$, $j = 1,\dots,n$, which are a basis of $\mathbb{R}^{2n}$, are in the closure of the image of $DF_4(\psi)$. It follows that $DF_4(\psi) : C^4 \to \mathbb{R}^n \times \mathbb{R}^n \approx \mathbb{R}^{2n}$ is surjective. As in the proof of [55, Proposition 5.1], this is sufficient to show that $\mathcal{M}_4$ is a C^1–submanifold of C^4, with codimension $2n$.

We prove the statement about $T_0\mathcal{M}_4$ now: For $v \in T_0\mathcal{M}_4$ there exists $\varepsilon > 0$ and a curve $\gamma : (-\varepsilon,\varepsilon) \to \mathcal{M}_4$ differentiable at 0, with $\gamma(0) = 0, \dot{\gamma}(0) = v$. It follows that

$$0 = \frac{d}{dt}\Big|_{t=0} F_4(\gamma(t)) = DF_4(0)\dot{\gamma}(0) = DF_4(0)v.$$

Hence $T_0\mathcal{M}_4 \subset \ker DF_4(0)$. Since both spaces have codimension $2n$ in C^3, they are equal. Now since $D^2 g_2(0,0)[(0,0),(\chi'',\chi')] = 0$ for all $\chi \in C^4$, we have (see (5.7))

$$\begin{aligned}T_0\mathcal{M}_4 &= \ker DF_4(0)\\ &= \left\{\chi \in C^4 \mid \dot{\chi}(0) = Dg_1(0,0)(\dot{\chi},\chi),\ \ddot{\chi}(0) = Dg_1(0,0)(\chi'',\chi')\right\}.\end{aligned}$$ □

Remark 5.6.

1) The continuous extension property for $D^2 g_2$ that is actually used in the proof of (5.10) above is weaker than assumption **($\mathbf{D^2g_2}$)**, because the proof uses only that $D_e^2 g_2(\psi',\psi)[(\psi'',\psi'),(0,\delta_m)] \to 0$ as $|\delta_m|_{C^1} \to 0$.

2) Convenient application of the chain rule to obtain C^1 smoothness of the map F_4 above was the main reason for constructing the manifold $\mathcal{M}_4$ as a subset of C^4.

3) With the last lemma, we have

$$X_1 \supset X_{1,+} \supset X_1 \cap C^2 = \mathcal{M}_2 \supset X_2 \supset X_2 \cap C^4 = \mathcal{M}_4,$$

where the X_j are invariant, but not smooth submanifolds of C^j, and the $\mathcal{M}_j$ are C^1-submanifolds of C^j, but not invariant under the semiflow on $X_{1,+}$. It seems that, based on increasingly higher smoothness assumptions on g, one could continue this construction to obtain a decreasing sequence of invariant subsets X_j containing (non-invariant) submanifolds $\mathcal{M}_j$, such that the semiflow restricted to X_j has higher order smoothness properties with increasing j. We do not pursue this here.

Recall the semigroup $\{S^0(t)\}_{t\geq 0}$ defined by the solutions of (3.1). Of course, a $\mathbb{C}^n$-valued solution $a+ib$ of (3.1) is to be understood in the sense that its real part a and its imaginary part b are $\mathbb{R}^n$-valued solutions of (3.1).

Proposition 5.7. *Assume* $(\mathbf{D^2g_2})$. *If* $y:\mathbb{R}\to\mathbb{C}^n$ *is a* C^4 *solution of equation* (3.1) *then* $\zeta := \operatorname{Re}(y)$ *solves also* $\dot\zeta(t) = Dg_1(0,0)[(\dot\zeta)_t,\zeta_t]$ $(t\in\mathbb{R})$, *and all segments* ζ_t *are contained in* $T_0\mathcal{M}_4$.

Proof. Set $\zeta := \operatorname{Re}(y)$. The maps $p_0 : t\mapsto \zeta_t\in C^1$ and $p_1 : t\mapsto \dot\zeta_t \in C^1$ are of class C^1, with $\dot p_0(t) = (\dot\zeta)_t,\ \dot p_1(t) = (\ddot\zeta)_t$, and in the equation $\frac{d}{dt}(\zeta(t) - L\zeta_t) = R\zeta_t$, the terms in the bracket are both individually differentiable w.r.t. to t. Thus we have

$$\begin{aligned}\dot\zeta(t) &= L(\dot\zeta)_t + R\zeta_t = D_{1,e}g_1(0,0)(\dot\zeta)_t + D_{2,e}g_1(0,0)(\zeta)_t\\ &= D_e g_1(0,0)[(\dot\zeta)_t,\zeta_t] = Dg_1(0,0)[(\dot\zeta)_t,\zeta_t],\end{aligned} \tag{5.11}$$

where the last equality holds because $(\dot\zeta)_t$ and ζ_t are in C^1 (even in C^3). Since ζ is of class C^4, differentiation of (5.11) gives for $t\in\mathbb{R}$: $\ddot\zeta(t) = Dg_1(0,0)[(\ddot\zeta)_t,(\dot\zeta)_t]$. It follows that for $t\in\mathbb{R}$ the segment $\chi := \zeta_t$ satisfies

$$\dot\chi(0) = Dg_1(0,0)(\chi',\chi) \quad\text{and}\quad \ddot\chi(0) = Dg_1(0,0)(\chi'',\chi').$$

We also have $\chi\in C^4$, and from Lemma 5.5 we see that $\chi\in T_0\mathcal{M}_4$. □

Corollary 5.8. *Under assumption* $(\mathbf{D^2g_2})$, *the following hold:*

a) If $\lambda\in\mathbb{C}$ *is an eigenvalue of the infinitesimal generator* $A^0_{\mathbb{C}}$ *of the (complexified) semigroup* $\{S^0(t)\}_{t\geq 0}$ *then the corresponding finite-dimensional generalized eigenspace* $\mathcal{G}_{\lambda,A^0_{\mathbb{C}}}$ *(see Lemma 3.2 and Lemma 3.14) satisfies*

$$\operatorname{Re}\,(\mathcal{G}_{\lambda,A^0_{\mathbb{C}}}) \subset T_0\mathcal{M}_4.$$

b) In the situation of Corollary 3.16, one has $E^+\subset T_0\mathcal{M}_4$.

Proof. Ad a): (We omit the subscript $\mathbb{C}$ in the proof.) For $\varphi\in\mathcal{G}_{\lambda,A^0}$ there exists a solution $y:\mathbb{R}\to\mathbb{C}^n$ of equation 3.1 of the form $y(t) = e^{\lambda t}\cdot p(t)$, where p is a polynomial with coefficients in $\mathbb{C}^n$, with $y_0 = \varphi$. (A solution of the finite-dimensional ODE generated by $A^0|_{\mathcal{G}_{\lambda,A^0}}$ on $\mathcal{G}_{\lambda,A^0}$.) Application of Proposition 5.7 to y at $t=0$ gives $\operatorname{Re}(\varphi) = \operatorname{Re}(y_0)\in T_0\mathcal{M}_4$.

Assertion b) follows from a) since $E^+ = \operatorname{Re}\big[\bigoplus_{\lambda\in\sigma(A^0)\cap\mathbf{S}_{(\beta,\infty)}} \mathcal{G}_{\lambda,A^0}\big]$. □

Under assumption $(\mathbf{D^2g_2})$, the submanifold $\mathcal{M}_4$ of C^4 is locally a graph over its tangent space. Hence there exist neighborhoods W_4 of zero in C^4, U_4 of zero in $T_0\mathcal{M}_4$, and a C^1 function (w.r.t. to $|\ |_{C^4}$) $m_4 : U_4\to C^4$ with $m_4(0)=0, Dm_4(0)=0$ such that

$$\mathcal{M}_4\cap W^4 = \Big\{\psi + m_4(\psi)\ \Big|\ \psi\in U_4\Big\}.$$

If we add the assumptions of Corollary 3.16, so the space E^+ is defined, then $E^+\subset T_0\mathcal{M}_4$ (Corollary 5.8 b)), and the set $U_4\cap E^+$ is a neighborhood of zero in E^+. We can then set $m_4^+ := m_4|_{U_4\cap E^+}$ and define

$$\mathcal{M}_4^+ := \Big\{\psi + m_4^+(\psi)\ \Big|\ \psi\in U_4\cap E^+\Big\},$$

which is a submanifold of $\mathcal{M}_4$ tangent to E^+ at zero in the C^4-topology. Clearly $\mathcal{M}_4 \subset C^1$, and in view of the decompositions (5.3) and (3.22), we have

$$C^1 = E^+ \oplus (T^1 \cap E^-) \oplus C^1_*, \tag{5.12}$$

where $C^1_* := \bigoplus_{j=1}^n \mathbb{R} \cdot \psi_j$. Hence we can assume that m_4^+ is a map $m_4^+ : U_4 \cap E^+ \to (T^1 \cap E^-) \oplus C^1_*$.

Corollary 5.9. *Assume* $(\mathbf{D^2 g_2})$ *and the conditions of Corollary 3.16. There exists a bounding function* ρ_4 *such that all* $\varphi \in \mathcal{M}_4^+$ *have a representation (in the sense of (5.12))*

$$\varphi = \varphi^+ + \varphi^- + \varphi_*, \quad \text{with } \max\{|\varphi^-|_{C^1}, |\varphi_*|_{C^1}\} \le \rho_4(|\varphi^+|_{C^1}) \cdot |\varphi^+|_{C^1}.$$

Proof. First, the properties $m_4^+(0) = 0$ and $Dm_4^+(0) = 0$ imply that for $\varphi \in \mathcal{M}_4^+, \varphi = \varphi^+ + \varphi^- + \varphi_*$, where $\varphi^- + \varphi_* = m_4^+(\varphi^+)$, one has $|\varphi^- + \varphi_*|_{C^4} = \tilde{\rho}_4(|\varphi^+|_{C^4}) \cdot |\varphi^+|_{C^4}$, with a bounding function $\tilde{\rho}_4$. Equivalence of the C^4 and the C^1 norms on the finite dimensional space E^+ gives a related bounding function $\hat{\rho}_4$ such that

$$|\varphi^- + \varphi_*|_{C^1} \le |\varphi^- + \varphi_*|_{C^4} \le \hat{\rho}_4(|\varphi^+|_{C^1}) \cdot |\varphi^+|_{C^1}.$$

Since the spaces in (5.12) are closed subspaces w.r. to $|\ |_{C^1}$, the corresponding projections are continuous w.r. to this norm, and the C^1-norm on $(T^1 \cap E^-) \oplus C^1_*$ is equivalent to the norm defined by $\psi^- + \psi_* \mapsto \max\{|\psi_-|_{C^1}, |\varphi_*|_{C^1}\}$ on this space. The asserted estimate with a third bounding function ρ_4 follows. □

6 The linearized instability theorem

Before using the preparations from the previous sections to prove our main theorem, we found it worth while to state an 'abstract' version of the main arguments in the lemma below, which reveals the essential structures. It is an adapted version of Lemma 3.3 from [30], p. 5389 and, like the latter, inspired by [3].

Lemma 6.1. *Let* $(E, |\ |)$ *be a Banach space. We make the subsequent assumptions:*

(i) *E has a decomposition $E = E_u \oplus E_s \oplus E_*$ into subspaces closed w.r. to $|\ |$, so the corresponding projections π_u, π_s, π_* are continuous as maps from $(E, |\ |)$ into itself. (We use the notation $x = x_u + x_s + x_*$ in obvious meaning.)*

(ii) *$X \subset E$ is a subset and $P : X \to E$ is a map which takes the form*

$$P(x) = P_L(\pi_u x + \pi_s x) + P_N(x), \text{ with a map } P_L : E_u \oplus E_s \to E_u \oplus E_s$$

satisfying the subsequent properties.

(iii) *There exist a norm $\|\ \|$ on $E_u \oplus E_s$ equivalent to $|\ ||_{E_u \oplus E_s}$ and numbers a, b with $a < b$ and $b > 1$ such that for $x_u \in E_u$, $x_s \in E_s$ one has $\|x_u + x_s\| = \max\{\|x_u\|, \|x_s\|\}$, and*

$$\|\pi_u P_L(x_u + x_s)\| \ge b\|x_u\|, \ \|\pi_s P_L(x_u + x_s)\| \le a\|x_s\|.$$

Under these assumptions there exists $c > 0$ such that for $x_s \in E_s$, $x_u \in E_u$ and $x \in E$ one has

$$\|x_s\| \le c|x_s|, \ \|x_u\| \le c|x_u|, \ |x_u + x_s| \le c\|x_u + x_s\|, \text{ and } |\pi_u x| \le c|x|, \ |\pi_s x| \le c|x|.$$

With such a number c, define now $\kappa := \min\left\{1/2, \frac{b-a}{4c^3}, \frac{b-1}{4c^3}\right\}$.

If then $x = x_s + x_u + x_* \in X$ satisfies

$$\|x_u\| \ge \|x_s\| \quad \text{and} \quad \max\{|\pi_* x|, |P_N(x)|\} \le \kappa \cdot |x|, \tag{6.1}$$

and $y = P(x) = y_u + y_s + y_*$, then also $\|y_u\| \ge \|y_s\|$ (cone invariance), and with $q := \frac{b+1}{2}$ one has $\|y_u\| \ge q\|x_u\|$ (expansion).

Proof. The existence of $c > 0$ as above is clear from equivalence of the norms and continuity of the projections. For x as in the assertion one has

$$|x| = |x_s + x_u + \pi_* x| \le |x_s + x_u| + |\pi_* x| \le |x_s + x_u| + \kappa|x|,$$

so $\kappa \le 1/2$ and (6.1) imply

$$|x| \le \frac{1}{1-\kappa}|x_s + x_u| \ \le 2|x_s + x_u| \le 2c\|x_s + x_u\| = 2c\max\{\|x_u\|, \|x_s\|\} = 2c\|x_u\|. \tag{6.2}$$

Further,

$$\begin{aligned}\|y_u\| &= \|\pi_u P(x)\| = \|\pi_u P_L(x_u + x_s) + \pi_u P_N(x)\| \ge b\|x_u\| - \|\pi_u P_N(x)\| \\ &\ge b\|x_u\| - c|\pi_u P_N(x)| \ge b\|x_u\| - c^2|P_N(x)| \ge b\|x_u\| - c^2\kappa|x| \\ &\ge (b - 2\kappa c^3)\|x_u\|,\end{aligned}$$

where we used (6.2) in the last estimate. Similarly,

$$\|y_s\| \le a\|x_s\| + 2\kappa c^3\|x_s + x_u\| = a\|x_s\| + 2\kappa c^3\|x_u\| \le (a + 2\kappa c^3)\|x_u\|.$$

The choice of κ implies that $b - 2\kappa c^3 \ge a + 2\kappa c^3$ and also $b - 2\kappa c^3 \ge b - (b-1)/2 = (b+1)/2 = q$, from which the assertions follow. □

For simplicity, we chose the cone defined by $\|x_u\| \ge \gamma\|x_s\|$ with $\gamma = 1$ above; similar arguments are possible with different cones. The theorem below is the main result of the present work:

Theorem 6.2 (Linearized Instability Principle)**.** *Consider equation (1.1), and assume that g satisfies conditions* **(g0)**, $\widetilde{\textbf{(g1)}}$, **(g2)**, **(g3)**,**(g6)**, **(g7)**, $\widetilde{\textbf{(g8)}}$, *and* **($\mathbf{D^2 g_2}$)**.

Further, assume that the operator L is as in (3.5), and that with appropriate numbers $\alpha < \beta$, the spectrum of the generator $A^0_{\mathbb{C}}$ (given by the zeroes of the characteristic function χ) splits into $\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(-\infty,\alpha]}$ and $\sigma(A^0_{\mathbb{C}}) \cap \mathbf{S}_{(\beta,\infty)}$ as in Theorem 3.15. With $\mathcal{M}^+_4$ as in Corollary 5.9, the zero equilibrium is then unstable for the semiflow Φ on X_2 in the following sense:

There exists a ball $\mathcal{B}_2$ around zero in C^2 such that for all nonzero $\varphi \in \mathcal{B}_2 \cap \mathcal{M}^+_4 \subset \mathcal{B}_2 \cap X_2$, there exists a time $t(\varphi) > 0$ with $\Phi(t(\varphi), \varphi) \notin \mathcal{B}_2$.

Proof. 1. in view of the spectral splitting assumption, Corollary 3.16 gives the decomposition $T^1 = E^+ \oplus (T^1 \cap E^-)$, and choosing $\tilde{\alpha} \in (\alpha, \beta)$ we have estimates analogous to (3.20) and (3.21) for the semigroup S^1 and $|\ |_{C^1}$. There exists a norm $\|\ \|$ equivalent to $|\ |_{C^1}|_{T^1}$ such that the estimates hold with K and $\tilde{K}$ replaced by 1 w.r. to this norm, and that $\|\varphi^+ + \varphi^-\| =$

$\max\{\|\varphi^+\|, \|\varphi^-\|\}$ for $\varphi^+ \in E^+$, $\varphi^- \in E^- \cap T^1$. (Extend S^1 to a group on E^+, define $\|\varphi^+\| := \sup_{t\le 0} \exp(\beta|t|)|S^1(t)\varphi^+|_{C^1}$ for $\varphi^+ \in E^+$, and $\|\varphi^-\| := \sup_{t\ge 0} \exp(-\tilde{\alpha}t)|S^1(t)\varphi^+|_{C^1}$ for $\varphi^+ \in T^1 \cap E^-$. Compare e.g. [3], Lemma 2.1, p. 10.) In particular, for $t := \Delta/2$ we obtain for the linear map $P_L := S^1(\Delta/2)$, which has E^+ and $T^1 \cap E^-$ as invariant subspaces, and with $b := \exp(\beta\Delta/2) > 1$, $a := \exp(\tilde{\alpha}\Delta/2) < b$ that

$$\|P_L\varphi^+\| \ge b\|\varphi^+\| \text{ for } \varphi^+ \in E^+, \ \|P_L\varphi^-\| \le a\|\varphi^-\| \text{ for } \varphi^- \in T^1 \cap E^-. \tag{6.3}$$

2. We want to apply Lemma 6.1 with $(C^1, |\,|_{C^1})$ in place of $(E, |\,|)$, and with the decomposition (5.12) in place of $E = E_u \oplus E_s \oplus E_*$. We see from (6.3) that the new norm, P_L, a and b are as required in Lemma 6.1. Thus we obtain numbers $c, \kappa > 0$ as in that lemma. Consider now δ_2 and the bounding function ρ from Lemma 5.4, and the bounding function ρ_* from Lemma 5.1. Choose $\delta_2^* \in (0, \delta_2]$ such that

$$\rho(\delta_2^*) \le \kappa \quad \text{and} \quad \rho_*(\delta_2^*) \le \kappa.$$

Next choose $\hat{\delta}_2 \in (0, \delta_2^*]$ such that with $B := B_{|\,|_{C^2}}(0, \hat{\delta}_2)$, for every $\varphi \in B \cap X_2$, the corresponding solution x^φ (with segments $x_t^\varphi \in X_2$) is defined at least on $[-h, \Delta/2]$, and satisfies

$$|x_t^\varphi|_{C^2} < \delta_2^* \qquad (t \in [0, \Delta/2]). \tag{6.4}$$

This is possible since the semiflow is continuous w.r.t. to $|\,|_{C^2}$ (Theorem 4.1); see also e.g. [1], Lemma (10.5), p. 125 and the obvious modification for semiflows, for the lower semicontinuity of the existence time. Then the map

$$P : B \cap X_2 \to X_2 \subset C^1, \qquad \varphi \mapsto x^\varphi_{\Delta/2}$$

is well-defined. For $\varphi \in B \cap X_2$ we have, in view of Lemma 5.1, $\varphi = \bar{\varphi} + \varphi_* \in T^1 \oplus C^1_*$, with

$$|\varphi_*|_{C^1} \le \rho_*(|\varphi|_{C^2}) \cdot |\varphi|_{C^1} \le \rho_*(\delta_2^*) \cdot |\varphi|_{C^1} \le \kappa|\varphi|_{C^1}.$$

Further, for such $\varphi = \bar{\varphi} + \varphi_*$, property (6.4) and Lemma 5.4 show that

$$P(\varphi) = x^\varphi_{\Delta/2} = S^1(\Delta/2)\bar{\varphi} + P_N(\varphi) = P_L(\bar{\varphi}) + P_N(\varphi),$$

with $|P_N(\varphi)|_{C^1} \le \rho(\delta_2^*) \cdot |\varphi|_{C^1} \le \kappa|\varphi|_{C^1}$.

3. We have proved in step 2 that for all $\varphi \in B \cap X_2$ the second condition in (6.1) is satisfied. In order to find initial functions $\varphi = \varphi^+ + \varphi^- + \varphi_* \in B$ which also satisfy the first condition in (6.1), that is, $\|\varphi^+\| \ge \|\varphi^-\|$, we employ Corollary 5.9, which first shows that for $\varphi \in B \cap \mathcal{M}_4^+$ one has $|\varphi^-|_{C^1} \le \rho_4(|\varphi^+|_{C^1}) \cdot |\varphi^+|_{C^1}$. The equivalence of the norms $\|\,\|$ and $|\,|_{C^1}$ on T^1 implies that, with a related bounding function $\tilde{\rho}_4$, one also has $\|\varphi^-\| \le \tilde{\rho}_4(|\varphi^+|_{C^1}) \cdot \|\varphi^+\|$ for these φ. Now we can choose a ball $\mathcal{B}_2 \subset B$ w.r. to $|\,|_{C^2}$ such that for $\varphi \in \mathcal{B}_2 \cap \mathcal{M}_4^+$ one has $\tilde{\rho}_4(|\varphi|_{C^1}) \le 1$. For these φ then $\|\varphi^-\| \le \|\varphi^+\|$, i.e., the first condition in (6.1) also holds.

4. We prove now that the subset $\mathcal{B}_2 \cap \mathcal{M}_4^+$ of X_2 has the asserted property: For $\varphi \ne 0$ in this set, $P(\varphi)$ is defined, invariance of X_2 under the semiflow gives that also $\psi := P(\varphi) \in X_2$, and Lemma 6.1 shows that $\psi = \psi^+ + \psi^- + \psi_*$ again satisfies the first condition in (6.1), and $\|\psi^+\| \ge q\|\varphi^+\|$. In case that still $\psi \in \mathcal{B}_2$, also the second condition from (6.1) holds for ψ, and we can apply Lemma 6.1 again to obtain $P(\psi) = P^2(\varphi)$ with $\|P(\psi)^+\| \ge q\|\psi^+\| \ge q^2\|\varphi^+\|$, and $P(\psi)$ again allows application of that lemma, in case $P(\psi) \in \mathcal{B}_2$. As long as this iteration is possible, we obtain a sequence $P^j(\varphi), j = 1, 2, \ldots$ with exponentially growing E^+- component. Thus there must exist a $j \in \mathbb{N}$ such that $P^j(\varphi)$ is defined, but not in $\mathcal{B}_2$, which implies the assertion. □

Remark 6.3.

1. In the above proof the manifold $\mathcal{M}_4^+$ was needed only to satisfy the first condition in (6.1) in the beginning – it is then preserved under iteration. The proof also shows that for nonzero $\varphi \in \mathcal{B}_2 \cap \mathcal{M}_4^+$, the corresponding trajectory has to leave the ball B (not only $\mathcal{B}_2$).

2. It would be interesting to know if in the situation of Theorem 6.2 solutions can stay in small C^0-neighborhood of zero, with only the C^2-norm growing such that the ball B_2 from above is left; for example, solutions with segments x_t even going to zero in the C^0-norm, but the C^2-norm growing (which would require rapid oscillations). We do at present not have an example.

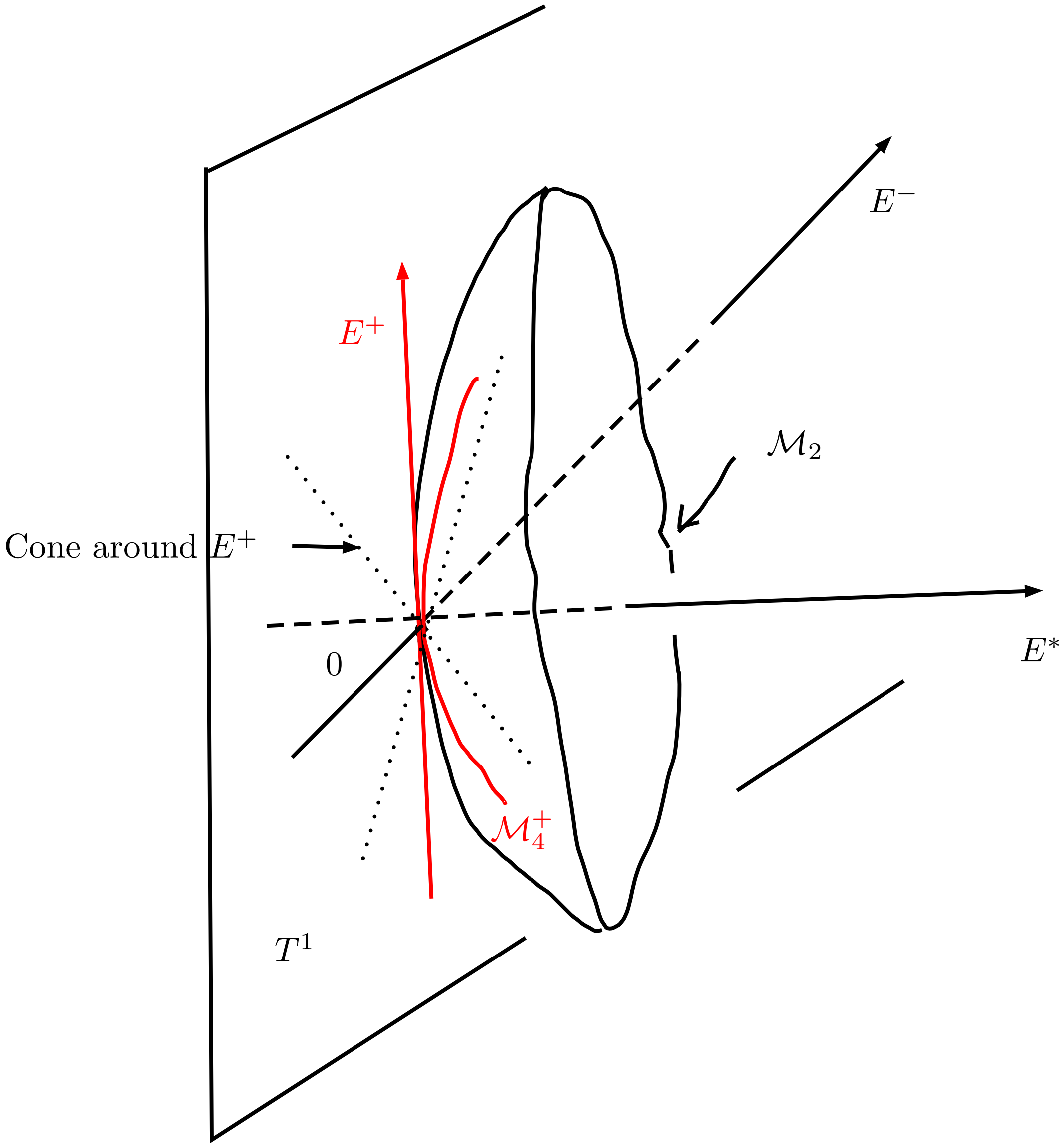

Figure 6.1: Symbolic illustration of some of the geometric objects

7 Application to examples

We show that generalizations of the mechanical example from [38] fit into our framework. In [38], the system below was considered as a model for hybrid experimental testing of a mechanical system, built by suspending a pendulum on a mass-spring-damper (MSD) system. The 'hybrid' testing consists of replacing the MSD by a computer simulation plus an actuator, which exerts the calculated force upon the pendulum, and is the source of delay in the system. $y(t)$ describes the (calculated, vertical) motion of the MSD system, while $\theta(t)$ describes the (angular) motion of the pendulum, see Fig. 1 in [38].

The equations used were (in the absence of external forcing)

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) + m\ddot{y}(t-\tau) + m\ell[\ddot{\theta}(t-\tau)\sin(\theta(t-\tau)) + \dot{\theta}^2(t-\tau)\cos(\theta(t-\tau))] = 0,$$
$$m\ell^2\ddot{\theta}(t-\tau) + \kappa\dot{\theta}(t-\tau) + mg\ell\sin(\theta(t-\tau)) + m\ell\ddot{y}(t-\tau)\sin(\theta(t-\tau)) = 0.$$

This corresponds to eq. (2.2) on p. 1274 in [38] with $k = 0$, with positive constants M, C, K, m, ℓ, κ. Here C and κ are friction coefficients, M and m are the masses of the MSD system and the pendulum, and ℓ is the pendulum length. The terms with a factor m in the first equation represent the inertial reaction force, the force from the angular acceleration, and from the radial acceleration of the pendulum mass, in this order. In the second equation, the first three terms correspond to the pendulum with fixed point of suspension, and the last term represents the force coming from the (in this case, simulated) MSD system. Obviously, $t-\tau$ may be replaced by t in all terms of the second equation.

So far, the delay τ is a fixed number, but one can imagine situations where it is state-dependent and of the form $\tau = \tau(y(t), \theta(t), \dot{y}(t), \dot{\theta}(t))$, with a maximal value $h > 0$ and a minimal value $\Delta \in (0, h]$. In addition, the coupling terms with delayed derivatives may involve nonlinearities, for example present in the devices providing measurements to the simulating computer. Then, rewriting the above system as a four-dimensional system of first order, one could for example obtain

$$\begin{cases} \dot{y}(t) = v(t) \\ \dot{\theta}(t) = \omega(t) \\ M\dot{v}(t) = -Cv(t) - Ky(t) - mf_1(\dot{v}(t-\tau)) \\ \qquad\qquad - m\ell f_2[\dot{\omega}(t-\tau)\sin(\theta(t-\tau)), \omega^2(t-\tau)\cos(\theta(t-\tau))], \\ m\ell^2\dot{\omega}(t) = -\kappa\omega(t) - mg\ell\sin(\theta(t)) - m\ell\dot{v}(t)\sin(\theta(t)), \end{cases} \tag{7.1}$$

with suitably smooth functions $\tau = \tau[y(t), \theta(t), v(t), \omega(t)]$ and $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \mathbb{R}^2 \to \mathbb{R}$, with $f_1(0) = f_2(0,0) = 0$. The function f_2 would be irrelevant for the linear approximation, and we can assume $f_1'(0) = 1$. (The terms $-Cv(t) - Ky(t)$ could also be replaced by corresponding nonlinear terms with derivative $-C$ and $-K$ at zero; we do not pursue this obvious generalization.)

System (7.1) is then an equation of the class as described in equation (2.5), with dimension $n = 4$. Formal linearization of that system at the zero solution $y(t) = \theta(t) = v(t) = \omega(t) = 0$ in the sense of Remark 2.2 (using the 'frozen delay principle') gives a linear system, with y and v decoupled from θ, ω, since all terms in (2.5) coupling these variables are of second order. The y-equation of that system, written again as second order equation, is the neutral equation with constant delay

$$M\ddot{y}(t) + C\dot{y}(t) + Ky(t) + m\ddot{y}(t-\tau_0) = 0, \tag{7.2}$$

where $\tau_0 = \tau(0,0,0,0)$ (the value of the state-dependent delay at $(0,0,0,0) \in \mathbb{R}^4$). The second equation of the linearized system is just the equation of a harmonic oscillator with friction, hence contributes only to the stable part of the spectrum and will not be considered.

The first-order version of eq. (7.2) is

$$\begin{cases} \dot{y}(t) = v(t) \\ \dot{v}(t) = \dfrac{1}{M}[-Cv(t) - Ky(t) - m \cdot \dot{v}(t-\tau_0)] \end{cases} \tag{7.3}$$

and generates a semigroup S^0 on $C^0([-h,0],\mathbb{R}^2)$. In analogy to eq. (3.1) the last equation can be rewritten as

$$\frac{d}{dt}\left[\begin{pmatrix} y \\ v \end{pmatrix}(t) + \begin{pmatrix} 0 & 0 \\ 0 & m/M \end{pmatrix} \cdot \begin{pmatrix} y \\ v \end{pmatrix}(t-\tau_0)\right] = \begin{pmatrix} 0 & 1 \\ -K/M & -C/M \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix}(t). \tag{7.4}$$

We see from Lemma 3.2 that the spectrum of its infinitesimal generator A consists only of isolated eigenvalues of finite multiplicity, and that these eigenvalues coincide with the solutions of the characteristic equation obtained from the exponential ansatz $y(t) = \exp(\lambda t) \cdot y(0)$ for solutions of equation (7.2). As in [38], we introduce the positive parameters

$$\omega_2 := \sqrt{K/M}, \qquad p := m/M, \qquad \zeta := \frac{C}{2\sqrt{MK}}, \qquad \hat{\tau} := \omega_2 \tau_0 \tag{7.5}$$

and set $z(t) := y(t/\omega_2)$. Then equation (7.2) is equivalent to the equation

$$\ddot{z}(t) + 2\zeta\dot{z}(t) + z(t) + p\ddot{z}(t-\hat{\tau}) = 0, \tag{7.6}$$

and the characteristic equation associated to the latter is

$$\chi(\lambda) := \lambda^2 + 2\zeta\lambda + 1 + p\lambda^2 \exp(-\lambda\hat{\tau}) = 0 \quad \text{(eq. (3.3) in [38])}. \tag{7.7}$$

This equation is analyzed in detail in [38], with a number of precursors, e.g. [4], [5] and [9]. We repeat some results from [38], adding additional pieces of information. For a nonzero complex number w we denote by $\arg(w)$ the unique angle $\varphi \in [0, 2\pi)$ with $w = |w| \exp(i\varphi)$.

From now on we make the following assumptions on the parameters:

$$p < 1, \quad \zeta < 1/\sqrt{2} \quad \text{and} \quad 1 - p^2 < (1 - 2\zeta^2)^2, \tag{7.8}$$

so that with the abbreviations $z := 1 - 2\zeta^2$, $q := 1 - p^2$ we have

$$z > 0, \qquad q < z^2. \tag{7.9}$$

First we show that for fixed parameters C, M, m, K, and hence for fixed ω_2, p and ζ, the following is true: For all small enough $\tau_0 > 0$, all zeroes of χ have negative real part. This is natural because equation (7.6) for $\tau_0 = 0$ is a harmonic oscillator with friction (see the corresponding remark after formula (3.3) on p. 1275 of [38]). However, the perturbation from delay zero to positive delay is not completely harmless, so we include a proof here.

Lemma 7.1. *For $\tau_0 > 0$ close enough to zero, all zeroes of χ have negative real part.*

Proof. Choose $R_1 > 0$ such that
$R_1(R_1(1-p)-2\zeta)-1>0$. If $\mathrm{Re}(\lambda)\geq 0$ and $|\lambda|>R_1$ then for all $\tau\geq 0$ one has

$$|\chi(\lambda)| \geq |\lambda|^2(1-p) - 2\zeta|\lambda| - 1 \geq R_1(R_1(1-p)-2\zeta)-1>0.$$

Choose $r_1 \in (0, R_1)$ such that $1-[r_1^2(1+p)+2\zeta r_1]>0$. If $\mathrm{Re}(\lambda)\geq 0$ and $|\lambda|<r_1$ then for all $\tau \geq 0$

$$|\chi(\lambda)| \geq 1-[r_1^2(1+p)+2\zeta r_1]>0.$$

On the compact set $K_1 := \left\{\lambda\in\mathbb{C} \;\middle|\; \mathrm{Re}(\lambda)\geq 0,\ r_1\leq|\lambda|\leq R_1\right\}$ the function χ converges uniformly to the function χ^* given by $\chi^*(\lambda)=\lambda^2(1+p)+2\zeta\lambda+1$ as $\hat{\tau}\to 0$, and hence also as $\tau_0\to 0$. The zeroes of χ^* have negative real parts (depending on p and ζ), so it follows that for all sufficiently small τ_0 the characteristic function χ also has no zeroes in K_1, and hence no zeroes in the closed right half plane. □

Lemma 7.2. *Assume the inequalities* (7.8)*. Then*

(i) $\lambda = i\omega$ *is a purely imaginary zero of the characteristic function* χ *with* $\omega>0$ *if and only if* $\omega\hat{\tau}=\arg(1-\omega^2-2i\zeta\omega)+2\pi n$ *for some* $n\in\mathbb{N}_0$*, and* $(1-p^2)\omega^4+(4\zeta^2-2)\omega^2+1=0$.

(ii) *In the situation of* (i) *one has* $\chi'(i\omega)\neq 0$*, so that the eigenvalue* $\lambda=i\omega$ *can be locally continued as a* C^1 *function of the parameters, in particular, of* $\hat{\tau}$.

(iii) *The second equation in* (i) *has exactly two positive solutions* $\omega_+>\omega_->0$ *(depending on p and* ζ*, but not on* $\hat{\tau}$*), and* $\mathrm{Re}(\lambda(\cdot))$ *has a positive derivative with respect to* $\hat{\tau}$ *at* $\hat{\tau}$*, if* $\lambda(\hat{\tau})=i\omega_+$*, and a negative derivative if* $\lambda(\hat{\tau})=i\omega_-$.

(iv) *In the situation of* (iii)*, the angles*

$$\varphi_\pm := \arg(1-\omega_\pm^2-2i\zeta\omega_\pm) \quad \textit{are equal to} \quad 2\pi-\arccos\left(\frac{1-\omega_\pm^2}{p\omega_\pm^2}\right)$$

and both contained in $(\pi,2\pi)$*, with* $\varphi_->\varphi_+$*. The corresponding* $\hat{\tau}$*-values obtained from the first equation in* (i) *are*

$$\tau_\pm(n) := \frac{\varphi_\pm+2\pi n}{\omega_\pm}, \qquad n=0,1,2,\ldots$$

Proof. Ad (i): For $\omega>0$, $\chi(i\omega)=0$ is equivalent to $-\omega^2+2\zeta i\omega+1-p\omega^2\exp(-i\omega\hat{\tau})=0$, and hence to

$$\omega\hat{\tau} = \underbrace{\arg(1-\omega^2-2i\zeta\omega)}_{\in(\pi,2\pi)}+2\pi n, \qquad n\in\{0,1,2,3,\ldots\}, \quad \text{and} \tag{7.10}$$

$$p^2\omega^4=(1-\omega^2)^2+4\zeta^2\omega^2, \quad \text{or} \quad (1-p^2)\omega^4+(4\zeta^2-2)\omega^2+1=0. \tag{7.11}$$

Ad (ii): We have for $\lambda\in\mathbb{C}$

$$\begin{aligned}\chi'(\lambda) &= 2\lambda+2\zeta+2\lambda p\exp(-\lambda\hat{\tau})-\hat{\tau}p\lambda^2\exp(-\lambda\hat{\tau})\\ &= 2(\lambda+\zeta)+p\lambda(2-\lambda\hat{\tau})\exp(-\lambda\hat{\tau}).\end{aligned} \tag{7.12}$$

If $\chi(\lambda) = 0$ then $\lambda \neq 0$ and $p\lambda^2 \exp(-\lambda\hat{\tau}) = -(\lambda^2 + 2\zeta\lambda + 1)$, and hence $p\lambda \exp(-\lambda\hat{\tau}) = -(\lambda^2 + 2\zeta\lambda + 1)/\lambda$ and $\chi'(\lambda) = 2(\lambda + \zeta) - \frac{(\lambda^2+2\zeta\lambda+1)(2-\lambda\hat{\tau})}{\lambda}$, so if also $\chi'(\lambda) = 0$ then

$$2(\lambda^2 + \lambda\zeta) = (\lambda^2 + 2\zeta\lambda + 1)(2 - \lambda\hat{\tau}), \quad \text{and hence}$$

$$0 = 2\lambda\zeta + 2 - \lambda\hat{\tau}(\lambda^2 + 2\zeta\lambda + 1).$$

In particular, if this would occur for $\lambda = i\omega$ then (from the real part) $0 = 2 + 2\zeta\omega^2\hat{\tau}$, which is impossible. Thus $\chi'(i\omega) \neq 0$ if $\chi(i\omega) = 0$; the remaining statement follows from the implicit function theorem.

Ad (iii): Writing u for ω^2, equation (7.11) gives $(1 - p^2)u^2 + (4\zeta^2 - 2)u + 1 = 0$. With the notation from (7.9), we obtain the solutions

$$u_\pm = \frac{1 - 2\zeta^2 \pm \sqrt{(1-2\zeta^2)^2 - (1-p^2)}}{1 - p^2} = \frac{z \pm \sqrt{z^2 - q}}{q}, \tag{7.13}$$

and thus the corresponding two solutions $\omega_\pm = \sqrt{u_\pm}$ with

$$0 < \omega_- < \omega_+.$$

In view of (ii), if $\chi(i\omega_*) = 0$ (where $* = +$ or $* = -$) for some values of the parameters $\hat{\tau}, \zeta$ and p, then, in particular, this eigenvalue can be locally viewed as a C^1 function of $\hat{\tau}$, so we can consider $\frac{d}{d\hat{\tau}} \operatorname{Re}(\lambda(\hat{\tau}))$. The assertion that the sign of this expression coincides with $*$ is contained in [38] (proof of Lemma 3.2, p. 1277 there), with the details of the calculation omitted, and with a misprint ($e^{-\lambda\tau}$ instead of $e^{\lambda\tau}$) in the formula for $\left(\frac{d\lambda}{d\tau}\right)^{-1}$. Therefore we show the main steps, and for this purpose we omit the hat in the symbol $\hat{\tau}$. Whenever an eigenvalue $\lambda(\neq 0)$ satisfies $\chi'(\lambda) \neq 0$ and is hence locally a unique C^1 function of τ, differentiation of the characteristic equation gives

$$0 = \chi'(\lambda)\frac{d\lambda}{d\tau} + p\lambda^2(-\lambda)e^{-\lambda\tau}, \quad \text{so} \quad \frac{d\lambda}{d\tau} = \frac{p\lambda^3 e^{-\lambda\tau}}{\chi'(\lambda)}.$$

Then, using (7.12), one gets

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2(\lambda+\zeta) + p\lambda(2-\tau\lambda)e^{-\lambda\tau}}{p\lambda^3 e^{-\lambda\tau}} = 2\frac{(\lambda+\zeta)e^{\lambda\tau} + p\lambda}{p\lambda^3} - \frac{\tau}{\lambda} \\ &= 2\left[(\lambda+\zeta)\frac{e^{\lambda\tau}}{p\lambda^3} + \frac{1}{\lambda^2}\right] - \frac{\tau}{\lambda} \quad \text{(compare also formula (3.9), p. 75 in [37]).}\end{aligned}$$

Since for a complex number $w \neq 0$ one has $\operatorname{sign}(\operatorname{Re}(w)) = \operatorname{sign}(\operatorname{Re}(w^{-1}))$, and since for $\lambda = i\omega$ the term τ/λ is imaginary, we get (omitting the factor 2)

$$\operatorname{sign}\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)|_{\lambda=i\omega} = \operatorname{sign}\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}|_{\lambda=i\omega} = \operatorname{sign}\operatorname{Re}\left((\lambda+\zeta)\frac{e^{\lambda\tau}}{p\lambda^3} + \frac{1}{\lambda^2}\right)|_{\lambda=i\omega}.$$

Substituting $e^{\lambda\tau}$ by $\frac{-p\lambda^2}{\lambda^2+2\zeta\lambda+1}$ (from the characteristic equation), inserting $\lambda = i\omega$, and multiplying by $\omega > 0$, the last expression is transformed to

$$\operatorname{sign}\left\{\frac{-\omega(1-\omega^2) + 2\zeta^2\omega}{(1-\omega^2)^2 + 4\zeta^2\omega^2} - \frac{1}{\omega}\right\}.$$

In view of eq. (7.11), the denominator of the first fraction equals $p^2\omega^4$ (if $\chi(i\omega) = 0$), and so multiplying with this factor one obtains

$$\operatorname{sign}\{-\omega(1-\omega^2) + 2\zeta^2\omega - p^2\omega^3\} = \operatorname{sign}\{2\zeta^2 - 1 + \omega^2(1-p^2)\}.$$

According to whether $\omega = \omega_+$ or $\omega = \omega_-$, the last expression equals in the notation of (7.13) $\operatorname{sign}\{-z + q \cdot u_\pm\} = \operatorname{sign}\{\pm\sqrt{z^2 - q}\}$. It is now obvious that this sign is positive for $\omega = \omega_+$ and negative for $\omega = \omega_-$.

Ad (iv): Since the imaginary parts are negative, the angles $\varphi_\pm = \arg(1 - \omega_\pm^2 - 2i\zeta\omega_\pm)$ are given by (consider the antipodal complex numbers)

$$\varphi_\pm = \arccos\left(\frac{\omega_\pm^2 - 1}{\sqrt{(\omega_\pm^2 - 1)^2 + 4\zeta^2\omega_\pm^2}}\right) + \pi \in (\pi, 2\pi),$$

which in view of equation (7.11) coincides with

$$\varphi_\pm = \arccos\left(\frac{\omega_\pm^2 - 1}{p\omega_\pm^2}\right) + \pi = 2\pi - \arccos\left(\frac{1 - \omega_\pm^2}{p\omega_\pm^2}\right).$$

(Compare [38], the passage after formula (3.7) on p. 1276 there.) Now since arccos is strictly decreasing and $\frac{d}{du}\left[\frac{u-1}{pu}\right] = \frac{1}{pu^2} > 0$, we see that $\omega_- < \omega_+$ implies $\varphi_+ < \varphi_-$. The assertion on the corresponding $\hat{\tau}$-values is clear. □

Corollary 7.3. *Assume in addition to (7.8) the followimg condition (which is more restrictive than the second inequality of (7.9)):*

$$\frac{q}{z^2} < \frac{17}{81}, \quad \textit{i.e.,} \quad \frac{1-p^2}{(1-2\zeta^2)^2} < \frac{17}{81}. \tag{7.14}$$

Then the numbers $\tau_\pm(n)$ from Lemma 7.2 satisfy

$$\tau_+(n) < \tau_+(n+1) < \tau_-(n) < \tau_-(n+1) \qquad (n \in \mathbb{N}_0),$$

where the first and last inequality are true independently of (7.14).

Proof. $\frac{q}{z^2} < \frac{17}{81}$ implies $\sqrt{1 - \frac{q}{z^2}} > \sqrt{\frac{64}{81}} = 8/9$, and hence

$$\frac{u_+}{u_-} = \frac{z + \sqrt{z^2 - q}}{z - \sqrt{z^2 - q}} = \frac{1 + \sqrt{1 - q/z^2}}{1 - \sqrt{1 - q/z^2}} > \frac{1 + 8/9}{1 - 8/9} = 17,$$

so with $\varphi_\pm \in (\pi, 2\pi)$ one sees that

$$\frac{\omega_+}{\omega_-} = \sqrt{\frac{u_+}{u_-}} > 4 = \frac{4\pi}{\pi} > \frac{\varphi_+ + 2\pi}{\varphi_-},$$

or $\omega_-(\varphi_+ + 2\pi) < \omega_+\varphi_-$, which in view of $0 < \omega_- < \omega_+$ implies for $n \in \mathbb{N}_0$

$$\omega_-\varphi_+ + \omega_- 2\pi + \omega_- 2\pi n < \omega_+\varphi_- + \omega_+ 2\pi n.$$

Thus we obtain

$$\omega_-(\varphi_+ + 2\pi(n+1)) < \omega_+(\varphi_- + 2\pi n), \quad \text{or}$$

$$\tau_+(n+1) = \frac{\varphi_+ + 2\pi(n+1)}{\omega_+} < \frac{\varphi_- + 2\pi n}{\omega_-} = \tau_-(n).$$

The remaining two inequalities are obvious since $\tau_\pm(n) = \frac{\varphi_\pm + 2\pi n}{\omega_\pm}$. □

Corollary 7.4 (Instability)**.** *Under the conditions of Corollary 7.3, the total number $N_+(\hat\tau)$ of zeroes of χ in the right half plane (counted with multiplicity) is even and satisfies*

$$N_+(\hat\tau) \geq 2 \text{ if } \hat\tau > \tau_+(0) = \frac{\varphi_+}{\omega_+}.$$

Proof. We know from Lemma 7.1 that for $\tau_0 > 0$ close to zero, and correspondingly, $\hat\tau > 0$ close to zero, all zeroes lie in the left half plane. If we keep p and ζ satisfying the conditions of Lemma 7.2 fixed and increase $\hat\tau$ from zero to positive values, we obtain the following from Lemma 7.2: At every value $\tau_+(n)$ $(n \in \mathbb{N}_0)$ a simple eigenvalue (and its conjugate) cross the imaginary axis from left to right at $\pm i\omega_+$, and at every value $\tau_-(n)$ $(n \in \mathbb{N}_0)$ a zero and its conjugate cross the imaginary axis from right to left at $\pm\omega_-$, and these are the only $\hat\tau$-values where such crossings happen. Corollary 7.3 shows that, in particular, $\tau_+(0) < \tau_-(0)$. If $\hat\tau \in (\tau_+(0), \tau_-(0))$ then $N_+(\hat\tau) \geq 2$. For $\hat\tau \geq \tau_-(0)$ the set $\left\{n \in \mathbb{N}_0 \mid \tau_-(n) \leq \hat\tau\right\}$ is not empty, and Corollary 7.3 shows that it is contained in the set $\left\{n \in \mathbb{N}_0 \mid \tau_+(n) \leq \hat\tau\right\}$. We can thus define $c_-(\hat\tau) := \max\left\{n \in \mathbb{N}_0 \mid \tau_-(n) \leq \hat\tau\right\}$. Then one sees from Corollary 7.3 that the number $c_+(\hat\tau) := \max\left\{n \in \mathbb{N}_0 \mid \tau_+(n) \leq \hat\tau\right\}$ satisfies $c_+(\hat\tau) \geq c_-(\hat\tau) + 1$, and hence we have

$$N_+(\hat\tau) = 2(c_+(\hat\tau) - c_-(\hat\tau)) \geq 2.$$ □

Remark 7.5. The situation described in Corollary 7.4 corresponds to p-values larger than p_1 in Figure 2 on p. 1276 of [38], and to $\hat\tau$-values larger than $\tau_+(0)$ (calculated for p and ζ with (7.8) and (7.14)), so that the point $(\hat\tau, p)$ lies in the non-shaded region of Figure 2 of [38]. The lower estimate for p corresponding to condition (7.14) is explicit, but will be larger than p_1 from [38].

For the statement of the theorem below we recollect the assumptions on the parameters, expressing them in a fashion slightly closer to the original parameters. Recall that $\hat\tau = \omega_2\tau_0 = \sqrt{K/M}\tau_0$. In this notation, the assumptions made above read as follows:

$$m < M, \quad \text{and that} \quad q := 1 - \left(\frac{m}{M}\right)^2 \quad \text{and} \quad z := 1 - \frac{C^2}{2MK} \quad \text{satisfy} \quad z > 0 \quad \text{and} \quad \frac{q}{z^2} < \frac{17}{81}.$$

Further, setting

$$u_+ := \frac{z + \sqrt{z^2 - q}}{q} \quad \text{and} \quad \varphi_+ := \arccos\left[\frac{M(u_+ - 1)}{m u_+}\right] + \pi$$

we assume that with the delay function τ one has

$$\tau_0 = \tau(0,0,0,0) > \frac{\varphi_+}{\sqrt{u_+}\sqrt{K/M}}.$$

Theorem 7.6. *Consider system (7.1) with f_1, f_2 and the delay function τ of class C^2, and with $f_1(0) = f_2(0) = 0, f_1'(0) = 1$. Also assume the above conditions on the parameter values. Then the spectrum of the infinitesimal generator of the semigroup generated by eq. (7.3) splits as required in Theorem 6.2, with an even number of eigenvalues in the right half plane. Hence the zero solution of system (7.1) is then unstable as described in that theorem.*

Proof. We see from Corollary 7.4 that under the given assumptions we can split the zeroes of χ in the ones with real part less or equal zero, and the even nonzero number of zeroes with real part larger than, e.g., $\tilde{\beta} := \frac{1}{2} \min \left\{ \operatorname{Re}(\lambda) \mid \chi(\lambda) = 0, \operatorname{Re}(\lambda) > 0 \right\}$. Due to the time rescaling in going from eq. (7.3) to eq. (7.6), the eigenvalues of the generator of the semigroup S^0 differ from the zeros of χ only by the factor $\omega_2 = \sqrt{K/M}$, and hence allow a splitting as required in Theorem 6.2 with $\alpha := 0$ and $\beta := \omega_2 \tilde{\beta}$. Next, by restricting to a suitable neighborhood, we can assume that τ takes values only in an interval of the form $[\Delta, h]$, where $0 < \Delta < \tau_0 < h$. Also, in view of Prop 2.1 b), system (7.1) fits in the framework of Theorem 6.2, from which the result follows. □

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Multiple positive solutions for a fractional Kirchhoff type equation with logarithmic and singular nonlinearities

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Abstract. In this paper, we study the following fractional Kirchhoff type equation

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u = |u|^{q-2}u\ln|u|^2+\frac{\lambda}{u^\gamma}, & \text{in } \Omega,\\ u>0, & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $0<s<1<p$, $0<\gamma<1$, $a>0, b\geq 0$, $N>ps$, $2p<q<q+2<p_s^*$, $p_s^*=\frac{Np}{N-ps}$ is the fractional critical exponent, $\lambda>0$ is a real parameter. By using the critical point theory for nonsmooth functionals and analytic techniques, the existence and multiplicity of positive solutions are obtained.

Keywords: Kirchhoff type equation, singular nonlinearity, positive solution.

2020 Mathematics Subject Classification: 35J20, 35B33, 58E05.

1 Introduction and main result

We consider the following fractional Kirchhoff type equation involving singular nonlinearity

$$\begin{cases} \left(a+b\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u = |u|^{q-2}u\ln|u|^2+\frac{\lambda}{u^\gamma}, & \text{in } \Omega,\\ u>0, & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $0<s<1<p$, $0<\gamma<1$, $a>0, b\geq 0$, $N>ps$, $2p<q<q+2<p_s^*$, $p_s^*=\frac{Np}{N-ps}$ is the fractional critical exponent, $\lambda>0$ is a real parameter.

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Problem (1.1) was proposed by Kirchhoff in [12] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings, which have the following stationary analogue of the Kirchhoff equation

$$-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=f(x,u).$$

Recently a great attention has been focused on studying the fractional problems, which are derived from the study of optimization, finance, phase transitions, stratified materials, anomalous diffusion, ultra-relativistic limits of quantum mechanics, water waves and so on, we can see [19] for more details. Many authors are interested in the existence of solutions for the fractional Kirchhoff type equation with logarithmic or singular terms. In [6], the authors dealt with the fractional p-Laplacian Choquard logarithmic equation involving critical and subcritical nonlinearities, they proved the existence and multiplicity of nontrivial solutions by using genus theory and the mountain pass lemma. Fan et al. in [7, 8] studied the fractional critical Schrödinger equation with logarithmic nonlinearity, by applying the Nehari manifold and the variational methods, the existence of positive ground state solutions and ground state sign-changing solutions were showed. Truong studied the fractional p-Laplacian equation with logarithmic nonlinearity on whole space, by the Nehari manifold method, the author obtained the existence of nontrivial solutions in [23].

In particular, the authors considered the following fractional Kirchhoff equation with logarithmic and critical nonlinearities

$$\begin{cases} M([u]^p_{s,p})(-\Delta)^s_p u=\lambda h(x)|u|^{q-2}u\ln|u|^2+|u|^{p^*_s-2}u, & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega,\end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $N>ps$ with $s\in(0,1)$, $p>1$ and

$$[u]^p_{s,p}=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy.$$

When $M([u]^p_{s,p})=a+b[u]^p_{s,p}$ and $h(x)=1$, by using constraint variational methods, Liang and Rădulescu in [15] dealt with the existence and least energy sign-changing solutions of (1.2). Under some assumptions on M and $p\geq 2$, the authors [14] obtained the existence of solutions in the case of high perturbations of (1.2) for λ sufficiently large. When $h(x)>0$, Lv and Zheng in [17] showed the existence of a nontrivial ground state solution for λ sufficiently small. When $M([u]^p_{s,p})=[u]^{(\theta-1)p}_{s,p}$ with $\theta\geq 1$, the authors [24] established the least energy solutions for (1.2) with $\theta p<q<p^*_s$ and $h(x)>0$ and two local least energy solutions with $1<q<\theta p$ and $h(x)$ is a sign-changing function by the Nehari manifold approach.

In [9], Fiscella and Mishra studied the following fractional Kirchhoff type equation with singular and critical growths

$$\begin{cases} M\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u=\lambda f(x)u^{-\gamma}+g(x)|u|^{2^*_s-2}u, & \text{in } \Omega,\\ u=0, & \text{in } \mathbb{R}^N\backslash\Omega,\end{cases} \tag{1.3}$$

where $N>2s$ with $s\in(0,1)$, $0<\gamma<1$, $2^*_s=\frac{2N}{N-2s}$ is the fractional critical Sobolev exponent, by the Nehari manifold method, they proved that (1.3) has at least two positive solutions for λ sufficiently small. In [21], by the variational methods and truncation arguments, the authors obtained the existence of multiple positive solutions for (1.3) with singular and Choquard critical nonlinearities. In addition, the existence of positive solutions for the fractional problems involving singular nonlinearity has been paid much attention by many authors, we can see [1,3,10,11,22,25,26] and so on.

Recently, Lei et al. in [13] investigated the following logarithmic elliptic equation with singular nonlinearity

$$\begin{cases} -\Delta u = u\log|u|^2 + \frac{\lambda}{u^\gamma}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $\gamma \in (0,1)$, by using the variational methods and the critical point theory for a nonsmooth functional, they obtained the existence of two positive solutions. In [20], the authors proved the existence of positive solutions for a logarithmic Schrödinger–Poisson system with singular nonlinearity.

Define the fractional Sobolev space $W^{s,p}(\Omega)$ is given by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy < \infty \right\},$$

with respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$

Let $Q = \mathbb{R}^{2N} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega$, we define

$$X = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable, } u|_\Omega \in L^p(\Omega) \text{ and } \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy < \infty \right\}.$$

The space X is endowed with the norm

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}},$$

where the norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. The space X_0 is defined as $X_0 = \{u \in X : u = 0 \text{ on } \mathcal{C}\Omega\}$, for all $p > 1$, it is a uniformly convex Banach space endowed with the norm

$$\|u\| := \|u\|_{X_0} = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy \right)^{\frac{1}{p}}. \tag{1.4}$$

The dual space of X_0 will be denoted by X_0^*. Since $u = 0$ in $\mathbb{R}^N \setminus \Omega$, the integral in (1.4) can be extended to $\mathbb{R}^N \times \mathbb{R}^N$. We denote by S_ρ (respectively, B_ρ) the sphere (respectively, the closed ball) of center zero and radius ρ, i.e. $S_\rho = \{u \in X_0 : \|u\| = \rho\}$, $B_\rho = \{u \in X_0 : \|u\| \leq \rho\}$.

Let S be the best fractional Sobolev constant

$$S = \inf_{u \in X_0 \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy}{\left(\displaystyle\int_\Omega |u|^{p_s^*} dx \right)^{p/p_s^*}}.$$

The energy functional associated with (1.1) has the form

$$I_\lambda(u) = \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} + \frac{2}{q^2}\int_\Omega |u|^q dx - \frac{1}{q}\int_\Omega |u|^q \ln |u|^2 dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma} dx.$$

Since the energy functional fails to be finite and loses C^1 smoothness on its natural Sobolev spaces, the classical critical point theory can not be applied directly, we overcome this hurdle by the critical point theory for nonsmooth functionals. Moreover, logarithmic nonlinearity is sign-changing, it becomes much more difficult than usual to obtain estimates of the energy functional. Our difficulties are as follows: (i) The singular term leads to the non-differentiability of the energy functional I_λ corresponding to (1.1) in [13]; (ii) The appearance of logarithmic and singular nonlinearities makes it more difficult for us to prove the convergence of the (PS) sequence; (iii) The fractional p-Laplacian operators also cause great difficulties for the existence of positive solutions.

Now we state our main result.

Theorem 1.1. *Assume that $0 < \gamma < 1$ and $2p < q < q+2 < p_s^*$ hold, there exists $\Lambda_0 > 0$ such that for all $\lambda \in (0, \Lambda_0)$, equation (1.1) has at least two positive solutions.*

2 Preliminaries

In this section, we first recall some concepts adapted from critical point theory for nonsmooth functionals in [4,16].

Definition 2.1. Let (Y, d) be a complete metric space, $f : Y \to \mathbb{R}$ be a continuous functional in Y. Denote by $|Df|(u)$ the supremum of κ in $[0, \infty)$ such that there exist $\delta > 0$ and a continuous map $\sigma : B_\delta(u) \times [0, \delta] \to Y$ satisfying

$$\begin{cases} f(\sigma(z,t)) \le f(z) - \kappa t, & (z,t) \in B_\delta(u) \times [0,\delta], \\ d(\sigma(z,t), z) \le t, & (z,t) \in B_\delta(u) \times [0,\delta]. \end{cases} \tag{2.1}$$

The extended real number $|Df|(u)$ is called the weak slope of f at u.

Definition 2.2. A sequence $\{u_n\}$ of Y is called (PS) sequence of the functional f, if $|Df|(u_n) \to 0$ as $n \to \infty$ and $f(u_n)$ is bounded. We say that $u \in Y$ is a critical point of f if $|Df|(u) = 0$. Since $u \to |Df|(u)$ is lower semicontinuous, any accumulation point of a (PS) sequence is clearly a critical point of f.

Since we are looking for positive solutions of (1.1), we consider the functional I_λ as defined on the closed positive cone P of X_0

$$P = \{u \mid u \in X_0,\ u(x) \ge 0,\ \text{a.e. } x \in \Omega\}.$$

P is a complete metric space and I_λ is a continuous functional on P. Then we have the following lemma.

Lemma 2.3. *Suppose that $|DI_\lambda|(u) < +\infty$ holds, then for all $v \in P$ such that*

$$\begin{aligned} &(a + b\|u\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[(v-u)(x)-(v-u)(y)]}{|x-y|^{N+ps}} dxdy \\ &\quad - \int_\Omega |u|^{q-2}u(v-u)\ln|u|^2 dx + |DI_\lambda|(u)\|v-u\| \ge \lambda \int_\Omega \frac{(v-u)}{u^\gamma} dx. \end{aligned} \tag{2.2}$$

Proof. Let $|DI_\lambda|(u) < \mu$, $\delta < \frac{1}{2}\|v-u\|$, $v \in P$ and $v \neq u$. Define the mapping $\sigma : B_\delta(u) \times [0,\delta] \to P$ by

$$\sigma(z,t) = z + t\frac{v-z}{\|v-z\|}.$$

Thus, we have $\|\sigma(z,t) - z\| = t$, by (2.1), there exists a pair $(z,t) \in B_\delta(u) \times [0,\delta]$ such that

$$I_\lambda(\sigma(z,t)) > I_\lambda(z) - \mu t.$$

Consequently, we assume that there exist sequences $\{u_n\} \subset P$ and $\{t_n\} \subset [0,\infty)$, such that $u_n \to u$, $t_n \to 0^+$, and

$$I_\lambda\left(u_n + t_n\frac{v-u_n}{\|v-u_n\|}\right) \geq I_\lambda(u_n) - \mu t_n,$$

that is

$$I_\lambda(u_n + s_n(v-u_n)) \geq I_\lambda(u_n) - \mu s_n\|v-u_n\|, \tag{2.3}$$

where $s_n = \frac{t_n}{\|v-u_n\|} \to 0^+$ as $n \to \infty$. Divided by s_n in (2.3), we have

$$\begin{aligned}
&\frac{a}{p}\frac{\|u_n + s_n(v-u_n)\|^p - \|u_n\|^p}{s_n} + \frac{b}{2p}\frac{\|u_n + s_n(v-u_n)\|^{2p} - \|u_n\|^{2p}}{s_n} \\
&\quad + \int_\Omega \frac{f(u_n + s_n(v-u_n)) - f(u_n)}{s_n}dx + \mu\|v-u_n\| \\
&\geq \frac{\lambda}{1-\gamma}\int_\Omega \frac{|u_n + s_n(v-u_n)|^{1-\gamma} - |u_n|^{1-\gamma}}{s_n}dx,
\end{aligned}$$

where

$$f(u_n) = \frac{2}{q^2}\int_\Omega |u_n|^q dx - \frac{1}{q}\int_\Omega |u_n|^q \ln|u_n|^2 dx.$$

Notice that

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \frac{f(u_n + s_n(v-u_n)) - f(u_n)}{s_n}dx \\
&= \lim_{n\to\infty}\frac{2}{q^2}\int_\Omega \frac{|u_n + s_n(v-u_n)|^q - |u_n|^q}{s_n}dx \\
&\quad - \lim_{n\to\infty}\frac{1}{q}\int_\Omega \frac{(|u_n + s_n(v-u_n)|^q - |u_n|^q)\ln|u_n + s_n(v-u_n)|^2}{s_n}dx \\
&\quad - \lim_{n\to\infty}\frac{1}{q}\int_\Omega \frac{|u_n|^q(\ln|u_n + s_n(v-u_n)|^2 - \ln|u_n|^2)}{s_n}dx \\
&= \frac{2}{q}\int_\Omega |u|^{q-2}u(v-u)dx - \int_\Omega |u|^{q-2}u(v-u)\ln|u|^2dx - \frac{2}{q}\int_\Omega |u|^{q-2}u(v-u)dx \\
&= -\int_\Omega |u|^{q-2}u(v-u)\ln|u|^2dx.
\end{aligned}$$

In fact, from [15], for all $r \in (q, p_s^*)$ and $2p < q < p_s^*$, we have that

$$\lim_{t\to 0}\frac{|t|^{q-1}\ln|t|^2}{|t|^{p-1}} = 0, \quad \text{and} \quad \lim_{t\to\infty}\frac{|t|^{q-1}\ln|t|^2}{|t|^{r-1}} = 0.$$

Then, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|t|^{q-1}\ln|t|^2 \leq \varepsilon|t|^{p-1} + C_\varepsilon|t|^{r-1}. \tag{2.4}$$

It follows from $u_n(x) \to u(x)$ a.e in Ω and $u_n \to |u_n|^q \ln |u_n|^2$ is continuous that

$$|u_n(x)|^q \ln |u_n(x)|^2 \to |u(x)|^q \ln |u(x)|^2, \quad \text{a.e. in } \Omega.$$

Thus, by the Lebesgue dominated convergence theorem and (2.4), we get

$$\int_\Omega |u_n|^q \ln |u_n|^2 dx \to \int_\Omega |u|^q \ln |u|^2 dx, \quad \text{as } n \to \infty.$$

Set

$$I_{1,n} = \int_\Omega \frac{|u_n + s_n(v - u_n)|^{1-\gamma} - |(1 - s_n)u_n|^{1-\gamma}}{s_n(1-\gamma)} dx,$$

and

$$I_{2,n} = \frac{(1-s_n)^{1-\gamma} - 1}{s_n(1-\gamma)} \int_\Omega |u_n|^{1-\gamma} dx.$$

Notice that

$$I_{1,n} = \int_\Omega \frac{\xi_n^{-\gamma} s_n v}{s_n} dx = \int_\Omega \xi_n^{-\gamma} v dx,$$

where $\xi_n \in (u_n - s_n u_n, u_n + s_n(v - u_n))$, which implies that $\xi_n \to u$ $(u_n \to u)$ as $s_n \to 0^+$. Since $I_{1,n} \geq 0$ for all n, by the Fatou lemma, we obtain that

$$\liminf_{n\to\infty} I_{1,n} \geq \int_\Omega \frac{v}{u^\gamma} dx,$$

for all $v \in P$. For $I_{2,n}$, by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} I_{2,n} = -\int_\Omega u^{1-\gamma} dx.$$

From the above information, we get

$$\begin{aligned}
&(a + b\|u\|^p) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))[(v-u)(x) - (v-u)(y)]}{|x-y|^{N+ps}} dxdy \\
&\quad - \int_\Omega |u|^{q-2} u(v-u) \ln |u|^2 dx + \mu \|v - u\| \\
&\geq \liminf_{n\to\infty} (I_{1,n} + I_{2,n}) \geq \lambda \int_\Omega \frac{(v-u)}{u^\gamma} dx,
\end{aligned}$$

for every $v \in P$. Since $|DI_\lambda|(u) < \mu$ is arbitrary. The proof is complete. □

Lemma 2.4. *Let $2p < q < q + 2 < p_s^*$, there exist constants $\alpha, \rho, \Lambda_0 > 0$, for all $\lambda \in (0, \Lambda_0)$. Then the functional I_λ satisfies the following conditions:*

(i) $I_\lambda|_{u\in S_\rho} \geq \alpha > 0$; $\inf_{u\in B_\rho} I_\lambda(u) < 0$;

(ii) *There exists $e \in X_0$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$.*

Proof. (i) Since $\ln|u|^2 \le |u|^2$, by the Hölder and Sobolev inequalities, we have

$$\begin{aligned} I_\lambda(u) &= \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} + \frac{2}{q^2}\int_\Omega |u|^q dx - \frac{1}{q}\int_\Omega |u|^q \ln|u|^2 dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\ &\ge \frac{a}{p}\|u\|^p - \frac{1}{q}\int_\Omega |u|^{q+2}dx - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\ &\ge \frac{a}{p}\|u\|^p - \frac{1}{q}|\Omega|^{\frac{p_s^*-q-2}{p_s^*}}\left(\int_\Omega |u|^{p_s^*}dx\right)^{\frac{q+2}{p_s^*}} - \frac{\lambda}{1-\gamma}|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}}\left(\int_\Omega |u|^{p_s^*}dx\right)^{\frac{1-\gamma}{p_s^*}} \\ &\ge \frac{a}{p}\|u\|^p - \frac{1}{q}|\Omega|^{\frac{p_s^*-q-2}{p_s^*}} S^{-\frac{q+2}{p}}\|u\|^{q+2} - \frac{\lambda}{1-\gamma}|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}} S^{-\frac{1-\gamma}{p}}\|u\|^{1-\gamma} \\ &= \|u\|^{1-\gamma}\left(\frac{a}{p}\|u\|^{p-1+\gamma} - \frac{1}{q}|\Omega|^{\frac{p_s^*-q-2}{p_s^*}} S^{-\frac{q+2}{p}}\|u\|^{q+1+\gamma} - \frac{\lambda}{1-\gamma}|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}} S^{-\frac{1-\gamma}{p}}\right). \end{aligned}$$

Set

$$h(t) = \frac{a}{p}t^{p-1+\gamma} - \frac{1}{q}|\Omega|^{\frac{p_s^*-q-2}{p_s^*}} S^{-\frac{q+2}{p}} t^{q+1+\gamma}$$

for $t > 0$, thus, there exists a constant

$$\rho = \left[\frac{aq(p-1+\gamma)S^{\frac{q+2}{p}}}{p(q+1+\gamma)|\Omega|^{\frac{p_s^*-q-2}{p_s^*}}}\right]^{\frac{1}{q+2-p}} > 0,$$

such that $\max_{t>0} h(t) = h(\rho) > 0$. Let

$$\Lambda_0 = \frac{h(\rho)(1-\gamma)S^{\frac{1-\gamma}{p}}}{|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}}},$$

thus, $I_\lambda|_{u\in S_\rho} \ge \alpha > 0$ for all $\lambda \in (0, \Lambda_0)$. Moreover, for $u \in X_0\backslash\{0\}$, we get

$$\lim_{t\to 0^+} \frac{I_\lambda(tu)}{t^{1-\gamma}} = -\frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx < 0.$$

Therefore, we obtain that $I_\lambda(tu) < 0$ for t small enough. Consequently, for $\|u\|$ small enough, we have

$$d \triangleq \inf_{u\in B_\rho} I_\lambda(u) < 0. \tag{2.5}$$

(ii) For all $u \in X_0\backslash\{0\}$ and $t > 0$, we have

$$\begin{aligned} I_\lambda(tu) &= \frac{at^p}{p}\|u\|^p + \frac{bt^{2p}}{2p}\|u\|^{2p} + \frac{2t^q}{q^2}\int_\Omega |u|^q dx - \frac{t^q}{q}\int_\Omega |u|^q \ln|tu|^2 dx \\ &\quad - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\ &= \frac{at^p}{p}\|u\|^p + \frac{bt^{2p}}{2p}\|u\|^{2p} + \frac{2t^q}{q^2}\int_\Omega |u|^q dx - \frac{2t^q}{q}\int_\Omega |u|^q \ln|u| dx \\ &\quad - \frac{2t^q}{q}\int_\Omega |u|^q \ln t dx - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \to -\infty \end{aligned}$$

as $t \to +\infty$, which implies that $I_\lambda(tu) < 0$ for $t > 0$ large enough. Thus, we can find $e \in X_0$ with $\|e\| > \rho$ such that $I_\lambda(e) < 0$. The proof is complete. □

Lemma 2.5. *Suppose that $2p < q < p^*$ and $0 < \gamma < 1$ hold, the functional I_λ satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset P$ be a (PS) sequence for I_λ at the level c, that is

$$I_\lambda(u_n) \to c, \quad \text{and} \quad |DI_\lambda|(u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.6}$$

It follows from (2.2) and (2.6) that

$$\begin{aligned}(a+b\|u_n\|^p)&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}}\\ &\quad\times[(v-u_n)(x)-(v-u_n)(y)]dxdy\\ &\quad-\int_\Omega |u_n|^{q-2}u_n(v-u_n)\ln|u_n|^2dx+|DI_\lambda|(u_n)\|v-u_n\|\\ &\geq \lambda\int_\Omega\frac{(v-u_n)}{u_n^\gamma}dx.\end{aligned} \tag{2.7}$$

Choosing $v = 2u_n \in P$ in (2.7), we obtain that

$$(a+b\|u_n\|^p)\|u_n\|^p - \int_\Omega |u_n|^q \ln|u_n|^2dx + |DI_\lambda|(u_n)\|u_n\| \geq \lambda\int_\Omega u_n^{1-\gamma}dx. \tag{2.8}$$

Combining with (2.6), (2.8) and the Hölder inequality, there exists a constant $C > 0$, we get

$$\begin{aligned}c+1+o(\|u_n\|) &\geq I_\lambda(u_n)+\frac{1}{q}|DI_\lambda|(u_n)\|u_n\|\\ &\geq a\left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p+b\left(\frac{1}{2p}-\frac{1}{q}\right)\|u_n\|^{2p}+\frac{2}{q^2}\int_\Omega|u_n|^qdx\\ &\quad-\lambda\left(\frac{1}{1-\gamma}-\frac{1}{q}\right)\int_\Omega|u_n|^{1-\gamma}dx\\ &\geq a\left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|^p-\lambda\left(\frac{1}{1-\gamma}-\frac{1}{q}\right)|\Omega|^{\frac{p_s^*-1+\gamma}{p_s^*}}S^{-\frac{1-\gamma}{p}}\|u_n\|^{1-\gamma}.\end{aligned}$$

Since $1-\gamma < 1 < p$, we deduce that $\{u_n\}$ is bounded in X_0. Therefore, we may assume up to a subsequence, still denoted by $\{u_n\}$, there exists $u \in X_0$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } X_0,\\ u_n \to u, & \text{strongly in } L^r(\Omega)\ (1 \leq r < p_s^*),\\ u_n(x) \to u(x), & \text{a.e. in } \Omega,\end{cases} \tag{2.9}$$

as $n \to \infty$. Taking $v = u_m$ in (2.7), we have

$$\begin{aligned}(a+b\|u_n\|^p)&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+ps}}\\ &\quad\times[(u_m-u_n)(x)-(u_m-u_n)(y)]dxdy\\ &\quad-\int_\Omega |u_n|^{q-2}u_n(u_m-u_n)\ln|u_n|^2dx+o(1)\|u_m-u_n\|\\ &\geq \lambda\int_\Omega\frac{(u_m-u_n)}{u_n^\gamma}dx.\end{aligned} \tag{2.10}$$

By changing the role of u_m and u_n in (2.10), we have a similar inequality. By adding the two inequalities, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y)) - |u_m(x)-u_m(y)|^{p-2}(u_m(x)-u_m(y))]}{|x-y|^{N+ps}} \\
&\quad \times [(u_n-u_m)(x)-(u_n-u_m)(y)]dxdy \\
&\leq \int_\Omega \left(\frac{|u_n|^{q-2}u_n \ln|u_n|^2}{a+b\|u_n\|^p} - \frac{|u_m|^{q-2}u_m \ln|u_m|^2}{a+b\|u_m\|^p} \right)(u_n-u_m)dx \\
&\quad + \lambda \int_\Omega \left(\frac{u_n^{-\gamma}}{a+b\|u_n\|^p} - \frac{u_m^{-\gamma}}{a+b\|u_m\|^p} \right)(u_n-u_m)dx + o(1)\|u_m-u_n\| \\
&\leq \int_\Omega \left(\frac{|u_n|^{q-2}u_n \ln|u_n|^2}{a+b\|u_n\|^p} - \frac{|u_m|^{q-2}u_m \ln|u_m|^2}{a+b\|u_m\|^p} \right)(u_n-u_m)dx + o(1)\|u_m-u_n\|.
\end{aligned} \tag{2.11}$$

With the help of (2.4), (2.9) and $\{u_n\}$ is bounded in X_0, for all $r \in (q, p_s^*)$, we have

$$\begin{aligned}
&\left| \int_\Omega \frac{|u_n|^{q-2}u_n \ln|u_n|^2}{a+b\|u_n\|^p}(u_n-u_m)dx \right| \\
&\quad \leq C \left| \int_\Omega |u_n|^{q-2}u_n \ln|u_n|^2 (u_n-u_m)dx \right| \\
&\quad \leq C\varepsilon \int_\Omega |u_n|^{p-1}|u_n-u_m|dx + C_\varepsilon \int_\Omega |u_n|^{r-1}|u_n-u_m|dx \\
&\quad \leq C\varepsilon \left(\int_\Omega |u_n|^p dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |u_n-u_m|^p dx \right)^{\frac{1}{p}} \\
&\qquad + C_\varepsilon \left(\int_\Omega |u_n|^r dx \right)^{\frac{r-1}{r}} \left(\int_\Omega |u_n-u_m|^r dx \right)^{\frac{1}{r}} \\
&\quad \leq C\varepsilon \|u_n-u_m\|_p + C_\varepsilon \|u_n-u_m\|_r \to 0,
\end{aligned} \tag{2.12}$$

as $n \to \infty$. By a similar calculation in (2.12), one has

$$\left| \int_\Omega \frac{|u_m|^{q-2}u_m \ln|u_m|^2}{a+b\|u_m\|^p}(u_n-u_m)dx \right| \leq C\varepsilon \|u_n-u_m\|_p + C_\varepsilon \|u_n-u_m\|_r \to 0, \tag{2.13}$$

as $n \to \infty$. It follows from (2.12) and (2.13) that

$$\lim_{n\to\infty} \int_\Omega \left(\frac{|u_n|^{q-2}u_n \ln|u_n|^2}{a+b\|u_n\|^p} - \frac{|u_m|^{q-2}u_m \ln|u_m|^2}{a+b\|u_m\|^p} \right)(u_n-u_m)dx = 0. \tag{2.14}$$

Therefore, by (2.11) and (2.14), we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y)) - |u_m(x)-u_m(y)|^{p-2}(u_m(x)-u_m(y))]}{|x-y|^{N+ps}} \\
&\quad \times [(u_n-u_m)(x)-(u_n-u_m)(y)]dxdy = 0.
\end{aligned} \tag{2.15}$$

Let us now recall the well-known Simon inequalities, for all $\xi, \zeta \in \mathbb{R}$ such that

$$|\xi-\zeta|^p \leq \begin{cases} c_p(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi-\zeta), & \text{for } p \geq 2, \\ C_p[(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi-\zeta)]^{\frac{p}{2}}(|\xi|^p+|\zeta|^p)^{\frac{2-p}{2}}, & \text{for } 1 < p < 2, \end{cases} \tag{2.16}$$

where $c_p, C_p > 0$ depending only on p. From which we distinguish two cases:
Case (i): if $p \geq 2$, it follows from (2.15) and (2.16) as $n \to \infty$ that

$$
\begin{aligned}
&\|u_n - u_m\|^p \\
&\quad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u_n-u_m)(x)-(u_n-u_m)(y)|^p}{|x-y|^{N+ps}}dxdy \\
&\quad \leq c_p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u_m(x)-u_m(y)|^{p-2}(u_m(x)-u_m(y))]}{|x-y|^{N+ps}} \\
&\qquad \times [(u_n-u_m)(x)-(u_n-u_m)(y)]dxdy \to 0.
\end{aligned}
$$

Case (ii): if $1 < p < 2$, since $\|u_n\|^p$ and $\|u_m\|^p$ are bounded in X_0, by the subadditivity inequality, for all $\xi, \zeta \geq 0$, we have

$$
(\xi + \zeta)^{\frac{2-p}{2}} \leq \xi^{\frac{2-p}{2}} + \zeta^{\frac{2-p}{2}}.
$$

Letting $\xi = u_n(x) - u_n(y)$ and $\zeta = u_m(x) - u_m(y)$ in (2.16) as $n \to \infty$, we obtain

$$
\begin{aligned}
&\|u_n - u_m\|^p \\
&\quad \leq C_p\Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u_m(x)-u_m(y)|^{p-2}(u_m(x)-u_m(y))]}{|x-y|^{N+ps}} \\
&\qquad \times [(u_n-u_m)(x)-(u_n-u_m)(y)]dxdy\Bigg]^{\frac{p}{2}} (\|u_n\|^p + \|u_m\|^p)^{\frac{2-p}{2}} \\
&\quad \leq C_p\Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u_m(x)-u_m(y)|^{p-2}(u_m(x)-u_m(y))]}{|x-y|^{N+ps}} \\
&\qquad \times [(u_n-u_m)(x)-(u_n-u_m)(y)]dxdy\Bigg]^{\frac{p}{2}} (\|u_n\|^{\frac{p(2-p)}{2}} + \|u_m\|^{\frac{p(2-p)}{2}}) \\
&\quad \leq C\Bigg[\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))-|u_m(x)-u_m(y)|^{p-2}(u_m(x)-u_m(y))]}{|x-y|^{N+ps}} \\
&\qquad \times [(u_n-u_m)(x)-(u_n-u_m)(y)]dxdy\Bigg]^{\frac{p}{2}} \to 0,
\end{aligned}
$$

where the constant $C > 0$. Thus, we can deduce that $u_n \to u$ in X_0. The proof is complete. □

Lemma 2.6. *If $|DI_\lambda|(u) = 0$, then u is a weak solution of (1.1). That is, $u^{-\gamma}\varphi \in L^1(\Omega)$ for all $\varphi \in X_0$ such that*

$$
\begin{aligned}
&(a + b\|u\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy \\
&\quad = \int_\Omega |u|^{q-2}u\varphi \ln|u|^2 dx + \lambda\int_\Omega u^{-\gamma}\varphi dx.
\end{aligned}
\tag{2.17}
$$

Proof. Since $|DI_\lambda|(u) = 0$, by Lemma 2.3, for all $v \in P$, we have

$$
\begin{aligned}
&(a + b\|u\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[(v-u)(x)-(v-u)(y)]}{|x-y|^{N+ps}}dxdy \\
&\quad - \int_\Omega |u|^{q-2}u(v-u)\ln|u|^2 dx \geq \lambda\int_\Omega \frac{(v-u)}{u^\gamma}dx.
\end{aligned}
\tag{2.18}
$$

Letting $t \in \mathbb{R}$, $\varphi \in X_0$, and taking $v = (u+t\varphi)^+ \in P$ in (2.18), for any $\varphi \in X_0$, we get

$$\begin{aligned}
0 \leq\ & (a+b\|u\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \\
& \times [((u+t\varphi)^+-u)(x)-((u+t\varphi)^+-u)(y)]dxdy \\
& -\int_\Omega |u|^{q-2}u((u+t\varphi)^+-u)\ln|u|^2dx-\lambda\int_\Omega\frac{((u+t\varphi)^+-u)}{u^\gamma}dx \\
\leq\ & t\Big[(a+b\|u\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}dxdy \\
& -\int_\Omega |u|^{q-2}u\varphi\ln|u|^2dx-\lambda\int_\Omega\frac{\varphi}{u^\gamma}dx\Big] \\
& -(a+b\|u\|^p)\int_{u+t\varphi<0}\int_{u+t\varphi<0}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} \\
& \times [(u+t\varphi)(x)-(u+t\varphi)(y)]dxdy \\
& +\int_{u+t\varphi<0}|u|^{q-2}u(u+t\varphi)\ln|u|^2dx+\lambda\int_{u+t\varphi<0}\frac{u+t\varphi}{u^\gamma}dx \\
\leq\ & t\Big[(a+b\|u\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}dxdy \\
& -\int_\Omega |u|^{q-2}u\varphi\ln|u|^2dx-\lambda\int_\Omega\frac{\varphi}{u^\gamma}dx\Big] \\
& -t(a+b\|u\|^p)\int_{u+t\varphi<0}\int_{u+t\varphi<0}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}dxdy \\
& +\int_{u+t\varphi<0}|u|^{q-2}u(u+t\varphi)\ln|u|^2dx.
\end{aligned}$$

Since $u(x) = 0$ for a.e. $x \in \Omega$ and

$$\text{meas}\{x\in\Omega|u(x)+t\varphi(x)<0,u(x)>0\}\to 0,\quad \text{as } t\to 0,$$

we have

$$\begin{aligned}
& (a+b\|u\|^p)\int_{u+t\varphi<0}\int_{u+t\varphi<0}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}dxdy \\
& = (a+b\|u\|^p)\int_{u+t\varphi<0,u>0}\int_{u+t\varphi<0,u>0}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}dxdy \\
& \to 0,
\end{aligned}$$

and

$$\int_{u+t\varphi<0}|u|^{q-2}u(u+t\varphi)\ln|u|^2dx=\int_{u+t\varphi<0,u>0}|u|^{q-2}u(u+t\varphi)\ln|u|^2dx\to 0,$$

as $t \to 0$. Therefore, we have that

$$\begin{aligned}
0\leq\ & t\Big[(a+b\|u\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}dxdy \\
& -\int_\Omega |u|^{q-2}u\varphi\ln|u|^2dx-\lambda\int_\Omega\frac{\varphi}{u^\gamma}dx\Big]+o(t).
\end{aligned}$$

Consequently, one has

$$(a+b\|u\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))[\varphi(x)-\varphi(y)]}{|x-y|^{N+ps}}dxdy$$
$$-\int_\Omega |u|^{q-2}u\varphi\ln|u|^2dx-\lambda\int_\Omega\frac{\varphi}{u^\gamma}dx\geq 0.$$

By the arbitrariness of the sign of φ, we can obtain that (2.18) holds. The proof is complete. □

3 Proof of Theorem 1.1

Theorem 3.1. *Suppose that $0<\lambda<\Lambda_0$ (Λ_0 is as in Lemma 2.4), then equation (1.1) has a positive solution u_* satisfying $I_\lambda(u_*)<0$.*

Proof. According to Lemma 2.4 and the definition of d in (2.5), there exists a minimizing sequence $\{u_n\}\subset B_\rho\subset P$ such that $\lim_{n\to\infty}I_\lambda(u_n)=d<0$. Obviously, $\{u_n\}$ is bounded in B_ρ, up to a subsequence, still denoted by $\{u_n\}$, there exists $u_*\in X_0$ such that

$$\begin{cases} u_n\rightharpoonup u_*, & \text{weakly in } X_0,\\ u_n\to u_*, & \text{strongly in } L^r(\Omega),\ 1\leq r<p_s^*,\\ u_n(x)\to u_*(x), & \text{a.e. in } \Omega,\end{cases}$$

as $n\to\infty$. Next, we prove that $u_n\to u_*$ as $n\to\infty$ in X_0. Let $w_n=u_n-u_*$, by the Brézis–Lieb lemma, there holds

$$\|u_n\|^p=\|w_n\|^p+\|u_*\|^p+o(1).$$

Therefore, by Lemma 2.5, we have

$$\begin{aligned} d&=\lim_{n\to\infty}I_\lambda(u_n)\\ &=I_\lambda(u_*)+\lim_{n\to\infty}\left[\frac{a}{p}\|w_n\|^p+\frac{b}{2p}(\|w_n\|^{2p}+2\|w_n\|^p\|u_*\|^p)\right]\\ &\geq I_\lambda(u_*)\geq d,\end{aligned}$$

which implies that $\|w_n\|\to 0$ as $n\to\infty$. Since B_ρ is closed and convex, we have $u_*\in B_\rho$. Thus, we can deduce that $I_\lambda(u_*)=d<0$, which implies that u_* is a local minimizer of I_λ and $u\not\equiv 0$ in Ω. For $v\in P$ and $t>0$ small enough such that $u_*+t(v-u_*)\in B_\rho$, similar to the proof of Lemma 2.6, we get

$$(a+b\|u_*\|^p)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_*(x)-u_*(y)|^{p-2}(u_*(x)-u_*(y))[(v-u_*)(x)-(v-u_*)(y)]}{|x-y|^{N+ps}}dxdy$$
$$-\int_\Omega |u_*|^{q-2}u_*(v-u_*)\ln|u_*|^2dx$$
$$\geq\lambda\int_\Omega\frac{(v-u_*)}{u_*^\gamma}dx.$$

Therefore, u_* is a critical point of I_λ, by Lemma 2.6, we obtain that $u_*\in P$ is a solution of (1.1) with $I_\lambda(u_*)=d<0$, which implies that $u_*\geq 0$ and $u_*\not\equiv 0$. We claim that

$$g(t)=2\ln t+\frac{\lambda}{t^{q-1+\gamma}}.$$

Notice that

$$\lim_{t\to 0^+} g(t) = +\infty, \quad \text{and} \quad \lim_{t\to +\infty} g(t) = +\infty.$$

Therefore, g achieves its minimum at

$$t_* = \left[\frac{\lambda(q-1+\gamma)}{2}\right]^{\frac{1}{q-1+\gamma}},$$

which implies that

$$\min_{t>0} g(t) = g(t_*) = \frac{2}{q-1+\gamma}\ln\frac{\lambda(q-1+\gamma)}{2} + \frac{2}{q-1+\gamma} \triangleq C.$$

Consequently, we obtain that

$$(-\Delta)^s_p u_* = \frac{1}{a+b\|u_*\|^p}\left(u_*^{q-1}\ln u_*^2 + \frac{\lambda}{u_*^{\gamma}}\right) \geq \frac{Cu_*^{q-1}}{a+b\|u_*\|^p} \geq 0,$$

where $a > 0, b \geq 0$. By using the strong maximum principle in [5,18], we deduce that $u_* \in P$ is a positive solution of (1.1). The proof is complete. □

Theorem 3.2. *Suppose that* $0 < \lambda < \Lambda_0$, *then equation* (1.1) *has a positive solution* v_* *such that* $I_\lambda(v_*) > 0$.

Proof. Applying the mountain pass lemma in [2] and Lemma 2.4, there exists a sequence $\{u_n\} \subset X_0$ such that

$$I_\lambda(u_n) \to c, \quad \text{and} \quad |DI_\lambda|(u_n) \to 0 \quad \text{as } n \to \infty,$$

where

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

and

$$\Gamma = \{\gamma \in C([0,1], X_0) : \gamma(0) = 0, \gamma(1) = e\}.$$

According to Lemma 2.5, we know that $\{u_n\} \subset X_0$ has a convergent subsequence, still denoted by $\{u_n\}$, we may assume that $u_n \to v_*$ in X_0 as $n \to \infty$, we have

$$I_\lambda(v_*) = \lim_{n\to\infty} I_\lambda(u_n) \geq \alpha > 0,$$

which implies that $v_* \not\equiv 0$. It is similar to Theorem 3.1 that $v_* > 0$, we obtain that v_* is a positive solution of equation (1.1) such that $I_\lambda(v_*) > 0$. Combining the above facts with Theorem 3.1 the proof of Theorem 1.1 is complete. □

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Structural stability for scalar reaction-diffusion equations

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Abstract. In this paper, we prove the structural stability for a family of scalar reaction-diffusion equations. Our arguments consist of using invariant manifold theorem to reduce the problem to a finite dimension and then, we use the structural stability of Morse–Smale flows in a finite dimension to obtain the corresponding result in infinite dimension. As a consequence, we obtain the optimal rate of convergence of the attractors and estimate the Gromov–Hausdorff distance of the attractors using continuous ε-isometries.

Keywords: Morse–Smale semiflows, rate of convergence of attractors, structural stability, invariant manifolds, Gromov–Hausdorff distance.

2020 Mathematics Subject Classification: 37D15, 34D30, 35B41, 35B42.

1 Introduction and statement of the results

The continuity of attractors is an important feature to study the stability of the semilinear evolution equations. For a family of attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ the continuity at $\varepsilon = 0$ means that the symmetric Hausdorff distance $d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \to 0$ as $\varepsilon \to 0$. The work [8] obtained positive results in the class of gradient systems, assuming structural conditions on the unperturbed attractor, together with information on the continuity of unstable manifolds of equilibria. In particular, if $\{u_*^\varepsilon\}_{\varepsilon\in[0,1]}$ is the family of equilibrium points then $d(u_*^\varepsilon, u_*^0) \to 0$ as $\varepsilon \to 0$ for the phase space metric d.

There is a natural question, as follows.

Question 1. Is the order in which $d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0)$ goes to zero the same as $d(u_*^\varepsilon, u_*^0)$?

There are many works concerning the rate of convergence of attractors to different situations, as we can see in [1, 3, 6] and [7]. The case of reaction-diffusion equation in a smooth domain, [1] has been shown that

$$d(u_*^\varepsilon, u_*^0) \sim \varepsilon \quad \text{and} \quad d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \sim \varepsilon^\beta, \quad 0 < \beta < 1. \tag{1.1}$$

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In [3], the authors have analyzed the reaction-diffusion equation in a thin domain under perturbations, where they have obtained

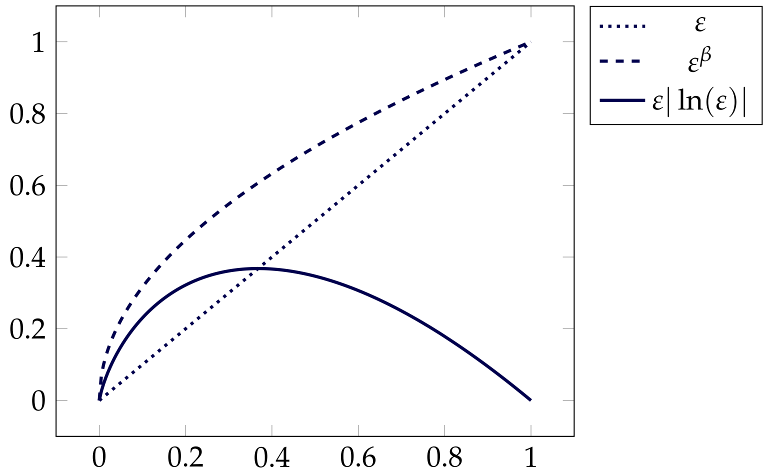
$$d(u_*^\varepsilon, u_*^0) \sim \varepsilon \quad \text{and} \quad d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \sim \varepsilon|\ln(\varepsilon)|. \tag{1.2}$$

Notice that booth above problems does not provide an answer to Question 1 because the rate of convergence of attractors is worse than equilibria.

The work [6] was able to answer Question 1 considering the reaction-diffusion equation where the diffusion coefficient becomes large in all domains when $\varepsilon \to 0$. The optimal rate state

$$d(u_*^\varepsilon, u_*^0) \sim \varepsilon \quad \text{and} \quad d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \sim \varepsilon. \tag{1.3}$$

The figure below shows (1.2) is better than (1.1) and (1.3) improves (1.2) as the parameter ε goes to zero.



The main argument to obtain (1.2) and (1.3) is the existence of a finite-dimensional invariant manifold that allows us to reduce the problem to finite dimension and, then we can use properties of Morse–Smale dynamical systems in finite-dimensional closed manifolds. For instance, [3] have used that in a neighborhood of the attractor, a Morse–Smale flow has the Lipschitz Shadowing property to estimate $d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0)$ by the continuity of the solution $T_\varepsilon(\cdot) \to T_0(\cdot)$ in a neighborhood of the $\cup_\varepsilon \mathcal{A}_\varepsilon$.

The purpose of this paper is to prove that the rate of convergence of the attractors for the scalar reaction-diffusion equations is optimal. Inspired by the optimal rate obtained in [6] and using the framework proposed by [3] we can reduce the problem to Morse–Smale flows in finite dimension and we use the structural stability of Morse–Smale flows in a finite dimension to obtain the corresponding result in infinite dimension. As a consequence, we obtain the optimal rate of convergence of the attractors. We observe that our arguments can be carried over to the problem addressed in [3] under appropriate adaptations. Another consequence of the structural stability is the estimate of the Gromov–Hausdorff distance of the attractors $d_{GH}(\mathcal{A}_\varepsilon, \mathcal{A}_0)$. This subject has been introduced by reaction-diffusion equation under perturbation of the domain in the paper [10]. Since structural stability means that there is a topological equivalence $\kappa_\varepsilon : \mathcal{A}_\varepsilon \to \mathcal{A}_0$ close to identity conjugating the flows, we have κ_ε a continuous ε-isometry between the attractors. This is enough requirement that we need to estimate $d_{GH}(\mathcal{A}_\varepsilon, \mathcal{A}_0)$.

Consider the following family of scalar reaction-diffusion equations

$$\begin{cases} u_t^\varepsilon - (a_\varepsilon(x)u_x^\varepsilon)_x = f(u^\varepsilon), & (t,x) \in (0,\infty) \times (0,\pi) \\ u^\varepsilon(t,0) = 0 = u^\varepsilon(t,\pi), & t \in (0,\infty), \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), & x \in (0,\pi), \end{cases} \tag{1.4}$$

where $\varepsilon \in [0,\varepsilon_0]$ is a parameter, $0 < \varepsilon_0 < 1$, the diffusion coefficients $a_\varepsilon \in C^1([0,\pi],[m_0,M_0])$, $m_0, M_0 > 0$, are continuous functions satisfying

$$\|a_\varepsilon - a\|_\infty := \|a_\varepsilon - a\|_{L^\infty(0,\pi)} \to 0 \quad \text{as } \varepsilon \to 0 \tag{1.5}$$

and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function such that,

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < 0. \tag{1.6}$$

It follows from [5, Theorem 14.2] that for each $\varepsilon \in [0,\varepsilon_0]$, the solutions of (1.4) defines a nonlinear gradient semigroup $T_\varepsilon(\cdot)$ having a global attractor $\mathcal{A}_\varepsilon$ such that

$$\sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{w\in\mathcal{A}_\varepsilon} \|w\|_{H_0^1(0,\pi)} < \infty \quad \text{and} \quad \sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{w\in\mathcal{A}_\varepsilon} \|w\|_{L^\infty(0,\pi)} < \infty. \tag{1.7}$$

Moreover, we assume that the equilibrium points of (1.4) with $\varepsilon = 0$ is hyperbolic. Hence, there are finitely many equilibrium points and we denote them by $\mathcal{E}_0 = \{u_*^{1,0}, \dots, u_*^{p,0}\}$.

Under the above assumption, we have from [5, Chapter 14] that, for ε_0 sufficiently small, the semigroup $T_\varepsilon(\cdot)$ has exactly p equilibria that we denote $\mathcal{E}_\varepsilon = \{u_*^{1,\varepsilon}, \dots, u_*^{p,\varepsilon}\}$ and the global attractors are given by $\mathcal{A}_\varepsilon = \cup_{i=1}^p W^u(u_*^{i,\varepsilon})$ and $\mathcal{A}_0 = \cup_{i=1}^p W^u(u_*^{i,0})$, where W^u denotes the unstable manifold. The main results of [5, Chapter 14] and [1] state that the convergence of equilibria can be estimate by

$$\|u_*^{i,\varepsilon} - u_*^{i,0}\|_{H_0^1(0,\pi)} \le C\|a_\varepsilon - a_0\|_\infty \tag{1.8}$$

and the continuity of the global attractors can be estimated by

$$d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \le C\|a_\varepsilon - a_0\|_\infty^\beta, \tag{1.9}$$

where $C > 0$ and $0 < \beta < 1$ are constants independent of ε and d_H denotes the Hausdorff distance in $H_0^1(0,\pi)$, that is,

$$d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) = \max\left\{ \sup_{u^\varepsilon\in\mathcal{A}_\varepsilon} \inf_{u^0\in\mathcal{A}_0} \|u^\varepsilon - u^0\|_{H_0^1(0,\pi)}, \sup_{u^0\in\mathcal{A}_0} \inf_{u^\varepsilon\in\mathcal{A}_\varepsilon} \|u^\varepsilon - u^0\|_{H_0^1(0,\pi)} \right\}. \tag{1.10}$$

Finally, we assume that $T_\varepsilon(\cdot)|_{\mathcal{A}_\varepsilon}$ is a group. It is well-known that under standard conditions the solutions of (1.4) are backward uniquely defined inside the attractor.

The main result of this paper states as follows.

Theorem 1.1. *The equation* (1.4) *is structurally stable at $\varepsilon = 0$. That is, given $\eta > 0$ there is $\varepsilon_\eta > 0$ such that for $\varepsilon \in (0,\varepsilon_\eta]$, there is a homeomorphism $\kappa_\epsilon : \mathcal{A}_\varepsilon \to \mathcal{A}_0$ such that*

$$\sup_{u^\varepsilon\in\mathcal{A}_\varepsilon} \|\kappa_\varepsilon(u^\varepsilon) - u^\varepsilon\|_{H_0^1(0,\pi)} < C(\|a_\varepsilon - a_0\|_\infty + \eta) \quad \textit{and} \quad \kappa_\varepsilon(T_\varepsilon(\tau_\varepsilon(t,u^\varepsilon))u^\varepsilon) = T_0(t)\kappa_\varepsilon(u^\varepsilon),$$

where $t \in \mathbb{R}$, $u^\varepsilon \in \mathcal{A}_\varepsilon$, $C > 0$ is a constant independent of ε and $\tau_\varepsilon : \mathbb{R} \times \mathcal{A}_\varepsilon \to \mathbb{R}$ is a function such that, $\tau_\varepsilon(0,u^\varepsilon) = 0$ and $\tau_\varepsilon(\cdot,u^\varepsilon)$ is a increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$.

As an immediate consequence of the Theorem 1.1 and (1.10) we have the following result.

Corollary 1.2. *For $\eta > 0$ there is $\varepsilon_\eta > 0$ such that for $\varepsilon \in (0, \varepsilon_\eta]$, the Hausdorff distance between the attractors can be estimated by*

$$d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \le C(\|a_\varepsilon - a_0\|_\infty + \eta), \tag{1.11}$$

where C is a constant independent of ε.

We say that a map $i_\varepsilon : \mathcal{A}_\varepsilon \to \mathcal{A}_0$ is a ε-isometry between $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ if

$$\left| \|i_\varepsilon(u^\varepsilon) - i_\varepsilon(v^\varepsilon)\|_{H_0^1(0,\pi)} - \|u^\varepsilon - v^\varepsilon\|_{H_0^1(0,\pi)} \right| \le \varepsilon, \qquad u^\varepsilon, v^\varepsilon \in \mathcal{A}_\varepsilon \tag{1.12}$$

and $B(i_\varepsilon(\mathcal{A}_\varepsilon), \varepsilon) = \mathcal{A}_0$, where $B(i_\varepsilon(\mathcal{A}_\varepsilon), \varepsilon) = \{u^0 \in \mathcal{A}_0 : \|i_\varepsilon(u^\varepsilon) - u^0\|_{H_0^1(0,\pi)} \le \varepsilon$, for some $u^\varepsilon \in \mathcal{A}_\varepsilon\}$. Analogously we can define a ε-isometry between $\mathcal{A}_0$ and $\mathcal{A}_\varepsilon$. The Gromov–Hausdorff distance $d_{GH}(\mathcal{A}_\varepsilon, \mathcal{A}_0)$ between $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ is defined as the infimum of $\varepsilon > 0$ for which there are ε-isometries $i_\varepsilon : \mathcal{A}_\varepsilon \to \mathcal{A}_0$ and $l_\varepsilon : \mathcal{A}_0 \to \mathcal{A}_\varepsilon$.

We have the following result as an immediate consequence of the Theorem 1.1.

Corollary 1.3. *For $\eta > 0$ there is $\varepsilon_\eta > 0$ such that for $\varepsilon \in (0, \varepsilon_\eta]$, the Gromov–Hausdorff distance between the attractors can be estimated by*

$$d_{GH}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \le C(\|a_\varepsilon - a_0\|_\infty + \eta), \tag{1.13}$$

where C is a constant independent of ε.

This paper is organized as follows. In Section 2 we introduce the functional setting to deal with (1.4). In Section 3 we use invariant manifolds to reduce the problem to finite dimension. In Section 4 we prove the Theorem 1.1.

2 Functional setting and technical results

Let $\varepsilon \in [0, \varepsilon_0]$. We define the operator $A_\varepsilon : D(A_\varepsilon) \subset L^2(0,\pi) \to L^2(0,\pi)$ by

$$\begin{cases} D(A_\varepsilon) = H^2(0,\pi) \cap H_0^1(0,\pi), \\ A_\varepsilon u = -(a_\varepsilon(x) u_x)_x, \quad u \in D(A_\varepsilon). \end{cases} \tag{2.1}$$

It is well-known that A_ε is a self-adjoint operator with compact resolvent. Hence, we can define the fractional power spaces X_ε^α, $0 < \alpha \le 1$, where $X_\varepsilon^0 = L^2(0,\pi)$, $X_\varepsilon^1 = D(A_\varepsilon)$ and $X_\varepsilon^{\frac{1}{2}} = H_0^1(0,\pi)$ with the inner product

$$\langle u, v \rangle_{X_\varepsilon^{\frac{1}{2}}} = \int_0^\pi a_\varepsilon(x) u_x v_x \, dx \tag{2.2}$$

which produces norms uniformly equivalent to the standard $H_0^1(0,\pi)$ norm, since a_ε is uniformly bounded in ε. Therefore, estimates on $X_\varepsilon^{\frac{1}{2}}$ are transported to $H_0^1(0,\pi)$ uniformly in ε.

Since there are many estimates in the paper, we will let C be a generic constant which is independent of ε, but which may depend on $m_0, M_0, u_0^0, \mathcal{E}_0$.

We summarize in the next theorem some useful estimates that can be proved as in [1] and [5, Chapter 14].

Theorem 2.1. *Let $\varepsilon \in [0, \varepsilon_0]$. The operators A_ε satisfy the following properties.*

(i) $\sup_{\varepsilon\in[0,\varepsilon_0]} \|A_\varepsilon^{-1}\|_{\mathcal{L}(L^2(0,\pi),H_0^1(0,\pi))} \leq C$.

(ii) $\|A_\varepsilon^{-1}u^\varepsilon - A_0^{-1}u^0\|_{H_0^1(0,\pi)} \leq C(\|u^\varepsilon - u^0\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty)$, $\quad u^\varepsilon, u^0 \in L^2(0,\pi)$.

(iii) $\|(\mu + A_\varepsilon)^{-1}u^\varepsilon - (\mu + A_0)^{-1}u^0\|_{H_0^1(0,\pi)} \leq C(\|u^\varepsilon - u^0\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty)$*, for μ in the resolvent set of A_ε and A_0 and $u^\varepsilon, u^0 \in L^2(0,\pi)$.*

Here, $C > 0$ is a constant independent of ε.

Proof. The proof has been done in [1, Section 3] and [5, Chapter 14]. Since there is a difference between these works due to the presence of an exponent $\frac{1}{2}$, we outline the proof of item (ii) here.

Let $u^\varepsilon, u^0 \in L^2(0,\pi)$ and let v^ε, v^0 be the respective solution of $A_\varepsilon v^\varepsilon = u^\varepsilon$ and $A_0 v^0 = u^0$. Then,

$$\int_0^\pi a_\varepsilon v_x^\varepsilon \varphi_x \, dx = \int_0^\pi u^\varepsilon \varphi \, dx, \quad \text{and} \quad \int_0^\pi a_0 v_x^0 \varphi_x \, dx = \int_0^\pi u^0 \varphi \, dx, \quad \varphi \in H_0^1(0,\pi). \tag{2.3}$$

Taking $\varphi = v^\varepsilon - v^0$, we obtain

$$\int_0^\pi a_\varepsilon v_x^\varepsilon (v_x^\varepsilon - v_x^0) \, dx - \int_0^\pi a_0 v_x^0 (v_x^\varepsilon - v_x^0) \, dx = \int_0^\pi (u^\varepsilon - u^0)(v^\varepsilon - v^0) \, dx.$$

which implies

$$\int_0^\pi a_\varepsilon (v_x^\varepsilon - v_x^0)^2 \, dx + \int_0^\pi (a_\varepsilon - a_0) v_x^0 (v_x^\varepsilon - v_x^0) \, dx = \int_0^\pi (u^\varepsilon - u^0)(v^\varepsilon - v^0) \, dx.$$

By (2.2) and the uniformity between the $X_\varepsilon^{\frac{1}{2}}$ norm and $H_0^1(0,\pi)$ norm, we get

$$\|v^\varepsilon - v^0\|_{H_0^1(0,\pi)} \leq C(\|u^\varepsilon - u^0\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty),$$

for some positive constant C independent of ε.

Finishing we notice that $A_\varepsilon v^\varepsilon = u^\varepsilon$ and $A_0 v^0 = u^0$ implies $v^\varepsilon = A_\varepsilon^{-1}u^\varepsilon$ and $v^0 = A_0^{-1}u^0$. □

We write (1.4) as an evolution equation in $L^2(0,\pi)$ in the following way

$$\begin{cases} u_t^\varepsilon + A_\varepsilon u^\varepsilon = f(u^\varepsilon), \\ u^\varepsilon(0) = u_0^\varepsilon, \end{cases} \tag{2.4}$$

where we have used the same notation f for the nonlinearity of (1.4) and its functional $f_I : H_0^1(0,\pi) \to L^2(0,\pi)$ given by $f_I(u)(x) = f(u(x))$.

We denote the spectra of the divergence operator $-A_\varepsilon$, $\varepsilon \in [0,\varepsilon_0]$, ordered and counting multiplicity by

$$\cdots < -\lambda_m^\varepsilon < -\lambda_{m-1}^\varepsilon < \cdots < -\lambda_1^\varepsilon$$

and we let $\{\varphi_i^\varepsilon\}_{i=1}^\infty$ be the corresponding eigenfunctions.

The resolvent convergence $\|A_\varepsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(L^2(0,\pi),H^1(0,\pi))} \to 0$ as $\varepsilon \to 0$ imply the convergence of eigenvalues, that is, $\lambda_m^\varepsilon \to \lambda_m^0$ as $\varepsilon \to 0$, $m = 1, 2, \ldots$ as we can see in [1, Proposition 3.3]. Moreover, by [1, Corollary 3.6], we obtain a constant $C > 0$ independent of ε such that,

$$|\lambda_m^\varepsilon - \lambda_m^0| \leq C\|a_\varepsilon - a\|_\infty, \qquad m = 1, 2, \ldots \tag{2.5}$$

We take a closed curve Γ_m contained in the resolvent set of $-A_0$ around $\{-\lambda_1^0, \dots, -\lambda_m^0\}$. By (2.5) we can take ε sufficiently small for that Γ_m be contained in the resolvent set of $-A_\varepsilon$ around $\{-\lambda_1^\varepsilon, \dots, -\lambda_m^\varepsilon\}$. Thus, we can define

$$P_\varepsilon^m = \frac{1}{2\pi i} \int_{\Gamma_m} (\mu + A_\varepsilon)^{-1} \, d\mu, \qquad \varepsilon \in [0, \varepsilon_0], \tag{2.6}$$

which is the spectral projection onto the space generated by the first m eigenfunctions of A_ε. It follows from (2.6) and Theorem 2.1 that there is a constant $C > 0$ independent of ε such that,

$$\|P_\varepsilon^m u^\varepsilon - P_0^m u^0\|_{H_0^1(0,\pi)} \leq C(\|u^\varepsilon - u^0\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty), \qquad u^\varepsilon, u^0 \in H_0^1(0,\pi) \tag{2.7}$$

and

$$\sup_{\varepsilon \in [0,\varepsilon_0]} \sup_{u^\varepsilon \in L^2(0,\pi)} \|P_\varepsilon^m u^\varepsilon\|_{H_0^1(0,\pi)} \leq C.$$

In the next section, we will fix m sufficiently large to obtain conditions for the invariant manifold theorem. Thus, to avoid heavy notation, we omit the dependency of m on P_ε^m and we denote $Q_\varepsilon = (I - P_\varepsilon)$ the projection over its orthogonal complement.

3 Invariant manifold and reduction of the dimension

The resolvent convergence $\|A_\varepsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(L^2(0,\pi), H^1(0,\pi))} \to 0$ as $\varepsilon \to 0$ guarantees the spectral convergence of the eigenvalues $\lambda_m^\varepsilon \to \lambda_m^0$ as $\varepsilon \to 0$, $m = 1, 2, \dots$. But, the operator A_0 has a gap on its eigenvalues, that is, $\lambda_{m+1}^0 - \lambda_m^0 \to \infty$ as $m \to \infty$. Thus, for ε_0 sufficiently small, we have a similar gap on the eigenvalues of A_ε. This fact, enables us to construct inertial manifolds of the same dimension given by $\text{rank}(P_\varepsilon) = \text{span}[\varphi_1^\varepsilon, \dots, \varphi_m^\varepsilon]$, where according with the previous section, φ_i^ε is the associated eigenfunction to the eigenvalue λ_i^ε, $m = 1, 2, \dots$.

For each $\varepsilon \in [0, \varepsilon_0]$, we decompose $H_0^1(0,\pi) = Y_\varepsilon \oplus Z_\varepsilon$, where $Y_\varepsilon = P_\varepsilon(H_0^1(0,\pi))$ and $Z_\varepsilon = Q_\varepsilon(H_0^1(0,\pi))$ and we define $A_\varepsilon^+ = A_\varepsilon|_{Y_\varepsilon}$ and $A_\varepsilon^- = A_\varepsilon|_{Z_\varepsilon}$. Using this decomposition we rewrite (2.4) as the following coupled equation

$$\begin{cases} v_t^\varepsilon + A_\varepsilon^+ v^\varepsilon = P_\varepsilon f(v^\varepsilon + z^\varepsilon) := H_\varepsilon(v^\varepsilon, z^\varepsilon), \\ z_t^\varepsilon + A_\varepsilon^- z^\varepsilon = Q_\varepsilon f(v^\varepsilon + z^\varepsilon) := G_\varepsilon(v^\varepsilon, z^\varepsilon). \end{cases} \tag{3.1}$$

The invariant manifold theorem whose proof can be found in [5, Chapter 8], states as follows.

Theorem 3.1. *For sufficiently large m and $\varepsilon_0 > 0$ small, there is an invariant manifold $\mathcal{M}_\varepsilon$ for (2.4) given by*

$$\mathcal{M}_\varepsilon = \{u^\varepsilon \in H_0^1(0,\pi)\,;\, u^\varepsilon = P_\varepsilon u^\varepsilon + s_*^\varepsilon(P_\varepsilon u^\varepsilon)\}, \qquad \varepsilon \in [0, \varepsilon_0],$$

where $s_*^\varepsilon : Y_\varepsilon \to Z_\varepsilon$ is a Lipschitz continuous map satisfying

$$\|s_*^\varepsilon(\tilde{v}^\varepsilon) - s_*^0(\tilde{v}^0)\|_{H_0^1(0,\pi)} \leq C(\|\tilde{v}^\varepsilon - \tilde{v}^0\|_{H_0^1(0,\pi)} + \|a_\varepsilon - a_0\|_\infty |\log(\|a_\varepsilon - a_0\|_\infty)|), \tag{3.2}$$

where $\tilde{v}^\varepsilon \in Y_\varepsilon$, $\tilde{v}^0 \in Y_0$ and C is a positive constant independent of ε. The invariant manifold $\mathcal{M}_\varepsilon$ is exponentially attracting and the global attractor $\mathcal{A}_\varepsilon$ of the problem (2.4) lies in $\mathcal{M}_\varepsilon$. The flow of $u_0^\varepsilon \in \mathcal{M}_\varepsilon$ is given by

$$T_\varepsilon(t) u_0^\varepsilon = v^\varepsilon(t) + s_*^\varepsilon(v^\varepsilon(t)), \qquad t \in \mathbb{R}, \tag{3.3}$$

where $v^\varepsilon(t)$ *satisfies*

$$\begin{cases} v_t^\varepsilon + A_\varepsilon^+ v^\varepsilon = H_\varepsilon(v^\varepsilon, s_*^\varepsilon(v^\varepsilon)), & t \in \mathbb{R}, \\ v^\varepsilon(0) = P_\varepsilon u_0^\varepsilon \in Y_\varepsilon. \end{cases} \tag{3.4}$$

For the proof of Theorem 3.1 we refer [5]. To see how obtain the estimate (3.2), we refer [1,4].

Now, we use the theory developed in [4] to identify (3.4) as an ordinary differential equation in $\mathbb{R}^m$. This identification is made by an isomorphism between Y_ε and $\mathbb{R}^m$. Since our aim in the next section will be to construct a ε-isometry between the attractors, it is convenient to make the isomorphism $Y_\varepsilon \approx \mathbb{R}^m$ an isometry. To accomplish this we follow the ideas of [4] that modify the basis of Y_ε.

Let $\varepsilon \in [0, \varepsilon_0]$. We consider in Y_ε the following set $\{P_\varepsilon \varphi_1^0, \ldots, P_\varepsilon \varphi_m^0\}$. It has been proved in [4] that this set is a basis for Y_ε. We define $L_\varepsilon : Y_\varepsilon \to \mathbb{R}^m$ by $L_\varepsilon(\sum_{i=1}^m \alpha_i P_\varepsilon \varphi_i^0) = \sum_{i=1}^m \alpha_i e_i$, where $\{e_i\}_{i=1}^m$ is the canonical basis of $\mathbb{R}^m$. This choices make L_ε a isometry between Y_ε and $\mathbb{R}^m$ and if we denote $\mathbb{R}^m_\varepsilon$ the $\mathbb{R}^m$ with the norm $\|x\|_{\mathbb{R}^m_\varepsilon} = (\sum_{i=1}^m x_i^2 \lambda_i^\varepsilon)^{\frac{1}{2}}$, then $\|\tilde{u}^0\|_{H_0^1(0,\pi)} = \|L_0 \tilde{u}^0\|_{\mathbb{R}^m_0}$.

Proposition 3.2. *The following statements hold true:*

(i) If $\tilde{u}^\varepsilon \in Y_\varepsilon$ *and* $\tilde{u}_0 \in Y_0$ *are such that* $\|\tilde{u}^\varepsilon\|_{H_0^1(0,\pi)} < \bar{C}$ *and* $\|\tilde{u}^\varepsilon\|_{H_0^1(0,\pi)} < \bar{C}$*, where* $\bar{C}$ *is a constant independent of* ε*. Then* $\|L_\varepsilon \tilde{u}^\varepsilon - L_0 \tilde{u}^0\|_{\mathbb{R}^m} \leq C(\|\tilde{u}^\varepsilon - \tilde{u}_0\|_{H_0^1(0,\pi)} + \|a_\varepsilon - a_0\|_\infty)$*, for a constant* $C > 0$ *independent of* ε*.*

(ii) If $\bar{u}^\varepsilon, \bar{u}^0 \in \mathbb{R}^m$ *are such that* $\|\bar{u}^\varepsilon\|_{\mathbb{R}^m} < \bar{C}$ *and* $\|\bar{u}^\varepsilon\|_{\mathbb{R}^m} < \bar{C}$*, where* $\bar{C}$ *is a constant independent of* ε*. Then* $\|L_\varepsilon^{-1}\bar{u}^\varepsilon - L_0^{-1}\bar{u}^0\|_{H_0^1(0,\pi)} \leq C(\|\bar{u}^\varepsilon - \bar{u}_0\|_{\mathbb{R}^m} + \|a_\varepsilon - a_0\|_\infty)$*, for a constant* $C > 0$ *independent of* ε*.*

Proof. The proof of item (i) follows as Lemma 5.4 of [4]. We prove (ii) using similar arguments.

Let $\bar{u}^\varepsilon = (\alpha_1^\varepsilon, \ldots, \alpha_m^\varepsilon)$ and $\bar{u}^0 = (\alpha_1^0, \ldots, \alpha_m^0)$ in $\mathbb{R}^m$. Then,

$$\begin{aligned} L_\varepsilon^{-1}\bar{u}^\varepsilon - L_0^{-1}\bar{u}^0 &= \sum_{i=1}^m \alpha_i^\varepsilon P_\varepsilon \varphi_i^0 - \sum_{i=1}^m \alpha_i^0 P_0 \varphi_i^0 \\ &= (P_\varepsilon - P_0)\sum_{i=1}^m \alpha_i^\varepsilon \varphi_i^0 + \sum_{i=1}^m (\alpha_i^\varepsilon - \alpha_i^0) P_0 \varphi_i^0 \end{aligned}$$

which implies,

$$\|L_\varepsilon^{-1}\bar{u}^\varepsilon - L_0^{-1}\bar{u}^0\|_{H_0^1(0,\pi)} \leq C\|a_\varepsilon - a_0\|_\infty + \|\bar{u}^\varepsilon - \bar{u}_0\|_{\mathbb{R}^m_0}$$

□

The map $s_*^\varepsilon : Y_\varepsilon \to Z_\varepsilon$ is obtained as the fixed point of the contraction $\Phi_\varepsilon : \Sigma_\varepsilon \to \Sigma_\varepsilon$ given by

$$\begin{cases} \Sigma_\varepsilon = \left\{ s^\varepsilon : Y_\varepsilon \to Z_\varepsilon \,;\, \|s^\varepsilon\|_\infty \leq D \text{ and } \|s^\varepsilon(v) - s^\varepsilon(\tilde{v})\|_{H_0^1(0,\pi)} \leq \Delta \|v - \tilde{v}\|_{H_0^1(0,\pi)} \right\}, \\ \Phi_\varepsilon(s^\varepsilon)(\eta) = \displaystyle\int_{-\infty}^0 e^{-A_\varepsilon^- r} G_\varepsilon(v^\varepsilon(r), s^\varepsilon(v^\varepsilon(r)))\, dr, \end{cases}$$

where D and Δ are positive constants independent of ε and $v^\varepsilon(r) \in Y_\varepsilon$ is the global solution of (3.4) with $\eta = P_\varepsilon u_0^\varepsilon$. With the aid of L_ε, we can define new invariant manifolds $\mathcal{N}_\varepsilon$, given by

$$\mathcal{N}_\varepsilon = \{L_\varepsilon^{-1}(x) + \theta_\varepsilon(x) : x \in \mathbb{R}^m\},$$

where $\theta : \mathbb{R}^m \to Z_\varepsilon$ is given by $\theta_*^\varepsilon = s_*^\varepsilon \circ L_\varepsilon^{-1}$. Therefore, θ_*^ε is a fixed point of

$$\theta_*^\varepsilon(x) = \int_{-\infty}^0 e^{-A_\varepsilon^- r} G_\varepsilon(v^\varepsilon(r), \theta_*^\varepsilon(L_\varepsilon v^\varepsilon(r)))\, dr,$$

such that

$$\|\theta_*^\varepsilon - \theta_*^0\|_\infty \leq C\|a_\varepsilon - a_0\|_\infty |\log(\|a_\varepsilon - a_0\|_\infty)|,$$

for some constant $C > 0$ independent of ε.

By Theorem 3.1 the semigroup $T_\varepsilon(\cdot)$ restrict to $\mathcal{M}_\varepsilon$ is a flow whose behavior is dictate by solutions of (3.4) that can be transposed to $\mathbb{R}^m$ as

$$\begin{cases} x_t^\varepsilon + L_\varepsilon A_\varepsilon^+ L_\varepsilon^{-1}(x^\varepsilon) = L_\varepsilon H_\varepsilon(L_\varepsilon^{-1}(x^\varepsilon), \theta_*^\varepsilon(x^\varepsilon)), & t \in \mathbb{R}, \\ x^\varepsilon(0) = L_\varepsilon P_\varepsilon u_0^\varepsilon := x_0^\varepsilon \in \mathbb{R}^m. \end{cases} \tag{3.5}$$

Theorem 3.3. *The solutions of (3.5) generate a Morse–Smale flow in $\mathbb{R}^m$.*

Proof. Since all equilibrium points of (1.4) are hyperbolic, the author in [9] has proved that the semigroup $T_\varepsilon(\cdot)$ is Morse–Smale. Therefore, $T_\varepsilon(\cdot)|_{\mathcal{N}_\varepsilon}$ is a Morse–Smale semigroup. Following [12, Chapter 3] we obtain that the projected semiflow $\bar{T}_\varepsilon(\cdot)$ of $T_\varepsilon(\cdot)$ in $\mathbb{R}^m$ is Morse–Smale. □

In what follows we prove several technical results that will be essential to prove the results in the next section. Here is the moment that we take a different way of [3].

Proposition 3.4. *The projection P_ε restrict to $\mathcal{M}_\varepsilon$ is an injective map and $P_\varepsilon^{-1}|_{\mathcal{M}_\varepsilon}$ restrict to the set $\tilde{\mathcal{A}}_\varepsilon := P_\varepsilon \mathcal{A}_\varepsilon$ is uniformly bounded in ε and*

$$\|P_\varepsilon^{-1}\tilde{u}^\varepsilon - P_0^{-1} q_\varepsilon(\tilde{u}^\varepsilon)\| \leq C(\|\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{L^2(0,\pi)} + \|a_\varepsilon - a_0\|_\infty), \qquad \tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon, \tag{3.6}$$

for any homeomorphism $q_\varepsilon : \tilde{\mathcal{A}}_\varepsilon \to \tilde{\mathcal{A}}_0$.

Proof. Let $u^\varepsilon, v^\varepsilon \in \mathcal{M}_\varepsilon$ such that $P_\varepsilon u^\varepsilon = P_\varepsilon v^\varepsilon$, then $u^\varepsilon = P_\varepsilon u^\varepsilon + s_*^\varepsilon(P_\varepsilon u^\varepsilon) = P_\varepsilon v^\varepsilon + s_*^\varepsilon(P_\varepsilon v^\varepsilon) = v^\varepsilon$. By (1.7), we have a positive constant C independent of ε such that,

$$\sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{\tilde{u}^\varepsilon \tilde{\mathcal{A}}_\varepsilon} \|P_\varepsilon^{-1}\tilde{u}^\varepsilon\|_{H^1(0,\pi)} \leq \sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{u^\varepsilon \in \mathcal{A}_\varepsilon} \|u^\varepsilon\|_{H^1(0,\pi)} \leq C.$$

Finally, if $\tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$ and $q_\varepsilon : \tilde{\mathcal{A}}_\varepsilon \to \tilde{\mathcal{A}}_0$ is a homeomorphism, then

$$\begin{aligned}
&\|P_\varepsilon^{-1}\tilde{u}^\varepsilon - P_0^{-1}q_\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} = \|P_0^{-1}P_0P_\varepsilon^{-1}\tilde{u}^\varepsilon - P_0^{-1}q_\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} \\
&\quad \leq \|P_0^{-1}\|_{\mathcal{L}(H_0^1(0,\pi),L^2(0,\pi))}\|P_0P_\varepsilon^{-1}\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} \\
&\quad \leq \|P_0^{-1}\|_{\mathcal{L}(H_0^1(0,\pi),L^2(0,\pi))}\|P_0P_\varepsilon^{-1}\tilde{u}^\varepsilon - P_\varepsilon P_\varepsilon^{-1}\tilde{u}^\varepsilon + P_\varepsilon P_\varepsilon^{-1}\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} \\
&\quad \leq \|P_0^{-1}\|_{\mathcal{L}(H_0^1(0,\pi),L^2(0,\pi))}\|(P_0 - P_\varepsilon)P_\varepsilon^{-1}\tilde{u}^\varepsilon\|_{H_0^1(0,\pi)} + \|P_0^{-1}\|_{\mathcal{L}(H_0^1(0,\pi),L^2(0,\pi))}\|\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} \\
&\quad \leq C(\|a_\varepsilon - a_0\|_\infty + \|\tilde{u}^\varepsilon - q_\varepsilon(\tilde{u}^\varepsilon)\|_{L^2(0,\pi)}),
\end{aligned}$$

where we have used (2.7) to obtain a positive constant C independent of ε. □

In what follows, we denote P_ε^{-1} the inverse of $P_\varepsilon|_{\mathcal{M}_\varepsilon} : \mathcal{M}_\varepsilon \to Y_\varepsilon$.

Proposition 3.5. *Let $\bar{T}_\varepsilon(\cdot)$ be the flow given by solutions of (3.5) and $\tilde{T}_\varepsilon(\cdot)$ be the flow given by solutions of (3.4). Then, it is valid the following properties* *(i)* $L_\varepsilon^{-1}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon = \tilde{T}_\varepsilon(t)L_\varepsilon^{-1}\bar{u}^\varepsilon, \quad \bar{u}^\varepsilon \in \mathbb{R}^m, \quad t \in \mathbb{R}.$

(ii) $\bar{T}_\varepsilon(t)L_\varepsilon\tilde{u}^\varepsilon = L_\varepsilon\tilde{T}_\varepsilon(t)\tilde{u}^\varepsilon, \quad \tilde{u}^\varepsilon \in Y_\varepsilon, \quad t \in \mathbb{R}.$

(iii) $P_\varepsilon T_\varepsilon(t)u^\varepsilon = \tilde{T}_\varepsilon(t)P_\varepsilon u^\varepsilon, \quad u^\varepsilon \in H_0^1(0,\pi), \quad t \geq 0.$

(iv) $P_\varepsilon^{-1}\tilde{T}_\varepsilon(t)\tilde{u}^\varepsilon = T_\varepsilon(t)P_\varepsilon^{-1}\tilde{u}^\varepsilon, \quad \tilde{u}^\varepsilon \in Y_\varepsilon, \quad t \geq 0.$

(v) Given a function $\tau_\varepsilon : \mathbb{R} \times \mathcal{A}_\varepsilon \to \mathbb{R}$ such that, $\tau_\varepsilon(0, u^\varepsilon) = 0$ and $\tau_\varepsilon(\cdot, u^\varepsilon)$ is a increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$, there exist a function $\tilde{\tau}_\varepsilon : \mathbb{R} \times \tilde{\mathcal{A}}_\varepsilon \to \mathbb{R}$ such that, $\tilde{\tau}_\varepsilon(0, P_\varepsilon u^\varepsilon) = 0$ and $\tilde{\tau}_\varepsilon(\cdot, P_\varepsilon u^\varepsilon)$ is a increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$ such that

$$P_\varepsilon T_\varepsilon(\tau_\varepsilon(t, u^\varepsilon))u^\varepsilon = \tilde{T}_\varepsilon(\tilde{\tau}_\varepsilon(t, P_\varepsilon u^\varepsilon))P_\varepsilon u^\varepsilon, \qquad u^\varepsilon \in \mathcal{A}_\varepsilon,\ t \in \mathbb{R}.$$

(vi) Given a function $\tilde{\tau}_\varepsilon : \mathbb{R} \times \tilde{\mathcal{A}}_\varepsilon \to \mathbb{R}$ such that, $\tilde{\tau}_\varepsilon(0, \tilde{u}^\varepsilon) = 0$ and $\tilde{\tau}_\varepsilon(\cdot, \tilde{u}^\varepsilon)$ is a increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$, there exist a function $\bar{\tau}_\varepsilon : \mathbb{R} \times \bar{\mathcal{A}}_\varepsilon \to \mathbb{R}$ such that, $\bar{\tau}_\varepsilon(0, L_\varepsilon\tilde{u}^\varepsilon) = 0$ and $\bar{\tau}_\varepsilon(\cdot, L_\varepsilon\tilde{u}^\varepsilon)$ is a increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$ such that

$$L_\varepsilon\tilde{T}_\varepsilon(\tilde{\tau}_\varepsilon(t, \tilde{u}^\varepsilon))\tilde{u}^\varepsilon = \bar{T}_\varepsilon(\bar{\tau}_\varepsilon(t, L_\varepsilon\tilde{u}^\varepsilon))L_\varepsilon\tilde{u}^\varepsilon, \qquad \tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon,\ t \in \mathbb{R}.$$

Proof. Let $\bar{u}^\varepsilon \in \mathbb{R}^m$, then $L_\varepsilon^{-1}\bar{u}^\varepsilon \in Y_\varepsilon$ and $\tilde{T}_\varepsilon(t)L_\varepsilon^{-1}\bar{u}^\varepsilon$ is a solution of

$$\begin{cases} v_t^\varepsilon + A_\varepsilon^+ v^\varepsilon = H_\varepsilon(v^\varepsilon, s_*^\varepsilon(v^\varepsilon)), \quad t \in \mathbb{R}, \\ v^\varepsilon(0) = L_\varepsilon^{-1}\bar{u}^\varepsilon \in Y_\varepsilon. \end{cases} \tag{3.7}$$

Defining $\varphi^\varepsilon(t) = L_\varepsilon^{-1}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon$, we have $\varphi^\varepsilon(0) = L_\varepsilon^{-1}\bar{T}_\varepsilon(0)\bar{u}^\varepsilon = L_\varepsilon^{-1}\bar{u}^\varepsilon$ and

$$\begin{aligned} \varphi_t^\varepsilon + A_\varepsilon^+\varphi^\varepsilon(t) &= L_\varepsilon^{-1}\frac{\partial}{\partial t}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon + A_\varepsilon^+ L_\varepsilon^{-1}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon \\ &= L_\varepsilon^{-1}(\frac{\partial}{\partial t}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon + L_\varepsilon A_\varepsilon^+ L_\varepsilon^{-1}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon). \end{aligned}$$

Since $x_\varepsilon(t) := \bar{T}_\varepsilon(t)\bar{u}^\varepsilon$ is a solution of

$$\begin{cases} x_t^\varepsilon + L_\varepsilon A_\varepsilon^+ L_\varepsilon^{-1}(x^\varepsilon) = L_\varepsilon H_\varepsilon(L_\varepsilon^{-1}(x^\varepsilon), \theta_*^\varepsilon(x^\varepsilon)), \quad t \in \mathbb{R}, \\ x^\varepsilon(0) = \bar{u}^\varepsilon \in \mathbb{R}^m, \end{cases} \tag{3.8}$$

we obtain

$$\varphi_t^\varepsilon + A_\varepsilon^+\varphi^\varepsilon(t) = H_\varepsilon(\varphi^\varepsilon(t), \theta_*^\varepsilon(\varphi^\varepsilon(t))).$$

The bijection between θ_*^ε and s_*^ε enables us to conclude that $\varphi_\varepsilon(t)$ is also a solution of (3.7). The result follows from the well-posedness of (3.7).

In the same way, we proof item (ii).

Item (iii) is immediate from (3.3) and (3.4) by noticing that $P_\varepsilon T_\varepsilon(t)u^\varepsilon = v^\varepsilon(t)$ and we are denoting $v^\varepsilon(t) = \tilde{T}_\varepsilon(t)P_\varepsilon u^\varepsilon$. Item (iv) follows from (iii) using that $P_\varepsilon u^\varepsilon = \tilde{u}^\varepsilon$ if only if $u^\varepsilon = P_\varepsilon^{-1}\tilde{u}^\varepsilon$, for some $\tilde{u}^\varepsilon \in Y_\varepsilon$. Item (v) follows from (iii) defining $\tilde{\tau}_\varepsilon(t, P_\varepsilon u^\varepsilon) = \tau_\varepsilon(t, u^\varepsilon)$. In the same way, we obtain (vi). □

Proposition 3.6. *The set $\tilde{\mathcal{A}}_\varepsilon = P_\varepsilon\mathcal{A}_\varepsilon$ is the global attractor for the semigroup $\tilde{T}_\varepsilon(\cdot)$ given by solutions of* (3.4).

Proof. Since $\mathcal{A}_\varepsilon$ is compact and P_ε is continuous, we have $\tilde{\mathcal{A}}_\varepsilon = P_\varepsilon \mathcal{A}_\varepsilon$ a compact set in Y_ε. Proving the attraction, let $B \subset Y_\varepsilon$ a bounded set and let $v^\varepsilon \in B$. Then $v^\varepsilon + s_*^\varepsilon(v^\varepsilon) \in \mathcal{M}_\varepsilon$ and $T_\varepsilon(t)w^\varepsilon = \tilde{T}_\varepsilon(t)v^\varepsilon + s_*^\varepsilon(\tilde{T}_\varepsilon(t)v^\varepsilon)$, for $t > 0$ and $w^\varepsilon \in P_\varepsilon^{-1}(v^\varepsilon)$. But $T_\varepsilon(t)$ is a gradient semigroup, then there is $u^\varepsilon \in \mathcal{A}_\varepsilon$ such that, $\|T_\varepsilon(t)w^\varepsilon - u^\varepsilon\|_{H_0^1(0,\pi)} \to 0$ as $t \to \infty$. In fact, the attraction property of the global attractor is uniform for the solutions starting at B. Hence, there is a neighborhood of $\mathcal{A}_\varepsilon$ containing all trajectory starting at B after a time t_B. We take $\tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$ such that $\tilde{u}^\varepsilon = P_\varepsilon u^\varepsilon$. Thus,

$$\begin{aligned}\|\tilde{T}_\varepsilon(t)v^\varepsilon - \tilde{u}^\varepsilon\|_{H_0^1(0,\pi)} &\leq \|\tilde{T}_\varepsilon(t)v^\varepsilon - \tilde{u}^\varepsilon\|_{H_0^1(0,\pi)} + \|s_*^\varepsilon(\tilde{T}_\varepsilon(t)v^\varepsilon) - s_*^\varepsilon(\tilde{u}^\varepsilon)\|_{H_0^1(0,\pi)} \\ &= C\|\tilde{T}_\varepsilon(t)v^\varepsilon + s_*^\varepsilon(\tilde{T}_\varepsilon(t)v^\varepsilon) - P_\varepsilon u^\varepsilon - s_*^\varepsilon(P_\varepsilon u^\varepsilon)\|_{H_0^1(0,\pi)} \\ &= C\|T_\varepsilon(t)w^\varepsilon - u^\varepsilon\|_{H_0^1(0,\pi)} \to 0 \quad \text{as } t \to \infty,\end{aligned}$$

for a constant $C > 0$ independent of ε, where the attraction property is also uniform for the solutions starting at bounded sets.

It remains to prove that $\tilde{\mathcal{A}}_\varepsilon$ is invariant. Let $\tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$ and $t \geq 0$. Writing $w^\varepsilon = P_\varepsilon \tilde{u}^\varepsilon$ for some $w^\varepsilon \in \mathcal{A}_\varepsilon$, we have by the invariance of $\mathcal{A}_\varepsilon$, that there is $\hat{w}^\varepsilon \in \mathcal{A}_\varepsilon$ such that $T_\varepsilon(\bar{t})\hat{w}^\varepsilon = w^\varepsilon$, for some $\bar{t} \geq 0$. Thus,

$$\tilde{u}^\varepsilon + s_*^\varepsilon(\tilde{u}^\varepsilon) = P_\varepsilon w^\varepsilon + s_*^\varepsilon(P_\varepsilon w^\varepsilon) = w^\varepsilon = T_\varepsilon(\bar{t})\hat{w}^\varepsilon = \tilde{T}_\varepsilon(\bar{t})P_\varepsilon\hat{w}^\varepsilon + s_*^\varepsilon(\tilde{T}_\varepsilon(\bar{t})P_\varepsilon\hat{w}^\varepsilon),$$

which implies $\tilde{u}^\varepsilon = \tilde{T}_\varepsilon(\bar{t})P_\varepsilon\hat{w}^\varepsilon$, where $P_\varepsilon\hat{w}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$. □

Proposition 3.7. *The set $\bar{\mathcal{A}}_\varepsilon = L_\varepsilon P_\varepsilon \mathcal{A}_\varepsilon$ is the global attractor for the semigroup $\bar{T}_\varepsilon(\cdot)$ given by solutions of* (3.5)*.*

Proof. Since L_ε is continuous and $P_\varepsilon \mathcal{A}_\varepsilon$ is compact, we have $\bar{\mathcal{A}}_\varepsilon = L_\varepsilon P_\varepsilon \mathcal{A}_\varepsilon$ a compact set in $\mathbb{R}^m$. Let B a bounded set in $\mathbb{R}^m$ and $\bar{u}^\varepsilon \in B$, then $L_\varepsilon^{-1}\bar{u}^\varepsilon \in L_\varepsilon^{-1}B$ which is a bounded set in Y_ε. Since $\tilde{T}_\varepsilon(\cdot)$ is gradient, there is $\tilde{w}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$ such that, $\|\tilde{T}_\varepsilon(t)L_\varepsilon^{-1}\bar{u}^\varepsilon - \tilde{w}^\varepsilon\|_{H_0^1(0,\pi)} \to 0$ as $t \to \infty$, where the attraction property is uniform for the solutions starting at bounded sets. Hence, $L_\varepsilon\tilde{w}^\varepsilon \in \bar{\mathcal{A}}_\varepsilon$ is such that,

$$\begin{aligned}\|\bar{T}_\varepsilon(t)\bar{u}^\varepsilon - L_\varepsilon\tilde{w}^\varepsilon\|_{\mathbb{R}_0^m} &= \|L_\varepsilon^{-1}\bar{T}_\varepsilon(t)\bar{u}^\varepsilon - \tilde{w}^\varepsilon\|_{H_0^1(0,\pi)} \\ &= \|\tilde{T}_\varepsilon(t)L_\varepsilon^{-1}\bar{u}^\varepsilon - \tilde{w}^\varepsilon\|_{H_0^1(0,\pi)} \to 0, \quad \text{as } t \to \infty,\end{aligned}$$

where we have used that L_ε is a isometry and Proposition 3.5.

It remains to prove that $\bar{\mathcal{A}}_\varepsilon$ is invariant. Let $\bar{u}^\varepsilon \in \bar{\mathcal{A}}_\varepsilon$. Then $L_\varepsilon^{-1}\bar{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$ which is invariant. Thus, there is $\tilde{w}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$ and $\bar{t} > 0$ such that $\tilde{T}_\varepsilon(\bar{t})\tilde{w}^\varepsilon = L_\varepsilon^{-1}\bar{u}^\varepsilon$. Thus, $L_\varepsilon\tilde{T}_\varepsilon(\bar{t})\tilde{w}^\varepsilon = \bar{u}^\varepsilon$ and by Proposition 3.5, we have $\bar{T}_\varepsilon(\bar{t})L_\varepsilon\tilde{w}^\varepsilon = \bar{u}^\varepsilon$. □

4 Proof of Theorem 1.1

In this section, we prove the main result of this paper, the Theorem 1.1.

Theorem 4.1. *The equation* (3.5) *is structurally stable at $\varepsilon = 0$. That is, for each $\eta > 0$ there is $\varepsilon_\eta > 0$ and for $\varepsilon \in (0, \varepsilon_\eta]$ there is a homeomorphism $h_\varepsilon : \bar{\mathcal{A}}_\varepsilon \to \bar{\mathcal{A}}_0$ such that,*

$$\sup_{\bar{u}^\varepsilon \in \bar{\mathcal{A}}_\varepsilon} \|h_\varepsilon(\bar{u}^\varepsilon) - \bar{u}^\varepsilon\|_{\mathbb{R}^m} < \eta \quad \text{and} \quad h_\varepsilon(\bar{T}_\varepsilon(\bar{\tau}_\varepsilon(t, \bar{u}^\varepsilon))\bar{u}^\varepsilon) = \bar{T}_0(t)h_\varepsilon(\bar{u}^\varepsilon), \tag{4.1}$$

where $\bar{u}^\varepsilon \in \bar{\mathcal{A}}_\varepsilon$, $t \in \mathbb{R}$ and $\bar{\tau}_\varepsilon : \mathbb{R} \times \bar{\mathcal{A}}_\varepsilon \to \mathbb{R}$ is function such that, $\bar{\tau}_\varepsilon(0, \bar{u}^\varepsilon) = 0$ and $\bar{\tau}_\varepsilon(\cdot, \bar{u}^\varepsilon)$ is a increasing function mapping $\mathbb{R}$ onto $\mathbb{R}$.

Proof. The works [1] and [5, Chapter 14] have obtained the continuity of the semigroups $T_\varepsilon(\cdot) \to T_0(\cdot)$ as $\varepsilon \to 0$ in the $H^1_0(0,\pi)$ norm. Following [2] we obtain $\bar{T}_\varepsilon(\cdot) \to \bar{T}_0(\cdot)$ as $\varepsilon \to 0$ in the C^1 norm, since the invariant manifolds $\mathcal{M}_\varepsilon$ and $\mathcal{M}_0$ are close in the C^1 topology. Thus, $\bar{T}_\varepsilon(\cdot)$ is a small C^1 perturbation of $\bar{T}_0(\cdot)$ which is a Morse–Smale semigroup $\mathbb{R}^m$. The main property of Morse–Smale flows in finite dimension stated in [11, 14] and [13] is the structural stability, that is, for each $\eta > 0$ there is $\varepsilon_\eta > 0$ and for $\varepsilon \in (0, \varepsilon_\eta]$ there is a homeomorphism $h_\varepsilon : \bar{\mathcal{A}}_\varepsilon \to \bar{\mathcal{A}}_0$ such that, (4.1) is valid. □

Theorem 4.2. *The equation* (3.4) *is structurally stable at* $\varepsilon = 0$. *That is, for each* $\eta > 0$ *there is* $\varepsilon_\eta > 0$ *and for* $\varepsilon \in (0, \varepsilon_\eta]$ *there is a homeomorphism* $j_\varepsilon : \tilde{\mathcal{A}}_\varepsilon \to \tilde{\mathcal{A}}_0$ *such that,*

$$\sup_{\tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon} \|j_\varepsilon(\tilde{u}^\varepsilon) - \tilde{u}^\varepsilon\|_{H^1_0(0,\pi)} < C(\|a_\varepsilon - a_0\|_\infty + \eta) \quad \text{and} \quad j_\varepsilon(\tilde{T}_\varepsilon(\tilde{\tau}_\varepsilon(t, \tilde{u}^\varepsilon))\tilde{u}^\varepsilon) = \tilde{T}_0(t) j_\varepsilon(\tilde{u}^\varepsilon), \tag{4.2}$$

where $\tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$, $t \in \mathbb{R}$ *and* $\tilde{\tau}_\varepsilon : \mathbb{R} \times \tilde{\mathcal{A}}_\varepsilon \to \mathbb{R}$ *is function such that,* $\tilde{\tau}_\varepsilon(0, \tilde{u}^\varepsilon) = 0$ *and* $\tilde{\tau}_\varepsilon(\cdot, \tilde{u}^\varepsilon)$ *is a increasing function mapping* $\mathbb{R}$ *onto* $\mathbb{R}$.

Proof. We define the map $j_\varepsilon : \tilde{\mathcal{A}}_\varepsilon \to \tilde{\mathcal{A}}_0$ by $j_\varepsilon = L_0^{-1} \circ h_\varepsilon \circ L_\varepsilon$. Then, for $\tilde{u}^\varepsilon \in \tilde{\mathcal{A}}_\varepsilon$ it follows from Proposition 3.2 and (4.1) that

$$\begin{aligned}
\|j_\varepsilon(\tilde{u}^\varepsilon) - \tilde{u}^\varepsilon\|_{H^1_0(0,\pi)} &= \|L_0^{-1} h_\varepsilon(L_\varepsilon(\tilde{u}^\varepsilon)) - \tilde{u}^\varepsilon\|_{H^1_0(0,\pi)} \\
&= \|L_0^{-1} h_\varepsilon(L_\varepsilon(\tilde{u}^\varepsilon)) - L_\varepsilon^{-1} L_\varepsilon \tilde{u}^\varepsilon\|_{H^1_0(0,\pi)} \\
&\le C(\|h_\varepsilon(L_\varepsilon(\tilde{u}^\varepsilon)) - L_\varepsilon \tilde{u}^\varepsilon\|_{\mathbb{R}^m} + \|a_\varepsilon - a_0\|_\infty) \\
&\le C(\eta + \|a_\varepsilon - a_0\|_\infty).
\end{aligned}$$

Moreover, by (4.1) and Proposition 3.5, we obtain

$$\begin{aligned}
j_\varepsilon(\tilde{T}_\varepsilon(\tilde{\tau}_\varepsilon(t, \tilde{u}^\varepsilon))\tilde{u}^\varepsilon) &= L_0^{-1} \circ h_\varepsilon \circ L_\varepsilon(\tilde{T}_\varepsilon(\tilde{\tau}_\varepsilon(t, \tilde{u}^\varepsilon))\tilde{u}^\varepsilon) \\
&= L_0^{-1}(h_\varepsilon(\bar{T}_\varepsilon(\bar{\tau}_\varepsilon(t, L_\varepsilon \tilde{u}^\varepsilon)) L_\varepsilon \tilde{u}^\varepsilon)) \\
&= L_0^{-1} \bar{T}_0(t) h_\varepsilon(L_\varepsilon \tilde{u}^\varepsilon) \\
&= \tilde{T}_0(t) L_0^{-1} h_\varepsilon(L_\varepsilon \tilde{u}^\varepsilon) \\
&= \tilde{T}_0(t) j_\varepsilon(\tilde{u}^\varepsilon).
\end{aligned}$$

Hence, j_ε is a homeomorphism between $\tilde{\mathcal{A}}_\varepsilon$ and $\tilde{\mathcal{A}}_0$ satisfying (4.2). □

Now, we are in a condition to prove the Theorem 1.1.

Proof. of Theorem 1.1. We define the map $\kappa_\varepsilon : \mathcal{A}_\varepsilon \to \mathcal{A}_0$ by $\kappa_\varepsilon = P_0^{-1} \circ j_\varepsilon \circ P_\varepsilon$. Similarly to the proof of Theorem 4.2, we can prove that κ_ε is a homeomorphism between $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ satisfying

$$\|\kappa_\varepsilon(u^\varepsilon) - u^\varepsilon\|_{H^1_0(0,\pi)} \le C(\eta + \|a_\varepsilon - a_0\|_\infty)$$

and

$$\kappa_\varepsilon(T_\varepsilon(\tau_\varepsilon(t, u^\varepsilon))u^\varepsilon) = T_0(t)\kappa_\varepsilon(u^\varepsilon).$$ □

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Global existence and blow-up of solution to a class of fourth-order equation with singular potential and logarithmic nonlinearity

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Abstract. In this paper, we consider the well-posedness and asymptotic behavior of Dirichlet initial boundary value problem for a fourth-order equation with strong damping and logarithmic nonlinearity. We establish the local solvability by the technique of cut-off combining with the method of Faedo–Galerkin approximation. By means of potential well method and Rellich inequality, we obtain the global existence and the decay estimate of global solutions under some appropriate conditions. Furthermore, we prove the finite time blow-up results of weak solutions, and establish the upper and lower bounds for blow-up time.

Keywords: fourth-order, singular potential, logarithmic nonlinearity, global existence, blow-up.

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1 Introduction

In this paper, we are concerned with the fourth-order parabolic problem

$$\begin{cases} \dfrac{u_t}{|x|^4} + \Delta^2 u - \Delta u_t = |u|^{p-2} u \ln |u|, & (x,t) \in \Omega \times (0,T); \\ u(x,t) = \Delta u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T); \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N\,(N > 4)$ is a bounded domain with smooth boundary $\partial\Omega$, $0 < T \leq \infty$, $u_0 \in H_0^2(\Omega)$, $x = (x_1, x_2, \dots, x_N) \in R^N$ with $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$, and parameter p satisfies the following

$$2 < p < \bar{p} = \begin{cases} \dfrac{8}{N} + 2, & N \geq 8, \\ \dfrac{4}{N-4} + 2, & 4 < N < 8. \end{cases}$$

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Many scholars have been devoted to the topic on the global existence and blow-up phenomena of the second-order partial differential equations (or a system of partial differential equations) and there have been fruitful results (see [1–4,7,10]). However, there are fewer studies on higher order equations (see [6,15] and references therein). In particular, the fourth-order parabolic partial differential equations have some applications in the fields such as materials science, engineering, biological mathematics, image analysis, etc.

King et al. [11] investigated the fourth-order semilinear parabolic equation modeling epitaxial thin film growth

$$u_t + \Delta^2 u - \nabla \cdot (f(\nabla u)) = g, \qquad (x,t) \in \Omega \times (0,+\infty),$$

by using the technique of semi-discrete approximation, they obtained existence, uniqueness and regularity of the weak solutions under appropriate conditions and derived the long-time asymptotic behavior.

The authors of [5,16] considered the following p-biharmonic parabolic equation with logarithmic nonlinearity

$$\begin{cases} u_t - \Delta u_t + \Delta\left(|\Delta u|^{p-2}\Delta u\right) = |u|^{q-2}u \ln u, & x \in \Omega,\ t > 0; \\ u(x,t) = \Delta u(x,t) = 0, & x \in \partial\Omega,\ t > 0; \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Cömert and Pişkin [5] considered the case of $2 < p < q < p\left(1 + \frac{4}{N}\right)$, they obtained the existence of the unique global weak solutions and decay polynomially of solutions by using the potential wells method and logarithmic Sobolev inequality. For $\max\left\{1, \frac{2N}{N+4}\right\} < p \le q < p\left(1 + \frac{4}{N}\right)$, Liu and Fang [16] established the local and global solvability, infinite and finite time blow-up phenomena of weak solutions in different energy levels. Moreover, the life span in different energy cases and extinction phenomenon are discussed.

Do et al. [8] considered the following higher-order reaction-diffusion parabolic problem

$$\begin{cases} \dfrac{u_t}{|x|^4} + \Delta^2 u = k(t)|u|^{p-1}u, & x \in \Omega,\ t > 0; \\ u(x,t) = \Delta u(x,t) = 0, & x \in \partial\Omega,\ t > 0; \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

The main difficulty is that the methods of [9,14,19] are no longer valid due to the presentation of singular potential. To overcome this difficulty, the authors of [8] combined the technique of cut-off with Hardy–Sobolev inequality and Faedo–Galerkin approximation to establish the local well-posedness. They also obtained the existence and decay estimation of a global weak solution. What is more, they discussed the upper and lower bounds on the blow-up time of a weak solution in [18].

In view of the works mentioned above, we consider the problem (1.1) with strong damping and logarithmic nonlinearity. In fact, the third derivative term Δu_t can be regarded as a damping term, which has effect on the qualitative properties such as blow-up, decay and so on. Mathematically, the logarithmic nonlinearity does not satisfy monotonicity and may change signs, thus the problem with logarithmic nonlinearity is more difficult than the one with power source. To the best our knowledge, this is the first work in the literature that takes into account a singular fourth-order equation with strong damping and logarithmic nonlinearity.

The paper is organized as follows. In Section 2, we introduce some potential wells, basic definitions and important lemmas. In section 3, we prove the local existence and uniqueness theorem. In Section 4, we prove the global existence and discuss the asymptotic behavior of solutions. Finally, in Section 5, we discuss the finite time blow-up of weak solutions and give the upper and lower bounds for blow-up time.

2 Preliminaries

In this section, we introduce some notations, basic definitions and lemmas that will be used throughout the paper. For convenience, we denote the norms

$$\|u\|_p := \|u\|_{L^p(\Omega)}, \qquad \|u\|_2 := \|u\|_{L^2(\Omega)}, \qquad \|\Delta u\|_2 := \|u\|_{H_0^2(\Omega)}.$$

For $u \in H_0^2(\Omega)$, we define the potential energy functional and Nehari functional as

$$J(u) = \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p^2}\|u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln|u|\,dx, \tag{2.1}$$

$$I(u) = \|\Delta u\|_2^2 - \int_\Omega |u|^p \ln|u|\,dx. \tag{2.2}$$

Then it follows from (2.1) and (2.2) that

$$J(u) = \frac{1}{p}I(u) + \left(\frac{1}{2} - \frac{1}{p}\right)\|\Delta u\|_2^2 + \frac{1}{p^2}\|u\|_p^p. \tag{2.3}$$

Furthermore, we introduce the following sets

$$\begin{aligned}
&W_1 = \left\{u \in H_0^2(\Omega) \mid J(u) < d\right\}, \quad W_2 = \left\{u \in H_0^2(\Omega) \mid J(u) = d\right\}, \quad W = W_1 \cup W_2,\\
&W_1^+ = \left\{u \in H_0^2(\Omega) \mid J(u) < d, I(u) > 0\right\}, \quad W_2^+ = \left\{u \in H_0^2(\Omega) \mid J(u) = d, I(u) > 0\right\},\\
&W_1^- = \left\{u \in H_0^2(\Omega) \mid J(u) < d, I(u) < 0\right\}, \quad W_2^- = \left\{u \in H_0^2(\Omega) \mid J(u) = d, I(u) < 0\right\},\\
&W^+ = W_1^+ \cup W_2^+, \quad W^- = W_1^- \cup W_2^-,
\end{aligned}$$

and the Nehari manifold

$$\mathcal{N} = \left\{u \in H_0^2(\Omega)\backslash\{0\},\ I(u) = 0\right\}.$$

The depth of potential well is defined as

$$d = \inf_{u \in \mathcal{N}} J(u).$$

Next, we give some definitions.

Definition 2.1 (Weak solution)**.** Let $T > 0$, the function $u \in L^\infty(0, T; H_0^2(\Omega))$ is a weak solution of problem (1.1) on $\Omega \times [0, T)$, if

$$u_t \in L^2(0, T; H_0^1(\Omega)), \qquad \frac{u_t}{|x|^2} \in L^2(0, T; L^2(\Omega)),$$

$u(x, 0) = u_0(x) \in H_0^2(\Omega)$ and $u(x, t)$ satisfies

$$\left\langle \frac{u_t}{|x|^4}, v\right\rangle + \langle \Delta u, \Delta v\rangle + \langle \nabla u_t, \nabla v\rangle = \left\langle |u|^{p-2}u\ln|u|, v\right\rangle,$$

for any $v \in H_0^2(\Omega)$ and $t \in [0, T)$.

Definition 2.2 (Maximal existence time [20])**.** Let $u(x,t)$ be a weak solution of problem (1.1), we define the maximal existence time T_{max} as follows

$$T_{max} = \sup\{T > 0;\ u(x,t) \text{ exists on } [0,T]\}.$$

(i) If $T_{max} = +\infty$, we say that the solution $u(t)$ is global;

(ii) If $T_{max} < +\infty$, we say that the solution $u(t)$ blows up in finite time and T_{max} is the blow-up time.

Definition 2.3 (Finite time blow-up)**.** Let $u(x,t)$ is a weak solution of problem (1.1). We say $u(x,t)$ blows up in finite time if the maximal existence time T_{max} is finite and

$$\lim_{t\to T_{max}^-}\int_0^t\left(\left\|\frac{u(\tau)}{|x|^2}\right\|_2^2 + \|\nabla u(\tau)\|_2^2\right)d\tau = +\infty.$$

In order to deal with the singular potential, we introduce the cut-off function

$$\rho_n(x) = \min\left\{|x|^{-4}, n\right\}, \qquad n \in N^+,$$

and the following Lemma 2.4.

Lemma 2.4 (Rellich's inequality [8])**.** *Let $N > 4$ and $u \in H_0^2(\Omega)$. Then $\frac{u}{|x|^2} \in L^2(\Omega)$ and there exists a constant $R_N > 0$ such that*

$$\int_\Omega \frac{|u|^2}{|x|^4}dx \le \frac{16}{N^2(N-4)^2}\int_\Omega |\Delta u|^2 dx =: R_N\int_\Omega |\Delta u|^2 dx.$$

Next, in Lemma 2.5, we describe some basic properties of the fiber mapping $J(\lambda u)$ that can be verified directly.

Lemma 2.5 ([5])**.** *Assume that $u \in H_0^2(\Omega)\backslash\{0\}$, then*

(i) $\lim_{\lambda\to 0^+} J(\lambda u) = 0$, $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$.

(ii) There exists a unique $\lambda^* = \lambda^*(u) > 0$ such that $\frac{d}{d\lambda}J(\lambda u)|_{\lambda=\lambda^*} = 0$.

(iii) $J(\lambda u)$ is increasing on $0 < \lambda < \lambda^*$, decreasing on $\lambda^* < \lambda < +\infty$, and attains the maximum at $\lambda = \lambda^*$.

(iv) $I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < +\infty$, and $I(\lambda^* u) = 0$.

We introduce the following inequality to deal with the logarithmic nonlinearity.

Lemma 2.6. *Let μ be a positive number. Then we have the following inequalities:*

$$s^p \ln s \le (e\mu)^{-1}s^{p+\mu}, \quad \text{for all } s \ge 1,$$

and

$$|s^p \ln s| \le (ep)^{-1}, \quad \text{for all } 0 < s < 1.$$

The next result is the Gagliardo–Nirenberg inequality.

Lemma 2.7. *For any $u \in H_0^2(\Omega)$, it holds that:*

$$\|u\|_{p+\mu}^{p+\mu} \le C_G \|\Delta u\|_2^{(p+\mu)\theta} \|u\|_2^{(1-\theta)(p+\mu)},$$

where $\theta \in (0,1)$ is determined by $\theta = \frac{N(p+\mu-2)}{4(p+\mu)}$, $0 < \mu < \frac{8}{N} + 2 - p$, and the constant $C_G > 0$ depends on N, p.

In order to prove the decay estimation of weak solutions, we will introduce the following lemma.

Lemma 2.8 ([12])**.** *let $f : R^+ \to R^+$ be a nonincreasing function and σ be a positive constant such that:*

$$\int_t^{+\infty} f^{1+\sigma}(s)ds \le \frac{1}{\omega} f^{\sigma}(0) f(t), \quad \forall\, t \ge 0.$$

Then we have

(i) $f(t) \le f(0)e^{1-\omega t}$*, for all $t \ge 0$, whenever $\sigma = 0$.*

(ii) $f(t) \le f(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}}$*, for all $t \ge 0$, whenever $\sigma > 0$.*

The following is the concavity lemma.

Lemma 2.9 ([13])**.** *Suppose that a positive, twice-differentiable function $\Psi(t)$ satisfies the inequality*

$$\Psi''(t)\Psi(t) - (1+\theta)(\Psi'(t))^2 \ge 0,$$

where $\theta > 0$. If $\Psi(0) > 0$ and $\Psi'(0) > 0$, then $\Psi(t) \to \infty$ as

$$t \to t_* \le t^* = \frac{\Psi(0)}{\theta \Psi'(0)}.$$

3 Local existence

In this section, we prove the local existence and uniqueness of weak solution to problem (1.1).

Lemma 3.1 ([16])**.** *Let $N > 4$, $2 < p < \bar{p}$. Then, for any $n \in N^+$ and any initial date $u_{n0} \in C_0^\infty(\Omega)$, there exists a unique weak solution $u_n \in L^\infty(0,T;H_0^2(\Omega))$ and $u_{nt} \in L^2(0,T;H_0^1(\Omega))$ satisfying the following equation*

$$\begin{cases} \rho_n(x)(u_n)_t + \Delta^2 u_n - \Delta(u_n)_t = |u_n|^{p-2}u_n \ln|u_n|, & x \in \Omega, t > 0; \\ u_n(x,t) = \Delta u_n(x,t) = 0, & x \in \partial\Omega, t > 0; \\ u_n(x,0) = u_{n0}, & x \in \Omega. \end{cases} \tag{3.1}$$

Theorem 3.2. *Let $u_0 \in H_0^2(\Omega) \setminus \{0\}$, $2 < p < \bar{p}$. Then there exist $T > 0$ and a unique weak solution $u(x,t) \in L^\infty(0,T;H_0^2(\Omega))$ of problem (1.1) with $u_t \in L^2(0,T;H_0^1(\Omega))$, $\frac{u_t}{|x|^2} \in L^2(0,T;L^2(\Omega))$ satisfying $u(x,0) = u_0(x)$. Moreover, $u(x,t)$ satisfies the energy equality*

$$\int_0^t \left(\left\| |x|^{-2} u_t \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + J(u(t)) = J(u_0), \qquad 0 \le t \le T.$$

Proof. We divide the proof of Theorem 3.2 into 3 steps.

Step 1. Local existence

We use Lemma 3.1 and approximation to prove the local existence of weak solutions to problem (1.1).

By Lemma 3.1, we know that $u_{n0} \in C_0^\infty(\Omega)$ such that

$$u_{n0} \to u_0(x) \quad \text{strongly in } H_0^2(\Omega), \tag{3.2}$$

and

$$\langle \rho_n(x)u_{nt}, \varphi\rangle + \langle \Delta u_n, \Delta\varphi\rangle + \langle \nabla u_{nt}, \nabla\varphi\rangle = \left\langle |u_n|^{p-2}u_n \ln|u_n|, \varphi \right\rangle. \tag{3.3}$$

Especially, taking $\varphi = u_n$ in (3.3), we get

$$\langle \rho_n(x)u_{nt}, u_n\rangle + \langle \Delta u_n, \Delta u_n\rangle + \langle \nabla u_{nt}, \nabla u_n\rangle = \left\langle |u_n|^{p-2}u_n \ln|u_n|, u_n \right\rangle. \tag{3.4}$$

Integrating the above equation over $[0,t]$, we have

$$\begin{aligned}
&\frac{1}{2}\left\| |\rho_n(x)|^{\frac{1}{2}} u_n(t) \right\|_2^2 + \int_0^t \|\Delta u_n(\tau)\|_2^2 d\tau + \frac{1}{2}\|\nabla u_n(t)\|_2^2 \\
&= \frac{1}{2}\left\| |\rho_n(x)|^{\frac{1}{2}} u_n(0) \right\|_2^2 + \frac{1}{2}\|\nabla u_n(0)\|_2^2 + \int_0^t \int_\Omega |u_n(\tau)|^p \ln|u_n(\tau)| dx d\tau.
\end{aligned}$$

Let

$$S_n(t) = \frac{1}{2}\left\| |\rho_n(x)|^{\frac{1}{2}} u_n(t) \right\|_2^2 + \frac{1}{2}\|\nabla u_n(t)\|_2^2 + \int_0^t \|\Delta u_n(\tau)\|_2^2 d\tau. \tag{3.5}$$

We observe that

$$S_n(t) = S_n(0) + \int_0^t \int_\Omega |u_n|^p \ln|u_n| \, dx d\tau. \tag{3.6}$$

From Lemma 2.6, we get

$$\begin{aligned}
\int_\Omega |u_n|^p \ln|u_n| \, dx &= \int_{\Omega_1 = \{x\in\Omega; |u_n(x)|\ge 1\}} |u_n|^p \ln|u_n| \, dx \\
&\quad + \int_{\Omega_2 = \{x\in\Omega; |u_n(x)|<1\}} |u_n|^p \ln|u_n| \, dx \\
&\le (e\mu)^{-1} \int_{\Omega_1 = \{x\in\Omega; |u_n(x)|\ge 1\}} |u_n|^{p+\mu} dx \\
&\le (e\mu)^{-1} \|u_n\|_{p+\mu}^{p+\mu}.
\end{aligned} \tag{3.7}$$

Then, by Lemma 2.7, Young's inequality and (3.7), we obtain

$$\begin{aligned}
\int_\Omega |u_n|^p \ln|u_n| \, dx &\le (e\mu)^{-1} \|u_n\|_{p+\mu}^{p+\mu} \\
&\le (e\mu)^{-1} C_G \|\Delta u_n\|_2^{\theta(p+\mu)} \|u_n\|_2^{(1-\theta)(p+\mu)} \\
&\le (e\mu)^{-1} C_G \varepsilon \|\Delta u_n\|_2^2 + (e\mu)^{-1} C_G C(\varepsilon) \|u_n\|_2^{\frac{2(1-\theta)(p+\mu)}{2-\theta(p+\mu)}} \\
&\le (e\mu)^{-1} C_G \varepsilon \|\Delta u_n\|_2^2 + (e\mu)^{-1} C_G C(\varepsilon) B_1 \|\nabla u_n\|_2^{\frac{2(1-\theta)(p+\mu)}{2-\theta(p+\mu)}},
\end{aligned} \tag{3.8}$$

where B_1 is the best constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, $\varepsilon \in (0,1)$, and we choose $\mu > 0$ with $2 < p + \mu < \frac{8}{N} + 2$. Substituting (3.8) into (3.6), we get

$$S_n(t) \le C_1 + C_2 \int_0^t [S_n(\tau)]^\alpha d\tau, \tag{3.9}$$

where $C_1 = \frac{S_n(0)}{1-(e\mu)^{-1}C_G\varepsilon}$, $C_2 = \frac{(e\mu)^{-1}C_G C(\varepsilon)2^\alpha B_1}{1-(e\mu)^{-1}C_G\varepsilon}$, $\alpha = \frac{4p+4\mu-Np-N\mu+2N}{8-N(p+\mu-2)} = 1 + \frac{4(p+\mu)-8}{2(4+N)-N(p+\mu)} > 1$. By direct calculation, we obtain

$$S_n(t) \le C_T, \tag{3.10}$$

where C_T is a positive constant dependent on T.

Now, multiplying the first equation of problem (1.1) by u_{nt} and integrating on $\Omega \times (0,t)$, we obtain

$$\int_0^t \left(\left\| |\rho_n(x)|^{\frac{1}{2}} u_{nt} \right\|_2^2 + \|\nabla u_{nt}\|_2^2 \right) d\tau + J(u_n(t)) = J(u_{n0}), \qquad 0 \le t \le T. \tag{3.11}$$

By the continuity of the functional $J(u)$ in $H_0^2(\Omega)$ and (3.2), there exists a constant $C > 0$ satisfying

$$J(u_{n0}) \le C, \quad \text{for any positive integer n.} \tag{3.12}$$

Applying (2.1), (3.5), (3.8), (3.11) and (3.12), we obtain

$$\begin{aligned} C \ge J(u_{n0}) \ge J(u_n(t)) &= \frac{1}{2}\|\Delta u_n\|_2^2 + \frac{1}{p^2}\|u_n\|_p^p - \frac{1}{p}\int_\Omega |u_n|^p \ln|u_n|\,dx \\ &\ge \frac{1}{2}\|\Delta u_n\|_2^2 + \frac{1}{p^2}\|u_n\|_p^p - \frac{C_G\varepsilon}{pe\mu}\|\Delta u_n\|_2^2 - \frac{C_G C(\varepsilon) B_1}{pe\mu}\|\nabla u_n\|_2^{2\alpha} \\ &\ge \left(\frac{1}{2} - \frac{C_G\varepsilon}{pe\mu}\right)\|\Delta u_n\|_2^2 + \frac{1}{p^2}\|u_n\|_p^p - \frac{C_G C(\varepsilon) B_1 2^\alpha}{pe\mu}(C_T)^\alpha, \end{aligned}$$

namely

$$\|\Delta u_n\|_2^2 + \|u_n\|_p^p \le C. \tag{3.13}$$

From (3.11)–(3.13), it follows that

$$\|u_n(t)\|_{L^\infty(0,T;H_0^2(\Omega))} \le C, \quad \text{for any positive integer } n, \tag{3.14}$$

$$\|u_n(t)\|_{L^\infty(0,T;L^p(\Omega))} \le C, \quad \text{for any positive integer } n, \tag{3.15}$$

$$\|u_{nt}(t)\|_{L^2(0,T;H_0^1(\Omega))} \le C, \quad \text{for any positive integer } n, \tag{3.16}$$

$$\left\| |\rho_n(x)|^{\frac{1}{2}} u_{nt} \right\|_{L^2(0,T;L^2(\Omega))} \le C, \quad \text{for any positive integer } n. \tag{3.17}$$

By (3.14), (3.16) and the Aubin–Lions–Simon lemma (see [17], Corollary 4), we get

$$u_n \to u \quad \text{in } C(0,T;L^2(\Omega)). \tag{3.18}$$

Therefore, $u_n(x,0) \to u(x,0) = u_0(x)$ in $L^2(\Omega)$. By (3.18), we have $u_n \to u$ a.e. $(x,t) \in \Omega \times (0,T)$, this implies

$$|u_n|^{p-2} u_n \ln|u_n| \to |u|^{p-2} u \ln|u| \quad \text{a.e. } (x,t) \in \Omega \times (0,T).$$

On the other hand, by a direct calculation and the Sobolev inequality, we have

$$\begin{aligned} \int_\Omega \left| |u_n|^{p-2} u_n \ln|u_n| \right|^2 dx &= \int_{\Omega_1 = \{x\in\Omega; |u_n(x)|\ge 1\}} \left| |u_n|^{p-2} u_n \ln|u_n| \right|^2 dx \\ &\quad + \int_{\Omega_2 = \{x\in\Omega; |u_n(x)|< 1\}} \left| |u_n|^{p-2} u_n \ln|u_n| \right|^2 dx \\ &\le (e\mu)^{-2} \|u_n\|_{2(p-1+\mu)}^{2(p-1+\mu)} + [e(p-1)]^{-2}|\Omega| \\ &\le (e\mu)^{-2} B_2 \|\Delta u_n\|_2^{2(p-1+\mu)} + [e(p-1)]^{-2}|\Omega| < C, \end{aligned}$$

where B_2 is the best constant of the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^{2(p-1+\mu)}(\Omega)$. Here we choose $0 < \mu \leq \frac{4}{N-4} + 2 - p$, $p < \frac{4}{N-4} + 2$, we know that

$$\left\| |u_n|^{p-2} u_n \ln |u_n| \right\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad \text{for any positive integer } n. \tag{3.19}$$

By (3.14)–(3.19), there exist functions u and a subsequence of $\{u_n\}_{n=1}^{\infty}$ which we still denote it by $\{u_n\}_{n=1}^{\infty}$ such that

$$\begin{aligned}
u_n &\to u && \text{weakly star in } L^\infty(0,T;H_0^2(\Omega)), && (3.20)\\
u_{nt} &\to u_t && \text{weakly star in } L^2(0,T;H_0^1(\Omega)), && (3.21)\\
|\rho_n(x)|^{\frac{1}{2}} u_{nt} &\to |x|^{-2} u_t && \text{weakly star in } L^2(0,T;L^2(\Omega)), && (3.22)\\
|u_n|^{p-2} u_n \ln |u_n| &\to |u|^{p-2} u \ln |u| && \text{weakly star in } L^\infty(0,T;L^2(\Omega)). && (3.23)
\end{aligned}$$

By (3.20)–(3.23), passing to the limit in (3.3), as $n \to +\infty$, it follows that u satisfies the initial condition $u(0) = u_0$,

$$\left\langle |x|^{-4} u_t, \varphi \right\rangle + \langle \Delta u, \Delta \varphi \rangle + \langle \nabla u_t, \nabla \varphi \rangle = \left\langle |u|^{p-2} u \ln |u|, \varphi \right\rangle,$$

for all $\varphi \in H_0^2(\Omega)$, and for a.e. $t \in [0,T]$.

Step 2. Uniqueness

Suppose there are two solutions u_1 and u_2 to the problem (1.1) with the same initial condition $u_1(x,0) = u_2(x,0) = u_0(x) \in H_0^2(\Omega)$, we have

$$\left\langle |x|^{-4} u_{1t}, v \right\rangle + \langle \Delta u_1, \Delta v \rangle + \langle \nabla u_{1t}, \nabla v \rangle = \left\langle |u_1|^{p-2} u_1 \ln |u_1|, v \right\rangle, \tag{3.24}$$

and

$$\left\langle |x|^{-4} u_{2t}, v \right\rangle + \langle \Delta u_2, \Delta v \rangle + \langle \nabla u_{2t}, \nabla v \rangle = \left\langle |u_2|^{p-2} u_2 \ln |u_2|, v \right\rangle. \tag{3.25}$$

Let $w = u_1 - u_2$ and $w(0) = 0$, then by subtracting the (3.24) and (3.25), we can derive

$$\int_\Omega |x|^{-4} w_t v dx + \int_\Omega \Delta w \Delta v dx + \int_\Omega \nabla w_t \nabla v dx = \int_\Omega \left(|u_1|^{p-2} u_1 \ln |u_1| - |u_2|^{p-2} u_2 \ln |u_2| \right) v dx,$$

Let $v = w$ and integrating it on $[0,t]$, we obtain

$$\frac{1}{2} \left\| |x|^{-2} w \right\|_2^2 + \int_\Omega \|\Delta w\|_2^2 dx + \frac{1}{2} \|\nabla w\|_2^2 = \int_0^t \int_\Omega \frac{|u_1|^{p-2} u_1 \ln |u_1| - |u_2|^{p-2} u_2 \ln |u_2|}{w} w^2 dx d\tau,$$

then

$$\|\nabla w\|_2^2 \leq 2 \int_0^t \int_\Omega \frac{f(u_1) - f(u_2)}{w} w^2 dx d\tau,$$

where $f(s) = |s|^{p-2} s \ln |s|$. By the Lipschitz continuity of $f : R^+ \to R^+$, we have

$$\|\nabla w\|_2^2 \leq 2 C_U \int_0^t \|\nabla w\|_2^2 d\tau.$$

Employing the Gronwall's inequality, the above inequality yields that $\|\nabla w\|_2^2 = 0$. Thus, we have $w = 0$ a.e. in $\Omega \times (0,T)$. Therefore, the uniqueness of problem (1.1) can be deduced.

Step 3. Energy equality

Multiplying (1.1) with u_t and integrating over $\Omega \times (0,t)$ to obtain the energy equality

$$\int_0^t \left(\left\| |x|^{-2} u_t \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + J(u(t)) = J(u_0), \qquad 0 \leq t \leq T. \tag{3.26}$$

The proof of Theorem 3.2 is completed □

4 Global existence and decay rate

In this section, we are concerned with the existence of global weak solutions to problem (1.1) and show that the norm $\|u(t)\|_{H_0^2(\Omega)}$ decays exponentially.

Theorem 4.1. *Assume that $u_0 \in W^+$, then problem (1.1) admits a global weak solution $u \in L^\infty(0,\infty;H_0^2(\Omega))$, $u_t \in L^2(0,\infty;H_0^1(\Omega))$ with $\frac{u_t}{|x|^2} \in L^2(0,\infty;L^2(\Omega))$, and $u(t) \in W^+$ for $0 \leq t \leq \infty$. Moreover, if $u_0 \in {W_1}^+$, then*

$$\|\Delta u\|_2^2 \leq \|\Delta u_0\|_2^2 e^{1-\frac{C_3}{C_4}t}, \quad t \geq 0,$$

where $C_3 = 1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2}{p}-1}$, $C_4 = \frac{R_N + B_1}{2}$, B_1 is the best embedding constant.

Step 1. Global existence

Proof. In order to prove the existence of global weak solutions, we consider two following cases.

Case 1. The initial data $u_0 \in W_1^+$.

Combining $J(u_0) < d$ with (3.26), then we get

$$\int_0^t \left(\left\| \frac{u_t}{|x|^2} \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + J(u(t)) = J(u_0) < d, \qquad 0 \leq t \leq T_{max}, \tag{4.1}$$

where T_{max} is the maximal existence time of solution $u(t)$, we shall prove that $T_{max} = +\infty$. Next, we will show that

$$u(x,t) \in W_1^+ \quad \text{for all } 0 \leq t \leq T_{max}. \tag{4.2}$$

In fact, assume that (4.2) does not hold and let t_* be the smallest time for which $u(t_*) \notin W_1^+$. Then, by the continuity of $u(t)$, we have $u(t_*) \in \partial W_1^+$. Hence, it follows that

$$J(u(t_*)) = d, \tag{4.3}$$

or

$$I(u(t_*)) = 0. \tag{4.4}$$

Nevertheless, it is clear that (4.3) is invalid by (4.1). On the other hand, if (4.4) holds, by the definition of d, we have

$$J(u(t_*)) \geq \inf_{u \in \mathcal{N}} J(u) = d,$$

which also contradicts with (4.1). Hence, we have $u(x,t) \in W_1^+$ such that $I(u(t)) > 0$. Consequently, it follows from this fact and (2.3) that

$$\int_0^t \left(\left\| \frac{u_t}{|x|^2} \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + \frac{1}{p} I(u) + \left(\frac{1}{2} - \frac{1}{p} \right) \|\Delta u\|_2^2 + \frac{1}{p^2} \|u\|_p^p < d, \tag{4.5}$$

namely

$$\int_0^t \left(\left\| \frac{u_t}{|x|^2} \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + \left(\frac{1}{2} - \frac{1}{p} \right) \|\Delta u\|_2^2 + \frac{1}{p^2} \|u\|_p^p < d. \tag{4.6}$$

This estimation allows us to take $T_{max} = +\infty$. Hence, we can conclude that there is a unique global weak solution $u(t) \in W_1^+$ of the problem (1.1).

Case 2. The initial data $u_0 \in W_2^+$.

Firstly, we choose a sequence $\{\theta_m\}_{m=1}^{\infty} \subset (0,1)$ such that $\lim\limits_{m\to\infty} \theta_m = 1$. Then we consider the following problem

$$\begin{cases} \dfrac{u_t}{|x|^4} + \Delta^2 u - \Delta u_t = |u|^{p-2} u \ln |u|, & (x,t) \in \Omega \times (0,T); \\ u(x,t) = \Delta u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T); \\ u(x,0) = u_{0m} = \theta_m u_0(x), & x \in \Omega. \end{cases} \tag{4.7}$$

Due to $I(\lambda(u_0)) = I(u_0) > 0$, we have $\lambda = 1$. From lemma 2.5, it follows that $\lambda^* > \lambda = 1$. Hence, from $\theta_m < 1 < \lambda^*$ we can deduce that $I(u_{0m}) = I(\theta_m u_0) > 0$ and $J(u_{0m}) = J(\theta_m u_0) < J(u_0) = d$, which means $u_{0m} \in W_1^+$. Using the similar arguments as the Case 1. We find that problem (4.7) admits a global weak solution u.

Step 2. Decay estimate

From $u_0 \in W_1^+$ and the conclusions of the global weak solutions, we know that $u(t) \in W_1^+$. Hence, by (2.3), we have

$$\left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u\|_2^2 + \frac{1}{p^2} \|u\|_p^p \le J(u(t)) \le J(u_0) < d. \tag{4.8}$$

Through a direct calculation, we arrive that

$$\lambda_0 \left[\left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u\|_2^2 + \frac{1}{p^2} \|u\|_p^p\right] \ge J(\lambda^* u(t)) \ge d, \tag{4.9}$$

where $\lambda_0 = \max\{(\lambda^*)^2, (\lambda^*)^p\}$. Combining with (4.8), we get

$$\lambda_0 \ge \frac{d}{J(u_0)} > 1, \tag{4.10}$$

so we can infer that $\lambda_* > 1$, it implies

$$\lambda^* \ge \left(\frac{d}{J(u_0)}\right)^{\frac{1}{p}} > 1. \tag{4.11}$$

From (2.2), we have

$$\begin{aligned} 0 = I(\lambda^* u) &= (\lambda^*)^2 \|\Delta u\|_2^2 - (\lambda^*)^p \int_\Omega |u|^p \ln |u| \, dx - (\lambda^*)^p \ln(\lambda^*) \|u\|_p^p \\ &= (\lambda^*)^p I(u) - \left[(\lambda^*)^p - (\lambda^*)^2\right] \|\Delta u\|_2^2 - (\lambda^*)^p \ln(\lambda^*) \|u\|_p^p. \end{aligned} \tag{4.12}$$

In view of (4.11) and (4.12) we have

$$I(u) = \|u\|_p^p \ln(\lambda^*) + \left[1 - (\lambda_*)^{2-p}\right] \|\Delta u\|_2^2 \ge C_3 \|\Delta u\|_2^2, \tag{4.13}$$

where $C_3 = 1 - \left(\frac{d}{J(u_0)}\right)^{\frac{2}{p}-1}$.

According to equation (2.2) and lemma 2.4, we obtain

$$\begin{aligned}\int_t^T I(u)\,ds &= \int_t^T \left(\|\Delta u\|_2^2 - \int_\Omega |u|^p \ln|u|\,dx\right) ds\\ &= -\frac{1}{2}\int_t^T \left(\frac{d}{dt}\left\|\frac{u}{|x|^2}\right\|_2^2 + \frac{d}{dt}\|\nabla u\|_2^2\right) ds\\ &= \frac{1}{2}\left(\left\|\frac{u(t)}{|x|^2}\right\|_2^2 + \|\nabla u(t)\|_2^2\right) - \frac{1}{2}\left(\left\|\frac{u(T)}{|x|^2}\right\|_2^2 + \|\nabla u(T)\|_2^2\right)\\ &\le \frac{1}{2}\left(\left\|\frac{u(t)}{|x|^2}\right\|_2^2 + \|\nabla u(t)\|_2^2\right)\\ &\le \left(\frac{R_N+B}{2}\right)\|\Delta u(t)\|_2^2 = C_4\|\Delta u(t)\|_2^2,\end{aligned} \tag{4.14}$$

where $C_4 = \frac{R_N+B_1}{2}$, B_1 is the best embedding constant.

By (4.13) and (4.14), we get

$$\int_t^T \|\Delta u(s)\|_2^2 ds \le \frac{C_4}{C_3}\|\Delta u(t)\|_2^2, \quad \text{for all } t \in [0,T], \tag{4.15}$$

let $T \to +\infty$ in (4.15), by virtue of lemma 2.8, it follows that

$$\|\Delta u(t)\|_2^2 \le \|\Delta u_0\|_2^2 e^{1-\frac{C_3}{C_4}t}.$$

The proof of Theorem 4.1 is completed. □

5 Blow-up phenomena of weak solutions

In this section, we consider the finite time blow-up results of weak solutions with $u_0 \in W^-$, and give the upper and lower bounds for blow up time to problem (1.1). For simplicity, we shall write

$$L(t) = \frac{1}{2}\left(\left\|\frac{u(t)}{|x|^2}\right\|_2^2 + \|\nabla u(t)\|_2^2\right).$$

5.1 Upper bound for blow-up time

Theorem 5.1. *Assume that $u_0 \in W^-$, $2 < p < \bar{p}$. Then the weak solution $u(t)$ of problem (1.1) blows up in finite time, the upper bound for blow-up time T_{max} is given by*

$$T_{max} \le \frac{\beta b^2}{(p-2)\beta b - \left(\left\|\frac{u_0}{|x|^2}\right\|_2^2 + \|\nabla u_0\|_2^2\right)},$$

where

$$\beta \in \left(0, \frac{p(d-J(u_0))}{p-1}\right], \qquad b > \max\left\{0, \frac{\left\|\frac{u_0}{|x|^2}\right\|_2^2 + \|\nabla u_0\|_2^2}{(p-2)\beta}\right\}.$$

Proof. We will divide the proof into two cases.

Case 1: $u_0 \in W_1^-$.

We claim that $u(t) \in W_1^-$ for $t \in [0, T_{max}]$ provided that $u_0 \in W_1^-$. Indeed, by contradiction, there exists a $t_0 \in (0, T_{max})$ such that $I(u(t)) > 0$ for $t \in [0, t_0)$ and $I(u(t_0)) = 0$. Recalling the definition of d, it is clear that $J(u(t_0)) \geq d$ which contradicts with $J(u(t)) \leq J(u_0) < d$. Hence, we get $u(t) \in W_1^-$ for $t \in [0, T_{max}]$.

From lemma 2.5, as $I(u(t)) < 0$, there is a $\lambda^* < 1$ such that $I(\lambda^* u) = 0$. Then

$$\begin{aligned} d \leq J(\lambda^* u) &= \frac{1}{p} I(\lambda^* u) + (\lambda^*)^2 \left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u\|_2^2 + \frac{(\lambda^*)^p}{p^2} \|u\|_p^p \\ &< \left(\frac{1}{2} - \frac{1}{p}\right) \|\Delta u\|_2^2 + \frac{1}{p^2} \|u\|_p^p . \end{aligned} \tag{5.1}$$

We show that $T_{max} < +\infty$. For any $T \in [0, T_{max})$, define the positive function

$$F(t) = \int_0^t L(t)\, d\tau + (T - t) L(0) + \frac{\beta}{2}(t + b)^2, \tag{5.2}$$

where $\beta > 0, b > 0$. We compute the first-order differential and second-order differential of $F(t)$, respectively, as follows:

$$\begin{aligned} F'(t) &= L(t) - L(0) + \beta(t + b) \\ &= \int_0^t \frac{d}{dt} L(t)\, d\tau + \beta(t + b) \\ &= \int_0^t \left(\int_\Omega \frac{u \cdot u_t}{|x|^4} dx + \int_\Omega \nabla u \cdot \nabla u_t dx \right) d\tau + \beta(t + b), \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} F''(t) &= L'(t) + \beta = -I(u) + \beta \\ &= -pJ(u) + \left(\frac{p}{2} - 1\right) \|\Delta u\|_2^2 + \frac{1}{p} \|u\|_p^p + \beta. \end{aligned} \tag{5.4}$$

From (5.2)–(5.4), through a direct calculation, we have

$$\begin{aligned} &F(t)F''(t) - (1+\theta)\left[F'(t)\right]^2 = F(t) F''(t) \\ &+ (1+\theta) \left\{ H(t) - [2F(t) - 2(T - t) L(0)] \left[\int_0^t \left(\left\| \frac{u_t}{|x|^2} \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + \beta \right] \right\}, \end{aligned} \tag{5.5}$$

the definition of $H(t)$ is following

$$\begin{aligned} H(t) = &\left[\int_0^t \left(\left\| \frac{u}{|x|^2} \right\|_2^2 + \|\nabla u\|_2^2 \right) d\tau + \beta(t+b)^2 \right] \cdot \left[\int_0^t \left(\left\| \frac{u_t}{|x|^2} \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau + \beta \right] \\ &- \left[\int_0^t \int_\Omega \left(\frac{u u_t}{|x|^4} + \nabla u \cdot \nabla u_t \right) dx d\tau + \beta(t + b) \right]^2 . \end{aligned}$$

Applying the Cauchy–Schwarz inequality, Young's inequality and Hölder's inequality, it is easy to verify that $H(t) \geq 0$ for any $t \in (0, T)$. Therefore, choosing $\theta = \frac{p-2}{2} > 0$, there are

$$\begin{aligned}
&F(t) F''(t) - \frac{p}{2}\left[F'(t)\right]^2 \\
&\geq F(t) F''(t) - \frac{p}{2}\left[2F(t) - 2(T-t)L(0)\right]\left[\int_0^t \left(\left\|\frac{u_t}{|x|^2}\right\|_2^2 + \|\nabla u_t\|_2^2\right) d\tau + \beta\right] \\
&\geq F(t)\left[F''(t) - p\int_0^t \left(\left\|\frac{u_t}{|x|^2}\right\|_2^2 + \|\nabla u_t\|_2^2\right) d\tau - p\beta\right] \\
&= F(t)\left[-pJ(u) + \left(\frac{p}{2} - 1\right)\|\Delta u\|_2^2 + \frac{1}{p}\|u\|_p^p\right. \\
&\qquad \left. - p\int_0^t \left(\left\|\frac{u_t}{|x|^2}\right\|_2^2 + \|\nabla u_t\|_2^2\right) d\tau + (1-p)\beta\right] \\
&= F(t)\,\varsigma(t),
\end{aligned}$$

we denote $\varsigma(t)$ as follows

$$\varsigma(t) = -pJ(u) + \left(\frac{p}{2} - 1\right)\|\Delta u\|_2^2 + \frac{1}{p}\|u\|_p^p - p\int_0^t \left(\left\|\frac{u_t}{|x|^2}\right\|_2^2 + \|\nabla u_t\|_2^2\right) d\tau + (1-p)\beta.$$

From (5.1) and $u(t) \in W_1^-$, when we chose $\beta \in \left(0, \frac{p(d-J(u_0))}{p-1}\right]$, we have

$$\begin{aligned}
\varsigma(t) = & -pJ(u_0) + \left(\frac{p}{2} - 1\right)\|\Delta u\|_2^2 + \frac{1}{p}\|u\|_p^p + (1-p)\beta \\
\geq & \; p(d - J(u_0)) + (1-p)\beta \geq 0.
\end{aligned}$$

Hence, by the above discussions, (5.5) becomes that

$$F(t) F''(t) - (1+\theta)\left[F'(t)\right]^2 \geq 0.$$

Therefore, Lemma 2.9 guarantees that $F(0) > 0$, $F'(0) = \beta b > 0$, then there is a T_1 satisfies $0 < T_1 < \frac{2F(0)}{(p-2)F'(0)}$ such that $F(t) \to \infty$, $t \to T_1$, we can obtain that

$$T_{max} \leq \frac{\beta b^2}{(p-2)\beta b - \left(\left\|\frac{u_0}{|x|^2}\right\|_2^2 + \|\nabla u_0\|_2^2\right)},$$

where

$$b > \max\left\{0, \frac{\left\|\frac{u_0}{|x|^2}\right\|_2^2 + \|\nabla u_0\|_2^2}{(p-2)\beta}\right\}.$$

Case 2: $u_0 \in W_2^-$

By the similar arguments as those in the proof of Case 1. When $u_0 \in W_2^-$, by continuity we see that there exists a $t_1 > 0$ such that $I(u(t_1)) < 0$, $\|\frac{u_t}{|x|^2}\|_2^2 > 0$ and $\|\nabla u_t\|_2^2 > 0$ for all $t \in [0, t_1)$. From energy equality we get

$$J(u(t_1)) \leq J(u_0) - \int_0^{t_1} \left(\left\| \frac{u_t}{|x|^2} \right\|_2^2 + \|\nabla u_t\|_2^2 \right) d\tau < J(u_0) = d.$$

The remainder of the proof is the same as Case 1. □

5.2 Lower bound for blow-up time

In this subsection, we shall derive a lower bound for the blow-up time T_{max}.

Theorem 5.2. *Assume that $u_0 \in W^-$, $2 < p < \bar{p}$. Then the weak solution $u(t)$ of problem (1.1) blows up in finite time, the lower bound for blow-up time T_{max} is given by*

$$T_{max} \geq \frac{L^{1-\alpha}(0)}{C_L(\alpha - 1)},$$

where $C_L = 2^\alpha (e\mu)^{-1} C_G C(\varepsilon) B_2$, $\alpha = \frac{4p+4\mu-Np-N\mu+2N}{8-N(p+\mu-2)} > 1$.

Proof. According to the proof proof of Theorem 5.1, we can get $u(t) \in W^-$. From problem (1.1) and equation (2.2), we obtain

$$\begin{aligned} L'(t) &= \int_\Omega \frac{u \cdot u_t}{|x|^4} dx + \int_\Omega \nabla u \cdot \nabla u_t dx \\ &= -\|\Delta u\|_2^2 + \int_\Omega |u|^p \ln |u| \, dx \\ &= -I(u) > 0. \end{aligned} \tag{5.6}$$

Recalling the inequality (3.8) and combining (3.8) and (5.6), it follows that

$$L'(t) \leq \left[(e\mu)^{-1} C_G \varepsilon - 1 \right] \|\Delta u\|_p^p + (e\mu)^{-1} C_G C(\varepsilon) B_2 \|\nabla u\|_2^{2\alpha}. \tag{5.7}$$

In view of $(e\mu)^{-1} C_G \varepsilon - 1 < 0$, $\alpha = \frac{4p+4\mu-Np-N\mu+2N}{8-N(p+\mu-2)} > 1$ and the definition of $L(t)$, we get

$$L'(t) \leq (e\mu)^{-1} C_G C(\varepsilon) B_2 \|\nabla u\|_2^{2\alpha} \leq C_L L^\alpha(t), \tag{5.8}$$

where $C_L = 2^\alpha (e\mu)^{-1} C_G C(\varepsilon) B_2$. Integrating (5.8) over $[0, t)$, we get

$$\frac{1}{1-\alpha} \left[L^{1-\alpha}(t) - L^{1-\alpha}(0) \right] \leq C_L t.$$

Since $\alpha > 1$, letting $t \to T_{max}$ in the above inequality and recalling that $\lim_{t \to T_{max}} L(t) = +\infty$, we obtain

$$T_{max} \geq \frac{L^{1-\alpha}(0)}{C_L(\alpha - 1)}.$$

The proof of Theorem 5.1 and Theorem 5.2 are finished. □

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Multiple subharmonic solutions with prescribed minimal periods for a class of second order impulsive differential systems

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Abstract. In this paper, we obtain three subharmonic solutions with different prescribed minimal periods for a class of second order impulsive differential systems. Our proof is based on level estimates and the least action principle in critical point theory.

Keywords: subharmonic solution, impulsive system, minimal period, variational method.

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1 Introduction and main result

In many evolution processes, the states of systems are changed abruptly at certain instants, which leads to impulsive behaviors in dynamical systems [2, 11, 13]. In recent years, the investigation of differential equations with impulses got particular attention by a lot of scholars, because of the widespread application of these impulsive differential systems in biology, mechanics, engineering and chaos theory, etc. [5, 6, 13, 21, 22].

Some classical approaches, such as the method of upper and lower solutions with the monotone iterative technique, the coincidence degree theory of Mawhin and the fixed point theory, were used to study impulsive problems [2, 11]. Especially, in the remarkable work of Nieto and O'Regan [13], by constructing a variational structure, they converted the problem of finding solutions for a second order impulsive equation to that of the existence of critical points for the corresponding energy functional [19]. After that, the variational methods and critical point theory were applied to prove the existence and multiplicity of solutions for second order, fourth order and fractional order impulsive differential equations by more and more researchers, see [1–4, 7–11, 13, 14, 16–20].

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Our purpose is to investigate the existence of multiple subharmonic solutions for the following second order impulsive systems

$$\begin{cases} \ddot{x}(t) + \nabla V(t, x(t)) = 0, \\ \Delta\left(\dot{x}^i\left(t_j\right)\right) = \dot{x}^i\left(t_j^+\right) - \dot{x}^i\left(t_j^-\right) = I_{ij}\left(x^i\left(t_j\right)\right), i = 1, 2, \ldots, N, j = 1, 2, \ldots, l, \\ x(0) = x(pT),\ \dot{x}(0) = \dot{x}(pT), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^N$, $\nabla V \in C([0,T] \times \mathbb{R}^N, \mathbb{R}^N)$ denotes the gradient of V in x, $I_{ij} \in C\left(\mathbb{R}^N, \mathbb{R}\right)$ and impulses occur at instants t_j with $j \in \mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$, $0 < t_1 < \cdots < t_l < T$ and $t_{j+l} = t_j$.

In [11], Luo, Xiao and Xu studied the existence of subharmonic solutions for the equation with non-negative impulses as follows

$$\begin{cases} \ddot{x}(t) + f(t, x(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}\backslash\{t_k \mid k \in \mathbb{Z}^*\}, \\ \Delta\dot{x}(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad k \in \mathbb{Z}^*, \end{cases}$$

where $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $I_k \geq 0$ are impulses that happen at instants t_k. Bai and Wang [2] generalized the results in [11] to allow a negative impulse term. Here, motivated by [2,11], we investigate the existence of multiple subharmonic solutions for system (1.1).

By a classical solution of (1.1), we mean a function

$$x \in \left\{w \in C([0,pT], \mathbb{R}^N) : w|_{[t_j, t_{j+1}]} \in H^2([t_j, t_{j+1}], \mathbb{R}^N),\ j = 1, 2, \ldots, l\right\},$$

which satisfies the differential equation in (1.1) and the boundary conditions $x(0) = x(pT)$, $\dot{x}(0) = \dot{x}(pT)$, the limits $\dot{x}^i(t_j^+), \dot{x}^i(t_j^-), i = 1, 2, \ldots, N, j = 1, 2, \ldots, l$, exist and verify the impulsive conditions in (1.1).

Now we state our main result.

Theorem 1.1. Suppose that $V(t, x)$ and $I_{ij}(x)$ satisfy the following conditions.

(H1) $V(t, x) = V(-t, x) = V(t, -x) = V(t + \frac{T}{2}, x)$ for every $(t, x) \in [0, T] \times \mathbb{R}^N$.

(H2) For every $x \in \mathbb{R}^N, t \in [0, T]$, there exist constants $\delta > 0$ and $A > \bar{A} > 0$ such that

$$V(t, x) \geq \frac{\bar{A}}{2}|x|^2, \qquad |x| \leq \delta$$

and

$$V(t, x) - (\nabla V(t, 0), x) \leq \frac{A}{2}|x|^2.$$

(H3) For $i = 1, 2, \ldots, N, j = 1, 2, \ldots, l$, there exist constants $d_{ij} \geq 0$ such that

$$I_{ij}(x) \leq d_{ij}|x|, \qquad x \in \mathbb{R}^N.$$

(H4) There exists an integer $p > 1$ such that

$$1 - 2\rho p^2 T > 0, \quad \frac{4\omega^2}{\bar{A} - 4D/T} < p^2 < \frac{\omega^2 s_p^2}{2\rho T \omega^2 s_p^2 + A + 2\rho/T}$$

and

$$\left(\frac{\omega^2}{2p^2} - \rho T \omega^2 - \frac{\rho}{T}\right)|x|^2 - V(t, x) \to +\infty, \quad \text{as } |x| \to \infty,$$

where $\omega = \frac{2\pi}{T}, \rho = \sum_{j=1}^{l} \sum_{j=1}^{N} d_{ij}, D = \sum_{j=1}^{l} \left(\sum_{i=1}^{N} d_{ij}^2\right)^{\frac{1}{2}}$ and s_p is the smallest prime factor of p.

(H5) ∇V satisfies

$$\int_0^T |\nabla V(t,0)|^2 \,\mathrm{d}t < \delta^2 \min\left\{K_1(K_2-K_3), \left(K_1 - \frac{3\omega^2 T}{p^2}\right)(4K_2-K_3)\right\},$$

$$K_1 = T\left(\bar{A} - \frac{\omega^2}{p^2}\right) - 4D, \qquad K_2 = \frac{\omega^2 s_p^2}{2p^2}\left(1 - 2\rho p^2 T\right), \qquad K_3 = \frac{A}{2} + \frac{\rho}{T}.$$

(H6) Suppose that q_0 is rational. If both $x(t)$ and $\nabla V(t,x)$ have minimal period $\frac{q_0 T}{2}$, then q_0 is an integer.

Then the impulsive system (1.1) possesses at least three periodic solutions. Two of them have minimal period pT and the other one has minimal period $\frac{pT}{2}$.

Remark 1.2. In [11], Luo, Xiao and Xu investigated second order impulsive differential equations with a non-negative impulse term and obtained the existence of at least one solution with minimal period pT. Bai and Wang [2] generalized the results of [11] by proving the existence of at least one solution with prescribed minimal period for second order impulsive systems allowing negative impulse terms. Here, we also do not have to assume that the impulse term is non-negative. Giving a suitable range of p and $\int_0^T |\nabla V(t,0)|^2 \mathrm{d}t$, we find three solutions with prescribed minimal periods for system (1.1).

2 Proof of the theorem

In the first place, we recall some basic notations. Let $p > 1$ is an integer, $T > 0$. We denote the inner product on $\mathbb{R}^N$ by $(\cdot,\cdot)$. $H^1_{pT}(\mathbb{R}^N)$ is a Hilbert space, which defined as

$$H^1_{pT} = \{x : [0,pT] \to \mathbb{R}^N \mid x \text{ is absolutely continuous}, \ x(0) = x(pT),\ \dot{x} \in L^2(0,pT;\mathbb{R}^N)\}.$$

Let $\langle\cdot,\cdot\rangle$ be the inner product on H^1_{pT}, i.e.

$$\langle x,y\rangle = \int_0^{pT} (\dot{x},\dot{y})\mathrm{d}t + \int_0^{pT} (x,y)\mathrm{d}t, \qquad x,y \in H^1_{pT},$$

which induces the norm $\|x\| = \langle x,x\rangle$. Additionally, the energy functional corresponds to system (1.1) is

$$\varphi(x) = \int_0^{pT} \left[\frac{1}{2}|\dot{x}|^2 - V(t,x)\right]\mathrm{d}t + \sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{x^i(t_j)} I_{ij}(s)\mathrm{d}s, \qquad x \in H^1_{pT}.$$

It follows that

$$\begin{aligned}\langle \varphi'(x),y\rangle = &\int_0^{pT}(\dot{x},\dot{y})\mathrm{d}t - \int_0^{pT}(\nabla V(t,x),y)\,\mathrm{d}t \\ &+ \sum_{j=1}^{pl}\sum_{j=1}^{N} I_{ij}\left(x^i(t_j)\right) y^i(t_j), \quad x,y \in H^1_{pT}.\end{aligned} \tag{2.1}$$

Definition 2.1. A function x is called a weak pT-periodic solution of (1.1) if and only if the following equation holds

$$\int_0^{pT}(\dot{x},\dot{y})\mathrm{d}t + \sum_{j=1}^{pl}\sum_{j=1}^{N} I_{ij}\left(x^i(t_j)\right) y^i(t_j) = \int_0^{pT}(\nabla V(t,x),y)\,\mathrm{d}t, \qquad \forall y \in H^1_{pT}.$$

The critical points of φ correspond to periodic solutions of impulsive system (1.1). Indeed, suppose x is a critical point of φ, by (2.1) and the Definition 2.1, x is a weak pT-periodic solution of (1.1). Moreover, for every $y \in H^1_{pT}$, we have

$$\begin{aligned} \langle \varphi'(x), y \rangle &= \int_0^{pT} (\dot{x}, \dot{y}) \mathrm{d}t - \int_0^{pT} (\nabla V(t,x), y) \, \mathrm{d}t + \sum_{j=1}^{pl} \sum_{j=1}^{N} I_{ij} \left(x^i(t_j) \right) y^i \left(t_j \right) \\ &= - \int_0^{pT} (\ddot{x}, y) \mathrm{d}t - \int_0^{pT} (\nabla V(t,x), y) \, \mathrm{d}t. \end{aligned} \tag{2.2}$$

It follows from (2.2) that

$$\ddot{x}(t) + \nabla V(t,x) = 0, \quad \text{a.e. } t \in [t_j, t_{j+1}].$$

Then we get $x \in H^2([t_j, t_{j+1}], \mathbb{R}^N)$ and

$$\ddot{x}(t) + \nabla V(t,x) = 0, \quad \text{a.e. } t \in [0, pT].$$

Multiplying the above equation by $y \in H^1_{pT}$ and integrating over $[0, pT]$, we obtain

$$\sum_{j=1}^{pl} \sum_{j=1}^{N} \Delta \left(\dot{x}^i \left(t_j \right) \right) y^i(t_j) = \sum_{j=1}^{pl} \sum_{j=1}^{N} I_{ij} \left(x^i \left(t_j \right) \right) y^i(t_j).$$

Thus, $\Delta \left(\dot{x}^i \left(t_j \right) \right) = I_{ij} \left(x^i \left(t_j \right) \right)$ for $i = 1, 2, \ldots, N, j = 1, 2, \ldots, l$, and the impulsive conditions in (1.1) are verified.

For the sake of convenience, let us define a couple of subspaces of H^1_{pT}. Set

$$X = \left\{ x \in H^1_{pT} \mid x(t) = -x(-t) \right\}, \qquad Y = \left\{ x \in H^1_{pT} \mid x \left(t + \frac{pT}{2} \right) = -x(t) \right\},$$

then we can define

$$\begin{aligned} X_1 &= X \cap Y, \qquad & X_2 &= X \cap Y^{\perp}, \\ Y_1 &= X^{\perp} \cap Y, \qquad & Y_2 &= X^{\perp} \cap Y^{\perp}. \end{aligned}$$

Clearly, we have $H^1_{pT} = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2$. In the following, we denote the norm of x on L^2_{pT} and L^∞_{pT} by $\|x\|_{L^2}$ and $\|x\|_\infty$ respectively.

Lemma 2.2 ([12])**.** Suppose that W is a reflexive Banach space, $\varphi : W \to \mathbb{R}$ is weakly lower semi-continuous and coercive on W, then φ attains its minimum on W.

Lemma 2.3. Under condition (H1), critical points of φ on X_1 (or X_2, Y_1, Y_2) are also critical points of φ on H^1_{pT}. The minimal period of such a critical point is an integer multiple of $\frac{T}{2}$.

Proof. On the one hand, if x is a critical point of φ on X, then

$$\langle \varphi'(x), y \rangle = 0, \qquad \forall y \in X.$$

Let $y \in X^{\perp}$, it can be deduced from (H1) and (2.2) that

$$\begin{aligned}\langle \varphi'(x), y\rangle &= -\int_{-\frac{pT}{2}}^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_{-\frac{pT}{2}}^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\ &= -\int_0^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_0^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\ &\quad -\int_0^{\frac{pT}{2}} (\ddot{x}(-t), y(-t))\mathrm{d}t - \int_0^{\frac{pT}{2}} (\nabla V(-t,x(-t)), y(-t))\,\mathrm{d}t \\ &= -\int_0^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_0^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\ &\quad -\int_0^{\frac{pT}{2}} (-\ddot{x}(t), y(t))\mathrm{d}t - \int_0^{\frac{pT}{2}} (-\nabla V(t,x(t)), y(t))\,\mathrm{d}t \\ &= 0.\end{aligned} \tag{2.3}$$

Thus, $\langle \varphi'(x), y\rangle = 0$, for all $y \in H^1_{pT}$.

On the other hand, providing that x is a critical point of φ on X_1, set $y \in X_2$, we find

$$\begin{aligned}\langle \varphi'(x), y\rangle &= -\int_{-\frac{pT}{2}}^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_{-\frac{pT}{2}}^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\ &= -\int_0^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_0^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\ &\quad -\int_0^{\frac{pT}{2}} \left(\ddot{x}\left(t-\frac{pT}{2}\right), y\left(t-\frac{pT}{2}\right)\right)\mathrm{d}t \\ &\quad -\int_0^{\frac{pT}{2}} \left(\nabla V\left(t-\frac{pT}{2}, x\left(t-\frac{pT}{2}\right)\right), y\left(t-\frac{pT}{2}\right)\right)\mathrm{d}t \\ &= -\int_0^{\frac{pT}{2}} (\ddot{x}, y)\mathrm{d}t - \int_0^{\frac{pT}{2}} (\nabla V(t,x), y)\,\mathrm{d}t \\ &\quad -\int_0^{\frac{pT}{2}} (-\ddot{x}(t), y(t))\mathrm{d}t - \int_0^{\frac{pT}{2}} (\nabla V(t,-x(t)), y(t))\,\mathrm{d}t \\ &= 0.\end{aligned}$$

It follows that x is a critical point of φ on X. From (2.3) we know that x is a critical point of φ on H^1_{pT}.

By a similar discussion, one can prove the cases of X_2, Y_1, Y_2 alike.

If the minimal period of $x(t)$ is $\frac{pT}{2q}$, where q is an integer. From (1.1) we have

$$\ddot{x}\left(t+\frac{pT}{2q}\right) + \nabla V\left(t+\frac{pT}{2q}, x\left(t+\frac{pT}{2q}\right)\right) = 0. \tag{2.4}$$

It follows from (2.4) that $\nabla V(t, x(t))$ has minimal period $\frac{pT}{2q}$. Then by $(H6)$, $\frac{p}{q}$ is an integer, which means that the minimal period of $x(t)$ is an integer multiple of $\frac{T}{2}$. □

Lemma 2.4 ([15]). Suppose that $H(t,x) \in C^1([0,T] \times \mathbb{R}^N, \mathbb{R})$ with $H(t,x) \to +\infty$ as $|x| \to \infty$ uniformly in $t \in [0,T]$, then there exist a real function $\gamma \in L^1([0,T], \mathbb{R})$ and a subadditive function $G : \mathbb{R}^N \to \mathbb{R}$, i.e.

$$G(x+y) \le G(x) + G(y), \qquad x, y \in \mathbb{R}^N, \tag{2.5}$$

such that

$$H(t,x) \geq G(x) + \gamma(t), \qquad x \in \mathbb{R}^N, \tag{2.6}$$

$$G(x) \to +\infty \quad \text{as } |x| \to \infty, \tag{2.7}$$

$$0 \leq G(x) \leq |x| + 1, \qquad x \in \mathbb{R}^N. \tag{2.8}$$

Lemma 2.5. Under condition (H4), φ is coercive on X_1 (or X_2, Y_1).

Proof. From Lemma 2.4 and (H4), there exist $G(x)$ and $\gamma(t) \in L^1([0,T],\mathbb{R})$ such that

$$\left(\frac{\omega^2}{2p^2} - \rho T\omega^2 - \frac{\rho}{T}\right)|x|^2 - V(t,x) \geq G(x) + \gamma(t). \tag{2.9}$$

To begin with, we claim that $\int_0^{pT} G(x)\mathrm{d}t$ is coercive on

$$X_1^1 = \left\{ r\sin\frac{\omega t}{p} \mid r \in \mathbb{R}^N \right\} \subset X_1.$$

Providing that $\{x_n\}$ is a sequence in X_1^1 with $\|x_n\| \to \infty$ as $n \to \infty$, we can set $x_n(t) = r_n \sin\frac{\omega t}{p}$, where $r_n \in \mathbb{R}^N$ and $|r_n| \to +\infty$ as $n \to \infty$. By (2.7), for every $L > 0$, there exists $M > 0$ such that

$$G(x) \geq L, \qquad |x| \geq M. \tag{2.10}$$

Since $|r_n| \to +\infty$ as $n \to \infty$, there exists $N_0 > 0$ such that $|r_n| > 2M$ for $n > N_0$. Furthermore, it is clear that

$$|x_n(t)| > M, \qquad \forall t \in \left[\frac{pT}{12}, \frac{5pT}{12}\right] \cup \left[\frac{7pT}{12}, \frac{11pT}{12}\right], \quad n > N_0. \tag{2.11}$$

From (2.10) and (2.11) we have

$$\int_0^{pT} G(x_n)\mathrm{d}t > \frac{2pLT}{3}, \qquad n > N_0.$$

The coercivity of $\int_0^{pT} G(x)\mathrm{d}t$ follows from the arbitrariness of L and $\{x_n\}$.

Let $x \in X_1$ and $x = x_1 + x_2$, where $x_1 \in X_1^1, x_2 \in (X_1^1)^\perp \cap X_1$. By the Parseval equality,

$$\|\dot{x}_1\|_{L^2}^2 = \frac{\omega^2}{p^2}\|x_1\|_{L^2}^2, \qquad \|\dot{x}_2\|_{L^2}^2 \geq \frac{9\omega^2}{p^2}\|x_2\|_{L^2}^2, \qquad \|\dot{x}_2\|_{L^2}^2 \geq \frac{9\omega^2}{9\omega^2 + p^2}\|x_2\|^2. \tag{2.12}$$

Additionally, (2.5) implies that

$$G(x_1) = G(x - x_2) \leq G(x) + G(-x_2). \tag{2.13}$$

It can be deduced from (H3) that

$$\begin{aligned} \left|\sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{y^i(t_j)} I_{ij}(s)\mathrm{d}s\right| &\leq \sum_{j=1}^{pl}\sum_{i=1}^{N}\frac{1}{2}d_{ij}\left|y\left(t_j\right)\right|^2 \\ &\leq \sum_{j=1}^{pl}\sum_{i=1}^{N} d_{ij}\left(\frac{\|y\|_{L^2}^2}{pT} + pT\|\dot{y}\|_{L^2}^2\right) \\ &\leq \frac{\rho}{T}\|y\|_{L^2}^2 + \rho p^2 T\|\dot{y}\|_{L^2}^2, \qquad y \in H_{pT}^1. \end{aligned} \tag{2.14}$$

To the best of our knowledge, the formula (2.14) was first proved in [2]. For more details, please refer to (2.9) in [2]. From (H4), (2.9), (2.12), (2.13) and (2.14), we have

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\int_0^{pT}|\dot{x}|^2\mathrm{d}t - \int_0^{pT}V(t,x)\mathrm{d}t + \sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{x^i(t_j)}I_{ij}(s)\mathrm{d}s \\ &\geq \left(\frac{1}{2}-\rho p^2 T\right)\int_0^{pT}|\dot{x}|^2\mathrm{d}t - \frac{\rho}{T}\int_0^{pT}|x|^2\mathrm{d}t - \int_0^{pT}V(t,x)\mathrm{d}t \\ &= \left(\frac{1}{2}-\rho p^2 T\right)\int_0^{pT}\left(|\dot{x}_1|^2+|\dot{x}_2|^2-\frac{\omega^2}{p^2}|x_1|^2-\frac{\omega^2}{p^2}|x_2|^2\right)\mathrm{d}t \\ &\quad + \int_0^{pT}\left[\left(\frac{\omega^2}{2p^2}-\rho\omega^2 T-\frac{\rho}{T}\right)|x|^2 - V(t,x)\right]\mathrm{d}t \\ &\geq \frac{8}{9}\left(\frac{1}{2}-\rho p^2 T\right)\int_0^{pT}|\dot{x}_2|^2\,\mathrm{d}t + \int_0^{pT}\left[G(x)+\gamma(t)\right]\mathrm{d}t \\ &\geq \frac{8}{9}\left(\frac{1}{2}-\rho p^2 T\right)\|\dot{x}_2\|_{L^2}^2 + \int_0^{pT}G(x_1)\,\mathrm{d}t - \int_0^{pT}G(-x_2)\,\mathrm{d}t + \int_0^{pT}\gamma(t)\mathrm{d}t \\ &\geq \frac{8}{9}\left(\frac{1}{2}-\rho p^2 T\right)\|\dot{x}_2\|_{L_2}^2 + \int_0^{pT}G(x_1)\,\mathrm{d}t - pT(1+\|x_2\|_\infty+\|\gamma\|_\infty) \\ &\geq \frac{8\omega^2}{9\omega^2+p^2}\left(\frac{1}{2}-\rho p^2 T\right)\|x_2\|^2 + \int_0^{pT}G(x_1)\,\mathrm{d}t - C_1\|x_2\| - C_2,\end{aligned} \tag{2.15}$$

where C_1 and C_2 are positive constants. With $\int_0^{pT}G(x_1)\mathrm{d}t$ being coercive on X_1 and

$$\|x\|\to\infty \quad \text{if and only if} \quad (\|x_1\|^2+\|x_2\|^2)^{\frac{1}{2}}\to\infty,$$

it follows from (2.15) that $\varphi(x)$ is coercive on X_1.

Through replacing X_1^1 with

$$X_2^1 = \left\{b\sin\frac{2\omega t}{p} \mid b\in\mathbb{R}^N\right\},$$
$$Y_1^1 = \left\{c\cos\frac{\omega t}{p} \mid c\in\mathbb{R}^N\right\},$$

repeating the above arguments with a small modification, one can prove the coercivity of φ on X_2 and Y_1. □

Proof of Theorem 1.1. According to Lemma 2.2, Lemma 2.3 and Lemma 2.5, there exists $x_1^*\in X_1$ such that

$$\langle\varphi'(x_1^*),y\rangle = 0, \qquad \forall y\in H^1_{pT}. \tag{2.16}$$

In what follows, we show that the minimal period of x_1^* is pT by contradiction. Suppose that x_1^* has minimal period $\frac{pT}{q_1}$, where $q_1>1$ is an integer. From Lemma 2.3, the minimal period of x_1^* is multiple of $\frac{T}{2}$, which means that $q_1=2$ or $q_1\geq s_p$.

If $q_1=2$, by Fourier expansion,

$$x_1^* = \sum_{k=1}^{+\infty}a_k^*\sin\frac{2k\omega t}{p}, \qquad a_k^*\in\mathbb{R}^N.$$

However, for every $x \in X_1$, we have

$$x = \sum_{k=1}^{+\infty} a_k \sin \frac{(2k-1)\omega t}{p}, \qquad a_k \in \mathbb{R}^N,$$

which implies $x_1^* = 0$. It contradicts that x_1^* has minimal period $\frac{pT}{2}$. So we get $q_1 \geq s_p$ and

$$x_1^* = \sum_{k=1}^{+\infty} a_k^* \sin \frac{kq_1\omega t}{p}, \qquad a_k^* \in \mathbb{R}^N. \tag{2.17}$$

It can be deduced from Parseval's equality and (2.17) that

$$\|\dot{x}_1^*\|_{L^2} \geq \frac{q_1\omega}{p}\|x_1^*\|_{L^2}. \tag{2.18}$$

Now, from (H2), (H4), (2.14) and (2.18), we have

$$\begin{aligned}
\varphi(x_1^*) &= \frac{1}{2}\int_0^{pT} |\dot{x}_1^*|^2 \mathrm{d}t - \int_0^{pT} V(t, x_1^*)\mathrm{d}t - \sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{(x_1^*)^i(t_j)} I_{ij}(s)\mathrm{d}s \\
&\geq \frac{1}{2}\|\dot{x}_1^*\|_{L^2}^2 - \int_0^{pT} \left[V(t,x_1^*) - (\nabla V(t,0), x_1^*)\right] \mathrm{d}t - \int_0^{pT} (\nabla V(t,0), x_1^*)\, \mathrm{d}t \\
&\quad - \frac{\rho}{T}\|x_1^*\|_{L^2}^2 - \rho p^2 T \|\dot{x}_1^*\|_{L^2}^2 \\
&\geq \left(\frac{1}{2} - \rho p^2 T\right)\|\dot{x}_1^*\|_{L^2}^2 - \left(\frac{A}{2} + \frac{\rho}{T}\right)\|x_1^*\|_{L^2}^2 - \|\nabla V(t,0)\|_{L^2}\,\|x_1^*\|_{L^2} \\
&\geq \left(\frac{\omega^2 q_1^2}{2p^2} - \rho T\omega^2 q_1^2 - \frac{A}{2} - \frac{\rho}{T}\right)\|x_1^*\|_{L^2}^2 - \|\nabla V(t,0)\|_{L^2}\,\|x_1^*\|_{L^2}.
\end{aligned} \tag{2.19}$$

It follows from (H4) and $q_1 \geq s_p$ that

$$\frac{\omega^2 q_1^2}{2p^2} - \rho T\omega^2 q_1^2 - \frac{A}{2} - \frac{\rho}{T} > 0,$$

which combined with (2.19) yields to

$$\varphi(x_1^*) \geq -\frac{1}{4}\left(\frac{\omega^2 q_1^2}{2p^2} - \rho T\omega^2 q_1^2 - \frac{A}{2} - \frac{\rho}{T}\right)^{-1} \|\nabla V(t,0)\|_{L^2}^2. \tag{2.20}$$

Choosing $\bar{x}_1(t) = (\delta \sin \frac{\omega t}{p}, 0, \ldots, 0) \in X_1$, where δ defined in (H2), the minimal period of $\bar{x}_1(t)$ is pT. According to mean value theorem, the Cauchy–Schwarz inequality and (H3), we have

$$\sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{(\bar{x}_1)^i(t_j)} I_{ij}(s)\mathrm{d}s \leq \sum_{j=1}^{pl}\sum_{i=1}^{N} d_{ij}|\theta| \left|\delta \sin \frac{\omega t_j}{p}\right| \leq p\delta^2 \sum_{j=1}^{l}\left(\sum_{i=1}^{N} d_{ij}^2\right)^{\frac{1}{2}}, \tag{2.21}$$

where $\theta \in \left(0, \delta \sin \frac{\omega t_j}{p}\right)$. In view of (2.21) and (H2), we get

$$\begin{aligned}
\varphi(\bar{x}_1) &= \frac{1}{2}\int_0^{pT} |\dot{\bar{x}}_1|^2 \mathrm{d}t - \int_0^{pT} V(t,\bar{x}_1)\mathrm{d}t + \sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{(\bar{x}_1)^i(t_j)} I_{ij}(s)\mathrm{d}s \\
&\le \frac{1}{2}\int_0^{pT} \delta^2\frac{\omega^2}{p^2}\cos^2\frac{\omega t}{p}\mathrm{d}t - \int_0^{pT} V\left(t, \delta\sin\frac{\omega t}{p}\right)\mathrm{d}t + p\delta^2\sum_{j=1}^{l}\left(\sum_{i=1}^{N} d_{ij}^2\right)^{\frac{1}{2}} \\
&\le \frac{1}{4}\delta^2\frac{\omega^2}{p}T - \int_0^{pT}\frac{\bar{A}}{2}\delta^2\sin^2\frac{\omega t}{p}\mathrm{d}t + p\delta^2\sum_{j=1}^{l}\left(\sum_{i=1}^{N} d_{ij}^2\right)^{\frac{1}{2}} \qquad (2.22)\\
&= \frac{1}{4}\delta^2\frac{\omega^2}{p}T - \frac{\bar{A}}{4}\delta^2 pT + p\delta^2\sum_{j=1}^{l}\left(\sum_{j=1}^{N} d_{ij}^2\right)^{\frac{1}{2}} \\
&= -\frac{1}{4}\left[\delta^2 pT\left(\bar{A} - \frac{\omega^2}{p^2}\right) - 4p\delta^2\sum_{j=1}^{l}\left(\sum_{i=1}^{N} {d_{ij}}^2\right)^{\frac{1}{2}}\right].
\end{aligned}$$

By (H4), (H5), (2.16), (2.20) and (2.22), we find

$$\inf_{x\in X_1}\varphi(x) = \varphi(x_1^*) > \varphi(\bar{x}_1).$$

That is a contradiction. Hence, there exists a critical point $x_1^* \in X_1$ of φ with minimal period pT.

Similarly, we can find $x_2^* \in X_2$ such that $\langle\varphi'(x_2^*), y\rangle = 0$, for every $y \in H^1_{pT}$. If the minimal period of x_2^* is not equal to $\frac{pT}{2}$, then there exists $q_2 > 1$ such that x_2^* has minimal period $\frac{pT}{2q_2}$. Lemma 2.3 implies that $q_2 \ge s_p$. Additionally, we have

$$x_2^* = \sum_{k=1}^{+\infty} b_k^* \sin\frac{2kq_2\omega t}{p}, \qquad b_k^* \in \mathbb{R}^N$$

and

$$\|\dot{x}_2^*\|_{L^2} \ge \frac{2q_2\omega}{p}\|x_2^*\|_{L^2}. \qquad (2.23)$$

It follows from (H2), (H4), (2.14) and (2.23) that

$$\begin{aligned}
\varphi\left(x_2^*\right) &= \frac{1}{2}\int_0^{pT}|\dot{x}_2^*|^2\mathrm{d}t - \int_0^{pT} V(t,x_2^*)\mathrm{d}t - \sum_{j=1}^{pl}\sum_{i=1}^{N}\int_0^{(x_2^*)^i(t_j)} I_{ij}(s)\mathrm{d}s \\
&\ge -\frac{1}{4}\left(\frac{2\omega^2q^2}{p^2} - 4\rho T\omega^2q^2 - \frac{A}{2} - \frac{\rho}{T}\right)^{-1}\|\nabla V\left(t,0\right)\|^2_{L^2}.
\end{aligned} \qquad (2.24)$$

Let $\bar{x}_2(t) = \left(\delta\sin\frac{2\omega t}{p}, 0, \ldots, 0\right) \in X_2$, then $\bar{x}_2(t)$ has minimal period $\frac{pT}{2}$. After a computation like (2.22), we get

$$\varphi(\bar{x}_2) \le -\frac{1}{4}\left[\delta^2 pT\left(\bar{A} - \frac{4\omega^2}{p^2}\right) - 4p\delta^2\sum_{j=1}^{l}\left(\sum_{i=1}^{N}{d_{ij}}^2\right)^{\frac{1}{2}}\right]. \qquad (2.25)$$

Taking (H4), (H5), (2.24) and (2.25) into consideration, we obtain

$$\inf_{x\in X_2}\varphi(x) = \varphi(x_2^*) > \varphi(\bar{x}_2).$$

This contradiction leads to the fact that the minimal period of x_2^* is $\frac{pT}{2}$.

Using a similar argument, a critical points y_1^* of φ with minimal period pT can be found on Y_1. It is clear that x_1^*, x_2^* and y_1^* are nonzero, which combines with $H^1_{pT} = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2$ to give that three points are different. □

3 Example

To show how our theorem applies in practice, we give the following example.

Example 3.1. Consider the impulsive system (1.1) with $T = 1, N = l = p = 3$,

$$I_{ij}(s) = \frac{|s|}{3240}, \qquad i, j \in \{1, 2, 3\},$$

and

$$V(t,x) = \begin{cases} \frac{7-\cos 4\pi t}{18}\omega^2|x|^2, & |x| \le 1, \\ \left(\frac{68\pi^2-1}{162} - \frac{7-\cos 4\pi t}{18}\omega^2 - \frac{1}{2}\right)|x|^2 & \\ \quad + \left(\frac{7-\cos 4\pi t}{9}\omega^2 - \frac{68\pi^2-1}{162} + \frac{1}{2}\right)(2|x| - 1), & 1 < |x| \le 2, \\ \frac{68\pi^2-1}{324}|x|^2 - \ln|x|^2 + \frac{7-\cos 4\pi t}{9}\omega^2 - \frac{68\pi^2-1}{162} + 2\ln 2 - \frac{1}{2}, & |x| > 2. \end{cases}$$

We can take

$$\omega = 2\pi, \qquad \rho = \frac{1}{4p^2 s_p^2} = \frac{1}{324},$$

then

$$\left(1 - 2\rho p^2 T\right)\frac{\omega^2}{2p^2} - \frac{\rho}{T} = \frac{68\pi^2 - 1}{324}.$$

It is easy to verify that $V(t,x)$ satisfies (H1). Let

$$d_{ij} = \frac{\rho}{9}, \qquad i, j \in \{1, 2, 3\},$$

and

$$A = \frac{8}{9}\omega^2, \qquad \bar{A} = \frac{2}{3}\omega^2, \qquad \delta = 1,$$

then $D = \frac{\rho}{3}$. One can easily check that (H3) is true and $V(t,x)$ satisfies (H2), (H4) and (H5). By Theorem 1.1, Example 3.1 possesses at least three periodic solutions. Two of them have minimal period 3 and the other one has minimal period 1.5.

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Global bifurcation curves of nonlocal elliptic equations with oscillatory nonlinear term

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Abstract. We study the one-dimensional nonlocal elliptic equation of Kirchhoff type with oscillatory nonlinear term. We establish the precise asymptotic formulas for the bifurcation curves $\lambda(\alpha)$ as $\alpha \to \infty$ and $\alpha \to 0$, where $\alpha := \|u_\lambda\|_\infty$ and u_λ is the solution associated with λ. We show that the second term of $\lambda(\alpha)$ is oscillatory as $\alpha \to \infty$.

Keywords: nonlocal elliptic equations, oscillatory bifurcation curves, asymptotic formulas.

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1 Introduction

We consider the following one-dimensional nonlocal elliptic equation

$$\begin{cases} -(b\|u'\|_2^2 + 1)u''(x) = \lambda(u(x)^p + u(x)\sin^2 u(x)), \quad x \in I := (0,1), \\ u(x) > 0, \quad x \in I, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $p > 1, b \geq 0$ are given constants, $\lambda > 0$ is a bifurcation parameter and $\|\cdot\|_2$ denotes the usual L^2-norm.

The purpose of this paper is to establish the asymptotic formulas for bifurcation curves $\lambda = \lambda(\alpha)$ of (1.1) as $\alpha \to \infty$ to understand well how the oscillatory term gives effect to the bifurcation curves. Here $\alpha := \|u_\lambda\|_\infty$ and u_λ is a solution of (1.1) associated with $\lambda > 0$. When we consider the case where $b = 0$, we use the following notation to avoid the confusion:

$$\begin{cases} -v''(x) = \mu(v(x)^p + v(x)\sin^2 v(x)), \quad x \in I, \\ v(x) > 0, \quad x \in I, \\ v(0) = v(1) = 0, \end{cases} \tag{1.2}$$

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where $\mu > 0$ is the bifurcation parameter. A solution pair of (1.2) is usually represented as $(\mu, v_\mu) \in \mathbb{R}_+ \times C^2(\bar{I})$, where v_μ is a solution of (1.2) associated with μ. In this paper, we adopt the explicit expression of the solution pair of (1.2), which was introduced in [12, Theorem 2.1]. That is, the solution pair $(\mu, v_\mu) \in \mathbb{R}_+ \times C^2(\bar{I})$ of (1.2) is parametrized by using a new parameter $\alpha > 0$. More precisely, let $\alpha > 0$ be an arbitrary given constant. Then by using the time map argument, we are able to obtain the unique solution pair $(\mu, v_\mu) \in \mathbb{R}_+ \times C^2(\bar{I})$ of (1.2) satisfying $\alpha = \|v_\mu\|_\infty$. Besides, μ is parametrized by α, namely, $\mu = \mu(\alpha)$ and it is a continuous function of α. The important point is that the solution pair (μ, v_μ) satisfying $\alpha = \|v_\mu\|_\infty$ is parametrized by the supremum norm $\alpha = \|v_\mu\|_\infty$ such as $(\mu(\alpha), v_{\mu(\alpha)})$. For simplicity, we write $v_\alpha := v_{\mu(\alpha)}$ in what follows.

Equation (1.1) is the nonlocal elliptic problem of Kirchhoff type motivated by the problem in [7]:

$$\begin{cases} -A\left(\int_0^1 (u'(x))^q dx\right) u''(x) = \lambda f(u(x)), & x \in I, \\ u(0) = u'(1) = 0, \end{cases} \tag{1.3}$$

where $A = A(y)$, which is called Kirchhoff function (cf. [10, 15]), is a continuous function of $y \geq 0$. Nonlocal problems have been investigated by many authors and there are quite many manuscripts which treated the problems with the backgrounds in physics, biology, engineering and so on. We refer to [1–4, 6–9, 11, 13, 14], and the references therein. One of the main interests there are existence, nonexistence and the number of positive and nodal solutions. However, there seems to be a few works which considered (1.3) from a view-point of bifurcation problems. We refer to [16–21] and the references therein. As far as the author knows, there are no works which treat the nonlinear oscillatory eigenvalue problem such as (1.2). Therefore, there seems no works which treat nonlocal bifurcation problems with oscillatory nonlinear term, so our results here seem to be novel. Our approach are mainly the time-map method and the complicated calculation of definite integrals.

The relationship between $\lambda(\alpha)$ and $\mu(\alpha)$ is as follows. Let $\alpha > 0$ be an arbitrary given constant. Assume that there exists a solution pair $(\lambda(\alpha), u_\alpha) \in \mathbb{R} \times C^2(\bar{I})$ with $\|u_\alpha\|_\infty = \alpha$. Then we have

$$-u_\alpha''(x) = \frac{\lambda(\alpha)}{b\|u_\alpha'\|_2^2 + 1}(u_\alpha(x)^p + u_\alpha(x)\sin^2 u_\alpha(x)). \tag{1.4}$$

We note that $\|u_\alpha\|_\infty = \alpha$. Then we find that $u_\alpha = v_\alpha$ and $\frac{\lambda(\alpha)}{b\|u_\alpha'\|_2^2+1} = \mu(\alpha)$, since the solution pair $(\mu(\alpha), v_\alpha) \in \mathbb{R}_+ \times C^2(\bar{I})$ of (1.2) with $\|v_\alpha\|_\infty = \alpha$ is unique (cf. [12]). This implies

$$\lambda(\alpha) = (b\|v_\alpha'\|^2 + 1)\mu(\alpha). \tag{1.5}$$

Therefore, to obtain $\lambda(\alpha)$, we need to obtain both $\mu(\alpha)$ and $\|v_\alpha'\|_2$.

Now we state our results. We first consider the case $p > 2$.

Theorem 1.1. *Consider* (1.2)*. Let $p > 2$. Then as $\alpha \to \infty$,*

$$\mu(\alpha) = 2(p+1)\alpha^{1-p}\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{1-p} + \frac{1}{2}(C_{12} + C_{21})\alpha^{-p} + \frac{1}{2}C_{22}\alpha^{-(p+1)} + (C_2 + C_3)\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\}^2, \tag{1.6}$$

where

$$
\begin{aligned}
C_{0,p} &:= \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}}ds, \\
C_1 &:= -\frac{p+1}{8}\int_0^1 \frac{1-s^2}{(1-s^{p+1})^{3/2}}ds, \\
C_{11} &:= \frac{2}{p+1}\int_0^{\pi/2} \cos(2\alpha \sin^{2/(p+1}\theta) \sin^{(3-p)/(p+1)} d\theta, \\
C_{12} &:= \frac{p-1}{2(p+1)}\int_0^{\pi/2} (\sin 2\alpha - \sin(2\alpha \sin^{2/(p+1)}\theta)) \sin^{(1-p)/(p+1)}\theta d\theta \\
&\quad + \frac{p+1}{4}\int_0^1 \frac{1-s}{(1-s^{p+1})^{3/2}}\sin(2\alpha s)ds, \\
C_{21} &:= -\frac{1}{p+1}\int_0^{\pi/2} \sin(2\alpha \sin^{2/(p+1)}\theta) \sin^{(3-p)/(p+1)}\theta d\theta \\
C_{22} &:= \frac{4(p-1)}{p+1}\int_0^{\pi/2} (\cos 2\alpha - \cos(2\alpha \sin^{2/(p+1)}\theta)) \sin^{(1-p)/(p+1)}\theta d\theta, \\
C_2 &:= \frac{3(p+1)^2}{128}\int_0^1 \frac{(1-s^2)^2}{(1-s^{p+1})^{5/2}}ds, \\
C_3 &:= -\frac{3}{32}(p+1)^2\int_0^1 \left(\int_0^s \frac{1-y^2}{(1-y^{p+1})^{5/2}}dy\right)\cos(2\alpha s)ds.
\end{aligned}
\tag{1.7}
$$

Theorem 1.2. *Consider (1.2). Let $p > 2$ and $\alpha \gg 1$. Then the following asymptotic formula for $\|v'_\alpha\|_2^2$ holds.*

$$
\|v'_\alpha\|_2^2 = 4\alpha^2\{G_0 + G_1\alpha^{1-p} + G_2\alpha^{-p} + G_3\alpha^{-(p+1)} + G_4\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\},
$$

where

$$
\begin{aligned}
G_0 &:= C_{0,p}E_{0,p}, \\
G_1 &:= C_{0,p}E_1 + \left(C_1 + \frac{1}{2}C_{11}\right)E_{0,p}, \\
G_2 &:= \frac{1}{2}\left(C_{12} + C_{12}\right)E_{0,p} + C_{0,p}E_2, \\
G_3 &:= \frac{1}{2}C_{22}E_{0,p} + C_{0,p}E_3, \\
G_4 &:= (C_2 + C_3)E_{0,p} + C_{0,p}E_4 + \left(C_1 + \frac{1}{2}C_{11}\right)E_1, \\
E_{0,p} &:= \int_0^1 \sqrt{1-s^{p+1}}ds, \\
E_1 &:= \frac{p+1}{8}\int_0^1 \frac{1-s^4}{\sqrt{1-s^{p+1}}}ds, \\
E_2 &:= -\frac{1}{4}\int_0^{\pi/2} \{\sin 2\alpha - \sin^{2/(p+1)}\theta \sin(2\alpha \sin^{2/(p+1)}\theta)\} \sin^{(1-p)/(p+1)}\theta d\theta, \\
E_3 &:= -\frac{1}{8}\int_0^1 \{\cos 2\alpha - \cos(2\alpha \sin^{2/(p+1)}\theta)\} \sin^{(1-p)/(p+1)} d\theta, \\
E_4 &:= -\frac{(p+1)^2}{128}\int_0^1 \frac{(1-s^2)^2}{(1-s^{p+1})^{3/2}}ds, \\
E_5 &:= \frac{2}{p+1}\int_0^1 \frac{1-s^{p+1}}{\sqrt{1-s^4}}ds.
\end{aligned}
$$

Remark 1.3. We should note that the order of the lower terms of $\mu(\alpha)$ in (1.6) changes according to p. Indeed, if we expand the bracket of the r.h.s. of (1.6), then the terms with

$$C_{0,p}^2, \alpha^{1-p}, \alpha^p, \alpha^{-(p+1)}, \alpha^{2(1-p)}, \alpha^{1-2p}$$

appear. Then for $\alpha \gg 1$, clearly, the first term is $C_{0,p}^2$ and the second is α^{1-p}. Besides, we have

$$\begin{cases} \alpha^{2(1-p)} \gg \alpha^{-p} \gg \alpha^{1-2p} \gg \alpha^{-(p+1)} & (1 < p < 2), \\ \alpha^{-p} \sim \alpha^{2(1-p)} \gg \alpha^{-(p+1)} \sim \alpha^{1-2p} & (p = 2), \\ \alpha^{-p} \gg \alpha^{2(1-p)} \gg \alpha^{-(p+1)} \gg \alpha^{1-2p} & (2 < p < 3), \\ \alpha^{-p} \gg \alpha^{-(p+1)} \sim \alpha^{2(1-p)} \gg \alpha^{1-2p} & (p = 3), \\ \alpha^{-p} \gg \alpha^{-(p+1)} \gg \alpha^{2(1-p)} \gg \alpha^{1-2p} & (p > 3). \end{cases} \tag{1.8}$$

Therefore, if $p > 2$, then the third term in the bracket of the r.h.s. of (1.6) is α^{-p}. However, if $1 < p < 2$, then the third term is $\alpha^{2(1-p)}$. Moreover, if p is very close to 1, then $1 - p \doteqdot 0$. Therefore, we have the sequence of the lower term, which are greater than α^{-p} in (1.6). In principle, it is possible to calculate them precisely. However, since the calculation is long and tedious, we do not carry out here.

Theorem 1.4. *Consider* (1.2)*.*

(i) Let $1 < p < 2$*. Then as* $\alpha \to \infty$*,*

$$\mu(\alpha) = 2(p+1)\alpha^{1-p}\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{1-p} + (C_2 + C_3)\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\}^2.$$

(ii) Let $p = 2$*. Then as* $\alpha \to \infty$*,*

$$\mu(\alpha) = 6\alpha^{-1}\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{-1} + \left(\frac{1}{2}C_{12} + \frac{1}{2}C_{21} + C_2 + C_3\right)\alpha^{-2} + o(\alpha^{2(1-p)})\right\}^2.$$

Theorem 1.5. *Consider* (1.2)*.*

(i) Let $1 < p < 2$*. Then as* $\alpha \to \infty$*,*

$$\|v_\alpha'\|_2^2 = 4\alpha^2\{G_0 + G_1\alpha^{1-p} + G_4\alpha^{2(1-p)} + G_2\alpha^{-p} + o(\alpha^{2(1-p)})\}.$$

(ii) Let $p = 2$*. Then as* $\alpha \to \infty$*,*

$$\|v_\alpha'\|_2^2 = 4\alpha^2\{G_0 + G_1\alpha^{-1} + (G_2 + G_4)\alpha^{-2} + o(\alpha^{-2})\}.$$

Theorems 1.4 and 1.5 are obtained directly from Theorems 1.1 and 1.2. So we omit the proofs.

We now consider (1.1).

Theorem 1.6. *Consider* (1.1) *with* $b > 0$*.*

(i) Let $p > 2$ *and* $\alpha \gg 1$*. Then the following asymptotic formula for* $\lambda(\alpha)$ *holds.*

$$\begin{aligned}\lambda(\alpha) = 2(p+1)\alpha^{1-p}&\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{1-p}\right.\\ &\left.+\frac{1}{2}(C_{12} + C_{21})\alpha^{-p} + \frac{1}{2}C_{22}\alpha^{-(p+1)} + C_2\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\}^2\\ &\times\left\{4b\alpha^2\{G_0 + G_1\alpha^{1-p} + G_2\alpha^{-p} + G_3\alpha^{-(p+1)} + G_4\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\} + 1\right\}.\end{aligned}$$

(ii) Let $p = 2$. Then as $\alpha \to \infty$,

$$\lambda(\alpha) = 6\alpha^{-1}\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{-1} + \left(\frac{1}{2}C_{12} + \frac{1}{2}C_{21} + C_2 + C_3\right)\alpha^{-2} + o(\alpha^{2(1-p)})\right\}^2 \times \left\{4b\alpha^2\{G_0 + G_1\alpha^{-1} + (G_2 + G_4)\alpha^{-2} + o(\alpha^{-2})\} + 1\right\}.$$

(iii) Let $1 < p < 2$ Then as $\alpha \to \infty$,

$$\lambda(\alpha) = 2(p+1)\alpha^{1-p}\left\{C_{0,p} + \left(C_1 + \frac{1}{2}C_{11}\right)\alpha^{1-p} + C_2\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\}^2 \times \left\{4b\alpha^2\{G_0 + G_1\alpha^{1-p} + G_4\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\} + 1\right\}.$$

We see from Theorem 1.6 that, roughly speaking, the asymptotic behaviors of $\lambda(\alpha)$ as $\alpha \to \infty$ are:

$$\lambda(\alpha) \sim \alpha^{3-p}.$$

We obtain Theorem 1.6 immediately by (1.5), Theorems 1.1, 1.2, 1.4 and 1.5. So we omit the proof.

Now we establish the asymptotic formulas for $\mu(\alpha)$ as $\alpha \to 0$ to understand the entire structure of $\mu(\alpha)$. We put

$$H_2 := -\frac{2}{p+1}\int_0^1 \frac{1 - s^{p+1}}{(1-s^4)^{3/2}}ds,$$
$$H_n := -2^{2n-2}(-1)^n\left\{\frac{1}{(2n-1)!}\int_0^1 \frac{1 - s^{2n-1}}{(1-s^4)^{3/2}}ds - \frac{1}{(2n)!}\int_0^1 \frac{1 - s^{2n}}{(1-s^4)^{3/2}}ds\right\}$$

for $n \geq 3$. Furthermore, let

$$L_1 := -\frac{p+1}{8}\int_0^1 \frac{1 - s^4}{(1-s^{p+1})^{3/2}}ds, \tag{1.9}$$
$$L_2 := -\frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-s^{p+1}}}K(s)ds, \tag{1.10}$$
$$K(s) := -2^3(p+1)\left\{\frac{1}{5!}\frac{1-s^5}{1-s^{p+1}} - \frac{1}{6!}\frac{1-s^6}{1-s^{p+1}} + O(\alpha^{7-p})\right\}. \tag{1.11}$$

Theorem 1.7. *Consider (1.2).*

(i) Let $1 < p < 3$. Then as $\alpha \to 0$,

$$\mu(\alpha) = 2(p+1)\alpha^{1-p}\left\{C_{0,p} + L_1\alpha^{3-p} + L_2\alpha^{5-p} + O(\alpha^{7-p})\right\}^2. \tag{1.12}$$

(ii) Let $p = 3$. Then as $\alpha \to 0$,

$$\mu(\alpha) = 4\alpha^{-2}\left\{C_{0,3} + \frac{1}{2}H_3\alpha^2 + O(\alpha^4)\right\}^2. \tag{1.13}$$

(iii) Let $3 < p \leq 5$. Then as $\alpha \to 0$,

$$\mu(\alpha) = 8\alpha^{-2}\left\{C_{0,3} + H_2\alpha^{p-3} + H_3\alpha^2 + O(\alpha^4)\right\}^2. \tag{1.14}$$

(iv) Assume that $p > 5$. *Then as* $\alpha \to 0$,

$$\mu(\alpha) = 8\alpha^{-2}\left\{C_{0,3} + H_3\alpha^2 + o(\alpha^2)\right\}^2. \tag{1.15}$$

Finally, we establish the asymptotic formulas for $\lambda(\alpha)$ as $\alpha \to 0$.

Theorem 1.8. *Consider* (1.1).

(i) Let $1 < p < 3$. *Then as* $\alpha \to 0$,

$$\lambda(\alpha) = 2(p+1)\alpha^{1-p}\left\{C_{0,p} + L_1\alpha^{3-p} + L_2\alpha^{5-p} + O(\alpha^{7-p})\right\}^2 \times \left\{4b\alpha^2\left\{E_{0,p}C_{0,p} + (E_{0,p}L_1 + C_{0,p}E_1)\alpha^{3-p} + o(\alpha^{3-p})\right\} + 1\right\}.$$

(ii) Let $p = 3$. *Then as* $\alpha \to 0$,

$$\lambda(\alpha) = 4\alpha^{-2}(1 + 4bE_{0.3}C_{0,3}\alpha^2 + o(\alpha^2))\left\{C_{0,3} + \frac{1}{2}H_3\alpha^2 + O(\alpha^4)\right\}^2.$$

(iii) Let $3 < p \le 5$. *Then as* $\alpha \to 0$,

$$\lambda(\alpha) = 8\alpha^{-2}\left\{C_{0,3} + H_2\alpha^{p-3} + H_3\alpha^2 + O(\alpha^4)\right\}^2 \times \left[4b\alpha^2\left\{C_{0,3} + H_2\alpha^{p-3} + H_3\alpha^2 + O(\alpha^4)\right\}\left\{E_{0,3} + E_5\alpha^{p-3}(1 + o(1))\right\} + 1\right].$$

(iv) Let $p > 5$. *Then as* $\alpha \to 0$,

$$\lambda(\alpha) = 8\alpha^{-2}\left\{C_{0,3} + H_3\alpha^2 + o(\alpha^2)\right\}^2 \times \left[4b\alpha^2\left\{C_{0,3} + H_3\alpha^2 + o(\alpha^2)\right\}\left\{E_{0,3} + E_5\alpha^{p-3}(1 + o(1))\right\} + 1\right].$$

By Theorem 1.8, we see that as $\alpha \to 0$,

$$\lambda(\alpha) \sim \begin{cases} \alpha^{1-p} & (1 < p \le 3), \\ \alpha^{-2} & (p > 3). \end{cases}$$

By Theorems 1.1, 1.4, 1.6 and 1.7, we obtain the qualitative shapes of $\mu(\alpha)$ and $\lambda(\alpha)$.

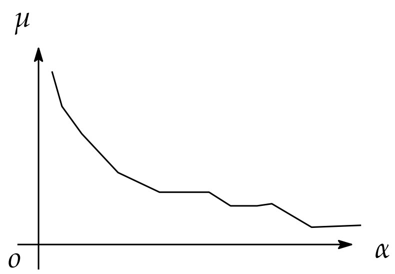

Figure 1.1: The graph of $\mu(\alpha)$

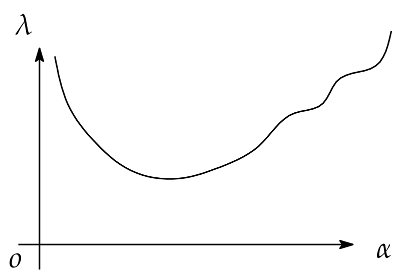

Figure 1.2: The graph of $\lambda(\alpha)$ $(1 < p < 3)$

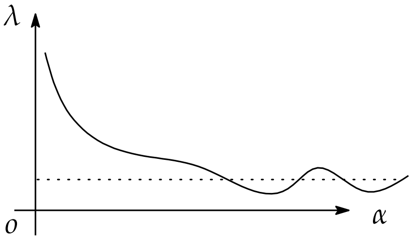

Figure 1.3: The graph of $\lambda(\alpha)$ $(p = 3)$

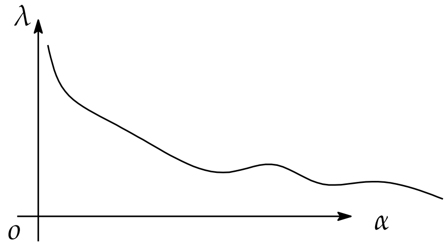

Figure 1.4: The graph of $\lambda(\alpha)$ $(p > 3)$

2 Proofs of Theorems 1.1 and 1.2

In this section, let $p > 2$ and we consider (1.2). In what follows, C denotes various positive constants independent of $\alpha \gg 1$. By [5], we know that if v_α is a solution of (1.2), then v_α satisfies

$$v_\alpha(x) = v_\alpha(1-x), \quad 0 \le x \le \frac{1}{2}, \tag{2.1}$$

$$\alpha := \|v_\alpha\|_\infty = v_\alpha\left(\frac{1}{2}\right), \tag{2.2}$$

$$v_\alpha'(x) > 0, \qquad 0 \le x < \frac{1}{2}. \tag{2.3}$$

We put

$$f(\theta) := \theta^p + \theta \sin^2\theta, \tag{2.4}$$

$$F(\theta) := \int_0^\theta f(y)dy = \frac{1}{p+1}\theta^{p+1} + \frac{1}{4}\theta^2 - \frac{1}{4}\theta\sin 2\theta - \frac{1}{8}\cos 2\theta + \frac{1}{8}. \tag{2.5}$$

Let $\alpha > 0$ be an arbitrary given constant. We write $\mu = \mu(\alpha)$ and $v_\alpha := v_{\mu(\alpha)}$ in what follows. By (1.2), for $x \in \bar{I}$, we have

$$\{v_\alpha''(x) + \mu f(v_\alpha(x))\}v_\alpha'(x) = 0.$$

By this and (2.2), for $x \in \bar{I}$, we have

$$\frac{1}{2}v_\alpha'(x)^2 + \mu F(v_\alpha(x)) = \text{constant} = \mu F\left(v_\alpha\left(\frac{1}{2}\right)\right) = \mu F(\alpha).$$

By this and (2.3), for $0 \le x \le 1/2$, we have

$$v_\alpha'(x) = \sqrt{2\mu(F(\alpha) - F(v_\alpha(x)))} \tag{2.6}$$

$$= \sqrt{\frac{2\mu}{p+1}}\sqrt{(\alpha^{p+1} - v_\alpha(x)^{p+1}) + \frac{p+1}{4}(\alpha^2 - v_\alpha(x)^2) - A_\alpha(v_\alpha(x)) - B_\alpha(v_\alpha(x))},$$

where

$$A_\alpha(v_\alpha(x)) := \frac{p+1}{4}(\alpha \sin 2\alpha - v_\alpha(x)\sin(2v_\alpha(x))), \tag{2.7}$$

$$B_\alpha(v_\alpha(x)) := \frac{p+1}{8}(\cos 2\alpha - \cos(2v_\alpha(x))). \tag{2.8}$$

Note that $A_\alpha(v_\alpha(x)) \ll \alpha^2, B_\alpha(v_\alpha(x)) \ll \alpha^2$. By this and putting $v_\alpha(x) = \alpha s$, we have

$$\begin{aligned}
\frac{1}{2} &= \int_0^{1/2} 1dx \\
&= \sqrt{\frac{p+1}{2\mu}} \int_0^{1/2} \frac{v_\alpha'(x)dx}{\sqrt{(\alpha^{p+1} - v_\alpha(x)^p) + \frac{p+1}{4}(\alpha^2 - v_\alpha(x)^2) - A_\alpha(v_\alpha(x)) - B_\alpha(v_\alpha(x))}} \\
&= \sqrt{\frac{p+1}{2\mu}} \alpha^{(1-p)/2} \int_0^1 \frac{ds}{\sqrt{(1-s^{p+1}) + \frac{p+1}{4}\alpha^{1-p}(1-s^2) - \frac{1}{\alpha^{p+1}}A_\alpha(\alpha s) - \frac{1}{\alpha^{p+1}}B_\alpha(\alpha s)}} \\
&= \sqrt{\frac{p+1}{2\mu}} \alpha^{(1-p)/2} \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}} \frac{ds}{\sqrt{1 + \frac{p+1}{4}\alpha^{1-p}\frac{1-s^2}{1-s^{p+1}} - \frac{1}{\alpha^{p+1}}\frac{A_\alpha(\alpha s)}{1-s^{p+1}} - \frac{1}{\alpha^{p+1}}\frac{B_\alpha(\alpha s)}{1-s^{p+1}}}}.
\end{aligned}$$

This along with Taylor expansion implies that

$$\begin{aligned}
\sqrt{\mu} &= \sqrt{2(p+1)}\alpha^{(1-p)/2} \\
&\quad \times \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}} \left\{ 1 - \frac{p+1}{8}\alpha^{1-p}\frac{1-s^2}{1-s^{p+1}} + \frac{1}{2}\frac{1}{\alpha^{p+1}}\frac{A_\alpha(\alpha s)}{1-s^{p+1}} + \frac{1}{2}\frac{1}{\alpha^{p+1}}\frac{B_\alpha(\alpha s)}{1-s^{p+1}} \right. \\
&\quad \left. + \frac{3}{8}\left(\frac{p+1}{4}\alpha^{1-p}\frac{1-s^2}{1-s^{p+1}}\right)^2 - \frac{3}{16}(p+1)\alpha^{-2p}\frac{1-s^2}{(1-s^{p+1})^2}A_\alpha(\alpha s) + o(\alpha^{2(1-p)}) \right\} ds \\
&= \sqrt{2(p+1)}\alpha^{(1-p)/2}\left[C_{0,p} + C_1\alpha^{1-p} + I + II + C_2\alpha^{2(1-p)} + III + o(\alpha^{2(1-p)})\right],
\end{aligned} \tag{2.9}$$

where

$$I = \frac{1}{2}\alpha^{-(p+1)}I_1 := \frac{1}{2}\alpha^{-(p+1)}\int_0^1 \frac{A_\alpha(\alpha s)}{(1-s^{p+1})^{3/2}}ds, \tag{2.10}$$

$$II = \frac{1}{2}\alpha^{-(p+1)}II_1 := \frac{1}{2}\alpha^{-(p+1)}\int_0^1 \frac{B_\alpha(\alpha s)}{(1-s^{p+1})^{3/2}}ds, \tag{2.11}$$

$$III = -\frac{3}{16}(p+1)\alpha^{-2p}\int_0^1 \frac{1-s^2}{(1-s^{p+1})^{5/2}}A_\alpha(\alpha s)ds. \tag{2.12}$$

Lemma 2.1. *Let $\alpha \gg 1$. Then*

$$I_1 = \int_0^1 \frac{A_\alpha(\alpha s)}{(1-s^{p+1})^{3/2}}ds = C_{11}\alpha^2 + C_{12}\alpha, \tag{2.13}$$

$$II_1 = \int_0^1 \frac{B_\alpha(\alpha s)}{(1-s^{p+1})^{3/2}}ds = C_{21}\alpha + C_{22}. \tag{2.14}$$

Proof. We first note that the definite integrals C_{11}, C_{12}, C_{21}, C_{22} exist, since we have $-1 < (1-p)/(p+1) < (3-p)/(p+1)$. We first prove (2.13). We put $s := \sin^{2/(p+1)}\theta$.

Then by integration by parts, we have

$$
\begin{aligned}
I_1 &= \frac{p+1}{4}\alpha \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}} \frac{\sin 2\alpha - \sin(2\alpha s)}{1-s^{p+1}} ds \\
&\quad + \frac{p+1}{4}\alpha \int_0^1 \frac{(1-s)}{(1-s^{p+1})^{3/2}} \sin(2\alpha s) ds \\
&= \frac{1}{2}\alpha \int_0^{\pi/2} \frac{1}{\cos^2\theta} \left[\left\{ \sin 2\alpha - \sin(2\alpha \sin^{2/(p+1)}\theta) \right\} \sin^{(1-p)/(p+1)}\theta \right] d\theta \\
&\quad + \frac{p+1}{4}\alpha \int_0^1 \frac{(1-s)}{(1-s^{p+1})^{3/2}} \sin(2\alpha s) ds \\
&= \frac{1}{2}\alpha \int_0^{\pi/2} (\tan\theta)' \left[\left\{ \sin 2\alpha - \sin(2\alpha \sin^{2/(p+1)}\theta) \right\} \sin^{(1-p)/(p+1)}\theta \right] d\theta \\
&\quad + \frac{p+1}{4}\alpha \int_0^1 \frac{(1-s)}{(1-s^{p+1})^{3/2}} \sin(2\alpha s) ds \\
&= \frac{1}{2}\alpha \left[\tan\theta \left[\left\{ \sin 2\alpha - \sin(2\alpha \sin^{2/(p+1)}\theta \right\} \sin^{(1-p)/(p+1)}\theta \right] \right]_0^{\pi/2} \qquad (*) \\
&\quad - \frac{1}{2}\alpha \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \left\{ -\frac{4}{p+1}\alpha \cos(2\alpha \sin^{2/(p+1}\theta) \sin^{(2-2p)/(p+1)}\theta \cos\theta \right. \\
&\qquad\qquad \left. - \frac{p-1}{p+1} (\sin 2\alpha - \sin(2\alpha \sin^{2/(p+1)}\theta)) \sin^{-2p/(p+1)}\theta \cos\theta \right\} d\theta \\
&\quad + \frac{p+1}{4}\alpha \int_0^1 \frac{(1-s)}{(1-s^{p+1})^{3/2}} \sin(2\alpha s) ds \\
&= \frac{2}{p+1}\alpha^2 \int_0^{\pi/2} \cos(2\alpha \sin^{2/(p+1}\theta) \sin^{(3-p)/(p+1)} d\theta \\
&\quad + \frac{p-1}{2(p+1)}\alpha \int_0^{\pi/2} (\sin 2\alpha - \sin(2\alpha \sin^{2/(p+1)}\theta)) \sin^{(1-p)/(p+1)}\theta d\theta \\
&\quad + \frac{p+1}{4}\alpha \int_0^1 \frac{(1-s)}{(1-s^{p+1})^{3/2}} \sin(2\alpha s) ds \\
&=: C_{11}\alpha^2 + C_{12}\alpha.
\end{aligned} \tag{2.15}
$$

We remark that by l'Hôpital's rule and direct calculation, we easily obtain that (∗) in (2.15) and (∗∗) in (2.16) below are equal to 0. Next, we put $s := \sin^{2/(p+1)}\theta$. Then by integration by parts, we have

$$
\begin{aligned}
II_1 &= \frac{1}{4} \int_0^{\pi/2} \frac{1}{\cos^2\theta} \left\{ \cos 2\alpha - \cos(2\alpha \sin^{2/(p+1)}\theta) \right\} \sin^{(1-p)/(p+1)}\theta d\theta \\
&= \frac{1}{4} \int_0^{\pi/2} (\tan\theta)' \left\{ \cos 2\alpha - \cos(2\alpha \sin^{2/(p+1)}\theta) \right\} \sin^{(1-p)/(p+1)}\theta d\theta \\
&= \frac{1}{4} \left[\tan\theta \left\{ \cos 2\alpha - \cos(2\alpha \sin^{2/(p+1)}\theta) \right\} \sin^{(1-p)/(p+1)}\theta \right]_0^{\pi/2} \qquad (**) \\
&\quad - \frac{1}{p+1}\alpha \int_0^{\pi/2} \sin(2\alpha \sin^{2/(p+1)}\theta) \sin^{(3-p)/(p+1)}\theta d\theta \\
&\quad + \frac{4(p-1)}{p+1} \int_0^{\pi/2} (\cos 2\alpha - \cos(2\alpha \sin^{2/(p+1)}\theta)) \sin^{(1-p)/(p+1)}\theta d\theta \\
&=: C_{21}\alpha + C_{22}.
\end{aligned} \tag{2.16}
$$

Thus the proof is complete. □

Lemma 2.2. *Let $\alpha \gg 1$. Then*

$$III = C_3\alpha^{2(1-p)} + o(\alpha^{2(1-p)}). \tag{2.17}$$

Proof. by (2.7) and (2.12), we have

$$\begin{aligned} III = & -\frac{3}{64}(p+1)^2\alpha^{-2p}\int_0^1 \frac{1-s^2}{(1-s^{p+1})^{5/2}}\{\alpha\sin 2\alpha - \alpha s\sin(2\alpha s)\}\,ds \\ = & -\frac{3}{64}(p+1)^2\alpha^{-2p+1}\int_0^1 \frac{1-s^2}{(1-s^{p+1})^{5/2}}\{\sin 2\alpha - \sin(2\alpha s)\}\,ds \\ & -\frac{3}{64}(p+1)^2\alpha^{-2p+1}\int_0^1 \frac{(1-s^2)(1-s)}{(1-s^{p+1})^{5/2}}\sin(2\alpha s)ds. \\ =: & -\frac{3}{64}(p+1)^2\alpha^{-2p+1}III_1 + O(\alpha^{-2p+1}). \end{aligned} \tag{2.18}$$

We show that $III_1 \sim \alpha$. We note that $(1-y^2)/(1-y^{p+1})^{5/2} \le (1-y^2)^{-3/2}$ for $0 \le y \le 1$. By this and integration by parts, we have

$$\begin{aligned} III_1 = & \lim_{\epsilon\to 0}\int_0^{1-\epsilon}\frac{d}{ds}\left(\int_0^s \frac{1-y^2}{(1-y^{p+1})^{5/2}}dy\right)\{\sin 2\alpha - \sin(2\alpha s)\}\,ds \\ = & \lim_{\epsilon\to 0}\left[\left(\int_0^s \frac{1-y^2}{(1-y^{p+1})^{5/2}}dy\right)\{\sin 2\alpha - \sin(2\alpha s)\}\right]_0^{1-\epsilon} \\ & + 2\alpha\lim_{\epsilon\to 0}\int_0^{1-\epsilon}\left(\int_0^s \frac{1-y^2}{(1-y^{p+1})^{5/2}}dy\right)\cos(2\alpha s)ds \\ = & 2\alpha(1+o(1))\int_0^1\left(\int_0^s \frac{1-y^2}{(1-y^{p+1})^{5/2}}dy\right)\cos(2\alpha s)ds. \end{aligned}$$

By this and (2.18), we have (2.17). Thus the proof is complete. □

Proof of Theorem 1.1. By (2.9) and Lemma 2.1, for $\alpha \gg 1$, we obtain

$$\begin{aligned} \sqrt{\mu} = & \sqrt{2(p+1)}\alpha^{(1-p)/2}\left\{C_{0,p} + (C_1 + \frac{1}{2}C_{11})\alpha^{1-p}\right. \\ & \left. + \frac{1}{2}(C_{12}+C_{21})\alpha^{-p} + \frac{1}{2}C_{22}\alpha^{-(p+1)} + (C_2+C_3)\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\right\}. \end{aligned} \tag{2.19}$$

By this, we obtain Theorem 1.1. Thus the proof is complete. □

We next prove Theorem 1.2.

Lemma 2.3. *Let v_α be the solution of (1.2) associated with $\mu > 0$ such that $\|v_\alpha\|_\infty = \alpha > 0$. Then for $\alpha \gg 1$*

$$\|v'_\alpha\|_2^2 = 4\alpha^2\{G_0 + G_1\alpha^{1-p} + G_2\alpha^{-p} + G_3\alpha^{-(p+1)} + G_4\alpha^{2(1-p)} + o(\alpha^{2(1-p)})\}. \tag{2.20}$$

Proof. By (2.6), putting $v_\alpha(x) = \alpha s$ and Taylor expansion, we obtain

$$\begin{aligned}
\|v_\alpha'\|_2^2 &= 2\int_0^{1/2} v_\alpha'(x)v_\alpha'(x)dx \qquad (2.21)\\
&= 2\sqrt{\frac{2\mu}{p+1}}\\
&\quad \times \int_0^{1/2}\sqrt{(\alpha^{p+1}-v_\alpha(x)^p)+\frac{p+1}{4}(\alpha^2-v_\alpha(x)^2)-A_\alpha(v_\alpha(x))-B_\alpha(v_\alpha(x))}v_\alpha'(x)dx\\
&= 2\sqrt{\frac{2\mu}{p+1}}\alpha^{(p+3)/2}\int_0^1\sqrt{1-s^{p+1}}\\
&\quad \times \sqrt{1+\frac{p+1}{4}\alpha^{1-p}\frac{1-s^2}{1-s^{p+1}}-\frac{1}{\alpha^{p+1}}\frac{A_\alpha(\alpha s)}{1-s^{p+1}}-\frac{1}{\alpha^{p+1}}\frac{B_\alpha(\alpha s)}{1-s^{p+1}}}ds\\
&= 2\sqrt{\frac{2\mu}{p+1}}\alpha^{(p+3)/2}\int_0^1\sqrt{1-s^{p+1}}\left\{1+\frac{p+1}{8}\alpha^{1-p}\frac{1-s^2}{1-s^{p+1}}-\frac{1}{2\alpha^{p+1}}\frac{A_\alpha(\alpha s)}{1-s^{p+1}}\right.\\
&\quad -\frac{1}{2\alpha^{p+1}}\frac{B_\alpha(\alpha s)}{1-s^{p+1}}-\frac{(p+1)^2}{128}\alpha^{2(1-p)}\left(\frac{1-s^2}{1-s^{p+1}}\right)^2\\
&\quad \left.+\frac{1}{64}(p+1)^2\alpha^{-2p}\frac{1-s^2}{1-s^{p+1}}(\alpha\sin 2\alpha-\alpha s\sin(2\alpha s))+o(\alpha^{2(1-p)})\right\}ds.
\end{aligned}$$

By putting $s = \sin^{2/(p+1)}\theta$, we have

$$\begin{aligned}
\int_0^1\frac{A_\alpha(\alpha s)}{\sqrt{1-s^{p+1}}}ds &= \frac{p+1}{4}\alpha\int_0^1\frac{\sin 2\alpha - s\sin(2\alpha s)}{\sqrt{1-s^{p+1}}}ds \qquad (2.22)\\
&= \frac{1}{2}\alpha\int_0^{\pi/2}\{\sin 2\alpha-\sin^{2/(p+1)}\theta\sin(2\alpha\sin^{2/(p+1)}\theta)\}\sin^{(1-p)/(p+1)}\theta d\theta,\\
\int_0^1\frac{B_\alpha(\alpha s)}{\sqrt{1-s^{p+1}}}ds &= \frac{p+1}{8}\int_0^1\frac{\cos 2\alpha-\cos(2\alpha s)}{\sqrt{1-s^{p+1}}}ds \qquad (2.23)\\
&= \frac{1}{4}\int_0^1\{\cos 2\alpha-\cos(2\alpha\sin^{2/(p+1)}\theta)\}\sin^{(1-p)/(p+1)}d\theta.
\end{aligned}$$

By (2.21)–(2.23), we have

$$\|v_\alpha'\|_2^2 = 2\sqrt{\frac{2\mu}{p+1}}\alpha^{(p+3)/2}\left\{E_{0,p}+E_1\alpha^{1-p}+E_2\alpha^{-p}+E_3\alpha^{-(p+1)}+E_4\alpha^{2(1-p)}+o(\alpha^{2(1-p)})\right\}. \qquad (2.24)$$

By this, (2.19)–(2.24), we have

$$\begin{aligned}
\|v_\alpha'\|_2^2 &= 4\alpha^2\left\{C_{0,p}+\left(C_1+\frac{1}{2}C_{11}\right)\alpha^{1-p}+\frac{1}{2}(C_{12}+C_{21})\alpha^{-p}\right.\\
&\qquad \left.+\frac{1}{2}C_{22}\alpha^{-(p+1)}+(C_2+C_3)\alpha^{2(1-p)}+o(\alpha^{2(1-p)})\right\}\\
&\quad \times\left\{E_{0,p}+E_1\alpha^{1-p}+E_2\alpha^{-p}+E_3\alpha^{-(p+1)}+E_4\alpha^{2(1-p)}+o(\alpha^{2(1-p)})\right\}\\
&= 4\alpha^2\{G_0+G_1\alpha^{1-p}+G_2\alpha^{-p}+G_3\alpha^{-(p+1)}+G_4\alpha^{2(1-p)}+o(\alpha^{2(1-p)})\}.
\end{aligned}$$

This implies (2.20). Thus the proof is complete. □

3 Proof of Theorem 1.7

In this section, let $0 < \alpha \ll 1$. We put $w_\alpha := v_\alpha/\alpha$. By (2.5) and Taylor expansion, we have

$$
\begin{aligned}
F(\alpha) &= \frac{1}{p+1}\alpha^{p+1} + \frac{1}{4}\alpha^2 - \frac{1}{4}\alpha\left\{2\alpha - \frac{1}{3!}(2\alpha)^3 + \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}(2\alpha)^{2n-1}\right\} \\
&\quad - \frac{1}{8}\left\{1 - \frac{1}{2!}(2\alpha)^2 + \frac{1}{4!}(2\alpha)^4 + \sum_{n=3}^{\infty}\frac{(-1)^n}{(2n)!}(2\alpha)^{2n}\right\} + \frac{1}{8}, \\
F(v_\alpha) &= \frac{1}{p+1}\alpha^{p+1}w_\alpha^{p+1} + \frac{1}{4}\alpha^2 w_\alpha(x)^2 \\
&\quad - \frac{1}{4}\alpha w_\alpha\left\{2\alpha w_\alpha - \frac{1}{3!}(2\alpha w_\alpha)^3 + \sum_{n=3}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}(2\alpha w_\alpha)^{2n-1}\right\} \\
&\quad - \frac{1}{8}\left\{1 - \frac{1}{2!}(2\alpha w_\alpha)^2 + \frac{1}{4!}(2\alpha w_\alpha)^4 + \sum_{n=3}^{\infty}\frac{(-1)^n}{(2n)!}(2\alpha w_\alpha)^{2n}\right\} + \frac{1}{8}.
\end{aligned}
$$

By the same argument as that to obtain (2.6), for $0 \le x \le 1$, we have

$$
\begin{aligned}
\frac{1}{2}\alpha^2 w_\alpha'(x)^2 = \mu\Bigg\{&\frac{1}{p+1}\alpha^{p+1}(1 - w_\alpha(x)^{p+1}) + \frac{1}{4}\alpha^4(1 - w_\alpha(x)^4) \\
&+ \frac{1}{4}\alpha\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n-1)!}2^{2n-1}\alpha^{2n-1}(1 - w_\alpha(x)^{2n-1}) \\
&- \frac{1}{8}\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n)!}2^{2n}\alpha^{2n}(1 - w_\alpha(x)^{2n})\Bigg\}.
\end{aligned}
$$

We put

$$
\begin{aligned}
H_\alpha(w_\alpha) &:= \frac{1}{4}\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n-1)!}2^{2n-1}\alpha^{2n}(1 - w_\alpha(x)^{2n-1}) \\
&= \sum_{n=3}^{\infty}\frac{(-1)^n}{(2n-1)!}2^{2n-3}\alpha^{2n}(1 - w_\alpha(x)^{2n-1}), \\
J_\alpha(w_\alpha) &= -\frac{1}{8}\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n)!}2^{2n}\alpha^{2n}(1 - w_\alpha(x)^{2n}) \\
&= -\sum_{n=3}^{\infty}\frac{(-1)^n}{(2n)!}2^{2n-3}\alpha^{2n}(1 - w_\alpha(x)^{2n}).
\end{aligned}
$$

We put

$$
\begin{aligned}
M_\alpha(w_\alpha) &:= H_\alpha(w_\alpha) + J_\alpha(w_\alpha(x)) \\
&= \sum_{n=3}^{\infty}(-1)^n 2^{2n-3}\left\{\frac{1}{(2n-1)!}(1 - w_\alpha(x)^{2n-1}) - \frac{1}{(2n)!}(1 - w_\alpha(x)^{2n})\right\}\alpha^{2n}.
\end{aligned}
$$

By this and (2.3), for $0 \le x \le 1/2$, we have

$$
w_\alpha'(x) = \sqrt{2\mu}\,\alpha^{-1}\sqrt{\frac{1}{p+1}\alpha^{p+1}(1 - w_\alpha(x)^{p+1}) + \frac{1}{4}\alpha^4(1 - w_\alpha(x)^4) + M_\alpha(w_\alpha)}. \tag{3.1}
$$

(i) Let $1 < p < 3$. Then by (3.1), we have

$$w_\alpha'(x) = \sqrt{2\mu\alpha^{-2}}\sqrt{\frac{\alpha^{p+1}}{p+1}}\sqrt{1-w_\alpha(x)^{p+1}}\sqrt{1+\frac{p+1}{4}\alpha^{3-p}\frac{1-w_\alpha(x)^4}{1-w_\alpha(x)^{p+1}}+K(w_\alpha)\alpha^{5-p}}, \quad (3.2)$$

where

$$K(w_\alpha(x)) := -2^3(p+1)\left\{\frac{1}{5!}\frac{1-w_\alpha(x)^5}{1-w_\alpha(x)^{p+1}} - \frac{1}{6!}\frac{1-w_\alpha(x)^6}{1-w_\alpha(x)^{p+1}}\right\}.$$

By (3.2) and Taylor expansion, we have

$$\begin{aligned}\sqrt{\frac{\mu}{2(p+1)}}\alpha^{(p-1)/2} &= \int_0^{1/2}\frac{w_\alpha'(x)}{\sqrt{1-w_\alpha(x)^{p+1}}\sqrt{1+\frac{p+1}{4}\alpha^{3-p}\frac{1-w_\alpha(x)^4}{1-w_\alpha(x)^{p+1}}+K(w_\alpha(x))\alpha^{5-p}}}dx\\ &= \int_0^1\frac{1}{\sqrt{1-s^{p+1}}}\left\{1-\frac{p+1}{8}\alpha^{3-p}\frac{1-s^4}{1-s^{p+1}}-\frac{1}{2}K(s)\alpha^{5-p}+O(\alpha^{5-p})\right\}ds.\end{aligned}$$

This implies from (1.7), (1.9) and (1.10) that

$$\sqrt{\mu} = \sqrt{2(p+1)}\alpha^{-(p-1)/2}\left\{C_{0,p}+L_1\alpha^{3-p}+L_2\alpha^{5-p}+O(\alpha^{7-p})\right\}. \quad (3.3)$$

This implies (1.12).

(ii) Let $p = 3$. Then by (3.1), we have

$$\begin{aligned}w_\alpha'(x) &= \sqrt{2\mu}\alpha^{-1}\sqrt{\frac{1}{2}\alpha^4(1-w_\alpha(x)^4)+M_\alpha(w_\alpha(x))}\\ &= \sqrt{\mu}\alpha\sqrt{1-w_\alpha(x)^4}\sqrt{1+2\alpha^{-4}\frac{M_\alpha(w_\alpha(x))}{1-w_\alpha(x)^4}}.\end{aligned}$$

This along with Taylor expansion implies that

$$\begin{aligned}\frac{1}{2}\sqrt{\mu} &= \alpha^{-1}\int_0^{1/2}\frac{w_\alpha'(x)}{\sqrt{1-w_\alpha(x)^4}\sqrt{1+2\alpha^{-4}\frac{M_\alpha(w_\alpha(x))}{1-w_\alpha(x)^4}}}dx\\ &= \alpha^{-1}\int_0^1\frac{1}{\sqrt{1-s^4}}\left\{1-\alpha^{-4}\frac{M_\alpha(s)}{1-s^4}+O(\alpha^4)\right\}ds.\end{aligned}$$

By this, we obtain

$$\begin{aligned}\sqrt{\mu} &= 2\alpha^{-1}\int_0^1\frac{1}{\sqrt{1-s^4}}\left\{1+8\alpha^2\left(\frac{1}{5!}\frac{1-s^5}{1-s^4}-\frac{1}{6!}\frac{1-s^6}{1-s^4}\right)+O(\alpha^4)\right\}ds\\ &= 2\alpha^{-1}\left\{C_{0,3}+\frac{1}{2}H_3\alpha^2+O(\alpha^4)\right\}.\end{aligned}$$

This implies (1.13).

(iii) Let $3 < p \le 5$. Then by (3.1), we have

$$\frac{1}{2}\sqrt{2\mu} = 2\alpha\int_0^{1/2}\frac{w_\alpha'(x)}{\alpha^2\sqrt{1-w_\alpha(x)^4}\sqrt{1+\frac{4}{p+1}\alpha^{p-3}\frac{1-w_\alpha(x)^{p+1}}{1-w_\alpha(x)^4}+Q_\alpha(w_\alpha(x))}}dx,$$

where

$$Q_\alpha(w_\alpha) := 4\alpha^{-4}\frac{M_\alpha(w_\alpha)}{1-w_\alpha(x)^4}.$$

By this and Taylor expansion, we have

$$\begin{aligned}\sqrt{\frac{\mu}{2}} &= 2\alpha^{-1}\int_0^1 \frac{1}{\sqrt{1-s^4}} \\ &\quad\times\left\{1-\frac{2}{p+1}\alpha^{p-3}\frac{1-s^{p+1}}{1-s^4}+\alpha^2\frac{2^4}{5!}\frac{1-s^5}{1-s^4}-\alpha^2\frac{2^4}{6!}\frac{1-s^6}{1-s^4}+O(\alpha^4)\right\}ds \\ &= 2\alpha^{-1}\left\{C_{0,3}+H_2\alpha^{p-3}+H_3\alpha^2+O(\alpha^4)\right\}.\end{aligned} \tag{3.4}$$

This implies (1.14).

(iv) Assume that $p > 5$. Then by (3.4), we have

$$\sqrt{\frac{\mu}{2}} = 2\alpha^{-1}\left\{C_{0,3}+H_3\alpha^2+o(\alpha^2)\right\}. \tag{3.5}$$

This implies (1.15). Thus the proof of Theorem 1.7 is complete. □

4 Proof of Theorem 1.8

In this section, we assume that $0 < \alpha \ll 1$. By Taylor expansion, we have

$$\begin{aligned}v_\alpha(x)\sin^2 v_\alpha(x) &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n-1}}{(2n)!}v_\alpha(x)^{2n+1} \\ &= v_\alpha(x)^3-\frac{1}{3}v_\alpha(x)^5+\frac{2}{45}v_\alpha(x)^7+O(v_\alpha(x)^9).\end{aligned} \tag{4.1}$$

(i) Let $1 < p < 3$. Then by (2.6), (4.1), Taylor expansion and putting $v_\alpha = \theta = \alpha s$, we have

$$\begin{aligned}\|v_\alpha'\|_2^2 &= 2\int_0^{1/2} v_\alpha'(x)v_\alpha'(x)dx \\ &= 2\sqrt{2\mu}\int_0^{1/2}\sqrt{\frac{1}{p+1}(\alpha^{p+1}-v_\alpha(x)^{p+1})+\frac{1}{4}(\alpha^4-v_\alpha(x)^4)(1+o(1))}v_\alpha'(x)dx \\ &= 2\sqrt{2\mu}\int_0^{\alpha}\sqrt{\frac{1}{p+1}(\alpha^{p+1}-\theta^{p+1})+\frac{1}{4}(\alpha^4-\theta^4)(1+o(1))}d\theta \\ &= 2\sqrt{\frac{2\mu}{p+1}}\alpha^{(p+3)/2}\int_0^1\sqrt{1-s^{p+1}}\sqrt{1+\frac{p+1}{4}\alpha^{3-p}\frac{1-s^4}{1-s^{p+1}}(1+o(1))}ds \\ &= 2\sqrt{\frac{2\mu}{p+1}}\alpha^{(p+3)/2}\int_0^1\sqrt{1-s^{p+1}}\left\{1+\frac{p+1}{8}\alpha^{3-p}\frac{1-s^4}{1-s^{p+1}}(1+o(1))\right\}ds \\ &= 2\sqrt{\frac{2}{p+1}}\sqrt{\mu}\alpha^{(p+3)/2}\left\{E_{0,p}+E_1\alpha^{3-p}+o(\alpha^{3-p})\right\}.\end{aligned} \tag{4.2}$$

By this and (3.3), we have

$$\begin{aligned}\|v_\alpha'\|_2^2 &= 2\sqrt{\frac{2}{p+1}}\alpha^{(p+3)/2}\left\{E_{0,p} + E_1\alpha^{3-p} + o(\alpha^{3-p})\right\} \\ &\quad \times \sqrt{2(p+1)}\alpha^{-(p-1)/2}\left\{C_{0,p} + L_1\alpha^{3-p} + L_2\alpha^{5-p} + O(\alpha^{7-p})\right\} \\ &= 4\alpha^2\left\{E_{0,p}C_{0,p} + (E_{0,p}L_1 + C_{0,p}E_1)\alpha^{3-p} + o(\alpha^{3-p})\right\}.\end{aligned}$$

By this, (1.5) and Theorem 1.7 (i), we have

$$\begin{aligned}\lambda(\alpha) &= 2(p+1)\alpha^{1-p}\left\{C_{0,p} + L_1\alpha^{3-p} + L_2\alpha^{5-p} + O(\alpha^{7-p})\right\}^2 \\ &\quad \times \left\{4b\alpha^2\left\{E_{0,p}C_{0,p} + (E_{0,p}L_1 + C_{0,p}E_1)\alpha^{3-p} + o(\alpha^{3-p})\right\} + 1\right\}.\end{aligned}$$

(ii) Let $p = 3$. Then by (4.2) and putting $s = v_\alpha(x)/\alpha$, we have

$$\begin{aligned}\|v_\alpha'\|_2^2 &= 2\sqrt{\mu}(1+o(1))\int_0^{1/2}\sqrt{\alpha^4 - v_\alpha(x)^4}\,v_\alpha'(x)dx \\ &= 2\sqrt{\mu}(1+o(1))\alpha^3\int_0^1\sqrt{1-s^4}ds \\ &= 2\sqrt{\mu}\alpha^3E_{0,3}(1+o(1)).\end{aligned}$$

By this, (1.5) and Theorem 1.7 (ii), we have

$$\begin{aligned}\|v_\alpha'\|_2^2 &= 2\alpha^3E_{0,3}(1+o(1))2\alpha^{-1}\left\{C_{0,3} + \frac{1}{2}H_3\alpha^2 + O(\alpha^4)\right\} \\ &= 4\alpha^2E_{0,3}C_{0,3}(1+o(1)).\end{aligned}$$

By this and Theorem 1.7 (ii), we have

$$\lambda(\alpha) = 4\alpha^{-2}(1 + 4bE_{0.3}C_{0,3}\alpha^2 + o(\alpha^2))\left\{C_{0,3} + \frac{1}{2}H_3\alpha^2 + O(\alpha^4)\right\}^2.$$

We next consider the case $p > 3$. By (4.1), for $0 < x < 1/2$, we have

$$\frac{1}{2}v_\alpha'(x)^2 + \mu\left\{\frac{1}{4}v_\alpha(x)^4 + \frac{1}{p+1}v_\alpha(x)^{p+1}(1+o(1))\right\} = \mu\left\{\frac{1}{4}\alpha^4 + \frac{1}{p+1}\alpha^{p+1}(1+o(1))\right\}.$$

By this, for $0 \le x \le 1/2$, we have

$$v_\alpha'(x) = \sqrt{\frac{\mu}{2}}\sqrt{\alpha^4 - v_\alpha(x)^4}\sqrt{1 + \frac{4}{p+1}\frac{\alpha^{p+1} - v_\alpha(x)^{p+1}}{\alpha^4 - v_\alpha(x)^4}(1+o(1))}.$$

By this, (3.5) and the same calculation as that of (4.2) and putting $v_\alpha(x) = \alpha s$, we have

$$\begin{aligned}\|v_\alpha'\|_2^2 &= \sqrt{2\mu}\alpha^3\int_0^1\sqrt{1-s^4}\sqrt{1 + \frac{4}{p+1}\alpha^{p-3}\frac{1-s^{p+1}}{1-s^4}(1+o(1))}ds \\ &= \sqrt{2\mu}\alpha^3\left\{E_{0,3} + E_5\alpha^{p-3}(1+o(1))\right\} \\ &= 4\alpha^2\left\{C_{0,3} + H_2\alpha^{p-3} + H_3\alpha^2 + O(\alpha^4)\right\}\left\{E_{0,p} + E_5\alpha^{p-3}(1+o(1))\right\}.\end{aligned} \tag{4.3}$$

(iii) Let $3 < p \leq 5$. Then by (1.5), (3.5) and (4.3), we have

$$\lambda(\alpha) = 8\alpha^{-2}\left\{C_{0,3} + H_2\alpha^{p-3} + H_3\alpha^2 + O(\alpha^4)\right\}^2 \times \left[4b\alpha^2\left\{C_{0,3} + H_2\alpha^{p-3} + H_3\alpha^2 + O(\alpha^4)\right\}\left\{E_{0,3} + E_5\alpha^{p-3}(1+o(1))\right\} + 1\right].$$

(iv) Let $p > 5$. Then by (1.8), (3.5) and (4.3), we have

$$\lambda(\alpha) = 8\alpha^{-2}\left\{C_{0,3} + H_3\alpha^2 + o(\alpha^2)\right\}^2 \times \left[4b\alpha^2\left\{C_{0,3} + H_3\alpha^2 + o(\alpha^2)\right\}\left\{E_{0,3} + E_5\alpha^{p-3}(1+o(1))\right\} + 1\right].$$

Thus the proof of Theorem 1.8 is complete. □

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Global boundedness and stabilization in a predator-prey model with cannibalism and prey-evasion

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Abstract. This paper is concerned with a predator-prey model with cannibalism and prey-evasion. The global existence and boundedness of solutions to the system in bounded domains of 1D and 2D are proved for any prey-evasion sensitivity coefficient. It is also shown that prey-evasion driven Turing instability when the prey-evasion coefficient surpasses the critical value. Besides, the existence of Hopf bifurcation, which generates spatiotemporal patterns, is established. And, numerical simulations demonstrate the complex dynamic behavior.

Keywords: predator-prey, cannibalism, prey-evasion, global existence, Turing instability, Hopf bifurcation.

2020 Mathematics Subject Classification: 35Q92, 35A01, 35K59, 35B35, 35B36.

1 Introduction

Cannibalism, adult preying on juveniles of the same species, has an effective impact on the regulation and equilibration of population density [7, 23]. Numerous mathematical modeling and analysis of cannibalism have been developed rapidly over the past few decades [5, 8]. These analyses focused mainly on the stabilizing-destabilizing effect of cannibalism, which seems to strongly depend on the form of the model. For example, Kohlmeier and Ebenhöh [13] found that cannibalism can stabilize population cycles. A high cannibalism rate may cause the internal steady state to change from being unstable to stable due to the interaction between logistic population growth of the prey and a Beddington–DeAngelis functional response. In 1999, Magnússon [18] proposed an age-structured predator-prey model and showed that cannibalism has a destabilizing effect. If the mortality rate of juveniles is high and/or the recruitment rate to the mature population is low, then the equilibrium will be stable for low levels of cannibalism. However, a loss of stability by the Hopf bifurcation will take place as the level of cannibalism increases, and numerical studies indicate that a stable limit cycle exists.

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In 2006, Buonomo and Lacitignola [3] introduced a predator-prey model with age structure and cannibalism in the predator population

$$\begin{cases} \dfrac{dA}{dt} = MJ - d_A A, \\ \dfrac{dJ}{dt} = \eta_1 \delta AP - (1-\eta_c)\sigma AJ - (M+d_J)J, \\ \dfrac{dP}{dt} = r_1 P - r_2 P^2 - \delta AP, \end{cases} \tag{1.1}$$

where $A(t)$ and $J(t)$ represent the densities of individuals of mature and immature predator populations at time t, respectively, and $P(t)$ denotes the number of individuals of prey population. Further, M is the constant maturation rate from juveniles to adults; δ is the inter-specific competition rate; σ is the cannibalism attack rate; η_1 and η_c denote the coefficients in converting preys into new immature predators (juveniles), and juveniles into new juveniles, respectively. r_1 and r_2 are the logistic coefficients, d_A and d_J are the death rates.

By the following non-dimensional variables

$$u = \delta A/d_A, \qquad v = M\delta J/d_A^2, \qquad w = r_2 P/d_A,\ \tau = d_A t,$$

and denoting τ by t again, system (1.1) becomes

$$\begin{cases} \dfrac{du}{dt} = v - u, \\ \dfrac{dv}{dt} = auw - \gamma uv - cv, \\ \dfrac{dw}{dt} = rw - w^2 - uw, \end{cases} \tag{1.2}$$

where $a = \frac{\eta_1 M\delta}{r_2 d_A}$, $\gamma = \frac{\sigma(1-\eta_c)}{\delta}$, $c = \frac{M+d_J}{d_A}$, $r = \frac{r_1}{d_A}$. Obviously, if $ar > c$, then system (1.2) has a unique positive equilibrium point $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v}, \tilde{w})$, where

$$\tilde{u} = \frac{ar-c}{a+\gamma}, \qquad \tilde{v} = \frac{ar-c}{a+\gamma}, \qquad \tilde{w} = \frac{\gamma r + c}{a+\gamma}. \tag{1.3}$$

Buonomo and Lacitignola derived that cannibalism is a stabilizing mechanism in the model (1.2). That is, when cannibalism attack rate increases to a level that exceeds the critical value, the coexistence steady state changes from being unstable to stable. Moreover, they provided numerical simulations to demonstrate the mathematical analysis. The same conclusion has been pointed out by Buonomo and coauthors [4]. They also found that the effects of cannibalism and prey growth are opposite. Besides, numerical simulations showed that the higher the uptake of prey by predators, the higher the critical value of cannibalism.

Recently, Jia *et al*. [10] discussed the corresponding pure diffusion system of (1.2) and obtained the result that the effects of prey growth and predator cannibalism rate on the stability of nonnegative constant steady state are opposite. They also proved the nonexistence and existence of nonconstant positive solutions and found that diffusion can cause a periodic solution of spatial inhomogeneity which occurs in unstable area (also the unstable area of ODE). Very recently, in another paper, we investigated the temporal, spatial and spatiotemporal patterns of the corresponding cross-diffusion system of (1.2) in detail. We showed that cannibalism is no longer a stabilizing effect, and cross-diffusion is the decisive factor of destabilizing positive steady state.

From biological characteristics, it can be seen that in addition to the random diffusion of predators, the spatial movements between predators and prey can also be pursuit and evasion,that is to say, predators pursuing preys and preys escaping from predators. Such movement is not random but directed, that is predators move toward the gradient direction of prey distribution (called "prey-taxis"), and/or preys move opposite to the gradient of predator distribution (called "prey-evasion" or "predator-taxis") [28]. These processes are well known to be important in biological control and ecological balance such as regulating prey (pest) population or incipient outbreaks of prey or forming large scale aggregation for survival [20,31].

Tsyganov and coauthors [22] proposed a predator-prey model with both prey-taxis and predator-taxis, and found that the taxis terms change the shape of the propagating waves and increase the propagation speed. Since then, there are many mathematical literatures demonstrating and explaining the pursuit-evasion phenomenon. Meanwhile, various reaction-diffusion models with prey-taxis and (or) predator-taxis have been proposed to study global existence, traveling wave, pattern formation, and bifurcation analysis [11,12,14,15,17,19,24,27,30]. Recently, Wu and coauthors [28] considered a reaction-diffusion predator-prey model system with predator-taxis, which is a similar situation occurs when susceptible population avoids the infected ones in epidemic spreading. They proved the global existence and boundedness of solutions in bounded domains of arbitrary spatial dimension and any predator-taxis sensitivity coefficient. It is also shown that a smaller predator-taxis effect can destabilize the positive constant steady state and generate non-constant spatial pattern.

Inspired by the above discussion, the main aim of this paper is to investigate the global existence and dynamical behavior in a predator-prey model with both cannibalism and prey-evasion

$$\begin{cases} u_t - d_1\Delta u = -u + v, & x \in \Omega,\ t > 0, \\ v_t - d_2\Delta v = auw - \gamma uv - cv, & x \in \Omega,\ t > 0, \\ w_t - d_3\Delta w - \xi\nabla\cdot(w\nabla u) = rw - w^2 - uw, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where $-\xi\nabla\cdot(w\nabla u)$ is prey-evasion, which shows the tendency of prey moving toward the opposite direction of the increasing predator gradient direction. $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. ν is the outer normal directional derivative on $\partial\Omega$. The homogeneous Neumann boundary condition indicates that this system is self-contained with zero population flux across the boundary. The initial values $u_0(x), v_0(x), w_0(x)$ are nonnegative smooth functions which are not identically zero.

Our main results on the global existence and boundedness of solutions of system (1.4) are as follows.

Theorem 1.1. *Let $n = \{1,2\}$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. For any $(u_0, v_0, w_0) \in [W^{1,p}(\Omega)]^3$ where $p > n$, satisfying $u_0(x) \geq 0, v_0(x) \geq 0, w_0(x) \geq 0$ for $x \in \Omega$, the system (1.4) has a unique nonnegative and bounded global classical solution $(u(x,t), v(x,t), w(x,t))$, and $(u,v,w) \in \big(C([0,\infty); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty))\big)^3$.*

The rest of the paper is organized as follows. In Section 2, we obtain some preliminary results. Section 3 is devoted to prove the global existence and uniform boundedness of the classical solution of (1.4). The dynamical behavior and pattern formation of the prey-evasion system are studied in Section 4. And, numerical simulations are emphasized the theoretical results. The last section is a brief discussion.

2 Preliminaries

2.1 Existence and uniqueness of local solutions

We first give a claim concerning the local-in-time existence of a classical solution to (1.4).

Lemma 2.1. *Assume that the initial data u_0, v_0, and w_0 be nonnegative and satisfy $(u_0, v_0, w_0) \in [W^{1,p}(\Omega)]^3$ for $p > n$. Then the following statements for the model (1.4) hold.*

(1) There exists a positive constant $T_{\max}$ (the maximal existence time) such that the problem (1.4) has a unique local in time classical solution $(u(x,t), v(x,t), w(x,t))$ satisfying

$$(u,v,w) \in \left(C([0,T_{\max}); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max}))\right)^3.$$

Moreover, u, v, and w satisfy the inequalities

$$u > 0, \qquad v > 0, \qquad w > 0 \quad \text{in } \Omega \times (0, T_{max}). \tag{2.1}$$

(2) If for each $T > 0$ there exists a constant $C(T)$ (depending on T and $\|(u_0, v_0, w_0)\|_{W^{1,p}(\Omega)}$ only) such that

$$\|(u(t), v(t), w(t))\|_{L^\infty} \le C(T), \qquad 0 < t < \min\{T, T_{\max}\}, \tag{2.2}$$

then $T_{\max} = +\infty$.

(3) The total mass of $u(x,t), v(x,t)$ and $w(x,t)$ satisfies

$$\int_\Omega w dx \le m_1 := \max\left\{\int_\Omega w_0 dx, r|\Omega|\right\}, \qquad t \in (0, T_{\max}), \tag{2.3}$$

$$\int_\Omega v dx \le m_2 := \max\left\{\int_\Omega (v_0 + a w_0) dx, \frac{a(r+c)}{c} m_1\right\}, \qquad t \in (0, T_{\max}), \tag{2.4}$$

$$\int_\Omega u dx \le m_3 := \max\left\{\int_\Omega u_0 dx, m_2\right\}, \qquad t \in (0, T_{\max}). \tag{2.5}$$

Proof. We first let $\eta = (u, v, w)^T$, then the system (1.4) can be reformulated as the abstract form

$$\begin{cases} \eta_t - \nabla \cdot (\mathcal{A}(\eta) \nabla \eta) = \mathcal{F}(\eta), & x \in \Omega, t > 0, \\ \frac{\partial \eta}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ \eta(\cdot, 0) = (u_0, v_0, w_0)^T, & x \in \Omega, \end{cases} \tag{2.6}$$

where

$$\mathcal{A}(\eta) = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ \xi w & 0 & d_3 \end{pmatrix}, \qquad \mathcal{F}(\eta) = \begin{pmatrix} -u + v \\ auw - \gamma uv - cv \\ rw - w^2 - uw \end{pmatrix}.$$

System (2.6) is normally parabolic since all the eigenvalues of $\mathcal{A}(\eta)$ are positive. Then from Theorem 7.3 and Corollary 9.3 in Ref. [1] or Theorem 14.4 and 14.6 in Ref. [2], we obtain that there exists a unique classical solution. Next, the estimates (2.1) follow from the maximum principle.

Furthermore, since the system (2.6) is a lower triangular system, then we can invoke Theorem 15.5 of Ref. [2] to conclude that $T_{\max} = \infty$ if (2.2) holds.

Finally, we show that the solution $(u(x,t), v(x,t), w(x,t))$ is bounded in $L^1(\Omega)$. Integrating the third equation in (1.4) over Ω and using the Cauchy–Schwarz inequality we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega w dx &= r\int_\Omega w dx - \int_\Omega w^2 dx - \int_\Omega uw dx \\ &\le r\int_\Omega w dx - \frac{1}{|\Omega|}\left(\int_\Omega w dx\right)^2, \qquad t \in (0, T_{\max}).\end{aligned}$$

By an ODE comparison principle, we derive

$$\int_\Omega w dx \le \max\left\{\int_\Omega w_0 dx, r|\Omega|\right\} =: m_1.$$

Then we have

$$\begin{aligned}\int_\Omega (v_t + aw_t) dx &= \frac{d}{dt}\int_\Omega (v + aw) dx \\ &= \int_\Omega [d_2\Delta v + d_3 a\Delta w + \xi a\nabla\cdot(w\nabla u)] dx + \int_\Omega (raw - aw^2 - \gamma uv - cv) dx \\ &= \int_\Omega [raw + acw - aw^2 - \gamma uv - c(v + aw)] dx \\ &\le \int_\Omega [aw(r + c) - c(v + aw)] dx\end{aligned}$$

since $\int_\Omega w dx \le m_1$, it gets

$$\int_\Omega v dx \le \int_\Omega (v + aw) dx \le \max\left\{\int_\Omega (v_0 + aw_0) dx, \frac{a(r+c)}{c} m_1\right\} =: m_2.$$

Similarly, it can be derived

$$\int_\Omega u dx \le \max\left\{\int_\Omega u_0 dx, m_2\right\} =: m_3.$$

This completes the proof of part (3). □

2.2 Relationship between bounds for u, ∇v and w in the case $n \ge 2$

In this subsection, by using appropriate smoothing estimates for the Neumann heat semigroup to the system (1.4), which have been inspired by Winkler [26], we establish some relationships between the quantities

$$\sup_{s\in(0,t)} \|u(\cdot,s)\|_{L^\infty}, \qquad \sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|_{L^q}, \qquad \sup_{s\in(0,t)} \|w(\cdot,s)\|_{L^p}, \qquad t \in (0, T_{\max}),$$

for suitably wide ranges of the free parameters $p \in (1, \infty]$ and $q \in (1, \infty)$ when $n \ge 2$.

Lemma 2.2. *Assume that $n \ge 2$ and $q > \max\{1, \frac{n}{3}\}$. Then for any $\varepsilon > 0$, there exists $C(\varepsilon, q) > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C(\varepsilon,q) + C(\varepsilon,q)\cdot\left\{\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}\right\}^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon}, \qquad t \in (0, T_{\max}). \quad (2.7)$$

Proof. Since $q > \frac{n}{3}$, without loss of generality we may assume that ε satisfies $(n+1-\frac{n}{q})\varepsilon < 2$ and $(n+1-\frac{n}{q})q\varepsilon < 3q-n$. Here the former property ensures that

$$r \equiv r(\varepsilon, q) := \frac{n}{2-(n+1-\frac{n}{q})\varepsilon}$$

is a positive number satisfying $r > \frac{n}{2} \geq 1$ as well as

$$\frac{(n-q)r}{n} = \frac{n-q}{2-(n+1-\frac{n}{q})\varepsilon} < \frac{n-q}{2-\frac{3q-n}{q}} = q.$$

Hence, the Gagliardo–Nirenberg inequality gives $c_1 = c_1(\varepsilon, q) > 0$ such that with $a := a(\varepsilon, q) := \frac{n-\frac{n}{r}}{n+1-\frac{n}{q}} \in (0,1)$ we have

$$\|\phi\|_{L^r(\Omega)} \leq c_1 \|\nabla\phi\|^a_{L^q(\Omega)} \|\phi\|^{1-a}_{L^1(\Omega)} + c_1\|\phi\|_{L^1}, \qquad \phi \in W^{1,q}(\Omega), \tag{2.8}$$

and moreover we can employ smoothing estimates for the Neumann heat semi-group $(e^{t\Delta})_{t\leq 0}$ [25] to find $c_2 = c_2(\varepsilon, q) > 0$ fulfilling

$$\|e^{t\Delta}\phi\|_{L^\infty(\Omega)} \leq c_2(1+t^{-\frac{n}{2r}})\|\phi\|_{L^r(\Omega)}, \qquad t > 0,\ \phi \in L^r(\Omega). \tag{2.9}$$

As Lemma 2.1 provides that with some $m_2 > 0$ we have $\|v(\cdot,t)\|_{L^1(\Omega)} \leq m_2$ for all $t \in (0, T_{\max})$, based on a variation-of-constants representation we can combine (2.8) with (2.9) to see that due to the maximum principle,

$$\begin{aligned}
&\|u(\cdot,t)\|_{L^\infty(\Omega)} \\
&\quad = \|e^{t(d_1\Delta-1)}u_0 + \int_0^t e^{(t-s)(d_1\Delta-1)}v(\cdot,s)ds\|_{L^\infty(\Omega)} \\
&\quad \leq e^{-t}\|u_0\|_{L^\infty(\Omega)} + c_2\int_0^t (1+(t-s)^{-\frac{n}{2r}})e^{-(t-s)}\|v(\cdot,s)\|_{L^r(\Omega)}ds \\
&\quad \leq \|u_0\|_{L^\infty(\Omega)} + c_1c_2\int_0^t (1+(t-s)^{-\frac{n}{2r}})e^{-(t-s)}\|\nabla v(\cdot,s)\|^a_{L^q(\Omega)}\|v(\cdot,s)\|^{1-a}_{L^1(\Omega)}ds \\
&\qquad + c_1c_2\int_0^t (1+(t-s)^{-\frac{n}{2r}})e^{-(t-s)}\|v(\cdot,s)\|_{L^1}ds \\
&\quad \leq \|u_0\|_{L^\infty(\Omega)} + \{c_1c_2m_2^{1-a}\|\nabla v\|^a_{L^\infty((0,t);L^q(\Omega))} + c_1c_2m_2\} \cdot \int_0^t (1+(t-s)^{-\frac{n}{2r}})e^{-(t-s)}ds \\
&\quad \leq \|u_0\|_{L^\infty(\Omega)} + \{c_1c_2m_2^{1-a}\|\nabla v\|^a_{L^\infty((0,t);L^q(\Omega))} + c_1c_2m_2\} \cdot \left(1+\Gamma\left(1-\frac{n}{2r}\right)\right)
\end{aligned}$$

for all $t \in (0, T_{\max})$. Here $\Gamma(1-\frac{n}{2r})$ is the Gamma function which is positive and real-valued according to $r > \frac{n}{2}$, this already entails (2.7) due to the fact that

$$a = \frac{n-(2-(n+1-\frac{n}{q})\varepsilon)}{n+1-\frac{n}{q}} = \frac{n-2}{n+1-\frac{n}{q}} + \varepsilon$$

by definition of a and r. □

A similar argument shows that the regularity of ∇v depends on L^p bounds for w and L^∞ bounds for u.

Lemma 2.3. *Let $n \geq 2$. Assume that $p \in (1, \infty]$ and $q > \frac{n}{n-1}$ be such that $(n-p)q < np$. Then for each $\varepsilon > 0$ there exists $C(\varepsilon, p, q) > 0$ such that*

$$
\begin{aligned}
&\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \\
&\quad \leq C(\varepsilon,p,q) + C(\varepsilon,p,q)\cdot\left\{1+\sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^p(\Omega)}\right\}^{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})}+\varepsilon}\cdot\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^\infty(\Omega)} \\
&\qquad + C(\varepsilon,p,q)\cdot\left\{1+\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}\right\}^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon}\cdot\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^\infty(\Omega)} \\
&\qquad + C(\varepsilon,p,q)\cdot\left\{1+\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}\right\}^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon}, \qquad t\in(0,T_{\max}). \qquad (2.10)
\end{aligned}
$$

Proof. Since $(n-p)q < np$ and thus $\frac{1}{q}+\frac{1}{n}-\frac{1}{p} > 0$, we assume that apart from $(1-\frac{1}{p})\varepsilon < \frac{1}{n}$ the inequality $(1-\frac{1}{p})\varepsilon < \frac{1}{q}+\frac{1}{n}-\frac{1}{p}$ holds about ε, so that

$$\lambda \equiv \lambda(\varepsilon,p,q) := \frac{1}{\frac{1}{q}+\frac{1}{n}-(1-\frac{1}{p})\varepsilon}$$

is a positive number satisfying $\lambda < q$. Moreover

$$\lambda > \frac{1}{\frac{1}{q}+\frac{1}{n}} > 1 \qquad (2.11)$$

thanks to the condition $q > \frac{n}{n-1}$.

By applying Duhamel representation and smoothing properties of the Neumann heat semigroup, for all $t \in (0, T_{\max})$ one can estimate

$$
\begin{aligned}
&\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \\
&\quad = \|\nabla e^{t(d_2\Delta-1)}v_0 + a\int_0^t \nabla e^{(t-s)(d_2\Delta-1)}u(\cdot,s)w(\cdot,s)ds - \gamma\int_0^t \nabla e^{(t-s)(d_2\Delta-1)}u(\cdot,s)v(\cdot,s)ds \\
&\qquad + (1-c)\nabla e^{(t-s)(d_2\Delta-1)}v(\cdot,s)ds\|_{L^q(\Omega)} \\
&\quad \leq c_1e^{-t}\|v_0\|_{L^q(\Omega)} + c_2a\int_0^t(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}\|u(\cdot,s)w(\cdot,s)\|_{L^\lambda(\Omega)}ds \\
&\qquad + c_2\gamma\int_0^t(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}\|u(\cdot,s)v(\cdot,s)\|_{L^\lambda(\Omega)}ds \\
&\qquad + c_2|1-c|\int_0^t(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}\|v(\cdot,s)\|_{L^\lambda(\Omega)}ds. \qquad (2.12)
\end{aligned}
$$

Furthermore, by the Hölder inequality, since $\lambda < p$ we have

$$
\begin{aligned}
\|u(\cdot,s)w(\cdot,s)\|_{L^\lambda(\Omega)} &\leq \|w(\cdot,s)\|^{a_1}_{L^p(\Omega)}\|w(\cdot,s)\|^{1-a_1}_{L^1}\|u(\cdot,s)\|_{L^\infty(\Omega)} \\
&\leq m_1^{1-a_1}\|w(\cdot,s)\|^{a_1}_{L^p(\Omega)}\|u(\cdot,s)\|_{L^\infty(\Omega)}, \qquad s\in(0,T_{\max})
\end{aligned}
$$

with $a_1 = a_1(\varepsilon,p,q) := \frac{1-\frac{1}{\lambda}}{1-\frac{1}{p}} \in (0,1)$, and with $m_1 := \sup_{t\in(0,T_{\max})} \|w(\cdot,t)\|_{L^1(\Omega)}$ being finite according to Lemma 2.1. And the Gagliardo–Nirenberg inequality yields

$$\begin{aligned}\|v(\cdot,s)\|_{L^\lambda(\Omega)} &\le \|v(\cdot,s)\|_{L^r(\Omega)}\\ &\le c_3\|\nabla v(\cdot,s)\|^{a_2}_{L^q(\Omega)}\|v(\cdot,s)\|^{1-a_2}_{L^1(\Omega)} + c_3\|v(\cdot,s)\|_{L^1(\Omega)}\\ &\le c_3 m_2^{1-a_2}\|\nabla v(\cdot,s)\|^{a_2}_{L^q(\Omega)} + c_3 m_2\end{aligned}$$

with $a_2 \equiv a_2(\varepsilon,p,q) := \frac{n-\frac{n}{\lambda}}{n+1-\frac{n}{q}} \in (0,1)$, and $\lambda < r$ which is given in Lemma 2.7.

Therefore, for all $t \in (0,T_{\max})$, (2.12) can be simplified as follows

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le c_1\|v_0\|_{W^{1,\infty}(\Omega)} &+ ac_2 m_3^{1-a_1}\cdot\left\{\sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^p(\Omega)}\right\}^{a_1}\\ &\cdot\left\{\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\right\}\cdot\int_0^t (1+(t-s)^{-\frac12-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}ds\\ &+(c_2\gamma + c_2|1-c|)\left(c_3 m_2^{1-a_2}\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|^{a_2}_{L^q(\Omega)} + c_3 m_2\right)\\ &\cdot\left\{\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\right\}\cdot\int_0^t (1+(t-s)^{-\frac12-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}ds.\end{aligned}$$

Noting that for all $t > 0$ we have

$$\begin{aligned}\int_0^t (1+(t-s)^{-\frac12-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-(t-s)}ds \le c_4(\varepsilon,p,q) &:= \int_0^t (1+\sigma^{-\frac12-\frac{n}{2}(\frac{1}{\lambda}-\frac{1}{q})})e^{-\sigma}d\sigma\\ &= \Gamma\left(\frac12-\frac{n}{2}\left(\frac{1}{\lambda}-\frac{1}{q}\right)\right),\end{aligned}$$

that $c_4 < \infty$ thanks to the inequality $\frac{1}{\lambda} < \frac{1}{q}+\frac{1}{n}$ contained in (2.11), and then

$$a_1 = \frac{1-\left\{\frac{1}{q}+\frac{1}{n}-\left(1-\frac{1}{p}\right)\varepsilon\right\}}{1-\frac{1}{p}} = \frac{n-1-\frac{n}{q}}{n\left(1-\frac{1}{p}\right)}+\varepsilon,$$

we conclude as intended. □

Combining the previous two lemmata allows us to eliminate the dependence on u in (2.10) as follows.

Lemma 2.4. *Let $2 \le n < 5$. Assume that $p \in (1,\infty]$ and that $q > \frac{n}{n-1}$ satisfy $q > \frac{n}{5-n}$ and $(n-p)q < np$. Then for all $\varepsilon > 0$ there exists $C(\varepsilon,p,q) > 0$ with the property that*

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le C(\varepsilon,p,q)\cdot\left\{1+\sup_{s\in(0,t)}\|u(\cdot,s)\|_{L^p(\Omega)}\right\}^{\frac{(n-1-\frac{n}{q})(n+1-\frac{n}{q})}{n(1-\frac{1}{p})(5-n-\frac{n}{q})}+\varepsilon}, \qquad t\in(0,T_{\max}). \tag{2.13}$$

Proof. We note that $n+1-\frac{n}{q} > 2(n-2)$ since the assumption that $q > \frac{n}{5-n}$, and that $n-1-\frac{n}{q} > 0$ due to $q > \frac{n}{n-1}$. Then, there exists $\tilde{\varepsilon} = \tilde{\varepsilon}(p,q) > 0$ such that

$$\theta(\varepsilon_1) := \left\{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})}+\varepsilon\right\}\cdot\frac{n+1-\frac{n}{q}}{(n+1-\frac{n}{q})(1-2\varepsilon_1)-2(n-2)}$$

is well-defined for all $\varepsilon_1 \in (0, \tilde{\varepsilon})$, with

$$\theta(\varepsilon_1) \to \theta_0 := \frac{(n-1-\frac{n}{q})(n+1-\frac{n}{q})}{n(1-\frac{1}{p})(5-n-\frac{n}{q})} \quad \text{as } \varepsilon_1 \searrow 0.$$

For $\varepsilon > 0$, we can find $\varepsilon_1 = \varepsilon_1(\varepsilon, p, q) \in (0, \tilde{\varepsilon})$ such that

$$\theta(\varepsilon_1) \le \theta_0 + \varepsilon, \tag{2.14}$$

and then from Lemma 2.2 and Lemma 2.3 provide $c_1 = c_1(\varepsilon, q) > 0$ and $c_2 = c_2(\varepsilon, p, q) > 0$ such that

$$L(t) := 1 + \sup_{s \in (0,t)} \|w(\cdot, s)\|_{L^p(\Omega)}, \qquad t \in (0, T_{\max}),$$

and

$$M(t) := \sup_{s \in (0,t)} \|\nabla v(\cdot, s)\|_{L^q(\Omega)}, \qquad t \in (0, T_{\max}),$$

as well as

$$N(t) := \sup_{s \in (0,t)} \|u(\cdot, s)\|_{L^\infty}(\Omega), \qquad t \in (0, T_{\max}),$$

satisfy

$$N(t) \le c_1 + c_1 M^{\frac{n-2}{n+1-\frac{n}{q}} + \varepsilon_1}(t), \qquad t \in (0, T_{\max}) \tag{2.15}$$

and

$$M(t) \le c_2 + c_2 L^{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})} + \varepsilon_1}(t) M(t) + c_2 M^{\frac{n-2}{n+1-\frac{n}{q}} + \varepsilon_1} N(t) + c_2 M^{\frac{n-2}{n+1-\frac{n}{q}} + \varepsilon_1}, \qquad t \in (0, T_{\max}). \tag{2.16}$$

In the case of $t \in (0, T_{\max})$ and $M(t) \ge 1$, from (2.15) we obtain that

$$N(t) \le 2c_1 M^{\frac{n-2}{n+1-\frac{n}{q}} + \varepsilon_1}(t)$$

and by (2.16),

$$\begin{aligned} M(t) &\le c_2 + 2c_1c_2 L^{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})} + \varepsilon_1}(t) M^{\frac{n-2}{n+1-\frac{n}{q}} + \varepsilon_1}(t) + 2c_1c_2 M^{\frac{2(n-2)}{n+1-\frac{n}{q}} + 2\varepsilon_1}(t) + c_2 M^{\frac{n-2}{n+1-\frac{n}{q}} + \varepsilon_1}(t) \\ &\le (2c_2 + 4c_1c_2) L^{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})} + \varepsilon_1}(t) M^{\frac{2(n-2)}{n+1-\frac{n}{q}} + 2\varepsilon_1}(t), \end{aligned}$$

because $L(t) \ge 1$ by definition. Since for any such t we therefore have

$$M^{1-2\varepsilon_1 - \frac{2(n-2)}{n+1-\frac{n}{q}}}(t) \le (2c_2 + 4c_1c_2) L^{\frac{n-1-\frac{n}{q}}{n(1-\frac{1}{p})} + \varepsilon_1}(t),$$

and since

$$1 - 2\varepsilon_1 - \frac{2(n-2)}{n+1-\frac{n}{q}} = \frac{(n+1-\frac{n}{q})(1-2\varepsilon_1) - 2(n-2)}{n+1-\frac{n}{q}} > 0$$

by positivity of $\theta(\varepsilon_1)$, from this we can infer that actually for arbitrary $t \in (0, T_{\max})$, regardless of the sign of $M(t) - 1$,

$$M(t) \le c_3 L^{\theta(\varepsilon_1)}(t)$$

with $c_3 \equiv c_3(\varepsilon, p, q) := \max\left\{1, (2c_2 + 4c_1c_2)^{\frac{n+1-\frac{n}{q}}{(n+1-\frac{n}{q})(1-2\varepsilon_1)-2(n-2)}}\right\} > 0$. Once again since $L(t) \ge 0$ for all $t \in (0, T_{\max})$, in view of (2.14) this establishes (2.13). □

Lemma 2.5. *Let $n = 2$. Then whenever $p \in (\frac{n}{n-1}, \infty]$ and $q > n$, for all $\varepsilon > 0$ there exists $C(\varepsilon, p, q) > 0$ such that*

$$\|w(\cdot,t)\|_{L^p(\Omega)} \leq C(\varepsilon,p,q) + C(\varepsilon,p,q) \cdot \left\{ \sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|_{L^q(\Omega)} \right\}^{\frac{1-\frac{1}{p}}{\frac{2}{n}-\frac{1}{q}}+\varepsilon} \tag{2.17}$$

for all $t \in (0, T_{\max})$.

Proof. Firstly, we observe that $\frac{1}{q} < \frac{1}{n} < \frac{1}{n} + \frac{1}{p} < 1$ thanks to the assumption that $p > \frac{n}{n-1}$ and $q > n$. Then the interval $J_1 := \left(\frac{1}{q}, \frac{1}{n} + \frac{1}{p}\right]$ is not empty and

$$\psi_1(\zeta) := \frac{1}{\zeta - \frac{1}{q}}, \qquad \zeta \in J_1,$$

defines a positive function ψ_1 on J_1 which satisfies

$$\frac{\psi_1\left(\frac{1}{n} + \frac{1}{p}\right)}{p} = \frac{\frac{1}{p}}{\frac{1}{n} + \frac{1}{p} - \frac{1}{q}} < \frac{\frac{1}{p}}{\frac{1}{q} + \frac{1}{p} - \frac{1}{q}} = 1. \tag{2.18}$$

Next, since $q > n$ together with the inequality $p \geq 1$ infer that $\frac{1}{p} + \frac{1}{q} < \frac{1}{n} + \frac{1}{p}$, similarly, it follows that $J_2 := \left(\frac{1}{p} + \frac{1}{q}, \frac{1}{n} + \frac{1}{p}\right] \neq \emptyset$, and

$$\psi_2(\zeta) := \frac{1 - \frac{1}{p}}{\zeta - \frac{1}{p} - \frac{1}{q}}, \qquad \zeta \in J_2,$$

is well-defined and nonnegative with

$$\psi_2\left(\frac{1}{n} + \frac{1}{p}\right) = \frac{1 - \frac{1}{p}}{\frac{1}{n} - \frac{1}{q}}. \tag{2.19}$$

According to (2.18), (2.19) and continuity of ψ_1 and ψ_2, we thereby see that for any $\varepsilon > 0$ it is possible to pick $\zeta = \zeta(\varepsilon, p, q) \in J_1 \cap J_2 = J_2$ such that $\zeta < \frac{1}{n} + \frac{1}{p}$ and that $\psi_1(\zeta) < p$ as well as $\psi_2(\zeta) \leq \frac{1-\frac{1}{p}}{\frac{2}{n}-\frac{1}{q}} + \varepsilon$, where we can clearly moreover achieve that $\zeta > \frac{1}{p}$.

Setting $\mu \equiv \mu(\varepsilon, p, q) := \frac{1}{\zeta}$, we can find a positive number μ simultaneously fulfilling

$$\mu < p, \qquad \mu < q, \qquad \frac{1}{\mu} > \frac{1}{p} + \frac{1}{q}, \qquad \frac{1}{\mu} < \frac{1}{n} + \frac{1}{q}, \quad \text{and} \quad \frac{1}{\mu} < \frac{1}{n} + \frac{1}{p} \tag{2.20}$$

as well as

$$\frac{q\mu}{q - \mu} < p \tag{2.21}$$

and

$$\frac{1 - \frac{1}{p}}{\frac{1}{\mu} - \frac{1}{p} - \frac{1}{q}} \leq \frac{1 - \frac{1}{p}}{\frac{2}{n} - \frac{1}{q}} + \varepsilon. \tag{2.22}$$

Furthermore, $\mu > 1$ since $p > \frac{n}{n-1}$ and the rightmost property in (2.20).

Keeping this parameter μ fixed henceforth, using a Duhamel representation, for all $t \in (0, T_{\max})$, we can estimate

$$
\begin{aligned}
&\|\nabla u(\cdot,t)\|_{L^q(\Omega)} \\
&= \|\nabla e^{t(d_1\Delta-1)}u_0 + \int_0^t \nabla e^{(t-s)(d_1\Delta-1)}v(\cdot,s)ds\|_{L^q(\Omega)} \\
&\le c_2 e^{-t}\|u_0\|_{L^q(\Omega)} + c_3\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{q})})e^{-(t-s)}\|v(\cdot,s)\|_{L^\mu(\Omega)}ds \\
&\le c_4 + \left\{c_4 \sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|^{a_0}_{L^q(\Omega)}\|v(\cdot,s)\|^{1-a_0}_{L^1(\Omega)} + c_4\|v(\cdot,s)\|_{L^1(\Omega)}\right\} \\
&\quad \cdot \left(1+\Gamma\left(\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\mu}-\frac{1}{q}\right)\right)\right) \\
&\le c_4 + \left(c_4 m_2^{1-a_0}\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|^{a_0}_{L^q(\Omega)} + c_4 m_2\right)\left(1+\Gamma\left(\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\mu}-\frac{1}{q}\right)\right)\right) \\
&\le c_5\left(1+\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|^{a_0}_{L^q(\Omega)}\right) \\
&\le c_6\left(1+\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}\right)^{a_0} \\
&\le c_6 + c_6\sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}
\end{aligned}
$$

where $a_0 = \frac{n-\frac{n}{\mu}}{n+1-\frac{n}{q}} \in (0,1)$ since $q > n$, and $\Gamma(\frac{1}{2}-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{q})) < \infty$ due to $\frac{1}{\mu} < \frac{1}{n}+\frac{1}{q}$. Apart from that, by the first inequality in (2.20) and regularization features of the Neumann heat semigroup ([25, Lemma 1.3], [29, Lemma 3.3]) one can pick $c_1 = c_1(\varepsilon, p, q) > 0$ satisfying

$$\|e^{t\Delta}\nabla\cdot\phi\|_{L^p(\Omega)} \le c_1\left(1+t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p})}\right)\|\phi\|_{L^\mu(\Omega)}$$

for all $t > 0$ and each $\phi \in C^1(\bar{\Omega};\mathbb{R}^n)$ such that $\phi\cdot\nu = 0$ on $\partial\Omega$, which shows that for all $t \in (0, T_{\max})$,

$$
\begin{aligned}
&\int_0^t \|e^{(t-s)(d_3\Delta-1)}\nabla\cdot(w(\cdot,s)\nabla u(\cdot,s))\|_{L^p(\Omega)}ds \\
&\qquad \le c_1\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p})})e^{-(t-s)}\|w(\cdot,s)\nabla u(\cdot,s)\|_{L^\mu(\Omega)}ds. \qquad (2.23)
\end{aligned}
$$

Hence due to the second relation in (2.20), we may employ the Hölder inequality shows that again writing $L(t) := 1+\sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^p(\Omega)}$ and $M(t) := \sup_{s\in(0,t)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)}$, $t \in (0, T_{\max})$, for any such t we have

$$
\begin{aligned}
\|w(\cdot,s)\nabla u(\cdot,s)\|_{L^\mu(\Omega)} &\le \|w(\cdot,s)\|^{\alpha}_{L^p(\Omega)}\|w(\cdot,s)\|^{1-\alpha}_{L^1(\Omega)}\|\nabla u(\cdot,s)\|_{L^q(\Omega)} \\
&\le m_1^{1-\alpha}\|w(\cdot,s)\|^{\alpha}_{L^p(\Omega)}(c_6 + c_6\|\nabla v(\cdot,s)\|_{L^q(\Omega)}) \\
&\le c_6 m_1^{1-\alpha}L^{\alpha}(t) + c_6 m_1^{1-\alpha}L^{\alpha}(t)M(t), \qquad s \in (0,t)
\end{aligned}
$$

with $\alpha = \alpha(\varepsilon, p, q) := \frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}} \in (0,1)$.

The relation (2.23) indicates that with some $c_7 = c_7(\varepsilon, p, q) > 0$,

$$\int_0^t \|e^{(t-s)(d_3\Delta-1)}\nabla\cdot(w(\cdot,s)\nabla u(\cdot,s))\|_{L^p(\Omega)}ds \le c_7 L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}}(t) + c_7 L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}}(t)M(t) \tag{2.24}$$

for all $t \in (0, T_{\max})$. In order to make appropriate use of this, we observe that from the third equation of (1.4),

$$w_t \le d_3\Delta w - w + \xi\cdot\nabla(w\nabla u) + (r+1)w \quad \text{in } \Omega\times(0,T_{\max}).$$

In view of the nonnegativity of w and an associated variation-of-constants formula, one can obtain that

$$\begin{aligned}
&\|w(\cdot,t)\|_{L^p(\Omega)} \\
&\le \left\| e^{t(d_3\Delta-1)}w_0 + \xi\int_0^t e^{(t-s)(d_3\Delta-1)}\nabla\cdot(w(\cdot,s)\nabla u(\cdot,s))ds + (r+1)\int_0^t e^{(t-s)(d_3\Delta-1)}w ds\right\|_{L^p(\Omega)} \\
&\le e^{-t}\|w_0\|_{L^p(\Omega)} + |\xi|\int_0^t \|e^{(t-s)(d_3\Delta-2)}\nabla\cdot(w(\cdot,s)\nabla u(\cdot,s))\|_{L^p(\Omega)}ds \\
&\quad + (r+1)c_8\int_0^t (1+t^{-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p})})e^{-(t-s)}\|w(\cdot,s)\|_{L^\mu(\Omega)}ds, \qquad t\in(0,T_{\max}).
\end{aligned} \tag{2.25}$$

Using the Hölder inequality, we have

$$\|w(\cdot,s)\|_{L^\mu(\Omega)} \le \|w(\cdot,s)\|^{\beta}_{L^p(\Omega)}\|w(\cdot,s)\|^{1-\beta}_{L^1(\Omega)} \le m_1^{1-\beta}\|w(\cdot,s)\|^{\beta}_{L^p(\Omega)},$$

where $\beta = \frac{1-\frac{1}{\mu}}{1-\frac{1}{p}}$. Therefore, by the Young inequality, we obtain that

$$\begin{aligned}
&(r+1)c_8\int_0^t (1+t^{-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p})})e^{-(t-s)}\|w(\cdot,s)\|_{L^\mu(\Omega)}ds \\
&\quad\le (r+1)c_8 m_1^{1-\beta}\left\{\sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^p(\Omega)}\right\}^{\beta}\left(1+\Gamma\left(1-\frac{n}{2}\left(\frac{1}{\mu}-\frac{1}{p}\right)\right)\right) \\
&\quad\le \frac{1}{2}\sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^p(\Omega)} + c_9
\end{aligned}$$

where $c_9 = \frac{1}{2}(r+1)c_8 m_1^{1-\beta}(1+\Gamma(1-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p})))$ and $\Gamma(1-\frac{n}{2}(\frac{1}{\mu}-\frac{1}{p}))$ is positive and real-valued due to $\frac{1}{\mu} < \frac{2}{n}+\frac{1}{p}$.

In conjunction with (2.25) and (2.24), this infers the existence of $c_{10} = c_{10}(\varepsilon, p, q) > 0$ such that

$$L(t) \le c_{10} + c_{10}L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}}(t) + c_{10}L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}}(t)M(t), \qquad t\in(0,T_{\max}),$$

where the third inequality in (2.20) ensures that $\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}} < 1$ and Young inequality so as to provide

$$c_{10}L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}} \le \frac{1}{4}L(t) + c_{11},$$

and

$$c_{10}L^{\frac{1+\frac{1}{q}-\frac{1}{\mu}}{1-\frac{1}{p}}}M(t) \le \frac{1}{4}L(t) + c_{12}M^{\frac{1-\frac{1}{p}}{\frac{1}{\mu}-\frac{1}{p}-\frac{1}{q}}}(t), \qquad t \in (0, T_{\max}).$$

In light of (2.22), this yields (2.17). □

3 Proof of Theorem 1.1

3.1 Boundedness when $n = 2$

Lemma 3.1. *Let $n = 2$. Then there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C, \qquad t \in (0, T_{\max}). \tag{3.1}$$

Proof. Without loss of generality assuming that $p < n$. Let

$$\theta(\zeta,\varepsilon) := \left\{\frac{1-\frac{1}{p}}{\frac{2}{n}-\zeta}+\varepsilon\right\}\left\{\frac{(n-1-n\zeta)(n+1-n\zeta)}{n(1-\frac{1}{p})(5-n-n\zeta)}+\varepsilon\right\},$$

$$\zeta \in J := \left(0, \frac{n-1}{n}\right], \qquad \varepsilon > 0,$$

noting that θ is well-defined because $\frac{n-1}{n} < \frac{5-n}{n}$. Since evidently $\theta(\frac{n-1}{n}, 0) = 0$, and since apart from that clearly $\frac{1}{p} - \frac{1}{n} < \frac{n-1}{n}$, by means of a continuity argument we can choose $\zeta \in J$ and $\varepsilon > 0$ such that $\zeta < \frac{n-1}{n}$ and

$$\zeta > \frac{1}{p} - \frac{1}{n} \tag{3.2}$$

and that

$$\theta(\zeta,\varepsilon) < 1, \tag{3.3}$$

and thus $\zeta < \frac{1}{n}$. Writing $q := \frac{1}{\zeta}$, therefore one can find that $q > \frac{n}{n-1}$ and $(n-p)q < np$ as well as $q > n$, where the latter relation together with the inequality $p > \frac{n}{n-1}$ enables us to invoke Lemma 2.5, thus inferring the existence of $c_1 > 0$ such that for $L(t) := 1 + \sup_{s\in(0,t)} \|w(\cdot,s)\|_{L^p(\Omega)}$ and $M(t) := \sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|_{L^q(\Omega)}$, $t \in (0, T_{\max})$, we have

$$L(t) \le c_1 + c_1 M^{\frac{1-\frac{1}{p}}{\frac{2}{n}-\frac{1}{q}}+\varepsilon}(t), \qquad t \in (0, T_{\max}). \tag{3.4}$$

On the other hand, using that $(n-p)q < np$ and $q > \frac{n}{n-1}$, and that thus also $q > \frac{n}{5-n}$, we may employ Lemma 2.4 to find $c_2 > 0$ such that

$$M(t) \le c_2 L^{\frac{(n+1-\frac{n}{q})(n-1-\frac{n}{q})}{n(1-\frac{1}{p})(5-n-\frac{n}{q})}+\varepsilon}(t), \qquad t \in (0, T_{\max}). \tag{3.5}$$

Combined with (3.4), this provides that

$$L(t) \le c_1 + c_1 c_2^{\frac{1-\frac{1}{p}}{\frac{2}{n}-\frac{1}{q}}} L^{\theta(\frac{1}{q},\varepsilon)}(t), \qquad t \in (0, T_{\max})$$

and thus shows that with some $c_3 > 0$ we have

$$L(t) \le c_3, \qquad t \in (0, T_{\max}),$$

because $\theta(\frac{1}{q}, \varepsilon) < 1$ by (3.3). Through (3.5), the latter entails boundedness of $(0, T_{\max}) \ni t \mapsto \|\nabla v(\cdot,t)\|_{L^q(\Omega)}$, so that Lemma 2.2 establishes the claim. □

Lemma 3.2. *Let $n = 2$. Then for all $q > n$ there exists $C(q) > 0$ fulfilling*

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le C(q), \qquad t \in (0, T_{\max}). \tag{3.6}$$

Proof. For each fixed $q > n$,

$$\frac{n-1-\frac{n}{q}}{n(\frac{2}{n}-\frac{1}{q})} = \frac{n-1-\frac{n}{q}}{2-\frac{n}{q}} < 1,$$

by a continuity argument we can pick $\varepsilon = \varepsilon(q) > 0$ appropriately small such that still

$$\theta := \left\{ \frac{1}{\frac{2}{n}-\frac{1}{q}} + \varepsilon \right\} \cdot \left\{ \frac{n-1-\frac{n}{q}}{n} + \varepsilon \right\} < 1.$$

Then from Lemma 3.1, we may employ Lemma 2.3 with $p := \infty$ to find $c_1 = c_1(q) > 0$ such that $L(t) := 1 + \sup_{s\in(0,t)} \|w(\cdot,s)\|_{L^p(\Omega)}$ and $M(t) := \sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|_{L^q(\Omega)}$, $t \in (0, T_{\max})$, satisfy

$$M(t) \le c_1 L^{\frac{n-1-\frac{n}{q}}{n}}(t), \qquad t \in (0, T_{\max}) \tag{3.7}$$

which we combine with the outcome of Lemma 2.5, applicable since the inequality $q > n$, which namely yields $c_2 = c_2(q) > 0$ fulfilling

$$L(t) \le c_2 + c_2 M^{\frac{1}{\frac{2}{n}-\frac{1}{q}}+\varepsilon}(t), \qquad t \in (0, T_{\max}).$$

Therefore

$$L(t) \le c_2 + c_1^{\frac{1}{\frac{2}{n}-\frac{1}{q}}+\varepsilon} c_2 L^{\theta}(t), \qquad t \in (0, T_{\max}),$$

so that the inequality $\theta < 1$ guarantees boundedness of L and thus, by (3.7), also derives boundedness of M. □

3.2 Boundedness in the one-dimensional case

Lemma 3.3. *Let $n = 1$. Then for all $q > 1$ there exists $C(q) > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot,t)\|_{L^q(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \le C(q), \qquad t \in (0, T_{\max}). \tag{3.8}$$

Proof. In view of the boundedness of $(0, T_{\max}) \ni t \mapsto \|v(\cdot,t)\|_{L^1(\Omega)}$ asserted by Lemma 2.1, straightforward application of L^1–L^∞ smoothing estimates for the Neumann heat semigroup in the present one-dimensional situation entails $c_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le c_1, \qquad t \in (0, T_{\max}), \tag{3.9}$$

which again thanks to Lemma 2.1 ensures boundedness of $(0, T_{\max}) \ni t \mapsto \|u(\cdot,t)w(\cdot,t)\|_{L^1(\Omega)}$ and $(0, T_{\max}) \ni t \mapsto \|u(\cdot,t)v(\cdot,t)\|_{L^1(\Omega)}$. Accordingly, standard L^∞–$W^{1,q}$ regularization properties of $(e^{t\Delta})_{t\ge 0}$ guarantee the existence of $c_2 = c_2(q) > 0$ fulfilling

$$\|v_x(\cdot,t)\|_{L^q(\Omega)} \le c_2, \qquad t \in (0, T_{\max}), \tag{3.10}$$

therefore $\|u_x(\cdot,t)\|_{L^q(\Omega)} \le c_3$.

To establish $L^\infty(\Omega)$ bound for w, we can find some $\mu = \mu(q) \in (1, q)$ for any q, and again combine the maximum principle with a known smoothing feature of the heat semigroup to fix $c_4, c_5 > 0$ such that

$$\begin{aligned}
\|w(\cdot,t)\|_{L^\infty(\Omega)} \le \|e^{t(d_3\Delta-1)}w_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(d_3\Delta-1)}\partial_x(w(\cdot,s)u_x(\cdot,s)\|_{L^\infty(\Omega)}ds \\
+ (r+1)\int_0^t \|e^{(t-s)(d_3\Delta-1)}w(\cdot,s)\|_{L^\infty(\Omega)}ds \\
\le e^{-t}\|w_0\|_{L^\infty(\Omega)} + c_4\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{1}{2\mu}})e^{-(t-s)}\|w(\cdot,s)u_x(\cdot,s)\|_{L^\mu(\Omega)}ds \\
+ c_5\int_0^t (1+(t-s)^{-\frac{n}{2\mu}})e^{-(t-s)}\|w(\cdot,s)\|_{L^\mu(\Omega)}ds, \qquad t\in(0,T_{\max}), \qquad (3.11)
\end{aligned}$$

where by the Hölder inequality, for all $s \in (0, T_{\max})$ one can estimate

$$\begin{aligned}
\|w(\cdot,s)u_x(\cdot,s)\|_{L^\mu(\Omega)} &\le \|w(\cdot,s)\|_{L^{\frac{\mu q}{q-\mu}}(\Omega)}\|u_x(\cdot,s)\|_{L^q(\Omega)} \\
&\le \|w(\cdot,s)\|^{\gamma}_{L^\infty(\Omega)}\|w(\cdot,s)\|^{1-\gamma}_{L^1(\Omega)}\|u_x(\cdot,s)\|_{L^q(\Omega)}
\end{aligned}$$

with $\gamma := \frac{\mu q - q + \mu}{\mu q} \in (0,1)$ since $q > \mu$. And

$$\|w(\cdot,s)\|_{L^\mu(\Omega)} \le \|w(\cdot,s)\|^{\delta}_{L^\infty(\Omega)}\|w(\cdot,s)\|^{1-\delta}_{L^1(\Omega)} \le c_6\|w(\cdot,s)\|_{L^\infty(\Omega)} + c_7$$

where $c_6 := \frac{1}{2c_5 m^{1-\delta}(1+\Gamma(1-\frac{n}{2\mu}))}$, $c_7 := \frac{1}{4c_6}$. In view of (3.10) and Lemma 2.1, from (3.11) we thus infer the existence of $c_8, c_9 > 0$ such that if now we let $L(t) := 1 + \sup_{s\in(0,t)}\|w(\cdot,s)\|_{L^\infty(\Omega)}$, $t \in (0, T_{\max})$, then

$$L(t) \le c_8 + c_8 \cdot \left\{\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{1}{2\mu}})e^{-(t-s)}ds\right\} \cdot L^\gamma(t) + \frac{1}{2}L(t)$$

thus

$$L(t) \le 2c_8 + 2c_8c_9L^\delta(t), \qquad t \in (0, T_{\max}),$$

where $c_9 \le \int_0^\infty (1+\sigma^{-\frac{1}{2}-\frac{1}{2\mu}})e^{-\sigma}d\sigma = 1 + \Gamma(\frac{1}{2}-\frac{1}{2\mu})$ is finite since $\mu > 1$. As $\gamma < 1$, this indicates boundedness of w and hence completes the proof. □

3.3 Proof of Theorem 1.1

Proof of Theorem 1.1. Using (2.3)–(2.5) and Lemma 3.3 when $n = 1$; combining Lemma 3.1 and Lemma 3.2 when $n = 2$, the conclusion of Theorem 1.1 is obtained immediately. □

4 Dynamical behavior of prey-evasion system

In this section, we investigate the dynamic behavior of the system (1.4). We first consider the local stability of the constant equilibrium solutions by linearized stability analysis. According to the principle of linearized stability for quasi-linear parabolic problems (see [21] Th 8.6, [6] Th 5.2), we know that the constant equilibrium $(\tilde{u}, \tilde{v}, \tilde{w})$ is locally asymptotically stable with respect to (1.4) if and only if all the eigenvalues of the linearized elliptic problem of (1.4) at an equilibrium are of negative real parts. To this end, we introduce the asymptotic stability of $(\tilde{u}, \tilde{v}, \tilde{w})$ of kinetic system (1.2) in [3].

Proposition 4.1. *Suppose that $ar > c$. Let*

$$f(\tilde{w}) = a(a+1)\tilde{w}^3 + (a^2+3a+1)\tilde{w}^2 + (a+1-ac-ar)\tilde{w} - c. \tag{4.1}$$

Then there exists a unique γ^*, such that $\tilde{\mathbf{u}}$ is asymptotically stable if $\gamma > \gamma^*$ and is unstable if $0 < \gamma < \gamma^*$, where $\gamma^* = \frac{a\tilde{w}-c}{r-\tilde{w}}$, $f(\tilde{w}) = 0$.

Linearizing the system (1.4) at an equilibrium solution (u, v, w), we obtain that

$$\begin{pmatrix} \varphi_t \\ \phi_t \\ \psi_t \end{pmatrix} = \mathcal{L}(\xi) \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix} = D \begin{pmatrix} \Delta\varphi \\ \Delta\phi \\ \Delta\psi \end{pmatrix} + J_{(u,v,w)} \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix} \tag{4.2}$$

where

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ \xi w & 0 & d_3 \end{pmatrix}, \qquad J_{(u,v,w)} = \begin{pmatrix} -1 & 1 & 0 \\ aw-\gamma v & -\gamma u - cv & au \\ -w & 0 & r-2w-u \end{pmatrix}. \tag{4.3}$$

The stability of $\tilde{\mathbf{u}}$ is determined by the following eigenvalue problem

$$\mathcal{L} \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix} = \lambda \begin{pmatrix} \varphi \\ \phi \\ \psi \end{pmatrix},$$

that is

$$\begin{cases} d_1\Delta\varphi - \varphi + \phi = \lambda\varphi, & x \in \Omega, \\ d_2\Delta\phi + (aw-\gamma v)\varphi - (\gamma u + c)\phi + au\psi = \lambda\phi, & x \in \Omega, \\ \xi w\Delta\varphi + d_3\Delta\psi - w\varphi + (r-2w-u)\psi = \lambda\psi, & x \in \Omega, \\ \dfrac{\partial\varphi}{\partial\nu} = \dfrac{\partial\phi}{\partial\nu} = \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

Let $-\Delta$ have eigenvalues $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots$ and $\lim_{i\to\infty} = \infty$ under the Neumann boundary condition, and let $y_i(x)$ be the normalized eigenfunction corresponding to μ_i. Suppose that λ is an eigenvalue of (4.4) with corresponding eigenfunction (φ, ϕ, ψ), therefore according to the Fourier expansion, there exists $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ such that

$$\varphi(x) = \sum_{i=0}^{\infty} a_i\varphi_i(x), \qquad \phi(x) = \sum_{i=0}^{\infty} b_i\phi_i(x), \qquad \psi(x) = \sum_{i=0}^{\infty} c_i\psi_i(x).$$

By a straightforward computation, we have

$$\mathcal{L}_i(\xi) \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \lambda \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}, \qquad i = 0, 1, 2, \ldots$$

with

$$\mathcal{L}_i(\xi) = \begin{pmatrix} -d_1\mu_i - 1 & 1 & 0 \\ aw-\gamma v & -d_2\mu_i - \gamma u - c & au \\ -\xi w\mu_i - w & 0 & -d_3\mu_i + r - 2w - u \end{pmatrix}. \tag{4.5}$$

Therefore, the local stability of positive constant steady states of the system (1.4) is given by the following lemma.

Lemma 4.2. *Assume that $ar > c, \gamma > \gamma^*, d_i > 0$ $(i = 1,2,3)$, $\xi > 0$. Then for system* (1.4), $(\tilde{u}, \tilde{v}, \tilde{w})$ *is locally asymptotically stable if* $0 < \xi < \xi_0$ *and is unstable if* $\xi > \xi_0$, *where*

$$\xi_0 = \frac{1}{a\tilde{u}\tilde{w}\mu_i}(\beta_1\mu_i^3 + \beta_2\mu_i^2 + \beta_3\mu_i + \beta_4) > 0,$$

β_i $(i = 1,2,3,4)$ *will be given in the following proof.*

Proof. If constant equilibrium solution $(u, v, w) = (\tilde{u}, \tilde{v}, \tilde{w})$, then

$$\mathcal{L}_i(\xi) = \begin{pmatrix} -d_1\mu_i - 1 & 1 & 0 \\ c & -d_2\mu_i - a\tilde{w} & a\tilde{u} \\ -\xi\tilde{w}\mu_i - \tilde{w} & 0 & -d_3\mu_i - \tilde{w} \end{pmatrix}, \tag{4.6}$$

and the characteristic equation of $\mathcal{L}_i$ is

$$\Phi(\lambda) = |\lambda I - \mathcal{L}_i| = \lambda^3 + \alpha_1(\xi)\lambda^2 + \alpha_2(\xi)\lambda + \alpha_3(\xi) = 0 \tag{4.7}$$

with

$$\begin{aligned} \alpha_1 &= (d_1 + d_2 + d_3)\mu_i + a\tilde{w} + \tilde{w} + 1, \\ \alpha_2 &= (d_1d_2 + d_1d_3 + d_2d_3)\mu_i^2 + ((d_1 + d_3)a\tilde{w} + (d_1 + d_2)\tilde{w} + d_2 + d_3)\mu_i + a\tilde{w}^2 + \gamma\tilde{u} + \tilde{w}, \quad (4.8) \\ \alpha_3 &= d_1d_2d_3\mu_i^3 + (d_1d_3a\tilde{w} + d_1d_2\tilde{w} + d_2d_3)\mu_i^2 + (d_1a\tilde{w}^2 + a\tilde{u}\tilde{w}\xi + d_3\gamma\tilde{u} + d_2\tilde{w})\mu_i + (ar - c)\tilde{w}. \end{aligned}$$

Obviously, $\alpha_j > 0$ $(j = 1,2,3)$ for all $i = 0,1,2,\dots,$ and

$$B(\xi) := \alpha_1\alpha_2 - \alpha_3 = \beta_1\mu_i^3 + \beta_2\mu_i^2 + (\beta_3 - a\tilde{u}\tilde{w}\xi)\mu_i + \beta_4,$$

where

$$\begin{aligned} \beta_1 &= (d_1 + d_3)(d_1 + d_2)(d_2 + d_3), \\ \beta_2 &= (d_1 + d_3)(d_1 + 2d_2 + d_3)a\tilde{w} + (d_1 + d_2)(d_1 + d_2 + 2d_3)\tilde{w} + (d_2 + d_3)(2d_1 + d_2 + d_3), \\ \beta_3 &= (d_1 + d_3)a^2\tilde{w}^2 + 2(d_1 + d_2 + d_3)a\tilde{w}^2 + (d_1 + d_2 + 2d_3)a\tilde{w} + (d_1 + d_2)(\gamma\tilde{u} + \tilde{w}^2) \\ &\quad + 2(d_1 + d_2 + d_3)\tilde{w} + d_2 + d_3, \\ \beta_4 &= a(a + 1)\tilde{w}^3 + (a^2 + 3a + 1)\tilde{w}^2 + (a + 1 - ac - ar)\tilde{w} - c. \end{aligned}$$

It is easy to see that $B(\xi)$ is monotonically decreasing with respect to ξ, that is $B(\xi) > 0$ if $\xi < \xi_0$, on the contrary $B(\xi) < 0$ if $\xi > \xi_0$, where $B(\xi_0) = 0$ with

$$\xi_0 = \frac{1}{a\tilde{u}\tilde{w}\mu_i}(\beta_1\mu_i^3 + \beta_2\mu_i^2 + \beta_3\mu_i + \beta_4) > 0 \tag{4.9}$$

thanks to $\beta_4 = f(\tilde{w}) > 0$ when $\gamma > \gamma^*$. By the Routh–Hurwitz criterion or Corollary 2.2 in [16], the proof is completed, that is $(\tilde{u}, \tilde{v}, \tilde{w})$ is locally asymptotically stable if $0 < \xi < \xi_0$ and is unstable if $\xi > \xi_0$. □

To illustrate our analysis of Lemma 4.2, we present the following numerical example.

Example 4.3. For (1.4), let $n = 1, \Omega = (0, 7)$ and set

$$a = 2, \quad c = 1, \quad r = 2, \gamma = 0.5, \quad d_1 = 0.3, \quad d_2 = 0.2, \quad d_3 = 0.3.$$

Then the equilibrium point $(\tilde{u}, \tilde{v}, \tilde{w}) = (1.2, 1.2, 0.8)$. According to the Lemma 4.2, $(\tilde{u}, \tilde{v}, \tilde{w})$ is asymptotically stable if $\xi < \xi_0 = 8.06$ $(k = 3)$, see Figure 4.1, and $(\tilde{u}, \tilde{v}, \tilde{w})$ is unstable if $\xi > \xi_0 = 8.06$ $(k = 3)$, see Figure 4.2.

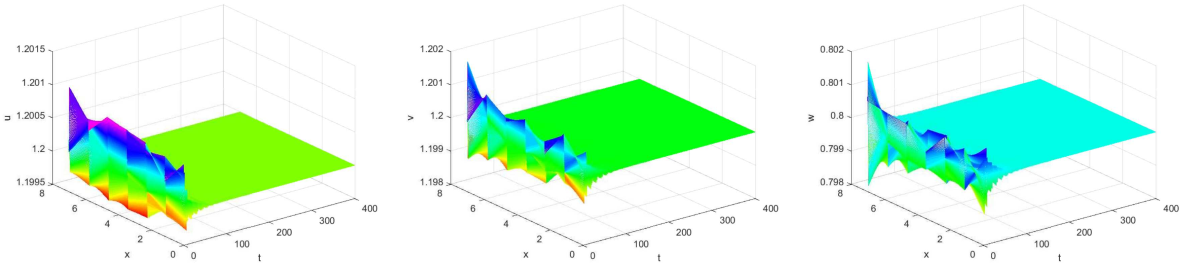

Figure 4.1: Stable behavior with $\chi = 7 < \chi_0 = 8.06$ for the model (1.4).

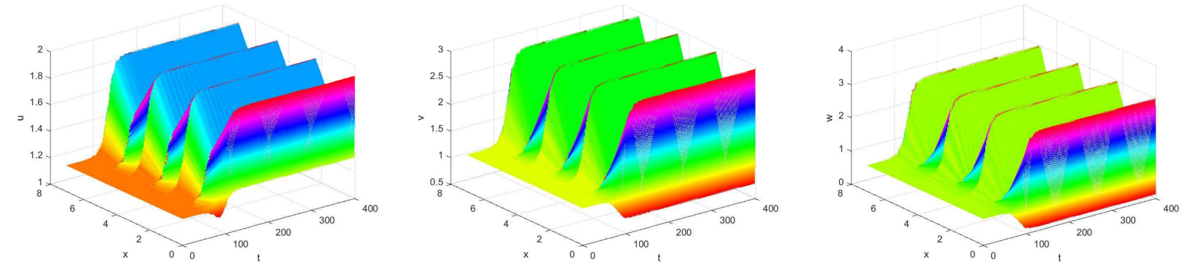

Figure 4.2: Unstable behavior with $\chi = 9 > \chi_0 = 8.06$ for the model (1.4).

Remark 4.4. Lemma 4.2 illustrates that prey-evasion has a destabilizing effect.

Remark 4.5. Lemma 4.2 implies that there is no steady state bifurcation curve near $(\tilde{u}, \tilde{v}, \tilde{w})$ since $\alpha_3 > 0$.

According to the proof of Lemma 4.2, we know that the linearized equation (4.4) has a pair of purely imaginary eigenvalues at $\xi = \xi_0$, then a Hopf bifurcation generating a family of periodic orbits of (1.4) occurs if some transversality conditions are met. We next show that the existence of periodic orbits of (1.4) for a certain parameter range.

To apply the Hopf bifurcation theorem (Theorem 6.1 of [16]), we first let the three roots of (4.6) be $\theta_{1,2} = \sigma(\xi) \pm i\delta(\xi)$ and θ_3 satisfying $\sigma(\xi_0) = 0$, $\delta(\xi_0) > 0$ when $\xi \in (\xi_0 - \varepsilon, \xi_0 + \varepsilon)$. From (4.7), we have

$$\begin{cases} -\alpha_1(\xi) = 2\sigma(\xi) + \theta_3(\xi), \\ \alpha_2(\xi) = \sigma^2(\xi) + \delta^2(\xi) + 2\sigma(\xi)\theta_3(\xi), \\ -\alpha_3(\xi) = (\sigma^2(\xi) + \delta^2(\xi))\theta_3(\xi). \end{cases} \tag{4.10}$$

Differentiating (4.10) with respect to ξ and using (4.8), we obtain

$$\begin{aligned} 2\sigma'(\xi) + \theta_3'(\xi) &= 0, \\ 2\sigma(\xi)\sigma'(\xi) + 2\delta(\xi)\delta'(\xi) + 2\sigma'(\xi)\theta_3(\xi) + 2\sigma(\xi)\theta_3'(\xi) &= 0, \\ (2\sigma(\xi)\sigma'(\xi) + 2\delta(\xi)\delta'(\xi))\theta_3(\xi) + (\sigma^2(\xi) + \delta^2(\xi))\theta_3'(\xi) &= -a\tilde{u}\tilde{w}\mu_i. \end{aligned} \tag{4.11}$$

Solving (4.11) with $\xi = \xi_0$ by Cramer's rule, we derive that

$$\theta_3'(\xi_0) = -\frac{a\tilde{u}\tilde{w}\mu_i}{\delta^2 + \theta_3^2} < 0,$$

and

$$\sigma'(\xi_0) = -\frac{1}{2}\theta_3'(\xi_0) > 0. \tag{4.12}$$

Moreover, it is easy to see that $\alpha_3 > 0$ for all $i \in \mathbb{N}$ if $\xi > 0$, then 0 cannot be an eigenvalue for (4.4) when $\xi = \xi_0$. Besides, in order to illustrate that $\theta = \pm i\delta(\xi_0)$ are a pair of simple eigenvalues of (4.4) for $\delta(\xi_0) > 0$, we need to assume that $\xi_{0k} \neq \xi_{0j}$, $j \neq k$. Then this shows that (4.4) has no eigenvalues of the form $k\delta(\xi_0)i$ for $k \in \mathbb{Z} \setminus \{\pm 1\}$.

Therefore the existence of nontrivial periodic orbits of (1.4) would be stated in the following theorem.

Theorem 4.6. *Let $ar > c$, $\gamma > \gamma^*$ and $\xi_{0k} \neq \xi_{0j}, j \neq k$. For some $i \in \mathbb{N}$, assume that μ_i is a simple eigenvalue of $-\Delta$ in Ω with Neumann boundary condition, and the corresponding eigenfunction is $y_i(x)$. Then*

i) (1.4) *has a unique one-parameter family $\{p(\tau) : 0 < \tau < \varepsilon\}$ of nontrivial periodic orbits near $(\xi, u, v, w) = (\xi_0, \tilde{u}, \tilde{v}, \tilde{w})$. More precisely, there exist $\varepsilon > 0$ and C^∞ function $\tau \mapsto (\mathbf{u}_i(\tau), T_i(\tau), \xi_i(\tau))$ from $\tau \in (-\varepsilon, \varepsilon)$ to $C^1(\mathbb{R}, X^3) \times (0, \infty, \mathbb{R})$ satisfying*

$$(\mathbf{u}_i(0), T_i(0), \xi_i(0)) = ((\tilde{u}, \tilde{v}, \tilde{w}), 2\pi/\delta_0, \xi_0),$$

and

$$\mathbf{u}_i(\tau, x, t) = (\tilde{u}, \tilde{v}, \tilde{w}) + \tau y_i(x) \left(V_i^+ e^{i\delta_0 t} + V_i^- e^{-i\delta_0 t} \right) + o(\tau), \tag{4.13}$$

where

$$\delta_0 = \sqrt{(d_1 d_2 + d_1 d_3 + d_2 d_3)\mu_i^2 + ((d_1 + d_3)a\tilde{w} + (d_1 + d_2)\tilde{w} + d_2 + d_3)\mu_i + a\tilde{w}^2 + \gamma\tilde{u} + \tilde{w}},$$

and $V_i^\pm$ is an eigenvector satisfying $\mathcal{L}_i(\xi)V_i^\pm = i\delta_0 V_i^\pm$;

ii) for $0 < |\tau| < \varepsilon$, $p(\tau) = p(\mathbf{u}_i(\tau)) = \{\mathbf{u}_i(\tau, \cdot, t) : t \in \mathbb{R}\}$ is a nontrivial periodic orbit of (1.4) *of period $T_i(\tau)$;*

iii) if $0 < \tau_1 < \tau_2 < \varepsilon$, then $p(\tau_1) \neq p(\tau_2)$;

iv) there exists $\iota > 0$ such that if (1.4) *has a nontrivial periodic solution $\bar{\mathbf{u}}(x, t)$ of period T for some $\xi \in \mathbb{R}$ with*

$$|\xi - \xi_{0i}| < \iota, \qquad |T - 2\pi/\delta_0| < \iota, \qquad \max_{t \in \mathbb{R}, x \in \bar{\Omega}} |\bar{\mathbf{u}}(x, t) - (\tilde{u}, \tilde{v}, \tilde{w})| < \iota,$$

then $\xi = \xi_0(\tau)$ and $\bar{\mathbf{u}}(x, t) = \mathbf{u}_i(\tau, x, t + \omega)$ for some $\tau \in (0, \varepsilon)$ and some $\omega \in \mathbb{R}$.

We carry out numerical simulation in one-dimension to demonstrate the analytical results of Theorem 4.6.

Example 4.7. For (1.4), let $n = 1$, $\Omega = (0, 8)$, and choose $a = 2$, $r = 2$, $c = 0.1$, $\gamma = 0.5$, $d_1 = 0.3$, $d_2 = 0.2$, $d_3 = 0.3$. Then the equilibrium point $(\tilde{u}, \tilde{v}, \tilde{w}) = (1.56, 1.56, 0.44)$. It can be calculated that Hopf bifurcation value $\xi = 5.33 (k = 3)$. This parameter set shows that the occurrence of a Hopf bifurcation at $(\tilde{u}, \tilde{v}, \tilde{w}, \xi)$, and the expression (4.13) gives the oscillation frequency, the eigenfunction $y_i(x) = \cos \frac{\pi j x}{l}$ gives the spatial profile of the oscillation, see Figure 4.3.

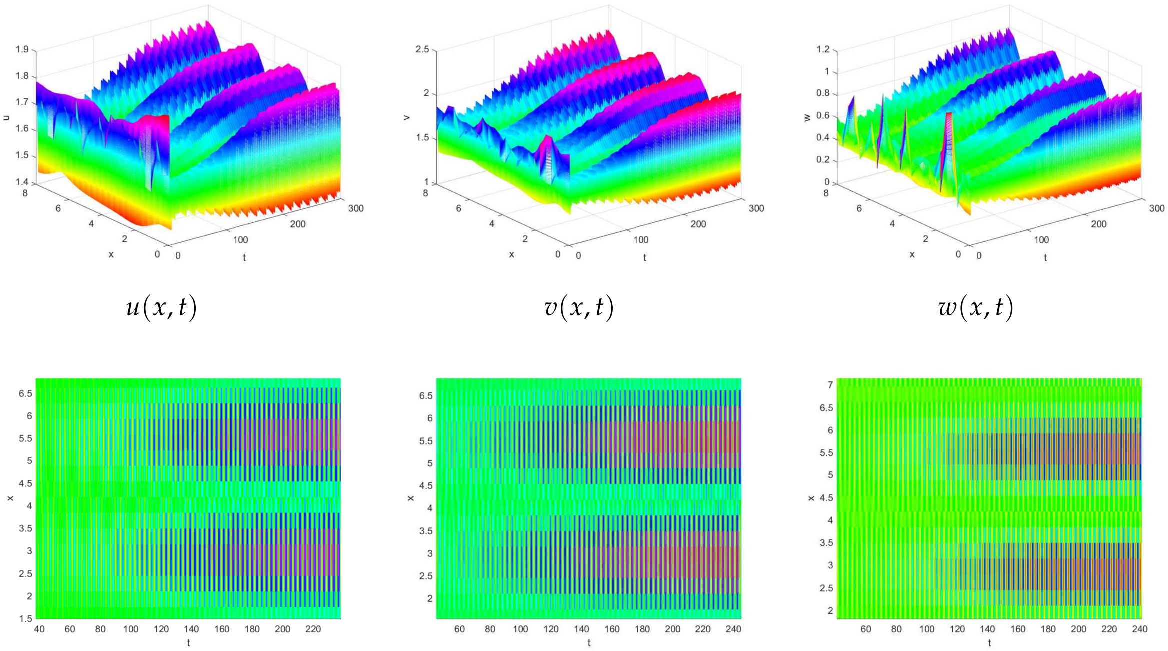

Figure 4.3: Spatiotemporal patterns of (1.4).

5 Conclusions

In this paper, a predator-prey system with both cannibalism and prey-evasion is considered. We first investigate the global existence and boundedness of the unique classical solution in 1D and 2D. The core steps are to establish some inequalities relating certain powers of the quantities

$$\sup_{s\in(0,t)} \|u(\cdot,s)\|_{L^\infty}, \qquad \sup_{s\in(0,t)} \|\nabla v(\cdot,s)\|_{L^q}, \qquad \sup_{s\in(0,t)} \|w(\cdot,s)\|_{L^p}, \qquad t\in(0,T_{\max}),$$

for suitably wide ranges of the free parameters $p \in (1,\infty]$ and $q \in (1,\infty)$ when $n \geq 2$.

Then we obtain the result that Turing instability occurs when prey-evasion sensitive coefficient ξ surpasses the threshold value ξ_0. We also show the existence of periodic orbits of (1.4) by treating prey-evasion ξ as a bifurcation parameter, which gives spatiotemporal patterns. This means that prey-evasion is the decisive factor in destabilizing positive steady state and cannibalism is no longer a stabilizing effect.

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Existence and uniqueness of solutions of a fourth-order boundary value problem with non-homogeneous boundary conditions

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Abstract. Let $m \geq 2$ and $a, b, c > 0$. We consider the existence and uniqueness of solutions for the fourth order iterative boundary value problem,

$$x^{(4)}(t) = -f(t, x(t), x^{[2]}(t), \dots, x^{[m]}(t)), \qquad -a \leq t \leq a$$

where $x^{[2]}(t) = x(x(t))$ and for $j = 3, \dots, m$, $x^{[j]}(t) = x(x^{[j-1]}(t))$, with solutions satisfying one of the following sets of conjugate boundary conditions:

$$x(-a) = -a, \qquad x'(-a) = b, \qquad x''(-a) = c, \qquad x(a) = a,$$
$$x(-a) = -a, \qquad x(a) = a, \qquad x'(a) = b, \qquad x''(a) = c.$$

The main tool used is the Schauder fixed point theorem.

Keywords: differential equations, iterative differential equations, boundary value problems.

2020 Mathematics Subject Classification: 34B15, 34K10, 39B05.

1 Introduction

In this paper we consider existence and uniqueness of solutions for the fourth-order iterative boundary value problem,

$$x^{(4)}(t) = -f(t, x(t), x^{[2]}(t), \dots, x^{[m]}(t)), \qquad -a \leq t \leq a \tag{1.1}$$

where $x^{[2]}(t) = x(x(t))$, and for $j = 3, \dots, m, x^{[j]}(t) = x(x^{[j-1]})(t)$, with solutions satisfying one of the boundary conditions:

$$x(-a) = -a, \qquad x'(-a) = b, \qquad x''(-a) = c, \qquad x(a) = a, \tag{1.2}$$
$$x(-a) = -a, \qquad x(a) = a, \qquad x'(a) = b, \qquad x''(a) = c. \tag{1.3}$$

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We assume throughout that $f : [-a, a] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Iterative differential equations are a special case of state-dependent differential equations. They have applications in a wide variety of fields, including climate change [12], economics [6], electrodynamics [4], infectious diseases [8], mechanical models [9], neural networks [2], and population dynamics [13].

One of the earliest works in iterative differential equations was by Petuhov [14] who, in 1965, studied existence and uniqueness of solutions of $x''(t) = \lambda x(x(t))$ with the condition x maps the interval $[-T, T]$ into itself, and that $x(0) = x(T) = \alpha$. Eder [5] then studied the existence, uniqueness, and analyticity of solutions of $x'(t) = x(x(t))$, proving that every solution is either monotonic or vanishes. In 1990, Wang [16] obtained a solution to $x'(t) = f(x(x(t))), x(a) = a$ using Schauder's fixed point theorem, and in 1993 Fečkan [7] used the Contraction Mapping Principle to show existence of local solutions of $x'(t) = f(x(x(t)), x(0) = 0$.

More recently, Kaufmann [10] established existence and uniqueness results for the second-order boundary value problem $x''(t) = f(t, x(t), x^{[m]}(t)), x(a) = a,\ x(b) = b$ using Schauder's fixed point theorem and the Contraction Mapping Principle. In 2020, Cheraiet, Bouakkaz, and Khemis [3] studied the third-order equation $x''(t) + f(t, x(t), x^{[2]}(t) \dots, x^{[n]}(t)) = 0$ with conditions $x(0) = x''(0) = 0, \alpha \int_0^{\eta} x(t)\, dt = x(T)$ with $\eta \in (0, T)$. Meanwhile, in 2022, Kaufmann [11] considered the fourth-order equation $x^{(4)}(t) = f(t, x(t), x^{[2]}(t), \dots, x^{[m]}(t))$ subject to the Lidstone conditions $x(a) = x(-a) = x''(a) = x''(-a) = 0$, establishing conditions for existence and uniqueness of solutions. The main goal of this paper is to further the results of [11].

In Section 2, we will rewrite (1.1), (1.2) as an integral equation, and state conditions under which the solution of the integral equation will be a solution of the boundary value problem. We will also state properties of the Green's function and of the norm of the difference of two iterative functions. In Section 3, we will state and prove results concerning the existence and uniqueness of solutions of (1.1), (1.2). In Section 4, we present the equivalent inversion of (1.1), (1.3) and state, without proof, the analogous existence and uniqueness results. Examples will be included to illustrate results.

2 Preliminaries

Our main goal of Section 2 is to invert (1.1), (1.2) into an integral equation. We will accomplish this by first inverting the non-homogeneous equation with homogeneous boundary conditions, and then solving the homogeneous equation with non-homogeneous boundary conditions. The inversion of (1.1), (1.2) will be the sum of the two expressions. We will end the section with a lemma on the Green's function and the norm of the difference of iterations, and then a statement of Schauder's fixed point theorem.

We will begin the inversion by considering

$$x^{(4)}(t) = -g(t), \quad -a \le t \le a, \tag{2.1}$$

$$x(-a) = x'(-a) = x''(-a) = x(a) = 0. \tag{2.2}$$

Integrate $x^{(4)}(t) = -g(t))$ from $-a$ to t twice and apply the boundary condition $x''(-a) = 0$.

$$x''(t) = x'''(-a)(t + a) - \int_{-a}^{t} (t - s) g(s)\, ds. \tag{2.3}$$

Integrating (2.3) and applying the condition $x'(-a) = 0$ yields,

$$x'(t) = x'''(-a)\frac{(t+a)^2}{2} - \int_{-a}^{t} \frac{(t-s)^2}{2} g(s)\,ds.$$

When we integrate once more and apply the condition $x(-a) = 0$, we obtain,

$$x(t) = x'''(-a)\frac{(t+a)^3}{6} - \int_{-a}^{t} \frac{(t-s)^3}{6} g(s)\,ds. \tag{2.4}$$

The constant $x'''(-a)$ is found by applying the condition $x(a) = 0$,

$$x'''(-a) = \int_{-a}^{a} \frac{(a-s)^3}{8a^3} g(s)\,ds. \tag{2.5}$$

When we plug (2.5) into (2.4) we get,

$$x(t) = \int_{-a}^{a} \frac{(a-s)^3(t+a)^3}{48a^3} g(s)\,ds - \int_{-a}^{t} \frac{(t-s)^3}{6} g(s)\,ds.$$

Finally, we can split the first integral and combine it with the second to obtain

$$x(t) = \int_{-a}^{t} \frac{(a-s)^3(t+a)^3 - 8a^3(t-s)^3}{48a^3} g(s)\,ds + \int_{t}^{a} \frac{(a-s)^3(t+a)^3}{48a^3} g(s)\,ds.$$

Thus, we have shown that if x is a solution to (2.1), (2.2), then x satisfies the integral equation

$$x(t) = \int_{-a}^{a} G(t,s)g(s)\,ds \tag{2.6}$$

where

$$G(t,s) = \frac{1}{48a^3}\begin{cases} (a-s)^3(t+a)^3 - 8a^3(t-s)^3, & -a \le s \le t \le a, \\ (a-s)^3(t+a)^3, & -a \le t \le s \le a. \end{cases} \tag{2.7}$$

It is easy to show that if x is a solution of

$$x^{(4)}(t) = 0,$$
$$x(-a) = -a, \quad x'(-a) = b, \quad x''(-a) = c, \quad x(a) = a,$$

then x is given by

$$x(t) = -a + b(t+a) + \frac{c}{2}(t+a)^2 + \frac{1-b-ac}{4a^2}(t+a)^3. \tag{2.8}$$

Consequently, if x is a solution of (1.1), (1.2), then x will then be the sum of (2.7) and (2.8). That is, x is a solution of the integral equation

$$\begin{aligned} x(t) = {} & \int_{-a}^{a} G(t,s) f(s, x(s), x^{[2]}(s), \ldots, x^{[m]}(s))\,ds \\ & - a + b(t+a) + \frac{c}{2}(t+a)^2 + \frac{1-b-ac}{4a^2}(t+a)^3, \end{aligned}$$

where $G(t,s)$ is given in (2.7).

In order for solutions to be well-defined, we also require the image of x be in the interval $[-a,a]$; that is, in order for $x(x^{[m]})(t)$ to be defined, we need $-a \le x(t) \le a$ for all $t \in [-a,a]$. Knowing this, we can show that if $x \in C[-a,a]$, satisfies $-a \le x(t) \le a$ for all t, and satisfies the integral equation (**??**), then it satisfies (1.1), (1.2). This gives us the following lemma.

Lemma 2.1. *The function $x \in C^4[-a,a]$ is a solution of* (1.1), (1.2) *if and only if $x \in C[-a,a]$ satisfies $-a \le x(t) \le a$, and the integral equation*

$$x(t) = \int_{-a}^{a} G(t,s) f(s, x(s), x^{[2]}(s), \ldots, x^{[m]}(s))\, ds \\ - a + b(t+a) + \frac{c}{2}(t+a)^2 + \frac{1-b-ac}{4a^2}(t+a)^3,$$

where $G(t,s)$ is defined in (2.7).

In order to prove the existence and uniqueness of solutions of (1.1), (1.2), we will need to have a bound for our Green's functions. For that, we will use the following lemma.

Lemma 2.2. *The Green's function given in* (2.7) *satisfies the following inequality:*

$$0 \le G(t,s) \le \frac{4a^3}{3}.$$

Proof. First note that $(a-s)^3(t+a)^3$ is an increasing function of t, so $(a-s)^3(t+a)^3 \le (a-s)^3(s+a)^3$. Since $\max_{s\in[-a,a]}(a-s)^3(s+a)^3$ occurs when $s=0$, and equals a^6, then $\frac{(a-s)^3(t+a)^3}{48a^3} \le \frac{a^3}{48}$.

Now consider the function $(a-s)^3(t+a)^3 - 8a^3(t-s)^3, s \le t$. Since $(t-s)^3$ is positive when $s \le t$, then $(a-s)^3(t+a)^3 - 8a^3(t-s)^3 \le (a-s)^3(t+a)^3$. Now, $(a-s)^3(t+a)^3$ is an increasing function of t, so $(a-s)^3(t+a)^3 \le 8a^3(a-s)^3$. But, $8a^3(a-s)^3$ is a decreasing function of s for $-a \le s$, so $8a^3(a-s)^3 \le 64a^6$. That is, $\frac{(a-s)^3(t+a)^3 - 8a^3(t-s)^3}{48a^3} \le \frac{64a^6}{48a^3} = \frac{4a^3}{3}$. Finally, since $\frac{a^3}{48} < \frac{4a^3}{3}$, we obtain that the upper bound on our Green's function is $\frac{4a^3}{3}$.

Similar procedures can be used to obtain the lower bound on our Green's function, that $0 \le G(t,s)$. □

We will use the Banach space $\Phi = (C[-a,a], \|\cdot\|)$ with the norm $\|x\| = \max_{t\in[-a,a]}|x(t)|$. Define the operator $T : C[-a,a] \to C[-a,a]$ by

$$(Tx)(t) = \int_{-a}^{a} G(t,s) f(s, x(s), x^{[2]}(s), \ldots, x^{[m]}(s))\, ds \\ - a + b(t+a) + \frac{c}{2}(t+a)^2 + \frac{1-b-ac}{4a^2}(t+a)^3 \tag{2.9}$$

where $G(t,s)$ is defined in (2.7).

We will also need the subspace

$$\Phi(J,M) = \{x \in \Phi : \|x\| \le J, |x(t_2) - x(t_1)| \le M|t_1 - t_2|, t_1, t_2 \in [-a,a]\}.$$

as well as the following lemma, which is proved in [17], [15].

Lemma 2.3. *If $x, y \in \Phi(J,M)$, then*

$$|x^{[m]}(t_1) - x^{[m]}(t_2)| \le M^m |t_1 - t_2|, \qquad m = 0,1,2,\ldots,$$

for all $t_1, t_2 \in [-a,a]$ and

$$\|x^{[m]}(t_1) - x^{[m]}(t_2)\| \le \sum_{k=0}^{m-1} M^k \|x-y\|.$$

We end this section by stating Schauder's fixed point theorem [1].

Theorem 2.4 (Schauder). *Let A be a nonempty compact convex subset of a Banach space and let $T : A \to A$ be continuous. Then T has a fixed point in A.*

3 Existence and uniqueness results for (1.1), (1.2)

In this section, we will state and prove our existence and uniqueness results for (1.1), (1.2). Let $T : C[-a,a] \to C[-a,a]$ be defined as (2.9). Throughout the section we will assume the following conditions hold.

(H1) There exists an $\alpha_\ell \in L[-a,a], \ell = 1,2,\dots,m+1$, such that

$$|f(t,x_1,\dots,x_{m+1}) - f(t,y_1,\dots,y_{m+1})| \le \sum_{\ell=1}^{m+1} \alpha_\ell(t)\|x_\ell - y_\ell\|$$

for all $t \in [-a,a]$ and $x_i, y_i \in \mathbb{R}, i = 1,2,\dots,m+1$.

(H2) There exists a $K \in \mathbb{R}$ such that $0 < K < \frac{3(1-b-ac)}{a^3}$ and

$$-K \le f\big(t, x(t), x^{[2]}(t), \dots, x^{[m]}(t)\big) < 0$$

for all $t \in [-a,a]$.

Notice that $(H2)$ puts further conditions on b and c, namely that $1 > b + ac > 0$.

Theorem 3.1. *Suppose that condition (H1) and (H2) holds. Then there exists a solution to* (1.1), (1.2).

Proof. Consider the convex, compact nonempty set $\Phi(a,M)$, where

$$M = |3 - 2b - ac| + K\left(6a^3 + \frac{1}{18}\right).$$

To use the Schauder fixed point theorem, we need for $T : \Phi(a,M) \to \Phi(a,M)$. We first show that $-a \le (Tx)(t) \le a$ for all $t \in [-a,a]$.

$$\begin{aligned}
(Tx)'(t) &= \frac{1}{16a^3}\int_{-a}^{a}(a-s)^3(t+a)^2 f(s)\,ds - \frac{1}{2}\int_{-a}^{a}(t-s)^2 f(s)\,ds \\
&\quad + \frac{3(1-b-ac)}{4a^2}(t+a)^2 + c(t+a) + b \\
&\ge \frac{-K}{16a^3}\int_{-a}^{a}(a-s)^3(t+a)^2\,ds + \frac{3(1-b-ac)}{4a^2}(t+a)^2 + c(t+a) + b \\
&\ge \frac{-Ka}{4}(t+a)^2 + \frac{3(1-b-ac)}{4a^2}(t+a)^2 + c(t+a) + b \\
&= \frac{-Ka^3 + 3(1-b-ac)}{4a^2}(t+a)^2 + c(t+a) + b.
\end{aligned}$$

Since $(H2)$ holds, then

$$\frac{-Ka^3 + 3(1-b-ac)}{4a^2}(t+a)^2 + c(t+a) + b > 0$$

for all $t \in [-a,a]$. That is, $(Tx)'(t) > 0$ and hence $(Tx)(t)$ is a strictly increasing function of t. Since $(Tx)(\pm a) = \pm a$, then $-a \le (Tx)(t) \le a$ and furthermore $\|Tx\| \le a$.

We need to show that for given $t_1, t_2 \in [-a, a]$, $\big|(Tx)(t_2) - (Tx)(t_1)\big| \leq M|t_2 - t_1|$, where M is defined as above. We may assume, without loss of generality, that $t_2 \leq t_1$. To this end we first note that

$$\begin{aligned}
|(Tx)(t_2) - (Tx)(t_1)| = & \int_{-a}^{a} \big|G(t_2,s) - G(t_1,s)\big| |f(s, x(s), x^{[2]}(s), \dots, x^{[m]}(s)|\, ds \\
& + b\big((t_2+a) - (t_1+a)\big) + \frac{c}{2}\big((t_2+a)^2 - (t_1+a)^2\big) \\
& + \frac{1-b-ac}{4a^2}\big((t_2+a)^3 - (t_1+a)^3\big) \\
\leq & \; K \int_{-a}^{a} \big|G(t_2,s) - G(t_1,s)\big|\, ds \\
& + b|t_2 - t_1| + 2ac|t_2 - t_1| + \frac{12a^2(1-b-ac)}{4a^2}|t_2 - t_1| \\
\leq & \; K \int_{-a}^{a} |G(t_2,s) - G(t_1,s)|\, ds + (3 - 2b - ac)|t_2 - t_1|.
\end{aligned}$$

Now consider $\int_{-a}^{a} |G(t_2,s) - G(t_1,s)|\, ds$. Since $t_2 \leq t_1$, we can rewrite the integral as

$$\begin{aligned}
\int_{-a}^{a} \big|G(t_2,s) - G(t_1,s)\big|\, ds \leq & \int_{-a}^{t_1} \big|G(t_2,s) - G(t_1,s)\big|\, ds + \int_{t_1}^{t_2} \big|G(t_2,s) - G(t_1,s)\big|\, ds \\
& + \int_{t_2}^{a} \big|G(t_2,s) - G(t_1,s)\big|\, ds.
\end{aligned}$$

Given that $t_1 \leq t_2$, the first term on the right satisfies

$$\begin{aligned}
& \int_{-a}^{t_1} |G(t_2,s) - G(t_1,s)|\, ds \\
& \leq \frac{1}{48a^3} \int_{-a}^{t_1} \big|(a-s)^3\big((t_2+a)^3 - (t_1+a)^3\big)\big| + 8a^3\big|(t_2-s)^3 - (t_1-s)^3\big|\, ds \\
& \leq \frac{1}{48a^3}\left(\left(4a^4 - \frac{(a-t_1)^4}{4}\right)(12a^2)\right)|t_2 - t_1| + \frac{1}{48a^3}(192a^6)|t_2 - t_1| \\
& \leq 5a^3|t_2 - t_1|.
\end{aligned}$$

Also, due to the bound on our Green's function,

$$\begin{aligned}
\int_{t_1}^{t_2} \big|G(t_2,s) - G(t_1,s)\big|\, ds \leq & \frac{1}{48a^3}\left(\frac{8a^3}{3}\right)|t_2 - t_1| \\
\leq & \frac{1}{18}|t_2 - t_1|.
\end{aligned}$$

And finally,

$$\begin{aligned}
\int_{t_2}^{a} |G(t_2,s) - G(t_1,s)|\, ds \leq & \frac{1}{48a^3} \int_{t_2}^{a} \big|(a-s)^3\big((t_2+a)^3 - (t_1+a)^3\big)\big|\, ds \\
\leq & \frac{1}{48a^3}\left(\frac{(a-t_2)^4}{4}(12a^2)\right)|t_2 - t_1| \\
\leq & \; a^3|t_2 - t_1|.
\end{aligned}$$

That is,

$$\int_{-a}^{a} |G(t_2,s) - G(t_1,s)|\, ds \leq \left(6a^3 + \frac{1}{18}\right)|t_2 - t_1|.$$

Consequently,

$$|(Tx)(t_2) - (Tx)(t_1)| \leq \left(|3 - 2b - ac| + K\left(6a^3 + \frac{1}{18}\right)\right)|t_2 - t_1|$$
$$= M|t_2 - t_1|.$$

Therefore, $T : \Phi(a, M) \to \Phi(a, M)$.

Lastly, it can be shown through standard arguments that T is continuous. Hence, by Schauder's fixed point theorem, there is a fixed point x of T, $(Tx)(t) = x(t)$, which by Lemma 2.1 is a solution of (1.1), (1.2). □

Example 3.2. *Consider the following boundary value problem with parameter k.*

$$x^{(4)}(t) = kt^2 \cos(x^{[2]}(t)) \tag{3.1}$$
$$x\left(-\frac{\pi}{3}\right) = -\frac{\pi}{3}, \quad x\left(\frac{\pi}{3}\right) = \frac{\pi}{3}, \tag{3.2}$$
$$x'\left(-\frac{\pi}{3}\right) = \frac{2}{3}, \quad x''\left(-\frac{\pi}{3}\right) = \frac{1}{\pi^2}. \tag{3.3}$$

Here, $m = 2$ and $f(t, x, x^{[2]}) = -kt^2 \cos(x^{[2]})$. Let $\alpha(t) = kt^2$. Then,

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq \alpha(t)|x_2 - y_2| \tag{3.4}$$

for all $t \in [-\frac{\pi}{3}, \frac{\pi}{3}]$. Also, $-k \leq f(t, x, x^{[2]}) \leq 0$. So, for all $0 < k < \frac{3(1-b-ac)}{a^3} = \frac{27(\pi-1)}{\pi^4} \approx 0.5936099$, there exists a solution to (3.1), (3.2), (3.3), according to Theorem 3.1.

We are now ready for our uniqueness result.

Theorem 3.3. *Suppose that $(H1)$ and $(H2)$ hold and that*

$$\frac{4a^3}{3} \sum_{\ell=1}^{m+1} \int_{-a}^{a} \alpha_\ell(s)\, ds \sum_{k=0}^{\ell-1} M^k < 1. \tag{3.5}$$

Then, there exists a unique solution to (1.1), (1.2).

Proof. By Theorem 3.1 and Lemma 2.1, there exists a solution of (1.1), (1.2), which is a fixed point of T. Assume x and y are two distinct fixed points of T. Then,

$$\begin{aligned} \|x - y\| &= |(Tx)(t) - (Ty)(t)| \\ &\leq \frac{4a^3}{3} \int_{-a}^{a} \sum_{\ell=1}^{m+1} \alpha_\ell(s) \|x^{[\ell]} - y^{[\ell]}\|\, ds \\ &\leq \left(\frac{4a^3}{3} \int_{-a}^{a} \sum_{\ell=1}^{m+1} \alpha_\ell(s) \sum_{k=0}^{l-1} M^k\, ds\right) \|x - y\| \\ &< \|x - y\| \end{aligned}$$

This contradiction implies $x = y$, and our fixed point is unique. □

It should be noted that the results in Theorem 3.3 can also be obtained using the Banach fixed point theorem.

Example 3.4. To illustrate our uniqueness result, again consider the boundary value problem (3.1), (3.2), (3.3). Again, note that

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le kt^2|x_2 - y_2|$$

for all $t \in [-\frac{\pi}{3}, \frac{\pi}{3}]$. So, $\alpha_0(t) = \alpha_1(t) = 0$ and $\alpha_2(t) = kt^2$. The left side of (3.5) becomes

$$\begin{aligned} \frac{4a^3}{3}\sum_{\ell=1}^{m+1}\int_{-a}^{a}\alpha_\ell(s)\,ds\sum_{k=0}^{\ell-1}M^k &= \frac{4(\frac{\pi}{3})^3}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} ks^2\,ds\,(1+M+M^2) \\ &= \frac{4\pi^3}{81}\frac{2k\pi^3}{81}\left(1+M+M^2\right) \\ &= \frac{8\pi^6}{6561}\left(1+M+M^2\right)k. \end{aligned}$$

In our case, $M < 5.8958889$. So, whenever $k < \frac{6561}{8\pi^6(1+M+M^2)} \approx 0.020478$, there exists a unique solution to (3.1), (3.2), (3.3) according to Theorem 3.3.

4 Other results

In this section, we give the corresponding results from Section 3 for (1.1), (1.3). The proof of the results in this section are similar to those found in Section 3. As such, we only point out the main differences. We begin by considering the boundary value problem (1.1), (1.3).

As in Section 2, we can show that if x is a solution of (1.1), (1.3), then $x(t)$ satisfies the integral equation

$$\begin{aligned} x(t) = &\int_{-a}^{a} G(t,s) f(s, x(s), x^{[2]}(s), \dots, x^{[m]}(s))\,ds \\ &+ a - b(a-t) + \frac{c}{2}(a-t)^2 - \frac{1-b+ac}{4a^2}(a-t)^3, \end{aligned} \tag{4.1}$$

where

$$G(t,s) = \frac{1}{48a^3}\begin{cases} (a-t)^3(s+a)^3, & -a \le s \le t \le a, \\ (a-t)^3(s+a)^3 - 8a^3(s-t)^3, & -a \le t \le s \le a. \end{cases} \tag{4.2}$$

The Green's function $G(t,s)$ in (4.2) satisfies Lemma 2.2.

In addition to $(H1)$, we will need the following condition.

(H3) There exists an $L \in \mathbb{R}$ such that $0 < L < \frac{3(1-b+ac)}{a^3}$ and

$$0 < f(t, x(t), x^{[2]}(t), \dots, x^{[m]}(t)) < L$$

for all $t \in [-a, a]$.

Notice that $(H3)$ puts further conditions on b and c, namely that $1 + ac > b > 0$.

Theorem 4.1. *Suppose that conditions $(H1)$ and $(H3)$ hold. Then there exists a solution to (1.1), (1.3).*

Proof. For this proof, the space $\Phi(a, M)$ where $M = |b| + L\left(\frac{1}{18}\right)$ is needed. The rest of the proof follows the sames steps as Theorem 3.1. □

Example 4.2. *Consider the following boundary value problem with parameter k.*

$$x^{(4)}(t) = -kt^2 \cos(x^{[2]}(t)) \tag{4.3}$$

$$x\left(-\frac{\pi}{3}\right) = -\frac{\pi}{3}, \quad x\left(\frac{\pi}{3}\right) = \frac{\pi}{3}, \tag{4.4}$$

$$x'\left(\frac{\pi}{3}\right) = \frac{1}{2}, \quad x''\left(\frac{\pi}{3}\right) = \frac{1}{\pi^2}. \tag{4.5}$$

Here, $m = 2$ *and* $f(t, x, x^{[2]}) = kt^2 \cos(x^{[2]})$. *Let* $\alpha(t) = kt^2$. *Then,*

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq \alpha(t)|x_2 - y_2|$$

for all $t \in [-\frac{\pi}{3}, \frac{\pi}{3}]$. *Also,* $0 \leq f(t, x, x^{[2]}) \leq L$. *So, for all* $0 < k < \frac{3(1-b+ac)}{a^3} = \frac{243\pi+162}{6\pi^4} \approx 1.583369$, *there exists a solution to* (4.3), (4.4), (4.5), *according to Theorem* 4.1.

Theorem 4.3. *Suppose that* (H1) *and* (H3) *hold and that*

$$\frac{4a^3}{3} \sum_{\ell=1}^{m+1} \int_{-a}^{a} \alpha_\ell(s)\, ds \sum_{k=0}^{l-1} M^k < 1. \tag{4.6}$$

Then, there exists a unique solution to (1.1), (1.3).

Example 4.4. *To illustrate our uniqueness result, again consider the boundary value problem* (4.3), (4.4), (4.5). *Again, note that*

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq kt^2|x_2 - y_2|$$

for all $t \in [-\frac{\pi}{3}, \frac{\pi}{3}]$. *So,* $\alpha_0(t) = \alpha_1(t) = 0$ *and* $\alpha_2(t) = kt^2$. *The left side of* (4.6) *becomes*

$$\begin{aligned}\frac{4a^3}{3} \sum_{\ell=1}^{m+1} \int_{-a}^{a} \alpha_\ell(s)\, ds \sum_{k=0}^{\ell-1} M^k &= \frac{4(\frac{\pi}{3})^3}{3} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} ks^2\, ds\, (1 + M + M^2) \\ &= \frac{4\pi^3}{81} \frac{2k\pi^3}{81} (1 + M + M^2) \\ &= \frac{8\pi^6}{6561} (1 + M + M^2)\, k.\end{aligned}$$

In this example, $M < .5879649$. *So, whenever* $k < \frac{6561}{8\pi^6(1+M+M^2)} \approx 0.4411629$, *there exists a unique solution to* (4.3), (4.4), (4.5) *according to Theorem* 4.3.

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